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Spectra problems in the higher Baire spaces

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Abstract

The cardinal characteristics of the continuum are already extensively studied. Their natural analogues at the uncountable, the cardinal characteristics of ${}^{\kappa}\kappa$ and $[\kappa]^{\kappa}$ for uncountable κ, have been studied only recently in the general framework of higher Baire space combinatorics.

This thesis aims to give a systematic presentation of the κ-almost disjointness number $\mathfrak{a}(\kappa)$, κ-dominating number $\mathfrak{d}(\kappa)$ and κ-unbounding number $\mathfrak{b}(\kappa)$ in this generalized and global setting, for an uncountable regular κ. These cardinal characteristics and some first results about their sizes will be introduced, followed by an investigation of them in the Cohen extension. We prove the existence of an indestructible κ-mad family. Afterwards, the cardinal characteristics and the global continuum function are evaluated in the Easton extension. The thesis ends with a forcing construction that allows us to add maximal κ-almost disjoint families of any given size.

Zusammenfassung

Die Kardinalzahlcharakteristiken des Kontinuums sind bereits umfassend untersucht worden. Ihre natürlichen Verallgemeinerungen im Überabzählbaren, das heißt, die Kardinalzahlcharakteristiken von ${}^\kappa\kappa$ und $[\kappa]^\kappa$ für überabzählbares κ, werden erst seit Kurzem im allgemeinen Rahmen der Kombinatorik höherer Baire-Räume erforscht.

Diese Arbeit gibt eine systematische Darstellung der κ-fast-Disjunktheitszahl $\mathfrak{a}(\kappa)$ (engl. *κ-almost disjointness number*), der κ-Dominierungszahl $\mathfrak{d}(\kappa)$ (engl. *κ-dominating number*) und der κ-Unbegrenztheitszahl $\mathfrak{b}(\kappa)$ (engl. *κ-unbounding number*) in dieser verallgemeinerten und globalen Umgebung für ein überabzählbares und reguläres κ. Diese drei Kardinalzahlcharakteristiken sowie erste Ergebnisse über ihre Größen werden vorgestellt, gefolgt von ihrer Untersuchung in der Cohen-Erweiterung. Wir beweisen die Existenz einer unzerstörbaren maximalen κ-fast-disjunkten Familie (engl. *indestructible κ-mad family*). Anschließend werden die Kardinalzahlcharakteristiken und die globale Kontinuumsfunktion in der Easton-Erweiterung ausgewertet. Die Arbeit schließt mit einer Forcing-Konstruktion, die es erlaubt, maximale κ-fast-disjunkte Familien beliebig vorgegebener Größe hinzuzufügen.

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Contents

1 Introduction

Cardinal characteristics of the continuum are cardinals that are characterized by combinatorial objects associated with the continuum. For example, the smallest cardinality of a family of infinite subsets of the natural numbers, such that any two of those subsets have a finite intersection (the almost disjointness number). Their study helps us to better understand the continuum and its size, denoted by $\mathfrak{c}$. By the continuum we mean the real line $\mathbb{R}$. In set theory, however, it is common to identify the continuum with the following spaces, which are equivalent for our purposes — the power set $\mathcal{P}(\omega)$ of the natural numbers ω, which yields $\mathfrak{c} = 2^{\aleph_0}$, or the Baire space of all functions from ω to ω, from which we obtain $\mathfrak{c} = \aleph_0^{\aleph_0}$, where $\aleph_0$ is the smallest infinite cardinal, the cardinality of ω. [Bla09]

Problems of Baire category and Lebesgue measure, as well as studies of the growth rate of functions, enriched the theory of the cardinal characteristics. This led to studying the functions in ${}^{\omega}\omega$ ordered by eventual dominance, which is the relation that one function is greater than or equal to the other for all sufficiently large inputs. Examples of cardinal characteristics that arose from this are the smallest cardinality of a family of functions that contains an eventually dominating function for any given function (the dominating number), or the smallest cardinality of a family of functions such that no function eventually dominates every family member (the unbounding number). For more details on the historical development, see [Ste12].

Georg Cantor showed that the continuum is strictly larger than the cardinality of an infinite yet countable set, that is, $\mathfrak{c} = 2^{\aleph_0} > \aleph_0$, which marks the first theorem about cardinal characteristics of the continuum [Bla09]. This led him to suspect that the continuum is the smallest uncountable cardinal and to formulate the Continuum Hypothesis $2^{\aleph_0} = \aleph_1$. Studying the continuum regained attention after David Hilbert put Cantor's Continuum Hypothesis at the top of his famous list of problems, which he presented at the International Congress of Mathematicians in Paris in 1900 [Ste12].

The Generalized Continuum Hypothesis states that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for any ordinal α. After Kurt Gödel showed the consistency of the Generalized Continuum Hypothesis in 1940, Paul J. Cohen introduced the method of forcing, with which he showed the independence of the Continuum Hypothesis from ZFC, see [Coh63]. Only one year later, in 1964, William B. Easton modified Cohen's construction and introduced a setting where the global continuum function $\alpha \mapsto 2^{\aleph_\alpha}$ can take on almost any value, as long as it is not contradicting Königs's theorem or the monotonicity (that is, the observation that $2^{\aleph_\alpha} \leq 2^{\aleph_\beta}$ if $\alpha < \beta$), see [Eas64].

After the cardinal characteristics of the continuum were extensively studied (see, e.g., [Bla09]), James Cummings and Saharon Shelah published one of the first papers exploring them above the continuum in 1995, their paper [CS95]. By now, most of the cardinal characteristics have been generalized to the higher Baire spaces (see, e.g., [BHZ][Bro+][RS18]) — that is, to combinatorial objects now associated with the collection of functions from κ to κ or with the collection of all the subsets of κ that have size κ for some uncountable cardinal κ.

This thesis aims to give a systematic presentation of the three examples mentioned above in this generalized and global setting, the κ-almost disjointness number (denoted by $\mathfrak{a}(\kappa)$), κ-dominating number ($\mathfrak{d}(\kappa)$) and κ-unbounding number ($\mathfrak{b}(\kappa)$) of the higher Baire spaces for an uncountable regular κ.

These cardinal characteristics and some first results about their sizes will be introduced in Chapter 2, including the theorem by Raghavan and Shelah [RS18] stating that $\mathfrak{b}(\kappa) = \kappa^+$ implies $\mathfrak{a}(\kappa) = \kappa^+$. The remaining three chapters are structured similarly. Each chapter introduces a different forcing poset, establishes its key properties, and then proves the main results obtained via that forcing. In Chapter 3, we investigate the κ-unbounding and κ-dominating numbers in the Cohen extension, the model we obtain via Cohen forcing, and find the following:

Theorem. *If GCH is satisfied in the ground model M, λ is a regular cardinal with $\lambda > \kappa^+$, $Fn_{<\kappa}(\lambda \times \kappa, \kappa)$ is the forcing poset and G is a generic filter, then*

$$M[G] \models \mathfrak{b}(\kappa) = \kappa^+ < \mathfrak{d}(\kappa) = \lambda.$$

Then, we show the existence of a $\mathbb{P}$-indestructible maximal κ-almost disjoint family of cardinality κ^+ for any forcing poset $\mathbb{P}$ of size κ, given that $2^\kappa = \kappa^+$. We end the chapter by also investigating the κ-almost disjointness number and its spectrum in the Cohen extension. The spectrum of $\mathfrak{a}(\kappa)$ is the set of all possible cardinalities of a maximal κ-almost disjoint family.

Theorem. *Assume* GCH *at and below* κ. *Let* λ *be a cardinal such that* $\lambda^\kappa = \lambda$, *let* $Fn_{<\kappa}(\lambda, 2)$ *be the forcing poset and* G *any generic filter. Then*

$$M[G] \models \mathfrak{sp}(\mathfrak{a}(\kappa)) = \{\kappa^+, \lambda\}.$$

Following the chronology of mathematics of the 20th century, Chapter 4 discusses Easton forcing. Evaluating the cardinal characteristics and the global continuum function in the Easton extension gives us:

Theorem. *Let* M *be a ctm, assume* GCH *and let* E *be an Easton function such that* $E(\kappa) > \kappa^+$ *for all* $\kappa \in dom(E)$. *If* $\mathbb{P}(E)$ *is the Easton poset and* G *a generic filter,*

$$M[G] \models \forall\kappa \in dom(E)(\mathfrak{a}(\kappa) = \mathfrak{b}(\kappa) = \kappa^+ < \mathfrak{d}(\kappa) = E(\kappa) = 2^\kappa).$$

Where $\mathfrak{d}(\kappa) = E(\kappa)$ *only holds if* $E(\kappa)$ *is regular.*

The thesis ends with Chapter 5, where we first define a forcing poset $\mathbb{H}_A$ for some index set A and then show that this poset adjoins a κ-almost disjoint family of the size of A, which is maximal if $|A| \geq \kappa^+$. We proceed to define the $<\kappa$-supported product of these forcing posets, and prove a theorem about the positive requirements for the spectrum of the almost disjointness number $\mathfrak{a}(\kappa)$:

Theorem. *Let* D *be a closed set of cardinals such that* $\min(D) \geq \kappa^+$ *and let* $\mathbb{P} = \prod^{<\kappa}_{\mu\in D} \mathbb{H}_{\{\mu\}\times\mu}$ *the* $<\kappa$*-supported product of the posets* $\mathbb{H}_{\{\mu\}\times\mu}$ *for* $\mu \in D$. *Let* G *be any* $\mathbb{P}$*-generic filter over* V. *Then,*

$$V[G] \models D \subseteq \mathfrak{sp}(\mathfrak{a}(\kappa)),$$

that is, $\mathbb{P}$ *adjoins a* κ*-mad family of cardinality* μ *for each* $\mu \in D$.

2 Higher Baire spaces cardinal characteristics

We will start by introducing the main characters of this thesis, the almost disjointness, unbounding and dominating number at the uncountable. Unless specified otherwise, κ is a regular uncountable cardinal.

Notation. The collection of all subsets of κ that are of cardinality κ will be denoted by $[\kappa]^\kappa$, for all subsets of κ of cardinality less than κ we write $[\kappa]^{<\kappa}$. The collection of all functions from κ to κ will be denoted by ${}^\kappa\kappa$.

Definition 2.1. [Fisc] The sets $A, B \in [\kappa]^\kappa$ are called **almost disjoint**, if their intersection has size less than κ, $|A \cap B| < \kappa$. A family $\mathcal{A} \subseteq [\kappa]^\kappa$ is **κ-almost disjoint**, if any two elements of $\mathcal{A}$ are almost disjoint. We call a family **κ-mad**, that is, maximal κ-almost disjoint, if it is κ-almost disjoint and maximal under inclusion. This means there is no larger κ-almost disjoint family that properly contains it. The κ-almost disjointness number $\mathfrak{a}(\kappa)$ is the minimal cardinality of a κ-mad family of cardinality at least κ,

$$\mathfrak{a}(\kappa) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq [\kappa]^\kappa,\ |\mathcal{A}| \geq \kappa,\ \mathcal{A} \text{ is } \kappa\text{-mad}\}.$$

Definition 2.2. [Fisc] A function $f \in {}^\kappa\kappa$ **eventually dominates** the function $g \in {}^\kappa\kappa$, if there is an $\alpha \in \kappa$ such that for all $\beta \geq \alpha$, $g(\beta) \leq f(\beta)$. We denote this by $g \leq^* f$. A family $\mathcal{B} \subseteq {}^\kappa\kappa$ is **unbounded**, if there is no $f \in {}^\kappa\kappa$ that eventually dominates all $g \in \mathcal{B}$. The κ-unbounding number $\mathfrak{b}(\kappa)$ is the minimal cardinality of an unbounded family in ${}^\kappa\kappa$,

$$\mathfrak{b}(\kappa) = \min\{|\mathcal{B}| : \mathcal{B} \subseteq {}^\kappa\kappa \text{ unbounded}\}.$$

A family $\mathcal{D} \subseteq {}^\kappa\kappa$ is **dominating**, if for all $f \in {}^\kappa\kappa$ there is a $g \in \mathcal{D}$ with $f \leq^* g$. The

κ-dominating number $\mathfrak{d}(\kappa)$ is the minimal cardinality of a dominating family in ${}^\kappa\kappa$,

$$\mathfrak{d}(\kappa) = \min\{|\mathcal{D}| : \mathcal{D} \subseteq {}^\kappa\kappa \text{ dominating}\}.$$

Next, we will show some first inequalities between $\mathfrak{a}(\kappa)$, $\mathfrak{b}(\kappa)$ and $\mathfrak{d}(\kappa)$ and show that they lie between κ^+ and 2^κ.

Lemma 2.3.

1. $\kappa^+ \leq \mathfrak{b}(\kappa), \mathfrak{d}(\kappa), \mathfrak{a}(\kappa) \leq 2^\kappa$
2. $\mathfrak{b}(\kappa) \leq \mathfrak{d}(\kappa)$
3. $\mathfrak{b}(\kappa) \leq \mathfrak{a}(\kappa)$

Proof.

1. First, we show the inequality $\kappa^+ \leq \mathfrak{b}(\kappa)$ as seen in [CS95]. Note that $\kappa^+ \leq \mathfrak{d}(\kappa), \mathfrak{a}(\kappa)$ will then follow with 2. and 3., respectively. Let $\mathcal{F} = \{f_\alpha : \alpha < \kappa\} \subseteq {}^\kappa\kappa$ be any family of functions of size κ. Then there is a function $f \in {}^\kappa\kappa$ that eventually dominates $\mathcal{F}$, that is, for all $\alpha < \kappa$, $f_\alpha <^* f$. Define the function f by $f(\beta) = \sup\{f_\alpha(\beta) + 1 : \alpha < \beta\}$ for $\beta < \kappa$. Now, fix an arbitrary $\alpha < \kappa$. Then for all $\alpha < \beta < \kappa$, we have $f_\alpha(\beta) < f(\beta)$ by the definition of our f, thus $f_\alpha <^* f$. This means that $\mathcal{F}$ is not unbounded, so $\mathfrak{b}(\kappa)$ must be at least κ^+.

   Next, we take care of $\mathfrak{b}(\kappa), \mathfrak{d}(\kappa) \leq 2^\kappa$. This holds as ${}^\kappa\kappa$ itself is an unbounded, as well as a dominating family of cardinality 2^κ. (That $|{}^\kappa\kappa| = 2^\kappa$ follows from Lemma I.13.7. in [Kun11].)

   It remains to show that also $\mathfrak{a}(\kappa) \leq 2^\kappa$. (This proof is an adaptation from the countable case in [Kun11], Lemma III.1.16) We assume that $2^{<\kappa} = \kappa$ and look at the set $T = {}^{<\kappa}2$ of all sequences of zeros and ones of length $< \kappa$. For each $f \in {}^\kappa 2$, let $X_f = \{f \restriction \lambda : \lambda \in \kappa\}$. Then X_f is a subset of T and $|X_f| = \kappa$. Whenever $f \neq g$ in ${}^\kappa 2$, there is a smallest $\mu \in \kappa$ such that $f(\mu) \neq g(\mu)$, and then $X_f \cap X_g = \{f \restriction \lambda : \lambda \leq \mu\}$. So X_f and X_g are κ-almost disjoint whenever $f \neq g$. Let $\mathcal{A}$ be the family of all sets X_f for $f \in {}^\kappa 2$, $\mathcal{A} = \{X_f\}_{f \in {}^\kappa 2}$. Then $\mathcal{A}$ is a κ-almost disjoint family of size 2^κ.

2. To show that $\mathfrak{b}(\kappa) \leq \mathfrak{d}(\kappa)$ we adapt the proof for $\mathfrak{b}(\omega) \leq \mathfrak{d}(\omega)$ seen in [Fisb]. We show that every dominating family is unbounded, which implies $\mathfrak{b}(\kappa) \leq \mathfrak{d}(\kappa)$. Let $\mathcal{D} \subseteq {}^\kappa\kappa$ be a dominating family and suppose that it is not unbounded. Then there is some $f \in {}^\kappa\kappa$ such that for all $g \in \mathcal{D}$, $g <^* f$. Since $\mathcal{D}$ is dominating, there is

a $g \in \mathcal{D}$ for every $h \in {}^\kappa\kappa$ such that $h <^* g$. But that means that there is also a $g \in \mathcal{D}$ for the f from above such that $f <^* g$, contradicting that $g <^* f$ for all $g \in \mathcal{D}$.

3. Finally, we will prove $\mathfrak{b}(\kappa) \leq \mathfrak{a}(\kappa)$ inspired by [Bro+]. We fix a κ-almost disjoint family of size λ, where $\lambda < \mathfrak{b}(\kappa)$, and show that it is not maximal. Let $\mathcal{A} = \{A_\alpha : \alpha < \lambda\}$ be a κ-almost disjoint family of size $\lambda < \mathfrak{b}(\kappa)$. Note that $\lambda \geq \kappa$ by definition. Define $\tilde{A}_\alpha = A_\alpha \setminus \bigcup_{\delta<\alpha}(A_\alpha \cap A_\delta)$ for each $\alpha < \kappa$. As $\mathcal{A}$ is a κ-almost disjoint family, $|A_\alpha \cap A_\delta| < \kappa$ for all $\delta < \kappa$, and so $\tilde{A}_\alpha$ has cardinality κ. The sets $\tilde{A}_\alpha$ are pairwise disjoint, that is, $\tilde{A}_\alpha \cap \tilde{A}_\beta = \emptyset$ for $\alpha \neq \beta,\ \alpha, \beta < \kappa$. In addition, they are almost equal to the sets A_α, that is, $\tilde{A}_\alpha =^* A_\alpha$ for all $\alpha < \kappa$, where almost equal ($=^*$) means modulo a set of size less than κ.

   For every $g \in {}^\kappa\kappa$, let $e_g^\alpha = \mathrm{next}(\tilde{A}_\alpha, g(\alpha))$ be the least ordinal in $\tilde{A}_\alpha$ that is greater than $g(\alpha)$. Let $E_g = \{e_g^\alpha : \alpha < \kappa\}$. This will be the element of $[\kappa]^\kappa$ showing that $\mathcal{A}$ is not maximal. Our next goal is to show that we can add it to $\mathcal{A}$ and that it does not destroy the almost disjointness. Since E_g contains one element of each $\tilde{A}_\alpha$, it is unbounded in κ, that is, $\sup(E_g) = \kappa$. Also, it contains at most $< \kappa$ elements of each A_α, so we have $|E_g \cap A_\alpha| < \kappa$ for all $\alpha < \kappa$. It remains to show that E_g is also κ-almost disjoint to the rest of the family, the A_α for $\kappa \leq \alpha < \lambda$. Fix an α such that $\kappa \leq \alpha < \lambda$. For each $\gamma < \kappa$ the intersection $A_\alpha \cap A_\gamma$ has size less than κ. Define the function $f_\alpha \in {}^\kappa\kappa$ by stipulating $f_\alpha(\gamma) = \sup(A_\alpha \cap A_\gamma) + 1$. Then all elements of $A_\alpha \cap A_\gamma$ are less than $f_\alpha(\gamma)$ for all $\gamma < \kappa$. Now $\{f_\alpha : \kappa \leq \alpha < \lambda\}$ builds a family of $\lambda < \mathfrak{b}(\kappa)$ functions. So, this family is not unbounded and there is a $g \in {}^\kappa\kappa$ such that $f_\alpha <^* g$ for all $\kappa \leq \alpha < \lambda$. Fix such a function g. If $e_g^\gamma \in E_g \cap A_\alpha$ then $e_g^\gamma \in \tilde{A}_\alpha$ and $e_g^\gamma > g(\gamma)$ (by definition of e_g^γ). Since $e_g^\gamma \in \tilde{A}_\alpha$ and $e_g^\gamma \in E_g \cap A_\alpha$, $e_g^\gamma \in A_\alpha \cap A_\gamma$ and so $e_g^\gamma < f_\alpha(\gamma)$. But then $f_\alpha(\gamma) > e_g^\gamma > g(\gamma)$. Since $f_\alpha <^* g$, this is only possible for less than κ values for γ. Therefore, $|E_g \cap A_\alpha| < \kappa$ for all $\alpha < \lambda$. With this we have shown that $\mathcal{A}$ was not maximal.

□

The following theorem by Blass, Hyttinen and Zhang [BHZ] gives a negative answer to the uncountable analogue of Roitman's problem, which asks whether it is consistent that $\mathfrak{d}(\omega) < \mathfrak{a}(\omega)$. While Shelah showed the consistency of $\mathfrak{d}(\omega) = \aleph_2 < \aleph_3 = \mathfrak{a}(\omega)$ in [She04], Roitman's problem remains open if $\mathfrak{d}(\omega) = \aleph_1$ [RS18].

Theorem 2.4. *For regular and uncountable κ, $\mathfrak{d}(\kappa) = \kappa^+$ implies $\mathfrak{a}(\kappa) = \kappa^+$.*

A proof of this theorem can be found in [BHZ]. We will end this chapter with a theorem by Raghavan and Shelah [RS18], stating that $\mathfrak{b}(\kappa) = \kappa^+$ implies $\mathfrak{a}(\kappa) = \kappa^+$ for regular uncountable κ. This is a refinement of Theorem 2.4, since $\mathfrak{d}(\kappa) = \kappa^+$ implies $\mathfrak{b}(\kappa) = \kappa^+$ by the above Lemma 2.3. Moreover, the result shows that ω is the only regular cardinal κ for which $\mathfrak{b}(\kappa) = \kappa^+ < \kappa^{++} = \mathfrak{a}(\kappa)$ (see [She84]) is consistent. For its proof we need a relative of $\mathfrak{b}(\kappa)$, the κ-club-unbounding number $\mathfrak{b}_{\mathrm{cl}}(\kappa)$, so we first recall what a club subset is.

Definition 2.5. [Kun11] A subset $C \subseteq \kappa$ is **club** in κ if and only if it is **closed** (that is, for all limit $\nu < \kappa$, if $\sup(C \cap \nu) = \nu$ then $\nu \in C$) and **unbounded** (that is, $\sup(C) = \kappa$). A subset $X \subseteq \kappa$ is **stationary** in κ if and only if it is not an element of the nonstationary ideal, $X \notin \mathrm{nonstat}_\kappa = \{X \subseteq \kappa : \exists C \subseteq \kappa\ (C \text{ is club} \wedge C \cap X = \emptyset)\}$. The nonstationary ideal $\mathrm{nonstat}_\kappa$ is the dual ideal to the club filter $\mathrm{club}_\kappa = \{X \subseteq \kappa : \exists C \subseteq \kappa\ (C \text{ is club} \wedge C \subseteq X)\}$, that is $\mathrm{nonstat}_\kappa = \mathrm{club}_\kappa^* = \{X \subseteq \kappa : \kappa \setminus X \in \mathrm{club}_\kappa\}$.

Definition 2.6. For functions $f, g \in {}^\kappa\kappa$, we say that g **club-dominates** f, denoted by $f \leq_{\mathrm{cl}} g$, if $\{\alpha \in \kappa : g(\alpha) < f(\alpha)\} \in \mathrm{nonstat}_\kappa$. Thus, $f \leq_{\mathrm{cl}} g$ means that g dominates f on a club subset of κ. Similarly to an unbounded family (Definition 2.2), we call a family $\mathcal{F} \subseteq {}^\kappa\kappa$ of functions **club-unbounded** if there is no $g \in {}^\kappa\kappa$ such that $f \leq_{\mathrm{cl}} g$ for all $f \in \mathcal{F}$. The κ-club-unbounding number $\mathfrak{b}_{\mathrm{cl}}(\kappa)$ is the minimal cardinality of a club-unbounded family in ${}^\kappa\kappa$,

$$\mathfrak{b}_{\mathrm{cl}}(\kappa) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq {}^\kappa\kappa,\ \mathcal{F} \text{ is club-unbounded}\}.$$

Cummings and Shelah introduce the κ-club-unbounding number in their paper [CS95], where they also show that it is essentially the same as the κ-unbounding number.

Lemma 2.7. For regular and uncountable κ, $\mathfrak{b}(\kappa) = \mathfrak{b}_{\mathrm{cl}}(\kappa)$.

A proof of this lemma can be found in [CS95]. Now, we are ready to prove this theorem by Raghavan and Shelah, as seen in their paper [RS18].

