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Group Chunks in a Model Theoretic Setting

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Benjamin Riff BSc

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Abstract

In 1955 Weil showed how to reconstruct a group from a set equipped with partially defined group operations, a so called *group chunk*. Since then, this type of result has provided a useful tool for model theorists to characterize definable groups in various settings. First steps in this direction were taken by Hrushovski in 1986 and van den Dries in 1990.

In this thesis we generalize van den Dries' approach to a purely filter theoretic setting and then use these results to prove three variants of the Group Chunk Theorem. First we give an exhibition of van den Dries' topological Group Chunk Theorem and show how it relates to the language developed in the filter theoretic setting. Next, we provide a similar treatment of Lewenberg's measurable group chunks from an unpublished part of his PhD thesis from 1995. Finally, we apply the filter theoretic results to obtain a Group Chunk Theorem for structures equipped with an existential matroid.

Zusammenfassung

Weil zeigte 1955, wie man aus einer Menge mit partiell definierten Gruppenoperationen, einem sogenannten *group chunk*, eine Gruppe rekonstruieren kann. Seitdem wurde diese Methode zu einem wichtigen Werkzeug für Modelltheoretiker, um in verschiedenen Strukturen definierbare Gruppen zu charakterisieren. Dies wurde erstmals von Hrushovski (1986) und van den Dries (1990) ausgearbeitet.

Die vorliegende Arbeit verallgemeinert van den Dries' Zugang in einem filtertheoretischen Rahmen. Diese Resultate werden in weiterer Folge verwendet um drei Varianten des „Group Chunk Theorems" herzuleiten. Zuerst wird van den Dries' „Topological Group Chunk Theorem" präsentiert und gezeigt, wie sich die topologischen Eigenschaften in die filtertheoretische Sprache übersetzen lassen. Danach werden Lewenbergs „measurable group chunks", welche er in einem unveröffentlichten Teil seiner PhD Arbeit in 1995 vorstellte, auf eine ähnliche Art bearbeitet. Letztlich werden die filtertheoretischen Resultate angewendet, um ein „Group Chunk Theorem" für Strukturen mit einem „existential matroid" zu beweisen.

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Contents

Notations and conventions

Natural numbers and indices

In this thesis we let $\mathbb{N} = \{0, 1, 2 \dots\}$ denote the set of natural numbers and let k, l, m, n range over $\mathbb{N}$.

Partial functions

Let X, Y be sets and $f \colon X \rightharpoonup Y$ be a partial function. The **domain** of f is the set $\operatorname{dom} f$ of all $x \in X$ such that $f(x)$ is defined. For $X_0 \subseteq X$ the **image** of X_0 under f is defined as

$$f[X_0] \;\coloneqq\; \{f(x) : x \in X_0 \cap \operatorname{dom} f\}.$$

The **preimage** of $Y_0 \subseteq Y$ under f is defined as

$$f^{-1}[Y_0] \;\coloneqq\; \{x \in \operatorname{dom} f : f(x) \in Y_0\}.$$

We say f is **injective**, if

$$\forall x, x' \in \operatorname{dom} f \;:\; f(x) = f(x') \;\Rightarrow\; x = x'.$$

If $g \colon Y \rightharpoonup Z$ is another partial function, then the **composition** $g \circ f$ is defined as the partial function $X \rightharpoonup Z$ where $(g \circ f)(x) = g(f(x))$ is defined whenever $x \in \operatorname{dom} f$ and $f(x) \in \operatorname{dom} g$.

Topology

Let $X = (X, \tau)$ be a topological space and $U \subseteq X$. We denote by $\bar{U}$ the **closure** of U and by $\operatorname{int}(U)$ the **interior** of U, with respect to τ.

Extended Reals

We denote with $\mathbb{R}_{\pm\infty}$ the set of extended reals, that is, the set $\mathbb{R} \cup \{-\infty, +\infty\}$ with the following additional operations for $x \in \mathbb{R}$:

$$
\begin{aligned}
(\pm\infty) + (\pm\infty) = (\pm\infty) + x = x + (\pm\infty) &= (\pm\infty). \\
(\pm\infty)(\pm\infty) &= +\infty \\
(\pm\infty)(\mp\infty) &= -\infty \\
x(\pm\infty) = (\pm\infty)x &= \begin{cases} (\pm\infty) \text{ if } x > 0, \\ 0 \text{ if } x = 0, \\ (\mp\infty) \text{ if } x < 0. \end{cases}
\end{aligned}
$$

Model theoretic background

We assume that the reader is familiar with basic model theory. If not, the relevant background can be found in [ADH17a, Appendix B] or in [TZ12]. We fix some of the model theoretic notation used in this thesis.

