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Operational Reconstruction of Generalized Particle Statistics


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# Abstract

In this Master's thesis we investigate the equivalence of transtatistics and $R$-parastatistics, two frameworks for generalized quantum particle statistics. While the former arises from an operational reconstruction of indistinguishable quantum particles, the latter is introduced by postulating generalized commutation relations for creation and annihilation operators that are parametrized by solutions of the Yang-Baxter equation. We formalize the definition of the Fock space for transtatistics and demonstrate how it can be described as a quadratic algebra, that is, the quotient space of the tensor algebra of a single particle over commutation relations. Furthermore, making use of the calculus of Gelfand-Tsetlin patterns, general transformations in the Fock space can be described. We implement this as a numerical algorithm in order to facilitate calculations for larger systems. Using this algorithm, we demonstrate that the original notion of locality can lead to probability distributions not behaving independently for disconnected subsystems, something that we consider unphysical. By introducing a stronger notion of locality in transtatistics, this can be mitigated, and solutions to the Yang-Baxter equation can be obtained as representations of the two-mode swap. Conversely, we also prove that $R$-parastatistics satisfies this condition. Finally, we introduce a generalized symmetrization postulate that the Fock space is a quadratic algebra generated by the action of swapping modes. We motivate this axiom operationally by relating it to the notion of indistinguishable particles. With this additional assumption, the equivalence of transtatistics to $R$-parastatistics is shown.

# Kurzfassung

In dieser Masterarbeit wird die Äquivalenz von Transtatistik und $R$-Parastatistik untersucht, zwei Theorien generalisierter Quantenteilchenstatistik. Während erstere einer operationalen Rekonstruktion der Quantenmechanik entstammt, wird letztere durch Vertauschungsrelationen von Erzeugungs- und Vernichtungsoperatoren postuliert, welche durch Lösungen der Yang-Baxter-Gleichung parametrisiert sind. Wir formalisieren den transtatistischen Fockraum und zeigen wie er als quadratische Algebra beschrieben werden kann, nämlich als Quotientenraum der Tensoralgebra des Hilbertraums eines einzelnen Teilchens modulo der Vertauschungsrelationen. Zudem können allgemeine Transformationen im Fockraum mithilfe von Gelfand-Tsetlin-Mustern beschrieben werden. Wir implementieren dies als einen numerischen Algorithmus, um Berechnungen auch für große Systeme durchzuführen. Mit diesem Algorithmus zeigen wir, dass der ursprüngliche Begriff der Lokalität zu Wahrscheinlichkeitsverteilungen führen kann, bei denen sich getrennte Subsysteme nicht unabhängig verhalten, was wir als unphysikalisch erachten. Durch das Einführen eines stärkeren Lokalitätsbegriffs in der Transtatistik kann dies verhindert werden und man erhält Lösungen der Yang-Baxter-Gleichung als Darstellungen des Vertauschens zweier Moden. Andererseits beweisen wir, dass $R$-Parastatistik dieser Bedingung genügt. Schließlich führen wir ein verallgemeinertes Symmetrisierungspostulat ein, nämlich dass der Fockraum eine quadratische Algebra erzeugt durch durch das Vertauschen zweier Moden ist. Wir motivieren dieses Axiom, indem wir es dem Begriff ununterscheidbarer Teilchen gleichstellen. Mit dieser zusätzlichen Annahme wird die Äquivalenz von Transtatistik und $R$-Parastatistik gezeigt.

# Contents

# 1. Introduction

Since the advent of quantum mechanics, it became clear from phenomenological observations that all physical particles fall into just two categories: bosons and fermions. Mathematically, they are described by two analogous but complementary constructions. While the creation and annihilation operators bosons obey commutation relations, the algebra of fermions is governed by anticommutation relations. This difference in sign has far-reaching consequences on the particles' properties, such as the allowed occupation numbers and behavior under particle exchange. Despite the well-established theoretical framework, the initial ad-hoc postulation is sometimes regarded as artificial and not sufficiently well motivated.

Over the years, various attempts have been made at generalizing Fermi-Dirac and Bose-Einstein statistics. Perhaps the most prominent model is Green's parastatistics [1], where he imposes commutation relations that are trilinear in the field operators. Subsequently, this formalism has been refined [2–6] and ultimately deemed incompatible with the symmetries of Minkowski spacetime [7]. Although there have also been other generalizations such as quons [8] or Gentile statistics [9], they did not gain traction as a physical theory. In recent decades, generalized quantum statistics have again shifted into the center of scientific attention in the form of topological generalizations. Anyons, quasiparticles that can only exist in less than three dimensions [10–12], may be able to serve as a platform for fault-tolerant quantum computing [13–17].

Recently, there have been two further independent attempts at formalizing generalized quantum statistics. *Transtatistics*, as proposed by Sánchez and Dakić [18, 19], roots in an operational axiomatization of indistinguishable particles. Starting from a device-independent approach and in line with the program of reconstruction of quantum mechanics [20], they investigate the possible structures of Fock spaces under an assumption of locality using representation theory. They find a necessary and sufficient condition on the Fock space of transtatistics, namely that it must exhibit a generalized Pauli exclusion principle according to a so-called totally positive sequence. Yet, their theory is not constructive in the sense as there are no creation and annihilation operators and that they give no concrete procedure to compute the action of unitaries on states.
On the other hand, Wang and Hazzard [21] introduced the formalism of *R-parastatistics* by generalizing the canonical commutation relations. By taking a solution to the Yang-Baxter equation—a so-called $R$-matrix—they define a concrete algebra of creation and annihilation operators that gives rise to a Fock space with a generalized Pauli exclusion principle. These operators also encompass an extra degree of freedom for particles, which behaves under particle swaps precisely as described by the $R$-matrix. With that, they are able to show that $R$-parastatistics can emerge as quasiparticles coming from specific Hamiltonians using a generalization of the Jordan-Wigner transform. However, the defining commutation relations are postulated ad-hoc and are not derived from an underlying physical principle.

At first glance, these two models seem completely unrelated apart from the fact that they generalize quantum statistics. However, under closer investigation, striking connections appear: For every $R$-matrix, it is possible to define a *Hilbert series*, which is proven to be a totally positive sequence [22]. Furthermore, both theories exhibit a notion of "internal" degrees of freedom of particles. This suggests that both generalizations build on the same underlying mathematical objects, hinting even to a possible equivalence. In this Master's thesis we investigate the precise relationship between transtatistics and $R$-parastatistics, whether one theory is contained in the other and if they are equivalent under some further restrictions.

In order to tackle these questions, we resort to both analytic and computational methods. Primarily, we analyze transtatistics in the light of representation theory and aim to find a constructive description of its Fock space. For this, we heavily rely on well-established literature in the field that we specialize to our concrete case. In order to also make concrete calculations possible for larger systems, we perform them computationally. To this end, we implement them in the computer algebra system `SageMath` [23].

This Master's thesis is composed as follows: In Chapter 2 we begin by introducing the formalism necessary for further discussions. We give a brief summary of the mathematical tools (representation theory, Young diagrams, Schur polynomials, and totally positive sequences) and review the formalism of transtatistics and $R$-parastatistics.
Chapter 3 is concerned with a formal definition of the Fock space of transtatistics. We postulate two complementary constructions and describe a general procedure for computing the action of unitaries in the Fock space. There we also discuss how to implement nonunitarity, a hallmark feature of $R$-parastatistics.
In Chapter 4 we proceed by investigating how the concept of locality manifests itself in both transtatistics and $R$-parastatistics. We introduce a stronger notion of locality and motivate it by showing that transtatistics exhibit unphysical behaviors without it. We also show that it is a necessary condition for the equivalence of both theories.
We conclude with a summary of this work in Chapter 5 and provide a list of open questions that need further treatment in the future.

# 2. Review of the formalism

## 2.1. Representation theory of Lie groups and Lie algebras

The main arguments of this Master's thesis involve the mathematical tools of Lie groups, Lie algebras, and their representations. Therefore, in this section, we briefly introduce the core concepts, where we mainly rely on standard textbooks on these topics, such as [24–26]. When full mathematical generality is not needed, the concepts outlined here are adjusted to the special cases necessary for the reasoning and understanding of the arguments.

A Lie group $G$ is a group $G$ whose elements also form a differentiable manifold. In particular, the group operation also has to be smooth.

A real (complex) Lie algebra $g$ is a vector space $g$ over $\mathbb{R}$ (resp. $\mathbb{C}$) equipped with a map $[\cdot,\cdot]: \; g \times g \to g$ (Lie bracket) such that

$$\begin{aligned}
&[\cdot,\cdot] \text{ is bilinear,}\\
&[x,y] = -[y,x] \text{ for all } x,y \in g,\\
&[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \text{ for all } x,y,z \in g \quad \text{(Jacobi identity)}\ .
\end{aligned} \tag{2.1}$$

For each Lie group, there is an associated Lie algebra, which can be thought of as the tangent space of the manifold at the point of identity. The topology of the manifold, the group structure and the behavior of the Lie bracket are deeply connected, which makes them a powerful tool in mathematics and theoretical physics.

Of particular importance for this work is the matrix Lie group $GL(d)$ as well as its subgroups $U(d)$ and $SU(d)$ with their respective Lie algebras $gl(d)$, $u(d)$ and $su(d)$.
The general linear group $GL(d)$ consists of all the complex invertible $d \times d$ matrices with the group operation being ordinary matrix multiplication. Often it is also customary to write $GL(V)$ where $V$ is a complex vector space of dimension $d$ to refer to the set of invertible linear transformations on $V$, which is isomorphic to $GL(d)$ when we fix a basis of $V$. The Lie algebra $gl(d)$ of $GL(d)$ is the set of all $d \times d$ complex matrices where the Lie bracket is just the matrix commutator. For matrix Lie groups such as $GL(d)$, the relationship between an element $g$ of the Lie group and $x$, an element of Lie algebra takes the following form:

$$g = e^x. \tag{2.2}$$

Basis vectors of $gl(d)$ are called generators $e_{ij}$ and correspond to a matrix with a 1 entry in the $i$-th row and $j$-th column and zeros elsewhere. For the generators of $gl(d)$, we have the defining relation

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj}, \tag{2.3}$$

which also holds for all subsets of $gl(d)$ and in particular also their representations.

The unitary group $U(d)$ is the subgroup of $GL(d)$ that contains all unitary matrices ($U^{-1} = U^{\dagger}$). Its associated Lie algebra $u(d)$ is the set of all skew-hermitian $d \times d$ matrices ($H^{\dagger} = -H$ ).

The special unitary group $SU(d)$ is the subgroup of $U(d)$ of matrices that have a determinant equal to one. Its Lie algebra $su(d)$ is composed of all skew-hermitian matrices $H^{\dagger} = -H$ that further satisfy Tr $H = 0$.

One is often concerned with the way that Lie groups can act on vectors, which is described by representation theory. This notion of "action of a group" is encompassed in the following definition.

**Definition 1.** *A representation $\Delta$ of a Lie group $G$ on a finite-dimensional vector space $V$ is a map $\Delta : G \to GL(V)$ satisfying*

$$\begin{aligned} &\Delta(gh) = \Delta(g)\Delta(h) \ , \\ &\Delta(\mathbb{1}_G) = \mathbb{1}_V \qquad \textit{for all } g, h \in G. \end{aligned} \tag{2.4}$$

*Furthermore, if $V$ is endowed with an inner product, a representation $\Delta : G \to GL(V)$ is called unitary if $\Delta(g^{-1}) = \Delta(g)^{\dagger}$ for all $g$ in $G$.*

If $G$ is a Lie group with a representation $\Delta$ on $V$, then there exists a unique representation $\delta$ of its Lie algebra $g$ on $V$, that is, the linear map $\delta : g \to GL(V)$ such that

$$\delta([x, y]) = [\delta(x), \delta(y)] \ , \tag{2.5}$$

$$\delta(gxg^{-1}) = \Delta(g)\delta(x)\Delta(g)^{-1} \ , \tag{2.6}$$

$$\delta(x) = \frac{d}{dt}\Delta(e^{tx})\Big|_{t=0} \qquad \text{for all } x, y \in g,\ g \in G. \tag{2.7}$$

In particular, the relation

$$\Delta(e^x) = e^{\delta(x)}. \tag{2.8}$$

holds for all $x$ in $g$.

An important object associated with a representation is the *character*.

**Definition 2.** *Let $\Delta : G \to GL(V)$ be a representation of a Lie group $G$. The associated character $\chi_{\Delta} : G \to \mathbb{C}$ is the function*

$$\chi_{\Delta}(g) = \textit{Tr}\ \Delta(g). \tag{2.9}$$

This function plays a crucial role in the classification of representations of Lie groups.

A core problem of representation theory is to identify if a representation $\Delta$ is 'elementary' or if it is a composite of 'smaller' representations. In the following paragraphs, we will introduce the formalism necessary to describe these concepts.

For a Lie group (algebra) representation $\Delta$ or $\delta$ respectively, of $G$ ($g$) on $V$, we call a subspace $W$ of $V$ an invariant subspace of $\Delta$ if $\Delta(g)w \in W$ for all $g \in G$ and $w \in W$.

If $V$ has no other invariant subspaces other than $V$ itself and $\{0\}$, we call $\Delta$ (or $\delta$) an *irreducible representation* or 'irrep' for short.
The definition of an irrep captures the notion of an elementary representation as $\Delta$ acts in a nontrivial manner on all vectors in the space, meaning that there is no direct substructure apparent.

On the other hand, it is possible to construct composite representations from others (possibly irreps), using direct sums and tensor products.
If $\Delta$ and $\Phi$ are representations of $G$ on spaces $V$ and $W$ respectively, then we can define the following representations:

- The *direct sum* representation

$$(\Delta \oplus \Phi) : G \to GL(V \oplus W), \tag{2.10}$$
$$(\Delta \oplus \Phi)(g) = \Delta(g) \oplus \Phi(g). \tag{2.11}$$

- The *tensor product* representation

$$(\Delta \otimes \Phi) : G \to GL(V \otimes W), \tag{2.12}$$
$$(\Delta \otimes \Phi)(g) = \Delta(g) \otimes \Phi(g). \tag{2.13}$$

This works analogously for Lie algebras in the case of a direct sum. However, the tensor product of two Lie algebra representations $\delta$ and $\phi$ is given by

$$(\delta \otimes \phi)(x) = \delta(x) \otimes \mathbb{1}_W + \mathbb{1}_V \otimes \phi(x). \tag{2.14}$$

In general, the tensor product of irreducible representations is no longer irreducible.

Furthermore, two representations $\Delta$ and $\Phi$ of $G$ (or $g$, respectively) on $V$ and $W$ are called *isomorphic*, if there exists an isomorphism $\varphi : V \to W$ such that

$$\varphi(\Delta(g)v) = \Phi(g)\varphi(v) \quad \text{for all } g \in G \text{ and } v \in V. \tag{2.15}$$

The notions of irreducibility and isomorphism interplay between representations $\Delta$, $\Phi$ of Lie groups and $\delta$, $\phi$ of their respective Lie algebras:

- $\Delta$ is irreducible if and only if $\delta$ is irreducible.

- $\Delta$ and $\Phi$ are isomorphic if and only if $\delta$ and $\phi$ are isomorphic.

Perhaps the most important theorem in representation theory is the following:

**Theorem 1.** ***(Decomposition into irreps)*** *Let $\Delta$ be a representation of a compact Lie group $G$ with an associated Lie algebra representation $\delta$ on a finite-dimensional vector space $V$. Then $V$ can be decomposed into invariant sectors $V_\lambda^{(i)}$ as*

$$V = \bigoplus_{\lambda,i} V_\lambda^{(i)}, \tag{2.16}$$

*labeled by an index* $\lambda$ *and each with multiplicity* $1 \leq i \leq m_\lambda \in \mathbb{N}$.
*The representations* $\Delta$ *and* $\delta$ *then decompose as*

$$\Delta = \bigoplus_{\lambda,i} \Delta_\lambda^{(i)}, \qquad \text{and} \quad \delta = \bigoplus_{\lambda,i} \delta_\lambda^{(i)}, \tag{2.17}$$

*where* $\Delta_\lambda^{(i)}$ *and* $\delta_\lambda^{(i)}$ *are each irreducible representations and act only on the according sector* $V_\lambda^{(i)}$.

Often, it is customary to leave out the multiplicity index and to denote subspaces with the same index $\lambda$ as, e.g., $V_\lambda^{\oplus m_\lambda}$. Although the above theorem is in principle only valid for compact groups (such as $U(d)$ or $SU(d)$), it also applies to $GL(d)$, when restricted to "polynomial representations", which is what we restrict ourselves to. Here, we are especially concerned with the irreducible representations of the unitary group. It is well-known that the irreps of $U(d)$ and $SU(d)$ are labeled by certain particular combinatorial objects: Young diagrams (see Definition 3 in Section 2.2). These serve as index $\lambda$ in the decomposition of representations of $U(d)$ according to Theorem 1.

Furthermore, the characters $\chi_\lambda(U)$ of irreps $\Delta_\lambda(U)$ of $U(d)$ are given by the Schur polynomials (Definition 5) $s_\lambda$ evaluated on the eigenvalues of $U$. An immediate implication of Theorem 1 is that the character of any representation of $U(d)$ is the sum of Schur polynomials.

Since $gl(d)$ is the so-called *complexification* of $u(d)$, the corresponding Lie groups share various properties. In particular, also the irreps of $GL(d)$ are indexed by Young diagrams and its characters are Schur polynomials.

## 2.2. Young diagrams and semistandard Young tableaux

(a) Young diagram of shape (2,2,1) (b) Young diagram of shape (3,1)

Figure 2.1.: Examples of Young diagrams

Young diagrams and tableaux are combinatorial objects that are intimately related to the representation theory of $GL(d)$ and its subgroups. The following definitions can be found, e.g., in [26, 27].

**Definition 3.** *A Young diagram (YD)* $\lambda$ *of a sequence of non-increasing nonnegative integers* $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_r$ *is a shape of* $r$ *rows with* $\lambda_i$ *left-adjusted boxes in row* $i$ *reading from top to bottom. The size* $|\lambda|$ *is the total number of boxes. The empty diagram is denoted as* $\bigcirc$.

This is illustrated in Figure 2.1.

**Definition 4.** *A semistandard Young tableau (SSYT) $\Lambda$ of shape $\lambda$ with maximum entry $1 \leq d \leq r$ is the $r$-row Young diagram $\lambda$ that is filled with integers from 1 to $d$ such that:*

1. *entries weakly increase in a row*
2. *entries strongly increase in a column*

*The weight $\mathrm{wt}_i \Lambda$ is the number of times that the entry $i$ appears in $\Lambda$.*
*Furthermore, $SSYT(\lambda)$ denotes the set of all semistandard Young tableaux of shape $\lambda$.*

The semistandard Young tableau to the YD in Figure 2.1b are displayed in Figure 2.2.

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$$

Figure 2.2.: All SSYTs of shape $\lambda = (2, 1)$ with $d = 2$

## 2.3. Schur polynomials

Another tightly related mathematical object are Schur functions. These are special symmetric polynomials that arise as the characters of irreducible representations of Lie groups [24, 28], in particular of $GL(d)$ and its subgroups. We give the following definition from [27]:

**Definition 5.** *Let $\lambda$ be a Young diagram. Then the Schur function $s_\lambda$ in $d$ variables is given as*

$$s_\lambda(x_1, ...x_d) = \sum_{\Lambda \in SSYT(\lambda)} \vec{x}^\Lambda, \tag{2.18}$$

*where the sum ranges over all SSYTs with maximum entry $d$ and $\vec{x}^\Lambda$ is given as*

$$\vec{x}^\Lambda = \prod_{i=1}^{d} x_i^{\mathrm{wt}_i \Lambda}. \tag{2.19}$$

In particular, there is a relation between the Schur function $s_\lambda$ and the number of SSYTs of shape $\lambda$ with maximal entry $d$:

$$|SSYT(\lambda)| = s_\lambda(\underbrace{1, 1, ..., 1}_{d}). \tag{2.20}$$

Schur functions are symmetric functions, meaning that $s_\lambda(\sigma\vec{x}) = s_\lambda(\vec{x})$ for any permutation $\sigma$ of $d$ variables. In addition, they form an orthonormal basis for the space of symmetric polynomials of degree $\leq |\lambda|$: Every symmetric polynomial $p(\vec{x})$ in $d$ variables can be uniquely expanded as a linear combination of Schur polynomials

$$p(\vec{x}) = \sum_{\lambda} \langle p(\vec{x}) | s_\lambda(\vec{x}) \rangle \, s_\lambda(\vec{x}), \tag{2.21}$$

with respect to the *Hall inner product* (see e.g., [27])

$$\left\langle s_\lambda(\vec{x}) \middle| s_\mu(\vec{x}) \right\rangle = \delta_{\lambda,\mu}. \tag{2.22}$$

Being a basis, a product of $s_\lambda(\vec{x})$ and $s_\mu(\vec{x})$ is again a sum of Schur polynomials

$$s_\lambda(\vec{x}) \cdot s_\mu(\vec{x}) = \sum_{\lambda} c^{\nu}_{\lambda,\mu} s_\nu(\vec{x}), \tag{2.23}$$

with the expansion coefficients $c^{\nu}_{\lambda,\mu} \equiv \left\langle s_\lambda(\vec{x}) \cdot s_\mu(\vec{x}) \middle| s_\nu(\vec{x}) \right\rangle$ being called Hall-Littlewood coefficients.

## 2.4. Totally positive sequences

Totally positive sequences (TPS, sometimes denoted as Pólya frequency sequences) arise in various applications, ranging from statistics to combinatorics. We give the following definition [29]:

**Definition 6.** *A sequence $a_n$ for $n \in \mathbb{N}$ with $a_0 = 1$ is called totally positive if the (infinitely large) matrix $A$ constructed by*

$$A_{ij} = a_{i-j}, \qquad a_{n \leq 0} = 0, \tag{2.24}$$

*only has non-negative minors*

$$(A_{ij})_{i \in X, j \in Y} \geq 0 \qquad \textit{for all } X, Y \subset \mathbb{N}. \tag{2.25}$$

It is a well-known fact [29, p. 413] about totally positive sequences that their generating function $a(x)$ can always be written as

$$a(x) = \sum_{i=0}^{\infty} a_i x^i = \frac{\prod_{j=1}^{\infty}(1+\alpha_j x)}{\prod_{j=1}^{\infty}(1-\beta_j x)} e^{\gamma x}, \tag{2.26}$$

with $\alpha_j \geq 0$, $\beta_j \geq 0$,$\sum_i \alpha_i < \infty$, $\sum_i \beta_i < \infty$ and $\gamma \geq 0$. Also, every function $a(x)$ of the form (2.26) gives rise to a TPS. In the following, we will refer to a TPS and its generating function interchangeably.

Totally positive sequences $a_n$ (respectively $a(x)$) exhibit the following properties:

- $a_n = 0 \implies a_{n+1} = 0$,
- $\dfrac{a_n}{a_{n-1}} \geq \dfrac{a_{n+1}}{a_n}$ (log-concavity [29, p. 394]), (2.27)

- If $a(x)$ is a TPS, so is $1/a(-x)$,
- If $a(x),\ b(x)$ are TPS, so is $a(x)b(x)$.

For our purposes, it will be sufficient to consider only integral TPS ($a_n \in \mathbb{N}^0$), which is further specified by Lemma 1 in Ref. [18]. In that case, it follows that $\gamma = 0$ and only a finite number of $\alpha_j, \beta_j$ are nonzero.

Two simple but crucial examples of TPS are

$$\chi_{\mathrm{f}}(x) = 1 + x \qquad \equiv (1, 1, 0, 0, \dots), \tag{2.28}$$

and

$$\chi_{\mathrm{b}}(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \qquad \equiv (1, 1, 1, 1, \dots), \tag{2.29}$$

which correspond to bosons and fermions, respectively, and satisfy $\chi_{\mathrm{f}}(x) = 1/\chi_{\mathrm{b}}(-x)$.

## 2.5. Transtatistics

In the paper "Reconstruction of Quantum Particle Statistics: Bosons, Fermions, and Transtatistics" [18], Sánchez and Dakić investigate quantum particle statistics using methods from representation theory. Fitting within the program of operational reconstruction of quantum mechanics (pioneered by Hardy [20]) from axiomatic principles, they aim to find axioms to derive bosonic and fermionic Fock spaces. Their assumptions lead not only to bosons and fermions, but also to generalizations that they call *transtatistics*. This section provides a brief summary of their framework.

They formalize the following gedankenexperiment: Assume that there is an optical device acting on $d$ modes. A single-particle state is transformed via the unitary matrix $U \in U(d)$ describing the device. This single-particle transformation is independent of the particle's statistics (bosonic or fermionic), as a single particle is not affected by exchange statistics. They then proceed to ask: If several indistinguishable particles of unspecified statistics are inserted into the device, what are all the possible transformations, outcomes, and statistical behaviors?

