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Localizable Operators

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Contents

1	Preliminaries	13
1.1	Nets	13
1.1.1	Definition of a net	13
1.1.2	Nets in a locally convex Hausdorff space	15
1.1.3	Cauchy nets	17
1.1.4	Closure and sequential closure of a set	21
1.2	Continuous linear operators	22
1.2.1	Projections	27
1.2.2	Isometries	27
1.2.3	Completion of $\mathcal{T}$	28
1.3	Metrization	30
1.4	Infinite sums	36
1.5	Weighted ℓ^p spaces	40
1.5.1	Synthesis operators	44
1.5.2	Analysis operators	46
1.5.3	Weak-* topologies	47
1.6	Infinite matrices	50
1.6.1	Formulas for specific norms	54
1.6.2	The adjoint	57
1.7	Banach algebras	60
1.7.1	Inversion	61
1.7.2	The spectrum	63
1.8	Operators on normed spaces	65
1.8.1	The inverse operator	66
1.8.2	The Adjoint	68
1.8.3	Projections	70
1.8.4	Pseudo inverse operator	70
2	Frames	81
2.1	Riesz bases	85
2.2	Applying operators to frames	88
2.3	Banach frames	89
3	Localized Frames	91
4	$\mathcal{H}_w^p$ space	95
4.1	The Gramian matrix	95
4.2	The construction of the co-orbit space	96
4.2.1	The case $p < \infty$	97
4.2.2	The case $p = \infty$	98

4.3	Continuity in $\mathcal{H}_w^p$	98
4.4	Intersection of $\mathcal{H}$ and $\mathcal{H}_w^p$	99
4.5	Other coefficient operators	100
4.6	The synthesis operator	101
4.6.1	The special synthesis operator	101
4.6.2	Other synthesis operators	103
4.7	Completeness of $\mathcal{H}_w^p$	106
4.8	Riesz bases	106
5	Duality of $\mathcal{H}_w^p$	109
5.1	Banach frames on $\mathcal{H}_w^p$	113
6	Localization of operators	115
6.1	Matrix representation of operators	115
6.2	Localization	119
6.3	Induced norm	121
6.4	Localization on Banach spaces	124
6.5	Connection to localized frames	125

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Abstract

In dieser Arbeit geht es um Resultate zur Lokalisierung von Operatoren in Hilberträumen, wobei eine allgemeine, solide Spektralmatrixalgebra verwendet wird. Genauer gesagt, ich analysiere die Struktur der Menge der lokalisierbaren Operatoren bezüglich zweier fixierter Frames. Einige dieser Resultate wurden bereits für spezielle solide Spektralmatrixalgebren bewiesen. Das heißt, ich verallgemeinere in dieser Arbeit existierende Resultate. Zusätzlich erkläre ich detailliert die notwendigen Informationen über Topologie und Funktionalanalysis und beschreibe die relevanten Schritte für die Konstruktion und die Eigenschaften des Koorbitraums $\mathcal{H}_w^p$.

Abstract

This thesis provides some results on the localization of operators in Hilbert spaces using a general solid spectral matrix algebra. More precisely, I analyze the structure of the set of localizable operators with respect to two fixed frames. Many of those results have already been proved for the case of specific solid spectral algebras, which means that in this thesis, I generalize some existing results. In addition, I provide a detailed explanation of the necessary knowledge in topology and functional analysis, and review the relevant construction steps and properties of the co-orbit spaces $\mathcal{H}_w^p$.

Introduction

In signal processing research, it is often useful to decompose a signal into elementary parts that can be analyzed individually. A common way to do it with a periodic signal f is to use its corresponding Fourier series

$$f = \sum_{k \in \mathbb{Z}} c_k f_k,$$

where $(c_k)_{k \in \mathbb{Z}}$ are the Fourier coefficients and $(f_k)_{k \in \mathbb{Z}} = (e^{-2\pi i k(\cdot)})_{k \in \mathbb{Z}}$ are harmonic signals. If f is an L^2 -function, then this decomposition exists and is unique. Moreover, $(f_k)_{k \in \mathbb{Z}}$ is an orthonormal basis. Although the existence of such a decomposition is essential, uniqueness can be a disadvantage. To better understand this argument, one must be aware of real-world limitations. In practice, only finite sums can be computed, which means that one must approximate the infinite sum with a finite sum by dropping all summands with a sufficiently small coefficient in magnitude. Depending on the coefficients, this implies that the approximating sum can have many summands. Additionally, owing to numerical limitations, one must ensure that the approximation is sufficiently stable. In some cases, having multiple ways to decompose an element makes it possible to choose a decomposition that best suits the computational effort and stability. This is where frames come into play. A frame $(f_k)_{k \in X}$ can be considered a sequence with certain properties, which allows a (not necessarily unique) decomposition of an element. Frames like Gabor Frames have been shown to be useful. They also have nice mathematical properties.

In the next chapter, we define a matrix algebra $\mathcal{A}$, which intuitively consists of matrices with a certain off-diagonal decay. In some literature, this algebra is explicitly defined; however, I define $\mathcal{A}$ using axioms to cover more cases. Subsequently, a frame can be defined to be localized if its corresponding Gramian matrix is an element of $\mathcal{A}$. Intuitively, a localized frame is a frame in which the elements whose indices are far apart are "almost orthogonal" to each other. This allows for more stable frame reconstruction. More generally, two frames are localized if their cross Gramian matrix is in $\mathcal{A}$.

Additionally, a localized frame allows us to define co-orbit spaces. This is similar to the following: If we compute the Fourier coefficient of an L^2 -element, we obtain an ℓ^2 sequence. Focusing on functions whose Fourier coefficient sequence is in ℓ^1 makes it easier to compute an approximate decomposition, as the Fourier coefficients decay faster, which reduces the number of significant summands. However, this drastically reduces the function space. If we choose instead $p > 2$, then we can extend the space of functions to the space, where the sequence of Fourier coefficients is in ℓ^p . This extension results in the unfortunate fact that the Fourier coefficient sequence decays more slowly, which increases the computation time. This entire idea can be transferred to the general setting of frames. To make the math more general, we use weighted ℓ_w^p spaces.

We also define the notion of a localizable operator. If ψ, ϕ are two frames, we say

that T is (ψ, ϕ) -localizable if the sequences $\psi, T\phi$ are localizable. Intuitively, if we think of the frames ψ, ϕ as coordinate systems, then T is (ψ, ϕ) -localizable if it maps ϕ to a coordinate system close to ψ . This raises the question of the structure of the set of localizable operators. This thesis provides a detailed answer to this question.

The chapters provide the following information: In the first chapter, we discuss the relevant results of the functional analysis. Some of these are well known, while others may be new to the reader. To provide information as gap-free as possible, this chapter contains proofs for most of the results. In Chapter 2, we discuss the basics of Frame Theory. Because the content can be found in [18], I do not provide many proofs. In the third chapter, we define the localization of a frame. The fourth chapter introduces the co-orbit spaces. We also discuss their properties in detail. In the fifth chapter, we show a duality result similar to the case of the ℓ_w^p spaces. In the final chapter, we discuss the localizability of operators in detail.

1 Preliminaries

Since Frame Theory is a branch of functional analysis, this chapter provides some background information. The idea is that it contains detailed information, while the other chapters use those results. This makes it possible to obtain the relevant information from the other chapters more clearly, as they do not contain many details. If the reader is not interested in the details, they can skip this chapter or read parts of it while working through the other chapters to fill the gaps.

This chapter begins with an introduction to nets and topology. In the second part, continuous linear functions are introduced. In the third section, we show that nets can be replaced by sequences in certain cases. Subsequently, we rigorously define infinite sums in locally convex Hausdorff spaces. The fifth part introduces weighted ℓ^p -spaces. Subsequently, the fundamentals of Banach algebras are introduced. Finally, operators in normed spaces are analyzed.

For every result and in this chapter, I searched for literature to cite it. However, for some results, I could not find any literature. In this case, I do not claim to be the first one to prove these results.

1.1 Nets

Sequences in metric spaces are useful for characterizing closed sets and continuous functions. However, if we try to extend this characterization to more general topological spaces, we must differentiate between continuous and sequentially continuous functions. Because there are sequentially continuous functions that are not continuous, sequences are not optimal in this setting. To solve this problem, we need to introduce nets, which are generalizations of sequences. An introduction to nets in topological spaces can be found in [1, Chapter 5.12]

1.1.1 Definition of a net

The index set of a sequence is $\mathbb{N}$ (with 0) or $\mathbb{N}^*$ (without 0), which is totally ordered with respect to $\leq$. For a net, we need a directed set, which is a set that satisfies certain properties.

Definition. [41, Definition 1.1.7] Let I be a nonempty set, and let $\leq$ be a relation on I . We call $(I, \leq)$ (or I) a *directed set* if the following statements are true:

1. Reflexivity: For all $i \in I$, we have

$$i \leq i.$$

2. Transitivity: For all $i, j, k \in I$, we have

$$(i \leq j) \wedge (j \leq k) \Rightarrow i \leq k.$$

3. For all $i, j \in I$, there exists a $k \in I$ such that

$$i \leq k \wedge j \leq k.$$

To make the definition simpler to understand, we present some simple examples.

Example. Many examples are easily verified.

1. $(\mathbb{N}, \leq)$ is a directed set. Indeed, we already know that $\leq$ is reflexive and transitive. In addition, for $i, j \in \mathbb{N}$, we choose $k := \max(i, j)$, which satisfies $i \leq k$ and $j \leq k$. Therefore, $(\mathbb{N}, \leq)$ is a directed set.
2. Similarly, one can show that $(\mathbb{N}^*, \leq)$, $(\mathbb{N}, \geq)$ and $(\mathbb{R}, \leq)$ are directed sets.
3. Let M be a nonempty set and P its power set. Then, $(P, \subseteq)$ is a directed set. Indeed, $\subseteq$ is reflexive and transitive. For $A, B \in P$, we choose $C := A \cup B$ to obtain $A \subseteq C$ and $B \subseteq C$, showing that $(P, \subseteq)$ is directed.
4. Let M be a nonempty set and P_F be the set of finite subsets of M . Then, $(P_F, \subseteq)$ is a directed set. This follows from the fact that the union of two finite sets is finite.
5. Let S be a set and F be the set of functions from S to $\mathbb{R}$. For $f, g \in F$, we define the relation

$$f \leq g \Leftrightarrow f(x) \leq g(x) \quad \forall x \in S.$$

Again, reflexivity and transitivity are easily verified. For the third property, let

$$h : S \rightarrow \mathbb{R}, x \mapsto \max(f(x), g(x)).$$

Then, $f \leq h$ and $g \leq h$.

If we consider the Cartesian product of two directed sets, we can naturally obtain a new directed set.

Proposition 1.1.1. [2] *Let I and J be directed sets. Then, $I \times J$ is a directed set with respect to the following relation:*

$$(i_1, j_1) \leq (i_2, j_2) :\Leftrightarrow (i_1 \leq i_2) \wedge (j_1 \leq j_2).$$

Proof. We show that these three properties hold.

1. If $i \in I$ and $j \in J$, then $i \leq i$ and $j \leq j$ and so $(i, j) \leq (i, j)$ showing reflexivity.
2. If $(i_1, j_1) \leq (i_2, j_2) \leq (i_3, j_3)$ then $i_1 \leq i_2 \leq i_3$ and $j_1 \leq j_2 \leq j_3$. Using the transitivity of I and J , we obtain $i_1 \leq i_3$ and $j_1 \leq j_3$, which yields $(i_1, j_1) \leq (i_3, j_3)$ proving transitivity.

3. Let $(i_1, j_1), (i_2, j_2) \in I \times J$. Choose $i \in I$ such that $i_1 \leq i$ and $i_2 \leq i$. Choose $j \in J$ such that $j_1 \leq j$ and $j_2 \leq j$. Then $(i_1, j_1) \leq (i, j)$ and $(i_2, j_2) \leq (i, j)$.

□

If we compute the Cartesian product of two directed sets without specifying the order, we use the one of the proposition. After going through the index set, we can now define a net.

Definition. [10, Definition 1.1.7] Let I be a directed set and S be a set. We then define $(x_i)_{i \in I} \subseteq S$ to be a net.

To make the nets useful, we must introduce a space with a certain topology.

1.1.2 Nets in a locally convex Hausdorff space

For our purposes, it is sufficient to work with a special class of topological vector spaces. Because most of the results work with $\mathbb{R}$ and $\mathbb{C}$, we define $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ to be fixed for the whole thesis. In addition, for the entire thesis, we assume for the topological space $\mathcal{T}$ the following:

- $\mathcal{T}$ is a non-trivial $\mathbb{F}$ -vector space (i.e., $\mathcal{T}$ consists of more than one element).
- $\mathcal{T}$ is equipped with the topology generated by a set of seminorms $\mathcal{P}$.
- $\mathcal{T}$ is a Hausdorff space. This means that if $p(x) = 0$ for all $p \in \mathcal{P}$, then $x = 0$.

If we require multiple such spaces, we write $\mathcal{T}_1, \mathcal{T}_2, \dots$, whose sets of seminorms are $\mathcal{P}_1, \mathcal{P}_2, \dots$ respectively. In some cases, we need an $\mathbb{F}$ -vector space without any topology. In this case, we use the letter $\mathcal{V}$. After setting up our space, we can define the convergence of the net.

Definition. [10, Definition 1.1.8] Let $(x_i)_{i \in I} \subseteq \mathcal{T}$ be a net. We say that $(x_i)_{i \in I}$ converges to some $x \in \mathcal{T}$ (or it is a converging net) if for all $p \in \mathcal{P}$ and $\epsilon > 0$ there exists an $i_0 \in I$ such that

$$p(x_i - x) < \epsilon \quad \forall i \geq i_0.$$

In this case, we use the notation

$$\lim_{i \in I} x_i = x$$

or

$$x_i \xrightarrow{i \in I} x.$$

It is easy to see that

$$\lim_{i \in I} x_i = x \Leftrightarrow \lim_{i \in I} p(x_i - x) = 0 \quad \forall p \in \mathcal{P}.$$

Since we assume that $\mathcal{T}$ is a Hausdorff space, the limits are unique. For consistency, we show that the limit is compatible with the Cartesian product of two directed sets.

Proposition 1.1.2. [45, Remark 2.5] Let I, J be directed sets and $(x_i)_{i \in I}$ be a converging net. Then

$$\lim_{(i,j) \in I \times J} x_i = \lim_{i \in I} x_i.$$

Proof. Let $x := \lim_{i \in I} x_i$. Let $\epsilon > 0$ and $p \in \mathcal{P}$. Choose $i_0 \in I$ such that

$$p(x_i - x) < \epsilon \quad \forall i \geq i_0.$$

Now, let $j_0 \in J$. Then

$$p(x_i - x) < \epsilon \quad \forall (i, j) \geq (i_0, j_0).$$

This proves our statement. $\square$

Similar to sequences, the limits are linear.

Proposition 1.1.3. [2, Proposition 2.1] Let $(x_i)_{i \in I} \subseteq \mathcal{T}$ be a net converging to x , $(y_j)_{j \in J} \subseteq \mathcal{T}$ converge to y and $\lambda \in \mathbb{F}$. Then, the following statements hold:

1. $\lim_{(i,j) \in I \times J} (x_i + y_j) = x + y$.
2. $\lim_{i \in I} (\lambda x_i) = \lambda x$.

Proof. The proofs are similar to those of the case with sequences.

1. Let $p \in \mathcal{P}$ and $\epsilon > 0$. Then there exist $i_0 \in I, j_0 \in J$ such that

$$p(x_i - x) < \frac{\epsilon}{2} \quad \forall i \geq i_0$$

and

$$p(y_j - y) < \frac{\epsilon}{2} \quad \forall j \geq j_0.$$

Then

$$p((x_i + y_j) - (x + y)) \leq p(x_i - x) + p(y_j - y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \forall (i, j) \geq (i_0, j_0).$$

2. If $\lambda = 0$, the statement is trivial. Otherwise, choose $i_0 \in I$ such that

$$p(x_i - x) < \frac{\epsilon}{|\lambda|} \quad \forall i \geq i_0.$$

Then

$$p(\lambda x_i - \lambda x) = |\lambda| p(x_i - x) < |\lambda| \frac{\epsilon}{|\lambda|} = \epsilon.$$

$\square$

Similar to sequences, we can also define accumulation points.

Definition. [11, Definition 11] Let $(x_i)_{i \in I} \subseteq \mathcal{T}$ be a net. We say that $x \in \mathcal{T}$ is an accumulation point of $(x_i)_{i \in I}$ if for all $p \in \mathcal{P}, \epsilon > 0$ and $i_0 \in I$ there exists an $i \geq i_0$ such that

$$p(x_i - x) < \epsilon.$$

1.1.3 Cauchy nets

Similar to sequences, we can also define Cauchy nets.

Definition. [2, Definition 2.2] Let $(x_i)_{i \in I} \subseteq \mathcal{T}$ be a net. We say that $(x_i)_{i \in I}$ is a Cauchy net if for all $p \in \mathcal{P}, \epsilon > 0$ there exists an $i_0 \in I$ such that

$$p(x_i - x_j) < \epsilon \quad \forall i, j \geq i_0.$$

Equivalently, $(x_i)_{i \in I}$ is a Cauchy net if for all $p \in \mathcal{P}$ it holds

$$\lim_{(i,j) \in I \times I} p(x_i - x_j) = 0.$$

We already know that any converging sequence is a Cauchy sequence. A similar statement holds for nets. However, the opposite is not true. However, we can show the following:

Proposition 1.1.4. [2, Corollary 2.5] Let $(x_i)_{i \in I} \subseteq \mathcal{T}$ be a Cauchy net with an accumulation point $x \in \mathcal{T}$. Then $(x_i)_{i \in I}$ converges to x . This means that every Cauchy net has at most one accumulation point.

Proof. Let $p \in \mathcal{P}$ and $\epsilon > 0$. Then choose $i_0 \in I$ such that

$$p(x_i - x_j) < \frac{\epsilon}{2} \quad \forall i, j \geq i_0.$$

We also can find an $i_1 \geq i_0$ such that

$$p(x_{i_1} - x) < \frac{\epsilon}{2}.$$

Combining those two inequalities, gives

$$p(x_i - x) \leq p(x_i - x_{i_1}) + p(x_{i_1} - x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \forall i \geq i_0.$$

□

Now, we define how two nets (with possibly different index sets) can be considered equivalent. This is useful because, later, we show that each locally convex topological Hausdorff space can be completed using Cauchy nets. To make the notation more intuitive, we write $\lim_{i \in I, j \in J}$ instead of $\lim_{(i,j) \in I \times J}$.

Definition. [2] Let $(x_i)_{i \in I}, (y_j)_{j \in J} \subseteq \mathcal{T}$ be nets. We call them equivalent if

$$\lim_{i \in I, j \in J} (x_i - y_j) = 0.$$

In this case, we use the notation $(x_i)_{i \in I} \sim (y_j)_{j \in J}$.

If two Cauchy nets have the same index set, then equivalence can be characterized as follows:

Proposition 1.1.5. [45, Remark 2.5], [2, Proposition 2.1] Let $(x_i)_{i \in I}, (y_i)_{i \in I} \subseteq \mathcal{T}$ be Cauchy nets. Then those are equivalent if and only if

$$\lim_{i \in I} (x_i - y_i) = 0.$$

Proof. Suppose that

$$\lim_{i \in I} (x_i - y_i) = 0.$$

Then, using Proposition 1.1.2 and linearity (Proposition 1.1.3), we get

$$\lim_{i \in I, j \in I} (x_i - y_j) = \lim_{i \in I, j \in I} (x_i - x_j + x_j - y_j) = \lim_{i \in I, j \in I} (x_i - x_j) + \lim_{j \in I} (x_j - y_j) = 0.$$

The other direction is trivial in this case. $\square$

As the name suggests, we obtain the following:

Proposition 1.1.6. [2] Being equivalent Cauchy nets is an equivalence relation in the class of all Cauchy nets.

Proof. Simple computations show the following:

1. Reflexivity: Since $(x_i)_{i \in I}$ is a Cauchy net,

$$\lim_{i \in I, j \in I} (x_i - x_j) = 0,$$

implying reflexivity.

2. Symmetry: If $(x_i)_{i \in I}$ is equivalent to $(y_j)_{j \in J}$ then

$$\lim_{i \in I, j \in J} (x_i - y_j) = 0.$$

By linearity (Proposition 1.1.3),

$$\lim_{j \in J, i \in I} (y_j - x_i) = - \lim_{i \in I, j \in J} (x_i - y_j) = 0.$$

3. Transitivity: Suppose that $(x_i)_{i \in I} \sim (y_j)_{j \in J}$ and $(y_j)_{j \in J} \sim (z_k)_{k \in K}$. Then again by Proposition 1.1.2 and linearity (Proposition 1.1.3)

$$\lim_{i \in I, k \in K} (x_i - z_k) = \lim_{i \in I, j \in J, k \in K} (x_i - y_j + y_j - z_k) = \lim_{i \in I, j \in J} (x_i - y_j) + \lim_{j \in J, k \in K} (y_j - z_k) = 0.$$

$\square$

If two nets are equivalent and one of them is a Cauchy net, then the other is also a Cauchy net.

Proposition 1.1.7. [2, Proposition 2.1] Let $(x_i)_{i \in I} \subseteq \mathcal{T}$ be a Cauchy net and $(y_j)_{j \in J} \subseteq \mathcal{T}$ be such that $(x_i)_{i \in I} \sim (y_j)_{j \in J}$. Then, $(y_j)_{j \in J}$ is a Cauchy net.

Proof. Let $\epsilon > 0$ and $p \in \mathcal{P}$. Choose $i_0 \in I, j_0 \in J$ such that

$$p(y_j - x_i) < \frac{\epsilon}{2} \quad \forall j \geq j_0 \quad \forall i \geq i_0.$$

This implies that

$$p(y_{j_1} - y_{j_2}) \leq p(y_{j_1} - x_{i_0}) + p(x_{i_0} - y_{j_2}) < \epsilon \quad \forall j_1, j_2 \geq j_0.$$

□

Converging nets with identical limits are equivalent.

Proposition 1.1.8. [45, Remark 2.5], [2, Proposition 2.1] Let $(x_i)_{i \in I}, (y_j)_{j \in J} \subseteq \mathcal{T}$ be two nets converging to the same x . Then, these are equivalent.

Proof. By Proposition 1.1.2 and linearity (Proposition 1.1.3)

$$\lim_{i \in I, j \in J} (x_i - y_j) = \lim_{i \in I, j \in J} x_i - \lim_{i \in I, j \in J} y_j = x - x = 0.$$

□

If a net converges, then each equivalent net converges to the same limit.

Proposition 1.1.9. [45, Remark 2.5], [2, Proposition 2.1] Let $(x_i)_{i \in I}, (y_j)_{j \in J} \subseteq \mathcal{T}$ be equivalent nets and $(x_i)_{i \in I}$ converge to some $x \in \mathcal{T}$. Then, $(y_j)_{j \in J}$ converges to x .

Proof. By Proposition 1.1.2 and linearity (Proposition 1.1.3)

$$\lim_{j \in J} y_j = \lim_{i \in I, j \in J} (y_j - x_i + x_i) = \lim_{i \in I, j \in J} (y_j - x_i) + \lim_{i \in I, j \in J} x_i = x.$$

□

The equivalence of nets is also compatible with vector addition and scaling.

Proposition 1.1.10. [2] Let $(x_i)_{i \in I} \sim (x'_j)_{j \in J}, (y_k)_{k \in K} \sim (y'_l)_{l \in L}$ in $\mathcal{T}$ and $\lambda \in \mathbb{F}$. Then

$$(\lambda x_i + y_k)_{(i,k) \in I \times K} \sim (\lambda x'_j + y'_l)_{(j,l) \in J \times L}.$$

Proof. We easily compute using Proposition 1.1.2 and linearity (Proposition 1.1.3)

$$\lim_{i \in I, j \in J, k \in K, l \in L} (\lambda x_i + y_k - \lambda x'_j + y'_l) = \lambda \lim_{i \in I, j \in J} (x_i - x'_j) + \lim_{k \in K, l \in L} (y_k - y'_l) = 0.$$

□

Equivalence is also compatible with seminorms.

Proposition 1.1.11. [21, Remark 3.1 d)] Let $(x_i)_{i \in I} \subseteq \mathcal{T}$ be a Cauchy net and $p \in \mathcal{P}$. Then $(p(x_i))_{i \in I}$ converges in $\mathbb{R}$. If $(y_j)_{j \in J}$ is an equivalent Cauchy net then

$$\lim_{i \in I} p(x_i) = \lim_{j \in J} p(y_j).$$

Proof. Let $\epsilon > 0$. By the definition of a Cauchy net, there exists an $i_0 \in I$ such that

$$p(x_i - x_{i'}) < \epsilon \quad \forall i, i' \geq i_0.$$

Since $|p(x_i) - p(x_{i'})| \leq p(x_i - x_{i'})$ because of the reverse triangular inequality, $(p(x_i))_{i \in I}$ is a Cauchy net and therefore converges. Since both nets are equivalent, we get

$$\lim_{i \in I, j \in J} |p(x_i) - p(y_j)| \leq \lim_{i \in I, j \in J} p(x_i - y_j) = 0$$

proving the statement. $\square$

A good thing about sequences is that the index set is always the same for all sequences. This is no longer true for nets, which makes the computations difficult. Our goal is to show that we can assume, without loss of generality, that each Cauchy net has the same index set. More precisely, we show that for each Cauchy net, there exists an equivalent Cauchy net having this specific index set. We can also ensure that the net members get close to each other quickly in a certain sense. To make notation simpler, we define

$$\hat{\mathcal{P}} := \{P \subseteq \mathcal{P} : |P| \text{ is finite}\}.$$

Recall that $\hat{\mathcal{P}}$ is a directed set with respect to $\subseteq$.

Theorem 1.1.12. [46, Theorem 2.1] Let $(x_i)_{i \in I} \subseteq \mathcal{T}$ be a Cauchy net. Then there exists an equivalent Cauchy net $(y_j)_{j \in J}$ such that $J = \mathbb{N}^* \times \hat{\mathcal{P}}$ and for all $\hat{p} \in \hat{\mathcal{P}}$ and $N \in \mathbb{N}^*$, we have

$$p(y_j - y_k) < \frac{1}{N} \quad \forall p \in \hat{p} \quad \forall j, k \geq (N, \hat{p}).$$

Proof. Let $n \in \mathbb{N}^*$ and $\hat{p} \in \hat{\mathcal{P}}$. Since $(x_i)_{i \in I}$ is a Cauchy net and $\hat{p}$ is a finite set, we can find an index $i_{n, \hat{p}} \in I$ such that

$$p(x_i - x_j) < \frac{1}{n} \quad \forall i, j \geq i_{n, \hat{p}} \quad \forall p \in \hat{p}. \quad (*)$$

For a fixed $(m, \hat{q}) \in J$, there are only finitely many $(n, \hat{p}) \in J$ such that $(n, \hat{p}) \leq (m, \hat{q})$. Therefore, we can assume without loss of generality that $i_{n, \hat{p}} \leq i_{m, \hat{q}}$ if $(n, \hat{p}) \leq (m, \hat{q})$. Now we define the net

$$(y_{(n, \hat{p})})_{(n, \hat{p}) \in J} := (x_{i_{n, \hat{p}}})_{(n, \hat{p}) \in J}.$$

Next, we show that this new net satisfies the formula of the theorem. Let $(N, \hat{p}) \in \mathbb{N}^* \times \hat{\mathcal{P}}$. Let $(m, \hat{q}), (n, \hat{r}) \geq (N, \hat{p})$. Using (*), we get

$$p(y_{(m, \hat{q})} - y_{(n, \hat{r})}) = p(x_{i_{m, \hat{q}}} - x_{i_{n, \hat{r}}}) < \frac{1}{N} \quad \forall p \in \hat{p}.$$

Finally, we show that $(x_i)_{i \in I} \sim (y_j)_{j \in J}$. Let $p \in \mathcal{P}$ and $\epsilon > 0$. Choose $n > \frac{1}{\epsilon}$. Let $i \geq i_{n, \{p\}}$ and $(m, \hat{p}) \geq (n, \{p\})$. Then, thanks to (*), we get

$$p(x_i - y_{(m, \hat{p})}) = p(x_i - x_{i_{m, \hat{p}}}) < \frac{1}{n} < \epsilon.$$

This implies that $(x_i)_{i \in I} \sim (y_j)_{j \in J}$ and by Proposition 1.1.7, $(y_j)_{j \in J}$ is a Cauchy net. $\square$

With this result, we can assume that the index set of every Cauchy net is $\mathbb{N}^* \times \hat{\mathcal{P}}$ without essential restrictions. This result is also important for set-theoretical reasons. If we assume that E is an equivalence class of Cauchy nets and also a set, then this equivalence class might contain a net of the form $(x_i)_{i \in \{E\}}$. In this case, we have a relation of the form

$$E \ni (x_i)_{i \in \{E\}} \ni \cdots \ni E$$

This means that E is a cyclic set, which is not allowed in set theory. To solve this problem, we restrict ourselves to nets with the index set $\mathbb{N}^* \times \hat{\mathcal{P}}$. So, for any Cauchy net $(x_i)_{i \in I}$, we define

$$[(x_i)_{i \in I}] := \{(y_j)_{j \in \mathbb{N}^* \times \hat{\mathcal{P}}} \subseteq \mathcal{T} : (y_j)_{j \in \mathbb{N}^* \times \hat{\mathcal{P}}} \sim (x_i)_{i \in I}\}$$

which is a set.

1.1.4 Closure and sequential closure of a set

Having set up the basic properties of nets, we go through the definition of a closed set, the closure, and a complete topological space using a net. We also compare these concepts with the use of sequences.

Definition. [45] Let $\mathcal{S} \subseteq \mathcal{T}$. We say that $\mathcal{S}$ is closed in $\mathcal{T}$ if for every net $(x_i)_{i \in I} \subseteq \mathcal{S}$ converging to some $x \in \mathcal{T}$, it holds $x \in \mathcal{S}$.

A sequentially closed subset is defined in a similar manner.

Definition. [39] Let $\mathcal{S} \subseteq \mathcal{T}$. We say that $\mathcal{S}$ is sequentially closed in $\mathcal{T}$ if for every sequence $(x_n)_{n \in \mathbb{N}} \subseteq \mathcal{S}$ converging to some $x \in \mathcal{T}$, it holds $x \in \mathcal{S}$.

It is easy to see that every closed subset is sequentially closed. However, the converse is not generally true. Next, we define the closure of a set, which can be defined in two equivalent ways.

Definition. Let $\mathcal{S} \subseteq \mathcal{T}$. Then we define $\overline{\mathcal{S}}$ to be the closure of $\mathcal{S}$ if one of the following equivalent conditions holds:

1. [2] It holds

$$\overline{\mathcal{S}} = \left\{ x \in \mathcal{T} : \exists (x_i)_{i \in I} \subseteq \mathcal{S} : \lim_{i \in I} x_i = x \right\}.$$

2. [49] The set $\overline{\mathcal{S}}$ is the intersection of all closed subsets of $\mathcal{T}$ containing $\mathcal{S}$.

One can show that both definitions are equivalent. In this case, one can use a diagonal net argument similar to the proof of Theorem 1.2.12. In the case of sequences, we use the following definitions.

Definition. [39] Let $\mathcal{S} \subseteq \mathcal{T}$.

1. We define the sequential hull $[S]^s$ of S as

$$[S]^s = \left\{ x \in \mathcal{T} : \exists (x_n)_{n \in \mathbb{N}} \subseteq S : \lim_{n \rightarrow \infty} x_n = x \right\}.$$

2. We define the sequential closure $\overline{S}^s$ of S as the intersection of all sequentially closed subsets containing S .

It is easy to see that $[S]^s \subseteq \overline{S}^s$, but the converse is not true in general. We now show the following.

Proposition 1.1.13. [29, Lemma 4] Let $S \subseteq \mathcal{T}$ be a subspace. Then, $\overline{S}$ and $[S]^s$ are also subspaces.

Proof. The $\mathbf{0}$ -vector is contained in both the sets.

1. Let $x, y \in \overline{S}$ and $\lambda \in \mathbb{F}$. Then, there exist nets $(x_i)_{i \in I}, (y_j)_{j \in J} \subseteq S$ converging to x and y respectively. By Proposition 1.1.3, we get

$$\lambda x + y = \lim_{i \in I, j \in J} (\lambda x_i + y_j) \in \overline{S}.$$

2. For $[S]^s$, we do the same as before, but we use sequences instead.

□

Finally, we define the density in terms of nets and sequences.

Definition. Let $S \subseteq \mathcal{T}$.

1. [45] We say that S is dense in $\mathcal{T}$ if $\overline{S} = \mathcal{T}$.
2. [19] We say that S is sequentially dense in $\mathcal{T}$ if $[S]^s = \mathcal{T}$.

One might define a different kind of density, where $\overline{S}^s$ has to be $\mathcal{T}$, but we do not need this definition here. Because every sequence is a net, each sequentially dense subset is also dense.

1.2 Continuous linear operators

Using nets, we can define the continuity of functions.

Definition. [41, Theorem 1.1.9] Let $f : \mathcal{T}_1 \rightarrow \mathcal{T}_2$. We say that f is continuous if for every $x \in \mathcal{T}_1$ and every net $(x_i)_{i \in I} \subseteq \mathcal{T}_1$ converging to x , the net $(f(x_i))_{i \in I}$ converges to $f(x)$.

Similar to metric spaces, continuity can also be characterized using open sets, neighborhoods, and filters, as in [49] however, we will not focus on that. We show that each seminorm that constructs the topology of $\mathcal{T}$ is continuous.

Proposition 1.2.1. [49, Proposition 7.3] Every seminorm $p \in \mathcal{P}$ is continuous.

Proof. Let $(x_i)_{i \in I} \in \mathcal{T}$ be a net converging to some $x \in \mathcal{T}$. This implies that

$$\lim_{i \in I} p(x_i - x) = 0.$$

Then, by linearity (Proposition 1.1.3)

$$\lim_{i \in I} (p(x_i) - p(x)) = \lim_{i \in I} (p(x_i - x + x) - p(x)) \leq \lim_{i \in I} p(x_i - x) + p(x) - p(x) = 0,$$

showing the continuity of p . $\square$

Similar to normed spaces, the continuity of a linear operator can be characterized by continuity at 0 .

Proposition 1.2.2. [49, Proposition 4.7] Let $T : \mathcal{T}_1 \rightarrow \mathcal{T}_2$ be linear. Then, T is continuous if and only if for every net $(x_i)_{i \in I} \subseteq \mathcal{T}_1$ converging to 0 , the net $(Tx_i)_{i \in I}$ converges to 0 .

Proof. One direction is trivial. For the other, let $(x_i)_{i \in I} \subseteq \mathcal{T}_1$ be a net converging to some $x \in \mathcal{T}_1$. Let $(y_i)_{i \in I} := (x_i - x)_{i \in I}$. Then

$$\lim_{i \in I} y_i = \lim_{i \in I} (x_i - x) = x - x = 0.$$

By our assumption, $\lim_{i \in I} Ty_i = 0$, therefore, by linearity (Proposition 1.1.3),

$$\lim_{i \in I} Tx_i = \lim_{i \in I} (Tx_i - Tx + Tx) = \lim_{i \in I} T(x_i - x) + Tx = \lim_{i \in I} Ty_i + Tx = Tx.$$

$\square$

There is also another way to characterize the continuity of a linear operator, however, we first introduce the notation for the set of all linear combinations of seminorms, where the coefficients are non-negative

$$\tilde{\mathcal{P}} := \left\{ \sum_{i=1}^N c_i p_i : N \in \mathbb{N}, (c_i)_{i=1}^N \subseteq \mathbb{R}_{\geq 0}, (p_i)_{i=1}^N \subseteq \mathcal{P} \right\}.$$

One can easily show that each element of $\tilde{\mathcal{P}}$ is a seminorm. Moreover, $\mathcal{P}$ is a directed set with respect to $\leq$. Indeed, reflexivity and transitivity are clear. Let

$$p := \sum_{i=1}^N c_i p_i, q := \sum_{i=1}^N d_i p_i,$$

where we assume, without loss of generality, that both seminorms use the same tuple $(p_i)_{i=1}^N$ by using 0 -coefficients. Then

$$\sum_{i=1}^N \max(c_i, d_i) p_i$$

provides an upper bound for the two seminorms. Our next goal is to characterize the continuity of a linear operator similar to that in normed spaces.

Theorem 1.2.3. [49, Proposition 7.7] Let $A : \mathcal{T}_1 \rightarrow \mathcal{T}_2$ be linear. Then A is continuous if and only if for all $q \in \mathcal{P}_2$, there exists a $\tilde{p} \in \tilde{\mathcal{P}}_1$ such that

$$q(Ax) \leq \tilde{p}(x) \quad \forall x \in \mathcal{T}_1.$$

Proof. We show both directions here.

$\Rightarrow$: Choose the directed set $I := \mathbb{N}^* \times \tilde{\mathcal{P}}_1$ with the combined order $(\leq, \leq)$. Let $q \in \mathcal{P}_2$. Suppose that the second statement does not hold for any linear combination $\tilde{p} \in \tilde{\mathcal{P}}_1$. Let $(n, \tilde{p}) \in \mathbb{N}^* \times \tilde{\mathcal{P}}_1$. Then, $n\tilde{p} \in \tilde{\mathcal{P}}_1$. By our assumption, there exists an $x_{n,\tilde{p}} \in \mathcal{T}_1$ such that

$$q(Ax_{n,\tilde{p}}) > n\tilde{p}(x_{n,\tilde{p}}).$$

Note that this inequality still holds if we replace $x_{n,\tilde{p}}$ by $\lambda x_{n,\tilde{p}}$ for any $\lambda > 0$. This implies that we can assume without loss of generality that $q(Ax_{n,\tilde{p}}) = 1$. Now we show that $(x_{n,\tilde{p}})_{(n,\tilde{p}) \in \mathbb{N}^* \times \tilde{\mathcal{P}}_1}$ converges to 0 . Let $p \in \mathcal{P}_1$. Then, for all $\tilde{p} \in \tilde{\mathcal{P}}_1$ such that $\tilde{p} \geq p$, we get

$$1 > n\tilde{p}(x_{n,\tilde{p}}) \geq np(x_{n,\tilde{p}})$$

implying

$$\frac{1}{n} > p(x_{n,\tilde{p}}).$$

Going to the limit with n and $\tilde{p}$ shows that our net converges to 0 . On the other hand, by continuity of A and q (Proposition 1.2.1),

$$\lim_{i \in I} q(Ax_i) = q(A0) = 0,$$

giving a contradiction.

$\Leftarrow$: By Proposition 1.2.2, it suffices to check continuity at 0 . Let $(x_i)_{i \in I}$ be a net converging to 0 . Let $q \in \mathcal{P}_2$. Then there exists a $\tilde{p} \in \tilde{\mathcal{P}}_1$ such that

$$q(Ax_i) \leq \tilde{p}(x_i) \quad \forall i \in I.$$

Since the right side goes to 0 , the left side does as well.