Theorem 2.8. *For regular and uncountable κ, $\mathfrak{b}(\kappa) = \kappa^+$ implies $\mathfrak{a}(\kappa) = \kappa^+$.*

Proof. Suppose $\mathfrak{b}(\kappa) = \kappa^+$. With the previous lemma, we know that $\mathfrak{b}(\kappa) = \mathfrak{b}_{\mathrm{cl}}(\kappa)$, so there is a club-unbounded family $\{f_\delta\}_{\delta<\kappa^+}$ of functions in ${}^\kappa\kappa$ of size κ^+. Thus, for any function $g \in {}^\kappa\kappa$, there is a $\delta < \kappa^+$ such that the set $\{\alpha < \kappa : g(\alpha) < f_\delta(\alpha)\}$ is stationary in κ.

For any subset $E \subseteq \kappa$ that is of cardinality κ, let $\langle \mu_{E,\xi} : \xi < \kappa \rangle$ be the increasing enumeration of E. Further, define for every $\delta < \kappa^+$ the set $C_\delta = \{\alpha < \kappa : \alpha \text{ is closed under } f_\delta\}$, where α is closed under f_δ if $f_\delta(\beta) < \alpha$ for all $\beta < \alpha$. Note that C_δ is club in κ: As $f_\delta \in {}^\kappa\kappa$, we have $f_\delta(\beta) \in \kappa$ for all $\beta < \kappa$, so $\sup(C_\delta) = \kappa$, that is, C_δ is unbounded. Now, let $\nu < \kappa$ be a limit and $\sup(C_\delta \cap \nu) = \nu$, that is, the lowest upper bound of all $\alpha < \nu$ that are closed under f_δ is ν, but then ν itself is closed under f_δ, so $\nu \in C_\delta$.

Let $\langle e_\delta : \kappa \leq \delta < \kappa^+ \rangle$ be a sequence of bijections $e_\delta : \kappa \to \delta$. Now, we will concentrate on the sequence $\langle (A_\delta, E_\delta) : \delta < \kappa^+ \rangle$ satisfying the following conditions for each $\delta < \kappa^+$:

1. $A_\delta \in [\kappa]^\kappa$ and $E_\delta \subseteq C_\delta$ is club in κ.

2. $\forall \gamma < \delta (|A_\gamma \cap A_\delta| < \kappa)$.

3. If $\kappa \leq \delta$, then $A_\delta = \bigcup_{\xi<\kappa} B_{\delta,\xi}$, where for each $\xi < \kappa$, we define $B_{\delta,\xi} = \{\mu_{E_\delta,\xi} \leq \alpha < \mu_{E_\delta,\xi+1} : \forall \nu < \mu_{E_\delta,\xi} (\alpha \notin A_{e_\delta(\nu)})\}$.

We will show first why this sequence is desired and then that such a sequence can, in fact, be constructed. Suppose that we constructed the sequence $\langle (A_\delta, E_\delta) : \delta < \kappa^+ \rangle$ satisfying the three conditions above, and let $\mathcal{A} = \{A_\delta\}_{\delta<\kappa^+}$. By the first two conditions, $\mathcal{A}$ is an almost disjoint family in $[\kappa]^\kappa$ of size κ^+. Note that our final goal is to show that $\mathfrak{a}(\kappa) = \kappa^+$ and that we already know $\kappa^+ \leq \mathfrak{a}(\kappa)$ (Lemma 2.3), so it suffices to show that there is a maximal almost disjoint family of size κ^+, that is, $\mathfrak{a}(\kappa) \leq \kappa^+$.

Claim 2.9. The almost disjoint family $\mathcal{A}$ is maximal.

Proof of claim. Fix an arbitrary $B \in [\kappa]^\kappa$. Let $g : \kappa \to \kappa$ be the function defined by

$$g(\mu) = \sup(\{\min(B \setminus (\mu + 1))\} \cup \{f_\nu(\mu) : \nu \leq \mu\})$$

for each $\mu \in \kappa$. Find a $\delta < \kappa^+$ such that the set $S = \{\mu \in \kappa : g(\mu) < f_\delta(\mu)\}$ is stationary in κ. This δ exists because of the club-unbounded family $\{f_\delta\}_{\delta<\kappa^+}$. By the definition of g, for all $\mu \in \kappa$, we have $g(\mu) \geq f_\nu(\mu)$ for $\nu \leq \mu$, thus, the δ we found must be $\delta \geq \kappa$. Therefore, because of the third condition above, $A_\delta = \bigcup_{\xi<\kappa} B_{\delta,\xi}$. Let

$$I = \{\xi < \kappa : B_{\delta,\xi} \cap B \neq \emptyset\}.$$

If I has cardinality κ, then $|A_\delta \cap B| = \kappa$, so $\mathcal{A} \cup \{B\}$ is not almost disjoint, and since B was arbitrary, this shows the maximality of $\mathcal{A}$. Thus, we will now suppose that $|I| < \kappa$.

Then $X = \{\mu_{E_\delta,\xi} : \xi \in I\} \subseteq E_\delta \subseteq \kappa$ and $|X| \leq |I| < \kappa$, so the supremum of the set $\sup(X) = \nu_0$ is strictly smaller than κ.

Now the set $\{\mu \in E_\delta : \mu > \nu_0\}= E_\delta \setminus \nu_0\subseteq \kappa$ is club in κ (as E_δ is club, $\sup(E_\delta) = \kappa$ and so $\sup(E_\delta \setminus \nu_0) = \kappa$. Also, for $\gamma < \kappa$ limit, if $\sup((E_\delta \setminus \nu_0) \cap \gamma) = \gamma$, then also $\sup(E_\delta \cap \gamma) = \gamma$, but then $\gamma \in E_\delta$ because E_δ is club, and thus $\gamma \in E_\delta \setminus \nu_0$). Further, the set

$$T = S \cap \{\mu \in E_\delta : \mu > \nu_0\}$$

is stationary in κ (since $E_\delta \setminus \nu_0$ is club, there is a club subset with which T has a nonempty intersection, therefore $T \notin \text{nonstat}_\kappa$). Consider an arbitrary $\mu \in T$. Then there is a $\xi \in \kappa \setminus I$ with $\mu = \mu_{E_\delta,\xi}$. Now $B_{\delta,\xi} \cap B = \emptyset$, because the ξ is not in I. But we also have that

$$\mu_{E_\delta,\xi} = \mu < \min(B \setminus (\mu + 1)) \leq g(\mu) < f_\delta(\mu) < \mu_{E_\delta,\xi+1},$$

where the first inequality holds as $B\setminus(\mu+1) = \{\alpha \in B : \alpha > \mu+1\}$, the second inequality by the definition of g, the third one as $\mu \in T$ and thus $\mu \in S = \{\mu \in \kappa : g(\mu) < f_\delta(\mu)\}$, and the last inequality holds because $\mu_{E_\delta,\xi+1} \in E_\delta \subseteq C_\delta$, so $\mu_{E_\delta,\xi+1}$ is closed under f_δ, so $f_\delta(\mu) < \mu_{E_\delta,\xi+1}$ as $\mu = \mu_{E_\delta,\xi} < \mu_{E_\delta,\xi+1}$.

Recall that $B \cap B_{\delta,\xi} = \emptyset$ and note that $\min(B \setminus (\mu + 1)) \notin B_{\delta,\xi}$. Then, it follows from the definition of $B_{\delta,\xi} = \{\mu_{E_\delta,\xi} \leq \alpha < \mu_{E_\delta,\xi+1} : \forall \nu < \mu_{E_\delta,\xi}(\alpha \notin A_{e_\delta(\nu)})\}$ that there is a $\nu < \mu$ such that $\min(B \setminus (\mu + 1)) \in A_{e_\delta(\nu)}$. This shows that for each $\mu \in T$, there is a $\nu < \mu$ and a $\beta \in B$ such that $\mu < \beta$ and $\beta \in A_{e_\delta(\nu)}$. So for each $\mu \in T$ there is a member of $\mathcal{A}$ that B has a nonempty intersection with. But we need a member to have an intersection of size κ with B.

As T is stationary in κ, there is a subset $T^* \subseteq T$ and a $\nu \in \kappa$ such that T^* is stationary in κ, $\nu < \mu$ for each $\mu \in T^*$, and there is a $\beta \in B$ such that $\mu < \beta$ and $\beta \in A_{e_\delta(\nu)}$. Thus, we found one $A_{e_\delta(\nu)} \in \mathcal{A}$ such that for each $\mu \in T^*$, there is a $\beta \in B \cap A_{e_\delta(\nu)}$.

As T^* is stationary, it meets every club set. For each $\alpha < \kappa$, the set $\kappa \setminus \alpha$ is club, so T^* contains elements larger than any given $\alpha \in \kappa$, which means that T^* is unbounded. If $|T^*| < \kappa$, then by the regularity of κ, the supremum of T^* would have to be $\sup(T^*) < \kappa$, but that would mean T^* is not unbounded, a contradiction. Thus, $|T^*| = \kappa$. But then $|B \cap A_{e_\delta(\nu)}| = \kappa$, so $\mathcal{A} \cup \{B\}$ is not almost disjoint. Since we fixed B arbitrarily, this shows that $\mathcal{A}$ is maximal almost disjoint. $\square$

Now we construct the sequence $\langle (A_\delta, E_\delta) : \delta < \kappa^+ \rangle$ which satisfies the three given conditions above for each $\delta < \kappa^+$.

Let $\{A_\gamma\}_{\gamma \in \kappa}$ be some partition of κ into κ many pieces of size κ that are pairwise disjoint. Let $E_\gamma = C_\gamma$ for each $\gamma < \kappa$. Then $\langle (A_\gamma, E_\gamma) : \gamma < \kappa \rangle$ satisfies the three conditions above, so we already built the first segment of the desired sequence.

Now fix $\kappa \leq \delta < \kappa^+$ and assume the segment of the desired sequence $\langle (A_\gamma, E_\gamma) : \gamma < \delta \rangle$ until the fixed δ is given. We construct the next pair of the sequence (A_δ, E_δ) as follows: Let θ be a sufficiently large cardinal and let $x = \{\kappa, \{f_\delta\}_{\delta < \kappa^+}, \{C_\delta\}_{\delta < \kappa^+}, \langle e_\delta : \kappa \leq \delta < \kappa^+ \rangle, \delta, \langle (A_\gamma, E_\gamma) : \gamma < \delta \rangle\}$ be this finite set of all the important parameters. Further, let $\langle N_\xi : \xi < \kappa \rangle$ be a sequence of models such that:

4. $\forall \xi < \kappa \; (N_\xi \prec H(\theta) \wedge x \in N_\xi)$

   that is, for all $\xi < \kappa$, N_ξ elementarily embeds into $H(\theta)$, which contains all models that model a (for us) big enough part of set theory (ZFC)

5. $\forall \xi < \kappa \; (|N_\xi| < \kappa \wedge \mu_\xi = N_\xi \cap \kappa \in \kappa)$

6. $\forall \xi < \xi + 1 < \kappa \; (\langle N_\zeta : \zeta \leq \xi \rangle \in N_{\xi+1})$

7. $\forall \xi < \kappa \; (\xi \text{ is a limit ordinal} \rightarrow N_\xi = \bigcup_{\zeta < \xi} N_\zeta)$

These conditions imply that for all $\zeta < \xi < \kappa$, $N_\zeta \in N_\xi$ and $N_\zeta \subseteq N_\xi$. Define E_δ to be $E_\delta = \{\mu_\xi : \xi < \kappa\}$. Then E_δ is club in κ and already in increasing order, that is, $\mu_{E_\delta, \xi} = \mu_\xi$ for all $\xi < \kappa$. These conditions also imply that $C_\delta \in N_\xi$ for each $\xi < \kappa$. And since C_δ is club in κ, it also follows that $\mu_\xi \in C_\delta$. Therefore, $E_\delta \subseteq C_\delta$, so we have E_δ as desired.

Now we will define A_δ by $A_\delta = \bigcup_{\xi < \kappa} B_{\delta,\xi}$, where $B_{\delta,\xi} = \{\mu_\xi \leq \alpha < \mu_{\xi+1} : \forall \nu < \mu_\xi \; (\alpha \notin A_{e_\delta(\nu)})\}$ for each $\xi < \kappa$. It follows that A_δ satisfies the third condition and that $A_\delta \subseteq \kappa$. For (A_δ, E_δ) to satisfy all three conditions, it remains to check that $|A_\delta| = \kappa$ and that for all $\gamma < \delta$, $|A_\gamma \cap A_\delta| < \kappa$.

Claim 2.10. $\forall \gamma < \delta \; (|A_\gamma \cap A_\delta| < \kappa)$

Proof of claim. Fix an arbitrary $\gamma < \delta$. Then there is a $\nu \in \kappa$ such that $e_\delta(\nu) = \gamma$, because $e_\delta : \kappa \rightarrow \delta$ is a bijection. Fix a $\zeta < \kappa$ with $\nu < \mu_\zeta$. Now, consider any $\xi < \kappa$ with $\zeta \leq \xi$. Then $\nu < \mu_\zeta \leq \mu_\xi$ and $A_\gamma \cap B_{\delta,\xi} = A_{e_\delta(\nu)} \cap B_{\delta,\xi} = \emptyset$ (recall that $A_\gamma = A_{e_\delta(\nu)}$, $\nu < \mu_\xi$ and $B_{\delta,\xi} = \{\mu_\xi \leq \alpha < \mu_{\xi+1} : \forall \nu < \mu_\xi \; (\alpha \notin A_{e_\delta(\nu)})\}$, so there cannot be an α that is in both sets). Therefore, it follows that

$$A_\gamma \cap A_\delta = \bigcup_{\xi<\kappa} (A_\gamma \cap B_{\delta,\xi}) = \bigcup_{\xi<\zeta} (A_\gamma \cap B_{\delta,\xi}) \subseteq \bigcup_{\xi<\zeta} B_{\delta,\xi},$$

since $A_\gamma \cap B_{\delta,\xi} = \emptyset$ for all $\zeta \leq \xi < \kappa$. Since $B_{\delta,\xi} \subseteq \mu_{\xi+1} \setminus \mu_\xi$ and $\mu_\xi, \mu_{\xi+1} \in \kappa$, we have that $|B_{\delta,\xi}| < \kappa$ for each $\xi < \zeta$. But then $\bigcup_{\xi<\zeta} B_{\delta,\xi}$ is a union of at most $|\zeta|$, which is $|\zeta| \leq \zeta < \kappa$ many sets, where each set has size $< \kappa$. And since κ is regular, this means $|\bigcup_{\xi<\zeta} B_{\delta,\xi}| < \kappa$ and with that $|A_\gamma \cap A_\delta| < \kappa$ follows. □

Claim 2.11. $|A_\delta| = \kappa$

Proof of claim. Since we defined $A_\delta = \bigcup_{\xi<\kappa} B_{\delta,\xi}$, it suffices to check that for each $\xi < \kappa$, $B_{\delta,\xi}$ is not empty.

Fix an arbitrary $\xi < \kappa$ and observe that for each $\nu < \mu_\xi$, $|A_{e_\delta(\mu_\xi)} \cap A_{e_\delta(\nu)}| < \kappa$ (as $e_\delta(\mu_\xi), e_\delta(\nu) \in \delta$ and $\langle (A_\gamma, E_\gamma) : \gamma < \delta \rangle$ satisfy the conditions 1.-3.). Let $R_\xi = \bigcup_{\nu<\mu_\xi} (A_{e_\delta(\mu_\xi)} \cap A_{e_\delta(\nu)})$. Then R_ξ is the union of at most $|\mu_\xi|$, that is $|\mu_\xi| \leq \mu_\xi < \kappa$ many sets, where each set is of size $< \kappa$. Again, by the regularity of κ, we have $|R_\xi| < \kappa$. Since $|A_{e_\delta(\mu_\xi)}| = \kappa$, there is thus an $\alpha \in A_{e_\delta(\mu_\xi)} \setminus R_\xi$ with $\mu_\xi \leq \alpha$. Recall that $N_{\xi+1} \prec H(\theta)$ and that all the relevant parameters belong to $N_{\xi+1}$ (as $x \in N_{\xi+1}$). Therefore, there exists an $\alpha \in N_{\xi+1}$ such that $\alpha \in \kappa$, $\mu_\xi \leq \alpha$, and for all $\nu < \mu_\xi$, $\alpha \notin A_{e_\delta(\nu)}$. Finally, we have that $\mu_\xi \leq \alpha < \mu_{\xi+1}$ and so $\alpha \in B_{\delta,\xi}$, which means that $B_{\delta,\xi}$ is not empty. □

This showed that we can, in fact, construct a sequence satisfying the conditions 1.-3., which we showed earlier implies that $\mathfrak{a}(\kappa) \leq \kappa^+$. □

3 κ-Cohen reals

This chapter follows Chapter 4 of [Fisc]. Some proofs were added or adapted to the uncountable case. Unless explicitly stated otherwise, κ is a regular uncountable cardinal. We begin with some basic definitions for a forcing poset $(\mathbb{P}, \leq, \mathbb{1})$.

Definition 3.1. [Fisa]

- Two conditions $p, q \in \mathbb{P}$ are **compatible**, and we write $p \not\perp q$, if and only if they have a common extension $r \in \mathbb{P}$, that is, $r \leq p$ and $r \leq q$. The conditions are **incompatible**, denoted by $p \perp q$, if and only if they are not compatible.

- A subset $A \subseteq \mathbb{P}$ of pairwise incompatible conditions is an **antichain**.

- A subset $D \subseteq \mathbb{P}$ is **dense** in $\mathbb{P}$ if and only if for every $p \in \mathbb{P}$, there is a $q \in D$ with $q \leq p$.

- A **filter** on $\mathbb{P}$ is a non-empty subset $G \subseteq \mathbb{P}$ such that $\mathbb{1} \in G$, any $p, q \in G$ have a common extension $r \in G$, and G is upwards closed (that is, for $p, q \in \mathbb{P}$, if $q \leq p$ and $q \in G$, then also $p \in G$).

- Let M be a ctm and $\mathbb{P} \in M$. Then the filter G is $(M, \mathbb{P})$-generic, or simply **generic**, if it meets all dense subsets $D \subseteq \mathbb{P}$ that are in M, that is, for each such subset D, we have $D \cap G \neq \emptyset$.

- The forcing poset $\mathbb{P}$ has the **$\boldsymbol{\kappa}^+$-chain condition** (κ^+-cc), if every antichain $A \subseteq \mathbb{P}$ has cardinality $|A| < \kappa^+$.

- Let $\delta < \kappa$ and $\langle p_\xi : \xi < \delta \rangle \subseteq \mathbb{P}$ be a sequence such that $p_{\xi_2} \leq p_{\xi_1}$ for all $\xi_1 < \xi_2 < \delta$. If there exists a $q \in \mathbb{P}$ with $q \leq p_\xi$ for all $\xi < \delta$, then $\mathbb{P}$ is **$\boldsymbol{\kappa}$-closed**.

- A **nice name** for a subset of the $\mathbb{P}$-name τ is a name of the form $\bigcup\{\{\sigma\} \times A_\sigma : \sigma \in \operatorname{dom}(\tau)\}$, where A_σ is an antichain for all $\sigma \in \operatorname{dom}(\tau)$.

Now we can introduce the Cohen forcing with the poset $\mathrm{Fn}_{<\kappa}(\lambda \times \kappa, \kappa)$ that we will be using throughout this chapter, and the κ-Cohen reals, which this poset adds.

Definition 3.2. The poset $\mathrm{Fn}_{<\kappa}(\lambda \times \kappa, \kappa)$ consists of all functions $p : \mathrm{dom}(p) \to \kappa$ such that $\mathrm{dom}(p) \subseteq \lambda \times \kappa$ and $|\mathrm{dom}(p)| < \kappa$, with extension relation superset: $p \leq q$ if and only if $p \supseteq q$. The largest element is $\mathbb{1} = \emptyset$.

Note that $p, q \in \mathrm{Fn}_{<\kappa}(\lambda \times \kappa, \kappa)$ are compatible, $p \not\perp q$, if and only if they agree on their shared domain $p \restriction \mathrm{dom}(p) \cap \mathrm{dom}(q) = q \restriction \mathrm{dom}(p) \cap \mathrm{dom}(q)$, because then their union is a common extension $p \cup q \leq p, q$.

For each $\alpha \in \lambda$, we will define $f_\alpha : \kappa \to \kappa$ and call it a **κ-Cohen real**. So we will end up with λ many κ-Cohen reals. Let $\alpha \in \lambda$ be such that there is a ξ such that $(\alpha, \xi) \in \mathrm{dom}(p)$ and define $p^\alpha : \kappa \to \kappa$, $\mathrm{dom}(p^\alpha) = \{\xi \in \kappa : \ (\alpha, \xi) \in \mathrm{dom}(p)\}$ with $p^\alpha(\xi) = p(\alpha, \xi)$. Let $\mathrm{pr}_1(\mathrm{dom}(p)) = \{\alpha : \exists \xi \ ((\alpha, \xi) \in \mathrm{dom}(p))\} \subseteq \lambda$. If G is a generic filter, we have $p^\alpha \subseteq f_\alpha$ for all $\alpha \in \mathrm{pr}_1(\mathrm{dom}(p))$. In this notation, we can define

$$f_\alpha = \bigcup \{p^\alpha : p \in G\},$$

where we let $p^\alpha = \emptyset$ if $\alpha \notin \mathrm{pr}_1(\mathrm{dom}(p))$.

Before we start forcing with the poset, we will show some properties of $\mathrm{Fn}_{<\kappa}(\lambda \times \kappa, \kappa)$ and the κ-Cohen real f_α.

Lemma 3.3.

1. $\mathrm{Fn}_{<\kappa}(\lambda \times \kappa, \kappa)$ is κ^+-cc.

2. $\mathrm{Fn}_{<\kappa}(\lambda \times \kappa, \kappa)$ is κ-closed.

3. The κ-Cohen real is unbounded over the ground model. That is, for a generic filter G and some $\alpha \in \lambda$, the above-defined $f_\alpha = \bigcup\{p^\alpha : p \in G\} \in {}^\kappa\kappa$ is not eventually dominated by any $h \in {}^\kappa\kappa \cap M$.

Proof. 1. We need the assumption that $\kappa^{<\kappa} = \kappa$. Let $\{p_\alpha\}_{\alpha<\kappa^+}$ be a family of κ^+ many conditions in $\mathrm{Fn}_{<\kappa}(\lambda \times \kappa, \kappa)$. Also, let $\{\mathrm{dom}(p_\alpha)\}_{\alpha<\kappa^+}$ be the family of the corresponding domains. Each domain is of size $< \kappa$, and so we can use the Delta System Lemma (see [Kun11], Lemma III.6.15). Thus, there is an index set $I \subseteq \kappa^+$, $|I| = \kappa^+$ such that $\{\mathrm{dom}(p_\alpha)\}_{\alpha \in I}$ is a Delta system with root R. That is, for $\alpha \neq \beta$ in I, $\mathrm{dom}(p_\alpha) \cap \mathrm{dom}(p_\beta) = R$. Since R must be of size $< \kappa$, and $|{}^R\kappa| \leq \kappa^{<\kappa} = \kappa$, there cannot be more than κ different functions from R to κ. So, there are $\alpha \neq \beta$ in I with $p_\alpha \restriction R = p_\beta \restriction R$, and in this case, they have a common extension. Therefore, $\{p_\alpha\}_{\alpha<\kappa^+}$ is not an antichain and $\mathrm{Fn}_{<\kappa}(\lambda \times \kappa, \kappa)$ is κ^+-cc.

2. [Fisa] Let $\delta < \kappa$ and $\langle p_\xi : \xi < \delta \rangle$ a sequence in $\mathrm{Fn}_{<\kappa}(\lambda \times \kappa, \kappa)$ such that for all $\xi_1 < \xi_2 < \ \delta$, $p_{\xi_2} \leq p_{\xi_1}$. Then $q = \bigcup p_\xi$ is a common extension.