For a given (one-sorted) structure $\boldsymbol{M} = (M; \dots)$, we say that a set $X \subseteq M^n$ is **definable** in $\boldsymbol{M}$ if X is definable in $\boldsymbol{M}$ *possibly with parameters* from M. For $A \subseteq M$ and a variable $x = (x_1, \dots, x_n)$, we denote by $\mathrm{St}_x(A)$ the **space of complete x-types over** A.

0 Introduction

The notion of a group chunk was first introduced in 1955 by Weil [Wei55] in an algebraic geometry context and has since then been generalized to various settings as a useful tool to characterize definable groups in different model theoretic structures, see Section 0.2. Roughly speaking, a group chunk is a set X together with partial maps $p\colon X \times X \rightharpoonup X$ and $i\colon X \rightharpoonup X$ that behave like group multiplication and inversion on a “large” part of X, where the notion of largeness depends on the specific setting. The goal is to then embed this group chunk into a group in a way that is compatible with the group chunk operations and the notion of largeness.

0.1 Strategy and main results

The first of two main results of this thesis is a generalization of the Group Chunk Theorem to a purely filter-theoretic setting (Theorem 1.21) using the axioms introduced by van den Dries [Dri90]. This allows us to apply van den Dries' blueprint in similar settings. In particular, we give an exposition of the topological Group Chunk Theorem and the measurable Group Chunk Theorem, which appears in an unpublished part of Lewenberg's PhD thesis [Lew95]. As the abstract part of the theorem is covered by the filter theoretic Group Chunk Theorem, it suffices to prove the topological and measure theoretical properties.

The second main result of this thesis (Theorem 4.31) is a Group Chunk Theorem for structures equipped with an existential matroid (cf. [For11]). Here we are again able to apply the filter theoretic theorem, which plays a substantial role in the proof. This setting is closely related to work by Eleftheriou [Ele21a] and Pillay, Peterzil and Point [PPP23], although we use different axioms.

An abstract Group Chunk Theorem

In Chapter 1, we work with a set X equipped with a filter $\mathcal{F} \subseteq \mathcal{P}(X)$ and call the pair $(X, \mathcal{F})$ a **filtered space**. A set in the filter is called **large** and a subset of X is called **stationary** if it has non-empty intersection with every large set. A **generic bijection** between two filtered spaces is a partial map such that the image and preimage of any large set is again large in the corresponding filtered space. Using these notions we are able to formally define group chunks on a filtered space.

Definition. A **group chunk** on the filtered space $(X, \mathcal{F})$ is a triple $(X; p, i)$, where

$$p\colon X \times X \rightharpoonup X \quad \text{and} \quad i\colon X \rightharpoonup X$$

satisfy the following conditions:

(I) For every $x \in X$ left multiplication $\lambda_x := p(x, -)\colon X \rightharpoonup X$ is a generic bijection;

(II) the inversion map i is a generic bijection;

(III) for every $x \in X$, the set of z for which $p(p(x, z), i(z))$ is defined is stationary;

(IV) for $x, y, z \in X$ the following identities hold whenever both sides are defined:

$$\begin{aligned} p(p(x,y),z) &= p(x,p(y,z)), \\ p(p(x,z),i(z)) &= x, \\ p(i(z),p(z,x)) &= x. \end{aligned}$$

The aim is to embed the group chunk into a group such that the notion of largeness is preserved and the embedding commutes with the group chunk operations. We call a pair (G, h) a **realization** of the group chunk $X = (X; p, i)$ if G is a group chunk which is also a group, and $h\colon X \rightharpoonup G$ is a generic bijection such that $h(p(x, y)) = h(x) \cdot h(y)$ for every $(x, y) \in \operatorname{dom} p$ such that $x, y, p(x, y) \in \operatorname{dom} h$.

We now have the necessary terminology to state the first main result of this thesis.

Theorem. *Let $X = (X; p, i)$ be a group chunk.*

(1) *There exists a realization (G, h) of X with $\operatorname{dom} h = X$.*

(2) *For any two realizations (G', h'), (G'', h'') of X there exists a unique group morphism $\gamma\colon G' \to G''$ such that $\gamma \circ h' = h''$ holds on a large set. Moreover, $\gamma \circ h' = h''$ holds on $\operatorname{dom} h' \cap \operatorname{dom} h''$, and γ is a group isomorphism and a generic bijection.*

The proof of the theorem (see Section 1.5) uses the same strategy as van den Dries for his topological Group Chunk Theorem [Dri90]. We consider the equivalence relation of two generic bijections $X \rightharpoonup X$ agreeing on a large set and define the realization as the group generated by the equivalence classes of left multiplication maps λ_x $(x \in X)$.

