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Abstract

In this thesis, we lay the foundations for the Gaussian beam approximation of high-frequency electromagnetic wave packets on globally hyperbolic spacetimes. After an introduction to the necessary mathematical notions, we review the theory of Gaussian beams and present the current state of the art, which treats the scalar case on general Lorentzian manifolds. This is done in considerable detail, since the methods developed there are very general and can be adapted to the electromagnetic case with only minor modifications. Next, we present the initial value formulation of Maxwell's equations in curved spacetime and discuss the well-posedness of the Cauchy problem for both the vector potential and the field strength tensor. We then prepare some basic results on bitensors, i.e. tensors that depend on two spacetime points. Such objects arise naturally in the study of observables such as linear and angular momentum. We provide a geometric definition of these observables based on the concept of local approximate symmetries. Furthermore, we introduce the method of stationary phase, together with a slight adaptation to the curved spacetime setting, which will later be used to evaluate the observables under consideration. In the final chapter, we introduce our approximation scheme and provide the necessary tools to construct approximate solutions to arbitrary order in the frequency. Finally, we prove an energy estimate for the difference between the approximate and exact solution. This estimate demonstrates that, under a suitable choice of timelike vector field with respect to which the energy is defined, the energy of the residual solution can be bounded in terms of the residual error in the divergence term of Maxwell's equations.

Kurzfassung

In dieser Arbeit legen wir die Grundlagen für die Gauß-Strahl-Approximation hochfrequenter elektromagnetischer Wellenpakete auf global hyperbolischen Raumzeiten. Nach einer Einführung in die notwendigen mathematischen Begriffe geben wir einen Überblick über die Theorie Gaußscher Strahlen und stellen den aktuellen Stand der Forschung dar, welcher den skalaren Fall auf allgemeinen Lorentz-Mannigfaltigkeiten behandelt. Wir präsentieren die Resultate ausführlich, da die dort entwickelten Methoden sehr allgemein sind und sich mit nur geringfügigen Anpassungen auf den elektromagnetischen Fall übertragen lassen. Anschließend diskutieren wir das Anfangswertproblem der Maxwell-Gleichungen in gekrümmter Raumzeit und zeigen, dass das Cauchy-Problem sowohl für das Vektorpotential als auch für den Feldstärketensor wohlgestellt ist. Danach erarbeiten wir einige grundlegende Resultate zu Bitensoren, also Tensoren, die von zwei Raumzeitpunkten abhängen. Solche Objekte treten in natürlicher Weise bei der Untersuchung von Observablen wie linearem Impuls und Drehimpuls auf. Wir geben eine geometrische Definition dieser Observablen auf Grundlage des Konzepts lokaler approximativer Symmetrien. Darüber hinaus führen wir die Methode der stationären Phase ein, zusammen mit einer leichten Anpassung an den Fall gekrümmter Raumzeiten, welche später zur Auswertung der betrachteten Observablen verwendet wird. Im abschließenden Kapitel führen wir unser Approximationsschema ein und stellen die notwendigen Werkzeuge zur Konstruktion approximativer Lösungen beliebiger Ordnung in der Frequenz bereit. Schließlich beweisen wir eine Energieabschätzung für die Differenz zwischen der approximativen und der exakten Lösung. Diese Abschätzung zeigt, dass unter einer geeigneten Wahl eines zeitartigen Vektorfeldes, bezüglich dessen die Energie definiert wird, die Energie der Residuenlösung durch den Residualfehler im Divergenzterm der Maxwell-Gleichungen abgeschätzt werden kann.

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1. Introduction

Propagation of waves is governed by hyperbolic partial differential equations, which admit solutions concentrated near distinguished curves in spacetime. In the high-frequency regime, approximate solutions to the scalar wave equation

$$\square_g \psi = g^{\mu\nu} \nabla_\mu \nabla_\nu \psi = 0 \quad (1.1)$$

are often constructed using a *geometric optics ansatz* of the form

$$\tilde{\psi}_\omega = e^{i\omega\phi} (a_0 + \omega^{-1}a_1 + \dots + \omega^{-N}a_N), \quad (1.2)$$

where $N \in \mathbb{N}$, ϕ is a real-valued function called the *eikonal*, the a_k are complex-valued functions, and $\omega > 0$ denotes the frequency of the wave. Substituting this ansatz into the wave equation yields a hierarchy of equations that can be solved order by order in ω . At leading order one obtains the eikonal equation for ϕ , while the next-to-leading order gives a transport equation for a_0 along the bicharacteristic flow generated by ϕ .

It is well known that the time up to which one has good control over the solution cannot be chosen arbitrarily large, since the approximation breaks down at caustics, i.e. where geodesics meet. To circumvent this problem, one chooses a complex-valued eikonal ϕ in the ansatz (1.2), which is real-valued along a bicharacteristic γ and whose imaginary part grows away from this bicharacteristic. Consequently, one obtains exponential decay in ω away from the bicharacteristic. Moreover, the equations obtained from the asymptotic expansion are imposed only up to a prescribed order of vanishing along γ . That is, the corresponding quantities and their derivatives are required to vanish on γ up to some order. The resulting approximate solution is known as a *Gaussian beam*. The advantage of the Gaussian beam approximation is that the simple ansatz above does not break down at caustics and yields an approximation valid up to arbitrarily large but finite times. A general mathematical theory of Gaussian beams was developed by Maslov [Mas94]. A detailed construction for traveling wave packets is given by Ralston [Ral83].

The approximation by Gaussian beams on curved spacetimes can be carried out under minimal assumptions. One requires a well-posed initial value problem, which is ensured by assuming that the spacetime is globally hyperbolic. Furthermore, one needs an appropriate energy estimate. This is guaranteed by imposing a suitable condition on the timelike, future-directed vector field with respect to which the energy is defined.

For the scalar case, Sbierski [Sbi15] studies the behavior of Gaussian beams on general globally hyperbolic Lorentzian manifolds. It is shown that the error between the exact and

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approximate solution, measured in an appropriate energy norm, can be made arbitrarily small up to a finite, but arbitrarily long, time by choosing ω sufficiently large. The construction of the functions a_k and ϕ in (1.2) allows one to restrict the support of the amplitude to a small neighborhood of a given null geodesic. Thus, from the fact that $\tilde{\psi}_\omega$ is an approximate solution with respect to a suitable energy norm, one can infer the existence of exact solutions to the wave equation whose energy remains localized along a given null geodesic up to some finite, but arbitrarily long, time. An analysis of the Gaussian beam approximation for electromagnetic waves in inhomogeneous media is presented in [COS26]. There, the authors derive a system of ordinary differential equations incorporating the leading-order correction to geodesic motion.

Outline of the analysis. In this thesis, we study the Gaussian beam approximation for electromagnetic fields on curved spacetimes governed by Maxwell's equations. In particular, we analyze observables such as the energy, linear momentum, and angular momentum of the wave packet, all constructed from its stress-energy tensor. This work provides the foundation for extending the results of [COS26] to curved spacetime backgrounds. More precisely, we consider a globally hyperbolic Lorentzian manifold (M, g) that is foliated by a one-parameter family of spacelike Cauchy hypersurfaces $\{\Sigma_\tau\}_{\tau \in \mathbb{R}}$ and we seek to construct an approximate vector potential $\hat{A}_\alpha$ such that, for some $T > 0$ and $p > 0$,

$$\begin{aligned} \sup_{\tau \in [0, T]} \|\square_g \hat{A}_\alpha - R_\alpha{}^\beta \hat{A}_\beta\|_{L^2(\Sigma_\tau, \mathfrak{g})} &= \mathcal{O}(\omega^{-p}), \\ \sup_{\tau \in [0, T]} \|\nabla_\beta \nabla^\alpha \hat{A}_\alpha\|_{L^2(\Sigma_\tau, \mathfrak{g})} &= \mathcal{O}(\omega^{-p}). \end{aligned} \tag{1.3}$$

Here $\mathfrak{g}$ denotes an auxiliary Riemannian metric on M , constructed from the original Lorentzian metric g and $R_\alpha{}^\beta$ denotes the Ricci tensor of (M, g) . Our Gaussian beam ansatz takes the form

$$\hat{A}_\alpha = \Re \left[\omega^{-1/4} \sum_{j=0}^N \omega^{-j} a_\alpha^{(j)} e^{i\omega\phi} \right], \tag{1.4}$$

where the $a_\alpha^{(j)}$ ($j = 1, \dots, N$) are compactly supported 1-forms and ϕ is a complex-valued phase. The main result, see theorem 6.2 is that, upon choosing the timelike vector field u , with respect to which the energy E^u is defined, to be Killing, then the energy associated to the difference of exact and approximate solution can be estimated as

$$\frac{d}{d\tau} \sqrt{E^u(\tau)} \leq C(\tau, u) \|f_\beta\|_{L^2(\Sigma_\tau, \mathfrak{g})}. \tag{1.5}$$

Here f_β is given by the residual error in the divergence term of Maxwell's equations, i.e. $\nabla^\alpha \hat{F}_{\alpha\beta} = f_\beta$ and $C(\tau, u)$ is a constant depending on τ and u .

The thesis is organized as follows. In chapter 2, we introduce the necessary mathematical background from causality theory, including the $3 + 1$ decomposition of spacetime, global

hyperbolicity, Cauchy surfaces, as well as Jacobi and Killing fields. In chapter 3, we formulate Maxwell’s equations using the language of differential forms and introduce the geometric objects required for later developments. We then discuss the prescription of initial data on a Cauchy surface and summarize existing results on the well-posedness of the Cauchy problem for both the vector potential and the field strength tensor. In chapter 4, we present an introduction to the theory of bitensors and define the van Vleck–Morette determinant, which relates the manifold measure to the corresponding measure on the tangent space. In chapter 5, we give geometric definitions of linear and angular momentum that remain meaningful even in spacetimes without symmetries. We also introduce the stationary phase approximation, which will later be used to evaluate the observables associated with the Gaussian beam. Finally, in the last chapter, we formulate a precise notion of approximate solutions to Maxwell’s equations and derive the constraints on the amplitudes appearing in eq. (1.4) required for eq. (1.3) to hold. The thesis concludes with the energy estimate controlling the difference between the exact and approximate solutions.

2. Gaussian beams on Lorentzian manifolds

After recalling some basic notions of Lorentzian geometry, this section introduces Gaussian beams and reviews the state of the art in their analysis on curved spacetimes.

2.1. Preliminaries on Lorentzian geometry and causality

Throughout this thesis, let (M, g) be a real, oriented and smooth pseudo-Riemannian manifold of finite dimension $d \in \mathbb{N}$, equipped with its Levi-Civita connection ∇ . We denote by $C^\infty(M, \mathbb{K})$ the ring of smooth $\mathbb{K}$ -valued functions on M and by $C_0^\infty(M, \mathbb{K})$ the subspace of smooth functions with compact support. If it is clear from the context, we will omit explicitly specifying the base field $\mathbb{K}$. For a given vector bundle $E \rightarrow M$ over M , we denote by $\Gamma(E)$ the $C^\infty(M)$ -module of smooth sections. Furthermore, we use the notation

$$\mathfrak{X}(M) := \Gamma(TM), \quad \Omega^k(M) := \Gamma(\Lambda^k T^*M), \quad (2.1)$$

for the $C^\infty(M)$ -modules of smooth vector fields and differential k -forms on M , respectively. For the Riemann and Ricci curvature tensor of (M, g) , we adopt the sign and index conventions

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)v^\rho =: R^\rho_{\sigma\mu\nu}v^\sigma \quad \forall v \in \mathfrak{X}(M), \quad R_{\alpha\beta} := R^\lambda_{\alpha\lambda\beta}. \quad (2.2)$$

Let (M, g) be a smooth 4-dimensional Lorentzian manifold with signature $(-, +, +, +)$. The Lorentzian metric g allows for a classification of tangent vectors into three categories: a tangent vector $v \in T_p M$ at some point $p \in M$ is called

- (i) *spacelike* if $g_p(v, v) > 0$ or if $v = 0$,
- (ii) *null* or *lightlike* if $g_p(v, v) = 0$ and $v \neq 0$,
- (iii) *timelike* if $g_p(v, v) < 0$.

A vector is called *causal*, if it is either timelike or lightlike. A Lorentzian manifold M is called *time-orientable*, if there exists a smooth timelike vector field on M . Now, for a time-oriented Lorentzian manifold, we can define for every $p \in M$ its *chronological future/past* by

$$I^\pm(p) := \{q \in M : \exists \text{ future/past directed timelike curve from } p \text{ to } q\}.$$

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If we replace the word “timelike” with “causal”, we obtain the *causal future* and *past*, denoted by $J^\pm(p)$. For a subset $A \subseteq M$, we further define

$$I^\pm(A) := \bigcup_{p \in A} I^\pm(p) \quad \text{and} \quad J^\pm(A) := \bigcup_{p \in A} J^\pm(p).$$

Let $\tilde{M}$ be a submanifold of a pseudo-Riemannian manifold (M, g) with inclusion $i : \tilde{M} \hookrightarrow M$. Then the pull back i^*g of the metric g to the submanifold $\tilde{M}$ at some $p \in \tilde{M}$ is just the restriction of g_p to the subspace $T_p\tilde{M} \subseteq T_pM$. If g is Riemannian, this restriction is positive definite and so $i^*g =: g|_{\tilde{M}}$ turns $\tilde{M}$ into a Riemannian manifold. However, if M is only pseudo-Riemannian the induced “metric” $g|_{\tilde{M}}$ need not be non-degenerate. For the rest of this subchapter we will closely follow [Lan21].

Definition 2.1 (Hypersurfaces). *Let (M, g) be a Lorentzian manifold and $\tilde{M} \subset M$ a hypersurface with nonzero normal vector field $n \in \mathfrak{X}(M)$. $\tilde{M}$ is called*

- (i) *spacelike if $g_p(n_p, n_p) < 0$ for each $p \in \tilde{M}$. This is the case iff the induced metric $g|_{\tilde{M}}$ is positive definite, so that $(\tilde{M}, g|_{\tilde{M}})$ is a 3-dimensional Riemannian manifold.*
- (ii) *timelike or Lorentzian if $g_p(n_p, n_p) > 0$ for each $p \in \tilde{M}$. This is the case iff the induced metric $g|_{\tilde{M}}$ is Lorentzian with signature $(-, +, +)$.*
- (iii) *null if $g_p(n_p, n_p) = 0$ for each $p \in \tilde{M}$. This is the case iff $g|_{\tilde{M}}$ is degenerate.*

We will now define a special class of connected, oriented, time-oriented smooth Lorentzian manifolds, or *spacetimes* for short, for which the Cauchy problem of hyperbolic operators is well posed.

Definition 2.2 (Global hyperbolicity). *A spacetime (M, g) is called globally hyperbolic if*

- (i) *(M, g) is causal, i.e. there do not exist closed causal curves, and*
- (ii) *for every $p, q \in M$ the causal diamond $J^+(p) \cap J^-(q)$ is compact.*

The second condition implies that whenever $q \in J^+(p)$, there exists a causal geodesic from p to q that extremizes the proper time among causal curves connecting them [O’N83]. It is easy to prove that a globally hyperbolic spacetime is necessarily non-compact [Sch26]. It turns out that a globally hyperbolic manifold possesses a particularly simple topological structure: it is diffeomorphic to a product $\mathbb{R} \times \Sigma$, where Σ is a spacelike hypersurface. This surface functions as a sort of snapshot and shows how causality is organized within the spacetime framework.

Definition 2.3. *A Cauchy (hyper)surface is a subset $\Sigma \subset M$ that is met by any inextendible timelike curve precisely once.*

The following theorem makes precise the above remark about the topology and causal structure on globally hyperbolic spacetimes.

2.1. Preliminaries on Lorentzian geometry and causality

Theorem 2.1. *A spacetime (M, g) is globally hyperbolic iff it has a smooth spacelike Cauchy surface. In that case:*

- (i) *There exists a smooth surjection $t : M \rightarrow \mathbb{R}$ with timelike past-directed gradient $\nabla^\mu t$, called a temporal function, such that M is foliated as*

$$M = \bigcup_{\tau \in \mathbb{R}} \Sigma_\tau,$$

where each $\Sigma_\tau = \{p \in M \mid t(p) = \tau\}$ is a smooth spacelike Cauchy surface and all Σ_τ are diffeomorphic.

- (ii) *M is diffeomorphic to $\mathbb{R} \times \Sigma$, where Σ is diffeomorphic to Σ_τ for any $\tau \in \mathbb{R}$.*

- (iii) *(M, g) is isometric to $(\mathbb{R} \times \Sigma, g')$, where the metric g' is given by*

$$g' = -N^2 dt^2 + h,$$

where $N : \mathbb{R} \times \Sigma \rightarrow (0, \infty)$ is a smooth function, called the lapse function, and h is a Riemannian metric on the Cauchy surface Σ .

The first part about the characterization of global hyperbolicity is due to Geroch [Ger70], while the rest is due to Bernal and Sánchez [BS05, BS06]. In view of theorem 2.1(ii), a foliation is equivalent to a diffeomorphism $F : \mathbb{R} \times \Sigma \rightarrow M$ with the property that every subspace $\Sigma_\tau := F_\tau(\Sigma)$ is spacelike, where we have used the notation $F_\tau : \Sigma \rightarrow M$, $F_\tau(x) := F(\tau, x)$. With this diffeomorphism at hand we define the lapse N and the shift $\vec{N}$ of the foliation in a coordinate-independent way by the decomposition

$$\frac{dF_\tau}{d\tau} =: N n_{\Sigma_\tau} + \vec{N}. \quad (2.3)$$

Here we have $N \in C^\infty(M)$ being the lapse function, $n_{\Sigma_\tau} \in \mathfrak{X}(M)$ is the normal future-directed vector field to Σ_τ and $\vec{N} \in \mathfrak{X}(M)$ is the orthogonal projection of $dF_\tau/d\tau$ onto $T_{F_\tau(x)}\Sigma_\tau$, hence tangent to Σ_τ . Using the temporal function, we have the relations

$$N = \frac{1}{\sqrt{-g(\nabla t, \nabla t)}} \quad \text{and} \quad n_{\Sigma_\tau} = -N \nabla t. \quad (2.4)$$

From this, together with theorem 2.1 (iii), the induced Riemannian metric on Σ_τ is given by

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu, \quad (2.5)$$

where we have abbreviated the unit normal n_{Σ_τ} by n .

Another concept that we will need in section 2.3 is that of Jacobi fields and Jacobi tensors [JKK95]. Let $I \subset \mathbb{R}$ be an open interval containing 0, and let $\gamma : I \rightarrow M$ be a geodesic. Since γ is an immersion, it induces a vector bundle

$$T\gamma := \bigcup_{t \in I} \{t\} \times T_{\gamma(t)}M$$

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over I , whose fiber at $t \in I$ is given by $T_{\gamma(t)}M$. The *normal bundle* of γ , denoted by $N(\gamma)$, is the subbundle of $T\gamma$ whose fiber over $t \in I$ is

$$N(\gamma)_t = \{v \in T_{\gamma(t)}M \mid g(v, \dot{\gamma}(t)) = 0\}. \quad (2.6)$$

The Levi-Civita connection on M induces connections on $T\gamma$, $N(\gamma)$, and related bundles such as $\text{End}(N(\gamma))$, which we also denote by ∇ . The Riemann curvature tensor R induces a bundle endomorphism $R_\gamma \in \Gamma(\text{End}(N(\gamma)))$ defined by

$$R_\gamma(t)v = R(v, \dot{\gamma}(t))\dot{\gamma}(t), \quad v \in N(\gamma)_t.$$

A vector field X along γ satisfying

$$\nabla_{\dot{\gamma}}^2 X + R(X, \dot{\gamma})\dot{\gamma} = 0 \quad (2.7)$$

is called a *Jacobi vector field* along γ .

Definition 2.4 (Jacobi tensor). *A tensor field $J \in \Gamma(\text{End}(N(\gamma)))$ is called a Jacobi tensor along γ if it satisfies*

$$\nabla_{\dot{\gamma}}^2 J + R_\gamma \circ J = 0. \quad (2.8)$$

One can easily see that for every parallel section $X \in \Gamma(N(\gamma))$, i.e. $\nabla_{\dot{\gamma}} X = 0$, the vector field

$$(JX)(t) := J(t)X(t)$$

is a Jacobi vector field along γ . Note that the Jacobi equation eq. (2.7) is a linear second-order ordinary differential equation along γ . Hence, a Jacobi field J is uniquely determined by the initial data $J(0)$ and $\nabla_{\dot{\gamma}} J(0)$. Let $t_0, t_1 \in I$ with $t_0 \neq t_1$. If there exists a non-trivial Jacobi field X along γ with $X(t_0) = 0 = X(t_1)$, then the points $\gamma(t_0)$ and $\gamma(t_1)$ are called *conjugate points* along γ . Note that Minkowski space contains no conjugate points, while on the sphere the north and south poles are conjugate points along every geodesic connecting them [dC16].

We conclude this section with the notion of Killing fields, which will play an important role in later chapters. A vector field $X \in \mathfrak{X}(M)$ is called a *Killing vector field* if

$$\mathcal{L}_X g = 0,$$

where $\mathcal{L}_X$ denotes the Lie derivative with respect to X . In local coordinates, this condition is equivalent to the Killing equation

$$\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0.$$

Killing vector fields are the infinitesimal generators of isometries of a spacetime. More generally, a vector field X is called a *conformal Killing vector field* if

$$\mathcal{L}_X g = \lambda g$$

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for some scalar function λ . Killing vector fields are closely related to Jacobi fields. Indeed, one can show that the restriction of a Killing vector field to any geodesic is a Jacobi field along that geodesic [O’N83, Lem. 9.26]. In particular, a Killing vector field is uniquely determined globally by its value and derivative at a single point. On an n -dimensional pseudo-Riemannian manifold, the dimension of the space of Killing vector fields is at most $\frac{1}{2}n(n+1)$, so that in four-dimensional spacetime there can be at most 10 linearly independent Killing vector fields. This maximal case is realized, for instance, by Minkowski space.

2.2. The Gaussian beam approximation for the scalar wave equation

Wave propagation is governed by hyperbolic partial differential equations. It is well known that linear hyperbolic equations admit highly localized solutions. As an example, consider the equation

$$P\psi = 0 \text{ on } \mathbb{R}^n \times \mathbb{R}, \quad \text{where } P = \frac{\partial}{\partial t}.$$

It is clear that its solutions are precisely the functions that do not depend on t . Thus, $P\psi = 0$ admits solutions that vanish outside a small neighborhood of $\{(x_0, t) : t \in \mathbb{R}\} \subseteq \mathbb{R}^n \times \mathbb{R}$.

A more interesting problem is the Cauchy problem for the wave equation

$$\square\psi = \frac{\partial^2\psi}{\partial t^2} - \sum_{k=1}^n \frac{\partial^2\psi}{\partial x_k^2} = 0 \text{ on } \mathbb{R}^n \times \mathbb{R},$$

with initial data

$$\psi(x, t_0) = g_0(x), \quad \frac{\partial\psi}{\partial t}(x, t_0) = g_1(x).$$

This problem is *well-posed*, in the sense that there always exists a unique solution depending continuously on the initial data [Eva10]. Suppose that one starts with stationary, localized, oscillatory initial data, where the frequency is denoted by $\omega > 0$. To this end, choose a bump function $\chi \in C_0^\infty(\mathbb{R}^n)$ such that $\chi(x) = 1$ for $|x| < R$, and let $\Psi(x, t, \omega)$ be the solution to the Cauchy problem

$$\begin{cases} \Psi(x, t_0, \omega) = \chi(x)e^{i\omega x \cdot \xi} e^{-\frac{\omega}{2}|x-x_0|^2}, \\ \frac{\partial^k \Psi}{\partial t^k}(x, t_0, \omega) = 0, \quad k = 0, 1, \quad \xi \in \mathbb{R}^n \setminus \{0\}. \end{cases} \quad (2.9)$$

Now, the main idea is to construct a function $\psi(x, t, \omega; x_0, \xi)$ such that for some $T > t_0$

$$\|\psi(x, T, \omega) - \Psi(x, T, \omega)\| = \mathcal{O}(\omega^{-p}),$$

where $\|\cdot\|$ denotes a suitable Sobolev norm. A general mathematical treatment of approximate solutions to differential equations which decay, in the high-frequency limit,

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outside a neighborhood of certain curves was given in [Mas94]. In work dating back to 1977, referred to as the *complex WKB method for nonlinear equations*, Maslov showed that such solutions are almost everywhere well approximated by functions of the form

$$a(x)e^{i\omega\phi(x)}, \quad (2.10)$$

where $\phi(x)$ is a complex-valued function satisfying $\Im(\phi(x)) \geq 0$. The name “complex WKB method” comes from the fact that when $\phi(x)$ is real-valued, the method for constructing approximate solutions of this type is known in quantum mechanics as the WKB method.

A detailed construction of Gaussian beams as approximate solutions to the wave equation on $\mathbb{R}^n \times \mathbb{R}$ can be found in [Ral83]. The construction starts with an ansatz of the form

$$\psi_\omega(x) = e^{i\omega\phi(x)} (a_0(x) + a_1(x)\omega^{-1} + \dots + a_N(x)\omega^{-N}) \quad (N \in \mathbb{N}), \quad (2.11)$$

where we have introduced the notation $x = (x_0, x_1, \dots, x_n)$ with $x_0 = t$. Given some $p > 0$, the complex-valued functions $\phi(x), a_0(x), \dots, a_N(x)$ are constructed such that

$$\|\Box\psi_\omega\| \leq C_p \omega^{-p}.$$

The Gaussian beam approximation for the wave equation on Lorentzian manifolds is treated in [Sbi15]. We give a fairly detailed summary, since the concepts used there are quite general and can be applied, with only minor modifications, to the setting considered in this thesis. Consider a time-oriented globally hyperbolic Lorentzian manifold with time function t , foliated by $\Sigma_\tau = \{t = \tau\}$. Given some $T > 0$, we set

$$R_{[0,T]} := \bigcup_{0 \leq \tau \leq T} \Sigma_\tau.$$

The future-directed unit normal to the slice Σ_τ will be denoted by n_{Σ_τ} . We define the stress-energy tensor T of ψ by

$$T_{\mu\nu}[\psi] = \frac{1}{2} (\partial_\mu \psi \overline{\partial_\nu \psi} + \partial_\nu \psi \overline{\partial_\mu \psi}) - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \psi \overline{\partial_\beta \psi}.$$

Given a future-directed timelike vector field $u \in \mathfrak{X}(M)$, we define

$$E_\tau^u[\psi] := \int_{\Sigma_\tau} T_{\mu\nu} u^\mu n_{\Sigma_\tau}^\nu d\mu_{\Sigma_\tau} \quad (2.12)$$

to be the u -energy of ψ . Furthermore, let γ be a null geodesic intersecting the Cauchy hypersurface Σ_0 , and let $\mathcal{N}$ be a neighborhood of γ . Similarly to the flat spacetime case described above, we seek to construct approximate solutions to the scalar wave equation

$$\Box_g \psi = 0, \quad (2.13)$$

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which are localized along γ , where $\square_g = g_{\mu\nu} \nabla^\mu \nabla^\nu$ denotes the d'Alembert operator. If the underlying metric is clear from the context, we will simply write $\square$. In analogy with eq. (2.10), our Gaussian beam ansatz will be a function $\psi_\omega \in C^\infty(M, \mathbb{C})$ of the form

$$\psi_\omega(x) = a(x) e^{i\omega\phi(x)} \quad (\omega > 0), \quad (2.14)$$

where $a, \phi \in C^\infty(M, \mathbb{C})$ are independent of ω and $a = a_{\mathcal{N}}$ depends on $\mathcal{N}$. Note that the phase function ϕ , also called the *eikonal*, is a complex-valued function. In section 2.3 we will construct a and ϕ such that ψ_ω satisfies the following three conditions:

$$\|\square\psi_\omega\|_{L^2(R_{[0,T]})} \leq C(T), \text{ where } C(T) \text{ does not depend on } \omega, \quad (2.15a)$$

$$E_0^u[\psi_\omega] \rightarrow \infty \text{ for } \omega \rightarrow \infty, \text{ and} \quad (2.15b)$$

$$\psi_\omega \text{ is supported in } \mathcal{N}. \quad (2.15c)$$

Once this has been achieved, we normalize our ansatz by setting

$$\tilde{\psi}_\omega := \frac{\psi_\omega}{\sqrt{E_0^u[\psi_\omega]}},$$

which yields

$$\|\square\tilde{\psi}_\omega\|_{L^2(R_{[0,T]})} \rightarrow 0 \text{ for } \omega \rightarrow \infty.$$

We now define the exact solution Ψ_ω of the wave equation, approximated by $\tilde{\psi}_\omega$, to be the solution of the initial value problem

$$\begin{aligned} \square\Psi &= 0, \\ \Psi|_{\Sigma_0} &= \tilde{\psi}_\omega|_{\Sigma_0}, \\ n_{\Sigma_0}\Psi|_{\Sigma_0} &= n_{\Sigma_0}\tilde{\psi}_\omega|_{\Sigma_0}. \end{aligned}$$

One can then show, [Sbi15, Thm. 2.1], that

$$E_\tau^u[\Psi_\omega - \tilde{\psi}_\omega] \leq C(T, u, \Sigma_\tau) \|\square\tilde{\psi}_\omega\|_{L^2(R_{[0,T]})}^2 \text{ for all } 0 \leq \tau \leq T,$$

which converges to zero as $\omega \rightarrow \infty$. Thus, from the fact that $\tilde{\psi}_\omega$ is an approximate solution with respect to a suitable energy norm, one obtains the existence of exact solutions to the wave equation whose energy remains localized along a given null geodesic for finite, but arbitrarily long, times.

