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Titel | Title

Hydrodynamic stability of the linearized two-dimensional Navier-Stokes equations near Poiseuille flow

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# Abstrakt

Diese Arbeit untersucht die hydrodynamische Stabilität der linearisierten zweidimensionalen Navier-Stokes-Gleichungen in der Nähe der Poiseuille-Strömung. Wir konzentrieren uns auf den Bereich niedriger Viskosität in der Geometrie der Ebene. Anders als im periodischen Fall kann die Wellenzahl in Strömungsrichtung auf der Ebene beliebig klein sein, weshalb die Abklingmechanismen wesentlich vom Verhältnis zwischen Wellenzahl und Viskosität abhängen. Mithilfe einer auf Hypokoerzivität basierenden Methode, die an die Poiseuille-Linearisierung angepasst wurde, leiten wir Abklingabschätzungen für von der Strömungsrichtung abhängige Lösungen her, welche die klassische diffusive Zeitskala verbessern. Wir zeigen, dass Lösungen bei höheren Frequenzen eine verstärkte Dissipation aufweisen. Umgekehrt wird die Relaxation bei niedrigeren Frequenzen durch Taylor-Dispersion bestimmt.

## Abstract

This thesis investigates the hydrodynamic stability of the linearized two-dimensional Navier–Stokes equations near Poiseuille flow. We focus on the low-viscosity regime in the planar geometry $\mathbb{R} \times \mathbb{R}$. Unlike in periodic settings, the streamwise wave number $\xi$ can be arbitrarily small on the plane, and the decay mechanisms therefore depend essentially on the relation between $\xi$ and $\nu$. Using a hypocoercivity-based method adapted to the Poiseuille linearization, we derive decay estimates for $x$-dependent solutions that improve on the classical diffusive timescale $\nu^{-1}$. We demonstrate that for higher frequencies $|\xi| \gtrsim \nu$, solutions experience enhanced dissipation with a decay rate proportional to $\nu^{1/2}|\xi|^{1/2}$. Conversely, for lower frequencies $|\xi| \lesssim \nu$, the relaxation is governed by Taylor dispersion with a decay rate of $\xi^2/\nu$.

# Contents



# Statement on the Use of AI Tools

During the preparation and revision of this thesis, I used GenAI for language polishing, grammar and syntax corrections, and LaTeX formatting suggestions. The tool was not used to generate original mathematical results, proofs, or citations.

After using this tool, I reviewed and edited the relevant content as needed and independently verified all mathematical arguments and references. I take full responsibility for the content of this thesis.

# 1 Introduction to the Navier-Stokes equations

This thesis studies the hydrodynamic stability of the two-dimensional Poiseuille flow on the plane. The main analytical difficulty of the planar setting $\mathbb{R}_x \times \mathbb{R}_y$ is that the streamwise frequency can be arbitrarily small, so the decay mechanism changes between a high-frequency enhanced-dissipation regime and a low-frequency Taylor-dispersion regime. After introducing the Navier–Stokes framework and the basic operators needed later, we use the Couette flow as a model case to explain the mechanisms of enhanced dissipation and inviscid damping. We then turn to the linearized Poiseuille problem and prove modewise decay estimates that distinguish the regimes $|\xi| \gtrsim \nu$ and $|\xi| \lesssim \nu$.

## 1.1 Derivation of the incompressible Navier–Stokes equations

A standard starting point in fluid mechanics is the *continuum hypothesis*: rather than tracking individual molecules, we model the fluid at a macroscopic level as a fluid continuum by vector fields such as the velocity $u(t, x)$, pressure $p(t, x)$, and (if relevant) density $\rho(t, x)$. If one does not adopt the continuum hypothesis, then the macroscopic fields cannot be treated as well-defined pointwise quantities, and the appropriate description becomes microscopic or kinetic. A standard dimensionless measure for the validity of the continuum approximation is the *Knudsen number*, as reviewed in [Gad99]

$$Kn := \frac{\lambda}{L},$$

where $\lambda$ is the mean free path (or, more generally, a microscopic length scale) and $L$ is a characteristic macroscopic length scale of the flow (for instance, a pipe diameter or boundary-layer thickness). When $Kn \ll 1$, each "fluid particle" contains many molecules and local averaging is meaningful; in this regime, the continuum equations (Euler/Navier–Stokes) provide an accurate effective model. By contrast, when $Kn$ is not small (as in rarefied gases, high-altitude flows, or micro-/nano-scale devices), non-continuum effects such as slip at boundaries and significant molecular fluctuations may appear, and kinetic descriptions become necessary. In most classical laboratory and engineering situations involving liquids such as water, and also gases like air at standard conditions, and macroscopic length scales, one typically has $Kn \ll 1$, which explains why the continuum hypothesis is an excellent approximation in these regimes.

Although real fluids evolve in three spatial dimensions, it is often natural to consider reduced-dimensional models. In particular, when the flow exhibits a large aspect ratio or strong geometric/physical constraints (for example, in certain large-scale atmospheric or oceanic regimes), the leading-order dynamics may be well approximated by an effectively two-dimensional model. A rigorous example is the quasigeostrophic approximation for large-scale atmospheric and oceanic flow [BB94]. Such reductions can substantially simplify both analysis and computation. For this reason, it is useful to keep in mind both the cases $d = 2$ and $d = 3$ in what follows, depending on which idealization best captures the phenomena under consideration.

In this section, we will derive the Navier-Stokes equations for a Newtonian fluid, starting from the basic principles of continuum mechanics:

- *Conservation of mass*: mass cannot be created or destroyed, and
- *Newton's second law of motion*: the rate of change of momentum of a fluid volume equals the sum of forces acting on it.

We will give a more mathematical definition below. Let us set the scene first. We consider a fluid occupying a domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, and denote by

- $u = u(t, x) : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ the velocity vector field with units $L/T$ (units of length/units of time)
- $\rho = \rho(t, x) : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ the density scalar field with units $[\rho] = ML^{-d}$,
- $p = p(t, x) : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ the pressure scalar field with units $[p] = ML^{-d+2}T^{-2}$,

Throughout, derivatives are taken with respect to the variables $(t, x)$.

### 1.1.1 Incompressibility

We will consider two viewpoints of fluid dynamics: the Lagrangian and Eulerian regimes. The Lagrangian viewpoint for fluid motion considers a particle moving with the fluid, where the Eulerian standpoint considers a stationary point $x$ at time $t$. Mathematically, this equals looking for the particle trajectories as unknowns: $X(t, a) = (X_1, \ldots, X_d)(t, a)$ where

$$X(t, \cdot) : a \to X(t, a)$$

is called the flow map. The connection between looking at stationary points vs particle trajectories is revealed in the following ordinary differential equation:

$$\partial_t X(t, a) = u(t, X(t, a)), \quad X(0, a) = a. \tag{1}$$

For a volume element $V \subset \Omega$ in the fluid, we define the pushforward under the flow map $X(t, .)$ as

$$V(t) = X(t, V) = X(t, a) : a \in V$$

**Definition 1.** *A vector field* u *is* incompressible *if the flow map is volume–preserving:*

$$|V| = |V(t)| \textit{ for all } V \subset \Omega \textit{ and all } t \in \mathbb{R}$$

By standard multivariable calculus, it is straightforward to show the following Lemma.

**Lemma 1.** *Let $u \in C_t Lip_x, ([-T, T], \Omega)$ then the following are equivalent*

- *$u$ is incompressible*
- *$u$ is divergence-free $\nabla \cdot u = 0$ for every $(t, x)$.*

In contrast to the standard derivative at a fixed point, the Eulerian view, the convective derivative measure the change along the flow, the Lagrangian view,

$$\partial_t(f(t,X(t,a))) = (D_t f)(t,(X(t,a)))$$

From the Chain rule and the equation (1) (assuming $C^1$) we observe that

$$D_t f = \partial_t f + u \cdot \nabla f.$$

The convective derivative describes how fluid quantities are transported by the flow and is highlighted in the following theorem.

**Theorem 1** (Transport theorem)**.** *Let $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be a $C^1$ function and let $u$ be a $C^1$ velocity field generating the flow map $X(\cdot,t)$. If $V(t)$ denotes the pushforward of a volume $V$ under the flow, then*

$$\frac{d}{dt}\int_{V(t)} f(t,x)\,dx = \int_{V(t)} \big(\partial_t f + \nabla \cdot (uf)\big)(t,x)\,dx. \tag{2}$$

*Proof.* By the change of variables $x = X(t,a)$, we write

$$\int_{V(t)} f(t,x)\,dx = \int_V f(t,X(t,a))\,J(t,a)\,da,$$

where $J(t,a) = \det \nabla_a X(t,a)$. Differentiating in time yields

$$\frac{d}{dt}\int_{V(t)} f(t,x)\,dx = \int_V \Big(D_t f(t,X(t,a))\,J(t,a) + f(t,X(t,a))\,\partial_t J(t,a)\Big)\,da.$$

Using the identities

$$D_t f = \partial_t f + u \cdot \nabla f, \qquad \partial_t J = (\nabla \cdot u)\,J,$$

we obtain

$$\frac{d}{dt}\int_{V(t)} f(t,x)\,dx = \int_V \big(\partial_t f + \nabla \cdot (uf)\big)(t,X(t,a))\,J(t,a)\,da.$$

Changing variables back to $x$ concludes the proof. □

### 1.1.2 Conservation of mass

We now formulate the first physical principle in mathematical terms. The mass contained in a volume $V \subset \mathbb{R}^d$ at time $t$ is defined by

$$m(t,V) = \int_V \rho(t,x)\,dx,$$

where $\rho(t,x)$ denotes the fluid density. Our basic modelling assumption is that mass is neither created nor destroyed. Hence, for every material volume $V(t)$ moving with the flow,

$$\frac{d}{dt} m(t,V(t)) = 0.$$

By the transport theorem, this is equivalent to

$$0 = \frac{d}{dt} m(t, V(t)) = \int_{V(t)} \big(\partial_t \rho + \nabla \cdot (\rho u)\big)\, dx.$$

Since this holds for every material volume, we obtain the continuity equation

$$\partial_t \rho + \nabla \cdot (\rho u) = 0.$$

A fluid is called *homogeneous* if its density is the same at every point in space. Thus,

$$\rho(t, x) \equiv \rho_0$$

for some constant $\rho_0 > 0$. Substituting this into the continuity equation gives

$$\nabla \cdot u = 0.$$

Hence, for a homogeneous fluid, conservation of mass reduces to the incompressibility condition.

### 1.1.3 Momentum balance and the Newtonian stress tensor

We now turn to the second physical principle, Newton's second law: the rate of change of momentum of a material volume $V(t)$ equals the sum of the forces acting on it. Neglecting external body forces, only internal surface forces remain, and these are described by the *Cauchy stress tensor*.

Cauchy's theorem states that there exists a symmetric tensor field $\sigma(t, x) \in \mathbb{R}^{d\times d}$ such that the traction which is the internal force per unit area exerted across a surface element by the fluid on one side on the fluid on the other, is a linear function of the outward unit normal $n$ on $\partial V(t)$,

$$T(n) = \sigma n;$$

see the classical account in Serrin [Ser59]. The balance of momentum for a material volume $V(t)$ then reads

$$\frac{d}{dt} \int_{V(t)} \rho\, u(t, x)\, dx = \int_{\partial V(t)} \sigma(t, x)\, n(t, x)\, dS(x), \tag{3}$$

where $dS$ denotes the surface measure on $\partial V(t)$.

The stress tensor splits into a pressure part and a viscous part,

$$\sigma = -pI + \tau.$$

The sign convention is chosen so that a positive pressure $p$ corresponds to a compressive normal stress. Whether we obtain the Euler or the Navier–Stokes equations is decided by $\tau$. For an ideal fluid $\tau \equiv 0$, so $\sigma = -pI$: forces between two adjacent regions of fluid are purely normal, and layers of fluid may slide across one another without experiencing any tangential force, which yields the Euler equations. In every real fluid there is internal friction, called viscosity, and the viscous stress $\tau$ accounts for the resulting shear contribution.

We work with *Newtonian fluids*: the viscous stress $\tau$ depends linearly on the velocity gradient $\nabla u$ and is invariant under rotations (isotropic). Linearity and isotropy force $\tau$ to depend only on the symmetric part of the velocity gradient,

$$D(u) = \tfrac{1}{2}\big(\nabla u + (\nabla u)^T\big).$$

For an incompressible fluid with constant dynamic viscosity $\mu > 0$ this gives

$$\tau = 2\mu\, D(u) = \mu\big(\nabla u + (\nabla u)^T\big), \qquad \text{hence} \qquad \sigma = -pI + \mu\big(\nabla u + (\nabla u)^T\big).$$

More details can again be found in Serrin [Ser59].

Applying the divergence theorem, the surface integral in (3) becomes a volume integral,

$$\int_{\partial V(t)} \sigma n\, dS = \int_{V(t)} \nabla \cdot \sigma\, dx.$$

For the pressure part,

$$\nabla \cdot (-pI) = -\nabla p,$$

while for the viscous part the $i$-th component of the divergence is, with constant $\mu$,

$$\big[\nabla \cdot \big(\mu(\nabla u + (\nabla u)^T)\big)\big]_i = \mu \sum_{j=1}^{d} \partial_j\big(\partial_j u_i + \partial_i u_j\big) = \mu\Big(\Delta u_i + \partial_i(\nabla \cdot u)\Big).$$

The incompressibility condition $\nabla \cdot u = 0$ cancels the last term, so that $\nabla \cdot \big(\mu(\nabla u + (\nabla u)^T)\big) = \mu \Delta u$ and therefore

$$\nabla \cdot \sigma = -\nabla p + \mu \Delta u.$$

The momentum balance (3) thus reads

$$\frac{d}{dt} \int_{V(t)} \rho\, u(t,x)\, dx = \int_{V(t)} \big(-\nabla p + \mu \Delta u\big)\, dx. \tag{4}$$

For the left-hand side we move the time derivative inside the integral, using the transport theorem (1) together with conservation of mass.

**Corollary 1** (Density transport)**.** *Let $f, \rho \in C^1$, and assume $\rho$ satisfies the continuity equation $\partial_t \rho + \nabla \cdot (\rho u) = 0$. Then for the pushforward $V(t) = X(t,V)$ of an open set $V \subset \Omega$ under the flow map and for all $t \geq 0$,*

$$\frac{d}{dt} \int_{V(t)} \rho\, f\, dx = \int_{V(t)} \rho\, (D_t f)\, dx, \qquad D_t f := \partial_t f + u \cdot \nabla f.$$

*Proof.* Apply the transport theorem (2) to $g := \rho f$ and expand the flux term as $\nabla \cdot (\rho f\, u) = f\, \nabla \cdot (\rho u) + \rho\, u \cdot \nabla f$:

$$\partial_t(\rho f) + \nabla \cdot (\rho f\, u) = \rho\big(\partial_t f + u \cdot \nabla f\big) + f\big(\partial_t \rho + \nabla \cdot (\rho u)\big) = \rho\, D_t f,$$

since the last bracket vanishes by the continuity equation. □

Applying Corollary (1) componentwise with $f = u$ (and, for a homogeneous fluid, constant $\rho$),

$$\frac{d}{dt} \int_{V(t)} \rho\, u\, dx = \int_{V(t)} \rho\, (\partial_t u + u \cdot \nabla u)\, dx.$$

Inserting this into (4), and since the identity holds for every material volume $V(t)$, we obtain the pointwise momentum equation

$$\rho\big(\partial_t u + u \cdot \nabla u\big) = -\nabla p + \mu \Delta u. \tag{5}$$

Together with conservation of mass, this leads to the incompressible Navier–Stokes equations, assembled in the next section.

### 1.1.4 The incompressible Navier–Stokes equations

Combining the balance of mass with the momentum equation (5), an incompressible Newtonian fluid of constant density $\rho > 0$ and viscosity $\mu > 0$ obeys the *dimensional* system

$$\begin{cases} \rho\big(\partial_t u + u \cdot \nabla u\big) = -\nabla p + \mu \Delta u, \\ \nabla \cdot u = 0. \end{cases} \tag{6}$$

For stability analysis we pass to dimensionless variables. This collapses the physical parameters $\rho, \mu, U, L$ into a single number, the Reynolds number. Let $U > 0$ and $L > 0$ be a characteristic velocity and length scale of the flow, and set

$$x = Lx', \qquad t = \frac{L}{U}\, t', \qquad u = Uu', \qquad p = \rho U^2 p'.$$

By the chain rule $\partial_t = \frac{U}{L}\partial_{t'}$, $\nabla_x = \frac{1}{L}\nabla_{x'}$ and $\Delta_x = \frac{1}{L^2}\Delta_{x'}$. Substituting into the momentum equation of (6) and dividing by $\rho U^2/L$ gives

$$\partial_{t'} u' + u' \cdot \nabla_{x'} u' + \nabla_{x'} p' = \frac{\mu}{\rho U L}\, \Delta_{x'} u', \qquad \nabla_{x'} \cdot u' = 0.$$

The prefactor of the Laplacian is the inverse of the *Reynolds number*

$$\mathrm{Re} := \frac{\rho U L}{\mu} = \frac{UL}{\mu/\rho},$$

where $\mu/\rho$ is the kinematic viscosity. The Reynolds number measures inertial (convective) effects against viscous diffusion: a large Re means weak diffusion compared to inertia, which is precisely the regime in which shear flows such as Couette and Poiseuille flow exhibit the subtle stability and transition phenomena studied in this thesis.

Dropping the primes and writing $\nu := 1/\mathrm{Re}$ for the single remaining parameter, we arrive at the non-dimensional incompressible Navier–Stokes equations used throughout this thesis,

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u, \\ \nabla \cdot u = 0, \end{cases} \qquad 0 < \nu < 1. \tag{7}$$

In contrast to (6), here $\nu$ is not a physical viscosity but the dimensionless parameter $1/\mathrm{Re}$; the low-viscosity regime $\nu \to 0$ corresponds to high Reynolds number.

## 1.2 The pressure

The Navier-Stokes equation has $d+1$ unknowns, the velocity components $u = (u_1; \dots, u_d)$ and the pressure $p$. Because the equation contains no time derivative for $p$, we will try to find an equation $p = p(u)$ that allows us to eliminate the pressure from equation (7). In order to do that, we start by taking the divergence of the momentum equation

$$\partial_t u + (u \cdot \nabla) u = -\nabla p + \nu \Delta u,$$

and use $\partial_t(\nabla \cdot u) = 0$ together with $\Delta(\nabla \cdot u) = 0$ to obtain

$$-\Delta p = \nabla \cdot \big((u \cdot \nabla) u\big).$$

Expanding in components and appealing again to $\partial_i u_i = 0$, we compute

$$\nabla \cdot \big((u\cdot\nabla)u\big) = \partial_i\big(u_j\,\partial_j u_i\big) = (\partial_i u_j)(\partial_j u_i) + u_j\,\partial_j(\partial_i u_i) = \sum_{i,j} \partial_i u_j\,\partial_j u_i,$$

and hence arrive at the Poisson equation for the pressure,

$$-\Delta p = \sum_{i,j} \partial_i u_j\,\partial_j u_i.$$

The classical approach to solving this equation is to use the Newtonian Potential, which can be found in the book by Evans ([Eva10]). We will take a slightly different, still connected, approach that will be useful to understand the vorticity form of the Navier-Stokes equation.

From now on, we will use the Einstein convention summing over equal indices. Hence, taking the Fourier transform in $\mathbb{R}^d$ with the convention

$$\widehat{g}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi}\, g(x)\,dx, \qquad g(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix\cdot\xi}\,\widehat{g}(\xi)\,d\xi,$$

we obtain

$$\widehat{-\Delta p}(\xi) = (\partial_i \widehat{u_j\,\partial_j} u_i)(\xi).$$

Using the differentiation and convolution properties of the Fourier Transform, this becomes

$$|\xi|^2\,\widehat{p}(\xi) = -\xi_i\xi_j\widehat{(u_iu_j)}(\xi),$$

and hence after inverting $|\xi|^2$ and the Fourier transform, we end up with:

$$p(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \frac{-\xi_i\xi_j}{|\xi|^2}\widehat{(u_iu_j)}(\xi)e^{ix\cdot\xi}\,d\xi.$$

Up to this point, the computations have been purely formal. In particular, the integral above raises an immediate analytical question: how should one interpret the singular behaviour at the zero frequency $\xi = 0$ (and, more generally, the non-integrable kernels that appear after inverting the Laplacian)? To address this, we briefly recall the harmonic-analytic framework of *singular integral operators*, with emphasis on *Calderón–Zygmund operators*. This theory provides a rigorous interpretation of these operators and the mapping estimates needed to control the pressure term in Navier–Stokes, and it will be used again later in the discussion of the Biot–Savart law.

### 1.2.1 Calderon-Zygmund Operators

The canonical one-dimensional example of a Calderón–Zygmund operator is the *Hilbert transform* on $\mathbb{R}$, defined by

$$Hf(x) := \frac{1}{\pi}\,\text{p.v.}\int_{\mathbb{R}} \frac{f(y)}{x-y}\,dy.$$

Here p.v. denotes the *Cauchy principal value*: since the kernel $(x-y)^{-1}$ is not locally integrable at $y = x$, the integral is understood as the symmetric truncation limit

$$\text{p.v.}\int_{\mathbb{R}} \frac{f(y)}{x-y}\,dy := \lim_{\varepsilon\to 0^+} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y}\,dy$$

In this form, one sees that $H$ is a convolution operator $Hf = (K * f)$ with kernel

$$K(t) = \text{p.v.}\frac{1}{\pi t}.$$

This kernel highlights the three structural features that underlie Calderón–Zygmund theory:

- **Singular size bound:** the kernel has a non-integrable singularity of the correct order,

$$|K(x)| \lesssim |x|^{-d}, \qquad x \neq 0.$$

- **Cancellation:** the singularity is compensated by symmetry. Since $K$ is odd, one has for all $0 < \varepsilon < R$,

$$\int_{\varepsilon<|t|<R} K(t)\,dt = 0.$$

- **Regularity away from the origin:** the kernel does not oscillate too wildly at large scales; this is captured by a Hörmander-type smoothness estimate (stated precisely below).

This motivates the following definition of the Calderon-Zygmund kernel and operators on the Class of the Schwartz functions.

**Definition 2** (Calderón–Zygmund kernel and operator)**.** *A function $K : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ is called a* Calderón–Zygmund kernel *if:*

*(i) (*size*) there exists $C > 0$ such that*

$$|K(x)| \leq \frac{C}{|x|^d}, \qquad x \neq 0;$$

*(ii) (*Hörmander regularity*) there exists $C > 0$ such that for all $h \neq 0$,*

$$\int_{|x|>2|h|} |K(x-h) - K(x)|\,dx \leq C;$$

*(iii) (*cancellation*) for every $0 < \varepsilon < R$,*

$$\int_{\varepsilon<|x|<R} K(x)\,dx = 0.$$

*Given such a kernel $K$, the associated* Calderón–Zygmund singular integral operator *is defined on $f \in \mathcal{S}(\mathbb{R}^d)$ by*

$$Tf(x) := \text{p.v.}\int_{\mathbb{R}^d} K(x-y)\,f(y)\,dy := \lim_{\varepsilon\to 0^+}\int_{|x-y|>\varepsilon} K(x-y)\,f(y)\,dy,$$

*whenever the limit exists.*

By using the fundamental theorem of calculus one can show that the following simple condition already guarantees the Hörmander regularity:

$$|\nabla K(x)| \lesssim |x|^{-d-1}.$$

The derivative decay $|x|^{-d-1}$ is just strong enough that the accumulated change from a translation by $h$ remains uniformly bounded when integrated over all scales. This guarantees that the Kernel cannot oscillate so violently that translations would destroy the cancellation structure of the integral.

The main result is the following Calderón–Zygmund boundedness theorem; see Muscalu and Schlag [MS13, Chapter 7].

**Theorem 2.** *Let $T$ be a singular integral operator as in Definition 2. Then for every $1 < p < \infty$, one can extend $T$ to a bounded operator on $L^p(\mathbb{R}^d)$ with the bound*

$$\|T\|_{L^p \to L^p} \leq C\, B,$$

*where $C = C(p, d)$.*

Now we will connect the Calderon-Zygmund operators with Fourier analysis, in order to do that we start by defining Fourier multipliers.