Topological and measurable group chunks

In Chapter 2 we give an exposition of van den Dries' topological Group Chunk Theorem and show how to relate the topological notions from his setting into the filter theoretic language introduced in Chapter 1. In this case a group chunk is defined on a topological space. The filter consists of all sets containing a dense open set and the relevant generic bijections are homeomorphisms between dense open subsets of the topological space.

We then analogously present Lewenberg's measurable Group Chunk Theorem from an unpublished part of his thesis [Lew95] in Chapter 3, as an application of the abstract Group Chunk Theorem. Here we consider a group chunk on a measure space, the filter consists of sets containing a "thick" set and the generic bijections are measure isomoprhisms between thick sets.

Existential matroids

In Chapter 4 we study structures equipped with an existential matroid as introduced by Fornasiero (cf. [For11]). We show how an existential matroid gives rise to a dimension function which in turn yields a notion of largeness. The definition of dimension used in this thesis extends the one used by Fornasiero, as it allows for any set to be given a dimension, not just definable sets (cf. Section 4.3). We work in a fixed monster model (with respect to some sufficiently large cardinal κ) of a complete theory that admits only infinite models.

Having introduced the filter of large sets coming from the dimension function on our model, we are able to apply the abstract Group Chunk Theorem from Chapter 1. For a small parameter set A, we call a group chunk $(X; p, i)$ A-definable if X, p and i are all A-definable in the model.

We now state the second main result of this thesis, a definable version of the Group Chunk Theorem. Here T is a finite generic sequence, necessary for technicalities in the proof, see Section 4.5.

Theorem. *Let $X = (X; p, i)$ be an A-definable group chunk.*

(1) *There exists an AT-definable realization $(\tilde{G}, \tilde{h})$ of X with $\operatorname{dom} \tilde{h} = X$.*

(2) *Let (G', h') be a B-definable and (G'', h'') a C-definable realization of X. Then the unique isomorphism $\gamma \colon G' \to G''$ from the abstract Group Chunk Theorem is ABC-definable.*

For the construction of the definable realization we use the same strategy as Peterzil, Pillay and Point in [PPP23]. The group is defined on the cartesian product $X \times X$

and consists of pairs (x, t), where $x \in X$ and $t \in T$. We heavily use the results from Chapter 1.

In Section 4.6 we axiomatically define a dimension function on a structure according to van den Dries [Dri89] and show how to obtain an existential matroid at the cost of adding a small amount of parameters. In Section 4.7 we compare the definition of dimension induced by a pregeometry in this thesis with another approach by Ángel and van den Dries, showing that the two notions yield the same dimension.

0.2 History of group chunks

The first Group Chunk Theorem was introduced by Weil in 1955 in an algebraic geometry context, where he proved how to embed a partially defined birational group law into an algebraic group [Wei55, p. 375]. Probably its first generalization came in 1970, when Artin expanded Weil's theorem to a scheme theoretic version [Art70].

The first application to model theory came in order to answer a question first posed by Poizat: *Is any group that is first-order definable in the theory of algebraically closed fields definably isomorphic to an algebraic group?*

Van den Dries - Topological group chunks (1990–1995)

A positive solution to this problem was given in 1990 by van den Dries, where he introduced a topological variant of the Group Chunk Theorem [Dri90]. He showed how to directly derive Weil's theorem from his topological Group Chunk Theorem in characteristic 0 and a quasi-algebraic Group Chunk Theorem for the case of positive characteristic. He then showed how to obtain an algebraic (or quasi-algebraic) group chunk from a definable group (in the theory of algebraically closed fields). The desired algebraic group could then be attained by using the corresponding Group Chunk Theorem. Van den Dries also derived a differential algebraic Group Chunk Theorem.

Following the same strategy as van den Dries, Lewenberg proved a measurable Group Chunk Theorem in an unpublished part of his PhD thesis [Lew95].