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First we calculate

$$\begin{aligned} \square\psi_\omega &= g^{\mu\nu} \nabla_\mu \nabla_\nu (a e^{i\omega\phi}) \\ &= \left[\square a + 2i\omega g^{\mu\nu} (\nabla_\mu a) (\nabla_\nu \phi) + i\omega a \square \phi - \omega^2 a g^{\mu\nu} (\nabla_\mu \phi) (\nabla_\nu \phi) \right] e^{i\omega\phi}. \end{aligned} \quad (2.16)$$

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In order for eq. (2.15a) to be satisfied, we could require

$$g^{\mu\nu}(\nabla_\mu\phi)(\nabla_\nu\phi) = 0 \quad (\text{eikonal equation}), \quad (2.17)$$

as well as

$$2g^{\mu\nu}(\nabla_\mu a)(\nabla_\nu\phi) + a\Box\phi = 0 \quad (\text{transport equation}). \quad (2.18)$$

This is usually done in the context of *geometric optics approximation*. However, it turns out that the solution of the eikonal equation might break down at some point along γ due to the formation of *caustics* [Sbi15]. To see this, note that the eikonal equation can be solved using the method of characteristics. Define

$$H(x, k) := \frac{1}{2}g^{\mu\nu}k_\mu k_\nu, \quad (2.19)$$

where $k_\mu := \nabla_\mu\phi$. The characteristic system associated to $H(x, k)$ is given by Hamilton's equations on T^*M :

$$\begin{aligned} \dot{x}^\mu &= \frac{\partial H}{\partial k_\mu} = g^{\mu\nu}(x) k_\nu, \\ \dot{k}_\mu &= -\frac{\partial H}{\partial x^\mu} = -\frac{1}{2}\frac{\partial g^{\nu\lambda}}{\partial x^\mu}(x) k_\nu k_\lambda. \end{aligned} \quad (2.20)$$

The cotangent bundle T^*M carries its canonical symplectic form

$$\omega = dx^\mu \wedge dk_\mu.$$

A submanifold $\Lambda \subset T^*M$ is called *Lagrangian* if

$$\omega|_\Lambda = 0 \quad \text{and} \quad \dim \Lambda = \frac{1}{2} \dim T^*M = \dim M.$$

Let $\Sigma_0 \subset M$ be a non-characteristic hypersurface and let initial data $\phi|_{\Sigma_0}$ be given. We choose the normal derivative $n_{\Sigma_0}^\mu \nabla_\mu\phi$ such that the eikonal equation

$$g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi = 0$$

is satisfied on Σ_0 , and assume that $d\phi$ is transversal to Σ_0 . This determines an initial submanifold

$$\Lambda_0 = \{(x, d\phi_x) : x \in \Sigma_0\} \subset T^*M.$$

Since $d(d\phi) = 0$, the canonical symplectic form restricts to zero on Λ_0 , so Λ_0 is a Lagrangian submanifold. Geometrically, it is the space of initial 1-jets of the phase function ϕ . Let Φ_t denote the Hamiltonian flow generated by H which is defined as the one-parameter family of diffeomorphisms satisfying

$$\frac{d}{dt}\Phi_t(x) = X_H(\Phi_t(x)), \quad \Phi_0 = \text{id}_M.$$

2.3. The construction of Gaussian beams

Because Hamiltonian flows preserve the symplectic form¹, $\Lambda_t := \Phi_t(\Lambda_0)$ remains Lagrangian for as long as the flow is defined. If the projection $\pi : \Lambda_t \rightarrow M$ is a local diffeomorphism, then Λ_t can be written locally as the graph of an exact one-form,

$$\Lambda_t = \{(x, d\phi(x))\},$$

for some smooth function ϕ . This is a special case of the general Hamilton–Jacobi existence theorem for involution systems of first-order PDEs, see [Dui11, Thm. 3.6.3]. In this case ϕ satisfies the eikonal equation

$$H(x, d\phi(x)) = 0,$$

and Λ_t is foliated by bicharacteristic curves, whose projections onto M are null geodesics with tangent vector $\dot{\gamma}^\mu = \nabla^\mu \phi$. Indeed, using $k_\nu = g_{\nu\lambda} \dot{x}^\lambda$, we find

$$\dot{k}_\mu = \frac{d}{ds}(g_{\mu\nu} \dot{x}^\nu) = \partial_\lambda g_{\mu\nu} \dot{x}^\lambda \dot{x}^\nu + g_{\mu\nu} \ddot{x}^\nu.$$

On the other hand, the second characteristic equation gives

$$\dot{k}_\mu = -\frac{1}{2} \partial_\mu g^{\nu\lambda} g_{\nu\rho} g_{\lambda\sigma} \dot{x}^\rho \dot{x}^\sigma.$$

Using $\partial_\mu g^{\nu\lambda} = -g^{\nu\alpha} g^{\lambda\beta} \partial_\mu g_{\alpha\beta}$, we obtain

$$\dot{k}_\mu = \frac{1}{2} \partial_\mu g_{\rho\sigma} \dot{x}^\rho \dot{x}^\sigma.$$

Hence

$$g_{\mu\nu} \ddot{x}^\nu + \partial_\rho g_{\mu\sigma} \dot{x}^\rho \dot{x}^\sigma - \frac{1}{2} \partial_\mu g_{\rho\sigma} \dot{x}^\rho \dot{x}^\sigma = 0.$$

Since $\dot{x}^\rho \dot{x}^\sigma$ is symmetric in ρ and σ , we may symmetrize the second term as

$$g^{\mu\kappa} \partial_\rho g_{\mu\sigma} \dot{x}^\rho \dot{x}^\sigma = \frac{1}{2} g^{\mu\kappa} (\partial_\rho g_{\mu\sigma} + \partial_\sigma g_{\mu\rho}) \dot{x}^\rho \dot{x}^\sigma.$$

¹Let (M, ω) be a symplectic manifold and $H \in C^\infty(M)$. Let X_H be the Hamiltonian vector field defined by

$$\iota_{X_H} \omega = dH.$$

By Cartan’s magic formula,

$$\mathcal{L}_{X_H} \omega = d(\iota_{X_H} \omega) + \iota_{X_H} d\omega = d(dH) + 0 = 0.$$

Hence,

$$\frac{d}{dt}(\Phi_t^* \omega) = \Phi_t^*(\mathcal{L}_{X_H} \omega) = 0.$$

Therefore $\Phi_t^* \omega$ is constant in t , and since $\Phi_0 = \text{id}$,

$$\Phi_t^* \omega = \omega.$$

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Therefore,

$$\ddot{x}^\kappa + \frac{1}{2}g^{\mu\kappa}(\partial_\rho g_{\mu\sigma} + \partial_\sigma g_{\mu\rho} - \partial_\mu g_{\rho\sigma})\dot{x}^\rho\dot{x}^\sigma = 0,$$

which is the geodesic equation

$$\ddot{x}^\kappa + \Gamma_{\rho\sigma}^\kappa \dot{x}^\rho \dot{x}^\sigma = 0. \quad (2.21)$$

Moreover, ϕ is constant along these geodesics. Let $\gamma(s)$ be a characteristic curve. Then

$$\frac{d}{ds}\phi(\gamma(s)) = \dot{\gamma}^\mu(s)\nabla_\mu\phi(\gamma(s)) = \nabla^\mu\phi(\gamma(s))\nabla_\mu\phi(\gamma(s)),$$

which is zero due to the eikonal equation. The solution breaks down once two characteristic geodesics meet. Such an intersection is called a caustic. Since ϕ is constant along characteristic geodesics, each geodesic carries the value of ϕ assigned to it by the initial data on Σ_0 . Different geodesics therefore generally carry different values of ϕ . If two such geodesics intersect, then the function ϕ would have to take two different values at the same spacetime point. Hence the solution to the eikonal equation cannot be extended smoothly beyond the caustic. Also note that by eq. (2.18) a is transported along the geodesics determined by ϕ . Hence, the transport equation has the same domain of existence as the eikonal equation, and thus also breaks down at caustics.

Also note that the breakdown of solutions to the eikonal equation does not necessarily occur in every spacetime. For example, in flat Minkowski spacetime one can always choose suitable initial data for ϕ so as to avoid the formation of caustics. A sufficient condition guaranteeing that no such choice of initial data exists is the presence of conjugate points along the null geodesic γ , see [Sbi15, Thm. C.1]. Therefore, instead of requiring the eikonal and transport equations to hold exactly, we only require approximate solutions in the sense that $g^{\mu\nu}(\nabla_\mu\phi)(\nabla_\nu\phi)$ and $2g^{\mu\nu}(\nabla_\mu a)(\nabla_\nu\phi) + a\Box\phi$ vanish on γ to some order. Furthermore, we also require the imaginary part of ϕ to be non-negative, at least locally. To this end, we impose the conditions

$$\phi|_\gamma \text{ and } \nabla_\mu\phi|_\gamma \text{ are real valued, and} \quad (H2.1)$$

$$\begin{aligned} \Im(\nabla\nabla\phi|_\gamma) \text{ is positive definite on a 3-dimensional subspace} \\ \text{transversal to } \dot{\gamma}. \end{aligned} \quad (H2.2)$$

Here, for $x \in M$, the expression $\Im(\nabla\nabla\phi|_x)$ denotes the imaginary part of the bilinear map

$$\nabla\nabla\phi|_x : T_x M \times T_x M \rightarrow \mathbb{C}, \quad \langle \nabla\nabla\phi|_x v, w \rangle := \langle \nabla_v \text{grad}\phi|_x, w \rangle.$$

Now, since M is globally hyperbolic, it follows that γ is an embedded submanifold of M . Thus, there exists a chart (U, φ) satisfying

$$\varphi(\text{Im}(\gamma) \cap U) = \{x_1 = x_2 = x_3 = 0\},$$

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called *slice coordinates*, such that

$$\Im(\phi)(x) \geq c \cdot (x_1^2 + x_2^2 + x_3^2). \quad (2.22)$$

We introduce the notation $\Re(\phi) =: \phi_1$ and $\Im(\phi) =: \phi_2$, so that $\phi = \phi_1 + i\phi_2$. Then eq. (2.16) becomes

$$\begin{aligned} \square\psi_\omega &= -\omega^2 \underbrace{g^{\mu\nu}(\nabla_\mu\phi)(\nabla_\nu\phi)}_{\text{underbraced}} a e^{i\omega\phi_1} e^{-\omega\phi_2} \\ &\quad + i\omega \underbrace{[2g^{\mu\nu}(\nabla_\mu a)(\nabla_\nu\phi) + \square\phi \cdot a]}_{\text{underbraced}} e^{i\omega\phi_1} e^{-\omega\phi_2} \\ &\quad + \square a e^{i\omega\phi_1} e^{-\omega\phi_2}. \end{aligned} \quad (2.23)$$

As discussed previously, if the underbraced terms vanish on γ to sufficiently high order, then, due to the exponential suppression arising from the factors $e^{-\omega\phi_2}$, one expects $\|\square\psi_\omega\|_{L^2(R_{[0,T]})}$ to become small for sufficiently large ω . The following lemma provides an explicit formula for calculating the order to which the underbraced terms must vanish.

Lemma 2.1. *Let $f \in C_0^\infty([0, T] \times \mathbb{R}^3, \mathbb{C})$ vanish along $\{x_1 = x_2 = x_3 = 0\}$ to order S , i.e. all partial derivatives up to and including order S of f vanish along $\{x_1 = x_2 = x_3 = 0\}$, and let $c > 0$ be a constant. We then have*

$$\int_{[0,T] \times \mathbb{R}^3} |f(x)|^2 e^{-\omega \cdot c(x_1^2 + x_2^2 + x_3^2)} dx \leq C \omega^{-(S+1) - \frac{3}{2}} \quad (2.24)$$

where C depends on f and T .

Proof. We will follow [Sbi15, Lem. 2.17] while being more explicit. Let $x = (x_0, \underline{x}) \in [0, T] \times \mathbb{R}^3$. We introduce stretched coordinates

$$y_0 := x_0, \quad y_i := \sqrt{\omega} x_i \quad \text{for } i \in \{1, 2, 3\}.$$

Since f vanishes on $\{x_1 = x_2 = x_3 = 0\}$ we have

$$\partial_{\underline{x}}^\alpha f(x_0, 0) = 0 \quad \text{for any multi-index with } |\alpha| \leq S. \quad (2.25)$$

We can now Taylor-expand f around $0 \in \mathbb{R}^3$ as

$$f(x_0, \underline{x}) = \sum_{|\alpha| \leq S} \frac{1}{\alpha!} \partial_{\underline{x}}^\alpha f(x_0, 0) \underline{x}^\alpha + R_{S+1}(x_0, \underline{x}),$$

where the remainder term R_{S+1} is given by

$$R_{S+1}(x_0, \underline{x}) = (S+1) \sum_{|\alpha|=S+1} \frac{\underline{x}^\alpha}{\alpha!} \int_0^1 (1-t)^S \partial_{\underline{x}}^\alpha f(x_0, t\underline{x}) dt.$$

From eq. (2.25) we can now estimate

$$|f(x_0, \underline{x})| \leq C_1 \sum_{|\alpha|=S+1} |\underline{x}^\alpha| \sup_{t \in [0,1]} |\partial_{\underline{x}}^\alpha f(x_0, t\underline{x})|.$$

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Since f has compact support we also have that

$$\sup_{x \in [0, T] \times \mathbb{R}^3} |\partial_{\underline{x}}^\alpha f(x)| \leq C_\alpha.$$

Noting that $|\underline{x}^\alpha| \leq |\underline{x}|^{S+1}$ for $|\alpha| = S+1$ we obtain

$$|f(x_0, \underline{x})| \leq C_2 |\underline{x}|^{S+1} \quad \text{for all } (x_0, \underline{x}) \in [0, T] \times \mathbb{R}^3.$$

From this follows that

$$\left| f\left(y_0, \frac{\underline{y}}{\sqrt{\omega}}\right) \right| \leq C_2 \frac{|\underline{y}|^{S+1}}{\omega^{\frac{S+1}{2}}}.$$

This yields

$$\begin{aligned} \int_{[0, T] \times \mathbb{R}^3} |f(x)|^2 e^{-\omega c |\underline{x}|^2} dx &\leq C_2^2 \int_{[0, T] \times \mathbb{R}^3} \frac{1}{\omega^{3/2}} \frac{|\underline{y}|^{2(S+1)}}{\omega^{S+1}} e^{-c |\underline{y}|^2} dy \\ &\leq C_2^2 \int_{[0, T] \times \mathbb{R}^3} |\underline{y}|^{2(S+1)} |e^{-c |\underline{y}|^2}|^2 dy \omega^{-(S+1) - \frac{3}{2}} \\ &\leq C \omega^{-(S+1) - \frac{3}{2}}. \end{aligned}$$

□

Remark 2.1. Note that, if $S < 0$, the condition that f vanishes to order S along $\{x_1 = x_2 = x_3 = 0\}$ is understood to be vacuous, i.e. no vanishing is imposed. In this case, the estimate still holds in the weaker form

$$\int_{[0, T] \times \mathbb{R}^3} |f(x)|^2 e^{-\omega c |\underline{x}|^2} dx \leq C \omega^{-3/2}.$$

Indeed, for any $f \in C_0^\infty([0, T] \times \mathbb{R}^3)$,

$$\int_{[0, T] \times \mathbb{R}^3} |f(x)|^2 e^{-\omega c |\underline{x}|^2} dx \leq T \|f\|_{L^\infty([0, T] \times \mathbb{R}^3)}^2 \int_{\mathbb{R}^3} e^{-\omega c |\underline{x}|^2} d\underline{x}.$$

Performing the change of variables $\underline{y} = \sqrt{\omega} \underline{x}$ gives

$$\int_{\mathbb{R}^3} e^{-c \omega |\underline{x}|^2} d\underline{x} = \omega^{-3/2} \int_{\mathbb{R}^3} e^{-c |\underline{y}|^2} d\underline{y},$$

which proves the claim.

Remark 2.2. The above lemma also applies to compactly supported one-forms on M . Indeed, let $a \in \Omega_0^1(M)$ and let $\{e^0, \dots, e^3\}$ be a smooth orthonormal coframe. Then

$$a = \sum_{b=0}^3 a_b e^b,$$

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where $a_b \in C_0^\infty([0, T] \times \mathbb{R}^3, \mathbb{C})$. Since the coframe is smooth and pointwise invertible, a vanishes to order S along $\{x_1 = x_2 = x_3 = 0\}$ if and only if each coefficient a_b does. Moreover,

$$|a|_g^2 := g^{\mu\nu} a_\mu \overline{a_\nu} = \sum_{b=0}^3 |a_b|^2.$$

Applying lemma 2.1 to each coefficient a_b and summing over b yields the corresponding estimate for a . Also note that the “vanishing to some order” condition can be expressed both in terms of standard partial derivatives and covariant derivatives. Indeed,

$$A_\alpha|_\gamma = 0 \quad \text{and} \quad \partial_\mu A_\alpha|_\gamma = 0$$

implies $\nabla_\mu A_\alpha|_\gamma = 0$, since

$$\nabla_\mu A_\alpha|_\gamma = \partial_\mu A_\alpha|_\gamma - \Gamma_{\mu\alpha}^\beta|_\gamma A_\beta|_\gamma.$$

An inductive argument also shows this for higher order tensors.

We now want to find the minimal order up to which the underbraced terms in eq. (2.23) need to vanish such that their L^2 -norms including the corresponding factors of ω go to zero as $\omega \rightarrow \infty$. We find that

$$g^{\mu\nu}(\nabla_\mu \phi)(\nabla_\nu \phi) \text{ has to vanish to second order along } \gamma, \text{ and} \quad (\text{H2.3})$$

$$2g^{\mu\nu}[(\nabla_\mu a)(\nabla_\nu \phi) + \square \phi \cdot a] \text{ has to vanish to zeroth order along } \gamma. \quad (\text{H2.4})$$

These constraints follow directly from lemma 2.1 observing that

$$\begin{aligned} \left(\omega^4 \int_{[0, T] \times \mathbb{R}^3} |g^{\mu\nu}(\nabla_\mu \phi)(\nabla_\nu \phi)|^2 e^{-\omega c(x_1^2 + x_2^2 + x_3^2)} \right)^{1/2} &\leq C\omega^2 \cdot \omega^{-3/2-3/4} \\ &= C\omega^{-1/4} \rightarrow 0 \quad (\omega \rightarrow \infty), \end{aligned}$$

as well as

$$\begin{aligned} \left(\omega^2 \int_{[0, T] \times \mathbb{R}^3} \left| 2g^{\mu\nu}[(\nabla_\mu a)(\nabla_\nu \phi) + \square \phi \cdot a] \right|^2 e^{-\omega c(x_1^2 + x_2^2 + x_3^2)} \right)^{1/2} &\leq C\omega \cdot \omega^{-1/2-3/4} \\ &= C\omega^{-1/4} \rightarrow 0 \quad (\omega \rightarrow \infty). \end{aligned}$$

Assume now that we are given $a, \phi \in C^\infty(M, \mathbb{C})$ satisfying (H2.1), (H2.2), (H2.3), and (H2.4), as well as

$$a(\text{Im}(\gamma) \cap \Sigma_0) \neq 0 \quad \text{and} \quad \nabla^\mu \phi(\text{Im}(\gamma) \cap \Sigma_0) \neq 0. \quad (\text{H2.5})$$

It is straightforward to show that, given a neighborhood $\mathcal{N}$ of γ , we can multiply by a suitable bump function $\chi_{\mathcal{N}}$ with $\text{supp } \chi_{\mathcal{N}} \subseteq \mathcal{N}$ such that

$$\psi_\omega = \chi_{\mathcal{N}} \cdot a e^{i\omega \phi}$$

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satisfies eq. (2.15a), eq. (2.15b), and eq. (2.15c), see [Sbi15, Lem. 2.19]. In the following, we explicitly construct a Gaussian beam along γ satisfying all five assumptions stated above.

First note that all of the conditions only constrain the values of ϕ , a , and their derivatives along γ . Therefore, in order to satisfy these conditions, it suffices to construct suitable first and second derivatives of ϕ , together with a function a , *along* γ . To obtain globally defined functions on M , we then write down a formal Taylor expansion up to second order for ϕ and up to zeroth order for a . This yields smooth functions on M whose derivatives of interest coincide with the values constructed previously. Such a construction is possible by *Borel's lemma*, see [Hö03, Thm. 1.2.6].

To begin, we require the first and second derivatives of ϕ to be *compatible*, in the sense that

$$\nabla_\mu \nabla_\nu \phi(\gamma(s)) \dot{\gamma}^\nu(s) = \frac{d}{ds} \nabla_\mu \phi(\gamma(s)). \quad (\text{H2.6})$$

Now let $s \mapsto \gamma(s)$ be an affinely parametrized, future-directed null geodesic such that $\gamma(0) \in \Sigma_0$ and $\phi(0) \in \mathbb{R}$. Set

$$\nabla_\mu \phi(s) := \dot{\gamma}_\mu(s), \quad (2.26)$$

which implies that $\phi(s) \in \mathbb{R}$ for all s , and hence (H2.1) is satisfied. Since $\dot{\gamma}$ is null, it follows from eq. (2.26) that $g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi = 0$ along γ .

At this point we switch from covariant derivatives to partial derivatives. Note that the two operations coincide when acting on scalars, and second covariant derivatives on scalars commute since the Levi-Civita connection is torsion-free. The reason for this switch is that we will differentiate expressions in which the metric appears explicitly. If one were to use covariant derivatives throughout, then the identity $\nabla g = 0$ would eliminate these terms automatically.

We choose slice coordinates (φ, U) along γ and define

$$f(x) := \frac{1}{2} g^{\mu\nu}(x) \partial_\mu \phi(x) \partial_\nu \phi(x) \quad (x \in U).$$

We now derive the equations implied by requiring f to vanish to second order. The zeroth-order condition has already been discussed above and implies that the eikonal equation holds along γ . At first order we obtain

$$0 = \partial_\kappa f|_\gamma = -(\partial_\kappa \dot{\gamma}) + \dot{\gamma}^\nu \partial_\nu \partial_\kappa \phi. \quad (2.27)$$

Requiring f to vanish to second order yields

$$\begin{aligned} 0 = & \frac{1}{2} (\partial_\kappa \partial_\rho g^{\mu\nu}) (\partial_\mu \phi) (\partial_\nu \phi)|_\gamma + (\partial_\kappa g^{\mu\nu}) (\partial_\rho \partial_\mu \phi) (\partial_\nu \phi)|_\gamma + (\partial_\rho g^{\mu\nu}) (\partial_\kappa \partial_\nu \phi) (\partial_\mu \phi)|_\gamma \\ & + g^{\mu\nu} (\partial_\rho \partial_\mu \phi) (\partial_\kappa \partial_\nu \phi)|_\gamma + g^{\mu\nu} \partial_\mu \phi (\partial_\rho \partial_\kappa \partial_\nu \phi)|_\gamma. \end{aligned} \quad (2.28)$$

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One can readily verify that eq. (2.28) can be rewritten as a (non-linear) *Riccati equation*

$$0 = A + BM + MB^T + MCM + \frac{d}{ds}M, \quad (2.29)$$

where the matrices are given by

$$\begin{aligned} A_{\kappa\rho}(s) &:= \frac{1}{2}(\partial_\kappa \partial_\rho g^{\mu\nu}) \partial_\mu \phi \partial_\nu \phi(\gamma(s)), \\ B_{\kappa\rho}(s) &:= \partial_\kappa g^{\rho\nu} \partial_\nu \phi(\gamma(s)), \\ C_{\kappa\rho}(s) &:= g^{\kappa\rho}(\gamma(s)), \\ M_{\kappa\rho}(s) &:= \partial_\kappa \partial_\rho \phi(\gamma(s)). \end{aligned}$$

Our aim is to show that there exists a global solution for M satisfying (H2.2) and (H2.6). It turns out that one can solve eq. (2.29) by constructing suitable Jacobi tensors V and J along γ satisfying the matrix ODE system

$$\begin{cases} \dot{J} = B^T J + CV, \\ \dot{V} = -AJ - BV. \end{cases} \quad (2.30)$$

We claim that if J is invertible, then $M := VJ^{-1}$ solves eq. (2.29). Indeed, differentiating the identity $J(s)J^{-1}(s) = I_4$ yields

$$\dot{J}^{-1} = -J^{-1} \dot{J} J^{-1}.$$

Using this identity, we compute

$$\begin{aligned} \dot{M} &= \dot{V}J^{-1} - VJ^{-1}\dot{J}J^{-1} = (-AJ - BV)J^{-1} - VJ^{-1}(B^T J + CV)J^{-1} \\ &= -A - BM - MB^T - MCM, \end{aligned}$$

which proves the claim. Thus, it suffices to show that one can always choose initial data such that J remains invertible for all time.

We refer the reader to [Sbi15] and only sketch the main idea of the proof. The key point is that constructing a matrix solution (J, V) to eq. (2.30) with invertible J is possible if and only if one can find four everywhere linearly independent Jacobi fields along γ . Note that if we required J to be real-valued, corresponding to a real-valued eikonal, then the solution would break down at caustics, as discussed previously. It turns out that choosing $M(0)$ such that

- (i) $M(0)$ is symmetric,
- (ii) $M(0)_{\mu\nu} \dot{\gamma}^\nu = (\partial_\mu \phi)(0)$,
- (iii) $\Im(M(0)_{\mu\nu}) dx^\mu|_{\gamma(0)} \otimes dx^\nu|_{\gamma(0)}$ is positive definite on a three-dimensional subspace of $T_{\gamma(0)}M$ which is transversal to $\dot{\gamma}$,

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and solving eq. (2.30) with initial data

$$\begin{pmatrix} J(0) \\ V(0) \end{pmatrix} = \begin{pmatrix} I_4 \\ M(0) \end{pmatrix}$$

yields invertibility of $J(s)$ for all s and a matrix M which is symmetric, solves eq. (2.29), and satisfies both the compatibility condition (H2.6) and condition (H2.2). In particular, eq. (H2.3) is satisfied. Condition (H2.4) follows from the fact that it is a linear first-order ordinary differential equation, and hence existence of solutions is always guaranteed. Applying Borel's lemma finally yields functions $a, \phi \in C^\infty(U, \mathbb{C})$ which also satisfy eq. (H2.5).

Covering $\text{Im}(\gamma)$ with slice coordinates (U_k, φ_k) , $k \in \mathbb{N}$, and extending $\{U_k\}_{k \in \mathbb{N}}$ to an open cover of M , we may patch together the local constructions via a partition of unity subordinate to that cover to obtain functions $a, \phi \in C^\infty(M, \mathbb{C})$ satisfying all of our assumptions.