□

To simplify the notation, we define

$$L(\mathcal{T}_1, \mathcal{T}_2) := \{T : \mathcal{T}_1 \rightarrow \mathcal{T}_2 : T \text{ is linear and continuous}\}.$$

If the domain and codomain is the same, we use the notation

$$L(\mathcal{T}) := L(\mathcal{T}, \mathcal{T}).$$

We define

$$\mathcal{T}' := L(\mathcal{T}, \mathbb{F})$$

to be the topological dual space of $\mathcal{T}$. A very useful result is that a continuous linear operator maps the Cauchy nets to Cauchy nets.

Proposition 1.2.4. [50, Proposition 1.18] Let $A \in L(\mathcal{T}_1, \mathcal{T}_2)$ and $(x_i)_{i \in I} \subseteq \mathcal{T}_1$ be a Cauchy net. Then, $(Ax_i)_{i \in I}$ is also a Cauchy net.

Proof. Let $p_2 \in \mathcal{P}_2$ and $\tilde{p}_1 \in \tilde{\mathcal{P}}_1$ according to the continuity of A (see Theorem 1.2.3). Then

$$p_2(Ax_i - Ax_j) = p_2(A(x_i - x_j)) \leq \tilde{p}_1(x_i - x_j).$$

This shows that $(Ax_i)_{i \in I}$ is a Cauchy net. $\square$

Similar to Banach spaces, $\mathcal{T}$ can also be considered complete under certain conditions.

Definition. We define two different types of completeness:

- [2] We call $\mathcal{T}$ complete if every Cauchy net converges.
- [43] We call $\mathcal{T}$ sequentially complete if every Cauchy sequence converges.

It is easy to see that if $\mathcal{T}$ is complete, then $\mathcal{T}$ is sequentially complete. The other direction is generally incorrect. See, for instance, Theorem 1.5.14 and Proposition 1.5.15. In some cases, we want to extend a densely defined continuous linear operator to the entire space. This is possible because of the following results:

Theorem 1.2.5. [49, Theorem 5.1] Let $\mathcal{T}_2$ be complete. Let $V \subseteq \mathcal{T}_1$ be a dense linear subspace and $A \in L(V, \mathcal{T}_2)$. Then, there exists a unique $\bar{A} \in L(\mathcal{T}_1, \mathcal{T}_2)$ such that $A \subseteq \bar{A}$.

Proof. The idea is to define

$$\bar{A}x = \lim_{i \in I} Ax_i$$

for any $x \in \mathcal{T}_1$ and net $(x_i)_{i \in I} \subseteq V$ converging to x . However, it is not clear whether this is well-defined and fulfills the desired properties. Therefore, we proceed stepwise.

1. Existence of the limit: If $(x_i)_{i \in I} \subseteq V$ converges in $\mathcal{T}_1$ then it is a Cauchy net. Using Proposition 1.2.4, $(Ax_i)_{i \in I}$ is a Cauchy net. Since $\mathcal{T}_2$ is complete, it is also convergent, therefore the limit

$$\lim_{i \in I} Ax_i$$

exists.

2. Independence of the nets: Now, we have to make sure that the limit Ax_i does only depend on the limit of $(x_i)_{i \in I}$. Let $(x_i)_{i \in I}$ and $(y_i)_{i \in I} \subseteq V$ be two nets converging to $x \in \mathcal{T}_1$. Then, by Proposition 1.1.8, these nets are equivalent. So we get by linearity (Proposition 1.1.3)

$$\lim_{i \in I} Ax_i - \lim_{i \in I} Ay_i = \lim_{i \in I} (Ax_i - Ay_i) = A \lim_{i \in I} (x_i - y_i) = 0.$$

So $\lim_{i \in I} Ax_i = \lim_{i \in I} Ay_i$ showing that $\bar{A}$ is well defined.

3. Linearity: Suppose $x_i \xrightarrow{i \in I} x, y_i \xrightarrow{i \in I} y$ and $\lambda \in \mathbb{F}$. Then, by Proposition 1.1.3,

$$\lim_{i \in I} (\lambda x_i + y_i) = \lambda x + y.$$

So

$$\begin{aligned} \bar{A}(\lambda x + y) - \lambda \bar{A}x - \bar{A}y &= \lim_{i \in I} A(\lambda x_i + y_i) - \lambda \lim_{i \in I} Ax_i - \lim_{i \in I} Ay_i \\ &= \lim_{i \in I} (A(\lambda x_i + y_i) - \lambda Ax_i - Ay_i) = 0. \end{aligned}$$

4. We show that $\bar{A}$ is continuous. Let $q \in \mathcal{P}_2$. Choose $\tilde{p} \in \tilde{\mathcal{P}}_1$ according to the continuity of A (see Theorem 1.2.3). Let $x \in \mathcal{T}_1$ and $(x_i)_{i \in I} \subseteq V$ converge to x . Then

$$q(\bar{A}x) = q\left(\lim_{i \in I} Ax_i\right) = \lim_{i \in I} q(Ax_i) \leq \lim_{i \in I} \tilde{p}(x_i) = \tilde{p}(x),$$

where we use the continuity of the seminorm (see Proposition 1.2.1).

5. Uniqueness: Suppose that $\bar{B}$ is another such extension. Let $x \in \mathcal{T}_1$. By density, there exists a net $(x_i)_{i \in I} \subseteq V$ converging to x . Then

$$\bar{B}x = \lim_{i \in I} Ax_i = \bar{A}x,$$

proving that $\bar{B} = \bar{A}$. This implies that the extension is unique in this case.

□

We can tweak the conditions to obtain a similar result using sequences instead of nets. To make the result more general, we introduce the following notation. For a subset $S \subseteq \mathcal{T}$, we write $S \preceq \mathcal{T}$ if every Cauchy sequence in S converges in $\mathcal{T}$. This is particularly true if $\mathcal{T}$ is sequentially complete. For $A \in L(\mathcal{T}_1, \mathcal{T}_2)$, we define the range of A to be

$$R(A) := \{y \in \mathcal{T}_2 : \exists x \in \mathcal{T}_1 : Ax = y\}.$$

Corollary 1.2.6. [50, Proposition 1.18] *Let $V \subseteq \mathcal{T}_1$ be a sequentially dense subspace and $A \in L(V, \mathcal{T}_2)$ such that $R(A) \preceq \mathcal{T}_2$. Then, there exists a unique $\bar{A} \in L(\mathcal{T}_1, \mathcal{T}_2)$ such that $A \subseteq \bar{A}$.*

Proof. The proof is very similar to the one of Theorem 1.2.5, except that we define

$$\bar{A}x := \lim_{n \rightarrow \infty} Ax_n,$$

where $(x_n)_{n \in \mathbb{N}} \subseteq V$ converges to $x \in \mathcal{T}_1$. Because $R(A) \preceq \mathcal{T}_2$, the limit exists in $\mathcal{T}_2$. □

In some cases, we want to extend the continuous linear functionals. The following result provides us with a tool.

Theorem 1.2.7 (Hahn-Banach). [51, Theorem 1] *Let $V \subseteq \mathcal{T}$ be a subspace and $l \in V'$. Then, there exists an extension $L \in \mathcal{T}'$ of l .*

1.2.1 Projections

A special class of operators is the class of projections. They are defined as follows:

Definition. [50, Definition 4.14] An operator $P \in L(\mathcal{T})$ is called a projection if $PP = P$.

It is easy to see that if $x \in R(P)$ then $Px = x$. Indeed, let $y \in \mathcal{T}$ such that $Py = x$. Then

$$Px = PP_y = Py = x.$$

This allows us to prove the following theorem.

Proposition 1.2.8. [19, Proposition 3.2] Let $P \in L(\mathcal{T})$ be a projection. Then, $R(P)$ is a closed subspace.

Proof. Let $(x_i)_{i \in I} \subseteq R(P)$ converge to some $x \in \mathcal{T}$. Then

$$x = \lim_{i \in I} x_i = \lim_{i \in I} Px_i = Px \in R(P).$$

□

To check whether a continuous operator is a projection, it is sufficient to check this property on a dense subspace.

Proposition 1.2.9. [5, Lemma 8] Let $P \in L(\mathcal{T})$ and $V \subseteq \mathcal{T}$ be a dense subspace such that

$$PPv = Pv \quad \forall v \in V.$$

Then, P is a projection.

Proof. Let $x \in \mathcal{T}$ and $(x_i)_{i \in I} \subseteq V$ converge to x . Then

$$PPx = PP \lim_{i \in I} x_i = \lim_{i \in I} PPx_i = \lim_{i \in I} Px_i = Px.$$

□

1.2.2 Isometries

From normed spaces, we are familiar with isometries. This concept can be extended to locally convex, Hausdorff spaces. Here, we have to clarify what we mean by the fact that seminorms are preserved, since we have multiple seminorms in different spaces.

Definition. [16] Let $T \in L(\mathcal{T}_1, \mathcal{T}_2)$. We say that T is an isometry if there exists a bijection $\mu : \mathcal{P}_1 \rightarrow \mathcal{P}_2$ such that

$$q(x) = \mu(q)(Tx) \quad \forall x \in \mathcal{T}_1, q \in \mathcal{P}_1.$$

Since we are in a Hausdorff space, the isometries are injective. Indeed, if $Tx = 0$, then $q(x) = \mu(q)(Tx) = 0$ for all $q \in \mathcal{P}$, so $x = 0$. It is also easy to see that if an isometry is bijective, then the inverse is also an isometry. Next, we show that the extension of a densely defined isometry is an isometry.

Proposition 1.2.10. [18, Lemma 5.1.7] Let $V \subseteq \mathcal{T}_1$ be a dense (or sequentially dense) subspace and $T \in L(V, \mathcal{T}_2)$ be an isometry such that there exists a continuous extension $\bar{T} \in L(\mathcal{T}_1, \mathcal{T}_2)$. Then, $\bar{T}$ is an isometry.

Proof. Let $q \in \mathcal{P}_1$ and $x \in \mathcal{T}_1$. Choose a net (or sequence) $(x_i)_{i \in I} \subseteq V$ converging to x . Then by Proposition 1.2.1

$$\mu(q)(\bar{T}x) = \lim_{i \in I} \mu(q)(Tx_i) = \lim_{i \in I} q(x_i) = q(x).$$

□

If we have a linear map, where the domain is a vector space and the codomain is equipped with seminorms, then this map induces seminorms to the domain.

Lemma 1.2.11. [37, Lemma 3.2] Let $T : \mathcal{V} \rightarrow \mathcal{T}$ be an injective linear map. Then for each $q \in \mathcal{P}$, let

$$q' : \mathcal{V} \rightarrow \mathcal{T}, x \mapsto q(Tx).$$

Then, q' is a seminorm. Furthermore, $\{q \circ T : q \in \mathcal{P}\}$ generate a locally convex Hausdorff topology on $\mathcal{V}$ and T is an isometry.

Proof. The non-negativity is evident. For the absolute homogeneity, we get

$$q'(\lambda x) = q(T\lambda x) = |\lambda|q(Tx) = |\lambda|q'(x)$$

and we get the triangular inequality by

$$q'(x + y) = q(T(x + y)) = q(Tx + Ty) \leq q(Tx) + q(Ty) = q'(x) + q'(y).$$

The isometry property follows from this definition. If $q'(x) = 0$ for every such seminorm q' then $q(Tx) = 0$ for all $q \in \mathcal{P}$. Since $\mathcal{T}$ is Hausdorff, $Tx = 0$. By the injectivity of T , $x = 0$, showing that $\mathcal{V}$ equipped with the induced seminorms is a Hausdorff space. □

1.2.3 Completion of $\mathcal{T}$

From Topology, it is well known that every metric space can be completed. This is also true for locally convex Hausdorff spaces. The proof is similar to that of the metric space, but having multiple seminorms and nets complicates the proof, although the idea is the same. A proof of the completion of a general topological vector space can be found in [49, Theorem 5.2]. Here, we focus on the case where the topology is generated by seminorms. This requires an extra step in which the seminorms are extended to the completion. This is possible because of [49, Theorem 5.1].

Theorem 1.2.12. [49, Theorem 5.2] There exists a (up to an isometric bijection) unique complete locally convex Hausdorff space $\bar{\mathcal{T}}$ and an isometric embedding $\iota \in L(\mathcal{T}, \bar{\mathcal{T}})$ such that $R(\iota) \subseteq \bar{\mathcal{T}}$ is dense.

Proof. We will go through these steps.

1. Definition: Let

$$\overline{\mathcal{T}} := \{[(x_i)_{i \in \mathbb{N}^* \times \hat{\mathcal{P}}}] : (x_i)_{i \in \mathbb{N}^* \times \hat{\mathcal{P}}} \subseteq \mathcal{T} \text{ is a Cauchy net}\}.$$

Thanks to Theorem 1.1.12, for every Cauchy net $(x_i)_{i \in I} \subseteq \mathcal{T}$, $[(x_i)_{i \in I}] \in \overline{\mathcal{T}}$. Therefore, in the entire proof, we assume without loss of generality that $I = \mathbb{N}^* \times \hat{\mathcal{P}}$.

2. Addition and scalar multiplication: We define

$$[(x_i)_{i \in I}] + [(y_i)_{i \in I}] := [(x_i + y_i)_{i \in I}]$$

and

$$\lambda[(x_i)_{i \in I}] := [(\lambda x_i)_{i \in I}].$$

According to Proposition 1.1.10, both operations are well defined.

3. Seminorms: For $x = [(x_i)_{i \in I}] \in \overline{\mathcal{T}}$ and $p \in \mathcal{T}$, we define

$$\bar{p}([(x_i)_{i \in I}]) := \lim_{i \in I} p(x_i),$$

which is, by Proposition 1.1.11, well-defined. In addition, if $\bar{p}(x) = 0$ for all $\bar{p}$, then $x = 0$ by the equivalence of the Cauchy sequences. Therefore, this space has the Hausdorff property.

4. Embedding: Let

$$\iota : \mathcal{T} \rightarrow \overline{\mathcal{T}}, x \mapsto [(x)_{i \in I}].$$

Then, ι is trivially linear. Also, because of

$$\bar{p}(\iota(x)) = \bar{p}([(x)_{i \in I}]) = \lim_{i \in I} p(x) = p(x)$$

ι is an isometry, and

5. Density: Let $x := [(x_i)_{i \in I}] \in \overline{\mathcal{T}}$ and $(N, \{p\}) \in \mathbb{N}^* \times \hat{\mathcal{P}}_1$. According to Theorem 1.1.12, we can assume that for $j \geq (N, \{p\})$, we have

$$\bar{p}(x - \iota(x_j)) = \lim_{i \in I} p(x_i - x_j) \leq \frac{1}{N}. \quad (*)$$

Applying the limit to j , we get

$$\lim_{j \in I} \bar{p}(x - \iota(x_j)) \leq \frac{1}{N}.$$

Letting N approach ∞ , we get

$$\lim_{j \in I} \bar{p}(x - \iota(x_j)) = 0.$$

This implies that $(\iota(x_i))_{i \in I}$ converges to x .

6. Completeness: Also, in this step, we assume that all nets are of the form of Theorem 1.1.12. Let $(x^i)_{i \in \mathbb{N}^* \times \hat{\mathcal{P}}} \subseteq \overline{\mathcal{T}}$ be a Cauchy net. For a fixed $i \in \mathbb{N}^* \times \hat{\mathcal{P}}$, x^i is

an equivalence class of Cauchy nets. Let $(x_j^i)_{j \in \mathbb{N}^* \times \hat{\mathcal{P}}}$ be one of its representatives. Now choose the diagonal net

$$y := (y_i)_{i \in \mathbb{N}^* \times \hat{\mathcal{P}}} := (x_i^i)_{i \in \mathbb{N}^* \times \hat{\mathcal{P}}}.$$

Then, y is a Cauchy net. Indeed, let $(N, \{p\}) \in \mathbb{N}^* \times \hat{\mathcal{P}}$ and $i, j \geq (N, \{p\})$. Then

$$\bar{p}(\iota(x_i^i) - x^i) \leq \frac{1}{N}$$

because of $(*)$ in step 5. Also

$$p(x_i^i - x_j^j) = \bar{p}(\iota(x_i^i) - \iota(x_j^j)) \leq \bar{p}(\iota(x_i^i) - x^i) + \bar{p}(x^i - x^j) + \bar{p}(x^j - \iota(x_j^j)) \leq \frac{3}{N}.$$

This shows that y is a Cauchy net. The equivalence class of this net is the limit of $(x^i)_{i \in \mathbb{N}^* \times \hat{\mathcal{P}}}$, since for $(N, \{p\}) \in \mathbb{N}^* \times \hat{\mathcal{P}}$ and $i \geq (N, \{p\})$, we have

$$\bar{p}(x^i - [(x_j^j)_{j \in I}]) \leq \underbrace{\bar{p}(x^i - \iota(x_i^i))}_{\leq \frac{1}{N}} + \underbrace{\bar{p}(\iota(x_i^i) - [(x_j^j)_{j \in I}])}_{\leq \frac{1}{N}} \stackrel{(*)}{\leq} \frac{2}{N}.$$

7. Uniqueness: Suppose, there is another completion $\tilde{\mathcal{T}}$ with the seminorms $\mathcal{P}'$ and the embedding $\hat{\iota}$. now let

$$l : R(\iota) \rightarrow \tilde{\mathcal{T}}, x \mapsto \hat{\iota}(\iota^{-1}(x)).$$

Then, l is the composition of two linear isometries, and therefore, a linear isometry. By Theorem 1.2.5, there exists a continuous linear extension $\bar{l} : \overline{\mathcal{T}} \rightarrow \tilde{\mathcal{T}}$, which is, by Proposition 1.2.10, an isometry. Similarly, we can find an isometry $\bar{m} : \tilde{\mathcal{T}} \rightarrow \overline{\mathcal{T}}$. The composition of $\bar{l}$ and $\bar{m}$ in any order restricted to a dense subspace is the identity. By Theorem 1.2.5, $\bar{l}$ and $\bar{m}$ are inverses with respect to each other, implying that $\overline{\mathcal{T}}$ and $\tilde{\mathcal{T}}$ are isometrically isomorphic.

□

As seen in the proof, $\mathcal{T}$ can be embedded in $\overline{\mathcal{T}}$. By abuse of notation, we treat $\mathcal{T}$ as a subset of $\overline{\mathcal{T}}$. One might ask whether we can perform a similar operation with sequences. Similar to the sequential hull and sequential closure of the set, defining a space consisting of equivalence classes of Cauchy sequences does not necessarily provide a sequentially complete space. Therefore, we must be careful when defining the sequential completion of a $\mathcal{T}$.

1.3 Metrization

In this section, we show that $\mathcal{T}$ is metrizable if $\mathcal{P}$ is countable. This is very useful because, in this setting, working with sequences instead of nets is fine. Our goal is to construct a metric that induces the same topology as $\mathcal{P}$. First, we need to define a pseudo-metric.

Definition. [15] For any set M , let $d : M \times M \rightarrow \mathbb{R}_{\geq 0}$. We call d a pseudometric if it satisfies the following conditions.

1. $d(x, x) = 0$ for all $x \in M$.
2. $d(x, y) = d(y, x)$ for all $x, y \in M$.
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in M$.

Seminorms induce pseudometrics.

Example. Let p be a seminorm. Then

$$d(x, y) := p(x - y)$$

defines a pseudometric.

A proof that a locally convex Hausdorff space generated by countably many seminorms is metrizable can be found in [49, Proposition 8.1]. We provide a similar proof here. To make things easier to understand, we provide three lemmas as stepping stones before we arrive at the main result.

Lemma 1.3.1. [49, Proposition 8.1] *The function*

$$f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}, t \mapsto \frac{t}{1+t}$$

is strictly increasing. Suppose that there exists a net $(t_i)_{i \in I} \subseteq \mathbb{R}_{\geq 0}$ such that $\lim_{i \in I} f(t_i) = 0$. Then $\lim_{i \in I} t_i = 0$.

Proof. We compute

$$f'(t) = \frac{(1+t) - t}{(1+t)^2} = \frac{1}{(1+t)^2} > 0,$$

showing that f is strictly increasing. For the second statement, we have

$$\frac{f(t)}{1-f(t)} = \frac{f(t)}{1 - \frac{t}{1+t}} = \frac{f(t)}{\frac{1+t-t}{1+t}} = t.$$

So if $f(t) \rightarrow 0$ then $t \rightarrow 0$. □

In some cases, it is useful to modify a pseudometric so that it becomes bounded from above without changing the topological structure.

Lemma 1.3.2. [49, Proposition 8.1] *Let d be a pseudometric on M . Then*

$$\tilde{d}(x, y) := \frac{d(x, y)}{1 + d(x, y)},$$

also defines a pseudometric. Moreover, $\tilde{d}(x, y) \leq 1$ for all $x, y \in M$ and $\tilde{d}$ induces the same topology as d .

Proof. We first check that $\tilde{d}$ is a pseudometric. Obviously,

$$\tilde{d}(x, x) = \frac{d(x, x)}{1 + d(x, x)} = 0$$

and

$$\tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)} = \frac{d(y, x)}{1 + d(y, x)} = \tilde{d}(y, x).$$

To show the triangular inequality, note that

$$f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}, t \mapsto \frac{t}{1 + t}$$

is increasing (see Lemma 1.3.1). So

$$\begin{aligned} \tilde{d}(x, z) &= f(d(x, z)) \leq f(d(x, y) + d(y, z)) \\ &= \frac{d(x, y) + d(y, z)}{1 + d(x, y) + d(y, z)} \\ &= \frac{d(x, y)}{1 + d(x, y) + d(y, z)} + \frac{d(y, z)}{1 + d(x, y) + d(y, z)} \\ &\leq \frac{d(x, y)}{1 + d(x, y)} + \frac{d(y, z)}{1 + d(y, z)} \\ &= \tilde{d}(x, y) + \tilde{d}(y, z). \end{aligned}$$

Also,

$$f(t) = \frac{t}{1 + t} \leq \frac{1 + t}{1 + t} \leq 1,$$

showing the boundedness of the pseudometric. Finally, suppose that a net $(x_i)_{i \in I}$ converges to some x with respect to d . Then

$$\lim_{i \in I} \tilde{d}(x_i, x) = \lim_{i \in I} \frac{d(x_i, x)}{1 + d(x_i, x)} = 0.$$

On the other hand, if we have convergence with respect to $\tilde{d}$, then

$$\lim_{i \in I} f(d(x_i, x)) = 0,$$

implying

$$\lim_{i \in I} d(x_i, x) = 0.$$

□

We can use this tool to combine countable infinitely many pseudometrics.

Lemma 1.3.3. [49, Proposition 8.1] Let $(d_n)_{n \in \mathbb{N}}$ be pseudometrics uniformly bounded by 1. Then

$$d(x, y) := \sum_{n \in \mathbb{N}} \frac{1}{2^n} d_n(x, y)$$

is a pseudometric. Moreover, this pseudometric induces the same topology as the pseudometrics $d_1, d_2, \dots$ together, i.e. for every net $(x_i)_{i \in I} \subseteq M$ and $x \in M$, we have

$$\lim_{i \in I} d(x_i - x) = 0 \Leftrightarrow \lim_{i \in I} d_n(x_i - x) = 0 \quad \forall n \in \mathbb{N}.$$

Proof. Because the pseudometrics are bounded by 1, the sum converges. The pseudometric properties can be easily verified. Suppose a net $(x_i)_{i \in I}$ converges to some x with respect to d . Then for $m \in \mathbb{N}$, we have

$$d_m(x_i, x) \leq 2^m \sum_{n \in \mathbb{N}} \frac{1}{2^n} d_n(x_i, x) = 2^m d(x_i, x) \rightarrow 0.$$

However, let $(x_i)_{i \in I}$ converge with respect to all d_n . Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that

$$\sum_{n=N+1}^{\infty} \frac{1}{2^n} < \epsilon.$$

Then

$$\lim_{i \in I} d(x_i, x) = \lim_{i \in I} \sum_{n=0}^N \frac{1}{2^n} d_n(x_i, x) + \lim_{i \in I} \sum_{n=N+1}^{\infty} \frac{1}{2^n} d_n(x_i, x) < \sum_{n=0}^N \frac{1}{2^n} \lim_{i \in I} d_n(x_i, x) + \epsilon = \epsilon.$$

Since ϵ is arbitrary, the statement follows. $\square$

We now present the main theorem.

Theorem 1.3.4. [49, Proposition 8.1] Suppose that the topology of a locally convex Hausdorff space $\mathcal{T}$ is generated by a countable set $\mathcal{P}$ of seminorms. Then, $\mathcal{T}$ is metrizable.

Proof. Let $(p_n)_{n \in \mathbb{N}}$ be the seminorms. Then, using Lemma 1.3.2 and Lemma 1.3.3, we get a pseudometric with

$$d(x, y) = \sum_{n \in \mathbb{N}} \frac{1}{2^n} \frac{p_n(x - y)}{1 + p_n(x - y)}.$$

Since we are in a Hausdorff space, d is a metric. $\square$

Having a metrizable space is very useful because it allows us to work with sequences instead of nets, which is much more convenient. First, we need the following lemma as a stepping stone.

Lemma 1.3.5. [49, Proposition 8.2] Let $\mathcal{T}$ be metrizable with metric d . Let $(x_i)_{i \in I} \subseteq \mathcal{T}$ be a Cauchy net and $S := \{x_i\}_{i \in I}$. Then there exists a sequence $(y_n)_{n \in \mathbb{N}} \subseteq S$, which is equivalent to $(x_i)_{i \in I}$.

Proof. Let $n \in \mathbb{N}^*$. Since $(x_i)_{i \in I}$ is a Cauchy net, there exists some $i_n \in I$ such that

$$d(x_i, x_j) < \frac{1}{n} \quad \forall i, j \geq i_n.$$

Therefore, we choose $y_n := x_{i_n}$. Ensuring that $\dots \leq i_{n-1} \leq i_n \leq i_{n+1} \dots$ completes the proof. $\square$

Now, we can explicitly show what it means to replace nets with sequences.

Theorem 1.3.6. *Let $\mathcal{T}$ be metrizable.*

1. [49, Proposition 8.2] *For any $\mathcal{S} \subseteq \mathcal{T}$, the closure of $\mathcal{S}$ coincides with the sequential closure and the sequential hull.*
2. [49, Proposition 8.2] *The completion of $\mathcal{T}$, coincides with its sequential completion, i.e., using equivalence classes of sequences instead of nets produces the same space.*
3. [49, Proposition 8.5] *A linear operator $A : \mathcal{T} \rightarrow \mathcal{T}_2$ is continuous if and only if it is sequentially continuous.*

Proof. For all proofs, we use the previous lemma.

1. Let $x \in \overline{\mathcal{S}}$. Then, there exists a net in $\mathcal{S}$ that converges to it. By Lemma 1.3.5, we can use an equivalent sequence instead.
2. Each equivalence class of Cauchy nets contains a sequence according to Lemma 1.3.5.
3. Suppose A is sequentially continuous but not continuous. Then there exists a net $(x_i)_{i \in I} \subseteq \mathcal{T}$ converging to x and $p \in \mathcal{P}_1$ such that

$$p(Ax_i - Ax)$$

does not converge to $\mathbf{0}$. This means that there exists some $\epsilon > \mathbf{0}$ such that for all $i \in I$, there exists some $j_i \in I$ with $j_i \geq i$ such that

$$p(Ax_{j_i} - Ax) \geq \epsilon.$$

Then, we define a net such that $y_i := x_{j_i}$, which is equivalent to $(x_i)_{i \in I}$. This has the property that

$$p(Ay_i - Ax) \geq \epsilon \quad \forall i \in I.$$

Now, thanks to Lemma 1.3.5, there exists an equivalent sequence $(z_n)_{n \in \mathbb{N}}$, giving us

$$p(Ay_i - Ax) \leq p(A(y_i - z_n)) + p(A(z_n - x)).$$

Since the right-hand side approaches $\mathbf{0}$, we obtain a contradiction. $\square$

Metrizable spaces are very useful because they allow us to prove certain theorems. A very useful result is the uniform boundedness principle, which is usually proved in the setting of normed spaces. Here, we prove a more general form.

Theorem 1.3.7 (Uniform boundedness principle). [22, Chapter 5] Let $\mathcal{T}_1$ be complete and metrizable. Suppose $F \subseteq L(\mathcal{T}_1, \mathcal{T}_2)$ such that

$$\sup_{T \in F} q(Tx) < \infty \quad \forall x \in \mathcal{T}_1, q \in \mathcal{P}_2.$$

Then for every $q \in \mathcal{P}_2$, there exists a $\tilde{p} \in \tilde{\mathcal{P}}_1$ such that

$$\sup_{T \in F} q(Tx) \leq \tilde{p}(x) \quad \forall x \in \mathcal{T}_1.$$

Proof. Let $q \in \mathcal{P}_2$. For every $n \in \mathbb{N}$, let

$$X_n := \left\{ x \in \mathcal{T}_1 : \sup_{T \in F} q(Tx) \leq n \right\} = \bigcap_{T \in F} \underbrace{\{x \in \mathcal{T}_1 : q(Tx) \leq n\}}_{(*)}.$$

By continuity, the sets marked with (*) are closed. In addition, X_n is closed because it is the intersection of closed subsets. By pointwise boundedness,

$$\bigcup_{n \in \mathbb{N}} X_n = \mathcal{T}_1 \neq \emptyset.$$

By Baire's theorem (see, for instance, [49, Proposition 8.3]), there exists some $m \in \mathbb{N}$ such that X_m contains a non-empty open set O . Since X_m can be written as

$$X_m = \left\{ x \in \mathcal{T}_1 : \sup_{T \in F} q(Tx) \leq m \right\} = \left\{ x \in \mathcal{T}_1 : \sup_{T \in F} q\left(T \frac{1}{m}x\right) \leq 1 \right\} = mX_1,$$

X_1 also contains a non-empty open set O . By symmetry, it also contains $-O$. In addition, X_1 is convex (i.e. $\lambda X_1 + (1-\lambda)X_1 \subseteq X_1$ for all $\lambda \in [0, 1]$), because $q \circ T$ is convex (by the triangular inequality), and the intersection of convex sets is convex. The convexity implies that

$$0 \in \frac{1}{2}O + \frac{1}{2}(-O) \subseteq X_1.$$

Since the topology of $\mathcal{T}$ is generated by the preimage of open sets in $\mathbb{R}_{\geq 0}$ with respect to the semi norms, we can find a semi norm $\tilde{p} \in \tilde{\mathcal{P}}_1$ (using a scaling argument) such that

$$U := \tilde{p}^{-1}([0, 1)) \subseteq X_1.$$

This implies that

$$q(Ty) \leq 1 \quad \forall T \in F, y \in \overline{U}.$$

For $x \in \mathcal{T}_1 \setminus \{0\}$, we get

$$q(Tx) = \tilde{p}(x) \underbrace{q\left(T \frac{1}{\tilde{p}(x)}x\right)}_{\in \overline{U}} \leq \tilde{p}(x).$$

□

Using this theorem, we immediately obtain the following:

Corollary 1.3.8. [22, Chapter 5] Let $\mathcal{T}_1$ be complete and metrizable. Let $(T_n)_{n \in \mathbb{N}} \subseteq L(\mathcal{T}_1, \mathcal{T}_2)$ converge pointwise to some T . Then $T \in L(\mathcal{T}_1, \mathcal{T}_2)$.

Proof. Since the sequence converges pointwise, it holds

$$\sup_{n \in \mathbb{N}} q(T_n x) < \infty \quad \forall x \in \mathcal{T}_1, q \in \mathcal{P}_2.$$

Let $q \in \mathcal{P}_2$. By the uniform boundedness principle (Theorem 1.3.7), there exists some $\tilde{p} \in \tilde{\mathcal{P}}_1$ such that

$$q(Tx) = \lim_{n \rightarrow \infty} q(T_n x) \leq \sup_{n \in \mathbb{N}} q(T_n x) \leq \tilde{p}(x) \quad \forall x \in \mathcal{T}_1.$$

Finally, we use Theorem 1.2.3 to prove that T is continuous. □

Note that this corollary cannot be generalized using nets instead of sequences, as the following result shows.

Proposition 1.3.9. [44, Proposition 6 (e)] Let $T : \mathcal{T} \rightarrow \mathbb{F}$ be linear. Then there exists a net $(T_i)_{i \in I} \subseteq \mathcal{T}'$ such that

$$\lim_{i \in I} T_i x = Tx \quad \forall x \in \mathcal{T}.$$

Proof. Let

$$I := \{V \subseteq \mathcal{N} : V \text{ is a finite-dimensional subspace}\}.$$

Then, for each $V \in I$, $T|_V$ has a finite dimensional domain, so it is continuous. Using the Hahn-Banach extension theorem (see Theorem 1.2.7), we can extend it to some $T_V \in L(\mathcal{T}, \mathbb{F})$. Now, if $v \in V$, then

$$T_V v = Tv$$

This means that $(T_V v)_{V \in I}$ converges to Tv . □

1.4 Infinite sums

In this section, we analyze infinite sums in $\mathcal{T}$. First, we need the following tool to define an infinite sum:

Definition. [5] Let $(F_n)_{n \in \mathbb{N}}$ be a sequence of finite subsets of X such that $F_n \subseteq F_{n+1}$ for all $n \in \mathbb{N}$ and

$$X = \bigcup_{n \in \mathbb{N}} F_n.$$

Then, we call $(F_n)_{n \in \mathbb{N}}$ a nested approximation of X .

All statements in this chapter are trivial if X is finite. Therefore, we restrict the proofs to the cases where X is infinite. Now, we can define infinite sums.

Definition. [18] Let $(x_k)_{k \in X} \subseteq \mathcal{T}$. Then we say that the sum

$$\sum_{k \in X} x_k$$

converges unconditionally, if for any nested approximation $(F_n)_{n \in \mathbb{N}}$ of X the limit

$$\lim_{n \rightarrow \infty} \sum_{k \in F_n} x_k$$

exists and is independent of the choice of nested approximation. If the limit exists for at least one nested approximation such that $|F_n| = n$ for all $n \in \mathbb{N}$, we say that the sum converges. If a sum converges but does not converge unconditionally, then we say that it converges conditionally.

One might ask the following: If a sum converges for any nested approximation, is the limit always the same? The answer is yes. To prove this, we reduce the general case to the case in $\mathbb{F}$ using the Hahn-Banach theorem.

Theorem 1.4.1. [31, Theorem 1.3.1] Let $(x_k)_{k \in X} \subseteq \mathcal{T}$. Suppose that for any nested approximation $(F_n)_{n \in \mathbb{N}}$ of X , the limit

$$\lim_{n \rightarrow \infty} \sum_{k \in F_n} x_k$$

exists. Then, the sum converges unconditionally.

Proof. Suppose there are two different limits, x and y for different nested approximations. Then, there exists some $\hat{l} \in (\text{span}\{x, y\})'$ such that $\hat{l}(x) \neq \hat{l}(y)$. By the Hahn Banach extension theorem (see Theorem 1.2.7), there exists some $l \in \mathcal{T}'$ such that $l(x) \neq l(y)$. Then

$$l(x) = l \left(\lim_{n \rightarrow \infty} \sum_{k \in F_n} x_k \right) = \lim_{n \rightarrow \infty} l \left(\sum_{k \in F_n} x_k \right) = \lim_{n \rightarrow \infty} \sum_{k \in F_n} l(x_k)$$

and

$$l(y) = l \left(\lim_{n \rightarrow \infty} \sum_{k \in \bar{F}_n} x_k \right) = \lim_{n \rightarrow \infty} l \left(\sum_{k \in \bar{F}_n} x_k \right) = \lim_{n \rightarrow \infty} \sum_{k \in \bar{F}_n} l(x_k).$$

On the right-hand side, we have convergence in $\mathbb{F}$. Here, we already know that if a series converges conditionally, then there exists a nested approximation of X where the series does not converge (see [31, Theorem 1.1.3.2]). This is a contradiction; therefore, we have unconditional convergence. $\square$

Similar to infinite sums in the real case, the terms of a converging sum decrease in a certain sense.

Proposition 1.4.2. [48, Theorem 1] Let $(x_k)_{k \in X} \subseteq \mathcal{T}$ such that the sum

$$\sum_{k \in X} x_k$$

converges unconditionally. Then for any $\epsilon > 0$ and $p \in \mathcal{P}$, there exists a finite set $F \subseteq X$ such that

$$p(x_k) < \epsilon \quad \forall k \in X \setminus F.$$

Proof. Choose a nested approximation $(F_n)_{n \in \mathbb{N}}$ of X with $|F_n| = n$ for all $n \in \mathbb{N}$. Because the sequence of partial sums converges, it is a Cauchy sequence. So, there exists an $N \in \mathbb{N}$ such that

$$p \left(\sum_{k \in F_{n+1} \setminus F_n} x_k \right) < \epsilon \quad \forall n \geq N.$$

Choosing $F := F_N$, the statement follows because $F_{n+1} \setminus F_n$ is a singleton. $\square$

Next, we define absolute convergence using seminorms:

Definition. [33, Definition 1.1] Let $(x_k)_{k \in X} \subseteq \mathcal{T}$ such that the sum

$$\sum_{k \in X} p(x_k)$$

converges for all $p \in \mathcal{P}$. Then we say that the sum

$$\sum_{k \in X} x_k$$

converges absolutely.

Note that the order of the summands of

$$\sum_{k \in X} p(x_k)$$

does not matter because we only add non-negative real numbers. Now, we show that convergence in combination with absolute convergence implies unconditional convergence.

Lemma 1.4.3. [31, Theorem 1.2.2] Let $(x_k)_{k \in X} \subseteq \mathcal{T}$ such that the sum

$$\sum_{k \in X} x_k$$

converges absolutely and converges. Then, it converges unconditionally.

Proof. Let $(F_n)_{n \in \mathbb{N}}$ be a nested approximation of X such that the sum converges with respect to this approximation, and let $(\tilde{F}_n)_{n \in \mathbb{N}}$ be another nested approximation of X . Let $p \in \mathcal{P}$. We now check different cases.

1. If $\tilde{F}_n \subseteq F_n$ for all $n \in \mathbb{N}$ then

$$p \left(\sum_{k \in F_n} x_k - \sum_{k \in \tilde{F}_n} x_k \right) = p \left(\sum_{k \in F_n \setminus \tilde{F}_n} x_k \right) \leq \sum_{k \in F_n \setminus \tilde{F}_n} p(x_k) \xrightarrow{n \rightarrow \infty} 0.$$

This implies that both the sequences of partial sums converge to the same point.

2. If $F_n \subseteq \tilde{F}_n$ for all $n \in \mathbb{N}$, we do the same as before, except that F_n and $\tilde{F}_n$ are swapped.
3. Otherwise, let $\hat{F}_n := F_n \cap \tilde{F}_n$. Then, $(\hat{F}_n)_{n \in \mathbb{N}}$ is also a nested approximation of X . Also, $\hat{F}_n \subseteq F_n$ and $\hat{F}_n \subseteq \tilde{F}_n$. By the previous cases, we get

$$\lim_{n \rightarrow \infty} \sum_{k \in F_n} x_k = \lim_{n \rightarrow \infty} \sum_{k \in \hat{F}_n} x_k = \lim_{n \rightarrow \infty} \sum_{k \in \tilde{F}_n} x_k.$$

□

Similar to the real case, absolute convergence implies unconditional convergence. However, in our case, we have to assume that the space is complete.