Hrushovski - Stable group chunks and group configurations (1986–1990)

Another approach to solve the problem posed by Poizat, without the Group Chunk Theorem, was given by Hrushovski in his thesis in 1986 [Hru86, §3.1]. However, he also proved his own version of a Group Chunk Theorem for stable theories, which he used as a step to prove that unidimensional theories are superstable [Hru86, §3.4]. This last result was published in a seperate paper by Hrushovski in 1990 [Hru90]. Expositions of the stable Group Chunk Theorem can be found in [Poi87, §5], [Bou89b], and [Pil96, §5.4].

A prominent application of Hrushovski's Group Chunk Theorem is the Group Configuration Theorem for stable theories. The goal is to extract a (type) definable group from purely abstract geometric independencies. The proof follows two main steps: First, a group chunk is produced from the independencies, then the corresponding Group Chunk Theorem gets applied to obtain the group. The stable version was shown by Hrushovski in his thesis, although the first idea of extracting a definable group from such a configuration (in the strongly minimal countably categorical setting) is originally due to Zil'ber, see [Zil84a, Lemma 3.3] and [Zil84b, Proposition 5.1]. A comprehensible exhibition of the stable Group Configuration Theorem was given by Bouscaren [Bou89c].

Evans and Hrushovski used the Group Configuration Theorem to prove results on the combinatorial pregeometry obtained from the algebraic closure in algebraically closed fields [EH91] and its automorphism group [EH95]. Konnnerth generalized the latter to differentially closed fields of characteristic 0 in 2002 [Kon02].

Non-stable analogues: Kowalski and Pillay proved a slight generalization of Hrushovski's group configuration theorem in 2001 to analyze definable groups in difference fields [KP02, Proposition 2.1]. Another version of the Group Configuration Theorem was used by Baro and Martin-Pizzaro in 2021 to prove results concerning small groups in dense pairs of topological structures [BM21].

An abelian version of the stable Group Configuration Theorem was proven in 2017 by Bays, Hills and Moosa in [BHM17, Theorem C.2] and generalized to higher arity in 2024 by Chernikov, Peterzil and Starchenko [CPS24]. Peterzil also gave a proof of the Group Configuration Theorem for o-minimal structures in 2020 [Pet20].

Pillay - Applications in differential algebra (1988–1997)

In 1988 Pillay followed Hrushovski's strategy to prove an o-minimal Group Chunk Theorem, which he used to show that any group definable in an o-minimal structure can be definably equipped with a topology, making it a topological group [Pil88]. Pillay gave another adaptation of Hrushovski's stable Group Chunk Theorem in 1990, when he showed that any group definable in a differentially closed field of characterstic 0 can be embedded into a differential algebraic group [Pil90].

Pillay generalized this in 1997 [Pil97] to produce a pro-algebraic Group Chunk Theorem which he uses to give a positive answer to a variant of Poizat's problem, posed by Kolchin in the preface of his book [Kol85]: *Does any definable group in a differentially closed field of characteristic* 0 *definably embed into an algebraic group?* As the theory of differentially closed fields is stable, Pillay was again able to apply Hrushovski's tools. A generalization of this result to an unstable context was recently accomplished by Pillay, Point and Rideau-Kikuchi [PPR25].

A similar result was proven for groups definable in separably closed fields of finite imperfection degree by Bouscaren and Delon in 2001 directly using Weil's algebraic Group Chunk Theorem [BD02].

Hrushovski, Pillay - Pseudo-finite and other fields (1994–2011)

In 1994 Hrushovski and Pillay [HP94] characterized groups that are first-order definable in the reals and p-adics, or in pseudo-finite fields, using and generalizing the characterization of groups definable in algebraically closed fields due to van den Dries and Hrushovski. One of the results, namely that a connected affine Nash group is Nash isogeneous to the connected component of a real algebraic group, conatined a mistake which was fixed in 2011 [HP11]. The new proof allowed for the result to apply not only to Nash groups over the reals, but an arbitrary real closed field.

Hrushovski and Pillay also used the algebraic Group Chunk Theorem and strategies similar to the Group Configuration Theorem to characterize definable groups in perfect pseudofinite algebraically closed fields in 1995 [HP95].

Ben-Yaacov, Tomašić, Wagner - Simple theories (2001–2004)

In 2001 Wagner generalized Hrushovski's Group Chunk Theorem to simple theories to reconstruct a hyperdefinable group from generically given data [Wag01]. Shortly after, Ben-Yaacov provided a first step towards generalizing the Group Configuration Theorem to simple theories [Ben02]. However, the group chunks obtained from the configurations did not satisfy the conditions for the simple Group Chunk Theorem to apply. This gap was bridged by Ben-Yaacov, Tomašić and Wagner in 2004 [BTW04]. Applications of the simple Group Configuration Theorem can be found in [BTW02], [BW04] and [Tom01].