Remark 2.3. *Observe that one can rewrite eq. (2.28) in a covariant form. Recall the notation $k_\mu = \partial_\mu \phi = \nabla_\mu \phi$. Replacing partial derivatives by covariant derivatives, at the expense of additional connection terms, we find that eq. (2.28) can be equivalently rewritten as*

$$R_{\mu\rho\nu\kappa} k^\mu k^\nu + (\nabla_\rho k_\mu)(\nabla_\kappa k^\mu) + (\nabla_\mu \nabla_\rho k_\kappa) k^\mu = 0. \quad (2.31)$$

Proof. First note that by the fact that k_μ is null, we have

$$k^\nu (\nabla_\sigma k_\nu) = 0.$$

Now, using

$$\begin{aligned} \partial_\rho k_\mu &= \nabla_\rho k_\mu + \Gamma_{\rho\mu}^\sigma k_\sigma, \\ \partial_\rho g^{\mu\nu} &= -\Gamma_{\rho\sigma}^\mu g^{\sigma\nu} - \Gamma_{\rho\sigma}^\nu g^{\mu\sigma}, \\ \partial_\rho \partial_\kappa k_\nu &= \nabla_\rho \nabla_\kappa k_\nu + (\partial_\rho \Gamma_{\kappa\nu}^\sigma) k_\sigma + \Gamma_{\kappa\nu}^\sigma (\nabla_\rho k_\sigma) + \Gamma_{\kappa\nu}^\sigma \Gamma_{\rho\sigma}^\lambda k_\lambda + \Gamma_{\rho\kappa}^\sigma (\nabla_\sigma k_\nu) + \Gamma_{\rho\nu}^\sigma (\nabla_\kappa k_\sigma), \end{aligned}$$

we can rewrite eq. (2.28) as

$$\begin{aligned} 0 &= g^{\mu\nu} (\partial_\rho \Gamma_{\kappa\nu}^\sigma) k_\sigma k_\nu - g^{\sigma\nu} (\partial_\kappa \Gamma_{\rho\sigma}^\mu) k_\mu k_\nu + g^{\mu\nu} \Gamma_{\kappa\nu}^\sigma \Gamma_{\rho\sigma}^\lambda k_\lambda k_\mu - g^{\mu\sigma} \Gamma_{\rho\sigma}^\nu \Gamma_{\kappa\nu}^\lambda k_\lambda k_\mu \\ &\quad + g^{\mu\nu} (\nabla_\rho k_\mu)(\nabla_\kappa k_\nu) + (\nabla_\rho \nabla_\kappa k_\nu) k^\nu. \end{aligned}$$

By the definition of the Riemann curvature tensor this simplifies to

$$0 = R_{\mu\nu\rho\kappa} k^\mu k^\nu + g^{\mu\nu} (\nabla_\rho k_\mu)(\nabla_\kappa k_\nu) + (\nabla_\rho \nabla_\kappa k_\nu) k^\nu.$$

The last term can be reformulated as

$$(\nabla_\rho \nabla_\kappa k_\nu) k^\nu = (\nabla_\rho \nabla_\nu k_\kappa) k^\nu = (\nabla_\nu \nabla_\rho k_\kappa) k^\nu - R_{\mu\kappa\rho\nu} k^\mu k^\nu,$$

2.3. The construction of Gaussian beams

where we have used that

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)k_\lambda = -R_{\sigma\lambda\mu\nu}k^\sigma.$$

Thus we obtain

$$0 = (R_{\mu\nu\rho\kappa} - R_{\mu\kappa\rho\nu})k^\mu k^\nu + g^{\mu\nu}(\nabla_\rho k_\mu)(\nabla_\kappa k_\nu) + (\nabla_\rho \nabla_\kappa k_\nu)k^\nu.$$

By the symmetry properties of the Riemann tensor and the Bianchi identity, it now follows that

$$R_{\mu\nu\rho\kappa} - R_{\mu\kappa\rho\nu} = R_{\mu\nu\rho\kappa} + R_{\mu\kappa\nu\rho} = -R_{\mu\rho\kappa\nu} = R_{\mu\rho\nu\kappa},$$

which gives the claim. □

3. Maxwell's equations in curved spacetime

Before being able to talk about the formulation of Maxwell's equations in curved spacetime, we will need to introduce some operators acting on differential forms.

3.1. Operations on differential forms

The conventions used here coincide with those of [Ham17]. Let (M, g) be an n -dimensional oriented pseudo-Riemannian manifold of signature (s, t) . To avoid repeatedly introducing complex conjugation, we treat the bundle of exterior k -forms $\Lambda^k T^*M$ as a real vector bundle. However, in chapter 6 we will work with complex-valued differential forms, and complex conjugation will then have to be taken into account.

If $\omega \in \Omega^k(M)$ is a k -form, we write it in local coordinates as

$$\omega = \frac{1}{k!} \omega_{\mu_1 \dots \mu_k} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_k} = \sum_{0 \leq \mu_1 < \dots < \mu_k \leq n} \omega_{\mu_1 \dots \mu_k} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_k},$$

where $\omega_{\mu_1 \dots \mu_k} := \omega(\partial_{\mu_1}, \dots, \partial_{\mu_k})$.

On M there exists a uniquely determined family of local maps $d^{(k)} : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$, denoted simply by d , which is called *exterior derivative*. It is $\mathbb{R}$ -linear, satisfies $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$ for $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$, coincides with the standard tangent map on $\Omega^0(M) = C^\infty(M)$ and satisfies $d \circ d = 0$. Given $\omega \in \Omega^k(M)$, in local coordinates we have

$$(d\omega)_{\mu_1 \dots \mu_{k+1}} = (k+1) \partial_{[\mu_1} \omega_{\mu_2 \dots \mu_{k+1}]}$$

The metric g together with the orientation of M define a *canonical volume form* $d\mu_g$. It is characterized by $d\mu_g(e_1, \dots, e_n) = +1$, where $\{e_1, \dots, e_n\}$ is an oriented, orthonormal basis of $T_p M$. In local coordinates it is given by

$$d\mu_g = \sqrt{|g|} dx^1 \wedge \dots \wedge dx^n,$$

where $|g| = |\det(g_{\mu\nu})|$. We equip $\Omega^k(M)$ with a $C^\infty(M)$ -bilinear form $\langle \cdot, \cdot \rangle : \Omega^k(M) \times \Omega^k(M) \rightarrow C^\infty(M)$ as follows: for $\alpha, \beta \in \Omega^k(M)$ we set

$$\langle \alpha, \beta \rangle := \frac{1}{k!} (g^\sharp)^{\otimes k}(\alpha, \beta) = \frac{1}{k!} \alpha_{\mu_1 \dots \mu_k} \beta^{\mu_1 \dots \mu_k}.$$

We define the *Hodge $\star$ -operator* to be the linear operator $\star : \Omega^k(M) \rightarrow \Omega^{n-k}(M)$ uniquely determined by the condition

$$\alpha \wedge \star \beta = \langle \alpha, \beta \rangle d\mu_g \tag{3.1}$$

3. Maxwell's equations in curved spacetime

for all $\alpha, \beta \in \Omega^k(M)$. Explicitly we have

$$\star(\alpha^{m_1} \wedge \dots \wedge \alpha^{m_k}) = g^{m_1 m_1} \dots g^{m_k m_k} \epsilon_{m_1 \dots m_k m_{k+1} \dots m_n} \alpha^{m_{k+1}} \wedge \dots \wedge \alpha^{m_n}, \quad (3.2)$$

where ϵ is totally antisymmetric with $\epsilon_{123\dots n} = 1$ and no summation over indices on the right hand side is assumed.

Let $\Omega_0^k(M)$ denote the set of differential k -forms with compact support on M . Then we define the L^2 -scalar product of forms $\langle \cdot, \cdot \rangle_{L^2} : \Omega_0^k(M) \times \Omega_0^k(M) \rightarrow \mathbb{R}$ by

$$\langle \alpha, \beta \rangle_{L^2} := \int_M \langle \alpha, \beta \rangle d\mu_g.$$

We further define the *codifferential* $\delta : \Omega^{k+1}(M) \rightarrow \Omega^k(M)$ by

$$\delta = (-1)^{t+nk+1} \star d \star.$$

One can show, see [Ham17, Thm. 7.2.10], that the codifferential δ is the formal adjoint of the differential d with respect to the L^2 -scalar product on forms with compact support, i.e.

$$\langle d\alpha, \beta \rangle_{L^2} = \langle \alpha, \delta\beta \rangle_{L^2} \quad (3.3)$$

for all $\alpha \in \Omega_0^k(M)$ and $\beta \in \Omega_0^{k+1}(M)$. Given $\omega \in \Omega^k(M)$, in local coordinates we have

$$(\delta\omega)_{\mu_1 \dots \mu_{k-1}} = -\nabla^\lambda \omega_{\lambda \mu_1 \dots \mu_{k-1}},$$

where ∇ denotes the Levi-Civita connection on (M, g) . Further we have [Sch26]

$$\star\star = (-1)^{k(d-k)s} \quad \text{acting on } \Omega^k(M), \quad (3.4)$$

where $d = \dim M$ and $s = 1$ for Riemannian manifolds and $s = -1$ for Lorentzian manifolds. With this notation, we define the *de Rham-Hodge d'Alembertian* by

$$\square_{\text{dRH}} := d\delta + \delta d.$$

It turns out that $\square_{\text{dRH}}$ and the connection d'Alembertian $\square_g = g^{\mu\nu} \nabla_\mu \nabla_\nu$ are related by means of the *Weitzenböck identity*

$$\square_{\text{dRH}} = -\square_g + \mathcal{R}_k \quad \text{on } \Omega^k(M),$$

where $\mathcal{R}_k$ is a 0th-order operator constructed out of the curvature of (M, g) . For $f \in \Omega^0(M)$ we have $\square_{\text{dRH}} f = -\square_g f$ while for $\omega \in \Omega^1(M)$ we have

$$\square_{\text{dRH}} \omega = -\square_g \omega + \text{Ric}_g(\omega)$$

with $\text{Ric}_g(\omega)_\alpha := (\text{Ric}_g)_\alpha^\beta \omega_\beta$. For a general formula for higher forms in the Riemannian case, see [Rha84, Chap. 27].

3.2. The Cauchy problem for Maxwell's equations

In Minkowski space, Maxwell's equations can be compactly written down using the electromagnetic field strength tensor F . In the absence of charges and currents they read

$$\partial_{[a}F_{bc]} = 0 \quad \text{and} \quad \partial^a F_{ab} = 0.$$

The generalization of Maxwell's equations to curved spacetime is accomplished by replacing partial derivatives with covariant ones. The equations then become

$$\nabla_{[\mu}F_{\nu\sigma]} = 0 \quad \text{and} \quad \nabla^\mu F_{\mu\nu} = 0. \quad (3.5)$$

With the mathematical language introduced in section 3.1, Maxwell's equations can be elegantly written in coordinate-independent form. Let (M, g) be a globally hyperbolic spacetime. By theorem 2.1, there exists a one-parameter family of Cauchy surfaces $\{\Sigma_\tau\}_{\tau \in \mathbb{R}}$, giving the spacetime the topology $M \cong \mathbb{R} \times \Sigma$. Each Cauchy surface inherits a Riemannian metric h , and the future-directed unit normal field to Σ will be denoted by n . The Hodge $\star$ -operator with respect to g acting on compactly supported forms will be denoted by

$$\star^{(4)} : \Omega_0^k(M) \rightarrow \Omega_0^{4-k}(M).$$

On (M, g) we also have the exterior derivative $d^{(4)}$ and codifferential

$$\delta^{(4)} = \star^{(4)} d^{(4)} \star^{(4)} : \Omega_0^k(M) \rightarrow \Omega_0^{k-1}(M).$$

On (Σ, h) we have the Hodge $\star$ -operator

$$\star^{(3)} : \Omega_0^k(\Sigma) \rightarrow \Omega_0^{3-k}(\Sigma),$$

as well as the exterior derivative $d^{(3)}$ and codifferential

$$\delta^{(3)} = (-1)^k \star^{(3)} d^{(3)} \star^{(3)} : \Omega_0^k(\Sigma) \rightarrow \Omega_0^{k-1}(\Sigma).$$

Let $n^\flat$ denote the metric dual one-form of n . Then the electromagnetic field strength decomposes as a differential 2-form into

$$F := E \wedge n^\flat + B, \quad (3.6)$$

where the electric field E is a spatial one-form and the magnetic field B is a spatial two-form satisfying $\iota_n E := E(n) = 0$ and $\iota_n B := B(n, \cdot) = 0$. The *Hodge dual* of F is given by

$$\star^{(4)} F = (\star^{(3)} B) \wedge n^\flat - \star^{(3)} E, \quad (3.7)$$

which follows immediately from eq. (3.2). Note that $\star^{(3)} E \in \Omega_0^2(\Sigma)$ and $\star^{(3)} B \in \Omega_0^1(\Sigma)$. By abuse of notation, we will not distinguish between the spatial magnetic 2-form $B \in \Omega_0^2(\Sigma)$, its spatial Hodge dual $\star^{(3)} B \in \Omega_0^1(\Sigma)$, and the spatial projection of $(\star^{(4)} F)$ defined by $(\star^{(4)} F)^{\alpha\mu} n_\mu$, as all of these are naturally identified under the 3 + 1 decomposition

3. Maxwell's equations in curved spacetime

determined by n . To this end, given a field strength tensor F , we define the electric and magnetic fields relative to n^μ by

$$E^\alpha := F^{\alpha\mu} n_\mu \quad \text{and} \quad B^\alpha := (\star F)^{\alpha\mu} n_\mu, \quad (3.8)$$

where we have abbreviated $\star F := \star^{(4)} F$. This definition is consistent with the decomposition of F in terms of E and B from eq. (3.6). Indeed, for the first relation we compute

$$\iota_n(E \wedge n^\flat + B) = (\iota_n E) \wedge n^\flat + (-1)^1 E \wedge \iota_n n^\flat + \iota_n B = E,$$

where we used that the insertion map ι_n is a graded $\wedge$ -derivation and that $g(n, n) = -1$. The second relation follows analogously.

Now, the vacuum Maxwell equations can be written elegantly as

$$d^{(4)} F = 0 \quad \text{and} \quad \delta^{(4)} F = 0. \quad (3.9)$$

To see that these equations reproduce the classical Maxwell equations in Minkowski space, we choose the standard observer $n = \partial_t$ and compute

$$\begin{aligned} 0 = d^{(4)} F &= d^{(4)}(E \wedge dt + B) = d^{(3)} E \wedge dt + d^{(3)} B + \partial_t B \wedge dt \\ &= (d^{(3)} E + \partial_t B) \wedge dt + d^{(3)} B. \end{aligned}$$

From this we immediately conclude

$$d^{(3)} E + \partial_t B = 0, \quad d^{(3)} B = 0,$$

which correspond to

$$\text{curl } E + \partial_t B = 0, \quad \text{div } B = 0,$$

under the standard identifications between differential forms and vector fields. Abbreviating $\star^{(3)}$ by $\star$ and using eq. (3.7), we find

$$\begin{aligned} 0 = \delta^{(4)} F &= \star^{(4)} d^{(4)} \star^{(4)} F \\ &= \star^{(4)} \left[(d^{(3)} \star B) \wedge dt - d^{(3)} \star E - (\partial_t \star E) \wedge dt \right] \\ &= \star^{(4)} \left[(d^{(3)} \star B - \partial_t \star E) \wedge dt - d^{(3)} \star E \right]. \end{aligned}$$

Since we are in vacuum (otherwise $d^{(4)} \star^{(4)} F = \star^{(4)} J$, with $\star^{(4)} J := j \wedge dt - Q$), we conclude

$$d^{(3)} \star B - \partial_t \star E = 0, \quad d^{(3)} \star E = 0,$$

3.2. The Cauchy problem for Maxwell's equations

which corresponds to the remaining Maxwell equations

$$\operatorname{curl} B - \partial_t E = 0, \quad \operatorname{div} E = 0.$$

At least locally, it is common to introduce the vector potential A_μ , which is related to the field strength tensor by $F_{\mu\nu} = \nabla_{[\mu} A_{\nu]}$. Reformulated in terms of the vector potential, the first Maxwell equation from eq. (3.5) is trivially satisfied, due to the first Bianchi identity. The Maxwell equations in both flat and curved spacetime exhibit gauge freedom. This freedom arises because one can always add the gradient of an arbitrary scalar function to the vector potential without changing the physical fields. Since covariant derivatives – like ordinary partial derivatives – commute when acting on scalar functions, this additional gradient term cancels out when forming the field strength tensor. As a result, one is free to impose a specific gauge condition, such as the *Lorenz gauge* $\nabla^\mu A_\mu = 0$, to simplify calculations.

A classical vector potential may be described as a one-form $A \in \Omega^1(M)$. The field strength tensor is then given as $F = dA \in \Omega^2(M)$. In this case we may write Maxwell's equations as

$$ddA = 0 \quad \text{and} \quad \delta dA = 0. \quad (3.10)$$

Note that by $d \circ d = 0$, the first equation is trivial, so we only need to study the second equation. The Lorenz gauge condition can be expressed with the codifferential as $\delta A = 0$. In this case, the second Maxwell equation reduces to

$$-\square_{\text{dRH}} A_\alpha = \square_g - R_\alpha{}^\beta A_\beta = 0, \quad (3.11)$$

where $R_\alpha{}^\beta$ denote the components of the Ricci tensor of g .

The Cauchy problem for Maxwell's equations was intensively studied, with different topological assumptions on the spacetime. While [Fur99] assumes compact Cauchy surfaces, more recent papers show that the bare assumption of global hyperbolicity is enough to obtain satisfactory results. While [SDH14] presents a comprehensive study of the distributional Cauchy problem for the vector potential, [Pfe09] provides an analysis thereof in the classical setting. The Cauchy problem for the field strength is discussed in [DL12]. Although not directly treating Maxwell's equations, [BGP08] establishes fundamental results on wave equations on Lorentzian manifolds and their quantization. In the following we will summarize the most important results.

What we now want to study is the Cauchy problem for eq. (3.11). First we need to deal with the initial data. We introduce a series of operators that map a solution to its data. For this, choose any Cauchy surface, say $\{0\} \times \Sigma$, and let $i : \Sigma \hookrightarrow M$ be its smooth

3. Maxwell's equations in curved spacetime

embedding. We define the operators

$$\rho_{(0)} := i^* \quad (\text{pullback}) \quad (3.12a)$$

$$\rho_{(d)} := -\star^{(3)} i^* \star^{(4)} d^{(4)} \quad (\text{forward normal derivative}) \quad (3.12b)$$

$$\rho_{(\delta)} := i^* \delta^{(4)} \quad (\text{pullback of divergence}) \quad (3.12c)$$

$$\rho_{(n)} := -\star^{(3)} i^* \star^{(4)} \quad (\text{forward normal}), \quad (3.12d)$$

where $\rho_{(0)}, \rho_{(d)} : \Omega_0^k(M) \rightarrow \Omega_0^k(\Sigma)$ and $\rho_{(\delta)}, \rho_{(n)} : \Omega_0^k(M) \rightarrow \Omega_0^{k-1}(\Sigma)$.

Next, let $A \in \Omega_0^1(M)$ and define

$$A_{(0)} := \rho_{(0)} A, \quad A_{(d)} := \rho_{(d)} A, \quad A_{(\delta)} := \rho_{(\delta)} A, \quad A_{(n)} := \rho_{(n)} A. \quad (3.13)$$

It is not difficult to see that specifying $A_{(0)}, A_{(d)}, A_{(\delta)}$ and $A_{(n)}$ is equivalent to specifying Cauchy data for A . Thus, the operators from eq. (3.12) collectively map a solution A to its Cauchy data $(A_\mu, n^\nu \nabla_\nu A_\mu)|_{\tau=0}$.

Theorem 3.1 (Cauchy problem for the vector potential). *Let (M, g) be a globally hyperbolic spacetime with Cauchy surface $\Sigma \subset M$. Let $(A_{(0)}, A_{(d)}, A_{(\delta)}, A_{(n)})$ specify initial data on Σ . Then there exists a unique solution $A \in \Omega_0^1(M)$ to $\square_{\text{dRH}} A = 0$ which satisfies*

$$\rho_{(0)} A = A_{(0)}, \quad \rho_{(d)} A = A_{(d)}, \quad \rho_{(\delta)} A = A_{(\delta)}, \quad \text{and} \quad \rho_{(n)} A = A_{(n)}. \quad (3.14)$$

Furthermore, A depends continuously on the Cauchy data.

Within the initial value formulation for the electromagnetic field strength, the electric and magnetic fields are commonly viewed as a one-form and a two-form, respectively. As initial data one chooses

$$B = \rho_{(0)} F \in \Omega_0^2(\Sigma), \quad E = \rho_{(n)} F \in \Omega_0^1(\Sigma).$$

The fact that this data indeed corresponds to the magnetic and electric fields is immediate in the first case. For the second case we compute

$$\begin{aligned} \rho_{(n)} F &= -\star^{(3)} i^* \star^{(4)} (n^\flat \wedge E + B) \\ &= -\star^{(3)} i^* (-\star^{(3)} E + \star^{(3)} B \wedge n^\flat) \\ &= \star^{(3)} \star^{(3)} E = E, \end{aligned}$$

where we used eq. (3.4) and the fact that Σ is Riemannian, so that

$$\star^{(3)} \star^{(3)} = (-1)^{1(3-1)} \text{id}_{\Omega_0^1(\Sigma)} = \text{id}_{\Omega_0^1(\Sigma)}.$$

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As initial data, E and B must satisfy constraint equations. We find that

$$\begin{aligned}
\delta^{(3)} E &= -\star^{(3)} d^{(3)} \star^{(3)} E \\
&= -\star^{(3)} d^{(3)} \star^{(3)} (-\star^{(3)} i^* \star^{(4)} F) \\
&= \star^{(3)} d^{(3)} i^* \star^{(4)} F \\
&= \star^{(3)} i^* d^{(4)} \star^{(4)} F \\
&= \star^{(3)} i^* \star^{(4)} \star^{(4)} d^{(4)} \star^{(4)} F \\
&= \star^{(3)} i^* \star^{(4)} (\underbrace{\delta^{(4)} F}_{=0}) = 0,
\end{aligned}$$

where we used that the exterior derivative commutes with pullbacks, as well as

$$\star^{(4)} \star^{(4)} = (-1)^{3(4-3)} (-1) \text{id}_{\Omega_0^3(M)} = \text{id}_{\Omega_0^3(M)}.$$

Furthermore,

$$d^{(3)} B = d^{(3)} i^* F = i^* \underbrace{d^{(4)} F}_{=0} = 0.$$

With this, we record that the Cauchy problem for the electromagnetic field strength is also well-posed:

Theorem 3.2 (Cauchy problem for the field strength). *Let (M, g) be a globally hyperbolic spacetime¹ with a Cauchy surface $\Sigma \subset M$. For each pair (E, B) such that $E \in \Omega_0^1(\Sigma)$, $B \in \Omega_0^2(\Sigma)$, satisfying the constraints*

$$\delta^{(3)} E = 0 \quad \text{and} \quad d^{(3)} B = 0,$$

there exists a unique solution $F \in \Omega^2(M)$ of the initial value problem

$$\begin{cases} d^{(4)} F = 0, & \delta^{(4)} F = 0, \\ \rho_{(n)} F = E, & \rho_{(0)} F = B. \end{cases}$$

Furthermore, F depends linearly and continuously on the initial data (E, B) . Each solution also satisfies

$$\text{supp}(F) \subset J^+(X) \cup J^-(X),$$

where $X := \text{supp}(E) \cup \text{supp}(B)$.

¹The original theorem in [DL12] further assumes that M has at most finitely many connected components. In our convention, a spacetime is always connected. Also note that the initial magnetic field differs by a sign, which is due to an opposite sign convention.

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The connection between the field strength F and the vector potential A is straightforward in spacetimes with trivial topology. Indeed, consider Maxwell's equations on a Minkowski background M , i.e.

$$dF = 0, \quad \delta F = 0. \quad (3.15)$$

Since M is diffeomorphic to $\mathbb{R}^4$, the Poincaré lemma implies that every closed 2-form is exact. Hence, there exists $A \in \Omega^1(M)$ such that

$$F = dA. \quad (3.16)$$

However, on a general spacetime, it is no longer true that every closed 2-form can be written globally as the exterior derivative of a 1-form. A possible way to retain the relation $F = dA$ would be to restrict attention to spacetimes whose second de Rham cohomology group is trivial. Recall that the k -th *de Rham cohomology group* is defined as the quotient vector space

$$H_{dR}^k(M) := \mathcal{Z}^k(M) / \mathcal{B}^k(M),$$

where

$$\begin{aligned} \mathcal{Z}^k(M) &:= \ker(d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)) = \{\text{closed } k\text{-forms on } M\}, \\ \mathcal{B}^k(M) &:= \text{im}(d : \Omega^{k-1}(M) \rightarrow \Omega^k(M)) = \{\text{exact } k\text{-forms on } M\}. \end{aligned}$$

Thus, $H_{dR}^2(M) = 0$ means that every closed 2-form is exact. However, this assumption is too restrictive from a physical point of view, as it excludes relevant spacetimes such as the Schwarzschild spacetime [DL12]. In general, if $[F] \in H_{dR}^2(M)$ is a non-trivial cohomology class, then it consists of all elements of the form

$$[F] = \{F + dA \mid A \in \Omega^1(M)\},$$

where $F \in \Omega^2(M)$ is any fixed closed representative. This shows that a field strength $[F]$ can be decomposed into two contributions. More precisely, given a representative $F \in [F]$, one can write

$$F = F_{\text{top}} + F_{\text{ex}},$$

where F_{top} is closed and non-exact while F_{ex} is exact, i.e. can be written as $F_{\text{ex}} = dA$. The term F_{top} captures qualitative information, such as topological charges, and depends on the specific chosen spacetime. In contrast, the exact part F_{ex} encodes the local, dynamical properties associated with a vector potential. Consequently, even in the presence of non-trivial topology, one can recover a well-posed Cauchy problem for the vector potential by restricting the dynamical analysis to the exact part of the field strength, while treating F_{top} as a fixed background contribution. In particular, if one prescribes initial data only for the exact part, one can still apply theorem 3.1. The following argument makes the

3.2. The Cauchy problem for Maxwell's equations

above more precise: Let (M, g) be a globally hyperbolic spacetime and let $F \in \Omega^2(M)$ be a field strength satisfying $dF = 0$. For a fixed $t_0 \in \mathbb{R}$ define

$$r_{t_0} : \mathbb{R} \times \Sigma \rightarrow \Sigma_{t_0}, \quad r_{t_0}(\tau, x) := (t_0, x),$$

and let

$$i_{t_0} : \Sigma_{t_0} \hookrightarrow \mathbb{R} \times \Sigma$$

denote the inclusion map. Then

$$r_{t_0} \circ i_{t_0} = \text{id}_{\Sigma_{t_0}}.$$

Now define $H : [0, 1] \times (\mathbb{R} \times \Sigma) \rightarrow \mathbb{R} \times \Sigma$ by

$$H(s, (\tau, x)) = ((1-s)\tau + st_0, x).$$

This satisfies

$$H(0, \cdot) = \text{id}_{\mathbb{R} \times \Sigma}, \quad H(1, \cdot) = i_{t_0} \circ r_{t_0}.$$

Moreover,

$$H(s, (t_0, x)) = (t_0, x)$$

for all $s \in [0, 1]$. Hence Σ_{t_0} is a strong deformation retract of M . Therefore the inclusion $i_{t_0} : \Sigma_{t_0} \hookrightarrow M$ is a homotopy equivalence. Since pullback commutes with the exterior derivative, i.e. $i_{t_0}^*(d\omega) = d(i_{t_0}^*\omega)$ for all $\omega \in \Omega^k(M)$, it follows that $i_{t_0}^*$ maps closed forms to closed forms and exact forms to exact forms. Indeed, if $\omega \in \Omega^k(M)$ satisfies $d\omega = 0$, then

$$d(i_{t_0}^*\omega) = i_{t_0}^*(d\omega) = 0,$$

so $i_{t_0}^*\omega$ is closed. If moreover $\omega = d\eta$ for some $\eta \in \Omega^{k-1}(M)$, then

$$i_{t_0}^*\omega = i_{t_0}^*(d\eta) = d(i_{t_0}^*\eta),$$

implying that $i_{t_0}^*\omega$ is exact. Hence $i_{t_0}^*$ descends to a well-defined map on the de Rham cohomology. In particular,

$$i_{t_0}^*[F] = [i_{t_0}^*F].$$

One can further show (see [Lee13, Thm. 17.11], for example) that

$$i_{t_0}^* : H_{\text{dR}}^k(M) \longrightarrow H_{\text{dR}}^k(\Sigma_{t_0})$$

is an isomorphism for every k . If the initial data are topologically trivial, i.e.

$$i_{t_0}^*F = dA \quad \text{for some } A \in \Omega^1(\Sigma_{t_0}),$$

then

$$[i_{t_0}^*F] = 0 \quad \text{in } H_{\text{dR}}^2(\Sigma_{t_0}).$$

Since $i_{t_0}^* : H_{\text{dR}}^2(M) \rightarrow H_{\text{dR}}^2(\Sigma_{t_0})$ is an isomorphism, it follows that

$$[F] = 0 \quad \text{in } H_{\text{dR}}^2(M).$$

Therefore, if the initial Maxwell data belong to the trivial cohomology class on one Cauchy hypersurface, then the evolved Maxwell field remains globally exact for all time.