3. This proof works like the one for the countable case in [Fisc]. We have a model M and a $\mathrm{Fn}_{<\kappa}(\lambda \times \kappa, \kappa)$-generic filter G. Fix $\alpha \in \lambda$ and let $f_\alpha = \bigcup\{p^\alpha : p \in G\}$ be a κ-Cohen real. Let h be any function in ${}^\kappa\kappa \cap M$. With a density argument, we will show that this arbitrary h is not eventually dominating f_α. For each $\beta \in \kappa$, define

$$D^\alpha_\beta = \{p \in \mathrm{Fn}_{<\kappa}(\lambda \times \kappa, \kappa) : \exists \gamma > \beta(\gamma \in \mathrm{dom}(p^\alpha) \wedge p^\alpha(\gamma) > h(\gamma))\}.$$

Then D^α_β is a dense subset of $\mathrm{Fn}_{<\kappa}(\lambda \times \kappa, \kappa)$ in M: Let $q \in \mathrm{Fn}_{<\kappa}(\lambda \times \kappa, \kappa)$. If q is not in D^α_β, we have to find a condition p that is in D^α_β and that extends q. Assume $q \notin D^\alpha_\beta$. Then there is no $\gamma > \beta$ with $\gamma \in \mathrm{dom}(q^\alpha)$ and $q^\alpha(\gamma) > h(\gamma)$. Since $|\mathrm{dom}(q)| < \kappa$, it is impossible that all $\gamma > \beta$ are in $\mathrm{dom}(q^\alpha)$ with $q^\alpha(\gamma) \leq h(\gamma)$. So there is a $\gamma > \beta$ with $\gamma \notin \mathrm{dom}(q^\alpha)$. Define $p \in \mathrm{Fn}_{<\kappa}(\lambda \times \kappa, \kappa)$ to be such that $p^\alpha \restriction \mathrm{dom}(q^\alpha) = q^\alpha$, $\mathrm{dom}(p^\alpha) = \mathrm{dom}(q^\alpha) \cup \{\gamma\}$ and $p^\alpha(\gamma) = h(\gamma) + 1$. Then $p \in D^\alpha_\beta$ and $p \leq q$. Now, since the set D^α_β is dense for all $\beta \in \kappa$, and since the generic filter G meets all dense sets, we have $M[G] \models \forall \beta \in \kappa \exists \gamma > \beta\ (h(\gamma) < f_\alpha(\gamma))$, and so $h \not\geq^* f_\alpha$.

□

Now we are ready to evaluate the κ-unbounding and κ-dominating number in the Cohen extension via $\mathrm{Fn}_{<\kappa}(\lambda \times \kappa, \kappa)$.

Theorem 3.4. *If GCH is satisfied in the ground model M, λ is regular with $\lambda > \kappa^+$, $\mathbb{P} = Fn_{<\kappa}(\lambda \times \kappa, \kappa)$ and G is a $(M, \mathbb{P})$-generic filter, then*

$$M[G] \models \mathfrak{b}(\kappa) = \kappa^+ < \mathfrak{d}(\kappa) = \lambda.$$

Proof. By Lemma 2.3 we know $\kappa^+ \leq \mathfrak{b}(\kappa) \leq \mathfrak{d}(\kappa)$, so it remains to show that $M[G] \models \mathfrak{b}(\kappa) \leq \kappa^+$ and $M[G] \models \mathfrak{d}(\kappa) = \lambda$. Since the GCH is satisfied in the ground model M, the family ${}^\kappa\kappa \cap M$ has cardinality κ^+. Thus, the following claim shows that in $M[G]$, there is an unbounded family of cardinality κ^+, so $M[G] \models \mathfrak{b}(\kappa) \leq \kappa^+$.

Claim 3.5. There is no $h \in {}^\kappa\kappa$ in $M[G]$ which eventually dominates each element of $M \cap {}^\kappa\kappa$.

Proof of claim. This proof is an adaptation from the countable setting seen in [Fisc]. Let $W = M \cap {}^\kappa\kappa$ and assume towards a contradiction that there is some $h \in M[G] \cap {}^\kappa\kappa$ such that $M[G] \models \forall x \in W(x \leq^* h)$. Let $p \in G$ be a condition such that $p \Vdash \forall \check{x} \in \check{W}(\check{x} \leq^* \dot{h})$, where $\dot{h}$ is a name for h. Further, let $\{r_\alpha\}_{\alpha \in I}$ be an enumeration of $\{q \in \mathbb{P} : q \leq p\}$. For each $\alpha \in I$, consider the set

$$E_\alpha = \{\delta \in \kappa : \exists \beta < \alpha(r_\beta \Vdash \check{\delta} \leq \dot{h}(\alpha))\}.$$

Now define $f \in W$ by setting $f(\alpha) = (\sup E_\alpha)^+$. By our assumption $p \Vdash \check{f} \leq^* \dot{h}$, so there exists a $\gamma \in \kappa$ and a condition q extending p such that $q \Vdash \forall \eta \geq \gamma(\check{f}(\eta) \leq \dot{h}(\eta))$. Since $q \leq p$, we have $q = r_\beta$ for some β. Fix an $\alpha \in \kappa$ with $\alpha > \gamma$ and $\alpha > \beta$. Then $r_\beta \Vdash \check{f}(\alpha) \leq \dot{h}(\alpha)$. Let $\delta = f(\alpha)$. Then, since $\beta < \alpha$ and $r_\beta \Vdash \check{\delta} = \check{f}(\alpha) \leq \dot{h}(\alpha)$, δ is an element of E_α and so $\delta < f(\alpha)$ by the definition of f. But this brings $f(\alpha) < f(\alpha)$, a contradiction. $\square$

The following two claims show that the κ-dominating number is λ in the generic extension.

Claim 3.6. $M[G] \models \mathfrak{d}(\kappa) \geq \lambda$.

Proof of claim. The proof is a generalization of the countable case seen in [Fisc]. For each $\alpha \in \lambda$, let $h_\alpha \in {}^\kappa\kappa \cap M[G]$ be the function defined by

$$h_\alpha(\delta) = \gamma \text{ iff } \exists p \in G((\alpha, \delta) \in \operatorname{dom}(p) \wedge p(\alpha, \delta) = \gamma).$$

Consider $\mathcal{F} = \{f_i\}_{i \in I} \subseteq {}^\kappa\kappa \cap M[G]$, an arbitrary family of functions of cardinality $|\mathcal{F}| < \lambda$. Since we can also view each f_i as a subset of $\kappa \times \kappa$, we can assign a nice name τ_i to each f_i. That is, for each $s_i \subseteq \kappa \times \kappa$, there is an antichain $A_{s_i} \subseteq \mathbb{P}$ such that $\tau_i = \bigcup_{s_i \in \kappa \times \kappa} \{\check{s}_i\} \times A_{s_i}$. For each $i \in I$, let $C_i \subseteq \lambda$ be the set of all $\alpha \in \lambda$ such that for some $\delta \in \kappa$ the pair (α, δ) appears in the domain of a condition in one of the antichains A_{s_i} associated with τ_i, that is, $C_i = \{\alpha \in \lambda : \exists \delta \in \kappa\ \exists s_i \in \kappa \times \kappa\ \exists p \in A_{s_i}((\alpha, \delta) \in \operatorname{dom}(p))\}$. Then each C_i is of cardinality at most κ, so the union over all of them $I_0 = \bigcup_{i \in I} C_i$ is of cardinality $|I_0| < \lambda$. To split up the generic extension, let $I_1 = \lambda \setminus I_0$ and define the filter $G_0 = G \cap \operatorname{Fn}_{<\kappa}(I_0 \times \kappa, \kappa)$ and $G_1 = G \cap \operatorname{Fn}_{<\kappa}(I_1 \times \kappa, \kappa)$. Then $M[G] = M[G_0][G_1]$ and for each $i \in I$, the evaluations of the name with respect to G and G_0 are equal, $(\tau_i)_G = (\tau_i)_{G_0}$, so the family $\mathcal{F}$ lies in $M[G_0]$.

Now, we fix an arbitrary $\alpha \in I_1$ and show that h_α is not dominated by any element of $\mathcal{F}$. Note that $\operatorname{Fn}_{<\kappa}(I_1 \times \kappa, \kappa) \cong \operatorname{Fn}_{<\kappa}(\{\alpha\} \times \kappa, \kappa) \times \operatorname{Fn}_{<\kappa}((I_1 \setminus \{\alpha\}) \times \kappa, \kappa)$, and use this to split up the generic extension further with the filter $G_1^0 = G_1 \cap \operatorname{Fn}_{<\kappa}(\{\alpha\} \times \kappa, \kappa)$ and $G_1^1 = G_1 \cap \operatorname{Fn}_{<\kappa}((I_1 \setminus \{\alpha\}) \times \kappa, \kappa)$ into $M[G_0][G_1] = M[G_0][G_1^0][G_1^1]$. Observe that

$h_\alpha(\delta) = \gamma$ if and only if there is a $p \in G_1^0$ with $(\alpha, \delta) \in \mathrm{dom}(p)$ and $p(\alpha, \delta) = \gamma$. Thus, since $\mathcal{F} \subseteq M[G_0]$ and the fixed h_α is only defined in $M[G_0][G_1^0]$, there is no element in $\mathcal{F}$ that dominates h_α. So the family $\mathcal{F}$ with $|\mathcal{F}| < \lambda$ is not dominating in $M[G]$, hence $M[G] \models \mathfrak{d}(\kappa) \geq \lambda$. □

Claim 3.7. $M[G] \models \mathfrak{d}(\kappa) \leq \lambda$.

Proof of claim. To show that a dominating family can be at most of size λ in the generic extension, we will show that $M[G] \models \kappa^\kappa = \lambda$.

Note that our poset $\mathbb{P} = \mathrm{Fn}_{<\kappa}(\lambda \times \kappa, \kappa)$ has cardinality $|\mathbb{P}| = \lambda$ and is κ^+-cc, so every antichain has size $\leq \kappa$. Thus, there are at most $|[\mathbb{P}]^\kappa| = \lambda^\kappa = \lambda$ many antichains. Now, we count how many nice names for functions in ${}^\kappa\kappa$ there are in the generic extension. Every function $f \in {}^\kappa\kappa$ can be identified by κ many antichains. For each $\alpha \in \kappa$, there is an antichain whose conditions decide the value for $f(\alpha)$. Thus, there are $\lambda \cdot \kappa = \lambda$ many nice names for functions in ${}^\kappa\kappa$, and therefore, $M[G] \models \kappa^\kappa \leq \lambda$.

That $M[G] \models \kappa^\kappa \geq \lambda$ will follow with an injection from λ to ${}^\kappa\kappa$. The functions $p \in \mathbb{P}$, $p : \lambda \times \kappa \to \kappa$ are partial and G is a $\mathbb{P}$-generic filter. With a density argument, it follows that the function $f_G = \bigcup G : \lambda \times \kappa \to \kappa$ is total: For each pair $(\alpha, \xi) \in \lambda \times \kappa$, let

$$D_{(\alpha,\xi)} = \{p \in \mathbb{P} : (\alpha, \xi) \in \mathrm{dom}(p)\}$$

and show that this is a dense set. Look at an arbitrary $p \in \mathbb{P}$. If $(\alpha, \xi) \in \mathrm{dom}(p)$, p itself is in $D_{(\alpha,\xi)}$. So assume $(\alpha, \xi) \notin \mathrm{dom}(p)$. Then, let q be the condition $q = p \cup \{(\alpha, \xi, \eta)\}$, where $\eta \in \kappa$ was chosen arbitrarily. Now $q \leq p$ and $q \in D_{(\alpha,\xi)}$, so the set $D_{(\alpha,\xi)}$ is indeed dense in $\mathbb{P}$ and thus f_G is a total function. For each $\alpha \in \lambda$, define $f_\alpha \in {}^\kappa\kappa$ to be the function with $f_\alpha(\xi) = f_G(\alpha, \xi)$. Again with a density argument, we will show that for each $\alpha \neq \beta$ in λ, the functions f_α and f_β are different: Fix $\alpha, \beta \in \lambda$ with $\alpha \neq \beta$ and show that

$$D_{\alpha,\beta} = \{p \in \mathbb{P} : \exists \xi \in \kappa \ ((\alpha, \xi) \in \mathrm{dom}(p) \wedge (\beta, \xi) \in \mathrm{dom}(p) \wedge p(\alpha, \xi) \neq p(\beta, \xi))\}$$

is dense. Look at an arbitrary $p \in \mathbb{P}$ and note that $|\mathrm{dom}(p)| < \kappa$. Thus, we can find a $\xi \in \kappa$ such that there is no η for which $(\eta, \xi) \in \mathrm{dom}(p)$. Now, take $q = p \cup \{(\alpha, \xi, \rho_1)\} \cup \{(\beta, \xi, \rho_2)\}$ where $\rho_1, \rho_2 \in \kappa$ with $\rho_1 \neq \rho_2$. Then $q \leq p$ and $q \in D_{\alpha,\beta}$, so $D_{\alpha,\beta}$ is indeed a dense set. This shows that the mapping $\lambda \to {}^\kappa\kappa,\ \alpha \mapsto f_\alpha$ is injective, so $M[G] \models \lambda \leq \kappa^\kappa$. □

This concludes the proof that $M[G] \models \mathfrak{b}(\kappa) = \kappa^+ < \mathfrak{d}(\kappa) = \lambda$. □

The next theorem shows the existence of an indestructible κ-mad family in the generic extension by a forcing poset of cardinality κ.

Theorem 3.8. *Let $2^\kappa = \kappa^+$ and let the forcing poset $\mathbb{P}$ in M be of cardinality κ. Then, there is a κ-mad family $\mathcal{A}$ of cardinality κ^+ in M such that*

$$M[G] \models \mathcal{A} \textit{ is } \kappa\textit{-mad}$$

for every $(M, \mathbb{P})$-generic filter G.

Proof. [Fisc] We will recursively define a family $\mathcal{A} = \{A_\xi\}_{\xi<\kappa^+}$ and show that this is a maximal κ-almost disjoint family in any generic extension.

Let the first κ elements of the family $\{A_\xi\}_{\xi<\kappa}$ be a partition of κ into sets of size κ. Let $\{(p_\xi, \tau_\xi)\}_{\kappa\leq\xi<\kappa^+}$ be an enumeration of all pairs (p, τ), where $p \in \mathbb{P}$ and τ is a nice $\mathbb{P}$-name for a subset of κ. We can fix this enumeration because we assumed $2^\kappa = \kappa^+$. Fix an ordinal ξ with $\kappa \leq \xi < \kappa^+$ and suppose all family members A_η for $\eta < \xi$ were already defined. We will now choose $A_\xi \subseteq \kappa$ satisfying the following two conditions:

1. A_ξ is almost disjoint with all the already defined sets,
$$\forall \eta < \xi \ (|A_\eta \cap A_\xi| < \kappa).$$
2. If p_ξ forces that what τ_ξ names has size κ and is almost disjoint from all the already defined sets, then there is a condition $r \geq q$ and $\beta \geq \alpha$ such that $\beta \in A_\xi$ and $r \Vdash \beta \in \tau_\xi$, for all $\alpha < \kappa$ and all conditions $q < p_\xi$,
$$p_\xi \Vdash |\tau_\xi| = \kappa \ \wedge \forall \eta < \xi \ (p_\xi \Vdash |\tau_\xi \cap A_\xi| < \kappa) \longrightarrow$$
$$\forall \alpha < \kappa \ \forall q \leq p_\xi \ \exists r \geq q \ \exists \beta \geq \alpha \ (\beta \in A_\xi \wedge r \Vdash \beta \in \tau_\xi).$$

This choice of A_ξ is possible: As $\kappa^+ \leq \mathfrak{a}(\kappa)$ (Lemma 2.3), there are no κ-mad families of cardinality κ, and so it is easy to find an A_ξ that satisfies the first condition. Assume the premise of the second condition, that is, $p_\xi \Vdash |\tau_\xi| = \kappa$ and $p_\xi \Vdash |\tau_\xi \cap A_\xi| < \kappa$ for all $\eta < \xi$. As we fixed $\xi < \kappa^+$, we can enumerate the already defined part of the family $\{A_\eta\}_{\eta<\xi}$ with $\{B_i\}_{i\in\kappa}$. Also fix the enumeration $\{(\alpha_i, q_i)\}_{i\in\kappa}$ for the product $\kappa \times \{q : q \leq p_\xi\}$. Since $q_i \leq p_\xi$ forces everything p_ξ does, that is, $q_i \Vdash |\tau_\xi| = \kappa$ and, with the new enumeration, $q_i \Vdash |\tau_\xi \cap B_j| < \kappa$ for all $j \leq i$, we have $q_i \Vdash |\tau_\xi \setminus \bigcup_{j\leq i} B_j| = \kappa$ for each $i \in \kappa$. Thus, we can choose a condition $r \leq q_i$ and a $\beta_i \geq \alpha_i$ such that $\beta_i \notin \bigcup_{j\leq i} B_j$

and $r \Vdash \beta_i \in \tau_\xi$. By letting A_ξ be the set of all these β_i for $i \in \kappa$, it satisfies both conditions that we wanted. This construction gives us the family $\mathcal{A} = \{A_\xi\}_{\xi<\kappa^+}$.

Finally, we show by contradiction that this family $\mathcal{A}$ is in fact κ-mad in every generic extension. Let G be any $(M, \mathbb{P})$-generic filter and suppose that $\mathcal{A}$ is not maximal in $M[G]$, that is, there is some (p_ξ, τ_ξ) with $p_\xi \in G$ and $p_\xi \Vdash \forall x \in \mathcal{A}\ (|\tau_\xi \cap x| < \kappa)$. This implies $p_\xi \Vdash |\tau_\xi \cap A_\xi| < \kappa$, so we can find a condition q extending p_ξ and some $\alpha < \kappa$ such that $q \Vdash (\tau_\xi \cap A_\xi \subseteq \alpha)$. Since A_ξ satisfies the second condition above, we know that there is a condition $r \leq q$ and a $\beta \geq \alpha$ such that $\beta \in A_\xi$ and $r \Vdash \beta \in \tau_\xi$. But that means there is a $\beta \geq \alpha$ with $\beta \in \tau_\xi \cap A_\xi \subseteq \alpha$, a contradiction. □

Recall that we already showed $\kappa^+ \leq \mathfrak{a}(\kappa) \leq 2^\kappa$ (Lemma 2.3) and that if $2^\kappa = \kappa^+$ and $\mathbb{P}$ is a forcing notion of cardinality κ, there is an indestructible κ-mad family of cardinality κ^+, so $\mathfrak{a}(\kappa) = \kappa^+$ in that model (Theorem 3.8). Using isomorphism types of names (which will be introduced in Claim 3.10), the next and last theorem of this chapter shows that there is no other cardinality for a κ-mad family than κ^+ and 2^κ in the generic extension, so the spectrum of the κ-almost disjointness number is $\mathfrak{sp}(\mathfrak{a}(\kappa)) = \{\kappa^+, 2^\kappa\}$.

Theorem 3.9. *Assume GCH at and below κ. Let λ be a cardinal such that $\lambda^\kappa = \lambda$, let $Fn_{<\kappa}(\lambda, 2)$ be the forcing poset and G any generic filter. Then*

$$M[G] \models \mathfrak{sp}(\mathfrak{a}(\kappa)) = \{\kappa^+, \lambda\}.$$

Proof. [Fisc] First of all, note that $\lambda = 2^\kappa$ in the generic extension $M[G]$. Fix μ with $\kappa^{++} \cdot 2 \leq \mu < 2^\kappa$. We start with a κ-almost disjoint family $\mathcal{B} = \{\dot{A}_\alpha\}_{\alpha<\mu}$ of $\mathrm{Fn}_{<\kappa}(\lambda, 2)$-names for subsets of κ and show that this family is not maximal, which implies $M[G] \models \mathfrak{sp}(\mathfrak{a}(\kappa)) \subseteq \{\kappa^+, 2^\kappa\}$.

Each name $\dot{A}_\alpha$ of the family $\mathcal{B}$ can be identified with κ many maximal antichains $\{\mathcal{X}^\alpha_\gamma\}_{\gamma\in\kappa}$, where $\mathcal{X}^\alpha_\gamma = \{p^\alpha_{\gamma,\delta}\}_{\delta\in\kappa} \subseteq \mathrm{Fn}_{<\kappa}(\lambda, 2)$, and κ many binary sequences $\{\mathcal{K}^\alpha_\gamma\}_{\gamma\in\kappa}$, where $\mathcal{K}^\alpha_\gamma = \{k^\alpha_{\gamma,\delta}\}_{\delta\in\kappa} \subseteq \{0, 1\}$. These antichains and sequences are chosen such that for each $\gamma, \delta \in \kappa$,

$$(p^\alpha_{\gamma,\delta} \Vdash \check{\gamma} \in \dot{A}_\alpha \text{ iff } k^\alpha_{\gamma,\delta} = 1) \text{ and } (p^\alpha_{\gamma,\delta} \Vdash \check{\gamma} \notin \dot{A}_\alpha \text{ iff } k^\alpha_{\gamma,\delta} = 0).$$

That way, we can identify each $\dot{A}_\alpha$ with a $(\kappa \times \kappa)$-matrix of pairs $\{(p^\alpha_{\gamma,\delta}, k^\alpha_{\gamma,\delta})\}_{\gamma\in\kappa,\delta\in\kappa}$.

Define $B^\alpha = \bigcup\{\mathrm{dom}(p^\alpha_{\gamma,\delta}) : \gamma, \delta \in \kappa\}$. Then $B^\alpha \subseteq \lambda$, $\dot{A}_\alpha$ is a $\mathrm{Fn}_{<\kappa}(B^\alpha, 2)$-name for a subset of κ and $|B^\alpha| = \kappa$. Observe that $\{B^\alpha\}_{\alpha<\mu}$ forms a family of at least κ^{++} many sets, each of cardinality κ. Thus, by the Delta System Lemma (see [Kun11], Lemma

III.6.15), there is an index set $I \subseteq \mu$ with $|I| = \kappa^{++}$, such that $\{B^\alpha\}_{\alpha\in I}$ forms a Delta system with root R, which is some subset of λ. That means for all $\alpha, \beta \in I$ with $\alpha \neq \beta$, $B_\alpha \cap B_\beta = R$. For each $\alpha, \beta \in I$ let $\varphi_{\alpha,\beta} : B^\alpha \to B^\beta$ be a bijection that fixes the root, $\varphi_{\alpha,\beta} \restriction R = \mathrm{id}$. This bijection induces an isomorphism $\psi_{\alpha,\beta} : \mathrm{Fn}_{<\kappa}(B^\alpha, 2) \to \mathrm{Fn}_{<\kappa}(B^\beta, 2)$, where $\psi_{\alpha,\beta}(p) = q$ if and only if

$$\mathrm{dom}(q) = \varphi_{\alpha,\beta}(\mathrm{dom}(p)) \ \wedge\ \forall\delta \in \mathrm{dom}(q)\ (q(\delta) = p(\varphi^{-1}_{\alpha,\beta}(\delta))).$$

Note that, since each B^α has cardinality κ, each $\mathrm{Fn}_{<\kappa}(B^\alpha, 2)$ is isomorphic to $\mathrm{Fn}_{<\kappa}(\kappa, 2)$.

Claim 3.10. There are only κ^+ many isomorphism types of names for subsets of κ.

Proof of claim. Recall that $2^\kappa = \kappa^+$ in the ground model. The name $\dot{A}_\alpha$ is identified with κ many binary sequences $\{\mathcal{K}^\alpha_\gamma\}_{\gamma\in\kappa}$, where $\mathcal{K}^\alpha_\gamma = \{k^\alpha_{\gamma,\delta}\}_{\delta\in\kappa} \subseteq \{0,1\}$. With the bijection $j : \kappa\times\kappa \to \kappa$ we have $\{k^\alpha_{\gamma,\delta}\}_{\gamma,\delta\in\kappa} \cong \{k^\alpha_{j(\gamma,\delta)}\}_{\gamma,\delta\in\kappa} = \{\tilde{k^\alpha_\zeta}\}_{\zeta\in\kappa} \subseteq \{0,1\}$, where $\tilde{k^\alpha_\zeta} := k^\alpha_{j(\gamma,\delta)}$ for $\zeta = j(\gamma,\delta)$. Thus, the name $\dot{A}_\alpha$ can be identified by a function $\tilde{k^\alpha} : \kappa \to \{0,1\}$, $\tilde{k^\alpha}(\zeta) = \tilde{k^\alpha_\zeta}$. We say that two names $\dot{A}_\alpha, \dot{A}_\beta$ have the same isomorphism type if $\tilde{k^\alpha} = \tilde{k^\beta}$. Since these are functions $\kappa \to 2$ and $2^\kappa = \kappa^+$, there can only be κ^+ distinct names. $\square$

But that means that there is some index set $J \subseteq I$ with $|J| \geq \kappa^{++}$ such that for any $\alpha, \beta \in J$, the isomorphism $\psi_{\alpha,\beta}$ identifies the names $\dot{A}^\alpha$ and $\dot{A}^\beta$ with each other, meaning that $\psi_{\alpha,\beta}(p^\alpha_{\gamma,\delta}) = p^\beta_{\gamma,\delta}$ and $k^\alpha_{\gamma,\delta} = k^\beta_{\gamma,\delta}$ for all $\gamma, \delta \in \kappa$. We can see that $k^\alpha_{\gamma,\delta}$ is independent from α, so we fix $k^\alpha_{\gamma,\delta} = k_{\gamma,\delta}$ for all $\alpha \in J$. We can assume that $J = \kappa^{++}$ and reindex the family $\mathcal{B}$, so that the first κ^{++} indices correspond to the ones in J,

$$\mathcal{B} = \{\dot{A}_\alpha\}_{\alpha\in J} \cup \{\dot{A}_\alpha\}_{\kappa^{++}\leq\alpha<\mu}.$$

Now we are ready to define a new name $\dot{A}_\mu$, so that the largest element of $\mathrm{Fn}_{<\kappa}(\lambda, 2)$ will force that $\dot{A}_\mu$ has an intersection of size less than κ with every element of $\mathcal{B}$, showing that $\mathcal{B}$ is not maximal.