In mathematical terms, this can be described as asking for all possible Fock spaces that are compatible with the group structure of $U(d)$ and the indistinguishability of particles: Let $\mathcal{F}_1$ denote the Fock space of an arbitrary number of particles in just a single mode and $\mathcal{F}_d$ the Fock space in $d$ modes. Furthermore, let $U \in U(d)$ be the unitary matrix that describes a physical transformation of one particle in $d$ modes. Under the assumption that transformations of many particles in $d$ modes can be extrapolated from the single-particle behavior, the group representation

$$\Delta_d : U(d) \to GL(\mathcal{F}_d), \tag{2.30}$$

of an arbitrary number of particles is introduced as the central object of the construction. In the model, the particle numbers $n_i$ in mode $i$ are the only measurable quantities, which are incorporated by $d$ detectors that can measure the state after a transformation. The probability of observing particle numbers $n_1, \ldots, n_d$ is given by Born's rule

$$p_{n_1,\ldots,n_d} = |\langle n_1, \ldots, n_d|\psi\rangle|^2 \,, \tag{2.31}$$

where the number states $|n_1, \ldots, n_d\rangle$—together with other degeneracies, as shown later in the article—span the Fock space $\mathcal{F}_d$, which is assumed to be the $d$-fold tensor product

$$\mathcal{F}_d = \mathcal{F}_1^{\otimes d}. \tag{2.32}$$

A crucial assumption is the so-called "locality assumption": If the transformation device consists of $d$ independent phase shifters with parameter $\theta_i$, then the representation $\Delta_d$ thereof acts locally in each subspace. It is formalized as

$$\Delta_d(U) = \Delta_1(e^{i\theta_1}) \otimes \Delta_1(e^{i\theta_2}) \otimes \ldots \otimes \Delta_1(e^{i\theta_d}), \tag{2.33}$$

for $U = \mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_d})$. This directly leads to the condition

$$\chi_d(U) = \prod_{i=1}^{d} \chi_1(x_i), \tag{2.34}$$

on the character $\chi_d \equiv \chi_{\Delta_d}$ and $x_i$ being the $i$-th eigenvalue of $U$. By introducing a generalized number operator $\tilde{N}$ such that $\Delta_1(e^{i\theta}) = e^{i\theta\tilde{N}}$ as well as

$$\tilde{N}_k = \mathbb{1}^{(k-1)} \otimes \tilde{N} \otimes \mathbb{1}^{d-k}, \tag{2.35}$$

for $1 \leq k \leq d$ in $d$ modes with $\tilde{N} = \sum_{n=0} f_n \,|n\rangle\,\langle n|$, the action of a diagonal matrix becomes

$$\Delta_d(U) = e^{i\theta_1\tilde{N}_1 + i\theta_1\tilde{N}_2 + \ldots + i\theta_d\tilde{N}_d}, \tag{2.36}$$

for $U = \mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_d})$, with $f_n \in \mathbb{N}$ denoting the possible occupation numbers. In accordance with Theorem 1, the full Fock space can be written as

$$\mathcal{F}_d = \bigoplus_{\lambda} V_\lambda^{\oplus m_\lambda}, \tag{2.37}$$

namely as a decomposition into irreps of $U(d)$. The task is now to find criteria for the occurring Young diagrams $\lambda$ and their respective multiplicities $m_\lambda$ such that equation (2.34) holds, i.e., the locality assumption is satisfied. Since the characters of the irreducible representations of $U(d)$ are given as Schur polynomials (see Definition 5), this amounts to finding the one-mode character $\chi_1(x)$ such that

$$\prod_{i=1}^{d} \chi_1(x_i) = \sum_{\lambda} m_\lambda s_\lambda(\vec{x}). \tag{2.38}$$

The solution to this combinatorial problem was obtained by applying a proof from [30] and states that $\chi_1$ has to be a totally positive sequence. This is formalized in the following theorem on the "classification of transtatistics":

**Theorem 2.** ***(Classification of transtatistics)***
*For $\chi_1(x) = \sum_{s=0}^{\infty} a_s x^s$ with $a_0 > 0$, the function $\chi_d(\vec{x}) = \prod_{i=1}^{d} \chi_1(x_i)$ is a valid character of $U(d)$ for all $d$ if and only if the single mode character is of the form*

$$\chi_1(x) = \frac{Q_-(x)}{Q_+(x)}, \tag{2.39}$$

*with $Q_\pm(x)$ being integral polynomials with only positive (negative) real roots. Furthermore, $Q_+(0) = 1$.*

When additionally only cases are considered where the one-mode Fock space is not a tensor product ($\mathcal{F}_1 \neq \mathcal{F}_1^A \otimes \mathcal{F}_1^B$), i.e. when $\chi_1$ cannot be factored in two $U(1)$ characters, then $\chi_1$ can only be

$$\chi_1(x) = \frac{1}{Q_+(x)}, \qquad \text{or } \chi_1(x) = Q_-(x), \tag{2.40}$$

with $Q_\pm(x)$ being irreducible over the $\mathbb{Z}[x]$. These two different cases are called "transbosons", respectively "transfermions". This enables a classification of transtatistics, which are labeled by a finite totally positive sequence $[d_i]_\pm$, with

$$Q_\pm(x) = \sum_{i=0}^{\deg\, Q_\pm} (\mp)^i d_i x^i \equiv [d_0, d_1, ...]_\pm. \tag{2.41}$$

Additionally, deg $Q_\pm$ is referred to as the order of statistics and the integer $p$ is called the maximum occupation number. For transfermions it is $p = \deg\, Q_-$, while it is set as $p = \infty$ for transbosons. $p$ refers to the maximum eigenvalue of the number operator and thus also describes a generalized Pauli exclusion principle.

One of the main goals of the article is to find suitable operational axioms for the Fock space such that only ordinary bosons and fermions remain as valid particle statistics. Such a condition is formalized as followed and dubbed "irreducibility condition": *All symmetries of the system of indistinguishable particles are determined by the $U(d)$ group.* In turn, this implies that all the multiplicities $m_\lambda$ in the decomposition (2.37) are at most one, which means that there are additional symmetries present.
Moreover, if one considers transtatistics of linear order ($[1, \alpha]_+$ and $[1, \beta]_-$ with $\alpha, \beta \in \mathbb{N}^*$), then they show that their respective Fock space $\mathcal{F}_d$ is isomorphic to

$$\mathcal{F}_d \simeq \bigoplus V_\pm^{(N)} \otimes \mathcal{H}_A^{\otimes N}, \tag{2.42}$$

where $V_\pm^{(N)}$ denotes the bosonic (resp. fermionic) irreps, which are described by a single horizontal (resp. vertical) strip of size $d$, and the space $\mathcal{H}_A$ subsumes internal degrees of freedom of the particles. However, this factorization is not possible for higher-order statistics, as necessarily neither bosonic nor fermionic irreps will appear in the decomposition.

Sánchez and Dakić proceed in establishing a connection to thermodynamics: The grand canonical partition sum of $d$ modes for per-mode energies $\epsilon_i$, Boltzmann factor $\beta = 1/k_B T$ and chemical potential $\mu$ is given as

$$\mathcal{Z}_d = \prod_{i=1}^{d} \chi_1\left(e^{-\beta(\epsilon_i - \mu)}\right). \tag{2.43}$$

Finally, the article considers the case where negative occupation numbers are also allowed in the particle number measurement. This generalization could enable the extension of the formalism to also include antiparticles in the framework of transtatistics. In this case, a valid character of a single mode $\tilde{\chi}_1(x)$ can be one of the following classes:

1. transtatistical type: $\tilde{\chi}_1(x) = x^k \chi_1(x)$ or $\tilde{\chi}_1(x) = x^k \chi_1(1/x)$ for $k \in \mathbb{Z}$ and $\chi_1$ being a valid transtatistical character by Theorem 2.
2. exceptional type: $\tilde{\chi}_1(x) = a \sum_{s\in\mathbb{Z}} \rho^s x^s$ for $a, \rho \in \mathbb{N}$.

Nevertheless, the formalism of transtatistics, as presented here, is not constructive. Most notably, the authors do not give a concrete method to implement the representation of $U(d)$ on the Fock space $\mathcal{F}_d$, only the frequency of the occurring irreps are specified. Moreover, it is not apparent how the generalization of ordinary creation and annihilation looks like for transtatistics, or if it is possible to define them consistently in the first place.

## 2.6. R-parastatistics

Wang and Hazzard [21] introduced a concrete algebraic approach to quantum particle statistics that generalizes bosons and fermions, called $R$-parastatistics. In this section, we briefly review the basic formalism. Central to their construction is a so-called R-matrix, that is a complex rank four tensorial object $R^{ab}_{cd}$ of dimension $m \times m \times m \times m$, which is a solution to the constant Yang-Baxter equation [31, 32]

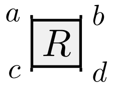
$$(R \otimes \mathbb{1})(\mathbb{1} \otimes R)(R \otimes \mathbb{1}) = (\mathbb{1} \otimes R)(R \otimes \mathbb{1})(\mathbb{1} \otimes R), \tag{2.44}$$

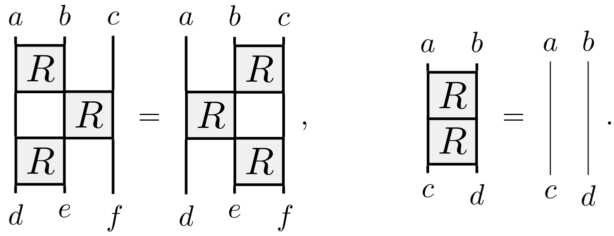
in $GL(\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m)$, or equivalently, in index notation

$$\sum_{hij} R^{ab}_{hi} R^{ic}_{jf} R^{hj}_{de} = \sum_{hij} R^{bc}_{hi} R^{ah}_{dj} R^{ji}_{ef}. \tag{2.45}$$

Furthermore, they demand that the R-matrix is involutive, which means that

$$\sum_{cd} R^{ab}_{cd} R^{cd}_{ef} = \delta_{ae}\delta_{bf}, \tag{2.46}$$

or $R^2 = \mathbb{1}$ for short. Equations (2.45) and (2.46) can also be written in tensor network notation, with $R^{ab}_{cd} \equiv$  , as follows:



(2.47)

They proceed by introducing creation operators $\hat{\psi}^{+}_{i,a}$ and annihilation operators $\hat{\psi}^{-}_{i,a}$ that depend not only on the external *mode* $i$, but also on an internal *flavor* $a$ ranging from $1 \leq a \leq m$. These creation and annihilation operators are then subject to the commutation relations

$$\hat{\psi}^{-}_{i,a}\hat{\psi}^{+}_{j,b} = \sum_{cd} R^{ac}_{bd}\hat{\psi}^{+}_{j,c}\hat{\psi}^{-}_{i,d} + \delta_{ab}\delta_{ij}, \tag{2.48}$$

$$\hat{\psi}^{+}_{i,a}\hat{\psi}^{+}_{j,b} = \sum_{cd} R^{cd}_{ab}\hat{\psi}^{+}_{j,c}\hat{\psi}^{+}_{i,d}, \tag{2.49}$$

$$\hat{\psi}^{-}_{i,a}\hat{\psi}^{-}_{j,b} = \sum_{cd} R^{ba}_{dc}\hat{\psi}^{-}_{j,c}\hat{\psi}^{-}_{i,d}, \tag{2.50}$$

which generalize the canonical commutation relations of bosons and fermions. Bosons and fermions can be recovered in this formalism by setting $R = +1$ or $R = -1$ ($m = 1$), respectively.

Analogously to ordinary bosons and fermions, one can define a Fock space for $R$-parastatistics. The starting point is a (normalized) vacuum state $|0\rangle$ which is sent to zero by all annihilation operators ($\hat{\psi}^{-}_{i,a}|0\rangle = 0$). The full Fock space $\mathcal{F}_d$ of $d$ modes is then the vector space spanned by

$$\mathcal{F}_d = \operatorname{span}\left\{\hat{\psi}^{+}_{i_1,a_1}\hat{\psi}^{+}_{i_2,a_2}\ldots\hat{\psi}^{+}_{i_k,a_k}|0\rangle\right\}. \tag{2.51}$$

However, note that the above expression is overcomplete and not a basis as the commutation relation (2.49) introduces linear dependencies between the $\hat{\psi}^{+}_{i,a}$. A proper basis of $\mathcal{F}_d$ is given by vectors of the form

$$\left|\begin{matrix} n_1 \\ \alpha_1 \end{matrix}, \begin{matrix} n_2 \\ \alpha_2 \end{matrix}, \ldots, \begin{matrix} n_d \\ \alpha_d \end{matrix}\right\rangle = \hat{\Psi}^{(1)+}_{n_1,\alpha_1}\hat{\Psi}^{(2)+}_{n_2,\alpha_2}\ldots\hat{\Psi}^{(d)+}_{n_d,\alpha_d}|0\rangle, \tag{2.52}$$

where the independent $n_i > 0$ indicate the number of particles in mode $i$ together with a degeneracy index $1 \leq \alpha_i \leq d_{n_i}$ that fully specifies the state, where the integers $d_n$ denote the degeneracy of the subspace. The auxiliary operators $\hat{\Psi}^{(i)+}_{n,\alpha}$ can be expanded into creation operators as

$$\hat{\Psi}^{(i)+}_{n,\alpha} = \frac{1}{\sqrt{n!}} \sum_{a_1 a_2 \ldots a_n} \Psi^{\alpha}_{a_1 a_2 \ldots a_n}\hat{\psi}^{+}_{i,a_1}\hat{\psi}^{+}_{i,a_2}\ldots\hat{\psi}^{+}_{i,a_n}, \tag{2.53}$$

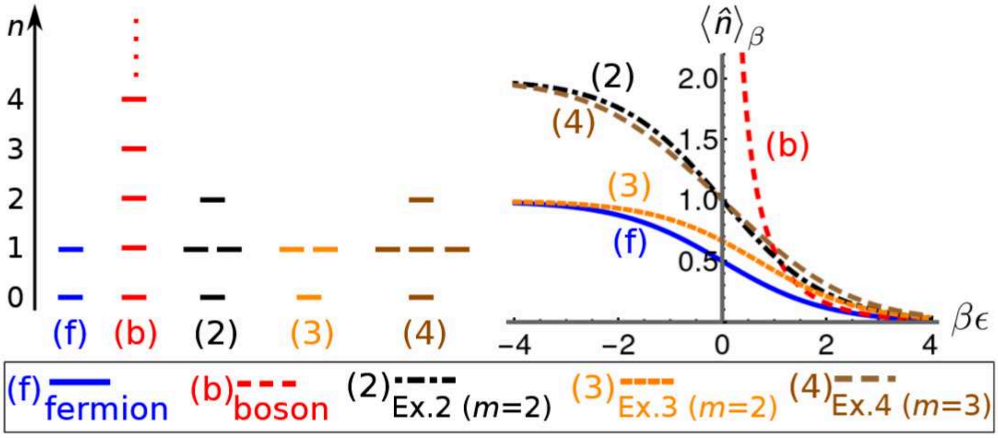



Figure 2.3.: Generalized exclusion principle (left) and thermal average particle number (right) for bosons, fermions and different examples of $R$-parastatistics. Figure taken from Ref. [21] and licensed under the Creative Commons BY license (`https://creativecommons.org/licenses/by/4.0/`).

with the expansion coefficients being symmetric functions under $R$ in $n$ arguments. That is, the set $\left\{\Psi^{\alpha}_{a_1 a_2 \dots a_n}\right\}_{\alpha=1}^{d_n}$ are the $d_n$ linearly independent solutions to the system of equations

$$\Psi_{a_1 \dots a_j a_{j+1} \dots a_n} = \sum_{b_j b_{j+1}} R^{a_j a_{j+1}}_{b_j b_{j+1}} \Psi_{a_1 \dots b_j b_{j+1} \dots a_n}, \tag{2.54}$$

for $1 \leq j \leq n-1$. The dimensions $d_n$ are coefficients of the so-called *Hilbert series* $h_R(x)$ of the $R$-matrix [22, 33]. The coefficients $d_n$ also specify the generalized exclusion statistics of $R$-parastatistics: In every mode, the state with particle number $n$ is $d_n$-fold degenerate. This also leads to observably different thermodynamics compared to ordinary bosons and fermions. This is indicated in figure Figure 2.3.

Aside from a construction of the Fock space of $R$-parastatistics, the authors describe how to implement a Lie algebra therein, in order to enable physical transformations of $R$-paraparticles. They introduce the bilinear contraction of creation and annihilation operators

$$\hat{e}_{ij} = \sum_{a=1}^{m} \hat{\psi}^{+}_{i,a} \hat{\psi}^{-}_{j,a}, \tag{2.55}$$

as the representations of the generators and proceed in showing that these satisfy the defining relation (2.3):

$$\left[\hat{e}_{ij}, \hat{e}_{kl}\right] = \delta_{jk}\hat{e}_{il} - \delta_{il}\hat{e}_{kj}, \tag{2.56}$$

of the Lie algebra $gl(d)$. Furthermore, these operators satisfy the following relations:

$$[\hat{e}_{ij}, \hat{\psi}^{+}_{k,b}] = \delta_{jk}\hat{\psi}^{+}_{i,b}, \qquad [\hat{e}_{ij}, \hat{\psi}^{-}_{k,b}] = -\delta_{ik}\hat{\psi}^{-}_{j,b}. \tag{2.57}$$

An important subset of operators are the number operators defined as

$$\hat{n}_i = \hat{e}_{ii}, \tag{2.58}$$

which obey

$$\hat{n}_i \left|\begin{smallmatrix} n_1 & n_2 & & n_d \\ \alpha_1, & \alpha_2, & \cdots, & \alpha_d \end{smallmatrix}\right\rangle = n_i \left|\begin{smallmatrix} n_1 & n_2 & & n_d \\ \alpha_1, & \alpha_2, & \cdots, & \alpha_d \end{smallmatrix}\right\rangle . \tag{2.59}$$

A particular feature of the formalism of $R$-parastatistics is that it does not make the overall assumption of unitarity and Hermiticity of operators or orthogonality of states, since an $R$-matrix can in general also be non-unitary. For example, only in the case of an unitary $R$-matrix is the basis (2.52) orthonormal or it holds that

$$\left(\hat{\psi}^{\pm}_{i,a}\right)^{\dagger} = \psi^{\mp}_{i,a}. \tag{2.60}$$

In the general, non-unitary case, this is not true. Nevertheless, it is possible to define a scalar product—by utilizing the root vectors of the Lie algebra $gl(d)$—that obeys

$$\hat{e}^{\dagger}_{ij} = \hat{e}_{ji}, \tag{2.61}$$

which in turn makes all observables constructed from combinations of $\hat{e}_{ij}$ Hermitian in the usual sense.

Wang and Hazzard also give a generalization of the Jordan-Wigner transform (JWT) [34] for parastatistics, which they utilize to map a system of free $R$-parabosons and $R$-parafermions to a string of local, non-interacting systems. With that, it is possible to construct quadratic Hamiltonians out of the operators $\hat{\psi}^{\pm}_{i,a}$, which are diagonalizable by mapping them to free $R$-paraparticles. To this end, they introduce the operators $\hat{x}^{\pm}_{i,a}$ and $\hat{y}^{\pm}_{i,a}$, which correspond to the single-site $\sigma^{\pm}$ matrices in the case of ordinary Fermions. As opposed to $\hat{\psi}^{\pm}_{i,a}$, the operators $\hat{x}^{\pm}_{i,a}$, $\hat{y}^{\pm}_{i,a}$ are local, i.e., for $i \neq j$ it holds that

$$\left[\hat{x}^{\pm}_{i,a}, \hat{x}^{\pm}_{j,b}\right] = \left[\hat{y}^{\pm}_{i,a}, \hat{y}^{\pm}_{j,b}\right] = \left[\hat{x}^{\pm}_{i,a}, \hat{y}^{\pm}_{j,b}\right] = 0. \tag{2.62}$$

On the same site, they obey the following commutation relations:

$$\begin{aligned} \hat{y}^{-}_{a}\hat{y}^{+}_{b} &= \sum_{c,d} R^{ac}_{bd}\hat{y}^{+}_{c}\hat{y}^{-}_{d} + \delta_{ab}, & \hat{x}^{-}_{a}\hat{x}^{+}_{b} &= \sum_{c,d} R^{ca}_{db}\hat{x}^{+}_{c}\hat{x}^{-}_{d} + \delta_{ab}, \\ \hat{y}^{-}_{a}\hat{y}^{-}_{b} &= \sum_{c,d} R^{ba}_{dc}\hat{y}^{+}_{c}\hat{y}^{-}_{d}, & \hat{x}^{-}_{a}\hat{x}^{-}_{b} &= \sum_{c,d} R^{ab}_{cd}\hat{x}^{+}_{c}\hat{x}^{-}_{d}, \\ \hat{y}^{+}_{a}\hat{y}^{+}_{b} &= \sum_{c,d} R^{cd}_{ab}\hat{y}^{+}_{c}\hat{y}^{+}_{d}, & \hat{x}^{+}_{a}\hat{x}^{+}_{b} &= \sum_{c,d} R^{dc}_{ba}\hat{x}^{+}_{c}\hat{x}^{+}_{d}. \end{aligned} \tag{2.63}$$

It should be noted that—on the same site—the operators $\hat{y}^{\pm}_{i,a}$ precisely mimic the commutation relations (2.48)-(2.50) of $\hat{\psi}^{\pm}_{i,a}$, while the $\hat{x}^{\pm}_{i,a}$ obey slightly altered ones. With these local operators, it is possible to formalize a generalized JWT by introducing the operators

$$\hat{T}^{\pm}_{ab} = \mp \left[\hat{y}^{\pm}_{a}, \hat{x}^{\mp}_{b}\right]. \tag{2.64}$$

The (nonlocal) creation and annihilation operators $\hat{\psi}^{\pm}_{i,a}$ can then be expressed as a combination of the local $\hat{y}^{\pm}_{a}$ and the aforementioned $\hat{T}^{\pm}_{ab}$, thus mapping a system of free $R$-paraparticles into a spin chain. It is even possible to consistently define exactly solvable models with $R$-paraparticles as excitations in two dimensions by using square lattices and extending the Jordan-Wigner transform accordingly.

# 3. On the Fock space and its transformations

In this chapter, we will analyze the Fock space of transtatistics and attempt a formal construction thereof. We will start by introducing the number basis as a basis that naturally respects the notion of particle numbers. Then we will shift our focus to the description of $U(d)$ transformations in the space, for which we will resort to a decomposition of the space into irreducible representations. We then describe an alternative construction in terms of quotients of a tensor algebra and conclude with a note about orthogonality.

Analogously to transtatistics, we begin our construction with the assumption of locality (4.1). As demonstrated in [18], this leads to the single-mode character $\chi_1$ being a TPS. We take this sequence as a starting point in our construction: Let $d_n$ be a finite (with $p$ being the index of the highest nonzero coefficient) or infinite ($p = \infty$) totally positive sequence of integers. We then define the single-mode Fock space as

$$\mathcal{F}_1 := \bigoplus_{n=0}^{p} \mathcal{F}_1^{(n)} = \bigoplus_{n=0}^{p} \mathbb{C}^{d_n}. \tag{3.1}$$

This decomposition makes $\mathcal{F}_1$ a graded vector space where each sector $\mathcal{F}_1^{(n)}$ corresponds to a subspace containing $n$ particles. In this context, the series $h_{\mathcal{F}}(x) = \sum_i d_i x^i$ is referred to as the *Hilbert series* [33] of the Fock space, and per definition it is equal to the single-mode character $\chi_1(x)$. In addition, we introduce a basis of $\mathcal{F}_1$ that is naturally consistent with the graded structure as

$$\mathcal{F}_1 = \operatorname{span}\left\{ \left|{n \atop \alpha}\right\rangle \mid n = 0, \dots, p,\ \alpha = 1, \dots, d_n \right\}, \tag{3.2}$$

where the integer $n$ that ranges from 0 to the maximum occupation number $p$ specifies the particle number and the auxiliary degeneracy index $\alpha$ labels the $n$-particle subspace with dimension $d_n$. That enables us to encompass internal degrees of freedom of arbitrary nature—such as the ones appearing in [21]—in our construction. If $n = 1$, we adopt the convention to label the degeneracy of the subspace with a Latin index $(a, b, \dots)$ instead of a Greek one, and we denote $d_1 \equiv m$. Since we demand $d_0 = 1$, this directly leads to a unique vacuum state $|0\rangle := \left|{0 \atop 1}\right\rangle$.

One direct implication of $h_{\mathcal{F}}(x)$ being a TPS is the maximal growth rate of the dimension of the $n$-particle sector of the Fock space. Using induction and $d_0 = 1$ it is easy to show that from the log-concavity property (2.27), it follows that

$$d_n \leq m^n. \tag{3.3}$$

This fact can be regarded as a manifestation of the principle that $n$ indistinguishable particles have at most as many degrees of freedom as $n$ distinguishable ones, which also appears in Section 3.3.

The Fock space of $d$ modes is then defined as the tensor product

$$\mathcal{F}_d := \mathcal{F}_1^{\otimes d}, \tag{3.4}$$

that admits the natural extension of the single-mode basis (3.2), which we will denote as the *number basis*

$$\left|\begin{matrix} n_1 & n_2 & \dots & n_d \\ \alpha_1 & \alpha_2 & \dots & \alpha_d \end{matrix}\right\rangle \equiv |\vec{n};\vec{\alpha}\rangle\,, \tag{3.5}$$

Note that the number basis is, in general, not required to be orthonormal. However, we impose the condition

$$\left\langle \begin{matrix} n \\ \alpha \end{matrix} \middle| \begin{matrix} m \\ \beta \end{matrix} \right\rangle = 0 \quad \text{for } n \neq m, \tag{3.6}$$

in $\mathcal{F}_1$, which therefore extends to

$$\left\langle \vec{n};\vec{\alpha} \middle| \vec{m};\vec{\beta} \right\rangle = 0 \quad \text{for } \vec{n} \neq \vec{m}, \tag{3.7}$$

for vectors in $\mathcal{F}_d$. We continue by introducing the number operators $n_i$ for $i = 1, ..., d$ that correspond to measuring the particle number in mode $i$ as

$$\hat{n}_i\, |\vec{n};\vec{\alpha}\rangle = n_i\, |\vec{n};\vec{\alpha}\rangle\,. \tag{3.8}$$

Since these operators are simultaneously diagonal in the number basis, they are mutually commutative (i.e., $[\hat{n}_i, \hat{n}_j] = 0$).

One defining feature of the Fock space $\mathcal{F}_d$ is the possibility of applying unitary transformations of the modes to the states. Mathematically, this is reflected in the existence of a representation

$$\Delta_d : U(d) \to GL(\mathcal{F}_d), \tag{3.9}$$

acting on $\mathcal{F}_d$. By Theorem 1, this implies that the Fock space can be decomposed into $U(d)$ irreps as

$$\mathcal{F}_d = \bigoplus_\lambda V_\lambda^{\oplus m_\lambda}, \tag{3.10}$$

where $\lambda$ ranges over Young Diagrams and $m_\lambda$ denotes the multiplicity of the irrep $V_\lambda$. In particular, the $m_\lambda$ are the expansion coefficients of $\chi_d$ into Schur polynomials $s_\lambda$, as described in equation (2.38).