**Definition 3** (Fourier multiplier)**.** *Let $m : \mathbb{R}^d \to \mathbb{C}$ be a measurable function (the* symbol*). The associated* Fourier multiplier operator *$T_m$ is defined initially on the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ by*

$$T_m f := \mathcal{F}^{-1}\big(m(\xi)\, \widehat{f}(\xi)\big), \qquad \textit{i.e.} \qquad \widehat{T_m f}(\xi) = m(\xi)\, \widehat{f}(\xi).$$

*We say that $m$ is an $L^p$*-multiplier *if $T_m$ extends to a bounded linear operator on $L^p(\mathbb{R}^d)$.*

We give a short, non-rigorous derivation of the Fourier multiplier representation of the Hilbert transform, with the sole purpose of highlighting the underlying intuition.

Recall that, at least formally,

$$Hf(x) = \frac{1}{\pi}\, \text{p.v.} \int_{\mathbb{R}} \frac{f(y)}{x-y}\, dy = \text{p.v.} \int_{\mathbb{R}} f(x-t)\, \frac{1}{\pi t}\, dt = (f * K)(x), \qquad K(t) = \text{p.v.}\, \frac{1}{\pi t}.$$

Since the Fourier transform should interchange convolution and pointwise multiplication, we expect

$$\widehat{Hf}(\xi) = \widehat{f * K}(\xi) = \widehat{f}(\xi)\, \widehat{K}(\xi),$$

so the problem reduces to identifying the distributional Fourier transform of the principal value kernel $K$.

The key heuristic is that p.v.$(1/x)$ is, up to constants, the distribution obtained by "dividing" by $x$, and division by $x$ corresponds on the Fourier side to taking an antiderivative in $\xi$. Indeed, with our convention, one has

$$\widehat{1}(\xi) = \delta(\xi),$$

and differentiation in $x$ corresponds to multiplication by $\xi$ in frequency:

$$\widehat{\partial_x 1}(\xi) = i\xi\, \widehat{1}(\xi) = i\xi\, \delta(\xi) = 0.$$

Interpreting this relation in the sense of distributions and "integrating in $\xi$" means choosing a primitive of $\delta$, namely $\frac{1}{2}\mathrm{sgn}(\xi)$, since

$$\partial_\xi\Big(\tfrac{1}{2}\,\mathrm{sgn}(\xi)\Big) = \delta(\xi).$$

Tracing the Fourier differentiation identities backwards (equivalently, using the duality between multiplication by $x$ and differentiation in $\xi$) leads to the classical distributional identity

$$\widehat{\frac{1}{-2\pi i x}}(\xi) = \frac{1}{2}\,\mathrm{sgn}(\xi), \qquad \text{hence} \qquad \widehat{\mathrm{p.v.}\,\frac{1}{\pi x}}(\xi) = -\,i\,\mathrm{sgn}(\xi).$$

Substituting this into the convolution–product formula yields the multiplier representation

$$\widehat{Hf}(\xi) = \big(-\,i\,\mathrm{sgn}(\xi)\big)\,\widehat{f}(\xi).$$

In particular, the Hilbert transform is an order-zero Fourier multiplier with symbol $m(\xi) = -i\,\mathrm{sgn}(\xi)$.

The previous example shows that many singular integral operators admit an equivalent description as Fourier multipliers. The following Hörmander–Mikhlin criterion gives a convenient sufficient condition for such multipliers to extend to bounded operators on $L^p$.

**Theorem 3** (Hörmander–Mikhlin multiplier theorem)**.** *Let $m : \mathbb{R}^d \to \mathbb{C}$ obey the homogeneous symbol estimates of order* 0*:*

$$|\nabla^k m(\xi)| \lesssim |\xi|^{-k} \qquad \textit{for all } \xi \neq 0 \textit{ and } 0 \leq k \leq d+2.$$

*Then $T_m$ is a Calderón–Zygmund operator and, for every $1 < p < \infty$, there exists a constant $C_{p,d} > 0$ such that*

$$\|T_m f\|_{L^p(\mathbb{R}^d)} \leq C_{p,d}\,\|f\|_{L^p(\mathbb{R}^d)}$$

Why is this relevant for inverting the Laplacian? The connection becomes transparent once we pass to the $d$-dimensional analogue of the Hilbert transform, namely the *Riesz transforms.* For $j = 1, \ldots, d$ they are given (as Calderón–Zygmund operators) by the principal value singular integral

$$R_j f(x) := \frac{1}{C(d)}\,\mathrm{p.v.}\int_{\mathbb{R}^d} \frac{x_j - y_j}{|x-y|^{d+1}}\,f(y)\,dy,$$

and, equivalently, by the Fourier multiplier representation

$$\widehat{R_j f}(\xi) = -\,i\,\frac{\xi_j}{|\xi|}\,\widehat{f}(\xi), \qquad \xi \neq 0.$$

The key point is that the operators arising from *differentiating* the inverse Laplacian can be expressed in terms of Riesz transforms. Indeed, as Fourier multipliers, one has the identity

$$\partial_i\partial_j(-\Delta)^{-1} = R_i R_j,$$

since both sides have symbol $\xi_i\xi_j/|\xi|^2$.

Applying this to the pressure Poisson equation $-\Delta p = \partial_i\partial_j(u_iu_j)$ (for divergence-free $u$) yields the representation

$$p(x) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} \frac{\xi_i\xi_j}{|\xi|^2}\,\widehat{(u_iu_j)}(\xi)\,e^{ix\cdot\xi}\,d\xi = R_iR_j(u_iu_j) = (-\Delta)^{-1}\partial_i\partial_j(u_iu_j). \tag{8}$$

This viewpoint is essential for two reasons. First, it makes the *nonlocality* of the equation precise: $p(x)$ is obtained from $u \otimes u = (u_i u_j)_{i,j=1}^d$ by applying an order-zero singular integral operator, so it depends on the velocity field at all spatial locations, and through the pressure, also the time derivative of $u(x,t)$. Second, Calderón–Zygmund theory provides the appropriate *a priori bounds* for these nonlocal operators: for every $1 < p < \infty$,

$$\|R_i R_j f\|_{L^p} \lesssim_{p,d} \|f\|_{L^p},$$

so, at the level of estimates, applying $\partial_i \partial_j (-\Delta)^{-1}$ does not worsen the $L^p$ integrability of the input. This is precisely the analytic mechanism that allows one to control the pressure term in Navier–Stokes and, later on, to treat the Biot–Savart law in the same unified framework.

#### 1.2.2 Sobolev Spaces

This also allows us to define Sobolev spaces in terms of frequency weights. For $s \in \mathbb{R}$, we introduce the operator $\langle \nabla \rangle^s := (I - \Delta)^{s/2}$, which acts as a Fourier multiplier with symbol $(1 + |\xi|^2)^{s/2}$. This leads to the following characterization of the Sobolev Norm

$$\|f\|_{H^s} = \|\langle \nabla \rangle^s f\|_{L^2},$$

which is equivalent to the classical Sobolev norm for $s \geq 0$ and extends naturally to all real $s$. In the same spirit, the homogeneous Sobolev space is defined via the multiplier $|\nabla|^s = (-\Delta)^{s/2}$ with symbol $|\xi|^s$, yielding

$$\|f\|_{\dot{H}^s} := \||\nabla|^s f\|_{L^2}.$$

#### 1.2.3 Definition of the Leray projector

A different, but also very useful, way of looking at the nonlocality is by viewing the pressure coming from a projection to the subspace of divergence-free vector fields. Let $n \geq 2$. Denote by

$$L^2_\sigma(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : \nabla \cdot u = 0\}$$

the closed subspace of divergence-free vector fields.

We want to find an orthogonal decomposition of the vector field $\phi \in L^2$ into its divergence-free and the gradient part

$$\phi = \mathbb{P}\phi + \nabla p, \qquad \nabla \cdot \mathbb{P}\phi = 0 \text{ and } p \in \dot{H}^1 \tag{9}$$

This is called the Helmholtz-Decomposition for an $L^p$-based review and its relation to the Navier–Stokes projection framework, see Maremonti [Mar18].

By applying the divergence to this representation $-\Delta p = -\nabla \cdot \phi$ and formally inverting the Laplacian, with a final application of the divergence, yields $\nabla p = -\nabla(-\Delta)^{-1}\nabla \cdot \phi$. Plugging this into (9), we derive the Leray projector given by the matrix operator

$$\mathbb{P} = Id + \nabla(-\Delta)^{-1}\nabla \cdot$$

By using that differentiation commutes and by invoking (8), we can represent the $j$-th component of the operator in Fourier variables as

$$\widehat{(\mathbb{P}\varphi)}_k(\xi) = \widehat{\varphi}_k(\xi) - \frac{\xi_k \xi_j}{|\xi|^2}\,\widehat{\varphi}_j(\xi), \tag{10}$$

or write the Fourier symbol of $\mathbb{P}$ as

$$\widehat{\mathbb{P}}(\xi) = 1 - \frac{\xi \otimes \xi}{|\xi|^2}, \qquad \text{for all } \xi \neq 0. \tag{11}$$

In the Fourier representation, it is easy to check that this operator projects into the subspace of divergence-free vector fields

$$(\widehat{\nabla \cdot \mathbb{P}\varphi})(\xi) = i\xi_k\, \widehat{(\mathbb{P}\varphi)}_k(\xi) = i\xi_k\, \widehat{\varphi}_k(\xi) - \frac{i\xi_k\xi_k\xi_j}{|\xi|^2}\, \widehat{\varphi}_j(\xi) = 0, \tag{12}$$

and that it is an orthogonal projection, because gradients are annihilated:

$$\widehat{(\mathbb{P}\varphi)}_k(\xi) = \widehat{(\mathbb{P}\nabla p)}_k(\xi) = i\xi_k\widehat{p}(\xi) - \frac{\xi_k\xi_j}{|\xi|^2}\, i\xi_j\widehat{p}(\xi) = i\xi_k\widehat{p}(\xi) - i\xi_k\widehat{p}(\xi) = 0. \tag{13}$$

By the theorem (3) and the obvious boundedness of the Fourier symbol, it follows that the operator $\mathbb{P}$ is bounded from $L^p \to L^p$ for all $p \in (1, \infty.)$

### 1.2.4 Pressure elimination

This allows us to eliminate the pressure term in the incompressible Navier–Stokes equation

$$\partial_t u + \mathcal{N}(u) + \nabla p = \nu \Delta u, \qquad \nabla \cdot u = 0,$$

where we write $\mathcal{N}(u) = (u \cdot \nabla)u$ for the nonlinear term. Applying the Leray projector $\mathbb{P}$ and using that $\mathbb{P}\nabla p = 0$, together with the fact that on $\mathbb{R}^d$ the Laplacian commutes with $\mathbb{P}$, yields the projected equation

$$\partial_t u + \mathbb{P}\mathcal{N}(u) = \nu \Delta u. \tag{14}$$

Hence, for divergence-free initial data $u_0 \in L^2_\sigma$, the evolution remains entirely in the divergence-free subspace. Therefore, we have reduced the Navier-Stokes equations to a single a single equation without the pressure term. The pressure can then be recovered a posteriori, as we have seen, by the Poisson equation

$$-\Delta p = \nabla \cdot \mathcal{N}(u) = \sum_{i,j=1}^{d} \partial_i \partial_j (u_i u_j).$$

In other words, the pressure gradient is precisely the gradient part of the nonlinearity,

$$\nabla p = (I - \mathbb{P})\mathcal{N}(u).$$

### 1.2.5 Vorticity formulation in two dimensions

Now we will study a different way to remove the pressure from the equation and automatically incorporate the divergence-free constraint. In two space dimensions, the incompressible Navier–Stokes equations admit a particularly convenient scalar formulation in terms of the vorticity that achieves exactly that

$$\partial_t u + (u \cdot \nabla)u = \nu \Delta u - \nabla p, \qquad \nabla \cdot u = 0, \qquad u|_{t=0} = u_{\text{in}}, \tag{15}$$

where $u = (u_1, u_2)$ denotes the two-dimensional velocity field. The vorticity is defined by

$$\omega = \nabla^\perp \cdot u = -\partial_y u_1 + \partial_x u_2, \qquad \nabla^\perp := (-\partial_y, \partial_x). \tag{16}$$

Applying the operator $\nabla^\perp\cdot$ to (15) yields

$$\partial_t(\nabla^\perp \cdot u) + \nabla^\perp \cdot \big((u \cdot \nabla)u\big) = \nu\Delta(\nabla^\perp \cdot u) - \nabla^\perp \cdot (\nabla p).$$

Since $\nabla^\perp \cdot (\nabla p) = 0$, and using incompressibility,

$$\nabla^\perp \cdot \big((u \cdot \nabla)u\big) = u \cdot \nabla(\nabla^\perp \cdot u),$$

we obtain the scalar vorticity equation

$$\partial_t\omega + (u \cdot \nabla)\omega = \nu\Delta\omega. \tag{17}$$

To close this equation, we need to recover the velocity field from the vorticity. In two dimensions, this is encoded in the div–curl system

$$\nabla \cdot u = 0, \qquad \nabla^\perp \cdot u = \omega. \tag{18}$$

Motivated by the divergence-free condition, we introduce the stream function $\psi$ by

$$u = \nabla^\perp\psi. \tag{19}$$

This automatically enforces incompressibility, since

$$\nabla \cdot u = \nabla \cdot \nabla^\perp\psi = 0.$$

Substituting (19) into (16), we find

$$\omega = \nabla^\perp \cdot u = \nabla^\perp \cdot \nabla^\perp\psi = \Delta\psi.$$

Hence, the stream function solves the Poisson equation

$$\Delta\psi = \omega. \tag{20}$$

Using the inverse Laplacian introduced earlier, we may therefore write

$$\psi = \Delta^{-1}\omega, \qquad u = \nabla^\perp\Delta^{-1}\omega. \tag{21}$$

This is the Biot–Savart law in operator form.

It is precisely here that the Calderón–Zygmund theory enters again. Indeed, differentiating (21), we obtain

$$\nabla u = \nabla\nabla^\perp\Delta^{-1}\omega,$$

so $\nabla u$ is given by a singular integral operator of order zero acting on $\omega$. In Fourier variables,

$$\widehat{\nabla u}(\xi) = -\frac{\xi \otimes \xi^\perp}{|\xi|^2}\,\widehat{\omega}(\xi), \qquad \xi^\perp := (-\xi_2, \xi_1), \tag{22}$$

which is a homogeneous multiplier of degree zero. Therefore, by the theorem (2) or equivalently the Mikhlin theorem, for every $1 < p < \infty$, the velocity gradient is bounded by the vorticity

$$\|\nabla u\|_{L^p(\mathbb{R}^2)} \lesssim \|\omega\|_{L^p(\mathbb{R}^2)}. \tag{23}$$

If one inserts the explicit Newtonian potential in two dimensions, one recovers the classical kernel form of the Biot–Savart law,

$$u(x) = \int_{\mathbb{R}^2} K(x-y)\,\omega(y)\,dy, \qquad K(z) = \frac{z^\perp}{2\pi|z|^2}, \tag{24}$$

where $z^\perp = (-z_2, z_1)$. We therefore can rewrite the incompressible Navier–Stokes equations in two dimensions as

$$\partial_t\omega + u\cdot\nabla\omega = \nu\Delta\omega, \qquad u = \nabla^\perp\Delta^{-1}\omega. \tag{25}$$

This formulation is particularly useful in two dimensions, as it reduces the number of unknowns from three (the two components of $u$ and the pressure $p$) to two (the vorticity $\omega$ and the stream function $\psi$), while automatically enforcing the incompressibility condition. This also allows us to derive the enstrophy balance law, by multiplying the equation (25) by $\omega$ and integrating:

$$\frac{1}{2}\frac{d}{dt}\|\omega\|^2 = \int \omega(u\cdot\nabla\omega)\,dx + \nu\int \omega\Delta\omega\,dx$$
$$\frac{1}{2}\frac{d}{dt}\|\omega\|^2 = -\int \omega^2\nabla\cdot u\,dx - \nu\|\nabla\omega\|^2 = -\nu\|\nabla\omega\|^2,$$

where we used integration by parts, and if we use the incompressibility condition $\nabla\cdot u = 0$ and integrate from 0 to $t$, we obtain

$$\frac{1}{2}\|\omega(t)\|^2 + \nu\int_0^t \|\nabla\omega\|^2\,dt = \frac{1}{2}\|\omega_0\|^2 \tag{26}$$

So the enstrophy is not conserved but rather dissipated due to the viscosity. Here we only have decay on the heat timescale, it is a purely diffusive law. The goal later in the thesis is to show a quicker decay, enhanced dissipation.

## 1.3 Duhamel formula and Scaling Criticality

Returning to the incompressible Navier–Stokes equations on the divergence-free subspace $L^2_\sigma$, after elimination of the pressure by the Leray projection $\mathbb{P}$, as we have seen, the system takes the form

$$\partial_t u - \nu\Delta u = -\mathbb{P}\mathcal{N}(u), \tag{27}$$

where

$$\mathcal{N}(u) = (u\cdot\nabla)u.$$

In this formulation, the equation may be regarded as a semilinear heat equation, with the nonlinear transport term acting as an inhomogeneity.

If $\|u\|_X < \varepsilon$ is small, then the nonlinear term is very small ($O(\varepsilon^2)$) compared to the linear term ($O(\varepsilon)$) and we can treat it as negligible. Therefore, we can view equation (27) as a perturbation of the corresponding homogeneous linear problem

$$\partial_t u - \nu\Delta u = 0, \qquad u|_{t=0} = u_{\text{in}},$$

whose solution is given by the heat semigroup,

$$u(t) = e^{\nu t\Delta} u_{\text{in}}.$$

To derive the perturbation of the linear problem for equation (27), consider first the abstract inhomogeneous problem

$$\partial_t u - \nu \Delta u = F(t), \qquad u|_{t=0} = u_{\text{in}}.$$

Motivated by the homogeneous solution, we make the ansatz

$$u(t) = e^{\nu t\Delta} v(t),$$

where $v(t)$ is an unknown function. Differentiating in time yields

$$\partial_t u = \nu \Delta e^{\nu t\Delta} v(t) + e^{\nu t\Delta} \partial_t v(t).$$

Since also

$$\nu \Delta u = \nu \Delta e^{\nu t\Delta} v(t),$$

substitution into the equation gives

$$e^{\nu t\Delta} \partial_t v(t) = F(t).$$

Applying $e^{-\nu t\Delta}$ to both sides, we obtain

$$\partial_t v(t) = e^{-\nu t\Delta} F(t).$$

By applying the Fundamental Lemma of Calculus, we find

$$v(t) = v(0) + \int_0^t e^{-\nu s\Delta} F(s)\, ds.$$

Since $u(0) = u_{\text{in}}$, the ansatz implies $v(0) = u_{\text{in}}$, and hence

$$v(t) = u_{\text{in}} + \int_0^t e^{-\nu s\Delta} F(s)\, ds.$$

Multiplying by $e^{\nu t\Delta}$ and using the semigroup property

$$e^{\nu t\Delta} e^{-\nu s\Delta} = e^{\nu(t-s)\Delta},$$

we arrive at

$$u(t) = e^{\nu t\Delta} u_{\text{in}} + \int_0^t e^{\nu(t-s)\Delta} F(s)\, ds.$$

In the Navier–Stokes case,

$$F(s) = -\mathbb{P}\big((u \cdot \nabla) u\big)(s),$$

so that the integral formulation becomes

$$u(t) = e^{\nu t\Delta} u_{\text{in}} - \int_0^t e^{\nu(t-s)\Delta}\, \mathbb{P}\big((u \cdot \nabla) u\big)(s)\, ds.$$

This is called Duhamel's formula and is a general principle for semilinear evolution equations, for a standard treatment in this setting, see Bedrossian and Vicol [BV22]. A function satisfying this identity is called a *mild solution*. At the outset, this identity is understood in a distributional sense. The well-posedness theory developed below will show that, under suitable assumptions on the initial data, mild solutions can be constructed and controlled in $L^p$-based spaces.

At this point, it is natural to ask in which function spaces the nonlinear term can indeed be treated perturbatively. For notational convenience, we set

$$\Phi(f,g)(t) := \int_0^t e^{\nu(t-s)\Delta}\,\mathbb{P}\big((f\cdot\nabla)g\big)(s)\,ds.$$

The guiding principle is the scaling symmetry of the Navier–Stokes equations: if $u(t,x)$ is a solution, then for every $\lambda > 0$,

$$u_\lambda(t,x) := \lambda^{-1}u(\lambda^{-2}t, \lambda^{-1}x)$$

is again a solution on the corresponding rescaled time interval.

Let $X$ be a translation-invariant Banach space on $\mathbb{R}^d$, and let $X_T$ be a translation-invariant Banach space on $[0,T]\times\mathbb{R}^d$. We say that $X$ has a scaling exponent $p$ if

$$\|f_\lambda\|_X = \lambda^p\|f\|_X, \qquad f_\lambda(x) := \lambda^{-1}f(\lambda^{-1}x),$$

and that $X_T$ has scaling exponent $p$ if

$$\|f_\lambda\|_{X_T} = \lambda^p\|f\|_{X_{\lambda^2T}}, \qquad f_\lambda(t,x) := \lambda^{-1}f(\lambda^{-2}t,\lambda^{-1}x).$$

Accordingly, such spaces are called subcritical, critical, or supercritical according as $p<0$, $p=0$, or $p>0$.

From the mild formulation, one expects a fixed-point argument in $X_T$ to close only if the Duhamel term satisfies a bilinear estimate of the form

$$\|\Phi(f,g)\|_{X_T} \le C\,\|f\|_{X_T}\|g\|_{X_T},$$

with a constant $C>0$ independent of $T$. To understand in which spaces such an estimate can be expected, one considers the scaling of the operator $\Phi$. Under the Navier–Stokes scaling, one can shot that

$$\Phi(f_\lambda, g_\lambda) = \big(\Phi(f,g)\big)_\lambda.$$

Hence, applying the bilinear estimate to $f_\lambda$ and $g_\lambda$ yields

$$\|\Phi(f,g)\|_{X_{\lambda^2T}} \le C\,\lambda^p\|f\|_{X_{\lambda^2T}}\|g\|_{X_{\lambda^2T}}.$$

Thus, such a bilinear estimate is scale-invariant only in the critical case $p=0$, and hence one can expect it to be a constant $C$, independent of $T$, only in critical spaces. In the subcritical regime $p<0$, one may still obtain a comparable estimate, but typically with a constant $C(T)$ that tends to zero as $T\to0$, thereby providing the smallness needed for a local fixed-point argument. In the supercritical regime $p>0$, however, the scaling is unfavourable, so the nonlinear term is generally too strong to be treated perturbatively, even on short time intervals.

## 1.4 Well-posedness in critical spaces

The mild formulation derived in the previous subsection suggests that local well-posedness should be proved in function spaces in which the Duhamel term

$$\Phi(f,g)(t) := \int_0^t e^{\nu(t-s)\Delta}\,\mathbb{P}\big((f\cdot\nabla)g\big)(s)\,ds$$

is perturbative. The scaling of the Navier–Stokes equations identifies the threshold regularity for such a fixed-point argument: in $\mathbb{R}^d$,it can be calculated that the critical homogeneous Sobolev space for the velocity is

$$\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d).$$

Consequently, the natural critical spaces are

$$L^2(\mathbb{R}^2) \quad \text{in two dimensions,} \qquad \dot{H}^{1/2}(\mathbb{R}^3) \quad \text{in three dimensions.}$$

In vorticity variables, the corresponding critical space in two dimensions is $L^1(\mathbb{R}^2)$.

This already reflects a major difference between the two cases: in two dimensions, the velocity-critical space is the energy space $L^2$, whereas in three dimensions the critical space lies strictly above the finite-energy level. As a result, the global theory is much stronger in two dimensions than in three dimensions.

