



MASTERARBEIT | MASTER'S THESIS

Titel | Title

Modular laws for chromatic quasisymmetric functions and LLT
polynomials

verfasst von | submitted by

Sebastian Redl BSc

angestrebter akademischer Grad | in partial fulfilment of the requirements for the degree of
Master of Science (MSc)

Wien | Vienna, 2026

Studienkennzahl lt. Studienblatt | Degree
programme code as it appears on the
student record sheet:

UA 066 821

Studienrichtung lt. Studienblatt | Degree
programme as it appears on the student
record sheet:

Masterstudium Mathematik

Betreut von | Supervisor:

Univ.-Prof. Dr. Anton Mellit Privatdoz.

Abstract

Recently, Hikita proved the Stanley–Stembridge conjecture, utilizing the so-called *modular law*, a linear relation satisfied by the chromatic (quasi-)symmetric function of unit interval graphs or unicellular LLT polynomials. This master’s thesis concerns itself with different aspects of the modular law, culminating in describing a generalization of it which produces linear relations amongst bigger numbers of unicellular LLT polynomials behaving in a similar fashion to the modular law (see chapter 4).

Furthermore, this thesis provides a new proof for Lee’s result of unicellular LLT polynomials (and thus chromatic quasisymmetric functions of unit interval graphs) satisfying the modular law in chapter 2. Chapter 3 offers a new proof on the termination of the algorithm used to prove Abreu and Nigro’s characterization of functions satisfying the modular law by their values on complete graphs.

Zusammenfassung

Vor Kurzem wurde von Hikita die Vermutung von Stanley–Stembridge bewiesen. Dabei wurde das sogenannte „modular law“ genutzt, eine lineare Relation, die von den chromatisch (quasi-)symmetrischen Funktionen von „Einheitsintervallgraphen“ oder „einzelligen“ LLT Polynomen erfüllt wird. Diese Masterarbeit beschäftigt sich mit unterschiedlichen Aspekten des modular law und kulminiert in Kapitel 4 in der Beschreibung einer Generalisierung dieser Relation, die weitere lineare Relationen zwischen einer größeren Anzahl an einzelligen LLT Polynomen liefert, die ihrerseits alle ein dem ursprünglichen modular law ähnliches Verhalten aufweisen.

Darüber hinaus liefert diese Arbeit in Kapitel 2 einen neuen Beweis dafür, dass einzellige LLT Polynome (und damit chromatisch quasisymmetrische Funktionen von Einheitsintervallgraphen) das modular law erfüllen. Kapitel 3 liefert einen neuen Beweis dafür, dass der von Abreu und Nigro verwendete Algorithmus zur Charakterisierung von das modular law erfüllenden Funktionen mit ihren Werten bei vollständigen Graphen für alle Eingaben nach endlich vielen Schritten terminiert.

Acknowledgments

The author would like to thank his supervisor Anton Mellit for introducing him to Hikita's paper and then patiently guiding him during the work on this thesis, recognizing interesting problems suitable for a master's thesis and giving helpful pointers throughout the time spent working on it.

He would also like to thank his parents for supporting him during his studies and naturally all of his remaining family and friends, especially his brother Konstantin and Felix Hinterreiter, with whom he conquered most classes in his bachelor's and master's studies.

Additionally, the author is grateful for his music teachers at *Musikschule Wien* for enabling him to find a balance to his studies.

Finally, it is not a given for an ice cream parlour to offer relatively cheap ice cream of high quality after the last few economically rather turbulent years and the author is immensely grateful to *Eissalon Tichy* for achieving this miracle — with Kurt Tichy's descendents continuing to enrich the world with their wonderful ice cream creations, all conceived under the legacy of his fabulous *Eismarillenknödel*.

Contents

1	Introduction	1
1.1	The structure of this thesis	1
1.2	(Semi-)Standard Young Tableaux	2
1.3	Dyck paths, Hessenberg functions and unit-interval graphs	3
1.4	q -integers and the chromatic (quasi-)symmetric function	4
2	The modular law	6
2.1	The chromatic symmetric function and the modular law	8
2.2	LLT polynomials	11
2.3	The modular law for the chromatic <i>quasisymmetric</i> function	19
3	Abreu and Nigro's characterization	22
3.1	Preliminary propositions	22
3.2	Abreu–Nigro's algorithm	28
4	Interlacing filters	32
4.1	From the modular law to <i>interlacing filters</i>	32
4.2	The Dyck paths used by interlacing filters	33
4.3	Interlacing filters and direct formulae	35
	Bibliography	40

1 Introduction

The aim of this work is to study relations between LLT polynomials, which are certain generating functions of tuples of semistandard Young tableaux. We focus on the special case of unicellular LLT polynomials, closely related to the chromatic quasisymmetric functions as shown by Carlsson and Mellit [CM18, Proposition 3.5]. In this case, a connection between the relations of unicellular LLT polynomials and the relations between chromatic quasisymmetric functions of unit interval graphs has already been established. An especially powerful example of such relations, the so-called *modular law*, first described by Guay-Paquet [Gua13] and generalized to the quasisymmetric case by Lee [Lee21], was recently used by Hikita in his proof of the Stanley–Stembridge conjecture.

We try to further generalize the modular law to produce a family of identities sharing similar characteristics which we will call *interlacing filters* (see chapter 4). Put more concretely, we produce relations between LLT polynomials that filter out all terms corresponding to non-interlacing permutations and where the remaining terms take the same value except for their sign changing depending on the permutation. Here, an interlacing permutation is a permutation of variables from two sets whose sizes differ by at most 1 and no two elements of a single set directly follow each other. E.g., for the sets $\{a\}$ and $\{b, c\}$, the interlacing permutations of these variables are bac and cab .

1.1 The structure of this thesis

The starting point for the work on this thesis was Hikita’s proof of the Stanley–Stembridge Conjecture found in [Hik25], which lead the author to concern himself with some literature on the modular law used in the proof. This also reflects in the thesis presented here, with each chapter being related to the modular law in a different way.

In this introduction, we give an overview of the thesis and fix the notation later used in different parts of the thesis.

The next chapter, chapter 2, concerns itself with stating the modular law and proving that the chromatic (quasi-)symmetric functions satisfy their respective versions of it. To this end, we first follow Guay-Paquet’s proof for the chromatic symmetric function presented in [Gua13] and then introduce all necessary concepts to give a self-contained proof (Theorem 2.19) different from the proof by Lee in [Lee21].

Following this, in chapter 3, we turn to an application (Theorem 3.1) of the modular law used to express values of functions defined on Dyck paths, or equivalently unit interval graphs, as products of their values on complete graphs. This application was discovered by Abreu and Nigro in [AN21]. We mostly follow their paper in

our exposition, but use a modified algorithm (Algorithm 3.8) and provide our own proof (Corollary 3.12) for its termination which also proves the termination of their formulation of the algorithm, though in a less technical way.

Finally, in chapter 4, we formulate generalizations of the modular law which we will call *interlacing filters* (Definition 4.5). In these terms, the modular law can be associated to the 12-interlacing filter, with other interlacing filters being examples of linear relations between a bigger number of different variations of Dyck paths. We will not be able to prove their existence in this thesis, but will instead, based on our observations of computationally calculated examples, formulate Conjecture 4.7, which provides formulae for directly calculating interlacing filters and which we intend to prove in our future work.

1.2 (Semi-)Standard Young Tableaux

An extensive treatment of the combinatorial objects described below can be found in [Sta11] and [Sta23].

A *composition* $\lambda = (\lambda_1, \dots, \lambda_l)$ of an integer n is a sequence of integers $\lambda_i \in \mathbb{Z}^+$, also called parts of λ , with $\sum_i \lambda_i = n$. A *partition* is a composition λ where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l$ also holds. We write $\lambda \vdash n$ to express that λ is a partition of n . The *Young diagram* corresponding to a partition $\lambda \vdash n$ is an array of n boxes (also called cells), where row i contains λ_i boxes and all rows are left-justified. It is drawn in *english notation*, if the topmost row corresponds to λ_1 , the next row corresponds to λ_2 continuing like this with the last row corresponding to λ_l . If a Young diagram is drawn in *french notation*, the order of the rows is reversed, i.e. the topmost row is the shortest one with the other rows following in the ascending order given by the parts of λ . This thesis will use the french notation unless stated otherwise. To talk about specific boxes we may reference them by coordinates, i. e. the box $(5, 2)$ in the Young diagram of the partition $\lambda = (5, 5, 3, 1) \vdash 14$ is the fifth box of the row corresponding to the part $\lambda_2 = 5$.

A *Semistandard Young Tableau* (SSYT) of shape $\lambda = (\lambda_i)_{i=1, \dots, l}$ with λ being a partition, is a map from the boxes of the Young diagram of λ to the integers $\mathbb{Z}^+$, where rows are weakly increasing and columns strictly increase, i. e. the entry of $(1, 1)$ is strictly smaller than the entry of $(1, 2)$, whilst the entries of $(1, 1)$ and $(2, 1)$ may be equal. A SSYT is visualized by placing the appropriate numbers into the boxes of the Young diagram.

To get the definition of a *Standard Young Tableau* (SYT) we put additional constraints on the definition of a SSYT: A SYT of shape $\lambda \vdash n$ is a map from the boxes of the Young diagram of λ to $[n] := \{1, 2, \dots, n\}$ (instead of $\mathbb{Z}^+$), where now the rows also increase strictly.

We denote the sets of all (Semi-)Standard Young Tableaux by SSYT respectively SYT and we also allow for (Semi-)Standard Young Tableaux to be of *skew shape*, meaning that the underlying shape consists of the boxes remaining after removing the boxes associated to a partition whose Young diagram is contained in the Young

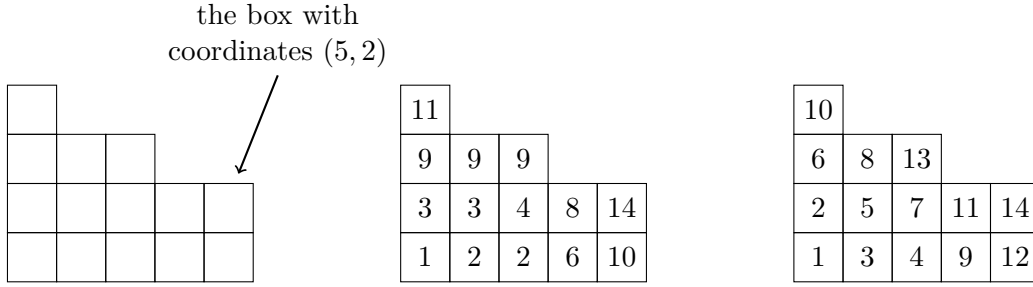


Figure 1.1: From left to right: The Young diagram, a semistandard Young tableau and a standard Young tableau all corresponding to the partition $\lambda = (5, 5, 3, 1)$. In both the semistandard and standard Young tableau the box $(5, 2)$ has the entry 14.

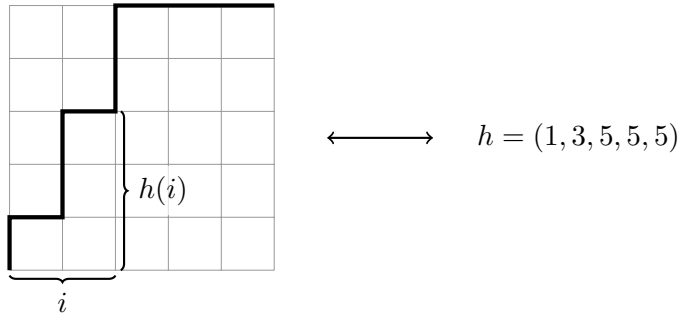


Figure 1.2: A dyck path of size 5 and its corresponding Hessenberg function.

diagram of the initial partition. E.g. λ/μ with $\mu \subseteq \lambda$, consists of the boxes contained in λ but not in μ .

Figure 1.1 provides examples of these objects written in french notation.

1.3 Dyck paths, Hessenberg functions and unit–interval graphs

For more information on the objects presented in this section, the reader can refer to the survey respectively introduction of [Hik25] or [AN21].

A *Dyck path* of length n is a path in the lattice $\mathbb{Z} \times \mathbb{Z}$ starting from $(0, 0)$ and ending in (n, n) that only utilizes n north and n east steps and never steps below the diagonal, i.e. the only possible steps to take from (a, b) are the steps to the east, corresponding to $(a + 1, b)$ (provided $a + 1 < b$), or north, corresponding to $(a, b + 1)$. The set of all Dyck paths is denoted by $\mathcal{D}$, the set of all Dyck paths of size n is denoted by $\mathcal{D}_n$. See Figure 1.2 for an example of a Dyck path in $\mathcal{D}_5$.

Hessenberg functions of length n are maps $h : [n] \rightarrow [n]$ satisfying $h(i + 1) \geq h(i)$ and $h(i) \geq i$. All Hessenberg functions are denoted by $\mathbb{H}$ with those of length n being

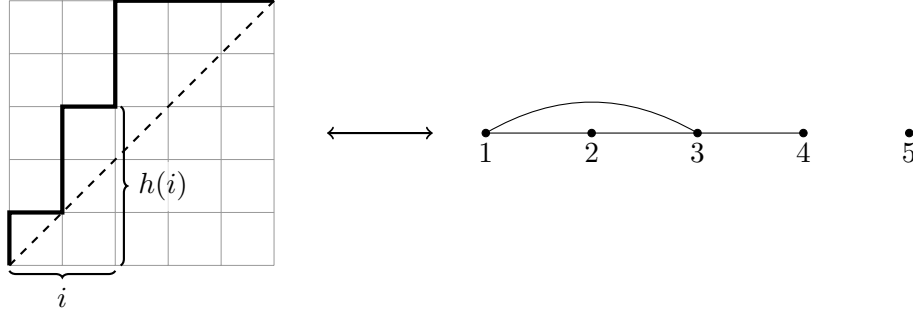


Figure 1.3: The Hessenberg function $h = (1, 3, 5, 5, 5)$ and its associated unit interval graph.

denoted by $\mathbb{H}_n$. It is easy to see that there is a bijection between Dyck paths and Hessenberg functions: Every Dyck path from $\mathcal{D}$ defines a Hessenberg function from $\mathbb{H}_n$ when interpreted as the “graph” of a function and vice versa. We say that a Hessenberg function is *complete* if $h(i) = n$ for all $i \in [n]$, because under the bijection explained below, it is associated to a complete graph. Finally, we define the product of $h_1 \in \mathbb{H}_n$ and $h_2 \in \mathbb{H}_m$, $h = h_1 h_2 \in \mathbb{H}_{n+m}$, via

$$h(i) = \begin{cases} h_1(i) & \text{if } i \leq n \\ h_2(i - n) & \text{if } n < i \leq n + m \end{cases}.$$

See Figure 1.2 for a Hessenberg function and its related Dyck path.