Since by (2.20) the dimension of $\dim V_\lambda = s_\lambda(1, \dots, 1)$ is equal to $s_\lambda(1, ...1)$ and thus the number of semistandard Young tableaux with maximum entry $d$, the decomposition (3.10) admits a natural basis: We say that each $V_\lambda$ is spanned by a basis labeled by SSYTs $\Lambda$, then with the addition of a multiplicity index $\kappa$, $\mathcal{F}_d$ has the basis

$$\mathcal{F}_d = \operatorname{span}\left\{|\Lambda, \kappa\rangle \;\middle|\; \Lambda \in \mathrm{SSYT}(\lambda),\ \kappa = 1, ..., m_\lambda \ \text{ for } \ \lambda \text{ appearing in } \chi_d\right\}. \tag{3.11}$$

We call this basis the “Young basis”. Since it is compatible with the irreducible representations of $\Delta_d$, it is a natural choice to study the action of the representation in the Fock space. Furthermore, we impose orthonormality on the Young basis, meaning that

$$\langle \Lambda, \kappa | \Lambda', \kappa' \rangle = \delta_{\Lambda\Lambda'}\ \delta_{\kappa\kappa'}. \tag{3.12}$$

## 3.1. Action of the Lie algebra

In order to formalize the concrete action of unitaries on the irreps, we resort to the formalism of 'Gelfand-Tsetlin patterns' (GTPs), which were introduced by Gelfand and Tsetlin [35] for this very reason. These combinatorial objects are defined as (see also [36–38]):

**Definition 7.** *A Gelfand-Tsetlin pattern of size $d$ is a triangular pattern of nonnegative integers $r_{k,j}$ of the form*

$$\begin{pmatrix} r_{1,d} & & r_{2,d} & & r_{3,d} & & \cdots & & r_{d,d} \\ & r_{1,d-1} & & r_{2,d-1} & & \cdots & & r_{d-1,d-1} & \\ & & r_{1,d-2} & & \cdots & & r_{d-2,d-2} & & \\ & & & \ddots & & \ddots & & & \\ & & & & r_{1,1} & & & & \end{pmatrix}, \tag{3.13}$$

*with the condition that*

$$r_{k,j} \geq r_{k,j-1} \geq r_{k+1,j}, \quad \textit{for } 1 \leq k < j \leq d. \tag{3.14}$$

Remarkably, not only GTPs are a suitable tool for calculating transformations in irreps, but they also have a one-to-one relationship with SSYTs [26]. In particular, the GTPs with top row $r_{j,d}$ correspond to SSYTs with the maximum entry $d$ and shape $\lambda = (r_{1,1}, \dots, r_{1,j}, \dots, r_{1,d})$. The algorithm for the conversion is outlined in Appendix A. Due to this bijection, we will drop the distinction between them and regard an SSYT $\Lambda$ and its associate GTP as the same object.

We proceed by specifying not the direct action on states $|\Lambda, \kappa\rangle$ of representations $\Delta_d$, but of their associate Lie algebra representation $\delta_d : gl(d) \to gl(\mathcal{F}_d)$. Let us—in accordance with Ref. [21]—denote the representations of generators $e_{ij}$ as

$$\hat{e}_{ij} \equiv \delta_d(e_{ij}). \tag{3.15}$$

Gelfand and Tsetlin give explicit formulae for the action of diagonal and first off-diagonal generators [35]. They are specified in the Young basis where the combinatorial structure of SSYTS—or GTPs, respectively—encapsulates this action directly:

$$\hat{e}_{kk} |\Lambda, \kappa\rangle = \left( \sum_{i=1}^{k} r_{i,k} - \sum_{i=1}^{k-1} r_{i,k-1} \right) |\Lambda, \kappa\rangle \,, \tag{3.16}$$

$$\hat{e}_{k-1,k} |\Lambda, \kappa\rangle = \sum_{j=1}^{k-1} \sqrt{(-1)^{k-1} \frac{\prod_{i=1}^{k}(l_{i,k} - l_{j,k-1}) \prod_{i=1}^{k-2}(l_{i,k-2} - l_{j,k-1} - 1)}{\prod_{i\neq j}^{k-1}(l_{i,k-1} - l_{j,k-1})(l_{i,k-1} - l_{j,k-1} - 1)}} \left|\Lambda + \delta_{j,k-1}, \kappa\right\rangle , \tag{3.17}$$

$$\hat{e}_{k,k-1}\left|\Lambda,\kappa\right\rangle = \sum_{j=1}^{k-1}\sqrt{(-1)^{k-1}\frac{\prod_{i=1}^{k}(l_{i,k}-l_{j,k-1}+1)\prod_{i=1}^{k-2}(l_{i,k-2}-l_{j,k-1})}{\prod_{i\neq j}^{k-1}(l_{i,k-1}-l_{j,k-1}+1)(l_{i,k-1}-l_{j,k-1})}}\left|\Lambda-\delta_{j,k-1},\kappa\right\rangle . \tag{3.18}$$

Here $\Lambda \pm \delta_{j,k-1}$ is obtained by increasing (decreasing) the entry $r_{j,k-1}$ in the GTP $\Lambda$ by 1. If the result is not a valid GTP, then $\left|\Lambda \pm \delta_{i,k-1},\kappa\right\rangle$ is understood to be equal to the zero vector. The variable $l_{i,k} := r_{i,k} - k$ for GTP entries $r_{k,j}$ of $\Lambda$ is introduced for compactness of notation.
It can be derived straightforwardly from (2.3) that all other generators $\hat{e}_{i,j}$ with $|i-j| \geq 2$ can be expressed recursively using nested commutators which at some point arrive at $\hat{e}_{k,k\pm 1}$:

$$\begin{aligned}\hat{e}_{ij} &= [\hat{e}_{i,i+1},\hat{e}_{i+1,j}] \quad \text{for } i<j,\\ \hat{e}_{ij} &= [\hat{e}_{i,i-1},\hat{e}_{i-1,j}] \quad \text{for } j<i.\end{aligned} \tag{3.19}$$

The relations (3.16)-(3.18) for the diagonal and first off-diagonals as well as (3.19) for all the other $\hat{e}_{ij}$ give rise to a concrete action of the Lie algebra representation $\delta_d$ on $\mathcal{F}_d$.

In this section, we will now investigate the action of $\hat{e}_{ij}$ further. As all others can be expressed in them, we will focus on the diagonal and first off-diagonal ones. Firstly, consider the $\hat{e}_{kk}$: From equation (3.16) it directly follows that $|\Lambda,\kappa\rangle$ are a simultaneous eigenbasis of $\hat{e}_{kk}$. The corresponding eigenvalue also has a simple interpretation. It can be shown (see e.g. [27]) that the eigenvalue can be written as

$$\hat{e}_{kk}\left|\Lambda,\kappa\right\rangle = \mathrm{wt}_k\,\Lambda\left|\Lambda,\kappa\right\rangle , \tag{3.20}$$

that is, as the number of times the number $k$ appears in the SSYT $\Lambda$. It is also customary to define the weight of a GTP as the weight of the corresponding SSYT, further weakening the distinction between these objects. The action of the off-diagonal elements changes the GTPs and therefore the according SSYTs. The application of an element $\hat{e}_{k,k+1}$ or $\hat{e}_{k,k-1}$ on a GTP $\Lambda$ gives according to (3.17) and (3.18) a linear combination of GTPs $\Lambda \pm \delta_{j,k\pm 1}$, which, in turn, changes the weight.

In general, we have, for the Lie algebra generators $\hat{e}_{ij}$

$$\hat{e}_{ij}\left|\Lambda,\kappa\right\rangle = \sum_{\Lambda'\in\{\Lambda_{j\to i}\}} c_{\Lambda'}\left|\Lambda',\kappa\right\rangle , \tag{3.21}$$

with coefficients $c_{\Lambda'}$ given by (3.16)-(3.19) and where the set $\{\Lambda_{j\to i}\}$ denotes all valid SSYTs where one entry $j$ has been changed to $i$ [25, 27]. If $\Lambda$ has weight $\mathrm{wt}\,\Lambda$, then the weight of the states $\Lambda'$ obtained by applying $\hat{e}_{ij}$ is

$$\mathrm{wt}_k\,\Lambda' = \begin{cases}\mathrm{wt}_k\,\Lambda + 1, & k=i\\ \mathrm{wt}_k\,\Lambda - 1, & k=j\\ \mathrm{wt}_k\,\Lambda, & \text{otherwise or if } i=j.\end{cases} \tag{3.22}$$

The action of the diagonal elements $\hat{e}_{kk}$ in (3.20) can be regarded as a particular case of the above equation where we set $\Lambda_{i\to i} := \{\Lambda\}$ and $c_\Lambda = \mathrm{wt}_k\,\Lambda$.

The concrete expression for the Lie algebra elements $\hat{e}_{ij}$ (3.16)-(3.19) give rise to the irreducible representation of $h \in gl(d)$ indexed by YD $\lambda$ just by linear extension as

$$\delta_\lambda(h) = \sum_{i,j=1}^{d} h_{ij}\hat{e}_{ij}. \tag{3.23}$$

By restricting the $d \times d$ matrix $h$ to a skew-hermitian or skew-hermitian traceless matrix, these are also the irreducible representations of the Lie groups $u(d)$ and $su(d)$. The representation $\Delta_d$ of the Lie group $U(d)$ can therefore be obtained by exponentiation:

$$\Delta_d(e^{iH}) = e^{i\delta(H)} = e^{i\bigoplus_\lambda \delta_\lambda(H)^{\oplus m_\lambda}}. \tag{3.24}$$

Here we have adopted the convention to write elements $h$ of $gl(d)$ as $iH$ with $H$ being a $d \times d$ Hermitian matrix. Furthermore, we will denote the representations of operators with diacritic circumflexes, such as $\delta(H) = \hat{H}$.

One peculiar feature of representations of $GL(d)$, and in turn also $U(d)$, is the following: Irreps $V_\lambda$ and $V_{\lambda'}$ are isomorphic as vector spaces if the YDs $\lambda$ and $\lambda'$ differ only by an initial block of height $d$. This can be visualized in the following way: The only way in which a SSYT that is a vertical strip of height $d$—denoted by $(1)^d$—can be filled with $d$ numbers is in increasing order ($|SSYT((1)^d)| = 1$). Therefore, prepending any YD with such a strip does not increase the number of according SSYTs, i.e.,

$$|SSYT((1)^d + \lambda)| = |SSYT(\lambda)|. \tag{3.25}$$

Moreover, the irreducible representations $\Delta_\lambda$ differ only in powers of determinants for such prepended strips (or blocks) [25]. Due to the vague resemblance with Fermi spheres in condensed matter physics [39], we refer to them as “Fermi blocks”. In principle, the blocks can also be of negative width, but this exceeds the definition of Young diagrams and requires proper mathematical treatment, which we will not discuss here. The concept is illustrated in Figure 3.1. Speculatively, those negative Fermi blocks could also be

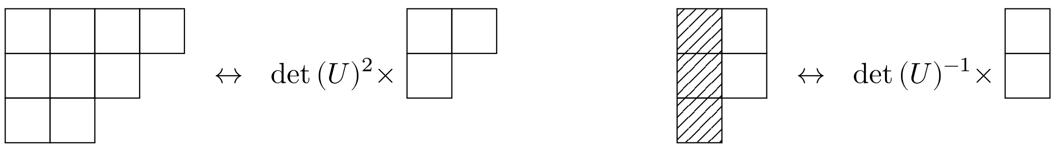



Figure 3.1.: Examples of “Fermi blocks” for $d = 3$. Hatched boxes represent negative particles

regarded as antiparticles—as quasiparticles similar to holes in condensed matter physics or even as fundamental ones—with negative occupation numbers as described in [18]. However, we defer this discussion to future work and conclude only with this superficial remark.

## 3.2. The action of the swap

In this section, we describe an immediate application of the previously introduced formalism of irreducible representations. Namely, we calculate the action of the swap

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.26}$$

to the wt $= (0, 1)$ and wt $= (1, 1)$ particle sectors, i.e., we compute the representation $\Delta_2(X)$ restricted to these sectors. For $d = 2$ modes, we have exactly two SSYTs of size one, which are simply $\boxed{1}$ and $\boxed{2}$, corresponding to the GTPs (1) and (2), respectively. The four SSYTs of size two (of which one is fermionic and three are bosonic) correspond to the following GTPs:

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \equiv \begin{pmatrix} 1 & & 1 \\ & 1 & \end{pmatrix},$$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array} \equiv \begin{pmatrix} 2 & & 0 \\ & 2 & \end{pmatrix}, \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \equiv \begin{pmatrix} 2 & & 0 \\ & 1 & \end{pmatrix}, \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline \end{array} \equiv \begin{pmatrix} 2 & & 0 \\ & 0 & \end{pmatrix}. \tag{3.27}$$

Using the expressions (3.16)-(3.18) for the actions of the representations of the lie algebra $gl(2)$, we obtain the following relations (where we suppress the multiplicity index):

$$\hat{e}_{11} \left|\boxed{1}\right\rangle = \left|\boxed{1}\right\rangle, \quad \hat{e}_{22} \left|\boxed{2}\right\rangle = \left|\boxed{2}\right\rangle, \qquad \hat{e}_{11} \left|\boxed{2}\right\rangle = \hat{e}_{22} \left|\boxed{1}\right\rangle = 0,$$

$$\hat{e}_{12} \left|\boxed{2}\right\rangle = \left|\boxed{1}\right\rangle, \quad \hat{e}_{21} \left|\boxed{1}\right\rangle = \left|\boxed{2}\right\rangle, \qquad \hat{e}_{12} \left|\boxed{1}\right\rangle = \hat{e}_{21} \left|\boxed{2}\right\rangle = 0, \tag{3.28}$$

$$\hat{e}_{11} \left|\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}\right\rangle = \hat{e}_{22} \left|\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}\right\rangle = \left|\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}\right\rangle, \qquad \hat{e}_{12} \left|\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}\right\rangle = \hat{e}_{21} \left|\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}\right\rangle = 0, \tag{3.29}$$

$$\begin{aligned} &\hat{e}_{11} \left|\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}\right\rangle = \hat{e}_{22} \left|\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}\right\rangle = \left|\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}\right\rangle, \\ &\hat{e}_{12} \left|\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}\right\rangle = \sqrt{2} \left|\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array}\right\rangle, \qquad \hat{e}_{21} \left|\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}\right\rangle = \sqrt{2} \left|\begin{array}{|c|c|}\hline 2 & 2 \\ \hline \end{array}\right\rangle, \end{aligned} \tag{3.30}$$

$$\begin{aligned} &\hat{e}_{11} \left|\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array}\right\rangle = 2 \left|\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array}\right\rangle, \qquad \hat{e}_{22} \left|\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array}\right\rangle = 0, \\ &\hat{e}_{12} \left|\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array}\right\rangle = 0, \qquad \hat{e}_{21} \left|\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array}\right\rangle = \sqrt{2} \left|\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}\right\rangle, \end{aligned} \tag{3.31}$$

$$\begin{aligned} &\hat{e}_{11} \left|\begin{array}{|c|c|}\hline 2 & 2 \\ \hline \end{array}\right\rangle = 0, \qquad \hat{e}_{22} \left|\begin{array}{|c|c|}\hline 2 & 2 \\ \hline \end{array}\right\rangle = 2 \left|\begin{array}{|c|c|}\hline 2 & 2 \\ \hline \end{array}\right\rangle, \\ &\hat{e}_{12} \left|\begin{array}{|c|c|}\hline 2 & 2 \\ \hline \end{array}\right\rangle = \sqrt{2} \left|\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}\right\rangle, \qquad \hat{e}_{21} \left|\begin{array}{|c|c|}\hline 2 & 2 \\ \hline \end{array}\right\rangle = 0. \end{aligned} \tag{3.32}$$

The swap matrix $X$ can be written as

$$X = e^{iH}, \qquad \text{with } H = \frac{\pi}{2} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{3.33}$$

Neglecting multiplicity, we can decompose the representations of the swap using (3.24) as

$$\begin{aligned} \Delta_2(X) = e^{i\delta_2(H)} &= \Delta_{\square}(X) \oplus \Delta_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}(X) \oplus \Delta_{\square\square}(X) \oplus \cdots \\ &= e^{i\delta_{\square}(H)} \oplus e^{i\delta_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}(H)} \oplus e^{i\delta_{\square\square}(H)} \oplus \cdots, \end{aligned} \tag{3.34}$$

where $\delta(H) = \frac{\pi}{2}(\hat{e}_{12} + \hat{e}_{21} - \hat{e}_{11} - \hat{e}_{22})$ spans all sectors. Using relations (3.28), we can write the representation of $H$ in the subspace spanned by $\{|\boxed{1}\rangle, |\boxed{2}\rangle\}$ as

$$\delta_{\square}(H) = \frac{\pi}{2}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{3.35}$$

which leads by exponentiation to

$$\Delta_2(X)\,|\boxed{1}\rangle = |\boxed{2}\rangle, \qquad \Delta_2(X) = |\boxed{2}\rangle = |\boxed{1}\rangle. \tag{3.36}$$

Since we have with relations (3.29) that the fermionic state is an eigenvector

$$\delta(H)\left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\right\rangle = -\pi\left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\right\rangle, \tag{3.37}$$

it directly follows by exponentiation that the one-dimensional fermionic irrep is stabilized under the swap as

$$\Delta_2(X)\left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\right\rangle = -\left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\right\rangle. \tag{3.38}$$

For the bosonic irrep, we see that the Lie algebra representation can be written as

$$i\delta_{\square\square}(H) = \frac{i\pi}{2}\begin{pmatrix} -2 & \sqrt{2} & 0 \\ \sqrt{2} & -2 & \sqrt{2} \\ 0 & \sqrt{2} & -2 \end{pmatrix}, \tag{3.39}$$

in the basis $\{|\boxed{1}\boxed{1}\rangle, |\boxed{1}\boxed{2}\rangle, |\boxed{2}\boxed{2}\rangle\}$. By calculating the matrix exponential, we obtain the representation of the swap in the bosonic sector:

$$\Delta_{\square\square}(X) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{3.40}$$

This calculation shows that the action of a swap stabilizes the bosonic and fermionic SSYTs $|\boxed{1}\boxed{2}\rangle$ and $\left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\right\rangle$ with eigenvalue $+1$ and $-1$, respectively, and cycles between $|\boxed{1}\boxed{1}\rangle$ and $|\boxed{2}\boxed{2}\rangle$. These results in the framework of SSYTs are—as expected—in complete accordance with the exchange statistics of ordinary bosons and fermions.

## 3.3. The Fock space via the tensor algebra

We now shift our focus of the construction of $\mathcal{F}_d$ from a 'manual' composition of irreducible representations towards the tensorial structure of the Fock space. In particular, we aim to give an alternative construction in terms of tensor products of a single-particle space.

Let $\mathcal{H}$ be the Hilbert space of a single particle. The Tensor algebra $\mathcal{T}(\mathcal{H})$ [33] of $\mathcal{H}$—sometimes regarded as the 'free' Fock space—is defined as

$$\mathcal{T}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}, \tag{3.41}$$

that is, as the infinite direct sum of $n$-particle Hilbert spaces $\mathcal{H}^{\otimes n}$. In this sense, it corresponds to the first quantization picture of $n$ identical particles, since the notion of an individual particle in the $n$-particle sector is well defined. Instead of just setting $\mathcal{H} = \mathbb{C}^d$, as appears in the constructions for bosons and fermions, we also admit $m$ flavors as a further internal degree of freedom, i.e. $\mathcal{H} = \mathbb{C}^d \otimes \mathbb{C}^m$. There is a natural representation $\Delta_d^\star : U(d) \to GL(\mathcal{T}(\mathcal{H}))$ of $U(d)$ acting on the free Fock space as

$$\Delta_d^\star(U) = \bigoplus_{n=0}^{\infty} (U \otimes \mathbb{1}_m)^{\otimes n} . \tag{3.42}$$

According to Theorem 1, it can be decomposed into irreps $\Delta_\lambda$ as

$$\Delta_d^\star(U) = \bigoplus_{\lambda} \Delta_\lambda(U)^{\oplus r_\lambda}, \tag{3.43}$$

where the index runs over all YTs and the multiplicities $r_\lambda$ are non-negative integers to be determined.
We now raise the question if the representation $\Delta_d$ on $\mathcal{F}_d$ can be regarded as a subrepresentation of $\Delta_d^\star$ on $\mathcal{T}(\mathcal{H})$. This is equivalent to having $m_\lambda \leq r_\lambda$ for all YDs $\lambda$, which is answered affirmatively in the following lemma.

**Lemma 1.** *The representation $\Delta_d$ on $\mathcal{F}_d$ is isomorphic to a subrepresentation of $\Delta_d^\star$ on the free Fock space $\mathcal{T}(\mathcal{H})$, i.e., there exists a projector $P$ such that $\Delta_d \cong P\Delta_d^\star$.*

*Proof.* It is necessary to show that the irreps $\Delta_\lambda$ appear with a multiplicity $r_\lambda$ in $\Delta_d^\star$ higher than or equal to $m_\lambda$ in $\Delta_\lambda$, i.e., $m_\lambda \leq r_\lambda$ for all $\lambda$. The projector $P$ can then be constructed by projecting on the $m_\lambda$ out of $r_\lambda$ irreps for each YD. To this end, we consider the character $\bar{\chi}$ of $\Delta_d^\star$:

$$\bar{\chi}_d(U) = \mathrm{Tr}\ \Delta_d^\star(U) = \sum_{n=0}^{\infty} (\mathrm{Tr}[U \otimes \mathbb{1}_m])^n = \sum_{n=0}^{\infty} (m\ \mathrm{Tr}\ U)^n = \sum_{n=0}^{\infty} \left( m \sum_{i=1}^{d} x_d \right)^n , \tag{3.44}$$

where $x_i$ are the eigenvalues of $U$. As the multiplicities are given as $r_\lambda = \langle s_\lambda | \bar{\chi}_d(x_1, ..., x_d) \rangle$ and $m_\lambda = \langle s_\lambda | \chi_d(x_1, ..., x_d) \rangle$, together with $\chi_1(x) = \chi_1(x_1)\chi_1(x_2) \cdots \chi_1(x_d)$ we have

$$r_\lambda - m_\lambda = \langle s_\lambda(\vec{x}) | \bar{\chi} - \chi_d \rangle = \left\langle s_\lambda(\vec{x}) \middle| \sum_{n=0}^{\infty} \left( m \sum_{i=1}^{d} x_d \right)^n - \prod_{i=1}^{d} \chi(x_i) \right\rangle . \tag{3.45}$$

As the Hall inner product with $s_\lambda$ is only nonzero for monomials of degree $|\lambda|$ and using the multinomial expansion, we have

$$\begin{aligned}
r_\lambda - m_\lambda &= \left\langle s_\lambda(\vec{x}) \middle| \left( m \sum_{i=1}^{d} x_d \right)^{|\lambda|} - \sum_{n_1+\ldots+n_d=|\lambda|} \prod_{i=1}^{d} d_i^{n_i} x_i^{n_i} \right\rangle \\
&\geq \left\langle s_\lambda(\vec{x}) \middle| m^{|\lambda|} \sum_{n_1+\ldots+n_d=|\lambda|} \frac{|\lambda|!}{n_1! \cdot \ldots \cdot n_d!} \prod_{i=1}^{d} x_i^{n_i} - m^{|\lambda|} \sum_{n_1+\ldots+n_d=|\lambda|} \prod_{i=1}^{d} x_i^{n_i} \right\rangle \\
&= m^{|\lambda|} \sum_{n_1+\ldots+n_d=|\lambda|} \left( \frac{|\lambda|!}{n_1! \cdot \ldots \cdot n_d!} - 1 \right) \left\langle s_\lambda(\vec{x}) \middle| \prod_{i=1}^{d} x_i^{n_i} \right\rangle \\
&= m^{|\lambda|} \sum_{\substack{\mu \in \mathrm{YD} \\ |\mu|=|\lambda|}} \underbrace{\left( \frac{|\lambda|!}{|\mu_1|! \cdot \ldots \cdot |\mu_d|!} - 1 \right)}_{\geq 0} \underbrace{\big\langle s_\lambda(\vec{x}) \big| m_\mu(\vec{x}) \big\rangle}_{K_{\lambda\mu \geq 0}} \geq 0. \qquad (3.46)
\end{aligned}$$

Here we also used the fact that for any TPS $d_i \leq d_1^i = m^i$, which can be deduced from the log-concavity property (2.27) by induction. Furthermore, we introduce the monomial symmetric polynomials $m_\mu(\vec{x}) = \sum_{\sigma \in S_d} \vec{x}^\sigma(\mu)$. As is known from the theory of symmetric functions, the Hall inner product $\big\langle s_\lambda(\vec{x}) \big| m_\mu(\vec{x}) \big\rangle$ is a non-negative integer, called the Kostka number $K_{\lambda\mu}$ [28, p. 101]. This shows that all irreps occur with a multiplicity smaller than or equal to $\Delta_d^\star$ in $\Delta_d$. $\square$

This indicates that we can regard the Fock space of transtatistics $\mathcal{F}_d$ as a subspace of the free Fock space $\mathcal{T}(\mathcal{H})$. Moreover, the representation $\Delta_d$ for every totally positive character $\chi_1$ is (isomorphic to) a subrepresentation of $\Delta_d^\star$ on the free Fock space. However, the proof of Lemma 1 is not constructive in the sense that there is no concrete construction of the projector $P$ and the isomorphism of its image with $\mathcal{F}_d$.

In a more recent work, Sánchez and Dakić [19] study a construction very similar to the above incorporation of $d$ *external* (modes) and $m$ *internal* (flavors) degrees of freedom in the single particle Hilbert space $\mathcal{H}$. There they give a construction of the Fock space $\mathcal{F}_d$ after imposing additional conditions from the tensor algebra $\mathcal{T}(\mathcal{H})$. In particular, there the quotient with an 'quadratic ideal' plays the role of the previously introduced projector $P$.

To facilitate further discussions, we introduce the following notion of a Fock space that is generated by taking the free Fock space $\mathcal{T}(\mathcal{H})$ and imposing some quadratic relations $W$. In full mathematical rigor, this construction has to be treated within the formalism of *quadratic algebras* [33], but for the sake of simplicity, we reduce it to the following definition.

**Definition 8.** *Let $\mathcal{H}$ be a vector space and let $W \subset \mathcal{H} \otimes \mathcal{H}$ be the so-called 'relation space'. We call the Fock space $\mathcal{F}_d$ quadratically generated if it is the quotient space*

$$\mathcal{F}_d = \mathcal{T}(\mathcal{H}) / \langle W \rangle. \qquad (3.47)$$

*Here, $\langle W \rangle \subset \mathcal{T}(\mathcal{H})$ denotes the ideal generated by $W$ and is identified with zero in the quotient construction. We have $v \in \langle W \rangle$ iff $v$ is of the form*

$$v = x \otimes w \otimes z, \qquad \text{with } x \in \mathcal{H}^{\otimes k}, w \in W, z \in \mathcal{H}^{\otimes m}, \tag{3.48}$$

*for some $k, m \geq 0$.*

The above definition is inspired by the work of Segal [40,41] and encompasses the procedure of going from first to second quantization (symmetrization and antisymmetrization in the case of bosons and fermions): Out of all possible states of $N$ particles the combinations that involve relations between pairs of particles, forming a basis of $W$ are sent to zero ("quotiented out").