We begin by choosing a new domain B^μ of cardinality κ such that it only shares R with all of the existing B^α, that is, $B^\mu \cap (\bigcup_{\alpha<\mu} B^\alpha) = R$. Then we define for each $\alpha < \kappa^{++}$ a bijection $\varphi_{\alpha,\mu} : B^\alpha \to B^\mu$ that, as the others, fixes the root $\varphi_{\alpha,\mu} \restriction R = \mathrm{id}$. These bijections again induce isomorphisms $\psi_{\alpha,\mu} : \mathrm{Fn}_{<\kappa}(B^\alpha, 2) \to \mathrm{Fn}_{<\kappa}(B^\mu, 2)$. Fix an arbitrary $\alpha < \kappa^{++}$ and define for each $\gamma, \delta \in \kappa$

$$p^\mu_{\gamma,\delta} = \psi_{\alpha,\mu}(p^\alpha_{\gamma,\delta}) \text{ and } k^\mu_{\gamma,\delta} = k^\alpha_{\gamma,\delta} = k_{\gamma,\delta}.$$

This way we make a copy of a name on our new domain. The matrix $\{(p^{\mu}_{\gamma,\delta}, k_{\gamma,\delta})\}_{\gamma\in\kappa,\delta\in\kappa}$ now uniquely determines the name $\dot{A}_{\mu}$. It remains to show that $\dot{A}_{\mu}$ turned out the way we wanted.

Fix an arbitrary $\beta < \mu$ and consider the set of intersections $\{B^{\beta} \cap B^{\alpha}\}_{\alpha<\kappa^{++}}$. Since $|B^{\beta}| = \kappa$ and $2^{\kappa} = \kappa^{+}$, there are only κ^{+} different subsets of B^{β} and so there must be $\alpha, \tilde{\alpha} \in \kappa^{++},\ \alpha \neq \tilde{\alpha}$ such that

$$B^{\beta} \cap B^{\alpha} = B^{\beta} \cap B^{\tilde{\alpha}} =: \Delta.$$

In particular, Δ is a subset of $B^{\alpha} \cap B^{\tilde{\alpha}} = R$ and we can fix an $\alpha < \kappa^{++}$ such that $B^{\beta} \cap B^{\alpha} \subseteq R$.

Claim 3.11. $B^{\mu} \cap B^{\beta} = B^{\alpha} \cap B^{\beta}$

Proof of claim. Recall that $\{B^{\alpha}\}_{\alpha<\kappa^{++}}$ forms a delta system with root R, so $R \subseteq B^{\alpha}$. Therefore $R \cap B^{\beta} \subseteq B^{\alpha} \cap B^{\beta} = \Delta \subseteq R$ and since $B^{\alpha} \cap B^{\beta} \subseteq B^{\beta}$, we obtain $R \cap B^{\beta} \subseteq B^{\alpha} \cap B^{\beta} \subseteq R \cap B^{\beta}$. This gives us the equality $R \cap B^{\beta} = B^{\alpha} \cap B^{\beta}$. By our choice of B^{μ} earlier, $R \subseteq B^{\mu}$ and $(B^{\mu} \setminus R) \cap B^{\beta} = \emptyset$. Thus,

$$\begin{aligned}
B^{\mu} \cap B^{\beta} &= (R \cup B^{\mu} \setminus R) \cap B^{\beta} && \text{by splitting up } B^{\mu} \text{ into } R \text{ and } B^{\mu} \setminus R \\
&= R \cap B^{\beta} \cup (B^{\mu} \setminus R \cap B^{\beta}) && \text{by distributivity of } \cap, \cup \\
&= R \cap B^{\beta} && \text{because } (B^{\mu} \setminus R) \cap B^{\beta} = \emptyset \\
&= B^{\alpha} \cap B^{\beta}. && \text{follows from above}
\end{aligned}$$

□

With this, we can write

$$\begin{aligned}
B^{\beta} \cup B^{\alpha} &= \Delta \sqcup (R \setminus \Delta) \sqcup (B^{\beta} \setminus R) \sqcup (B^{\alpha} \setminus R) \text{ and} \\
B^{\beta} \cup B^{\mu} &= \Delta \sqcup (R \setminus \Delta) \sqcup (B^{\beta} \setminus R) \sqcup (B^{\mu} \setminus R).
\end{aligned}$$

Define $\varphi : B^{\beta} \cup B^{\alpha} \to B^{\beta} \cup B^{\mu}$ by setting $\varphi(x) := \begin{cases} x & \text{if } x \in B^{\beta} \cup R \\ \varphi_{\alpha,\mu}(x) & \text{otherwise (if } x \in B^{\alpha} \setminus R). \end{cases}$
This function φ is a bijection, which extends the identity map on B^{β} and the bijection $\varphi_{\alpha,\mu}$. It also induces an isomorphism $\psi : \mathrm{Fn}_{<\kappa}(B^{\beta} \cup B^{\alpha}, 2) \to \mathrm{Fn}_{<\kappa}(B^{\beta} \cup B^{\mu}, 2)$. Since

$\mathbb{1}_{\mathrm{Fn}_{<\kappa}(B^\beta \cup B^\alpha, 2)} \Vdash |\dot{A}_\alpha \cap \dot{A}_\beta| < \kappa$, the isomorphism gives us

$$\psi(\mathbb{1}_{\mathrm{Fn}_{<\kappa}(B^\beta \cup B^\alpha, 2)}) \Vdash |\psi(\dot{A}_\alpha) \cap \psi(\dot{A}_\beta)| < \kappa.$$

The isomorphism acts in the following way:

- $\psi(\mathbb{1}_{\mathrm{Fn}_{<\kappa}(B^\beta \cup B^\alpha, 2)}) = \mathbb{1}_{\mathrm{Fn}_{<\kappa}(B^\beta \cup B^\mu, 2)}$
- $\psi(\dot{A}_\beta) = \dot{A}_\beta$
- $\psi(\dot{A}_\alpha) = \psi_{\alpha,\mu}(\dot{A}_\alpha) = \dot{A}_\mu$,

therefore we have for any arbitrary $\beta < \mu$

$$\mathbb{1}_{\mathrm{Fn}_{<\kappa}(B^\beta \cup B^\mu, 2)} \Vdash |\dot{A}_\mu \cap \dot{A}_\beta| < \kappa,$$

and so, it follows that

$$\mathbb{1}_{\mathrm{Fn}_{<\kappa}(\lambda, 2)} \Vdash |\dot{A}_\mu \cap \dot{A}_\beta| < \kappa.$$

That shows that $\dot{A}_\mu$ is indeed as desired, the largest element of $\mathrm{Fn}_{<\kappa}(\lambda, 2)$ forces it to have an intersection of size less than κ with every element of $\mathcal{B}$. That means, in the final generic extension $M[G]$, the family $\mathcal{B} \cup \{\dot{\mathcal{A}}_\mu\}$ is κ-almost disjoint, so $\mathcal{B}$ is not maximal. Thus, we know that in $M[G]$ we have $\mathfrak{sp}(\mathfrak{a}(\kappa)) \subseteq \{\kappa^+, 2^\kappa\}$.

Since there is always a κ-mad family of cardinality 2^κ, we have that $2^\kappa \in \mathfrak{sp}(\mathfrak{a}(\kappa))$. By the previous Theorem 3.8, there exists a $\mathrm{Fn}_{<\kappa}(\kappa, 2)$-indestructible κ-mad family of cardinality κ^+. This family remains maximal in $M[G]$ (see [Fisc] and [BFF]). Therefore also $\kappa^+ \in \mathfrak{sp}(\mathfrak{a}(\kappa))$, so $\mathfrak{sp}(\mathfrak{a}(\kappa)) = \{\kappa^+, 2^\kappa\}$. $\square$

4 Easton extensions

We follow Chapter 5 of [Fisc], as well as Section V.2 of [Kun11], and start by introducing the Easton forcing we will use. In the second section of this chapter, we will further refine these concepts and show which values $\mathfrak{a}(\kappa)$, $\mathfrak{b}(\kappa)$, $\mathfrak{d}(\kappa)$ and 2^κ take on in the Easton extension.

4.1 Introduction to Easton forcing

To define the Easton poset, we need the Easton index function.

Definition 4.1. The function E is an **Easton index function** if the domain $\mathrm{dom}(E)$ is a set of regular cardinals and the following holds:

1. For each $\kappa \in \mathrm{dom}(E)$, the value $E(\kappa)$ is an infinite cardinal such that $\mathrm{cf}(E(\kappa)) > \kappa$, and
2. for all $\kappa, \kappa' \in \mathrm{dom}(E)$, if $\kappa < \kappa'$ then $E(\kappa) \leq E(\kappa')$.

Notation. Let $\mathbb{R} = \prod_{i \in I} \mathbb{P}_i$, where I is an index set and $\mathbb{P}_i$ is a forcing poset for each $i \in I$. Recall that the product $\mathbb{R}$ is the poset of all sequences $p \in \prod_{i \in I} \mathbb{P}_i$ with extension relation $q \leq p$ if and only if $q(i) \leq p(i)$ for all $i \in I$. The largest element is the sequence $\mathbb{1}_{\mathbb{R}} = \langle \mathbb{1}_i : i \in I \rangle$. For $\mathbb{R}$, we fix the following notation:

- The **support** of a condition $p \in \mathbb{R}$ is the set

$$\mathrm{supp}(p) = \{i \in I : p(i) \neq \mathbb{1}_{\mathbb{P}_i}\}.$$

- Some $p \in \mathbb{R}$ satisfies the **Easton condition**, or has **Easton support**, if for all regular cardinals λ, $|\mathrm{supp}(p) \cap \lambda| < \lambda$.
- If $\mathbb{R} = \prod_{i \in I} \mathbb{P}_i$ is given, where I is some set of cardinals, let

$$\prod_{i \in I,\ \text{Easton}} \mathbb{P}_i$$

be the suborder of $\mathbb{R}$ consisting only of those conditions $p \in \mathbb{R}$ with Easton support.

Definition 4.2. Let E be an Easton index function and let $\mathbb{R}$ be the product of forcing posets $\mathbb{R} = \prod_{\kappa \in \mathrm{dom}(E)} \mathrm{Fn}_{<\kappa}(E(\kappa), 2)$. Then the **Easton poset** $\mathbb{P}(E)$ is the suborder of $\mathbb{R}$ of all conditions p such that for all regular cardinals λ, $|\{\kappa \in \lambda \cap \mathrm{dom}(E) : p(\kappa) \neq \mathbb{1}_{\mathrm{Fn}_{<\kappa}(E(\kappa),2)}\}| < \lambda$. In the above notation,

$$\mathbb{P}(E) = \prod_{\kappa \in \mathrm{dom}(E),\ \text{Easton}} \mathrm{Fn}_{<\kappa}(E(\kappa), 2).$$

Next, we will show that Easton poset has the λ^+-chain condition.

Lemma 4.3. Let λ be a regular cardinal such that $2^{<\lambda} = \lambda$ and let E be an Easton index function such that $\mathrm{dom}(E) \subseteq \lambda^+$. Then $\mathbb{P}(E)$ is λ^+-cc.

Proof. [Kun11] To show that the given $\mathbb{P}(E)$ is λ^+-cc, we look at an arbitrary family $W = \{p_\alpha\}_{\alpha<\lambda^+}$ of λ^+ many conditions in $\mathbb{P}(E)$ and show that this is not an antichain.

For each $\alpha < \lambda^+$, the condition $p_\alpha \in W$ is a sequence of partial functions $p_\alpha(\gamma) \in \mathrm{Fn}_{<\gamma}(E(\gamma), 2)$, for $\gamma \in \mathrm{dom}(p_\alpha)$. Also, $|\mathrm{dom}(p_\alpha(\gamma))| < \gamma \leq \lambda$ for every $\gamma \in \mathrm{dom}(p_\alpha) \subseteq \mathrm{dom}(E) \subseteq \lambda^+$. By definition of $\mathbb{P}(E)$, we have $|\{\gamma \in \mathrm{dom}(E) \cap \lambda : p_\alpha(\gamma) \neq \emptyset\}| < \lambda$ for each $\alpha < \lambda^+$ and so for each condition p_α, $|\{\gamma \in \mathrm{dom}(E) : p_\alpha(\gamma) \neq \emptyset\}| < \lambda$. Let $D_\alpha = \bigcup\{\{\gamma\} \times \mathrm{dom}(p_\alpha(\gamma)) : \gamma \in \mathrm{dom}(E)\}$, then also $|D_\alpha| < \lambda$. Therefore, $\{D_\alpha\}_{\alpha<\lambda^+}$ is a family of λ^+ many sets, each of size less than λ, and so we can apply the Delta System Lemma (see [Kun11], Lemma III.6.15). This gives us an index set $B \subseteq \lambda^+$ such that $|B| = \lambda^+$ and the family $\{D_\alpha\}_{\alpha \in B}$ forms a Delta system with root R. Note that each condition p_α is a function from D_α to $\{0, 1\}$, with $p_\alpha((\gamma, x)) = p_\alpha(\gamma)(x)$ for $(\gamma, x) \in \{\gamma\} \times \mathrm{dom}(p_\alpha(\gamma)) \subseteq D_\alpha$. Since R is the intersection of any two D_α, D_β for $\alpha \neq \beta \in B$ and $|D_\alpha| < \lambda$ for all α, R also has cardinality $|R| < \lambda$. Together with the premise $2^{<\lambda} = \lambda$, we see that there cannot be more than λ many distinct functions from R to $\{0, 1\}$. So, there are $\alpha_1 \neq \alpha_2$ such that $p_{\alpha_1} \restriction R = p_{\alpha_2} \restriction R$, which implies that p_{α_1} and p_{α_2} are compatible and that W is not an antichain. $\square$

With the following definition, we can split the Easton poset at some fixed ordinal into two parts.

Definition 4.4. For an Easton index function E and any ordinal λ, we define

$$E_\lambda^- = E \restriction \{\kappa \in \mathrm{dom}(E) : \kappa \leq \lambda\} \text{ and}$$
$$E_\lambda^+ = E \restriction \{\kappa \in \mathrm{dom}(E) : \kappa > \lambda\}.$$

The posets arising with this definition have the following properties.

Lemma 4.5. Let λ be a regular cardinal such that $2^{<\lambda} = \lambda$ and let E be an Easton index function with $\mathrm{dom}(E) \subseteq \lambda^+$. Then:

1. $\mathbb{P}(E) \cong \mathbb{P}(E^-_\lambda) \times \mathbb{P}(E^+_\lambda)$.

2. $\mathbb{P}(E^-_\lambda)$ is λ^+-cc.

3. $\mathbb{P}(E^+_\lambda)$ is λ^+-closed.

Proof. 1. Looking at the definition of $\mathbb{P}(E) = \prod_{\kappa \in \mathrm{dom}(E),\text{ Easton}} \mathrm{Fn}_{<\kappa}(E(\kappa), 2)$, we can see that the product can be "split up" in the claimed way:

$$\begin{aligned}\mathbb{P}(E) &\cong \prod_{\kappa\in\mathrm{dom}(E),\ \kappa\leq\lambda,\text{ Easton}} \mathrm{Fn}_{<\kappa}(E(\kappa),2) \times \prod_{\kappa\in\mathrm{dom}(E),\ \kappa>\lambda,\text{ Easton}} \mathrm{Fn}_{<\kappa}(E(\kappa),2) \\ &= \mathbb{P}(E^-_\lambda) \times \mathbb{P}(E^+_\lambda)\end{aligned}$$

2. By definition, the domain of $E^-_\lambda = E \restriction \{\kappa \in \mathrm{dom}(E) : \kappa \leq \lambda\}$ is a subset of λ^+. By the previous lemma, it follows immediately that $\mathbb{P}(E^-_\lambda)$ is λ^+-cc.

3. Let $\delta < \lambda^+$ and $\langle p_\xi : \xi < \delta\rangle$ a sequence in $\mathbb{P}(E^+_\lambda)$ such that for all $\xi_1 < \xi_2 < \delta$, $p_{\xi_2} \leq p_{\xi_1}$. It is to show that there exists a $q \in \mathbb{P}(E^+_\lambda)$ such that $q \leq p_\xi$ for all $\xi < \delta$. Recall that the conditions p_ξ are sequences of partial functions $p_\xi(\kappa) \in \mathrm{Fn}_{<\kappa}(E(\kappa), 2)$ for $\kappa \in \mathrm{dom}(E^+_\lambda)$ and that $p_{\xi_2} \leq p_{\xi_1}$ if and only if $p_{\xi_2}(\kappa) \leq p_{\xi_1}(\kappa)$ for all $\kappa \in \mathrm{dom}(E^+_\lambda)$. Fix an arbitrary index κ. For all $\xi_1 < \xi_2 < \delta$ we have $p_{\xi_2}(\kappa) \leq p_{\xi_1}(\kappa)$, that is, $p_{\xi_2}(\kappa) \supseteq p_{\xi_1}(\kappa)$, and so for $q(\kappa) = \bigcup_{\xi<\delta} p_\xi(\kappa) \in \mathrm{Fn}_{<\kappa}(E(\kappa), 2)$ we have $p_\xi(\kappa) \subseteq q(\kappa)$ for all $\xi < \delta$. Let q be the sequence $\langle q(\kappa) : \kappa \in \mathrm{dom}(E^+_\lambda)\rangle$, then $q \in \mathbb{P}(E^+_\lambda)$ and $q \leq p_\xi$ for all $\xi < \delta$.

□

To prove the subsequent lemma, we need the Approximation Lemma for the uncountable setting, so we adapted it and its proof from [Fisa].

Lemma 4.6 (Approximation Lemma)**.** Let λ be a regular uncountable cardinal, $\mathbb{P}$ a forcing poset in M with $(\mathbb{P}$ is λ^+-cc$)^M$ and let $A, B \in M$. Further, let G be a $(M, \mathbb{P})$-generic filter and let $f : A \to B$ be a function in $M[G]$. Then there is a function $F : A \to \mathcal{P}(B)$ in M such that for all $a \in A$, $f(a) \in F(a)$ and $(|F(a)| \leq \lambda)^M$.

Proof. We have $M[G] \models f : A \to B$. By the Truth Lemma (see [Kun11], Lemma IV.2.24) there is a name $\dot{f}$ in M with $\dot{f}_G = f$ and a condition $p \in G$ such that $p \Vdash \dot{f} : \check{A} \to \check{B}$. Let $F : A \to \mathcal{P}(B)$ be the function defined by $F(a) = \{b \in B : \exists q \leq p \ (q \Vdash \dot{f}(\check{a}) = \check{b}\}$ for all $a \in A$. By the Definability Lemma (see [Kun11], Lemma IV.2.25), this function is in the ground model, $F \in M$. Suppose that $M[G] \models f(a) = b$. Again with the Truth Lemma, there is then a condition $q \leq p$ such that $q \Vdash f(a) = b$ and by our definition of F, $b \in F(a)$. Thus, $M[G] \models f(a) \in F(a)$.

Now we fix an arbitrary $a \in A$ and use that $(\mathbb{P}$ is λ^+-cc$)^M$ to show that $(|F(a)| \leq \lambda)^M$. Note that we can choose $q_b \leq p$ such that $q_b \Vdash \dot{f}(\check{a}) = \check{b}$ for each $b \in F(a)$. If $c \neq b$ in $F(a)$, the conditions q_b and q_c must be incompatible: Otherwise, they would have a common extension $r \leq q_b, q_c$ forcing $\dot{f}(\check{a}) = \check{b} \wedge \dot{f}(\check{a}) = \check{c}$. Since $\mathbb{P}$ is λ^+-cc, there can only be less than λ^+ many incompatible conditions below p and so $|F(a)| \leq \lambda$. $\square$

Now we can show that the Easton forcing preserves cofinalities and cardinalities.

Lemma 4.7. Let M be a ctm with $M \models \mathsf{GCH}$. If $(E$ is an Easton function$)^M$ and $\mathbb{P} = \mathbb{P}(E)$ is the Easton poset, then $\mathbb{P}$ preserves cofinalities, and so $\mathbb{P}$ also preserves cardinalities.

Proof. [Fisc] Recall that every $\mathbb{P} \in M$ that preserves cofinalities also preserves cardinals. Also, that $\mathbb{P}$ preserves cardinals if and only if the regularity of any limit β is preserved in any generic extension (see [Kun11], Lemma IV.3.3). So, it is sufficient to show that any uncountable regular cardinal in M is also regular in any generic extension. Towards a contradiction, we assume that there is a regular uncountable cardinal θ in M and a $(M, \mathbb{P})$-generic filter K such that θ does not remain regular in the generic extension, that is, $\lambda = (\mathrm{cf}(\theta))^{M[K]} < \theta$. The cofinality is always a regular cardinal $\mathrm{cf}(\mathrm{cf}(\theta)) = \mathrm{cf}(\theta)$, and so λ is a regular cardinal in $M[K]$. Since regularity is downwards absolute, λ is also regular in M. There is a function $f : \lambda \to \theta$ in $M[K]$ with $\sup(\mathrm{ran}(f)) = \theta$, because $\lambda = \mathrm{cf}(\theta)$ in $M[K]$.

We have seen that $\mathbb{P} = \mathbb{P}(E) \cong \mathbb{P}(E_\lambda^-) \times \mathbb{P}(E_\lambda^+)$ and we will use that to rewrite the generic extension as $M[K] = M[H][G]$, where H is $\mathbb{P}(E_\lambda^+)$-generic over M and G is $\mathbb{P}(E_\lambda^-)$-generic over $M[H]$. The poset $\mathbb{P}(E_\lambda^+)$ is λ^+-closed, so there are no new sequences of length λ in $M[H]$ and so $M[H] \models 2^{<\lambda} = \lambda$. With that, it follows that $\mathbb{P}(E_\lambda^-)$ is λ^+-cc in $M[H]$ and, by seeing the function $f : \lambda \to \theta$ above as a λ-sequence, that $f \in M[K] = M[H][G]$ above cannot belong to $M[H]$. Thus, f must belong to a generic extension of $M[H]$ via a λ^+-cc forcing, namely the forcing poset $\mathbb{P}(E_\lambda^-)$ and extension $(M[H])[G]$. Using the Approximation Lemma, there is a function $F : \lambda \to \mathcal{P}(\theta)$ in

$M[H]$ such that $|F(\alpha)| \leq \lambda$ and $f(\alpha) \in F(\alpha)$ for all $\alpha \in \lambda$. The function F, again seen as a λ-sequence, must be in the ground model because the generic extension $M[H]$ was obtained with a λ^+-closed forcing.

Now, since $F \in M$, we have $(|F(\alpha)| \leq \lambda)^M$ for each $\alpha \in \lambda$ and so $\bigcup_{\alpha<\lambda} F(\alpha)$ is a set of cardinality at most λ, which is unbounded in θ, that is, $\sup(\bigcup_{\alpha<\lambda} F(\alpha)) = \theta$. But that would mean that θ is not regular in M, a contradiction. □

This theorem shows Easton's result from 1964 [Eas64], where he developed Cohen's method of forcing further to show that the global continuum function $\{2^\kappa : \kappa \in \mathrm{dom}(E)\}$ can take on almost any value, as long as it does not contradict König's Theorem, which states that $\mathrm{cf}(2^\kappa) > \kappa$ for infinite κ (see [Kun11], Theorem I.13.12) or the monotonicity (that $\kappa < \kappa'$ implies $2^\kappa \leq 2^{\kappa'}$). König's theorem corresponds to the first item, and the monotonicity to the second item in Definition 4.1 of the Easton index function.