Another object in bijection with Hessenberg functions and Dyck paths are *unit interval graphs*. Given n intervals of length 1, we can construct a unit interval graph by assigning a vertex to every interval and connecting exactly the vertices with edges whose associated intervals intersect. Note that, instead of treating intervals of unit length as underlying objects, we can also consider points on the number line as underlying objects, where the vertices corresponding to two points are connected if the distance between them is at most 1. One bijection between Hessenberg functions and Dyck paths can be defined in the following way: For $h \in \mathbb{H}_n$, construct a graph with n vertices, where the vertex associated to i is labeled with $n + 1 - i$ and connect two vertices labeled $a = n + 1 - i$ and $b = n + 1 - j$, $a < b$ with an edge exactly if $j < i \leq h(j)$. See Figure 1.3 for a Hessenberg function and its associated unit interval graph.

The above bijection is similar to the bijection we will use in Section 2.2, some examples of other bijections can be found in [Hik25, section 2.3.1]

1.4 q -integers and the chromatic (quasi-)symmetric function

For any integer n we define the corresponding q -integer $[n]_q$ for a real variable $q \in (0, 1)$ by

$$[n]_q := \frac{1 - q^n}{1 - q}.$$

Symmetric functions are formal power series in the infinitely many variables $(x_i)_{i \in \mathbb{Z}^+}$ such that, given a such a series f , we have that $f(x_1, x_2, \dots) = f(x_{\sigma(1)}, x_{\sigma(2)}, \dots)$ for any permutation σ of $\mathbb{Z}^+$. A *quasisymmetric function* is a formal power series in the infinitely many variables $(x_i)_{i \in \mathbb{Z}^+}$ such that for a given series f , the coefficient of the term $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ is equal to the coefficient of the term $x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_n}^{a_n}$ whenever $i_1 < i_2 < \dots < i_n$, with $a_i \in \mathbb{Z}^+$. For more information we refer the reader to [Sta23, Chapter 7]. We sometimes write X instead of $x_1, x_2, x_3, \dots$ as a shorthand.

Given a graph G with n vertices and a proper coloring of this graph $\kappa : G \rightarrow \mathbb{Z}^+$, proper meaning that two connected vertices have different color, we define the *chromatic symmetric function* $\text{csf}(X; G)$ of the graph G by

$$\begin{aligned} \text{csf}(X; G) &:= \sum_{\kappa} x_{\kappa(v_1)} x_{\kappa(v_2)} x_{\kappa(v_3)} \cdots x_{\kappa(v_n)} \\ &= \sum_{\kappa} x_1^{|\kappa^{-1}(1)|} x_2^{|\kappa^{-1}(2)|} x_3^{|\kappa^{-1}(3)|} \cdots, \end{aligned}$$

which was first described by Stanley [Sta95]. We then define the *chromatic quasisymmetric function* by

$$\begin{aligned} \text{csf}_q(X; G) &:= \sum_{\kappa} q^{\text{asc}(\kappa)} x_{\kappa(v_1)} x_{\kappa(v_2)} x_{\kappa(v_3)} \cdots x_{\kappa(v_n)} \\ &= \sum_{\kappa} q^{\text{asc}(\kappa)} x_1^{|\kappa^{-1}(1)|} x_2^{|\kappa^{-1}(2)|} x_3^{|\kappa^{-1}(3)|} \cdots \end{aligned}$$

where $\text{asc}(\kappa) := |\{v_i, v_j \in G : i < j \text{ and } \kappa(v_i) < \kappa(v_j)\}|$ and which was first described by Shareshian and Wachs [SW16]. For both the chromatic symmetric function and the chromatic quasisymmetric function it is easy to see that they are a symmetric respectively quasisymmetric function.

The survey of [Hik25] and introduction of [Gua13] also give a short overview of the chromatic (quasi-)symmetric function of a graph.

When speaking of linear maps or linear relations in this thesis, we work over the base field $\mathbb{Q}(q)$ unless another field is clear from the context.

2 The modular law

The so-called *modular law* was first described by Guay–Paquet [Gua13, Proposition 3.1] for the chromatic symmetric function of a $(3+1)$ -free poset. For a poset to be $(3+1)$ -free it is not allowed to contain a subposet consisting of a chain with three elements and a fourth vertex incomparable to each of the other three. A q -analogue was later found by Lee [Lee21, Theorem 3.5] which can be translated to the chromatic quasisymmetric function of unit interval graphs.

We state the modular law here as it is stated in [AN21], using Hessenberg functions. The formulation below is used in the quasisymmetric case, for the symmetric case see 2.7 and 2.10. By the bijection given in section 1.3 this is equivalent to stating it directly for unit interval graphs.

Definition 2.1 (modular law, [AN21, Definition 2.1]). A function $f : \mathbb{H} \rightarrow \mathcal{A}$, where $\mathcal{A}$ is a $\mathbb{Q}(q)$ -algebra, satisfies the *modular law* if

$$(1 + q)f(h_1) = qf(h_0) + f(h_2) \quad (2.1)$$

for a triple of Hessenberg functions (h_0, h_1, h_2) that in turn satisfies one of the following two conditions:

- (1) There exists $i \in [n - 1]$ such that $h_1(i - 1) < h_1(i) < h_1(i + 1)$ and $h_1(h_1(i)) = h_1(h_1(i) + 1)$ or $h_1(i) = n$. Moreover, h_0 and h_2 satisfy $h_k(j) = h_1(j)$ for every $j \neq i$ and $k = 0, 2$, while $h_k(i) = h_1(i) - 1 + k$.
- (2) There exists $i \in [n - 1]$ such that $h_1(i + 1) = h_1(i) + 1$ and $h_1^{-1}(i) = \emptyset$. Moreover, h_0 and h_2 satisfy $h_k(j) = h_1(j)$ for every $j \neq i, i + 1$ and $k = 0, 2$, while $h_0(i) = h_0(i + 1) = h_1(i)$ and $h_2(i) = h_2(i + 1) = h_1(i + 1)$.

See Figure 2.1 and Figure 2.2 for examples of triples satisfying condition (1) respectively condition (2).

As mentioned before, in [Lee21] a similar relation for unicellular LLT polynomials is found that, together with a result from [CM18], proves that the chromatic quasisymmetric function of unit interval graphs satisfies the modular law as stated above.

In the following sections we will state and prove the modular law for the chromatic symmetric function of a $(3 + 1)$ -free poset respectively a unit interval graph, which is then followed by a short introduction into unicellular LLT polynomials in order to state and give an alternative proof of Lee’s [Lee21, Theorem 3.5] result. It is our intention to collect as self-contained proofs as possible of both formulations in the same work.

The sections following section 2.1 do not depend on it, therefore the reader interested just in the quasisymmetric case may skip to section 2.2.

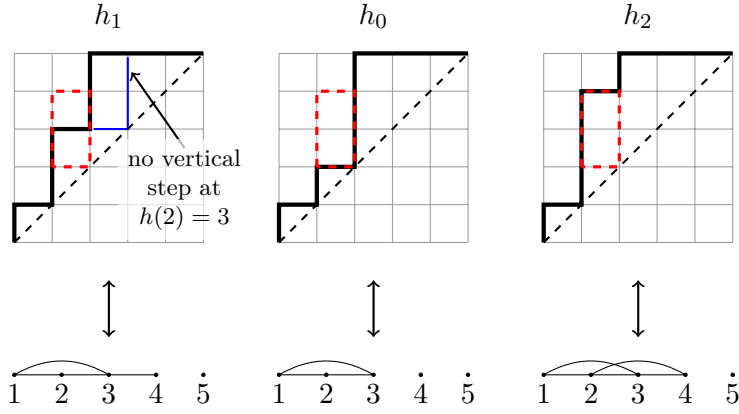


Figure 2.1: An example of a triple $h_0, h_1, h_2 \in \mathbb{H}_5$ satisfying condition (1) of Definition 2.1 and their associated unit interval graphs.

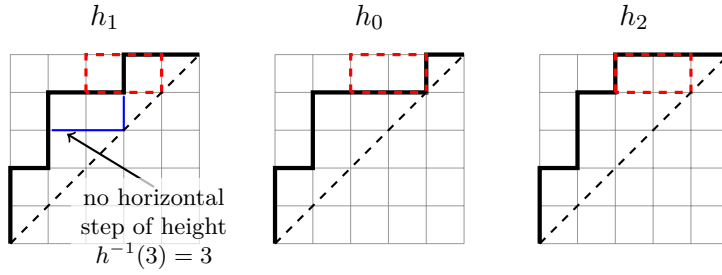


Figure 2.2: An example of a triple $h_0, h_1, h_2 \in \mathbb{H}_5$ satisfying condition (2) of Definition 2.1.

2.1 The chromatic symmetric function and the modular law

In this section, we closely follow [Gua13], omitting the parts not required for our purposes and providing additional details on some proofs.

First, we note that the chromatic symmetric function of a poset matches the chromatic symmetric function of its incomparability graph. A colouring of a poset is called proper, if every colour class is a chain. Thus, the same colouring interpreted as the colouring of the incomparability graph also is a proper colouring, since elements from a chain are by definition disconnected in this graph. Conversely, a proper colouring of the incomparability graph of a poset P also is a proper colouring when viewed as the colouring of P , since vertices not connected in the incomparability graph are comparable in P and thus the subposet induced by the colour classes forms a chain.

In order to state the modular law, we now introduce a different way of representing $(3+1)$ -free posets.

Definition 2.2 (part listings, [Gua13, p. 3]). A *part listing* is an ordered list of *parts*. All *parts* are arranged on positive integer *levels*, with a single part being either a single vertex or a bipartite graph where both colour classes are each arranged on a level directly adjacent to the other colour class and no edge of the graph connects two vertices with matching colour.

A part listing can also be given as a word from the alphabet

$$\Sigma = \{v_i : i \in \mathbb{Z}^+\} \cup \{b_{i,i+1}(G) : i \in \mathbb{Z}^+, G \text{ a bipartite graph}\},$$

where v_i corresponds to a single vertex on level i and $b_{i,i+1}(G)$ corresponds to a copy of the bipartite graph G whose colour classes are placed on level i and $i + 1$ respectively and no edge connects two vertices of the same colour.

To construct the poset corresponding to a part listing L , we use the following rules for every two vertices x and y in L (sticking to [Gua13, p. 4]): We say $x < y$ if

- (i) x is at least two levels below y , or
- (ii) x is exactly one level below y and x is listed strictly to the left of the part containing y , or
- (iii) x is exactly one level below y and they are joined by a bipartite graph edge.

Remark 2.3. One poset may admit multiple part listings L and L' . E.g. if in a part listing L , the vertex x is two or more levels below y , with x and y both not being part of a bipartite graph and there is no other vertex between them, it is possible to exchange the position of x and y in the word defining the part listing to get a different part listing L' that still yields the same poset.

Two part listings giving rise to the same poset are called *poset-equivalent*. In [Gua13] multiple transformations of part listings yielding poset-equivalent part listings are described, in order to, in combination with the modular law, reduce the Stanley–Stembridge–Conjecture from $(3+1)$ -free posets to $(3+1)$ - and $(2+2)$ -free posets which are in bijection with unit interval graphs.

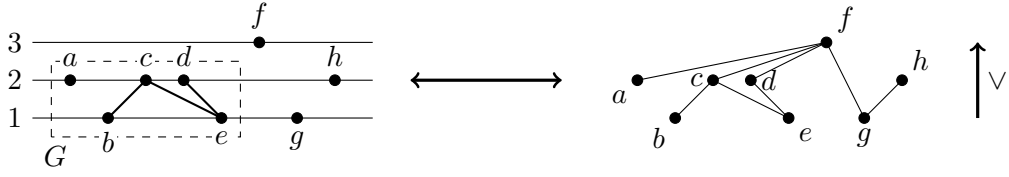


Figure 2.3: An example of a part listing and its associated poset. The order relations given by the edges in the poset on the right are to be read from the bottom to the top.

Example 2.4. Figure 2.3 illustrates the part listing given by $b_{12}(G)v_3v_1v_2$ — where G is a bipartite graph of 5 vertices — and its associated poset.

The next two propositions show that it suffices to consider part listings when working with $(3+1)$ -free posets.

Proposition 2.5 ([Gua13, Proposition 2.4]). *Given any part listing L , the associated poset P is $(3+1)$ -free.*

The proof stated here uses the same approach as in [Gua13] whilst being more explicit.

Proof. We give a proof by contradiction. Suppose there exists a chain $x < y < z$ and a vertex w that is incomparable to each of x , y and z in P .

First, we observe that w has to be on the same level as y since otherwise it is comparable to at least one of x , y or z . Furthermore, x and z cannot be in the same part, as such a part would have to extend over more than two levels.

Now, the condition $x < y < z$ forces them to be listed in alphabetical order in the part listing provided they each are in distinct parts. In case where x and y are in the same part, the part containing z has to be to the right of their part. In case where y and z are in the same part, the part containing x has to be to the left of their part.

Considering each case it is easy to see that there is no way to place w such that it is incomparable to the other three vertices:

If x , y and z are in distinct parts, w cannot be in an additional distinct part, because any placement would make it comparable to x or z . For the same reason, it is not possible to place w inside a part containing one of the other three vertices.

If x and y or y and z are in the same part, w cannot be outside of the part since placing it there makes it comparable to x or z . It also cannot be in the same part as x and y or y and z , since this would force $w < z$ or $x < w$ respectively.