In Ref. [19], Sánchez and Dakić proved that under the further assumption that one can impose a total order of modes *and* flavors, one can consistently define an algebra of creation and annihilation operators for transtatistics. Concretely, they assume that non only is there an order of modes, but they also impose the existence of an order of flavors. The corresponding Fock space is then quadratically generated in the sense of Definition 8, and the algebra of creation and annihilation operators gives rise to solutions to the YBE. Therefore, $R$-parastatistics can be obtained by the construction in Ref. [19]. However, we will not continue with the discussion of their formalism here, as we do not assume the order of flavors in our construction.

Nevertheless, the assumption that the single-particle Fock space $\mathcal{F}_1$ is quadratically generated can also be motivated by Theorem 1.1 in [42], which we restate in a slightly modified manner:

**Theorem 3.** ***(Adaption of Thm. 1.1 in [42])***
*If $d_i$ is a totally positive sequence, then there exists a quadratically generated space $\mathcal{F}$ which has $d_i$ as its Hilbert series: $h_{\mathcal{F}}(x) = \sum_i d_i x^i$.*

In fact, the above theorem makes an even stricter statement, namely it imposes the space to be generated by a *Koszul Algebra* [33], a subclass of quadratic algebras which obey several momentous properties. Perhaps most prominently, they exhibit the notion of "Koszul duality". This implies that there exists a dual space (respectively, algebra) $\mathcal{F}^!$, whose Hilbert series is in a reciprocal relationship:

$$h_{\mathcal{F}^!}(x) = \frac{1}{h_{\mathcal{F}}(-x)}. \tag{3.49}$$

This duality manifests itself for bosons and fermions. It can be seen directly that the Hilbert series of the respective Fock spaces are dual in the sense of equation (3.49) (see Section 2.4).

## 3.4. Relationship between the Young and number basis

So far, we have given a construction that yields representations of $U(d)$ in the Fock space $\mathcal{F}_d$. However, this is of little use in the operational sense as it relies on a specific basis deliberately constructed for the representation. This Young basis has no direct connection to the operationally accessible number basis. In this section, we will explore this connection.

In particular, we want to find an appropriate change of basis such that the representations of unitaries are compatible with a few general assumptions. To this end, we introduce the following linear relation between these bases as

$$|\vec{n};\vec{\alpha}\rangle = \sum_{\Lambda,k} M_{(\Lambda,\kappa),(\vec{n},\vec{\alpha})} |\Lambda,\kappa\rangle , \tag{3.50}$$

where the matrix $M$ captures the transformation between the Young and number basis. Here $\Lambda$ ranges over all SSYTs and $\kappa$ over all multiplicities in the decomposition of the Fock space. It should be noted that we (a priori) do not make the requirement for $M$ to be unitary. Since the Young basis is an orthonormal basis, we multiply (3.50) with $|\Lambda,\kappa\rangle$ and set the matrix elements as

$$M_{(\Lambda,\kappa),(\vec{n},\vec{\alpha})} = \langle\Lambda,\kappa|\vec{n};\vec{\alpha}\rangle , \tag{3.51}$$

in order to avoid indices. In Section 3.1 we have described a concrete form for the representation $\Delta_d$ (or $\delta_d$) as matrices in the Young basis. In order to obtain the representations of unitaries (or their Lie algebra), we have to perform a change of basis. Thus, by letting $M_d$ denote $M$ in $d$ modes, we have

$$\hat{\Delta}_d(U) = M_d^{-1}\Delta_d(U)M_d, \qquad \text{and} \quad \hat{\delta}_d(H) = M_d^{-1}\delta_d(H)M_d, \tag{3.52}$$

where $\hat{\Delta}_d$ and $\hat{\delta}_d$ refer to representations as matrices in the number basis. If it is clear that we are working in the number basis, we shall omit the diacritic hats.

We go forth to seek conditions on $M$ such that the representations in the Fock space remain operationally consistent. One observation concerns the particle number, of which we have a clear interpretation in the (defining) number basis as well as the Young basis. Namely, we ask for the following: The representation of the diagonal matrix $h_{ii} = \mathrm{diag}(0,0,...,1,...,0)$ with a one as $i$-th entry should have the same eigen-decomposition as the number operators $\hat{n}_i$ defined in Eq. (3.8). This then leads to the following condition of $M$:

**Lemma 2.** *For a representation $\Delta_d$ that is local in the sense of Eq. (2.33), the representations of diagonal elements can be identified with the number operators, $\hat{\delta}_d(h_{ii}) \equiv \hat{n}_i$, and we have that*

$$\langle\Lambda,\kappa|\vec{n};\vec{\alpha}\rangle = 0 \qquad \textit{if } \vec{n} \neq \mathrm{wt}\,\Lambda. \tag{3.53}$$

*Proof.* From the locality assumption, we have that $\Delta_d(e^{ith_{kk}}) = \mathbb{1}^{k-1} \otimes \Delta_1(e^{it\hat{n}}) \otimes \mathbb{1}^{d-k}$ for some $t \in \mathbb{R}$ and thus $\hat{\delta}(h_{ii}) = \hat{n}_{ii}$ follows from differentiation by $t$ and Eq. (2.7). By definition of the number basis we have

$$\hat{\delta}_d(h_{ii}) \begin{vmatrix} n_1 & n_2 \dots n_d \\ \alpha_1 & \alpha_2 \dots \alpha_d \end{matrix}\Big\rangle = n_i \begin{vmatrix} n_1 & n_2 \dots n_d \\ \alpha_1 & \alpha_2 \dots \alpha_d \end{matrix}\Big\rangle . \tag{3.54}$$

By multiplying with $\langle \Lambda, \kappa|$ from the left, we obtain

$$\mathrm{wt}_i \, \Lambda \, \langle \Lambda, \kappa | \vec{n}; \vec{\alpha} \rangle = n_i \, \langle \Lambda, \kappa | \vec{n}; \vec{\alpha} \rangle , \tag{3.55}$$

by using equation (3.20). For $(n_1, n_2, ...n_d) \neq \mathrm{wt}\, \Lambda$, equation (3.55) can only hold if the corresponding matrix element $\langle \vec{n}; \vec{\alpha} | \Lambda, \kappa \rangle$ is zero. □

This lemma implies the following: The matrix $M$ cannot couple number states to arbitrary SSYTs $\Lambda$, but the weights (i.e., the frequency of the fillings) of $\Lambda$ have to precisely match the particle numbers that can be observed for a number state. This in turn implies that the transformation $M$ can be decomposed into smaller blocks

$$M = \bigoplus_{\vec{n}} M_{\vec{n}}. \tag{3.56}$$

If appropriate orderings are chosen for the Young and number basis, then $M$ is of block-diagonal form as a matrix. A simple combinatorial argument for counting the possible vectors $\vec{n}$ shows that there are $\chi_1(1)^d = \dim \mathcal{F}_d$ decoupled sectors or blocks of $M_{\vec{n}}$, each of which is of dimension

$$\dim M_{\vec{n}} = \prod_{i=1}^{d} d_{n_i}, \tag{3.57}$$

with the coefficients $d_k$ of the single mode character $\chi_1$.

The action of $M$ can also be regarded as the following: for each state in the Fock space which can be partly determined by observing the particle numbers $\vec{n}$, the according sector $M_{\vec{n}}$ mediates the remaining degrees of freedom in the number and Young basis. In the number basis, these degrees of freedom are collected into the labels $\vec{\alpha}$. Via $M$, they are coupled to a linear combination of states $|\Lambda, \kappa\rangle$ whose (a priori) non-observable degrees of freedom consist of

1. The multiplicity index $\kappa$ of each irrep.
2. The order in which the entries with frequencies $\vec{n}$ appear in the SSYT $\Lambda$.
3. The shape of the SSYT $\Lambda$.

We formalize these considerations in the following lemma, which gives an alternative expression for the dimension of the sectors $M_{\vec{n}}$ in terms of quantities that appear in the Young basis.

**Lemma 3.** *The dimensions of the* $\mathrm{wt} = \vec{n}$ *sectors* $M_{\vec{n}}$ *are given as*

$$\dim M_{\vec{n}} = \prod_{i=1}^{d} d_{\mu_i} = \sum_{\lambda \in YD} m_\lambda K_{\lambda\mu}, \tag{3.58}$$

*with $\mu$ being the YD with row sizes corresponding to the ordered entries of $\vec{n}$, the Kostka numbers $K_{\lambda\mu}$ and the multiplicities $m_\lambda$ of the YD $\lambda$ in $\chi_d$ .*

*Proof.* The Kostka number $K_{\lambda\mu}$ is equal to the number of semistandard Young tableaux of shape $\lambda$ that have weights described by any permutation of the YD $\mu$ [27]. The number of SSYTs with weight $\vec{n}$ in the Fock space $\mathcal{F}_d$ is then precisely given by the RHS in equation (3.58). The LHS is equal to the dimension of the subspace $|\vec{n};\vec{\alpha}\rangle$ for a fixed $\vec{n}$. □

## 3.5. Unitarity and the scalar product

Previously, we have introduced a scalar product $\langle\cdot|\cdot\rangle$ on the Fock space that establishes the Young basis $\{|\Lambda,\kappa\rangle\}$ to be an orthonormal basis. In addition, we did not impose the condition of unitarity on the matrix $M$, but left the explicit possibility open that $M$ is not unitary. This in turn implies that the number basis $|\vec{n},\vec{\alpha}\rangle$ is, in general, not an orthonormal basis. At first glance, this seems to starkly contradict the conventional formalism of quantum mechanics, where bases, observables and transformations are always presumed to be orthonormal, Hermitian and unitary. Usually, relaxing these properties is regarded as giving rise to many undesirable consequences, most prominently the violation of probability conservation [43]. However, as we will see, this deviation from the formalism cannot only be made consistent with operational requirements, as also achieved by Wang and Hazzard [21], but also appears to be a necessity of the mathematical structure. In particular, Lechner *et. al.* [44] proved that $R$-matrices with Hilbert series that have irrational roots cannot be unitary and involutive. From this perspective, it is desirable to extend the formalism to encompass also such cases.

Although the number basis is not orthonormal, it is possible to obtain expressions for scalar products. By multiplying with the adjoint of Eq. (3.50), we get

$$\left\langle\vec{n},\vec{\alpha}\middle|\vec{m},\vec{\beta}\right\rangle = (M^\dagger M)_{\vec{n}\vec{\alpha},\vec{m}\vec{\beta}}. \tag{3.59}$$

In this sense, the matrix $M^\dagger M$ plays the role of a Gram matrix (or a metric tensor) $g$. This expression of the scalar product is fully compatible with Eq. (3.7) insofar, as by Lemma 2 $M$ is 'block diagonal' with respect to different weights and therefore also $M^\dagger M$ is zero for different particle numbers. We thus have the decomposition of the Gram matrix

$$g_{(\vec{n}\vec{\alpha})(\vec{m}\vec{\beta})} = (M^\dagger M)_{\vec{n}\vec{\alpha},\vec{m}\vec{\beta}} = \delta_{\vec{n}\vec{m}} g^{(\vec{n})}_{\vec{\alpha}\vec{\beta}}, \tag{3.60}$$

which ensures orthogonality in the number basis for different particle numbers. Alternatively, it can be useful to introduce the *dual basis* (see e.g., [45, 46]) of the number basis denoted with turtleshell brackets $(\cdot|$ and satisfying the defining relation

$$\left(\vec{n};\vec{\alpha}\middle|\vec{m};\vec{\beta}\right\rangle = \delta_{\vec{n},\vec{m}}\delta_{\vec{\alpha},\vec{\beta}}. \tag{3.61}$$

In general, for non-orthogonal bases, the adjoint and dual bases are not the same but are related by the gram matrix:

$$(\vec{n};\vec{\alpha}| = \sum_{\vec{m},\vec{\beta}} \left(g^{-1}\right)_{(\vec{n}\vec{\alpha})(\vec{m}\vec{\beta})} \left\langle \vec{m};\vec{\beta}\right| . \tag{3.62}$$

For the orthonormal Young basis $(\Lambda,\kappa| = \langle\Lambda,\kappa|$ and $|\Lambda,\kappa) = |\Lambda,\kappa\rangle$ always holds. Only in the case that $M$ is unitary, the dual basis and "regular" bras coincide ( $\langle\cdot| = (\cdot|$ ) also for the number basis. Consequently, (unitary) operators that are defined by their action on the Young basis also change their form when changing to the number basis according to the transformations in (3.52). It can be directly observed that unitaries in the Young basis are non-unitary matrices in the number basis for a general $M$.

The probability $p(\vec{n}|\,|\psi\rangle\,)$ of observing a particle number distribution $\vec{n}$ given a state $|\psi\rangle$ is given by the Born's rule

$$p(\vec{n}|\,|\psi\rangle\,) = \sum_{\substack{\Lambda,\ \kappa \\ \mathrm{wt}\,\Lambda=\vec{n}}} \left|\,\langle\Lambda,\kappa|\psi\rangle\,\right|^2, \tag{3.63}$$

where the sum ranges over all multiplicities $1 \leq \kappa \leq m_\lambda$ and SSYTs $\Lambda$ that appear in the decomposition of the Fock space and have the desired weight.

By Ref. [35], since the Young basis is chosen to be orthonormal, the generators $\hat{e}_{ij}$ defined in (3.16)-(3.19) obey

$$\hat{e}_{ij}^{\dagger} = \hat{e}_{ji}, \tag{3.64}$$

which is also the case for $R$-parastatistics. The scalar product defined here via the orthonormality, therefore, agrees with the scalar product in [21]. Following the argument therein, this means that even though the number basis is non-orthonormal, physical observables (representations of Hermitian matrices) remain Hermitian.

Similarly, the non-unitary form of, e.g., time evolution operators in the number basis appears to be a stark violation of Wigner's theorem, stating that all physical transformations have to be (anti)unitary [47, pp. 251–254]. However, this contradiction might be resolved by assuming that the flavors corresponding to the internal degrees of freedom of the particles are not detectable by any means. Therefore, any non-unitary transformation that happens solely within the subspace of flavors has no measurable effect and should thus also not lead to a violation of probability conservation. As a more speculative note, this phenomenon could perhaps also be phrased in the language of generalized probabilistic theories (GPTs). In that framework, it might be related to the relaxation of the so-called *no-restriction hypothesis* [48, 49], which corresponds to only admitting a subset of all possible measurements. However, this needs a more thorough mathematical justification, which we shall not pursue here.

# 4. On locality

In their paper [18], Sánchez and Dakić center their discussion on their assumption on locality, which we will call *phase locality* for further discussions:

$$\Delta_d(U) = \Delta_1(e^{i\theta_1}) \otimes \Delta_1(e^{i\theta_2}) \otimes ... \otimes \Delta_1(e^{i\theta_d}), \tag{4.1}$$

for a diagonal $U = \mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, ..., e^{i\theta_d})$, which they operationally interpret as the action of only phase shifters acting locally. This condition immediately implies that the characters factorize. The factorization of the characters in turn leads to the restriction of them being totally positive sequences. Under the assumption that the representation of diagonal matrices correspond to the number operators $\hat{n}_i$ in the number basis, we were also able to derive a condition on $M$ (the relationship between number and Young basis) as formalized in Lemma 2.

However, we will try to make the point in the following that only the assumption of phase locality (4.1) is not strict enough to ensure 'locality' in a broader physical sense. This is because phase locality ensures statistical independence only in the case of phase shifters but not for general independent devices. To this end, we introduce the following notion that ensures that also the representation of block-diagonal matrices perform in a statistically independent manner. It is designed to apply to experimental scenarios such as the one illustrated in Figure 4.1, where there are devices that act only on subsystems independently.

**Definition 9.** *A family of representations $\Delta_d$ of $U(d)$ on $\mathcal{F}_d$ is called 'separable' if*

$$p(\vec{n}_A \oplus \vec{n}_B | \Delta_d(U_A \oplus U_B) |\psi_A\rangle |\psi_B\rangle ) = p(\vec{n}_A | \Delta_{d_A}(U_A) |\psi_A\rangle ) p(\vec{n}_B | \Delta_{d_B}(U_B) |\psi_B\rangle ), \tag{4.2}$$

*holds for all $d$, subsystems $A$ and $B$ with $d_A + d_B = d$, $U_A \in U(d_A)$, $U_B \in U(d_B)$, $|\psi_A\rangle \in \mathcal{F}_A$, $|\psi_B\rangle \in \mathcal{F}_B$.*

One might be tempted to directly conclude that separability implies that the representation $\Delta_d$ has to factor into tensor products $\Delta_{d_A} \otimes \Delta_{d_B}$ as the probabilities factorize for particle number measurements. However, this conclusion might not be true as the states for a given particle number $\vec{n}$ are degenerate by construction and thus the particle number measurements cannot distinguish states with different internal flavors.

Nevertheless, we—inspired by the tensor product structure of the Fock space and suggested by [50]—also introduce this even stricter definition of locality, which we call 'strong locality'.

**Definition 10.** *A family of representations $\Delta_d$ of $U(d)$ on $\mathcal{F}_d$ is called 'strongly local' if*

$$\Delta_d(U_A \oplus U_B) = \Delta_{d_A}(U_A) \otimes \Delta_{d_B}(U_B), \tag{4.3}$$

*holds for all $d$, $d_A + d_B = d$ and $U_A \in U(d_A)$, $U_B \in U(d_B)$.*

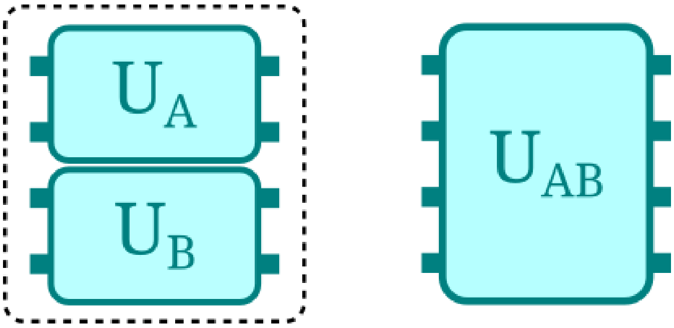



Figure 4.1.: Experimental motivation of Definition 9: Two unitary devices $U_A \oplus U_B$ (left) are operationally equivalent to one combined device $U_{AB}$ (right).

The notation $U_A \oplus U_B$ stands for the block-diagonal composite matrix $\mathrm{diag}(U_A, U_B)$.

It can be observed that strong locality (4.3) directly implies separability (4.2) via the form of the tensor product as well as phase locality (4.1) by specializing to $d \times d$ diagonal unitaries. But can strong locality also be derived purely from separability by combining it with a further assumption? Perchance. We will leave this question open for future discussion. In any case, separability and therefore strong locality cannot be derived from phase locality, for which we also provide a counterexample in Section 4.1. We continue by investigating strong locality due to its far-reaching implications on the mathematical structure of representations.

Definition 10, which imposes strong conditions on the representations of the unitary groups, also has an analog at the level of Lie algebras: By taking the logarithm of Eq. (4.3) (where $\Delta_d(U) = \exp(i\delta_d(H))$ and the block-diagonal structure is preserved), we obtain

$$\delta_d(H_A \oplus 0_B) + \delta_d(0_A \oplus H_B) = \delta_{d_A}(H_A) \otimes \mathbb{1}_B + \mathbb{1}_A \otimes \delta_{d_B}(H_B), \tag{4.4}$$

for all subsystems $A$ and $B$. This can be inverted by taking the exponential, so Eq. (4.3) and (4.4) are equivalent conditions to specify strong locality.

Equations (4.3) and (4.4) also serve as an implicit definition of the number basis. Explicitly, when the representations $\Delta_d$ and $\delta_d$ are specified as functions in the Young basis as described in section Section 3.4, setting $H_B = 0_B$ in (4.4), strong locality implies

$$\delta_d(H_A \oplus 0_B) = M_d \left( (M_{d_A}^{-1} \delta_{d_A}(H_A) M_{d_A}) \otimes \mathbb{1}_B \right) M_d^{-1}. \tag{4.5}$$

The representation $\delta_d(H_A + 0_B)$ can be regarded as an embedding of $\delta_{d_A}(H_A)$ acting on $\mathcal{F}_{d_A}$ into the larger $\mathcal{F}_d$. This can be denoted as

$$\delta_d(H_A \oplus 0_B) = \mathcal{E}\left[\delta_{d_A}(H_A)\right], \tag{4.6}$$

where the map $\mathcal{E}$ acting on the generators of $gl(d)$ has to be compatible with their concrete expressions (3.20)-(3.19) in the Young basis of both Fock spaces $\mathcal{F}_{d_A}$ and $\mathcal{F}_d$. This appears to be a difficult combinatorial problem. In a sense, it seems to encompass the act of extending Young diagrams and tableaux [27], but it has to be consistent with the action of $gl(d_A)$ and the multiplicative nature of the Hilbert series of $\mathcal{F}_d$. Hypothetically, by matching the concrete form of $\mathcal{E}$ with the RHS of (4.5), one might be able to obtain a recursive expression for $M_d$ in terms of smaller $M_{d_A}$. However, solving this problem is beyond the scope of this work.

## 4.1. Phase-local representations can be non-separable

We will now discuss the necessity of introducing the Definitions 9 and 10 in the first place. To this end, we will construct a representation $\Delta_d$ which satisfies phase locality, but violates the separability condition (4.2) and thus also strong locality: Consider the character

$$\chi_1(x) = 1 + 3x + x^2, \tag{4.7}$$

which is associated to the totally positive sequence $[1, 3, 1]_-$ and thus of transfermionic type. It also appears as “Ex. 4” in Figure 2.3. The expansion coefficients of the character into Schur polynomials according to Eq. (2.37) an thus the decomposition of the Fock space into irreps is given in Table 4.1 for $d = 1, 2, 3$. On the other hand, constructing the

| $\lambda$ | ○ | $(1)$ | $(1,1)$ | $(2)$ | $(1,1,1)$ | $(2,1)$ | $(2,1,1)$ | $(2,2)$ | $(2,2,1)$ | $(2,2,2)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $d = 1$ | | | | | | | | | | |
| $m_\lambda$ | 1 | 3 | | 1 | | | | | | |
| dim $V_\lambda$ | 1 | 1 | | 1 | | | | | | |
| $d = 2$ | | | | | | | | | | |
| $m_\lambda$ | 1 | 3 | 8 | 1 | | 3 | | 1 | | |
| dim $V_\lambda$ | 1 | 2 | 1 | 3 | | 2 | | 1 | | |
| $d = 3$ | | | | | | | | | | |
| $m_\lambda$ | 1 | 3 | 8 | 1 | 21 | 3 | 8 | 1 | 3 | 1 |
| dim $V_\lambda$ | 1 | 3 | 3 | 6 | 1 | 8 | 3 | 6 | 3 | 1 |

Table 4.1.: Multiplicities and dimensions of irreps $\lambda$ for character $1 + 3x + x^2$ for $d = 1, 2, 3$

number basis is straightforward. For the single mode Fock space $\mathcal{F}_1$, we set the basis as

$$\mathcal{F}_1 = \text{span}\{ |0\rangle, |A\rangle, |B\rangle, |C\rangle, |2\rangle \}, \tag{4.8}$$

where we slightly deviated from the previously defined notation for reasons of compactness: As the number states $|0\rangle, |2\rangle$ are non-degenerate, we leave out the degeneracy label. And since the only degeneracy occurs for $n = 1$, we just write the degeneracy labels $A, B, C$.

Nevertheless, multi-mode states are just defined as the ordinary tensor product—which we denote simply by juxtaposition—e.g., $|A,2,0\rangle$.

The crucial part now lies in relating the number basis and the Young basis of SSYTs through a suitable matrix $M$ as in Eq. (3.50). This $M = M(d)$ can, in principle, be different for any number of modes.

We proceed by defining a concrete matrix $M$ that satisfies Lemma 2, but that will lead to a violation of separability (Definition 9). For this, we simply choose $M$ to associate SSYTs with number basis states in a one-to-one manner, i.e., we set $M$ to be a permutation matrix that respects the weights. Such a pairing is shown for $d = 1,2,3$ in Tables C.1-C.3 and always exists for higher $d$.

The representation $\Delta_d$ defined by this concrete $M$ satisfies phase locality, but we will show that it violates separability and thus also strong locality. To this end, consider a beam splitter in the first two modes and identity in the third mode, which is the representation of the block-diagonal unitary

$$U_{\text{tot}} = U_{\text{BS}} \oplus \mathbb{1}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}. \tag{4.9}$$

The unitary $U_{\text{BS}}$ of the two-mode beam-splitter is generated by the Hamiltonian

$$H_{\text{BS}} = \frac{\pi}{2\sqrt{2}} \begin{pmatrix} \sqrt{2}-1 & -1 \\ -1 & \sqrt{2}+1 \end{pmatrix}. \tag{4.10}$$

To show that this setup violates separability, we will compute both the RHS and LHS of Eq. (4.2) where $U_A = U_{\text{BS}}$ and $U_B = \mathbb{1}_1$ for different number basis vectors and measurement outcomes. For the subsystem $B$ of the third mode, we directly have

$$p(\vec{n}_B|\Delta_{d_B}(U_B)\,|\psi_B\rangle) = p(\vec{n}_B|\Delta_1(\mathbb{1})\,|\psi\rangle_B) = \delta_{\vec{n}_B,\text{wt}\,|\psi\rangle_B}, \tag{4.11}$$

for any number basis state $|\psi\rangle_B$ of $\mathcal{F}_1$. The computation of the probabilities of the system $A$ and the joint system $AB$ is more involved, as it is hardly tractable to obtain the explicit form of the representation by manual calculation. Therefore, we will resort to computer algebra systems for this task. The algorithm to compute the probabilities using the software package `sagemath` [23] is outlined in Appendix B. The probabilities of measurement outcomes for the two-mode system $A$ (with $\Delta_2(U_{\text{BS}})$) are displayed in Table C.5. The probabilities of the joint system $AB$ obtained from the representation $\Delta_3(U_{\text{tot}})$—that correspond to the LHS in Eq. (4.2)—are shown in Table C.6. There, also the probabilities obtained by the RHS are indicated. There are some input states for which the values $p_{AB}$ and $p_A p_B$ are not equal. Therefore, there exist choices of $M$ which define representations that are phase local but non-separable.

## 4.2. R-parastatistics are strongly local

As $R$-parastatistics defined by Wang and Hazzard [21] and summarized in Section 2.6 show many features similar to transtatistics, we now strive to investigate if they satisfy the previously defined notions of locality.