Theorem 1.4.4. [31, Theorem 1.2.2] Suppose that $\mathcal{T}$ is complete. Let $(x_k)_{k \in X} \subseteq \mathcal{T}$ such that the sum

$$\sum_{k \in X} x_k$$

converges absolutely. Then, it converges unconditionally.

Proof. Let $(F_n)_{n \in \mathbb{N}}$ be a nested approximation of X . Let $p \in \mathcal{P}$. Then, for all $m, n \in \mathbb{N}$ with $m \geq n$ we have

$$p \left(\sum_{k \in F_m \setminus F_n} x_k \right) \leq \sum_{k \in F_m \setminus F_n} p(x_k).$$

This implies that the sequence of partial sums is a Cauchy sequence. It converges by completeness. Since $(F_n)_{n \in \mathbb{N}}$ is arbitrary, we have unconditional convergence. □

The opposite of this statement is false. Indeed, suppose that $\mathcal{T} = \ell^\infty(\mathbb{N}^*)$ and we analyze the series

$$\sum_{k \in \mathbb{N}^*} \frac{1}{k} \delta_k,$$

where $(\delta_k)_l = \delta_{k,l}$ for all $k, l \in \mathbb{N}^*$. Then this sum converges unconditionally to the element

$$\left(1, \frac{1}{2}, \frac{1}{3}, \dots \right),$$

but

$$\sum_{k \in \mathbb{N}^*} \left\| \frac{1}{k} \delta_k \right\|_{\ell^\infty} = \sum_{k \in \mathbb{N}^*} \frac{1}{k} = \infty,$$

Therefore, it does not converge absolutely.

Convention: From now on, if the convergence type of sum is not explicitly stated then we implicitly mean unconditional convergence.

1.5 Weighted ℓ^p spaces

In this section, we introduce weighted ℓ^p - spaces. We assume that X is a countable index set. First, we define the vector space of sequences as

$$\ell := \{(\alpha_k)_{k \in X} \subseteq \mathbb{F}\}.$$

Next, we introduce a positive weight function $w : X \rightarrow \mathbb{R}_{>0}$. If multiple weight functions are required, we use $\bar{w}$ and $\tilde{w}$ as potentially different weight functions. Then, for any sequence $\alpha = (\alpha_k)_{k \in X}$ and any $p \in [1, \infty]$, we define the weighted ℓ^p -norm as

$$\|\alpha\|_{\ell_w^p} := \|\alpha w\|_{\ell^p} = \|(\alpha_k w_k)_{k \in X}\|_{\ell^p}. [5]$$

For $p = 0$ we use the same norm as for $p = \infty$. For $p \in [1, \infty]$, We define the corresponding spaces

$$\ell_w^p := \{\alpha \in \ell : \|\alpha\|_{\ell_w^p} < \infty\}.$$

For $p = 0$, we define

$$\ell_w^0 := \{\alpha \in \ell_w^\infty : \forall \epsilon > 0 : \exists \text{ finite } F \subseteq X : |\alpha_k| w_k < \epsilon \forall k \in X \setminus F\}. [5]$$

For any $\alpha \in \ell_w^p$ and $Y \subseteq X$, we define the restriction of α on Y as

$$(\alpha|_Y)_k := \begin{cases} \alpha_k, & k \in Y \\ 0, & \text{otherwise} \end{cases} [17].$$

We define

$$\ell^{00} := \{\alpha \in \ell : \exists \text{ finite } F \subseteq X : \alpha_k = 0 \forall k \in X \setminus F\}. [5]$$

For $k \in X$, we define δ_k such that

$$(\delta_k)_l := \delta_{k,l} = \begin{cases} 1, & l = k \\ 0, & \text{otherwise.} \end{cases} \quad \forall l \in X.$$

We can easily define an isometric isomorphism between a weighted ℓ^p space and the corresponding unweighted ℓ^p - space, defining the map

$$\omega : \ell_w^p \rightarrow \ell^p, \alpha \mapsto \alpha w.$$

Now, we show that these spaces are complete.

Theorem 1.5.1. [5] Let $p \in \{0\} \cup [1, \infty]$. Then, ℓ_w^p is a Banach space.

Proof. We know that the ℓ^p spaces are Banach spaces. Let $(\alpha^n)_{n \in \mathbb{N}} \subseteq \ell_w^p$ be a Cauchy sequence. Then

$$\|\alpha^m - \alpha^n\|_{\ell_w^p} = \|\alpha^m w - \alpha^n w\|_{\ell^p},$$

Therefore, $(\alpha^n w)_{n \in \mathbb{N}}$ is a Cauchy sequence in ℓ^p , so it converges to some $\beta \in \ell^p$. This implies that $\beta w^{-1} \in \ell_w^p$, so we conclude,

$$\lim_{n \rightarrow \infty} \|\alpha^n - \beta w^{-1}\|_{\ell_w^p} = \lim_{n \rightarrow \infty} \|\alpha^n w - \beta\|_{\ell^p} = 0.$$

□

The dominated convergence theorem is well known in the unweighted case. The following theorem shows that it also holds for the weighted case.

Theorem 1.5.2 (Dominated convergence). [14] Let $p \in [1, \infty)$. Let $(\alpha^n)_{n \in \mathbb{N}} \subseteq \ell_w^p$ converge pointwise to some $\alpha \in \ell_w^p$. Assume that there exists a majorant $\beta \in \ell_w^p$ such that $|\alpha^n| \leq \beta$ (componentwise) for all $n \in \mathbb{N}$. Then

$$\lim_{n \rightarrow \infty} \alpha^n = \alpha.$$

Proof. By definition, $\alpha^n w \in \ell^p$ for all $n \in \mathbb{N}$ and $\beta w \in \ell^p$. In addition, $\alpha^n w$ converges pointwise to αw as $n \rightarrow \infty$ and βw dominates $(\alpha^n w)_{n \in \mathbb{N}}$. So, by dominated convergence,

$$\lim_{n \rightarrow \infty} \|\alpha^n - \alpha\|_{\ell_w^p} = \lim_{n \rightarrow \infty} \|\alpha^n w - \alpha w\|_{\ell^p} = 0.$$

□

Using this theorem, we can show that each α can be approximated by its restrictions to the finite subsets of X .

Corollary 1.5.3. [5] Let $p \in \{0\} \cup [1, \infty)$, $\alpha \in \ell_w^p$ and $(F_n)_{n \in \mathbb{N}}$ be a nested approximation of X . Then

$$\lim_{n \in \mathbb{N}} \alpha|_{F_n} = \alpha.$$

In particular,

$$\alpha = \sum_{k \in X} \alpha_k \delta_k.$$

Proof. First, we analyze the case $p \in [1, \infty)$. Let $\alpha^n := \alpha|_{F_n}$. Then, $(\alpha^n)_{n \in \mathbb{N}}$ converges pointwise to α and is dominated by α . We obtain the result by dominated convergence (see Theorem 1.5.2). If $p = 0$, let $\epsilon > 0$. Then there exists a finite $F \subseteq X$ such that

$$\|\alpha|_{X \setminus F}\|_{\ell_w^0} < \epsilon.$$

Then for large $n \in \mathbb{N}$, we have $F_n \supseteq F$ and we get

$$\|\alpha - \alpha^n\|_{\ell_w^0} = \|\alpha|_{X \setminus F_n}\|_{\ell_w^0} < \epsilon.$$

This proves the above statement. □

Unfortunately, this does not hold in the case of $p = \infty$, as one can see with the sequence $(1)_{k \in X}$. However, we can prove a slightly weaker statement that covers $p = \infty$.

Corollary 1.5.4. [17] Let $p \in \{0\} \cup [1, \infty]$, $\alpha \in \ell_w^p$ and $(F_n)_{n \in \mathbb{N}}$ be a nested approximation of X . Then

$$\lim_{n \rightarrow \infty} \|\alpha|_{F_n}\|_{\ell_w^p} = \|\alpha\|_{\ell_w^p}.$$

Proof. For the case $p < \infty$, we use Corollary 1.5.3 and the continuity of the norm. Otherwise, note that for $m \leq n$, we have

$$\|\alpha|_{F_m}\|_{\ell_w^\infty} \leq \|\alpha|_{F_n}\|_{\ell_w^\infty} \leq \|\alpha\|_{\ell_w^\infty}.$$

This shows that the limit exists and one of the inequality directions. By way of contradiction, assume that

$$\lim_{n \rightarrow \infty} \|\alpha|_{F_n}\|_{\ell_w^\infty} < \|\alpha\|_{\ell_w^\infty}.$$

Then, there exists some $k \in X$ such that

$$\lim_{n \rightarrow \infty} \|\alpha|_{F_n}\|_{\ell_w^\infty} < |\alpha_k|w_k.$$

For large n , it holds $k \in F_n$, contradicting the strict inequality. $\square$

Next, we show that the Hölder inequality also holds in the weighted case. To do this, we have to introduce the Hölder conjugate

$$p' := \begin{cases} 1, & \text{if } p \in \{0, \infty\} \\ \infty, & \text{if } p = 1 \\ \frac{p}{p-1}, & \text{otherwise.} \end{cases}$$

Theorem 1.5.5 (Hölder inequality). [50, Proposition 2.23] Let $p \in \{0\} \cup [1, \infty]$ and p' be the Hölder conjugate of p . Then, for all $\alpha \in \ell_w^p, \beta \in \ell_{w^{-1}}^{p'}$, we have

$$\|\alpha\beta\|_{\ell^1} \leq \|\alpha\|_{\ell_w^p} \|\beta\|_{\ell_{w^{-1}}^{p'}}.$$

Proof. By the Hölder inequality in the unweighted case, we get

$$\|\alpha\beta\|_{\ell^1} = \|\alpha w \beta w^{-1}\|_{\ell^1} \leq \|\alpha w\|_{\ell^p} \|\beta w^{-1}\|_{\ell^{p'}} = \|\alpha\|_{\ell_w^p} \|\beta\|_{\ell_{w^{-1}}^{p'}}.$$

$\square$

Here, we introduce a method to combine sequences.

Definition. [5] Let $p \in \{0\} \cup [1, \infty]$. Let $\alpha \in \ell_w^p, \beta \in \ell_{w^{-1}}^{p'}$, we define

$$\langle \alpha, \beta \rangle_{\ell_w^p, \ell_{w^{-1}}^{p'}} := \sum_{k \in X} \alpha_k \overline{\beta_k}.$$

According to the Hölder inequality (Theorem 1.5.5), this sum converges absolutely and, therefore, unconditionally. This also defines a sesquilinear form.

Our next goal is to introduce an isometric isomorphism between the dual space of the ℓ^p space and a weighted version of it.

Lemma 1.5.6. [50, Proposition 4.27 (1)] Let $p \in \{0\} \cup [1, \infty]$ and

$$\omega' : (\ell^p)' \rightarrow (\ell_w^p)', \beta' \mapsto \beta' \circ \omega.$$

Then, ω' is an isometric isomorphism.

Proof. Linearity is clear. On the one hand,

$$\|\omega'(\beta')\| = \|\beta' \circ \omega\| \leq \|\beta'\| \|\omega\| = \|\beta'\|.$$

On the other hand

$$\|\beta'\| = \|\beta' \circ \omega \circ \omega^{-1}\| \leq \|\beta' \circ \omega\| \|\omega^{-1}\| = \|\beta' \circ \omega\| = \|\omega'(\beta')\|.$$

This implies that ω' is an isometry. We can construct the inverse by

$$\omega'^{-1} : (\ell_w^p)' \rightarrow (\ell^p)', \beta' \mapsto \beta' \circ \omega^{-1},$$

since

$$\omega'(\omega'^{-1}(\beta')) = \omega'(\beta' \circ \omega^{-1}) = \beta' \circ \omega^{-1} \circ \omega = \beta'$$

and

$$\omega'^{-1}(\omega'(\beta')) = \omega'^{-1}(\beta' \circ \omega) = \beta' \circ \omega \circ \omega^{-1} = \beta'.$$

□

Now, we want to characterize the dual spaces of the weighted ℓ^p spaces. We use ω and ω' to reduce the weighted case to the case without weights.

Theorem 1.5.7. [5] Let $p \in \{0\} \cup [1, \infty)$. Then, there is a bijective, isometric, conjugate linear map

$$\iota : \ell_{w^{-1}}^{p'} \rightarrow (\ell_w^p)'$$

with

$$\iota(\beta) = \langle \cdot, \beta \rangle_{\ell_w^p, \ell_{w^{-1}}^{p'}}.$$

Proof. Let

$$\hat{\iota} : \ell^{p'} \rightarrow (\ell^p)'$$

be the bijective, isometric, conjugate linear map given by

$$\hat{\iota}(\beta) = \langle \cdot, \beta \rangle_{\ell^p, \ell^{p'}}.$$

Now we define

$$\iota := \omega' \circ \hat{\iota} \circ \omega^{-1}.$$

This is a composition of three isometric bijections; therefore, it is an isometric bijection. This implies

$$\iota(\beta) = \omega'(\hat{\iota}(\omega^{-1}(\beta))) = \hat{\iota}(\omega^{-1}(\beta)) \circ \omega$$

and so

$$\iota(\beta)(\alpha) = \hat{\iota}(\omega^{-1}(\beta))(\omega(\alpha)) = \langle \omega(\alpha), \omega^{-1}(\beta) \rangle_{\ell^p, \ell^{p'}} = \langle \alpha, \beta \rangle_{\ell_w^p, \ell_{w^{-1}}^{p'}},$$

because the weights cancel. The conjugate linearity is easy to verify. $\square$

1.5.1 Synthesis operators

Since in frame theory, some operators are defined in terms of sums, we provide a theorem that gives us continuity for free if it is well defined.

Theorem 1.5.8. [18, Lemma 3.2.1] Let $p \in \{0\} \cup [1, \infty]$ and $(v_k)_{k \in X} \subseteq \mathcal{T}$ such that the operator

$$D_v : \ell_w^p \rightarrow \mathcal{T}, D_v \alpha := \sum_{k \in X} \alpha_k v_k$$

is well-defined. Then, D_v is continuous.

Proof. Let $(F_n)_{n \in \mathbb{N}}$ be a nested approximation of X . We define the operator

$$D_n : \ell_w^p \rightarrow \mathcal{T}, \alpha \mapsto \sum_{k \in F_n} \alpha_k v_k.$$

Then, D_n is linear and bounded for all $n \in \mathbb{N}$. Additionally, the family $(D_n)_{n \in \mathbb{N}}$ converges pointwise to D_v . By Corollary 1.3.8, D_v is continuous. $\square$

Next, we want to show that if such an operator is continuous on ℓ^{00} , then the sum even makes sense on the whole space.

Theorem 1.5.9. [49, Theorem 5.1] Let $p \in \{0\} \cup [1, \infty)$ and $\mathcal{T}$ be complete. Let $(v_k)_{k \in X} \subseteq \mathcal{T}$. Suppose the operator

$$D_v : \ell^{00} \rightarrow \mathcal{T}, \alpha \mapsto \sum_{k \in X} \alpha_k v_k$$

is well-defined and continuous with respect to ℓ_w^p in the domain. Then the unique continuous extension can be written as

$$\bar{D}_v : \ell_w^p \rightarrow \mathcal{T}, \alpha \mapsto \sum_{k \in X} \alpha_k v_k.$$

Proof. Since ℓ^{00} is dense in ℓ_w^p (Corollary 1.5.3), we obtain a unique extension $\bar{D}_v \in L(\ell_w^p, \mathcal{T})$. Let $(F_n)_{n \in \mathbb{N}}$ be a nested approximation of X and $\alpha^n = \alpha|_{F_n}$. Then according to Corollary 1.5.3,

$$\bar{D}_v \alpha = \bar{D}_v \lim_{n \rightarrow \infty} \alpha^n = \lim_{n \rightarrow \infty} D_v \alpha^n = \lim_{n \rightarrow \infty} \sum_{k \in F_n} \alpha_k v_k = \sum_{k \in X} \alpha_k v_k.$$

$\square$

We need to exclude $p = \infty$, since ℓ^{00} is not dense in ℓ_w^∞ . Additionally, if we choose $w = (1)_{k \in X}$, $\mathcal{T} = \ell^\infty$ and $(v_k)_{k \in X} = (\delta_k)_{k \in X}$, then the operator

$$D_v : \ell^{00} \rightarrow \ell^\infty, \alpha \mapsto \sum_{k \in X} \alpha_k \delta_k$$

is well-defined and continuous, because it is the identity. Unfortunately,

$$\tilde{D}_v : \ell^\infty \rightarrow \ell^\infty, \alpha \mapsto \sum_{k \in X} \alpha_k \delta_k$$

is not well-defined, since for $\alpha = (1)_{k \in X}$, the sum

$$\sum_{k \in X} \delta_k$$

does not converge in the norm in ℓ^∞ . If we assume that $\mathcal{T} = \mathbb{F}$ then we can include $p = \infty$.

Theorem 1.5.10. [38] Let $p \in \{0\} \cup [1, \infty]$. Let $(c_k)_{k \in X} \subseteq \mathbb{F}$. Suppose the operator

$$D_c : \ell^{00} \rightarrow \mathbb{F}, \alpha \mapsto \sum_{k \in X} \alpha_k c_k$$

is well-defined and continuous with respect to the ℓ_w^p -norm in the domain. Then the operator

$$\tilde{D}_c : \ell_w^p \rightarrow \mathbb{F}, \alpha \mapsto \sum_{k \in X} \alpha_k c_k$$

is well-defined and continuous.

Proof. Let $K := \|D_c\|$ with respect to the ℓ_w^p -norm on ℓ^{00} . Suppose that there exists some $\alpha \in \ell_w^p$ such that the sum does not converge unconditionally. This implies that the sum does not converge absolutely. Then there exists a finite set $F \subseteq X$ such that

$$\sum_{k \in F} |\alpha_k c_k| > K.$$

By scaling, we can assume, without loss of generality, that $\|\alpha\|_{\ell_w^p} = 1$. Choosing $\beta \in \ell_w^p$ such that $|\beta_k| = |\alpha_k|$ and $\beta_k c_k \geq 0$ for all $k \in X$, we get

$$\sum_{k \in F} \beta_k c_k > K.$$

This implies

$$|D_c \beta|_F > K = \|D_c\| \geq |D_c \beta| \geq |D_c \beta|_F,$$

This is a contradiction; therefore, $\tilde{D}_c$ is well-defined. By Theorem 1.5.8, $\tilde{D}_c$ is continuous. $\square$

We also require the following result:

Theorem 1.5.11. [50, Proposition 2.26] Let $p \in \{0\} \cup [1, \infty]$. Let $\alpha \in \ell$ such that for all $\beta \in \ell_w^p$ we have

$$\sum_{k \in X} \alpha_k \beta_k$$

converges unconditionally (or absolutely). Then, $\alpha \in \ell_{w^{-1}}^{p'}$.

Proof. First, we assume that $p < \infty$. Then let

$$T : \ell_w^p \rightarrow \mathbb{F}, \beta \mapsto \sum_{k \in X} \alpha_k \beta_k.$$

By Theorem 1.5.8, $T \in (\ell_w^p)'$. By duality (Theorem 1.5.7), $\alpha \in \ell_{w^{-1}}^{p'}$. For $p = \infty$, choose β such that $\alpha_k \beta_k \geq 0$ and $|\beta_k| w_k = 1$ for all $k \in X$. Then

$$\|\alpha\|_{\ell_{w^{-1}}^1} = \sum_{k \in X} |\alpha_k| w_k^{-1} = \sum_{k \in X} |\alpha_k| |\beta_k| = \sum_{k \in X} \alpha_k \beta_k < \infty.$$

□

1.5.2 Analysis operators

Next, we analyze operators, which are defined in terms of the elements of $\mathcal{T}'$. We want to show that for such an operator, well-definedness is sufficient to obtain continuity.

Theorem 1.5.12. [18, Exercise 3.7] Let $p \in \{0\} \cup [1, \infty]$ and $\mathcal{T}$ be complete and metrizable. Let $(v'_k)_{k \in X} \subseteq \mathcal{T}'$. Suppose the operator

$$C_{v'} : \mathcal{T} \rightarrow \ell_w^p, v \mapsto (v'_k(v))_{k \in X}$$

is well-defined. Then it is continuous.

Proof. Let $(F_n)_{n \in \mathbb{N}}$ be a nested approximation of X . We define

$$C_n : \mathcal{T} \rightarrow \ell_w^p, v \mapsto (v'_k(v))_{k \in X|F_n}.$$

Then, C_n is a finite linear combination of continuous operators and is therefore continuous. Also, for some fixed $v \in \mathcal{T}$, we have

$$\|C_n v\|_{\ell_w^p} \leq \|C_{v'} v\|_{\ell_w^p}.$$

Therefore, $(C_n)_{n \in \mathbb{N}}$ is pointwise bounded, so we apply the uniform boundedness principle (Theorem 1.3.7) to get $\tilde{p} \in \tilde{\mathcal{P}}$ such that

$$\sup_{n \in \mathbb{N}} \|C_n v\|_{\ell_w^p} \leq \tilde{p}(v) \quad \forall v \in \mathcal{T}.$$

Using Corollary 1.5.4, we get

$$\|C_{v'} v\|_{\ell_w^p} = \lim_{n \rightarrow \infty} \|C_n v\|_{\ell_w^p} \leq \tilde{p}(v).$$

□

Our next goal is to show that if such an operator is densely defined, then the continuous extension still has this form.

Theorem 1.5.13. [18, Lemma 3.2.6] Let $p \in \{0\} \cup [1, \infty]$. Let $(v'_k)_{k \in X} \subseteq \mathcal{T}'$. Let $V \subseteq \mathcal{T}$ be a dense subspace. Suppose the operator

$$C_{v'} : V \rightarrow \ell_w^p, v \mapsto (v'_k(v))_{k \in X}$$

is well-defined and continuous. Then its unique continuous extension is of the form

$$\bar{C}_{v'} : \mathcal{T} \rightarrow \ell_w^p, v \mapsto (v'_k(v))_{k \in X}.$$

Proof. Let $\bar{C}_{v'}$ be the unique continuous extension of $C_{v'}$. Let $v \in \mathcal{T}$ and $(v_i)_{i \in I} \subseteq V$ converge to v . Then for any $k \in X$, the net $(v'_k(v_i))_{i \in I}$ converges to $v'_k(v)$, so we have the pointwise limit

$$\lim_{i \in I}^{\text{pw}} T v_i = (v'_k(v))_{k \in X}.$$

On the other hand, $(C_{v'} v_i)_{i \in I}$ is a Cauchy net and converges in ℓ_w^p . Since convergence in ℓ_w^p implies pointwise convergence and the pointwise limit is unique, we conclude that $(C_{v'} v_i)_{i \in I}$ converges to $(v'_k(v))_{k \in X}$ in ℓ_w^p . $\square$

1.5.3 Weak-* topologies

In this chapter, we introduce the weak-* topologies of certain ℓ_w^p spaces and show some of their properties. For $p \in [1, \infty]$, we define

$$p^* := \begin{cases} 0, & \text{if } p = 1 \\ p', & \text{otherwise.} \end{cases}$$

Note that p^* is chosen such that $(\ell_{w^{-1}}^{p^*})'$ is isometrically isomorphic to ℓ_w^p . We introduce the notation

$$\langle \alpha, \beta \rangle_{\ell_w^p, \ell_{w^{-1}}^{p^*}} := \sum_{k \in X} \alpha_k \overline{\beta_k}. [5]$$

This allows us to define the weak-* topology.

Definition. [50] We define the weak-* topology on $\mathcal{T}'$ as the topology generated by the seminorms

$$x' \mapsto |x'(x)|, x \in \mathcal{T}.$$

So, for $p \in [1, \infty]$, the weak-* topology on ℓ_w^p is the topology generated by the seminorms

$$\left| \langle \cdot, \alpha \rangle_{\ell_w^p, \ell_{w^{-1}}^{p^*}} \right|, \alpha \in \ell_{w^{-1}}^{p^*}.$$

It is easy to see that if a net converges with respect to this topology, then it converges pointwise, since we can take $\delta_k \in \ell_{w^{-1}}^{p^*}$ for all $k \in X$. Our first goal is to show that these spaces are sequentially complete with respect to this topology, that is, each Cauchy sequence converges.

Theorem 1.5.14. [5] Let $p \in [1, \infty]$. Then, ℓ_w^p is sequentially complete with respect to the weak-* topology.

Proof. Let $(\alpha^n)_{n \in \mathbb{N}} \subseteq \ell_w^p$ be a Cauchy sequence. Then, for each α^n , we take the corresponding linear continuous functional l_n using duality (see Theorem 1.5.7). For each $\beta \in \ell_{w^{-1}}^{p^*}$, we obtain that $(l_n(\beta))_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{F}$ and therefore converges to some $l(\beta)$. It is easy to show that l is linear. By Corollary 1.3.8, l is continuous and can therefore be identified with some $\alpha \in \ell_w^p$, which is the weak-* limit of the sequence. $\square$

One might ask whether the spaces are complete (with respect to general nets). To answer this question, we first introduce the algebraic dual space of a vector space.

Definition. [32] We define $\mathcal{V}^*$ as the vector space of all linear functionals on $\mathcal{V}$.

Now, we can show the following proposition:

Proposition 1.5.15. [44, Proposition 6 (e)] The weak-* completion of $\mathcal{T}'$ is $\mathcal{T}^*$.

Proof. Let $(v'_i)_{i \in I}$ be a Cauchy net. Then,

$$v'(v) := \lim_{i \in I} v'_i(v) \quad \forall v \in \mathcal{T},$$

is well defined because $\mathbb{F}$ is complete. Linearity follows from the linearity of the limit (see Proposition 1.1.3). On the other hand, let $T \in \mathcal{T}^*$. Then, using Proposition 1.3.9, there exists a net in $\mathcal{T}'$ converging to T . $\square$

This directly implies that ℓ_w^p is not complete with respect to the weak-* topology if X is infinite, since the completion yields the set of linear functionals on the predual. Next, we show that each element can be approximated by restrictions on finite sets in the weak-* sense.

Theorem 1.5.16. [5] Let $p \in [1, \infty]$ and $(F_n)_{n \in \mathbb{N}}$ be a nested approximation of X and $\alpha \in \ell_w^p$. Then $(\alpha|_{F_n})_{n \in \mathbb{N}}$ converges to α in the weak-* sense. In particular, ℓ^{00} is dense in ℓ_w^p with respect to the weak-* topology.

Proof. For any $\beta \in \ell_{w^{-1}}^{p^*}$, we get

$$\left| \langle \alpha - \alpha|_{F_n}, \beta \rangle_{\ell_w^p, \ell_{w^{-1}}^{p^*}} \right| = \left| \sum_{k \in X \setminus F_n} \alpha_k \bar{\beta}_k \right| \xrightarrow{n \rightarrow \infty} 0.$$

$\square$

Because working with weak-* convergence can be complicated, we provide a simpler characterization.

Proposition 1.5.17. [50, Corollary 5.6], [22] Let $p \in [1, \infty]$ and $(\alpha^n)_{n \in \mathbb{N}} \subseteq \ell_w^p$. Then, this sequence converges with respect to the weak-* topology if and only if it converges pointwise and $(\|\alpha^n\|_{\ell_w^p})_{n \in \mathbb{N}}$ is a bounded sequence.

Proof. We show both directions.

$\Rightarrow$: Suppose the weak-* limit is α . Pointwise convergence follows by taking $\delta_k \in \ell_{w^{-1}}^{p*}$ for all $k \in X$, since

$$\alpha_k^n - \alpha_k = \langle \alpha^n - \alpha, \delta_k \rangle_{\ell_w^p, \ell_{w^{-1}}^{p*}} \xrightarrow{n \rightarrow \infty} 0.$$

Also,

$$\sup_{n \in \mathbb{N}} \left| \langle \alpha^n, \beta \rangle_{\ell_w^p, \ell_{w^{-1}}^{p*}} \right| < \infty \quad \forall \beta \in \ell_{w^{-1}}^{p*}.$$

By duality (see Theorem 1.5.7), the elements $\alpha^n \in \ell_w^p$ correspond to the elements $\langle \alpha^n, \cdot \rangle_{\ell_w^p, \ell_{w^{-1}}^{p*}} \in (\ell_{w^{-1}}^{p*})'$. By the uniform boundedness principle (see Theorem 1.3.7) and

$$\|\alpha^n\|_{\ell_w^p} = \left\| \langle \alpha^n, \cdot \rangle_{\ell_w^p, \ell_{w^{-1}}^{p*}} \right\|,$$

the sequence $(\|\alpha^n\|_{\ell_w^p})_{n \in \mathbb{N}}$ is bounded.

$\Leftarrow$: Suppose that α is the pointwise limit and $(\|\alpha^n\|_{\ell_w^p})_{n \in \mathbb{N}}$ is a bounded sequence. Since

$$\|\alpha^n - \alpha\|_{\ell_w^p} \leq \|\alpha^n\|_{\ell_w^p} + \|\alpha\|_{\ell_w^p},$$

the sequence $(\|\alpha^n - \alpha\|_{\ell_w^p})_{n \in \mathbb{N}}$ is also bounded. Let $\beta \in \ell_{w^{-1}}^{p*}$. Let $(F_m)_{m \in \mathbb{N}}$ be a nested approximation of X . Then

$$\left| \langle \alpha^n - \alpha, \beta \rangle_{\ell_w^p, \ell_{w^{-1}}^{p*}} \right| \leq \left| \langle \alpha^n - \alpha, \beta|_{F_m} \rangle_{\ell_w^p, \ell_{w^{-1}}^{p*}} \right| + \left| \langle \alpha^n - \alpha, \beta|_{X \setminus F_m} \rangle_{\ell_w^p, \ell_{w^{-1}}^{p*}} \right|.$$

let $\epsilon > 0$. Using the boundedness of $(\|\alpha^n - \alpha\|_{\ell_w^p})_{n \in \mathbb{N}}$, Corollary 1.5.3 and Hölder (Theorem 1.5.5), we get for large m ,

$$\left| \langle \alpha^n - \alpha, \beta|_{X \setminus F_m} \rangle_{\ell_w^p, \ell_{w^{-1}}^{p*}} \right| \leq \|\alpha^n - \alpha\|_{\ell_w^p} \|\beta|_{X \setminus F_m}\|_{\ell_{w^{-1}}^{p*}} < \epsilon.$$

Using pointwise convergence, we get

$$\langle \alpha^n - \alpha, \beta|_{X \setminus F_m} \rangle = \sum_{k \in F_m} (\alpha_k^n - \alpha_k) \overline{\beta_k} \xrightarrow{n \rightarrow \infty} 0,$$

since each of the finitely many summands approaches 0.

□

Recall that we have shown that if a synthesis operator is defined on ℓ^{00} and is continuous, then it can be continuously extended to ℓ_w^p , still having the sum representation. We can obtain a similar result using the weak-* topology. To do this, we introduce the notation

$$\lim^*$$

to indicate converges with respect to the weak-* topology.

Theorem 1.5.18. [49, Theorem 5.1] Let $p \in [1, \infty]$. Let $(v_k)_{k \in X} \subseteq \mathcal{T}$. Suppose that the operator

$$D_v : \ell^{00} \rightarrow \mathcal{T}, \alpha \mapsto \sum_{k \in X} \alpha_k v_k$$

is well-defined and continuous with respect to the weak-* topology on the domain and $R(D_v) \preceq \mathcal{T}$. Then,

$$\bar{D}_v : \ell_w^p \rightarrow \mathcal{T}, \alpha \mapsto \sum_{k \in X} \alpha_k v_k$$

is well-defined and continuous with respect to the weak-* topology on the domain.

Proof. By weak-* density (Theorem 1.5.16) and Corollary 1.2.6, there exists a unique weak-* continuous extension $\bar{D}_v$ of D_v . Let $(F_n)_{n \in \mathbb{N}}$ be a nested approximation of X and $\alpha^n := \alpha|_{F_n}$. Then

$$\bar{D}_v \alpha = \bar{D}_v \lim_{n \rightarrow \infty}^* \alpha^n = \lim_{n \rightarrow \infty} \bar{D}_v \alpha^n = \lim_{n \rightarrow \infty} D_v \alpha^n = \lim_{n \rightarrow \infty} \sum_{k \in F_n} \alpha_k v_k = \sum_{k \in X} \alpha_k v_k.$$

□

Now, we show that such a sesquilinear form is continuous in a certain sense with respect to both slots.

Proposition 1.5.19. [50, Proposition 2.23], [22] Let $p \in \{0\} \cup [1, \infty)$. Let $(\alpha^n)_{n \in \mathbb{N}} \subseteq \ell_w^p$ converge to $\alpha \in \ell_w^p$ in norm and $(\beta^n)_{n \in \mathbb{N}} \subseteq \ell_{w^{-1}}^{p'}$ converge to $\beta \in \ell_{w^{-1}}^{p'}$ in the weak-* topology. Then

$$\lim_{n \rightarrow \infty} \langle \alpha^n, \beta^n \rangle_{\ell_w^p, \ell_{w^{-1}}^{p'}} = \langle \alpha, \beta \rangle_{\ell_w^p, \ell_{w^{-1}}^{p'}}.$$

Proof. We compute

$$\begin{aligned} \langle \alpha^n, \beta^n \rangle_{\ell_w^p, \ell_{w^{-1}}^{p'}} - \langle \alpha, \beta \rangle_{\ell_w^p, \ell_{w^{-1}}^{p'}} &= \langle \alpha^n, \beta^n \rangle_{\ell_w^p, \ell_{w^{-1}}^{p'}} - \langle \alpha, \beta^n \rangle_{\ell_w^p, \ell_{w^{-1}}^{p'}} + \langle \alpha, \beta^n \rangle_{\ell_w^p, \ell_{w^{-1}}^{p'}} - \langle \alpha, \beta \rangle_{\ell_w^p, \ell_{w^{-1}}^{p'}} \\ &= \langle \alpha^n - \alpha, \beta^n \rangle_{\ell_w^p, \ell_{w^{-1}}^{p'}} + \langle \alpha, \beta^n - \beta \rangle_{\ell_w^p, \ell_{w^{-1}}^{p'}}. \end{aligned}$$

We apply Theorem 1.5.5 and Proposition 1.5.17, to get

$$\left| \langle \alpha^n - \alpha, \beta^n \rangle_{\ell_w^p, \ell_{w^{-1}}^{p'}} \right| \leq \|\alpha^n - \alpha\|_{\ell_w^p} \|\beta^n\|_{\ell_{w^{-1}}^{p'}} \lesssim \|\alpha^n - \alpha\|_{\ell_w^p} \rightarrow 0.$$

The second summand becomes small because of weak-* convergence. □

1.6 Infinite matrices

In this whole chapter, a Matrix has the form

$$M : X \times X \rightarrow \mathbb{F}, (k, l) \mapsto M_{k,l}.$$

It is well known that in the finite-dimensional case, matrices can be identified with linear operators on the entire vector space. This is no longer the case in the infinite-dimensional case. A major problem is that matrix-vector multiplication requires the computation of infinite sums. Because infinite sums do not always converge, we must be careful. If we use an infinite matrix as a linear operator, we must know which vectors are allowed for multiplication. This leads to the following definition:

Definition. [23] Let M be a matrix. Then we define the domain as

$$D(M) := \left\{ \alpha \in \ell : \sum_{l \in X} M_{k,l} \alpha_l \text{ converges unconditionally } \forall k \in X \right\}.$$

We can identify each matrix with a linear operator as

$$M : D(M) \rightarrow \ell, (M\alpha)_k := \sum_{l \in X} M_{k,l} \alpha_l.$$

In some cases, we want to identify a matrix with a linear operator. We can show that if a matrix can be identified with a well-defined linear operator on an ℓ^p space, then it is continuous.

Theorem 1.6.1. [23, Lemma 1.1 (a)] Let $p, q \in \{0\} \cup [1, \infty]$. Let M be a matrix such that $D(M) \supseteq \ell_w^p$ and $M\alpha \in \ell_w^q$ for all $\alpha \in \ell_w^p$. Then the operator

$$\ell_w^p \rightarrow \ell_w^q, \alpha \mapsto M\alpha$$

is continuous.

Proof. For $k \in X$, let

$$M_k : \ell_w^p \rightarrow \mathbb{F}, \alpha \mapsto (M\alpha)_k = \sum_{l \in X} M_{k,l} \alpha_l.$$

By Theorem 1.5.8, $M_k \in (\ell_w^p)'$ for all $k \in X$. So for $\alpha \in \ell_w^p$, we have

$$M\alpha = (M_k \alpha)_{k \in X}.$$

Applying Theorem 1.5.12 shows that the operator is continuous. $\square$

In the finite-dimensional case, the operator norm of a matrix is always well-defined. This is no longer the case in the infinite-dimensional case, but we can define it for some matrices.

Definition. [23] Let $p, q \in \{0\} \cup [1, \infty]$ and M be a matrix such that $D(M) \supseteq \ell_w^p$ and $M\alpha \in \ell_w^q$ for all $\alpha \in \ell_w^p$. Then we define the operator norm of a matrix M as

$$\|M\|_{L(\ell_w^p, \ell_w^q)} := \sup_{\|\alpha\|_{\ell_w^p} = 1} \|M\alpha\|_{\ell_w^q} = \sup_{\|\alpha\|_{\ell_w^p} \leq 1} \|M\alpha\|_{\ell_w^q} = \sup_{\alpha \in \ell_w^p \setminus \{0\}} \frac{\|M\alpha\|_{\ell_w^q}}{\|\alpha\|_{\ell_w^p}}.$$

Because Theorem 1.6.1, the norm is always finite.

If a matrix M can be interpreted as an operator in $L(\ell_w^p, \ell_{\tilde{w}}^q)$, we write $M \in \mathcal{M}(\ell_w^p, \ell_{\tilde{w}}^q)$. We also use the shorthand notation $\mathcal{M}(\ell_w^p) := \mathcal{M}(\ell_w^p, \ell_w^p)$. Now, we show that if the matrix operator is continuous on ℓ^{00} , it is well-defined and continuous on the entire space.

Theorem 1.6.2. [23, Lemma 1.1 (b)] Let $p, q \in \{0\} \cup [1, \infty)$ and M be a matrix such that $D(M) \supseteq \ell^{00}$ and the operator

$$M|_{\ell^{00}} : \ell^{00} \rightarrow \ell_{\tilde{w}}^q, \alpha \mapsto M\alpha$$

is continuous from ℓ_w^p to $\ell_{\tilde{w}}^q$. Then $M \in \mathcal{M}(\ell_w^p, \ell_{\tilde{w}}^q)$.