4. Bitensors and associated geometric structures

In the following chapters, we will work with tensorial quantities that depend on two points $(x, x') \in M \times M$. Such objects are called *bitensors*. We adopt the convention that unprimed indices refer to the point x , while primed indices refer to the point x' . More precisely, in the language of fiber bundles, a bitensor of type (r, s, r', s') is defined as a section of the exterior tensor product bundle

$$T_s^r M \boxtimes T_{s'}^{r'} M \rightarrow M \times M,$$

where $T_s^r M$ denotes the tensor bundle of type (r, s) and $\boxtimes$ the exterior tensor product. As an example, the bitensor $B^\mu_{\nu'\lambda'}(x, x')$ represents a vector at x and a rank-2 covariant tensor at x' , and thus assigns to each $(x, x') \in M \times M$ an element of $T_x M \otimes T_{x'}^* M \otimes T_{x'}^* M$. The first subsection is primarily based on [AMO⁺25, KS21], while the discussion of parallel propagators and the expansion of bitensors near coincidence closely follows [PPV11, Vis93].

4.1. Synge's world function

A smooth curve $c : [0, 1] \rightarrow M$ into a pseudo-Riemannian manifold M is called a *geodesic* if its acceleration is zero, i.e. $c'' = 0$, where the primes denote covariant derivatives along c . In local coordinates, this condition translates to a nonlinear system of ordinary differential equations of second order. It is thus an immediate consequence of standard existence and uniqueness results, that given some $x \in M$ and $v \in T_x M$ there exists an open interval $I \subset \mathbb{R}$ around 0 and a unique geodesic $c_v : I \rightarrow M$ with $c_v(0) = x$ and $c'_v(0) = v$. Furthermore it can be shown that the domain of c_v is indeed *maximal*, i.e. if $c : J \rightarrow M$ is any geodesic with $c(0) = x$ and $c'(0) = v$ then $J \subseteq I$ and $c_v|_J = c$. Define $\mathcal{D}_x := \{v \in T_x M : c_v \text{ is at least defined on } [0, 1]\}$. The *exponential map* of M at x is defined as

$$\exp_x : \mathcal{D}_x \rightarrow M, \quad \exp_x(v) := c_v(1).$$

It follows that $\exp_x(tv) = c_{tv}(1) = c_v(t)$, i.e. the exponential map maps straight lines $t \mapsto tv$ through 0 in $T_x M$ to geodesics $c_v(t)$ through x in M . It turns out that this is done in a diffeomorphic way, i.e. there exist neighborhoods $\tilde{U}$ of 0 in $T_x M$ and U of x in M such that $\exp_x : \tilde{U} \rightarrow U$ is a diffeomorphism. If in this case $\tilde{U}$ is star shaped, i.e. $v \in \tilde{U}$ and $t \in [0, 1]$ implies that $tv \in \tilde{U}$, then U is called a *normal neighborhood* of x . One can show that for each $x' \in U$ there exists a unique geodesic $c_r : [0, 1] \rightarrow U$ from x to x' called the *radial geodesic* from x to x' . A subset $C \subset M$ is called (*geodesically*) *convex*, if

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it is a normal neighborhood of all its points, i.e. for every $x, x' \in C$ there exists a unique geodesic segment contained entirely in C from x to x' . On such a set, we use the notation

$$x' - x := \exp_x^{-1}(x') \in T_x M$$

which is the tangent vector at x of the unique geodesic segment from x to x' that is contained entirely in C .

Definition 4.1 (Synge's world function). *Let (M, g) be a smooth pseudo-Riemannian manifold and C a geodesically convex open set. Then, the map $\sigma : C \times C \rightarrow \mathbb{R}$ defined by*

$$\sigma(x, x') := \frac{1}{2} g_x(x' - x, x' - x), \quad (x, x' \in C) \quad (4.1)$$

is called Synge's world function.

The unique geodesic segment in C connecting $x \in C$ with $x' \in C$ is given by $c(t) = \exp_x[t(x' - x)]$ and since this defines a smooth map on $[0, 1] \times C \times C$, we conclude that σ defines a smooth function on $C \times C$. Furthermore, if (M, g) is a Riemannian manifold or a Lorentzian manifold with signature $(-, +, \dots, +)$, then σ is related to the geodesic distance via

$$\sigma(x, x') = \frac{1}{2} s_{x, x'} \left[\int_a^b \sqrt{|g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))|} dt \right]^2, \quad (4.2)$$

where $s_{x, x'} = -1$ if γ is timelike and $s_{x, x'} = 1$ otherwise. In other words, σ is half the square of the signed geodesic length from x to x' .

If (M, g) is *globally* geodesically convex, which means that any pair of points of M can be connected by a unique geodesic segment, then σ is well defined globally on $M \times M$. In general, the world function σ depends on the choice of convex set C and therefore cannot be defined globally. However, one can always find an open neighborhood $\mathcal{U} \subset M \times M$ of the diagonal $\{(x, x) \in M \times M \mid x \in M\}$ on which σ is well defined.

The world function $\sigma(x, x')$ may be differentiated with respect to either of its arguments. We denote by $\sigma_\alpha := \partial\sigma/\partial x^\alpha$ its partial derivative with respect to x , and by $\sigma_{\alpha'} := \partial\sigma/\partial x^{\alpha'}$ its derivative with respect to x' . The quantity σ_α transforms as a covector at x but as a scalar at x' , whereas $\sigma_{\alpha'}$ behaves as a scalar at x and as a covector at x' . We further define $\sigma_{\alpha\beta} := \nabla_\beta \sigma_\alpha$ as the covariant derivative of σ_α with respect to x . This object is a rank-2 tensor at x and a scalar at x' , and since σ is a scalar field at x , it follows that $\sigma_{\alpha\beta} = \sigma_{\beta\alpha}$. Similarly, we define $\sigma_{\alpha\beta'} := \partial_{\beta'} \sigma_\alpha = \partial^2 \sigma / (\partial x^{\beta'} \partial x^\alpha)$, which is a covector at both x and x' . In the same way, $\sigma_{\alpha'\beta} := \partial_\beta \sigma_{\alpha'} = \partial^2 \sigma / (\partial x^\beta \partial x^{\alpha'})$. By the commutativity of partial derivatives, it follows that $\sigma_{\alpha\beta'} = \sigma_{\beta'\alpha}$. Finally, we define $\sigma_{\alpha'\beta'} := \nabla_{\beta'} \sigma_{\alpha'}$, which is a symmetric rank-2 tensor at x' and a scalar at x . This notation extends naturally to higher derivatives. For instance, $\sigma_{\alpha\beta\gamma\delta'} := \nabla_{\delta'} \nabla_\gamma \nabla_\beta \nabla_\alpha \sigma$ defines a bitensor that is of rank three at x and a covector at x' .

4.2. Parallel propagator

Let $x, x' \in C$ as in the previous section. Thus there exists a unique geodesic segment c , parametrized by λ , that connects x with x' . We want to construct an object that takes a vector at x' and parallel transports it to a vector at x along c . To this end, along c we introduce an orthonormal frame $e_a^\mu(z)$ ($a = 0, \dots, 3$) that is parallel transported along this segment, i.e.

$$g_{\mu\nu} e_a^\mu e_b^\nu = \eta_{ab}, \quad \text{and} \quad \frac{D e_a^\mu}{d\lambda} = 0.$$

The dual frame is defined as $e_\mu^a := \eta^{ab} g_{\mu\nu} e_b^\nu$ and is likewise parallel transported along c . Given now a vector field $A^\mu(z)$ along c , we can uniquely decompose it in terms of the frame as

$$A^\mu(z) = A^a(z) e_a^\mu(z),$$

where the components are given by $A^a(z) = A^\mu(z) e_\mu^a(z)$. Note that if A^μ is parallel transported along c , then its coefficients A^a are constant, i.e. $A^a = A^a(z)$ for any z on c . In particular, we have

$$A^a = A^a(x') = A^{\alpha'}(x') e_{\alpha'}^a(x').$$

From this it follows that the vector at x can be written as

$$A^\alpha(x) = J_{\alpha'}^\alpha(x, x') A^{\alpha'}(x'), \quad (4.3)$$

where $J_{\alpha'}^\alpha(x, x') = e_\alpha^a(x) e_{\alpha'}^a(x')$ is called the *parallel propagator*. Similarly, one obtains

$$A^{\alpha'}(x') = J_{\alpha}^{\alpha'}(x', x) A^\alpha(x), \quad (4.4)$$

with $J_{\alpha}^{\alpha'}(x', x) = e_{\alpha'}^a(x') e_\alpha^a(x)$, which performs the reverse operation of transporting a vector at x to a vector at x' . One readily checks that

$$J_{\alpha}^{\alpha'}(x, x') = J_{\alpha'}^\alpha(x', x) \quad \text{and} \quad J_{\alpha'}^\alpha(x', x) = J_{\alpha}^{\alpha'}(x, x').$$

The action of the parallel propagator on higher-rank tensors is immediate: each index is parallel transported separately. As an example, the rank-2 tensor $A^{\alpha\beta}$ at x is obtained by parallel transport from x' via

$$A^{\alpha\beta}(x) = J_{\alpha'}^\alpha(x, x') J_{\beta'}^\beta(x, x') A^{\alpha'\beta'}(x').$$

4.3. The Van Vleck-Morette determinant

Definition 4.2 (Van Vleck-Morette determinant). *Let (M, g) be a Riemannian or Lorentzian manifold, and $\mathcal{U}$ be an open neighborhood of the diagonal on which σ is well defined. Then, the Van Vleck-Morette determinant is defined as*

$$\Delta(x, x') := \epsilon_g \frac{\det [-\partial_\mu \partial_{\nu'} \sigma(x, x')]}{\sqrt{|g(x)|} \sqrt{|g(x')|}} \quad (4.5)$$

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for all $(x, x') \in \mathcal{U}$, where $g(x) := \det((g_x)_{\mu\nu})$ denotes the metric determinant at x and $\epsilon_g := \text{sgn} \det((g_x)_{\mu\nu}) = (-1)^q$, with q representing the number of negative eigenvalues of the metric.

The reason for introducing the Van Vleck-Morette determinant will become clear in what follows. In section 5.1 we will deal with integrals of bitensors over Cauchy hypersurfaces. These integrals will be of the form

$$Q(\tau) = \int_{\Sigma_\tau} T(x') P(x, x') d\mu_{\Sigma_\tau}(x'), \quad (4.6)$$

where T denotes the stress-energy tensor, P is a bitensor, x' is an auxiliary integration variable, and x denotes a point on a reference worldline Γ with respect to which the quantity Q is defined. What we want to do is evaluate these integrals using the stationary phase approximation in flat spacetime, as discussed in section 5.4. However, since our integrals are, in general, taken over hypersurfaces with nontrivial curvature, this flat-space result cannot be used directly. In order to apply it, we will have to lift the integrals in a covariant manner to the corresponding tangent space at a given point on Γ . To this end, note first that $\sigma^\mu = \nabla^\mu \sigma(x, x')$ gives (minus) the tangent vector of the unique geodesic between x and x' emanating from x . Comparing with the definition of the exponential map from section 4.1, we see that the function

$$\sigma_x^\mu : U(x) \subset \Sigma_\tau \rightarrow T_x \Sigma_\tau, \quad x' \mapsto \nabla^\mu \sigma(x, x') \quad (4.7)$$

is precisely the inverse of the exponential map. With this insight, the following result is not too hard to prove.

Proposition 4.1 (Pullback of the manifold measure under the exponential map). *Let (Σ, h) be an oriented Riemannian manifold and let $x \in \Sigma$. Assume we work in a geodesically convex normal neighborhood of x such that $\exp_x$ is a diffeomorphism onto its image. Then*

$$(\exp_x)^*(d\mu_\Sigma)(v) = \Delta(x, x')^{-1} d\mu_{T_x \Sigma}(v), \quad (4.8)$$

where

$$x' = \exp_x(v).$$

Proof. Fix $x \in \Sigma$ and work in a geodesically convex normal neighborhood of x such that

$$x' = \exp_x(v), \quad v \in T_x \Sigma,$$

is well-defined and invertible. Let

$$c_r(t) = \exp_x(tv)$$

be the unique geodesic segment from x to x' , with $c_r(0) = x$ and $c_r(1) = x'$. As pointed out above we have

$$\sigma^\mu(x, x') = \nabla^\mu \sigma(x, x') = -v^\mu.$$

4.3. The Van Vleck-Morette determinant

Hence

$$\sigma_\nu(x, x') = -v_\nu.$$

Differentiating with respect to x' gives

$$-\nabla_{\mu'} \nabla_\nu \sigma(x, x') = -\nabla_{\mu'} \sigma_\nu(x, x') = \partial_{\mu'} v_\nu.$$

Here we used that $\sigma_\nu(x, x')$ carries an index only at x , so the covariant derivative in the x' -variable reduces to an ordinary partial derivative. Since $\exp_x$ is a diffeomorphism in our neighborhood, the inverse function theorem yields

$$\frac{\partial v}{\partial x'} = (d \exp_x)_v^{-1}.$$

Therefore

$$\det[-\partial_{\mu'} \partial_\nu \sigma(x, x')] = \det(d \exp_x)_v^{-1}.$$

By definition of the van Vleck–Morette determinant,

$$\Delta(x, x') = \frac{\det[-\partial_{\mu'} \partial_\nu \sigma(x, x')]}{\sqrt{h(x)} \sqrt{h(x')}}.$$

Thus

$$\det(d \exp_x)_v = \frac{1}{\Delta(x, x') \sqrt{h(x)} \sqrt{h(x')}}.$$

Now we compute the pullback of the volume form

$$d\mu_\Sigma(x') = d \operatorname{vol}_h(x') = \sqrt{h(x')} d^n x'.$$

Since $\exp_x$ is a diffeomorphism onto its image, we use the general identity for volume forms under orientation-preserving diffeomorphisms, which reads

$$(\exp_x)^*(d \operatorname{vol}_h) = d \operatorname{vol}_{(\exp_x)^* h}.$$

In coordinates v^α on $T_x \Sigma$, the pullback metric is given by

$$((\exp_x)^* h)_{\alpha\beta}(v) = h_{\mu'\nu'}(\exp_x(v)) \frac{\partial(\exp_x(v))^{\mu'}}{\partial v^\alpha} \frac{\partial(\exp_x(v))^{\nu'}}{\partial v^\beta}. \quad (4.9)$$

The volume form associated with the pullback metric is

$$d \operatorname{vol}_{(\exp_x)^* h}(v) = \sqrt{\det((\exp_x)^* h)(v)} d^n v.$$

From eq. (4.9) it follows that

$$\det((\exp_x)^* h)(v) = \det(h(\exp_x(v))) (\det d(\exp_x)_v)^2.$$

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Taking square roots gives

$$\sqrt{\det((\exp_x)^*h)(v)} = \sqrt{h(\exp_x(v))} |\det d(\exp_x)_v|.$$

If $\exp_x$ is restricted to a sufficiently small neighborhood of 0, it is orientation preserving, so $\det d(\exp_x)_v > 0$, and therefore

$$d \operatorname{vol}_{(\exp_x)^*h} = \sqrt{h(\exp_x(v))} \det d(\exp_x)_v d^n v.$$

Thus

$$(\exp_x)^*(d\mu_\Sigma)(v) = \sqrt{h(x')} \det(d\exp_x)_v d^n v, \quad x' = \exp_x(v).$$

Substituting the expression for the determinant yields

$$(\exp_x)^*(d\mu_\Sigma)(v) = \frac{1}{\Delta(x, x') \sqrt{h(x)}} d^n v.$$

Since the metric at x induces the volume form

$$d\mu_{T_x\Sigma}(v) = \sqrt{h(x)} d^n v,$$

we conclude

$$(\exp_x)^*(d\mu_\Sigma)(v) = \Delta(x, x')^{-1} d\mu_{T_x\Sigma}(v).$$

□

With this result at hand, we can perform the following coordinate transformation: let $x \in \operatorname{Im}(\Gamma)$ a point on some reference worldline Γ and let $x' = \exp_x(v) \Leftrightarrow v = \exp_x^{-1}(x')$ on a normal neighborhood with $v \in T_x\Sigma_\tau$. Then

$$\int_{\Sigma_\tau} T(x') P(x, x') d\mu_{\Sigma_\tau}(x') = \int_{T_x\Sigma_\tau} T(\exp_x v) P(x, \exp_x(v)) \Delta^{-1}(x, \exp_x(v)) d\mu_{T_x\Sigma_\tau}(v). \quad (4.10)$$

Note that we are now integrating over the tangent space $T_x\Sigma_\tau \cong \mathbb{R}^3$ at some fixed $x \in \operatorname{Im}(\Gamma)$, which allows us to apply the standard stationary phase result, as we will do in section 6.3.

4.4. Expansion of bitensors near coincidence

It is sometimes useful to study a bitensor $B(x, x')$ under the assumption that x is close to x' . This leads to a generalized notion of a Taylor series, where the monomials $(x - x')^\alpha$ from the flat-spacetime case are replaced by the Synge world function $\sigma^{\alpha'}(x, x')$. We first study the limiting behaviour of a bitensor as x approaches x' . The *coincidence limit* of a bitensor $B_{\dots}$, where $\dots$ denotes the indices carried by B , is defined by

$$[B_{\dots}] := \lim_{x \rightarrow x'} B_{\dots}(x, x').$$

4.4. Expansion of bitensors near coincidence

Here, the limit is assumed to be independent of the direction from which x approaches x' and to yield a unique tensorial function at the point x' . Using *Synge's rule*

$$[\sigma_{\dots\alpha'}] = [\sigma_{\dots}]_{;\alpha'} - [\sigma_{\dots\alpha}], \quad (4.11)$$

one can derive a general procedure for computing coincidence limits. We refer the reader to [PPV11] for details and only summarize the most important results concerning the Synge world function and the van Vleck determinant, which will be used in later chapters. One finds

$$[\sigma] = 0, \quad [\sigma_\alpha] = 0 = [\sigma_{\alpha'}], \quad (4.12)$$

as well as

$$[\sigma_{\alpha\beta}] = g_{\alpha'\beta'} = [\sigma_{\alpha'\beta'}], \quad [\sigma_{\alpha\beta'}] = -g_{\alpha'\beta'} = [\sigma_{\alpha'\beta}]. \quad (4.13)$$

One can further show that the coincidence limit of any third derivative of the Synge world function, irrespective of whether derivatives are taken with respect to x or x' , always vanishes. The coincidence limit of the fourth derivatives of σ , again independently of the choice of base point, consists of a sum of two terms involving the Riemann tensor.

As already mentioned above, bitensors can also be expanded near coincidence as a generalized Taylor series in powers of $-\sigma^{\alpha'}$. Again, we refer the reader to [PPV11] for the general method and only record the special cases needed in the following. Note that the zeroth-order term in each expansion is simply the coincidence limit of the expanded object. One finds

$$\sigma_{\alpha'\beta'} = g_{\alpha'\beta'} - \frac{1}{3}R_{\alpha'\gamma'\beta'\delta'}\sigma^{\gamma'}\sigma^{\delta'} + \mathcal{O}(\sigma^3), \quad (4.14)$$

$$\sigma_{\alpha'\beta} = -J^{\beta'}_{\beta} \left(g_{\alpha'\beta'} + \frac{1}{6}R_{\alpha'\gamma'\beta'\delta'}\sigma^{\gamma'}\sigma^{\delta'} \right) + \mathcal{O}(\sigma^3), \quad (4.15)$$

$$\sigma_{\alpha\beta} = J^{\alpha'}_{\alpha} J^{\beta'}_{\beta} \left(g_{\alpha'\beta'} - \frac{1}{3}R_{\alpha'\gamma'\beta'\delta'}\sigma^{\gamma'}\sigma^{\delta'} \right) + \mathcal{O}(\sigma^3). \quad (4.16)$$

For the van Vleck determinant, we obtain

$$\Delta = 1 + \frac{1}{6}R_{\alpha'\beta'}\sigma^{\alpha'}\sigma^{\beta'} + \mathcal{O}(\sigma^3). \quad (4.17)$$

5. Relevant observables and the method of stationary phase

In this section, we introduce observables such as linear and angular momentum and present a method for evaluating them using the stationary phase approximation.

5.1. A geometric definition of linear and angular momentum

In this subsection we will give a geometric definition of linear and angular momentum in general relativity. A detailed analysis thereof can be found in [Har12, Har15]. We consider a relativistic extended test mass with stress-energy tensor $T^{\mu\nu}$ moving in a globally hyperbolic spacetime (M, g) . We denote its worldtube by $W := \text{supp } T^{\mu\nu}$ and assume the spacial slices $W \cap \Sigma_\tau$ to be compact and of finite 3-volume. Further we assume stress-energy conservation expressed as

$$\nabla_\mu T^{\mu\nu} = 0. \quad (5.1)$$

We define a *momentum functional*

$$P_\xi(\Sigma_\tau) := \int_{\Sigma_\tau} g_{\mu\nu} \xi^\mu T^{\nu\sigma} dS_\sigma, \quad (5.2)$$

where $dS_\sigma = (n_{\Sigma_\tau})_\sigma d\mu_{\Sigma_\tau}$ denotes the measure induced on Σ_τ by the metric g . This functional takes as input a Cauchy hypersurface Σ_τ with $\Sigma_\tau \cap W$ non-empty and a vector field ξ^μ that we will choose later. The term “momentum” in the functional comes from the fact that $P_\xi(\Sigma_\tau)$ returns the component of momentum conjugate to ξ^μ at a time τ defined by Σ_τ . Note that if ξ^μ is a Killing field, then $P_\xi(\Sigma_\tau)$ is independent of Σ_τ . Indeed, by energy-momentum conservation and the symmetry of T we have

$$\nabla_\sigma(\xi_\nu T^{\nu\sigma}) = T^{\nu\sigma}(\nabla_\sigma \xi_\nu) = \frac{1}{2} T^{\nu\sigma}(\nabla_\sigma \xi_\nu + \nabla_\nu \xi_\sigma) = \frac{1}{2} T^{\nu\sigma} \mathcal{L}_\xi g_{\sigma\nu}, \quad (5.3)$$

which equals to zero by the Killing equation. Now let Σ_1, Σ_2 be two Cauchy slices and denote by $\Omega(\Sigma_1, \Sigma_2)$ the region bounded by these two slices. Then using the compact support of spacial slices of T we have

$$P(\Sigma_2) - P(\Sigma_1) = \int_{\Omega(\Sigma_1, \Sigma_2)} \nabla_\sigma(\xi_\nu T^{\nu\sigma}) = 0,$$

which gives the claim.

Note that $\xi \mapsto P_\xi(\Sigma_\tau)$ is linear. Suppose for now that ξ^μ is a Killing field. Recall from section 2.1 that any Killing field is fixed everywhere by specifying it and its first

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derivative at a single point. Setting $\Xi_\mu := g_{\mu\nu}\xi^\nu$ we may thus express $P_\xi(\Sigma_\tau)$ as a linear combination of these values as

$$P_\xi(\Sigma_\tau) = p^\mu(\tau)\Xi_\mu(\Gamma_\tau) + \frac{1}{2}S^{\mu\nu}(\tau)\nabla_\mu\Xi_\nu(\Gamma_\tau), \quad (5.4)$$

where $\Gamma = (\Gamma_\tau)_{\tau \in \mathbb{R}}$ denotes some timelike worldline with $\text{Im}(\Gamma) \cap \Sigma_\tau$ non-empty. Note that knowledge of $P_\xi(\Sigma_\tau)$ for all possible ξ^μ is equivalent to the knowledge of p^μ and $S^{\mu\nu} = S^{[\mu\nu]}$. It will turn out that, in the flat spacetime case, the coefficients p^μ and $S^{\mu\nu}$ will closely correspond to classical textbook definitions for the linear and angular momentum, see section 5.3. However, in a general spacetime, one cannot expect that Killing fields exist and certainly not the $4 + 6 = 10$ required to define all components of p^μ and $S^{\mu\nu}$. The idea is to construct *Killing-type generalized affine collineations* or *generalized Killing fields* (GKFs), see [Har08] for more details. The key concept is that of *local symmetries*.

We introduce vector fields $\psi^{\alpha'}(x_0, x')$ which should generalize Killing fields in some reasonable way in a normal neighborhood U of a fixed reference point x_0 . We associate primed indices with spacetime points around x' in U while unprimed indices are associated with the preferred point x_0 . Without loss of generality we can chose $\psi^{\alpha'}$ such that

$$\nabla_{\alpha'} \mathcal{L}_\psi g_{\beta'\sigma'} \big|_{x_0} = 0.$$

We further assume that $\psi^{\alpha'}$ depends linearly on the initial data $\psi^\alpha(x_0, x_0)$ and $\nabla_\alpha \psi^\beta(x_0, x_0)$. After an anti-symmetrization operation, approximate Killing fields can be constructed by projecting symmetries of $T_{x_0}U$ into U using the exponential map. Consider the transformations

$$X^\alpha \rightarrow X^\alpha + B_\beta{}^\alpha X^\beta$$

in $T_{x_0}U$ and define the associated homogeneous transformations by

$$\Psi^\alpha = B_\beta{}^\alpha X^\beta. \quad (5.5)$$

Such transformations correspond to shifts

$$x' \rightarrow x' + \epsilon\psi$$

of spacetime points in U associated to X^α via $x' = \exp_{x_0} X$. It turns out that $\psi^{\alpha'}$ and Ψ^α are related via

$$\Psi^\alpha = -\sigma^\alpha_{\alpha'}(x_0, x) \psi^{\alpha'}.$$

Introducing

$$H^{\alpha'}_\alpha(x_0, x') = [-\sigma^\alpha_{\alpha'}(x_0, x')]^{-1},$$

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from eq. (5.5) it follows that

$$\psi^{\alpha'}(x_0, x') = -H^{\alpha'}_{\alpha}(x_0, x')\sigma_{\beta}(x_0, x')B^{\beta\alpha}. \quad (5.6)$$

Including the transformations shifting directly the base point x_0 , given by

$$x_0 \rightarrow x_0 + \epsilon A,$$

generates a 20-parameter family of so-called *approximate affine collineations*

$$\psi^{\alpha'} = H^{\alpha'}_{\alpha}(x_0, x')[\sigma^{\alpha}_{\beta}(x_0, x')A^{\beta} - \sigma_{\beta}(x_0, x')B^{\beta\alpha}]. \quad (5.7)$$

The initial data $A^{\alpha} = \psi^{\alpha}(x_0, x_0)$ and $B^{\alpha\beta} = \nabla^{\alpha}\psi^{\beta}(x_0, x_0)$ determine $\psi^{\alpha'}(x_0, x')$ throughout U . Singling out the subset of fields distinguished by antisymmetric $B_{\alpha\beta} = \nabla_{\alpha}\psi_{\beta}$, which, by abuse of notation, we will also call ψ , yields a 10-dimensional group of fields that can be thought of as infinitesimal generators of a *generalized Poincaré group*. In Minkowski space eq. (5.7) reduces to

$$\psi^{\alpha'} = A^{\alpha'} + (x' - x_0)^{\beta'} B_{\beta'}^{\alpha'},$$

which makes the name *affine* collineation apparent. One can show, see [Har08, Cor.1], that given any $\psi^{\alpha'}$ satisfying $\mathcal{L}_{\psi}g_{\alpha\beta}$ we have that

$$\mathcal{L}_{\psi}\sigma = \mathcal{L}_{\psi}\sigma_{\alpha} = \mathcal{L}_{\psi}\sigma_{\alpha'} = \mathcal{L}_{\psi}\sigma_{\alpha\alpha'} = \mathcal{L}_{\psi}H^{\alpha'\alpha} = 0, \quad (5.8)$$

where the Lie derivative is defined to act independently on each argument.