Thus, we have a contradiction to P containing a $(3+1)$ subposet. $\square$

Proposition 2.6 ([Gua13, Proposition 2.5]). *Given any $(3+1)$ -free poset P , there exists a part listing L for which the associated poset is P .*

Proof. See [Gua13] where this is proven by applying a result from [GMR14, Theorem 3.3]¹. $\square$

Now it is possible to state and prove the modular law for the chromatic symmetric function of a (3+1)-free poset.

Proposition 2.7 (csf satisfies the modular law, [Gua13, Proposition 3.1]). *Consider a part listing of the form $Ab_{i,i+1}(G)B$ where A and B are words over Σ and G is a bipartite graph satisfying the necessary conditions for $b_{i,i+1}(G)$ to be part of a part listing. Additionally, suppose G contains two edges e_1, e_2 incident to the same vertex y such that for some vertices x and z we have $e_1 = xy$ and $e_2 = yz$.*

Consider the following graphs and posets obtained by removing edges from G :

- $G_1 := (G \text{ without } e_1), P_1 := Ab_{i,i+1}(G_1)B,$
- $G_2 := (G \text{ without } e_2), P_2 := Ab_{i,i+1}(G_2)B, \text{ and}$
- $G_{12} := (G \text{ without } e_1 \text{ and } e_2), P_{12} := Ab_{i,i+1}(G_{12})B.$

Then,

$$\text{csf}(P) + \text{csf}(P_{12}) = \text{csf}(P_1) + \text{csf}(P_2). \quad (2.2)$$

Remark 2.8. Equation (2.2) is what is called the *modular law* for the chromatic symmetric function.

See Figure 2.4 for an illustration of the covering relations between the vertices x, y and z in the posets P, P_1, P_2 and P_{12} .

Proof. Reviewing the Definition of the chromatic symmetric function (see section 1.4) makes it obvious that it suffices to check that any colouring $\kappa : V \rightarrow \mathbb{Z}^+$ contributes the same amount to both sides of (2.2), since all involved posets have the same vertices.

Firstly, in order to make any contribution, it has to be proper for P . We assume this has been checked for all vertices and edges except for the relations amongst x, y and z .

For the vertices x, y and z this forces $\kappa(x) \neq \kappa(z)$ since x and z are incomparable. Also, $\kappa(y)$ may be equal to $\kappa(x)$ or $\kappa(z)$ or neither.

There are three cases to consider:

- If $\kappa(y) = \kappa(x)$, then κ is proper for P and P_2 (there *is* a chain connecting x to y), but not for P_1 and P_{12} (there is *no* chain connecting x to y).
- If $\kappa(y) = \kappa(z)$, then κ is proper for P and P_1 , but not for P_2 and P_{12} .
- If $\kappa(y)$ is distinct from both $\kappa(x)$ and $\kappa(z)$, κ is proper for all of P, P_1, P_2 and P_{12} .

Since there are no other cases left and the contribution to both sides of (2.2) has been checked to always be the same, we have proven that the chromatic symmetric function of (3+1)-free posets satisfies the modular law. $\square$

Remark 2.9. As it is noted in [Gua13], the proof of Proposition 2.7 only relies on the covering relations between x, y and z . Thus, we can state a more general proposition,

¹This paper seems to have been restructured, the result relevant here corresponds to Theorem 3.3 from the first version of the paper found on arXiv: <https://arxiv.org/abs/1303.3652v1>.

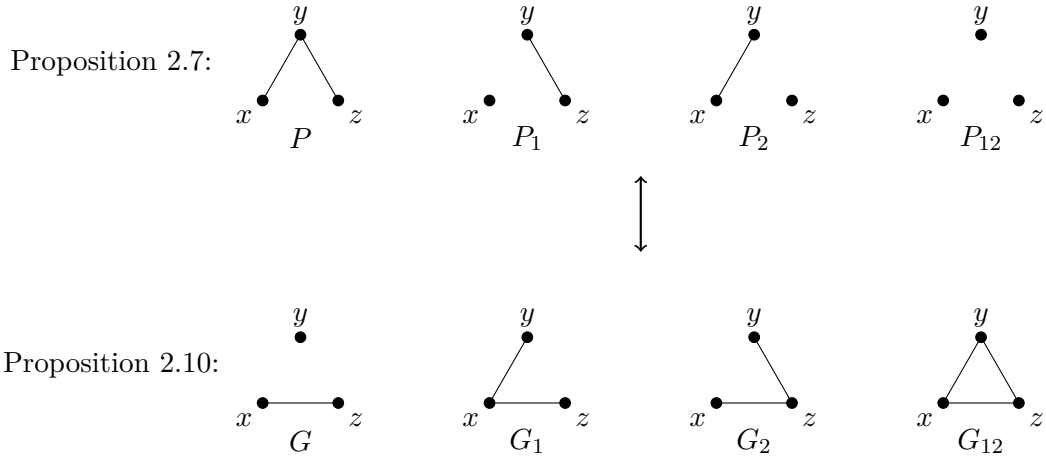


Figure 2.4: The relations between the vertices x , y and z in the posets respectively graphs appearing in Proposition 2.7 and 2.10.

now formulated in terms of graphs, about the chromatic symmetric function satisfying the modular law, by translating the covering relations into conditions for edges and paths connecting the three vertices in the incomparability graph.

Proposition 2.10. *Consider a graph G with vertices x , y and z , such that there is an edge connecting x to z and that there are no edges or paths connecting any of these three vertices in any other way. Then, we construct the graphs G_1 , G_2 and G_{12} by*

- *adding an edge connecting x and y to obtain G_1 ,*
- *adding an edge connecting z and y to obtain G_2 , and*
- *adding both the edges connecting x to y and z to y respectively to obtain G_{12} .*

Then,

$$\text{csf}(G) + \text{csf}(G_{12}) = \text{csf}(G_1) + \text{csf}(G_2).$$

Figure 2.4 illustrates the relations between the vertices x , y and z in the graphs G , G_1 , G_2 and G_{12} from the proposition above.

2.2 LLT polynomials

In this section, we will introduce LLT polynomials, which are used to prove that the chromatic quasisymmetric function satisfies the modular law, similar to how it is stated in Definition 2.1. Furthermore, we will explain how it is possible to transfer linear relations found for unicellular LLT polynomials to the chromatic quasisymmetric function justifying that the proof in the following section also proves the modular law for the chromatic quasisymmetric function. Leading up to our own proof of Lee's result, we will structure our work similarly to [Lee21], but use different formulations of intermediate results providing more detailed proofs and drawing from [Ass07], [Hag+05]

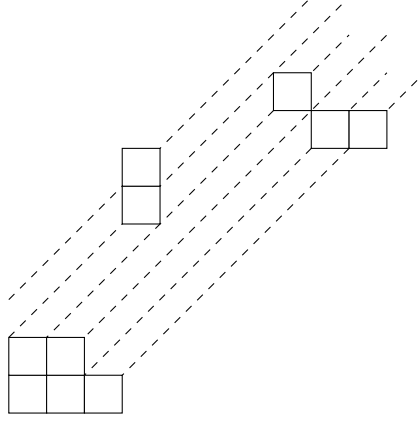


Figure 2.5: The diagram of the 3-tuple $\lambda = ((3, 2), (1, 1, 1, 1)/(1, 1), (3, 1)/(1))$.

and [HNY20]. We also try to give an explanation of the connection to the chromatic quasisymmetric function. Again, it is our goal to keep this work as self-contained as possible.

Lascoux, Leclerc and Thibon first described LLT polynomials in [LLT97]. In this thesis, we will use the formulation of LLT polynomials later found by Haglund et al. in [Hag+05].

Given a d -tuple $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(d)})$ of Young diagrams of the (possibly skew) shape $\lambda^{(i)}$, we define the sets

$$\begin{aligned} \text{SSYT}_d(\lambda) &= \{d\text{-tuples of SSYT of shape } \lambda\} \\ \text{SYT}_d(\lambda) &= \{\text{standard } d\text{-tuples of shape } \lambda\} \end{aligned}$$

where a standard d -tuple of shape λ is a d -tuple of shape λ where the cells have entries from 1 to n , where n is the total number of cells in λ , each number is used exactly once and in every component of λ , rows and columns are strictly increasing. Now, for a $T \in \text{SSYT}_d(\lambda)$ we denote by x^T the product $x^T := x_1^{\alpha_1} x_2^{\alpha_2} \dots$ where α_i is the number of cells in T with entry i . We think of the skew shapes forming λ as arranged diagonally from the southwest to the northeast, so that cells x of same content $c(x)$ lie on the same diagonal (if the cell x is the cell in row i and column j of its skew shape, $c(x) = j - i$).

Example 2.11. Figure 2.5 shows the diagram of the 3-tuple $\lambda = ((3, 2), (1, 1, 1, 1)/(1, 1), (3, 1)/(1))$ and Figure 2.6 an example of a $T \in \text{SSYT}_3(\lambda)$ and a $S \in \text{SYT}_3(\lambda)$.

As a final ingredient for the definition of LLT polynomials we require the notion of two cells of some $T \in \text{SSYT}_d(\lambda)$ being in inversion with each other. To this end, we define the content of a cell $x \in T$ as $\tilde{c}(x) := dc(x) + i$, with x being part of the component $\lambda^{(i)}$ of λ and $c(x)$ being the content of x inside $\lambda^{(i)}$ as explained above. The entry of the cell x is denoted by $T(x)$. Now, two cells x and y form an inversion,

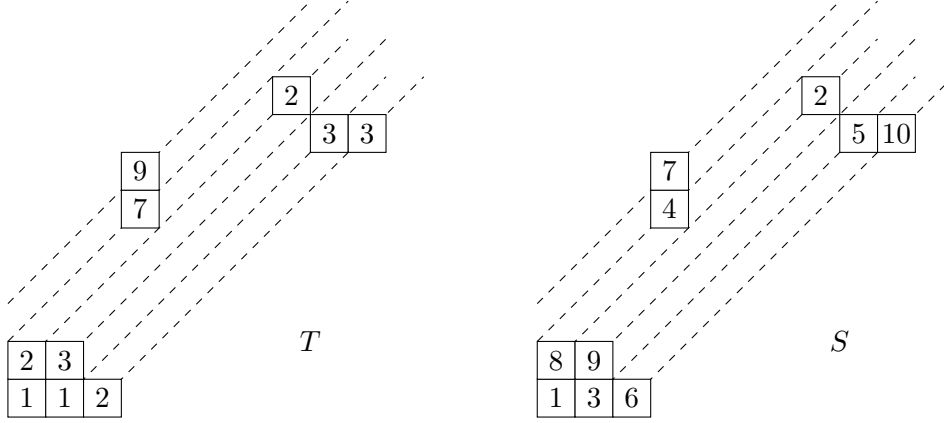


Figure 2.6: Examples of a 3-tuple of SSYT and a standard 3-tuple, both of shape $\lambda = ((3, 2), (1, 1, 1, 1)/(1, 1), (3, 1)/(1))$.

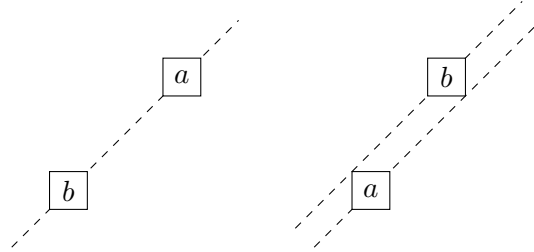


Figure 2.7: The two possible configurations of cells with entries $a < b$ that are in inversion.

if both $T(x) < T(y)$ and $0 < \tilde{c}(x) - \tilde{c}(y) < d$ hold. The number of pairs of cells of λ forming an inversion is then denoted by $\text{inv}(T)$.

Example 2.12. There are two possible ways for cells x and y to be in inversion. Suppose $a := T(x) < T(y) =: b$, then either $\tilde{c}(x) = \tilde{c}(y)$ and the cell y is in a component of λ of lesser index than the component of the cell x or $\tilde{c}(x) = \tilde{c}(y) + 1$ and the cell y is in a component of greater index than the component of x . See Figure 2.7 for an illustration of both options.

Looking at S from Figure 2.6 the cells with the following entries are in inversion: $\{(2, 8), (1, 2)\}$.

Now we can formulate the definition of LLT polynomials, as it can be found in [Lee21] or [HNY20, Definition 2.3]. A proof that the following definition matches the LLT polynomials initially discovered by Lascoux, Leclerc and Thibon when multiplied

by a suitable power of q can be found in [Hag+05, Corollary 5.2.4].

Definition 2.13. (LLT polynomials) Let $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \dots)$ be a d -tuple of Young diagrams. Then the LLT polynomial corresponding to λ is given by

$$\text{LLT}_\lambda(X; q) := \sum_{T \in \text{SSYT}(\lambda)} q^{\text{inv}(T)} x^T.$$

For a combinatorial proof that LLT polynomials are symmetric functions, one can consult the appendix of [HHL05], else, [LLT97, Theorem 6.1] provides a proof for the initial formulation of these polynomials.

Our next goal, similarly to [Lee21], is to find a formula expressing LLT polynomials in terms of a sum over SYTs of Gessel's fundamental quasisymmetric functions, which were first presented by Gessel in [Ges84]. We will diverge from [Lee21] and stick more closely to the formulations chosen by [Ass07]. Thus, our treatment of Gessel's fundamental quasisymmetric functions utilizes maps $\sigma : [n-1] \rightarrow \{+, -\}$ to define them as

$$F_\sigma(X) := \sum_{\substack{x_{i_1} \leq x_{i_2} \leq \dots \leq x_{i_n} \\ \sigma(j) = - \Rightarrow x_{i_j} < x_{i_{j+1}}}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

To sum over $T \in \text{SYT}_d(\lambda)$, we need to explain how to construct the maps σ depending on a given T . This requires the notion of the *content reading word* of T . It is obtained by reading the entries of each cell in T , starting with the diagonal with least content k and reading it from left to right, i.e. from the southwest to the northeast end and then continuing in the same way with the diagonal of content $k+1$ until finally the diagonal with highest content is read. We denote the content reading word of T by w_T and define

$$\sigma_T := \sigma(w_T) := \begin{cases} + & \text{if } i+1 \text{ appears to the right of } i \text{ in } w_T \\ - & \text{if } i+1 \text{ appears to the left of } i \text{ in } w_T, \end{cases}$$

similarly to how it can be found in [Ass07, p. 17]

The next proposition gives us our desired result. It is stated in a similar way in [Lee21, p. 975] and [Ass07, p. 35], with the latter providing some hints on how an explicit proof could look like. We will provide our own proof, utilizing the process for the standardization of tableaux described in [Hag+05, Lemma 5.2.3, Proposition 2.4.2].