Tao - A probability theoretic approach (2013)

Another approach to the algebraic Group Chunk and Group Configuration Theorems was given by Terry Tao in 2013, using notions from qualitative probability theory [Tao13].

Hrushovski, Rideau-Kikuchi - Metastable group chunks (2019–2021)

In 2019 Hrushovski and Rideau-Kikuchi introduced metastable theories and stated a corresponding Group Chunk Theorem [HR19, §3.4], generalizing Hrushovski's earlier work and Pillay's pro-group chunks. The result is used to study metastable groups and, as a special case, to characterize definable groups in the theory ACVF (algebraically closed nontrivially valued fields) in terms of group schemes over the valuation ring.

Rideau-Kikuchi then used the aforementioned results in 2021 to study groups definable in separably closed valued fields [Rid21]. This also expands on work of Bouscaren and Delon [BD02].

Eleftheriou - Group chunks from dimension (2021)

In 2021 Eleftheriou published two papers in which he obtains versions of the Group Chunk Theorem using dimension properties. In [Ele21a] the Group Chunk Theorem is used to prove Pillay's conjecture for groups definable in weakly o-minimal non-valuational structures, while in [Ele21b] it is used to characterize definable groups in tame expansions of an o-minimal structure.

Pillay, Peterzil, Point - Geometric theories (2023–2025)

In 2023 Pillay, Peterzil and Point proved a Group Chunk Theorem for geometric theories and used it to show that any group definable in a real closed field with a generic derivation can be definably embedded into a semialgebraic group [PPP23, Theorem 5.4]. They also showed how to adapt these results to related structures. Although Eleftheriou's axiomatic setting [Ele21a] covers geometric theories, [PPP23] gives a detailed construction which yields a deeper understanding of the group obtained via the Group Chunk Theorem, necessary for the desired applications.

A similar version of the Group Chunk Theorem was used by Point in 2025 to characterize groups definable in dp-minimal topological fields with a generic derivation [Poi25].

0.3 Questions and further research

We give a brief discussion on how to expand upon this thesis and state some questions that arose during the process of writing and in discourse.

Other axioms

We chose to use van den Dries' axioms for our filter theoretic setting in Chapter 1, but the same type of generalization to cover multiple settings with one strategy could be done with other axioms. One could use, for example, the axioms from [Ele21a] or try to generalize Hrushovksi's stable group chunks [Hru86] in a similar way.

On other settings

Some of the many other settings that admit a known Group Chunk Theorem might be related to this thesis. In particular, it is still unclear wether Eleftheriou's definable group chunks [Ele21a] or Pillay, Peterzil and Point's group chunks [PPP23] can be covered by our results from Chapter 4.

Application to Transseries

As discussed in Section 4.6, differential algebraic closure provides the unique existential matroid on monster models of the theory of transseries. This suggests Theorem 4.31 to be applicable to $\mathbb{T}$ (the field of transseries, cf. [ADH17a]), in order to study definable groups. For this, one would need to consider how to obtain definable group chunks in $\mathbb{T}$, satisfying our group chunk axioms. This could also yield a corresponding Group Configuration Theorem.

New settings for group chunks

An extension of the work done in this thesis would be to find new settings for which a Group Chunk Theorem might be of interest and to which the filter theoretic version, Theorem 1.21, can be applied to. Possible settings with a natural notion of largeness that came to mind are:

- The filter generated by co-Haar null sets (also known as *prevalent* sets) in Polish groups, see [EV15, EV17, Chr72].
- The filter of sets which contain a *club set* on a regular cardinal. (For more on regular cardinals and club sets, see [Kun11].)

Another question to ask is whether in the topological setting additional properties on the group chunk carry on to its realization. For example, in van den Dries' application the realization only needed to be a homogeneous group, as the topology in question was the Zariski topology. One could wonder under what assumptions on the group chunk the realization would be a topological group.

Upgrading a Lemma

Finally, a small remark on a possible enhancement of the thesis: The filter theoretic generation Lemma (Lemma 1.22) is not directly applicable to obtain the corresponding result in Chapter 4 (Lemma 4.32), although the proof is almost identical. Perhaps it is possible to capture both results in the filter theoretic setting.

1 An Abstract Group Chunk Theorem

In this chapter we define a generalized version of van den Dries' topological group chunks [Dri90] and prove the corresponding theorem about embedding such a group chunk into an abstract group. We will use this result in the next chapters to derive various versions of this theorem when the group chunk carries additional structure.