Theorem 4.8. *Let M be a ctm with $M \models \mathsf{GCH}$. In M, let E be an Easton index function and let $\mathbb{P} = \mathbb{P}(E)$ be the Easton poset. Let K be some $(M, \mathbb{P})$-generic filter. Then $\mathbb{P}$ preserves cofinalities and cardinalities, and*

$$M[K] \models \forall \kappa \in \mathit{dom}(E)\ (2^\kappa = E(\kappa)).$$

Proof. [Kun11] In the previous lemma, we saw that $\mathbb{P}$ preserves cofinalities and cardinalities. Fix $\kappa \in \mathrm{dom}(E)$ and let $F_\kappa : E(\kappa) \to \{0,1\}$ be the function defined by

$$F_\kappa(\delta) = i \ \text{ iff } \ \exists p \in K\ (\delta \in \mathrm{dom}(p(\kappa) \wedge p(\kappa)(\delta) = i).$$

Now, define $h_\alpha : \kappa \to \{0,1\}$ for each $\alpha \in E(\kappa)$ by setting

$$h_\alpha(\xi) = F_\kappa(\kappa \cdot \alpha + \xi).$$

We show that the functions $\{h_\alpha\}_{\alpha \in E(\kappa)}$ are pairwise distinct with a density argument: Let $\alpha, \beta \in E(\kappa)$ with $\alpha \neq \beta$. Now, h_α and h_β are distinct if and only if there is a $\xi \in \kappa$ such that $h_\alpha(\xi) \neq h_\beta(\xi)$. Note that $h_\alpha(\xi) \neq h_\beta(\xi)$ if and only if $F_\kappa(\kappa \cdot \alpha + \xi) \neq F_\kappa(\kappa \cdot \beta + \xi)$, which holds if and only if there is a $p \in K$ with $\kappa \cdot \alpha + \xi, \kappa \cdot \beta + \xi \in \mathrm{dom}(p(\kappa))$ such that $p(\kappa)(\kappa \cdot \alpha + \xi) \neq p(\kappa)(\kappa \cdot \beta + \xi)$. So, we define for the above fixed κ and α, β the set

$$\begin{aligned} D^\kappa_{\alpha,\beta} = \{&p \in \mathbb{P} : \\ &\exists \xi \in \kappa\ (\kappa \cdot \alpha + \xi, \kappa \cdot \beta + \xi \in \mathrm{dom}(p(\kappa)) \wedge p(\kappa)(\kappa \cdot \alpha + \xi) \neq p(\kappa)(\kappa \cdot \beta + \xi))\}, \end{aligned}$$

and claim that it is a dense subset of $\mathbb{P}$. Let $p \in \mathbb{P}$, then either $p \in D^\kappa_{\alpha,\beta}$ or $p \notin D^\kappa_{\alpha,\beta}$. If p is not already in $D^\kappa_{\alpha,\beta}$, we can extend p to p' by adding what is missing to the domain. Without loss of generality, assume $\kappa \cdot \alpha + \xi$ is not in the domain of $p(\kappa)$. Let $\mathrm{dom}(p'(\kappa)) = \mathrm{dom}(p(\kappa)) \cup \{\kappa \cdot \alpha + \xi\}$ and set $p'(\kappa)(\kappa \cdot \alpha + \xi) \neq p(\kappa)(\kappa \cdot \beta + \xi) \in \{0,1\}$. If neither of $\kappa \cdot \alpha + \xi, \kappa \cdot \beta + \xi$ are in the domain of $p(\kappa)$, add them both and let $p'(\kappa)$ map one of them to 0 and the other one to 1. That way we found an extension p' of p that is in $D^\kappa_{\alpha,\beta}$, so $D^\kappa_{\alpha,\beta}$ is dense. Since K is $(M, \mathbb{P})$-generic, there is a $p \in K \cap D^\kappa_{\alpha,\beta}$ forcing that there is $\xi \in \kappa$ such that $\kappa \cdot \alpha + \xi, \kappa \cdot \beta + \xi \in \mathrm{dom}(p(\kappa))$ and $p(\kappa)(\kappa \cdot \alpha + \xi) \neq p(\kappa)(\kappa \cdot \beta + \xi)$, which means that we have $h_\alpha \neq h_\beta$ in $M[K]$. So $\{h_\alpha\}_{\alpha \in E(\kappa)} \subseteq {}^\kappa 2$ are pairwise distinct, and this shows $M[K] \models 2^\kappa \geq E(\kappa)$.

It remains to show that $M[K] \models 2^\kappa \leq E(\kappa)$. For this, we will use nice names (recall Definition 3.1). Remember that $\mathbb{P}(E) \cong \mathbb{P}(E^-_\kappa) \times \mathbb{P}(E^+_\kappa)$ and that $\mathbb{P}(E^+_\kappa)$ is κ^+-closed, so all new subsets of κ must be added by $\mathbb{P}(E^-_\kappa)$. With $(M, \mathbb{P}(E^+_\kappa))$-generic H and $(M[H], \mathbb{P}(E^-_\kappa))$-generic G, where $\mathbb{P}(E^-_\kappa) \cong \mathbb{P}(E)$, we can see the generic extension $M[K]$ as an extension in two steps $M[K] = M[H][G]$. Only $E(\kappa)$ many antichains from $\mathbb{P}(E^-_\kappa)$ are in M and in $M[H]$, since an antichain is also just a subset and could not have been added by $\mathbb{P}(E^+_\kappa)$. So, there are at most $E(\kappa)$ many nice names for subsets of κ in $M[H]$. And therefore, $M[H][G] \models 2^\kappa \leq E(\kappa)$. □

4.2 Evaluating some cardinal characteristics in the Easton extension

After introducing Easton-like products, we will let them help us find a nice name for a function. This will be very useful in the following proofs.

Definition 4.9. Let I be a set of cardinals and J_i some index set for each $i \in I$. A poset of the form

$$\mathbb{P}(I) = \prod_{i \in I,\ \text{Easton}} \mathrm{Fn}_{<i}(J_i, 2)$$

is an **Easton-like product**.

Note that every Easton poset is an Easton-like product, but not vice versa. But, we will see later that some specific Easton-like products can be completely embedded into a given Easton poset.

Definition 4.10. The map $i : \mathbb{Q} \to \mathbb{R}$ is a **complete embedding** from forcing poset $\mathbb{Q}$ into forcing poset $\mathbb{R}$ if and only if:

1. $i(\mathbb{1}_{\mathbb{Q}}) = \mathbb{1}_{\mathbb{R}}$.
2. $\forall q_1, q_2 \in \mathbb{Q}\ (q_1 \leq_{\mathbb{Q}} q_2 \longrightarrow i(q_1) \leq_{\mathbb{R}} i(q_2))$.
3. $\forall q_1, q_2 \in \mathbb{Q}\ (q_1 \perp_{\mathbb{Q}} q_2 \longleftrightarrow i(q_1) \perp_{\mathbb{R}} i(q_2))$.
4. If A is a maximal antichain in $\mathbb{Q}$, then $i(A)$ is a maximal antichain in $\mathbb{R}$.

If $\mathbb{Q} \subseteq \mathbb{R}$ and the identity map is a complete embedding, then we denote this by $\mathbb{Q} \lessdot \mathbb{R}$.

Now, we will find a nice name for a given function.

Remark 4.11. Assume that we have a ctm M, an Easton index function E in M and a $(M, \mathbb{P}(E))$-generic filter K. Fix an arbitrary $\kappa \in \mathrm{dom}(E)$, let $f \in {}^{\kappa}\kappa \cap M[K]$ and partition the poset $\mathbb{P}(E) \cong \mathbb{P}(E_\kappa^-) \times \mathbb{P}(E_\kappa^+)$ as in Lemma 4.5. Then we can write the generic extension by K as

$$M[K] = M[H][G],$$

where H is $\mathbb{P}(E_\kappa^+)$-generic over M and G is $\mathbb{P}(E_\kappa^-)$-generic over $M[H]$. Remember that $\mathbb{P}(E_\kappa^+)$ is κ^+-closed, so there are no new functions from κ to κ in $M[H]$, that is, $M[H] \cap {}^{\kappa}\kappa = M \cap {}^{\kappa}\kappa$. Also note that in $M[H]$, the poset $\mathbb{P}(E_\kappa^-)$ is κ^+-cc. Now we can fix a nice name for f in $M[H]$ as follows:

For each $\alpha \in \kappa$, let $A_\alpha = \{p_{\alpha,\gamma}\}_{\gamma<\kappa} \subseteq \mathbb{P}(E_\kappa^-)$ be a maximal antichain of conditions deciding $\dot{f}(\alpha)$. That means for each $\gamma < \kappa$ there is an ordinal $\mu_{\alpha,\gamma}$ such that

$$p_{\alpha,\gamma} \Vdash \dot{f}(\alpha) = \check{\mu}_{\alpha,\gamma}.$$

Since the conditions $p_{\alpha,\gamma}$ are from $\mathbb{P}(E_\kappa^-)$, the support of $p_{\alpha,\gamma}$ is of cardinality less than κ, that is, $|\mathrm{supp}(p_{\alpha,\gamma})| = |\{\xi \in \mathrm{dom}(E) \cap \kappa^+ : p_{\alpha,\gamma}(\xi) \neq \emptyset\}| < \kappa$ for each $\alpha < \kappa$ and $\gamma < \kappa$. Define the set

$$I_f = \bigcup_{\alpha,\gamma\in\kappa} \mathrm{supp}(p_{\alpha,\gamma}).$$

Then $I_f \subseteq \mathrm{dom}(E) \cap \kappa^+$ and I_f is of size at most κ. Further, define for each $i \in I_f$

$$J_i = \bigcup_{\alpha,\gamma\in\kappa} \mathrm{dom}(p_{\alpha,\gamma}(i)).$$

Since $\mathrm{dom}(p_{\alpha,\gamma}(i)) \subseteq E(i)$ with $|\mathrm{dom}(p_{\alpha,\gamma}(i))| < i \leq \kappa$, we have $J_i \subseteq E(i)$ and $|J_i| \leq \kappa$. For each $\alpha \in \kappa$, the antichain A_α lies now in the Easton-like product

$$\mathbb{P}(I_f) = \prod_{i \in I_f,\ \text{Easton}} \mathrm{Fn}_{<i}(J_i, 2).$$

Note that this poset depends on $\dot{f}$ and has cardinality κ. Also, it is not an Easton poset, as I_f is not necessarily an Easton index function. Now, define the function $\bar{E} : \mathrm{dom}(E) \cap \kappa^+ \to \kappa^+$ by letting

$$\bar{E}(i) = \begin{cases} E(i) & \text{if } E(i) < \kappa \\ \kappa & \text{if } E(i) \geq \kappa. \end{cases}$$

Then $\bar{E}$ is not an Easton index function, but we can use it to build the following Easton-like products:

$$\mathbb{P}(\bar{E}, \kappa) = \prod_{i \in \mathrm{dom}(E_\kappa^-),\ \text{Easton}} \mathrm{Fn}_{<i}(\bar{E}(i), 2)$$
$$\mathbb{P}(\bar{E}, I_f) = \prod_{i \in I_f,\ \text{Easton}} \mathrm{Fn}_{<i}(\bar{E}(i), 2).$$

As $\mathbb{P}(E_\kappa^+)$ is κ^+-closed, the posets $\mathbb{P}(\bar{E}, \kappa)$ and $\mathbb{P}(\bar{E}, I_f)$ must lie in the ground model M. Also note that $\mathbb{P}(\bar{E}, \kappa)$ is of cardinality κ and that $\mathbb{P}(\bar{E}, I_f)$ is isomorphic to $\mathbb{P}(I_f)$. We can show one more very useful fact about $\mathbb{P}(\bar{E}, \kappa)$:

Claim 4.12. $\mathbb{P}(\bar{E}, \kappa) \lessdot \mathbb{P}(E_\kappa^-)$

Proof of claim. Recall that $E_\kappa^- = E \upharpoonright \{\gamma \in \mathrm{dom}(E) : \gamma \leq \kappa\}$, therefore, we have $\mathrm{dom}(E) \cap \kappa^+ = \mathrm{dom}(E_\kappa^-)$. By the definition of $\bar{E}$, for every $i \in \mathrm{dom}(E_\kappa^-)$ we have $\bar{E}(i) \subseteq E(i)$. Thus, any partial function $p(i) : \bar{E}(i) \to 2$ with $|\mathrm{dom}(p(i))| < i$ is also a partial function from $E(i)$ to 2 and so, $\mathbb{P}(\bar{E}, \kappa) \subseteq \mathbb{P}(E_\kappa^-)$.

It remains to show that the identity map $\mathrm{id} : \mathbb{P}(\bar{E}, \kappa) \to \mathbb{P}(E_\kappa^-)$ is a complete embedding: Clearly, $\mathrm{id}(\mathbb{1}_{\mathbb{P}(\bar{E},\kappa)}) = \mathbb{1}_{\mathbb{P}(E_\kappa^-)}$, as $\mathbb{1} = \emptyset$ in any poset of the form $\mathrm{Fn}_{<l}(I, J)$. Let $q, q' \in \mathbb{P}(\bar{E}, \kappa)$. Assume $q \leq_{\mathbb{P}(\bar{E},\kappa)} q'$, that is, $q(i) \leq_{\mathrm{Fn}_{<i}(\bar{E}(i),2)} q'(i)$ for all $i \in \mathrm{dom}(E_\kappa^-)$. This implies that also $q(i) \leq q'(i)$ in $\mathrm{Fn}_{<i}(E(i), 2)$ for all $i \in \mathrm{dom}(E_\kappa^-)$, so $q \leq_{\mathbb{P}(E_\kappa^-)} q'$. Now, assume $q \perp_{\mathbb{P}(\bar{E},\kappa)} q'$. That means, there is no $r \in \mathbb{P}(\bar{E}, \kappa)$ such that $r \leq_{\mathbb{P}(\bar{E},\kappa)} q, q'$. So, there is $i \in \mathrm{dom}(E_\kappa^-)$ with $r(i) \not\leq_{\mathbb{P}(\bar{E},\kappa)} q(i), q'(i)$ and this is the case if and only if $q(i), q'(i)$ do not agree on their shared domain. Particularly, they then also do not

agree on their shared domain as elements of $\mathbb{P}(E_\kappa^-)$. Hence, $q \perp_{\mathbb{P}(\bar{E},\kappa)} q'$ if and only if $q \perp_{\mathbb{P}(E_\kappa^-)} q'$.

Let $A = \{p^\alpha\}_{\alpha \in I}$ be a maximal antichain in $\mathbb{P}(\bar{E},\kappa)$. Since $p^\alpha \perp_{\mathbb{P}(\bar{E},\kappa)} p^\beta$ if and only if $p^\alpha \perp_{\mathbb{P}(E_\kappa^-)} p^\beta$, A is also an antichain in $\mathbb{P}(E_\kappa^-)$. Suppose towards a contradiction that A is not maximal in $\mathbb{P}(E_\kappa^-)$, so there is a $q \in \mathbb{P}(E_\kappa^-)$ with $q \neq p^\alpha$ for all $\alpha \in I$ such that $A \cup \{q\}$ is an antichain. That means $q \perp_{\mathbb{P}(E_\kappa^-)} p^\alpha$ for all $\alpha \in I$. Let $\tilde{q}$ be the sequence with $\tilde{q}(i) = q(i) \restriction \bar{E}(i)$ for all $i \in \mathrm{dom}(E_\kappa^-)$. Then $\tilde{q} \in \mathbb{P}(\bar{E},\kappa)$ and as A is a maximal antichain there, we can find a $p^\alpha \in A$ such that $\tilde{q} \not\perp_{\mathbb{P}(\bar{E},\kappa)} p^\alpha$. That means $\tilde{q}$ and p^α have a common extension, in particular, $\tilde{q}(i)$ and $p^\alpha(i)$ agree on their shared domain for all $i \in \mathrm{dom}(E_\kappa^-)$. Since nothing of $\mathrm{dom}(q(i)) \setminus \mathrm{dom}(\tilde{q}(i)$ appears in $\mathrm{dom}(p^\alpha(i))$ for any i, this means that also all the $q(i)$ and $p^\alpha(i)$ agree on their shared domain. But then q, p^α have a common extension in $\mathbb{P}(E_\kappa^-)$, $q \not\perp_{\mathbb{P}(E_\kappa^-)} p^\alpha$, a contradiction. □

With this claim, it follows that we, in fact, found a $\mathbb{P}(\bar{E},\kappa)$-nice name for f, up to isomorphism of the $\bar{E}$ function in the following sense:

Claim 4.13. Let E and F both be Easton index functions in M and let $\kappa \in \mathrm{dom}(E) \cap \mathrm{dom}(F)$. Define the functions $\bar{E} : \mathrm{dom}(E) \cap \kappa^+ \to \kappa^+$ and $\bar{F} : \mathrm{dom}(F) \cap \kappa^+ \to \kappa^+$, as well as the Easton-like products $\mathbb{P}(\bar{E},\kappa)$ and $\mathbb{P}(\bar{F},\kappa)$ as described above. Then $\bar{F} \cong \bar{E}$ implies $\mathbb{P}(\bar{E},\kappa) \cong \mathbb{P}(\bar{F},\kappa)$.

Proof of claim. Assume $\bar{E} \cong \bar{F}$ through a bijection $g : \mathrm{dom}(E) \cap \kappa^+ \to \mathrm{dom}(F) \cap \kappa^+$ with $\bar{F}(g(i)) = \bar{E}(i)$ for all $i \in \mathrm{dom}(E) \cap \kappa^+$. Using this bijection, we can see that the elements of the posets are also in bijection to each other. Let $p \in \mathbb{P}(\bar{E},\kappa)$, then

$$p = \langle p(i) : i \in \mathrm{dom}(E_\kappa^-), p(i) \in \mathrm{Fn}_{<i}(\bar{E}(i), 2) \rangle.$$

With the bijection g we can find a $p' = \langle p'(j) : j \in \mathrm{dom}(F_\kappa^-), p(j) \in \mathrm{Fn}_{<j}(\bar{F}(j), 2) \rangle$ in $\mathbb{P}(\bar{F},\kappa)$ such that

$$\forall i \in \mathrm{dom}(E_\kappa^-) \exists j \in \mathrm{dom}(F_\kappa^-)\ (g(i) = j \wedge p(i) = p'(j)).$$

For $q_1, q_2 \in \mathbb{P}(\bar{E},\kappa)$, we have $q_1 \leq q_2$ if and only if $q_1(i) \leq q_2(i)$ for all $i \in \mathrm{dom}(E_\kappa^-)$. But then

$$q_1'(j) = q_1'(g(i)) = q_1(i) \leq q_2(i) = q_2'(g(i)) = q_2'(j),$$

thus, we have $q_1'(j) \leq q_2'(j)$ for all $j \in \mathrm{dom}(F_\kappa^-)$, and so $q_1' \leq q_2'$ in $\mathbb{P}(\bar{F}, \kappa)$. This concludes $\mathbb{P}(\bar{E}, \kappa) \cong \mathbb{P}(\bar{F}, \kappa)$. □

In preparation for the last theorem of this chapter, we prove that $\mathbb{P}(\bar{E}, \kappa)$ can also be completely embedded into the Easton poset $\mathbb{P}(E)$.

Lemma 4.14. $\mathbb{P}(\bar{E}, \kappa) \lessdot \mathbb{P}(E)$

Proof. First, we observe that $\mathbb{P}(\bar{E}, \kappa)$ is indeed a subset of $\mathbb{P}(E)$, as

$$\begin{aligned}\mathbb{P}(\bar{E}, \kappa) &= \prod_{i \in \mathrm{dom}(E_\kappa^-),\ \text{Easton}} \mathrm{Fn}_{<i}(\bar{E}(i), 2) \\ &\subseteq \prod_{i \in \mathrm{dom}(E),\ \text{Easton}} \mathrm{Fn}_{<i}(E(i), 2) = \mathbb{P}(E).\end{aligned}$$

It remains to show that the identity map $\mathrm{id} : \mathbb{P}(\bar{E}, \kappa) \to \mathbb{P}(E)$ is a complete embedding: Let $q_1, q_2 \in \mathbb{P}(\bar{E}, \kappa)$ and assume that $q_1 \leq_{\mathbb{P}(\bar{E},\kappa)} q_2$. That is, $q_1(i) \supseteq q_2(i)$ for all $i \in \mathrm{dom}(E_\kappa^-)$. But then this is also the case for q_1 and q_2 as elements of $\mathbb{P}(E)$, so $q_1 \leq_{\mathbb{P}(E)} q_2$. Now, assume that $q_1 \perp_{\mathbb{P}(\bar{E},\kappa)} q_2$, that means, there is no $r \in \mathbb{P}(\bar{E}, \kappa)$ with $r(i) \supseteq q_1(i)$ and $r(i) \supseteq q_2(i)$ for all $i \in \mathrm{dom}(E_\kappa^-)$. This is equivalent to there being no $r \in \mathbb{P}(E)$ that extends both q_1 and q_2 in $\mathbb{P}(E)$. So, for $q_1, q_2 \in \mathbb{P}(\bar{E}, \kappa)$, $q_1 \perp_{\mathbb{P}(E)} q_2$ if and only if $q_1 \perp_{\mathbb{P}(\bar{E},\kappa)} q_2$. Also, $\mathbb{1}_{\mathbb{P}(\bar{E},\kappa)} = \langle p(i) = \emptyset : i \in \mathrm{dom}(E_\kappa^-) \rangle \mapsto \mathbb{1}_{\mathbb{P}(E)} = \langle p(i) = \emptyset : i \in \mathrm{dom}(E) \rangle$.

Finally, assume that $A = \{q_i\}_{i \in I}$ is a maximal antichain in $\mathbb{P}(\bar{E}, \kappa)$. We know that $q_i \perp_{\mathbb{P}(\bar{E},\kappa)} q_j$ if and only if $q_i \perp_{\mathbb{P}(E)} q_j$, so A is an antichain in $\mathbb{P}(E)$. Suppose A is not maximal in $\mathbb{P}(E)$, then there is a $p \in \mathbb{P}(E)$ such that $p \neq q_i$ and $p \perp_{\mathbb{P}(E)} q_i$ for all $i \in I$. Let $p' \in \mathbb{P}(\bar{E}, \kappa)$ be the sequence with $p'(i) = p(i) \restriction \bar{E}(i)$ for all $i \in \mathrm{dom}(E_\kappa^-)$. Since A is maximal in $\mathbb{P}(\bar{E}, \kappa)$, there is a $q_i \in A$ such that $p' \not\perp_{\mathbb{P}(\bar{E},\kappa)} q_i$. Thus, all elements of p' and q_i agree on their shared domain. But then all elements of q_i and p also agree on their shared domain, as $\mathrm{dom}(q_i(j)) \subseteq \bar{E}(j)$ we have $\mathrm{dom}(p'(j)) \cap \mathrm{dom}(q_i(j)) = \mathrm{dom}(p(j)) \cap \mathrm{dom}(q_i(j))$, which is a contradiction to $p \perp_{\mathbb{P}(E)} q_i$. Therefore, A is also a maximal antichain in $\mathbb{P}(E)$. □

Remark 4.15. The complete embedding of the above lemma induces a mapping of names $i^* : V^{\mathbb{P}(\bar{E},\kappa)} \to V^{\mathbb{P}(E)}$, which is recursively defined by $i^*(\tau) = \{\langle i^*(\sigma), p \rangle : \langle \sigma, p \rangle \in \tau\}$. Now, if G is a $(M, \mathbb{P}(E))$-generic filter we can set $\bar{G} = \mathbb{P}(\bar{E}, \kappa) \cap G$ and note that $(i^*(\tau))_G = (\tau)_{\bar{G}}$ for any $\mathbb{P}(\bar{E}, \kappa)$-name τ. In particular, this implies $M[\bar{G}] \subseteq M[G]$.

Now we are finally ready for the last theorem of this chapter, where we evaluate our cardinal characteristics $\mathfrak{a}(\kappa)$, $\mathfrak{b}(\kappa)$, $\mathfrak{d}(\kappa)$ and 2^κ in the Easton extension.