We quickly revise the definition of well-posedness: **Local and global well-posedness.** Let $X$ be a Banach space of divergence-free vector fields (or vorticity fields, depending on the formulation). We say that Navier–Stokes is

- *locally well-posed in* $X$ if for every initial datum $u_{\rm in} \in X$ there exists a time $T = T(\|u_{\rm in}\|_X) > 0$ and a unique solution
$$u \in C([0,T];X)$$
(possibly with additional regularity), such that the solution map $u_{\rm in} \mapsto u$ is continuous (often locally Lipschitz) from $X$ to $C([0,T];X)$;
- *globally well-posed in* $X$ if one may take $T = +\infty$, i.e. the solution exists uniquely for all $t \geq 0$ with continuous dependence on the data.

**Two-dimensional well-posedness.** For divergence-free data in $L^2$, the two-dimensional incompressible Navier–Stokes equations are globally well-posed, see the modern overview in Bedrossian and Vicol [BV22]. More precisely, for every

$$u_{\rm in} \in L^2_\sigma(\mathbb{R}^2),$$

there exists a unique global solution

$$u \in C([0,\infty);L^2_\sigma(\mathbb{R}^2)) \cap L^2_{\rm loc}([0,\infty);\dot{H}^1(\mathbb{R}^2)),$$

depending continuously on the initial data. In particular, the criticality of $L^2$ in two dimensions is compatible with a global theory. Equivalently, in the vorticity formulation, one may regard $L^1$ as the scale-invariant space.

**Three-dimensional well-posedness.** In three dimensions, the critical theory is more delicate. A classical critical-space result is the theorem of Fujita and Kato [FK64], which gives local well-posedness in the critical space $\dot{H}^{1/2}$ and global existence for small initial data.

**Theorem 4** (Fujita–Kato)**.** *Let $u_{\mathrm{in}} \in \dot{H}^{1/2}(\mathbb{R}^3)$ be divergence-free. Then there exists*

$$T = T(\|u_{\mathrm{in}}\|_{\dot{H}^{1/2}}) > 0$$

*and a unique mild solution to* (27) *such that*

$$u \in C([0,T]; \dot{H}^{1/2}(\mathbb{R}^3)) \cap L^2(0,T; \dot{H}^{3/2}(\mathbb{R}^3)).$$

*Moreover, the data-to-solution map is locally Lipschitz in $\dot{H}^{1/2}$. If, in addition,*

$$\|u_{\mathrm{in}}\|_{\dot{H}^{1/2}} < \varepsilon_0$$

*for some universal constant $\varepsilon_0 > 0$, then the solution exists globally in time.*

## 1.5 Stationary Solutions: Couette and Poiseuille flow

In this thesis, we study stability properties of two canonical steady shear flows: the *Couette flow* and the *Poiseuille flow*. Both belong to the class of *parallel* (or *shear*) flows, i.e. velocity fields of the form

$$u^S(x,y) = (U(y), 0), \tag{28}$$

where the velocity points in the streamwise $x$–direction and varies only in the transversal $y$–direction.

We briefly verify when such profiles solve the incompressible Navier–Stokes equations in two dimensions,

$$\partial_t v + v \cdot \nabla v + \nabla p = \nu \Delta v, \qquad \nabla \cdot v = 0, \tag{29}$$

with viscosity $\nu > 0$. For $u^S$ as in (28), incompressibility holds automatically: $\nabla \cdot u^S = \partial_x U(y) + \partial_y 0 = 0$. Moreover, the transport term vanishes, since $U$ does not depend on $x$:

$$(u^S \cdot \nabla) u^S = (U(y)\partial_x)(U(y), 0) = (0,0).$$

Finally,

$$\Delta u^S = (\partial_{yy} U(y), 0) = (U''(y), 0).$$

Hence, a *steady* solution must satisfy

$$\nabla p^S = \nu (U''(y), 0). \tag{30}$$

In particular, the second component implies $\partial_y p^S = 0$, so the pressure depends only on $x$. Therefore $\partial_x p^S$ is independent of $y$, and (30) forces $U''(y)$ to be constant.

The vorticity in the shear case reduces to

$$\omega^S(y) = -U'(y).$$

Thus, in the shear setting, the vorticity is (minus) the local shear rate: it measures how rapidly the streamwise velocity changes across the channel.

The only shear profiles of the form (28) that are exact steady solutions are linear (Couette-type) or quadratic (Poiseuille-type). We now derive these two profiles from the standard physical setups.

**No-slip boundary condition.** For viscous flows along a rigid boundary, the standard closure is the *no-slip* condition: the fluid velocity equals the wall velocity at the boundary. Mathematically, this is a Dirichlet condition for the Navier–Stokes equations. For a channel with walls at $y = 0$ and $y = h$ one prescribes

$$v(t, x, 0) = V_{\mathrm{wall}}^{(0)}, \qquad v(t, x, h) = V_{\mathrm{wall}}^{(h)}.$$

Physically, viscosity and molecular interactions in a thin layer near the wall prevent tangential sliding, so fluid particles adjacent to the boundary are carried along with the boundary motion, for the physical discussion and caveats surrounding this assumption, see [LBS07]. In the Couette setup, this corresponds to one moving and one stationary wall, while in the Poiseuille setup both walls are stationary.

### 1.5.1 Deriving the Couette flow (wall-driven)

Consider a channel bounded by two parallel plates at $y = 0$ and $y = h$. Suppose the lower plate is at rest and the upper plate moves with constant speed $U_0$ in the $x$–direction. Assuming a fully developed steady shear flow (28) and no imposed pressure gradient ($\partial_x p^S = 0$), (30) yields

$$0 = \nu U''(y),$$

so $U$ is linear. Imposing the no-slip boundary conditions

$$U(0) = 0, \qquad U(h) = U_0,$$

we obtain

$$U(y) = \frac{U_0}{h}\, y.$$

This is the Couette flow; it illustrates the notion of shear: adjacent fluid layers move with different streamwise speeds. Its vorticity is constant:

$$\omega(y) = -U'(y) = -\frac{U_0}{h}.$$

This reflects *uniform shear*: every fluid layer is strained at the same rate.

### 1.5.2 Deriving the Poiseuille flow (pressure-driven)

Now, assume both plates are stationary, so $U(0) = U(h) = 0$, but a constant pressure drop drives the fluid in the $x$–direction. Equivalently, we impose a constant streamwise pressure gradient

$$\partial_x p^S = -G \qquad (G > 0).$$

Then (30) becomes

$$-G = \nu U''(y),$$

so $U''(y) = -G/\nu$ is constant and $U$ is quadratic. Integrating twice and using $U(0) = U(h) = 0$ gives

$$U(y) = \frac{G}{2\nu}\, y(h - y).$$

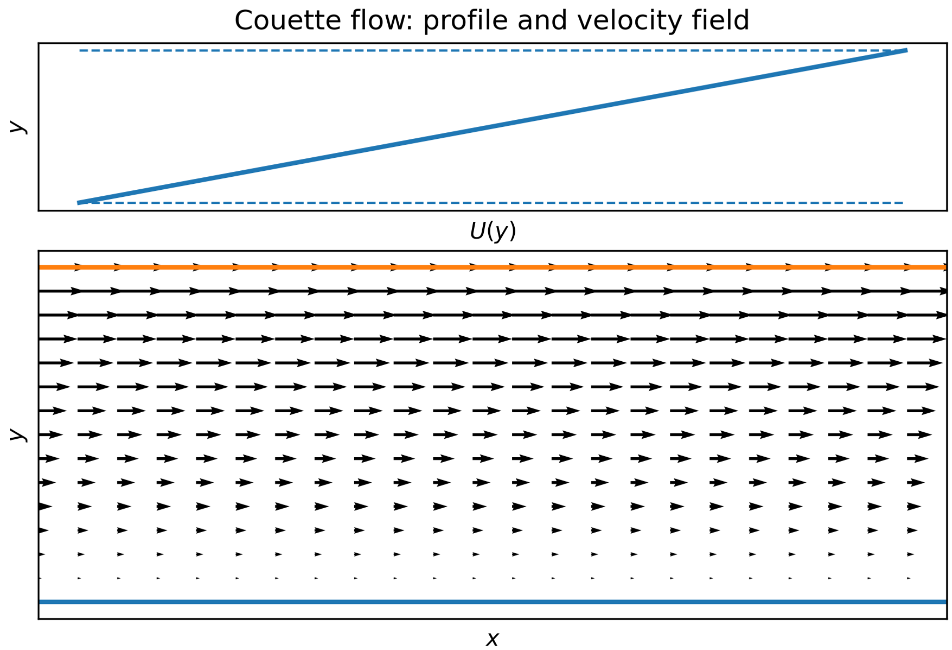


Figure 1: Couette flow in a channel: $u = (U(y), 0)$ with linear profile $U(y) = y$ (dimensionless).

This is the (plane) Poiseuille flow: a parabolic profile with maximal speed at the channel center $y = h/2$ and no-slip at the walls. Its vorticity is affine in $y$:

$$\omega(y) = -U'(y) = -\frac{G}{2\nu}(h - 2y) = \frac{G}{\nu}\Big(y - \frac{h}{2}\Big).$$

In particular, $\omega$ changes sign across the centerline $y = h/2$, indicating that the shear (and hence the local sense of rotation) is opposite in the upper and lower halves of the channel, for the general Navier–Stokes and shear-flow background, see Bedrossian and Vicol [BV22].

**Why Couette and Poiseuille flows require a perturbative formulation.** The classical well-posedness results above are formulated for the full velocity field in finite-energy or critical spaces such as

$$L^2(\Omega), \qquad H^s(\Omega), \qquad \dot{H}^{\frac{d}{2}-1}(\Omega).$$

However, on unbounded domains, the shear flows of interest here do not belong to these spaces. Indeed,

$$U_{\text{Couette}}(y) = (y, 0), \qquad U_{\text{Pois}}(y) = (y^2, 0)$$

have polynomial growth in $y$, and therefore are not elements of $L^2(\mathbb{R}^2)$ or of the usual Sobolev spaces built on $L^2$. Likewise, their vorticities,

$$\omega_{\text{Couette}} = -1, \qquad \omega_{\text{Pois}} = -2y,$$

do not lie in the natural critical vorticity spaces on $\mathbb{R}^2$, such as $L^1$ or $L^2$.

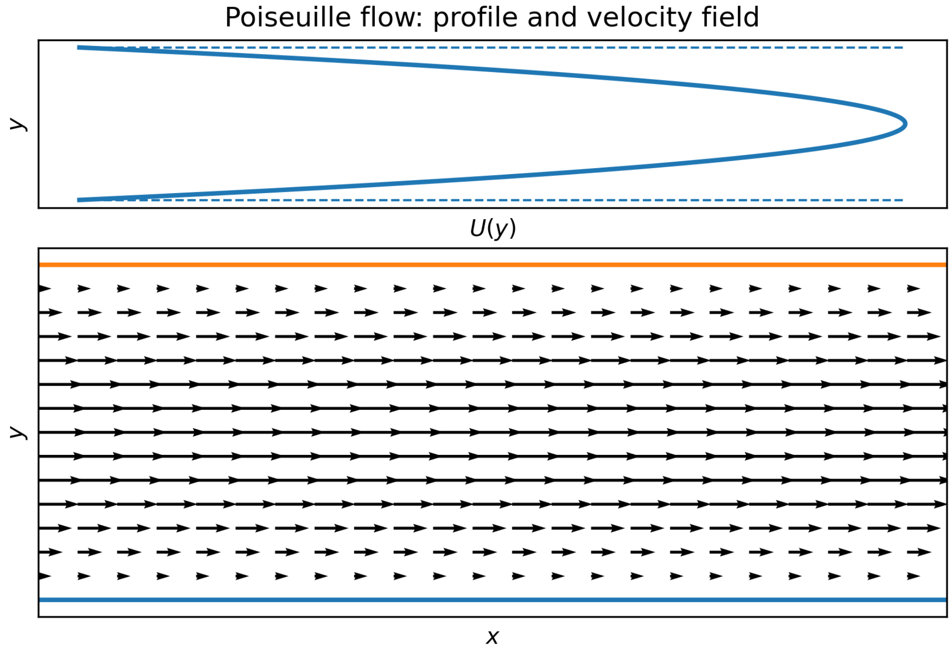


Figure 2: Poiseuille flow in a channel: $u = (U(y), 0)$ with parabolic profile $U(y) = 4y(1 - y)$ (dimensionless).

For this reason, one does not study well-posedness for the full solution $u$ in a finite-energy space. Instead, one writes

$$u = U + v,$$

where $U$ is the prescribed background shear and $v$ is the perturbation, and then proves well-posedness directly for $v$. In velocity form, the perturbation satisfies

$$\partial_t v + U \cdot \nabla v + v \cdot \nabla U - \nu \Delta v + \nabla p = -(v \cdot \nabla)v, \qquad \nabla \cdot v = 0.$$

Thus, the linear part is no longer the heat operator alone, but the Navier–Stokes linearization around the shear flow $U$. Correspondingly, the relevant mild formulation is built from the semigroup generated by this linearized operator.

This perturbative viewpoint is the natural framework for the stability problems considered later. The background flow $U$ is not itself a finite-energy object, but the perturbation $v$ may still be measured in finite-energy, or Sobolev spaces. It is precisely in this sense that one studies the stability of Couette and Poiseuille flow on unbounded domains.

**Why boundaries are hard** On boundaryless domains such as $\mathbb{T}^d$ or $\mathbb{R}^d$, the linearized operators can be treated by Fourier analysis and integration by parts produces no boundary terms, so one can build closed energy estimates mode-by-mode. In a bounded domain, however, integration by parts generates boundary contributions that must be controlled through trace estimates and the specific boundary conditions, which typically costs regularity and complicates hypocoercive constructions. More fundamentally, the inviscid limit $\nu \to 0$ is singular in a precise analytical sense in the presence of solid walls; see Kato's classical criterion [Kat84]. Navier–Stokes is naturally posed with no–slip, while Euler only enforces impermeability. The resulting mismatch forces the formation of thin boundary layers with large gradients, and these layers may become unstable and inject vorticity into the bulk. For this reason, we work primarily on $\mathbb{T} \times \mathbb{R}$ or $\mathbb{R}^2$, where the analysis isolates interior mixing and enhanced dissipation mechanisms without boundary-layer effects.

## 2 Linear Dynamics: Couette flow

In this chapter, we transition from the foundational properties of the Navier-Stokes equations to an analysis of their linear dynamics, specifically focusing on the Couette flow. Building upon the derivation of the Couette profile in Section 1.5 and the two-dimensional vorticity formulation introduced in Section 1.2.5, we analyze the system by decomposing the velocity field into a steady equilibrium state, which for the Couette flow is given by $u_E = (y, 0)$, and a small perturbation. By dropping the quadratic nonlinear term, we are able to derive the linearized problem. We then recast this linearized system into its vorticity formulation, providing a highly convenient scalar equation that inherently satisfies the fluid's incompressibility constraint. This structural framework provides the foundation to examine the crucial stabilizing phenomena inherent to the Couette flow, specifically enhanced dissipation and inviscid damping, which have been rigorously in the works of Bedrossian, Masmoudi, and Vicol [BMV16] and Bedrossian and Masmoudi [BM15] established. To rigorously analyze these mechanisms, we must first clarify what it means for such a flow to be stable; consequently, the following section will formally define the specific notions of linear and nonlinear stability that underpin our analysis.

### 2.1 Stability

Let us start by deriving an equation for the evolution of the perturbation by looking at the incompressible Navier-Stokes equation in two dimensions again:

$$\begin{aligned} \partial_t v + v \cdot \nabla v &= \nu \Delta v - \nabla p \\ \nabla \cdot v &= 0 \\ v_{|t=0} &= v_{in} \end{aligned}$$

Now set $v = u_E + u$, where E stands for equilibrium and $u$ for perturbation. Then we have

$$\begin{aligned} \partial_t u + u_E \cdot \nabla u + u \cdot \nabla u_E + u \cdot \nabla u &= \nu \Delta u - \nabla p \\ \nabla \cdot u &= 0 \\ u_{|t=0} &= u_{in} \end{aligned}$$

We call the linearized problem the one where we drop the quadratic term $u \cdot \nabla u$. In the Couette case we have $u_E = (y, 0)$. Noting that $u \cdot \nabla u_E = (u_2, 0)^T$, we get the linearized velocity equation:

$$\begin{aligned} \partial_t u + y \partial_x u + \begin{pmatrix} u_2 \\ 0 \end{pmatrix} &= \nu \Delta u - \nabla p \\ \nabla \cdot u &= 0 \\ u_{|t=0} &= u_{in} \end{aligned} \tag{31}$$

As we have seen, a very convenient way to write the equation is to look at the vorticity form, which is a scalar equation with built-in incompressibility and without the pressure term. To do this, we decompose the total vorticity into the steady equilibrium profile and a small perturbation: $w = w_E + \omega$.

For the Couette flow $u_E = (y, 0)$, the equilibrium vorticity is a constant $w_E = \nabla^\perp \cdot u_E = -1$. Because the background vorticity is uniform, its gradient vanishes ($\nabla w_E = 0$). Consequently, the term representing the perturbation velocity advecting the background vorticity ($u \cdot \nabla w_E$) disappears entirely. Expressing the perturbation velocity in terms of the scalar stream function $u = \nabla^{\perp}\phi$, we obtain the full **nonlinear** evolution equation for the perturbation vorticity $\omega$:

$$\begin{aligned} \partial_t\omega + y\partial_x\omega + \nabla^{\perp}\phi \cdot \nabla\omega &= \nu\Delta\omega \\ \omega &= \Delta\phi \end{aligned} \tag{32}$$

Here, the quadratic term $\nabla^{\perp}\phi \cdot \nabla\omega$ (which is equivalent to $u \cdot \nabla\omega$) captures the nonlinear self-advection of the perturbation.

We define the **linearized** problem by assuming the perturbation is small enough that this quadratic self-advection term can be neglected. Dropping $\nabla^{\perp}\phi \cdot \nabla\omega$ leaves us with the linearized vorticity equation:

$$\begin{aligned} \partial_t\omega + y\partial_x\omega &= \nu\Delta\omega \\ \omega &= \Delta\phi \end{aligned} \tag{33}$$

#### 2.1.1 Linear Stability

There are different ways to define stability in the case of the Navier-Stokes equation at high Reynolds number, where we differentiate between linear and nonlinear stability. By using the Leray Operator (1.2.3) we can define the linear Operator $\mathcal{L}_E = \mathbb{P}(-y\partial_x + \begin{pmatrix} u_2 \\ 0 \end{pmatrix} + \nu\Delta)$ which allows us to rewrite the linearized equation (31) in the following abstract form:

$$\begin{aligned} \partial_t u &= \mathcal{L}_E u \\ u_{|t=0} &= u_{in} \end{aligned}$$

With this formulation, we want to define the most classical metric of linear stability: **spectral stability**.But because our operator $\mathcal{L}_E$ involves both spatial derivatives ($\Delta$, $\partial_x$) and multiplication by the unbounded spatial coordinate $y$, $\mathcal{L}_E$ is not a bounded operator on our underlying Hilbert space. For such operators, the spectrum must be formally defined via the resolvent set.

**Definition 4.** *Let $H$ be a Hilbert space and let $A : D(A) \to H$ be a closed linear operator with domain $D(A) \subset H$. The resolvent set $\rho(A) \subset \mathbb{C}$ is given by*

$$\rho(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is invertible on } D(A) \to H\}.$$

*The spectrum is then defined as $\sigma(A) = \mathbb{C} \setminus \rho(A)$.*

This allows us to define spectral stability for our operator.

**Definition 5** (Spectral stability)**.** *Let $H$ be a Hilbert space (e.g. $H = L^2_\sigma$), and let $\sigma(\mathcal{L}_E)$ denote the spectrum of the operator obtained by linearizing about an equilibrium $u_E$. We call $u_E$* spectrally stable *in $H$ if*

$$\sigma(\mathcal{L}_E) \cap \{\, c \in \mathbb{C} : \operatorname{Re} c > 0 \,\} = \varnothing.$$

*It is* spectrally unstable *if*

$$\sigma(\mathcal{L}_E) \cap \{\, c \in \mathbb{C} : \operatorname{Re} c > 0 \,\} \neq \varnothing,$$

*and* neutrally stable *if it is spectrally stable but*

$$\sigma(\mathcal{L}_E) \cap \{\, c \in \mathbb{C} : \operatorname{Re} c = 0 \,\} \neq \varnothing.$$

Working with this linearized vorticity equation makes assessing spectral stability remarkably straightforward. If we define the linearized vorticity operator as $\widetilde{\mathcal{L}}_E = -y\partial_x + \nu\Delta$ and remember the enstrophy dissipation:

$$\langle \widetilde{\mathcal{L}}_E \omega, \omega \rangle \frac{1}{2}\frac{d}{dt}\|\omega\|_{L^2}^2 = -\nu\|\nabla\omega\|_{L^2}^2 \leq 0 \qquad \text{if } \omega \in H^1. \tag{34}$$

We can apply Grönwall's inequality to this differential inequality, which guarantees that the $L^2$-norm of the perturbation is monotonically non-increasing, meaning $\|\omega(t)\|_{L^2} \leq \|\omega_{in}\|_{L^2}$ for all $t \geq 0$. In the language of functional analysis, this non-increasing property means that our operator $\widetilde{\mathcal{L}}_E$ is strictly dissipative. By the Lumer-Phillips theorem, a direct consequence of the Hille-Yosida theorem for contraction semigroups, any densely defined, closed, dissipative operator generates a strongly continuous contraction semigroup. A fundamental property of such semigroups is that the spectrum of their generator is entirely confined to the left half of the complex plane ($\mathrm{Re}(c) \leq 0$). Therefore, it directly follows that the linearized Couette flow is spectrally stable on $H^1$.

A different notion of linear stability is the following definition.

**Definition 6** (Lyapunov stability)**.** *For two given norms $X$ and $Y$, the equilibrium $u_E$ is called* ***Lyapunov stable*** *if*

$$\|u\|_Y \lesssim \|u_{in}\|_X.$$

One might naturally assume that spectral stability guarantees Lyapunov stability, but this is a dangerous misconception—even for simple, finite-dimensional systems of ODEs. The two definitions are not equivalent when the underlying linear operator is not diagonalizable (the infinite-dimensional analog being a *non-normal* operator). A canonical example of this phenomenon, which is directly relevant to our hydrodynamic stability analysis, is given by the following linear system:

$$\partial_t \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{35}$$

The matrix $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ is not diagonalizable. Its spectrum is strictly $\{0\}$, making it neutrally spectrally stable. However, it is clearly *not* Lyapunov stable, as the exact solutions are given by:

$$\begin{cases} x_1(t) = x_1(0) + tx_2(0) \\ x_2(t) = x_2(0). \end{cases} \tag{36}$$

Because of the off-diagonal term, the solution for $x_1$ exhibits secular growth in time, violating Lyapunov stability.

We can modify this example to make it strictly spectrally stable by introducing a small dissipative parameter $\nu > 0$, mimicking the role of viscosity in the Navier-Stokes equations:

$$\partial_t \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\nu & 1 \\ 0 & -\nu \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{37}$$

For any $\nu > 0$, both eigenvalues are exactly $-\nu$. The system is now strictly spectrally stable and, ultimately, Lyapunov stable. However, as pointed out by Orr in 1907 [Orr07], the non-normality of the matrix implies that the system will experience massive *transient growth* before the exponential decay takes over. As $\nu \to 0$, the system degenerates, leading to the optimal upper bound:

$$\sup_{t\geq 0} \|(x_1(t), x_2(t))\| \lesssim \nu^{-1}\|(x_1(0), x_2(0))\|. \tag{38}$$

Hence, while the problem is linearly stable for all $\nu > 0$, the solutions can grow arbitrarily large as $\nu \to 0$. In high Reynolds number shear flows, this massive transient behavior can amplify infinitesimally small linear perturbations until they become large enough to activate the nonlinear terms of the Navier-Stokes equations, bypassing linear stability entirely and triggering a subcritical transition to turbulence. This mathematical framework beautifully reconciles theory with physical reality: Ever since the seminal experiments of Osborne Reynolds in 1883 [Rey83], laboratory observations have routinely demonstrated that shear flows transition to turbulence at high Reynolds numbers, despite being unconditionally linearly stable.

### 2.1.2 Nonlinear Stability

As we have seen, linear stability analysis is fundamentally limited for our shear flows. Because the linearized operator is highly non-normal, perturbations can experience transient amplification before viscous decay eventually takes over.