Now that we have constructed local symmetries around a particular point, we want to extend the notion to symmetries near a worldline, that corresponds to the path of some observer. In order to do that one needs to fix a metric, a timelike wordline $\Gamma = \{\Gamma_{\tau} \mid \tau \in \mathbb{R}\}$ and a timelike vector field $\mathbf{n}_{\tau}^{\mu}$ along Γ . The $\mathbf{n}_{\tau}^{\mu}$ fix a family of spacelike hypersurfaces Σ_{τ} that provide a time function $\Sigma_{\tau} \ni x \mapsto \tau$ inside W . At a fixed τ , Σ_{τ} is defined as the union of radially-emanating geodesics that pass through Γ_{τ} and are orthogonal to $\mathbf{n}_{\tau}^{\mu}$ at that point. We assume that each hypersurface is in a normal neighborhood of Γ_{τ} . One can show, see [Sch79a, Sch79b] that, in the massive case, under mild assumptions, Γ and $\mathbf{n}_{\tau}^{\mu}$ are uniquely specified by center of mass conditions. The case for a massless object is subject to ongoing work. The main idea now is to consider the function $s(x')$ that identifies which leaf of the foliation includes an arbitrary point x' in the worldtube W , i.e. it satisfies $x' \in \Sigma(s(x'))$. Then we set

$$\xi^{\alpha'}(x') = \psi^{\alpha'}(x', \Gamma(s(x'))). \quad (5.9)$$

Suppose now that $A_{\alpha}(\tau_0)$ and $B_{\alpha\beta}(\tau_0) = B_{[\alpha\beta]}(\tau_0)$ are given as initial data at some Γ_{τ_0} . The goal is to determine the $\xi^{\alpha'}(x')$ which satisfy

$$A^{\alpha}(\tau) = \xi^{\alpha}(\Gamma_{\tau}), \quad B^{\alpha\beta}(\tau) = \nabla^{\alpha}\xi^{\beta}(\Gamma_{\tau})$$

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at the appropriate point. To do that we solve the Killing transport equations

$$\frac{D}{d\tau} A_\alpha(\tau) - \dot{\Gamma}_\tau^\beta B_{\beta\alpha}(\tau) = 0, \quad (5.10a)$$

$$\frac{D}{d\tau} B_{\alpha\beta}(\tau) + R_{\alpha\beta\gamma}{}^\delta(\Gamma_\tau) \dot{\Gamma}_\tau^\gamma A_\delta(\tau) = 0, \quad (5.10b)$$

to uniquely extend A and B to all of Γ . Consider all pairs (Γ_τ, v^α) with $v^\alpha \in T_{\Gamma_\tau} M$. and $v_\alpha \mathbf{n}_\tau^\alpha = 0$. These pairs define a subset $T_\perp \Gamma$ of the tangent bundle TM . For each element of $T_\perp \Gamma$, one can associate an affinely parameterized geodesic $\omega \mapsto y(w)$ satisfying

$$y(0) = \Gamma_\tau, \quad \dot{y}^\alpha(0) = v^\alpha.$$

Provided that these geodesics admit sufficiently large extensions, the map

$$(\Gamma_\tau, v^\alpha) \mapsto y(1)$$

defines a smooth map from $T_\perp \Gamma$ into M . Its Jacobian is invertible, at least at every point $(\Gamma_\tau, 0)$. Hence, by the inverse function theorem, this map determines a local diffeomorphism on some neighborhood $\mathcal{W}$ of Γ . This neighborhood is precisely the region in which the generalized Killing fields are defined. We assume that the body under consideration remains entirely contained within this region, i.e. $\mathcal{W} \subseteq W$. We now define the generalized Killing field

$$\Xi_\alpha(x') := g_{\alpha\beta} \xi^\alpha(x') \quad (5.11)$$

corresponding to prescribed data $A_\alpha(\tau_0)$ and $B_{\alpha\beta}(\tau_0)$. The diffeomorphism described above uniquely associates each point $x \in W$ with a pair $(\Gamma_\tau, v^\alpha) \in T_\perp \Gamma$. Using this pair, we construct the geodesic $y(w)$ as above. The generalized Killing field is then obtained along $y(w)$ by solving the Jacobi equation

$$\frac{D^2 \Xi_\alpha}{dw^2} - R_{\alpha\beta\gamma}{}^\delta \dot{y}^\beta \dot{y}^\gamma \Xi_\delta = 0$$

with initial data

$$\begin{aligned} \Xi_\alpha(\Gamma_\tau) &= A_\alpha(\tau), \\ \frac{D \Xi_\alpha(\Gamma_\tau)}{dw} &= v^\beta B_{\beta\alpha}(\tau). \end{aligned}$$

By virtue of the Jacobi equation, GKF's are also globally specified by one value and its derivative at some point Γ_τ . Moreover by construction GKF's are ‘approximately Killing’, in the sense that

$$\nabla_{(\alpha} \Xi_{\beta)}|_\Gamma = 0 = \nabla_\alpha \nabla_{(\beta} \Xi_{\sigma)}|_\Gamma \Leftrightarrow \mathcal{L}_\xi g_{\alpha\beta}|_\Gamma = 0 = \nabla_\alpha \mathcal{L}_\xi g_{\beta\sigma}. \quad (5.12)$$

Linear and angular momenta $p^\alpha(\tau)$ and $S^{\alpha\beta}(\tau)$ may now be introduced as tensor fields along Γ as in eq. (5.4). Note that $S^{\alpha\beta}$ contains information about both the spin 1-form

5.2. The MPD equations

and the dipole moment of Newtonian physics, see section 5.3 for the details in Minkowski space. If $x' \in \Sigma_\tau$, then from eq. (5.7) we may relate $\Xi_\alpha(x')$ to $\Xi_\alpha(\Gamma_\tau)$ and $\nabla_\alpha \Xi_\beta(\Gamma_\tau)$ as

$$\xi^{\alpha'}(x') = g^{\alpha\beta}(\Gamma_\tau) \left[K^{\alpha'}{}_\alpha(\Gamma_\tau, x') \Xi_\beta(\Gamma_\tau) - H^{\alpha'\lambda}(\Gamma_\tau, x') \sigma_\alpha(\Gamma_\tau, x') \nabla_\beta \Xi_\lambda(\Gamma_\tau) \right], \quad (5.13)$$

where $K_\alpha{}^{\alpha'} := H^{\beta\alpha'} \nabla_\alpha \sigma_\beta$. Inserting the above expressing into eq. (5.2) and eq. (5.4) yields the following explicit expressions for the momenta:

$$p^\alpha(\tau) = g^{\alpha\beta}(\Gamma_\tau) \int_{\Sigma_\tau} g_{\alpha'\beta'}(x') K^{\alpha'}{}_\beta(\Gamma_\tau, x') T^{\beta'\sigma'}(x') dS_{\sigma'}, \quad (5.14)$$

$$S^{\alpha\beta}(\tau) = 2 \int_{\Sigma_\tau} g_{\alpha'\beta'}(x') H^{\alpha'[\alpha}(\Gamma_\tau, x') g^{\beta]\lambda}(\Gamma_\tau) \sigma_\lambda(\Gamma_\tau, x') T^{\beta'\lambda'}(x') dS_{\lambda'}. \quad (5.15)$$

5.2. The MPD equations

If we differentiate eq. (5.4) and use the identity

$$\nabla_\beta \nabla_\alpha \Xi_\lambda(\Gamma_\tau) = R_{\lambda\alpha\beta}{}^\delta(\Gamma_\tau) \Xi_\delta(\Gamma_\tau), \quad (5.16)$$

we find

$$\frac{d}{d\tau} P_\xi(\Sigma_\tau) = \left(\frac{Dp^\alpha}{d\tau} - \frac{1}{2} R_{\beta\lambda\delta}{}^\alpha S^{\beta\lambda} \dot{\Gamma}_\tau^\delta \right) \Xi_\alpha + \frac{1}{2} \left(\frac{DS^{\alpha\beta}}{d\tau} - 2p^{[\alpha} \dot{\Gamma}_\tau^{\beta]} \right) \nabla_\alpha \Xi_\beta. \quad (5.17)$$

Here $\dot{\Gamma}_\tau^\alpha$ denotes the tangent vector to the curve Γ at Γ_τ . On the other hand from eq. (5.3) we obtain

$$\frac{d}{d\tau} P_\xi(\Sigma_\tau) = \frac{1}{2} \int_{\Sigma_\tau} T^{\alpha\beta} \mathcal{L}_\xi g_{\alpha\beta} d\mu_{\Sigma_\tau}. \quad (5.18)$$

It is now clear, that if ξ is Killing, i.e. P_ξ is conserved, then from eq. (5.17) we obtain the following evolution equations for p^α and $S^{\alpha\beta}$, which are called the *MPD equations*:

$$\frac{Dp^\alpha}{d\tau} - \frac{1}{2} R_{\beta\lambda\delta}{}^\alpha S^{\beta\lambda} \dot{\Gamma}_\tau^\delta = 0, \quad (5.19a)$$

$$\frac{DS^{\alpha\beta}}{d\tau} - 2p^{[\alpha} \dot{\Gamma}_\tau^{\beta]} = 0. \quad (5.19b)$$

Named after Mathisson [Mat10], Papapetrou [Pap51, CP51] and Dixon [Dix70a, Dix70b, Dix74], these describe the motion of a spinning body moving in a gravitational field. Although they are most commonly applied in the massive case, the formalism also works for massless particles. If ξ is not Killing, then $\mathcal{L}_\xi g_{\alpha\beta}$ will not be zero and thus there will be correction terms. We define a net force $F^\alpha(\tau)$ and a net torque $N^{\alpha\beta}(\tau) = N^{[\alpha\beta]}(\tau)$ such that

$$\frac{Dp^\alpha}{d\tau} = \frac{1}{2} R_{\beta\lambda\delta}{}^\alpha S^{\beta\lambda} \dot{\Gamma}_\tau^\delta + F^\alpha, \quad (5.20a)$$

$$\frac{DS^{\alpha\beta}}{d\tau} = 2p^{[\alpha} \dot{\Gamma}_\tau^{\beta]} + N^{\alpha\beta}. \quad (5.20b)$$

Here F and N can be written as (formal) series, see [Har12, Har10] for explicit formulas.

5. Relevant observables and the method of stationary phase

5.3. The Minkowski case

It is enlightening to see how the observables defined above, reduce to the well-known quantities in flat spacetime. In Minkowski spacetime $(\mathbb{R}^{1+3}, \eta)$, geodesics are straight lines and parallel transport is trivial, so the Jacobi propagators reduce to identity maps:

$$K_{\alpha}^{\alpha'} \rightarrow \delta_{\alpha}^{\alpha'}, \quad H^{\alpha\alpha'} \rightarrow \eta^{\alpha\alpha'}.$$

The Synge world function becomes

$$\sigma(x, x') \rightarrow \frac{1}{2} \eta_{\alpha\beta} (x - x')^{\alpha} (x - x')^{\beta},$$

which yields for the Synge's separation vector the ordinary displacement

$$\sigma^{\alpha}(x, x') = (x - x')^{\alpha}.$$

Hence the linear momentum becomes

$$p_{\alpha} = \int_{\Sigma_{\tau}} T^{\beta}_{\alpha} dS_{\beta}.$$

Choosing a constant-time hypersurface $\{t = \tau\} \cong \mathbb{R}^3$, with

$$dS_{\beta} = n_{\beta} d^3x \quad \text{and} \quad n^{\beta} = (1, 0, 0, 0),$$

gives

$$p^{\mu} = \int_{\mathbb{R}^3} T^{0\mu} d^3x. \tag{5.21}$$

Therefore the components of p^{μ} are

$$p^0 = E = \int_{\mathbb{R}^3} T^{00} d^3x, \quad p^i = \int_{\mathbb{R}^3} T^{0i} d^3x,$$

so the four-momentum encodes both total energy and linear momentum. Similarly, the angular momentum tensor reduces to

$$S^{\alpha\beta} = \int_{\Sigma_{\tau}} \left[(x - x')^{\beta} T^{0\alpha}(x') - (x - x')^{\alpha} T^{0\beta}(x') \right] d^3x', \tag{5.22}$$

where $x^{\beta}(\tau)$ corresponds to some reference worldline. Its purely spatial components,

$$S^{ij} = \int_{\mathbb{R}^3} \left[(x - x')^j T^{0i}(x') - (x - x')^i T^{0j}(x') \right] d^3x',$$

correspond to the ordinary angular momentum, with

$$J_k = \epsilon_{kij} S^{ij}.$$

5.4. The stationary phase approximation

From eq. (5.22) one can easily derive the transformation behavior of S if we change the reference point x . Suppose we shift the reference point as

$$x^\mu \rightarrow x^\mu + \delta x^\mu.$$

Then we immediately find that

$$S^{\alpha\beta}(x + \delta x) = S^{\alpha\beta}(x) + 2\delta x^{[\beta} p^{\alpha]}, \quad (5.23)$$

where p^α is the momentum as given by eq. (5.21). The center-of-energy worldline is fixed by the condition

$$t_\alpha S^{\alpha\beta} = 0,$$

where $t^\alpha = (1, 0, 0, 0)$ denotes the standard observer in Minkowski spacetime. Evaluating eq. (5.22) on the hypersurface Σ_τ with $x^0 = x'^0 = \tau$ gives $(x - x')^0 = 0$, so that

$$S^{0i} = \int_{\Sigma_\tau} (x - x')^i T^{00}(x') d^3 x', \quad (5.24)$$

which represents the dipole moment. Imposing $S^{0i} = 0$ then yields

$$x^\beta = \frac{1}{E} \int_{\mathbb{R}^3} x'^\beta T^{00}(x') d^3 x'. \quad (5.25)$$

5.4. The stationary phase approximation

The observables p^α from eq. (5.14) and $S^{\alpha\beta}$ from eq. (5.15) are expressed in terms of integrals over Cauchy hypersurfaces. What we want to do is approximate those integrals using the stationary phase approximation. Note that in general spacetimes, these hypersurfaces will be of non-zero curvature, hence the integration is non-trivial. However, as pointed out in section 4.3, there is a covariant way to lift the integrals, based at some point Γ_τ on a reference worldline Γ , to the corresponding tangent space $T_{\Gamma_\tau} \Sigma_\tau \cong \mathbb{R}^3$ via the exponential map, see eq. (4.10). Now that we integrate over $\mathbb{R}^3$ we can apply the following

Theorem 5.1 (Stationary phase 1, [Hö03, Thm. 7.7.5]). *Let $K \subset \mathbb{R}^n$ be a compact subset, X an open neighborhood of K , k a positive integer and $\omega > 0$. If $u \in C_0^{2k}(K)$, $f \in C^{3k+1}(X)$ and $\Im m f \geq 0$ in X , $\Im m f(\gamma_\tau) = 0$, $f'(\gamma_\tau) = 0$, $\det f''(\gamma_\tau) \neq 0$, $f' \neq 0$ in $K \setminus \{\gamma_\tau\}$ then*

$$\left| \int_{\mathbb{R}^n} u(x) e^{i\omega f(x)} dx - e^{i\omega f(\gamma_\tau)} \left[\det \left(\frac{\omega}{2\pi i} A \right) \right]^{-1/2} \sum_{j < k} \omega^{-j} L_j u(\gamma_\tau) \right| \leq C \omega^{-k} \sum_{|\alpha| \leq 2k} \sup |D^\alpha u|. \quad (5.26)$$

5. Relevant observables and the method of stationary phase

In the above equation $A_{ab} = \partial_a \partial_b f(\gamma_\tau)$, $D^\alpha = (-i)^{|\alpha|} \partial^\alpha$,

$$g(x) = f(x) - f(\gamma_\tau) - \frac{1}{2} \partial_a \partial_b f(\gamma_\tau) (x - \gamma_\tau)^a (x - \gamma_\tau)^b, \quad (5.27)$$

and

$$L_j u(\gamma_\tau) = \sum_{\nu - \mu = j} \sum_{2\nu \geq 3\mu} \frac{1}{i^j 2^\nu \mu! \nu!} \left[-(A^{-1})^{ab} \partial_a \partial_b \right]^\nu (g^\mu u)(\gamma_\tau). \quad (5.28)$$

Remark 5.1. In section 6.3 we will use this result to approximate integrals over the energy-momentum tensor of our Gaussian beam. By construction, as in section 2.3, all of the assumptions of theorem 5.1 are fulfilled. In particular we are in the case where all the functions are infinitely differentiable.

Remark 5.2. Note that the point of stationary phase γ_τ does not have to coincide with the point Γ_τ on the reference worldline, see section 6.3.

We compute explicitly the action of L_j on u for $j = 0, 1$. Note that by the definition of g we have

$$g(\gamma_\tau) = 0, \quad \nabla^\alpha g(\gamma_\tau) = 0 \text{ for } |\alpha| \leq 2 \quad \text{and} \quad \partial^\alpha g(\gamma_\tau) = \partial^\alpha f(\gamma_\tau) \quad \forall |\alpha| \geq 3. \quad (5.29)$$

At lowest order ($j = 0$) we have

$$L_0 u(\gamma_\tau) = u(\gamma_\tau). \quad (5.30)$$

At first order ($j = 1$) we have to evaluate

$$L_1 u(\gamma_\tau) = \sum_{\nu - \mu = 1} \sum_{2\nu \geq 3\mu} \frac{1}{i 2^\nu \mu! \nu!} \left[-(A^{-1})^{ab} \partial_a \partial_b \right]^\nu (g^\mu u)(\gamma_\tau).$$

We need to find all pairs (ν, μ) such that $\nu - \mu = 1$ and $2\nu \geq 3\mu$. The only ones that work are $(1, 0)$, $(2, 1)$ and $(3, 2)$. Thus we find

$$\begin{aligned} L_1 u(\gamma_\tau) &= \frac{i}{2} (A^{-1})^{ab} \partial_a \partial_b u(\gamma_\tau) - \frac{i}{8} (A^{-1})^{ab} (A^{-1})^{cd} \partial_a \partial_b \partial_c \partial_d (gu)(\gamma_\tau) \\ &\quad + \frac{i}{96} (A^{-1})^{ab} (A^{-1})^{cd} (A^{-1})^{ef} \partial_a \partial_b \partial_c \partial_d \partial_e \partial_f (g^2 u)(\gamma_\tau). \end{aligned}$$

Note that by eq. (5.29) at least 3 derivatives need to hit g to get a non-zero term when evaluated at $x = \gamma_\tau$. In particular we can rewrite the second term from above as

$$\begin{aligned} -\frac{i}{8} (A^{-1})^{ab} (A^{-1})^{cd} \partial_a \partial_b \partial_c \partial_d (gu)(\gamma_\tau) &= -\frac{i}{8} \left[4 (A^{-1})^{ab} (A^{-1})^{cd} (\partial_a \partial_b \partial_c f) (\partial_d u) \right. \\ &\quad \left. + (A^{-1})^{ab} (A^{-1})^{cd} (\partial_a \partial_b \partial_c \partial_d f) u \right] (\gamma_\tau), \end{aligned}$$

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where the factor of 4 comes from the symmetry of A . So altogether we have

$$\begin{aligned} L_1 u(\gamma_\tau) &= \frac{i}{2}(A^{-1})^{ab}\partial_a\partial_b u(\gamma_\tau) - \frac{i}{2}(A^{-1})^{ab}(A^{-1})^{cd}(\partial_a\partial_b\partial_c f)(\gamma_\tau)(\partial_d u)(\gamma_\tau) \\ &\quad - \frac{i}{8}u(\gamma_\tau)(A^{-1})^{ab}(A^{-1})^{cd}(\partial_a\partial_b\partial_c\partial_d f)(\gamma_\tau) \\ &\quad + \frac{i}{96}(A^{-1})^{ab}(A^{-1})^{cd}(A^{-1})^{ef}\partial_a\partial_b\partial_c\partial_d\partial_e\partial_f(g^2 u)(\gamma_\tau). \end{aligned} \quad (5.31)$$

In section 6.3 we apply the stationary phase approximation to amplitudes which are formally bitensors. However, these bitensors are scalar-valued in the integration variable. More precisely, the oscillatory integrals under consideration involve bitensors $u(x, x')$ where the first variable $x \equiv \Gamma_\tau$ is fixed along a reference worldline Γ and the second variable x' is the integration variable. For fixed x , the dependence on x' is scalar-valued, so the integrand depends on x' through scalar functions of the form

$$\Sigma_\tau \ni x' \mapsto u(x, x').$$

Introducing exponential coordinates around x , we write

$$T_x \Sigma_\tau \ni v \mapsto u(x, \exp_x(v)).$$

Since stationary phase involves ordinary derivatives in Euclidean coordinates, these correspond to vertical derivatives with respect to $v \in T_x \Sigma_\tau$. In our setting, the critical point of the phase is located at γ_τ , rather than at x . We therefore introduce exponential coordinates centered at γ_τ by writing

$$\exp_x(v) = \exp_{\gamma_\tau}(w),$$

where $w = w(x, v) \in T_{\gamma_\tau} \Sigma_\tau$ is uniquely determined in a sufficiently small normal neighborhood U , as illustrated below.

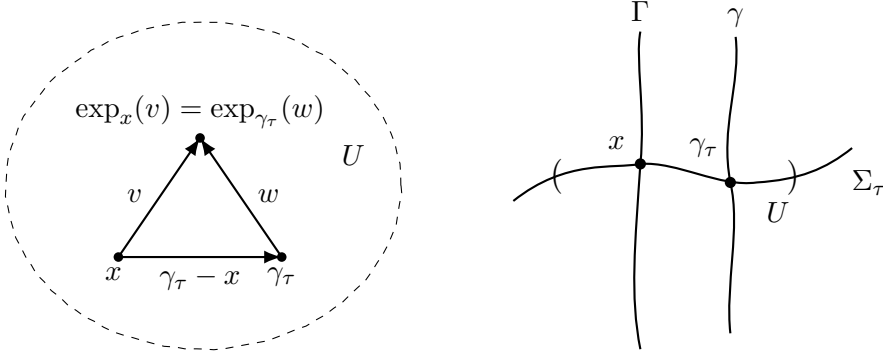


Figure 5.1.: Visual representation of the change in exponential coordinates. The points x , γ_τ , and w all lie within the normal neighborhood $U \subset \Sigma_\tau$, and $\gamma_\tau - x$, v , and w denote the corresponding (geodesic) displacement vectors between them. Γ denotes the reference worldline with respect to which the observables are defined, while γ is the null geodesic along which the Gaussian beam is constructed.

5. Relevant observables and the method of stationary phase

Applying the chain rule yields

$$\left. \frac{\partial}{\partial v^{\alpha'}} u(x, \exp_x(v)) \right|_{v=\gamma_\tau-x} = \frac{\partial w^{\beta'}}{\partial v^{\alpha'}} \left. \frac{\partial}{\partial w^{\beta'}} u(x, \exp_{\gamma_\tau}(w)) \right|_{w=0}.$$

Since u is scalar-valued in the integration variable, we obtain

$$\left. \frac{\partial}{\partial w^{\beta'}} u(x, \exp_{\gamma_\tau}(w)) \right|_{w=0} = \nabla_{\beta'} u(x, \gamma_\tau),$$

as justified by the following

Lemma 5.1 ([Sha05, Lemma 2.1]). *Let $u \in C^\infty(M)$. Then for every multi-index α ,*

$$\nabla^\alpha u(\gamma_\tau) = \partial_w^\alpha [u(\exp_{\gamma_\tau}(w))]_{w=0}.$$

In this formulation, all derivatives are taken in a fixed coordinate neighborhood after identification via the exponential map. Once evaluated at $w = 0$, the resulting expression involves only tensors at the single point γ_τ , and standard product rules for covariant derivatives in the second variable apply.

6. Approximate solutions to Maxwell's equations in curved spacetime

With all the necessary notions and methods collected in the previous sections, we can finally put them to use for studying electromagnetic Gaussian beams that satisfy Maxwell's equations.

6.1. The approximation scheme

Here we want to make precise our notion of approximate solutions to Maxwell's equations. We consider a globally hyperbolic Lorentzian manifold (M, g) that is foliated by a one-parameter family of spacelike Cauchy hypersurfaces $\{\Sigma_\tau\}_{\tau \in \mathbb{R}}$. The future directed unit normal vector to Σ_τ , denoted by n , satisfies $g(n, n) = -1$. We introduce an orthonormal frame $(e_a)^\mu$ with $e_0 = n$, such that

$$g_{\mu\nu} = \eta_{ab}(e^a)_\mu(e^b)_\nu = -n_\mu n_\nu + \sum_{j=1}^3 (e^j)_\mu (e^j)_\nu. \quad (6.1)$$

Recall from eq. (2.5) that the induced Riemannian spatial metric on Σ_τ is given by

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu. \quad (6.2)$$

In order to keep information of the data in the transverse direction of Σ_τ we introduce the following auxiliary Riemannian metric on M :

$$\mathfrak{g}_{\mu\nu} = g_{\mu\nu} + 2n_\mu n_\nu. \quad (6.3)$$

We consider Maxwell's equations

$$\nabla^\alpha F_{\alpha\beta} = 0, \quad (6.4)$$

$$\nabla_{[\mu} F_{\alpha\beta]} = 0, \quad (6.5)$$

where $F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha$. Our aim is to construct approximate solutions $\hat{F}_{\alpha\beta}(x; \omega)$ such that for some $T > 0$ and $p > 0$ we have

$$\sup_{\tau \in [0, T]} \|\nabla^\alpha \hat{F}_{\alpha\beta}\|_{L^2(\Sigma_\tau, \mathfrak{g})} = \mathcal{O}(\omega^{-p}), \quad (6.6a)$$

$$\sup_{\tau \in [0, T]} \|\nabla_{[\mu} \hat{F}_{\alpha\beta]}\|_{L^2(\Sigma_\tau, \mathfrak{g})} = \mathcal{O}(\omega^{-p}). \quad (6.6b)$$

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Note that by $F = dA$ and the fact that the exterior derivative satisfies $d^2 = 0$, the second equation from above is trivially fulfilled if we work with the potential A . Since $\hat{F}$ is an approximate solution, it differs from the exact solution by a residual, which we denote by f . So we have

$$\nabla^\alpha \hat{F}_{\alpha\beta} = f_\beta. \quad (6.7)$$

The norm from (6.6) is defined as

$$\|f_\beta\|_{L^2(\Sigma_\tau, \mathfrak{g})}^2 := \int_{\Sigma_\tau} \mathfrak{g}^{\mu\nu} f_\mu f_\nu d\mu_{\Sigma_\tau} = \int_{\Sigma_\tau} \left(|f(n)|^2 + \sum_{i=1}^3 |f(e_i)|^2 \right) d\mu_{\Sigma_\tau}. \quad (6.8)$$

As usual, the integration measure $d\mu_{\Sigma_\tau}$ is given by

$$d\mu_{\Sigma_\tau} = d\mu_h = \sqrt{h} dx^1 \wedge dx^2 \wedge dx^3,$$

where $h = \det(h_{ij})$. Note that if we consider the special case of flat Minkowski space with $n = \partial_t$, then we have

$$\|f_\beta\|_{L^2(\Sigma_\tau, \mathfrak{g})}^2 = \int_{\mathbb{R}^3} \left(|\nabla \cdot \hat{E}|^2 + |\partial_t \hat{E} - \nabla \times \hat{B}|^2 \right) d^3x.$$

Let us now continue to work with the vector potential A in Lorenz gauge $\nabla^\alpha A_\alpha = 0$. Then we have

$$\begin{aligned} \nabla^\mu F_{\mu\alpha} &= \nabla^\mu (\nabla_\mu A_\alpha - \nabla_\alpha A_\mu) = \square_g A_\alpha - \nabla^\mu \nabla_\alpha A_\mu \\ &= \square_g A_\alpha - \nabla_\alpha \nabla_\mu A^\mu - R^\mu_{\beta\mu\alpha} A^\beta \\ &= \square_g A_\alpha - R_\alpha^\beta A_\beta, \end{aligned}$$

where we have used that

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) A^\rho = R^\rho_{\sigma\mu\nu} A^\sigma$$

as well as the symmetry of the Ricci tensor. We want to construct $\hat{A}_\alpha(x; \omega)$ so that $\hat{F}_{\alpha\beta} = \nabla_\alpha \hat{A}_\beta - \nabla_\beta \hat{A}_\alpha$ is an approximate solution to Maxwell's equations, as defined in eq. (6.6a). We introduce the notation

$$\square_g \hat{A}_\alpha - R_\alpha^\beta \hat{A}_\beta = w_\alpha, \quad \nabla^\alpha \hat{A}_\alpha = G, \quad (6.9)$$

where w_α and G are residuals of the approximation. We then obtain

$$\begin{aligned} f_\beta &= \nabla^\alpha \hat{F}_{\alpha\beta} = \nabla^\alpha (\nabla_\alpha \hat{A}_\beta - \nabla_\beta \hat{A}_\alpha) = \square_g \hat{A}_\beta - \nabla^\alpha \nabla_\beta \hat{A}_\alpha \\ &= w_\beta + R_\beta^\alpha \hat{A}_\alpha - \nabla^\alpha \nabla_\beta \hat{A}_\alpha \\ &= w_\beta + R_\beta^\alpha \hat{A}_\alpha - (R^\alpha_\beta \hat{A}_\alpha + \nabla_\beta \nabla^\alpha \hat{A}_\alpha) \\ &= w_\beta - \nabla_\beta G. \end{aligned}$$

6.1. The approximation scheme

Consequently, at the level of norms, we have

$$\|f_\beta\|_{L^2(\Sigma_\tau, \mathfrak{g})} \leq \|w_\beta\|_{L^2(\Sigma_\tau, \mathfrak{g})} + \|\nabla_\beta G\|_{L^2(\Sigma_\tau, \mathfrak{g})}. \quad (6.10)$$

So, in order to obtain an approximate solution that satisfies eq. (6.6a), the approximate vector potential $\hat{A}$ needs to satisfy

$$\sup_{\tau \in [0, T]} \|\square_g \hat{A}_\alpha - R_\alpha^\beta \hat{A}_\beta\|_{L^2(\Sigma_\tau, \mathfrak{g})} = \mathcal{O}(\omega^{-p}), \quad (6.11a)$$

$$\sup_{\tau \in [0, T]} \|\nabla_\beta \nabla^\alpha \hat{A}_\alpha\|_{L^2(\Sigma_\tau, \mathfrak{g})} = \mathcal{O}(\omega^{-p}). \quad (6.11b)$$

We now make the following ansatz for our Gaussian beam,

$$\mathfrak{A}_\alpha(x) = \omega^{-1/4} a_\alpha(x) e^{i\omega\phi(x)} \quad (x \in M), \quad (6.12)$$

where $a \in \Omega_0^1(M, \mathbb{C})$ and $\phi \in C^\infty(M, \mathbb{C})$. The prefactor $\omega^{-1/4}$ was introduced such that the initial energy is of order ω^0 . This choice follows directly from theorem 5.1 as well as the fact that, without that factor, the energy-momentum tensor is of order ω^2 .