Theorem 4.16. *Let M be a ctm, assume GCH and let E be an Easton function such that $E(\kappa) > \kappa^+$ for all $\kappa \in dom(E)$. Then, for a $(M, \mathbb{P}(E))$-generic filter G,*

$$M[G] \models \forall\kappa \in dom(E)(\mathfrak{a}(\kappa) = \mathfrak{b}(\kappa) = \kappa^+ < E(\kappa) = 2^\kappa).$$

If additionally holds that $E(\kappa)$ is regular, then also

$$M[G] \models \mathfrak{d}(\kappa) = E(\kappa) = 2^\kappa.$$

Proof. [Fisc] Note that $\kappa^+ < E(\kappa)$ is given and $E(\kappa) = 2^\kappa$ was already shown in Theorem 4.8 and further, that we already know $\kappa^+ \leq \mathfrak{b}(\kappa) \leq \mathfrak{a}(\kappa)$ by Lemma 2.3. Thus, to show the first part of the theorem, $M[G] \models \forall\kappa \in \mathrm{dom}(E)(\mathfrak{a}(\kappa) = \mathfrak{b}(\kappa) = \kappa^+ < E(\kappa) = 2^\kappa)$, it is sufficient to show that in $M[G]$, there is a κ-mad family of cardinality κ^+ for an arbitrary but fixed $\kappa \in \mathrm{dom}(E)$.

Consider $\mathbb{P}(\bar{E}, \kappa)$ and note that this poset is in M and has cardinality κ (Remark 4.11). By Theorem 3.8, there exists a $\mathbb{P}(\bar{E}, \kappa)$ indestructible κ-mad family $\mathcal{A}_\kappa$ of cardinality κ^+, so $\mathcal{A}_\kappa$ is κ-mad in any generic extension via $\mathbb{P}(\bar{E}, \kappa)$.

Claim 4.17. The family $\mathcal{A}_\kappa$ is also κ-mad in the extension $M[G]$ by $\mathbb{P}(E)$.

Proof of claim. Let $X \in [\kappa]^\kappa \cap M[G]$. Just like we did for a function in Remark 4.11, we can fix a $\mathbb{P}(\bar{E}, \kappa)$-nice name τ for the set X (up to isomorphism). Because of Lemma 4.14 and Remark 4.15, with $\bar{G} = \mathbb{P}(\bar{E}, \kappa) \cap G$, we find that $M[\bar{G}] \subseteq M[G]$ and that $X \in [\kappa]^\kappa \cap M[\bar{G}]$. Since $\mathcal{A}_\kappa$ is $\mathbb{P}(\bar{E}, \kappa)$ indestructible, there is a witness in $\mathcal{A}_\kappa$ for its maximality, $M[\bar{G}] \models \exists B \in \mathcal{A}_\kappa(|B \cap X| = \kappa)$. But if $M[\bar{G}] \models |B \cap X| = \kappa$ then $M[G] \models |B \cap X| = \kappa$, therefore $\mathcal{A}_\kappa$ is also maximal κ-almost disjoint in the extension $M[G]$ by $\mathbb{P}(E)$. □

Let κ still be arbitrary but fixed. Assume that $E(\kappa)$ is regular and note that we already know $\mathfrak{d}(\kappa) \leq 2^\kappa = E(\kappa)$ by Lemma 2.3 and Theorem 4.8. Thus it is sufficient to show $M[G] \models \mathfrak{d}(\kappa) \geq E(\kappa)$ to prove the second part of the theorem, $M[G] \models \mathfrak{d}(\kappa) = E(\kappa) = 2^\kappa$.

Claim 4.18. Let $\mathcal{F} \subseteq {}^\kappa\kappa$ be an arbitrary family of functions such that $|\mathcal{F}| < E(\kappa)$. Then $\mathcal{F}$ is not dominating.

Proof of claim. For each $f \in \mathcal{F}$, let τ_f be a $\mathbb{P}(E^-_\kappa)$-nice name, as seen in Remark 4.11. Let $D(\tau_f) = \bigcup_{\alpha\in\kappa}\{\{\xi\} \times \mathrm{dom}(p(\xi)) : \xi \in \mathrm{supp}(p),\ p \in A^f_\alpha\}$ denote the domain of τ_f, where $\{A^f_\alpha\}_{\alpha<\kappa}$ is the family of maximal antichains whose conditions decide $\dot{f}(\alpha)$ for each $\alpha < \kappa$. Then $D(\tau_f) \subseteq \bigcup_{\alpha \in \mathrm{supp}(\tau_f)} \{\alpha\} \times E(\alpha)$ and $|D(\tau_f)| \leq \kappa$. Further, it follows that $|\bigcup_{f \in \mathcal{F}} D(\tau_f)| \leq |\mathcal{F}| < E(\kappa)$.

Define $\pi_\kappa(D(\tau_f)) = \{\beta \in E(\kappa) : (\kappa, \beta) \in D(\tau_f)\}$ and observe that $\pi_\kappa(D(\tau_f)) \subseteq E(\kappa)$ and $|\pi_\kappa(D(\tau_f))| < E(\kappa)$. Recall that we assumed $\mathrm{cf}(E(\kappa)) = E(\kappa)$, so $\sup(\pi_\kappa(D(\tau_f))) < E(\kappa)$. Fix an $\alpha \in E(\kappa)$ such that the ordinal product $\kappa \cdot \alpha$ lies strictly between $\sup(\pi_\kappa(D(\tau_f)))$ and $E(\kappa)$, that is, $\sup(\pi_\kappa(D(\tau_f))) < \kappa \cdot \alpha < E(\kappa)$.

Now, define $h_\alpha \in {}^\kappa\kappa$ to be the enumerating function of the set $\sigma_\alpha \subseteq \{\kappa \cdot \alpha + \xi\}_{\xi \in \kappa}$, which is defined through its characteristic function $\chi_{\sigma_\alpha} : \{\kappa \cdot \alpha + \xi\}_{\xi \in \kappa} \to 2$. For $\xi \in \kappa$, let $\chi_{\sigma_\alpha}(\kappa \cdot \alpha + \xi) := p(\kappa)(\kappa \cdot \alpha + \xi)$ where p is a condition in the filter G such that $\kappa \in \mathrm{supp}(p)$ and $\kappa \cdot \alpha + \xi \in \mathrm{dom}(p(\kappa))$. We finalize the proof that $\mathcal{F}$ is not dominating by showing with a density argument that h_α is unbounded by $\mathcal{F}$, meaning that no function in $\mathcal{F}$ eventually dominates h_α. For each $f \in \mathcal{F}$ let

$$\begin{aligned} D^f = \{p \in \mathbb{P}(E) : \kappa \in \mathrm{supp}(p) \wedge \exists \gamma \in \kappa \forall \xi \geq \gamma \\ (\kappa \cdot \alpha + \xi \in \mathrm{dom}(p(\kappa)) \wedge p(\kappa)(\kappa \cdot \alpha + \xi) = 1 \to f(\xi) < h_\alpha(\xi))\}. \end{aligned}$$

These sets are indeed dense subsets of $\mathbb{P}(E)$: Fix an arbitrary $f \in \mathcal{F}$ and let $q \in \mathbb{P}(E)$. Assume $q \notin D^f$. Then $\kappa \notin \mathrm{supp}(q)$ or there is no $\gamma \in \kappa$ such that for all $\xi \geq \gamma$, if h_α is defined, then $h_\alpha(\xi) > f(\xi)$. If $\kappa \notin \mathrm{supp}(q)$, we can find a condition $q' \in \mathbb{P}(E)$, where $q'(\alpha) = q(\alpha)$ for all $\alpha \in \mathrm{supp}(q)$ and $q'(\kappa)$ is a partial function such that there is a $\gamma \in \kappa$ so that for all $\xi \geq \gamma$, $\kappa \cdot \alpha + \xi \in \mathrm{dom}(q'(\kappa))$ with $q'(\kappa)(\kappa \cdot \alpha + \xi) = 1$ if $f(\xi) < h_\alpha(\xi)$ and $\kappa \cdot \alpha + \xi \notin \mathrm{dom}(q'(\kappa))$ or $q'(\kappa)(\kappa \cdot \alpha + \xi) = 0$ otherwise. Then $q' \in D^f$ and $q' \leq q$. Now, suppose $\kappa \in \mathrm{supp}(q)$ but for all $\gamma \in \kappa$ and $\xi \geq \gamma$, $h_\alpha(\xi)$ is defined and $h_\alpha(\xi) \leq f(\xi)$. But $q(\kappa) \in \mathrm{Fn}_{<\kappa}(E(\kappa), 2)$, so $|\mathrm{dom}(q(\kappa))| < \kappa$, so it is impossible that h_α is defined for all $\xi \in \kappa$. Thus, there must be a $\gamma \in \kappa$ such that for all $\xi \geq \gamma$, $h_\alpha(\xi)$ is not defined or $h_\alpha(\xi) > f(\xi)$. Hence, $q \in D^f$.

The filter G is $(M, \mathbb{P}(E))$-generic, so it has a nonempty intersection with D^f for each $f \in \mathcal{F}$. Thus, in $M[G]$, for all $f \in \mathcal{F}$ there is a $\gamma \in \kappa$ such that $h_\alpha(\xi) > f(\xi)$ for all ξ above γ for which h_α is defined. In particular, no function in $\mathcal{F}$ eventually dominates h_α, so $\mathcal{F}$ is not dominating. □

This concludes the proof that in $M[G]$, $\mathfrak{a}(\kappa) = \mathfrak{b}(\kappa) = \kappa^+ < \mathfrak{d}(\kappa) = E(\kappa) = 2^\kappa$, for all $\kappa \in \mathrm{dom}(E)$, where $\mathfrak{d}(\kappa) = E(\kappa)$ only holds if $E(\kappa)$ is regular. □

5 The spectrum of κ-mad families

Following Chapters 6 and 7 of [Fisc], we first take a quick look at two-step iterations of forcing posets. These will reappear in the next section, where we introduce a way to adjoin an arbitrarily large κ-mad family, before we end this chapter with a positive requirement for the spectrum of the κ-almost disjointness number.

5.1 Two-step iterations

Also here, κ is a regular uncountable cardinal unless stated otherwise.

Definition 5.1. Let $\mathbb{P}$ be a forcing notion. A $\mathbb{P}$-**name** for a forcing poset is a triple $(\dot{\mathbb{Q}}, \dot{\leq}_{\mathbb{Q}}, \dot{\mathbb{1}}_{\mathbb{Q}})$ such that $\dot{\mathbb{1}}_{\mathbb{Q}} \in \operatorname{dom}(\dot{\mathbb{Q}})$ and $\mathbb{1}_{\mathbb{P}}$ forces that "$\mathbb{1}_{\dot{\mathbb{Q}}} \in \operatorname{dom}(\dot{\mathbb{Q}})$ and $\dot{\leq}_{\mathbb{Q}}$ is a pre-order of $\dot{\mathbb{Q}}$ with largest element $\mathbb{1}_{\dot{\mathbb{Q}}}$".

Definition 5.2. Let $\mathbb{P}$ be a forcing notion and let $(\dot{\mathbb{Q}}, \dot{\leq}_{\mathbb{Q}}, \dot{\mathbb{1}}_{\mathbb{Q}})$ be a $\mathbb{P}$-name for a forcing poset. The **two-step iteration** $\mathbb{P} * \dot{\mathbb{Q}}$ is the poset consisting of the pairs $(p, \dot{q}) \in \mathbb{P} \times \operatorname{dom}(\dot{\mathbb{Q}})$ such that $p \Vdash \dot{q} \in \dot{\mathbb{Q}}$, with largest element $\mathbb{1} = (\mathbb{1}_{\mathbb{P}}, \dot{\mathbb{1}}_{\mathbb{Q}})$ and the following extension relation: $(p_1, \dot{q}_1) \leq (p_2, \dot{q}_2)$ if and only if $p_1 \leq_{\mathbb{P}} p_2$ and $p_1 \Vdash \dot{q}_1 \dot{\leq}_{\mathbb{Q}} \dot{q}_2$.

This lemma will be needed in the proof that the two-step iteration is κ-cc preserving.

Lemma 5.3. Let $\mathbb{P}$ be a forcing poset with the κ-cc and let $\dot{S}$ be a $\mathbb{P}$-name with $\mathbb{1}_{\mathbb{P}} \Vdash (\dot{S} \subseteq \kappa \wedge |\dot{S}| < \kappa)$. Then there is a $\beta < \kappa$ such that $\mathbb{1}_{\mathbb{P}} \Vdash \dot{S} \subseteq \check{\beta}$.

Proof. [Fisc] Define E to be the set of all $\alpha < \kappa$ such that there exists a $p \in \mathbb{P}$ with $p \Vdash \alpha = \sup(\dot{S})$. For each $\alpha \in E$ let p_α be the condition forcing $\alpha = \sup(\dot{S})$. Then $\{p_\alpha\}_{\alpha \in E}$ builds an antichain, so E has to be of size less than κ as $\mathbb{P}$ is κ-cc. Since κ is regular, $\sup(E) < \kappa$ and so there is a $\beta < \kappa$ such that $E \subseteq \beta$. Every κ-cc poset preserves cardinals $\geq \kappa$ and so $\mathbb{1}_{\mathbb{P}} \Vdash (\kappa \text{ is regular})$. Thus $\mathbb{1}_{\mathbb{P}} \Vdash \sup(\dot{S}) < \kappa$, as $\mathbb{1}_{\mathbb{P}} \Vdash (\dot{S} \subseteq \kappa \wedge |\dot{S}| < \kappa)$. We want to show that $\mathbb{1}_{\mathbb{P}} \Vdash \dot{S} \subseteq \check{\beta}$, so suppose this is not being forced. Hence, we can find a condition $q \in \mathbb{P}$ such that $q \Vdash \dot{S} \not\subseteq \beta$ and so $q \Vdash \sup(\dot{S}) \in \kappa \setminus \beta$. Note that then there is an $\alpha \in \kappa \setminus \beta$ and a condition p extending q such that $p \Vdash \sup(\dot{S}) = \alpha$. Thus $\alpha \in E$ and $\alpha \geq \beta$, contradicting $E \subseteq \beta$. $\square$

Now we can prove that the two-step iteration preserves the κ-chain condition.

Lemma 5.4. Let $\mathbb{P}$ be a κ-cc forcing poset and $\dot{\mathbb{Q}}$ a $\mathbb{P}$-name for a forcing poset such that $\mathbb{1}_{\mathbb{P}} \Vdash$"$\dot{\mathbb{Q}}$ is κ-cc". Then the two-step iteration $\mathbb{P} * \dot{\mathbb{Q}}$ is κ-cc.

Proof. [Fisc] Let M be a ctm and $\mathbb{P}, \kappa, \dot{\mathbb{Q}}$ elements of it. Suppose, in M, that $\mathbb{P} * \dot{\mathbb{Q}}$ is not κ-cc. Then there is an antichain $\{(p_\xi, \dot{q}_\xi)\}_{\xi<\kappa}$ of length κ. Let $\dot{S} = \{(\check{\xi}, p_\xi)\}_{\xi<\kappa}$. Then $\dot{S}$ is a $\mathbb{P}$-name and $\mathbb{1}_{\mathbb{P}} \Vdash \dot{S} \subseteq \kappa$.

Suppose towards a contradiction that there is a $\beta < \kappa$ with $\mathbb{1}_{\mathbb{P}} \Vdash \dot{S} \subseteq \check{\beta}$. Fix a ξ such that $\beta < \xi < \kappa$, which exists as κ is a limit ordinal. Then $p_\xi \leq \mathbb{1}_{\mathbb{P}}$ and $p_\xi \Vdash \check{\xi} \in \dot{S}$, so $p_\xi \Vdash \xi \leq \beta$, which yields the contradiction. Thus, there is no $\beta < \kappa$ with $\mathbb{1}_{\mathbb{P}} \Vdash \dot{S} \subseteq \check{\beta}$.

Fix a filter G that is $(M, \mathbb{P})$-generic and let ξ and η be distinct elements of $\dot{S}_G = \{\check{\xi}_G : \exists p \in G\ ((\check{\xi}, p) \in \dot{S})\}$. Since $\xi, \eta \in \dot{S}_G$, we find in particular that p_ξ and p_η are in G. We want to show that $\{(\dot{q}_\xi)_G\}_{\xi \in \dot{S}_G}$ is an antichain in $M[G]$. Suppose it is not and that $(\dot{q}_\xi)_G, (\dot{q}_\eta)_G$ are witnesses for it. That is, in $M[G]$, $(\dot{q}_\xi)_G, (\dot{q}_\eta)_G$ are compatible with common extension $\dot{q}_G$ for a $\dot{q} \in \mathrm{dom}(\dot{\mathbb{Q}})$. By the Truth Lemma (see [Kun11], Lemma IV.2.24), there is then a $p \in G$ which forces exactly that,

$$p \Vdash \dot{q} \in \dot{\mathbb{Q}} \wedge \dot{q} \leq \dot{q}_\xi \wedge \dot{q} \leq \dot{q}_\eta.$$

Because p_ξ and p_η are in G, we may assume that p is an extension of both of them. But then $(p, \dot{q})$ is an extension of both $(p_\xi, \dot{q}_\xi)$ and $(p_\eta, \dot{q}_\eta)$, contradicting that $\{(p_\xi, \dot{q}_\xi)\}_{\xi<\kappa}$ is an antichain. Therefore, $\{(\dot{q}_\xi)_G\}_{\xi \in \dot{S}_G}$ is an antichain in $M[G]$. However $M[G] \models |\dot{S}_G| < \kappa$, because $M[G] \models (\dot{\mathbb{Q}}_G$ is κ-cc). And $\mathbb{1}_{\mathbb{P}} \Vdash |\dot{S}| < \kappa$, because the filter G was chosen arbitrarily. Now, it follows with Lemma 5.3 that there actually is $\beta < \kappa$ such that $\mathbb{1}_{\mathbb{P}} \Vdash \dot{S} \subseteq \check{\beta}$, but this is a contradiction to the result from the second paragraph of this proof. And this concludes the proof that $\mathbb{P} * \dot{\mathbb{Q}}$ is κ-cc. ☐

5.2 Adjoining an arbitrarily large κ-mad family

Let κ be an uncountable regular cardinal such that $\kappa^{<\kappa} = \kappa$ and let us begin this section by defining the forcing poset we will use to add a κ-almost disjoint family. We will show that this poset is forcing equivalent to a two-step iteration, which will be a key fact in the proof that the adjoined family is maximal κ-almost disjoint.

Definition 5.5. Let A be an index set. The forcing poset $\mathbb{H}_A$ consists of all $p \subseteq A \times \kappa \times \{0, 1\}$ such that $|p| < \kappa$ and for all $a \in A$, the set $p_a = \{(\xi, i) : (a, \xi, i) \in p\}$ is a function, that is, for each ξ, there is at most one i such that $(\xi, i) \in p_a$. We call p_a the **projection** of p onto the coordinate a and let $\mathrm{dom}(p) = \{a \in A : p_a \neq \emptyset\}$ be the **domain** of p. For $q, p \in \mathbb{H}_A$, we define the extension relation as follows: $q \leq p$ if and only if

$$q \supseteq p \wedge \forall a, b \in \mathrm{dom}(p)(a \neq b \longrightarrow q_a^{-1}(1) \cap q_b^{-1}(1) = p_a^{-1}(1) \cap p_b^{-1}(1)).$$

The largest element is $\mathbb{1}_{\mathbb{H}_A} = \emptyset$. Sometimes we add κ to the notation and write $\mathbb{H}_A^{(\kappa)}$ instead of $\mathbb{H}_A$ to emphasize that the poset adjoins κ-reals.

Definition 5.6. A poset $\mathbb{P}$ has the **κ-Knaster property** if in every collection of κ many conditions from $\mathbb{P}$, there are κ many pairwise compatible conditions. [Fis15]

Remark 5.7. The κ-Knaster property implies the κ-chain condition: Let $\mathbb{P}$ be a κ-Knaster forcing poset and $A \subseteq \mathbb{P}$ an arbitrary subset of size κ. Then there is $B \subseteq A$ with $|B| = \kappa$ such that all conditions of B are pairwise compatible. Therefore, A can not be an antichain, and since A was arbitrary, it follows that all antichains must be of cardinality less than κ.

The poset $\mathbb{H}_A$ has the Knaster property and more.

Lemma 5.8.

1. $\mathbb{H}_A$ is κ^+-Knaster.

2. $\mathbb{H}_A$ is κ-closed.

3. If $B \subset A$, then $\mathbb{H}_B \lessdot \mathbb{H}_A$.

Proof. 1. Let $H \subseteq \mathbb{H}_A$ be a collection of κ^+ many conditions from $\mathbb{H}_A$. We have to show that there are κ^+ many pairwise compatible conditions in H. Recall that for every $p \in \mathbb{H}_A$ the domain is $\mathrm{dom}(p) = \{a \in A : p_a \neq \emptyset\}$ and $|p| < \kappa$, so the family of domains $\{\mathrm{dom}(p)\}_{p \in H}$ has size κ^+ and consists of sets where each is of size less than κ. With the Delta System Lemma (see [Kun11], Lemma III.6.15), there exists an $H' \subseteq H$ with $|H'| = \kappa^+$ such that $\{\mathrm{dom}(p)\}_{p \in H'}$ is a Delta system with root R, where $R \subseteq A, |R| < \kappa$. That is, for every $p \neq q$ in H', $\mathrm{dom}(p) \cap \mathrm{dom}(q) = R$.

We want the conditions to be pairwise compatible. For that, they have to agree as functions $p_a = q_a$ for all $a \in R$, because then their union $r = p \cup q$ is a condition in $\mathbb{H}_A$ and a common extension. Recall that $r \leq p$ if and only if $r \supseteq p$ and for all distinct $a, b \in \mathrm{dom}(p)$, $p_a^{-1}(1) \cap p_b^{-1}(1) = r_a^{-1}(1) \cap r_b^{-1}(1)$. For the second condition, it is necessary that they not only not disagree but actually are the same on their shared domain, $p \upharpoonright R = q \upharpoonright R$.

Since $|[R \times \kappa \times \{0,1\}]^{<\kappa}| = \kappa$, there are only κ many possibilities for the conditions $p \upharpoonright R$. Thus, we can take those κ many out and find $H'' \subseteq H', |H''| = \kappa^+$, such that for all $p, q \in H''$ and for all $a \in R$, $p_a = q_a$, that is $p \upharpoonright R = q \upharpoonright R$ for all $p, q \in H''$. Now, for any two conditions $p, q \in H''$, their union $r = p \cup q$ is a condition in $\mathbb{H}_A$ and a common extension of p and q. Thus, we found κ^+ many pairwise compatible conditions in H.

2. Let $\delta < \kappa$ and $\langle p_\xi : \xi < \delta \rangle$ be a sequence in $\mathbb{H}_A$ such that for all $\xi_1 < \xi_2 < \delta$ we have $p_{\xi_2} \leq p_{\xi_1}$. We want to show that then there exists a $q \in \mathbb{H}_A$ extending every condition of the sequence. Let $q = \bigcup_{\xi<\delta} p_\xi$. Then $q \in \mathbb{H}_A$ is the wanted condition with $q \leq p_\xi$ for all $\xi < \delta$:

   Indeed, $q \subseteq A \times \kappa \times \{0,1\}$ and, as $\delta < \kappa$ and $|p_\xi| < \kappa$ for each $\xi < \delta$, also $|q| < \kappa$. Fix $a \in A$ and assume that q_a is not a function. Then there is a $\gamma \in \kappa$ such that $(a, \gamma, 0) \in q$ and $(a, \gamma, 1) \in q$. But that would mean that there are $p_\xi, p_{\xi'}$ in the sequence with $(a, \gamma, 0) \in p_\xi$ and $(a, \gamma, 1) \in p_{\xi'}$. Without loss of generality $\xi < \xi'$, so $p_{\xi'} \leq p_\xi$, which is impossible if $(a, \gamma, 0) \in p_\xi$ and $(a, \gamma, 1) \in p_{\xi'}$. So q_a is a function for each $a \in A$ and thus, q is a condition in $\mathbb{H}_A$. Now, fix $\xi < \delta$. By the definition of q, we have $q \supseteq p_\xi$. Let $a, b \in \mathrm{dom}(p_\xi)$ with $a \neq b$, then obviously $p_{\xi,a}^{-1}(1) \cap p_{\xi,b}^{-1}(1) \subseteq q_a^{-1}(1) \cap q_b^{-1}(1)$. Assume there is a $\gamma \in \kappa$ such that $\gamma \in q_a^{-1}(1) \cap q_b^{-1}(1)$, but $\gamma \notin p_{\xi,a}^{-1}(1) \cap p_{\xi,b}^{-1}(1)$. That means there is a condition $p_{\xi'}$ in the sequence with $(a, \gamma, 1), (b, \gamma, 1) \in p_{\xi'}$. As the sequence is decreasing, it must be that $\xi < \xi'$ and $p_{\xi'} \leq p_\xi$, as $p_{\xi'} \supset p_\xi$. This implies $p_{\xi,a}^{-1}(1) \cap p_{\xi,b}^{-1}(1) = p_{\xi',a}^{-1}(1) \cap p_{\xi',b}^{-1}(1)$, but we have $\gamma \notin p_{\xi,a}^{-1}(1) \cap p_{\xi,b}^{-1}(1)$ and $\gamma \in p_{\xi',a}^{-1}(1) \cap p_{\xi',b}^{-1}(1)$, a contradiction. Thus, $p_{\xi,a}^{-1}(1) \cap p_{\xi,b}^{-1}(1) = q_a^{-1}(1) \cap q_b^{-1}(1)$ and therefore, $q \leq p_\xi$.