Proof. For $k \in X$, let

$$M_k^{00} : \ell^{00} \rightarrow \mathbb{F}, \alpha \mapsto (M\alpha)_k = \sum_{l \in X} M_{k,l} \alpha_l.$$

It is easy to see that for all $k \in X$, M_k^{00} is continuous. By Theorem 1.5.9, the unique continuous extension is M_k . We then have

$$M\alpha = (M_k \alpha)_{k \in X} \quad \forall \alpha \in \ell^{00}.$$

Applying Theorem 1.5.13 completes the proof. $\square$

Now, the question arises as to whether each linear continuous operator gives a corresponding matrix. This is indeed the case if $p < \infty$.

Theorem 1.6.3. [23, Lemma 1.1 (b)] Let $p \in \{0\} \cup [1, \infty)$ and $T \in L(\ell_w^p)$. Then there exists a unique matrix $M \in \mathcal{M}(\ell_w^p)$, such that

$$M\alpha = T\alpha \quad \forall \alpha \in \ell_w^p.$$

Proof. Let

$$M_{k,l} := (T\delta_l)_k.$$

Then for any $\alpha \in \ell_w^p$, we get by using Corollary 1.5.3 and continuity of T

$$T\alpha = T \sum_{l \in X}^{\ell_w^p} \alpha_l \delta_l = \sum_{l \in X}^{\ell_w^p} \alpha_l T\delta_l.$$

This gives us

$$(T\alpha)_k = \sum_{l \in X} \alpha_l (T\delta_l)_k = \sum_{l \in X} M_{k,l} \alpha_l.$$

Now, suppose there is another matrix $\tilde{M}$ with this property. Then

$$M_{m,l} = (T\delta_m)_k = \sum_{l \in X} \tilde{M}_{k,l} \delta_{l,m} = \tilde{M}_{m,l}$$

showing uniqueness. $\square$

This theorem does not hold for $p = \infty$. Indeed, let

$$\hat{l} : \text{span}(\{(1)_{k \in X}\} \cup \ell^{00}), \lambda(1)_{k \in X} + \sum_{k \in F} \beta_k \delta_k \mapsto \lambda.$$

Then, we use Hahn-Banach (Theorem 1.2.7) to obtain an extension $l \in (\ell^\infty)'$. Now, for fixed $k \in X$, we define

$$T : \ell^\infty \rightarrow \ell^\infty, \alpha \mapsto (l(\alpha)\delta_{k,l})_{l \in X}.$$

Note that $T|_{\ell^0} = 0$, so for $p = 0$, the corresponding matrix representation has to be the 0 -matrix. This implies that the matrix representation of T must be the 0 -matrix, which is a contradiction, since $T \neq 0$.

To define the corresponding continuous linear operator, only pointwise convergence is required. However, if $p < \infty$, we obtain convergence in ℓ_w^p .

Lemma 1.6.4. [23, Lemma 1.1 (a)] Let $p \in \{0\} \cup [1, \infty)$. Let $M \in \mathcal{M}(\ell_w^p)$. Then for all $\alpha \in \ell_w^p$, we have

$$M\alpha = \sum_{l \in X}^{\ell_w^p} (M_{k,l}\alpha_l)_{k \in X}.$$

Proof. Let $(F_n)_{n \in \mathbb{N}}$ be a nested approximation of X and α^n be the restriction on F_n of α . By Corollary 1.5.3, α^n converges to α in ℓ_w^p . This implies that

$$M\alpha = M \left(\lim_{n \rightarrow \infty}^{\ell_w^p} \alpha^n \right) = \lim_{n \rightarrow \infty}^{\ell_w^p} M(\alpha^n) = \lim_{n \rightarrow \infty}^{\ell_w^p} \sum_{l \in F_n} (M_{k,l}\alpha_l)_{k \in X} = \sum_{l \in X}^{\ell_w^p} (M_{k,l}\alpha_l)_{k \in X}.$$

□

This also does not work in ℓ_w^∞ , since one can simply take $M_{k,l} = \delta_{k,l}$, $w = (1)_{k \in X} = \alpha$ to obtain a counter example.

Similar to the finite-dimensional case, matrices can be multiplied; however, we must be careful with the convergence of the resulting entries.

Definition. [12] Let M, N be matrices. Then we define

$$(MN)_{j,l} := \sum_{k \in X} M_{j,k} N_{k,l}$$

if the sum converges unconditionally for all $j, l \in X$.

This multiplication also corresponds to the composition of two continuous linear operators.

Proposition 1.6.5. [34] Let $p, q, r \in \{0\} \cup [1, \infty)$. Let $N \in \mathcal{M}(\ell_w^p, \ell_{\tilde{w}}^q)$ and $M \in \mathcal{M}(\ell_{\tilde{w}}^q, \ell_{\tilde{w}}^r)$. Then, MN is well-defined and $MN \in \mathcal{M}(\ell_w^p, \ell_{\tilde{w}}^r)$. Moreover, MN is the matrix representation of the composition of the operators.

Proof. First, we show that the product is well-defined. We get

$$(N(\delta_m))_k = \sum_{l \in X} N_{k,l} \delta_{l,m} = N_{k,m}$$

and

$$(M(N(\delta_m)))_j = (M(N_{k,m})_{k \in X})_j = \sum_{k \in X} M_{j,k} N_{k,m} = (MN)_{j,m}.$$

Therefore

$$M(N(\delta_m)) = ((MN)_{j,m})_{j \in X} = (MN)\delta_m.$$

This also implies that $D(MN) \supseteq \ell^{00}$. We see that MN and the composition of M and N match on ℓ^{00} , showing that MN is bounded. Using Theorem 1.6.2, we obtain the result. $\square$

1.6.1 Formulas for specific norms

From linear algebra, we already know that the 1-norm of a matrix can be computed as the maximum of the absolute sum of the columns, and the ∞ -norm can be computed as the maximum of the absolute sum of the rows. We show that this also holds in the infinite-dimensional case. To simplify the discussion, we start with the unweighted case.

Theorem 1.6.6. [35, Theorem 2.13 (ii)] Let M be a matrix. Then $M \in \mathcal{M}(\ell^1)$ if and only if

$$\sup_{l \in X} \sum_{k \in X} |M_{k,l}| < \infty.$$

In this case,

$$\|M\|_{L(\ell^1)} = \sup_{l \in X} \sum_{k \in X} |M_{k,l}|.$$

Proof. We show both of these inequalities.

” $\geq$ ”: Suppose $M \in \mathcal{M}(\ell^1)$. Then we compute

$$\sup_{l \in X} \sum_{k \in X} |M_{k,l}| = \sup_{l \in X} \sum_{k \in X} |(M\delta_l)_k| \leq \sup_{\|\alpha\|_{\ell^1}=1} \sum_{k \in X} |(M\alpha)_k| = \sup_{\|\alpha\|_{\ell^1}=1} \|M\alpha\|_{\ell^1} = \|M\|_{L(\ell^1)}.$$

” $\leq$ ”: Suppose the supremum is finite. Let $\|\alpha\|_{\ell^1} = 1$. Then

$$\begin{aligned} \|M\alpha\|_{\ell^1} &= \sum_{k \in X} \left| \sum_{l \in X} M_{k,l} \alpha_l \right| \leq \sum_{k \in X} \sum_{l \in X} |M_{k,l}| |\alpha_l| = \sum_{l \in X} |\alpha_l| \sum_{k \in X} |M_{k,l}| \\ &\leq \underbrace{\sum_{l \in X} |\alpha_l|}_{=1} \sup_{m \in X} \sum_{k \in X} |M_{k,m}| = \sup_{m \in X} \sum_{k \in X} |M_{k,m}|. \end{aligned}$$

The swapping of sums is allowed because of Tonelli’s theorem. $\square$

We can show a similar statement with the ℓ^∞ norm.

Theorem 1.6.7. [35, Theorem 2.6] Let M be a matrix. Then $M \in \mathcal{M}(\ell^\infty)$ if and only if

$$\sup_{k \in X} \sum_{l \in X} |M_{k,l}| < \infty.$$

In this case,

$$\|M\|_{L(\ell^\infty)} = \sup_{k \in X} \sum_{l \in X} |M_{k,l}|.$$

Proof. We show both of these inequalities.

” $\geq$ ” : Suppose $M \in \mathcal{M}(\ell^\infty)$. For every $k \in X$ let $\alpha^k \in \ell^\infty$ such that $|\alpha_l^k| = 1$ and $M_{k,l} \alpha_l^k \geq 0$ for all $l \in X$. Then we compute

$$\begin{aligned} \sup_{k \in X} \sum_{l \in X} |M_{k,l}| &= \sup_{k \in X} (M\alpha^k)_k \leq \sup_{k \in X} \sup_{\|\alpha\|_{\ell^\infty}=1} |(M\alpha)_k| = \sup_{\|\alpha\|_{\ell^\infty}=1} \sup_{k \in X} |(M\alpha)_k| \\ &= \sup_{\|\alpha\|_{\ell^\infty}=1} \|M\alpha\|_{\ell^\infty} = \|M\|_{L(\ell^\infty)}. \end{aligned}$$

The third step is allowed, since supremums can be interchanged.

” $\leq$ ” : Suppose, the supremum of the sum is finite. Let $\|\alpha\|_{\ell^\infty} = 1$. Then

$$\|M\alpha\|_{\ell^\infty} = \sup_{k \in X} \left| \sum_{l \in X} M_{k,l} \alpha_l \right| \leq \sup_{k \in X} \sum_{l \in X} |M_{k,l}| |\alpha_l| \leq \sup_{k \in X} \sum_{l \in X} |M_{k,l}|.$$

□

Now, we analyze how the norms behave on weighted ℓ^p spaces. We introduce the matrix

$$W_{k,l} := w_k \delta_{k,l}$$

which is a diagonal matrix consisting of the elements of w . It is easy to see that

$$W\alpha = w(\alpha),$$

implying that $W \in \mathcal{M}(\ell_w^p, \ell^p)$. The inverse is of the form

$$(W^{-1})_{k,l} = w_k^{-1} \delta_{k,l}$$

and an element in $\mathcal{M}(\ell^p, \ell_w^p)$.

Proposition 1.6.8. [23, Theorem 3.3], [6, Proposition 3] Let M be a matrix. Then $M \in \mathcal{M}(\ell_w^p)$ if and only if $WMW^{-1} \in \mathcal{M}(\ell^p)$. In this case

$$WMW^{-1} = (w_k M_{k,l} w_l^{-1})_{k,l \in X}.$$

Proof. Using the diagram

$$\ell^p \xrightarrow{W^{-1}} \ell_w^p \xrightarrow{M} \ell_w^p \xrightarrow{W} \ell^p,$$

We can easily see that $M \in \mathcal{M}(\ell_w^p)$ implies $WMW^{-1} \in \mathcal{M}(\ell^p)$. On the other hand, the diagram

$$\ell_w^p \xrightarrow{W} \ell^p \xrightarrow{WMW^{-1}} \ell^p \xrightarrow{W^{-1}} \ell_w^p$$

gives us the converse. Now we compute

$$(WM)_{k,m} = \sum_{l \in X} W_{k,l} M_{l,m} = w_k M_{k,m}$$

and

$$(WMW^{-1})_{k,n} = \sum_{m \in X} (WM)_{k,m} W_{m,n}^{-1} = \sum_{m \in X} w_k M_{k,m} W_{m,n}^{-1} = w_k M_{k,n} w_n^{-1}.$$

□

Using this representation, we obtain the following formula for the ℓ_w^p -norm:

Proposition 1.6.9. [23, Theorem 3.3] For any matrix M , we get

$$\|M\|_{L(\ell_w^p)} = \|WMW^{-1}\|_{L(\ell^p)} = \|(w_k M_{k,l} w_l^{-1})_{k,l \in X}\|_{L(\ell^p)},$$

if one of these three expressions makes sense.

Proof. We easily compute

$$\|M\|_{L(\ell_w^p)} = \sup_{\alpha \in \ell_w^p \setminus \{0\}} \frac{\|M\alpha\|_{\ell_w^p}}{\|\alpha\|_{\ell_w^p}} = \sup_{\omega(\alpha) \in \ell^p \setminus \{0\}} \frac{\|\omega(M\alpha)\|_{\ell^p}}{\|\omega(\alpha)\|_{\ell^p}} = (*).$$

Substituting $\beta = \omega(\alpha)$ gives

$$(*) = \sup_{\beta \in \ell^p \setminus \{0\}} \frac{\|\omega(M\omega^{-1}(\beta))\|_{\ell^p}}{\|\beta\|_{\ell^p}} = \|WMW^{-1}\|_{L(\ell^p)}.$$

□

Now, we easily obtain the two corresponding formulas for the $L(\ell_w^1)$ -norm and the $L(\ell_w^\infty)$ -norm.

Theorem 1.6.10. [23, Theorem 3.3] The following formulas hold (if one side makes sense):

1. $\|M\|_{L(\ell_w^1)} = \sup_{l \in X} \sum_{k \in X} |w_k M_{k,l} w_l^{-1}|$
2. $\|M\|_{L(\ell_w^\infty)} = \sup_{k \in X} \sum_{l \in X} |w_k M_{k,l} w_l^{-1}|$

Proof. Using Proposition 1.6.8, Theorem 1.6.6 and Theorem 1.6.7, we easily see that

1. $\|M\|_{L(\ell_w^1)} = \|(w_k M_{k,l} w_l^{-1})_{k,l \in X}\|_{L(\ell^1)} = \sup_{l \in X} \sum_{k \in X} |w_k M_{k,l} w_l^{-1}|,$
2. $\|M\|_{L(\ell_w^\infty)} = \|(w_k M_{k,l} w_l^{-1})_{k,l \in X}\|_{L(\ell^\infty)} = \sup_{k \in X} \sum_{l \in X} |w_k M_{k,l} w_l^{-1}|.$

□

1.6.2 The adjoint

For a given Matrix M , we define its formal adjoint as

$$M_{k,l}^* := \overline{M_{l,k}}.$$

The connection between a matrix and its formal adjoint is the same as that between operators on Banach spaces.

Proposition 1.6.11. [5] Let $p \in [1, \infty]$. If $M \in \mathcal{M}(\ell_w^p)$ then $M^* \in \mathcal{M}(\ell_{w^{-1}}^{p^*}, \ell_{w^{-1}}^{p'})$. Additionally, for all $\alpha \in \ell_w^p$ and $\beta \in \ell_{w^{-1}}^{p^*}$, it holds

$$\langle M\alpha, \beta \rangle_{\ell_w^p, \ell_{w^{-1}}^{p^*}} = \langle \alpha, M^*\beta \rangle_{\ell_w^p, \ell_{w^{-1}}^{p'}}.$$

Proof. Note that $p^* \neq \infty$, therefore ℓ^{00} is norm-dense in $\ell_{w^{-1}}^{p^*}$. First, we assume that $\beta \in \ell^{00}$. Let $F \subseteq X$ be finite such that $\beta = \beta|_F$. We get

$$\begin{aligned} \langle M\alpha, \beta \rangle_{\ell_w^p, \ell_{w^{-1}}^{p^*}} &= \sum_{k \in F} (M\alpha)_k \overline{\beta_k} = \sum_{k \in F} \sum_{l \in X} M_{k,l} \alpha_l \overline{\beta_k} = \sum_{l \in X} \sum_{k \in F} \alpha_l \overline{M_{l,k}^* \beta_k} \\ &= \sum_{l \in X} \alpha_l \overline{(M^*\beta)_l} = \langle \alpha, M^*\beta \rangle_{\ell_w^p, \ell_{w^{-1}}^{p'}}, \end{aligned}$$

where the last equation follows from Theorem 1.5.11. Moreover,

$$\left| \langle \alpha, M^*\beta \rangle_{\ell_w^p, \ell_{w^{-1}}^{p'}} \right| = \left| \langle M\alpha, \beta \rangle_{\ell_w^p, \ell_{w^{-1}}^{p^*}} \right| \leq \|M\|_{L(\ell_w^p)} \|\alpha\|_{\ell_w^p} \|\beta\|_{\ell_{w^{-1}}^{p^*}},$$

according to Theorem 1.5.5. Now we take the supremum of all $\alpha \in \ell_w^{(p')^*} \subseteq \ell_w^p$ such that $\|\alpha\|_{\ell_w^{(p')^*}} = 1$, to get by duality (Theorem 1.5.7) and the Hölder inequality (Theorem 1.5.5)

$$\begin{aligned} \|M^*\beta\|_{\ell_{w^{-1}}^{p'}} &= \sup_{\|\alpha\|_{\ell_w^{(p')^*}} = 1} \left| \langle \alpha, M^*\beta \rangle_{\ell_w^p, \ell_{w^{-1}}^{p'}} \right| \leq \sup_{\|\alpha\|_{\ell_w^{(p')^*}} = 1} \|M\|_{L(\ell_w^p)} \|\alpha\|_{\ell_w^p} \|\beta\|_{\ell_{w^{-1}}^{p^*}} \\ &= \|M\|_{L(\ell_w^p)} \|\beta\|_{\ell_{w^{-1}}^{p^*}}. \end{aligned}$$

This shows that M^* is continuous on ℓ^{00} . By Theorem 1.6.2, we obtain $M^* \in \mathcal{M}(\ell_{w^{-1}}^{p^*}, \ell_{w^{-1}}^{p'})$. The remainder follows from the density. $\square$

Note that the result is useful for $p > 1$, since then $p' = p^*$, meaning that if $M \in \mathcal{M}(\ell_w^p)$ then $M^* \in \mathcal{M}(\ell_{w^{-1}}^{p^*})$. In the case $p = 1$, we get $1' = \infty$ and $1^* = 0$. This means that $M \in \mathcal{M}(\ell_w^1)$ implies $M^* \in \mathcal{M}(\ell_{w^{-1}}^0, \ell_{w^{-1}}^\infty)$. We would like to have $M^* \in \mathcal{M}(\ell_{w^{-1}}^0)$ instead. However, this is generally not the case, as one can see with a matrix having a row of ones and zeroes everywhere else. Let $X = \mathbb{N}$,

$$M_{k,l} := \delta_{k,0}$$

and $w = (1)_{k \in X}$. In this case, $M \in \mathcal{M}(\ell^1)$ as can be seen using the column sums (Theorem 1.6.6). However, $\delta_0 \in \ell^0$, but

$$(M^* \delta_0)_k = \sum_{l \in X} M_{k,l}^* \delta_{l,0} = \sum_{l \in X} M_{l,k} \delta_{l,0} = \sum_{l \in X} \delta_{l,0} \delta_{l,0} = \delta_{0,0} \delta_{0,0} = 1.$$

This implies that $M^* \delta_0 = (1)_{k \in X} \notin \ell^0$. We can slightly change the conditions to obtain the desired results.

Proposition 1.6.12. [5] *Let*

$$M \in \bigcap_{p \in \{0\} \cup [1, \infty]} \mathcal{M}(\ell_w^p).$$

Then

$$M^* \in \bigcap_{q \in \{0\} \cup [1, \infty]} \mathcal{M}(\ell_{w^{-1}}^q).$$

Additionally, it holds

$$\langle M\alpha, \beta \rangle_{\ell_w^p, \ell_{w^{-1}}^{p^*}} = \langle \alpha, M^* \beta \rangle_{\ell_w^p, \ell_{w^{-1}}^{p^*}} \quad \forall p \in [1, \infty], \alpha \in \ell_w^p, \beta \in \ell_{w^{-1}}^{p^*}.$$

Proof. We consider different cases.

- $q \in [1, \infty)$: According to Proposition 1.6.11, it holds

$$M^* \in \mathcal{M}(\ell_{w^{-1}}^{p^*}, \ell_{w^{-1}}^{p'}) \quad \forall p \in [1, \infty].$$

Choosing $p := q' \in (1, \infty]$ gives

$$p^* = (q')^* = q$$

and

$$p' = q'' = q.$$

This implies

$$M^* \in \mathcal{M}(\ell_{w^{-1}}^{p^*}, \ell_{w^{-1}}^{p'}) = \mathcal{M}(\ell_{w^{-1}}^q).$$

- $q = \infty$: We use the formula for the ∞ norm of a matrix (Theorem 1.6.10) to get

$$\|M^*\|_{L(\ell_{w^{-1}}^\infty)} = \sup_{k \in X} \sum_{l \in X} |w_k^{-1} M_{k,l}^* w_l| = \sup_{k \in X} \sum_{l \in X} |w_l M_{l,k} w_k^{-1}| = \|M\|_{L(\ell_w^1)} < \infty.$$

Therefore,

$$M^* \in \mathcal{M}(\ell_{w^{-1}}^\infty).$$

- $q = 0$: Because of Proposition 1.6.11, it holds

$$M^* \in \mathcal{M}(\ell_{w^{-1}}^0, \ell_{w^{-1}}^\infty).$$

For any $k \in X$, it holds

$$M^* \delta_k \in \ell_{w^{-1}}^1 \subseteq \ell_{w^{-1}}^0,$$

because $M^* \in \mathcal{M}(\ell_{w^{-1}}^1)$. This implies

$$M^* \ell_{w^{-1}}^{00} \subseteq \ell_{w^{-1}}^0$$

and by continuity

$$M^* \ell_{w^{-1}}^0 \subseteq \ell_{w^{-1}}^0,$$

showing that $M^* \in \mathcal{M}(\ell_{w^{-1}}^0)$.

□

Using the adjoint, we can easily show that continuity implies continuity in a weak-* sense.

Theorem 1.6.13. [24] Let $p \in (1, \infty]$. Let M be a matrix. If $M \in \mathcal{M}(\ell_w^p)$ then M is continuous in the weak-* sense.

Proof. Note that $p' = p^*$. Let $\beta \in \ell_{w^{-1}}^{p^*}$. Let $(\alpha^i)_{i \in I} \subseteq \ell_w^p$ converge to $\alpha \in \ell_w^p$ in a weak-* sense. Then according to Proposition 1.6.11,

$$\lim_{i \in I} \langle M \alpha^i, \beta \rangle_{\ell_w^p, \ell_{w^{-1}}^{p^*}} = \lim_{i \in I} \langle \alpha^i, M^* \beta \rangle_{\ell_w^p, \ell_{w^{-1}}^{p'}} = \langle \alpha, M^* \beta \rangle_{\ell_w^p, \ell_{w^{-1}}^{p'}} = \langle M \alpha, \beta \rangle_{\ell_w^p, \ell_{w^{-1}}^{p^*}}.$$

□

This also does not work for $p = 1$. Indeed, suppose $X = \mathbb{N}$, $w = (1)_{k \in \mathbb{N}}$ and let

$$M_{k,l} := \delta_{k,0}.$$

Then, the sequence $(\delta_k)_{k \in \mathbb{N}} \subseteq \ell^1$ converges to $\mathbf{0}$ with respect to the weak-* topology. However,

$$(M \delta_m)_k = \sum_{l \in X} M_{k,l} \delta_{l,m} = M_{k,m} = \delta_{k,0}.$$

This implies that $M \delta_m = \delta_0$ and the sequence $(\delta_0)_{m \in \mathbb{N}}$ does not converge to $\mathbf{0}$. If we change the conditions, we can include $p = 1$.

Theorem 1.6.14. [24] Let $p \in [1, \infty]$. Let

$$M \in \bigcap_{q \in \{0\} \cup [1, \infty]} \mathcal{M}(\ell_w^q)$$

Then M is continuous in a weak-* sense on ℓ_w^p .

Proof. Let $\beta \in \ell_{w^{-1}}^{p^*}$. Let $(\alpha^i)_{i \in I} \subseteq \ell_w^p$ converge to $\alpha \in \ell_w^p$ in a weak-* sense. Then according to Proposition 1.6.12

$$\lim_{i \in I} \langle M \alpha^i, \beta \rangle_{\ell_w^p, \ell_{w^{-1}}^{p^*}} = \lim_{i \in I} \langle \alpha^i, M^* \beta \rangle_{\ell_w^p, \ell_{w^{-1}}^{p^*}} = \langle \alpha, M^* \beta \rangle_{\ell_w^p, \ell_{w^{-1}}^{p^*}} = \langle M \alpha, \beta \rangle_{\ell_w^p, \ell_{w^{-1}}^{p^*}}.$$

□

1.7 Banach algebras

In this section, we recall the concept of Banach algebras. The idea is as follows: A certain set of square matrices can be considered as a vector space. Additionally, we can multiply two matrices to obtain another matrix. This multiplication satisfies certain properties. In addition, we can equip this space with a certain norm that satisfies certain properties. Finally, each matrix has its adjoint that satisfies certain properties. We can use all of these properties as axioms for some abstract sets with a structure that is relevant beyond the chosen case. First, we focus on the multiplication of two such objects.

Definition. [20] Let $\mathcal{D}$ be a $\mathbb{C}$ -vector space and $\cdot : \mathcal{D} \times \mathcal{D} \rightarrow \mathcal{D}$ be a bilinear and associative operation. Then, we call $\mathcal{D}$ an algebra.

In some cases, the algebras have certain elements or additional maps.

Definition. Let $\mathcal{D}$ be an algebra.

- [20] If $\mathcal{D}$ contains a neutral element I with respect to $\cdot$, then we call $\mathcal{D}$ a unital algebra.
- [20, Definition 1.1.2] If $\mathcal{D}$ is equipped with a norm $\|\cdot\|$ such that

$$\|AB\| \leq \|A\| \|B\| \quad \forall A, B \in \mathcal{D},$$

then we call $\mathcal{D}$ a normed algebra.

- [20, Definition 1.1.2] If a normed algebra is complete with respect to the norm, then it is called a Banach algebra.
- [20, Definition 3.5.1] If $\mathcal{D}$ has a conjugate linear map $*$: $\mathcal{D} \rightarrow \mathcal{D}$, such that

$$(A^*)^* = A \text{ and } (AB)^* = B^*A^* \quad \forall A, B \in \mathcal{D},$$

then $\mathcal{D}$ is called a $*$ -algebra.

- [30] If $\mathcal{D}$ is a Banach algebra and a $*$ -algebra such that $*$ is an isometry with respect to the norm then $\mathcal{D}$ is called an involutive Banach algebra or a Banach- $*$ algebra.

In some cases, we have an algebra that is not unital. In this case, we can extend it to obtain a similar algebra, which has a unit.

Definition (Unitization). [20] Let $\mathcal{D}$ be a non-unital algebra. Then we define

$$\mathcal{D}^+ := \mathcal{D} \times \mathbb{F}$$

with the multiplication

$$(A, \lambda) \cdot (B, \mu) := (AB + \mu A + \lambda B, \lambda\mu).$$

In this case, $(0, 1)$ is a unit.

1. If $\mathcal{D}$ is normed then we define

$$\|(A, \lambda)\| := \|A\| + |\lambda|.$$

2. If $\mathcal{D}$ is a $*$ -algebra then we define

$$(A, \lambda)^* := (A^*, \overline{\lambda}).$$

It is easy to see that $\mathcal{D}^+$ is a Banach algebra if $\mathcal{D}$ is one. A similar statement holds for $\mathcal{D}$ being a Banach $*$ -algebra.

In some cases, we have an involution $*$ on a Banach algebra $\mathcal{D}$, which is assumed to be continuous. In this case, one can define the norm

$$\|A\| := \max\{\|A\|, \|A^*\|\},$$

which is equivalent to $\|\cdot\|$ and makes $(\mathcal{D}, \|\cdot\|)$ a Banach- $*$ algebra [47]. This means that if we are only interested in the topological structure of a norm, we can assume, without loss of generality, that a continuous involution is automatically isometric by changing the norm.

From now on, we assume that $\mathcal{A}, \mathcal{A}_1, \mathcal{A}_2, \dots$ are unital Banach- $*$ algebras. We have the following classical example.

Example. Let $\mathcal{H}$ be a nontrivial Hilbert space. The vector space $L(\mathcal{H})$ is a unital Banach- $*$ -algebra. Indeed, multiplication corresponds to the composition of two operators. For the norm, we use the operator norm, and the map $*$ corresponds to the adjoint. We will leave the details for the reader.

1.7.1 Inversion

Similar to matrices, some elements have a multiplicative inverse. This leads to the following definition.

Definition. [20, Definition 1.4.1] We say that an element $A \in \mathcal{A}$ is invertible if there exists some $A^{-1} \in \mathcal{A}$ such that

$$AA^{-1} = I = A^{-1}A.$$

It is well known that if $A, B \in \mathcal{A}$ are invertible, then AB is invertible with

$$(AB)^{-1} = B^{-1}A^{-1}.$$

One can easily show that if $A \in \mathcal{A}$ is invertible then A^* is invertible with

$$(A^*)^{-1} = (A^{-1})^*.$$

To simplify the notation, we define

$$\mathcal{A}_{\text{inv}} := \{A \in \mathcal{A} : A \text{ is invertible}\}.$$

Now, we show that any element close to I is invertible.

Theorem 1.7.1 (Von Neumann series). [18, Theorem A.5.3] Let $A \in \mathcal{A}$ such that $\|I - A\| < 1$. Then A is invertible and

$$A^{-1} = \sum_{n \in \mathbb{N}} (I - A)^n.$$

Proof. First, we show that the sum converges absolutely. We get

$$\sum_{n \in \mathbb{N}} \|(I - A)^n\| \leq \sum_{n \in \mathbb{N}} \|I - A\|^n.$$

Since this is a geometric sum with $\|I - A\| < 1$, it converges. By Theorem 1.4.4, we obtain unconditional convergence. Now let $N \in \mathbb{N}$ and $B := I - A$. Then

$$A \sum_{n=0}^N B^n = (I - B) \sum_{n=0}^N B^n = \sum_{n=0}^N B^n - \sum_{n=1}^{N+1} B^n = I - B^{N+1}.$$

A similar computation shows that

$$\left(\sum_{n=0}^N B^n \right) A = I - B^{N+1}.$$

Letting N approach ∞ and using $B^{N+1} \xrightarrow{N \rightarrow \infty} 0$, we get

$$A \sum_{n \in \mathbb{N}} B^n = I = \left(\sum_{n \in \mathbb{N}} B^n \right) A.$$

□

We can immediately show that the inverses of operators close to the identity are uniformly bounded. For convenience, we introduce the following notation. We write $\lesssim$ to indicate that an inequality holds up to a positive constant factor. This factor might depend on the variables quantified before the inequality but not on those quantified after.

Corollary 1.7.2. [20, Theorem 1.4.2 (iv)] Let

$$\mathcal{U} := \{A \in \mathcal{A} : \|A - I\| < 1/2\}.$$

Then $\mathcal{U} \subseteq \mathcal{A}_{inv}$ and

$$\|A^{-1}\| \lesssim 1 \quad \forall A \in \mathcal{U}.$$

Proof. According to Theorem 1.7.1, every $A \in \mathcal{U}$ is invertible. Using the von Neumann series (Theorem 1.7.1), we get for all $A \in \mathcal{U}$,

$$\|A^{-1}\| = \left\| \sum_{n \in \mathbb{N}} (I - A)^n \right\| \leq \sum_{n \in \mathbb{N}} \|(I - A)^n\| \leq \sum_{n \in \mathbb{N}} \|I - A\|^n < \sum_{n \in \mathbb{N}} \left(\frac{1}{2}\right)^n = 2.$$

□

We can make this result slightly more general by replacing I with any other invertible element.

Lemma 1.7.3. [20, Theorem 1.4.2 (iv)] Let $B \in \mathcal{A}_{inv}$. Then there exists a neighborhood $\mathcal{U}_B \subseteq \mathcal{A}$ of B such that $\mathcal{U}_B \subseteq \mathcal{A}_{inv}$ and

$$\|A^{-1}\| \lesssim 1 \quad \forall A \in \mathcal{U}_B.$$

Proof. Let

$$\mathcal{U}_B := \left\{ A \in \mathcal{A} : \|A - B\| < \frac{1}{2\|B^{-1}\|} \right\}.$$

Then

$$\|AB^{-1} - I\| = \|(A - B)B^{-1}\| \leq \|A - B\| \|B^{-1}\| < \frac{1}{2}.$$

This implies that AB^{-1} is invertible and by Corollary 1.7.2

$$\|AB^{-1}\| \lesssim 1 \quad \forall A \in \mathcal{U}_B.$$

Since B is also invertible, $AB^{-1}B = A$ is also invertible. Next

$$\|A^{-1}\| = \|A^{-1}BB^{-1}\| \leq \|A^{-1}B\| \|B^{-1}\| \lesssim 1 \quad \forall A \in \mathcal{U}_B,$$

proving the statement. □

Now, we can show that the inversion is continuous.

Theorem 1.7.4. [20, Theorem 1.4.2 (iv)] Let

$$\Psi : \mathcal{A}_{inv} \rightarrow \mathcal{A}_{inv}, A \mapsto A^{-1}.$$

Then, Ψ is continuous.

Proof. Let $B \in \mathcal{A}_{inv}$. We show that Ψ is continuous at B . For any $A \in \mathcal{A}_{inv}$,

$$A^{-1} - B^{-1} = A^{-1}(I - AB^{-1}) = A^{-1}(B - A)B^{-1}.$$

Therefore

$$\|A^{-1} - B^{-1}\| \leq \|A^{-1}\| \|B - A\| \|B^{-1}\|.$$

We must ensure that the right-hand side becomes small. Since B is fixed, $\|B^{-1}\|$ is also fixed. By Lemma 1.7.3, $\|A^{-1}\|$ is bounded for all A close to B . Since we can make $\|B - A\|$ as small as we want, Ψ is continuous. □

1.7.2 The spectrum

Analyzing the eigenvalues of matrices is very useful. A similar definition can be used for unital algebras. To simplify the notation, we write $\lambda - A$ instead of $\lambda I - A$, if $\lambda \in \mathbb{C}$.

Definition. [20, Definition 2.1.1] Let $A \in \mathcal{A}$. We define the resolvent of A as

$$\rho_{\mathcal{A}}(A) := \{\lambda \in \mathbb{C} : (\lambda - A) \in \mathcal{A}_{\text{inv}}\}.$$

We define the spectrum as

$$\sigma_{\mathcal{A}}(A) := \mathbb{C} \setminus \rho_{\mathcal{A}}(A).$$

In some cases, we might omit the subscript $\mathcal{A}$ if there is no ambiguity in the context.

It can be shown that the spectrum is always nonempty and compact [20, Proposition 2.1.3]. We now show the following.

Lemma 1.7.5. [20] Let $A, B \in \mathcal{A}$ such that $I - AB \in \mathcal{A}_{\text{inv}}$. Then, $I - BA \in \mathcal{A}_{\text{inv}}$.

Proof. Let $C := (I - AB)^{-1}$. Then

$$\begin{aligned} (I - BA)(I + BCA) &= I + BCA - BA - BABCA = I + B(C - I - ABC)A \\ &= I - B((I - AB)C - I)A = I - B(I - I)A = I \end{aligned}$$

and

$$(I + BCA)(I - BA) = I + BCA - BA - BCABA = I + B(C - I - CAB)A = I.$$

This implies that $(I - BA)^{-1} = (I + BCA)$. $\square$

This lemma can be used to prove that the spectrum of a product does not change, except possibly at 0 , when we swap the factors.

Theorem 1.7.6. [20] We have

$$\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\} \quad \forall A, B \in \mathcal{A}.$$

Proof. Let $\lambda \in \rho(AB) \setminus \{0\}$. Then, $\lambda - AB$ is invertible. Dividing by λ gives the invertibility of $I - (\lambda^{-1}A)B$. Then $I - B(\lambda^{-1}A)$ is invertible because of Lemma 1.7.5, and so $\lambda - BA$ is invertible. This implies that

$$\sigma(AB) \setminus \{0\} \supseteq \sigma(BA) \setminus \{0\}$$

The other inclusion follows from the symmetry. $\square$

Since the spectrum of an element is nonempty and compact, there is always an element with the maximal absolute value. This yields the following definition:

Definition. [20, Definition 2.1.2] For $A \in \mathcal{A}$, we define the spectral radius to be

$$r_{\mathcal{A}}(A) := \max\{|\lambda| : \lambda \in \sigma(A)\}.$$

One can show that $r_{\mathcal{A}}(A) \leq \|A\|$ and

$$r_{\mathcal{A}}(A) = \lim_{n \rightarrow \infty} \|A^n\|^{\frac{1}{n}}.$$

See, for instance, [20, Proposition 2.1.3 (iv)].

Definition. [20] Let $\mathcal{D} \subseteq \mathcal{A}$ such that $\mathcal{D}$ is a unital $*$ -algebra with respect to the restricted operations of $\mathcal{A}$. Then, we call $\mathcal{D}$ a unital $*$ -subalgebra.

If we write $\mathcal{A}_1 \leq \mathcal{A}_2$, we assume that $\mathcal{A}_1$ is a unital $*$ -subalgebra of $\mathcal{A}_2$. However, we do not necessarily assume that the norm of $\mathcal{A}_1$ is the restriction of the norm of $\mathcal{A}_2$. To distinguish between the different norms, we sometimes write $\|\cdot\|_{\mathcal{A}_1}$ and $\|\cdot\|_{\mathcal{A}_2}$.

Next, we show that the spectrum becomes larger if the algebra is reduced.

Proposition 1.7.7. [20] If $\mathcal{A}_1 \leq \mathcal{A}_2$ and $A \in \mathcal{A}_1$ then

$$\sigma_{\mathcal{A}_2}(A) \subseteq \sigma_{\mathcal{A}_1}(A).$$

Proof. Let $\lambda \in \sigma_{\mathcal{A}_2}(A)$. Then, $\lambda - A$ is not invertible in $\mathcal{A}_2$. Therefore, it is not invertible in $\mathcal{A}_1$ and so $\lambda \in \sigma_{\mathcal{A}_1}(A)$. $\square$

From this, we immediately obtain the following.

Corollary 1.7.8. [20] If $\mathcal{A}_1 \leq \mathcal{A}_2$ and $A \in \mathcal{A}_1$ then

$$r_{\mathcal{A}_2}(A) \leq r_{\mathcal{A}_1}(A).$$

In some cases, we want to ensure that each element of a subalgebra that is invertible in the larger algebra has its inverse in the subalgebra.

Definition. [27] Let $\mathcal{A}_1 \leq \mathcal{A}_2$. We call $\mathcal{A}_1$ inverse closed in $\mathcal{A}_2$ if for all $A \in \mathcal{A}_1$ which are invertible in $\mathcal{A}_2$ we have

$$A^{-1} \in \mathcal{A}_1.$$

The following can be shown:

Proposition 1.7.9 (Hulanicki). [27, Proposition 3.2] Let $\mathcal{D} \subseteq \mathcal{A}$ be a unital subalgebra such that for each self adjoint $A \in \mathcal{D}$ (i.e., $A^* = A$), we have

$$r_{\mathcal{D}}(A) = r_{\mathcal{A}}(A).$$

Then for each $A \in \mathcal{D}$, we get

$$\sigma_{\mathcal{D}}(A) = \sigma_{\mathcal{A}}(A).$$

1.8 Operators on normed spaces

In this chapter, we discuss some properties of operators in normed spaces. First, we analyzed the invertible operators and properties of the inversion. Next, we discuss the basics of adjoint operators. Subsequently, we go through the projections. Finally, we discuss the definition and properties of the pseudo-inverse of an operator. Hereinafter, we use the following conventions:

- $\mathcal{N}, \mathcal{N}_1, \mathcal{N}_2, \dots$ are non-trivial normed spaces over $\mathbb{F}$.
- $\mathcal{B}, \mathcal{B}_1, \mathcal{B}_2, \dots$ are non-trivial Banach spaces over $\mathbb{F}$.
- $\mathcal{H}, \mathcal{H}_1, \mathcal{H}_2, \dots$ are non-trivial Hilbert spaces over $\mathbb{F}$. The inner product is conjugate and linear with respect to the second slot.