Proposition 2.14. *Using the same notation as above, we have*

$$\text{LLT}_\lambda(X; q) = \sum_{T \in \text{SYT}_d(\lambda)} q^{\text{inv}(T)} F_{\sigma_T}(X). \quad (2.3)$$

Proof. To prove this proposition we first describe how a $T \in \text{SSYT}_d(\lambda)$ gives rise to a unique $S \in \text{SYT}_d(\lambda)$, preserving $\text{inv}(T)$, i.e. $\text{inv}(T) = \text{inv}(S)$ and prove the injectivity of the map. Following this, we show how in turn each term of the right hand side of

$$\sum_{T \in \text{SYT}_d(\lambda)} q^{\text{inv}(T)} F_{\sigma(T)} = \sum_{T \in \text{SYT}_d(\lambda)} q^{\text{inv}(T)} \sum_{\substack{x_{i_1} \leq x_{i_2} \leq \dots \leq x_{i_n} \\ \sigma_T(j) = - \Rightarrow x_{i_j} < x_{i_{j+1}}}} x_{i_1} x_{i_2} \cdots x_{i_n}$$

corresponds to a unique $\tilde{T} \in \text{SSYT}_d(\boldsymbol{\lambda})$, proving the surjectivity of our map.

Starting with a $T \in \text{SSYT}_d(\boldsymbol{\lambda})$, we choose $S \in \text{SYT}_d(\boldsymbol{\lambda})$ such that $T \circ S^{-1}$ is weakly increasing (here, we interpret T and S as maps assigning the cells of $\boldsymbol{\lambda}$ to the entries in $\mathbb{Z}^+$), and if $T \circ S^{-1}(j) = T \circ S^{-1}(j+1) = \dots = T \circ S^{-1}(k)$, then $d(S) \cap \{j, j+1, \dots, k-1\} = \emptyset$. Note, that $d(S) := \{a : \text{if } S(x) = a, S(y) = a+1, \text{ then } \tilde{c}(x) > \tilde{c}(y)\}$ denotes the set of descents of S .

This choice of S is unique for every T . To see this, observe that the first condition, $T \circ S^{-1}$ being weakly increasing only allows for variations in the standardization in cells of T with equal entries. Now, considering that equal entries in T form horizontal strips in the SSYT_d constituting T , i. e. no two equal entries can appear in the same column of a component of $\boldsymbol{\lambda}$, the condition on the descents ensures a unique choice of standardization for the remaining cells.

This way of standardizing also preserves $\text{inv}(T)$: Equal entries in T are increasing in S in content reading order, generating no new inversions and the entries that are not equal remain in the same relation to each other as well. Thus $\text{inv}(T) = \text{inv}(S)$.

Also note, that x^T is a term appearing in F_{σ_S} : Equal entries j in T correspond to entries $i, i+1$ increasing in content reading order in S , meaning $\sigma_S(i) = +$. Thus, there is a term in F_{σ_S} with x_j reaching the required power.

Now, given distinct $T_1, T_2 \in \text{SSYT}_d(\boldsymbol{\lambda})$, they correspond to different terms on the right hand side of (2.3), because if they belong to the same $S \in \text{SYT}_d(\boldsymbol{\lambda})$, they will also differ in the entries used to fill them, since T_1 having the same entries as T_2 , albeit rearranged, contradicts them belonging to the same S , since $T_i \circ S^{-1}$ has to be weakly increasing.

The arguments above prove that we have a well-defined and injective map from the terms appearing on the left hand side to the terms appearing on the right hand side. It remains to show that this map is also surjective.

Indeed, since given an $S \in \text{SYT}_d(\boldsymbol{\lambda})$ and a term $x = x_{i_1} x_{i_2} \dots x_{i_n}$ of F_{σ_S} , we get a $\tilde{T} \in \text{SSYT}_d(\boldsymbol{\lambda})$ by reversing the map described above. This is done by replacing 1 in S with x_{i_1} , 2 with x_{i_2} , $\dots$ until n is replaced by x_{i_n} . Here, σ_S ensures that in this way we are constructing a well-defined $\tilde{T} \in \text{SSYT}_d(\boldsymbol{\lambda})$, since when we take a step to a cell of smaller content than the cell before, the number assigned to it will be strictly larger than every other number assigned until then by the definition of σ_S .

Thus, we have a bijection between the terms appearing on the left and right sides of the equation (2.3), proving the proposition. $\square$

In order to prove that the chromatic quasisymmetric function satisfies the modular law, we only require the LLT polynomials of shapes $\boldsymbol{\lambda}$ where the components of $\boldsymbol{\lambda}$ consist of single cells. These LLT polynomials are called *unicellular* LLT polynomials and from now on, all the LLT polynomials we use will be unicellular. See Figure 2.8 for an example of a d -tuple underlying a unicellular LLT polynomial.

We also want to give another reformulation of the formula defining unicellular LLT polynomials. To this end, we assign a skew shape, now using the english notation, $D_{\boldsymbol{\lambda}}$ to any $\boldsymbol{\lambda}$ which is contained in a staircase shape, i. e. it is a subset of the partition

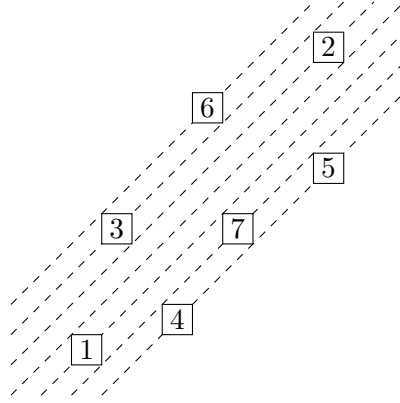


Figure 2.8: A schematic illustration of the d -tuple of an unicellular LLT polynomial.

defined by $(d-1, d-2, \dots, 1)$. The bijection we use here can be found in [HNY20, p. 5] which is in turn based on [CM18, Section 3.2].

Given a d -tuple λ of Young diagrams, we start with the staircase shape $(d-1, d-2, \dots, 1)$ and label the columns from 1 to d starting from the leftmost column and label the rows from d to 1, starting with the topmost row. It turns out to be convenient to label the diagonal entries $(i, d+1-i)$ with i for $i \in [d]$ which enables us to infer the row and column of a cell in the labeling described above since the label of the diagonal entry in the same row determines the row of the cell and analogously the label of the diagonal entry in the same column determines the column of the cell.

Next, we cross out the cells in the column labeled c and row labeled r whenever the c -th and r -th cell of λ in content reading order do not *attack* each other, i. e. cannot form an inversion no matter their entries according to our earlier definition. This leaves us with crossed out cells corresponding to a partition and a skew shape of cells that are not crossed out which we will denote by D_λ .

In chapter 4 it will be convenient to think of D_λ as rotated clockwise by a quarter turn such that the partition formed by the crossed out boxes is rotated and justified in the northeastern corner and the labels of the diagonal cells are increasing from 1 to d from the northwest to the southeast. Under this point of view, the information contained in D_λ may also be described by a 0-1-matrix whose entries are 0 when they are not the coordinates of a cell in D_λ and 1 if they are.

See Figure 2.9 for an example of a unicellular shape and its associated skew shape under the bijection above.

Using the D_λ we can give a simple reformulation of the formula for LLT polynomials in the unicellular case, similar in spirit to [Lee21, Proposition 3.2 or Definition 3.3], but continuing our own approach.

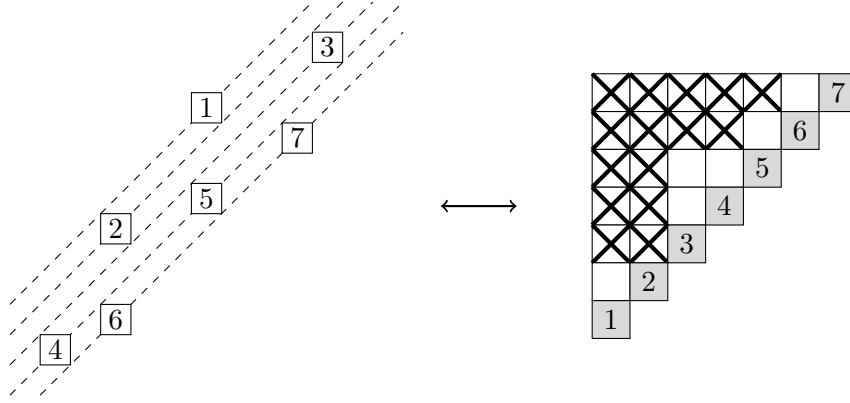


Figure 2.9: The shape λ of a unicellular LLT polynomial and the corresponding D_λ . (The cells of λ are labeled in content reading order.)

Proposition 2.15. *For a unicellular d -tuple of Young diagrams λ , we have*

$$\text{LLT}_\lambda(X; q) = \sum_{\pi \in \mathcal{S}_d} q^{\text{inv}(\pi, D_\lambda)} F_{\sigma_\pi},$$

where $\text{inv}(\pi, D_\lambda)$ is the number of inversions (i, j) , $i < j$ of π such that $(i, j) \in D_\lambda$ and σ_π is defined similarly as above, using the word constructed by writing down the values $\pi(i)$, $i \in [d]$, next to each other.

Proof. This is just a simple reformulation. Instead of summing over all $\text{SYT}_d(\lambda)$, we now sum over the content reading words π . The definition of D_λ and $\text{inv}(\pi, D_\lambda)$ amounts to considering exactly the same inversions we would have considered in the previous formulation for a $T \in \text{SYT}_d(\lambda)$ of content reading word π . $\square$

We now have everything in place to state and give our proof of Lee's result (cf. section 2.3), but first we want to elaborate how a linear relation of unicellular LLT polynomials translates to a linear relation of chromatic quasisymmetric functions of different graphs.

Again, we use a bijection described in [HNY20, p. 5] to associate a unit interval graph to a skew shape D_λ .

Given a D_λ as defined above, we construct a graph consisting of d vertices, each vertex associated to a cell in λ but label them in reverse content reading order, giving us the vertex set $V = \{v_i\}_{i \in [d]}$. The choice of labeling them in reverse content reading order is made so that the asc and inv statistics directly coincide later on. Finally, two vertices v_i and v_j will be connected by an edge whenever the cell in row i and column j is inside of D_λ .

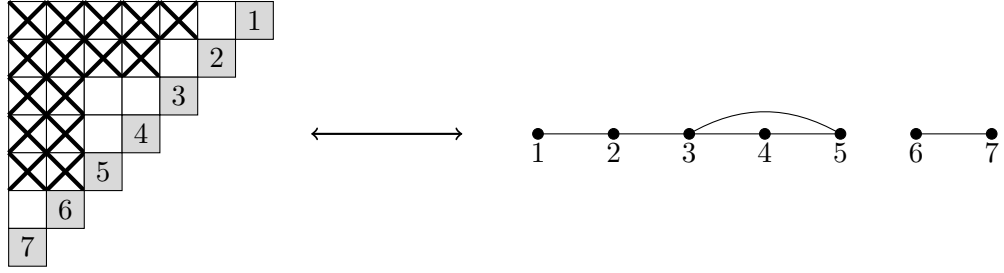


Figure 2.10: The skew shape D_λ and its associated unit interval graph G_λ .

A possibly simpler way to describe this bijection works as follows: Reverse the numbers written in the diagonal used in the construction of D_λ and then connect the vertices labeled i and j whenever the cell in row i and column j is in D_λ , now using the reversed diagonal entries to determine the row and column of a cell.

We denote the unit interval graph constructed via this bijection by G_λ if it is associated to D_λ .

See Figure 2.10 for an example of this bijection.

Remark 2.16. To see that this map is well defined, note that, using the latter labeling, if there is a cell (i, j) , $i < j$ in D_λ , meaning the vertices i and j are connected in G_λ , this implies that the points on the number line underlying G_λ are at most 1 apart. To obtain a valid unit interval graph, it is now necessary that all vertices l with $i < l < j$ are connected to i and j , since their associated points on the number line lie between those associated to i and j . But by construction of D_λ , this is clearly the case. It is now easy to convince oneself that this map also is a bijection.

Remark 2.17. Colouring the graph G_λ , one quickly sees that due to the way of constructing the graph, $\text{inv}(T)$ and $\text{asc}(T)$ coincide when the entries of the cells match the colour of their associated vertex. Thus, we may justify to write (cf. [HNY20, p. 5])

$$\text{LLT}_{G_\lambda}(X; q) := \text{LLT}_\lambda(X; q) = \sum_{\substack{\kappa \\ \text{colourings of } G}} q^{\text{asc}(\kappa)} x^\kappa$$

The formula in Remark 2.17 already is very similar to the formula defining the chromatic quasisymmetric function of a graph — it only differs in also admitting non-proper colourings of the graph. But the following result from [CM18] describes how they are related using the plethystic substitution.

Proposition 2.18 ([CM18, Proposition 3.5]). *Let G be a unit interval graph. Then*

$$\text{csf}_q(X; G) = (q - 1)^{-d} \text{LLT}_G[(q - 1)X; q].$$

Proof. We refer the reader to [CM18] for a proof of this proposition. $\square$

Using this, as stated in [AP18, Proposition 8.1], we have for unit interval graphs G_i

$$\sum_{l=0}^k c_l(q) \text{csf}_q(X, G_l) = 0 \quad \text{if and only if} \quad \sum_{l=0}^k c_l(q) \text{LLT}_{G_l}(X; q) = 0,$$

telling us that any linear relation found for unicellular LLT polynomials also holds for the chromatic quasisymmetric functions corresponding to them.