Physical experiments and mathematical theory both demonstrate that the permissible size of an initial perturbation strictly depends on the viscosity $\nu$. As $\nu \to 0$ (the high Reynolds number limit), the threshold for maintaining stability shrinks rapidly toward zero.

This critical dependence motivates us to explicitly quantify exactly how small the initial perturbation must be, relative to the viscosity, to guarantee that the flow not only avoids turbulence but eventually returns to equilibrium. This leads to the following definition:

**Definition 7** (Quantitative asymptotic stability)**.** *For two given norms $X$ and $Y$, the equilibrium $u_E$ is called **asymptotically stable with exponent** $\gamma$ if there exists a smallness condition relative to the viscosity, specifically:*

$$\|u_{in}\|_X \ll \nu^\gamma \implies \begin{cases} \|u(t)\|_Y \ll 1 & \text{for all } t > 0, \\ \|u(t)\|_Y \to 0 & \text{as } t \to \infty. \end{cases}$$

This definition establishes the concept of a *transition threshold*, scaling as $\mathcal{O}(\nu^\gamma)$. This immediately raises one of the most central questions in modern hydrodynamic stability theory: **What is the smallest exponent $\gamma$ such that the equilibrium $u_E$ is quantitatively asymptotically stable?**

To determine this optimal threshold, we must precisely quantify how the background shear interacts with viscosity to accelerate the decay of perturbations and counteract transient growth. The subsequent section will introduce the coordinate transformations required to rigorously analyze the most crucial stabilizing phenomenon in shear flows: *enhanced dissipation*.

## 2.2 Enhanced Dissipation

For the following discussion, we will restrict our domain to $\mathbb{T} \times \mathbb{R}$. In a loose sense, this geometry mimics the interior dynamics of a very long pipe or channel. The periodic streamwise direction ($x \in \mathbb{T}$) represents a fully developed flow. Meanwhile, taking the transverse direction to be unbounded ($y \in \mathbb{R}$) is a deliberate mathematical idealization to avoid the complications of boundary terms. While a physical pipe possesses solid walls, these boundaries generate thin boundary layers that can become unstable and inject vorticity into the bulk of the fluid. By removing the walls and working on $\mathbb{T} \times \mathbb{R}$, we effectively isolate the “pure” interior of the pipe.

f the initial data is small and the Reynolds number it is reasonably to assume that the transport term in the equation (34) takes over. Therefore, we introduce the following coordinate transformation to mode out the background flow:

$$
\begin{aligned}
x &= x - yt \\
Y &= y
\end{aligned}
$$

We use the convention that capital letters denote variables in the sheared coordinates $(X, Y)$, while lowercase denotes the original $(x, y)$:

$$
U(t, X, Y) = u(t, x, y), \qquad \Omega(t, X, Y) = \omega(t, x, y).
$$

In these coordinates, the operators become

$$
\nabla_L = \begin{pmatrix} \partial_X \\ \partial_Y - t\partial_X \end{pmatrix}, \qquad \Delta_L = \partial_X^2 + (\partial_Y - t\partial_X)^2,
$$

by the chain rule, so that

$$
\nabla u = \nabla_L U, \qquad \Delta u = \Delta_L U.
$$

Then the vorticity–stream function system reads

$$
\begin{aligned}
\partial_t \Omega - y\partial_X \Omega + y\partial_X \Omega + \nabla_L^\perp \Phi \cdot \nabla_L \Omega &= \nu \Delta_L \Omega, \\
\Omega &= \Delta_L \Phi,
\end{aligned}
$$

where the following cancellation occurs:

$$
\begin{aligned}
\nabla_L^\perp \Phi \cdot \nabla_L \Omega &= (-\partial_Y + t\partial_X)\Phi \partial_X \Omega + \partial_X \Phi (\partial_Y - t\partial_X)\Omega \\
&= -\partial_Y \Phi \partial_X \Omega + \partial_X \Phi \partial_Y \Omega \\
&= \nabla^\perp \Phi \cdot \nabla \Omega.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
\partial_t \Omega + \nabla^\perp \Phi \cdot \nabla \Omega &= \nu \Delta_L \Omega, \\
\Omega &= \Delta_L \Phi,
\end{aligned} \tag{39}
$$

and if we drop the nonlinear term, we obtain the simple linearized equation

$$
\partial_t \Omega = \nu \Delta_L \Omega, \tag{40}
$$

which is just a heat equation with a time-dependent diffusion operator.

In order to continue our analysis we will quickly define the necessary Fourier analysis to handle our exact geometry. Since our domain is periodic in the streamwise direction ($x \in \mathbb{T}$), it is natural to analyze the fluid's dynamics mode-by-mode. For any sufficiently regular function $f(t, x, y)$, we define its **streamwise Fourier decomposition** by

$$
f(t, x, y) = \sum_{j \in \mathbb{Z}} a_j(t, y)\, e^{ijx}, \qquad \text{where} \quad a_j(t, y) = \frac{1}{2\pi} \int_{\mathbb{T}} f(t, x, y)\, e^{-ijx}\, dx. \tag{41}
$$

To study the energy at specific spatial scales, it is useful to group these frequencies into **symmetric frequency bands** $f_k$ for non-negative wavenumbers $k \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$:

$$f_k(t,x,y) := \sum_{|j|=k} a_j(t,y)\, e^{ijx}. \tag{42}$$

Because physical quantities like velocity and vorticity are strictly real-valued, their Fourier coefficients satisfy the Hermitian symmetry $a_{-k} = \overline{a_k}$. This conveniently simplifies the non-zero frequency bands ($k \geq 1$) to strictly real functions localized at the frequencies $\{\pm k\}$:

$$f_k(t,x,y) = a_k e^{ikx} + \overline{a_k e^{ikx}} = 2\,\Re\big(a_k(t,y)\, e^{ikx}\big).$$

While analyzing the streamwise modes is often sufficient, we will also frequently utilize the **mixed Fourier transform**, combining a Fourier series in $x$ with a continuous Fourier transform in $y$:

$$\widehat{f}(t,k,\eta) := \frac{1}{2\pi} \int_{\mathbb{T}} \int_{\mathbb{R}} f(t,x,y)\, e^{-i(kx+\eta y)}\, dx\, dy, \qquad k \in \mathbb{Z},\ \eta \in \mathbb{R}. \tag{43}$$

The original function can be fully recovered via the corresponding inversion formula:

$$f(t,x,y) = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \widehat{f}(t,k,\eta)\, e^{i(kx+\eta y)}\, d\eta.$$

Finally, we define the **zero mode** (the $x$-mean) and the **non-zero mode projection**:

$$f_0(t,y) := \frac{1}{2\pi} \int_{\mathbb{T}} f(t,x,y)\, dx, \qquad \text{and} \qquad f_{\neq} := f - f_0 = \sum_{k\geq 1} f_k. \tag{44}$$

In the context of shear flows, the zero mode $f_0$ captures the purely transverse background profile, while $f_{\neq}$ isolates the oscillating, zero-mean perturbations that are actively stretched and mixed by the shear.

Coming back to the equation (40) for $\Omega_0 = \frac{1}{2\pi} \int \Omega\, dX$ we obtain:

$$\partial_t \Omega_0 = \nu \partial_{YY} \Omega_0.$$

If we look at the solution in the Fourier space

$$\partial_t \widehat{\Omega_0}(t,\eta) = -\,\nu\, \eta^2\, \widehat{\Omega_0}(t,\eta), \qquad \widehat{\Omega_0}(0,\eta) = \widehat{\Omega_0^{\mathrm{in}}}(\eta),$$

$$\widehat{\Omega_0}(t,\eta) = e^{-\nu\eta^2 t}\, \widehat{\Omega_0^{\mathrm{in}}}(\eta),$$

we see that these modes decay with the dissipation time-scale $\nu^{-1}$, which is especially slow for small $\nu$.

Applying the Fourier transform to the linearized equation (40) and projecting onto the non-zero modes ($k \neq 0$), we obtain the evolution equation in frequency space:

$$\partial_t \widehat{\Omega}_{\neq}(t,k,\eta) = -\nu(k^2 + (\eta - tk)^2)\widehat{\Omega}_{\neq}(t,k,\eta) \tag{45}$$

Since this is a first-order linear ordinary differential equation in time for each frequency pair $(k, \eta)$, we can integrate it directly to find the exact solution:

$$\widehat{\Omega}_{\neq}(t, k, \eta) = \exp\left(-\nu \int_0^t (k^2 + (\eta - ks)^2)\, ds\right) \widehat{\Omega}_{\neq}^{in}(k, \eta) \tag{46}$$

Evaluating the time integral in the exponent reveals a cubic dependence on $t$. Bounding this integral from below yields the following pointwise estimate in Fourier space:

$$|\widehat{\Omega}_{\neq}(t, k, \eta)| \leq \exp(-c\nu k^2 t^3)|\widehat{\Omega}_{\neq}^{in}(k, \eta)| \tag{47}$$

Because we are strictly considering the non-zero modes ($|k| \geq 1$), the exponential decay factor can be uniformly bounded by $\exp(-c\nu t^3)$. Applying Plancherel's theorem then translates this frequency-space bound directly into an $L^2$ decay estimate in physical space:

$$\|\Omega_{\neq}\| \leq e^{-c\nu t^3}\|\Omega_{in}\| \tag{48}$$

To obtain a corresponding decay estimate for the velocity field, we utilize the Biot-Savart law (1.2.5), which in the sheared Fourier space takes the form:

$$\widehat{U}(t, k, \eta) = \frac{i\xi_L}{|\xi_L|^2}\widehat{\Omega}(t, k, \eta) = \frac{i(\eta - kt, -k)}{k^2 + (\eta - kt)^2}\widehat{\Omega}(t, k, \eta) \tag{49}$$

Taking the absolute value and again exploiting the fact that $|k| \geq 1$ for the non-zero modes, we can uniformly bound the velocity multiplier:

$$\begin{aligned}
|\widehat{U}_{\neq}| &= \frac{\sqrt{k^2 + (\eta - kt)^2}}{k^2 + (\eta - kt)^2}|\widehat{\Omega}_{\neq}| \\
&= \frac{1}{\sqrt{(\eta - kt)^2 + k^2}}|\widehat{\Omega}_{\neq}| \\
&\leq \frac{1}{|k|}|\widehat{\Omega}_{\neq}| \\
&\leq |\widehat{\Omega}_{\neq}|
\end{aligned}$$

Combining this velocity bound with our previous vorticity estimate yields the fundamental enhanced dissipation estimate:

$$\|U_{\neq}\| + \|\Omega_{\neq}\| \lesssim e^{-c\nu t^3}\|\Omega_{\neq}^{in}\| \tag{50}$$

Physically, this $\mathcal{O}(\nu^{-1/3})$ decay arises because the Couette shear advects the perturbation, stretching its vorticity into thin horizontal layers known as filaments. As these structures become progressively thinner, the effective transverse wavenumber grows linearly with time ($|\eta - kt|$), and the viscous Laplacian ($\sim \nu(\eta - kt)^2$) rapidly annihilates them. Thus, the shear acts as a powerful catalyst: it transfers enstrophy into the high-frequency regime, allowing viscosity to destroy it far more efficiently than standard diffusion. This effect is hence also a lot weaker for a smaller wavenumber $k$.

To fully appreciate the stabilizing power of the background shear, it is instructive to compare this enhanced dissipation timescale against the standard diffusive timescale. We define the timescale $T$ as the time required for the perturbation norm to strictly decay below its initial value.

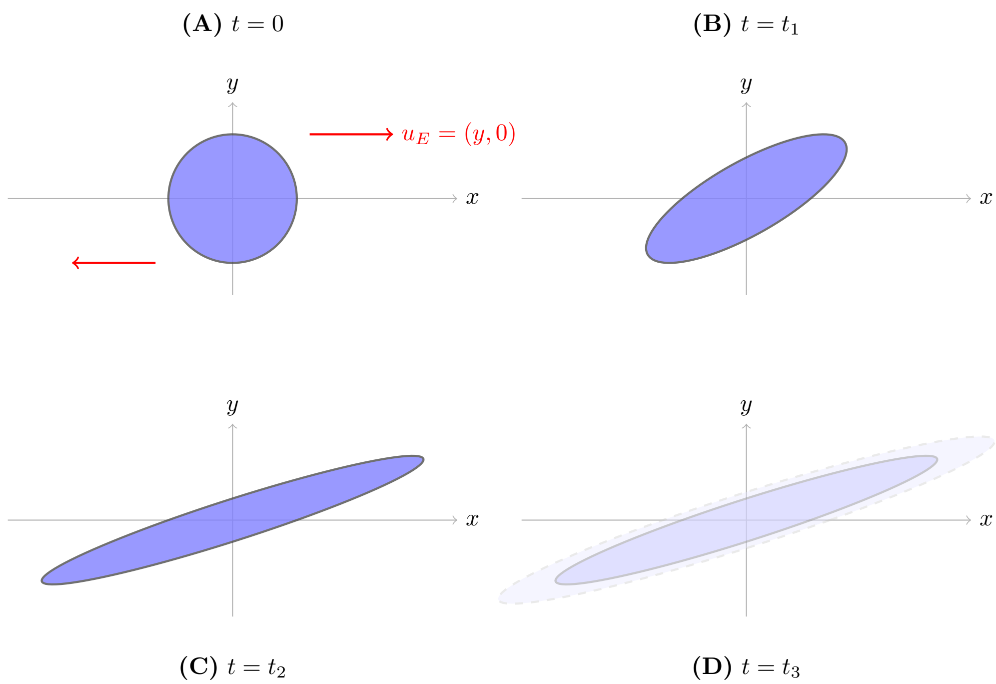


Figure 3: **Vorticity Filamentation and Enhanced Dissipation in 2D Shear Flow.** **(A)** $t = 0$: The process begins with a localized, smooth vorticity perturbation. The red arrows indicate the background Couette profile. **(B)** $t = t_1$: Kinematic shear ($y\partial_x$) stretches the blob horizontally and thins it in the transverse $y$-direction, conservatively transferring enstrophy to higher frequencies. **(C)** $t = t_2$: Full filamentation occurs. The transverse scale has shrunk proportionally to $1/t$, causing the gradients ($\partial_y \omega$) to explode. **(D)** $t = t_3$: Driven by these intense gradients, the effective viscous dissipation rate ($\sim \nu t^2$) rapidly annihilates the enstrophy, represented here by the fading and diffusing of the filaments.

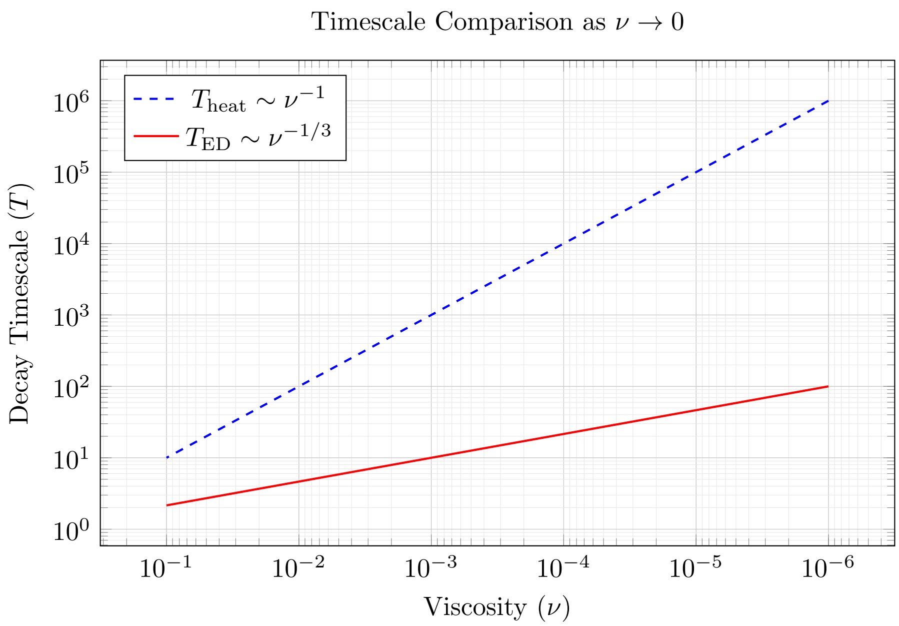


Figure 4: A log-log comparison of the standard heat dissipation timescale versus the enhanced dissipation timescale. The x-axis is reversed to represent increasing Reynolds number moving to the right.

For standard heat diffusion, this timescale scales as $T_{\text{heat}} \sim \mathcal{O}(\nu^{-1})$. However, for the non-zero modes in our sheared Couette flow ($|k| \geq 1$), the enhanced dissipation timescale scales as $T_{\text{ED}} \sim \mathcal{O}(\nu^{-1/3})$.

[BGM17], [**bedrossianSobolevStabilityThreshold2018**]

## 2.3 Inviscid Damping

The Biot-Savart law (21) reveals another fundamental phenomenon: *inviscid damping*. In sheared Fourier space, the velocity is given by:

$$\hat{U}(t,k,\eta) = \frac{i(\eta - kt, -k)}{k^2 + (\eta - kt)^2} \hat{\Omega}(t,k,\eta)$$

Substituting the enhanced dissipation estimate for the non-zero modes ($k \neq 0$), we obtain:

$$\hat{U}_{\neq}(t,k,\eta) \ \lesssim \ i \ \begin{pmatrix} \eta - kt \\ -k \end{pmatrix} \frac{1}{k^2 + (\eta - kt)^2} \, e^{-c\nu t^3} \, \hat{\Omega}^{in}_{\neq}(k,\eta)$$

This expression highlights two distinct regimes, jointly known as the **Orr mechanism**:

- As $t \to \infty$, the non-zero modes decay to zero uniformly with respect to the viscosity $\nu$.
- If $\nu t^3 \ll 1$ (before viscous effects dominate), the denominator reaches a strict minimum at the critical time $t_c = \eta/k$, leading to transient amplification of the velocity.

Using the Japanese bracket notation $\langle t\rangle := (1+t^2)^{1/2}$ and the Fourier characterization of Sobolev spaces (1.2.2) we derive polynomial decay estimates for the velocity components. For any integer $s \geq 0$, there exists a constant $C_s > 0$ such that using where we used that

$$\|U^1_{\neq}(t)\|_{H^s} = \|\langle\nabla\rangle^s U^1_{\neq}(t)\|_{L^2} \leq \frac{C_s}{\langle t\rangle}\,\|\langle\nabla\rangle^{s+1}\Omega_{\rm in}\|_{L^2} = \frac{C_s}{\langle t\rangle}\,\|\Omega_{\rm in}\|_{H^{s+1}}, \tag{51}$$

$$\|U^2_{\neq}(t)\|_{H^s} = \|\langle\nabla\rangle^s U^2_{\neq}(t)\|_{L^2} \leq \frac{C_s}{\langle t\rangle^2}\,\|\langle\nabla\rangle^{s+2}\Omega_{\rm in}\|_{L^2} = \frac{C_s}{\langle t\rangle^2}\,\|\Omega_{\rm in}\|_{H^{s+2}}. \tag{52}$$

In particular, the full velocity field satisfies:

$$\|U_{\neq}(t)\|_{H^s} \lesssim \frac{1}{\langle t\rangle}\,\|\Omega_{\rm in}\|_{H^{s+1}}. \tag{53}$$

Crucially, these decay rates are *independent of the Reynolds number* (they hold uniformly for $\nu \geq 0$).

As before, this effect is weakest at low streamwise wavenumbers. Physically, the Couette background flow advects the non-zero vorticity modes, shifting their enstrophy to progressively higher transverse frequencies through filamentation. Because the Biot-Savart operator acts as a smoothing inverse-Laplacian, it heavily penalizes these high-frequency structures. Consequently, the oscillatory components of the velocity field are damped out, leaving only the mean shear flow. This process mirrors an inverse energy cascade, as the energy in the $x$-dependent modes disappears and only the macroscopic Couette profile survives.

To mathematically quantify the transient growth predicted by the Orr mechanism, we consider the regime before enhanced dissipation takes over (i.e., for $k \neq 0$ and $\nu t^3 \ll 1$). Here, the sheared vorticity is essentially frozen in time:

$$\widehat{\Omega}(t,k,\eta) \approx \widehat{\Omega}_{\rm in}(k,\eta).$$

At the initial time $t = 0$, the velocity magnitude is:

$$\big|\widehat{U}(0,k,\eta)\big| = \frac{\sqrt{k^2+\eta^2}}{k^2+\eta^2}\,\big|\widehat{\Omega}_{\rm in}(k,\eta)\big| = \frac{1}{\sqrt{k^2+\eta^2}}\,\big|\widehat{\Omega}_{\rm in}\big|,$$

while at the Orr critical time $t_c := \eta/k$, the denominator shrinks, yielding:

$$\big|\widehat{U}(t_c,k,\eta)\big| = \frac{|k|}{k^2}\,\big|\widehat{\Omega}_{\rm in}(k,\eta)\big| = \frac{1}{|k|}\,\big|\widehat{\Omega}_{\rm in}\big|.$$

Therefore, the transient amplification factor between $t = 0$ and $t = t_c$ is precisely:

$$\frac{\big|\widehat{U}(t_c,k,\eta)\big|}{\big|\widehat{U}(0,k,\eta)\big|} \approx \frac{\sqrt{k^2+\eta^2}}{|k|}$$

In particular, if the initial perturbation is highly oscillating in the transverse direction ($|\eta| \gg |k|$), the velocity field experiences a massive but *transient* growth of magnitude $\sim |\eta|/|k|$. This amplification persists until enhanced dissipation eventually neutralizes the perturbation at times $t \gtrsim (\nu k^2)^{-1/3}$.

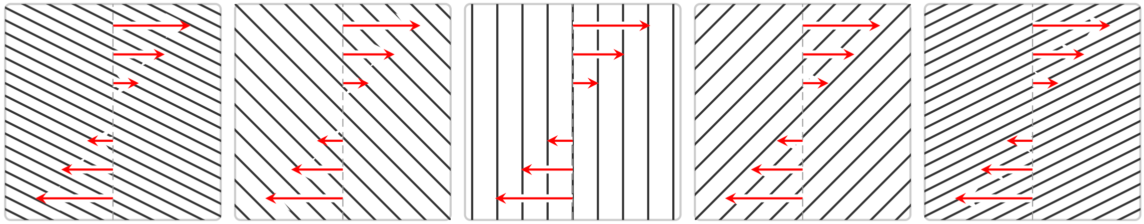


Figure 5: **A mode-by-mode visualization of the Orr mechanism.** The red arrows represent the background shear flow $u_E = (y, 0)$. The black stripes denote the level sets of the highly oscillatory transverse mode $e^{ik(x-ty)+i\eta y}$. Time advances from left to right. Initially, the perturbation is tilted against the shear. The shear actively "un-tilts" the phase lines until they become perfectly vertical at the critical time $t_c = \eta/k$ (center panel), aligning the velocity for transient amplification. Subsequently, the shear "remixes" the modes again, leading to the uniform decay described above.

### 2.4 Nonlinear Stability near Couette: energy method and bootstrap principle

In this section, we describe a stability result for perturbations of the Couette flow in the geometry $\mathbb{T} \times \mathbb{R}$ in Sobolev regularity, due to Bedrossian–Vicol–Wang [BVW18]. The theorem provides a quantitative stability threshold of order $\nu^{1/2}$ and offers a clean illustration of the energy method and the bootstrap principle in a genuinely hydrodynamic context.

We restrict to the Couette case $u_E(y) = y$, and work again in the sheared variables

$$X = x - ty, \qquad Y = y,$$

and

$$\Omega(t, X, Y) = \omega(t, x, y),$$

where $\omega$ denotes the vorticity of the perturbation. In these variables the 2D Navier–Stokes equations near Couette take the form

$$\partial_t \Omega + U \cdot \nabla_L \Omega = \nu \Delta_L \Omega, \qquad U = -\nabla_L^\perp (-\Delta_L)^{-1} \Omega, \tag{54}$$

with

$$\nabla_L = \begin{pmatrix} \partial_X \\ \partial_Y - t\partial_X \end{pmatrix}, \qquad \Delta_L = \partial_{XX} + (\partial_Y - t\partial_X)^2.$$

We denote by $\Omega_{\neq}$ the projection of $\Omega$ to nonzero $x$-Fourier modes (equivalently nonzero $X$-modes), and by $H^N$ the standard Sobolev space in $(X, Y)$.