We begin with a useful result.

Proposition 1.8.1. [18, Lemma A.7.1] Let $T \in L(\mathcal{H}_1, \mathcal{H}_2)$. Then $\hat{T} := T|_{K(T)^\perp}$ is injective and

$$R(T) = R(\hat{T}).$$

Proof. Let $x \in K(\hat{T})$. Then

$$0 = \hat{T}x = Tx,$$

so $x \in K(T)$. By the definition of $\hat{T}$, $x \in K(T)^\perp$, so $x = 0$, showing the injectivity of $\hat{T}$. Clearly, $R(\hat{T}) \subseteq R(T)$. Let $y \in R(T)$. Then there exists an $x \in \mathcal{H}_1$ such that

$$Tx = y.$$

Now we can write $x = k + o$, where $k \in K(T)$ and $o \in K(T)^\perp$. Then, we get

$$y = Tx = Tk + To = To = \hat{T}o,$$

proving $y \in R(\hat{T})$. □

1.8.1 The inverse operator

In some cases, it is useful to invert the operators. In this section, we analyze the invertibility of operators and some properties of the inverse of an operator. First, we define the inverse as follows:

Definition. [50] Let $T \in L(\mathcal{B}_1, \mathcal{B}_2)$ be bijective. Then we define the inverse T^{-1} of T to be the unique linear operator from $\mathcal{B}_2$ to $\mathcal{B}_1$ such that

$$TT^{-1} = \text{id}_{\mathcal{B}_2}$$

and

$$T^{-1}T = \text{id}_{\mathcal{B}_1}.$$

One might ask whether the inverse of an operator is always bounded. Because we are in the Banach space setting, the answer is yes, according to the following theorem.

Theorem 1.8.2. (Bounded inverse) [18, Theorem A.5.2] Let $T \in L(\mathcal{B}_1, \mathcal{B}_2)$ be invertible. Then $T^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$.

Now we introduce the notation

$$I(\mathcal{B}_1, \mathcal{B}_2) := \{T \in L(\mathcal{B}_1, \mathcal{B}_2) : T \text{ is invertible}\}$$

and

$$I(\mathcal{B}) := I(\mathcal{B}, \mathcal{B}).$$

It is well known that the composition of invertible operators is also invertible. More precisely, if $T \in I(\mathcal{B}_1, \mathcal{B}_2)$ and $S \in I(\mathcal{B}_2, \mathcal{B}_3)$ then

$$ST \in I(\mathcal{B}_1, \mathcal{B}_3)$$

and

$$(ST)^{-1} = T^{-1}S^{-1}.$$

Using the theory of inversion, we can characterize injective operators with a closed range.

Proposition 1.8.3. [19] Let $T \in L(\mathcal{B}_1, \mathcal{B}_2)$. Then, the following are equivalent:

1. T is injective and has closed range.
2. We have the inequality

$$\|Tx\| \gtrsim \|x\| \quad \forall x \in \mathcal{B}_1.$$

Proof. Both directions can be easily demonstrated.

$\Rightarrow$: Since T is injective, we can define the inverse

$$T^{-1} : R(T) \rightarrow \mathcal{B}_1.$$

Since $R(T)$ is closed, it is a Banach space, and by the bounded inverse theorem (Theorem 1.8.2), T^{-1} is bounded. This implies

$$\|x\| = \|T^{-1}Tx\| \lesssim \|Tx\| \quad \forall x \in \mathcal{B}_1.$$

$\Leftarrow$: If $Tx = 0$ then we get $x = 0$ by the assumed inequality, showing injectivity. Now let $(y_n)_{n \in \mathbb{N}} \subseteq R(T)$ converge to some $y \in \mathcal{B}_2$. Then choose the unique sequence $(x_n)_{n \in \mathbb{N}} \subseteq \mathcal{B}_1$ such that

$$Tx_n = y_n \quad \forall n \in \mathbb{N}.$$

By the assumed inequality,

$$\|x_m - x_n\| \lesssim \|T(x_m - x_n)\| = \|y_m - y_n\|.$$

Since $(y_n)_{n \in \mathbb{N}}$ is a Cauchy sequence, $(x_n)_{n \in \mathbb{N}}$ must also be a Cauchy sequence, and it converges to some $x \in \mathcal{B}_1$. By the boundedness of T , we get

$$y = \lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} Tx_n = Tx,$$

so $y \in R(T)$ showing the closedness of $R(T)$.

□

It suffices to show the lower bound on the dense subspace.

Proposition 1.8.4. [18, Lemma 5.1.7] Let $T \in L(\mathcal{B}_1, \mathcal{B}_2)$ and $V \subseteq \mathcal{B}_1$ be a dense subspace such that

$$\|Tx\| \gtrsim \|x\| \quad \forall x \in V.$$

Then, this inequality holds for all $x \in \mathcal{B}_1$ (with the same implicit constant factor).

Proof. Let $x \in \mathcal{B}_1$ and $(x_n)_{n \in \mathbb{N}} \subseteq V$ converge to x . Then

$$\|Tx\| = \lim_{n \rightarrow \infty} \|Tx_n\| \gtrsim \lim_{n \rightarrow \infty} \|x_n\| = \|x\|.$$

□

1.8.2 The Adjoint

In this section, we restrict ourselves to the setting of Hilbert spaces because, in this case, the definition of the adjoint of an operator is simpler.

Definition. [40] Let $T \in L(\mathcal{H}_1, \mathcal{H}_2)$. Then the adjoint of T is defined to be the unique operator $T^* \in L(\mathcal{H}_2, \mathcal{H}_1)$ such that

$$\langle Tx, y \rangle_{\mathcal{H}_2} = \langle x, T^*y \rangle_{\mathcal{H}_1} \quad \forall x \in \mathcal{H}_1, y \in \mathcal{H}_2.$$

It is also well known that

$$R(T)^\perp = K(T^*)$$

and

$$K(T)^\perp = \overline{R(T^*)},$$

where $K(T)$ is the kernel of T and $\perp$ indicates the orthogonal complement of a set. It is also known that for any $T \in L(\mathcal{H}_1, \mathcal{H}_2)$, it holds

$$\|T\|^2 = \|T^*T\| = \|T^*\|^2.$$

It is important to note that the adjoint of an operator depends on the codomain of the operator. Therefore, it is reasonable to introduce a notation. Let $\mathbf{A}, \mathbf{B}$ be sets and

$$f : \mathbf{A} \rightarrow \mathbf{B}.$$

Let $C \subseteq \mathbf{A}$ and $f(C) \subseteq D \subseteq \mathbf{B}$. Then, we define

$$f|_{C \rightarrow D} : C \rightarrow D, x \mapsto f(x).$$

We show that the adjoint behaves well if we restrict the domain of T to $K(T)^\perp$.

Proposition 1.8.5. Let $T \in L(\mathcal{H}_1, \mathcal{H}_2)$ then

$$(T|_{K(T)^\perp \rightarrow \mathcal{H}_2})^* = T^*|_{\mathcal{H}_2 \rightarrow K(T)^\perp}.$$

Proof. For all $x \in K(T)^\perp, y \in \mathcal{H}_2$, we have

$$\langle x, T^*|_{\mathcal{H}_2 \rightarrow K(T)^\perp} y \rangle_{K(T)^\perp} = \langle x, T^*y \rangle_{\mathcal{H}_1} = \langle Tx, y \rangle_{\mathcal{H}_2} = \langle T|_{K(T)^\perp \rightarrow \mathcal{H}_2} x, y \rangle_{\mathcal{H}_2}.$$

□

The adjoint also behaves well if the codomain of the operator is restricted to the closure of its range.

Proposition 1.8.6. *Let $T \in L(\mathcal{H}_1, \mathcal{H}_2)$. Then*

$$(T|_{\mathcal{H}_1 \rightarrow \overline{R(T)}})^* = T^*|_{\overline{R(T)} \rightarrow \mathcal{H}_1}.$$

Proof. For all $x \in \mathcal{H}_1, y \in \overline{R(T)}$, we have

$$\langle x, T^*|_{\overline{R(T)} \rightarrow \mathcal{H}_1} y \rangle_{\mathcal{H}_1} = \langle x, T^* y \rangle_{\mathcal{H}_1} = \langle Tx, y \rangle_{\mathcal{H}_2} = \langle T|_{\mathcal{H}_1 \rightarrow \overline{R(T)}} x, y \rangle_{\overline{R(T)}}.$$

□

Sometimes, it is useful for an operator to have a closed range. We show that if an operator has a closed range, then its adjoint also has a closed range.

Proposition 1.8.7. *[18, Lemma A.6.1 (ii)] Let $T \in L(\mathcal{H}_1, \mathcal{H}_2)$ have a closed range. Then, T^* also has a closed range.*

Proof. We go through different cases.

1. Assume that T is bijective. Then

$$(T^{-1})^* T^* = (TT^{-1})^* = \text{id}_{\mathcal{H}_2}^* = \text{id}_{\mathcal{H}_2}$$

and

$$T^*(T^{-1})^* = (T^{-1}T)^* = \text{id}_{\mathcal{H}_1}^* = \text{id}_{\mathcal{H}_1},$$

implying that T^* is bijective with $(T^*)^{-1} = (T^{-1})^*$. Therefore, T^* has a closed range.

2. If T is surjective then

$$T|_{K(T)^\perp \rightarrow \mathcal{H}_2}$$

is bijective and has the same range as T according to Proposition 1.8.1. Then, Proposition 1.8.5 yields

$$(T|_{K(T)^\perp \rightarrow \mathcal{H}_2})^* = T^*|_{\mathcal{H}_2 \rightarrow K(T)^\perp}.$$

Because of the previous case, both operators had a closed range. In addition, this operator has the same range as T^* , so T^* has a closed range.

3. In the general case,

$$T|_{\mathcal{H}_1 \rightarrow \overline{R(T)}}$$

is surjective. Using Proposition 1.8.6,

$$(T|_{\mathcal{H}_1 \rightarrow \overline{R(T)}})^* = T^*|_{\overline{R(T)} \rightarrow \mathcal{H}_1}$$

has a closed range because of the previous case. Moreover, this operator has the same range as T^* , since $R(T) = K(T^*)^\perp$ and because of Proposition 1.8.1. This implies that T^* has a closed range.

□

Composing a closed-range operator with its adjoint does not change its range.

Lemma 1.8.8. [40] Let $T \in L(\mathcal{H}_1, \mathcal{H}_2)$ have a closed range. Then

$$R(TT^*) = R(T),$$

and

$$R(T^*T) = R(T^*),$$

Proof. We begin with the first equation. Trivially,

$$R(TT^*) \subseteq R(T).$$

For the other inclusion, let $y \in R(T)$. Then, there exists some $x \in \mathcal{H}_1$ such that $Tx = y$. Now, we write x as $k + o$, where $k \in K(T)$ and $o \in K(T)^\perp = R(T^*)$. Then

$$y = T(k + o) = To.$$

Since $o \in R(T^*)$ and $y = To$, it holds,

$$y \in R(TT^*).$$

The second equation follows from swapping T with T^* and using Proposition 1.8.7. □

1.8.3 Projections

In this section, we discuss the definition of an orthogonal projection.

Definition. [18] Let $P \in L(\mathcal{H})$. We call P an orthogonal projection if it is a projection and one of the following equivalent statements holds:

- $P^* = P$.
- $K(P)^\perp = R(P)$ and $R(P)^\perp = K(P)$.

For a closed subspace $V \subseteq \mathcal{H}$, we define $\pi_{\mathcal{H} \rightarrow V} \in L(\mathcal{H})$ as the unique orthogonal projection such that $R(\pi_{\mathcal{H} \rightarrow V}) = V$.

1.8.4 Pseudo inverse operator

Because the inverse of an operator is very useful but not always defined, the pseudo-inverse operator might help in some cases. Although pseudo inversion has less useful properties than inversion, it is sometimes good enough in Frame Theory as it allows us

to show that the set of localized frames has a certain structure. More on that can be found in the next chapters. Therefore, we show how to construct and characterize the pseudo inverse of an operator.

Before proceeding, we introduce another notation. Let $A \subseteq B$ be sets. Then, we define

$$\text{id}_{A \rightarrow B} : A \rightarrow B, x \mapsto x.$$

We can now construct the pseudo-inverse of an operator with a closed range. We use the idea shown in [18, Lemma A.7.1]. Let $T \in L(\mathcal{H}_1, \mathcal{H}_2)$ have a closed range. Let $\hat{T} := T|_{K(T)^\perp \rightarrow R(T)}$. Then $\hat{T}$ is bijective so we get the inverse

$$\hat{T}^{-1} : R(T) \rightarrow K(T)^\perp.$$

Since $R(T)$ is closed, $\hat{T}^{-1}$ is bounded by the bounded inverse theorem (see Theorem 1.8.2). Finally, we define the pseudo inverse of T as

$$T^\dagger := \text{id}_{K(T)^\perp \rightarrow \mathcal{H}_1} \hat{T}^{-1} \pi_{\mathcal{H}_2 \rightarrow R(T)} \in L(\mathcal{H}_2, \mathcal{H}_1).$$

In particular, $T^\dagger = T^{-1}$ if T is bijective. Keep in mind that the domain of the pseudo-inverse is the codomain of T . Similar to the adjoint operator, in some cases, we must explicitly write the codomain of the restriction to avoid confusion. Now, we show that restricting an operator to the orthogonal complement of its kernel interacts well with the pseudo-inverse.

Proposition 1.8.9. [26] Let $T \in L(\mathcal{H}_1, \mathcal{H}_2)$ have a closed range and $\tilde{T} := T|_{K(T)^\perp \rightarrow \mathcal{H}_2}$. Then

$$\tilde{T}^\dagger = (T|_{K(T)^\perp \rightarrow \mathcal{H}_2})^\dagger = T^\dagger|_{\mathcal{H}_2 \rightarrow K(T)^\perp}.$$

Proof. Let $\tilde{\mathcal{H}}_1 := K(T)^\perp$ and $\hat{T} := T|_{\tilde{\mathcal{H}}_1 \rightarrow R(T)}$. Then

$$T^\dagger|_{\mathcal{H}_2 \rightarrow \tilde{\mathcal{H}}_1} = (\text{id}_{\tilde{\mathcal{H}}_1 \rightarrow \mathcal{H}_1} \hat{T}^{-1} \pi_{\mathcal{H}_2 \rightarrow R(T)})|_{\mathcal{H}_2 \rightarrow \tilde{\mathcal{H}}_1} = \hat{T}^{-1} \pi_{\mathcal{H}_2 \rightarrow R(T)}.$$

Using Proposition 1.8.1, we get

$$\hat{T}^{-1} \pi_{\mathcal{H}_2 \rightarrow R(T)} = \hat{T}^{-1} \pi_{\mathcal{H}_2 \rightarrow R(\tilde{T})}$$

and by injectivity

$$K(\tilde{T})^\perp = \{0\}^\perp = \tilde{\mathcal{H}}_1,$$

where $\perp$ denotes the orthogonal complement in $\tilde{\mathcal{H}}_1$. Therefore

$$T^\dagger|_{\mathcal{H}_2 \rightarrow \tilde{\mathcal{H}}_1} = \hat{T}^{-1} \pi_{\mathcal{H}_2 \rightarrow R(\tilde{T})} = \text{id}_{K(\tilde{T})^\perp \rightarrow \tilde{\mathcal{H}}_1} \hat{T}^{-1} \pi_{\mathcal{H}_2 \rightarrow R(\tilde{T})} = \tilde{T}^\dagger.$$

□

If we restrict the codomain of an operator to its range, it also interacts well with the pseudo-inverse.

Proposition 1.8.10. Let $T \in L(\mathcal{H}_1, \mathcal{H}_2)$ have closed range and $\tilde{T} := T|_{\mathcal{H}_1 \rightarrow R(T)}$. Then

$$\tilde{T}^\dagger = (T|_{\mathcal{H}_1 \rightarrow R(T)})^\dagger = T^\dagger|_{R(T) \rightarrow \mathcal{H}_1}.$$

Proof. Let $\tilde{\mathcal{H}}_2 := R(T)$. Since T and $\tilde{T}$ only differ by the codomain, we get

$$T^\dagger|_{\tilde{\mathcal{H}}_2 \rightarrow \mathcal{H}_1} = (\text{id}_{K(T)^\perp \rightarrow \mathcal{H}_1} \hat{T}^{-1} \pi_{\mathcal{H}_2 \rightarrow R(T)})|_{\tilde{\mathcal{H}}_2 \rightarrow \mathcal{H}_1} = \text{id}_{K(\tilde{T})^\perp \rightarrow \mathcal{H}_1} \hat{T}^{-1} \pi_{\tilde{\mathcal{H}}_2 \rightarrow R(\tilde{T})} = \tilde{T}^\dagger.$$

□

We now discuss some basic properties of the pseudo-inverse operator.

Proposition 1.8.11. Let $T \in L(\mathcal{H}_1, \mathcal{H}_2)$ have a closed range. Then, we obtain the following properties:

1. [18, (A.11)] $K(T)^\perp = R(T^\dagger)$, in particular, $T^\dagger$ has closed range.
2. [18, (A.11)] $R(T)^\perp = K(T^\dagger)$
3. [13, Theorem 2.1. (iii)] $TT^\dagger = \pi_{\mathcal{H}_2 \rightarrow R(T)}$
4. [18, Lemma A.7.2 (ii)] $T^\dagger T = \pi_{\mathcal{H}_1 \rightarrow R(T^\dagger)}$

Proof. Again, let

$$\hat{T} := T|_{K(T)^\perp \rightarrow R(T)}.$$

1. Since $\hat{T}$ is defined on $K(T)^\perp$ and bijective, we get

$$K(T)^\perp = R(\hat{T}^{-1}).$$

Since $\pi_{\mathcal{H}_2 \rightarrow R(T)}$ maps its domain surjectively to $R(T)$, which is the domain of $\hat{T}^{-1}$, we get

$$R(\hat{T}^{-1}) = R(\hat{T}^{-1} \pi_{\mathcal{H}_2 \rightarrow R(T)}).$$

Finally, $T^\dagger = \text{id}_{K(T)^\perp \rightarrow \mathcal{H}_1} \hat{T}^{-1} \pi_{\mathcal{H}_2 \rightarrow R(T)}$, which gives us

$$K(T)^\perp = R(T^\dagger).$$

2. We get

$$x \in R(T)^\perp \Leftrightarrow \pi_{\mathcal{H}_2 \rightarrow R(T)} x = 0 \Leftrightarrow \hat{T}^{-1} \pi_{\mathcal{H}_2 \rightarrow R(T)} x = 0 \Leftrightarrow T^\dagger x = 0 \Leftrightarrow x \in K(T^\dagger).$$

3. Let $x \in \mathcal{H}_1$. Then

$$TT^\dagger x = T\hat{T}^{-1} \pi_{\mathcal{H}_2 \rightarrow R(T)} x = \hat{T}\hat{T}^{-1} \pi_{\mathcal{H}_2 \rightarrow R(T)} x = \pi_{\mathcal{H}_2 \rightarrow R(T)} x.$$

4. We get

$$T^\dagger TT^\dagger T = T^\dagger \pi_{\mathcal{H}_2 \rightarrow R(T)} T = T^\dagger T,$$

Therefore, $T^\dagger T$ is a projection. Since

$$T^\dagger T = \text{id}_{K(T)^\perp \rightarrow \mathcal{H}_1} \hat{T}^{-1} \pi_{\mathcal{H}_2 \rightarrow R(T)} T = \text{id}_{K(T)^\perp \rightarrow \mathcal{H}_1} \hat{T}^{-1} T,$$

we get

$$R(T^\dagger T) = R(\hat{T}^{-1} T) = R(\hat{T}^{-1}) = K(T)^\perp,$$

where the equation in the middle follows from the fact that T maps $\mathcal{H}_1$ surjectively onto the domain of $\hat{T}^{-1}$. Also

$$K(T^\dagger T) = K(\hat{T}^{-1} T) = K(T),$$

where the last equation follows from the injectivity of $\hat{T}^{-1}$. Therefore, $T^\dagger T$ is the orthogonal projection onto $K(T)^\perp = R(T^\dagger)$.

□

It is useful to know which operators have the same range. Therefore, we propose the following:

Proposition 1.8.12. [18, (A.11)] Let $T \in L(\mathcal{H}_1, \mathcal{H}_2)$ have a closed range. Then

$$R(T^*) = R(T^\dagger)$$

and

$$R((T^*)^\dagger) = R(T) = R((T^\dagger)^*).$$

Proof. We use Proposition 1.8.11 and Proposition 1.8.7 to get

$$R(T^*) = K(T)^\perp = R(T^\dagger).$$

Similarly,

$$R((T^*)^\dagger) = K(T^*)^\perp = R(T) = K(T^\dagger)^\perp = R((T^\dagger)^*).$$

□

The pseudo-inverse can be characterized as follows:

Proposition 1.8.13. [18, (A.11)] Let $T \in L(\mathcal{H}_1, \mathcal{H}_2)$ have a closed range and $S \in L(\mathcal{H}_2, \mathcal{H}_1)$ such that the following hold:

1. $K(T)^\perp = R(S)$,
2. $R(T)^\perp = K(S)$,
3. $TSx = x$ for all $x \in R(T)$.

Then, $T^\dagger = S$.

Proof. Suppose that $x \in R(T)^\perp$. Then

$$T^\dagger x = \hat{T}^{-1} \pi_{\mathcal{H}_2 \rightarrow R(T)} x = \hat{T}^{-1} 0 = 0.$$

Since $R(T)^\perp = K(S)$,

$$Sx = 0.$$

This shows that $T^\dagger$ and S match on $R(T)^\perp$. Now let $x \in R(T)$. Then

$$TSx = x.$$

Applying $T^\dagger$ from left on the left-hand side, gives

$$T^\dagger TSx = \pi_{\mathcal{H}_1 \rightarrow R(T^\dagger)} Sx = \pi_{\mathcal{H}_1 \rightarrow K(T)^\perp} Sx = \pi_{\mathcal{H}_1 \rightarrow R(S)} Sx = Sx.$$

Applying $T^\dagger$ to the right-hand side gives

$$T^\dagger x.$$

This shows that $T^\dagger$ and S also match on $R(T)$. □

There is also a second method to characterize the pseudo-inverse. This is similar to the way the inverse is defined.

Theorem 1.8.14. [26] Let $T \in L(\mathcal{H}_1, \mathcal{H}_2)$ have closed range and $S \in L(\mathcal{H}_2, \mathcal{H}_1)$ have closed range such that

$$TS = \pi_{\mathcal{H}_2 \rightarrow R(T)}$$

and

$$ST = \pi_{\mathcal{H}_1 \rightarrow R(S)}.$$

Then, $S = T^\dagger$.

Proof. We prove this statement using the characterization of Proposition 1.8.13. Let

$$T_S := T|_{R(S) \rightarrow R(T)}$$

and

$$S_T := S|_{R(T) \rightarrow R(S)}.$$

Then, T_S and S_T are inverses with respect to each other. We get

$$T_S Sx = TSx = \pi_{\mathcal{H}_2 \rightarrow R(T)} x.$$

Multiplying $S_T = T_S^{-1}$ from the left gives

$$Sx = S_T \pi_{\mathcal{H}_2 \rightarrow R(T)} x.$$

This gives

$$K(S) = K(S_T \pi_{\mathcal{H}_2 \rightarrow R(T)}) = K(\pi_{\mathcal{H}_2 \rightarrow R(T)}) = R(T)^\perp.$$

Similarly,

$$K(T) = R(S)^\perp$$

or

$$K(T)^\perp = R(S).$$

We obtain the third property for free, so $S = T^\dagger$. $\square$

We know that the inverse of the inverse of a bijective operator is the operator itself. The same is true for the pseudo-inverse.

Proposition 1.8.15. [9, (4.7)] Let $T \in L(\mathcal{H}_1, \mathcal{H}_2)$ have a closed range. Then, $(T^\dagger)^\dagger = T$.

Proof. We check the three properties from Proposition 1.8.13.

1. Using Proposition 1.8.11, we get

$$K(T^\dagger)^\perp = R(T)^\perp = \overline{R(T)} = R(T).$$

2. Again, by Proposition 1.8.11, it holds

$$R(T^\dagger)^\perp = K(T)^\perp = K(T).$$

3. For $x \in R(T^\dagger)$, we get by Proposition 1.8.11

$$T^\dagger(T^\dagger)^\dagger x = \pi_{\mathcal{H}_1 \rightarrow R(T^\dagger)} x = x.$$

$\square$

Next, we will show that the pseudo-inverse behaves well if an operator is composed with its adjoint.

Proposition 1.8.16. [18, Lemma A.7.2 (iii)] Let $T \in L(\mathcal{H}_1, \mathcal{H}_2)$ have a closed range. Then

$$(T^*)^\dagger = (T^\dagger)^*.$$

Proof. We check the two properties of Theorem 1.8.14. By Proposition 1.8.12,

$$(T^\dagger)^* T^* = (T T^\dagger)^* = \pi_{\mathcal{H}_2 \rightarrow R(T)}^* = \pi_{\mathcal{H}_2 \rightarrow R(T)} = \pi_{\mathcal{H}_2 \rightarrow R((T^\dagger)^*)}.$$

Similarly,

$$T^* (T^\dagger)^* = (T^\dagger T)^* = \pi_{\mathcal{H}_1 \rightarrow R(T^\dagger)}^* = \pi_{\mathcal{H}_1 \rightarrow R(T^\dagger)} = \pi_{\mathcal{H}_1 \rightarrow R(T^*)}.$$

This shows that

$$(T^*)^\dagger = (T^\dagger)^*.$$

$\square$

Pseudo inversion works well with the composition of an operator with its adjoint.

Proposition 1.8.17. [36, Theorem 1.1 (9), (10)] Let $T \in L(\mathcal{H}_1, \mathcal{H}_2)$ have a closed range. Then, TT^* and T^*T have closed ranges. Moreover,

$$(TT^*)^\dagger = (T^*)^\dagger T^\dagger,$$

and

$$(T^*T)^\dagger = T^\dagger (T^*)^\dagger.$$

Proof. By Proposition 1.8.7, T^* has a closed range. By Lemma 1.8.8, TT^* and T^*T have closed ranges. We show the two properties of Theorem 1.8.14. A combination of Proposition 1.8.16, Proposition 1.8.12 and Lemma 1.8.8 gives

$$R((T^*)^\dagger) = R((T^\dagger)^*) = R((T^\dagger)^* T^\dagger) = R((T^*)^\dagger T^\dagger).$$

Therefore

$$\begin{aligned} (T^*)^\dagger T^\dagger TT^* &= (T^*)^\dagger \pi_{\mathcal{H}_1 \rightarrow R(T^\dagger)} T^* = (T^*)^\dagger \pi_{\mathcal{H}_1 \rightarrow R(T^*)} T^* = (T^*)^\dagger T^* \\ &= \pi_{\mathcal{H}_2 \rightarrow R((T^*)^\dagger)} = \pi_{\mathcal{H}_2 \rightarrow R((T^*)^\dagger T^\dagger)}. \end{aligned}$$

Similarly,

$$TT^* (T^*)^\dagger T^\dagger = T \pi_{\mathcal{H}_1 \rightarrow R(T^*)} T^\dagger = T \pi_{\mathcal{H}_1 \rightarrow R(T^\dagger)} T^\dagger = TT^\dagger = \pi_{\mathcal{H}_2 \rightarrow R(T)} = \pi_{\mathcal{H}_2 \rightarrow R(TT^*)}.$$

By Theorem 1.8.14, the first equation follows. Swapping the roles of T and T^* yields the second equation. $\square$

If we assume that T is injective or surjective, we obtain a slightly stronger result.

Corollary 1.8.18. [36, Theorem 1.1 (9), (10)] Let $T \in L(\mathcal{H}_1, \mathcal{H}_2)$ have a closed range.

1. If T is injective, then T^*T is invertible and

$$(T^*T)^{-1} = T^\dagger (T^*)^\dagger.$$

2. If T is surjective, then TT^* is invertible and

$$(TT^*)^{-1} = (T^*)^\dagger T^\dagger.$$

Proof. We show both cases.

1. Using Proposition 1.8.12

$$R(T^*T) = R(T^*) = K(T)^\perp = \mathcal{H}_1$$

and

$$K(T^*T) = R((T^*T)^*)^\perp = R(T^*T)^\perp = R(T^*)^\perp = K(T) = \{0\}.$$

This implies that T^*T is invertible. Using Proposition 1.8.17, the proof is complete.

2. We define $\mathcal{S} := T^*$. According to Proposition 1.8.7, $\mathcal{S}$ has a closed range. Applying the first case to $\mathcal{S}$ completes the proof.

□

Next, we prove that the pseudo inverse of any operator can be expressed in terms of the pseudo inverse of a self adjoint operator.

Lemma 1.8.19. [36, Corollary 2.8, 2.9] Let $T \in L(\mathcal{H}_1, \mathcal{H}_2)$ have a closed range. Then

$$T^\dagger = (T^*T)^\dagger T^* = T^*(TT^*)^\dagger.$$

Proof. We use Proposition 1.8.17 and Proposition 1.8.16 to get

$$(T^*T)^\dagger T^* = T^\dagger (T^*)^\dagger T^* = T^\dagger \pi_{\mathcal{H}_1 \rightarrow R((T^*)^\dagger)} = T^\dagger \pi_{\mathcal{H}_1 \rightarrow R((T^\dagger)^*)} = T^\dagger \pi_{\mathcal{H}_1 \rightarrow K(T^\dagger)^\perp} = T^\dagger.$$

Similarly, Proposition 1.8.12 yields

$$T^*(TT^*)^\dagger = T^*(T^*)^\dagger T^\dagger = \pi_{\mathcal{H}_1 \rightarrow R(T^*)} T^\dagger = \pi_{\mathcal{H}_1 \rightarrow R(T^\dagger)} T^\dagger = T^\dagger.$$

□

If T is injective or surjective, we obtain a simple formula for the pseudo-inverse.

Corollary 1.8.20. Let $T \in L(\mathcal{H}_1, \mathcal{H}_2)$ have a closed range.

1. [13, Theorem 4.2.] If T is injective, then

$$T^\dagger = (T^*T)^{-1}T^*.$$

2. [18, Lemma A.7.2 (iv)] If T is surjective, then

$$T^\dagger = T^*(TT^*)^{-1}.$$

Proof. Both statements follow directly from Corollary 1.8.18 and Lemma 1.8.19 □

Our goal is to construct a formula for the pseudo-inverse of T without requiring that T is injective or surjective. To do this, we need the following result as a stepping stone:

Lemma 1.8.21. [53, Remark 2.3 (iii)] Let $\delta > 0$ and $T \in L(\mathcal{H}_1, \mathcal{H}_2)$. Then, $T^*T + \delta$ is invertible.

Proof. First,

$$\langle (T^*T + \delta)x, x \rangle = \langle T^*Tx, x \rangle + \langle \delta x, x \rangle = \langle Tx, Tx \rangle + \delta \langle x, x \rangle = \|Tx\|^2 + \delta \|x\|^2 \geq \delta \|x\|^2,$$

The left-hand side is also real. On the other hand by Cauchy-Schwartz,

$$\langle (T^*T + \delta)x, x \rangle \leq \|(T^*T + \delta)x\| \|x\|.$$

Combining these two inequalities and dividing by $\|x\|$ gives us

$$\delta \|x\| \leq \|(T^*T + \delta)x\|.$$

Using Proposition 1.8.3, we obtain that $T^*T + \delta$ is injective and has a closed range. Obviously, $T^*T + \delta$ is self adjoint, therefore

$$R(T^*T + \delta) = K((T^*T + \delta)^*)^\perp = K(T^*T + \delta)^\perp = \{0\}^\perp = \mathcal{H}_1.$$

This implies that $T^*T + \delta$ is surjective and, therefore, invertible. $\square$

The second result is that $(T^*T + \delta)^{-1}T^*$ can be restricted nicely.

Lemma 1.8.22. *Let $T \in L(\mathcal{H}_1, \mathcal{H}_2)$ have a closed range, $\tilde{\mathcal{H}}_1 := K(T)^\perp$ and $\tilde{T} := T|_{\tilde{\mathcal{H}}_1 \rightarrow \mathcal{H}_2}$. Then*

$$((T^*T + \delta)^{-1}T^*)|_{\mathcal{H}_2 \rightarrow \tilde{\mathcal{H}}_1} = (\tilde{T}^*\tilde{T} + \delta)^{-1}\tilde{T}^*.$$

Proof. By Lemma 1.8.21, $T^*T + \delta$ is invertible. According to Proposition 1.8.1, $\tilde{T}$ is injective and, because of Proposition 1.8.5, we get

$$\tilde{T}^* = T^*|_{\mathcal{H}_2 \rightarrow \tilde{\mathcal{H}}_1}.$$

Then $\tilde{T}^*\tilde{T} + \delta \in I(\tilde{\mathcal{H}}_1)$ is invertible because of Lemma 1.8.21. Then

$$(T^*T + \delta)|_{\tilde{\mathcal{H}}_1 \rightarrow \mathcal{H}_1} = T^*\tilde{T} + \delta \text{id}_{\tilde{\mathcal{H}}_1 \rightarrow \mathcal{H}_1}$$

and because of $\tilde{\mathcal{H}}_1 = K(T)^\perp = R(T^*)$, we get

$$(T^*T + \delta)|_{\tilde{\mathcal{H}}_1 \rightarrow \tilde{\mathcal{H}}_1} = \tilde{T}^*\tilde{T} + \delta \text{id}_{\tilde{\mathcal{H}}_1 \rightarrow \tilde{\mathcal{H}}_1}.$$

This implies that

$$(T^*T + \delta)^{-1}|_{\tilde{\mathcal{H}}_1 \rightarrow \tilde{\mathcal{H}}_1} = (\tilde{T}^*\tilde{T} + \delta \text{id}_{\tilde{\mathcal{H}}_1 \rightarrow \tilde{\mathcal{H}}_1})^{-1}$$

and therefore

$$(T^*T + \delta)^{-1}T^*|_{\mathcal{H}_2 \rightarrow \tilde{\mathcal{H}}_1} = (\tilde{T}^*\tilde{T} + \delta \text{id}_{\tilde{\mathcal{H}}_1 \rightarrow \tilde{\mathcal{H}}_1})^{-1}\tilde{T}^*.$$

$\square$

For general T , we obtain a formula that is slightly more complicated but very useful.

Theorem 1.8.23. [52, Corollary 3.4] *Let $T \in L(\mathcal{H}_1, \mathcal{H}_2)$ have a closed range. Then we get*

$$T^\dagger = \lim_{\delta \searrow 0} (T^*T + \delta)^{-1}T^*,$$

where we have convergence with respect to the operator norm.

Proof. Let $\tilde{\mathcal{H}}_1 := R(T^*) = K(T)^\perp \subseteq \mathcal{H}_1$. Then, we define

$$\tilde{T} := T|_{\tilde{\mathcal{H}}_1 \rightarrow \mathcal{H}_2},$$

which is, according to Proposition 1.8.1, injective. Then, $\tilde{T}^*\tilde{T}$ is invertible according to Corollary 1.8.18, and $\tilde{T}^*\tilde{T} + \delta$ is invertible because of Lemma 1.8.21. Then by Lemma 1.8.22,

$$((T^*T + \delta)^{-1}T^*)|_{\mathcal{H}_2 \rightarrow \tilde{\mathcal{H}}_1} = (\tilde{T}^*\tilde{T} + \delta \text{id}_{\tilde{\mathcal{H}}_1 \rightarrow \tilde{\mathcal{H}}_1})^{-1}\tilde{T}^*.$$

Since the inversion is continuous (Theorem 1.7.4), we get

$$\lim_{\delta \searrow 0} (\tilde{T}^*\tilde{T} + \delta \text{id}_{\tilde{\mathcal{H}}_1 \rightarrow \tilde{\mathcal{H}}_1})^{-1} = (\tilde{T}^*\tilde{T})^{-1}.$$

Combining everything and using Corollary 1.8.20, we get

$$\tilde{T}^\dagger = (\tilde{T}^*\tilde{T})^{-1}\tilde{T}^* = \lim_{\delta \searrow 0} (\tilde{T}^*\tilde{T} + \delta \text{id}_{\tilde{\mathcal{H}}_1 \rightarrow \tilde{\mathcal{H}}_1})^{-1}\tilde{T}^* = \lim_{\delta \searrow 0} (T^*T + \delta)^{-1}T^*|_{\mathcal{H}_2 \rightarrow \tilde{\mathcal{H}}_1}.$$

By Proposition 1.8.9,

$$\tilde{T}^\dagger = T^\dagger|_{\mathcal{H}_2 \rightarrow \tilde{\mathcal{H}}_1}.$$

This implies that

$$T^\dagger = \lim_{\delta \searrow 0} (T^*T + \delta)^{-1}T^*.$$

□

One can show that an inverse closed unital $*$ -subalgebra of $L(\mathcal{H})$ is even pseudo-inverse closed. Because this proof is involved, we omit it.

Theorem 1.8.24. [27, Corollary 3.5] *Let $\mathcal{A} \leq L(\mathcal{H})$ be inverse closed. Then $\mathcal{A}$ is pseudo-inverse closed in $L(\mathcal{H})$, that is, for every $T \in \mathcal{A}$ with closed range, we have $T^\dagger \in \mathcal{A}$.*

Sometimes, we also need to compare the operator norm with another norm. This is possible because of the following reasons:

Theorem 1.8.25. [27, Proposition 3.2] *Let $\mathcal{A} \leq L(\mathcal{H})$. Then*

$$\|T\| \leq \|T\|_{\mathcal{A}} \quad \forall T \in L(\mathcal{H}).$$

Proof. First, we assume that T is self-adjoint. Then by Corollary 1.7.8 and [42],

$$\|T\| = r_{L(\mathcal{H})}(T) \leq r_{\mathcal{A}}(T) = \lim_{n \rightarrow \infty} \|T^n\|_{\mathcal{A}}^{\frac{1}{n}} \leq \|T\|_{\mathcal{A}}.$$

For the general case, we get

$$\|T\|^2 = \|T^*T\| \leq \|T^*T\|_{\mathcal{A}} \leq \|T^*\|_{\mathcal{A}} \|T\|_{\mathcal{A}} = \|T\|_{\mathcal{A}}^2.$$

□

2 Frames

Orthonormal bases in Hilbert spaces can be very useful because they provide a way to uniquely decompose an element into its basis parts. In some applications, this is too restrictive, and multiple ways to decompose an element are required. This leads us to frame theory. In this chapter, we introduce frames, which are a generalization of orthonormal bases, and their corresponding operators, that is, the coefficient operator, synthesis operator, and frame operator. The content of this chapter can be found in [18].