2.3 The modular law for the chromatic *quasisymmetric* function

Using the setup from the previous section, we can now state Lee's result that translates to the modular law for the chromatic quasisymmetric function. Note, that we think of D_λ as rotated by a quarter turn clockwise and consider the partitions right justified, i.e. we produce rotated shapes D_λ by removing right justified partitions from the right justified staircase shape.

Theorem 2.19 (cf. [Lee21, Theorem 3.5]). *Let μ^1 be a partition contained in a staircase shape $(d-1, d-2, \dots, 1)$ and let i be such that $\mu_{i+1}^1 < \mu_i^1 < \mu_{i-1}^1$ and $\mu_{n-\mu_i^1}^1 = \mu_{n-\mu_{i-1}^1}^1$. We define additional partitions μ^0 and μ^2 by $\mu_j^k = \mu_j^1$ if $i \neq j$ and $\mu_i^0 = \mu_i^1 + 1$ and $\mu_i^2 = \mu_i^1 - 1$.*

Furthermore, let λ^k be the unicellular d -tuples of Young diagrams such that the D_{λ^k} associated to the LLT polynomials LLT_{λ^k} are the skew shapes defined by removing the cells of μ^k from the staircase shape.

Then,

$$\text{LLT}_{\lambda^2}(X; q) - \text{LLT}_{\lambda^1}(X; q) = q(\text{LLT}_{\lambda^1}(X; q) - \text{LLT}_{\lambda^0}(X; q)). \quad (2.4)$$

Remark 2.20. Interpreting the D_{λ^k} as Hessenberg functions h_k by viewing the north-eastern border of the cells as a Dyck path, it is clear that the theorem above proves the first case of the modular law (Definition 2.1) for the chromatic quasisymmetric function. The statement corresponding to the second case is analogous to the statement above, as is the proof to the one presented below.

See Figure 2.11 for an illustration of some D_{λ^k} behaving as in Theorem 2.19. We now give our own proof.

Proof. Since unicellular LLT polynomials are given by the formula

$$\text{LLT}_\lambda(X; q) = \sum_{\pi \in \mathcal{S}_d} q^{\text{inv}(\pi, D_\lambda)} F_{\sigma_\pi},$$

it suffices to check that no matter how π permutes the indices of the rows and columns of the cells differing in D_{λ^0} , D_{λ^1} and D_{λ^2} , both sides of equation (2.4) remain equal.

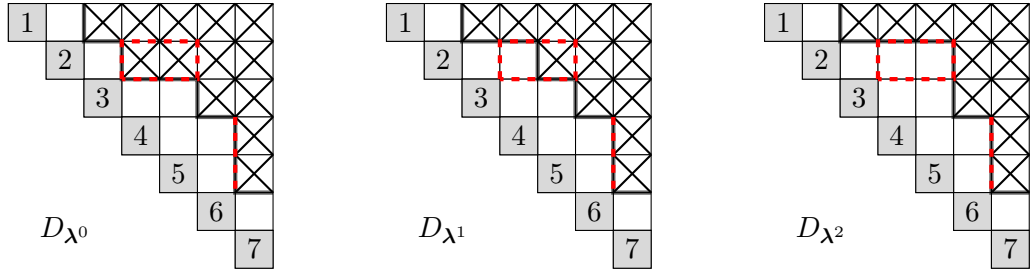


Figure 2.11: Some D_{λ^k} as in Theorem 2.19.

For the purposes of this proof, we view D_{λ^k} as rotated clockwise by a quarter turn, allowing us to describe the cells as if they were entries in a $(d \times d)$ -matrix.

Under this point of view, the cells differing in D_{λ^0} , D_{λ^1} and D_{λ^2} are the cells (x, y) and $(x, y - 1)$, where we use $x = i$ and $y = n - \mu_i^1$. Given a permutation $\pi \in \mathcal{S}_d$, we write $\pi(x) = a$, $\pi(y - 1) = b$ and $\pi(y) = c$.

Now, we list the factors contributed to $\text{inv}(\pi, D_{\lambda^k})$ by x , $y - 1$ and y , depending on the order relations between a , b and c , in the table following this paragraph. Expressed differently, this amounts to listing the values of $\text{inv}(\pi, D_{\lambda^k}) \cap \{(x, y - 1), (x, y), (y - 1, y)\}$. Note, that we do not have to consider what happens in the remaining parts of the rows $y - 1$ and y , since due to the condition $\mu_{n-\mu_i^1}^1 = \mu_{n-\mu_i^1-1}^1$ on μ , all possible configurations of a , b and c , for a π that is otherwise fixed, have equal contribution outside of the three cells we are interested in, which we will use later on in the proof.

	D_{λ^0}	D_{λ^1}	D_{λ^2}	$D_{\lambda^2} - D_{\lambda^1}$	$D_{\lambda^1} - D_{\lambda^0}$
$a < b < c$	1	1	1	0	0
$a < c < b$	q	q	q	0	0
$b < a < c$	1	q	q	0	$q - 1$
$b < c < a$	1	q	q^2	$q(q - 1)$	$q - 1$
$c < a < b$	q	q	q^2	$q(q - 1)$	0
$c < b < a$	q	q^2	q^3	$q^2(q - 1)$	$q(q - 1)$

Looking at the contributions of the differences corresponding to each side of equation (2.4) listed in the table above, we see that, since the right side is multiplied by q in the equation, the contributions match in all cases except when $b < a < c$ and $c < a < b$.

Thus, it is left to show, denoting by π_1 a permutation where $b < c$ and by π_2 the same permutation, except that the roles of b and c have been interchanged, that $F_{\sigma_{\pi_1}} = F_{\sigma_{\pi_2}}$.

Considering that b and c are the images of $y - 1$ and y under π_1 respectively π_2 , we see that they are listed right next to each other in the word used in the definition of

σ_{π_k} . Therefore, σ_{π_k} will not change when b and c swap places, provided that $|b - c| \geq 2$. But this is always the case, since we are working with permutations where $b < a < c$ and $c < a < b$, proving Theorem 2.19. $\square$

3 Abreu and Nigro's characterization

In this chapter, we will discuss the algorithm developed by Abreu and Nigro to prove their result in [AN21, Theorem 1.2] making it possible to characterize functions satisfying the modular law by their values on complete graphs.

Theorem 3.1 ([AN21, Theorem 1.2]). *Let $\mathcal{A}$ be a $\mathbb{Q}(q)$ -algebra and let $f : \mathbb{H} \rightarrow \mathcal{A}$ be a function that satisfies the modular law, as in Definition 2.1. Then f is determined by its values $f(k_{n_1}k_{n_2} \cdots k_{n_m})$ on the product of complete Hessenberg functions and these values are independent of the order the product is taken in.*

As alluded to before, the main ingredient to the proof of this theorem is an algorithm that allows to decompose the value $f(h)$ for arbitrary $h \in \mathbb{H}$ into values of f on complete Hessenberg functions.

We will present a slight variation of the algorithm and provide a proof different from the proof found in [AN21] for its termination, which still applies to the unmodified version of the algorithm.

3.1 Preliminary propositions

A few preliminary results are required to state the algorithm. The next two propositions lead up to the proposition used in the algorithm to decompose functions satisfying the modular law.

Proposition 3.2 ([AN21, Proposition 2.2]). *Let $h_1 \in \mathbb{H}_n$ be a Hessenberg function and $1 \leq i < j \leq n$ be integers such that*

- (1) *either $h_1(i-1) < h_1(i)$, or $i = 1$ and $h_1(1) > 1$,*
- (2) *$j-1 < h_1(i) = h_1(i+1) = \dots = h_i(j-1)$, and*
- (3) *$h_1^{-1}(\{i, \dots, j-2\}) = \emptyset$.*

Furthermore, let h_0 and h_2 be the Hessenberg functions defined by

$$h_0(l) := \begin{cases} h_1(l) - 1 & \text{if } l \in \{i, \dots, j-1\}, \\ h_1(l) & \text{otherwise} \end{cases}$$

and

$$h_2(l) := \begin{cases} h_1(l) - 1 & \text{if } l \in \{i, \dots, j-2\}, \\ h_1(l) & \text{otherwise.} \end{cases}$$

Then, for a function f satisfying the modular law (Definition 2.1), we get

$$f(h_1) = [j-i]_q f(h_2) + (1 - [j-i]_q) f(h_0).$$

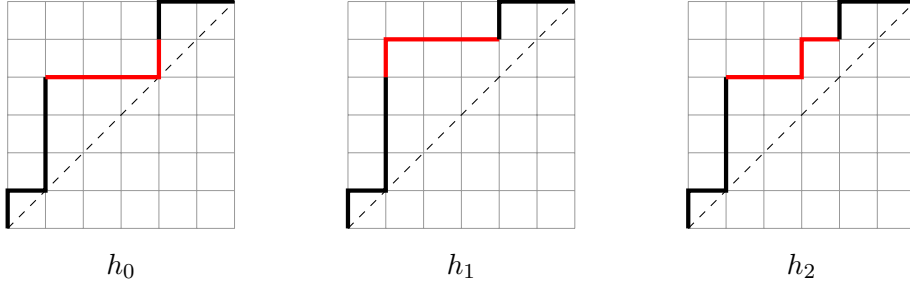


Figure 3.1: Examples of h_0 , h_1 , and h_2 as in Proposition 3.2.

See Figure 3.1 for an illustration of the Hessenberg functions used in Proposition 3.2. We give a proof following the outline described in [AN21], filling in some details.

Proof. In order to prove this Proposition, we will iteratively apply the modular law to functions $g_k \in \mathbb{H}_n$, $k \in \{0, \dots, j - i - 1\}$, defined by

$$g_k(l) := \begin{cases} h_1(l) - 1 & \text{if } l \in \{i, \dots, i + k\} \\ h_1(l) & \text{otherwise.} \end{cases}$$

Additionally, we define $g_{-1} := h_1$. Conditions (1) and (2) in the statement of this proposition ensure that all g_k are well defined.

Now, every triple (g_{k-1}, g_k, g_{k+1}) , $k \in \{0, \dots, j - i - 2\}$, satisfies condition (2) in the Definition of the modular law (Definition 2.1), with g_{k-1} , g_k and g_{k+1} taking the roles of h_0 , h_1 and h_2 respectively:

- $g_k(k + i + 1) = g_k(k + i) + 1$ by construction for $k \in \{0, \dots, j - i - 2\}$,
- $g_k^{-1}(k + i) = \emptyset$ for $k \in \{0, \dots, j - i - 2\}$ by condition (3) in the statement of this Proposition, and
- g_{k-1} and g_{k+1} satisfy $g_{k-1}(k + i) = g_{k-1}(k + i + 1) = g_k(k + i)$ and $g_{k+1}(k + i) = g_{k+1}(k + i + 1) = g_k(k + i + 1)$ by construction.

Thus, we can write

$$(1 + q)f(g_k) = qf(g_{k+1}) + f(g_{k-1}),$$

or, after rearranging,

$$f(g_{k-1}) = (1 + q)f(g_k) - qf(g_{k+1}). \tag{3.1}$$

Now, noticing that we can use q -integers to write $f(g_{-1}) = [2]_q f(g_0) - ([2]_q - 1)f(g_1)$

and using (3.1) to replace $f(g_k)$ in $[k+2]_q f(g_k) - ([k+2]_q - 1)f(g_{k+1})$, we get

$$\begin{aligned} & [k+2]_q f(g_k) - ([k+2]_q - 1)f(g_{k+1}) \\ &= [k+2]_q ((1+q)f(g_{k+1}) - qf(g_{k+2})) - ([k+2]_q - 1)f(g_{k+1}) \\ &= ((1+q)[k+2]_q - [k+2]_q + 1)f(g_{k+1}) - q[k+2]_q f(g_{k+2}) \\ &= [k+3]_q f(g_{k+1}) - ([k+3]_q - 1)f(g_{k+2}), \end{aligned}$$

where $(1+q)[n]_q - [n]_q + 1 = [n+1]_q$ and $q[n]_q = [n+1]_q - 1$ is used in the last step.

Observing that $g_{-1} = h_1$, $g_{j-i-2} = h_2$ and $g_{j-i-1} = h_0$, we get the statement of the Proposition by iteratively applying (3.1):

$$\begin{aligned} f(h_1) &= f(g_{-1}) = [j-i-2+2]_q f(g_{j-i-2}) - ([j-i-2+2]_q - 1)f(g_{j-i-1}) \\ &= [j-i]_q f(h_2) + (1 - [j-i]_q)f(h_0). \end{aligned}$$

□

Proposition 3.3 ([AN21, Proposition 2.3]). *Let $h_1 \in \mathbb{H}_n$ be a Hessenberg function and $1 \leq i \leq n$ be an integer such that*

- (1) *either $h_1(i-1) < h_1(i)$, or $i = 1$ and $h_1(1) > 1$ and*
- (2) *there exists $\max\{h_1(i-1), i\} \leq a < h_1(i)$ such that $h_1(a+1) = h_1(a+2) = \dots = h_1(h_1(i))$.*

Let h_0 and h_2 be Hessenberg functions defined by

$$h_0(l) := \begin{cases} a & \text{if } l = i, \\ h_1(l) & \text{otherwise} \end{cases}$$

and

$$h_2(l) := \begin{cases} a+1 & \text{if } l = i, \\ h_1(l) & \text{otherwise.} \end{cases}$$

Then, for a function f satisfying the modular law (Definition 2.1),

$$f(h_1) = [h_1(i) - a]_q f(h_2) + (h_1(i) - a)f(h_0).$$

Figure 3.2 illustrates an example of a triple of Hessenberg functions h_0 , h_1 , and h_2 as in the statement of the proposition above.

Proof. We proceed similarly to Proposition 3.2. First, we define Hessenberg functions g_k by

$$g_k(l) := \begin{cases} h_1(l) - k & \text{if } l = i \\ h_1(l) & \text{otherwise,} \end{cases}$$

for $k \in \{0, \dots, h_1(i) - a\}$. Their well-definedness is guaranteed by condition (2).