3. Since $B \subset A$, we have $\mathbb{H}_B \subseteq \mathbb{H}_A$ by the definition of the posets. The identity map $\mathrm{id} : \mathbb{H}_B \to \mathbb{H}_A$ maps $\mathbb{1}_{\mathbb{H}_B}$ to $\mathbb{1}_{\mathbb{H}_A}$, as $\mathbb{1}_{\mathbb{H}_B} = \emptyset = \mathbb{1}_{\mathbb{H}_A}$. For all $p_1, p_2 \in \mathbb{H}_B$, if $p_1 \leq_{\mathbb{H}_B} p_2$, then $p_1 \leq_{\mathbb{H}_A} p_2$, since being a member of $\mathbb{H}_A$ does not do anything about the extension relation for any $p_1, p_2 \in \mathbb{H}_B$. Further, for all $p_1, p_2 \in \mathbb{H}_B$, we have $p_1 \perp_{\mathbb{H}_B} p_2$ if and only if $p_1 \perp_{\mathbb{H}_A} p_2$, because if two conditions are incompatible in $\mathbb{H}_B$, it means that they have some shared domain on which they disagree, thus they do not have a common extension. But exactly in that case, there can also be no common extension in $\mathbb{H}_A$.

For any $p \in \mathbb{H}_A$, call $p^* \in \mathbb{H}_B$ a reduction of p to $\mathbb{H}_B$ if and only if for all $q \in \mathbb{H}_B$, $q \perp_{\mathbb{H}_A} p$ implies $q \perp_{\mathbb{H}_B} p^*$. Define for each $p \in \mathbb{H}_A$ the reduction $p^* := p \cap B \times \kappa \times \{0,1\} \in \mathbb{H}_B$. Let A be a maximal antichain in $\mathbb{H}_B$ and suppose it is not a maximal antichain in $\mathbb{H}_A$. Thus, there is a $p \in \mathbb{H}_A$ such that $A \cup \{p\}$ is an antichain, so $p \perp_{\mathbb{H}_A} r$ for all $r \in A$. Let p^* be the reduction of p to $\mathbb{H}_B$, then $p^* \perp_{\mathbb{H}_B} r$ for all $r \in A$. But that would mean $A \cup \{p^*\}$ is an antichain, a contradiction to A being maximal in $\mathbb{H}_B$.

□

The next lemma shows how we gain a κ-almost disjoint family of size $|A|$ with the poset $\mathbb{H}_A$.

Lemma 5.9. Let A be an index set and G an $\mathbb{H}_A$-generic filter. For each $a \in A$, let $X_a = \chi_a^{-1}(1) \in [\kappa]^\kappa$ be the set defined through the characteristic function $\chi_a = \bigcup_{p \in G} p_a$. Let $\mathcal{X} = \{X_a\}_{a \in A}$ be the family of these sets. Then $\mathcal{X}$ is κ-almost disjoint.

Proof. [Fisc] First, we show that X_a is unbounded in κ for each $a \in A$, so that X_a is indeed a subset of size κ. Fix $a \in A$ and for each $\xi \in \kappa$, let

$$D_\xi = \{p \in \mathbb{H}_A : \exists \eta > \xi \ ((a, \eta, 1) \in p)\}.$$

We will show that D_ξ is dense in $\mathbb{H}_A$. Because then, as G is generic, there is a condition $q \in D_\xi \cap G$ for each $\xi \in \kappa$. Thus, for each $\xi \in \kappa$, there is $\eta > \xi$ with $\eta \in \chi_a^{-1}(1) = X_a$, so X_a is unbounded in κ.

Fix an arbitrary $p \in \mathbb{H}_A$ and $\xi \in \kappa$. If $p \in D_\xi$, we are done, so assume $p \notin D_\xi$ and find $q \in D_\xi$ with $q \leq p$. By definition $|p| < \kappa$, so there is a $\xi^* \in \kappa$ with $\xi^* > \xi$ such that $\xi^* \notin \mathrm{dom}(p_b)$ for any $b \in \mathrm{dom}(p)$ (this could include our fixed a or not). Let $q = p \cup \{(a, \xi^*, 1)\} \cup \{(b, \xi^*, 0) : b \in \mathrm{dom}(p) \setminus \{a\}\}$. Then $q \in D_\xi$, $q \supseteq p$ and for all distinct $c, d \in \mathrm{dom}(p)$, $c \neq d \to p_c^{-1}(1) \cap p_d^{-1}(1) = q_c^{-1}(1) \cap q_d^{-1}(1)$ as $q_c = p_c$ for all $c \in \mathrm{dom}(p)$, so $q \leq p$. Note that this extension works in both cases, either with $a \in \mathrm{dom}(p)$ or $a \notin \mathrm{dom}(p)$. If $a \notin \mathrm{dom}(p)$ it would be sufficient to build the extension $q' = p \cup \{(a, \xi^*, 1)\} \in D_\xi$ to get $q' \leq p$.

Next, we want to show that the family $\mathcal{X} = \{X_a\}_{a \in A}$ is κ-almost disjoint. Fix arbitrary $a, b \in A$ with $a \neq b$. Let $D_{a,b} = \{p \in \mathbb{H}_A : a, b \in \mathrm{dom}(p)\}$. Then $D_{a,b}$ is dense in $\mathbb{H}_A$: For an arbitrary $p \in \mathbb{H}_A$, assume that $a, b \notin \mathrm{dom}(p)$, otherwise, p itself is in $D_{a,b}$ and we are done. Let $r_a \subseteq \kappa \times \{0,1\}$ and $r_b \subseteq \kappa \times \{0,1\}$ be functions with $|r_a| < \kappa$ and $|r_b| < \kappa$. Then $q = p \cup (\{a\} \times r_a) \cup (\{b\} \times r_b)$ is a condition in $\mathbb{H}_A$, moreover, $q \in D_{a,b}$

and $q \leq p$. If, without loss of generality, only $a \notin \operatorname{dom}(p)$ and $b \in \operatorname{dom}(p)$, then the extension $q' = p \cup (\{a\} \times r_a)$ would be sufficient.

Since $D_{a,b}$ is dense, any $\mathbb{H}_A$-generic filter G will contain a condition with a and b in the domain. Thus, we can now consider an arbitrary $p \in \mathbb{H}_A$ with $a, b \in \operatorname{dom}(p)$ and show

$$p \Vdash X_a \cap X_b = p_a^{-1}(1) \cap p_b^{-1}(1).$$

Suppose towards a contradiction that this does not hold. Then there is an $\mathbb{H}_A$-generic filter G with $p \in G$ and $\xi \notin p_a^{-1}(1) \cap p_b^{-1}(1)$, but $\xi \in X_a \cap X_b$. That $\xi \in X_a$ means that $\chi_a(\xi) = 1$ and χ_a was defined as the union of all q_a for $q \in G$. So there is a $q \in G$ with $(a, \xi, 1) \in q$. In the same sense $\xi \in X_b$ implies that there is a $\tilde{q} \in G$ with $(b, \xi, 1) \in \tilde{q}$. We now have $p, q, \tilde{q} \in G$ and since G is a filter, they have a common extension r. This r extends p and $a, b \in \operatorname{dom}(p)$, so $r_a^{-1}(1) \cap r_b^{-1}(1) = p_a^{-1}(1) \cap p_b^{-1}(1)$. Also, $r \leq q$, which means $r \supseteq q$ and with that $(a, \xi, 1) \in r$. Similarly, $(b, \xi, 1) \in r$ as $r \leq \tilde{q}$. But then $\xi \in r_a^{-1}(1) \cap r_b^{-1}(1) = p_a^{-1}(1) \cap p_b^{-1}(1)$, contradicting $\xi \notin p_a^{-1}(1) \cap p_b^{-1}(1)$. So p forces $X_a \cap X_b = p_a^{-1}(1) \cap p_b^{-1}(1)$ and since $|p| < \kappa$, this shows that $|X_a \cap X_b| < \kappa$ so $\mathcal{X}$ is indeed κ-almost disjoint. $\square$

Let us introduce another poset and show some of its properties.

Definition 5.10. Let $B \subsetneq A$ and let G_B be a $\mathbb{H}_B$-generic filter. For each $b \in B$, let $X_b = \chi_b^{-1}(1)$ be the set defined through $\chi_b = \bigcup_{p \in G_B} p_b$. In the generic extension $V[G_B]$, let $\mathbb{R}_{A,B}$ be the partial order of all pairs (p, F) such that $p \in \mathbb{H}_{A \setminus B}$ and $F \in [B]^{<\kappa}$, with the following extension relation: $(q, K) \leq (p, F)$ if and only if

$$q \leq_{\mathbb{H}_{A \setminus B}} p \ \wedge \ K \supseteq F \ \wedge \ \forall b \in F \forall a \in \operatorname{dom}(p) \ (q_a^{-1}(1) \cap X_b = p_a^{-1}(1) \cap X_b).$$

Recall that $q \leq_{\mathbb{H}_{A \setminus B}} p$ if and only if $q \supseteq p$ and for all distinct $a, b \in \operatorname{dom}(p)$, $q_a^{-1}(1) \cap q_b^{-1}(1) = p_a^{-1}(1) \cap p_b^{-1}(1)$. Additionally, here q_a and p_a share the same set of ξ with X_b that they map to one.

Lemma 5.11. The poset $\mathbb{R}_{A,B}$ is κ^+-Knaster and κ-closed.

Proof. First, we show that $\mathbb{R}_{A,B}$ is κ^+-Knaster. Let $R \subseteq \mathbb{R}_{A,B}$ be a set of κ^+ many conditions. The family $\{\operatorname{dom}(p) : (p, F) \in R\}$ consists of κ^+ many sets, each of size less than κ. Thus, by the Delta System Lemma (see [Kun11], Lemma III.6.15), there is $R' \subseteq R$ with $|R'| = \kappa^+$ and a root $C \subseteq A \setminus B$ such that $\operatorname{dom}(p) \cap \operatorname{dom}(q) = C$

for all $(p, F), (q, K) \in R'$ with $p \neq q$. As $|C| < \kappa$, there are only κ many possibilities for the conditions $p \upharpoonright C$ and so we can find $R'' \subseteq R'$, $|R''| = \kappa^+$ such that for all $(p, F), (q, K) \in R''$, the first coordinates p, q agree on their shared domain C, $p \upharpoonright C = q \upharpoonright C$. Now, fix $(p, F), (q, K) \in R''$ and let $r = p \cup q$. Then $r \in \mathbb{H}_{A \setminus B}$ and $r \leq_{\mathbb{H}_{A \setminus B}} p, q$. Let $L = F \cup K$. Then $L \supseteq F$ and $L \supseteq K$. Also, because of our choice of r, we have $r_a = p_a = q_a$ for all $a \in C$, as well as $r_a = p_a$ for all $a \in \mathrm{dom}(p) \setminus C$ and $r_a = q_a$ for all $a \in \mathrm{dom}(q) \setminus C$. Thus, $r_a^{-1}(1) \cap X_b = p_a^{-1}(1) \cap X_b$ for all $b \in F$ and all $a \in \mathrm{dom}(p)$, and $r_a^{-1}(1) \cap X_b = q_a^{-1}(1) \cap X_b$ for all $b \in K$ and all $a \in \mathrm{dom}(q)$. That means that we found a common extension $(r, L) \in \mathbb{R}_{A,B}$ for $(p, F), (q, K) \in R''$, and since they were chosen arbitrarily, $R'' \subseteq R$ is a subset of κ^+ many pairwise compatible conditions.

Now, we show that $\mathbb{R}_{A,B}$ is κ-closed. Let $\delta < \kappa$ and $\langle (p_\xi, F_\xi) : \xi < \delta \rangle$ be a sequence in $\mathbb{R}_{A,B}$ such that $(p_{\xi_2}, F_{\xi_2}) \leq (p_{\xi_1}, F_{\xi_1})$ for all $\xi_1 < \xi_2 < \delta$. Let $q = \bigcup_{\xi < \delta} p_\xi$ and $K = \bigcup_{\xi < \delta} F_\xi$. Since $p_{\xi_2} \leq_{\mathbb{H}_{A \setminus B}} p_{\xi_1}$ for all $\xi_1 < \xi_2 < \delta$, the q we defined is in fact a condition in $\mathbb{H}_{A \setminus B}$ and $q \leq_{\mathbb{H}_{A \setminus B}} p_\xi$ for all $\xi < \delta$. For each $\xi < \delta$, $F_\xi \in [B]^{<\kappa}$ and $\delta < \kappa$, so $K \in [B]^{<\kappa}$. Lastly, again by our choice of q, for each $\xi < \delta$ we have $q_a^{-1}(1) \cap X_b = p_{\xi,a}^{-1}(1) \cap X_b$ for all $b \in F_\xi$ and for all $a \in \mathrm{dom}(p_\xi)$. Thus, (q, K) extends all conditions of the sequence $\langle (p_\xi, F_\xi) : \xi < \delta \rangle$. $\square$

Now we will show that the poset $\mathbb{H}_A$ is, in fact, forcing equivalent to the two-step iteration $\mathbb{H}_B * \dot{\mathbb{R}}_{A,B}$, where B is some proper subset of A. This is the case if the respective generic extensions are equal. For the proof we need the definition of a dense embedding.

Definition 5.12. For forcing posets $\mathbb{Q}$ and $\mathbb{P}$, the map $i : \mathbb{Q} \to \mathbb{P}$ is a **dense embedding**, if the first three requirements for a complete embedding (Definition 4.10) hold and additionally, that $i(\mathbb{Q})$ is a dense subset of $\mathbb{P}$.

Lemma 5.13. *Let $B \subsetneq A$. Then $\mathbb{H}_A$ and $\mathbb{H}_B * \dot{\mathbb{R}}_{A,B}$ are forcing equivalent and we write $\mathbb{H}_A = \mathbb{H}_B * \dot{\mathbb{R}}_{A,B}$.*

Proof. We have the forcing notion $\mathbb{H}_B$ and the $\mathbb{H}_B$-generic filter G_B to bring us from V to the generic extension $V[G_B]$. In $V[G_B]$, we have the forcing notion $\mathbb{R}_{A,B}$ and some $\mathbb{R}_{A,B}$-generic filter G_R to form the final generic extension $V[G_B][G_R]$. Instead of forcing twice, we let $\dot{\mathbb{R}}_{A,B}$ be an $\mathbb{H}_B$-name for a forcing notion such that $(\dot{\mathbb{R}}_{A,B})_{G_B} = \mathbb{R}_{A,B}$ and look at the two-step iteration $\mathbb{H}_B * \dot{\mathbb{R}}_{A,B}$. There is an $\mathbb{H}_B * \dot{\mathbb{R}}_{A,B}$-generic filter K such that $V[G_B][G_R] = V[K]$. Now, we want to show that that $\mathbb{H}_A$ and $\mathbb{H}_B * \dot{\mathbb{R}}_{A,B}$ are forcing equivalent. So we want to show that there is an $\mathbb{H}_A$-generic filter G such that $V[K] = V[G]$. It suffices to show that there is a dense embedding $i : \mathbb{H}_A \to \mathbb{H}_B * \dot{\mathbb{R}}_{A,B}$ (this follows by Lemma IV.4.7. in [Kun11]).

Let the map $i : \mathbb{H}_A \to \mathbb{H}_B * \dot{\mathbb{R}}_{A,B}$ be defined by

$$i(p) = (p \restriction B, \dot{q}_p)$$

for $p \in \mathbb{H}_A$, where $\dot{q}_p$ is an $\mathbb{H}_B$-name for

$$q_p = (p \restriction (A \setminus B), \operatorname{dom}(p \restriction B)).$$

Since $p \in \mathbb{H}_A$ and $B \subsetneq A$, we know $p \restriction B \in \mathbb{H}_B$, $p \restriction (A \setminus B) \in \mathbb{H}_{A \setminus B}$ and $\operatorname{dom}(p \restriction B) \in [B]^{<\kappa}$. Thus, $q_p \in \mathbb{R}_{A,B}$ holds in $V[G_B]$ and by our choice of the names, we have $\dot{q}_p \in \dot{\mathbb{R}}_{A,B}$ in V, so we can write $p \restriction B \Vdash_{\mathbb{H}_B} \dot{q}_p \in \dot{\mathbb{R}}_{A,B}$. Thus, indeed, $i(p) \in \mathbb{H}_B * \dot{\mathbb{R}}_{A,B}$. Also note that $i(\mathbb{1}_{\mathbb{H}_A}) = (\mathbb{1}_{\mathbb{H}_B}, \dot{\mathbb{1}}_{\mathbb{R}_{A,B}})$.

Claim 5.14. $i(\mathbb{H}_A)$ is dense in $\mathbb{H}_B * \dot{\mathbb{R}}_{A,B}$.

Proof of claim. Fix an arbitrary $(s, \dot{t}) \in \mathbb{H}_B * \dot{\mathbb{R}}_{A,B}$ and show that there is a condition in $i(\mathbb{H}_A)$ that extends $(s, \dot{t})$ in $\mathbb{H}_B * \dot{\mathbb{R}}_{A,B}$. So we have $(s, \dot{t}) \in \mathbb{H}_B * \dot{\mathbb{R}}_{A,B}$, that is $s \in \mathbb{H}_B$ and $\dot{t}$ is a name for $t = (\tilde{t}, F) \in \mathbb{R}_{A,B}$, where $\tilde{t} \in \mathbb{H}_{A \setminus B}$ and $F \in [B]^{<\kappa}$. Now, we want to reconstruct from this a condition $r \in \mathbb{H}_A$ such that $i(r) = (r \restriction B, \dot{q}_r) \leq_{\mathbb{H}_B * \dot{\mathbb{R}}_{A,B}} (s, \dot{t})$, which is the case if and only if $r \restriction B \leq_{\mathbb{H}_B} s$ and $r \restriction B \Vdash_{\mathbb{H}_B} \dot{q}_r \leq_{\dot{\mathbb{R}}_{A,B}} \dot{t}$. As $B \subsetneq A$, there exists a condition $r \in \mathbb{H}_A$ such that

$$r \restriction B \leq_{\mathbb{H}_B} s \ \wedge \ F \subseteq \operatorname{dom}(r \restriction B) \ \wedge \ r \restriction (A \setminus B) = \tilde{t}.$$

If $F \subseteq \operatorname{dom}(s)$, we can simply choose $r \restriction B = s$, then obviously $r \restriction B \leq_{\mathbb{H}_B} s$ and $F \subseteq \operatorname{dom}(r \restriction B)$. If there are indices $\{a_i\}_{i \in I} \subseteq F$ that are not in $\operatorname{dom}(s)$, we let $r \restriction B = s \cup \{(a_i, \alpha, 0) : i \in I\}$ for some $\alpha \in \kappa$. Then $r \restriction B \in \mathbb{H}_B$ and we also have $F \subseteq \operatorname{dom}(r \restriction B)$ and $r \restriction B \leq_{\mathbb{H}_B} s$, as $r \restriction B = s \cup \{(a_i, \alpha, 0) : i \in I\} \supseteq s$ and for all $a \in \operatorname{dom}(s)$, $(r \restriction B)_a = s_a$. By our choice of $r \restriction (A \setminus B) = \tilde{t}$, in $V[G_B]$, we have $r \restriction (A \setminus B) \leq_{\mathbb{H}_{A \setminus B}} \tilde{t}$ and $\tilde{t}_a^{-1}(1) \cap X_b = r \restriction (A \setminus B)_a^{-1}(1) \cap X_b$ for all $b \in F$, $a \in \operatorname{dom}(\tilde{t})$, where $X_b = \chi_b^{-1}(1)$ and $\chi_b = \bigcup_{p \in G_B} p_b$. Therefore, in V, we have that $\dot{q}_r \leq_{\dot{\mathbb{R}}_{A,B}} \dot{t}$. Thus, we found a condition $i(r) \in i(\mathbb{H}_A)$ such that $i(r) \leq_{\mathbb{H}_B * \dot{\mathbb{R}}_{A,B}} (s, \dot{t})$, so $i(\mathbb{H}_A)$ is dense in $\mathbb{H}_B * \dot{\mathbb{R}}_{A,B}$. $\square$

Claim 5.15. For all $r, s \in \mathbb{H}_A$, if $r \leq_{\mathbb{H}_A} s$ then $i(r) \leq_{\mathbb{H}_B * \dot{\mathbb{R}}_{A,B}} i(s)$. Also, r and s are incompatible in $\mathbb{H}_A$ if and only if $i(r)$ and $i(s)$ are incompatible in $\mathbb{H}_B * \dot{\mathbb{R}}_{A,B}$.

Proof of claim. Fix arbitrary $r, s \in \mathbb{H}_A$. We will show that $i(r) \not\leq_{\mathbb{H}_B * \dot{\mathbb{R}}_{A,B}} i(s)$ implies $r \not\leq_{\mathbb{H}_A} s$. If $i(r) \not\leq_{\mathbb{H}_B * \dot{\mathbb{R}}_{A,B}} i(s)$, then $r \restriction B \not\leq_{\mathbb{H}_B} s \restriction B$ or $q_r \not\leq_{\mathbb{R}_{A,B}} q_s$. If $r \restriction B \not\leq_{\mathbb{H}_B} s \restriction B$, then $r \leq_{\mathbb{H}_A} s$ is impossible. If $q_r \not\leq_{\mathbb{R}_{A,B}} q_s$, then we are in (at least) one of the following three situations:

1. $r \upharpoonright (A \setminus B) \not\leq_{\mathbb{H}_{A\setminus B}} s \upharpoonright (A \setminus B)$,

2. $\mathrm{dom}(r \upharpoonright B) \not\supseteq \mathrm{dom}(s \upharpoonright B)$,

3. $\exists b \in \mathrm{dom}(s) \cap B\ \exists a \in \mathrm{dom}(s) \setminus B\ (r_a^{-1}(1) \cap X_b \neq s_a^{-1}(1) \cap X_b)$.

In the first case, $r \leq_{\mathbb{H}_A} s$ is impossible. In the second case, $r \supseteq s$ is impossible and so $r \leq_{\mathbb{H}_A} s$ is impossible. If we are in the third case, we can find $a, b \in \mathrm{dom}(s)$ with $a \neq b$ such that $r_a^{-1}(1) \cap r_b^{-1}(1) \neq s_a^{-1}(1) \cap s_b^{-1}(1)$ and so $r \leq_{\mathbb{H}_A} s$ is impossible.

Now fix arbitrary $r, s \in \mathbb{H}_A$ with $r \perp_{\mathbb{H}_A} s$. So there is no $p' \in \mathbb{H}_A$ such that $p' \leq_{\mathbb{H}_A} r$ and $p' \leq_{\mathbb{H}_A} s$. Assume towards a contradiction that there is $(s, \dot{t}) \in \mathbb{H}_B * \dot{R}_{A,B}$ that is a common extension of $i(r)$ and $i(s)$. Since $i(\mathbb{H}_A)$ is dense in $\mathbb{H}_B * \dot{R}_{A,B}$, there is a $p \in \mathbb{H}_A$ such that $i(p) \leq_{\mathbb{H}_B * \dot{R}_{A,B}} (s, \dot{t})$. Since $(s, \dot{t}) \leq_{\mathbb{H}_B * \dot{R}_{A,B}} i(r), i(s)$, we get that $i(p) \leq_{\mathbb{H}_B * \dot{R}_{A,B}} i(r), i(s)$. But then $p \upharpoonright B \leq_{\mathbb{H}_B} r \upharpoonright B, s \upharpoonright B$ and $p \upharpoonright (A \setminus B) \leq_{\mathbb{H}_{A\setminus B}} r \upharpoonright (A \setminus B), s \upharpoonright (A \setminus B)$, as $p \upharpoonright B \Vdash \dot{q}_p \leq_{\dot{\mathbb{R}}_{A,B}} \dot{q}_r, \dot{q}_s$. Putting these together, we get $p \leq_{\mathbb{H}_A} r$ and $p \leq_{\mathbb{H}_A} s$, a contradiction.