First, note that in the notation of $R$-parastatistics, the representation of Lie algebra $\delta_d(H)$ is defined implicitly through the images of the generators $e_{ij}$ of $gl(d)$:

$$\delta_d(e_{ij}) \equiv \hat{e}_{ij}. \tag{4.12}$$

The representation of unitaries $\Delta_d(e^{iH})$ is then obtained by exponentiation

$$\Delta_d(e^{iH}) = e^{i\delta_d(H)}. \tag{4.13}$$

The Fock space $\mathcal{F}_{R,d}$ of $R$-parastatistics is also defined implicitly as the span of all creation operators $\psi^+_{i,a}$—modulo the commutation relations—acting on the unique vacuum state $|0\rangle$ as shown in Eq. (2.51). For this definition, it is a priori not apparent if it has a straightforward tensor product structure, although the notation of state vectors strongly suggests this. However, this is answered affirmatively by lemma S2.4 in Ref. [21].

For the basis vectors of the Fock space of subsystems $A$ and $B$, respectively we set the tensor product as

$$\begin{aligned}
&\left|\begin{matrix} n_1 & n_2 & \dots & n_{d_A} \\ \alpha_1 & \alpha_2 & \dots & \alpha_{d_A}\end{matrix}\right\rangle \otimes \left|\begin{matrix} m_1 & m_2 & \dots & m_{d_B} \\ \beta_1 & \beta_2 & \dots & \beta_{d_B}\end{matrix}\right\rangle \\
&= \left(\hat{\Psi}^{(1)+}_{n_1,\alpha_1}\hat{\Psi}^{(2)+}_{n_2,\alpha_2}\dots\hat{\Psi}^{(d_A)+}_{n_{d_A},\alpha_{d_A}}|0\rangle\right) \otimes \left(\hat{\Psi}^{(1)+}_{m_1,\beta_1}\hat{\Psi}^{(2)+}_{m_2,\beta_2}\dots\hat{\Psi}^{(d_B)+}_{m_{d_B},\beta_{d_B}}|0\rangle\right) \\
&\equiv \hat{\Psi}^{(1)+}_{n_1,\alpha_1}\hat{\Psi}^{(2)+}_{n_2,\alpha_2}\dots\hat{\Psi}^{(d_A)+}_{n_{d_A},\alpha_{d_A}}\hat{\Psi}^{(d_A+1)+}_{m_1,\beta_1}\hat{\Psi}^{(d_A+2)+}_{m_2,\beta_2}\dots\hat{\Psi}^{(d_A+d_B)+}_{m_{d_B},\beta_{d_B}}|0\rangle \\
&= \left|\begin{matrix} n_1 & n_2 & \dots & n_{d_A} & m_1 & m_2 & \dots & m_{d_B} \\ \alpha_1 & \alpha_2 & \dots & \alpha_{d_A} & \beta_1 & \beta_2 & \dots & \beta_{d_B}\end{matrix}\right\rangle .
\end{aligned} \tag{4.14}$$

We proceed by checking that the Lie algebra version of strong locality (4.4) holds. Due to linearity, it suffices to check this for generators $\hat{e}_{ij}$ for $i, j \in A$ and $i, j \in B$. Using Eq. (2.57), we immediately have $[\hat{e}_{ij}, \hat{\Psi}^{(k)+}_{n,\alpha}] = 0$ for $k \neq j$. For the case of the same site, we obtain

$$\left[\hat{e}_{ij}, \hat{\Psi}^{(j)+}_{n,\alpha}\right] =: \hat{\Phi}^{(j\to i)+}_{n,\alpha}, \tag{4.15}$$

where it can be seen by repeated application of Eq. (2.57) that the operator $\hat{\Phi}^{(j\to i)+}_{n,\alpha}$ consists purely of sums and products of $\hat{\psi}^+_{i,a}$ and $\hat{\psi}^+_{j,b}$. With Eq. (4.15) and the fact that $\hat{e}_{ij}|0\rangle = \sum_a \hat{\psi}^+_{i,a}\hat{\psi}^-_{j,b}|0\rangle = 0$, we evaluate the application of $\hat{e}_{ij}$ on any basis vector of the

$R$-parastatistics Fock space:

$$
\begin{aligned}
&\hat{e}_{ij}\left|\begin{matrix} n_1 & n_2, \dots, n_d \\ \alpha_1 & \alpha_2, \dots, \alpha_d \end{matrix}\right\rangle = \hat{e}_{ij}\hat{\Psi}^{(1)+}_{n_1,\alpha_1}\hat{\Psi}^{(2)+}_{n_2,\alpha_2}\cdots\hat{\Psi}^{(d)+}_{n_d,\alpha_d}|0\rangle \\
&= \hat{\Psi}^{(1)+}_{n_1,\alpha_1}\hat{\Psi}^{(2)+}_{n_2,\alpha_2}\cdots\hat{e}_{ij}\hat{\Psi}^{(j)+}_{n_j,\alpha_j}\cdots\hat{\Psi}^{(d)+}_{n_d,\alpha_d}|0\rangle \\
&= \hat{\Psi}^{(1)+}_{n_1,\alpha_1}\hat{\Psi}^{(2)+}_{n_2,\alpha_2}\cdots\left(\hat{\Phi}^{(j\to i)+}_{n_j,\alpha_j} + \hat{\Psi}^{(j)+}_{n_j,\alpha_j}\hat{e}_{ij}\right)\cdots\hat{\Psi}^{(d)+}_{n_d,\alpha_d}|0\rangle \\
&= \hat{\Psi}^{(1)+}_{n_1,\alpha_1}\hat{\Psi}^{(2)+}_{n_2,\alpha_2}\cdots\hat{\Phi}^{(j\to i)+}_{n_j,\alpha_j}\cdots\hat{\Psi}^{(d)+}_{n_d,\alpha_d}|0\rangle + \hat{\Psi}^{(1)+}_{n_1,\alpha_1}\hat{\Psi}^{(2)+}_{n_2,\alpha_2}\cdots\hat{\Psi}^{(d)+}_{n_d,\alpha_d}\hat{e}_{ij}|0\rangle \\
&\quad = \hat{\Psi}^{(1)+}_{n_1,\alpha_1}\hat{\Psi}^{(2)+}_{n_2,\alpha_2}\cdots\hat{\Psi}^{(i-1)+}_{n_{i-1},\alpha_{i-1}}\left(\cdots\hat{\Phi}^{(j\to i)+}_{n_j,\alpha_j}\right)\hat{\Psi}^{(j+1)+}_{n_{j+1},\alpha_{j+1}}\cdots\hat{\Psi}^{(d)+}_{n_d,\alpha_d}|0\rangle \qquad \text{for } i \le j \\
&\quad = \hat{\Psi}^{(1)+}_{n_1,\alpha_1}\hat{\Psi}^{(2)+}_{n_2,\alpha_2}\cdots\hat{\Psi}^{(j-1)+}_{n_{j-1},\alpha_{j-1}}\left(\hat{\Phi}^{(j\to i)+}_{n_j,\alpha_j}\cdots\right)\hat{\Psi}^{(i+1)+}_{n_{i+1},\alpha_{i+1}}\cdots\hat{\Psi}^{(d)+}_{n_d,\alpha_d}|0\rangle \qquad \text{for } i > j \\
&= \left|\begin{matrix} n_1, \dots, n_{\min(i,j)-1} \\ \alpha_1, \dots, \alpha_{\min(i,j)-1} \end{matrix}\right\rangle \otimes |\psi\rangle \otimes \left|\begin{matrix} n_{\max(i,j)+1}, \dots, n_d \\ \alpha_{\max(i,j)+1}, \dots, \alpha_d \end{matrix}\right\rangle ,
\end{aligned}
\tag{4.16}
$$

where

$$
|\psi\rangle = \hat{e}_{ij}\left|\begin{matrix} n_{\min(i,j)}, \dots, n_{\max(i,j)} \\ \alpha_{\min(i,j)}, \dots, \alpha_{\max(i,j)} \end{matrix}\right\rangle , \tag{4.17}
$$

can be obtained by normal ordering (i.e., applying the commutation relations to reorder $\hat{\psi}^{\pm}_{i,a}$ on increasing mode index) $\hat{\Psi}^{(i)+}_{n_i,\alpha_i}\cdots\hat{\Psi}^{(j)+}_{n_j,\alpha_j}\hat{\Phi}^{(j\to i)+}_{n_j,\alpha_j}$ or $\hat{\Phi}^{(j\to i)+}_{n_j,\alpha_j}\hat{\Psi}^{(j)+}_{n_j,\alpha_j}\cdots\hat{\Psi}^{(i)+}_{n_i,\alpha_i}$, respectively.

The crucial insight in the above calculation is that $\hat{e}_{ij}$ only affects the sub-states in the interval enclosed by $i, j$. Setting $i, j \in A$ or $i, j \in B$, Eq. (4.16), and the fact that the number states are a basis of the Fock space immediately imply Eq. (4.4) and therefore Eq. (4.3) also holds. We have thus shown that $R$-parastatistics satisfies strong locality (Definition 10).

## 4.3. Strongly local representations give solutions to the Yang-Baxter equation

In this section, we show that strongly local representations give rise to solutions to the Yang-Baxter equations, i.e., $R$-matrices. To this end, we will consider the representations of permutations of modes. Let $P^{(d)}_{\sigma}$ denote the permutation matrix of $\pi \in S(d)$, and, in particular, let

$$
X \equiv P^{(2)}_{12} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P^{(3)}_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P^{(3)}_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad P^{(3)}_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{4.18}
$$

We will now show that the 2-mode representation of the swap $P^{(2)}_{12}$ is an $R$-matrix. Since $P^{(3)}_{12} = P^{(2)}_{12} \oplus \mathbb{1}_1$ and $P^{(3)}_{23} = \mathbb{1}_1 \oplus P^{(2)}_{12}$ as well as

$$
P^{(3)}_{12}P^{(3)}_{23}P^{(3)}_{13} = P^{(3)}_{23}P^{(3)}_{12}P^{(3)}_{23}, \tag{4.19}
$$

due to the $P_\sigma^{(d)}$ being permutations, it follows that for any strongly local representation $\Delta_2(X)$ is an $R$-matrix. This can be seen by direct calculation:

$$\begin{aligned}
&(\Delta_2(X)\otimes \mathbb{1})\,(\mathbb{1}\otimes \Delta_2(X))\,(\Delta_2(X)\otimes \mathbb{1})\\
&= \Delta_3\left(X\oplus \mathbb{1}_1\right)\ \Delta_3\left(\mathbb{1}_1\oplus X\right)\ \Delta_3\left(X\oplus \mathbb{1}_1\right)\\
&= \Delta_3\left(P_{12}^{(3)}P_{23}^{(3)}P_{12}^{(3)}\right) = \Delta_3\left(P_{23}^{(3)}P_{12}^{(3)}P_{23}^{(3)}\right)\\
&= \Delta_3\left(\mathbb{1}_1\oplus X\right)\Delta_3\left(X\oplus \mathbb{1}_1\right)\ \Delta_3\left(\mathbb{1}_1\oplus X\right)\\
&= \ (\mathbb{1}\otimes \Delta_2(X))\,(\Delta_2(X)\otimes \mathbb{1})\,(\mathbb{1}\otimes \Delta_2(X))\,.
\end{aligned} \tag{4.20}$$

The assumption of strong locality in Equation (4.3) is crucial in this argument as it connects the representations $\Delta_d$ for different $d$ in such a way that they are consistent with the structure of permutations of modes. It should also be noted that the $R$-matrix defined this way acts on the whole Fock space as

$$\Delta_2(X): \mathcal{F}\otimes\mathcal{F}\to\mathcal{F}\otimes\mathcal{F}. \tag{4.21}$$

In order to further align these results with Wang's formalism, we want to show the following: not only does $\Delta_2(X)$ satisfy the Yang-Baxter equation, but also the projection of the swap on the wt $=(1,1)$ particle sector is an $R$-matrix.

**Theorem 4.** *If $\Delta_d$ is a family of strongly local representations in the sense of Equation* (4.3)*, then the $m\times m\times m\times m$-matrix $R_{cd}^{ab}$ defined as*

$$\Delta_2(X)\left|\begin{smallmatrix}1&1\\a&b\end{smallmatrix}\right\rangle = \sum_{cd} R_{ab}^{cd}\left|\begin{smallmatrix}1&1\\c&d\end{smallmatrix}\right\rangle\,, \tag{4.22}$$

*satisfies the Yang-Baxter equation*

$$(\mathbb{1}\otimes R)(R\otimes\mathbb{1})(\mathbb{1}\otimes R) = (R\otimes\mathbb{1})(\mathbb{1}\otimes R)(R\otimes\mathbb{1}). \tag{4.23}$$

*Moreover, $R$ can (as an $m^2\times m^2$) matrix be written as*

$$R_{(cd)}^{(ab)} = M_{(1,1)}^{-1} D M_{(1,1)}, \tag{4.24}$$

*where $D$ is a diagonal matrix with entries $\pm1$ and $M_{(1,1)}$ denotes the Block of $M$ corresponding to the weight $\vec{n}=(1,1)$ in equation* (3.56)*.*

*Proof.* Since the representation $\Delta_2(X)$ as a whole satisfies the YBE as shown in equation (4.20), it remains to show that this also holds for the action of the swap in the wt $=(1,1)$ particle sector, i.e. the subspace of $\mathcal{F}_2$ that is spanned by vectors of the form $\left|\begin{smallmatrix}1&1\\a&b\end{smallmatrix}\right\rangle$. First we show that $\Delta_2(X)$ stabilizes this subspace: Due to Lemma 2, the matrix $M$ couples the basis of the wt $=(1,1)$ particle sector with the block $M_{(1,1)}$ purely to Young basis vectors of the form

$$\left|\begin{smallmatrix}\boxed{1}\\\boxed{2}\end{smallmatrix},\kappa\right\rangle\,,\qquad \text{or}\quad \left|\boxed{1}\boxed{2},\kappa\right\rangle\,, \tag{4.25}$$

with undetermined multiplicity indices $\kappa$, the action $\Delta_2(P^{(2)}_{12})$ trivially stabilizes the one-dimensional 'fermionic' irrep $\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$ and thus $\left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}, \kappa\right\rangle$ is an eigenvector of the swap. We already showed that $\left|\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}, \kappa\right\rangle$ is also stabilized (an eigenvector) by an explicit calculation in Eq. (3.40) in Section 3.2.

This implies that equation (4.22) is a valid expression, as the wt $=(1,1)$-sector of $\mathcal{F}_2$ is stabilized by the swap. Moreover, the matrix $M_{(1,1)}$ expresses this sector in terms of the vectors (4.25), which are all eigenvectors of the swap. Since $P^{(2)}_{12}$ and thus also its representation is an involution, the eigenvalues of $R$ are $\pm 1$. Thus, equation (3.52) directly yields the decomposition in equation (4.24).

Finally, to prove that the projection $R^{ab}_{cd}$ also satisfies the YBE, consider the wt $=(1,1,1)$ subspace of $\mathcal{F}_3$ spanned by vectors of the form $\left|\begin{smallmatrix}1&1&1\\a&b&c\end{smallmatrix}\right\rangle$. This subspace is preserved by both $\Delta_2(X)\otimes\mathbb{1}$ and $\mathbb{1}\otimes\Delta_2(X)$. Therefore we can write (using Einstein summation convention)

$$
\begin{aligned}
&(\mathbb{1}\otimes\Delta_2(X))\,(\Delta_2(X)\otimes\mathbb{1})\,(\mathbb{1}\otimes\Delta_2(X))\left|\begin{smallmatrix}1&1&1\\a&b&c\end{smallmatrix}\right\rangle\\
&\quad= R^{gh}_{bc}R^{ef}_{ag}R^{rs}_{fh}\left|\begin{smallmatrix}1&1&1\\e&r&s\end{smallmatrix}\right\rangle, \qquad\qquad \text{and}
\end{aligned}
\tag{4.26}
$$

$$
\begin{aligned}
&(\Delta_2(X)\otimes\mathbb{1})\,(\mathbb{1}\otimes\Delta_2(X))\,(\Delta_2(X)\otimes\mathbb{1})\left|\begin{smallmatrix}1&1&1\\a&b&c\end{smallmatrix}\right\rangle\\
&\quad= R^{ef}_{ab}R^{gh}_{fc}R^{rs}_{eg}\left|\begin{smallmatrix}1&1&1\\r&s&h\end{smallmatrix}\right\rangle = R^{hf}_{ab}R^{gs}_{fc}R^{er}_{hg}\left|\begin{smallmatrix}1&1&1\\e&r&s\end{smallmatrix}\right\rangle,
\end{aligned}
\tag{4.27}
$$

Since the LHSs of Eq. (4.26) and (4.27) are equal by Eq. (4.20), also the RHSs are equal. But this is precisely Eq. (2.45), the YBE in index notation. Furthermore, as the vectors $\left|\begin{smallmatrix}1&1&1\\a&b&c\end{smallmatrix}\right\rangle$ span the image of $R\otimes\mathbb{1}$ (where $\mathbb{1}$ acts in the single particle sector), the matrix $R$ indeed solves the Yang-Baxter equation (4.23). $\square$

With the statements from Section 4.2 and Theorem 4, we have thus established a connection between strongly local representations of transtatistics and $R$-parastatistics: The latter are strongly local, while from every strongly local representation we can construct an $R$-matrix which defines parastatistics. However, this is not a bijection in a direct sense. This is because we have not assured that the $R$-parastatistics constructed in this manner are the same—or at least isomorphic in some sense—to the original strongly local representation. We will tackle this problem in the next section.

## 4.4. Symmetrization of the Fock space

In the following, our aim is to connect our findings about strong locality with the structure of the Fock space. To this end, we postulate a generalization of the (anti)symmetrization postulate of ordinary fermionic and bosonic Fock spaces.

**Definition 11.** ***(Generalized symmetrization postulate)***
*We call a graded vector space $\mathcal{F}_1 = \bigoplus_{n=0}^{\infty} \mathcal{F}_1^{(n)}$ swap-symmetrized, if it is quadratically generated from the one-particle sector $\mathcal{F}_1^{(1)}$ with the relations being*

$$\mathcal{F}_1 = \frac{\mathcal{T}(\mathcal{F}_1^{(1)})}{\langle \mathrm{im}[\Delta_2(X) - \mathbb{1}]\big|_{\mathrm{wt}=(1,1)} \rangle}, \tag{4.28}$$

*where $\Delta_2(X)$ denotes the representation of the swap on the two-mode Fock space $\mathcal{F}_2 = \mathcal{F}_1 \otimes \mathcal{F}_1$.*
*We call $\mathcal{F}_d = \mathcal{F}_1^{\otimes d}$ swap-symmetrized iff $\mathcal{F}_1$ is swap-symmetrized.*

This assumption may seem very ad-hoc and heavy in formalism; however, it can be regarded as an incarnation of an operational view on indistinguishability of particles. In the usual sense—as required in the standard argument for bosons and fermions, for example—it requires the notion of swapping individual particles, which should not have any notable effect. However, in the formalism developed here, we do not have operational "access" to individual particles, only modes can be acted upon. Nevertheless, for the sake of argument, we can suppose that two particles are operationally indistinguishable in all regards, except residing in two different modes. Therefore, the action of swapping the modes ($\Delta_2(X)$), and thus the particles within, is operationally indistinguishable from not performing any action ($\mathbb{1}$). We then propose that when an observer cannot distinguish the modes (they reside in the same single-mode Fock space), this is the same situation as two distinguishable modes subject to the relation that a swap is not detectable. This is illustrated in Figure 4.2. These considerations can perhaps also be understood as a mechanism of coarse-graining: The way two modes can be merged into one is by forgetting about the *mode* degree of freedom. And within a mode, swapping two particles should not be detectable, which is why the coarse-graining process involves swap-symmetrizing.

Therefore, all states that differ only by an application of the swap should be considered to lie in the same equivalence class, as described by the quotient in Definition 11. Moreover, the principle should hold for arbitrary numbers of particles in a way such that the coarse-graining is "associative" for multiple modes, which is taken care of by the tensor algebra and the ideal appearing in (4.28). The introduction of this axiom now allows us to prove that the $R$-matrix of the swap, as defined in Theorem 4, determines the Hilbert series of the Fock space.

**Theorem 5.** *Let $\mathcal{F}_d = \mathcal{F}_1^{\otimes d}$ be a swap-symmetrized Fock space that is endowed with a strongly local representation $\Delta_d$. Then $\mathcal{F}_1$ is quadratically generated by an R-matrix:*

$$\mathcal{F}_1 = \frac{\mathcal{T}(\mathcal{H})}{\langle \mathrm{im}[R - \mathbb{1}] \rangle}. \tag{4.29}$$

*Proof.* Since the representation $\Delta_d$ is strongly local, by Theorem 4, the action of the

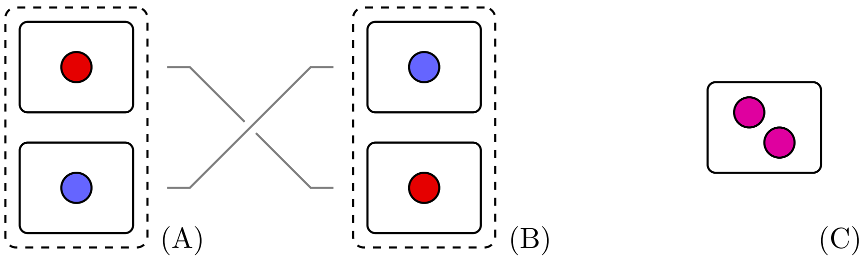



Figure 4.2.: Indistinguishable particles: Boxes represent modes, colors stand for potential non-detectable degrees of freedom. If the swapping of two modes containing particles (A) vs (B) is not detectable, the particles can be regarded as residing in a single mode (C).

swap in the wt $= (1,1)$-sector is an involutive $m \times m \times m \times m$ $R$-matrix:

$$\left\langle \begin{matrix}1 & 1\\ a & b\end{matrix} \right| \Delta_2(X) \left| \begin{matrix}1 & 1\\ c & d\end{matrix} \right\rangle = R^{ab}_{cd}. \tag{4.30}$$

Furthermore, it is customary to denote the one-particle sector $\mathcal{F}_1^{(1)}$ containing all the internal degrees of freedom of a single particle with $\mathcal{H} \equiv \mathbb{C}^m$. Combining this with the generalized symmetrization postulate (Definition 11), the quadratic structure of the single mode Fock space

$$\mathcal{F}_1 = \frac{\mathcal{T}(\mathcal{H})}{\langle \mathrm{im}[R - \mathbb{1}] \rangle}, \tag{4.31}$$

immediately follows. □

Although the proof of the above statement is rather straightforward, it shows that the assumption of the generalized symmetrization postulate is indeed so strong that it determines the structure of the entire Fock space from the action of the swap, which is shown earlier to be an $R$-matrix. In fact, Theorem 5 forces $\mathcal{F}_1$ to come from an *$R$-symmetric algebra* [51, 52].
On the other hand, the Fock space of $R$-parastatistics is also precisely such an algebra, as is demonstrated in equation (S15) in [21], albeit using a different notation. As shown in Appendix D, $\Delta_2(X) = R$, so the Fock space of $R$-parastatistics also exhibits the property of being swap-symmetrized.

**Corollary 1.** *Let $\Delta_d$ and $\Delta'_d$ be two strongly local representations acting on the swap-symmetrized Fock spaces $\mathcal{F}_d$ and $\mathcal{F}'_d$, with single mode sector dimensions $d_n$ and $d'_n$, respectively. If $d_1 = d'_1 = m$ and*

$$\Delta_2(X) \left| \begin{matrix}1 & 1\\ a & b\end{matrix} \right\rangle = \Delta_2(X)' \left| \begin{matrix}1 & 1\\ a & b\end{matrix} \right\rangle \qquad \textit{for all } a, b, \tag{4.32}$$

*then $\mathcal{F}_1 \cong \mathcal{F}'_1$ as graded vector spaces and $\Delta_d \cong \Delta'_d$ as representations.*

*Proof.* Condition (4.32) implies that $\Delta_2$ and $\Delta'_2$ give rise to the same $R$-matrix. Therefore, Theorem 5 directly states that $\mathcal{F}_d \simeq \mathcal{F}'_d$ as graded vector spaces. Furthermore, as the Hilbert series and the single-mode characters coincide ($h_{\mathcal{F}} = h_{\mathcal{F}'} = \chi_1 = \chi'_1$), also the $d$-mode characters are equal ($\chi_d = \chi'_d$). Since two representations are isomorphic if and only if their characters agree, it follows that $\Delta_d \cong \Delta'_d$. □

Thereof, a striking consequence emerges: Every strongly local representation $\Delta_d$ on a swap-symmetrized Fock space $\mathcal{F}_d$ is isomorphic to the representation induced by the construction of $R$-paraparticles by Wang and Hazzard [21], with $R$ being the restriction of the swap to the $\mathrm{wt} = (1,1)$-sector. In other words: **Transtatistics under the assumption of strong locality (Definition 10) and the generalized symmetrization postulate (Definition 11) is equivalent to $R$-parastatistics.**

# 5. Conclusion

We conclude this Master's thesis by a recapitulation and discussion of what was achieved. While Chapter 1 introduces the topic and research questions of the thesis, Chapter 2 provides a short overview of the most relevant mathematical tools. In Chapter 3, we give an explicit construction of the Fock space of transtatistics: Starting from a Hilbert series $h_{\mathcal{F}}$, we showed that there are two important bases of $\mathcal{F}_d$: Firstly, the number basis, which encompasses the tensor structure of the Fock space. It differs from the number basis of ordinary bosons and fermions, as it describes a generalized Pauli exclusion principle and allows internal degrees of freedom of particles, called "flavors". Second, it has proven useful, to also implement the Young basis on the Fock space. It naturally covers the structure of the irreducible representations of $U(d)$ and is formalized with semistandard Young tableaux (or Gelfand-Tsetlin patterns, respectively) together with an auxiliary index. Using the calculus of GTPs, it is possible to systematically describe and concretely calculate any unitary action on $\mathcal{F}_d$. These two bases are related by a matrix $M$, which is shown to obey a block-diagonal-like pattern, only coupling sectors that represent the same particle number. Lemma 3 furthermore gives an expression for the size of those sectors and connects them with the combinatorial Kostka numbers.
In Section 3.3, we proceed to present a different construction of the Fock space: As the quotient space of the tensor algebra with a quadratic ideal. This approach is motivated by the works of Segal [40, 41] and also pursued in a recent formalization of transtatistics [19]. This corresponds to taking the full first-quantized picture of distinguishable particles and imposing relations $W$ on the free Fock space. We could show that such a projective approach is always possible for any Fock space of transtatistics, but we could not derive concrete relations just from the assumption of (phase) locality. Notably, such quadratically generated Fock spaces carry an intrinsic notion of duality, of which the relationship between bosons and fermions is the perhaps simplest manifestation.
We conclude the chapter with some remarks on the scalar product and unitarity. By demanding that the Young basis is orthonormal, the scalar product of the number basis can be obtained by constructing a Gram matrix from $M$. whenever $M$ is non-unitary, the number basis is a non-orthogonal basis. Although this seems to starkly contradict the usual notion of quantum mechanics, calculations in such a basis have already been formalized [45, 46], and Wang and Hazzard also consider non-unitary $R$-matrices. To make calculations feasible, it is customary to introduce another set of dual vectors, which we denote as $(\cdot|$. It should be noted, however, that the non-unitary part only appears within number states of the same weight. If one regards the flavors of particles as being truly undetectable, then issues concerning violation of probability conservation are also mitigated.