Definition. [5] For a sequence $\psi = (\psi_k)_{k \in X} \subseteq \mathcal{H}$, we define the synthesis operator $D_\psi : \ell^2 \rightarrow \mathcal{H}$ (if possible) to be

$$D_\psi(\alpha_k)_{k \in X} := \sum_{k \in X} \alpha_k \psi_k.$$

Next, we introduce the coefficient operator.

Definition. [5] For a sequence $\psi = (\psi_k)_{k \in X} \subseteq \mathcal{H}$, we define the coefficient (or analysis) operator $C_\psi : \mathcal{H} \rightarrow \ell^2$ (if possible) to be

$$C_\psi f := (\langle f, \psi_k \rangle)_{k \in X}.$$

Having defined the coefficient operator, we can easily define a frame. To write it more compactly, we introduce the symbol $\simeq$. We write $A \simeq B$ to indicate $A \lesssim B$ and $B \lesssim A$.

Definition. [5] A sequence $\psi = (\psi_k)_{k \in X}$ is called a frame if $C_\psi f$ is well-defined and

$$\|C_\psi f\|^2 = \sum_{k \in X} |\langle f, \psi_k \rangle|^2 \simeq \|f\|^2, \quad \forall f \in \mathcal{H}.$$

There are many examples of frames.

Example (Gabor frames). Let $\mathcal{H} := L^2(\mathbb{R})$ and $g : \mathbb{R} \rightarrow \mathbb{C}$. Then

$$(e^{2\pi i m(\cdot)} g((\cdot) - n))_{(m,n) \in \mathbb{Z} \times \mathbb{Z}}$$

is a frame for $L^2(\mathbb{R})$ if g is chosen correctly. This is a special case. More general cases and details can be found in [25].

A benefit of a frame is that there are multiple ways to characterize them.

Proposition 2.0.1. *Let $\psi = (\psi_k)_{k \in X} \subseteq \mathcal{H}$ be a sequence. The following are equivalent.*

1. ψ is a frame.
2. [18, Lemma 5.5.1] *The synthesis operator is well-defined, bounded, and surjective.*

3. [18, Corollary 5.5.3, (i)] The coefficient operator is well-defined, bounded, injective and has closed range.

Because the synthesis operator must be surjective, $\mathcal{H}$ must be separable. Otherwise, no frames would exist. Therefore, we assume that $\mathcal{H}$ is separable. The coefficient operator and synthesis operator are adjoint to each other.

Proposition 2.0.2. [28] Let $\psi = (\psi_k)_{k \in X}$ be a frame. Then

$$C_\psi^* = D_\psi$$

and

$$D_\psi^* = C_\psi.$$

By composing the synthesis operator with the coefficient operator, we obtain the frame operator.

Definition. [5] For a frame $\psi = (\psi_k)_{k \in X} \subseteq \mathcal{H}$, we define the frame operator $S_\psi : \mathcal{H} \rightarrow \mathcal{H}$ to be

$$S_\psi f := D_\psi C_\psi f = \sum_{k \in X} \langle f, \psi_k \rangle \psi_k.$$

This operator has useful properties.

Theorem 2.0.3. [18, Lemma 5.1.5, (i)] The frame operator is self-adjoint and invertible. In addition, $(S_\psi^{-1} \psi_k)_{k \in X}$ is a frame.

Next, we use the frame operator to define the corresponding canonical dual frame for a certain frame.

Definition. [5] Let $\psi = (\psi_k)_{k \in X} \subseteq \mathcal{H}$ be a frame. We define its corresponding canonical dual frame as

$$\tilde{\psi} = (\tilde{\psi}_k)_{k \in X} := (S_\psi^{-1} \psi_k)_{k \in X}.$$

Recall from functional analysis that if we have an orthonormal basis $(e_k)_{k \in X}$, we can reconstruct any element $f \in \mathcal{H}$ using

$$f = \sum_{k \in X} \langle f, e_k \rangle e_k.$$

For frames, we have a similar formula using the canonical dual frame.

Theorem 2.0.4 (Reconstruction formula). [18, Lemma 5.1.5, (i)] Let $\psi = (\psi_k)_{k \in X} \subseteq \mathcal{H}$ be a frame. Then it holds

$$f = D_{\tilde{\psi}} C_\psi f = \sum_{k \in X} \langle f, \psi_k \rangle \tilde{\psi}_k$$

and

$$f = D_\psi C_{\tilde{\psi}} f = \sum_{k \in X} \langle f, \tilde{\psi}_k \rangle \psi_k.$$

In some cases, the reconstruction formula still holds if we replace $\tilde{\psi}$ with another frame $\bar{\psi}$. In this case, $\bar{\psi}$ is called the dual frame of ψ . From now on, we write $\bar{\psi}$ to indicate that $\bar{\psi}$ is a dual of ψ . We now introduce the Gramian matrix of two frames.

Definition. [5] Suppose that ψ, ϕ are frames. Then we define the (mixed) Gramian matrix as

$$(G_{\psi, \phi})_{k, l} := \langle \phi_l, \psi_k \rangle.$$

The Gramian matrix can be interpreted as an operator that is the composition of the coefficient operator and the analysis operator.

Proposition 2.0.5. [5] Let ψ, ϕ be frames. If the cross Gramian matrix $G_{\psi, \phi}$ is interpreted as an operator in $L(\ell^2)$, then

$$G_{\psi, \phi} = C_\psi D_\phi.$$

Proof. Let $\alpha \in \ell^2$. Then

$$C_\psi D_\phi \alpha = C_\psi \sum_{k \in X} \alpha_k \phi_k = \sum_{k \in X} \alpha_k C_\psi \phi_k = \sum_{k \in X} \alpha_k (\langle \phi_k, \psi_l \rangle)_{l \in X} = \sum_{k \in X}^{\text{pw}} ((G_{\psi, \phi})_{k, l} \alpha_k)_{l \in X} = G_{\psi, \phi} \alpha.$$

□

If two Gramian matrices are composed in a certain way, then the result is again a Gramian matrix.

Proposition 2.0.6. [18, Lemma 5.1.5, (i)], [5] Let ψ, ϕ, ξ be frames. Then

$$G_{\psi, \phi} G_{\phi, \xi} = G_{\psi, \xi}.$$

Proof. Using Proposition 2.0.5 and Theorem 2.0.4, we get

$$G_{\psi, \phi} G_{\phi, \xi} = C_\psi D_\phi C_\phi D_\xi = C_\psi D_\xi = G_{\psi, \xi}.$$

□

The Gramian matrix of a frame and its canonical dual frame has the following property:

Proposition 2.0.7. [5, Lemma 8] Let ψ be a frame. Then

$$G_{\psi, \bar{\psi}} = G_{\bar{\psi}, \psi}.$$

Proof. We have

$$(G_{\psi, \bar{\psi}})_{k, l} = \langle \bar{\psi}_l, \psi_k \rangle = \langle S_\psi^{-1} \psi_l, \psi_k \rangle = \langle \psi_l, S_\psi^{-1} \psi_k \rangle = \langle \psi_l, \tilde{\psi}_k \rangle = (G_{\bar{\psi}, \psi})_{k, l}.$$

□

Computing the adjoint of the Gramian matrix swaps the order of the frames.

Proposition 2.0.8. [28], [5] Let ψ, ϕ be frames. Then

$$G_{\psi, \phi}^* = G_{\phi, \psi}.$$

Proof. By Proposition 2.0.5 and Proposition 2.0.2,

$$G_{\psi, \phi}^* = (C_\psi D_\phi)^* = D_\phi^* C_\psi^* = C_\phi D_\psi = G_{\phi, \psi}.$$

□

The Gramian matrix of a frame and its canonical dual frame is an orthogonal projection.

Proposition 2.0.9. [5, Lemma 8] Let ψ be a frame. Then, $G_{\psi, \tilde{\psi}}$ is the orthogonal projection onto $R(C_\psi)$.

Proof. By Proposition 2.0.6, we get

$$G_{\psi, \tilde{\psi}} G_{\psi, \tilde{\psi}} = G_{\psi, \tilde{\psi}},$$

Therefore, $G_{\psi, \tilde{\psi}}$ is a projection. Also by Proposition 2.0.8 and Proposition 2.0.7,

$$G_{\psi, \tilde{\psi}}^* = G_{\tilde{\psi}, \psi} = G_{\psi, \tilde{\psi}},$$

Therefore, $G_{\psi, \tilde{\psi}}$ is an orthogonal projection. Since $D_{\tilde{\psi}}$ is surjective,

$$R(G_{\psi, \tilde{\psi}}) = R(C_\psi D_{\tilde{\psi}}) = R(C_\psi).$$

□

The coefficient and synthesis operator of a frame have properties similar to those of the canonical dual frame.

Corollary 2.0.10. [5, Lemma 8] Let ψ be a frame. Then

$$R(C_\psi) = R(C_{\tilde{\psi}})$$

and

$$K(D_\psi) = K(D_{\tilde{\psi}}).$$

Proof. Applying Proposition 2.0.9 and Proposition 2.0.7, gives

$$R(C_\psi) = R(G_{\psi, \tilde{\psi}}) = R(G_{\tilde{\psi}, \psi}) = R(C_{\tilde{\psi}}).$$

Next, we use Proposition 2.0.2 to get

$$K(D_\psi) = K(C_\psi^*) = R(C_\psi)^\perp = R(C_{\tilde{\psi}})^\perp = K(C_{\tilde{\psi}}^*) = K(D_{\tilde{\psi}}).$$

□

Because the coefficient and synthesis operators have closed ranges, we can compute their pseudo-inverses, which have a specific form.

Proposition 2.0.11. [18, Lemma 5.4.3] Let ψ be a frame. The following statements are true:

1. $D_\psi^\dagger = C_{\tilde{\psi}}$.
2. $C_\psi^\dagger = D_{\tilde{\psi}}$.

Proof. Both statements can easily be verified.

1. By the reconstruction formula (Theorem 2.0.4), we get

$$D_\psi C_{\tilde{\psi}} = \text{id}_{\mathcal{H}} = \pi_{\mathcal{H} \rightarrow R(D_\psi)}.$$

By Proposition 2.0.9,

$$C_{\tilde{\psi}} D_\psi = G_{\tilde{\psi}, \psi} = \pi_{\mathcal{H} \rightarrow R(C_{\tilde{\psi}})}.$$

Using Theorem 1.8.14, we conclude that $D_\psi^\dagger = C_{\tilde{\psi}}$.

2. Using Proposition 1.8.16 and Proposition 2.0.2, we get

$$C_\psi^\dagger = (D_\psi^*)^\dagger = (D_\psi^\dagger)^* = C_{\tilde{\psi}}^* = D_{\tilde{\psi}}.$$

□

The pseudo-inverse of the Gramian matrix also has a useful representation.

Proposition 2.0.12. [36, Theorem 1.1 (9), (10)], [18, Lemma 5.4.3] Let ψ, ϕ be frames. Then

$$G_{\psi, \phi}^\dagger = G_{\tilde{\phi}, \tilde{\psi}}.$$

Proof. A combination of Proposition 2.0.5, Proposition 2.0.2, Proposition 1.8.17 and Proposition 2.0.11 gives

$$G_{\psi, \phi}^\dagger = (D_\psi C_\phi)^\dagger = C_\phi^\dagger D_\psi^\dagger = D_{\tilde{\phi}} C_{\tilde{\psi}} = G_{\tilde{\phi}, \tilde{\psi}}.$$

□

2.1 Riesz bases

Riesz bases are a special case of frames. The only difference between an orthonormal basis and a Riesz bases is a bijective operator in-between.

Definition. [18, Definition 3.6.1] A frame ψ is called a Riesz basis if there exists an orthonormal basis $e = (e_k)_{k \in X} \subseteq \mathcal{H}$ and a bijective $T \in L(\mathcal{H})$ such that

$$Te_k = \psi_k \quad \forall k \in X.$$

Riesz bases can be characterized in several ways.

Proposition 2.1.1. [4] Let $\psi = (\psi_k)_{k \in X}$ be a sequence. The following are equivalent:

1. ψ is a Riesz basis.
2. C_ψ is well defined and bijective.
3. D_ψ is well defined and bijective.

The inverses of the synthesis and coefficient operators have useful representations.

Proposition 2.1.2. [4], [18, Lemma 5.4.3] Let ψ be a Riesz basis. Then the following statements hold:

1. $D_\psi^{-1} = C_{\tilde{\psi}}$.
2. $C_\psi^{-1} = D_{\tilde{\psi}}$.

This implies that $\tilde{\psi}$ is also a Riesz basis.

Proof. This statement follows from Proposition 2.0.11 and the fact that both operators are bijective (Proposition 2.1.1). $\square$

There are several methods for constructing Riesz bases. To obtain some examples, we introduce the left shift operator on $\ell^2(\mathbb{Z})$. For $c \in \ell^2(\mathbb{Z})$, we define

$$Lc := (c_{k-1})_{k \in \mathbb{Z}}.$$

It is easy to see that L is an isometry. For $n \in \mathbb{N}^*$, we get

$$L^n = \underbrace{L \dots L}_{k \text{ times}}.$$

For general $n \in \mathbb{Z}$, we naturally get

$$L^n c := (c_{k-n})_{k \in \mathbb{Z}}.$$

Such operators are sufficient to provide a set of Riesz bases.

Theorem 2.1.3. [18, Theorem A.5.3] Suppose that $X = \mathbb{Z}$ and L is the left shift operator. Let $(c_k)_{k \in \mathbb{Z}} \in \ell^1(\mathbb{Z})$ with $\|(c_k)_{k \in \mathbb{Z}}\|_{\ell^1} < 1$ and

$$T := id_{\mathcal{H}} + \sum_{k \in \mathbb{Z}} c_k L^k.$$

Then $T \in I(\mathcal{H})$. Therefore, the corresponding sequence $(Te_k)_{k \in \mathbb{N}}$ is a Riesz basis. The same holds for the right shift R instead of L .

Proof. Linearity is clear. For continuity, we have

$$\|T\| = \left\| \text{id}_{\mathcal{H}} + \sum_{k \in \mathbb{Z}} c_k L^k \right\| \leq 1 + \sum_{k \in \mathbb{Z}} |c_k| < 2.$$

Similarly,

$$\|\text{id}_{\mathcal{H}} - T\| \leq \left\| \sum_{k \in \mathbb{Z}} c_k L^k \right\| < 1,$$

Therefore, according to Theorem 1.7.1, T is invertible. $\square$

Using this theorem, we can construct an example and analyze it.

Example. We analyze the sequence

$$\psi := \left(\dots, e_{-1} + \frac{1}{2}e_0, e_0 + \frac{1}{2}e_1, e_1 + \frac{1}{2}e_2, e_2 + \frac{1}{2}e_3, \dots \right) = \left(e_k + \frac{1}{2}e_{k+1} \right)_{k \in \mathbb{Z}}.$$

By Theorem 2.1.3, ψ is a Riesz basis. The corresponding synthesis operator is $D_\psi = \text{id}_{\mathcal{H}} + \frac{1}{2}R$, whereas the coefficient operator is $C_\psi = \text{id} + \frac{1}{2}L$. The frame operator is

$$S_\psi = D_\psi C_\psi = \left(\text{id}_{\mathcal{H}} + \frac{1}{2}R \right) \left(\text{id}_{\mathcal{H}} + \frac{1}{2}L \right) = \frac{5}{4} \text{id}_{\mathcal{H}} + \frac{1}{2}L + \frac{1}{2}R.$$

We get the following inverse

$$D_\psi^{-1} = \sum_{n=0}^{\infty} \left(-\frac{1}{2}R \right)^n$$

since

$$\left(\text{id}_{\mathcal{H}} + \frac{1}{2}R \right) \sum_{n=0}^N \left(-\frac{1}{2}R \right)^n = \sum_{n=0}^N \left(\left(-\frac{1}{2}R \right)^n - \left(-\frac{1}{2}R \right)^{n+1} \right) = \text{id}_{\mathcal{H}} - \underbrace{\left(-\frac{1}{2}R \right)^{N+1}}_{\rightarrow 0 \text{ (} N \rightarrow \infty \text{)}}.$$

Swapping both side also give the identity. Similarly,

$$C_\psi^{-1} = \sum_{n=0}^{\infty} \left(-\frac{1}{2}L \right)^n$$

and

$$S_\psi^{-1} = \left(\text{id}_{\mathcal{H}} - \left(-\frac{1}{4} \text{id}_{\mathcal{H}} - \frac{1}{2}L - \frac{1}{2}R \right) \right)^{-1} = \sum_{n \in \mathbb{N}} \left(-\frac{1}{4} \text{id}_{\mathcal{H}} - \frac{1}{2}L - \frac{1}{2}R \right)^n.$$

2.2 Applying operators to frames

In this chapter, we apply operators to frame elements and analyze the results. Let ψ be a frame and $T \in L(\mathcal{H})$. We use the notation

$$T\psi := (T\psi_k)_{k \in X}.$$

Now, the question arises as to whether $T\psi$ is still a frame. To answer this question, we first need the following lemma.

Lemma 2.2.1. [5] *Let ψ be a frame and $T \in L(\mathcal{H})$. Then*

$$D_{T\psi} = TD_\psi.$$

Proof. Let $(c_k)_{k \in X} \in \ell^2$. Then

$$TD_\psi(c_k)_{k \in X} = T \sum_{k \in X} c_k \psi_k = \sum_{k \in X} c_k T\psi_k = D_{T\psi}(c_k)_{k \in X}.$$

□

It turns out that the question of whether $T\psi$ is still a frame can be easily characterized.

Theorem 2.2.2. [28, Theorem 6.1], [18, Lemma A.6.1] *Let ψ be a frame and $T \in L(\mathcal{H})$. Then, $T\psi$ is a frame if and only if T is surjective.*

Proof. We show both directions.

$\Rightarrow$: If $T\psi$ is a frame then $D_{T\psi}$ is surjective because of Proposition 2.0.1. This means that for any $f \in \mathcal{H}$, there exists $(\alpha_k)_{k \in X} \in \ell^2$ such that

$$D_{T\psi}(\alpha_k)_{k \in X} = f.$$

On the left-hand side, we get by Lemma 2.2.1

$$D_{T\psi}(\alpha_k)_{k \in X} = TD_\psi(\alpha_k)_{k \in X}.$$

This shows that $f \in R(T)$ proving surjectivity.

$\Leftarrow$: Suppose that T is surjective. By the Lemma 2.2.1,

$$D_{T\psi} = TD_\psi.$$

Because T and D_ψ are surjective, $D_{T\psi}$ is surjective, proving that $T\psi$ is a frame (Proposition 2.0.1).

□

We can also use the adjoint to characterize this statement.

Corollary 2.2.3. [28, Theorem 6.1] Let ψ be a frame and $T \in L(\mathcal{H})$. Then, $T\psi$ is a frame if and only if T^* is injective and has a closed range (or equivalently $\|T^*f\| \gtrsim \|f\| \ \forall f \in \mathcal{H}$).

Proof. We know that $T\psi$ is a frame if and only if T is surjective. In addition, T has a closed range if and only if T^* has a closed range (Proposition 1.8.7). Moreover,

$$K(T^*) = R(T)^\perp,$$

Therefore, $K(T^*) = \{0\}$ if and only if $R(T)$ is dense in $\mathcal{H}$. $\square$

The next question is the following: If ψ is a Riesz basis, is $T\psi$ also a Riesz basis. For this, we need an additional assumption to be made.

Theorem 2.2.4. [18, Definition 3.6.1] Let ψ be a Riesz basis and $T \in L(\mathcal{H})$. Then, $T\psi$ is a Riesz basis if and only if T is bijective.

Proof. By Theorem 2.2.2, we only need to focus on injectivity. If T is bijective, then $D_{T\psi} = TD_\psi$ is also bijective (Lemma 2.2.1 and Proposition 2.1.1), showing that $T\psi$ is a Riesz basis. On the other hand, if $T\psi$ is a Riesz basis, then $D_{T\psi} = TD_\psi$ is bijective. Since D_ψ is bijective, T must also be bijective. $\square$

2.3 Banach frames

So far, we have only analyzed frames in Hilbert spaces. Now, we extend the definition to Banach spaces. In this setting, frames are sequences in $\mathcal{B}'$ instead of $\mathcal{B}$. In addition, previously, we only worked with the sequence space ℓ^2 . In the Banach space setting, we use a more general sequence space $\mathcal{B}_d$.

Definition. [5] Let $\psi = (\psi_k)_{k \in X} \subseteq \mathcal{B}'$. We call ψ a Banach frame if there exists a Banach space $\mathcal{B}_d$ of $\mathbb{F}$ -valued sequences with index set X such that the following hold:

1. The coefficient operator

$$C_\psi : \mathcal{B} \rightarrow \mathcal{B}_d, f \mapsto (\psi_k(f))_{k \in X}$$

is well defined and bounded.

2. We have

$$\|f\|_{\mathcal{B}} \simeq \|C_\psi f\|_{\mathcal{B}_d} \ \forall f \in \mathcal{B}.$$

3. There exists a bounded operator $R : \mathcal{B}_d \rightarrow \mathcal{B}$ called the reconstruction operator, such that

$$R(C_\psi f) = f \ \forall f \in \mathcal{B}.$$

In the Hilbert space setting, we obtain the third condition for free, since the reconstruction operator is simply the synthesis operator. This is no longer the case in Banach space settings.

3 Localized Frames

In this section, we introduce the notion of localizable frames. The idea is that we call a pair of frames localized with respect to each other if their corresponding Gramian matrix fulfills certain properties. In some other literature, the Gramian matrix is expected to have a certain off-diagonal decay. One example of such a definition can be found in [28, Definition 2.1]. In our case, we define the localization property more generally using abstract axioms. We do this by defining a certain Banach- $*$ algebra. Our final goal is to show that certain results also hold in the more general case.

Definition. [5] We call a unital Banach $*$ -algebra $\mathcal{A}$ of matrices with index set X a solid spectral matrix algebra, if

1. $\mathcal{A} \subseteq \mathcal{M}(\ell^2)$.
2. $\mathcal{A}$ is inverse closed in $\mathcal{M}(\ell^2)$. This means that for any matrix $A \in \mathcal{A}$, which is invertible in $\mathcal{M}(\ell^2)$, $A^{-1} \in \mathcal{A}$.
3. $\mathcal{A}$ is solid, i.e., if $A = (a_{k,l})_{k,l \in X} \in \mathcal{A}$ and $|b_{k,l}| \leq |a_{k,l}|$ for all $k, l \in X$, then $B = (b_{k,l})_{k,l \in X} \in \mathcal{A}$ and $\|B\|_{\mathcal{A}} \leq \|A\|_{\mathcal{A}}$.

There are many examples of such structures, but first, we introduce a simple one.

Example. Let

$$\mathcal{A} := \{M \in \mathcal{M}(\ell^2) : M_{k,l} = 0 \text{ if } k \neq l\}$$

the subsets of all the diagonal matrices. It is easy to see that $\mathcal{A}$ is a linear subspace, closed under $*$ and contains the identity. Let $M, N \in \mathcal{A}$. Then

$$(MN)_{k,m} = \sum_{l \in X} M_{k,l} N_{l,m} = M_{k,k} N_{k,m}.$$

If $k \neq m$ then the right-hand side is 0, meaning that MN is a diagonal matrix. Therefore, we have a unital sub- $*$ -algebra. If

$$M_{k,l} = \delta_{k,l} m_k$$

one can easily see that M is invertible in $\mathcal{M}(\ell^2)$ if and only if

$$|m_k| \gtrsim 1 \quad \forall k \in X.$$

The inverse is given by

$$M_{k,l}^{-1} = \delta_{k,l} m_k^{-1},$$

since

$$(MM^{-1})_{j,l} = \sum_{k \in X} M_{j,k} M_{k,l}^{-1} = \sum_{k \in X} \delta_{j,k} m_j \delta_{k,l} m_k^{-1} = m_j \delta_{j,l} m_j^{-1} = \delta_{j,l}$$

and the other way around. Equipping $\mathcal{A}$ with the $L(\ell^2)$ norm, which can be computed by

$$\|M\|_{L(\ell^2)} = \sup_{k \in X} |M_{k,k}|,$$

$\mathcal{A}$ becomes a Banach-* algebra, which is also solid in this case.

Other examples can be found in the literature; however, showing that these examples fulfill the requirements is not trivial.

Example. For simplicity, we restrict ourselves to the special cases. More general cases can be found in the literature.

- [27] Let $X = \mathbb{Z}$ and $2 < s$. Then, the Jaffard norm is defined to be

$$\|A\|_{\mathcal{A}} := \sup_{k,l \in \mathbb{Z}} |A_{k,l}| (1 + |k - l|)^s.$$

In this case, $\mathcal{A}$ is defined as a subset of matrices such that the expression above is finite. This means that the matrices in $\mathcal{A}$ must have a certain off-diagonal decay. Indeed, for any entry far from the diagonal, $|k - l|$ becomes large, implying that $(1 + |k - l|)^s$ also becomes large. Therefore, $|A_{k,l}|$ must be small to ensure that the supremum is finite.

- [28, Definition 2.1] Let $X = \mathbb{Z}$. Then, the norm is defined to be

$$\|A\|_{\mathcal{A}} := \left\| \sup_{k \in \mathbb{Z}} |A_{k,k+(\cdot)}| \right\|_{\ell^1}.$$

This also implies that these matrices have a certain off-diagonal decay. However, unlike the Jaffard class, the decay is not bounded by a specific sequence up to a constant factor but is bounded by any ℓ^1 sequence.

Note that some norms, like the Jaffard norm, are not necessarily submultiplicative, as we only have the inequality

$$\|AB\|_{\mathcal{A}} \lesssim \|A\|_{\mathcal{A}} \|B\|_{\mathcal{A}} \quad \forall A, B \in \mathcal{A},$$

However, this is not a problem because it is possible to define a submultiplicative norm that is equivalent to $\|\cdot\|_{\mathcal{A}}$.

From now on, we assume that $\mathcal{A}$ is a solid spectral matrix algebra. This Banach-* algebra allows us to define the localization of the frames.

Definition. [5, Definition 1] Let ψ, ϕ be frames. We call them localized with respect to each other if $G_{\psi,\phi} \in \mathcal{A}$. In this case, we use the notation $\psi \sim \phi$. We call ψ self-localized if $G_{\psi,\psi} \in \mathcal{A}$.

It turns out that if a frame is self-localized, then this localization property is conveyed to its canonical dual frame.

Proposition 3.0.1. [5, Theorem 2] Let ψ be a self-localized frame. Let $\tilde{\psi}$ be the canonical dual frame of ψ . Then, $\psi \sim \tilde{\psi}$ and $\tilde{\psi} \sim \tilde{\psi}$.

Proof. Since $\mathcal{A}$ is inverse closed, it is also pseudo-inverse closed (Theorem 1.8.24). Therefore, by Proposition 2.0.12,

$$G_{\psi,\psi}^\dagger = G_{\tilde{\psi},\tilde{\psi}} \in \mathcal{A}.$$

By the closedness of the multiplication and Proposition 2.0.6, we get

$$G_{\tilde{\psi},\psi} = G_{\tilde{\psi},\tilde{\psi}} G_{\psi,\psi} \in \mathcal{A}.$$

□

As expected, the order of the frames does not matter in the definition of localization.

Lemma 3.0.2. [5, Lemma 8] Suppose a pair ψ, ϕ is localized, then ϕ, ψ is localized. In other words, localization is a symmetric relation.

Proof. Since $\mathcal{A}$ is closed under $*$ and by Proposition 2.0.8

$$G_{\phi,\psi} = G_{\psi,\phi}^* \in \mathcal{A},$$

we obtain $\phi \sim \psi$.

□

We also have the following result:

Lemma 3.0.3. [5] Let ψ, ϕ, ξ be frames such that $\psi \sim \phi$ and $\tilde{\phi} \sim \xi$. Then, $\psi \sim \xi$.

Proof. By definition, we obtain $G_{\psi,\phi} \in \mathcal{A}$ and $G_{\tilde{\psi},\xi} \in \mathcal{A}$. Therefore, by Proposition 2.0.6,

$$G_{\psi,\xi} = G_{\psi,\phi} G_{\tilde{\phi},\xi} \in \mathcal{A}.$$

This implies that $\psi \sim \xi$.

□

If we combine different results, we obtain transitivity.

Proposition 3.0.4. [5] Let ψ, ϕ, ξ be frames such that $\psi \sim \phi$ and $\phi \sim \xi$. Then, $\psi \sim \xi$.

Proof. By Proposition 3.0.1, $\tilde{\phi} \sim \tilde{\phi}$. We apply Lemma 3.0.3 to $\tilde{\phi} \sim \tilde{\phi}$ and $\phi \sim \xi$ to obtain $\tilde{\phi} \sim \xi$. Then, we apply Lemma 3.0.3 to $\psi \sim \phi$ and $\tilde{\phi} \sim \xi$ to obtain $\psi \sim \xi$ proving the result.

□

To simplify the notation, we let $\mathbf{Loc}(\mathcal{A})$ be the set of frames with index set X that are self-localized. Now, we obtain the following result:

Theorem 3.0.5. [5] Localization is an equivalence relation on $\mathbf{Loc}(\mathcal{A})$.

Proof. Reflexivity follows from the definition of $\mathbf{Loc}(\mathcal{A})$. Symmetry follows from Lemma 3.0.2. Transitivity follows from Proposition 3.0.4.

□

Finally, we show that self-localization can be conveyed to other frames.

Theorem 3.0.6. [5, Lemma 5] Let $\psi \in \mathbf{Loc}(\mathcal{A})$ and ϕ be a frame such that $\phi \sim \psi$. Then $\phi \in \mathbf{Loc}(\mathcal{A})$.

Proof. By symmetry (Lemma 3.0.2), $\psi \sim \phi$. By transitivity (see Proposition 3.0.4), $\phi \sim \phi$ completing the proof.

□

4 $\mathcal{H}_w^p$ space

In this section, we introduce the co-orbit space $\mathcal{H}_w^p$ using our Hilbert space $\mathcal{H}$. Here, p has a fixed value in $\{0\} \cup [1, \infty]$ and w is a fixed positive weight on X . We also have to ensure that the weight w satisfies a certain property.

Definition. [5] We say that w is admissible for $\mathcal{A}$ if

$$\mathcal{A} \subseteq \bigcap_{q \in \{0\} \cup [1, \infty]} \mathcal{M}(\ell_w^q).$$

For a special case of $\mathcal{A}$, an example of a class of admissible weights is as follows:

Example. Let $\mathcal{A}$ be the Jaffard class with some fixed $s > 2$. If the weight w satisfies the condition

$$w_{x+y} \lesssim (1 + |x|)^s w_y \quad \forall x, y \in \mathbb{Z}$$

then w is $\mathcal{A}$ -admissible. This follows from [27, Lemma 2.1].

For our purposes, we assume that w is $\mathcal{A}$ -admissible. This means that for any $\alpha \in \ell_w^q$, we get

$$M\alpha \in \ell_w^q \quad \forall q \in \{0\} \cup [1, \infty].$$

To distinguish between a matrix and an operator, we define

$$M^{q,w} : \ell_w^q \rightarrow \ell_w^q, \alpha \mapsto M\alpha.$$

Using Theorem 1.6.1, we conclude that $M^{q,w} \in L(\ell_w^q)$. By Theorem 1.6.14, $M^{q,w}$ is weak-* continuous for all $q \in [1, \infty]$. Using Proposition 1.6.12, we can even conclude that $M \in \mathcal{M}(\ell_{w^{-1}}^q)$, meaning that w^{-1} is $\mathcal{A}$ -admissible.

4.1 The Gramian matrix

For this whole chapter, we assume that ψ is a self localized frame. This means that we get a continuous operator using the Gramian Matrix, i.e.,

$$G_{\psi,\psi}^{p,w} \in L(\ell_w^p).$$

We can also show that the Gramian matrix of some different frames is continuous.

Proposition 4.1.1. [5, Lemma 8] Let ϕ, ξ be frames such that $\phi \sim \psi$ and $\xi \sim \psi$. Then

$$G_{\phi,\xi} \in \mathcal{M}(\ell_w^q) \quad \forall q \in \{0\} \cup [1, \infty].$$

Proof. By Theorem 3.0.6 and Theorem 3.0.5, we obtain $\phi \sim \xi$. This implies that

$$G_{\phi,\xi} \in \mathcal{A} \subseteq \mathcal{M}(\ell_w^q) \quad \forall q \in \{0\} \cup [1, \infty].$$

□

We also show a similar result for the weight w^{-1} .

Proposition 4.1.2. [5] *Let ϕ, ξ be frames such that $\phi \sim \psi$ and $\xi \sim \psi$. Then*

$$G_{\phi,\xi} \in \mathcal{M}(\ell_{w^{-1}}^q) \quad \forall q \in \{0\} \cup [1, \infty].$$

Proof. By Proposition 4.1.1 and closedness under involution, it holds

$$G_{\phi,\xi}^* \in \mathcal{A} \subseteq \mathcal{M}(\ell_w^q) \quad \forall q \in \{0\} \cup [1, \infty].$$

Using Proposition 1.6.12, we conclude that

$$G_{\phi,\xi} \in \mathcal{M}(\ell_{w^{-1}}^q) \quad \forall q \in \{0\} \cup [1, \infty].$$

□

We already know that $G_{\phi,\phi}$ is an orthogonal projection (see Proposition 2.0.9) on ℓ^2 . We can show a similar statement for ℓ_w^p .

Proposition 4.1.3. [5, Lemma 8] *Let ϕ be a frame such that $\phi \sim \psi$. Then, $G_{\phi,\phi}^{p,w}$ is well-defined and a projection in ℓ_w^p .*

Proof. A combination of Proposition 4.1.1 and Proposition 3.0.1 gives

$$G_{\phi,\phi} \in \mathcal{M}(\ell_w^p).$$

Because of Proposition 2.0.9, $G_{\phi,\phi}^{p,w}$ behaves like a projection on ℓ^{00} . Since ℓ^{00} is norm dense (Corollary 1.5.3) or sequentially weak-* dense (Theorem 1.5.16) in ℓ_w^p , we use Proposition 1.2.9 to conclude that $G_{\phi,\phi}^{p,w}$ is a projection onto its range. □

By Proposition 1.2.8, $R(G_{\psi,\tilde{\psi}})$ is norm closed in ℓ_w^p and is therefore norm complete. For $p \in [1, \infty]$, we can also use the weak-* topology to conclude that $R(G_{\psi,\tilde{\psi}})$ is weak-* closed. Additionally, because ℓ_w^p is sequentially weak-* complete and $R(G_{\psi,\tilde{\psi}})$ is sequentially weak-* closed, it is sequentially weak-* complete.

4.2 The construction of the co-orbit space

Our goal is to construct the co-orbit space $\mathcal{H}_w^p$. To make the concept simpler to understand, we start with an analogy using the ℓ_w^p space. Suppose one starts with the ℓ^2 space and wants to construct the ℓ_w^p space. The first step is to restrict ℓ^2 to ℓ^{00} . For $p < \infty$, the next step is to equip ℓ^{00} with the ℓ_w^p -norm. Finally, the ℓ_w^p -space can be defined as

the completion of ℓ^{00} with respect to this norm. For $p = \infty$, ℓ_w^∞ can be reached by first taking the weak-* completion of ℓ^{00} . Subsequently, one takes the sequential hull of ℓ^{00} inside the completion, yielding ℓ_w^∞ . To start, we first define a subspace similar to ℓ^{00} , namely [5]

$$\mathcal{H}^{00} := D_\psi \ell^{00} = \text{span}(\psi_k)_{k \in X}.$$

To construct the co-orbit space, we must ensure that the coefficient operator fulfills certain properties. For a frame ϕ , we use the notation

$$C_\phi^{00} := C_\phi|_{\mathcal{H}^{00}}.$$

First, we must ensure that its range is a subset of ℓ_w^p .

Proposition 4.2.1. [5] *Let ϕ be a frame such that $\phi \sim \psi$. Then*

$$R(C_\phi^{00}) \subseteq \ell^2 \cap \ell_w^p.$$

In particular, this statement holds for $\phi = \tilde{\psi}$.

Proof. Being a subset of ℓ^2 is clear. Let $l \in X$. Then by Proposition 2.0.5,

$$C_\phi^{00} \psi_l = C_\phi \psi_l = C_\phi D_\psi \delta_l = G_{\phi, \psi} \delta_l = G_{\phi, \psi}^{p, w} \delta_l.$$

Since $e_l \in \ell_w^p$ and $G_{\phi, \psi}^{p, w} \in L(\ell_w^p)$, $C_\phi^{00} \psi_l \in \ell_w^p$, the result follows. $\square$

The next step is to define a norm and a topology on $\mathcal{H}^{00}$ similar to the weak-* topology. First, for $p \in [1, \infty]$, we define the map

$$\|\cdot\|_{\mathcal{H}_w^p} : \mathcal{H}^{00} \rightarrow \mathbb{R}, \|f\|_{\mathcal{H}_w^p} := \left\| C_\psi^{00} f \right\|_{\ell_w^p},$$

which is a norm since the coefficient operator is injective. For $p \in [1, \infty]$, we can also equip $\mathcal{H}^{00}$ with seminorms, which we define as

$$q_\alpha : \mathcal{H}^{00} \rightarrow \mathbb{R}, f \mapsto \left| \langle C_\psi^{00} f, \alpha \rangle_{\ell_w^p, \ell_w^{p*}} \right| \quad \forall \alpha \in \ell_w^{p*}.$$

For simplicity, we call the topology generated by the seminorms the weak-* topology. According to Lemma 1.2.11, $\mathcal{H}^{00}$ is a Hausdorff space with respect to this topology, and C_ψ^{00} is an isometry with respect to both topologies. We now have to distinguish between $p < \infty$ and $p = \infty$.

4.2.1 The case $p < \infty$

For $p \in [1, \infty)$, we define $\mathcal{H}_w^p$ as the norm completion of $\mathcal{H}^{00}$ [5, Definition 2]. For $p = 0$, we use the $\|\cdot\|_{\mathcal{H}_w^\infty}$ norm. Since C_ψ^{00} is an isometry, we can use Theorem 1.2.5 and Proposition 1.2.10 to extend it to $\mathcal{H}_w^p$, calling the extension $C_\psi^{p, w}$ which is an isometry. We can also define the extended seminorms

$$q_\alpha : \mathcal{H}_w^p \rightarrow \mathbb{R}, f \mapsto \left| \langle C_\psi^{p, w} f, \alpha \rangle_{\ell_w^p, \ell_w^{p*}} \right| \quad \forall \alpha \in \ell_w^{p*}.$$

Here, we see that by definition $C_{\tilde{\psi}}^{p,w}$ is an isometry with respect to the weak-* topologies.

4.2.2 The case $p = \infty$

The case $p = \infty$ is more involved. First, we define $\overline{\mathcal{H}}$ to be the weak-* completion of $\mathcal{H}^{00}$. Afterward, we define $\mathcal{H}_w^\infty$ to be the sequential hull of $\mathcal{H}^{00}$ in $\overline{\mathcal{H}}$ with respect to this topology. We also use Corollary 1.2.6 and Proposition 1.2.10 to extend $C_{\tilde{\psi}}^{00}$ to the whole space

$$C_{\tilde{\psi}}^{\infty,w} : \mathcal{H}_w^\infty \rightarrow \ell_w^\infty,$$

getting an isometry. We also get a norm by

$$\|f\|_{\mathcal{H}_w^\infty} := \|C_{\tilde{\psi}}^{\infty,w} f\|_{\ell_w^\infty},$$

Therefore, $C_{\tilde{\psi}}^{\infty,w}$ is an isometry with respect to both topologies.