This time, all triples (g_{k-1}, g_k, g_{k+1}) , $k \in \{1, \dots, h_1(i) - a - 1\}$ satisfy condition (1) in the statement of the modular law (Definition 2.1), with g_{k-1} , g_k and g_{k+1} taking the roles of h_2 , h_1 and h_0 respectively:

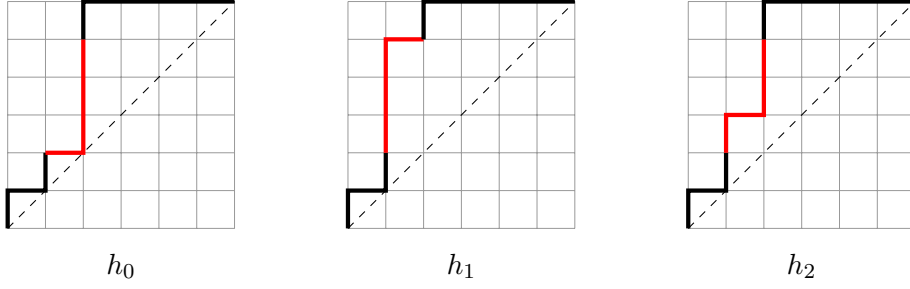


Figure 3.2: Examples of h_0 , h_1 , and h_2 as in Proposition 3.3.

- $g_k(i-1) < g_k(i) < g_k(i+1)$ by construction,
- $g_k(g_k(i)) = g_k(g_k(i)+1)$ or $g_k(i) = n$ by condition (2), and
- g_{k-1}, g_{k+1} match g_k for $l \neq i$ and $g_{k\pm 1}(i) = g_k(i) \pm 1$ by construction.

Now, by using the modular law as in the proof of Proposition 3.2 and since we now start with g_0 instead of g_1 , we get

$$\begin{aligned} f(h_1) &= f(g_0) = [h_1(i) - a]_q f(g_{h(i)-a-1}) + (1 - [h_1(i) - a]_q) f(g_{h(i)-a}) \\ &= [h_1(i) - a]_q f(h_2) + (1 - [h_1(i) - a]_q) f(h_0), \end{aligned}$$

where we used that $g_{h(i)-a-1} = h_2$ and $g_{h(i)-a} = h_0$. $\square$

Combining the previous two propositions, we are able to state and prove the next proposition.

Proposition 3.4 ([AN21, Proposition 2.4]). *Let $h_1 \in \mathbb{H}_n$ be a Hessenberg function and let $1 \leq i < j \leq n$ be integers such that*

- (1) *either $h_1(i-1) < h_1(i)$ or $i = 1$ and $h_1(1) > 1$,*
- (2) *$j-1 < h_1(i) = h_1(i+1) = \dots = h_1(j-1) < h_1(j)$,*
- (3) *$h_1^{-1}(\{i, \dots, j-2\}) = \emptyset$, and*
- (4) *there exists $1 \leq b \leq h_1(j) - h_1(i)$ such that $h_1(h_1(i)) = h_1(h_1(i)+1) = \dots = h_1(h_1(i)+b)$.*

Let h_0 and h_2 be the Hessenberg functions defined by

$$h_0(l) := \begin{cases} h_1(l) - 1 & \text{if } l \in \{i, \dots, j-1\}, \\ h_1(l) & \text{otherwise} \end{cases}$$

and

$$h_2(l) := \begin{cases} h_1(l) - 1 & \text{if } l \in \{i, \dots, j-2\}, \\ h_1(l) + b & \text{if } l = j-1, \\ h_1(l) & \text{otherwise.} \end{cases}$$

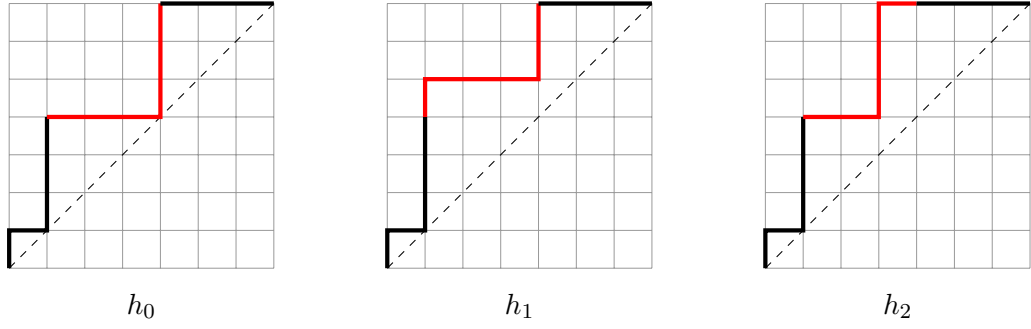


Figure 3.3: Examples of h_0 , h_1 , and h_2 as in Proposition 3.4.

Then,

$$[b+1]_q f(h_1) = [j-i]_q f(h_2) + ([b+1]_q - [j-i]_q) f(h_0).$$

See Figure 3.3 for an example of Hessenberg functions h_0 , h_1 , and h_2 as they appear in the proposition above.

Proof. We define $h' \in \mathbb{H}_n$ as

$$h'(l) := \begin{cases} h_1(l) - 1 & \text{if } l \in \{i, \dots, j-2\}, \\ h_1(l) & \text{otherwise.} \end{cases}$$

Applying Proposition 3.2 to h_1 , which is possible because of conditions (1)–(3), we get

$$f(h_1) = [j-i]_q f(h') + (1 - [j-i]_q) f(h_0),$$

and applying Proposition 3.3 to h_2 , with i from Proposition 3.3 corresponding to $j-1$ and $a := h_1(i) - b - 1$ and condition (1) and (4) ensuring its applicability, we get

$$f(h_2) = [b+1]_q f(h') + (1 - [b+1]_q) f(h_0).$$

Rearranging for $f(h')$, multiplying by $[b+1]_q$ and $[j-i]_q$ respectively and equating the equations, we get

$$\begin{aligned} & [b+1]_q f(h_1) - [b+1]_q f(h_0) + [b+1]_q [j-i] f(h_0) \\ &= [j-i]_q f(h_2) - [j-i]_q f(h_0) + [j-i]_q [b+1]_q f(h_0), \end{aligned}$$

which is equivalent to,

$$[b+1]_q f(h_1) - [b+1]_q f(h_0) = [j-i]_q f(h_2) - [j-i]_q f(h_0),$$

which can be rearranged to yield the statement of the proposition. $\square$

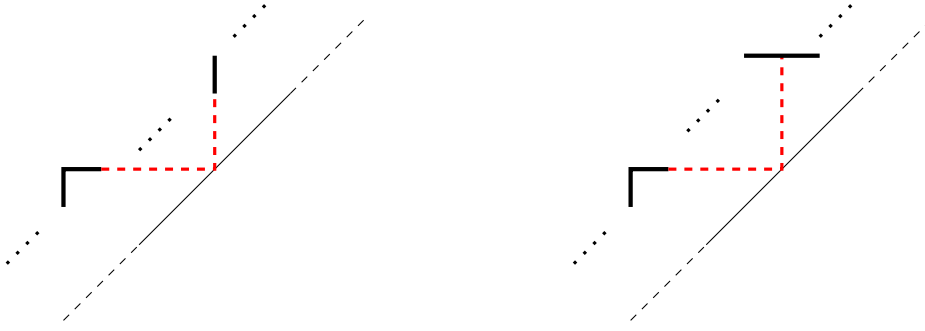


Figure 3.4: Examples of an aligned corner on the left and a non-aligned corner on the right.

Similarly to the notion of *aligned* Hessenberg used in [AN21], we define *aligned corners* that we will use in our alternative proof of the termination of Algorithm 3.8.

Definition 3.5 (aligned corner). Let $h \in \mathbb{H}_n$ be a Hessenberg function. A *corner* of h is a pair $(i, h(i))$ such that $h(i-1) < h(i)$ for $i \in \{1, \dots, n-1\}$, using $h(0) = 0$ if necessary. We say that a corner $(i, h(i))$ is an *aligned* corner if $h(h(i)) < h(h(i)+1)$, otherwise we say that it is *non-aligned*.

Figure 3.4 illustrates examples of aligned and non-aligned corners.

We also use this notion to define the next proposition, used to choose appropriate parameters for Proposition 3.4 in Algorithm 3.8, similar to [AN21, Propositions 2.5 and 2.6].

Proposition 3.6. Let $h \in \mathbb{H}_n$ be a Hessenberg function containing non-aligned corners and let $(l, h(l))$ be the non-aligned corner with maximal l such that, with $k := \min\{s : h(s) > h(l)\}$, we have $h^{-1}(\{l, \dots, k-2\}) = \emptyset$. Now, define $i := l$, $j := k$ and $b := \max\{m : m \geq 1, h(h(i)+m) = h(h(i)) \text{ and } h(i)+m \leq h(j)\}$.

Then, h together with i , j and b satisfies the conditions required for Proposition 3.4.

Proof. If such a corner exists, all conditions follow directly from the definition of i , j and b and the fact that $(i, h(i))$ is non-aligned.

To prove the existence, note that if there is a non-aligned $(l, h(l))$ corner and there is no $k := \min\{s : h(s) > h(l)\}$ such that we have $h^{-1}(\{l, \dots, k-2\}) = \emptyset$, there has to be a non-aligned corner $(l', h(l'))$ where $h(l') \in \{l, \dots, k-2\}$ and thus $l' < l$. But since Hessenberg functions are finite, there has to be a smallest non-aligned corner satisfying our requirements in order to not contradict its minimality. $\square$

Remark 3.7. Proposition 3.6 differs from Proposition 2.5 and Proposition 2.6 used in [AN21] to choose i , j and b : There, instead of always choosing the rightmost corner

satisfying the necessary conditions, the authors choose the leftmost corner $(i, h(i))$ whenever there is a non-aligned corner with $h(i) \neq n$.

This difference in choices will also be the part making our algorithm differ slightly from the algorithm in [AN21]. Our version seems to give decompositions of $h \in \mathbb{H}_n$ into fewer parts than Abreu and Nigro's algorithm for Hessenberg functions, provided the specific function actually leads to the algorithm taking different steps. This behaviour has been checked computationally for all $h \in \mathbb{H}_n$ for $n < 12$ and our proof of the termination of Algorithm 3.8 (which also proves the termination of Abreu and Nigro's algorithm since the proof does not depend on Proposition 3.6) will provide heuristic reasons for this difference in behaviour.

3.2 Abreu–Nigro's algorithm

Using propositions 3.4 and 3.6, we are finally able to state our variation of the algorithm.

Algorithm 3.8 (variation of [AN21, Algorithm 2.7]). Let $f : \mathbb{H} \rightarrow \mathcal{A}$ be a function satisfying the modular law (Definition 2.1). To compute $f(h)$ in terms of $f(k_{n_1} \cdots k_{n_m})$, or in terms of $f(k_i)$ if f is multiplicative (i.e. $f(h_1 h_2) = f(h_1) f(h_2)$), we proceed the following way:

- (1) Return $f(h)$ if $h = k_{n_1} \cdots k_{n_m}$ for $n_i \in \mathbb{Z}^+$. Else, go to step (2).
- (2) Choose i, j and b as in Proposition 3.6, then use Proposition 3.4 with i, j and b . Then, return to step (1) with h_0 and h_2 respectively.

Remark 3.9. Notice that a Hessenberg function containing non-aligned corners cannot be of the form $h = k_{n_1} \cdots k_{n_m}$, ensuring that at every step of the algorithm either step (1) or step (2) can be applied.

See Figure 3.5 for the algorithm applied to the Hessenberg function $h = (1, 3, 4, 5, 5) \in \mathbb{H}_5$.

For our proof of the termination of this algorithm, we introduce the notion of an *alignedness vector*.

Definition 3.10. (alignedness vector) We consider $h \in \mathbb{H}_n$ and call $h(i)$ the *height* for any corner $(i, h(i))$. Then, we associate numbers $a_i(h) \in \{0, 1\}$ to h defined by

$$a_i(h) := \begin{cases} 1 & \text{if there exists a non-aligned corner of height } i \text{ in } h, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we call $a(h) := (a_n(h), a_{n-1}(h), \dots, a_1(h))$ the *alignedness vector* of h , since it encodes the height of any non-aligned corners in h .

It turns out that applying Proposition 3.4 to a Hessenberg function always yields Hessenberg functions with strictly smaller alignedness vectors when ordering them lexicographically.

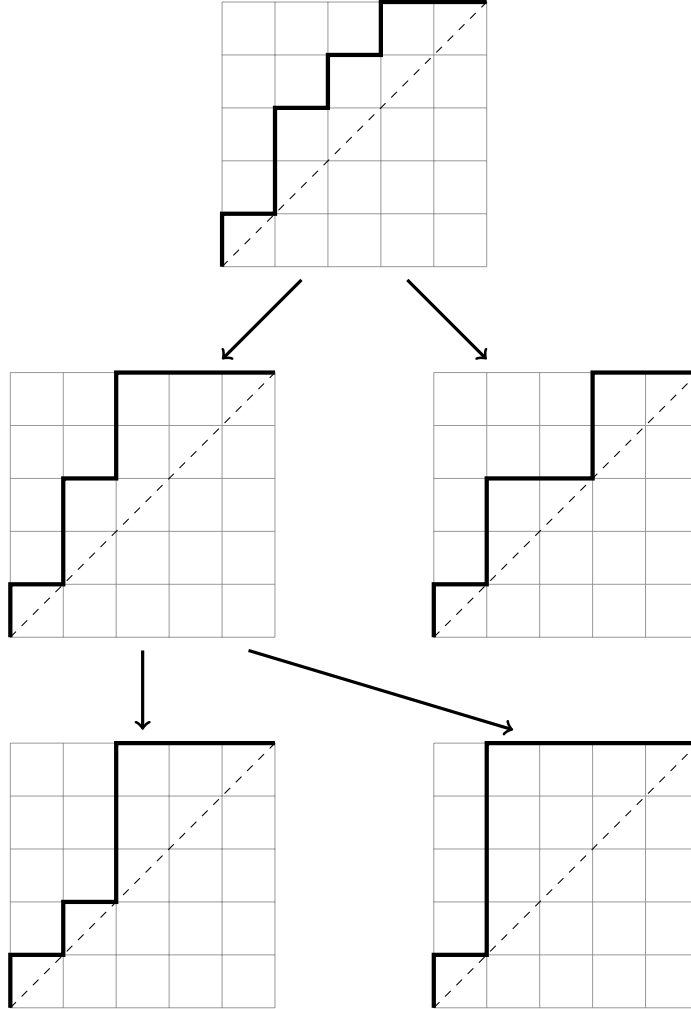


Figure 3.5: Algorithm 3.8 applied to the Hessenberg function $h = (1, 3, 4, 5, 5) \in \mathbb{H}_5$.