**Theorem 5** (Bedrossian–Vicol–Wang, nonlinear stability near Couette)**.** *Let $N > 1$ and $0 < \nu \leq 1$. There exists a constant $C_N > 0$ such that the following holds. Assume the initial vorticity $\Omega_{\text{in}}$ satisfies*

$$\|\Omega_{\text{in}}\|_{H^N} = \varepsilon$$

*with*

$$\varepsilon \leq C_N^{-1} \nu^{1/2}.$$

*Then the unique global solution $\Omega$ of* (54) *with $\Omega|_{t=0} = \Omega_{\text{in}}$ obeys*

$$\|\Omega(t)\|_{H^N} + \nu^{1/2}\|\nabla_L \Omega\|_{L^2_t H^N} \lesssim_N \varepsilon \quad \text{for all } t \geq 0,$$

*and the enhanced dissipation estimate*

$$\|\Omega_{\neq}\|_{L^2_t H^N} \lesssim_N \varepsilon\, \nu^{-1/6}.$$

*In particular, perturbations of size $\varepsilon \ll \nu^{1/2}$ remain globally small in $H^N$, and the nonzero $x$-modes experience enhanced dissipation on a time scale of order $\nu^{-1/3}$, faster than the standard diffusive time scale $\nu^{-1}$.*

**Energy functional with a time-dependent Fourier multiplier**

The key idea is to design an energy functional that is equivalent to the $H^N$-norm but "builds in" the mixing structure of the Couette flow and the enhanced dissipation. This is done via a time-dependent Fourier multiplier $M(t, k, \xi)$, where $(k, \xi)$ are the Fourier variables dual to $(X, Y)$.

Define

$$A(t, D) = M(t, D)\, \langle D \rangle^N, \qquad \langle D \rangle = \sqrt{1 + |D|^2},$$

and the corresponding energy

$$E(t) := \frac{1}{2}\|A(t)\Omega(t)\|^2_{L^2}.$$

The multiplier $M$ is chosen so that:

- **Normalization and uniform boundedness.** The multiplier starts from the unweighted energy and remains uniformly comparable to 1:

  $$M(0, k, \xi) = M(t, 0, \xi) = 1, \qquad 1 \geq M(t, k, \xi) \geq c.$$

  Thus, the modified energy does not distort the Sobolev norm; it only reweights frequencies by a controlled factor. In particular, the zero mode is left untouched, reflecting that the multiplier is only needed to capture the nontrivial mixing effects of the $x$-dependent modes.

- **Equivalence with $H^N$.** For all $t$ and all $\Omega$,

  $$c\,\|\Omega(t)\|_{H^N} \leq \|A(t)\Omega(t)\|_{L^2} \leq C\,\|\Omega(t)\|_{H^N},$$

  with constants independent of $\nu$. Hence, $E(t)$ is equivalent to $\|\Omega(t)\|^2_{H^N}$.

- **Ghost dissipation.** The time derivative $\dot{M}$ is negative on nonzero modes in a way that compensates for transient growth caused by shear, with the strongest effect near the critical Orr time $t \approx \xi/k$:

  $$-\frac{\dot{M}(t, k, \xi)}{M(t, k, \xi)} \gtrsim \frac{|k|}{k^2 + |\xi - kt|^2} \qquad \text{for } k \neq 0.$$

  This creates an additional positive term in the energy identity, often called *ghost dissipation*.

- **Mild dependence on the vertical frequency.** The multiplier varies only slowly with respect to $\xi$:

$$\left|\frac{\partial_\xi M(t,k,\xi)}{M(t,k,\xi)}\right| \lesssim \frac{1}{|k|} \qquad \text{for } k \neq 0,$$

  uniformly in $\xi$. Conceptually, this means that nearby vertical frequencies are weighted in a comparable way. This is important because nonlinear interactions shift frequencies, and one needs the multiplier to be stable under such shifts, rather than changing too abruptly from one frequency to the next.

- **Compatibility with enhanced dissipation.** The multiplier is also designed so that the nonzero modes can be controlled in spacetime norms with the characteristic loss $\nu^{-1/6}$, consistent with the enhanced dissipation scale $\nu^{-1/3}$:

$$1 \lesssim \nu^{-1/6}\left(\sqrt{-\dot{M}(t,k,\eta)\, M(t,k,\eta)} + \nu^{1/2}|k,\eta-kt|\right) \qquad \text{for } k \neq 0.$$

  In other words, the damping coming from the time dependence of $M$ combines with the usual viscous dissipation in the sheared variables to yield exactly the coercivity needed at the enhanced dissipation scale.

- **Consistency across nearby frequencies.** The dissipation weight $\sqrt{-\dot{M}\,M}$ at nearby frequencies is comparable up to the frequency gap:

$$\sqrt{-\dot{M}\,M(t,k,\eta)} \lesssim \langle \eta-\xi\rangle \sqrt{-\dot{M}\,M(t,k,\xi)}.$$

  This ensures that the ghost dissipation term behaves well under frequency interactions. At a conceptual level, it says that the dissipative effect built into the multiplier is not concentrated on an extremely thin set of frequencies; instead, it is sufficiently robust to control products and convolutions arising from the nonlinearity.

With this choice of $M$, differentiating the energy gives an identity of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) + \nu\|\nabla_L A\Omega\|_{L^2}^2 + \left\|\sqrt{-M\dot{M}}\,\langle D\rangle^N\Omega\right\|_{L^2}^2 = -\int A(U\cdot\nabla_L\Omega)\,A\Omega\,\mathrm{d}X\,\mathrm{d}Y. \tag{55}$$

The two positive terms on the left correspond to *viscous dissipation* and *ghost dissipation*. The remaining properties of $M$ ensure that this modified energy is not only coercive, but also stable under the frequency shifts and convolutions generated by the nonlinear transport term.

**Sketch of the proof of Theorem 5**

The proof reduces to establishing a global *a priori* estimate for the energy $E(t)$ under a smallness condition on the initial data.

**Step 1: A priori target estimate.** Using the equivalence of norms, it suffices to show

$$\|A\Omega\|_{L_t^\infty L_{X,Y}^2} + \nu^{1/2}\|\nabla_L A\Omega\|_{L_t^2 L_{X,Y}^2} + \left\|\sqrt{-M\dot{M}}\,\langle D\rangle^N\Omega\right\|_{L_t^2 L_{X,Y}^2} \lesssim \varepsilon. \tag{56}$$

From this, the statements of Theorem 5 follow immediately, by the Norm Equivalence and together with the specific way $M$ is chosen to encode enhanced dissipation.

**Step 2: Bootstrap assumption.** By local well-posedness in $H^N$ of the 2D NavierStokes equations (1.4), (56) holds on a small time interval with a constant, say, $2\varepsilon$. Define

$$T := \sup \Big\{ t > 0 : \|A\Omega\|_{L^\infty(0,t)L^2} + \nu^{1/2} \|\nabla_L A\Omega\|_{L^2(0,t)L^2} + \big\| \sqrt{-M\dot{M}}\, \langle D \rangle^N \Omega \big\|_{L^2(0,t)L^2} \leq 8\varepsilon \Big\},$$

and assume $T < \infty$ for contradiction. This inequality is the *bootstrap assumption.*

**Step 3: Energy inequality and decomposition of the nonlinear term.** Integrating (55) from 0 to $T$ yields

$$E(T) + \nu \|\nabla_L A\Omega\|^2_{L^2(0,T;L^2)} + \Big\| \sqrt{-M\dot{M}}\, \langle D \rangle^N \Omega \Big\|^2_{L^2(0,T;L^2)} = E(0) - \int_0^T \int A(U \cdot \nabla_L \Omega)\, A\Omega \, \mathrm{d}X \, \mathrm{d}Y.$$

The nonlinear term is split into zero and nonzero $x$-frequency contributions by decomposing the velocity

$$U = U_0 + U_{\neq}$$

and writing

$$-\int_0^T \int A(U \cdot \nabla_L \Omega)\, A\Omega \, \mathrm{d}X \, \mathrm{d}Y = T_0 + T_{\neq},$$

where $T_0$ involves $U_0$ and $T_{\neq}$ involves $U_{\neq}$.

**Step 4: Estimate of the nonzero-mode term $T_{\neq}$.** Using the Biot–Savart law in sheared coordinates and the algebra property of $H^N$, one shows

$$|T_{\neq}| \lesssim_N \Big\| \sqrt{-M\dot{M}}\, \langle D \rangle^N \Omega_{\neq} \Big\|_{L^2_t L^2} \|\nabla_L \Omega\|_{L^2_t H^N} \|A\Omega\|_{L^\infty_t L^2}.$$

Under the bootstrap bounds, this gives by using $|k| \geq 1$ and the Ghost Dissipation property of $M$:

$$|T_{\neq}| \lesssim_N \varepsilon^3 \nu^{-1/2}.$$

This is the term which dictates the smallness condition $\varepsilon \ll \nu^{1/2}$: for such $\varepsilon$, the contribution of $T_{\neq}$ can be absorbed into the dissipative left-hand side.

**Step 5: Estimate of the zero-mode term $T_0$.** $U_0$ depends only on $Y$, and one can exploit cancellations and commutator estimates for $A$. After some Fourier analysis, using the uniform bound in $\xi$ and the mean value theorem on the symbol $A(k, \xi)$, one obtains

$$|T_0| \lesssim_N \|\Omega_0\|_{L^\infty_t H^N} \|\Omega_{\neq}\|^2_{L^2_t H^N}.$$

The enhanced dissipation estimate for $\Omega_{\neq}$ then yields

$$|T_0| \lesssim_N \varepsilon^3 \nu^{-1/3},$$

which is again small compared to the dissipative terms for $\varepsilon$ sufficiently small and $N > 1$.

**Step 6: Closing the estimate and improving the bootstrap.** Combining the estimates for $T_0$ and $T_{\neq}$ and the initial energy $E(0) \sim \varepsilon^2$, one arrives at

$$\|A\Omega\|^2_{L^\infty(0,T;L^2)} + \nu \|\nabla_L A\Omega\|^2_{L^2(0,T;L^2)} + \left\| \sqrt{-M\dot{M}}\, \langle D \rangle^N \Omega \right\|^2_{L^2(0,T;L^2)} \leq 4\varepsilon^2,$$

provided $\varepsilon$ is chosen small enough in terms of $N$ and $\nu$. In particular, the bootstrap constant "8" can be improved to "4", contradicting the definition of $T$ unless $T = +\infty$.

Hence, the *a priori* bound (56) holds globally in time, and Theorem 5 follows.

### Abstract bootstrap principle

What we did above is a typical bootstrap argument: we assume that the solution satisfies a certain bound on a time interval, use this assumption to prove a strictly stronger bound, and then extend the estimate beyond the original interval. In this sense, the bootstrap method is a continuity argument, and may be viewed as the continuous analogue of mathematical induction.

One can formalize this as follows, as in [Tao06, Section 1.3]. Let $I \subset [0, \infty)$ be a time interval, and for each $t \in I$ consider two statements: a *bootstrap hypothesis* $H(t)$ and a *bootstrap conclusion* $C(t)$. Here $H(t)$ is the *a priori* bound that one is willing to assume temporarily, while $C(t)$ is the improved estimate derived from the equation under that assumption.

Assume that the following four properties hold:

- **Base case.** There exists $t_0 \in I$ such that $H(t_0)$ is true.

  This initializes the argument, typically from the initial data and local well-posedness.

- **Bootstrap implication.** If $H(t)$ holds for some $t \in I$, then $C(t)$ holds as well.

  This is the nonlinear step: one inserts the assumed bound into the equation and shows that the nonlinear terms are controlled well enough to yield the improved estimate.

- **Improvement / openness step.** If $C(t)$ holds for some $t \in I$, then $H(t')$ holds for all $t'$ in a neighbourhood of $t$.

  This means that the conclusion is stronger than the hypothesis, so by continuity the weaker bound continues to hold slightly beyond time $t$.

- **Closedness.** If $t_n \to t$ in $I$ and $C(t_n)$ holds for all $n$, then $C(t)$ also holds.

  This ensures that the conclusion is stable under limits of times.

Then $C(t)$ holds for every $t \in I$. Indeed, define

$$\Omega := \{\, t \in I : C(t) \text{ holds} \,\}.$$

By the base case and the bootstrap implication, $\Omega \neq \emptyset$. By the improvement step, $\Omega$ is open in $I$. By closedness, $\Omega$ is closed in $I$. Since $I$ is connected, it follows that $\Omega = I$, and hence $C(t)$ holds for all $t \in I$.

In applications, the key point is that the conclusion is *stronger* than the hypothesis. Thus one assumes a rough bound, proves a sharper one, and then uses continuity to extend the validity of the hypothesis beyond the current time. Repeating this mechanism yields control on the whole interval.

For the purposes of this thesis, Theorem 5 provides a concrete *nonlinear* stability result near the Couette flow in a Sobolev topology, with an explicit $\nu^{1/2}$ stability threshold, complementing the linear enhanced dissipation estimates discussed previously. It also showcases the energy method and the bootstrap principle in a nontrivial setting.

# 3 The Poiseuille flow case

At the end of this chapter in 3.6 we will prove an enhanced dissipation result for the linearized Poiseuille flow on $\mathbb{R}_x \times \mathbb{R}_y$. In order to do that, we will begin by linearizing the problem around the poiseuille flow and conducting a term-by-term analysis of the resulting equation 3.1. Then we will introduce the proof idea that we use in our result, namely the hypercoercivity method 3.2. In 3.4 we will analyze the $L-$periodic equation and prove an enhanced dissipation result for this case.

## 3.1 Linearization around Poiseuille flow

As we have seen in Section (2.1), a central question in hydrodynamic stability theory is whether and how quickly a flow returns to equilibrium after a small perturbation. The first step in analyzing this problem in our setting is to study the linearization around the poiseuille flow. Understanding the linear dynamics is particularly important in this case, since, unlike the Couette flow, the Poiseuille profile has non-uniform shear, which introduces additional coupling terms in the perturbation equation, as we will see below.

Formally, we decompose the velocity field into the steady Poiseuille flow and a perturbation,

$$v = u^S + u, \qquad u^S = (y^2, 0),$$

and correspondingly write the vorticity as

$$\omega^S = -2y + \omega.$$

Here $u^S$ is a stationary solution of the Navier–Stokes equations (7), with associated pressure $p^S(x) = 2\nu x$, and $u$ and $\omega$ denote the velocity and vorticity of the perturbation, which are assumed to be small in a suitable norm.

Substituting this decomposition into the vorticity equation yields the evolution equation for the perturbation vorticity,

$$\partial_t \omega + y^2 \partial_x \omega - 2u_y + \nu \Delta \omega = -\, u \cdot \nabla \omega.$$

The *linearized equation* is obtained by neglecting the quadratic term $-u \cdot \nabla \omega$.

Since the perturbation velocity can be expressed in terms of a stream function $u = \nabla^\perp \psi$, as we have seen in (1.2.5), the system can be written as

$$\begin{aligned} \partial_t \omega + y^2 \partial_x \omega - 2\partial_x \psi - \nu \Delta \omega = 0, \\ u = \nabla^\perp \psi, \\ \omega = \Delta \psi. \end{aligned} \tag{57}$$

The representation $u = \nabla^\perp \psi$ automatically enforces the incompressibility condition, since

$$\nabla \cdot u = -\partial_x \partial_y \psi + \partial_y \partial_x \psi = 0.$$

Now we will give an interpretation of each term of the equation and try to provide a holistic explanation.

**Interpretation of the equation.**

The stream function $\psi$ is a scalar potential whose level curves correspond to the flow's streamlines. It provides a convenient way to encode the incompressibility condition: the velocity field is obtained from $\psi$ by a $90^\circ$ rotation of its gradient, while the vorticity is linked to $\psi$ through the Poisson equation $\Delta\psi = \omega$. This coupling is **nonlocal**: the velocity at a given point depends on the entire vorticity field, which becomes explicit in the Biot–Savart law.

$$u(x) = \int_{\mathbb{R}\times\mathbb{R}} K(x-y)\,\omega(y)\,dy, \qquad K(z) = \frac{z^\perp}{2\pi|z|^2}.$$

**Term-by-term analysis.**

- The transport term $y^2\partial_x\omega$ arises from the advection of the perturbation vorticity by the background flow $(y^2, 0)$. Since the streamwise velocity grows quadratically in $y$, different vertical layers move at different speeds. This shear stretches perturbations into thin filaments.
- The term $-2\partial_x\psi = -2u_y$ originates from the interaction of the perturbation velocity with the gradient of the base vorticity. Because for $U(y) = (y^2, 0)$, the base vorticity is the scalar $w(y) = -2y$, so $\nabla w = (0, -2)$ and $u \cdot \nabla w = -2u_y = -2\partial_x\psi$. Intuitively, the vertical component of the perturbation velocity pushes fluid into regions with stronger shear. Because $\psi$ is linked to $\omega$ through the Poisson equation, this coupling is nonlocal: $\partial_x\psi$ at a point depends on $\omega$ everywhere. This mechanism mixes vorticity across layers in the $y$-direction.
- The viscous term $\nu\Delta\omega$ damps small-scale structures. Passing to Fourier variables in the streamwise direction,
$$\omega(x, y, t) = \sum_{k\in\mathbb{Z}} \hat{\omega}_k(y, t)e^{ikx},$$
one finds that each mode decays as
$$\hat{\omega}_k(t) = \hat{\omega}_k(0)e^{-\nu|k|^2 t}.$$
Hence, higher frequencies decay much faster than lower ones, exactly as in the heat equation.

## 3.2 An abstract view of the hypocoercivity method

We briefly present the hypocoercivity method in an abstract setting, following the framework introduced by Villani [Vil09], for hydrodynamic and shear-flow adaptations relevant to the present setting, see Bedrossian and Coti Zelati [BZ17] and Albritton, Beekie, and Novack [ABN22]. To do this, we will use the semigroup formalism introduced in chapter(1.3).

We consider linear evolution equations of the form

$$\partial_t f = -\mathcal{L}f, \qquad \mathcal{L} = A^*A + B,$$

posed on a real Hilbert space $\mathcal{H}$, where:

- $A$ is a (possibly unbounded) linear operator,
- $A^*A$ represents a diffusive mechanism,
- $B$ is skew-adjoint, i.e. $B^* = -B$, and typically corresponds to transport.

Such operators naturally arise in linearized advection–diffusion equations, including the linearized Poiseuille problem is studied in this chapter.

**Coercivity.** A self-adjoint operator $A$ is called *coercive* if there exists $c > 0$ such that

$$\langle Ax, x\rangle \geq c\|x\|^2 \qquad \text{for all } x \in \mathcal{H}.$$

In this case, the equation $\partial_t f = -Af$ admits exponential decay. Indeed, choosing the energy functional

$$E(f) := \frac{1}{2}\|f\|^2,$$

we compute

$$\frac{d}{dt}E(f) = \langle \partial_t f, f\rangle = -\langle Af, f\rangle \leq -c\|f\|^2 = -2cE(f),$$

and Grönwall's inequality yields

$$E(f(t)) \leq E(f(0))\, e^{-2ct}.$$

If we had a more complex energy functional we would need to show additionally that the energy functional is norm-equivalent,

$$c_1\|f\|^2 \leq E(f) \leq c_2\|f\|^2,$$

to infer that the decay of $E(f)$ implies decay of the solution $f$ itself.

**Failure of coercivity and hypocoercivity.** In many problems of interest, including shear flows, the generator $\mathcal{L} = A^*A + B$ is not coercive. The diffusion $A^*A$ typically acts only in some directions (e.g. spatial derivatives), while the skew-adjoint transport operator $B$ does not dissipate energy:

$$\langle Bf, f\rangle = 0.$$

As a result, standard coercivity arguments fail to capture the true decay mechanism.

For our problem, we define the linear operator

$$\mathcal{L} = y^2\partial_x - 2\partial_x\Delta^{-1} - \nu\Delta,$$

where $A^*A = -\nu\Delta$ and $B = y^2\partial_x - 2\partial_x^{-1}\Delta^{-1}$. The proof that this is not coercive, is encoded in the enstrophy balance law, which we will prove in the next chapter

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\omega\|^2 &= -\langle A^*A\omega, \omega\rangle \; - \; \langle B\omega, \omega\rangle \\ &= \nu\,\langle \Delta\omega, \omega\rangle + 2\langle \partial_x\psi, \omega\rangle - \langle y^2\partial_x\omega, \omega\rangle \\ &= -\nu\|\nabla\omega\|^2. \end{aligned} \tag{58}$$

In particular, the dissipation comes exclusively from $A$, while $B$ conserves the $L^2$-energy.

The operator $\mathcal{L}$ is nevertheless called *hypocoercive* if solutions still decay to equilibrium, despite the lack of direct coercivity. The key idea is that the transport operator $B$ mixes the variables in such a way that the diffusion $A^*A$ eventually acts on all components.

**Commutator structure.** The hypocoercivity method exploits this mixing by incorporating commutators between $A$ and $B$. Rather than studying $A^*A + B$ directly, one constructs a modified energy involving

$$A^*A + [A,B]^*[A,B],$$

and, if necessary, higher-order commutators

$$[A,B], \quad [[A,B],B], \quad [[[A,B],B],B], \ \ldots$$

If these operators together generate sufficient control, the modified energy becomes coercive, leading to quantitative decay estimates.

For our Problem following the hypocoercivity method, we will introduce a modified energy of the form

$$\Phi(t) \ \sim \ \|\omega\|_2^2 \ + \ \alpha \, \|A\omega\|_2^2 \ + \ \beta \, \langle A\omega, \, [A,B]\omega\rangle \ + \ \gamma \, (\|[A,B]\omega\|_2^2 + \text{elliptic correction term}), \tag{59}$$

up to the elliptic correction needed to treat the nonlocal part $-\partial_x \Delta^{-1}$. with suitably small $\alpha, \beta, \gamma > 0$.

In the context of the linearized Poiseuille equation, the transport operator induces strong mixing in the streamwise direction, while diffusion acts isotropically. The hypocoercivity framework allows us to rigorously capture the resulting enhanced dissipation and to quantify the rate at which perturbations decay. [ZG23], [ABN21]

## 3.3 Known results and motivation

There are already enhanced dissipation results for the boundaryless domain $\mathbb{T} \times \mathbb{R}$ due to Coti Zelati, Elgindi, and Widmayr [ZEW20], obtained using a norm $\|\cdot\|_X$ that is closely related to the one employed in the present work. These results were further refined by Del Zotto [Zot23], who introduced an energy functional with time-dependent weights, leading to a sharpened enhanced dissipation estimate in $\|\cdot\|_{L_2}$.

This naturally raises the question of why it is interesting to consider the fully planar case $\mathbb{R}\times\mathbb{R}$. From a mathematical perspective, working on a domain that is periodic in the streamwise variable $x$ has a significant advantage: it eliminates small $x$-frequencies. Since the enhanced dissipation mechanism becomes weaker at low streamwise frequencies; this simplification avoids one of the main technical difficulties of the problem.

In contrast, on the planar domain $\mathbb{R}\times\mathbb{R}$, arbitrarily small $x$-frequencies are present and must be controlled. In this regime, shear-induced mixing alone is not sufficient to yield uniform decay, and additional mechanisms are required. A key tool in this setting is Taylor-dispersion, described in the classical works of Taylor [Tay53] and Aris [Ari56], which captures how advection by a shear flow enhances diffusion at large scales. This phenomenon has been analyzed in detail by Coti Zelati and Gallay [ZG23] for scalar advection–diffusion equations, and will play a crucial role in our analysis of the Poiseuille flow. In [AB25] Arbon and Bedrossian prove Taylor-dispersion for the Couette flow in $\mathbb{R}\times\mathbb{R}$.

Before addressing the fully planar problem, we first study the $L$-periodic case. This intermediate setting allows us to isolate the effect of small streamwise frequencies and to clarify the transition from purely periodic domains to the planar geometry. In the next section, we therefore analyze the $L$-periodic problem and prove an enhanced dissipation estimate that highlights the role of low $x$-frequencies.