The case $\mathcal{H} = \ell^2$ provides a simple example.

Example. Suppose that $\mathcal{H} = \ell^2$ and $(\psi_k)_{k \in X} = (\delta_k)_{k \in X}$. Then $\tilde{\psi} = \psi$ and $C_{\tilde{\psi}} = \text{id}_{\ell^2} = D_\psi$. This implies that

$$\mathcal{H}_w^p \cong \ell_w^p \quad \forall p \in \{0\} \cup [1, \infty].$$

4.3 Continuity in $\mathcal{H}_w^p$

Since we have different topologies, we must be very precise when dealing with continuity. Therefore, we introduce the following term to cover both types of continuity: Let U, V be subspaces of ℓ_w^p or $\mathcal{H}_w^p$. We say that an operator $T : U \rightarrow V$ is super-continuous if the following hold:

1. T is norm continuous.
2. For $p \in [1, \infty]$, T is continuous with respect to the weak-* topologies.

First, we show that the Gramian matrix is super-continuous if the corresponding frames are localized in a certain sense.

Proposition 4.3.1. [24] *Let ϕ, ξ be frames such that $\phi \sim \psi$ and $\xi \sim \psi$. Then, $G_{\phi,\xi}^{p,w}$ is super-continuous.*

Proof. By Proposition 4.1.1, $G_{\phi,\xi}^{q,w}$ is norm continuous for all $q \in \{0\} \cup [1, \infty]$. Using Theorem 1.6.14, $G_{\phi,\xi}^{p,w}$ is weak-* continuous if $p \in [1, \infty]$. $\square$

We can easily see that $C_{\tilde{\psi}}^{p,w}$ is super-continuous.

Proposition 4.3.2. [5] *The coefficient operator $C_{\tilde{\psi}}^{p,w}$ is super-continuous.*

Proof. This follows from the fact that $C_{\tilde{\psi}}^{p,w}$ is an isometry with respect to both the topologies. $\square$

It is easy to see that the composition of super-continuous operators is super-continuous. We also obtain the following:

Theorem 4.3.3. [49, Proposition 4.3] Let U be a subspace of ℓ_w^p or $\mathcal{H}_w^p$. Let $V \in \{\ell_w^p, \mathcal{H}_w^p\}$. Let $S, T : U \rightarrow V$ be super-continuous operators. Suppose that $W \subseteq U$ is a norm dense or sequentially weak-* dense subspace of such that

$$S|_W = T|_W.$$

Then $S = T$.

Proof. This follows from Theorem 1.2.5 applied to $S|_W$. □

4.4 Intersection of $\mathcal{H}$ and $\mathcal{H}_w^p$

By the construction of $\mathcal{H}_w^p$,

$$\mathcal{H}^{00} \subseteq \mathcal{H}_w^p \cap \mathcal{H}.$$

In some cases, it is reasonable to identify more elements of $\mathcal{H}$ with elements in $\mathcal{H}_w^p$. Note that it holds,

$$C_{\tilde{\psi}}|_{\mathcal{H}^{00}} = C_{\tilde{\psi}}^{p,w}|_{\mathcal{H}^{00}}.$$

Since both coefficient operators are injective, it makes sense to identify $f \in \mathcal{H}$ with $g \in \mathcal{H}_w^p$ if

$$C_{\tilde{\psi}}f = C_{\tilde{\psi}}^{p,w}g.$$

This immediately gives us

$$C_{\tilde{\psi}}|_{\mathcal{H}_w^p \cap \mathcal{H}} = C_{\tilde{\psi}}^{p,w}|_{\mathcal{H}_w^p \cap \mathcal{H}}.$$

To better understand the intersection, we do the following. Let $f \in \mathcal{H}$. Then $f \in \mathcal{H}_w^p \cap \mathcal{H}$ if and only if there exists some $g \in \mathcal{H}_w^p$ such that

$$C_{\tilde{\psi}}f = C_{\tilde{\psi}}^{p,w}g.$$

We can equivalently write

$$\mathcal{H}_w^p \cap \mathcal{H} = \{f \in \mathcal{H} : C_{\tilde{\psi}}f \in R(C_{\tilde{\psi}}^{p,w})\}.$$

A similar argument gives us

$$\mathcal{H}_w^p \cap \mathcal{H} = \{f \in \mathcal{H}_w^p : C_{\tilde{\psi}}^{p,w}f \in R(C_{\tilde{\psi}})\}.$$

We can also characterize the intersection as follows:

Proposition 4.4.1. Let $f \in \mathcal{H}$ such that $C_{\tilde{\psi}}f \in \ell_w^p$. Then

$$f \in \mathcal{H}_w^p \cap \mathcal{H}.$$

Proof. Let $\alpha := C_{\tilde{\psi}} f \in \ell_w^p \cap \ell^2$. Let $(F_n)_{n \in \mathbb{N}}$ be a nested approximation of X and $\alpha^n := \alpha|_{F_n}$. Then, $(\alpha^n)_{n \in \mathbb{N}} \subseteq \ell^{00}$ converges to α in ℓ^2 and ℓ_w^p for $p < \infty$ (see Corollary 1.5.3). In the case $p = \infty$, we have weak-* convergence instead (see Theorem 1.5.16). Now let $f_n := D_{\psi} \alpha^n \in \mathcal{H}^{00}$. Then, it holds

$$\lim_{n \rightarrow \infty}^{\ell^2} f_n = \lim_{n \rightarrow \infty}^{\ell^2} D_{\psi} \alpha^n = D_{\psi} \alpha = D_{\psi} C_{\tilde{\psi}} f = f,$$

where the last equation follows from the reconstruction formula (see Theorem 2.0.4). Next, we show that $(f_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}_w^p$, where we use the norm topology if $p < \infty$ and the weak-* topology if $p = \infty$. Because of Proposition 2.0.5, it holds

$$C_{\tilde{\psi}}^{p,w} f_n = C_{\tilde{\psi}} f_n = C_{\tilde{\psi}} D_{\psi} \alpha^n = G_{\tilde{\psi}, \psi} \alpha^n = G_{\tilde{\psi}, \psi}^{p,w} \alpha^n.$$

Since $(\alpha^n)_{n \in \mathbb{N}}$ is a Cauchy sequence in ℓ_w^p and $G_{\tilde{\psi}, \psi}^{p,w}$ is super-continuous (see Proposition 4.3.1), $(G_{\tilde{\psi}, \psi}^{p,w} \alpha^n)_{n \in \mathbb{N}}$ is a Cauchy sequence (see Proposition 1.2.4). Since $C_{\tilde{\psi}}^{p,w}$ is an isometry, $(f_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}_w^p$. Because of $\mathcal{H}^{00} \lesssim \mathcal{H}_w^p$, $(f_n)_{n \in \mathbb{N}}$ converges to some $g \in \mathcal{H}_w^p$. Our final goal is to show that $f = g$. Recall that this means that

$$C_{\tilde{\psi}} f = C_{\tilde{\psi}}^{p,w} g.$$

We get

$$\begin{aligned} C_{\tilde{\psi}}^{p,w} g &= \lim_{n \rightarrow \infty}^{\ell_w^p} C_{\tilde{\psi}}^{p,w} f_n = \lim_{n \rightarrow \infty}^{\ell_w^p} C_{\tilde{\psi}}^{p,w} D_{\psi} \alpha^n = \lim_{n \rightarrow \infty}^{\ell_w^p} C_{\tilde{\psi}} D_{\psi} \alpha^n = \lim_{n \rightarrow \infty}^{\ell_w^p} G_{\tilde{\psi}, \psi}^{p,w} \alpha^n = G_{\tilde{\psi}, \psi}^{p,w} \alpha \\ &= G_{\tilde{\psi}, \psi} \alpha = C_{\tilde{\psi}} f. \end{aligned}$$

This implies $f = g$ and therefore $f \in \mathcal{H}_w^p \cap \mathcal{H}$. □

4.5 Other coefficient operators

By construction, $C_{\tilde{\psi}}^{p,w}$ behaves very well, as it is an isometry with respect to both topologies. For a different frame, this might not be the case, but if we have a frame such that it is localized with ψ then we obtain at least super continuity.

Proposition 4.5.1. [5, Lemma 8], [5, Proposition 3] Let ϕ be a frame such that $\phi \sim \psi$. Then the operator C_{ϕ}^{00} is super-continuous and has a unique super-continuous extension $C_{\phi}^{p,w}$ on $\mathcal{H}_w^p$, namely

$$C_{\phi}^{p,w} = G_{\phi, \psi}^{p,w} C_{\tilde{\psi}}^{p,w}.$$

Proof. According to Theorem 2.0.4 and Proposition 2.0.5,

$$C_{\phi} = C_{\phi} D_{\psi} C_{\tilde{\psi}} = G_{\phi, \psi} C_{\tilde{\psi}}.$$

Restricting both sides to $\mathcal{H}^{00}$ and using $R(C_{\tilde{\psi}}^{00}) \subseteq \ell^2 \cap \ell_w^p$ (Proposition 4.2.1), we get

$$C_{\phi}^{00} = G_{\phi,\psi}^{p,w} C_{\tilde{\psi}}^{00}.$$

Because both operators on the right side are super-continuous, the composition is as well. Using Theorem 4.3.3, the unique super-continuous extension is

$$C_{\phi}^{p,w} = G_{\phi,\psi}^{p,w} C_{\tilde{\psi}}^{p,w}.$$

□

Next, we show that C_{ϕ} and $C_{\phi}^{p,w}$ match on $\mathcal{H}_w^p \cap \mathcal{H}$.

Theorem 4.5.2. [5, Lemma 6] *Let $\phi \sim \psi$. Then*

$$C_{\phi}^{p,w}|_{\mathcal{H}_w^p \cap \mathcal{H}} = C_{\phi}|_{\mathcal{H}_w^p \cap \mathcal{H}}.$$

Proof. Let $f \in \mathcal{H}_w^p \cap \mathcal{H}$. Using Proposition 4.5.1, Theorem 2.0.4 and Proposition 2.0.5, we get

$$C_{\phi}^{p,w} f = G_{\phi,\psi}^{p,w} C_{\tilde{\psi}}^{p,w} f = G_{\phi,\psi} C_{\tilde{\psi}} f = C_{\phi} f.$$

□

4.6 The synthesis operator

After introducing the coefficient operator, we are ready to define the synthesis operator as follows: The idea is simple. We restrict D_{ϕ} to ℓ^{00} and extend it continuously to ℓ_w^p . We start with the special case of the frame ψ , and then define the synthesis operator for other frames.

4.6.1 The special synthesis operator

Recall that $R(G_{\tilde{\psi},\psi}) = R(C_{\tilde{\psi}})$ according to Proposition 2.0.9. We show a weaker statement for the case of $\mathcal{H}_w^p$.

Lemma 4.6.1. [5, Theorem 4 (i)] *It holds*

$$R(G_{\tilde{\psi},\psi}^{p,w}) \subseteq R(C_{\tilde{\psi}}^{p,w}).$$

Proof. By Proposition 2.0.5, it holds

$$R(G_{\tilde{\psi},\psi}^{00}) = R(C_{\tilde{\psi}}^{00} D_{\psi}^{00}) \subseteq R(C_{\tilde{\psi}}^{00}).$$

If $p < \infty$, we use the super continuity of $G_{\tilde{\psi},\psi}^{00}$ (Proposition 4.3.1) and $C_{\tilde{\psi}}^{00}$ (Proposition 4.3.2) to get

$$R(G_{\tilde{\psi},\psi}^{p,w}) \subseteq \overline{R(G_{\tilde{\psi},\psi}^{00})} \subseteq \overline{R(C_{\tilde{\psi}}^{00})} \subseteq R(C_{\tilde{\psi}}^{p,w}),$$

where we use the closure with respect to the $\mathcal{H}_w^p$ -norm. The last inclusion follows from the fact that $R(C_{\tilde{\psi}}^{p,w})$ is isometrically isomorphic to a norm-complete space. If $p = \infty$, we get

$$R(G_{\tilde{\psi},\psi}^{\infty,w}) \subseteq [R(G_{\tilde{\psi},\psi}^{00})]^s \subseteq [R(C_{\tilde{\psi}}^{00})]^s \subseteq R(C_{\tilde{\psi}}^{\infty,w}),$$

where we use the sequential hull with respect to the weak-* topology on $\mathcal{H}_w^\infty$. The last inclusion follows from $\mathcal{H}^{00} \preceq \mathcal{H}_w^\infty$ with respect to the weak-* topology and the fact that $C_{\tilde{\psi}}^{\infty,w}$ is an isometry with respect to the weak-* topology. $\square$

This result allows us to show that the synthesis operator can be uniquely and continuously extended to ℓ_w^p .

Proposition 4.6.2. [5, Lemma 8] *The restricted synthesis operator*

$$D_\psi^{00} := D_\psi|_{\ell^{00}}$$

is super-continuous with respect to ℓ_w^p and $\mathcal{H}_w^p$ and has a unique super-continuous extension, namely

$$D_\psi^{p,w} = (C_{\tilde{\psi}}^{p,w}|_{\mathcal{H}_w^p \rightarrow R(C_{\tilde{\psi}}^{p,w})})^{-1} G_{\tilde{\psi},\psi}^{p,w}.$$

Proof. By Proposition 3.0.1, $\tilde{\psi} \sim \psi$. Let $\alpha \in \ell^{00}$. Since $C_{\tilde{\psi}}^{p,w}$ is an isometry, it is injective with a super-continuous inverse on $R(C_{\tilde{\psi}}^{p,w})$. So we get

$$D_\psi^{00} = (C_{\tilde{\psi}}^{p,w}|_{\mathcal{H}_w^p \rightarrow R(C_{\tilde{\psi}}^{p,w})})^{-1} C_{\tilde{\psi}}^{p,w} D_\psi^{00} = (C_{\tilde{\psi}}^{p,w}|_{\mathcal{H}_w^p \rightarrow R(C_{\tilde{\psi}}^{p,w})})^{-1} G_{\tilde{\psi},\psi}^{00},$$

where the last equation follows from Proposition 2.0.5. By Lemma 4.6.1, $R(G_{\tilde{\psi},\psi}^{p,w}) \subseteq R(C_{\tilde{\psi}}^{p,w})$. Therefore,

$$D_\psi^{p,w} := (C_{\tilde{\psi}}^{p,w}|_{\mathcal{H}_w^p \rightarrow R(C_{\tilde{\psi}}^{p,w})})^{-1} G_{\tilde{\psi},\psi}^{p,w}$$

is well-defined and a super-continuous extension. By Theorem 4.3.3, this extension is unique. $\square$

According to Proposition 2.0.5, it holds

$$G_{\tilde{\psi},\psi} = C_{\tilde{\psi}} D_\psi.$$

A similar statement holds for $\mathcal{H}_w^p$.

Proposition 4.6.3. [5, Lemma 8] *It holds*

$$C_{\tilde{\psi}}^{p,w} D_\psi^{p,w} = G_{\tilde{\psi},\psi}^{p,w}.$$

Proof. According to Proposition 2.0.5, this equation holds on ℓ^{00} . Since both sides are super-continuous and ℓ^{00} is norm or weak-* dense in ℓ_w^p , we use Theorem 4.3.3 to conclude that this equation holds. $\square$

It is clear that D_ψ and $D_\psi^{p,w}$ coincide on ℓ^{00} . We can even show that they coincide in their common domain.

Proposition 4.6.4. [5, Lemma 7] We have

$$D_\psi|_{\ell_w^p \cap \ell^2} = D_\psi^{p,w}|_{\ell_w^p \cap \ell^2}.$$

Proof. Let $\alpha \in \ell_w^p \cap \ell^2$. Using Proposition 2.0.5 and Proposition 4.6.3, we get

$$C_{\tilde{\psi}} D_\psi \alpha = G_{\tilde{\psi}, \psi} \alpha$$

and

$$C_{\tilde{\psi}}^{p,w} D_\psi^{p,w} \alpha = G_{\tilde{\psi}, \psi}^{p,w} \alpha = G_{\tilde{\psi}, \psi} \alpha.$$

This implies that

$$D_\psi \alpha = D_\psi^{p,w} \alpha.$$

□

4.6.2 Other synthesis operators

We now want to look at the synthesis operator of frames not used to define the co-orbit space. Similar to before, the idea is to restrict D_ϕ to ℓ^{00} and extend it to ℓ_w^p . Here, we have to be careful, since it is not clear if $R(D_\phi^{00}) \subseteq \mathcal{H}_w^p$. Therefore, we need the following results.

Lemma 4.6.5. [5] Let ϕ be a frame such that $\phi \sim \psi$. Then for all $k \in X$, we have

$$\phi_k \in \mathcal{H}_w^p \cap \mathcal{H}.$$

Proof. Trivially, $\phi_k \in \mathcal{H}$. By Proposition 3.0.1 and Theorem 3.0.5, it holds $\tilde{\psi} \sim \phi$. We get by Proposition 2.0.5,

$$C_{\tilde{\psi}} \phi_k = C_{\tilde{\psi}} D_\phi \delta_k = G_{\tilde{\psi}, \phi} \delta_k = G_{\tilde{\psi}, \phi}^{p,w} \delta_k \in \ell_w^p,$$

because $G_{\tilde{\psi}, \phi} \in \mathcal{M}(\ell_w^p)$. This implies $\phi_k \in \mathcal{H}_w^p$ by Proposition 4.4.1. □

Later, we need a similar statement with the weight w^{-1} . Therefore, we prove this here.

Lemma 4.6.6. [5] Let ϕ be a frame such that $\phi \sim \psi$. Then for all $k \in X$, we have

$$\phi_k \in \mathcal{H}_{w^{-1}}^p \cap \mathcal{H}.$$

Proof. Trivially, $\phi_k \in \mathcal{H}$. By Proposition 3.0.1 and Theorem 3.0.5, it holds $\tilde{\psi} \sim \phi$. We get by Proposition 2.0.5,

$$C_{\tilde{\psi}} \phi_k = C_{\tilde{\psi}} D_\phi \delta_k = G_{\tilde{\psi}, \phi} \delta_k = G_{\tilde{\psi}, \phi}^{p, w^{-1}} \delta_k \in \ell_{w^{-1}}^p,$$

because of $G_{\tilde{\psi},\phi} \in \mathcal{M}(\ell_{w^{-1}}^p)$ (see Proposition 4.1.2). This implies $\phi_k \in \mathcal{H}_{w^{-1}}^p$ by Proposition 4.4.1. $\square$

This result allows us to define the synthesis operator for the other frames.

Proposition 4.6.7. [5, Lemma 8], [5, Proposition 3] Let $\phi \sim \psi$. Then

$$D_\phi^{00} : \ell^{00} \rightarrow \mathcal{H}_w^p$$

is well-defined and super-continuous. Moreover, it has a unique super-continuous extension, namely

$$D_\phi^{p,w} = D_\psi^{p,w} G_{\tilde{\psi},\phi}^{p,w}.$$

Proof. Because of Theorem 2.0.4 and Proposition 2.0.5 it holds

$$D_\phi = D_\psi G_{\tilde{\psi},\phi}$$

and

$$D_\phi^{00} = D_\psi^{p,w} G_{\tilde{\psi},\phi}^{00}.$$

Since both operators on the right-hand side are super-continuous (Proposition 4.6.2 and Proposition 4.3.1), D_ϕ^{00} is also super-continuous. We can write

$$D_\phi^{p,w} = D_\psi^{p,w} G_{\tilde{\psi},\phi}^{p,w}$$

to obtain a super-continuous extension, which is according to Theorem 4.3.3 unique. $\square$

In the next step, we recover the sum representation of the synthesis operator. Here, we must specify the topology used for the sum.

Proposition 4.6.8. [5, Lemma 7] Let $\phi \sim \psi$. Then the synthesis operator has the expression

$$D_\phi^{p,w} : \ell_w^p \rightarrow \mathcal{H}_w^p, \alpha \mapsto \sum_{k \in X} \alpha_k \phi_k.$$

For $p < \infty$, the sum converges in the norm. For $p \in [1, \infty]$, the sum converges in the weak-* sense.

Proof. Recall that $D_\phi^{p,w}$ is super-continuous according to Proposition 4.6.7. Because of the definition, D_ϕ^{00} has the sum representation, which is by Lemma 4.6.5 well defined. If $p < \infty$ then we use Theorem 1.5.9 to conclude that the unique continuous extension $D_\phi^{p,w}$ has the sum representation that converges in norm. If $p \in [1, \infty]$, we use Theorem 1.5.18 to conclude that $D_\phi^{p,w}$ has a sum representation that converges in a weak-* sense. $\square$

Note that the sum does not always converge in the norm if $p = \infty$. Indeed, suppose that $\mathcal{H} = \ell^2$, $w = (1)_{k \in X} = \alpha$, $\mathcal{H}^\infty = \ell^\infty$ and $\psi = (\delta_k)_{k \in X}$. Then

$$D_\psi^{\infty,w} \alpha = \sum_{l \in X} \alpha_l \delta_l = \sum_{l \in X} \delta_l$$

does not converge in the ℓ^∞ -norm because the summands do not converge to $\mathbf{0}$.

We also have to ensure that D_ϕ and $D_\phi^{p,w}$ match on their common domain.

Proposition 4.6.9. [5, Lemma 7] Let $\phi \sim \psi$. We then

$$D_\phi|_{\ell_w^p \cap \ell^2} = D_\phi^{p,w}|_{\ell_w^p \cap \ell^2}.$$

Proof. Let $\alpha \in \ell_w^p \cap \ell^2$. Then, using Theorem 2.0.4, Proposition 2.0.5 and Proposition 4.6.4

$$D_\phi \alpha = D_\psi G_{\tilde{\psi}, \phi} \alpha = D_\psi^{p,w} G_{\tilde{\psi}, \phi}^{p,w} \alpha = D_\phi^{p,w} \alpha.$$

□

Having defined the most important operators in the co-orbit space, we can recover some of their properties, which we know from the Hilbert space setting.

Theorem 4.6.10. [5, Lemma 8] Let ϕ, ξ be frames such that $\phi \sim \psi$ and $\xi \sim \psi$. Then

$$G_{\phi, \xi}^{p,w} = C_\phi^{p,w} D_\xi^{p,w}.$$

Proof. By Proposition 2.0.5, this equation holds on the dense subset ℓ^{00} . Since both sides are super-continuous operators, by Theorem 4.3.3, they are equal. □

We can also recover the reconstruction formula.

Theorem 4.6.11. [5, Proposition 3] Let ϕ be a frame such that $\phi \sim \psi$. Then

$$D_\phi^{p,w} C_\phi^{p,w} = id_{\mathcal{H}_w^p} = D_\phi^{p,w} C_\phi^{p,w}.$$

Proof. By Theorem 2.0.4, this equation holds on the dense subset ℓ^{00} . Since both sides are super-continuous operators, by Theorem 4.3.3, they are equal. □

In the space $\mathcal{H}$, the coefficient operator has a lower bound. We also show that this works in the co-orbit spaces.

Theorem 4.6.12. [5, Proposition 1] Let ϕ be a frame such that $\phi \sim \psi$. Then

$$\|f\|_{\mathcal{H}_w^p} \lesssim \|C_\phi^{p,w} f\|_{\ell_w^p} \quad \forall f \in \mathcal{H}_w^p.$$

Proof. By Theorem 4.6.11 and Theorem 4.6.10,

$$C_{\tilde{\psi}}^{p,w} = C_{\tilde{\psi}}^{p,w} D_{\tilde{\phi}}^{p,w} C_\phi^{p,w} = G_{\tilde{\psi}, \phi}^{p,w} C_\phi^{p,w}.$$

This gives us

$$\|f\|_{\mathcal{H}_w^p} = \|C_{\tilde{\psi}}^{p,w} f\|_{\ell_w^p} = \|G_{\tilde{\psi},\phi}^{p,w} C_{\phi}^{p,w} f\|_{\ell_w^p} \leq \|G_{\tilde{\psi},\phi}^{p,w}\|_{L(\ell_w^p)} \|C_{\phi}^{p,w} f\|_{\ell_w^p}.$$

□

4.7 Completeness of $\mathcal{H}_w^p$

So far, we only know that $\mathcal{H}_w^p$ is norm complete for $p < \infty$. To obtain more information about completeness, we show the following.

Lemma 4.7.1. [5, Theorem 4 (i)] It holds

$$R(C_{\tilde{\psi}}^{p,w}) = R(G_{\tilde{\psi},\psi}^{p,w}).$$

Proof. According to Theorem 4.6.11 and Theorem 4.6.10, we get

$$C_{\tilde{\psi}}^{p,w} = C_{\tilde{\psi}}^{p,w} D_{\psi}^{p,w} C_{\tilde{\psi}}^{p,w} = G_{\tilde{\psi},\psi}^{p,w} C_{\tilde{\psi}}^{p,w}.$$

This implies that $R(C_{\tilde{\psi}}^{p,w}) \subseteq R(G_{\tilde{\psi},\psi}^{p,w})$. The other inclusion follows from Lemma 4.6.1. This completes the proof. □

This allows us to show completeness.

Theorem 4.7.2. [22], [5, Proposition 2] The space $\mathcal{H}_w^p$ is norm-complete. For $p \in [1, \infty]$, $\mathcal{H}_w^p$ is sequentially weak-* complete.

Proof. We know that $C_{\tilde{\psi}}^{p,w}$ is an isometric isomorphism onto $R(C_{\tilde{\psi}}^{p,w})$. By Lemma 4.7.1, $R(C_{\tilde{\psi}}^{p,w}) = R(G_{\tilde{\psi},\psi}^{p,w})$. Since $R(G_{\tilde{\psi},\psi}^{p,w})$ is norm complete and for $p \in [1, \infty]$ sequentially weak-* complete, the same holds for $\mathcal{H}_w^p$. □

4.8 Riesz bases

In this subsection, we assume that ψ is a Riesz basis. In this case, we can also recover some properties that we know from the space $\mathcal{H}$.

Theorem 4.8.1. [5, Lemma 8], [5, Proposition 3], [5, Theorem 4 (i)] The operators $C_{\tilde{\psi}}^{p,w}$ and $D_{\psi}^{p,w}$ are bijective with $(C_{\tilde{\psi}}^{p,w})^{-1} = D_{\psi}^{p,w}$.

Proof. By Proposition 2.1.1, $C_{\tilde{\psi}}$ and D_{ψ} are bijective. Using Proposition 2.0.5 and Proposition 2.0.9, we get

$$C_{\tilde{\psi}} D_{\psi} = G_{\tilde{\psi},\psi} = \pi_{\ell^2 \rightarrow R(C_{\tilde{\psi}})} = \text{id}_{\ell^2}.$$

Therefore, $G_{\tilde{\psi},\psi}$ is the identity matrix, and $G_{\tilde{\psi},\psi}^{p,w}$ is the identity operator. Since Theorem 4.6.10 implies

$$C_{\tilde{\psi}}^{p,w} D_{\psi}^{p,w} = G_{\tilde{\psi},\psi}^{p,w} = \text{id}_{\ell_w^p}$$

and Theorem 4.6.11 implies

$$D_{\psi}^{p,w} C_{\tilde{\psi}}^{p,w} = \text{id}_{\mathcal{H}_w^p},$$

We obtain that $C_{\tilde{\psi}}^{p,w}$ and $D_{\psi}^{p,w}$ are invertible and inverses with respect to each other. $\square$

5 Duality of $\mathcal{H}_w^p$

As seen in one of the previous chapters, $(\ell_w^p)' \simeq \ell_{w^{-1}}^{p'}$ for $p \in \{0\} \cup [1, \infty)$. Our goal is to show that similarly, $(\mathcal{H}_w^p)'$ is isomorphic to $\mathcal{H}_{w^{-1}}^{p'}$. Similar to the previous chapter, we assume that ψ is a self-localized frame. First, we define a sesquilinear form similar to that in weighted ℓ^p spaces.

Definition. [5, Proposition 2] Let $p \in \{0\} \cup [1, \infty]$. For $f \in \mathcal{H}_w^p, g \in \mathcal{H}_{w^{-1}}^{p'}$, we define

$$\langle f, g \rangle_{\mathcal{H}_w^p, \mathcal{H}_{w^{-1}}^{p'}} := \langle C_{\tilde{\psi}}^{p,w} f, C_{\psi}^{p',w^{-1}} g \rangle_{\ell_w^p, \ell_{w^{-1}}^{p'}}.$$

A natural question is whether this sequilinear form coincides with the inner product on $\mathcal{H}$ on an appropriate domain. The following proposition provides the answer.

Proposition 5.0.1. [28] Let $p \in \{0\} \cup [1, \infty]$. Let $f \in \mathcal{H}_w^p \cap \mathcal{H}$ and $g \in \mathcal{H}_{w^{-1}}^{p'} \cap \mathcal{H}$. Then

$$\langle f, g \rangle_{\mathcal{H}_w^p, \mathcal{H}_{w^{-1}}^{p'}} = \langle f, g \rangle_{\mathcal{H}}.$$

Proof. Using Theorem 4.5.2, Proposition 2.0.2 and Theorem 2.0.4, we compute

$$\langle f, g \rangle_{\mathcal{H}_w^p, \mathcal{H}_{w^{-1}}^{p'}} = \langle C_{\tilde{\psi}}^{p,w} f, C_{\psi}^{p',w^{-1}} g \rangle_{\ell_w^p, \ell_{w^{-1}}^{p'}} = \langle C_{\tilde{\psi}} f, C_{\psi} g \rangle_{\ell^2} = \langle D_{\psi} C_{\tilde{\psi}} f, g \rangle_{\mathcal{H}} = \langle f, g \rangle_{\mathcal{H}}.$$

□

It turns out that this sequilinear form has properties similar to $\langle \cdot, \cdot \rangle_{\ell_w^p, \ell_{w^{-1}}^{p'}}$. For example, we can show that we also have a Hölder inequality in this case. However, we obtain an additional factor.

Theorem 5.0.2. [5, Proposition 2] Let $p \in \{0\} \cup [1, \infty]$. It holds

$$\left| \langle f, g \rangle_{\mathcal{H}_w^p, \mathcal{H}_{w^{-1}}^{p'}} \right| \lesssim \|f\|_{\mathcal{H}_w^p} \|g\|_{\mathcal{H}_{w^{-1}}^{p'}} \quad \forall f \in \mathcal{H}_w^p, g \in \mathcal{H}_{w^{-1}}^{p'}.$$

Proof. Using the Hölder inequality (Theorem 1.5.5) and the continuity of the coefficient operator (Proposition 4.3.2), we get

$$\left| \langle f, g \rangle_{\mathcal{H}_w^p, \mathcal{H}_{w^{-1}}^{p'}} \right| = \left| \langle C_{\tilde{\psi}}^{p,w} f, C_{\psi}^{p',w^{-1}} g \rangle_{\ell_w^p, \ell_{w^{-1}}^{p'}} \right| \leq \|C_{\tilde{\psi}}^{p,w} f\|_{\ell_w^p} \|C_{\psi}^{p',w^{-1}} g\|_{\ell_{w^{-1}}^{p'}} \lesssim \|f\|_{\mathcal{H}_w^p} \|g\|_{\mathcal{H}_{w^{-1}}^{p'}}.$$

□

This shows that the sesquilinear form is continuous with respect to both slots.

We can combine the norm continuity of the first slot with the weak-* continuity of the second slot to obtain the following:

Lemma 5.0.3. [50, Proposition 2.23], [22] Let $p \in \{0\} \cup [1, \infty)$ and $f \in \mathcal{H}_w^p, g \in \mathcal{H}_{w^{-1}}^{p'}$. Let $(f_n)_{n \in \mathbb{N}} \subseteq \mathcal{H}_w^p$ converge to some $f \in \mathcal{H}_w^p$ in the norm. Let $(g_n)_{n \in \mathbb{N}} \subseteq \mathcal{H}_{w^{-1}}^{p'}$ converge to some $g \in \mathcal{H}_{w^{-1}}^{p'}$ in the weak-* sense. Then

$$\lim_{n \rightarrow \infty} \langle f_n, g_n \rangle_{\mathcal{H}_w^p, \mathcal{H}_{w^{-1}}^{p'}} = \langle f, g \rangle_{\mathcal{H}_w^p, \mathcal{H}_{w^{-1}}^{p'}}.$$

Proof. Using Proposition 1.5.19 and super continuity of the coefficient operators, we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \langle f_n, g_n \rangle_{\mathcal{H}_w^p, \mathcal{H}_{w^{-1}}^{p'}} &= \lim_{n \rightarrow \infty} \langle C_{\tilde{\psi}}^{p,w} f_n, C_{\psi}^{p',w^{-1}} g_n \rangle_{\ell_w^p, \ell_{w^{-1}}^{p'}} = \langle C_{\tilde{\psi}}^{p,w} f, C_{\psi}^{p',w^{-1}} g \rangle_{\ell_w^p, \ell_{w^{-1}}^{p'}} \\ &= \langle f, g \rangle_{\mathcal{H}_w^p, \mathcal{H}_{w^{-1}}^{p'}}. \end{aligned}$$

□

In $\mathcal{H}$, we know that $C_{\psi}^* = D_{\psi}$. We can prove similar statements for $\mathcal{H}_w^p$.

Theorem 5.0.4. [5, Lemma 7] Let $\phi \sim \psi$ and $p \in \{0\} \cup [1, \infty]$. The following statements hold:

1. For all $f \in \mathcal{H}_w^p, \alpha \in \ell_{w^{-1}}^{p'}$, we have

$$\langle C_{\phi}^{p,w} f, \alpha \rangle_{\ell_w^p, \ell_{w^{-1}}^{p'}} = \langle f, D_{\phi}^{p',w^{-1}} \alpha \rangle_{\mathcal{H}_w^p, \mathcal{H}_{w^{-1}}^{p'}}.$$

2. For all $f \in \mathcal{H}_{w^{-1}}^{p'}, \alpha \in \ell_w^p$, we have

$$\langle D_{\phi}^{p,w} \alpha, f \rangle_{\mathcal{H}_w^p, \mathcal{H}_{w^{-1}}^{p'}} = \langle \alpha, C_{\phi}^{p',w^{-1}} f \rangle_{\ell_w^p, \ell_{w^{-1}}^{p'}}.$$

Proof. If $f \in \mathcal{H}^{00}$ and $\alpha \in \ell^{00}$ then both statements hold because of Proposition 5.0.1 and Proposition 2.0.2. Next, we show that the statement holds for $p < \infty$.

1. Let $(f_n)_{n \in \mathbb{N}} \subseteq \mathcal{H}^{00}$ norm converge to $f \in \mathcal{H}_w^p$ and $(\alpha^n)_{n \in \mathbb{N}} \subseteq \ell^{00}$ weak-* converge to $\alpha \in \mathcal{H}_{w^{-1}}^{p'}$. Then, because of Lemma 5.0.3,

$$\begin{aligned} \langle C_{\phi}^{p,w} f, \alpha \rangle_{\ell_w^p, \ell_{w^{-1}}^{p'}} &= \lim_{n \rightarrow \infty} \langle C_{\phi}^{p,w} f_n, \alpha^n \rangle_{\ell_w^p, \ell_{w^{-1}}^{p'}} = \lim_{n \rightarrow \infty} \langle f_n, D_{\phi}^{p',w^{-1}} \alpha^n \rangle_{\mathcal{H}_w^p, \mathcal{H}_{w^{-1}}^{p'}} \\ &= \langle f, D_{\phi}^{p',w^{-1}} \alpha \rangle_{\mathcal{H}_w^p, \mathcal{H}_{w^{-1}}^{p'}}, \end{aligned}$$

where we use the super continuity of the coefficient operator (Proposition 4.5.1) and the synthesis operator (Proposition 4.6.7).

2. Let $(\alpha^n)_{n \in \mathbb{N}} \subseteq \ell^{00}$ norm converge to α and $(f_n)_{n \in \mathbb{N}} \subseteq \mathcal{H}^{00}$ weak-* converge to $f \in \mathcal{H}_{w^{-1}}^{p'}$. Then, because of Lemma 5.0.3, the super continuity of the synthesis operator (Proposition 4.6.7) and the super continuity of the coefficient operator (Proposition 4.5.1),

$$\begin{aligned} \langle D_\phi^{p,w} \alpha, f \rangle_{\mathcal{H}_w^p, \mathcal{H}_{w^{-1}}^{p'}} &= \lim_{n \rightarrow \infty} \langle D_\phi^{p,w} \alpha^n, f_n \rangle_{\mathcal{H}} = \lim_{n \rightarrow \infty} \langle \alpha^n, C_\phi^{p',w^{-1}} f_n \rangle_{\ell^2} \\ &= \langle \alpha, C_\psi^{p',w^{-1}} f \rangle_{\mathcal{H}_w^p, \mathcal{H}_{w^{-1}}^{p'}}. \end{aligned}$$

For $p = \infty$, we swap the terms to reduce it to one of the previous results in the case of $p = 1$. First, we get

$$\langle C_\phi^{\infty,w} f, \alpha \rangle_{\ell_w^\infty, \ell_{w^{-1}}^1} = \overline{\langle \alpha, C_\phi^{\infty,w} f \rangle_{\ell_{w^{-1}}^1, \ell_w^\infty}} = \overline{\langle D_\phi^{1,w^{-1}} \alpha, f \rangle_{\mathcal{H}_{w^{-1}}^1, \mathcal{H}_w^\infty}} = \langle f, D_\phi^{1,w^{-1}} \alpha \rangle_{\mathcal{H}_w^\infty, \mathcal{H}_{w^{-1}}^1}.$$

The other case works similarly, namely

$$\langle D_\phi^{\infty,w} \alpha, f \rangle_{\mathcal{H}_w^\infty, \mathcal{H}_{w^{-1}}^1} = \overline{\langle f, D_\phi^{\infty,w} \alpha \rangle_{\mathcal{H}_{w^{-1}}^1, \mathcal{H}_w^\infty}} = \overline{\langle C_\phi^{1,w^{-1}} f, \alpha \rangle_{\ell_{w^{-1}}^1, \ell_w^\infty}} = \langle \alpha, C_\phi^{1,w^{-1}} f \rangle_{\ell_w^\infty, \ell_{w^{-1}}^1}.$$

□

By definition, $\langle \cdot, \cdot \rangle$ is non-degenerate. This means that for all $v \in \mathcal{H}$, it holds

$$\langle v, \cdot \rangle = 0 \Rightarrow v = 0 \text{ and } \langle \cdot, v \rangle = 0 \Rightarrow v = 0.$$

We want to show that the same holds for $\langle \cdot, \cdot \rangle_{\mathcal{H}_w^p, \mathcal{H}_{w^{-1}}^{p'}}$.

Lemma 5.0.5. [5, Proposition 2] Let $p \in \{0\} \cup [1, \infty]$. The sesquilinear form $\langle \cdot, \cdot \rangle_{\mathcal{H}_w^p, \mathcal{H}_{w^{-1}}^{p'}}$ is non-degenerate.