If $i(r), i(s)$ are incompatible in $\mathbb{H}_B * \dot{\mathbb{R}}_{A,B}$, then r, s also cannot have a common extension in $\mathbb{H}_A$. Otherwise, the image of this common extension would work as a common extension in $\mathbb{H}_B * \dot{\mathbb{R}}_{A,B}$ too and we would arrive at a contradiction. Thus, for all $r, s \in \mathbb{H}_A$, we showed that $r \perp_{\mathbb{H}_A} s$ if and only if $i(r) \perp_{\mathbb{H}_B * \dot{\mathbb{R}}_{A,B}} i(s)$. □

Thus, the above defined map $i : \mathbb{H}_A \to \mathbb{H}_B * \dot{\mathbb{R}}_{A,B}$ is a dense embedding, so we indeed have the forcing equivalence $\mathbb{H}_A = \mathbb{H}_B * \dot{\mathbb{R}}_{A,B}$. □

We are now ready to prove that the κ-almost disjoint family added by $\mathbb{H}_A$, which we saw in Lemma 5.9, is maximal if $|A| \geq \kappa^+$. In the proof, we will make use of the helpful fact from the above lemma, that $\mathbb{H}_A$ and $\mathbb{H}_B * \dot{\mathbb{R}}_{A,B}$ are forcing equivalent.

Lemma 5.16. Let A be an index set with $|A| \geq \kappa^+$, let G be an $\mathbb{H}_A$-generic filter and $\mathcal{X} = \{X_a\}_{a \in A}$ be defined as above. That is, for each $a \in A$, $X_a = \chi_a^{-1}(1)$, where $\chi_a = \bigcup_{p \in G} p_a$. Then $\mathcal{X}$ is a maximal κ-almost disjoint family.

Proof. [Fisc] Towards a contradiction, we suppose that in $V[G]$, $\mathcal{X} \cup \{X\}$ is κ-almost disjoint for some $X \in ([\kappa]^\kappa \setminus \mathcal{X}) \cap V[G]$. The poset $\mathbb{H}_A$ is κ^+-Knaster, so it is also κ^+-cc (see Remark 5.7). Therefore, we can identify a name $\dot{X}$ for X with a family of κ many maximal antichains $\{\mathcal{K}_\alpha\}_{\alpha \in \kappa}$, where the elements of each $\mathcal{K}_\alpha = \{p_{\alpha,\xi}\}_{\xi \in \kappa} \subseteq \mathbb{H}_A$ decide if α is in $\dot{X}$. Let $B = \bigcup_{\alpha\in\kappa,\xi\in\kappa} \mathrm{dom}(p_{\alpha,\xi}) \subseteq A$ and note that $|B| \leq \kappa$, as $|\mathrm{dom}(p_{\alpha,\xi})| < \kappa$ for all $\alpha \in \kappa, \xi \in \kappa$. We have $|A| \geq \kappa^+$, so we can fix $a \in A \setminus B$. Note that this a does not appear in any of the conditions in the $\dot{X}$ identifying antichains.

Since $B \subsetneq A$, and thus $\mathbb{H}_B \lessdot \mathbb{H}_A$ (see Lemma 5.8), $G_B = G \cap \mathbb{H}_B$ is an $\mathbb{H}_B$-generic filter. And since the conditions in the antichains $\mathcal{K}_\alpha$ are only working on B, we observe that $X \in V[G_B] \cap ([\kappa]^\kappa \setminus \mathcal{X})$. Our goal now is to show that the a we fixed is the index of the member of $\mathcal{X}$ taking care of the intruder X, that is, $|X_a \cap X| = \kappa$.

Fix an arbitrary $\xi \in \kappa$ and let $D_\xi = \{r \in \mathbb{R}_{A,B} : \exists \xi^* > \xi (r \Vdash_{\mathbb{R}_{A,B}} \xi^* \in X \cap \dot{X}_a)\}$. Then D_ξ is dense in $\mathbb{R}_{A,B}$: Let p^* be some condition in $\mathbb{R}_{A,B}$, so $p^* = (p, F)$, where $F \in [B]^{<\kappa}$ and $p \in \mathbb{H}_{A\setminus B}$. We assumed $\mathcal{X} \cup \{X\}$ is κ-almost disjoint in $V[G]$. Since $X \in V[G_B]$, this implies that only the members of $\mathcal{X}$ with indices in B together with X form a κ-almost disjoint family in $V[G_B]$, that is, $V[G_B] \models (\{X_b\}_{b\in B} \cup \{X\}$ is κ-almost disjoint). Then $V[G_B] \models \forall b \in B(|X \cap X_b| < \kappa)$. Note that $F \subseteq B$ and $|F| < \kappa$, so the set $X \setminus (\bigcup_{b\in F} X_b)$ is of cardinality κ and we can choose a $\xi^* \in X \setminus (\bigcup_{b\in F} X_b)$ such that $\xi^* > \xi$ and $\xi^* > \sup_{b\in\mathrm{dom}(p)}(\sup(\mathrm{dom}(p_b)))$. Now, let $q = p \cup \{(a, \xi^*, 1)\} \cup \{(b, \xi^*, 0) : b \in \mathrm{dom}(p) \setminus \{a\}\}$. We chose the ξ^* so it does not appear in any of the p_b yet, so adding $(b, \xi^*, 0)$ does not conflict with p_b being a function, so $q \in \mathbb{H}_{A\setminus B}$. Therefore, $q^* = (q, F) \in D_\xi$ and $q^* \leq_{\mathbb{R}_{A,B}} p^*$, so D_ξ is dense in $\mathbb{R}_{A,B}$.

Since $\mathbb{H}_A = \mathbb{H}_B * \dot{\mathbb{R}}_{A,B}$, there is a $\dot{\mathbb{R}}_{A,B}$-generic filter H such that $V[G] = V[G_B][H]$, that has a nonempty intersection with D_ξ for every $\xi \in \kappa$. Therefore, this density argument shows that $V[G] \models |X_a \cap X| = \kappa$, contradicting the assumption that $\mathcal{X} \cup \{X\}$ is κ-almost disjoint. So the family $\mathcal{X}$ is indeed maximal κ-almost disjoint in $V[G]$. $\square$

5.3 Positive requirements for the spectrum of $\mathfrak{a}(\kappa)$

We just saw how the forcing poset $\mathbb{H}_A$ adjoins a κ-almost disjoint family of size $|A|$, which is maximal given that $|A| \geq \kappa^+$. Now, we will look at a product of these posets over a closed set D of cardinals and show how this technique adds a κ-mad family of each cardinality given in the set D. In other words, we will show that D is a subset of the spectrum of $\mathfrak{a}(\kappa)$.

Notation. For the index set A that we used in Definition 5.5, we will now put the set $\{\xi\} \times \xi$, where ξ is some cardinal. With that we get the poset $\mathbb{H}_{\{\xi\}\times\xi}$ consisting of all $p \subseteq \{\xi\} \times \xi \times \kappa \times \{0, 1\}$ such that $|p| < \kappa$ and for all $(\xi, \zeta) \in \{\xi\} \times \xi$, the set $p_{\xi,\zeta} = \{(\gamma, i) \in \kappa \times \{0, 1\} : (\xi, \zeta, \gamma, i) \in p\}$ is a function. The first index ξ indicates the association to $\mathbb{H}_{\{\xi\}\times\xi}$.

The product $\prod_{\xi\in D}\mathbb{H}_{\{\xi\}\times\xi}$ is then the poset of all sequences $\bar{p}=\langle p_\xi : \xi\in D\rangle$ such that $p_\xi\in\mathbb{H}_{\{\xi\}\times\xi}$ for each $\xi\in D$, with the extension relation: $\bar{q}\leq\bar{p}$ if and only if $q_\xi\leq p_\xi$ for all $\xi\in D$. The largest element is the sequence of largest elements, $\mathbb{1}=\langle\mathbb{1}_{\mathbb{H}_{\{\xi\}\times\xi}} : \xi\in D\rangle$. Also recall that the support of a condition $\bar{p}$ is the set $\operatorname{supp}(\bar{p})=\{\xi\in D : p_\xi\neq\mathbb{1}_{\mathbb{H}_{\{\xi\}\times\xi}}\}$.

Definition 5.17. The **$<\kappa$-supported product** $\prod^{<\kappa}_{\xi\in D}\mathbb{H}_{\{\xi\}\times\xi}$ is the suborder of all conditions $\bar{p}\in\prod_{\xi\in D}\mathbb{H}_{\{\xi\}\times\xi}$ with $|\operatorname{supp}(\bar{p})|<\kappa$.

Before forcing with this poset, we show some of its key properties.

Lemma 5.18. Let D be a closed set of cardinals with $\min(D)\geq\kappa^+$. Then the $<\kappa$-supported product $\mathbb{P}=\prod^{<\kappa}_{\xi\in D}\mathbb{H}_{\{\xi\}\times\xi}$ is κ^+-Knaster and κ-closed, and therefore also κ^+-cc and cardinal preserving.

Proof. [Fisc] First, we will show that $\mathbb{P}$ is κ-closed: Let $\delta<\kappa$ and $\{\bar{p}^\alpha\}_{\alpha<\delta}$ a sequence in $\mathbb{P}$ such that for all $\alpha_1<\alpha_2<\delta$, $\bar{p}^{\alpha_2}\leq\bar{p}^{\alpha_1}$. We will define a condition $\bar{q}\in\mathbb{P}$ that extends every condition of the sequence. For that set $\operatorname{supp}(\bar{q})=\bigcup_{\alpha\in\delta}\operatorname{supp}(\bar{p}^\alpha)$ and for each $\xi\in\operatorname{supp}(\bar{q})$ define $q_\xi=\bigcup\{p^\alpha_\xi : \alpha<\delta\}$. Then $\bar{q}\in\mathbb{P}$ and indeed $\bar{q}\leq\bar{p}^\alpha$ for each $\alpha\in\delta$.

Next, we will show that $\mathbb{P}$ is κ^+-Knaster: Let $\mathcal{F}=\{\bar{p}^\alpha\}_{\alpha<\kappa^+}$ be a family of κ^+ many conditions in $\mathbb{P}$. We will show that there are κ^+ many pairwise compatible conditions in this family. The supports $\{\operatorname{supp}(\bar{p}^\alpha)\}_{\alpha<\kappa^+}$ of the conditions in $\mathcal{F}$ build a family of κ^+ many sets, where each is of size less than κ. With the Delta System Lemma (see [Kun11], Lemma III.6.15), we get $I\subseteq\kappa^+$ with $|I|=\kappa^+$ and a root $R\subseteq D$ with $|R|<\kappa$ such that

$$\forall\alpha,\beta\in I(\alpha\neq\beta\longrightarrow\operatorname{supp}(\bar{p}^\alpha)\cap\operatorname{supp}(\bar{p}^\beta)=R).$$

Fix $\mu\in R$. Then $p^\alpha_\mu\neq\emptyset$ for each $\alpha\in I$. Further, note that the domains $\{\operatorname{dom}(p^\alpha_\mu)\}_{\alpha\in I}$ then build a family of size κ^+, where each $|\operatorname{dom}(p^\alpha_\mu)|<\kappa$. Use the Delta System Lemma again to get $I^\mu\subseteq I$ with $|I^\mu|=\kappa^+$ and a root $R^\mu\subseteq\{\mu\}\times\mu$, $|R^\mu|<\kappa$ such that

$$\forall\alpha,\beta\in I^\mu(\alpha\neq\beta\longrightarrow\operatorname{dom}(p^\alpha_\mu)\cap\operatorname{dom}(p^\beta_\mu)=R^\mu).$$

That means, for any index $(\mu,\nu)\in R^\mu$, the function $p^\alpha_{\mu,\nu}\subseteq\kappa\times\{0,1\}$ is not trivial, for $\alpha\in I^\mu$. There are only κ many such functions as $|[\kappa\times\{0,1\}]^{<\kappa}|=\kappa$, so by the Pigeonhole Principle, there is $J^\mu\subseteq I^\mu, |J^\mu|=\kappa^+$ and an index $(\mu,\nu)\in R^\mu$ such that the conditions $\{\bar{p}^\alpha\}_{\alpha\in J^\mu}$ agree pairwise as functions at this index:

$$\forall\alpha,\beta\in J^\mu(\alpha\neq\beta\longrightarrow p^\alpha_{\mu,\nu}=p^\beta_{\mu,\nu}).$$

Now, using the same argument, there is an index set $I^*\subseteq I, |I^*|=\kappa^+$, such that for each $\mu\in R$ the family $\{\mathrm{dom}(p^\alpha_\mu)\}_{\alpha\in I^*}$ is a Delta system with root $R^\mu\subseteq\{\mu\}\times\mu$ and

$$\forall\alpha,\beta\in I^*\forall(\mu,\nu)\in R^\mu(\alpha\neq\beta\longrightarrow p^\alpha_{\mu,\nu}=p^\beta_{\mu,\nu}).$$

That is, for $\alpha,\beta\in I^*,\alpha\neq\beta$ the conditions $\bar{p}_\alpha,\bar{p}_\beta$ have a shared support $R\subseteq D$, where their entries p^α_μ are not trivial. Also, for every $\mu\in R$, the entries p^α_μ,p^β_μ have shared domain $R^\mu\subseteq\{\mu\}\times\mu$, where they are not trivial as functions $p^\alpha_{\mu,\nu}$. Finally, for every $(\mu,\nu)\in R^\mu$ they agree as functions $p^\alpha_{\mu,\nu}=p^\beta_{\mu,\nu}$. For this at most $\kappa=|[\kappa\times 2]^{<\kappa}|$ many conditions were sorted out, so the subfamily $\{\bar{p}^\alpha\}_{\alpha\in I^*}\subseteq\mathcal{F}$ still has size κ^+. It remains to show that this subfamily consists of pairwise compatible conditions.

Fix arbitrary $\alpha,\beta\in I^*$ with $\alpha\neq\beta$. We will show that there is a common extension for $\bar{p}^\alpha$ and $\bar{p}^\beta$. Define $\bar{q}\in\mathbb{P}=\prod^{<\kappa}_{\xi\in D}\mathbb{H}_{\{\xi\}\times\xi}$ as follows:

- If $\xi\in\mathrm{supp}(\bar{p}^\alpha)\setminus R$, then set $q_\xi:=p^\alpha_\xi$.
- If $\xi\in\mathrm{supp}(\bar{p}^\beta)\setminus R$, then set $q_\xi:=p^\beta_\xi$.
- If $\xi\in R=\mathrm{supp}(\bar{p}^\alpha)\cap\mathrm{supp}(\bar{p}^\beta)$, then let $\mathrm{dom}(q_\xi)=\mathrm{dom}(p^\alpha_\xi)\cup\mathrm{dom}(p^\beta_\xi)$ and define q_ξ through each $(\xi,\rho)\in\mathrm{dom}(q_\xi)$ by setting

$$q_{\xi,\rho}:=\begin{cases}p^\alpha_{\xi,\rho} & \text{if } (\xi,\rho)\in\mathrm{dom}(p^\alpha_\xi)\setminus R^\xi\\ p^\beta_{\xi,\rho} & \text{if } (\xi,\rho)\in\mathrm{dom}(p^\beta_\xi)\setminus R^\xi\\ p^\alpha_{\xi,\rho}\ (=p^\beta_{\xi,\rho}) & \text{if } (\xi,\rho)\in R^\xi.\end{cases}$$

Then $\mathrm{supp}(\bar{q})=\mathrm{supp}(\bar{p}^\alpha)\cup\mathrm{supp}(\bar{p}^\beta)$, so $|\mathrm{supp}(\bar{q})|<\kappa$ and for each $\xi\in D$ we have $q_\xi\in\mathbb{H}_{\{\xi\}\times\xi}$ and $q_\xi\leq p^\alpha_\xi$ and $q_\xi\leq p^\beta_\xi$. Thus, $\bar{q}\in\mathbb{P}$ and $\bar{q}$ is a common extension of $\bar{p}^\alpha$ and $\bar{p}^\beta$.

Note that the κ^+-Knaster property implies the κ^+-cc (Remark 5.7) and that a κ^+-cc poset preserves cardinals ([Kun11] Lemma IV.3.4, adapted to κ^+-cc posets with the help of Lemma 4.6). $\square$

Now, we are ready to show the positive requirement for the spectrum of the κ-almost disjointness number, the final theorem of this thesis.

Theorem 5.19. *Let D be a closed set of cardinals such that $\min(D) \geq \kappa^+$ and let $\mathbb{P} = \prod_{\mu \in D}^{<\kappa} \mathbb{H}_{\{\mu\}\times\mu}$ the $<\kappa$-supported product of the posets $\mathbb{H}_{\{\mu\}\times\mu}$ for $\mu \in D$. Let G be any $\mathbb{P}$-generic filter over V. Then,*

$$V[G] \models D \subseteq \mathfrak{sp}(\mathfrak{a}(\kappa)),$$

that is, for each $\mu \in D$, a κ-mad family of cardinality μ is adjoined by $\mathbb{P}$.

Proof. [Fisc] Fix $\mu \in D$ and define for each $\xi \in \mu$

$$\chi_{\mu,\xi} = \bigcup_{\bar{p} \in G} p_{\mu,\xi} = \{(\gamma, i) \in \kappa \times \{0,1\} : \exists \bar{p} \in G((\mu, \xi, \gamma, i) \in \bar{p})\},$$

which is the characteristic function defining the set

$$a_{\mu,\xi} = \chi_{\mu,\xi}^{-1}(1) = \{\gamma \in \kappa : \exists \bar{p} \in G((\mu, \xi, \gamma, 1) \in \bar{p})\}.$$

Gather these sets together to build the family $\mathcal{A}_\mu = \{a_{\mu,\xi}\}_{\xi \in \mu}$. By Lemma 5.9 and the fact that for each $\mu \in D$ we have $\mathbb{H}_{\{\mu\}\times\mu} \lessdot \mathbb{P} = \prod_{\mu \in D}^{<\kappa} \mathbb{H}_{\{\mu\}\times\mu}$, it follows that $\mathcal{A}_\mu$ is a κ-almost disjoint family. It remains to show that $\mathcal{A}_\mu$ is maximal:

Consider some $a \in [\kappa]^\kappa \cap V[G]$. Since $\mathbb{P}$ is κ^+-cc (by the above lemma), we can identify a $\mathbb{P}$-name $\dot{a}$ for a with a family of κ many antichains $\mathcal{K}_\xi = \{\bar{p}_{\xi,\zeta}\}_{\zeta \in \kappa}$ and sequences $\{k_{\xi,\zeta}\}_{\zeta \in \kappa} \subseteq \{0,1\}$, such that they decide over the elements of a in the following way, ($\bar{p}_{\xi,\zeta} \Vdash \xi \in \dot{a}$ iff $k_{\xi,\zeta} = 1$) and ($\bar{p}_{\xi,\zeta} \Vdash \xi \notin \dot{a}$ iff $k_{\xi,\zeta} = 0$). Let

$$X = \bigcup_{\xi \in \kappa, \zeta \in \kappa} \operatorname{dom}(p_{\xi,\zeta,\mu}) = \{(\mu, \nu) \in \{\mu\} \times \mu : \exists \xi \in \kappa \exists \bar{p}_{\xi,\zeta} \in \mathcal{K}_\xi(p_{\xi,\zeta,\mu,\nu} \neq \emptyset)\},$$

and call it the domain of $\dot{a}$. Then $X \subseteq \{\mu\} \times \mu$ and $|X| \leq \kappa$. Since we assumed $\min(D) \geq \kappa^+$, the fixed $\mu \in D$ must be $\mu \geq \kappa^+$. Thus, there is an $\alpha \in \mu$ such that $(\mu, \alpha) \notin X$. Now we will show that the set a cannot be added to $\mathcal{A}_\mu$ without destroying the almost disjointness, that is,

$$V[G] \models |a_{\mu,\alpha} \cap a| = \kappa,$$

where $a_{\mu,\alpha} = (\dot{a}_{\mu,\alpha})_G$ and $a = (\dot{a})_G$. Let $A = (\{\mu\} \times \mu) \cap X$ and let $B = (\{\mu\} \times \mu) \setminus A$. Further, let $\mathbb{P}' = \prod_{\nu \in D \setminus \{\mu\}}^{<\kappa} \mathbb{H}_{\{\nu\}\times\nu}$ and let $\mathbb{P}_\mu = \mathbb{H}_{\{\mu\}\times\mu}$. Note that $\mathbb{P}_\mu = \mathbb{H}_A * \dot{\mathbb{R}}_{B,A}$ (by

Lemma 5.13), and thus

$$\mathbb{P} = \mathbb{P}' \times \mathbb{P}_\mu = \mathbb{P}' \times (\mathbb{H}_A * \dot{\mathbb{R}}_{B,A}) = (\mathbb{P}' \times \mathbb{H}_A) * \dot{\mathbb{R}}_{B,A}.$$

So $\mathbb{P}$ is forcing equivalent to $(\mathbb{P}' \times \mathbb{H}_A) * \dot{\mathbb{R}}_{B,A}$.

Now, let G' be a $\mathbb{P}'$-generic filter, let $G_\mu = G \cap \mathbb{P}_\mu$, $H_A = G_\mu \cap \mathbb{H}_A$ and $H_{A,B} = G_\mu / H_A$, that is, let $H_{A,B}$ be a filter such that $G_\mu = H_A * H_{A,B} = \{(r, \dot{s}) \in \mathbb{H}_A * \dot{\mathbb{R}}_{B,A} : r \in H_A \wedge \dot{s}_{H_A} \in H_{A,B}\}$. Then

$$V[G] = V[G'][G_\mu] = (V[G'][H_A])[H_{A,B}].$$

Use W to denote the extension $W = V[G'][H_A]$. By our choice of A, which depends on the domain X of $\dot{a}$, we have $a \in W \cap [\kappa]^\kappa$.

Case 1, there is someone in the already evaluated part of the family that takes care of the intruder: If there is $b \in A$ such that $W \models |a \cap a_b| = \kappa$, then this also holds in the final extension $V[G]$. Thus, in $V[G]$, the family $\mathcal{A}_\mu \cup \{a\}$ is not κ-almost disjoint.

Case 2, there is no one like that in the already evaluated part of the family: That means, for all $b \in A$, we have $W \models |a \cap a_b| < \kappa$. For $\xi \in \kappa$ let

$$D_\xi = \{r \in \dot{\mathbb{R}}_{B,A} : \exists \xi^* > \xi \ (r \Vdash \xi^* \in \dot{a}_{\mu,\alpha} \cap \check{a})\}.$$

Then in W, the set D_ξ is dense in $\dot{\mathbb{R}}_{B,A}$: Fix an arbitrary $p^* \in \mathbb{R}_{B,A}$, so $p^* = (p, F)$, where $p \in \mathbb{H}_B$ and $F \in [A]^{<\kappa}$. The set $a \setminus \bigcup_{b \in F} a_b$ has size κ and the domain of $p_{\mu,\alpha}$ has size less than κ, so there is a ξ^* such that $\xi^* > \xi$, $\xi^* > \sup \operatorname{dom}(p_{\mu,\alpha})$ and $\xi^* \in a \setminus \bigcup_{b \in F} a_b$. With this ξ^*, we can now extend p to $q = p \cup \{(\mu, \alpha, \xi^*, 1)\}$. Recall that $(\mu, \alpha) \notin X$, so also not in A, and therefore $q \in \mathbb{H}_B$. Also, $q \leq_{\mathbb{H}_B} p$ and so $q^* = (q, F) \in \mathbb{R}_{B,A}$ with $q^* \leq_{\mathbb{R}_{B,A}} p^*$ and $q^* \Vdash_{\mathbb{R}_{B,A}} \xi^* \in \dot{a}_{\mu,\alpha} \cap \check{a}$. Thus $q^* \in D_\xi$, which shows that D_ξ is dense in $\dot{\mathbb{R}}_{B,A}$. As $\xi \in \kappa$ was also arbitrary, D_ξ is dense for every ξ. Therefore, since the filter $H_{A,B}$ and each D_ξ have a nonempty intersection, $W[H_{A,B}] \models |a_{\mu,\alpha} \cap a| = \kappa$. Thus, in the final extension $W[H_{A,B}] = V[G]$, the family $\mathcal{A}_\mu \cup \{a\}$ is not κ-almost disjoint.

This concludes the proof that in $V[G]$, for an arbitrary but fixed $\mu \in D$, the family $\mathcal{A}_\mu$ is of size μ and κ-mad. Therefore, $\mathbb{P}$ adjoins such a family for each cardinality belonging to the set D. $\square$

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