### 3.4 The L-periodic problem

The periodic-strip problem is a useful intermediate model. On $\mathbb{T}_L \times \mathbb{R}$, nonzero streamwise frequencies satisfy $|k| \geq 2\pi/L$, so arbitrarily small $x$-frequencies are absent. This removes the most delicate obstruction present on the plane, and allows the enhanced-dissipation mechanism to be seen more clearly. The planar problem will then be treated as a refinement in which low-$k$ modes are controlled by Taylor-dispersion.

We first consider $x$–periodic functions with period $L > 0$,

$$f(t,x,y) = \sum_{j\in\mathbb{Z}} a_j(t,y) e^{i(2\pi j/L)x}, \qquad a_j(t,y) = \frac{1}{L}\int_{\mathbb{T}_L} f(t,x,y)\, e^{-i(2\pi j/L)x}\, dx,$$

where $\mathbb{T}_L = [0, L)$. For this functions, it is convenient to define the Fourier projections onto the $j$-th $x$–mode by

$$(\mathbb{P}_j f)(t,x,y) := a_j(t,y)\, e^{i(2\pi j/L)x}, \qquad j \in \mathbb{Z},$$

so that

$$f = \sum_{j\in\mathbb{Z}} \mathbb{P}_j f \quad \text{in } L^2(\mathbb{T}_L \times \mathbb{R}).$$

In particular, the projection onto the zero Fourier mode (the $x$–average) is

$$(\mathbb{P}_0 f)(t,y) := a_0(t,y) = \frac{1}{L}\int_{\mathbb{T}_L} f(t,x,y)\, dx,$$

which is independent of $x$. We also denote by

$$\mathbb{P}_{\neq 0} f := (I - \mathbb{P}_0) f = \sum_{j\neq 0} \mathbb{P}_j f$$

the projection onto the nonzero $x$–modes (the mean–free part). The following identities are obtained by integration by parts, and will motivate the hypocoercive energy functional used throughout the chapter.

**Proposition 1.** *Let $\omega$ be a solution to the problem (57) on $\mathbb{T} \times \mathbb{R}$ then the following identities hold*

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\omega\|^2 &= -\nu\|\nabla\omega\|^2, \\
\frac{1}{2}\frac{d}{dt}\|\nabla\omega\|^2 &= -\nu\|\Delta\omega\|^2 - 2\langle y\partial_x\omega, \partial_y\omega\rangle, \\
\frac{d}{dt}\langle \partial_y\omega, y\partial_x\omega\rangle &= -2\|y\partial_x\omega\|^2 - 4\|\partial_{xy}\psi\|^2 - 2\nu\langle \Delta\omega, y\partial_{xy}\omega\rangle, \\
\frac{1}{2}\frac{d}{dt}[\|y\partial_x\omega\|^2 + 2\|\nabla\partial_x\psi\|^2] &= -\nu\|\partial_x\omega\|^2 - \nu\|y\partial_x\nabla\omega\|^2.
\end{aligned}$$

*Proof.* We will only need integration by parts and the antisymmetry property $\langle y^n\partial_x\omega, \omega\rangle = 0$ for $n \in \mathbb{N}_0$ to derive the identities above. Lets check that first, by considering $\partial_x(\omega^2) = 2\partial_x\omega\omega$:

$$
\begin{aligned}
\langle y^n \partial_x \omega,\, \omega \rangle &= \tfrac{1}{2} \int_{\mathbb{R}}\int_{\mathbb{T}} y^n\, \partial_x\big(\omega^2\big)\, dx\, dy \\
&= \tfrac{1}{2} \int_{\mathbb{R}} y^n\, [\omega^2]_{x=0}^{x=L}\, dy \\
&= 0, \qquad n \in \mathbb{N}_0.
\end{aligned}
$$

For the first property, we test the equation with $\omega$ and obtain it by using Integration by Parts

$$
\begin{aligned}
0 &= \langle \partial_t \omega,\, \omega \rangle + \left\langle y^2 \partial_x \omega,\, \omega \right\rangle - 2\, \langle \partial_x \psi,\, \omega \rangle - \nu\, \langle \Delta\omega,\, \omega \rangle \\
&= \tfrac{1}{2} \tfrac{d}{dt}\, \|\omega\|^2 - 2\, \langle \partial_x \psi,\, \Delta\psi \rangle + \nu\, \|\nabla \omega\|^2 .
\end{aligned}
$$

Using antisymmetry, we can see that

$$
\langle \partial_x \psi,\, \Delta\psi \rangle = \langle \nabla \partial_x \psi,\, \nabla \psi \rangle = \langle \partial_x (\nabla \psi),\, \nabla \psi \rangle = \langle \partial_x \omega,\, \omega \rangle = 0,
$$

hence

$$
\tfrac{1}{2} \tfrac{d}{dt}\, \|\omega\|^2 + \nu\, \|\nabla \omega\|^2 = 0.
$$

In order to obtain the second identity we test the equation with $\Delta\omega$ and use integration by parts again to get

$$
0 = \langle \partial_t \omega,\, \Delta\omega \rangle + \left\langle y^2 \partial_x \omega,\, \Delta\omega \right\rangle - 2\, \langle \partial_x \psi,\, \Delta\omega \rangle - \nu\, \langle \Delta\omega,\, \Delta\omega \rangle .
$$

For the transport part,

$$
\begin{aligned}
\left\langle y^2 \partial_x \omega,\, \Delta\omega \right\rangle &= \left\langle y^2 \partial_x \omega,\, \partial_{xx}\omega + \partial_{yy}\omega \right\rangle = \left\langle y^2 \partial_x \omega,\, \partial_{yy}\omega \right\rangle \\
&= -2\, \langle y\, \partial_x \omega,\, \partial_y \omega \rangle - \left\langle y^2 \partial_{xy} \omega,\, \partial_y \omega \right\rangle = -2\, \langle y\, \partial_x \omega,\, \partial_y \omega \rangle ,
\end{aligned}
$$

and for the $\psi$–term,

$$
\langle \partial_x \psi,\, \Delta\omega \rangle = \langle \partial_x \psi,\, \Delta\Delta\psi \rangle = \langle \partial_x \nabla\psi,\, \nabla\Delta\psi \rangle = \langle \partial_x \Delta\psi,\, \Delta\psi \rangle = 0.
$$

This allows us to conclude the second identity

$$
\langle \partial_t \omega,\, \Delta\omega \rangle - 2\, \langle y\, \partial_x \omega,\, \partial_y \omega \rangle - \nu\, \|\Delta\omega\|^2 = 0.
$$

For the third identity, we compute

$$
\begin{aligned}
\frac{d}{dt}\, \langle y\, \partial_x \omega,\, \partial_y \omega \rangle &= \left\langle y\, \partial_x \omega,\, \partial_y \left( -y^2 \partial_x \omega + 2\, \partial_x \psi + \nu\, \Delta\omega \right) \right\rangle + \left\langle y\, \partial_x \left( -y^2 \partial_x \omega + 2\, \partial_x \psi + \nu\, \Delta\omega \right),\, \partial_y \omega \right\rangle \\
&= \nu \Big( \langle y\, \partial_x \Delta\omega,\, \partial_y \omega \rangle + \langle y\, \partial_x \omega,\, \partial_y \Delta\omega \rangle \Big) \\
&\quad - \Big( \left\langle y^3 \partial_{xx} \omega,\, \partial_y \omega \right\rangle + \left\langle y\, \partial_x \omega,\, \partial_y (y^2 \partial_x \omega) \right\rangle \Big) \\
&\quad + 2 \Big( \langle y\, \partial_{xx} \psi,\, \partial_y \omega \rangle + \langle y\, \partial_x \omega,\, \partial_{xy} \psi \rangle \Big).
\end{aligned}
$$

The $\nu$–terms combine (Integration by parts) to

$$
\langle y\, \partial_x \Delta\omega,\, \partial_y \omega \rangle + \langle y\, \partial_x \omega,\, \partial_y \Delta\omega \rangle = -2\, \langle \Delta\omega,\; y\, \partial_{xy} \omega \rangle .
$$

The $y^2\partial_x$–transport terms give

$$\begin{aligned}\left\langle y^3\partial_{xx}\omega,\, \partial_y\omega\right\rangle + \left\langle y\,\partial_x\omega,\, \partial_y(y^2\partial_x\omega)\right\rangle &= -\left\langle y\,\partial_x\omega,\, y^2\partial_{xy}\omega\right\rangle + 2\,\|y\,\partial_x\omega\|^2 + \left\langle y\,\partial_x\omega,\, y^2\partial_{xy}\omega\right\rangle \\ &= 2\,\|y\,\partial_x\omega\|^2 .\end{aligned}$$

For the $\psi$–coupling, only the mixed derivatives survive:

$$\begin{aligned}\langle y\,\partial_{xx}\psi,\, \partial_y\omega\rangle + \langle y\,\partial_x\omega,\, \partial_{xy}\psi\rangle &= -\,\langle \Delta(y\,\partial_x\psi),\, \partial_{xy}\psi\rangle + \langle y\,\partial_x\Delta\psi,\, \partial_{xy}\psi\rangle \\ &= -\,\langle y\,\partial_{xxx}\psi,\, \partial_{xy}\psi\rangle - \langle y\,\partial_{yyx}\psi,\, \partial_{xy}\psi\rangle + \dots \\ &= -2\,\|\partial_{xy}\psi\|^2 ,\end{aligned}$$

where dots represent terms that vanish due to antisymmetry. Putting these together yields the third identity.

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\,\|y\,\partial_x\omega\|^2 &= \left\langle y\,\partial_x\omega,\; y\,\partial_x(\nu\Delta\omega - y^2\partial_x\omega + 2\,\partial_x\psi)\right\rangle \\ &= \nu\,\langle y\,\partial_x\omega,\; y\,\partial_x\Delta\omega\rangle + 2\,\langle y\,\partial_x\Delta\psi,\; y\,\partial_{xx}\psi\rangle \\ &= \nu\,\|\partial_x\omega\|^2 - \nu\,\|y\,\partial_x\Delta\omega\|^2 - 4\,\langle y\,\partial_{xy}\psi,\, \partial_{xx}\psi\rangle ,\end{aligned}$$

using antisymmetry in the last step. Multiplying equation (57) by $\partial_{xx}\psi$ gives

$$\langle \partial_t\omega,\, \partial_{xx}\psi\rangle + \left\langle y^2\partial_x\omega,\, \partial_{xx}\psi\right\rangle - 2\,\langle \partial_x\psi,\, \partial_{xx}\psi\rangle = \nu\,\langle \Delta\omega,\, \partial_{xx}\psi\rangle .$$

Note that

$$\left\langle y^2\partial_x\Delta\psi,\, \partial_{xx}\psi\right\rangle = -2\,\langle y\,\partial_{xy}\psi,\, \partial_{xx}\psi\rangle - \left\langle y^2\partial_x\nabla\psi,\, \nabla\partial_{xx}\psi\right\rangle = -2\,\langle y\,\partial_{xy}\psi,\, \partial_{xx}\psi\rangle ,$$

and an integration by parts yields

$$\frac{1}{2}\frac{d}{dt}\,\|\nabla\partial_x\psi\|^2 - 2\,\langle y\,\partial_{xy}\psi,\, \partial_{xx}\psi\rangle = -\,\nu\,\|\partial_x\omega\|^2 .$$

If we put everything together, this concludes the last identity.

□

After this technical proof, we briefly interpret each of the identities obtained in Proposition 1.

The first identity is the classical *enstrophy balance*. It shows that the viscous term dissipates enstrophy through the gradient $\nabla\omega$, while all transport terms vanish by antisymmetry. This decay occurs on the natural diffusive time scale $t \sim \nu^{-1}$ and does not yet reflect any mixing effects induced by the shear. This is the effect of the selfadjoined $AA^*$ part of the operator $\mathcal{L}$.

The second identity describes the evolution of the gradient norm $\|\nabla\omega\|$. In addition to viscous dissipation, it contains a shear-induced coupling term $\langle y\partial_x\omega, \partial_y\omega\rangle$, which has no definite sign. This term reflects the redistribution of gradients caused by the non-uniform shear and indicates that transport alone can transfer energy between directions without directly dissipating it. We can also think about this of controlling $\|A\omega\|^2$ by $\langle [A,B], A\rangle$ because the term $y\partial_x\omega$ is exactly the commutator of the shear part $y^2\partial_x$ with $\partial_y$:

$$[\partial_y,\; y^2\partial_x]\omega = \partial_y(y^2\partial_x\omega) - y^2\partial_x(\partial_y\omega) = 2y\,\partial_x\omega$$

The part $\partial_x\Delta^{-1}$ is translation invariant and therefore does not contribute to the commutator with $\nabla$.

The third identity aims to control $\langle \partial_y\omega, [A,B]\omega\rangle$ . Although this quantity is not a norm, its evolution reveals a hidden coercive structure: it generates negative contributions proportional to $\|y\partial_x\omega\|^2$ and $\|\partial_{xy}\psi\|^2$. This identity is the key mechanism through which shear-induced mixing becomes analytically accessible.

Finally, the last identity corresponds to a carefully chosen combination of weighted vorticity and velocity gradients. As we have seen in the proof of this identity, this exactly cancels the term $2\nu\langle\Delta\omega, y\partial_{xy}\omega\rangle$ from identity 3. For this quantity, all transport effects cancel, and only viscous dissipation remains. The term $\|\nabla\partial_x\psi\|^2$ controls the contribution from the nonlocal term $\partial_x\Delta^{-1}$. Because $u = \nabla^{\perp}\psi$, our term in the energy functional corresponds to controlling the norm of the velocity in $y-$direction $\nabla u_2$. This illustrates the central idea of the hypocoercivity method: while neither transport nor diffusion alone yields enhanced decay, their interaction produces a coercive energy functional, leading to faster relaxation toward equilibrium. This allows us to prove the following enhanced dissipation result for the $L-$periodic case.

The key new feature is the mixed term $\langle\partial_y\omega,\, y\partial_x\omega\rangle$, which encodes the commutator between the shear and $\partial_y$ and converts transport-induced mixing into a coercive estimate. However, the nonlocal term $-\partial_x\Delta^{-1}$ produces velocity contributions that cannot be controlled by weighted vorticity alone; this is why the energy includes an elliptic correction involving $\nabla\partial_x\psi$. With these ingredients, one obtains a time-weighted hypocoercive functional whose decay yields enhanced dissipation on $T_L\times\mathbb{R}$.

The following result introduces our energy functional in the style of (59) and we will show a type of coercivity and will estimate its decay in time. The time-dependent energy functional are adapted from the $\mathbb{T}\times\mathbb{R}$ analysis in [Zot21].

**Theorem 6** (Time-weighted hypocoercive energy; after [Zot21])**.** *Let $\omega$ solve the linear problem (57) on $\mathbb{T}_L\times\mathbb{R}$.*

*Fix $0<\varepsilon<\frac{1}{36}$ and set*

$$\alpha := \varepsilon^2, \qquad \beta := \varepsilon^3, \qquad \gamma := 16\,\varepsilon^4.$$

*Define, for $t\geq 0$,*

$$\Phi(t) := \frac{1}{2}\|\omega\|_2^2 + \frac{1}{2}\,\alpha\nu t\,\|\nabla\omega\|_2^2 + 2\beta\nu t^2\,\langle\partial_y\omega,\, y\partial_x\omega\rangle + \frac{1}{2}\,\gamma\nu t^3\Big(\|y\partial_x\omega\|_2^2 + 2\|\nabla\partial_x\psi\|_2^2\Big). \tag{60}$$

*Then $\Phi$ is coercive in the sense that, for all $t\geq 0$,*

$$\Phi(t) \;\geq\; \frac{1}{2}\|\omega\|_2^2 + \frac{1}{4}\,\alpha\nu t\,\|\nabla\omega\|_2^2 + \frac{1}{4}\,\gamma\nu t^3\Big(\|y\partial_x\omega\|_2^2 + 2\|\nabla\partial_x\psi\|_2^2\Big), \tag{61}$$

*and*

$$\Phi'(t) \;\leq\; -\gamma\nu t^3\,\|\partial_x\omega\|_2^2 \qquad \textit{for all } t\geq 0. \tag{62}$$

*Proof.* Coercivity of $\Phi$: Using Cauchy–Schwarz and Young's inequality, for any $\alpha>0$,

$$2\beta\nu t^2\,|\langle\partial_y\omega,\, y\partial_x\omega\rangle| \leq 2\beta\nu t^2\,\|\partial_y\omega\|_2\,\|y\partial_x\omega\|_2 \leq \frac{\alpha\nu t}{4}\|\partial_y\omega\|_2^2 + \frac{4\beta^2}{\alpha}\,\nu t^3\|y\partial_x\omega\|_2^2.$$

Inserting this bound into (60) and using $\|\partial_y\omega\|_2^2 \leq \|\nabla\omega\|_2^2$, we obtain

$$\Phi(t) \;\geq\; \frac{1}{2}\|\omega\|_2^2 + \frac{1}{4}\,\alpha\nu t\,\|\nabla\omega\|_2^2 + \frac{1}{2}\Big(\gamma - \frac{8\beta^2}{\alpha}\Big)\nu t^3\|y\partial_x\omega\|_2^2 + \gamma\nu t^3\|\nabla\partial_x\psi\|_2^2.$$

With the above choices

$$\alpha = \varepsilon^2 \qquad \beta = \varepsilon^3 \qquad \gamma = 16\varepsilon^4,$$

one checks that

$$\gamma - \frac{8\beta^2}{\alpha} = 16\varepsilon^4 - 8\varepsilon^4 = 8\varepsilon^4 > 0,$$

which yields (61).

Dissipation of $\Phi$

$$\begin{aligned}\frac{d}{dt}\Phi(t) = & -\nu\|\nabla\omega\|^2 + \alpha\nu\|\nabla\omega\|^2 + \alpha\nu t\Big(-\|\Delta\omega\|^2 - 2(\partial_y\omega, \partial_y\omega)\Big) + 4\beta\nu(\omega_y, \partial_y\omega) \\ & + 2\beta\nu t\Big(-2(\Delta\omega, \partial_y\omega) - 2\|\partial_y\omega\|^2 - 4\|\partial_{yy}\omega\|^2\Big) \\ & + \frac{3}{2}\gamma\nu t^2\Big(\|\partial_{yy}\omega\|^2 + 2\|\nabla\partial_y\omega\|^2\Big) + \gamma\nu\Big(-\|\nabla\omega\|^2 - \|\omega_y\|^2\Big).\end{aligned}$$

By rearranging the terms, we see that

$$\frac{d}{dt}\Phi(t) = I_1 + I_2 + I_3 + I_4 + I_5. \tag{63}$$

where

$$\begin{aligned} I_1 &= -\frac{\nu}{2}\|\nabla\omega\|^2 + \alpha\nu\|\nabla\omega\|^2 - \gamma\nu^2\|\partial_y\omega\|^2, \\ I_2 &= -\alpha\nu^2 t\|\Delta\omega\|^2 - 4\beta\nu^2 t(\Delta\omega, \partial_y\omega) - \gamma\nu^2 t\|\nabla\partial_y\omega\|^2, \\ I_3 &= -\frac{\nu}{2}\|\nabla\omega\|^2 - 2\alpha\nu t(\partial_y\omega, \partial_y\omega) - \beta\nu^2 t\|\partial_y\omega\|^2 - 2\beta\nu^2 t\|\partial_{yy}\omega\|^2, \\ I_4 &= -\frac{\nu}{2}\|\nabla\omega\|^2 + 4\beta\nu(\omega_y, \partial_y\omega) - \beta\nu^2 t\|\partial_y\omega\|^2 - 2\beta\nu^2 t\|\partial_{yy}\omega\|^2, \\ I_5 &= -2\beta\nu^2\|\omega_y\|^2 + 2\beta\nu\|\partial_y\omega\|^2 + \frac{3}{2}\gamma\nu^2 t^2\Big(\|\partial_{yy}\omega\|^2 + 2\|\nabla\partial_y\omega\|^2\Big). \end{aligned}$$

If we impose the follwoing conditions on the constants $\alpha, \beta, \gamma$

$$\alpha < \frac{1}{2}, \qquad 16\beta^2 \le \alpha\gamma, \qquad 4\alpha^2 < \beta < \frac{1}{16}\alpha, \qquad 9\gamma < 4\beta,$$

we can use the Cauchy–Schwarz inequality followed by the Young inequality to get

$$I_1 \le -\gamma\nu^2\|\partial_y\omega\|^2;$$

$$I_2 \le -\frac{1}{2}\alpha\nu^2 t\|\Delta\omega\|^2 - \left(\gamma - \frac{8\beta^2}{\alpha}\right)\nu^2 t\|\nabla\partial_y\omega\|^2 < 0;$$

$$I_3 \le -\frac{1}{4}\nu\|\nabla\omega\|^2 - (\beta - 2\alpha^2)\nu^2 t\|\partial_y\omega\|^2 - 2\beta\nu^2 t\|\partial_{yy}\omega\|^2 < 0;$$

$$I_4 \le -\frac{\nu}{2}(\alpha - 4\beta)\|\nabla\omega\|^2 - \frac{\beta}{2}\nu^2 t\|\partial_y\omega\|^2 - 2\beta\nu^2 t\|\partial_{yy}\omega\|^2 < 0.$$

Moreover, from

$$(\partial_{yy}\omega, \partial_y\omega) = -\frac{1}{2}\|\partial_{yy}\omega\|^2 + \frac{1}{2}\|\partial_y\omega\|^2$$

we can deduce that

$$\|\nabla \partial_y \omega\|^2 \leq \|\partial_{yy}\omega\|^2 + 3\|\partial_y \omega\|^2,$$

hence

$$I_5 \leq -\left(2\beta - \frac{9}{2}\gamma\right)\nu^2\left(\|\partial_{yy}\omega\|^2 + 2\|\nabla\partial_y\omega\|^2\right) \leq 0.$$

Putting everything together, we obtain the upper bound on $\Phi'(t)$.

□

The coercivity and the decay of our energy functional allow us to prove a semigroup estimate, which shows exponential decay for modes $k \neq 0$. In the linear enhanced dissipation result below, $L$, the size of our periodic strip, appears for the first time. We will discuss the implications after the proof.