Theorem 3.11. For h_1 , h_0 and h_2 and parameters (i, j, b) from Proposition 3.4, we have that

$$a(h_0) <_{\text{lex}} a(h_1)$$

and

$$a(h_2) <_{\text{lex}} a(h_1).$$

Proof. We look at the non-aligned corner $(i, h(i))$ (if there is no non-aligned corner, applying Proposition 3.4 is not possible).

To construct h_0 , we lower the value of h_1 for all l such that $h_1(l) = h_1(i)$ by 1. Thus, the non-aligned corner at height $h_1(i)$ vanishes and a new non-aligned corner may be introduced at height $h_0(i)$. But $h_0(i) < h_1(i)$, so $a(h_0) <_{\text{lex}} a(h_1)$.

In the case of h_2 , we repeat its construction in terms of h_1 :

$$h_2(l) := \begin{cases} h_1(i) - 1 & \text{if } l \in \{i, \dots, j-2\}, \\ h_1(i) + b & \text{if } l = j-1, \\ h_1(i) & \text{otherwise.} \end{cases}$$

Lowering the values at $l \in \{i, \dots, j-2\}$ may introduce a non-aligned corner of height $h_1(i) - 1$ but at the same time ensures that there is no non-aligned corner of the form $(l, h_1(i))$ present in h_2 .

Adding b to the value at $l = j-1$ ensures that there is no non-aligned corner $(j-1, h_1(i))$ present in h_2 . Additionally, no new non-aligned corner is introduced, because b is chosen such that either

$$h_1(h_1(i) + b) < h_1(h_1(i) + b + 1) \text{ and } h_1(i) + b \leq h_1(j)$$

or

$$h_1(h_1(i) + b) \leq h_1(h_1(i) + b + 1) \text{ and } h_1(i) + b = h_1(j).$$

In the first case, the newly introduced corner is aligned and in the second case, since $h_1(j) = h_2(j-1)$, the newly introduced corner, although perhaps non-aligned, introduces no new non-aligned corner of height $h_1(i) + b$ in h_2 compared to h_1 since such a corner would have already existed for h_1 due to $h_1(i) + b = h_1(j)$.

Thus, we get $a(h_2) <_{\text{lex}} a(h_1)$ as well. $\square$

Corollary 3.12. Algorithm 3.8 terminates.

Proof. There are only finitely many Hessenberg functions and their lexicographic ordering decreases with every step, thus forcing the algorithm to eventually terminate. $\square$

Remark 3.13. The proof now makes it possible to give a heuristic reason for the different number of parts in the decomposition of $h \in \mathbb{H}$ between Algorithm 3.8 and [AN21, Algorithm 2.7]: In contrast to the unmodified algorithm from Abreu and Nigro, we always choose the “highest” possible non-aligned corner, thus reducing the amount of places where non-aligned corners can appear during the execution of the algorithm faster.

Remark 3.14. Note that the algorithm alone does not prove Theorem 3.1. To complete the proof, one uses the following proposition.

Proposition 3.15 ([AN21, Proposition 2.14]). *For every sequence of positive integers $(c_1, \dots, c_m)$, a permutation $\sigma \in \mathcal{S}_m$ and $f : \mathbb{H} \rightarrow \mathcal{A}$ a function satisfying the modular law, we have*

$$f(k_{c_1} k_{c_2} \cdots k_{c_m}) = f(k_{\sigma(c_1)} k_{\sigma(c_2)} \cdots k_{\sigma(c_m)}).$$

Proof. For a proof we refer the reader to [AN21]. □

4 Interlacing filters

In this chapter we take the first steps outlining the direction of our future research. It is based on our own work with Anton Mellit pointing the author in the right directions.

The purpose of this chapter is to generalize the modular law to find similar linear relations for infinitely many skew shapes following certain rules to be specified in section 4.2.

Having this goal in mind, we initially review certain aspects of our proof of the chromatic quasisymmetric function satisfying the modular law we intend to generalize, continue by describing the Dyck paths used for *interlacing filters* in section 4.2 and finally state the definition of *interlacing filters* together with conjectured formulae for calculating them in section 4.3.

4.1 From the modular law to *interlacing filters*

Considering the proof of Theorem 2.19 and continuing to use its notation as well as sticking to the rotation chosen for D_λ , the first thing to notice is that the non-fixed sections of the shapes D_{λ^k} correlate to all possible sections of Dyck paths that consist of one south and two east steps. We will later call these sections a *12-block*. Note, that in this chapter a Dyck path of length n is considered to be a sequence of n south and n east steps, starting at the northwest corner of an $n \times n$ grid and ending in its southeast corner, with no part of the path dipping below the diagonal. Therefore, we can interpret the northeast border of D_{λ^k} as a Dyck path.

Next, the variables relevant for the proof correspond to the images of the row and column indices of the 12-block under the permutation π .

Finally, denote by c_1, c_2 and c_3 the coefficients $(c_1, c_2, c_3) = (1, -(q+1), q)$ corresponding to the Dyck paths of the sections $p_1 = ees$, $p_2 = ese$ and $p_3 = see$ (here, the string of e and s is meant to represent the east and south steps of the Dyck path D_{λ^k} inside the 12-block; also beware, that the numbering of the paths has been changed). When c_1, c_2 and c_3 are used in the linear combination

$$\sum_{i=1}^3 c_i q^{\text{inv}(\pi, p_i)}, \quad (4.1)$$

where $\text{inv}(\pi, p_i) = \text{inv}(\pi) \cap \{(x, y-1), (x, y), (y-1, y)\}$, it evaluates to 0 for all π except for those where $b < a < c$ or $c < a < b$. In these cases, the absolute value of the linear combination turns out to be the same. Furthermore, the sign of the linear combination correlates with $\text{sgn}(\pi)$, because in the case of $b < a < c$, equation 4.1

turns into $1 \cdot q - (1 + q) \cdot q + q \cdot 1 = -q^2 + q$ and in the case of $c < a < b$ it turns into $1 \cdot q^2 - (1 + q) \cdot q + q \cdot q = q^2 - q$.

Since a is associated to the row index and b, c to the column indices, we have two sets of variables that we say to be *interlacing* when $b < a < c$ or $c < a < b$, since variables from each set are listed alternatingly. These are the only two possibilities for variables of a 1-element and 2-element set to interlace.

Hence, expressed differently, the coefficients in the modular law act as a filter, filtering out all the permutations π where a, b and c are not interlacing. It therefore seems justified to call $(1, -(q + 1), q)$ the *12-interlacing filter*.

4.2 The Dyck paths used by interlacing filters

We now discuss the Dyck paths used to construct interlacing filters.

It is our goal to filter for the cases where two sets of variables interlace, one corresponding to adjacent row indices and the other one to adjacent column indices, already putting up restrictions for the possible sections of Dyck paths we have to consider. If the difference in size of each set is greater than or equal to 2, it is not possible to have them interlace, because two variables of one set would always end up directly next to each other. So it is only necessary to find useful sections of Dyck paths in the four cases where

- there is one more column than row involved, fixing a unique way of interlacing,
- the number of rows matches the number of columns and they interlace starting with a variable associated to a row,
- the number of rows matches the number of columns and they interlace starting with a variable associated to a column, and
- there is one more row than column involved, again fixing a unique way of interlacing.

When talking of the paths associated to r rows and c columns, we speak of an *rc-block*.

Notice that the first and fourth cases are similar in structure. Accordingly, we construct the paths used in the fourth case from the first case by replacing south steps with east steps and vice versa. Additionally, for the examples calculated in this thesis, we have reversed the order of the paths. The second and third possible case for variables to interlace only differ on the level of the permutation, the underlying paths used are the same. This leaves us to find suitable paths in the first and second case.

One final restriction for the paths used is that they have to avoid two consecutive south steps, because they would not contribute any useful information for reasons that will become apparent later (cf. Remark 4.6).

We are now in a position to describe the paths we want to use.

To construct the paths inside of a $n(n + 1)$ -block, start with a sequence of $n + 2$ south and $n + 1$ east steps that are taken alternatingly, i. e. start with *sese...ses*. Then, to create the paths one by one, delete two south steps. For the first path in the block, delete the first and second south step, for the second path, delete the first and third south step of the initial sequence and continue in this way until the last path is



Figure 4.1: Example of a path from the 23-block and the 22-block being constructed.

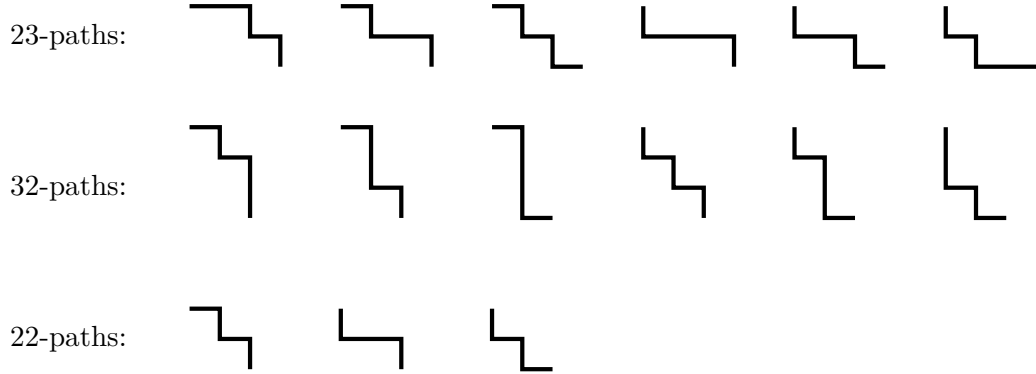


Figure 4.2: All the paths appearing in a 23-, 32- and 22-block, listed from top to bottom.

created by deleting the second to last and last south step. In this way, $\binom{n+2}{2}$ paths are created.

For the case of an nn -block, proceed similarly, this time starting with an initial path of $n + 1$ south and n east steps and only deleting one south step for each path. Therefore, $\binom{n+1}{1} = n + 1$ paths are constructed.

Example 4.1. Figure 4.1 gives an example of two paths being constructed by deleting south steps. For a complete list of the paths appearing in the 23-, 32- and 22-block, see Figure 4.2.

Finally, we would like to chain multiple instances of rc -blocks together, always starting with a 12-block, in order to create paths varying over much larger sections. Then, the indices of the columns of one block are the indices of the rows of the block directly following it.

The parts of the path which are the same for all paths no matter the paths chosen inside the blocks are of no interest to us, since they have no effect when used in linear

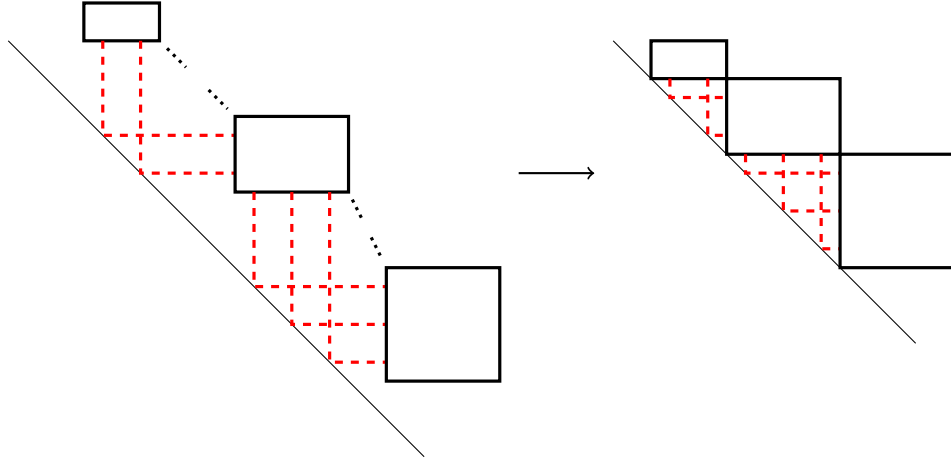


Figure 4.3: On the left: 12-, 23- and 33-blocks chained together with some sections of the path outside of boxes. On the right: 12-, 23- and 33-blocks directly chained together.

combinations similar to 4.1. Due to this, we will directly chain the blocks we choose together, ignoring the rest of the path, meaning that their south west corner always touches the diagonal. See Figure 4.3 for an example of blocks that are chained together.

Finally, to fix an order in which the paths created by chaining together smaller blocks are listed, we say that the first path is the path where the path constructed first in every block is chosen, the second path matches this path except for the last block where we chose the section constructed second. Continuing like this, choosing sections constructed later in the other blocks when necessary, we iterate through all the possible paths associated to the given chain of blocks, with the last path being the one where the section constructed last is chosen in every block.

4.3 Interlacing filters and direct formulae

Using the Dyck paths from the previous section, our next goal is to give a definition of interlacing filters.

First, more notation is required to better describe the interlacing cases of interest. Analogously to how the paths in a section of r rows and c columns are called an rc -block, we will also speak of the rc -interlacing cases when we want to express that the r variables of one set interlace with the c variables of another set. This is already unique whenever $|r - c| = 1$ and to distinguish the two cases that are possible when $r = c$, we call the rWc -interlacing cases the cases where a variable from the first set

(corresponding to the r variables) starts the interlacing pattern, and the rMc -interlacing cases the cases where a variable from the second set starts the interlacing pattern. The sequence rc with a possible W or M in between will be called a *block sequence*.