We proceed with a treatment of locality in Chapter 4. We begin by introducing two additional definitions that aim to supplement the assumption of (phase) locality as introduced by Sánchez and Dakić in [18]. Firstly, we define the notion of separability (Definition 9),

which is specified in an operational context: It demands that for transformations that come from independent devices (i.e., $U_{AB} = U_A \oplus U_B$), the particle number measurements in the subsystems $A$ and $B$ have to be statistically independent, meaning that joint probabilities factorize as $p_{AB} = p_A p_B$.
Since the notion of separability is mathematically difficult to work with as it does only make statements on the measurement probabilities, we also introduce a closely related concept at the level of representations, that is, strong locality. It demands that the representation $\Delta_d$ factorizes as

$$\Delta_{d_A+d_B}(U_A \oplus U_B) = \Delta_{d_A}(U_A) \otimes \Delta_{d_B}(U_B), \tag{5.1}$$

for block diagonal unitaries. These three different notions of locality are related to each other in a hierarchical sense. Namely, we show that strong locality implies both non-signaling as well as phase locality, making it the strictest definition. We leave it as an open question whether non-signaling also implies strong locality, rendering it as an equivalent statement.
In every case, we proceed to show that phase locality is a strictly weaker statement than non-signaling and therefore also strong locality. To this end, we construct an example of a representation of "2+1" block-diagonal unitary, parametrized by $M$, which we set to be a simple permutation matrix. Using numerical software, we compute all joint probabilities $p_{AB}$ and show in Appendix C that some cannot be factorized, violating the non-signaling condition. This shows that even though if the matrix $M$ respects phase locality via Lemma 3, strong locality not necessarily holds. Conversely, strong locality imposes additional constraints on $M$, which we state in equation (4.5). However, solving this for $M$ appears to be a difficult combinatorial problem.

In this work, we aim to make the point that the stricter notion of strong locality should be preferred over mere phase locality. We showed that the latter alone can give rise to arguably nonphysical behavior. We continue by linking strong locality to the constructive model of $R$-parastatistics. Showing that the representation defined via $\hat{e}_{ij} = \sum_a \hat{\psi}^+_{i,a} \hat{\psi}^-_{j,a}$ satisfies strong locality is done in Section 4.2. In the following section, we prove an important step of the converse statement in Theorem 4. Namely, for every strongly local representation $\Delta_d$, the action $\Delta_2(X)$ in the wt $= (1,1)$-sector of the two-mode Fock space is a solution to the Yang-Baxter equation, i.e., an (involutive) $R$-matrix as

$$\left\langle \begin{matrix} 1 & 1 \\ a & b \end{matrix} \right| \Delta_2(X) \left| \begin{matrix} 1 & 1 \\ c & d \end{matrix} \right\rangle = R^{cd}_{ab}. \tag{5.2}$$

Furthermore, the wt $= (1,1)$-sector of $M$ diagonalizes $R$ into a bosonic and a fermionic sector.

Finally, Section 4.4 aims to establish a formal equivalence between transtatistics and $R$-parastatistics under a further assumption: the generalized symmetrization postulate (Definition 11). It demands that the single-mode Fock space $\mathcal{F}_1$ is quadratically generated by the relation that swapping two modes is equivalent to identity. While the mathematical formulation of this postulate arguably appears to be very strict, it can nevertheless be regarded as being motivated by an operational manifestation of the indistinguishability of particles. Concretely, it gives a coarse-graining procedure on the modes, where two particles are said to reside in the same mode, if swapping them cannot be observed. This, together with strong locality, is shown in Theorem 5 and subsequently Corollary 1 to be equivalent to $R$-parastatistics with an $R$-matrix as given in (5.2). Concretely, it forces the underlying Fock space to be an $R$-symmetric algebra. In this sense, we have answered the initial question on a possible equivalence between transtatistics and $R$-parastatistics only to hold under further restrictions, i.e. strong locality and the generalized symmetrization postulate.

## 5.1. Outlook and open questions

In this thesis, we managed to shed some light on the connection between transtatistics and $R$-parastatistics. Nevertheless, many questions regarding this topic were not addressed or only partially. In the following, we conclude by presenting a non-exhaustive list of points that need more rigorous treatment or which remained completely elusive.

1. One question concerns a potential extension of the formalisms presented in this Master's thesis: It is possible to loosen the condition of involutivity to the so-called *Hecke condition*, meaning that

$$(R - q\mathbb{1})\left(R + q^{-1}\mathbb{1}\right) = 0, \tag{5.3}$$

for some $q \in \mathbb{C} \setminus \{0\}$. With this generalization, many of the underlying theorems in, e.g. Refs. [22, 51, 52] still apply. Thereof, the question arises, whether it is possible to consistently define Fock spaces as well as creation and annihilation algebras using Hecke $R$-matrices.
Speculatively, these extensions might be able to give rise to anyons, since the swap squared not being identity is a hallmark feature of these quasiparticles. This claim is supported by [53] and the references therein, which link the Schur polynomials to anyonic fusion rules.

2. One of the core assumptions in both transtatistics and $R$-parastatistics is that of finitely many modes. But in order to incorporate also degrees of freedom like a particle's position or momentum, one needs to extend the formalism to incorporate continuous variables as mode indices. A naïve approach would be to simply generalize the commutation relations of $R$-parastatistics to

$$\hat{\psi}^{-}_{x,a}\hat{\psi}^{+}_{y,b} = \sum_{cd} R^{bd}_{ac}\hat{\psi}^{+}_{y,c}\hat{\psi}^{-}_{x,d} + \delta_{ab}\delta(x-y), \tag{5.4}$$

for continuous $x, y$. However, it is not clear whether this ansatz directly leads to inconsistencies, especially convergence-related ones. Also, the formulation of strong locality might not be achievable for continuous variables as it seems to require infinite tensor products.

3. In equation (4.5) we specified an implicit condition on $M_d$ in order to define a strongly local representation, but did not attempt to find a general solution. It seems worth pursuing to find an expression for $M_d$ in terms of $M_{d-1}$. One path to a solution requires finding a procedure to "extend" SSYTs to more modes in a way that is consistent with the action of $\hat{e}_{ij}$ acting on subsystems.
   Related to this is the general solvability of equation (4.5), i.e., " Does $M_d$ always exist?" In other words, can every phase-local representation be made strongly local by the correct choice of basis? Answering this amounts to asking whether, for every TPS, there is an $R$-matrix which has it as its Hilbert series, which appears to still be an open problem in mathematics.
4. The generalized symmetrization postulate (Definition 11) is a very strong statement, as it directly forces $\mathcal{F}_1$ to be quadratically generated. One might attempt to relax this postulate, perhaps by utilizing Theorem 3, in order to find a *minimal* assumption in addition to strong locality that establishes an equivalence between transtatistics and $R$-parastatistics.
5. On the mathematical side, it might be desirable to formally treat negative occupation numbers. As suggested in Figure 3.1, these could correspond to Young diagrams with negative size. Possibly, this is already covered in the formalism when extending to rational representations of $GL(d)$ [25]. Moreover, the question remains of how exceptional statistics as presented in Ref. [18] fit into the picture. That is, can one make operational sense of a divergent character $\chi_1$, or is this just an artifact that should be discarded?

Although we have made progress on illuminating the relation between these two formalisms, there is still a variety of unanswered questions. We leave them open for future work.

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# A. Appendix: Bijection between SSYTs and GTPs

In this section, we discuss the bijection between semistandard Young tableaux and Gelfand-Tsetlin patterns. We summarize the arguments described in [26, 38] and show the bijection on an illustrative example.

Let $\Lambda$ be a SSYT. Then the $n$-th row $r_{k,n}$ (counted from below) of the corresponding GTP is given by to the shape of the Young diagram $\lambda^{\leq n}$ that encloses all squares of $\Lambda$ that have entries less than or equal to $n$. In particular, the top row $r_{k,d}$ indicates the shape of $\Lambda$. This shown in Figure A.1 for an SSYT with shape $\lambda = (3, 2, 2, 1)$ and $d = 4$. As can also be observed therein, the number of times the symbol $i$ appears in the SSYT, is precisely the difference of sizes $|\lambda^{\leq i}| - |\lambda^{\leq i-1}|$. Therefore, the weight is given by the formula

$$\mathrm{wt}_i \Lambda = \sum_k r_{k,i} - \sum_k r_{k,i-1}. \tag{A.1}$$

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 2 \\ \cline{1-2} 3 & 4 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array} \quad \xleftrightarrow{\leq 4} \quad \begin{pmatrix} 3 & & 2 & & 2 & & 1 \\ & 3 & & 2 & & 1 & \\ & & 2 & & 2 & & \\ & & & 2 & & & \end{pmatrix}$$

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 2 \\ \cline{1-2} 3 & 4 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array} \quad \xleftrightarrow{\leq 3} \quad \begin{pmatrix} 3 & & 2 & & 2 & & 1 \\ & 3 & & 2 & & 1 & \\ & & 2 & & 2 & & \\ & & & 2 & & & \end{pmatrix}$$

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 2 \\ \cline{1-2} 3 & 4 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array} \quad \xleftrightarrow{\leq 2} \quad \begin{pmatrix} 3 & & 2 & & 2 & & 1 \\ & 3 & & 2 & & 1 & \\ & & 2 & & 2 & & \\ & & & 2 & & & \end{pmatrix}$$

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 2 \\ \cline{1-2} 3 & 4 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array} \quad \xleftrightarrow{\leq 1} \quad \begin{pmatrix} 3 & & 2 & & 2 & & 1 \\ & 3 & & 2 & & 1 & \\ & & 2 & & 2 & & \\ & & & 2 & & & \end{pmatrix}$$

Figure A.1.: Example of the bijection of an SSYT and its according GTP for $d = 4$.

# B. Appendix: Numerical algorithm to compute representations

In this section, we give a brief overview of a numerical algorithm to compute the representations $\Delta_d(U)$ of unitaries acting in a Fock space $\mathcal{F}_d$ parametrized by a totally positive sequence $h_{\mathcal{F}}(x) = \sum_n d_n x^n$.

We present the skeletons of the relevant functions in Python-like pseudocode, where we make extensive use of subroutines implemented in `sagemath` [23], a software library providing a vast number of mathematical objects and algorithms. In particular, the package implements Schur polynomials, SSYTs, GTPs and various linear algebra routines. The code listed in this appendix is for illustrative purposes only and is not intended to be run. In particular, it features no input sanitization, data conversion, or performance optimizations.

In Listing B.1 we provide the functions to calculate the representation $\Delta_d(U) = \exp \delta_d(iH)$ of any $d \times d$ matrix on $\mathcal{F}_d$ in the Young basis. The auxiliary functions `multiply_character` and `modify_gtp` serve the purpose of expressing the character $\chi_d = h_{\mathcal{F}}^d$ in terms of Young diagrams $\lambda$ and multiplicities $m_\lambda$, as well as computing the incremented (decremented) GTP $\Lambda \pm \delta_{ij}$. The functions `representation_e_i_j` and `representation_e_band` implement the expressions for the representations lie algebra generators $\hat{e}_{ij}$ as given in (3.19) and (3.16)-(3.18), although only for a fixed YD $\lambda$. The full action of a matrix $\delta_d(H) = \sum_{i,j} h_{ij} \hat{e}_{ij}$ and its exponential on the entire Fock space parametrized by $\{(\lambda, m_\lambda)\}$ is calculated using the functions `repr_matrix_on_irrep` and `repr_matrix_exp`, which take linear combinations of $\hat{e}_{ij}$, combine them in the irrep sectors as block-diagonal matrices, and finally take the matrix exponential in order to obtain the representation $\Delta_d$ in the Young basis.

```
from sage.all import *
def multiply_character(coefficients, d):
    schur_fn = SymmetricFunctions(ZZ).s()
    R = PolynomialRing(ZZ, "m")
    P = PolynomialRing(ZZ,_VAR_ALPHABET[:d] if d<len(_VAR_ALPHABET) else [
    "y"+str(i) for i in range(d)] , Integer(d))
    x = P.gens()
    chi_1 = PolynomialRing(ZZ, "m")(coefficients)
    chi_d = product([chi_1(m=x[i]) for i in range(d)])
    schur_character = schur_fn.from_polynomial(chi_d)
    partitions = schur_character.support()
    sectors = [(schur_character.coefficient(lambda_), Partition(lambda_))
    for lambda_ in partitions if schur_fn(lambda_).expand(d) != 0]
    sectors.sort(key= lambda x: x[1].size())
    return sectors

def modify_gtp(gtp, p, q, summand):
    new_gt = deepcopy(gtp)
```

```
    new_gt[len(gtp[0]) - q -1 ][p] += summand
    try:
        new_gt.check()
        return new_gt
    except AssertionError:
        return None

def representation_e_i_j(i, j, shape, d):
    if abs(i - j) <= 1:
        SSYTs = SemistandardTableaux(shape, max_entry=d)
        gtp_basis = [GelfandTsetlinPatterns(n=d)(tableau) for tableau in SSYTs]
        return representation_e_band(i, j, gtp_basis, d)
    elif i > j + 1:
        A = representation_e_i_j(i, i-1, shape, d)
        B = representation_e_i_j(i-1, j, shape, d)
        return A * B - B * A
    elif i < j - 1:
        A = representation_e_i_j(i, i+1, shape, d)
        B = representation_e_i_j(i+1, j, shape, d)
        return A * B - B * A

def representation_e_band(k, j, gtp_basis, d):
    n = len(gtp_basis)
    if k == j:
        return diagonal_matrix(field, (gtp.weight()[k] for gtp in gtp_basis))
    elif j > k:
        sign  = +1
        shift = 0
        k = j
    else:
        sign  = -1
        shift = 1
    result = zero_matrix(field, n, n, sparse = True)
    for col_idx, gtp in zip(range(n), gtp_basis):
        def l(p, q):
            return gtp[d-q-1][p] - p
        for i in range(k):
            new_gtp = modify_gtp(gtp, i, k-1, sign)
            if new_gtp is not None:
                row_idx = gtp_basis.index(new_gtp)
                coefficient = sqrt( abs((-1)**(k%2) *
                    prod(( l(r,k)    - l(i,k-1) + shift          for r in range(k+1))) *
                    prod(( l(r,k-2)  - l(i,k-1) + shift -1       for r in range(k-1))) /
                    prod(((l(r,k-1)  - l(i,k-1) + shift) * (l(r,k-1) - l(i,k-1) + shift -1)
                        for r in range(k) if r != i)) )
                    )
                result[row_idx, col_idx] += coefficient
    return result
```

```python

def repr_matrix_on_irrep(mat, shape):
    d = mat.ncols()
    return sum(mat[i][j] * representation_e_i_j(i, j, d, shape)
                for i in range(d) for j in range(d) if not bool(mat[i][j] == 0))

def repr_matrix_exp(mat, sectors):
    def itr():
        for mult, shape in sectors:
            rep_lie = repr_matrix_on_irrep(mat, shape)
            rep = rep_lie.exp()
            for i in range(mult):
                yield rep
    return matrix.block_diagonal(list(itr()))
```

Listing B.1: Representation in the Young basis

In Listing B.2 we illustrate how the probabilities of measuring given particle numbers are calculated after applying unitary transformations to certain input states. The probabilities $p_A$ and $p_{AB}$ are computed for transformations generated by the Hamiltonians $H_{\mathrm{BS}}$ from equation (4.10) and $H_{\mathrm{tot}} = H_{\mathrm{BS}} \oplus 0$, respectively. The function `get_M_matrix` creates $M$ as a simple permutation matrix such that it is compliant with Lemma 2, i.e., matching the weights of Young and number basis vectors. The function `calculate_distribution` then computes the probabilities of measurement results $p(\vec{n}|\,|\psi\rangle)$ for all possible input states in the number basis. The distributions for probabilities $p_A$ and $p_{AB}$ are calculated via `calculate_distribution(U_BS, 2)` and `calculate_distribution(U_tot, 3)`, respectively. The results are displayed in Appendix C.

```python
import numpy as np
from sage.all import *
import itertools

coefficients = [1, 3, 1]
p = len(coefficients) - 1
H_BS = I*pi*matrix([[-1 + sqrt(2), -1], [-1, 1 + sqrt(2)]]) / (2*sqrt(2))
U_BS = repr_matrix_exp(H_BS, multiply_character(coefficients, 2))
H_tot = matrix.block_diagonal([H_BS, matrix.zero(1)])
U_tot = repr_matrix_exp(H_tot, multiply_character(coefficients, 3))

def get_M_matrix(number_basis, coefficients, d):
    perm_young = []
    perm_number = []
    sectors = multiply_character(coefficients, d)[1]
    young_basis = [ssyt for m, part in sectors for _ in range(m) for ssyt in SemistandardTableaux(part, max_entry=d)]
    def pad(v):
        return tuple(v + (d-len(v))*[0])
    for weight_vector in itertools.product(range(p + 1), repeat = d):
        perm_young += [i+1 for i, x in enumerate(young_basis) if pad(x.weight()) == weight_vector]
        perm_number += [i+1 for i, x in enumerate(number_basis) if tuple(
```

```
    map(lambda y: y[0],x)) == weight_vector]
    print(len(list(number_basis)))
    return Permutation(perm_number).to_matrix() * Permutation(perm_young).
    to_matrix().T

def calculate_distribution(U, d):
    number_basis = list(itertools.product([(n, alpha) for n in range(p +
    1) for alpha in range(coefficients[n])], repeat = d))
    M = get_M_matrix(number_basis, coefficients, d).change_ring(SR)
    U_in_nb = M * U * M.inverse()
    for i, psi in enumerate(number_basis):
        for outcome in itertools.product(range(p + 1), repeat = d):
            probability = 0
            for j, phi in enumerate(number_basis):
                if outcome == tuple(map(lambda x:x[0], phi)):
                    probability += abs(U_in_nb[j, i])**2
            if probability > 0: print(psi, outcome, probability)

calculate_distribution(U_BS, 2)
calculate_distribution(U_tot, 3)
```

Listing B.2: Calculation of the probabilities $p_A$ and $p_{AB}$

# C. Appendix: A phase local representation

In this appendix, we provide the numerical data for an explicit calculation of a phase local representation with character $1 + 3x + x^2$. First, we choose a matrix $M$ such that the basis vectors of the number basis—which are denoted according to (4.8)—are mapped one-to-one to SSYTs in the Young basis. The only criterion for this mapping is that the weights match. Such a mapping was obtained by searching over all permutations and is listed in Tables C.1, C.2 and C.3 for $d = 1$, 2 and $d = 3$, respectively.

| $\lvert\Lambda, \kappa\rangle$ | $M\lvert\Lambda, \kappa\rangle$ |
|---|---|
| $\lvert\bigcirc, 1\rangle$ | $\lvert 0\rangle$ |
| $\lvert\boxed{1}, 1\rangle$ | $\lvert A\rangle$ |
| $\lvert\boxed{1}, 2\rangle$ | $\lvert B\rangle$ |
| $\lvert\boxed{1}, 3\rangle$ | $\lvert C\rangle$ |
| $\lvert\boxed{1}\boxed{1}, 1\rangle$ | $\lvert 2\rangle$ |

Table C.1.: $M$ for $d = 1$

| $\lvert\Lambda, \kappa\rangle$ | $M\lvert\Lambda, \kappa\rangle$ |
|---|---|
| $\lvert\bigcirc, 1\rangle$ | $\lvert 0, 0\rangle$ |
| $\lvert\boxed{1}, 1\rangle$ | $\lvert A, 0\rangle$ |
| $\lvert\boxed{2}, 1\rangle$ | $\lvert 0, A\rangle$ |
| $\lvert\boxed{1}, 2\rangle$ | $\lvert B, 0\rangle$ |
| $\lvert\boxed{2}, 2\rangle$ | $\lvert 0, B\rangle$ |
| $\lvert\boxed{1}, 3\rangle$ | $\lvert C, 0\rangle$ |
| $\lvert\boxed{2}, 3\rangle$ | $\lvert 0, C\rangle$ |
| $\left\lvert\begin{smallmatrix}\boxed{1}\\\boxed{2}\end{smallmatrix}, 1\right\rangle$ | $\lvert A, A\rangle$ |
| $\left\lvert\begin{smallmatrix}\boxed{1}\\\boxed{2}\end{smallmatrix}, 2\right\rangle$ | $\lvert A, B\rangle$ |
| $\left\lvert\begin{smallmatrix}\boxed{1}\\\boxed{2}\end{smallmatrix}, 3\right\rangle$ | $\lvert A, C\rangle$ |
| $\left\lvert\begin{smallmatrix}\boxed{1}\\\boxed{2}\end{smallmatrix}, 4\right\rangle$ | $\lvert B, A\rangle$ |
| $\left\lvert\begin{smallmatrix}\boxed{1}\\\boxed{2}\end{smallmatrix}, 5\right\rangle$ | $\lvert B, B\rangle$ |
| $\left\lvert\begin{smallmatrix}\boxed{1}\\\boxed{2}\end{smallmatrix}, 6\right\rangle$ | $\lvert B, C\rangle$ |
| $\left\lvert\begin{smallmatrix}\boxed{1}\\\boxed{2}\end{smallmatrix}, 7\right\rangle$ | $\lvert C, A\rangle$ |
| $\left\lvert\begin{smallmatrix}\boxed{1}\\\boxed{2}\end{smallmatrix}, 8\right\rangle$ | $\lvert C, B\rangle$ |
| $\lvert\boxed{1}\boxed{1}, 1\rangle$ | $\lvert 2, 0\rangle$ |
| $\lvert\boxed{1}\boxed{2}, 1\rangle$ | $\lvert C, C\rangle$ |
| $\lvert\boxed{2}\boxed{2}, 1\rangle$ | $\lvert 0, 2\rangle$ |
| $\left\lvert\begin{smallmatrix}\boxed{1}&\boxed{1}\\\boxed{2}&\end{smallmatrix}, 1\right\rangle$ | $\lvert 2, A\rangle$ |
| $\left\lvert\begin{smallmatrix}\boxed{1}&\boxed{2}\\\boxed{2}&\end{smallmatrix}, 1\right\rangle$ | $\lvert A, 2\rangle$ |
| $\left\lvert\begin{smallmatrix}\boxed{1}&\boxed{1}\\\boxed{2}&\end{smallmatrix}, 2\right\rangle$ | $\lvert 2, B\rangle$ |
| $\left\lvert\begin{smallmatrix}\boxed{1}&\boxed{2}\\\boxed{2}&\end{smallmatrix}, 2\right\rangle$ | $\lvert B, 2\rangle$ |
| $\left\lvert\begin{smallmatrix}\boxed{1}&\boxed{1}\\\boxed{2}&\end{smallmatrix}, 3\right\rangle$ | $\lvert 2, C\rangle$ |
| $\left\lvert\begin{smallmatrix}\boxed{1}&\boxed{2}\\\boxed{2}&\end{smallmatrix}, 3\right\rangle$ | $\lvert C, 2\rangle$ |
| $\left\lvert\begin{smallmatrix}\boxed{1}&\boxed{1}\\\boxed{2}&\boxed{2}\end{smallmatrix}, 1\right\rangle$ | $\lvert 2, 2\rangle$ |

Table C.2.: $M$ for $d = 2$

In Tables C.5 and C.6 we show the probabilities of measuring different particle numbers $\vec{n}$ after applying a beam splitter given by (4.9) to the (sub)system $A$ or $AB$ of 2 or 3 modes, respectively. For the latter, we also indicate the joint probabilities $p_A p_B$ that are obtained from the RHS in (4.2) and highlight all lines where they differ from the LHS $p_{AB}$.