**Theorem 7.** *For $\omega_{in} \in L^2$ and $0 < \nu < 1$ we can find $C > 0$ and $c > 0$ such that each $k$-mode $k \neq 0$ experiences enhanced dissipation for $t \geq 0$.*

$$\|e^{\mathcal{L}_\nu t}\mathbb{P}_k(\omega_{in})\| \leq Ce^{-c\nu^{\frac{1}{2}}|k|^{\frac{1}{2}}L^{-\frac{1}{2}}t}\|\mathbb{P}_k(\omega_{in})\| \tag{64}$$

*We will write $\omega_k = \mathbb{P}_k(\omega)$ for ease of notation.*

*Proof.* By plugging in $\omega_k = e^{\mathcal{L}_\nu t}\omega_{\text{in}}$ in (61) and dropping every time-dependent term, we obtain

$$\frac{1}{2}\|e^{\mathcal{L}_\nu t}\omega_k\|^2 \leq \Phi_k(t).$$

Doing the same thing for (62), we get

$$\Phi_k^{'}(t) \leq -\gamma\nu^2 t^3\|\partial_x\omega_k\|^2 = -\gamma\nu^2(\frac{2\pi|k|}{L})^2 t^3\|e^{\mathcal{L}_\nu t}\omega_k\|^2$$

Hence we obtain

$$\begin{aligned}
\frac{1}{2}\left\|e^{\mathcal{L}_\nu t}\omega_k\right\|^2 &\leq \Phi_k(t) \\
&= \Phi_k(0) + \int_0^t \Phi_k'(s)\,ds \\
&\leq \Phi_k(0) - \gamma\nu^2\left(\frac{2\pi|k|}{L}\right)^2\int_0^t s^3\left\|e^{\mathcal{L}_\nu s}\omega_k\right\|^2 ds \\
&\leq \frac{1}{2}\|\omega_k\|^2 - \gamma\nu^2\left(\frac{2\pi|k|}{L}\right)^2\frac{t^4}{4}\left\|e^{\mathcal{L}_\nu t}\omega_k\right\|^2.
\end{aligned}$$

By rearranging, we obtain the following inequality

$$\|e^{\mathcal{L}_\nu t}\omega_k\|^2 \leq \frac{1}{1+\gamma\nu^2 2\pi^2\frac{|k|^2}{L^2}t^4}\|\omega_k\|^2 \tag{65}$$

Then we take $t_0$ such that

$$\frac{1}{1+\gamma\nu^2 2\pi^2\frac{|k|^2}{L^2}t_0^4} = \frac{1}{2}$$

So

$$t_0 = \frac{L^{1/2}}{2^{1/4}\gamma^{1/4}k^{1/2}\pi^{1/2}\nu^{1/2}}$$

Now we will write for everytime $t > t_0$ $t = \lfloor \frac{t}{t_0} \rfloor t_0 + t^*$ and $t^* \in [0, t_0)$. Because our equations are autonomous, we can use the semigroup property together with (65) for every $s \in [0, t_0]$

$$\|e^{\mathcal{L}_\nu(t_0+s)}\omega_k\|^2 \leq \frac{1}{1+\gamma\nu^2 2\pi^2 \frac{|k|^2}{L^2} t_0^4}\|e^{\mathcal{L}_\nu s}\omega_k\|^2 \leq \frac{1}{2}\|e^{\mathcal{L}_\nu s}\omega_k\|^2$$

We showed an uniform bound for a fixed time step, which allows us to use the semigroup property to bootstrap to global decay by applying the argument above iteratively to obtain

$$\|e^{\mathcal{L}_\nu(t)}\omega_k\|^2 \leq \|e^{\mathcal{L}_\nu \lfloor \frac{t}{t_0} \rfloor t_0}\omega_k\|^2 \leq (\frac{1}{2})^{\lfloor \frac{t}{t_0} \rfloor}\|\omega_k\|^2, \tag{66}$$

where we used the fact that $\|e^{\mathcal{L}_\nu t}\omega_k\|$ decays in $t$. Then, if we apply the following trick

$$\frac{1}{2}^{\lfloor \frac{t}{t_0} \rfloor} \leq \frac{1}{2}^{t/t_0 - 1} = 2 \cdot \frac{1}{2}^{t/t_0} = e^{-\ln(2)t/t_0}$$

and then plugging in for $t_0$ we get the desired result. □

**Interpretation.** Theorem (7) shows that each nonzero streamwise mode $k \neq 0$ decays on the enhanced dissipation time scale

$$t_{\mathrm{ED}}(k) \ \sim\ \nu^{\frac{1}{2}}|k|^{\frac{1}{2}}L^{-\frac{1}{2}},$$

which is much faster than the purely diffusive scale $\nu^{-1}$. In particular, for fixed $\nu$, the decay improves as $|k|$ increases: stronger oscillations in the streamwise direction are mixed more efficiently by the shear and are therefore damped more rapidly by viscosity. Conversely, the estimate degenerates as $|k| \to 0$. This precisely highlights the role of low streamwise frequencies: on a periodic domain one has the spectral gap $|k| \geq 2\pi/L$ for $k \neq 0$, so uniform enhanced dissipation follows. However, as $L \to \infty$ (or on $\mathbb{R}^2$ where $\xi \in \mathbb{R}$), arbitrarily small nonzero frequencies are present and the shear-induced mechanism alone is no longer sufficient to obtain a uniform decay rate. This is the obstruction that motivates the Taylor-dispersion arguments in the planar setting.

## 3.5 Taylor-dispersion

In Coti Zelati and Gallay [ZG23] the evolution of a passive scalar density $f = f(x, y, t)$ governed by the advection–diffusion equation

$$\partial_t f(x,y,t) + v(y)\,\partial_x f(x,y,t) \ =\ \nu\Delta f(x,y,t), \qquad (x,y) \in \Sigma,\ t > 0, \tag{67}$$

where $\nu > 0$ is the diffusivity coefficient and $\Delta = \partial_x^2 + \Delta_y$ acts on $(x, y)$ is studied on the infinite cylinder

$$\Sigma := \mathbb{R}_x \times \Omega_y \subset \mathbb{R}^{d+1}.$$

Let $v : \Omega \to \mathbb{R}$ be a smooth shear profile and set the divergence-free velocity field

$$u(x,y) := (v(y), 0)^\top.$$

Since (67) is invariant under translations in the $x$-direction, it is natural to take the (partial) Fourier transform in $x$, defined formally by

$$\widehat{f}(\xi, y, t) := \int_{\mathbb{R}} f(x, y, t)\, e^{-i\xi x}\, dx, \qquad \xi \in \mathbb{R},\ y \in \Omega,\ t > 0. \tag{68}$$

For each fixed horizontal wavenumber $\xi \in \mathbb{R}$, the function $\widehat{f}(\xi, \cdot, \cdot)$ solves

$$\partial_t \widehat{f}(\xi, y, t) + i\xi\, v(y)\, \widehat{f}(\xi, y, t) \ = \ \nu\big(-\xi^2 + \Delta_y\big)\widehat{f}(\xi, y, t), \qquad y \in \Omega,\ t > 0, \tag{69}$$

together with the same Neumann boundary condition in $y$.

In the Taylor-dispersion regime, $\xi$ is very small, so the horizontal diffusion term $-\nu\xi^2\widehat{f}$ plays only a minor role and can be removed by the change of unknown

$$\widehat{f}(\xi, y, t) = e^{-\nu\xi^2 t}\, q(\xi, y, t). \tag{70}$$

Substituting (70) into (69), we obtain the reduced equation

$$\partial_t q(\xi, y, t) + i\xi\, v(y)\, q(\xi, y, t) \ = \ \nu\, \Delta_y q(\xi, y, t), \qquad y \in \Omega,\ t > 0. \tag{71}$$

The operator only has *diffusion in* $y$, so it is not elliptic in $(x, y)$. Nevertheless, the transport term $ik\, v(y)$ couples $y$–regularity back into $x$–decay (after inverting the $x$–Fourier transform), and one observes smoothing/relaxation in directions that are not directly diffused. This is the typical setting of hypoellipticity/hypocoercivity: a skew-adjoint transport part does not dissipate by itself, but in combination with diffusion, it produces decay. With a Hypocoercivity argument and the energy functional $\Phi$ they were able to prove a more general version of the following enhanced and Taylor-dispersion result.

**Theorem 8.** *Let $v = y^2$, then there exist positive constants $\beta_0, C_3$ such that, for all $\nu > 0$, all $\xi \neq 0$, and all initial data $g_0 \in H_0^1(\Omega)$, the solution of (71) satisfies, for all $t \geq 0$,*

$$\Phi(t) \leq e^{-C_3 \lambda_{\nu,\xi} t}\, \Phi(0), \qquad \textit{where} \qquad \lambda_{\nu,\xi} := \begin{cases} \nu^{\frac{1}{2}}\, |\xi|^{\frac{1}{2}}, & \textit{if } 0 < \nu \leq \beta_0 |\xi|, \\ \dfrac{\xi^2}{\nu}, & \textit{if } 0 < \beta_0|\xi| \leq \nu. \end{cases} \tag{72}$$

**Remark 1.** *The geometry of $\Omega$ is either assumed to be the periodic strip $\Omega = \mathbb{T}$ or a smooth, bounded domain $\Omega \subset \mathbb{R}$ with homogeneous Dirichlet boundary conditions.*

Theorem (8) yields a mode-by-mode hypocoercive relaxation estimate: for every streamwise Fourier mode $\xi \neq 0$,

$$\Phi(t) \leq e^{-C_3 \lambda_{\nu,\xi} t}\, \Phi(0), \qquad \lambda_{\nu,\xi} \sim \begin{cases} \nu^{1/2}|\xi|^{1/2}, & \nu \lesssim |\xi| \quad \text{(enhanced dissipation)}, \\ \xi^2/\nu, & |\xi| \lesssim \nu \quad \text{(Taylor-dispersion)}. \end{cases}$$

The first regime reflects mixing by the Poiseuille shear, which transfers energy to small transverse scales where diffusion is efficient, giving the enhanced rate $\lambda_{\nu,\xi} \approx \sqrt{\nu|\xi|}$. In the opposite regime $|\xi| \ll \nu$, the streamwise variation is weak and the combined effect of cross-stream relaxation and diffusion produces an effective streamwise diffusion of size $\lambda_{\nu,\xi} \approx \xi^2/\nu$. The crossover occurs at $|\xi| \approx \nu/\beta_0$ (so in particular $\lambda_{\nu,\xi} \approx \nu$ when $|\xi| \approx \nu$ up to constants). For a fixed streamwise wavenumber $\xi \neq 0$, plain diffusion would give the decay rate $\nu\xi^2$ (time scale $\sim (\nu\xi^2)^{-1}$).

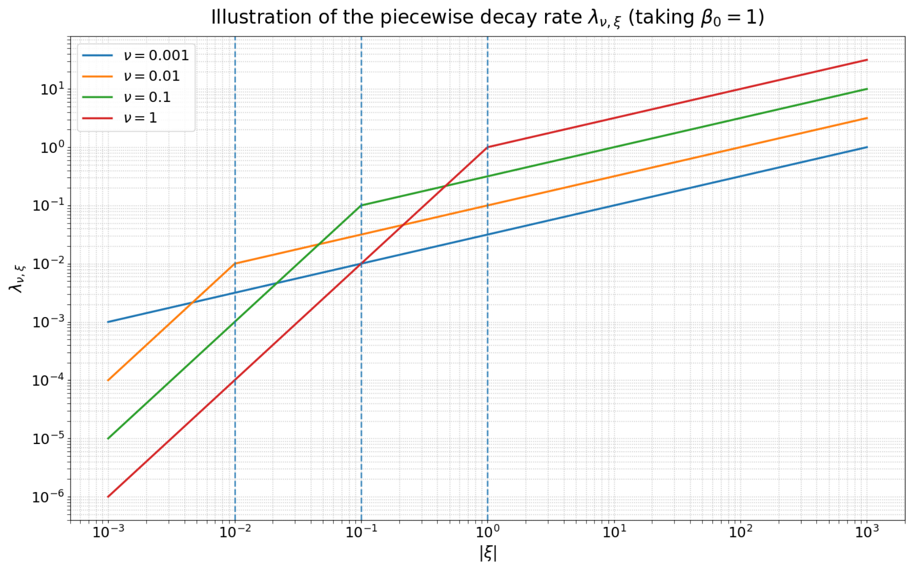


Figure 6: Illustration of the decay

## 3.6 On $\mathbb{R}_x \times \mathbb{R}_y$

We now pass from the $L$–periodic strip $\mathbb{T}_L \times \mathbb{R}_y$ to the fully planar domain $\mathbb{R}_x \times \mathbb{R}_y$. In contrast to the periodic setting, the streamwise Fourier variable is continuous and can be arbitrarily close to zero. Hence, the enhanced-dissipation estimate from the periodic case does not directly give a uniform decay estimate for all non-zero frequencies. The low-frequency regime has to be treated separately, and the relevant decay mechanism is Taylor dispersion.

In this section we prove a pointwise-in-frequency estimate for solutions of the linearized Poiseuille equation (57) in the Taylor-dispersion regime. We write

$$\omega_\xi(t, y) := \widehat{\omega}(t, \xi, y),$$

where the Fourier transform is taken only in the streamwise variable $x$. For a function $f = g(x, y)$, define

$$\|f\|_X^2 := \|f\|_{L^2}^2 + \|yf\|_{L^2}^2.$$

For $\xi \neq 0$ and $g_\xi(y)$, we also define the frequency-dependent norm

$$\|g_\xi\|_{\widetilde{X}_\xi}^2 := \|g_\xi\|_{L^2_y}^2 + \|yg_\xi\|_{L^2_y}^2 + |\xi|^{-2}\|g_\xi\|_{L^2_y}^2.$$

Equivalently, after integrating in $\xi$, this corresponds to the physical space norm

$$\|f\|_{\widetilde{X}}^2 = \|f\|_{L^2}^2 + \|yf\|_{L^2}^2 + \||\partial_x|^{-1} f\|_{L^2}^2.$$

**Theorem 9** (Pointwise Taylor-dispersion estimate)**.** *Fix the constant $\beta_0 > 0$. There exist constants $C_0 \geq 1$ and $\varepsilon > 0$, independent of $\nu$, $\xi$, $t$, and the initial datum, such that the following holds. Let $0 < \nu < 1$ and let $\xi \in \mathbb{R} \setminus \{0\}$ satisfy*

$$0 < \beta_0|\xi| \leq \nu.$$

*Let $\omega_\xi(t) = e^{t\mathcal{L}_\xi}\omega_{\xi,\mathrm{in}}$ be the solution of the Fourier-transformed linearized equation at frequency $\xi$. Then, for all $t \geq 0$,*

$$\|\omega_\xi(t)\|_{X_\xi} \leq C_0 \exp\left(-\varepsilon \frac{\xi^2}{\nu} t\right) \|\omega_{\xi,\mathrm{in}}\|_{\widetilde{X}_\xi}.$$

The estimate is pointwise in the fixed frequency $\xi$. In particular, the operator $L_\xi$ denotes the linearized operator after Fourier transform in $x$.

**Remark 2** (Limitation of Theorem (9))**.** *If we integrate the squared bound in Theorem ((9)) over the frequency interval $I$ we obtain*

$$\int_I \|e^{t\mathcal{L}_\xi}\omega_{\xi,\mathrm{in}}\|_X^2 \, d\xi \leq C_0^2 \int_I e^{-2c\xi^2 t/\nu} \|\omega_{\xi,\mathrm{in}}\|_{\widetilde{X}}^2 \, d\xi.$$

*If $0 \in \overline{I}$, then*

$$\sup_{\xi \in I} e^{-c\xi^2 t/\nu} = 1.$$

*Hence, the pointwise estimate does not imply a uniform exponential spectral-gap estimate on such a frequency band.*

We start by fixing $\xi \in \mathbb{R}$, then the operators become

$$\partial_x \;\mapsto\; i\xi, \qquad \Delta \;\mapsto\; \Delta_\xi := \partial_{yy} - \xi^2, \qquad \nabla \;\mapsto\; \nabla_\xi := (i\xi, \partial_y).$$

In these variables, the linearized Poiseuille system (57) reads

$$\partial_t \widehat{\omega} + i\xi\, y^2\, \widehat{\omega} - 2i\xi\, \widehat{\psi} - \nu \Delta_\xi \widehat{\omega} = 0, \tag{73}$$

$$\Delta_\xi \widehat{\psi} = \widehat{\omega}. \tag{74}$$

To highlight the dependence on the streamwise frequency $\xi$ and to ease the notation we write

$$\omega_\xi(y,t) := \widehat{\omega}(\xi, y, t), \qquad \psi_\xi(y,t) := \widehat{\psi}(\xi, y, t).$$

and all norms and scalar products in this section are taken in $L^2(\mathbb{R}_y)$. Now we state a lemma that is closely related to the Proposition (1).

**Lemma 2** (Modewise energy identities)**.** *Fix $\xi \in \mathbb{R} \setminus \{0\}$ and let $\omega_\xi$ solve (73) and (74)*

$$\partial_t \omega_\xi + i\xi y^2 \omega_\xi - 2i\xi\psi_\xi - \nu\Delta_\xi\omega_\xi = 0, \qquad \Delta_\xi \psi_\xi = \omega_\xi.$$

*Then the following identities hold*

$$\frac{1}{2}\frac{d}{dt}\|\omega_\xi\|^2 = -\nu\|\nabla_\xi \omega_\xi\|^2, \tag{75}$$

$$\frac{1}{2}\frac{d}{dt}\|\nabla_\xi\omega_\xi\|^2 = -\nu\|\Delta_\xi\omega_\xi\|^2 - 2\,\mathrm{Re}\langle yi\xi\omega_\xi, \partial_y\omega_\xi\rangle, \tag{76}$$

$$\frac{d}{dt}\,\mathrm{Re}\langle yi\xi\omega_\xi, \partial_y\omega_\xi\rangle = -2\|yi\xi\omega_\xi\|^2 - 4\|i\xi\partial_y\psi_\xi\|^2 - 2\nu\,\mathrm{Re}\langle\Delta_\xi\omega_\xi, yi\xi\partial_y\omega_\xi\rangle, \tag{77}$$

$$\frac{1}{2}\frac{d}{dt}\left(\|yi\xi\omega_\xi\|^2 + 2\|\nabla_\xi(i\xi\psi_\xi)\|^2\right) = -\nu\|i\xi\omega_\xi\|^2 - \nu\|yi\xi\nabla_\xi\omega_\xi\|^2. \tag{78}$$

*Sketch of proof.* After the partial Fourier transform in $x$ (using Plancherel), all inner products and norms are taken in $L^2(\mathbb{R}_y)$. The four identities are obtained exactly as the corresponding ones in Proposition (1), with two simplifications. First, there is no integration by parts in $x$ anymore: the transform has already turned $\int(\cdot)\,dx$ into multiplication by $i\xi$, so the passage $\mathbb{T}\to\mathbb{R}$ in the streamwise variable plays no role. Second, the antisymmetry property persists in the form

$$\operatorname{Re}\langle y^n\, i\xi\, f,\, f\rangle = \operatorname{Re}\Big(i\xi\int_{\mathbb{R}} y^n|f|^2\,dy\Big) = 0, \qquad n\in\mathbb{N}_0,$$

since $i\xi$ is skew-adjoint and the weight $y^n$ is real; this is why each identity carries a Re. The integration by parts in $y$ is identical to Proposition (1); the boundary terms at $y=\pm\infty$ vanish for $\omega_\xi$ in a dense class of sufficiently rapidly decaying functions, e.g. $\mathcal{S}(\mathbb{R}_y)$, the general case following by density. Substituting $\partial_t\omega_\xi = -i\xi y^2\omega_\xi + 2i\xi\psi_\xi + \nu\Delta_\xi\omega_\xi$ and $\omega_\xi = \Delta_\xi\psi_\xi$ and carrying out the same cancellations as in Proposition (1) then yields the four identities. $\square$

This allows us, to introduce the modewise hypocoercive energy functional

$$\Phi_\xi(t) := \frac{1}{2}\|\omega_\xi\|^2 + \frac{\alpha}{2}\|\nabla_\xi\omega_\xi\|^2 + 2\operatorname{Re}\beta\langle y\, i\xi\,\omega_\xi, \partial_y\omega_\xi\rangle + \frac{\gamma}{2}\|y\, i\xi\,\omega_\xi\|^2 + \gamma\|\nabla_\xi(i\xi\,\psi_\xi)\|^2 \tag{79}$$

In the Taylor–dispersion regime $\beta_0|\xi|\le\nu$, we prove a closed differential inequality for $\Phi_\xi$ which yields exponential decay at rate $|\xi|^2/\nu$.

**Theorem 10.** *Let $0<\nu<1$ and $\xi\in\mathbb{R}\setminus\{0\}$ with $\nu\,|\xi|^{-1}\ge\beta_0$. There exist constants $\alpha_1,\beta_1,\gamma_1>0$ and $\varepsilon>0$, depending only on $\beta_0$ (and in particular independent of $\nu$ and $\xi$), such that with $\alpha=\alpha_1$, $\beta=\beta_1/\nu$, $\gamma=\gamma_1/\xi^2$ the energy functional (79) satisfies the following differential inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_\xi + \varepsilon\,\frac{\xi^2}{\nu}\,\Phi_\xi + \frac{\alpha\nu}{2}\,\|\Delta_\xi\omega_\xi\|^2 + \frac{7}{8}\gamma\nu\,\|\,y\, i\xi\nabla_\xi\omega_\xi\|^2 + \gamma\nu\,\|i\xi\omega_\xi\|^2 \le 0 \qquad (t\ge 0).$$

*Proof.* This proof boils down to finding the appropriate constraints for the constants $\alpha,\beta,$ and $\gamma$. First we notice, that we can bound the inner product by the Cauchy-Schwarz Inequality followed by the Young-Inequality

$$2\beta|\xi||\langle y\omega_\xi, \partial_y\omega_\xi\rangle| \le 4\beta|\xi|\|y\omega_\xi\|\|\partial_y\omega_\xi\| \le \frac{\alpha}{2}\|\nabla_\xi\omega_\xi\|^2 + \frac{2\beta^2}{\alpha}\|yi\xi\omega_\xi\|^2.$$

If we assume

$$\frac{\beta^2}{\alpha\,\gamma} \le \frac{1}{64}, \tag{80}$$

then $\dfrac{4\beta^2}{\alpha}\le\dfrac{\gamma}{16}\le\dfrac{\gamma}{4}$, so the mixed term is dominated by $\frac{\alpha}{4}\|\nabla_\xi\omega_\xi\|^2+\frac{\gamma}{4}\|yi\xi\omega_\xi\|^2$, and we obtain the upper and lower bounds for our energy functional

$$\begin{aligned} 0 &\le \frac{1}{4}\Big(2\|\omega_\xi\|^2 + \alpha\|\nabla_\xi\omega_\xi\|^2 + \gamma\|y\, i\xi\omega_\xi\|^2 + 4\gamma\|\nabla_\xi(i\xi\psi_\xi)\|^2\Big) \le \Phi_\xi, \\ \Phi_\xi &\le \frac{1}{4}\Big(2\|\omega_\xi\|^2 + 3\alpha\|\nabla_\xi\omega_\xi\|^2 + 3\gamma\|y\, i\xi\omega_\xi\|^2 + 4\gamma\|\nabla_\xi(i\xi\psi_\xi)\|^2\Big). \end{aligned} \tag{81}$$

Using the modewise identities from lemma (2), we first obtain the raw differential identity

$$\frac{d}{dt}\Phi_\xi + \nu\|\nabla_\xi\omega_\xi\|^2 + \alpha\nu\|\Delta_\xi\omega_\xi\|^2 + 4\beta\|yi\xi\omega_\xi\|^2 + 8\beta\|i\xi\partial_y\psi_\xi\|^2 + \gamma\nu\|yi\xi\nabla_\xi\omega_\xi\|^2 + \gamma\nu\|i\xi\omega_\xi\|^2 \tag{82}$$

$$= -4\beta\nu\,\mathrm{Re}\langle\Delta_\xi\omega_\xi, yi\xi\partial_y\omega_\xi\rangle - 2\alpha\,\mathrm{Re}\langle yi\xi\omega_\xi, \partial_y\omega_\xi\rangle. \tag{83}$$

The first term on the right-hand side is again controlled by using Cauchy–Schwarz and Young's inequality, followed by using (80):

$$\begin{aligned}4\beta\nu\,|\mathrm{Re}\langle\Delta_\xi\omega_\xi, yi\xi\partial_y\omega_\xi\rangle| \le 4\beta\nu\|\Delta_\xi\omega_\xi\|\|yi\xi\partial_y\omega_\xi\| &\le \frac{\alpha\nu}{2}\|\Delta_\xi\omega_\xi\|^2 + \frac{8\beta^2\nu}{\alpha}\|yi\xi\nabla_\xi\omega_\xi\|^2\\ &\le \frac{\alpha\nu}{2}\|\Delta_\xi\omega_\xi\|^2 + \frac{\gamma\nu}{8}\|yi\xi\nabla_\xi\omega_\xi\|^2,\end{aligned}$$

The second term is estimated as

$$\begin{aligned}2\alpha\,|\mathrm{Re}\langle yi\xi\omega_\xi, \partial_y\omega_\xi\rangle| &\le 2\alpha\|yi\xi\omega_\xi\|\|\partial_y\omega_\xi\|\\ &\le \frac{\nu}{4}\|\nabla_\xi\omega_\xi\|^2 + \frac{4\alpha^2}{\nu}\|yi\xi\omega_\xi\|^2.\end{aligned}$$

It remains to relate the term involving $i\xi\partial_y\psi_\xi$ to $\|\nabla_\xi(i\xi\psi_\xi)\|$. Since $\Delta_\xi\psi_\xi = \omega_\xi$, we have

$$i\xi\omega_\xi = i\xi\Delta_\xi\psi_\xi = -\xi^2 i\xi\psi_\xi + i\xi\partial_{yy}\psi_\xi.$$

Equivalently, the missing component $-\xi^2\psi_\xi$ can be controlled by $yi\xi\omega_\xi$ and $i\xi\partial_y\psi_\xi$. More precisely,

$$\|\xi^2\psi_\xi\|^2 \le \|yi\xi\omega_\xi\|^2 + 2\|i\xi\partial_y\psi_\xi\|^2.$$

Therefore

$$\|\nabla_\xi(i\xi\psi_\xi)\|^2 = \|\xi^2\psi_\xi\|^2 + \|i\xi\partial_y\psi_\xi\|^2 \le \|yi\xi\omega_\xi\|^2 + 3\|i\xi\partial_y\psi_\xi\|^2.$$

Combining these estimates with the raw identity gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_\xi + \frac{3\nu}{4}\,\|\nabla_\xi\omega_\xi\|^2 + \frac{\alpha\nu}{2}\,\|\Delta_\xi\omega_\xi\|^2 + \left(2\beta - \frac{4\alpha^2}{\nu}\right)\|\,y\,i\xi\omega_\xi\|^2 \tag{84}$$

$$+\,2\beta\,\|\nabla_\xi(i\xi\psi_\xi)\|^2 + \frac{7}{8}\gamma\nu\,\|\,y\,i\xi\nabla_\xi\omega_\xi\|^2 + \gamma\nu\,\|i\xi\omega_\xi\|^2 \le 0. \tag{85}$$

We take

$$\beta = \frac{\beta_1}{\nu} \qquad \beta_1 \in (0, \beta_0^2], \qquad \gamma = \frac{\gamma_1}{\xi^2}, \qquad \alpha = \alpha_1$$

with $\beta_1$ independent of $\nu$ and $\xi$.