We will now derive conditions on a_α and ϕ such that eq. (6.11a) holds. In order to do that we will use lemma 2.1. Note that by remark 2.2, the lemma can still be used, even though our integrands are given as squared norms with respect to the auxiliary Riemannian metric $\mathfrak{g}$. This is due to the fact that on finite-dimensional vector spaces all norms are equivalent and that a_α is a compactly supported one-form on M . Concretely this means there is a constant $C > 0$ such that $\mathfrak{g}^{\mu\nu} a_\mu \bar{a}_\nu \leq C \sum_\mu |a_\mu|_g^2$. We have

$$\begin{aligned} \square \mathfrak{A}_\alpha - R_\alpha^\beta \mathfrak{A}_\beta &= e^{i\omega\phi} \left\{ -\omega^{7/4} a_\alpha g^{\mu\nu} (\nabla_\mu \phi) (\nabla_\nu \phi) \right. \\ &\quad + i\omega^{3/4} [2g^{\mu\nu} (\nabla_\mu \phi) (\nabla_\nu a_\alpha) + a_\alpha \square \phi] \\ &\quad \left. + \omega^{-1/4} [\square a_\alpha - R_\alpha^\beta a_\beta] \right\}. \end{aligned} \quad (6.13)$$

We make a formal expansion of a_α as

$$a_\alpha = \sum_{j=0}^N a_\alpha^{(j)} \omega^{-j} \quad (N \in \mathbb{N}), \quad (6.14)$$

where $a_\alpha^{(j)} \in \Omega_0^1(M)$ for $j = 0, \dots, N$. Our ansatz will thus be

$$\mathfrak{A}_\alpha = \omega^{-1/4} \sum_{j=0}^N a_\alpha^{(j)} \omega^{-j} e^{i\omega\phi}, \quad (6.15)$$

and for the approximate, physical potential we set

$$\hat{A}_\alpha := \Re(\mathfrak{A}_\alpha) = \frac{1}{2}(\mathfrak{A}_\alpha + \bar{\mathfrak{A}}_\alpha). \quad (6.16)$$

In the following we will derive constraints on the order up to which certain terms have to vanish, so that the chosen ansatz satisfies our approximation.

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Remark 6.1. Note that by the fact that $\hat{A}_\alpha = \Re(\mathfrak{A}_\alpha)$, letting differential operators act on it, will always give two contributions: some amplitude times $e^{i\omega\phi}$ plus the complex conjugate amplitude times $e^{-i\omega\bar{\phi}}$. Now, it is clear that $z = 0$ iff $\bar{z} = 0$ for every $z \in \mathbb{C}$. This means that in the following it is sufficient to perform computations with $\mathfrak{A}_\alpha$, since the contributions from the complex conjugate part do not contain further structure. In this sense, when deriving constraint equations that need to be satisfied as we will do in the following, it is always understood that the complex conjugated version also has to be satisfied.

Plugging the above ansatz into the expression from eq. (6.13) we obtain

$$\begin{aligned}
e^{-i\omega\phi}(\Box \mathfrak{A}_\alpha - R_\alpha{}^\beta \mathfrak{A}_\beta) &= \omega^{7/4} \left[-a_\alpha^{(0)} g^{\mu\nu} (\nabla_\mu \phi) (\nabla_\nu \phi) \right] \\
&\quad + \omega^{3/4} \left[-a_\alpha^{(1)} g^{\mu\nu} (\nabla_\mu \phi) (\nabla_\nu \phi) + 2ig^{\mu\nu} (\nabla_\mu \phi) (\nabla_\nu a_\alpha^{(0)}) + ia_\alpha^{(0)} \Box \phi \right] \\
&\quad + \omega^{-1/4} \left[-a_\alpha^{(2)} g^{\mu\nu} (\nabla_\mu \phi) (\nabla_\nu \phi) + 2ig^{\mu\nu} (\nabla_\mu \phi) (\nabla_\nu a_\alpha^{(1)}) \right. \\
&\quad \quad \left. + ia_\alpha^{(1)} \Box \phi + \Box a_\alpha^{(0)} - R_\alpha{}^\beta a_\beta^{(0)} \right] \\
&\quad + \sum_{j=3}^N \omega^{7/4-j} \left[-a_\alpha^{(j)} g^{\mu\nu} (\nabla_\mu \phi) (\nabla_\nu \phi) + 2ig^{\mu\nu} (\nabla_\mu \phi) (\nabla_\nu a_\alpha^{(j-1)}) \right. \\
&\quad \quad \left. + ia_\alpha^{(j-1)} \Box \phi + \Box a_\alpha^{(j-2)} - R_\alpha{}^\beta a_\beta^{(j-2)} \right] \\
&\quad + \text{lower orders.}
\end{aligned}$$

Now, using lemma 2.1, we analyze to what order the various terms from above have to vanish such that the approximation from eq. (6.11a) is fulfilled. We need that

$$g^{\mu\nu} (\nabla_\mu \phi) (\nabla_\nu \phi) \text{ vanishes to } (2p+1)\text{-th order along } \gamma, \quad (6.17a)$$

$$\begin{aligned}
&-a_\alpha^{(1)} g^{\mu\nu} (\nabla_\mu \phi) (\nabla_\nu \phi) + 2ig^{\mu\nu} (\nabla_\mu \phi) (\nabla_\nu a_\alpha^{(0)}) + ia_\alpha^{(0)} \Box \phi \text{ vanishes} \\
&\text{to } (2p-1)\text{-th order along } \gamma, \text{ and}
\end{aligned} \quad (6.17b)$$

$$\begin{aligned}
&-a_\alpha^{(j)} g^{\mu\nu} (\nabla_\mu \phi) (\nabla_\nu \phi) + 2ig^{\mu\nu} (\nabla_\mu \phi) (\nabla_\nu a_\alpha^{(j-1)}) + ia_\alpha^{(j-1)} \Box \phi + \Box a_\alpha^{(j-2)} - R_\alpha{}^\beta a_\beta^{(j-2)} \\
&\text{vanishes to } (1+2p-2j)\text{-th order along for } j \geq 2.
\end{aligned} \quad (6.17c)$$

Proposition 6.1. *In this setting, we have*

$$\sup_{\tau \in [0, T]} \|\nabla^\alpha \hat{A}_\alpha\|_{L^2(\Sigma_\tau, \mathfrak{g})} = \mathcal{O}(\omega^{-p}) \implies \sup_{\tau \in [0, T]} \|\nabla_\beta \nabla^\alpha \hat{A}_\alpha\|_{L^2(\Sigma_\tau, \mathfrak{g})} = \mathcal{O}(\omega^{-p+1}).$$

6.1. The approximation scheme

Proof. In view of remark 6.1 it suffices to perform computations with $\mathfrak{A}_\alpha$. We find

$$\begin{aligned} e^{-i\omega\phi}\nabla^\alpha\mathfrak{A}_\alpha &= \omega^{3/4}[ia_\alpha^{(0)}\nabla^\alpha\phi] \\ &\quad + \omega^{-1/4}[\nabla^\alpha a_\alpha^{(0)} + ia_\alpha^{(1)}\nabla^\alpha\phi] \\ &\quad + \omega^{-5/4}[\nabla^\alpha a_\alpha^{(1)} + ia_\alpha^{(2)}\nabla^\alpha\phi] \\ &\quad + \sum_{j=3}^N \omega^{3/4-j}[\nabla^\alpha a_\alpha^{(j-1)} + ia_\alpha^{(j)}\nabla^\alpha\phi] \\ &\quad + \text{lower orders.} \end{aligned}$$

Further we have $e^{-i\omega\phi}\nabla_\beta\nabla^\alpha\mathfrak{A}_\alpha =$

$$\begin{aligned} &= -\omega^{7/4}[a_\alpha^{(0)}\nabla^\alpha\phi\nabla_\beta\phi] \\ &\quad + \omega^{3/4}[i\nabla_\beta a_\alpha^{(0)}\nabla^\alpha\phi + ia_\alpha^{(0)}\nabla_\beta\nabla^\alpha\phi + i\nabla^\alpha a_\alpha^{(0)}\nabla_\beta\phi - a_\alpha^{(1)}\nabla^\alpha\phi\nabla_\beta\phi] \\ &\quad + \omega^{-1/4}[\nabla_\beta\nabla^\alpha a_\alpha^{(0)} + i\nabla_\beta a_\alpha^{(1)}\nabla^\alpha\phi + ia_\alpha^{(1)}\nabla_\beta\nabla^\alpha\phi \\ &\quad \quad + i\nabla^\alpha a_\alpha^{(1)}\nabla_\beta\phi - a_\alpha^{(2)}\nabla^\alpha\phi\nabla_\beta\phi] \\ &\quad + \sum_{j=3}^N \omega^{7/4-j}[\nabla_\beta\nabla^\alpha a_\alpha^{(j-2)} + i\nabla_\beta a_\alpha^{(j-1)}\nabla^\alpha\phi + ia_\alpha^{(j-1)}\nabla_\beta\nabla^\alpha\phi \\ &\quad \quad + i\nabla^\alpha a_\alpha^{(j-1)}\nabla_\beta\phi - a_\alpha^{(j)}\nabla^\alpha\phi\nabla_\beta\phi] \\ &\quad + \text{lower orders.} \end{aligned}$$

We apply lemma 2.1 to find that if $\sup_{\tau \in [0, T]} \|\nabla^\alpha \hat{A}_\alpha\|_{L^2(\Sigma_\tau, \mathfrak{g})} = \mathcal{O}(\omega^{-p})$ then we need that

$a_\alpha^{(0)}\nabla^\alpha\phi$ has to vanish to $(2p-1)$ -th order on γ ,

$\nabla^\alpha a_\alpha^{(0)} + ia_\alpha^{(1)}\nabla^\alpha\phi$ has to vanish to $(2p-3)$ -th order on γ , and

$\nabla^\alpha a_\alpha^{(j-1)} + ia_\alpha^{(j)}\nabla^\alpha\phi$ has to vanish to $(2p-2j-1)$ -th order on γ for $j \geq 2$.

For $p = 2$ we thus have

$$a_\alpha^{(0)}\nabla^\alpha\phi|_\gamma = 0, \tag{6.21a}$$

$$\nabla_\beta(a_\alpha^{(0)}\nabla^\alpha\phi)|_\gamma = 0 \Rightarrow \nabla_\beta a_\alpha^{(0)}\nabla^\alpha\phi|_\gamma = -a_\alpha^{(0)}\nabla_\beta\nabla^\alpha\phi|_\gamma, \tag{6.21b}$$

$$(\nabla^\alpha a_\alpha^{(0)} + ia_\alpha^{(1)}\nabla^\alpha\phi)|_\gamma = 0 \Rightarrow i\nabla^\alpha a_\alpha^{(0)}|_\gamma = a_\alpha^{(1)}\nabla^\alpha\phi|_\gamma, \tag{6.21c}$$

$$\nabla_\beta(\nabla^\alpha a_\alpha^{(0)} + ia_\alpha^{(1)}\nabla^\alpha\phi)|_\gamma = 0 \Rightarrow \nabla_\beta\nabla^\alpha a_\alpha^{(0)} = -i\nabla_\beta a_\alpha^{(1)}\nabla^\alpha\phi|_\gamma - ia_\alpha^{(1)}\nabla_\beta\nabla^\alpha\phi|_\gamma, \tag{6.21d}$$

$$\nabla_\mu\nabla_\beta a_\alpha^{(0)}\nabla^\alpha\phi|_\gamma + \nabla_\beta a_\alpha^{(0)}\nabla_\mu\nabla^\alpha\phi|_\gamma + \nabla_\mu a_\alpha^{(0)}\nabla_\beta\nabla^\alpha\phi|_\gamma + a_\alpha^{(0)}\nabla_\mu\nabla_\beta\nabla^\alpha\phi|_\gamma = 0, \tag{6.21e}$$

$$\begin{aligned} &\nabla_\sigma\nabla_\mu\nabla_\beta a_\alpha^{(0)}\nabla^\alpha\phi|_\gamma + \nabla_\mu\nabla_\beta a_\alpha^{(0)}\nabla_\sigma\nabla^\alpha\phi|_\gamma + \nabla_\sigma\nabla_\beta a_\alpha^{(0)}\nabla_\mu\nabla^\alpha\phi|_\gamma \\ &\quad + \nabla_\beta a_\alpha^{(0)}\nabla_\sigma\nabla_\mu\nabla^\alpha\phi|_\gamma + \nabla_\sigma\nabla_\mu a_\alpha^{(0)}\nabla_\beta\nabla^\alpha\phi|_\gamma + \nabla_\mu a_\alpha^{(0)}\nabla_\sigma\nabla_\beta\nabla^\alpha\phi|_\gamma \\ &\quad + \nabla_\sigma a_\alpha^{(0)}\nabla_\mu\nabla_\beta\nabla^\alpha\phi|_\gamma + a_\alpha^{(0)}\nabla_\sigma\nabla_\mu\nabla_\beta\nabla^\alpha\phi|_\gamma = 0. \end{aligned} \tag{6.21f}$$

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Applying lemma 2.1 in order to obtain $\sup_{\tau \in [0, T]} \|\nabla_\beta \nabla^\alpha \hat{A}_\alpha\|_{L^2(\Sigma_\tau, \mathfrak{g})} = \mathcal{O}(\omega^{-p+1})$ we need

$$a_\alpha^{(0)} \nabla^\alpha \phi \nabla_\beta \phi \text{ to vanish to } (2p-1)\text{-th order on } \gamma, \quad (6.22a)$$

$$i \nabla_\beta a_\alpha^{(0)} \nabla^\alpha \phi + i a_\alpha^{(0)} \nabla_\beta \nabla^\alpha \phi + i \nabla^\alpha a_\alpha^{(0)} \nabla_\beta \phi - a_\alpha^{(1)} \nabla^\alpha \phi \nabla_\beta \phi \text{ to vanish to } (2p-3)\text{-th order on } \gamma, \text{ and} \quad (6.22b)$$

$$\nabla_\beta \nabla^\alpha a_\alpha^{(j-2)} + i \nabla_\beta a_\alpha^{(j-1)} \nabla^\alpha \phi + i a_\alpha^{(j-1)} \nabla_\beta \nabla^\alpha \phi + i \nabla^\alpha a_\alpha^{(j-1)} \nabla_\beta \phi - a_\alpha^{(j)} \nabla^\alpha \phi \nabla_\beta \phi \text{ to vanish to } (2p-2j-1)\text{-th order on } \gamma \text{ for } j \geq 2. \quad (6.22c)$$

What we now have to do is show that the conditions in eq. (6.22) are satisfied, given the relations in eq. (6.21).

We start by checking eq. (6.22a). We first need that $a_\alpha^{(0)} \nabla^\alpha \phi \nabla_\beta \phi|_\gamma = 0$. Since within the Gaussian beam approximation we assume that $\nabla_\beta \phi|_\gamma$ is non-zero we obtain the desired result due to eq. (6.21a). Next we find

$$\nabla_\mu (a_\alpha^{(0)} \nabla^\alpha \phi \nabla_\beta \phi)|_\gamma = \underbrace{\nabla_\mu a_\alpha^{(0)} \nabla^\alpha \phi \nabla_\beta \phi|_\gamma}_{\stackrel{(6.21b)}{=} -a_\alpha^{(0)} \nabla_\mu \nabla^\alpha \phi|_\gamma} + a_\alpha^{(0)} \nabla_\mu \nabla^\alpha \phi \nabla_\beta \phi|_\gamma + \underbrace{a_\alpha^{(0)} \nabla^\alpha \phi \nabla_\mu \nabla_\beta \phi|_\gamma}_{\stackrel{(6.21a)_0}{=} 0} = 0.$$

We further compute $\nabla_\nu \nabla_\mu (a_\alpha^{(0)} \nabla^\alpha \phi \nabla_\beta \phi)|_\gamma =$

$$\begin{aligned} & \underbrace{\nabla_\nu \nabla_\mu a_\alpha^{(0)} \nabla^\alpha \phi \nabla_\beta \phi|_\gamma}_{(1.1)} + \underbrace{\nabla_\mu a_\alpha^{(0)} \nabla_\nu \nabla^\alpha \phi \nabla_\beta \phi|_\gamma}_{(1.2)} + \underbrace{\nabla_\mu a_\alpha^{(0)} \nabla^\alpha \phi \nabla_\nu \nabla_\beta \phi|_\gamma}_{(2.1)} \\ & + \underbrace{\nabla_\nu a_\alpha^{(0)} \nabla_\mu \nabla^\alpha \phi \nabla_\beta \phi|_\gamma}_{(1.3)} + \underbrace{a_\alpha^{(0)} \nabla_\nu \nabla_\mu \nabla^\alpha \phi \nabla_\beta \phi|_\gamma}_{(1.4)} + \underbrace{a_\alpha^{(0)} \nabla_\mu \nabla^\alpha \phi \nabla_\nu \nabla_\beta \phi|_\gamma}_{(2.2)} \\ & + \underbrace{\nabla_\nu a_\alpha^{(0)} \nabla^\alpha \phi \nabla_\mu \nabla_\beta \phi|_\gamma}_{(3.1)} + \underbrace{a_\alpha^{(0)} \nabla_\nu \nabla^\alpha \phi \nabla_\mu \nabla_\beta \phi|_\gamma}_{(3.2)} + \underbrace{a_\alpha^{(0)} \nabla^\alpha \phi \nabla_\nu \nabla_\mu \nabla_\beta \phi|_\gamma}_{\stackrel{(6.21a)_0}{=} 0}. \end{aligned}$$

We notice that the terms (1.1)–(1.4) all have the common factor $\nabla_\beta \phi|_\gamma$, hence they can be combined to give exactly the four terms in eq. (6.21e) which sum to zero. Similarly the terms (2.1)–(2.2) and (3.1)–(3.2) sum to zero due to eq. (6.21b). Thus this yields that also $\nabla_\nu \nabla_\mu (a_\alpha^{(0)} \nabla^\alpha \phi \nabla_\beta \phi)|_\gamma = 0$. For the case $p = 2$ it remains to show that $\nabla_\sigma \nabla_\nu \nabla_\mu (a_\alpha^{(0)} \nabla^\alpha \phi \nabla_\beta \phi)|_\gamma = 0$. The proof works analogously to the previous calculation: now we get 27 terms, 8 of which can be eliminated by eq. (6.21f), 3 sets of 4 terms vanish due to eq. (6.21e), 3 sets of 2 terms cancel due to eq. (6.21b) and one term vanishes due to eq. (6.21a). A detailed calculation can be found in chapter A.

6.2. Explicit form of the equations to order ω^{-2}

Next we analyze eq. (6.22b). We compute

$$\underbrace{i \nabla_\beta a_\alpha^{(0)} \nabla^\alpha \phi|_\gamma}_{\stackrel{(6.21b)}{=} -a_\alpha^{(0)} \nabla_\beta \nabla^\alpha \phi|_\gamma} + i a_\alpha^{(0)} \nabla_\beta \nabla^\alpha \phi|_\gamma + \underbrace{i \nabla^\alpha a_\alpha^{(0)} \nabla_\beta \phi|_\gamma}_{\stackrel{(6.21c)}{=} a_\alpha^{(1)} \nabla^\alpha \phi|_\gamma} - a_\alpha^{(1)} \nabla^\alpha \phi \nabla_\beta \phi|_\gamma = 0.$$

We further find that the first covariant derivative of eq. (6.22b) is given by

$$\begin{aligned} & \underbrace{i \nabla_\mu \nabla_\beta a_\alpha^{(0)} \nabla^\alpha \phi|_\gamma}_{(1.1)} + \underbrace{i \nabla_\beta a_\alpha^{(0)} \nabla_\mu \nabla^\alpha \phi|_\gamma}_{(1.2)} + \underbrace{i \nabla_\mu a_\alpha^{(0)} \nabla_\beta \nabla^\alpha \phi|_\gamma}_{(1.3)} + \underbrace{i a_\alpha^{(0)} \nabla_\mu \nabla_\beta \nabla^\alpha \phi|_\gamma}_{(1.4)} \\ & + \underbrace{i \nabla_\mu \nabla^\alpha a_\alpha^{(0)} \nabla_\beta \phi|_\gamma}_{(3.1)} + \underbrace{i \nabla^\alpha a_\alpha^{(0)} \nabla_\mu \nabla_\beta \phi|_\gamma}_{(2.1)} - \underbrace{\nabla_\mu a_\alpha^{(1)} \nabla^\alpha \phi \nabla_\beta \phi|_\gamma}_{(3.2)} - \underbrace{a_\alpha^{(1)} \nabla_\mu \nabla^\alpha \phi \nabla_\beta \phi|_\gamma}_{(3.3)} \\ & - \underbrace{a_\alpha^{(1)} \nabla^\alpha \phi \nabla_\mu \nabla_\beta \phi|_\gamma}_{(2.2)}. \end{aligned}$$

We notice that the terms (1.1)–(1.4) sum up to zero due to eq. (6.21e), the terms (2.1) and (2.2) vanish due to eq. (6.21c) and the terms (3.1)–(3.3) cancel due to eq. (6.21d).

For the case $p = 2$, the condition from eq. (6.22c) is trivial by remark 2.1. Similarly on argues for the case $p > 2$, which finishes the proof. $\square$

6.2. Explicit form of the equations to order ω^{-2}

We analyze the equations that we get by requiring

$$\sup_{\tau \in [0, T]} \|\square_g \hat{A}_\alpha - R_\alpha^\beta \hat{A}_\beta\|_{L^2(\Sigma_\tau, g)} = \mathcal{O}(\omega^{-2})$$

more closely. In particular this yields

$$g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi|_\gamma = 0, \quad (6.23)$$

which means that $\nabla_\mu \phi|_\gamma$ is null. Further we obtain

$$g^{\mu\nu} (\nabla_\beta \nabla_\mu \phi) (\nabla_\nu \phi)|_\gamma = 0. \quad (6.24)$$

Higher derivatives of $g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi|_\gamma$ lead to an analogue of eq. (2.29), which in this setting becomes an inhomogeneous *linear* ordinary differential equation along γ , see [Ral83]. Standard ODE theory then guarantees existence and uniqueness of solutions. The term from eq. (6.17b) reduces to

$$(2g^{\mu\nu} \nabla_\mu \phi \nabla_\nu a_\alpha^{(0)} + a_\alpha^{(0)} \square \phi)|_\gamma = 0, \quad (6.25)$$

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which is a transport equation for $a_\alpha^{(0)}$ along the bicharacteristic flow defined by ϕ . The term from eq. (6.17c) gives

$$\left(2ig^{\mu\nu}(\nabla_\mu\phi)(\nabla_\nu a_\alpha^{(1)}) + ia_\alpha^{(1)}\square\phi + \square a_\alpha^{(0)} - R_\alpha{}^\beta a_\beta^{(0)} \right) \Big|_\gamma = 0. \quad (6.26)$$

Note that both eq. (6.25) as well as eq. (6.26) are linear ODEs and can be easily solved analytically. Indeed, let $\gamma(s)$ be a bicharacteristic curve associated to the phase function ϕ , defined by

$$\frac{d\gamma^\mu}{ds} = g^{\mu\nu}\nabla_\nu\phi|_{\gamma(s)}.$$

Then for any component f ,

$$\frac{d}{ds}f(\gamma(s)) = \frac{d\gamma^\mu}{ds}\nabla_\mu f = g^{\mu\nu}(\nabla_\nu\phi)\nabla_\mu f = g^{\mu\nu}(\nabla_\mu\phi)\nabla_\nu f,$$

so that along γ the differential operator $g^{\mu\nu}(\nabla_\mu\phi)\nabla_\nu$ acts as d/ds . With this identification, eq. (6.25) becomes

$$2\frac{d}{ds}a_\alpha^{(0)} + \square\phi a_\alpha^{(0)} = 0,$$

hence

$$a_\alpha^{(0)}(s) = a_\alpha^{(0)}(s_0) \exp\left(-\frac{1}{2}\int_{s_0}^s \square\phi(\tau) d\tau\right). \quad (6.27)$$

Further, eq. (6.26) can be rewritten as

$$\frac{d}{ds}a_\alpha^{(1)} + \frac{1}{2}(\square\phi) a_\alpha^{(1)} = \frac{i}{2}\left(\square a_\alpha^{(0)} - R_\alpha{}^\beta a_\beta^{(0)}\right).$$

Thus

$$a_\alpha^{(1)}(s) = \mathcal{I}(s, s_0) a_\alpha^{(1)}(s_0) + \frac{i}{2} \int_{s_0}^s \mathcal{I}(s, t) \left(\square a_\alpha^{(0)} - R_\alpha{}^\beta a_\beta^{(0)} \right)(t) dt, \quad (6.28)$$

where we have defined

$$\mathcal{I}(s, t) := e^{-\frac{1}{2}\int_t^s \square\phi(\tau) d\tau}.$$

We know by proposition 6.1 that if we want

$$\sup_{\tau \in [0, T]} \|\nabla_\beta \nabla^\alpha \hat{A}_\alpha\|_{L^2(\Sigma_\tau, \mathfrak{g})} = \mathcal{O}(\omega^{-2})$$

to be fulfilled we need that

$$\sup_{\tau \in [0, T]} \|\nabla^\alpha \hat{A}_\alpha\|_{L^2(\Sigma_\tau, \mathfrak{g})} = \mathcal{O}(\omega^{-3}).$$

Using lemma 2.1 we get that

$$\begin{aligned} a_\alpha^{(0)}\nabla^\alpha\phi & \text{ has to vanish to 5th order on } \gamma, \\ \nabla^\alpha a_\alpha^{(0)} + ia_\alpha^{(1)}\nabla^\alpha\phi & \text{ has to vanish to 3rd order on } \gamma, \text{ and} \\ \nabla^\alpha a_\alpha^{(1)} + ia_\alpha^{(2)}\nabla^\alpha\phi & \text{ has to vanish to 1st order on } \gamma. \end{aligned}$$

6.3. The stationary phase approximation for the approximate solutions

In the following we will combine the results obtained previously to approximate observables defined in section 5.1 using the stationary phase approximation from section 5.4. Recall our ansatz

$$\hat{A}_\alpha = \Re \left[\omega^{-1/4} \sum_{j=0}^N \omega^{-j} a_\alpha^{(j)} e^{i\omega\phi} \right].$$

Let

$$\mathfrak{F}_{\alpha\mu} := \nabla_\alpha \mathfrak{A}_\mu - \nabla_\mu \mathfrak{A}_\alpha.$$