We justify the choice of “ W ” and “ M ” by imagining both sets of variables written as vertices in two rows, with the top row corresponding to the first set of variables and then connecting vertices with edges to form a chain read from left to right instead of using order relations, the initial segment of the interlacing cases starting with the top row will form a “ W ”-shape and conversely the cases starting with the second row form an “ M ” shape — assuming that the rows are written in increasing order.

Example 4.2 (interlacing cases). Given the two sets of variables $\{a, b\}$ and $\{c, d, e\}$, some 23-interlacing cases are “ $c < a < d < b < e$ ”, “ $d < a < e < b < c$ ” or “ $e < b < d < a < c$ ”. In total, there are 12 cases, since there are 2 ways to permute the first set and 6 ways to permute the second set.

If both sets have the same size, i. e. $\{a, b\}$ and $\{c, d\}$, there are 4 $2W2$ -interlacing cases, one of them being “ $a < c < b < d$ ” and 4 $2M2$ -interlacing cases, one of them being “ $c < a < d < b$ ”.

It is now straightforward to express the interlacing cases created by chaining multiple sets of variables together, one simply chains together their block sequences with the last number of one sequence being the first number of the following one. This process yields sequences of the form 23W34 or 34321.

Definition 4.3. (interlacing sequence) A sequence as above that starts with 1 is called an *interlacing sequence*. The length of an interlacing sequence I is denoted by $l(I)$, with $l(I) := \sum_i i$, where i runs over all integers appearing inside of I (e.g. the sequence 12W2M23 has length $1 + 2 + 2 + 2 + 3 = 10$).

From now on, for every interlacing sequence I we will consider the elements of the sets whose size is described by I to be the integers from 1 to $l(I)$, where, if $i_1, i_2, \dots$ are the integers appearing in I , the integers $1, 2, \dots, i_1$ correspond to the first set, the integers $i_1 + 1, i_1 + 2, \dots$ correspond to the second set etc.

A permutation $\pi \in \mathcal{S}_{l(I)}$ is said to be interlacing, if the integers corresponding to i_j, i_{j+1} , for all possible j , interlace according to their block sequence when ignoring all other integers.

Example 4.4. (interlacing sequence) Given the interlacing sequence $I = 12W23$, we are dealing with permutations in $\mathcal{S}_8$, with 21643758 being an example of an interlacing permutation and 12643758 being an example of a non-interlacing permutation, since the integers corresponding to the 12-block do not interlace.

We can finally state the definition of interlacing filters.

Definition 4.5. (interlacing filter) Let I be an interlacing sequence of the integers $i_1, \dots, i_k$ with equal integers separated by W or M . Let $p_\nu, \nu \in [n]$ be the paths created by chaining together the sections of paths corresponding to each block in I , where n is the number of all possible combinations of these sections of paths.

Let the positions of the integers $(m_1, m_2, \dots, m_{j_k})$ associated to the last integer in I be fixed to some positions $(s_1, s_2, \dots, s_{j_k})$ and let the possible positions of the integers I_α associated to the part i_α of the interlacing sequence be fixed to sets J_α . We call a tuple of coefficients $(c_1, c_2, \dots, c_n)$ an *interlacing filter*, if for all the possible fixed positions as above the linear combination

$$\sum_{\substack{\pi \in \mathcal{S}_{l(I)} \\ \pi|_{(m_1, m_2, \dots, m_{j_k})} = (s_1, s_2, \dots, s_{j_k}) \\ \pi|_{I_\alpha} = J_\alpha}} c_1 q^{\text{inv}(\pi, p_1)} + c_2 q^{\text{inv}(\pi, p_2)} + \dots + c_n q^{\text{inv}(\pi, p_n)},$$

evaluates to 0 if the permutations π are non-interlacing and to $\text{sgn}(\sigma)S_I$ for a S_I fixed and σ being the permutation of the integers in the last block, ignoring all other integers, if the permutations π are interlacing. (Note, that we sum over π where the variables of a set can always be found in the same positions, thus the “interlacingness” of the permutations does not change once the positions are fixed as above.)

Here, $\text{inv}(\pi, p_i)$ denotes the number of inversions (i, j) of π such that (i, j) is a box between p_i and the diagonal.

We call the interlacing filter corresponding to the interlacing sequence I the I -interlacing filter.

We would now like to prove the existence of interlacing filters and give explicit formulae for the coefficients c_i . Sadly, the former is out of the scope of this thesis, but we will spend the rest of this chapter further examining some properties and formulating a conjecture for directly calculating interlacing filters.

Computationally looking at examples for small numbers of variables using Python and Sage, leads us to believe that the coefficient of a path can be calculated by multiplying the coefficients of each of the sections that chain together to form the given path. Continuing forward, we will assume the multiplicativity of path sections due to this observation, enabling us to reduce the problem of finding coefficients for the whole path to finding coefficients for all the sections of the path.

Suppose we have an interlacing sequence I and have already found the interlacing filter associated to the sequence I' , with I' matching I up to the last set of variables appearing in I , which are missing in I' .

Then, with σ' now denoting the permutation of just the last set of variables I_j of I' , $|I_j| = k'$, and fixing their positions as $(s'_1, s'_2, \dots, s'_{k'})$, we know that the sum of linear combinations

$$\sum_{\substack{\pi' \in \mathcal{S}_{l(I')}} \\ \pi'|_{I_j} = (s'_1, s'_2, \dots, s'_{k'}) \\ \pi'|_{I_\alpha} = J_\alpha}} c'_1 q^{\text{inv}(\pi', p'_1)} + c'_2 q^{\text{inv}(\pi', p'_2)} + \dots + c'_n q^{\text{inv}(\pi', p'_n)}$$

is 0 if the permutations π' are non-interlacing and $\text{sgn}(\sigma')S_{I'}$ if the π' are interlacing.

Since we are assuming that the coefficients of the path are multiplicative and since the contributions to the number of inversions by the last block in I does not change

anything about the inversions counted by the earlier blocks, we need to find $\tilde{c}_l$ such that $c'_i \tilde{c}_j = c_m$ where the c_m form the interlacing filter corresponding to I . In other words, we want to find $\tilde{c}_l$ such that, fixing the positions $(s_1, s_2, \dots, s_k)$ of the last set of variables I_{j+1} in I , $|I_{j+1}| = k$, σ expressing the permutation of the last block of variables of I and $\tilde{p}_i, i \in \tilde{n}$ ($\tilde{n}$ denoting the number of possible $\tilde{p}_i$), expressing the parts of the Dyck path chosen in the last block,

$$\sum_{\substack{\pi \in \mathcal{S}_{l(I)} \\ \pi|_{I_{j+1}} = (s_1, s_2, \dots, s_k) \\ \pi|_{I_\alpha} = J_\alpha}} \text{sgn}(\sigma') S_{I'} (\tilde{c}_1 q^{\text{inv}(\sigma, \tilde{p}_1)} + \tilde{c}_2 q^{\text{inv}(\sigma, \tilde{p}_2)} + \dots + \tilde{c}_{\tilde{n}} q^{\text{inv}(\sigma, \tilde{p}_{\tilde{n}})})$$

is 0 if all the integers in $[l(I)]$ form a non-interlacing permutation and equal to some S_I multiplied by the sign of the permutation of the last set of variables.

With l enumerating the paths in the last block, this can be rewritten as

$$S_{I'} \cdot \sum_{\substack{\pi \in \mathcal{S}_{l(I)} \\ \pi|_{I_{j+1}} = (s_1, s_2, \dots, s_k)}} \left(\sum_l \tilde{c}_l \left(\sum_{\sigma' \text{ even}} q^{\text{inv}(\sigma, \tilde{p}_l)} - \sum_{\sigma' \text{ odd}} q^{\text{inv}(\sigma, \tilde{p}_l)} \right) \right).$$

Noticing that the summation over paths stays the same except for possible reorderings depending on π resulting in every summand of the outer sum being the same, we have found a formula allowing us to calculate the coefficients $\tilde{c}_l$ for a single block. To do this, set every sum over a path corresponding to the non-interlacing cases equal to 0 and solve the system of linear equations in the variables $\tilde{c}_l$.

Remark 4.6. This formula now also makes it obvious why paths with two consecutive south steps are avoided: For every path like this, it would be possible to simply swap the rows corresponding to the south steps in question to find a permutation with different sign but a matching number of inversions restricted to said path, leading to the sum always being 0.

Finally, equipped with a simpler way of calculating the coefficients $\tilde{c}_l$, it is possible to computationally generate more examples, leading to the next conjecture, marking the end of this thesis. We intend to prove these formulae in our future work.

Conjecture 4.7. For every block being a part of an interlacing sequence I , there exist coefficients $\tilde{c}_l$ for each section of the path inside the block, which, when multiplied together according to how their corresponding path sections form a full path for I , form an interlacing filter. They are given by the formulae

- (1) for $n(n+1)$ -blocks, $((-1)q^k \sum_{j=0}^i q^j)_{\substack{k=0, \dots, n \\ i=0, \dots, n-k}}$, or, written more explicitly, $(1, -q-1, q^2+q+1, \dots, (-1)^n \sum_{i=0}^n q^i, -q(q+1), \dots, (-1)^{n-1} q \sum_{i=0}^{n-1} q^i, \dots, q^n)$,
- (2) $((-1)^i q^i)_{i=0, \dots, n}$ for nWn -blocks,
- (3) $((-1)^i)_{i=0, \dots, n}$ for nMn -blocks and

- (4) $\tilde{c}_j = 1$ if the l -th path is the path alternating south and east steps, starting with a south step, and $\tilde{c}_l = 0$ else, for $n(n-1)$ -blocks.

Example 4.8. We look at the interlacing sequence $I = (12343)$. The coefficients of each block are $(1, -q-1, q)$, $(1, -q-1, q^2+q+1, q, -q^2-q, q^2)$, $(1, -q-1, q^2+q+1, -q^3-q^2-q-1, q, -q^2-q, q^3+q^2+q, q^2, -q^3-q^2, q^3)$ and $(0, 0, 0, 0, 0, 0, 1, 0, 0, 0)$, resulting in an interlacing filter with 1800 coefficients, some of which are $(0, \dots, 0, -q-1, 0, \dots, 0, q^2+q+1, 0, \dots, q^5+2q^4+3q^3+2q^2+q, \dots, -q^6-q^5, 0, \dots, 0, q^6, 0, 0, 0)$.

An example of a manageable size is $I = (12W2)$, yielding the coefficients for each block $(1, -q-1, q)$ and $(1, -q, q^2)$, resulting in the interlacing filter $(1, -q, q^2, -q-1, q^2+q, -q^3-q^2, q, -q^2, q^3)$.

The interlacing filter corresponding to $I = (123)$ is $(1, -q-1, q^2+q+1, q, -q^2-q, q^2, -q-1, q^2+2q+1, -q^3-2q^2-2q-1, -q^2-q, q^3+2q^2+q, -q^3-q^2, q, -q^2-q, q^3+q^2+q, q^2, -q^3-q^2, q^3)$, which can be calculated by using the coefficients for the blocks that are already listed in the first example.

Bibliography

- [AN21] Alex Abreu and Antonio Nigro. “Chromatic symmetric functions from the modular law”. In: *Journal of Combinatorial Theory, Series A* 180 (2021), p. 105407.
- [AP18] Per Alexandersson and Greta Panova. “LLT polynomials, chromatic quasisymmetric functions and graphs with cycles”. In: *Discrete Mathematics* 341.12 (2018), pp. 3453–3482.
- [Ass07] Sami Hayes Assaf. “Dual equivalence graphs, ribbon tableaux and Macdonald polynomials”. PhD thesis. University of California, Berkeley, 2007.
- [CM18] Erik Carlsson and Anton Mellit. “A proof of the shuffle conjecture”. In: *J. Amer. Math. Soc.* 31.3 (2018), pp. 661–697.
- [Ges84] Ira M. Gessel. “Multipartite P-partitions and inner products of skew Schur functions”. In: *Contemp. Math.* 34 (1984), pp. 289–317.
- [GMR14] Mathieu Guay-Paquet, Alejandro H. Morales, and Eric Rowland. “Structure and enumeration of $(3+1)$ -free posets”. In: *Ann. Comb.* 18 (2014), pp. 645–674.
- [Gua13] Mathieu Guay-Paquet. *A modular relation for the chromatic symmetric functions of $(3+1)$ -free posets*. 2013.
- [Hag+05] J. Haglund et al. “A Combinatorial Formula for the Character of the Diagonal Coinvariants”. In: *Duke Mathematical Journal* 126.2 (2005), pp. 195–232.
- [HHL05] J. Haglund, M. Haiman, and N. Loehr. “A Combinatorial Formula for Macdonald Polynomials”. In: *J. Amer. Math. Soc.* 18 (2005), pp. 735–761.
- [Hik25] Tatsuyuki Hikita. *A proof of the Stanley–Stembridge conjecture*. 2025.
- [HNY20] JiSun Huh, Sun-Young Nam, and Meesue Yoo. “Melting lollipop chromatic quasisymmetric functions and Schur expansion of unicellular LLT polynomials”. In: *Discrete Mathematics* 343.3 (2020), p. 111728.
- [Lee21] Seung Jin Lee. “Linear relations on LLT polynomials and their k -Schur positivity for $k=2$ ”. In: *Journal of algebraic combinatorics* 53.4 (2021), pp. 973–990.
- [LLT97] Alain Lascoux, Bernard Leclerc, and Jean-Yves Thibon. “Ribbon tableaux, Hall–Littlewood functions, quantum affine algebras, and unipotent varieties”. In: *Journal of Mathematical Physics* 38.2 (Feb. 1997), pp. 1041–1068.

- [Sta11] Richard P. Stanley. *Enumerative Combinatorics*. 2nd ed. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2011.
- [Sta23] Richard Stanley. *Enumerative Combinatorics*. 2nd ed. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2023.
- [Sta95] R.P. Stanley. “A Symmetric Function Generalization of the Chromatic Polynomial of a Graph”. In: *Advances in Mathematics* 111.1 (1995), pp. 166–194.
- [SW16] John Shareshian and Michelle L. Wachs. “Chromatic quasisymmetric functions”. In: *Advances in Mathematics* 295 (2016), pp. 497–551.