| $\lvert\Lambda,\kappa\rangle$ | $M\lvert\Lambda,\kappa\rangle$ |
|---|---|
| $\lvert\bigcirc,0\rangle$ | $\lvert 0,0,0\rangle$ |
| $\lvert\boxed{1},1\rangle$ | $\lvert A,0,0\rangle$ |
| $\lvert\boxed{2},1\rangle$ | $\lvert 0,A,0\rangle$ |
| $\lvert\boxed{3},1\rangle$ | $\lvert 0,0,A\rangle$ |
| $\lvert\boxed{1},2\rangle$ | $\lvert B,0,0\rangle$ |
| $\lvert\boxed{2},2\rangle$ | $\lvert 0,B,0\rangle$ |
| $\lvert\boxed{3},2\rangle$ | $\lvert 0,0,B\rangle$ |
| $\lvert\boxed{1},3\rangle$ | $\lvert C,0,0\rangle$ |
| $\lvert\boxed{2},3\rangle$ | $\lvert 0,C,0\rangle$ |
| $\lvert\boxed{3},3\rangle$ | $\lvert 0,0,C\rangle$ |
| $\left\lvert\begin{smallmatrix}1\\2\end{smallmatrix},1\right\rangle$ | $\lvert A,A,0\rangle$ |
| $\left\lvert\begin{smallmatrix}1\\3\end{smallmatrix},1\right\rangle$ | $\lvert A,0,A\rangle$ |
| $\left\lvert\begin{smallmatrix}2\\3\end{smallmatrix},1\right\rangle$ | $\lvert 0,A,A\rangle$ |
| $\left\lvert\begin{smallmatrix}1\\2\end{smallmatrix},2\right\rangle$ | $\lvert A,B,0\rangle$ |
| $\left\lvert\begin{smallmatrix}1\\3\end{smallmatrix},2\right\rangle$ | $\lvert A,0,B\rangle$ |
| $\left\lvert\begin{smallmatrix}2\\3\end{smallmatrix},2\right\rangle$ | $\lvert 0,A,B\rangle$ |
| $\left\lvert\begin{smallmatrix}1\\2\end{smallmatrix},3\right\rangle$ | $\lvert A,C,0\rangle$ |
| $\left\lvert\begin{smallmatrix}1\\3\end{smallmatrix},3\right\rangle$ | $\lvert A,0,C\rangle$ |
| $\left\lvert\begin{smallmatrix}2\\3\end{smallmatrix},3\right\rangle$ | $\lvert 0,A,C\rangle$ |
| $\left\lvert\begin{smallmatrix}1\\2\end{smallmatrix},4\right\rangle$ | $\lvert B,A,0\rangle$ |
| $\left\lvert\begin{smallmatrix}1\\3\end{smallmatrix},4\right\rangle$ | $\lvert B,0,A\rangle$ |
| $\left\lvert\begin{smallmatrix}2\\3\end{smallmatrix},4\right\rangle$ | $\lvert 0,B,A\rangle$ |
| $\left\lvert\begin{smallmatrix}1\\2\end{smallmatrix},5\right\rangle$ | $\lvert B,B,0\rangle$ |
| $\left\lvert\begin{smallmatrix}1\\3\end{smallmatrix},5\right\rangle$ | $\lvert B,0,B\rangle$ |
| $\left\lvert\begin{smallmatrix}2\\3\end{smallmatrix},5\right\rangle$ | $\lvert 0,B,B\rangle$ |
| $\left\lvert\begin{smallmatrix}1\\2\end{smallmatrix},6\right\rangle$ | $\lvert B,C,0\rangle$ |
| $\left\lvert\begin{smallmatrix}1\\3\end{smallmatrix},6\right\rangle$ | $\lvert B,0,C\rangle$ |
| $\left\lvert\begin{smallmatrix}2\\3\end{smallmatrix},6\right\rangle$ | $\lvert 0,B,C\rangle$ |
| $\left\lvert\begin{smallmatrix}1\\2\end{smallmatrix},7\right\rangle$ | $\lvert C,A,0\rangle$ |
| $\left\lvert\begin{smallmatrix}1\\3\end{smallmatrix},7\right\rangle$ | $\lvert C,0,A\rangle$ |
| $\left\lvert\begin{smallmatrix}2\\3\end{smallmatrix},7\right\rangle$ | $\lvert 0,C,A\rangle$ |
| $\left\lvert\begin{smallmatrix}1\\2\end{smallmatrix},8\right\rangle$ | $\lvert C,B,0\rangle$ |
| $\left\lvert\begin{smallmatrix}1\\3\end{smallmatrix},8\right\rangle$ | $\lvert C,0,B\rangle$ |
| $\left\lvert\begin{smallmatrix}2\\3\end{smallmatrix},8\right\rangle$ | $\lvert 0,C,B\rangle$ |
| $\lvert\begin{smallmatrix}1&1\end{smallmatrix},1\rangle$ | $\lvert 2,0,0\rangle$ |
| $\lvert\begin{smallmatrix}1&2\end{smallmatrix},1\rangle$ | $\lvert C,C,0\rangle$ |
| $\lvert\begin{smallmatrix}1&3\end{smallmatrix},1\rangle$ | $\lvert C,0,C\rangle$ |
| $\lvert\begin{smallmatrix}2&2\end{smallmatrix},1\rangle$ | $\lvert 0,2,0\rangle$ |
| $\lvert\begin{smallmatrix}2&3\end{smallmatrix},1\rangle$ | $\lvert 0,C,C\rangle$ |
| $\lvert\begin{smallmatrix}3&3\end{smallmatrix},1\rangle$ | $\lvert 0,0,2\rangle$ |
| $\left\lvert\begin{smallmatrix}1\\2\\3\end{smallmatrix},1\right\rangle$ | $\lvert A,A,A\rangle$ |
| $\left\lvert\begin{smallmatrix}1\\2\\3\end{smallmatrix},2\right\rangle$ | $\lvert A,A,B\rangle$ |
| $\left\lvert\begin{smallmatrix}1\\2\\3\end{smallmatrix},3\right\rangle$ | $\lvert A,A,C\rangle$ |
| $\left\lvert\begin{smallmatrix}1\\2\\3\end{smallmatrix},4\right\rangle$ | $\lvert A,B,A\rangle$ |
| $\left\lvert\begin{smallmatrix}1\\2\\3\end{smallmatrix},5\right\rangle$ | $\lvert A,B,B\rangle$ |
| $\left\lvert\begin{smallmatrix}1\\2\\3\end{smallmatrix},6\right\rangle$ | $\lvert A,B,C\rangle$ |
| $\left\lvert\begin{smallmatrix}1\\2\\3\end{smallmatrix},7\right\rangle$ | $\lvert A,C,A\rangle$ |
| $\left\lvert\begin{smallmatrix}1\\2\\3\end{smallmatrix},8\right\rangle$ | $\lvert A,C,B\rangle$ |
| $\left\lvert\begin{smallmatrix}1\\2\\3\end{smallmatrix},9\right\rangle$ | $\lvert A,C,C\rangle$ |
| $\left\lvert\begin{smallmatrix}1\\2\\3\end{smallmatrix},10\right\rangle$ | $\lvert B,A,A\rangle$ |
| $\left\lvert\begin{smallmatrix}1\\2\\3\end{smallmatrix},11\right\rangle$ | $\lvert B,A,B\rangle$ |
| $\left\lvert\begin{smallmatrix}1\\2\\3\end{smallmatrix},12\right\rangle$ | $\lvert B,A,C\rangle$ |
| $\left\lvert\begin{smallmatrix}1\\2\\3\end{smallmatrix},13\right\rangle$ | $\lvert B,B,A\rangle$ |
| $\left\lvert\begin{smallmatrix}1\\2\\3\end{smallmatrix},14\right\rangle$ | $\lvert B,B,B\rangle$ |
| $\left\lvert\begin{smallmatrix}1\\2\\3\end{smallmatrix},15\right\rangle$ | $\lvert B,B,C\rangle$ |
| $\left\lvert\begin{smallmatrix}1\\2\\3\end{smallmatrix},16\right\rangle$ | $\lvert B,C,A\rangle$ |
| $\left\lvert\begin{smallmatrix}1\\2\\3\end{smallmatrix},17\right\rangle$ | $\lvert B,C,B\rangle$ |
| $\left\lvert\begin{smallmatrix}1\\2\\3\end{smallmatrix},18\right\rangle$ | $\lvert B,C,C\rangle$ |
| $\left\lvert\begin{smallmatrix}1\\2\\3\end{smallmatrix},19\right\rangle$ | $\lvert C,A,A\rangle$ |
| $\left\lvert\begin{smallmatrix}1\\2\\3\end{smallmatrix},20\right\rangle$ | $\lvert C,A,B\rangle$ |
| $\left\lvert\begin{smallmatrix}1\\2\\3\end{smallmatrix},21\right\rangle$ | $\lvert C,A,C\rangle$ |
| $\left\lvert\begin{smallmatrix}1&1\\2\end{smallmatrix},1\right\rangle$ | $\lvert 2,A,0\rangle$ |
| $\left\lvert\begin{smallmatrix}1&1\\3\end{smallmatrix},1\right\rangle$ | $\lvert 2,0,A\rangle$ |
| $\left\lvert\begin{smallmatrix}1&2\\2\end{smallmatrix},1\right\rangle$ | $\lvert A,2,0\rangle$ |
| $\left\lvert\begin{smallmatrix}1&2\\3\end{smallmatrix},1\right\rangle$ | $\lvert C,B,A\rangle$ |
| $\left\lvert\begin{smallmatrix}1&3\\2\end{smallmatrix},1\right\rangle$ | $\lvert C,B,B\rangle$ |
| $\left\lvert\begin{smallmatrix}1&3\\3\end{smallmatrix},1\right\rangle$ | $\lvert A,0,2\rangle$ |
| $\left\lvert\begin{smallmatrix}2&2\\3\end{smallmatrix},1\right\rangle$ | $\lvert 0,2,A\rangle$ |
| $\left\lvert\begin{smallmatrix}2&3\\3\end{smallmatrix},1\right\rangle$ | $\lvert 0,A,2\rangle$ |
| $\left\lvert\begin{smallmatrix}1&1\\2\end{smallmatrix},2\right\rangle$ | $\lvert 2,B,0\rangle$ |
| $\left\lvert\begin{smallmatrix}1&1\\3\end{smallmatrix},2\right\rangle$ | $\lvert 2,0,B\rangle$ |
| $\left\lvert\begin{smallmatrix}1&2\\2\end{smallmatrix},2\right\rangle$ | $\lvert B,2,0\rangle$ |
| $\left\lvert\begin{smallmatrix}1&2\\3\end{smallmatrix},2\right\rangle$ | $\lvert C,B,C\rangle$ |
| $\left\lvert\begin{smallmatrix}1&3\\2\end{smallmatrix},2\right\rangle$ | $\lvert C,C,A\rangle$ |
| $\left\lvert\begin{smallmatrix}1&3\\3\end{smallmatrix},2\right\rangle$ | $\lvert B,0,2\rangle$ |
| $\left\lvert\begin{smallmatrix}2&2\\3\end{smallmatrix},2\right\rangle$ | $\lvert 0,2,B\rangle$ |
| $\left\lvert\begin{smallmatrix}2&3\\3\end{smallmatrix},2\right\rangle$ | $\lvert 0,B,2\rangle$ |
| $\left\lvert\begin{smallmatrix}1&1\\2\end{smallmatrix},3\right\rangle$ | $\lvert 2,C,0\rangle$ |
| $\left\lvert\begin{smallmatrix}1&1\\3\end{smallmatrix},3\right\rangle$ | $\lvert 2,0,C\rangle$ |
| $\left\lvert\begin{smallmatrix}1&2\\2\end{smallmatrix},3\right\rangle$ | $\lvert C,2,0\rangle$ |
| $\left\lvert\begin{smallmatrix}1&2\\3\end{smallmatrix},3\right\rangle$ | $\lvert C,C,B\rangle$ |
| $\left\lvert\begin{smallmatrix}1&3\\2\end{smallmatrix},3\right\rangle$ | $\lvert C,C,C\rangle$ |
| $\left\lvert\begin{smallmatrix}1&3\\3\end{smallmatrix},3\right\rangle$ | $\lvert C,0,2\rangle$ |
| $\left\lvert\begin{smallmatrix}2&2\\3\end{smallmatrix},3\right\rangle$ | $\lvert 0,2,C\rangle$ |
| $\left\lvert\begin{smallmatrix}2&3\\3\end{smallmatrix},3\right\rangle$ | $\lvert 0,C,2\rangle$ |
| $\left\lvert\begin{smallmatrix}1&1\\2\\3\end{smallmatrix},1\right\rangle$ | $\lvert 2,A,A\rangle$ |
| $\left\lvert\begin{smallmatrix}1&2\\2\\3\end{smallmatrix},1\right\rangle$ | $\lvert A,2,A\rangle$ |
| $\left\lvert\begin{smallmatrix}1&3\\2\\3\end{smallmatrix},1\right\rangle$ | $\lvert A,A,2\rangle$ |
| $\left\lvert\begin{smallmatrix}1&1\\2\\3\end{smallmatrix},2\right\rangle$ | $\lvert 2,A,B\rangle$ |
| $\left\lvert\begin{smallmatrix}1&2\\2\\3\end{smallmatrix},2\right\rangle$ | $\lvert A,2,B\rangle$ |
| $\left\lvert\begin{smallmatrix}1&3\\2\\3\end{smallmatrix},2\right\rangle$ | $\lvert A,B,2\rangle$ |
| $\left\lvert\begin{smallmatrix}1&1\\2\\3\end{smallmatrix},3\right\rangle$ | $\lvert 2,A,C\rangle$ |
| $\left\lvert\begin{smallmatrix}1&2\\2\\3\end{smallmatrix},3\right\rangle$ | $\lvert A,2,C\rangle$ |
| $\left\lvert\begin{smallmatrix}1&3\\2\\3\end{smallmatrix},3\right\rangle$ | $\lvert A,C,2\rangle$ |
| $\left\lvert\begin{smallmatrix}1&1\\2\\3\end{smallmatrix},4\right\rangle$ | $\lvert 2,B,A\rangle$ |
| $\left\lvert\begin{smallmatrix}1&2\\2\\3\end{smallmatrix},4\right\rangle$ | $\lvert B,2,A\rangle$ |
| $\left\lvert\begin{smallmatrix}1&3\\2\\3\end{smallmatrix},4\right\rangle$ | $\lvert B,A,2\rangle$ |
| $\left\lvert\begin{smallmatrix}1&1\\2\\3\end{smallmatrix},5\right\rangle$ | $\lvert 2,B,B\rangle$ |
| $\left\lvert\begin{smallmatrix}1&2\\2\\3\end{smallmatrix},5\right\rangle$ | $\lvert B,2,B\rangle$ |
| $\left\lvert\begin{smallmatrix}1&3\\2\\3\end{smallmatrix},5\right\rangle$ | $\lvert B,B,2\rangle$ |
| $\left\lvert\begin{smallmatrix}1&1\\2\\3\end{smallmatrix},6\right\rangle$ | $\lvert 2,B,C\rangle$ |
| $\left\lvert\begin{smallmatrix}1&2\\2\\3\end{smallmatrix},6\right\rangle$ | $\lvert B,2,C\rangle$ |
| $\left\lvert\begin{smallmatrix}1&3\\2\\3\end{smallmatrix},6\right\rangle$ | $\lvert B,C,2\rangle$ |
| $\left\lvert\begin{smallmatrix}1&1\\2\\3\end{smallmatrix},7\right\rangle$ | $\lvert 2,C,A\rangle$ |
| $\left\lvert\begin{smallmatrix}1&2\\2\\3\end{smallmatrix},7\right\rangle$ | $\lvert C,2,A\rangle$ |

Table C.3.: $M$ for $d=3$ (continued on next page)

| $\vert\Lambda,\kappa\rangle$ | $M\vert\Lambda,\kappa\rangle$ |
|---|---|
| $\left\vert\begin{ytableau}1&3\\2\\3\end{ytableau},7\right\rangle$ | $\vert C,A,2\rangle$ |
| $\left\vert\begin{ytableau}1&1\\2\\3\end{ytableau},8\right\rangle$ | $\vert 2,C,B\rangle$ |
| $\left\vert\begin{ytableau}1&2\\2\\3\end{ytableau},8\right\rangle$ | $\vert C,2,B\rangle$ |
| $\left\vert\begin{ytableau}1&3\\2\\3\end{ytableau},8\right\rangle$ | $\vert C,B,2\rangle$ |
| $\left\vert\begin{ytableau}1&1\\2&2\end{ytableau},1\right\rangle$ | $\vert 2,2,0\rangle$ |
| $\left\vert\begin{ytableau}1&1\\2&3\end{ytableau},1\right\rangle$ | $\vert 2,C,C\rangle$ |
| $\left\vert\begin{ytableau}1&1\\3&3\end{ytableau},1\right\rangle$ | $\vert 2,0,2\rangle$ |
| $\left\vert\begin{ytableau}1&2\\2&3\end{ytableau},1\right\rangle$ | $\vert C,2,C\rangle$ |
| $\left\vert\begin{ytableau}1&2\\3&3\end{ytableau},1\right\rangle$ | $\vert C,C,2\rangle$ |
| $\left\vert\begin{ytableau}2&2\\3&3\end{ytableau},1\right\rangle$ | $\vert 0,2,2\rangle$ |
| $\left\vert\begin{ytableau}1&1\\2&2\\3\end{ytableau},1\right\rangle$ | $\vert 2,2,A\rangle$ |
| $\left\vert\begin{ytableau}1&1\\2&3\\3\end{ytableau},1\right\rangle$ | $\vert 2,A,2\rangle$ |
| $\left\vert\begin{ytableau}1&2\\2&3\\3\end{ytableau},1\right\rangle$ | $\vert A,2,2\rangle$ |
| $\left\vert\begin{ytableau}1&1\\2&2\\3\end{ytableau},2\right\rangle$ | $\vert 2,2,B\rangle$ |
| $\left\vert\begin{ytableau}1&1\\2&3\\3\end{ytableau},2\right\rangle$ | $\vert 2,B,2\rangle$ |
| $\left\vert\begin{ytableau}1&2\\2&3\\3\end{ytableau},2\right\rangle$ | $\vert B,2,2\rangle$ |
| $\left\vert\begin{ytableau}1&1\\2&2\\3\end{ytableau},3\right\rangle$ | $\vert 2,2,C\rangle$ |
| $\left\vert\begin{ytableau}1&1\\2&3\\3\end{ytableau},3\right\rangle$ | $\vert 2,C,2\rangle$ |
| $\left\vert\begin{ytableau}1&2\\2&3\\3\end{ytableau},3\right\rangle$ | $\vert C,2,2\rangle$ |
| $\left\vert\begin{ytableau}1&1\\2&2\\3&3\end{ytableau},1\right\rangle$ | $\vert 2,2,2\rangle$ |

Table C.4.: $M$ for $d=3$ (continuation)

| $\vert\psi\rangle_A$ | $n_1$ | $n_2$ | p |
|---|---|---|---|
| $\vert 0,0\rangle$ | 0 | 0 | 1.0 |
| $\vert 0,A\rangle$ | 0 | 1 | 0.5 |
| | 1 | 0 | 0.5 |
| $\vert 0,B\rangle$ | 0 | 1 | 0.5 |
| | 1 | 0 | 0.5 |
| $\vert 0,C\rangle$ | 0 | 1 | 0.5 |
| | 1 | 0 | 0.5 |
| $\vert 0,2\rangle$ | 0 | 2 | 0.25 |
| | 1 | 1 | 0.5 |
| | 2 | 0 | 0.25 |
| $\vert A,0\rangle$ | 0 | 1 | 0.5 |
| | 1 | 0 | 0.5 |
| $\vert A,A\rangle$ | 1 | 1 | 1.0 |
| $\vert A,B\rangle$ | 1 | 1 | 1.0 |

| $\vert\psi\rangle_A$ | $n_1$ | $n_2$ | p |
|---|---|---|---|
| $\vert A,C\rangle$ | 1 | 1 | 1.0 |
| $\vert A,2\rangle$ | 1 | 2 | 0.5 |
| | 2 | 1 | 0.5 |
| $\vert B,0\rangle$ | 0 | 1 | 0.5 |
| | 1 | 0 | 0.5 |
| $\vert B,A\rangle$ | 1 | 1 | 1.0 |
| $\vert B,B\rangle$ | 1 | 1 | 1.0 |
| $\vert B,C\rangle$ | 1 | 1 | 1.0 |
| $\vert B,2\rangle$ | 1 | 2 | 0.5 |
| | 2 | 1 | 0.5 |
| $\vert C,0\rangle$ | 0 | 1 | 0.5 |
| | 1 | 0 | 0.5 |
| $\vert C,A\rangle$ | 1 | 1 | 1.0 |
| $\vert C,B\rangle$ | 1 | 1 | 1.0 |

| $\vert\psi\rangle_A$ | $n_1$ | $n_2$ | p |
|---|---|---|---|
| $\vert C,C\rangle$ | 0 | 2 | 0.5 |
| | 2 | 0 | 0.5 |
| $\vert C,2\rangle$ | 1 | 2 | 0.5 |
| | 2 | 1 | 0.5 |
| $\vert 2,0\rangle$ | 0 | 2 | 0.25 |
| | 1 | 1 | 0.5 |
| | 2 | 0 | 0.25 |
| $\vert 2,A\rangle$ | 1 | 2 | 0.5 |
| | 2 | 1 | 0.5 |
| $\vert 2,B\rangle$ | 1 | 2 | 0.5 |
| | 2 | 1 | 0.5 |
| $\vert 2,C\rangle$ | 1 | 2 | 0.5 |
| | 2 | 1 | 0.5 |
| $\vert 2,2\rangle$ | 2 | 2 | 1.0 |

Table C.5.: Probabilities to measure the particle numbers $\vec{n}_A=(n_1,n_2)$ after applying the beam splitter to number states $\vert\psi\rangle_A$ of two modes. Only outcomes with nonzero probabilities are shown.

| $\|\psi\rangle_{AB}$ | $n_1$ | $n_2$ | $n_3$ | $p_{AB}$ | $p_A p_B$ |
|---|---|---|---|---|---|
| $\|0,0,0\rangle$ | 0 | 0 | 0 | 1.0 | 1.0 |
| $\|0,0,A\rangle$ | 0 | 0 | 1 | 1.0 | 1.0 |
| $\|0,0,B\rangle$ | 0 | 0 | 1 | 1.0 | 1.0 |
| $\|0,0,C\rangle$ | 0 | 0 | 1 | 1.0 | 1.0 |
| $\|0,0,2\rangle$ | 0 | 0 | 2 | 1.0 | 1.0 |
| $\|0,A,0\rangle$ | 0 | 1 | 0 | 0.5 | 0.5 |
| $\|0,A,0\rangle$ | 1 | 0 | 0 | 0.5 | 0.5 |
| $\|0,A,A\rangle$ | 0 | 1 | 1 | 0.5 | 0.5 |
| $\|0,A,A\rangle$ | 1 | 0 | 1 | 0.5 | 0.5 |
| $\|0,A,B\rangle$ | 0 | 1 | 1 | 0.5 | 0.5 |
| $\|0,A,B\rangle$ | 1 | 0 | 1 | 0.5 | 0.5 |
| $\|0,A,C\rangle$ | 0 | 1 | 1 | 0.5 | 0.5 |
| $\|0,A,C\rangle$ | 1 | 0 | 1 | 0.5 | 0.5 |
| $\|0,A,2\rangle$ | 0 | 1 | 2 | 0.5 | 0.5 |
| $\|0,A,2\rangle$ | 1 | 0 | 2 | 0.5 | 0.5 |
| $\|0,B,0\rangle$ | 0 | 1 | 0 | 0.5 | 0.5 |
| $\|0,B,0\rangle$ | 1 | 0 | 0 | 0.5 | 0.5 |
| $\|0,B,A\rangle$ | 0 | 1 | 1 | 0.5 | 0.5 |
| $\|0,B,A\rangle$ | 1 | 0 | 1 | 0.5 | 0.5 |
| $\|0,B,B\rangle$ | 0 | 1 | 1 | 0.5 | 0.5 |
| $\|0,B,B\rangle$ | 1 | 0 | 1 | 0.5 | 0.5 |
| $\|0,B,C\rangle$ | 0 | 1 | 1 | 0.5 | 0.5 |
| $\|0,B,C\rangle$ | 1 | 0 | 1 | 0.5 | 0.5 |
| $\|0,B,2\rangle$ | 0 | 1 | 2 | 0.5 | 0.5 |
| $\|0,B,2\rangle$ | 1 | 0 | 2 | 0.5 | 0.5 |
| $\|0,C,0\rangle$ | 0 | 1 | 0 | 0.5 | 0.5 |
| $\|0,C,0\rangle$ | 1 | 0 | 0 | 0.5 | 0.5 |
| $\|0,C,A\rangle$ | 0 | 1 | 1 | 0.5 | 0.5 |
| $\|0,C,A\rangle$ | 1 | 0 | 1 | 0.5 | 0.5 |
| $\|0,C,B\rangle$ | 0 | 1 | 1 | 0.5 | 0.5 |
| $\|0,C,B\rangle$ | 1 | 0 | 1 | 0.5 | 0.5 |
| $\|0,C,C\rangle$ | 0 | 1 | 1 | 0.5 | 0.5 |
| $\|0,C,C\rangle$ | 1 | 0 | 1 | 0.5 | 0.5 |
| $\|0,C,2\rangle$ | 0 | 1 | 2 | 0.5 | 0.5 |
| $\|0,C,2\rangle$ | 1 | 0 | 2 | 0.5 | 0.5 |
| $\|0,2,0\rangle$ | 0 | 2 | 0 | 0.25 | 0.25 |
| $\|0,2,0\rangle$ | 1 | 1 | 0 | 0.5 | 0.5 |
| $\|0,2,0\rangle$ | 2 | 0 | 0 | 0.25 | 0.25 |
| $\|0,2,A\rangle$ | 0 | 2 | 1 | 0.25 | 0.25 |
| $\|0,2,A\rangle$ | 1 | 1 | 1 | 0.5 | 0.5 |
| $\|0,2,A\rangle$ | 2 | 0 | 1 | 0.25 | 0.25 |
| $\|0,2,B\rangle$ | 0 | 2 | 1 | 0.25 | 0.25 |