Then due to the linearity of $\Re$ we have

$$\hat{F}_{\alpha\mu} = \Re(\mathfrak{F}_{\alpha\mu}).$$

We find

$$\mathfrak{F}_{\alpha\mu} = \omega^{-1/4} e^{i\omega\phi} \left[\nabla_\alpha a_\mu - \nabla_\mu a_\alpha + i\omega \{ (\nabla_\alpha \phi) a_\mu - (\nabla_\mu \phi) a_\alpha \} \right].$$

In order to keep the notation shorter, we define

$$B_{\alpha\mu} := \nabla_\alpha a_\mu - \nabla_\mu a_\alpha + i\omega \{ (\nabla_\alpha \phi) a_\mu - (\nabla_\mu \phi) a_\alpha \}, \quad (6.30)$$

so that $\mathfrak{F}_{\alpha\mu} = \omega^{-1/4} e^{i\omega\phi} B_{\alpha\mu}$. Using that $\Re(z) = \frac{1}{2}(z + \bar{z})$ for any $z \in \mathbb{C}$, we compute

$$\begin{aligned} \hat{F}_{\alpha\mu} \hat{F}_\beta{}^\mu &= \Re(\mathfrak{F}_{\alpha\mu}) \Re(\mathfrak{F}_\beta{}^\mu) = \frac{\omega^{-1/2}}{4} (\mathfrak{F}_{\alpha\mu} + \bar{\mathfrak{F}}_{\alpha\mu})(\mathfrak{F}_\beta{}^\mu + \bar{\mathfrak{F}}_\beta{}^\mu) \\ &= \frac{\omega^{-1/2}}{4} \left[e^{2i\omega\phi} B_{\alpha\mu} B_\beta{}^\mu + e^{i\omega(\phi+\bar{\phi})} B_{\alpha\mu} \bar{B}_\beta{}^\mu + e^{-i\omega(\bar{\phi}-\phi)} \bar{B}_{\alpha\mu} B_\beta{}^\mu + e^{-2i\omega\bar{\phi}} \bar{B}_{\alpha\mu} \bar{B}_\beta{}^\mu \right] \\ &= \frac{\omega^{-1/2}}{4} e^{-2\omega\Im(\phi)} [B_{\alpha\mu} \bar{B}_\beta{}^\mu + \bar{B}_{\alpha\mu} B_\beta{}^\mu] + 2 \cdot \frac{\omega^{-1/2}}{4} \Re(e^{2i\omega\phi} B_{\alpha\mu} B_\beta{}^\mu) \\ &= \frac{\omega^{-1/2}}{2} \left\{ \Re[B_{\alpha\mu} \bar{B}_\beta{}^\mu] e^{-2\omega\Im(\phi)} + \Re[B_{\alpha\mu} B_\beta{}^\mu e^{2i\omega\Re(\phi)}] e^{-2\omega\Im(\phi)} \right\}. \end{aligned} \quad (6.31)$$

Thus the stress-energy tensor is given by

$$\begin{aligned} T_{\alpha\beta}[\hat{F}] &= \frac{\omega^{-1/2}}{2} \left\{ \Re[B_{\alpha\mu} \bar{B}_\beta{}^\mu] e^{-2\omega\Im(\phi)} + \Re[B_{\alpha\mu} B_\beta{}^\mu e^{2i\omega\Re(\phi)}] e^{-2\omega\Im(\phi)} \right\} \\ &\quad + \frac{\omega^{-1/2}}{8} \left\{ g_{\alpha\beta} |B|_g^2 e^{-2\omega\Im(\phi)} + \Re[B_{\mu\nu} B^{\mu\nu}] e^{-2\omega\Im(\phi)} \right\} \\ &= \frac{\omega^{-1/2}}{8} \left\{ 4\Re[B_{\alpha\mu} \bar{B}_\beta{}^\mu] e^{-2\omega\Im(\phi)} - g_{\alpha\beta} |B|_g^2 e^{-2\omega\Im(\phi)} \right\} \end{aligned} \quad (6.32)$$

$$+ \frac{\omega^{-1/2}}{8} \left\{ 4\Re[(B_{\alpha\mu} B_\beta{}^\mu - B_{\mu\nu} B^{\mu\nu}) e^{2i\omega\Re(\phi)}] e^{-2\omega\Im(\phi)} \right\}. \quad (6.33)$$

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Explicitly we have

$$\begin{aligned}
B_{\alpha\mu}\bar{B}_\beta{}^\mu &= (\nabla_\alpha a_\mu)(\nabla_\beta \bar{a}^\mu) - (\nabla_\alpha a_\mu)(\nabla^\mu \bar{a}_\beta) - i\omega(\nabla_\beta \bar{\phi})(\nabla_\alpha a_\mu)\bar{a}^\mu \\
&\quad + i\omega(\nabla^\mu \bar{\phi})(\nabla_\alpha a_\mu)\bar{a}_\beta - (\nabla_\mu a_\alpha)(\nabla_\beta \bar{a}^\mu) + (\nabla_\mu a_\alpha)(\nabla^\mu \bar{a}_\beta) \\
&\quad + i\omega(\nabla_\beta \bar{\phi})(\nabla_\mu a_\alpha)\bar{a}^\mu - i\omega(\nabla^\mu \bar{\phi})(\nabla_\mu a_\alpha)\bar{a}_\beta + i\omega(\nabla_\alpha \phi)(\nabla_\beta \bar{a}^\mu)a_\mu \\
&\quad - i\omega(\nabla_\alpha \phi)(\nabla^\mu \bar{a}_\beta)a_\mu + \omega^2(\nabla_\alpha \phi)(\nabla_\beta \bar{\phi})a_\mu \bar{a}^\mu - \omega^2(\nabla_\alpha \phi)(\nabla^\mu \bar{\phi})a_\mu \bar{a}_\beta \\
&\quad - i\omega(\nabla_\mu \phi)(\nabla_\beta \bar{a}^\mu)a_\alpha + i\omega(\nabla_\mu \phi)(\nabla^\mu \bar{a}_\beta)a_\alpha - \omega^2(\nabla_\mu \phi)(\nabla_\beta \bar{\phi})a_\alpha \bar{a}^\mu \\
&\quad + \omega^2(\nabla_\mu \phi)(\nabla^\mu \bar{\phi})a_\alpha \bar{a}_\beta.
\end{aligned} \tag{6.34}$$

From this follows that

$$\begin{aligned}
|B|_g^2 = B_{\alpha\mu}\bar{B}^{\alpha\mu} &= 2 \left\{ (\nabla_\alpha a_\mu)(\nabla^\alpha \bar{a}^\mu) - (\nabla_\alpha a_\mu)(\nabla^\mu \bar{a}^\alpha) \right. \\
&\quad + i\omega [(\nabla_\alpha a_\mu)(\nabla^\mu \bar{\phi})\bar{a}^\alpha + (\nabla_\alpha \phi)(\nabla^\alpha \bar{a}^\mu)a_\mu - (\nabla_\alpha a_\mu)(\nabla^\alpha \bar{\phi})\bar{a}^\mu \\
&\quad \left. - (\nabla_\alpha \phi)(\nabla^\mu \bar{a}^\alpha)a_\mu] \right. \\
&\quad \left. + \omega^2 [(\nabla_\alpha \phi)(\nabla^\alpha \bar{\phi})a_\mu \bar{a}^\mu - (\nabla_\alpha \phi)(\nabla^\mu \bar{\phi})a_\mu \bar{a}^\alpha] \right\}.
\end{aligned} \tag{6.35}$$

Setting $x \equiv \Gamma_\tau$, we first want to compute

$$\begin{aligned}
p_\alpha(\tau) &= \int_{\Sigma_\tau} K_\alpha{}^{\alpha'}(x, x') T_{\beta'\alpha'}(x') n^{\beta'} d\mu_{\Sigma_\tau}(x') \\
&= \int_{T_x \Sigma_\tau \cong \mathbb{R}^3} T_{\beta'\alpha'}(\exp_x(v)) K_\alpha{}^{\alpha'}(x, \exp_x(v)) \Delta^{-1}(x, \exp_x(v)) n^{\beta'}(\exp_x(v)) d^3v.
\end{aligned}$$

By theorem 5.1 we know that we have

$$\int_{\mathbb{R}^3} u(x) e^{i\omega f(x)} d^3x = \frac{e^{i\omega f(\gamma_\tau)}}{\sqrt{\det\left(\frac{\omega}{2\pi i} A\right)}} \sum_{j < k} \omega^{-j} L_j u(\gamma_\tau) + \mathcal{O}(\omega^{-k}), \tag{6.36}$$

where $A_{\mu\nu} = \nabla_\mu \nabla_\nu f(\gamma_\tau)$, $f = 2i\Im\phi$ and $\gamma_\tau = \gamma(\tau)$ denotes the geodesic along which our Gaussian beam is constructed.

We notice that the term in eq. (6.33) contains a factor of $e^{2i\omega\Re(\phi)}$ and hence decays to an arbitrarily high order in ω . This is justified by the following

Theorem 6.1 (Stationary phase 2, [Hö03, Thm. 7.7.1]). *Let $K \subset \mathbb{R}^n$ be a compact set, X a neighborhood of K and k a non-negative integer. If $u \in C_0^k(K)$ and $f \in C^{k+1}(X, \mathbb{R})$ then*

$$\omega^k \left| \int_{\mathbb{R}^n} u(x) e^{i\omega f(x)} d^n x \right| \leq C \sum_{|\alpha| \leq k} \sup |D^\alpha u| |f'|^{|\alpha|-2k}, \quad \omega > 0. \tag{6.37}$$

Here C is bounded when f stays in a bounded set in $C^{k+1}(X)$.

6.3. The stationary phase approximation for the approximate solutions

For convenience we define

$$\mathcal{T}_{\alpha\beta} := -\frac{1}{8}g_{\alpha\beta} |B|_g^2 + \frac{1}{2}\Re[B_{\alpha\mu}\bar{B}_\beta{}^\mu]. \quad (6.38)$$

Then, working in local coordinates, the corresponding integrand for the linear momentum is

$$u_\mu(x, \exp_x(v)) = \omega^{-1/2} \mathcal{T}_{\alpha'\beta'}(\exp_x(v)) n_{\exp_x(v)}^{\alpha'} K_\mu{}^{\beta'}(x, \exp_x(v)) \Delta^{-1}(x, \exp_x(v)). \quad (6.39)$$

For the angular momentum we have as integrand

$$u^{\mu\nu}(x, \exp_x(v)) = \omega^{-1/2} \mathcal{T}_{\alpha'\beta'}(\exp_x(v)) n_{\exp_x(v)}^{\alpha'} \sigma^{[\nu} H^{\mu]\beta'}(x, \exp_x(v)) \Delta^{-1}(x, \exp_x(v)). \quad (6.40)$$

In what follows we will assume that the point of reference x with respect to which the observables are defined is close to the point of stationary phase γ_τ . Hence, as discussed in section 4.4, it makes sense to expand the bitensors from eq. (6.39) and eq. (6.40) near coincidence. This will give a generalized Taylor series in powers of $-\sigma(x, \gamma_\tau)$.

Remark 6.2. Comparing with a similar analysis of electromagnetic wave packets propagating in an inhomogeneous medium [COS26], one expects that the Synge separation vector σ^α between the point of stationary phase and the point on the center of mass worldline with respect to which S and p are defined is of order ω^{-1} . In curved spacetime this fact is not as immediate since the center of energy is only implicitly defined by requiring that $t_\alpha S^{\alpha\beta} = 0$ for some timelike vector field t . We will give a proof here in the Minkowski case. Suppose that $x(\tau)$ is a point on the center of energy worldline Γ and γ_τ is the point of stationary phase evaluation. Then by eq. (5.23) we know that

$$S^{\alpha\beta}(x) = S^{\alpha\beta}(\gamma_\tau) + 2\delta x^{[\beta} p^{\alpha]},$$

where $\delta x = x - \gamma_\tau$. Contracting with t gives

$$0 = t_\alpha S^{\alpha\beta}(x) = t_\alpha S^{\alpha\beta}(\gamma_\tau) + 2t_\alpha \delta x^{[\beta} p^{\alpha]}. \quad (6.41)$$

Applying stationary phase to $S^{\alpha\beta}(\gamma)$ yields

$$S^{\alpha\beta}(\gamma_\tau) = 2(\gamma_\tau - \gamma_\tau)^{[\beta} \eta^{\alpha]\alpha'} T_{\alpha'}^0(\gamma_\tau) + \mathcal{O}(\omega^{-1}),$$

i.e. $S^{\alpha\beta}(\gamma_\tau) = \mathcal{O}(\omega^{-1})$. On the other hand if we apply stationary phase to the second term in eq. (6.41), we simply have to evaluate p^α on γ_τ . Note that at leading order in the stationary phase approximation we have a factor of $\omega^{-3/2}$ coming from the determinant term which cancels the order of T , which is of order $\omega^{3/2}$ by construction. So, in order for eq. (6.41) to be satisfied, we need that δx^β has to be order ω^{-1} .

Lemma 6.1. In our situation we have

$$\sigma^\mu(\gamma_\tau, x) = \mathcal{O}(\omega^{-1}). \quad (6.42)$$

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Proof. In the following, unprimed indices will always refer to the first argument of bitensors, while primed indices will refer to the second argument. Let $u^\mu := \sigma^\mu(\gamma_\tau, x) \in T_{\gamma_\tau}\Sigma_\tau$. For fixed x' within the support of our Gaussian beam we expand $\sigma^\alpha(x, x')$ around $x = \gamma_\tau$ as follows

$$\sigma^\alpha(x, x') = J^\alpha_\mu(x, \gamma_\tau) [\sigma^\mu(\gamma_\tau, x') + \sigma^\mu_{\nu}(\gamma_\tau, x') u^\nu + \frac{1}{2} \sigma^\mu_{\nu\rho}(\gamma_\tau, x') u^\nu u^\rho + \mathcal{O}(u^3)]$$

From eq. (4.16) we obtain that

$$\sigma^\mu_{\nu}(\gamma_\tau, x') = J^\mu_{\mu'}(\gamma_\tau, x') J_{\nu}^{\nu'}(\gamma_\tau, x') \left(\delta_{\nu'}^{\mu'} - \frac{1}{3} R^{\mu'}_{\lambda'\nu'\delta'} v^{\lambda'} v^{\delta'} + \mathcal{O}(Rv^3) \right),$$

with $v^{\mu'} := \sigma^{\mu'}(\gamma_\tau, x') \in T_{x'}\Sigma_\tau$. Further, by the fact that the coincidence limit of any third derivative of the Synge world function vanishes, we have

$$\sigma^\mu_{\nu\rho}(\gamma_\tau, x') = 0 + \mathcal{O}(Rv).$$

From this follows that

$$\sigma^\alpha(x, x') = J^\alpha_\mu(x, \gamma_\tau) \sigma^\mu(\gamma_\tau, x') + J^\alpha_\mu(x, \gamma_\tau) \sigma^\mu(\gamma_\tau, x) + \mathcal{O}(Rv^2u, Rvu^2).$$

Further we have

$$H^{\alpha\alpha'}(x, x') = [-\nabla_{\alpha'} \sigma_\alpha(x, x')]^{-1} = J^\alpha_\mu(x, \gamma_\tau) H^{\mu\alpha'}(\gamma_\tau, x') + \mathcal{O}(u).$$

Thus we obtain

$$\begin{aligned} S^{\alpha\beta}(x) &= 2 \int_{\Sigma_\tau} \left[J^{[\beta}_\mu(x, \gamma_\tau) \sigma^\mu(\gamma_\tau, x') + J^{[\beta}_\mu(x, \gamma_\tau) \sigma^\mu(\gamma_\tau, x) + \mathcal{O}(Rv^2u, Rvu^2, u^3) \right] \\ &\quad \cdot \left[J^{\alpha]}_\sigma(x, \gamma_\tau) H^{\sigma\alpha'}(\gamma_\tau, x') + \mathcal{O}(u) \right] T^{\beta'}_{\alpha'}(x') dS_{\beta'} \\ &= J^{[\beta}_\mu(x, \gamma_\tau) J^{\alpha]}_\sigma(x, \gamma_\tau) \int_{\Sigma_\tau} \sigma^\mu(\gamma_\tau, x') H^{\sigma\alpha'}(\gamma_\tau, x') T^{\beta'}_{\alpha'}(x') dS_{\beta'} \\ &\quad + 2 J^{[\beta}_\mu(x, \gamma_\tau) \sigma^\mu(\gamma_\tau, x) J^{\alpha]}_\sigma(x, \gamma_\tau) \int_{\Sigma_\tau} H^{\sigma\alpha'}(\gamma_\tau, x') T^{\beta'}_{\alpha'}(x') dS_{\beta'}(x') \\ &\quad + \mathcal{O}(Rv^2u^2, Ru^3v) \\ &= J^{[\beta}_\mu(x, \gamma_\tau) J^{\alpha]}_\sigma(x, \gamma_\tau) S^{\sigma\mu}(\gamma_\tau) \\ &\quad + 2u^\mu J^{[\beta}_\mu(x, \gamma_\tau) J^{\alpha]}_\sigma(x, \gamma_\tau) \int_{\Sigma_\tau} H^{\sigma\alpha'}(\gamma_\tau, x') T^{\beta'}_{\alpha'}(x') dS_{\beta'}(x') \\ &\quad + \mathcal{O}(Rv^2u^2, Ru^3v). \end{aligned} \tag{6.43}$$

Similarly to the Minkowski case discussed in remark 6.2 we have that $S^{\alpha\beta}(\gamma_\tau) = \mathcal{O}(\omega^{-1})$. Contracting with a timelike vector t we can rewrite eq. (6.43) as

$$F(\epsilon, u) := C_1\epsilon + C_2u + f(u) = 0,$$

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with appropriate constants that do not depend on $\epsilon := \omega^{-1}$ and a function f that encodes the u -dependence of the higher order terms. Note that $f(0) = 0$. We can also assume that $f'(u)|_{u=0} \neq -1$. We thus have

$$\left. \frac{\partial F}{\partial u} \right|_{(0,0)} = 1 + f'(u) \neq 0.$$

Now due to the implicit function theorem, locally, there exists a smooth function ϕ such that $\phi(0) = 0$ and such that $F(\epsilon, u = \phi(\epsilon)) = 0$. From this follows that

$$u = \phi(\epsilon) = 0 + \phi'(0)\epsilon + \mathcal{O}(\epsilon^2) = \mathcal{O}(\epsilon),$$

which concludes the proof. $\square$

Remark 6.3. As a consistency check, we analyze eq. (6.43) in flat Minkowski space. Recall that in this case parallel transport is trivial and $H^{\alpha\alpha'} \rightarrow \eta^{\alpha\alpha'}$. Choosing the standard Minkowski observer $n_\beta = (1, 0, 0, 0)$ and a constant-time hypersurface $\{t = \tau\} \cong \mathbb{R}^3$ we obtain

$$\begin{aligned} S^{\alpha\beta}(x) &= S^{\alpha\beta}(\gamma_\tau) + 2(x - \gamma_\tau)^{[\beta} \int_{\Sigma_\tau} \eta^{\alpha]\alpha'} T^{\beta'}_{\alpha'}(x') dS_{\beta'} \\ &= S^{\alpha\beta}(\gamma_\tau) + 2(x - \gamma_\tau)^{[\beta} \int_{\mathbb{R}^3} T^{\alpha]\beta'}(x') n_{\beta'} d^3x' \\ &= S^{\alpha\beta}(\gamma_\tau) + 2(x - \gamma_\tau)^{[\beta} \int_{\mathbb{R}^3} T^{\alpha]0}(x) d^3x \\ &= S^{\alpha\beta}(\gamma_\tau) + 2(x - \gamma_\tau)^{[\beta} p^{\alpha]}, \end{aligned}$$

which is consistent with eq. (5.23).

Now, we have a look at the expansion of the bitensors that appear in the definition of the linear and angular momentum. Using eq. (4.15) we find

$$\begin{aligned} H^{\alpha\alpha'}(x, x') &= [-\nabla_{\alpha'} \sigma_\alpha]^{-1} = [-\sigma_{\alpha'\alpha}]^{-1} \\ &= \left[J^{\beta'}_{\alpha} \left(g_{\alpha'\beta'} + \frac{1}{6} R_{\alpha'\gamma'\beta'\delta'} \sigma^{\gamma'} \sigma^{\delta'} + \mathcal{O}(\sigma^3) \right) \right]^{-1} \\ &= J^{\alpha}_{\beta'} \left(g^{\alpha'\beta'} - \frac{1}{6} R^{\alpha'}_{\gamma'}{}^{\beta'}_{\delta'} \sigma^{\gamma'} \sigma^{\delta'} \right) + \mathcal{O}(R^2 \sigma^4). \end{aligned} \quad (6.44)$$

For the inversion operation we used the following: suppose we are given a metric g that is perturbed by some metric h . Hence we can write

$$\tilde{g}_{ab} = g_{ab} + h_{ab}.$$

We want to find k^{ab} such that

$$(g^{ac} + k^{ac})(g_{cb} + h_{cb}) = \delta_b^a.$$

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Neglecting the kh term, we find that

$$k^{ab} = -g^{ac}g^{bd}h_{cd} + \mathcal{O}(hk).$$

For the second bitensor, using eq. (4.16), we find that

$$K_{\alpha}{}^{\alpha'} = H^{\beta\alpha'}\sigma_{\beta\alpha} = \delta_{\alpha}^{\alpha'} - \frac{1}{2}R^{\alpha'}{}_{\gamma'\alpha\delta'}\sigma^{\gamma'}\sigma^{\delta'} + \mathcal{O}(R^2\sigma^4), \quad (6.45)$$

where we have used the pair-symmetry of the Riemann tensor. Recall from eq. (4.17), that for the expansion of the van-Vleck Morette determinant we have

$$\Delta = 1 + \frac{1}{6}R_{\alpha'\beta'}\sigma^{\alpha'}\sigma^{\beta'} + \mathcal{O}(\sigma^3),$$

which yields the following expansion of the inverse

$$\Delta^{-1} = 1 - \frac{1}{6}R_{\alpha'\beta'}\sigma^{\alpha'}\sigma^{\beta'} + \mathcal{O}(R^2\sigma^4).$$

So, altogether we have

$$K_{\alpha}{}^{\alpha'}\Delta^{-1} = \delta_{\alpha}^{\alpha'} - \frac{1}{2}\left(R^{\alpha'}{}_{\gamma'\alpha\delta'} + \frac{1}{3}\delta_{\alpha}^{\alpha'}R_{\gamma'\delta'}\right)\sigma^{\gamma'}\sigma^{\delta'} + \mathcal{O}(\sigma^3).$$

The expression appearing in the definition of the angular momentum is given by

$$\begin{aligned} 2\sigma^{[\beta}H^{\alpha]}\Delta^{-1} &= \left(g^{\alpha\alpha'}\sigma^{\beta} - g^{\beta\alpha'}\sigma^{\alpha}\right)\left(1 - \frac{1}{6}R_{\gamma'\delta'}\sigma^{\gamma'}\sigma^{\delta'}\right) \\ &\quad - \frac{1}{6}\left(R^{\alpha'}{}_{\gamma'}{}^{\alpha}{}_{\delta'}\sigma^{\beta} - R^{\alpha'}{}_{\gamma'}{}^{\beta}{}_{\delta'}\sigma^{\alpha}\right)\sigma^{\gamma'}\sigma^{\delta'} + \mathcal{O}(\sigma^5). \end{aligned}$$

Applying now stationary phase yields

$$p_{\mu}(\tau) = \frac{\omega^{-3/2}}{\sqrt{\det\left(\frac{1}{2\pi i}A\right)}}\left(u_{\mu} + \omega^{-1}L_1u_{\mu}\right)\Big|_{v=\gamma_{\tau}-x} + \mathcal{O}(\omega^{-2}) \quad (6.46)$$

and

$$S^{\mu\nu}(\tau) = \frac{\omega^{-3/2}}{\sqrt{\det\left(\frac{1}{2\pi i}A\right)}}\left(u^{\mu\nu} + \omega^{-1}L_1u^{\mu\nu}\right)\Big|_{v=\gamma_{\tau}-x} + \mathcal{O}(\omega^{-2}). \quad (6.47)$$

First note that u_{μ} and $u^{\mu\nu}$ itself are of order $\omega^{3/2}$, since $\mathcal{T}_{\alpha\beta}$ is of order ω^2 . Thus, the prefactor $\omega^{-3/2}$ coming from the stationary phase gives that the first term in both the p_{μ} and $S^{\mu\nu}$ is of order 0 in ω . Also note that the evaluation of the above expression is at the vector

$$\gamma_{\tau} - x = \exp_x^{-1}(\gamma_{\tau}).$$

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This comes from the fact that we are integrating an expression of the form $u(x, \exp_x(v))$ with respect to v , which evaluates to

$$u(x, \exp_x(v)) \Big|_{v=\gamma_\tau-x} = u(x, \exp_x(\exp_x^{-1}(\gamma_\tau))) = u(x, \gamma_\tau),$$

as desired. Note that $u(x, \exp_x(v))$ is always a scalar at $\exp_x(v)$ since all primed indices are contracted, and it is a 1- or 2-form at $x \equiv \Gamma_\tau$ in the case of linear or angular momentum. Recall that derivatives of the form

$$\frac{\partial}{\partial v^{\alpha'}} u(x, \exp_x(v)) \Big|_{v=\gamma_\tau-x}$$

may be evaluated by introducing the vector $w = x - \gamma_\tau + v$ which yields

$$\frac{\partial}{\partial v^{\alpha'}} u(x, \exp_x(v)) \Big|_{v=\gamma_\tau-x} = \frac{\partial w^{\beta'}}{\partial v^{\alpha'}} \frac{\partial}{\partial w^{\beta'}} u(x, \exp_{\gamma_\tau}(w)) \Big|_{w=0}.$$

Thus we can apply lemma 5.1 to find that

$$\frac{\partial}{\partial w^{\lambda'}} u(x, \exp_{\gamma_\tau}(w)) \Big|_{w=0} = \nabla_{\lambda'} u(x, \gamma_\tau). \quad (6.48)$$

At zeroth order in ω we have for the linear momentum

$$p_\mu^{(0)}(\tau) = \frac{\omega^{-2}}{\sqrt{\det(\frac{1}{2\pi i} A)}} \frac{1}{2} \Re \left[B_{\alpha'\sigma'} \bar{B}_{\beta'}{}^{\sigma'} - \frac{1}{4} |B|_g^2 g_{\alpha'\beta'} \right] n^{\alpha'} \delta_\mu^{\beta'} \Big|_{\gamma_\tau}. \quad (6.49)$$

From eq. (6.34) we see that there are four terms of order ω^2 . After writing $a_\mu = a_\mu^{(0)} + \mathcal{O}(\omega^{-1})$, we note that the term

$$(\nabla_\mu \phi) (\nabla^\mu \bar{\phi}) a_\alpha^{(0)} \bar{a}_\beta^{(0)} \Big|_{\gamma_\tau}$$

vanishes, since $\nabla^\mu \phi|_\gamma$ is real-valued and ϕ satisfies the eikonal equation along γ . Further, the two terms of the form

$$(\nabla_\alpha \phi) (\nabla^\mu \bar{\phi}) a_\mu^{(0)} \bar{a}_\beta^{(0)} \Big|_{\gamma_\tau}$$

also vanish, since $\nabla_\mu \phi \bar{a}^{(0)\mu} = 0 = \nabla^\mu \bar{\phi} a_\mu^{(0)}$ by eq. (6.21a) as well as remark 6.1. Moreover, from eq. (6.35) there are two terms at order ω^2 , that vanish due to the above arguments as well. Thus, we record that

$$\mathcal{T}_{\alpha\beta} \Big|_{\gamma_\tau} = \frac{\omega^2}{2} (\nabla_\alpha \phi) (\nabla_\beta \phi) |a^{(0)}|_g^2 \Big|_{\gamma_\tau} + \mathcal{O}(\omega^1). \quad (6.50)$$

So, what remains is

$$\begin{aligned} p_\mu^{(0)}(\tau) &= \frac{1}{\sqrt{\det(\frac{1}{2\pi i} A)}} \frac{1}{2} (\nabla_\alpha \phi) (\nabla_\beta \phi) a_\nu^{(0)} \bar{a}^{(0)\nu} n^\alpha \delta_\mu^\beta \Big|_{\gamma_\tau} \\ &= \frac{1}{2\sqrt{\det(\frac{1}{2\pi i} A)}} |a^{(0)}|_g^2 (n \cdot k)_g k_\mu \Big|_{\gamma_\tau}. \end{aligned} \quad (6.51)$$

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For the angular momentum at highest order we have

$$\begin{aligned}
S^{(0)\mu\nu}(\tau) &= \frac{\omega^{-3/2}}{\sqrt{\det\left(\frac{1}{2\pi i}A\right)}} \omega^{-1/2} \frac{1}{2} \Re \left[B_{\alpha'\sigma'} \bar{B}_{\beta'}{}^{\sigma'} - \frac{1}{4} |B|_g^2 g_{\alpha'\beta'} \right] n^{\alpha'} (g^{\mu\beta'} \sigma^\nu - g^{\nu\beta'} \sigma^\mu) \Big|_{\gamma_\tau} \\
&= \frac{1}{2\sqrt{\det\left(\frac{1}{2\pi i}A\right)}} \Re \left[(\nabla_{\alpha'} \phi)(\nabla_{\beta'} \phi) |a^{(0)}|_g^2 n^{\alpha'} (g^{\mu\beta'} \sigma^\nu - g^{\nu\beta'} \sigma^\mu) \right] \Big|_{\gamma_\tau} \\
&= \frac{1}{2\sqrt{\det\left(\frac{1}{2\pi i}A\right)}} |a^{(0)}|_g^2 \Re \left[(n \cdot k)_g k^\mu \sigma^\nu - (n \cdot k)_g k^\nu \sigma^\mu \right] \Big|_{\gamma_\tau} \\
&= \frac{1}{\sqrt{\det\left(\frac{1}{2\pi i}A\right)}} |a^{(0)}|_g^2 (n \cdot k)_g k^{[\mu} \sigma^{\nu]} \Big|_{\gamma_\tau}. \tag{6.52}
\end{aligned}$$

6.4. Energy estimate

Our aim is to find an estimate for the energy of the difference between approximate and exact solution. To do that we have to first recall the notion of energy that we will use. The stress-energy tensor in the Maxwell case is given by

$$T_{\alpha\beta}[F] = F_{\alpha\mu} F_{\beta}{}^{\mu} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} g_{\alpha\beta}.$$

The stress-energy current associated with a future directed timelike vector field u^α is given by

$$(J^u)_\alpha = u^\mu T_{\mu\alpha}.$$

Then, as in section 2.3, the u -energy is defined by

$$E^u(\tau) = \int_{\Sigma_\tau} (J^u)_\alpha n^\alpha d\mu_{\Sigma_\tau}, \tag{6.53}$$

where n denotes the unit normal vector field to Σ_τ .