Integration by parts give us

$$\mathrm{Re}\langle y\omega_\xi, \partial_y\omega_\xi\rangle = -\frac{1}{2}\|\omega_\xi\|^2.$$

Hence, we can deduce by Cauchy-Schwarz and Young-inequality for $\sigma > 0$:

$$\|\omega_\xi\|^2 \le \sigma\|\partial_y\omega_\xi\|^2 + \frac{1}{\sigma}\|y\omega_\xi\|^2 = \sigma\|\partial_y\omega_\xi\|^2 + \frac{1}{\sigma}\frac{1}{\xi^2}\|yi\xi\omega_\xi\|^2. \tag{86}$$

Multiplying the inequality with $\frac{\xi^2\beta_1}{\nu}$ and chosing $\sigma = 1$ gives:

$$\frac{\xi^2\beta_1}{\nu}\|\omega_\xi\|^2 \le \frac{\xi^2\beta_1}{\nu}\|\partial_y\omega_\xi\|^2 + \frac{\beta_1}{\nu}\|yi\xi\omega_\xi\|^2$$

Because $\beta_1 \le \beta_0^2$ and $\frac{\nu}{|\xi|} \ge \beta_0$ we have

$$\frac{\xi^2\beta_1}{\nu} \le \frac{\xi^2\beta_0^2}{\nu} \le \nu,$$

which allows us to write

$$\beta\xi^2\|\omega_\xi\|^2 \le \nu\|\partial_y\omega_\xi\|^2 + \beta\|yi\xi\omega_\xi\|^2.$$

Using $\|\partial_y\omega_\xi\|^2 \le \|\nabla_\xi\omega_\xi\|^2$ and multiplying with $\frac{1}{2}$ we get:

$$\frac{1}{2}\beta\xi^2\|\omega_\xi\|^2 \le \frac{\nu}{2}\|\nabla_\xi\omega_\xi\|^2 + \beta\|yi\xi\omega_\xi\|^2$$

By estimating the corresponding terms in (84)-(85), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Phi_\xi + \frac{1}{2}\beta\xi^2\|\omega_\xi\|^2 + \frac{\nu}{4}\,\|\nabla_\xi\omega_\xi\|^2 + \frac{\alpha\nu}{2}\,\|\Delta_\xi\omega_\xi\|^2 + \left(\beta - \frac{4\alpha^2}{\nu}\right)\,\|\,y\,i\xi\omega_\xi\|^2 \\ +2\beta\,\|\nabla_\xi(i\xi\psi_\xi)\|^2 + \frac{7}{8}\gamma\nu\,\|\,y\,i\xi\nabla_\xi\omega_\xi\|^2 + \gamma\nu\,\|i\xi\omega_\xi\|^2 \le 0\,.\end{aligned}$$

Now we pull out $\frac{1}{4}\frac{\xi^2\beta_1}{\nu}$ from the terms that we can compare to $\Phi_\xi$ (81) and require

$$\beta_1 \ge 4\alpha_1^2$$

in order to group the non-local term and $\|yi\xi\omega_\xi\|^2$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Phi_\xi + \frac{1}{4}\frac{\xi^2\beta_1}{\nu}\Big[2\|\omega_\xi\|^2 + \frac{\nu^2}{\beta_1\xi^2}\,\|\nabla_\xi\omega_\xi\|^2 + \left(\frac{4\beta_1 - 16\alpha_1^2}{\beta_1\xi^2}\right)(\,\|\,y\,i\xi\omega_\xi\|^2 + 2\,\|\nabla_\xi(i\xi\psi_\xi)\|^2)\Big] \\ +\frac{\alpha\nu}{2}\,\|\Delta_\xi\omega_\xi\|^2 + \frac{7}{8}\gamma\nu\,\|\,y\,i\xi\nabla_\xi\omega_\xi\|^2 + \gamma\nu\,\|i\xi\omega_\xi\|^2 \le 0\,.\end{aligned}$$

This allows us to conclude: Since $\frac{\nu^2}{\beta_1\xi^2} \ge \frac{\beta_0^2}{\beta_1} \ge 1$, the bracket dominates $\Phi_\xi$ (via the upper bound in (81)) provided

$$\alpha_1 \le \frac{1}{3}, \qquad \frac{4\beta_1 - 16\alpha_1^2}{\beta_1} \ge \gamma_1, \qquad \frac{\beta_1^2}{\alpha_1\gamma_1} \le \frac{\beta_0^2}{64}, \qquad \beta_1 \ge 4\alpha_1^2.$$

Collecting the restrictions: with $\alpha = \alpha_1,\ \beta = \frac{\beta_1}{\nu},\ \gamma = \frac{\gamma_1}{\xi^2}$,

$$\alpha_1 \le \tfrac{1}{3}, \qquad \beta_1 \ge 4\alpha_1^2, \qquad \beta_1 \le \beta_0^2, \tag{87}$$

$$\frac{4\beta_1 - 16\alpha_1^2}{\beta_1} \ge \gamma_1, \qquad \frac{\beta_1^2}{\alpha_1\gamma_1} \le \frac{\beta_0^2}{64}. \tag{88}$$

The dimensional condition $\frac{\beta^2}{\alpha\gamma} \le \frac{1}{64}$ used in (80) follows from (88) and $\nu\,|\xi|^{-1} \ge \beta_0$, since

$$\frac{\beta^2}{\alpha\gamma} = \frac{\beta_1^2}{\alpha_1\gamma_1}\frac{\xi^2}{\nu^2} \le \frac{\beta_1^2}{\alpha_1\gamma_1}\beta_0^{-2} \le \frac{1}{64}.$$

We now choose the constants explicitly. Let $\varepsilon > 0$ be sufficiently small, explicitly

$$0 < \varepsilon \le \min\left\{\frac{2}{9},\, \frac{\beta_0^2}{4},\, \left(\frac{\beta_0^2}{1024\sqrt{2}}\right)^{2/3}\right\}.$$

Set

$$\alpha_1 = \sqrt{\frac{\varepsilon}{2}}, \qquad \beta_1 = 4\varepsilon, \qquad \gamma_1 = 1.$$

Then, since $\varepsilon \le 2/9$, we have

$$\alpha_1 = \sqrt{\frac{\varepsilon}{2}} \le \sqrt{\frac{1}{9}} = \frac{1}{3}.$$

Moreover,

$$4\alpha_1^2 = 4\cdot\frac{\varepsilon}{2} = 2\varepsilon \le 4\varepsilon = \beta_1,$$

and, since $\varepsilon \le \beta_0^2/4$,

$$\beta_1 = 4\varepsilon \le \beta_0^2.$$

Furthermore,

$$\frac{4\beta_1 - 16\alpha_1^2}{\beta_1} = \frac{16\varepsilon - 8\varepsilon}{4\varepsilon} = 2 \ge 1 = \gamma_1.$$

Finally,

$$\frac{\beta_1^2}{\alpha_1\gamma_1} = \frac{16\varepsilon^2}{\sqrt{\varepsilon/2}} = 16\sqrt{2}\,\varepsilon^{3/2}.$$

Therefore,

$$\frac{\beta_1^2}{\alpha_1\gamma_1} \le \frac{\beta_0^2}{64}$$

provided that

$$16\sqrt{2}\,\varepsilon^{3/2} \le \frac{\beta_0^2}{64},$$

which is equivalent to

$$\varepsilon \le \left(\frac{\beta_0^2}{1024\sqrt{2}}\right)^{2/3}.$$

Thus all restrictions are satisfied, and with the decay rate $\varepsilon = \beta_1/4$ we conclude

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_\xi + \varepsilon\frac{\xi^2}{\nu}\Phi_\xi + \frac{\alpha\nu}{2}\,\|\Delta_\xi\omega_\xi\|^2 + \frac{7}{8}\gamma\nu\,\|\,y\,i\xi\nabla_\xi\omega_\xi\|^2 + \gamma\nu\,\|i\xi\omega_\xi\|^2 \le 0\,.$$

□

In order to obtain an estimate for the semigroup, we will first show the decay of $Q_\xi(t)$ for all $t \ge 0$.

**Corollary 2.** *Let $0 < \nu \le 1$, then*

$$Q_\xi(t) := \frac{1}{2}\|\omega_\xi(t)\|^2 + \frac{\gamma_1}{4}\Big[\|y\omega_\xi(t)\|^2 + 2\|\nabla_\xi\psi_\xi(t)\|^2\Big]$$

*decays with the rate*

$$Q_\xi(t) \le C_0^2 Q_\xi(0)e^{-2\varepsilon\lambda_{\nu,\xi}t}, \quad t \ge 0,$$

*with*

$$\lambda_{\nu,\xi} = \frac{\xi^2}{\nu}, \qquad \textit{if } 0 \le \beta_0|\xi| \le \nu.$$

*Proof.* For $\beta_0|\xi| \le \nu$, set

$$\lambda_{\nu,\xi} := \frac{\xi^2}{\nu}.$$

We have from the previous theorem:

$$\frac{1}{2}\|\omega_\xi\|^2 + \frac{\alpha}{4}\|\nabla_\xi\omega_\xi\|^2 + \frac{\gamma_1}{4\xi^2}\Big[\|yi\xi\omega_\xi\|^2 + 2\|\nabla_\xi(i\xi\psi_\xi)\|^2\Big] \le \Phi_\xi.$$

We can compare this to $\Phi_\xi$ by using (81):

$$Q_\xi(t) := \frac{1}{2}\|\omega_\xi\|^2 + \frac{\gamma_1}{4}\Big[\|y\omega_\xi\|^2 + 2\|\nabla_\xi\psi_\xi\|^2\Big] \le \Phi_\xi.$$

By the monotonicity identities from Proposition (1) for $\|\omega_\xi\|^2$ and

$$\|yi\xi\omega_\xi\|^2 + 2\|\nabla_\xi(i\xi\psi_\xi)\|^2,$$

we obtain

$$Q_\xi(t) \le Q_\xi(0).$$

This gives the desired estimate for

$$0 \le t \le T_{\nu,\xi} := \frac{1}{\varepsilon\lambda_{\nu,\xi}},$$

since then $e^{-\varepsilon\lambda_{\nu,\xi}t} \ge e^{-1}$, and therefore

$$Q_\xi(t) \le Q_\xi(0) \le e\, e^{-\varepsilon\lambda_{\nu,\xi}t}Q_\xi(0).$$

For $t > T_{\nu,\xi}$, we integrate the energy identity to obtain

$$2\int_0^t \nu\|\nabla_\xi\omega_\xi(s)\|^2 ds = -\|\omega_\xi(t)\|^2 + \|\omega_\xi(0)\|^2 \le \|\omega_\xi(0)\|^2.$$

Hence, in particular,

$$\int_0^{T_{\nu,\xi}} \nu\|\nabla_\xi\omega_\xi(s)\|^2 ds \le \frac{1}{2}\|\omega_\xi(0)\|^2.$$

By the mean-value theorem for integrals, there exists $t^* \in (0, T_{\nu,\xi})$ such that

$$\nu\|\nabla_\xi\omega_\xi(t^*)\|^2 \le \frac{1}{T_{\nu,\xi}}\int_0^{T_{\nu,\xi}} \nu\|\nabla_\xi\omega_\xi(s)\|^2\, ds \le \frac{\varepsilon\lambda_{\nu,\xi}}{2}\|\omega_\xi(0)\|^2.$$

Thus

$$\|\nabla_\xi \omega_\xi(t^*)\|^2 \le \frac{\varepsilon \lambda_{\nu,\xi}}{2\nu} \|\omega_\xi(0)\|^2 = \frac{\varepsilon \xi^2}{2\nu^2} \|\omega_\xi(0)\|^2.$$

Since $\beta_0|\xi| \le \nu$, we have $\xi^2/\nu^2 \le 1/\beta_0^2$. Therefore

$$\frac{\alpha}{4} \|\nabla_\xi \omega_\xi(t^*)\|^2 \le \frac{\alpha\varepsilon}{8\beta_0^2} \|\omega_\xi(0)\|^2 \le \frac{\alpha\varepsilon}{4\beta_0^2} Q_\xi(0).$$

Hence, we get by (81)

$$\Phi_\xi(t^*) \le \frac{1}{2}\|\omega_\xi(t^*)\|^2 + \frac{\alpha}{4}\|\nabla_\xi \omega_\xi(t^*)\|^2 + \frac{\gamma_1}{4\xi^2}\Big[3\|yi\xi\omega_\xi(t^*)\|^2 + 4\|\nabla_\xi(i\xi\psi_\xi(t^*))\|^2\Big].$$

Using the estimate above we get

$$\Phi_\xi(t^*) \le \frac{1}{2}\|\omega_\xi(t^*)\|^2 + \frac{\alpha\varepsilon}{4\beta_0^2} Q_\xi(0) + \frac{\gamma_1}{4}\Big[3\|y\omega_\xi(t^*)\|^2 + 4\|\nabla_\xi \psi_\xi(t^*)\|^2\Big].$$

Since

$$\frac{\gamma_1}{4}\Big[3\|y\omega_\xi(t^*)\|^2 + 4\|\nabla_\xi \psi_\xi(t^*)\|^2\Big] \le 3Q_\xi(t^*),$$

and since $Q_\xi(t^*) \le Q_\xi(0)$, we obtain

$$\Phi_\xi(t^*) \le K_0 Q_\xi(0)$$

for some constant $K_0 > 0$ independent of $t, \nu, \xi$.

From the differential inequality for $\Phi_\xi$ and $0 < t^* < T_{\nu,\xi}$, it follows for $t \ge T_{\nu,\xi}$ that

$$Q_\xi(t) \le \Phi_\xi(t) \le e^{-\varepsilon \frac{\xi^2}{\nu}(t-t^*)} \Phi_\xi(t^*).$$

Therefore

$$Q_\xi(t) \le K_0 e^{-\varepsilon \frac{\xi^2}{\nu}(t-t^*)} Q_\xi(0) = K_0 e^{\varepsilon \frac{\xi^2}{\nu} t^*} e^{-\varepsilon \frac{\xi^2}{\nu} t} Q_\xi(0).$$

Since $t^* < T_{\nu,\xi} = 1/(\varepsilon \lambda_{\nu,\xi})$, we have

$$e^{\varepsilon \frac{\xi^2}{\nu} t^*} = e^{\varepsilon \lambda_{\nu,\xi} t^*} \le e.$$

Thus

$$Q_\xi(t) \le e K_0 e^{-\varepsilon \frac{\xi^2}{\nu} t} Q_\xi(0).$$

Combining the short-time and long-time estimates, we obtain

$$Q_\xi(t) \le C_0 e^{-\varepsilon \frac{\xi^2}{\nu} t} Q_\xi(0)$$

for all $t \ge 0$, where $C_0 > 0$ is independent of $t, \nu, \xi$. □

The following lemma measures the decay of the gradient of the stream function $\psi$ and is an result of looking at the planar problem. For the periodic case, we could use the estimate without the dependence on $\frac{1}{\xi}$.

**Lemma 3.** *Let $\xi \in \mathbb{R} \setminus \{0\}$ and let $\omega_\xi \in L^2(\mathbb{R}_y)$ be given and let $\psi_\xi$ solve*

$$\Delta_\xi \psi_\xi = \omega_\xi,$$

*where $\Delta_\xi = -\xi^2 + \partial_{yy}$. Then $\|\nabla_\xi \psi_\xi\| \le \frac{1}{\xi}\|\omega_\xi\|$.*

*Proof.* Take the Fourier transform in $y$ of the elliptic equation

$$(\partial_{yy} - \xi^2)\,\psi_\xi = \omega_\xi$$

yields, for every $\xi \in \mathbb{R}$,

$$(-\eta^2 - \xi^2)\,\widehat{\psi}_\xi(\eta) = \widehat{\omega}_\xi(\eta),$$

so that

$$\widehat{\psi}_\xi(\eta) = -\frac{\widehat{\omega}_\xi(\eta)}{\xi^2 + \eta^2}.$$

We now estimate $\|\nabla_\xi \psi_\xi\|_{L^2}$. Using Plancherel's theorem,

$$\|\nabla_\xi \psi_\xi\|^2_{L^2(\mathbb{R}_y)} = \int_{\mathbb{R}} \left(|\xi|^2 + |\eta|^2\right) |\widehat{\psi}_\xi(\eta)|^2 \, d\eta.$$

Substituting $\widehat{\psi}_\xi(\eta) = \widehat{\omega}_\xi(\eta)/(\xi^2 + \eta^2)$ gives

$$\|\nabla_\xi \psi_\xi\|^2_{L^2(\mathbb{R}_y)} = \int_{\mathbb{R}} \frac{|\xi|^2 + |\eta|^2}{(\xi^2 + \eta^2)^2} |\widehat{\omega}_\xi(\eta)|^2 \, d\eta = \int_{\mathbb{R}} \frac{1}{\xi^2 + \eta^2} |\widehat{\omega}_\xi(\eta)|^2 \, d\eta.$$

Since $\xi^2 + \eta^2 \ge \xi^2$ for all $\xi$ and $\eta$, we have

$$\frac{1}{\xi^2 + \eta^2} \le \frac{1}{\xi^2},$$

and therefore

$$\|\nabla_\xi \psi_\xi\|^2_{L^2(\mathbb{R}_y)} \le \frac{1}{\xi^2} \int_{\mathbb{R}} |\widehat{\omega}_\xi(\eta)|^2 \, d\eta = \frac{1}{\xi^2} \, \|\omega_\xi\|^2_{L^2(\mathbb{R}_y)}.$$

Taking square roots yields

$$\|\nabla_\xi \psi_\xi\|_{L^2(\mathbb{R}_y)} \le \frac{1}{|\xi|} \, \|\omega_\xi\|_{L^2(\mathbb{R}_y)},$$

which is the desired estimate.

□

Now we can put everything together to prove theorem (9).

*Proof.* Recall the functional

$$Q_\xi(t) := \frac{1}{2}\|\omega_\xi(t)\|^2_{L^2_y} + \frac{\gamma_1}{4}\Big(\|y\,\omega_\xi(t)\|^2_{L^2_y} + 2\|\nabla_\xi \psi_\xi(t)\|^2_{L^2_y}\Big).$$

Since $0 < \beta_0|\xi| \le \nu$, we are in the Taylor-dispersion regime and hence $\lambda_{\nu,\xi} = \xi^2/\nu$. Therefore, for all $t \ge 0$,

$$Q_\xi(t) \le C_0^2 \, e^{-2\varepsilon \frac{\xi^2}{\nu} t} \, Q_\xi(0). \tag{89}$$

*Step 1: $Q_\xi(t)$ controls the $X$–norm.* By definition of $Q_\xi$ we have

$$Q_\xi(t) \;\geq\; \frac{1}{2}\|\omega_\xi(t)\|_{L^2_y}^2 + \frac{\gamma_1}{4}\|y\,\omega_\xi(t)\|_{L^2_y}^2,$$

hence there is a constant $C_X > 0$ (depending only on $\gamma_1$) such that

$$\|\omega_\xi(t)\|_X^2 = \|\omega_\xi(t)\|_{L^2_y}^2 + \|y\,\omega_\xi(t)\|_{L^2_y}^2 \;\leq\; C_X\, Q_\xi(t). \tag{90}$$

*Step 2: $Q_\xi(0)$ is controlled by the $\tilde{X}$–norm.* By Lemma (3) (elliptic estimate) we have

$$\|\nabla_\xi \psi_\xi(0)\|_{L^2_y} \leq \frac{1}{|\xi|}\|\omega_\xi(0)\|_{L^2_y}.$$

Thus there is a constant $C_{\tilde{X}_\xi} > 0$ (again depending only on $\gamma_1$) such that

$$Q_\xi(0) \;\leq\; C_{\tilde{X}_\xi}\Big(\|\omega_\xi(0)\|_{L^2_y}^2 + \|y\,\omega_\xi(0)\|_{L^2_y}^2 + \|\xi^{-1}\omega_\xi(0)\|_{L^2_y}^2\Big) = \|C_{\tilde{X}_\xi}\,\omega_\xi(0)\|_{\tilde{X}_\xi}^2. \tag{91}$$

*Conclusion.* Combining (89)–(91) yields, for all $t \geq 0$,

$$\|\omega_\xi(t)\|_X^2 \leq C_X C_0^2 e^{-2\varepsilon\frac{\xi^2}{\nu}t}\, C_{\tilde{X}_\xi}\|\omega_\xi(0)\|_{\tilde{X}_\xi}^2.$$

Taking square roots and absorbing constants into $C_0$ (and $\varepsilon := \varepsilon$) gives

$$\|e^{t\mathcal{L}_\xi}\omega_{\xi,\mathrm{in}}\|_X = \|\omega_\xi(t)\|_X \leq C_0\, e^{-\varepsilon\frac{\xi^2}{\nu}t}\,\|\omega_{\xi,\mathrm{in}}\|_{\tilde{X}_\xi},$$

which is exactly the claimed bound. $\square$

# 4 Conclusion

This thesis has investigated the hydrodynamic stability of the linearized two-dimensional Navier-Stokes equations near a parabolic Poiseuille flow, focusing specifically on the low-viscosity regime ($\nu > 0$) within the unbounded planar geometry $\mathbb{R}_x \times \mathbb{R}_y$. Unlike periodic settings where the streamwise wavenumber $\xi$ is bounded away from zero, the fully planar domain allows $\xi$ to be arbitrarily small, necessitating a bifurcated analytical approach based on the precise relation between $\xi$ and $\nu$.

By employing a time-weighted hypocoercive energy functional $\Phi_\xi(t)$ adapted to the Poiseuille linearization, the analysis successfully demonstrated that perturbations experience decay at rates significantly faster than the classical diffusive timescale $\nu^{-1}$. Specifically, for higher frequencies where $|\xi| \gtrsim \nu$, the non-uniform shear stretches and mixes the perturbations, leading to an enhanced dissipation rate proportional to $\nu^{1/2}|\xi|^{1/2}$. Conversely, for lower frequencies where $|\xi| \lesssim \nu$, this shear-induced mixing degenerates; however, the relaxation is shown to be governed by Taylor dispersion, yielding a streamwise diffusion with a decay rate of $\xi^2/\nu$.

While operating on the unbounded domain $\mathbb{R} \times \mathbb{R}$ isolates the intrinsic interior mixing dynamics from the contaminating effects of solid boundary layers, it represents a mathematical idealization. Furthermore, by restricting the analysis to the linearized equations and neglecting the quadratic self-advection term, the analysis inherently assumes that perturbations remain infinitesimally small for all time.

A natural and critical direction for future research is to establish a quantitative nonlinear stability threshold for Poiseuille flow on the unbounded plane. As the periodic case suggests, this is where the main analytical difficulties lie. The challenge is no longer merely to identify the linear decay mechanisms, but to show that they remain strong enough to control the cumulative effect of the quadratic nonlinearity over long times.

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