| $\|\psi\rangle_{AB}$ | $n_1$ | $n_2$ | $n_3$ | $p_{AB}$ | $p_A p_B$ |
|---|---|---|---|---|---|
| $\|0,2,B\rangle$ | 1 | 1 | 1 | 0.5 | 0.5 |
| $\|0,2,B\rangle$ | 2 | 0 | 1 | 0.25 | 0.25 |
| $\|0,2,C\rangle$ | 0 | 2 | 1 | 0.25 | 0.25 |
| $\|0,2,C\rangle$ | 1 | 1 | 1 | 0.5 | 0.5 |
| $\|0,2,C\rangle$ | 2 | 0 | 1 | 0.25 | 0.25 |
| $\|0,2,2\rangle$ | 0 | 2 | 2 | 0.25 | 0.25 |
| $\|0,2,2\rangle$ | 1 | 1 | 2 | 0.5 | 0.5 |
| $\|0,2,2\rangle$ | 2 | 0 | 2 | 0.25 | 0.25 |
| $\|A,0,0\rangle$ | 0 | 1 | 0 | 0.5 | 0.5 |
| $\|A,0,0\rangle$ | 1 | 0 | 0 | 0.5 | 0.5 |
| $\|A,0,A\rangle$ | 0 | 1 | 1 | 0.5 | 0.5 |
| $\|A,0,A\rangle$ | 1 | 0 | 1 | 0.5 | 0.5 |
| $\|A,0,B\rangle$ | 0 | 1 | 1 | 0.5 | 0.5 |
| $\|A,0,B\rangle$ | 1 | 0 | 1 | 0.5 | 0.5 |
| $\|A,0,C\rangle$ | 0 | 1 | 1 | 0.5 | 0.5 |
| $\|A,0,C\rangle$ | 1 | 0 | 1 | 0.5 | 0.5 |
| $\|A,0,2\rangle$ | 0 | 1 | 2 | 0.5 | 0.5 |
| $\|A,0,2\rangle$ | 1 | 0 | 2 | 0.5 | 0.5 |
| $\|A,A,0\rangle$ | 1 | 1 | 0 | 1.0 | 1.0 |
| $\|A,A,A\rangle$ | 1 | 1 | 1 | 1.0 | 1.0 |
| $\|A,A,B\rangle$ | 1 | 1 | 1 | 1.0 | 1.0 |
| $\|A,A,C\rangle$ | 1 | 1 | 1 | 1.0 | 1.0 |
| $\|A,A,2\rangle$ | 1 | 1 | 2 | 1.0 | 1.0 |
| $\|A,B,0\rangle$ | 1 | 1 | 0 | 1.0 | 1.0 |
| $\|A,B,A\rangle$ | 1 | 1 | 1 | 1.0 | 1.0 |
| $\|A,B,B\rangle$ | 1 | 1 | 1 | 1.0 | 1.0 |
| $\|A,B,C\rangle$ | 1 | 1 | 1 | 1.0 | 1.0 |
| $\|A,B,2\rangle$ | 1 | 1 | 2 | 1.0 | 1.0 |
| $\|A,C,0\rangle$ | 1 | 1 | 0 | 1.0 | 1.0 |
| $\|A,C,A\rangle$ | 1 | 1 | 1 | 1.0 | 1.0 |
| $\|A,C,B\rangle$ | 1 | 1 | 1 | 1.0 | 1.0 |
| $\|A,C,C\rangle$ | 1 | 1 | 1 | 1.0 | 1.0 |
| $\|A,C,2\rangle$ | 1 | 1 | 2 | 1.0 | 1.0 |
| $\|A,2,0\rangle$ | 1 | 2 | 0 | 0.5 | 0.5 |
| $\|A,2,0\rangle$ | 2 | 1 | 0 | 0.5 | 0.5 |
| $\|A,2,A\rangle$ | 1 | 2 | 1 | 0.5 | 0.5 |
| $\|A,2,A\rangle$ | 2 | 1 | 1 | 0.5 | 0.5 |
| $\|A,2,B\rangle$ | 1 | 2 | 1 | 0.5 | 0.5 |
| $\|A,2,B\rangle$ | 2 | 1 | 1 | 0.5 | 0.5 |
| $\|A,2,C\rangle$ | 1 | 2 | 1 | 0.5 | 0.5 |
| $\|A,2,C\rangle$ | 2 | 1 | 1 | 0.5 | 0.5 |
| $\|A,2,2\rangle$ | 1 | 2 | 2 | 0.5 | 0.5 |

| $\|\psi\rangle_{AB}$ | $n_1$ | $n_2$ | $n_3$ | $p_{AB}$ | $p_Ap_B$ |
|---|---|---|---|---|---|
| $\|A,2,2\rangle$ | 2 | 1 | 2 | 0.5 | 0.5 |
| $\|B,0,0\rangle$ | 0 | 1 | 0 | 0.5 | 0.5 |
| $\|B,0,0\rangle$ | 1 | 0 | 0 | 0.5 | 0.5 |
| $\|B,0,A\rangle$ | 0 | 1 | 1 | 0.5 | 0.5 |
| $\|B,0,A\rangle$ | 1 | 0 | 1 | 0.5 | 0.5 |
| $\|B,0,B\rangle$ | 0 | 1 | 1 | 0.5 | 0.5 |
| $\|B,0,B\rangle$ | 1 | 0 | 1 | 0.5 | 0.5 |
| $\|B,0,C\rangle$ | 0 | 1 | 1 | 0.5 | 0.5 |
| $\|B,0,C\rangle$ | 1 | 0 | 1 | 0.5 | 0.5 |
| $\|B,0,2\rangle$ | 0 | 1 | 2 | 0.5 | 0.5 |
| $\|B,0,2\rangle$ | 1 | 0 | 2 | 0.5 | 0.5 |
| $\|B,A,0\rangle$ | 1 | 1 | 0 | 1.0 | 1.0 |
| $\|B,A,A\rangle$ | 1 | 1 | 1 | 1.0 | 1.0 |
| $\|B,A,B\rangle$ | 1 | 1 | 1 | 1.0 | 1.0 |
| $\|B,A,C\rangle$ | 1 | 1 | 1 | 1.0 | 1.0 |
| $\|B,A,2\rangle$ | 1 | 1 | 2 | 1.0 | 1.0 |
| $\|B,B,0\rangle$ | 1 | 1 | 0 | 1.0 | 1.0 |
| $\|B,B,A\rangle$ | 1 | 1 | 1 | 1.0 | 1.0 |
| $\|B,B,B\rangle$ | 1 | 1 | 1 | 1.0 | 1.0 |
| $\|B,B,C\rangle$ | 1 | 1 | 1 | 1.0 | 1.0 |
| $\|B,B,2\rangle$ | 1 | 1 | 2 | 1.0 | 1.0 |
| $\|B,C,0\rangle$ | 1 | 1 | 0 | 1.0 | 1.0 |
| $\|B,C,A\rangle$ | 1 | 1 | 1 | 1.0 | 1.0 |
| $\|B,C,B\rangle$ | 1 | 1 | 1 | 1.0 | 1.0 |
| $\|B,C,C\rangle$ | 1 | 1 | 1 | 1.0 | 1.0 |
| $\|B,C,2\rangle$ | 1 | 1 | 2 | 1.0 | 1.0 |
| $\|B,2,0\rangle$ | 1 | 2 | 0 | 0.5 | 0.5 |
| $\|B,2,0\rangle$ | 2 | 1 | 0 | 0.5 | 0.5 |
| $\|B,2,A\rangle$ | 1 | 2 | 1 | 0.5 | 0.5 |
| $\|B,2,A\rangle$ | 2 | 1 | 1 | 0.5 | 0.5 |
| $\|B,2,B\rangle$ | 1 | 2 | 1 | 0.5 | 0.5 |
| $\|B,2,B\rangle$ | 2 | 1 | 1 | 0.5 | 0.5 |
| $\|B,2,C\rangle$ | 1 | 2 | 1 | 0.5 | 0.5 |
| $\|B,2,C\rangle$ | 2 | 1 | 1 | 0.5 | 0.5 |
| $\|B,2,2\rangle$ | 1 | 2 | 2 | 0.5 | 0.5 |
| $\|B,2,2\rangle$ | 2 | 1 | 2 | 0.5 | 0.5 |
| $\|C,0,0\rangle$ | 0 | 1 | 0 | 0.5 | 0.5 |
| $\|C,0,0\rangle$ | 1 | 0 | 0 | 0.5 | 0.5 |
| $\|C,0,A\rangle$ | 0 | 1 | 1 | 0.5 | 0.5 |
| $\|C,0,A\rangle$ | 1 | 0 | 1 | 0.5 | 0.5 |
| $\|C,0,B\rangle$ | 0 | 1 | 1 | 0.5 | 0.5 |
| $\|C,0,B\rangle$ | 1 | 0 | 1 | 0.5 | 0.5 |

| $\|\psi\rangle_{AB}$ | $n_1$ | $n_2$ | $n_3$ | $p_{AB}$ | $p_Ap_B$ |
|---|---|---|---|---|---|
| $\|C,0,C\rangle$ | 0 | 1 | 1 | 0.5 | 0.5 |
| $\|C,0,C\rangle$ | 1 | 0 | 1 | 0.5 | 0.5 |
| $\|C,0,2\rangle$ | 0 | 1 | 2 | 0.5 | 0.5 |
| $\|C,0,2\rangle$ | 1 | 0 | 2 | 0.5 | 0.5 |
| $\|C,A,0\rangle$ | 1 | 1 | 0 | 1.0 | 1.0 |
| $\|C,A,A\rangle$ | 1 | 1 | 1 | 1.0 | 1.0 |
| $\|C,A,B\rangle$ | 1 | 1 | 1 | 1.0 | 1.0 |
| $\|C,A,C\rangle$ | 1 | 1 | 1 | 1.0 | 1.0 |
| $\|C,A,2\rangle$ | 1 | 1 | 2 | 1.0 | 1.0 |
| $\|C,B,0\rangle$ | 1 | 1 | 0 | 1.0 | 1.0 |
| $\|C,B,A\rangle$ | **0** | **2** | **1** | **0.5** | **0** |
| $\|C,B,A\rangle$ | **2** | **0** | **1** | **0.5** | **0** |
| $\|C,B,A\rangle$ | **1** | **1** | **1** | **0** | **1.0** |
| $\|C,B,B\rangle$ | 1 | 1 | 1 | 1.0 | 1.0 |
| $\|C,B,C\rangle$ | **0** | **2** | **1** | **0.5** | **0** |
| $\|C,B,C\rangle$ | **2** | **0** | **1** | **0.5** | **0** |
| $\|C,B,C\rangle$ | **1** | **1** | **1** | **0** | **1.0** |
| $\|C,B,2\rangle$ | 1 | 1 | 2 | 1.0 | 1.0 |
| $\|C,C,0\rangle$ | 0 | 2 | 0 | 0.5 | 0.5 |
| $\|C,C,0\rangle$ | 2 | 0 | 0 | 0.5 | 0.5 |
| $\|C,C,A\rangle$ | **0** | **2** | **1** | **0** | **0.5** |
| $\|C,C,A\rangle$ | **1** | **1** | **1** | **1.0** | **0** |
| $\|C,C,A\rangle$ | **2** | **0** | **1** | **0** | **0.5** |
| $\|C,C,B\rangle$ | 0 | 2 | 1 | 0.5 | 0.5 |
| $\|C,C,B\rangle$ | 2 | 0 | 1 | 0.5 | 0.5 |
| $\|C,C,C\rangle$ | **0** | **2** | **1** | **0** | **0.5** |
| $\|C,C,C\rangle$ | **1** | **1** | **1** | **1.0** | **0** |
| $\|C,C,C\rangle$ | **2** | **0** | **1** | **0** | **0.5** |
| $\|C,C,2\rangle$ | 0 | 2 | 2 | 0.5 | 0.5 |
| $\|C,C,2\rangle$ | 2 | 0 | 2 | 0.5 | 0.5 |
| $\|C,2,0\rangle$ | 1 | 2 | 0 | 0.5 | 0.5 |
| $\|C,2,0\rangle$ | 2 | 1 | 0 | 0.5 | 0.5 |
| $\|C,2,A\rangle$ | 1 | 2 | 1 | 0.5 | 0.5 |
| $\|C,2,A\rangle$ | 2 | 1 | 1 | 0.5 | 0.5 |
| $\|C,2,B\rangle$ | 1 | 2 | 1 | 0.5 | 0.5 |
| $\|C,2,B\rangle$ | 2 | 1 | 1 | 0.5 | 0.5 |
| $\|C,2,C\rangle$ | 1 | 2 | 1 | 0.5 | 0.5 |
| $\|C,2,C\rangle$ | 2 | 1 | 1 | 0.5 | 0.5 |
| $\|C,2,2\rangle$ | 1 | 2 | 2 | 0.5 | 0.5 |
| $\|C,2,2\rangle$ | 2 | 1 | 2 | 0.5 | 0.5 |
| $\|2,0,0\rangle$ | 0 | 2 | 0 | 0.25 | 0.25 |
| $\|2,0,0\rangle$ | 1 | 1 | 0 | 0.5 | 0.5 |

| $\vert\psi\rangle_{AB}$ | $n_1$ | $n_2$ | $n_3$ | $p_{AB}$ | $p_A p_B$ |
|---|---|---|---|---|---|
| $\vert 2,0,0\rangle$ | 2 | 0 | 0 | 0.25 | 0.25 |
| $\vert 2,0,A\rangle$ | 0 | 2 | 1 | 0.25 | 0.25 |
| $\vert 2,0,A\rangle$ | 1 | 1 | 1 | 0.5 | 0.5 |
| $\vert 2,0,A\rangle$ | 2 | 0 | 1 | 0.25 | 0.25 |
| $\vert 2,0,B\rangle$ | 0 | 2 | 1 | 0.25 | 0.25 |
| $\vert 2,0,B\rangle$ | 1 | 1 | 1 | 0.5 | 0.5 |
| $\vert 2,0,B\rangle$ | 2 | 0 | 1 | 0.25 | 0.25 |
| $\vert 2,0,C\rangle$ | 0 | 2 | 1 | 0.25 | 0.25 |
| $\vert 2,0,C\rangle$ | 1 | 1 | 1 | 0.5 | 0.5 |
| $\vert 2,0,C\rangle$ | 2 | 0 | 1 | 0.25 | 0.25 |
| $\vert 2,0,2\rangle$ | 0 | 2 | 2 | 0.25 | 0.25 |
| $\vert 2,0,2\rangle$ | 1 | 1 | 2 | 0.5 | 0.5 |
| $\vert 2,0,2\rangle$ | 2 | 0 | 2 | 0.25 | 0.25 |
| $\vert 2,A,0\rangle$ | 1 | 2 | 0 | 0.5 | 0.5 |
| $\vert 2,A,0\rangle$ | 2 | 1 | 0 | 0.5 | 0.5 |
| $\vert 2,A,A\rangle$ | 1 | 2 | 1 | 0.5 | 0.5 |
| $\vert 2,A,A\rangle$ | 2 | 1 | 1 | 0.5 | 0.5 |
| $\vert 2,A,B\rangle$ | 1 | 2 | 1 | 0.5 | 0.5 |
| $\vert 2,A,B\rangle$ | 2 | 1 | 1 | 0.5 | 0.5 |
| $\vert 2,A,C\rangle$ | 1 | 2 | 1 | 0.5 | 0.5 |
| $\vert 2,A,C\rangle$ | 2 | 1 | 1 | 0.5 | 0.5 |
| $\vert 2,A,2\rangle$ | 1 | 2 | 2 | 0.5 | 0.5 |
| $\vert 2,A,2\rangle$ | 2 | 1 | 2 | 0.5 | 0.5 |
| $\vert 2,B,0\rangle$ | 1 | 2 | 0 | 0.5 | 0.5 |
| $\vert 2,B,0\rangle$ | 2 | 1 | 0 | 0.5 | 0.5 |
| $\vert 2,B,A\rangle$ | 1 | 2 | 1 | 0.5 | 0.5 |
| $\vert 2,B,A\rangle$ | 2 | 1 | 1 | 0.5 | 0.5 |
| $\vert 2,B,B\rangle$ | 1 | 2 | 1 | 0.5 | 0.5 |
| $\vert 2,B,B\rangle$ | 2 | 1 | 1 | 0.5 | 0.5 |
| $\vert 2,B,C\rangle$ | 1 | 2 | 1 | 0.5 | 0.5 |
| $\vert 2,B,C\rangle$ | 2 | 1 | 1 | 0.5 | 0.5 |
| $\vert 2,B,2\rangle$ | 1 | 2 | 2 | 0.5 | 0.5 |
| $\vert 2,B,2\rangle$ | 2 | 1 | 2 | 0.5 | 0.5 |
| $\vert 2,C,0\rangle$ | 1 | 2 | 0 | 0.5 | 0.5 |
| $\vert 2,C,0\rangle$ | 2 | 1 | 0 | 0.5 | 0.5 |
| $\vert 2,C,A\rangle$ | 1 | 2 | 1 | 0.5 | 0.5 |
| $\vert 2,C,A\rangle$ | 2 | 1 | 1 | 0.5 | 0.5 |
| $\vert 2,C,B\rangle$ | 1 | 2 | 1 | 0.5 | 0.5 |
| $\vert 2,C,B\rangle$ | 2 | 1 | 1 | 0.5 | 0.5 |
| $\vert 2,C,C\rangle$ | 1 | 2 | 1 | 0.5 | 0.5 |
| $\vert 2,C,C\rangle$ | 2 | 1 | 1 | 0.5 | 0.5 |
| $\vert 2,C,2\rangle$ | 1 | 2 | 2 | 0.5 | 0.5 |
| $\vert 2,C,2\rangle$ | 2 | 1 | 2 | 0.5 | 0.5 |
| $\vert 2,2,0\rangle$ | 2 | 2 | 0 | 1.0 | 1.0 |
| $\vert 2,2,A\rangle$ | 2 | 2 | 1 | 1.0 | 1.0 |
| $\vert 2,2,B\rangle$ | 2 | 2 | 1 | 1.0 | 1.0 |
| $\vert 2,2,C\rangle$ | 2 | 2 | 1 | 1.0 | 1.0 |
| $\vert 2,2,2\rangle$ | 2 | 2 | 2 | 1.0 | 1.0 |

Table C.6.: Probabilities to measure the particle numbers $\vec{n}_{AB} = (n_1, n_2, n_3)$ after applying the beam splitter to number states $|\psi\rangle_{AB}$ of three modes. Only outcomes with nonzero probabilities are shown.

# D. Appendix: Evolution of 2-paraparticle state under general Hamiltonian

In this section, we derive an expression for evolution of a $R$-paraparticle state

$$e^{i\hat{H}}\left|\begin{matrix}1 & 1\\ a & b\end{matrix}\right\rangle = e^{i\hat{H}}\hat{\psi}^+_{1,a}\hat{\psi}^+_{2,b}\left|0\right\rangle, \tag{D.1}$$

under a general 2-mode Hamiltonian

$$\hat{H} = p\hat{n}_1 + q\hat{n}_2 + c\hat{e}_{12} + c^*\hat{e}_{21}, \tag{D.2}$$

with $p, q \in \mathbb{R}$ and $c \in \mathbb{C}$. Using the relation (2.57), we can calculate the commutator of $\hat{H}$ with $\hat{\psi}^+_{1,a}\hat{\psi}^+_{2,b}$:

$$\begin{aligned}\left[\hat{H}, \hat{\psi}^+_{1,a}\hat{\psi}^+_{2,b}\right] &= \hat{\psi}^+_{1,a}\left[\hat{H}, \hat{\psi}^+_{2,b}\right] + \left[\hat{H}, \hat{\psi}^+_{1,a}\right]\hat{\psi}^+_{2,b}\\ &= \hat{\psi}^+_{1,a}\left(c\hat{\psi}^+_{1,b} + q\hat{\psi}^+_{2,b}\right) + \left(c^*\hat{\psi}^+_{2,a} + p\hat{\psi}^+_{1,a}\right)\hat{\psi}^+_{2,b}\\ &= (p+q)\hat{\psi}^+_{1,a}\hat{\psi}^+_{2,b} + c\hat{\psi}^+_{1,a}\hat{\psi}^+_{1,b} + c^*\hat{\psi}^+_{2,a}\hat{\psi}^+_{2,b}.\end{aligned} \tag{D.3}$$

In complete analogy, we obtain

$$\left[\hat{H}, \hat{\psi}^+_{2,a}\hat{\psi}^+_{1,b}\right] = (p+q)\hat{\psi}^+_{2,a}\hat{\psi}^+_{1,b} + c\hat{\psi}^+_{1,a}\hat{\psi}^+_{1,b} + c^*\hat{\psi}^+_{2,a}\hat{\psi}^+_{2,b}, \tag{D.4}$$

$$\left[\hat{H}, \hat{\psi}^+_{1,a}\hat{\psi}^+_{1,b}\right] = c^*\hat{\psi}^+_{1,a}\hat{\psi}^+_{2,b} + c^*\hat{\psi}^+_{2,a}\hat{\psi}^+_{1,b} + 2p\hat{\psi}^+_{1,a}\hat{\psi}^+_{1,b}, \tag{D.5}$$

$$\left[\hat{H}, \hat{\psi}^+_{2,a}\hat{\psi}^+_{2,b}\right] = c\hat{\psi}^+_{1,a}\hat{\psi}^+_{2,b} + c\hat{\psi}^+_{2,a}\hat{\psi}^+_{1,b} + 2q\hat{\psi}^+_{2,a}\hat{\psi}^+_{2,b}. \tag{D.6}$$

We aim to evaluate expression (D.1) using a series expansion. Since $\hat{e}_{ij}\left|0\right\rangle = 0$, we can express powers of $H$ acting on $\left|\begin{matrix}1 & 1\\ a & b\end{matrix}\right\rangle$ in terms of nested commutators:

$$\begin{aligned}\hat{H}^n\hat{\psi}^+_{1,a}\hat{\psi}^+_{2,b}\left|0\right\rangle &= \hat{H}^{n-1}\left([\hat{H}, \hat{\psi}^+_{1,a}\hat{\psi}^+_{2,b}] + \hat{\psi}^+_{1,a}\hat{\psi}^+_{2,b}\hat{H}\right)\left|0\right\rangle = \hat{H}^{n-1}[\hat{H}, \hat{\psi}^+_{1,a}\hat{\psi}^+_{2,b}]\left|0\right\rangle\\ &= \hat{H}^{n-2}\left[\hat{H}, [\hat{H}, \hat{\psi}^+_{1,a}\hat{\psi}^+_{2,b}]\right]\left|0\right\rangle = \hat{H}^{n-3}\left[\hat{H}, [\hat{H}, [\hat{H}, \hat{\psi}^+_{1,a}\hat{\psi}^+_{2,b}]\right]\left|0\right\rangle = \ \dots\ =\\ &= \underbrace{\left[\hat{H}, [\hat{H}, [\dots, \hat{\psi}^+_{1,a}\hat{\psi}^+_{2,b}]\right]}_{n}\left|0\right\rangle.\end{aligned} \tag{D.7}$$

Using (D.3)-(D.6) we see that the commutator of $\hat{H}$ with a formal sum of operators $\hat{\psi}^+_{i,a}\hat{\psi}^+_{j,b}$ can be rewritten as:

$$\begin{aligned}&\left[\hat{H}, A_1\hat{\psi}^+_{1,a}\hat{\psi}^+_{2,b} + A_2\hat{\psi}^+_{2,a}\hat{\psi}^+_{1,b} + A_3\hat{\psi}^+_{1,a}\hat{\psi}^+_{1,b} + A_4\hat{\psi}^+_{2,a}\hat{\psi}^+_{2,b}\right] =\\ &= A_1\left[\hat{H}, \hat{\psi}^+_{1,a}\hat{\psi}^+_{2,b}\right] + A_2\left[\hat{H}, \hat{\psi}^+_{2,a}\hat{\psi}^+_{1,b}\right] + A_3\left[\hat{H}, \hat{\psi}^+_{1,a}\hat{\psi}^+_{1,b}\right] + A_4\left[\hat{H}, \hat{\psi}^+_{2,a}\hat{\psi}^+_{2,b}\right]\\ &= \left((p+q)A_1 + c^*A_3 + cA_4\right)\hat{\psi}^+_{1,a}\hat{\psi}^+_{2,b} + \left((p+q)A_2 + c^*A_3 + cA_4\right)\hat{\psi}^+_{2,a}\hat{\psi}^+_{1,b}\\ &\quad + \left(cA_1 + cA_2 + 2pA_3\right)\hat{\psi}^+_{1,a}\hat{\psi}^+_{1,b} + \left(c^*A_1 + c^*A_2 + 2qA_4\right)\hat{\psi}^+_{2,a}\hat{\psi}^+_{2,b} =\\ &= A_1'\hat{\psi}^+_{1,a}\hat{\psi}^+_{2,b} + A_2'\hat{\psi}^+_{2,a}\hat{\psi}^+_{1,b} + A_3'\hat{\psi}^+_{1,a}\hat{\psi}^+_{1,b} + A_4'\hat{\psi}^+_{2,a}\hat{\psi}^+_{2,b}.\end{aligned} \tag{D.8}$$

Since a sum of quadratic terms $\hat{\psi}^+_{i,a}\hat{\psi}^+_{j,b}$ is closed under the commutator with $\hat{H}$, the coefficients in (D.8) can be regarded to change according to a matrix $C$ when commuting with $\hat{H}$:

$$\begin{pmatrix} A'_1 \\ A'_2 \\ A'_3 \\ A'_4 \end{pmatrix} = \underbrace{\begin{pmatrix} p+q & 0 & c^* & c \\ 0 & p+q & c^* & c \\ c & c & 2p & 0 \\ c^* & c^* & 0 & 2q \end{pmatrix}}_{C} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix}. \tag{D.9}$$

In particular, the $n$-th nested commutator with $\hat{H}$ can therefore be obtained simply by $C^n$ acting on the coefficients. This—together with (D.7)—implies the following result for the initial expression (D.1):

$$\begin{aligned} e^{i\hat{H}} \left|\begin{smallmatrix} 1 & 1 \\ a & b \end{smallmatrix}\right\rangle &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \hat{H}^n \hat{\psi}^+_{1,a}\hat{\psi}^+_{2,b} \left|0\right\rangle = \sum_{n=0}^{\infty} \frac{i^n}{n!} \underbrace{\left[\hat{H}, [\hat{H}, [\dots, \hat{\psi}^+_{1,a}\hat{\psi}^+_{2,b}]\right]}_{n} \left|0\right\rangle \\ &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \begin{pmatrix} \hat{\psi}^+_{1,a}\hat{\psi}^+_{2,b} \\ \hat{\psi}^+_{2,a}\hat{\psi}^+_{1,b} \\ \hat{\psi}^+_{1,a}\hat{\psi}^+_{1,b} \\ \hat{\psi}^+_{2,a}\hat{\psi}^+_{2,b} \end{pmatrix} \cdot C^n \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \left|0\right\rangle = \begin{pmatrix} \hat{\psi}^+_{1,a}\hat{\psi}^+_{2,b} \\ \hat{\psi}^+_{2,a}\hat{\psi}^+_{1,b} \\ \hat{\psi}^+_{1,a}\hat{\psi}^+_{1,b} \\ \hat{\psi}^+_{2,a}\hat{\psi}^+_{2,b} \end{pmatrix} \cdot e^{iC} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \left|0\right\rangle , \end{aligned} \tag{D.10}$$

where the (real) scalar product is to be understood as expanding the operators $\hat{\psi}^+_{i,a}\hat{\psi}^+_{j,b}$ with the according coefficients, which are the first column of $e^{iC}$.

In particular, this enables us to calculate the action of a swap $\Delta(X) = e^{i\hat{H}}$ on a two-mode state, where

$$\hat{H} = \frac{\pi}{2}(\hat{e}_{12} + \hat{e}_{21} - \hat{n}_1 - \hat{n}_2). \tag{D.11}$$

By inserting into equation (D.10) and using (2.49), we obtain

$$\Delta(X) \left|\begin{smallmatrix} 1 & 1 \\ a & b \end{smallmatrix}\right\rangle = \hat{\psi}^+_{2,a}\hat{\psi}^+_{1,b} \left|0\right\rangle = \sum_{c,d} R^{cd}_{ab} \hat{\psi}^+_{1,c}\hat{\psi}^+_{2,d} = \sum_{c,d} R^{cd}_{ab} \left|\begin{smallmatrix} 1 & 1 \\ c & d \end{smallmatrix}\right\rangle . \tag{D.12}$$

This also agrees with expression (14) in Ref. [21].