Proof. We show this for both slots.

1. Let $f \in \mathcal{H}_w^p$ such that

$$\langle f, g \rangle_{\mathcal{H}_w^p, \mathcal{H}_{w^{-1}}^{p'}} = 0 \quad \forall g \in \mathcal{H}_{w^{-1}}^{p'}.$$

Then, for all $k \in X$, we use Theorem 5.0.4 to get

$$0 = \langle f, D_\psi^{p',w^{-1}} \delta_k \rangle_{\mathcal{H}_w^p, \mathcal{H}_{w^{-1}}^{p'}} = \langle C_\psi^{p,w} f, \delta_k \rangle_{\ell_w^p, \ell_{w^{-1}}^{p'}} = (C_\psi^{p,w} f)_k.$$

This implies $C_\psi^{p,w} f = 0$ and by injectivity (Theorem 4.6.12), $f = 0$.

2. Let $g \in \mathcal{H}_{w^{-1}}^{p'}$ such that

$$\langle f, g \rangle_{\mathcal{H}_w^p, \mathcal{H}_{w^{-1}}^{p'}} = 0 \quad \forall f \in \mathcal{H}_w^p.$$

Then for all $k \in X$, we use Theorem 5.0.4 to get

$$0 = \langle D_\psi^{p,w} \delta_k, g \rangle_{\mathcal{H}_w^p, \mathcal{H}_{w^{-1}}^{p'}} = \langle \delta_k, C_\psi^{p',w^{-1}} g \rangle_{\ell_w^p, \ell_{w^{-1}}^{p'}} = (C_\psi^{p',w^{-1}} g)_k.$$

This implies $C_\psi^{p',w^{-1}} g = 0$ and by injectivity (Theorem 4.6.12), $g = 0$.

□

This sesquilinear form allows us to define linear functionals naturally.

Lemma 5.0.6. [5, Proposition 2] Let $p \in \{0\} \cup [1, \infty]$. Then

$$\Phi : \mathcal{H}_{w^{-1}}^{p'} \rightarrow (\mathcal{H}_w^p)', g \mapsto \langle \cdot, g \rangle_{\mathcal{H}_w^p, \mathcal{H}_{w^{-1}}^{p'}}$$

is an injective, norm-continuous, and conjugate linear map.

Proof. The map is well-defined because the sesquilinear form is linear with respect to the first slot and continuous because of Theorem 5.0.2. Conjugate linearity is clear because the sesquilinear form is conjugate linear with respect to the second slot. Norm continuity follows from Theorem 5.0.2. The injectivity follows from Lemma 5.0.5. □

This lemma shows that any element in $\mathcal{H}_{w^{-1}}^{p'}$ can be identified with an element in $(\mathcal{H}_w^p)'$. We can also show the converse if we exclude $p = \infty$.

Lemma 5.0.7. [5, Proposition 2] Let $p \in \{0\} \cup [1, \infty)$ and $h' \in (\mathcal{H}_w^p)'$. Then there exists some $h \in \mathcal{H}_{w^{-1}}^{p'}$ such that

$$h'(f) = \langle f, h \rangle_{\mathcal{H}_w^p, \mathcal{H}_{w^{-1}}^{p'}} \quad \forall f \in \mathcal{H}_w^p.$$

Proof. We easily see that $h' D_\psi^{p,w} \in (\ell_w^p)'$. By duality (Theorem 1.5.7), there exists some $\beta \in \ell_{w^{-1}}^{p'}$ such that

$$h' D_\psi^{p,w} = \langle \cdot, \beta \rangle_{\ell_w^p, \ell_{w^{-1}}^{p'}}.$$

Now, let $h := D_{\tilde{\psi}}^{p',w^{-1}} \beta$. Then, using Theorem 5.0.4,

$$\langle f, h \rangle_{\mathcal{H}_w^p, \mathcal{H}_{w^{-1}}^{p'}} = \langle f, D_{\tilde{\psi}}^{p',w^{-1}} \beta \rangle_{\mathcal{H}_w^p, \mathcal{H}_{w^{-1}}^{p'}} = \langle C_{\tilde{\psi}}^{p,w} f, \beta \rangle_{\ell_w^p, \ell_{w^{-1}}^{p'}} = h' D_\psi^{p,w} C_{\tilde{\psi}}^{p,w} f = h'(f),$$

where the last equation follows from Theorem 4.6.11. □

Finally, we can show that $(\mathcal{H}_w^p)'$ can be identified with $\mathcal{H}_{w^{-1}}^{p'}$.

Theorem 5.0.8. [5, Proposition 2] Let $p \in \{0\} \cup [1, \infty)$. Then, there exists a bijective, conjugate linear map $\Phi : \mathcal{H}_{w^{-1}}^{p'} \rightarrow (\mathcal{H}_w^p)'$.

Proof. Combining Lemma 5.0.6 and Lemma 5.0.7 yields the desired result. □

5.1 Banach frames on $\mathcal{H}_w^p$

Recall that for any Banach space $\mathcal{B}$, we need a sequence $(\phi_k)_{k \in X}$ to be in the dual space $\mathcal{B}'$. In our case, this means that any Banach frame for $\mathcal{H}_w^p$ is in $(\mathcal{H}_w^p)'$. Using Theorem 5.0.8, we can use the space $\mathcal{H}_{w^{-1}}^{p'}$ instead. Moreover, we also investigated the intersection $\mathcal{H}_{w^{-1}}^{p'} \cap \mathcal{H}$. This means that for a frame $\phi \subseteq \mathcal{H}$, we can check if $\phi \subseteq (\mathcal{H}_w^p)'$ by the abuse of notation. If this is the case, one can also check if ϕ is a Banach frame for $\mathcal{H}_w^p$. First, we need the following result.

Lemma 5.1.1. [5, Lemma 6] Let $p \in \{0\} \cup [0, \infty]$. Let $\phi \sim \psi$. Then

$$C_\phi^{p,w} f = \left(\langle f, \phi_k \rangle_{\mathcal{H}_w^p, \mathcal{H}_{w^{-1}}^{p'}} \right)_{k \in X} \quad \forall f \in \mathcal{H}_w^p.$$

Proof. Because of Lemma 4.6.6, $\langle f, \phi_k \rangle_{\mathcal{H}_w^p, \mathcal{H}_{w^{-1}}^{p'}}$ is well defined. According to Proposition 5.0.1, this statement is valid for $f \in \mathcal{H}^{00}$. Let $k \in X$ and $f \in \mathcal{H}_w^p$. For $p < \infty$, let $(f_n)_{n \in \mathbb{N}} \subseteq \mathcal{H}^{00}$ converge to f in norm. Then, by super continuity (Proposition 4.5.1), it holds

$$(C_\phi^{p,w} f)_k = \lim_{n \rightarrow \infty} (C_\phi^{p,w} f_n)_k = \lim_{n \rightarrow \infty} \langle f_n, \phi_k \rangle_{\mathcal{H}_w^p, \mathcal{H}_{w^{-1}}^{p'}} = \langle f, \phi_k \rangle_{\mathcal{H}_w^p, \mathcal{H}_{w^{-1}}^{p'}}.$$

If $p = \infty$, choose the sequence $(f_n)_{n \in \mathbb{N}} \subseteq \mathcal{H}^{00}$ to converge to f with respect to the weak-* topology. In this case, $\infty' = 1 = \infty^*$; therefore, the same calculation works. $\square$

We now provide an answer to this question. This is a generalization of [28, Theorem 5.1].

Theorem 5.1.2. Let $p \in \{0\} \cup [1, \infty]$. Let $\psi, \phi \in \text{Loc}(\mathcal{A})$ such that $\psi \sim \phi$. Then, ϕ is a Banach frame for $\mathcal{H}_w^p(\psi)$.

Proof. Let $\mathcal{B}_d := \ell_w^p$. We go through the three steps of the definition of a Banach frame.

1. According to Lemma 5.1.1, the coefficient operator is of the form

$$C_\phi^{p,w} : \mathcal{H}_w^p(\psi) \rightarrow \ell_w^p, f \mapsto (\langle f, \phi_k \rangle_{\mathcal{H}_w^p, \mathcal{H}_{w^{-1}}^{p'}})_{k \in X}.$$

This is bounded by Proposition 4.5.1.

2. We get the lower bound from Theorem 4.6.12.
3. Using Theorem 4.6.11, we get $D_\phi^{p,w}$ as the reconstruction operator.

$\square$

6 Localization of operators

In this section, we analyze localizable operators. The idea is that any operator in $L(\mathcal{H})$ has a matrix representation in $\mathcal{M}(\ell^2)$. This matrix can either be an element in $\mathcal{A}$ or not. If it is, then we say that the operator is localizable. Then, some natural questions arise.

- If an operator is localizable, is the adjoint also localizable?
- Is the sum of two localizable operators also localizable?
- Is the composition of two localizable operators also localizable?

Our goal is to answer these questions and more. For this entire chapter, we assume that ψ, ϕ, ξ are frames. While the other chapters have provided different approaches to prove or state known results, in this chapter, we provide unknown results, generalizing [28].

6.1 Matrix representation of operators

In this chapter, we use frames to represent a linear continuous operator on $\mathcal{H}$ as a matrix in a certain sense, see also [3].

Definition. [28, Definition 3.1] Let $T \in L(\mathcal{H})$. We define the matrix representation of T as

$$\mathcal{M}^{\psi, \phi}(T) := (\langle T\phi_l, \psi_k \rangle)_{k, l \in X}.$$

If we interpret the matrix representation of an operator as an operator, we obtain the following:

Proposition 6.1.1. [28, Proposition 3.2] Let $T \in L(\mathcal{H})$ and $\alpha \in \ell^2$. Then

$$\mathcal{M}^{\psi, \phi}(T)\alpha = C_\psi T D_\phi \alpha.$$

Proof. On the left side, we get

$$(\langle T\phi_l, \psi_k \rangle)_{k, l \in X}(\alpha_l)_{l \in X} = \left(\sum_{l \in X} \langle T\phi_l, \psi_k \rangle \alpha_l \right)_{k \in X}.$$

Starting on the right side from the inside, we get

$$D_\phi \alpha = \sum_{l \in X} \alpha_l \phi_l$$

and

$$TD_\phi\alpha = T \sum_{l \in X} \alpha_l \phi_l = \sum_{l \in X} \alpha_l T\phi_l.$$

Then

$$C_\psi TD_\phi\alpha = (\langle TD_\phi\alpha, \psi_k \rangle)_{k \in X} = \left(\left\langle \sum_{l \in X} \alpha_l T\phi_l, \psi_k \right\rangle \right)_{k \in X} = \left(\left\langle \sum_{l \in X} T\phi_l, \psi_k \right\rangle \alpha_l \right)_{k \in X},$$

which is the same as the left-hand side. $\square$

Using Theorem 1.6.3, we identify $\mathcal{M}(\ell^2)$ with $L(\ell^2)$ for the rest of the chapter. Recall that this identification is consistent with matrix multiplication (see Proposition 1.6.5) and involution (see Proposition 1.6.11). Next, we analyze the operator $\mathcal{M}^{\psi, \phi} : L(\mathcal{H}) \rightarrow L(\ell^2)$.

Proposition 6.1.2. [3, Theorem 3.1] *It holds $\mathcal{M}^{\psi, \phi} \in L(L(\mathcal{H}), L(\ell^2))$.*

Proof. For all statements, we use Proposition 6.1.1 to represent the matrix.

1. Additivity: We compute

$$\mathcal{M}^{\psi, \phi}(S + T) = C_\psi(S + T)D_\phi = C_\psi S D_\phi + C_\psi T D_\phi = \mathcal{M}^{\psi, \phi}(S) + \mathcal{M}^{\psi, \phi}(T).$$

2. Homogeneity: We get

$$\mathcal{M}^{\psi, \phi}(\lambda T) = C_\psi(\lambda T)D_\phi = \lambda C_\psi T D_\phi = \lambda \mathcal{M}^{\psi, \phi}(T)$$

3. Continuity: We get

$$\|\mathcal{M}^{\psi, \phi}(T)\| = \|C_\psi T D_\phi\| \leq \|C_\psi\| \|T\| \|D_\phi\|,$$

showing continuity. $\square$

Next, we show the following.

Proposition 6.1.3. [3, Proposition 3.2] *The operator $\mathcal{M}^{\psi, \phi}$ is injective.*

Proof. We show that $\mathbf{K}(\mathcal{M}^{\psi, \phi}) = \{0\}$. Suppose that $\mathcal{M}^{\psi, \phi}(T) = 0$. Using Proposition 6.1.1, we get

$$C_\psi T D_\phi = 0.$$

Using Theorem 2.0.4, we get

$$0 = D_{\tilde{\psi}} 0 C_{\tilde{\phi}} = D_{\tilde{\psi}} C_\psi T D_\phi C_{\tilde{\phi}} = T.$$

$\square$

Next, we show that the matrix representation works well with involution.

Proposition 6.1.4. *Let $T \in L(\mathcal{H})$. Then*

$$\mathcal{M}^{\psi,\phi}(T^*) = (\mathcal{M}^{\phi,\psi}(T))^*.$$

Proof. We use Proposition 6.1.1 and Proposition 2.0.2 to get

$$\mathcal{M}^{\psi,\phi}(T^*) = C_\psi T^* D_\phi = (D_\phi^* T C_\psi^*)^* = (C_\phi T D_\psi)^* = (\mathcal{M}^{\phi,\psi}(T))^*.$$

□

Additionally, the matrix representation works well with the composition.

Proposition 6.1.5. *[3, Proposition 3.2] Let $S, T \in L(\mathcal{H})$. Then*

$$\mathcal{M}^{\xi,\psi}(ST) = \mathcal{M}^{\xi,\bar{\phi}}(S) \mathcal{M}^{\phi,\psi}(T).$$

Proof. We use Proposition 6.1.1 and Theorem 2.0.4 to get

$$\mathcal{M}^{\xi,\bar{\phi}}(S) \mathcal{M}^{\phi,\psi}(T) = D_\xi T C_{\bar{\phi}} D_\phi S C_\psi = D_\xi T S C_\psi = \mathcal{M}^{\xi,\psi}(ST).$$

□

In some cases, we obtain a useful representation of the range of the matrix representation of an operator.

Lemma 6.1.6. *[7, Lemma 3.3] Let $T \in L(\mathcal{H})$ be a surjective operator. Then*

$$R(\mathcal{M}^{\psi,\phi}(T)) = R(C_\psi).$$

Proof. By Proposition 6.1.1, we get

$$R(\mathcal{M}^{\psi,\phi}(T)) = R(C_\psi T D_\phi).$$

Since T and D_ϕ are surjective, it holds

$$R(C_\psi T D_\phi) = R(C_\psi),$$

which proves the claim. □

The condition that T is surjective cannot be omitted since $T = \mathbf{0}$ provides a counterexample. If we choose two frames that are canonical duals with respect to each other, then the matrix representation of the identity is an orthogonal projection.

Lemma 6.1.7. *[3, Example 3.1 (4)] It holds*

$$\mathcal{M}^{\psi,\tilde{\psi}}(id_{\mathcal{H}}) = \pi_{\ell^2 \rightarrow R(C_\psi)} = \pi_{\ell^2 \rightarrow R(C_{\tilde{\psi}})} = \mathcal{M}^{\tilde{\psi},\psi}(id_{\mathcal{H}}).$$

Proof. Using Proposition 2.0.5 and Proposition 2.0.9, we get

$$\mathcal{M}^{\psi, \tilde{\psi}}(\text{id}_{\mathcal{H}}) = C_{\psi} D_{\tilde{\psi}} = G_{\psi, \tilde{\psi}} = \pi_{\ell^2 \rightarrow R(C_{\psi})}.$$

Similarly,

$$\mathcal{M}^{\tilde{\psi}, \psi}(\text{id}_{\mathcal{H}}) = G_{\tilde{\psi}, \psi} = \pi_{\ell^2 \rightarrow R(C_{\tilde{\psi}})}.$$

Using Proposition 2.0.7 completes the proof. $\square$

Finally, we show that the matrix representation works well with inversion. Here, we must use the pseudo-inverse.

Proposition 6.1.8. [8, Theorem 4.1] *Let $T \in I(\mathcal{H})$. Then*

$$\mathcal{M}^{\psi, \phi}(T^{-1}) = (\mathcal{M}^{\tilde{\phi}, \tilde{\psi}}(T))^{\dagger}.$$

Proof. We use Proposition 6.1.5, Lemma 6.1.7 and Lemma 6.1.6 to get

$$\mathcal{M}^{\tilde{\phi}, \tilde{\psi}}(T) \mathcal{M}^{\psi, \phi}(T^{-1}) = \mathcal{M}^{\tilde{\phi}, \phi}(TT^{-1}) = \mathcal{M}^{\tilde{\phi}, \phi}(\text{id}_{\mathcal{H}}) = \pi_{\ell^2 \rightarrow R(C_{\tilde{\phi}})} = \pi_{\ell^2 \rightarrow R(\mathcal{M}^{\tilde{\phi}, \tilde{\psi}}(T))}.$$

Similarly,

$$\mathcal{M}^{\psi, \phi}(T^{-1}) \mathcal{M}^{\tilde{\phi}, \tilde{\psi}}(T) = \mathcal{M}^{\psi, \tilde{\psi}}(T^{-1}T) = \mathcal{M}^{\psi, \tilde{\psi}}(\text{id}_{\mathcal{H}}) = \pi_{\ell^2 \rightarrow R(C_{\psi})} = \pi_{\ell^2 \rightarrow R(\mathcal{M}^{\psi, \phi}(T^{-1}))}.$$

Using Theorem 1.8.14 completes this proof. $\square$

Note that this proposition shows that $\mathcal{M}^{\psi, \phi}(T)$ is invertible and

$$\mathcal{M}^{\tilde{\phi}, \tilde{\psi}}(T^{-1}) = (\mathcal{M}^{\psi, \phi}(T))^{-1}$$

if and only if ψ and ϕ are Riesz bases [7, Theorem 3.4]. Indeed, since

$$\mathcal{M}^{\psi, \phi}(T) = C_{\psi} T D_{\phi},$$

D_{ϕ} must necessarily be injective to make $\mathcal{M}^{\psi, \phi}(T)$ injective. Similarly, C_{ψ} must necessarily be surjective to make $\mathcal{M}^{\psi, \phi}(T)$ surjective. Therefore, C_{ψ} and D_{ϕ} must be bijective, being equivalent to ψ and ϕ being Riesz bases (Proposition 2.1.1). Note that these conditions are also sufficient because $\mathcal{M}^{\psi, \phi}(T)$ is the composition of three bijective operators.

One might ask the following question: If $T \in L(\mathcal{H})$ has closed range, do we get

$$\mathcal{M}^{\tilde{\phi}, \tilde{\psi}}(T^{\dagger}) = (\mathcal{M}^{\psi, \phi}(T))^{\dagger}?$$

The answer is no. To see this, let $\mathcal{H} = \mathbb{R}^2$. Let ψ be a Riesz basis such that

$$C_{\psi} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then, because of Proposition 2.1.2,

$$D_{\tilde{\psi}} = C_{\psi}^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Let

$$T := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, $T^{\dagger} = T$. Now, we compute

$$\begin{aligned} \mathcal{M}^{\psi, \psi}(T) \mathcal{M}^{\tilde{\psi}, \tilde{\psi}}(T^{\dagger}) &= \mathcal{M}^{\psi, \tilde{\psi}}(TT^{\dagger}) = \mathcal{M}^{\psi, \tilde{\psi}}(T) = C_{\psi} T D_{\tilde{\psi}} \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Note that the resulting matrix is not self-adjoint and therefore cannot be an orthogonal projection. According to Theorem 1.8.14, $\mathcal{M}^{\tilde{\psi}, \tilde{\psi}}(T^{\dagger})$ cannot be the pseudo-inverse of $\mathcal{M}^{\psi, \psi}(T)$.

6.2 Localization

Having set up the matrix representation, we can now define when an operator is localizable. Here, we need our solid spectral matrix algebra $\mathcal{A}$ -

Definition. Let ψ, ϕ be frames. Let $T \in L(\mathcal{H})$. We call T (ψ, ϕ) -localizable if

$$\mathcal{M}^{\psi, \phi}(T) \in \mathcal{A}.$$

In this case, we use the notation $T \in \mathbf{Loc}(\psi, \phi)$. If both frames are the same, we write $\mathbf{Loc}(\psi)$ instead of $\mathbf{Loc}(\psi, \psi)$.

We easily see that $\text{id}_{\mathcal{H}} \in \mathbf{Loc}(\psi, \phi)$ if and only if $\psi \sim \phi$. Now, we analyze the structure of $\mathbf{Loc}(\psi, \phi)$.

Proposition 6.2.1. *The set $\mathbf{Loc}(\psi, \phi)$ is a linear subspace of $L(\mathcal{H})$.*

Proof. Trivially,

$$\mathcal{M}^{\psi, \phi}(0) = 0 \in \mathcal{A}.$$

Let $S, T \in \mathbf{Loc}(\psi, \phi)$ and $\lambda \in \mathbb{F}$. By Proposition 6.1.2,

$$\mathcal{M}^{\psi, \phi}(\lambda S + T) = \lambda \mathcal{M}^{\psi, \phi}(S) + \mathcal{M}^{\psi, \phi}(T) \in \mathcal{A},$$

showing that $\lambda S + T \in \mathbf{Loc}(\psi, \phi)$. □

[28, Theorem 3.3] provides a result about the composition of two localizable operators. This statement also holds in the general setting.

Proposition 6.2.2. *Let $\psi, \phi, \xi \in \mathbf{Loc}(\mathcal{A})$ (i.e., be self-localized frames), $S \in \mathbf{Loc}(\psi, \tilde{\phi})$ and $T \in \mathbf{Loc}(\phi, \xi)$. Then, $ST \in \mathbf{Loc}(\psi, \xi)$.*

Proof. By Proposition 6.1.5, we get

$$\mathcal{M}^{\psi, \xi}(ST) = \mathcal{M}^{\psi, \tilde{\phi}}(S)\mathcal{M}^{\phi, \xi}(T).$$

The remainder follows from the closedness of multiplication in $\mathcal{A}$. $\square$

Next, we show that a frame can be replaced by a frame that is in the same localization equivalence class, without changing the localizability property of the operator, which is a generalization of [28, Theorem 4.1].

Theorem 6.2.3. *Let $\psi, \psi', \phi, \phi' \in \text{Loc}(\mathcal{A})$ be such that $\psi \sim \psi'$ and $\phi \sim \phi'$. Let $T \in \text{Loc}(\psi, \phi)$. Then*

$$T \in \text{Loc}(\psi', \phi').$$

Proof. Using Proposition 6.1.5, we get

$$\mathcal{M}^{\psi', \phi'}(T) = \mathcal{M}^{\psi', \psi}(\text{id}_{\mathcal{H}})\mathcal{M}^{\tilde{\psi}, \tilde{\psi}}(\text{id}_{\mathcal{H}})\mathcal{M}^{\psi, \phi}(T)\mathcal{M}^{\tilde{\phi}, \tilde{\phi}}(\text{id}_{\mathcal{H}})\mathcal{M}^{\phi, \phi'}(\text{id}_{\mathcal{H}}).$$

By our assumptions, all five matrices are in $\mathcal{A}$, so the product is also in $\mathcal{A}$. $\square$

Because self-localized frames are localized with their canonical dual frames (Proposition 3.0.1), we immediately obtain

Corollary 6.2.4. *For $\psi, \phi \in \text{Loc}(\mathcal{A})$, we get*

$$\text{Loc}(\psi, \phi) = \text{Loc}(\psi, \tilde{\phi}) = \text{Loc}(\tilde{\psi}, \phi) = \text{Loc}(\tilde{\psi}, \tilde{\phi}).$$

Using this corollary, we can rephrase Proposition 6.2.2 by omitting the canonical dual frame as follows.

Proposition 6.2.5. *Let $\psi, \phi, \xi \in \text{Loc}(\mathcal{A})$, $S \in \text{Loc}(\psi, \phi)$ and $T \in \text{Loc}(\phi, \xi)$. Then*

$$ST \in \text{Loc}(\psi, \xi).$$

Proof. By Corollary 6.2.4,

$$S \in \text{Loc}(\psi, \tilde{\phi}).$$

Using Proposition 6.2.2,

$$ST \in \text{Loc}(\psi, \xi).$$

$\square$

The localization property interacts well with the involution.

Proposition 6.2.6. *Let $T \in \text{Loc}(\psi, \phi)$. Then $T^* \in \text{Loc}(\phi, \psi)$.*

Proof. This follows from Proposition 6.1.4, i.e.,

$$\mathcal{M}^{\psi,\phi}(T^*) = \mathcal{M}^{\phi,\psi}(T)^*$$

and the fact that $\mathcal{A}$ is closed under $*$. $\square$

We summarize the results as follows.

Theorem 6.2.7. *The set $\mathbf{Loc}(\psi)$ is a unital $*$ -algebra,*

Proof. By Proposition 6.2.1, $\mathbf{Loc}(\psi)$ is a subspace. Since $\psi \in \mathbf{Loc}(\mathcal{A})$, $\mathbf{Loc}(\psi)$ contains the identity operator. By Proposition 6.2.6, $\mathbf{Loc}(\psi)$ is closed under an involution. By Proposition 6.2.5, $\mathbf{Loc}(\psi)$ is closed under composition. $\square$

Next, we generalize [28, Theorem 4.4], stating that localization works well with inverses.

Theorem 6.2.8. *Let $T \in \mathbf{Loc}(\psi, \phi)$ be invertible. Then $T^{-1} \in \mathbf{Loc}(\phi, \psi)$.*

Proof. According to Proposition 6.1.8, we have

$$\mathcal{M}^{\tilde{\phi}, \tilde{\psi}}(T^{-1}) = (\mathcal{M}^{\psi, \phi}(T))^{\dagger}.$$

Because of Theorem 1.8.24, the matrix is in $\mathcal{A}$, which implies that

$$T^{-1} \in \mathbf{Loc}(\tilde{\phi}, \tilde{\psi}).$$

By Corollary 6.2.4,

$$T^{-1} \in \mathbf{Loc}(\phi, \psi).$$

$\square$

6.3 Induced norm

So far, we know that $\mathbf{Loc}(\psi)$ is an inverse closed unital $*$ -algebra. Our goal is to equip this algebra with an appropriate norm such that it becomes a Banach algebra. In this case, we obtain pseudo-inverse closedness for free.

Definition. We define the map

$$\|\cdot\|_{\psi, \phi} : \mathbf{Loc}(\psi, \phi) \rightarrow \mathbb{R}, T \mapsto \|\mathcal{M}^{\psi, \phi}(T)\|_{\mathcal{A}}.$$

Now, we show that this is a norm.

Proposition 6.3.1. *The map $\|\cdot\|_{\psi, \phi}$ is a norm.*

Proof. We go through the three properties and use the fact that $\mathcal{M}^{\psi, \phi}$ is linear (see Proposition 6.1.2).

1. Positive definiteness follows from Proposition 6.1.3.

2. We compute

$$\|\lambda T\|_{\psi,\phi} = \|\mathcal{M}^{\psi,\phi}(\lambda T)\|_{\mathcal{A}} = \|\lambda \mathcal{M}^{\psi,\phi}(T)\|_{\mathcal{A}} = |\lambda| \|\mathcal{M}^{\psi,\phi}(T)\|_{\mathcal{A}} = |\lambda| \|T\|_{\psi,\phi},$$

showing positive homogeneity.

3. We also compute

$$\begin{aligned} \|S + T\|_{\psi,\phi} &= \|\mathcal{M}^{\psi,\phi}(S + T)\|_{\mathcal{A}} = \|\mathcal{M}^{\psi,\phi}(S) + \mathcal{M}^{\psi,\phi}(T)\|_{\mathcal{A}} \\ &\leq \|\mathcal{M}^{\psi,\phi}(S)\|_{\mathcal{A}} + \|\mathcal{M}^{\psi,\phi}(T)\|_{\mathcal{A}} \\ &= \|S\|_{\psi,\phi} + \|T\|_{\psi,\phi}, \end{aligned}$$

showing a triangular inequality.

□

For the next couple of results, we restrict ourselves to the space $\mathbf{Loc}(\psi)$. In this case, we use the notation

$$\|\cdot\|_{\psi} := \|\cdot\|_{\psi,\tilde{\psi}}.$$

Proposition 6.3.2. *The norm $\|\cdot\|_{\psi}$ is submultiplicative:*

Proof. We use Proposition 6.1.5, to get

$$\begin{aligned} \|ST\|_{\psi} &= \|\mathcal{M}^{\psi,\tilde{\psi}}(ST)\|_{\mathcal{A}} = \|\mathcal{M}^{\psi,\tilde{\psi}}(S)\mathcal{M}^{\psi,\tilde{\psi}}(T)\|_{\mathcal{A}} \leq \|\mathcal{M}^{\psi,\tilde{\psi}}(S)\|_{\mathcal{A}} \|\mathcal{M}^{\psi,\tilde{\psi}}(T)\|_{\mathcal{A}} \\ &= \|S\|_{\psi} \|T\|_{\psi}. \end{aligned}$$

□

We also show that our new norm dominates the operator norm.

Lemma 6.3.3. *It holds*

$$\|T\| \lesssim \|T\|_{\psi} \quad \forall T \in \mathbf{Loc}(\psi).$$

Proof. By the reconstruction formula (Theorem 2.0.4),

$$T = D_{\tilde{\psi}} C_{\psi} T D_{\tilde{\psi}} C_{\psi} = D_{\tilde{\psi}} \mathcal{M}^{\psi,\tilde{\psi}}(T) C_{\psi}$$

and therefore

$$\|T\| = \|D_{\tilde{\psi}} \mathcal{M}^{\psi,\tilde{\psi}}(T) C_{\psi}\| \leq \|D_{\tilde{\psi}}\| \|\mathcal{M}^{\psi,\tilde{\psi}}(T)\| \|C_{\psi}\| \lesssim \|\mathcal{M}^{\psi,\tilde{\psi}}(T)\|.$$

Using Theorem 1.8.25, we get

$$\|\mathcal{M}^{\psi,\tilde{\psi}}(T)\| \leq \|\mathcal{M}^{\psi,\tilde{\psi}}(T)\|_{\mathcal{A}} = \|T\|_{\psi}.$$

□

This shows that $\mathbf{Loc}(\psi)$ is a normed algebra. Next, we show that it is complete.

Proposition 6.3.4. *The algebra $\mathbf{Loc}(\psi)$ is complete with respect to $\|\cdot\|_\psi$.*

Proof. Let $(T_n)_{n \in \mathbb{N}} \subseteq \mathbf{Loc}(\psi)$ be a Cauchy sequence. Then, $(\mathcal{M}^{\psi, \tilde{\psi}}(T_n))_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{A}$ and therefore converges to some $M \in \mathcal{A}$. By Theorem 1.8.25, this convergence also holds for $\mathcal{M}(\ell^2)$. By Theorem 2.0.4, it holds

$$\|T_m - T_n\| = \|D_{\tilde{\psi}} C_\psi (T_m - T_n) D_{\tilde{\psi}} C_\psi\| \leq \|D_{\tilde{\psi}}\| \|\mathcal{M}^{\psi, \tilde{\psi}}(T_m - T_n)\| \|C_\psi\|.$$

This shows that $(T_n)_{n \in \mathbb{N}}$ is a Cauchy sequence with respect to the operator norm and converges to some $T \in L(\mathcal{H})$. This implies that

$$M = \lim_{n \rightarrow \infty}^{\mathcal{M}(\ell^2)} \mathcal{M}^{\psi, \tilde{\psi}}(T_n) = \lim_{n \rightarrow \infty}^{L(\ell^2)} C_\psi T_n D_{\tilde{\psi}} = C_\psi T D_{\tilde{\psi}} = \mathcal{M}^{\psi, \tilde{\psi}}(T).$$

So, $T \in \mathbf{Loc}(\psi)$. Finally,

$$\|T_n - T\|_\psi = \|\mathcal{M}^{\psi, \tilde{\psi}}(T_n) - \mathcal{M}^{\psi, \tilde{\psi}}(T)\|_{\mathcal{A}} = \|\mathcal{M}^{\psi, \tilde{\psi}}(T_n) - M\|_{\mathcal{A}}.$$

Since the right-hand side converges to 0, the left-hand side does as well, showing that $\mathbf{Loc}(\psi)$ is complete. $\square$

This yields the following result.

Theorem 6.3.5. *The set $\mathbf{Loc}(\psi)$ is a unital Banach-* algebra with respect to $\|\cdot\|_\psi$.*

Proof. We must show that the involution is continuous. Let

$$\|T\| := \max\{\|T\|_\psi, \|T\|_{\tilde{\psi}}\}.$$

It is simple to show that this map is a norm. Moreover,

$$\begin{aligned} \|ST\| &= \max\{\|ST\|_\psi, \|ST\|_{\tilde{\psi}}\} \leq \max\{\|S\|_\psi \|T\|_\psi, \|S\|_{\tilde{\psi}} \|T\|_{\tilde{\psi}}\} \\ &\leq \max\{\|S\| \|T\|, \|S\| \|T\|\} \\ &= \|S\| \|T\|, \end{aligned}$$

proving submultiplicativity. Next, we show that $\mathcal{A}$ is complete with respect to $\|\cdot\|$. Let $(T_n)_{n \in \mathbb{N}}$ be a Cauchy sequence with respect to $\|\cdot\|$. Then, it is a Cauchy sequence with respect to $\|\cdot\|_\psi$ and $\|\cdot\|_{\tilde{\psi}}$. By completeness (Proposition 6.3.4), there exist $S, T \in \mathcal{A}$ such that

$$\lim_{n \rightarrow \infty} \|T_n - T\|_\psi = 0$$

and

$$\lim_{n \rightarrow \infty} \|T_n - S\|_{\tilde{\psi}} = 0.$$

By Lemma 6.3.3, $(T_n)_{n \in \mathbb{N}}$ converges to S and T in the operator norm, implying that $S = T$. This also implies that the sequence converges to T in $\|\cdot\|$, thus proving completeness. By definition,

$$\|T\|_\psi \leq \|T\| \quad \forall T \in \mathbf{Loc}(\psi).$$

Using the bounded inverse theorem (Theorem 1.8.2) on the identity, we conclude that both the norms are equivalent. Since

$$\|T^*\|_\psi = \|C_\psi T^* D_{\tilde{\psi}}\|_{\mathcal{A}} = \|(C_{\tilde{\psi}} T D_\psi)^*\|_{\mathcal{A}} \lesssim \|C_{\tilde{\psi}} T D_\psi\|_{\mathcal{A}} = \|T\|_{\tilde{\psi}},$$

we get

$$\|T^*\| = \max\{\|T^*\|_\psi, \|T^*\|_{\tilde{\psi}}\} = \max\{\|T\|_{\tilde{\psi}}, \|T\|_\psi\} = \|T\|.$$

This shows that the involution is continuous with respect to $\|\cdot\|$ and by norm equivalence, it is continuous with respect to $\|\cdot\|_\psi$. $\square$

This result gives us pseudo-inverse closedness.

Lemma 6.3.6. *Let $T \in \text{Loc}(\psi)$ have a closed range. Then*

$$T^\dagger \in \text{Loc}(\psi).$$

Proof. We have shown that $\text{Loc}(\psi) \leq L(\mathcal{H})$ (see Theorem 6.3.5). Since $\text{Loc}(\psi)$ is inverse closed (Theorem 6.2.8), it is also pseudo-inverse closed (Theorem 1.8.24), so $T^\dagger \in \text{Loc}(\psi)$. $\square$

We can also show pseudo-inverse closedness if the two frames are different.

Theorem 6.3.7. *Let $T \in \text{Loc}(\psi, \phi)$ have a closed range. Then*

$$T^\dagger \in \text{Loc}(\phi, \psi).$$

Proof. By Proposition 6.2.6, $T^* \in \text{Loc}(\phi, \psi)$. By Proposition 6.2.5, $T^*T \in \text{Loc}(\psi)$. Then $(T^*T)^\dagger \in \text{Loc}(\psi)$ because of Lemma 6.3.6. Finally, by Lemma 1.8.19,

$$T^\dagger = (T^*T)^\dagger T^* \in \text{Loc}(\phi, \psi).$$

$\square$

6.4 Localization on Banach spaces

Note that any operator $T \in L(\mathcal{H})$ can be restricted to $\mathcal{H}^{00}$. The question is whether this restriction can be continuously extended to $\mathcal{H}_w^p$. We show that this is possible. The following result is a generalization of [28, Theorem 5.2].

Theorem 6.4.1. *Let $\psi \in \text{Loc}(\mathcal{A})$. Let $T \in \text{Loc}(\psi)$. Then for all $p \in \{0\} \cup [1, \infty]$,*

$$T^{00} : \mathcal{H}^{00} \rightarrow \mathcal{H}_w^p$$

is super-continuous with respect to $\mathcal{H}_w^p$ and can be uniquely extended to a super-continuous linear operator on $\mathcal{H}_w^p$.

Proof. By Theorem 2.0.4, we get

$$T = D_\psi C_{\tilde{\psi}} T D_\psi C_{\tilde{\psi}} = D_\psi \mathcal{M}^{\tilde{\psi}, \psi}(T) C_{\tilde{\psi}}.$$

Since $\mathcal{M}^{\tilde{\psi}, \psi}(T) \in \mathcal{A} \subseteq \mathcal{M}(\ell_w^p)$, this implies

$$T^{00} = D_\psi^{p, w} \mathcal{M}^{\tilde{\psi}, \psi}(T) C_{\tilde{\psi}}^{00}.$$

All operators on the right side are super continuous (Proposition 4.6.2), so T^{00} is also super continuous. Writing

$$T^{p, w} = D_\psi^{p, w} \mathcal{M}^{\tilde{\psi}, \psi}(T) C_{\tilde{\psi}}^{p, w},$$

we obtain a super continuous extension, which is according to Theorem 4.3.3 unique. $\square$

6.5 Connection to localized frames

Recall that for any operator $T \in L(\mathcal{H})$ and any frame ψ , $T\psi$ is a frame if and only if T is surjective. One might ask, in which cases $T\psi$ is a localized frame. The following result provides a characterization, which is a generalization of [28, Theorem 6.2].

Theorem 6.5.1. *Let $\psi \in \text{Loc}(\mathcal{A})$ and $T \in L(\mathcal{H})$ be surjective. Then, $T\psi \in \text{Loc}(\mathcal{A})$ if and only if $T^*T \in \text{Loc}(\psi)$.*

Proof. By Theorem 2.2.2, $T\psi$ is a frame. An easy calculation shows that

$$\mathcal{M}^{\psi, \psi}(T^*T) = (\langle T^*T\psi_l, \psi_k \rangle)_{k, l \in X} = (\langle T\psi_l, T\psi_k \rangle)_{k, l \in X} = G_{T\psi, T\psi}.$$

Since the left-hand side is in $\mathcal{A}$ if and only if the right-hand side is in $\mathcal{A}$, the proof is complete. $\square$

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