Proposition 6.2. *The u -energy is conserved if and only if u is conformally Killing.*

Proof. We calculate

$$\begin{aligned}
\nabla^\alpha (J^u)_\alpha &= \nabla^\alpha (u^\mu T_{\mu\alpha}) = (\nabla^\alpha u^\mu) T_{\mu\alpha} + u^\mu (\nabla^\alpha T_{\mu\alpha}) \\
&= (\nabla^\alpha u^\mu) T_{\mu\alpha},
\end{aligned}$$

since we know that $\nabla^\alpha T_{\mu\alpha} = 0$. Using the symmetry of T we can write

$$\nabla^\alpha (J^u)_\alpha = \frac{1}{2} T^{\mu\alpha} (\nabla_\alpha u_\mu + \nabla_\mu u_\alpha). \tag{6.54}$$

6.4. Energy estimate

Consider now the spacetime region $\mathcal{D} \subset M$ bounded by two Cauchy slices Σ_1 and Σ_2 . Assuming that F has compact support on each Σ_τ and using Stokes' theorem, we can write

$$\int_{\mathcal{D}} \nabla_\alpha (J^u)^\alpha d\mu_{\mathcal{D}} = \int_{\Sigma_2} (J^u)^\alpha n_\alpha d\mu_{\Sigma_2} - \int_{\Sigma_1} (J^u)^\alpha n_\alpha d\mu_{\Sigma_1} = E^u(\tau_2) - E^u(\tau_1). \quad (6.55)$$

Thus the u -energy is conserved if and only if $\nabla_\alpha (J^u)^\alpha = 0$.

Suppose now that u is conformally Killing, i.e.

$$\nabla_\alpha u_\mu + \nabla_\mu u_\alpha = \lambda g_{\alpha\mu}$$

for some scalar λ . Then by eq. (6.54) and the fact that T is trace-free, we find

$$\nabla^\alpha (J^u)_\alpha = \frac{\lambda}{2} T^{\mu\alpha} g_{\mu\alpha} = 0,$$

i.e. the u -energy is conserved. Conversely, if the u -energy is conserved, then again by eq. (6.54) we must have

$$T^{\mu\alpha}[F](\nabla_\alpha u_\mu + \nabla_\mu u_\alpha) = 0$$

for all fields F . Introduce the tensor $U_{\alpha\beta} := \nabla_\alpha u_\beta + \nabla_\beta u_\alpha$ and decompose it as

$$U_{\alpha\mu} = \left(U_{\alpha\mu} - \frac{1}{4} g_{\alpha\mu} (\text{tr } U) \right) + \frac{1}{4} g_{\alpha\mu} (\text{tr } U),$$

where the first term is the trace-free part of U , which we denote by $\tilde{U}$. Contracting U with T now gives

$$T^{\mu\alpha}[F] \tilde{U}_{\mu\alpha} = 0 \quad \text{for all } F.$$

We conclude that

$$\tilde{U}_{\mu\alpha} = 0 \Leftrightarrow \nabla_\alpha u_\mu + \nabla_\mu u_\alpha = \frac{1}{4} (\text{tr } U) g_{\alpha\mu},$$

which shows that u is conformally Killing. □

We now consider the difference of the approximate and exact solution and introduce the notation

$$\tilde{F} := F - \hat{F}, \quad (6.56)$$

where $F = dA$ and $\hat{F} = d\hat{A}$ such that

$$\nabla^\alpha \tilde{F}_{\alpha\beta} = f_\beta. \quad (6.57)$$

6. Approximate solutions to Maxwell's equations in curved spacetime

Lemma 6.2. *Using the notation introduced above we have*

$$\nabla_\alpha T^{\alpha\beta}[\tilde{F}] = \tilde{F}^{\beta\mu} f_\mu, \quad (6.58a)$$

$$\nabla_\alpha (J^u)^\alpha = \tilde{F}^{\alpha\beta} u_\alpha f_\beta + T^{\alpha\beta}[\tilde{F}] \nabla_{(\alpha} u_{\beta)}. \quad (6.58b)$$

Proof. We compute

$$\begin{aligned} \nabla_\alpha T^{\alpha\beta} &= \nabla_\alpha (\tilde{F}^\alpha_\mu \tilde{F}^{\beta\mu} - \frac{1}{4} g^{\alpha\beta} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu}) \\ &= f_\mu \tilde{F}^{\beta\mu} + \tilde{F}^\alpha_\mu (\nabla_\alpha \tilde{F}^{\beta\mu}) - \frac{1}{2} \tilde{F}^{\mu\nu} (\nabla^\beta \tilde{F}_{\mu\nu}). \end{aligned}$$

Now using the fact that $\nabla_{[\mu} \tilde{F}_{\alpha\beta]} = 0$ as well as the antisymmetry of F we find that

$$\tilde{F}^{\mu\alpha} \nabla_\mu \tilde{F}_{\alpha\beta} = -\frac{1}{2} \tilde{F}^{\mu\alpha} \nabla_\beta \tilde{F}_{\mu\alpha},$$

which yields eq. (6.58a). The second claim is a direct consequence of the first, noting that T is symmetric. $\square$

Since also $\tilde{F}$ has compact spatial support, from the proof of proposition 6.2 we know that

$$\int_{\mathcal{D}} \nabla_\alpha (J^u)^\alpha d\mu_{\mathcal{D}} = \int_{\Sigma_2} (J^u)^\alpha n_\alpha d\mu_{\Sigma_2} - \int_{\Sigma_1} (J^u)^\alpha n_\alpha d\mu_{\Sigma_1} = E^u(\tau_2) - E^u(\tau_1), \quad (6.59)$$

where $\mathcal{D} \subset M$ denotes the spacetime region bounded by the two Cauchy slices Σ_1 and Σ_2 . Based on the chosen foliation of M , we can write the spacetime volume element restricted to $\mathcal{D}$ as

$$d\mu_{\mathcal{D}} = N(\tau, x^i) d\tau d\mu_{\Sigma_\tau}(x^i),$$

where N is the lapse function. Using now eq. (6.59) as well as the fundamental theorem of calculus we obtain

$$\int_{\tau_1}^{\tau_2} \frac{d}{d\tau} E^u(\tau) d\tau = \int_{\tau_1}^{\tau_2} \int_{\Sigma_\tau} N \nabla_\alpha (J^u)^\alpha d\mu_{\Sigma_\tau} d\tau. \quad (6.60)$$

From lemma 6.2 we conclude that

$$\frac{d}{d\tau} E^u(\tau) = \int_{\Sigma_\tau} N \nabla_\alpha (J^u)^\alpha d\mu_{\Sigma_\tau} = \int_{\Sigma_\tau} N \left(\tilde{F}^{\alpha\beta} u_\alpha f_\beta + T^{\alpha\beta}[\tilde{F}] \nabla_{(\alpha} u_{\beta)} \right) d\mu_{\Sigma_\tau}. \quad (6.61)$$

We estimate both terms of the above integral by

$$\begin{aligned} \left| \int_{\Sigma_\tau} N \tilde{F}^{\alpha\beta} u_\alpha f_\beta d\mu_{\Sigma_\tau} \right| &\leq \int_{\Sigma_\tau} N |\tilde{F}^{\alpha\beta} u_\alpha|_{\mathfrak{g}} |f_\beta|_{\mathfrak{g}} d\mu_{\Sigma_\tau} \\ &\leq \|N\|_{L^\infty(\Sigma_\tau, \mathfrak{g})} \|\tilde{F}^{\alpha\beta} u_\alpha\|_{L^2(\Sigma_\tau, \mathfrak{g})} \|f_\beta\|_{L^2(\Sigma_\tau, \mathfrak{g})} \end{aligned}$$

and

$$\begin{aligned} \left| \int_{\Sigma_\tau} N T^{\alpha\beta}[\tilde{F}] \nabla_{(\alpha} u_{\beta)} d\mu_{\Sigma_\tau} \right| &\leq \int_{\Sigma_\tau} N |T^{\alpha\beta}[\tilde{F}]|_{\mathfrak{g}} |\nabla_{(\alpha} u_{\beta)}|_{\mathfrak{g}} d\mu_{\Sigma_\tau} \\ &\leq \|N\|_{L^\infty(\Sigma_\tau, \mathfrak{g})} \|\nabla_{(\alpha} u_{\beta)}\|_{L^\infty(\Sigma_\tau, \mathfrak{g})} \int_{\Sigma_\tau} |T^{\alpha\beta}[\tilde{F}]|_{\mathfrak{g}} d\mu_{\Sigma_\tau}. \end{aligned}$$

Proposition 6.3. *Suppose that u and n are consistently oriented, then we have*

$$\|\tilde{F}^{\alpha\beta} u_\alpha\|_{L^2(\Sigma_\tau, \mathfrak{g})} \leq C_1(\tau, u) \sqrt{E^u(\tau)} \quad (6.62)$$

and

$$\int_{\Sigma_\tau} |T^{\alpha\beta}[\tilde{F}]|_{\mathfrak{g}} d\mu_{\Sigma_\tau} \leq C_2(\tau, u) E^u(\tau). \quad (6.63)$$

Proof. We have

$$\|\tilde{F}^{\alpha\beta} u_\alpha\|_{L^2(\Sigma_\tau, \mathfrak{g})}^2 = \int_{\Sigma_\tau} \mathfrak{g}_{\mu\nu} \tilde{F}^{\alpha\mu} \tilde{F}^{\beta\nu} u_\alpha u_\beta d\mu_{\Sigma_\tau}.$$

We estimate

$$|\mathfrak{g}_{\mu\nu} \tilde{F}^{\alpha\mu} \tilde{F}^{\beta\nu} u_\alpha u_\beta| \leq |g_{\mu\nu} \tilde{F}^{\alpha\mu} \tilde{F}^{\beta\nu} u_\alpha u_\beta| + 2|n_\mu n_\nu \tilde{F}^{\alpha\mu} \tilde{F}^{\beta\nu} u_\alpha u_\beta|. \quad (6.64)$$

Recall that the energy-momentum tensor with respect to $\tilde{F}$ is given by

$$T_{\alpha\beta}[\tilde{F}] = \tilde{F}_{\alpha\mu} \tilde{F}_\beta{}^\mu - \frac{1}{4} g_{\alpha\beta} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu}.$$

From now on we will simply write $T_{\alpha\beta}$ instead of $T_{\alpha\beta}[\tilde{F}]$. From this follows that

$$\varepsilon(u) := T_{\alpha\beta} n^\alpha u^\beta = \tilde{F}_{\alpha\mu} \tilde{F}_\beta{}^\mu n^\alpha u^\beta - \frac{1}{4} (n \cdot u)_g \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} \geq 0,$$

where we have used that T satisfies the dominant energy condition. We decompose u into normal and tangential parts to the Cauchy slice as

$$u^\mu = a n^\mu + v^\mu \quad \text{with} \quad n_\mu v^\mu = 0 \quad \text{and} \quad a = -g_{\mu\nu} u^\mu n^\nu.$$

We define the electric field E and the magnetic field B relative to n by

$$E^\alpha := \tilde{F}^{\alpha\mu} n_\mu, \quad B^\alpha := (\star \tilde{F})^{\alpha\mu} n_\mu,$$

and we notice that $E^\alpha n_\alpha = 0 = B^\alpha n_\alpha$. In what follows, we denote the scalar product and norm induced by h simply by “ $\cdot$ ” and $|\cdot|$, respectively. The components of T are given by

$$\begin{aligned} T(n, n) &= \frac{1}{2} (|E|^2 + |B|^2), \\ T(n, v) &= (E \times B) \cdot v, \\ T(v, v) &= \frac{1}{2} (|E|^2 + |B|^2) |v|^2 - (v \cdot B)^2 - (v \cdot E)^2. \end{aligned}$$

6. Approximate solutions to Maxwell's equations in curved spacetime

Since u is timelike and consistently oriented with n we have

$$g(u, u) = g(an + v, an + v) = -a^2 + |v|^2 < 0 \Rightarrow a^2 > |v|^2 \Rightarrow a > |v|.$$

We compute

$$T(u, u) = T(an + v, an + v) = a^2 T(n, n) + 2aT(n, v) + T(v, v) \quad (6.65)$$

and

$$\varepsilon(u) = T(u, n) = T(an + v, n) = aT(n, n) + T(v, n). \quad (6.66)$$

Since $T(n, v) = (E \times B) \cdot v$ we have

$$|T(n, v)| \leq |E||B||v| \leq \frac{1}{2}(|E|^2 + |B|^2)|v| = T(n, n)|v|. \quad (6.67)$$

Also from $T(v, v) = \frac{1}{2}(|E|^2 + |B|^2)|v|^2 - (v \cdot E)^2 - (v \cdot B)^2$ we have

$$|T(v, v)| \leq T(n, n)|v|^2 + |v|^2(|E|^2 + |B|^2) = 3|v|^2 T(n, n). \quad (6.68)$$

So, using eq. (6.66) and eq. (6.65) as well as eq. (6.67) and eq. (6.68) we have

$$\begin{aligned} |T(u, u)| &= |a^2 T(n, n) + 2aT(n, v) + T(v, v)| \\ &\leq (a^2 + 2a|v| + 3|v|^2)T(n, n). \end{aligned}$$

Moreover it holds that

$$\varepsilon(u) \geq aT(n, n) - |T(n, v)| \geq (a - |v|)T(n, n),$$

from which follows that

$$T(n, n) \leq \frac{\varepsilon(u)}{a - |v|}, \quad \text{and} \quad |T(u, u)| \leq \frac{a + 2a|v| + 3|v|^2}{(a - |v|)} \varepsilon(u). \quad (6.69)$$

Also, we have that

$$F^2 = 2(|B|^2 - |E|^2) \leq 2(|E|^2 + |B|^2) = 4T(n, n). \quad (6.70)$$

Now, for the first term in eq. (6.64) we can use eq. (6.69) and eq. (6.70) to estimate

$$\begin{aligned} |g_{\mu\nu} \tilde{F}^{\alpha\mu} \tilde{F}^{\beta\nu} u_\alpha u_\beta| &= |\tilde{F}_{\alpha\mu} \tilde{F}_\beta{}^\mu u^\alpha u^\beta| = |T(u, u) + \frac{1}{4}(u \cdot u)_g F^2| \\ &\leq c(u)\varepsilon(u) + |(u \cdot u)_g| T(n, n) \\ &\leq \tilde{c}(u)\varepsilon(u). \end{aligned}$$

For the second term in eq. (6.64) we have

$$\begin{aligned} 2|n_\mu n_\nu \tilde{F}^{\alpha\mu} \tilde{F}^{\beta\nu} u_\alpha u_\beta| &= 2|(u \cdot E)_g|^2 = 2|(u \cdot E)_g|^2 \\ &\leq 2|u|_g^2 |E|_g^2 \leq 4|u|_g^2 T(n, n) \\ &\leq c''(u) \varepsilon(u), \end{aligned}$$

where we have used the Cauchy-Schwarz inequality for the Riemannian metric g as well as $|E|_g^2 = |E|^2 \leq |E|^2 + |B|^2 = 2T(n, n)$. So all together we have

$$\|\tilde{F}^{\alpha\beta} u_\alpha\|_{L^2(\Sigma_\tau, g)}^2 \leq c_1(u, \tau) \int_{\Sigma_\tau} \varepsilon(u) d\Sigma_\tau$$

which gives the first estimate from eq. (6.62). For the second estimate we chose an orthonormal frame (n, e_1, e_2, e_3) such that we have

$$|T^{\alpha\beta}[\tilde{F}]|_g^2 = |T(n, n)|^2 + 2 \sum_{i=1}^3 |T(n, e_i)|^2 + \sum_{i,j=1}^3 |T(e_i, e_j)|^2.$$

From eq. (6.67) we have

$$|T(n, e_i)|^2 \leq T(n, n)^2 \leq \bar{c}(u) \varepsilon(u)^2,$$

while from eq. (6.68) we obtain

$$|T(e_i, e_j)|^2 \leq 9T(n, n)^2 \leq 9\bar{c}(u) \varepsilon(u)^2.$$

This yields

$$\int_{\Sigma_\tau} |T^{\alpha\beta}[\tilde{F}]|_g d\mu_{\Sigma_\tau} \leq C_2(\tau, u) \int_{\Sigma_\tau} \varepsilon(u) d\tau_{\Sigma_\tau},$$

which also gives the second claim from eq. (6.63). $\square$

Putting everything together, we obtain the following

Theorem 6.2 (Energy estimate). *Let $E^u(\tau)$ be the u -energy computed from the stress-energy tensor $T_{\alpha\beta}[\tilde{F}]$, where $\tilde{F}$ is the difference between the exact and approximate field strength. Further, let $\nabla^\alpha \tilde{F}_{\alpha\beta} = f_\beta$ as in eq. (6.57) and denote by N the lapse function of the foliated spacetime. Then*

$$\frac{d}{d\tau} E^u(\tau) \leq A(\tau, u) \|f_\beta\|_{L^2(\Sigma_\tau, g)} \sqrt{E^u(\tau)} + B(\tau, u) E^u(\tau), \quad (6.71)$$

where $A(\tau, u) = C_1(\tau, u) \|N\|_{L^\infty(\Sigma_\tau, g)}$ and $B(\tau, u) = C_2(\tau, u) \|N\|_{L^\infty(\Sigma_\tau, g)} \|\nabla_{(\alpha} u_{\beta)}\|_{L^\infty(\Sigma_\tau, g)}$. We can also rewrite this as

$$\frac{d}{d\tau} \sqrt{E^u(\tau)} \leq \frac{1}{2} A(\tau, u) \|f_\beta\|_{L^2(\Sigma_\tau, g)} + \frac{1}{2} B(\tau, u) \sqrt{E^u(\tau)}. \quad (6.72)$$

6. Approximate solutions to Maxwell's equations in curved spacetime

Proof. This now follows immediately from lemma 6.2 as well as eq. (6.61). $\square$

Remark 6.4. We make the following observations. If u is a Killing vector field, then $B(\tau, u) = 0$ which implies that the change in energy is bounded above by the norm of the residual from eq. (6.57), i.e.

$$\frac{d}{d\tau} \sqrt{E^u(\tau)} \leq \frac{1}{2} A(\tau, u) \|f_\beta\|_{L^2(\Sigma_\tau, \mathfrak{g})}.$$

If u is not Killing, then by the lemma of Grönwall¹ we find

$$\begin{aligned} \sqrt{E^u(\tau)} &\leq \sqrt{E^u(\tau_0)} \exp \left(\frac{1}{2} \int_{\tau_0}^{\tau} B(s, u) ds \right) \\ &\quad + \frac{1}{2} \int_{\tau_0}^{\tau} A(s, u) \|f_\beta\|_{L^2(\Sigma_s, \mathfrak{g})} \exp \left(\frac{1}{2} \int_s^{\tau} B(r, u) dr \right) ds. \end{aligned}$$

Note that this gives an exponential growth in time. This does however not mean that our approximation breaks down, but is just a consequence of the fact that u is not Killing.

¹Let $I = [t_0, t_1]$ and $f, g \in C(I, \mathbb{R})$. If $\varphi \in C^1(I, \mathbb{R})$ satisfies the inequality

$$\varphi'(t) \leq f(t) + g(t)\varphi(t) \quad \forall t \in I,$$

then

$$\varphi(t) \leq \varphi(t_0) \exp \left(\int_{t_0}^t g(s) ds \right) + \int_{t_0}^t f(s) \exp \left(\int_s^t g(\tau) d\tau \right) ds \quad (t \in I).$$

A. Appendix

Here we provide the full calculation from the proof of proposition 6.1, showing that

$$\nabla_\sigma \nabla_\nu \nabla_\mu (a_\alpha^{(0)} \nabla^\alpha \phi \nabla_\beta \phi) \big|_\gamma = 0.$$

Proof. We calculate $\nabla_\sigma \nabla_\nu \nabla_\mu (a_\alpha^{(0)} \nabla^\alpha \phi \nabla_\beta \phi) \big|_\gamma =$

$$\begin{aligned} &= \underbrace{(\nabla_\sigma \nabla_\nu \nabla_\mu a_\alpha^{(0)}) (\nabla^\alpha \phi) (\nabla_\beta \phi) \big|_\gamma}_{(1.1)} + \underbrace{(\nabla_\nu \nabla_\mu a_\alpha^{(0)}) (\nabla_\sigma \nabla^\alpha \phi) (\nabla_\beta \phi) \big|_\gamma}_{(1.2)} + \underbrace{(\nabla_\nu \nabla_\mu a_\alpha^{(0)}) (\nabla^\alpha \phi) (\nabla_\sigma \nabla_\beta \phi) \big|_\gamma}_{(2.1)} \\ &+ \underbrace{(\nabla_\sigma \nabla_\mu a_\alpha^{(0)}) (\nabla_\nu \nabla^\alpha \phi) (\nabla_\beta \phi) \big|_\gamma}_{(1.3)} + \underbrace{(\nabla_\mu a_\alpha^{(0)}) (\nabla_\sigma \nabla_\nu \nabla^\alpha \phi) (\nabla_\beta \phi) \big|_\gamma}_{(1.4)} + \underbrace{(\nabla_\mu a_\alpha^{(0)}) (\nabla_\nu \nabla^\alpha \phi) (\nabla_\sigma \nabla_\beta \phi) \big|_\gamma}_{(2.2)} \\ &+ \underbrace{(\nabla_\sigma \nabla_\mu a_\alpha^{(0)}) (\nabla^\alpha \phi) (\nabla_\nu \nabla_\beta \phi) \big|_\gamma}_{(3.1)} + \underbrace{(\nabla_\mu a_\alpha^{(0)}) (\nabla_\sigma \nabla^\alpha \phi) (\nabla_\nu \nabla_\beta \phi) \big|_\gamma}_{(3.2)} + \underbrace{(\nabla_\mu a_\alpha^{(0)}) (\nabla^\alpha \phi) (\nabla_\sigma \nabla_\nu \nabla_\beta \phi) \big|_\gamma}_{(5.1)} \\ &+ \underbrace{(\nabla_\sigma \nabla_\nu a_\alpha^{(0)}) (\nabla_\mu \nabla^\alpha \phi) (\nabla_\beta \phi) \big|_\gamma}_{(1.5)} + \underbrace{(\nabla_\nu a_\alpha^{(0)}) (\nabla_\sigma \nabla_\mu \nabla^\alpha \phi) (\nabla_\beta \phi) \big|_\gamma}_{(1.6)} + \underbrace{(\nabla_\nu a_\alpha^{(0)}) (\nabla_\mu \nabla^\alpha \phi) (\nabla_\sigma \nabla_\beta \phi) \big|_\gamma}_{(2.3)} \\ &+ \underbrace{(\nabla_\sigma a_\alpha^{(0)}) (\nabla_\nu \nabla_\mu \nabla^\alpha \phi) (\nabla_\beta \phi) \big|_\gamma}_{(1.7)} + \underbrace{a_\alpha^{(0)} (\nabla_\sigma \nabla_\nu \nabla_\mu \nabla^\alpha \phi) (\nabla_\beta \phi) \big|_\gamma}_{(1.8)} + \underbrace{a_\alpha^{(0)} (\nabla_\nu \nabla_\mu \nabla^\alpha \phi) (\nabla_\sigma \nabla_\beta \phi) \big|_\gamma}_{(2.4)} \\ &+ \underbrace{(\nabla_\sigma a_\alpha^{(0)}) (\nabla_\mu \nabla^\alpha \phi) (\nabla_\nu \nabla_\beta \phi) \big|_\gamma}_{(3.3)} + \underbrace{a_\alpha^{(0)} (\nabla_\sigma \nabla_\mu \nabla^\alpha \phi) (\nabla_\nu \nabla_\beta \phi) \big|_\gamma}_{(3.4)} + \underbrace{a_\alpha^{(0)} (\nabla_\mu \nabla^\alpha \phi) (\nabla_\sigma \nabla_\nu \nabla_\beta \phi) \big|_\gamma}_{(5.2)} \\ &+ \underbrace{(\nabla_\sigma \nabla_\nu a_\alpha^{(0)}) (\nabla^\alpha \phi) (\nabla_\mu \nabla_\beta \phi) \big|_\gamma}_{(4.1)} + \underbrace{(\nabla_\nu a_\alpha^{(0)}) (\nabla_\sigma \nabla^\alpha \phi) (\nabla_\mu \nabla_\beta \phi) \big|_\gamma}_{(4.2)} + \underbrace{(\nabla_\nu a_\alpha^{(0)}) (\nabla^\alpha \phi) (\nabla_\sigma \nabla_\mu \nabla_\beta \phi) \big|_\gamma}_{(6.1)} \\ &+ \underbrace{(\nabla_\sigma a_\alpha^{(0)}) (\nabla_\nu \nabla^\alpha \phi) (\nabla_\mu \nabla_\beta \phi) \big|_\gamma}_{(4.3)} + \underbrace{a_\alpha^{(0)} (\nabla_\sigma \nabla_\nu \nabla^\alpha \phi) (\nabla_\mu \nabla_\beta \phi) \big|_\gamma}_{(4.4)} + \underbrace{a_\alpha^{(0)} (\nabla_\nu \nabla^\alpha \phi) (\nabla_\sigma \nabla_\mu \nabla_\beta \phi) \big|_\gamma}_{(6.2)} \\ &+ \underbrace{(\nabla_\sigma a_\alpha^{(0)}) (\nabla^\alpha \phi) (\nabla_\nu \nabla_\mu \nabla_\beta \phi) \big|_\gamma}_{(7.1)} + \underbrace{a_\alpha^{(0)} (\nabla_\sigma \nabla^\alpha \phi) (\nabla_\nu \nabla_\mu \phi) \big|_\gamma}_{(6.2)} + \underbrace{a_\alpha^{(0)} (\nabla^\alpha \phi) (\nabla_\sigma \nabla_\nu \nabla_\mu \nabla_\beta \phi) \big|_\gamma}_{(6.21a)_0}. \end{aligned}$$

Now, the terms (1.1)–(1.6) vanish due to eq. (6.21f), the terms (2.1)–(2.4), (3.1)–(3.4) and (4.1)–(4.4) vanish due to eq. (6.21e) while the terms (5.1)–(5.2), (6.1)–(6.2) and (7.1)–(7.2) sum to zero due to eq. (6.21b). $\square$

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