We begin by introducing the relevant language for our setting. A key aspect for the results in [Dri90] is the notion of a "large" subset of a given set. To generalize this we use the notion of a filter according to Bourbaki's *General Topology* [Bou89a, p. 57]. In particular, we will work with filtered spaces and consider maps preserving the structure of large sets, so called generic bijections. This will allow us to make the notion of an abstract group chunk precise and formulate the Group Chunk Theorem.

1.1 Large sets

Definition. Let X be a non-empty set. A **filter** on X is a set $\mathcal{F}$ of subsets of X that satisfies the following properties:

(F1) $X \in \mathcal{F}$, $\varnothing \notin \mathcal{F}$;

(F2) $\mathcal{F}$ is closed under finite intersections;

(F3) Every subset of X which contains a set in $\mathcal{F}$ belongs to $\mathcal{F}$.

Example 1.1. For an infinite set X, the set of cofinite subsets of X (all subsets whose complement in X is finite) form a filter on X, called the *Fréchet filter*.

Example 1.2. Let X be a topological space and $x \in X$. Then the set of all neighborhoods of x is a filter on X, called the *neighborhood filter of x*. (See [Bou89a].)

Example 1.3. Let X be a topological space. Then the collection of all subsets of X that contain a dense open set form a filter on X. This follows from the fact that the intersection of two dense open subsets of X is again dense open in X. For details see Lemma 2.2.

Definition. A **filtered space** is a pair $(X, \mathcal{F})$ consisting of a set X and a filter $\mathcal{F}$ on X. We denote a filtered space $(X, \mathcal{F})$ by just X when there is no risk of confusion.

We call $X_0 \subseteq X$ $\mathcal{F}$**-large** in X if $X_0 \in \mathcal{F}$. If the filter is clear from the context we simply say X_0 is **large** (in X).

We finish this section with a result on filtered groups (i.e., groups equipped with a filter) which we will need later.

Lemma 1.4. *Let G be a group equipped with a filter such that for every $g \in G$ and large subset $G_0 \subseteq G$ the left coset gG_0 is large in G. Then a subgroup $H \subseteq G$ is large in G if and only if $H = G$.*

Proof. Let $H \subseteq G$ be a subgroup of G and $g \in G$ be arbitrary. Then H and gH are large in G, thus $H \cap gH$ is large and in particular non-empty. Then $H = gH$, so $g \in H$. Since $g \in G$ was arbitrary we get $H = G$. $\square$

Next, we consider maps that preserve the structure of large sets on two given filtered spaces.

1.2 Generic bijections

Definition. Let $(X, \mathcal{F}_X)$ and $(Y, \mathcal{F}_Y)$ be filtered spaces. We call a partial map $f\colon X \rightharpoonup Y$ a **generic bijection** if f is injective and

(1) for every $X_0 \in \mathcal{F}_X$, $f[X_0] \in \mathcal{F}_Y$;

(2) for every $Y_0 \in \mathcal{F}_Y$, $f^{-1}[Y_0] \in \mathcal{F}_X$.

It follows immediately that the domain and codomain of any generic bijection are large. We give some examples of generic bijections.

Example 1.5. Let X, Y be homeomorphic topological spaces and $x \in X$, $y \in Y$. Let X, Y be equipped with the neighborhood filters of x and y, respectively (see Example 1.2). Then any homeomorphism $X \to Y$ that maps x to y is a generic bijection. This follows from the fact that a continuous function between two topological spaces takes neighborhoods of a point in its codomain to neighborhoods of its preimage.

Example 1.6. Let X, Y be topological spaces and consider the filters from Example 1.3 of dense open sets on X and Y. Then any homeomorphism $X_0 \to Y_0$ where X_0, Y_0 are dense open subsets of X, Y, respectively, is a generic bijection $X \rightharpoonup Y$. For details, see Lemma 2.6.

We can now gather some basic properties about generic bijections.

Lemma 1.7. *Let X, Y and Z be filtered spaces, and let $f\colon X \rightharpoonup Y$ and $g\colon Y \rightharpoonup Z$ be generic bijections. Then $f^{-1}\colon Y \rightharpoonup Z$ and $g \circ f\colon X \rightharpoonup Z$ are generic bijections.*

Proof. Note that $g \circ f$ and f^{-1} are both injective. Let X_0, Y_0, Z_0 be large in X, Y, Z, respectively. Then by definition $f^{-1}[Y_0]$ is large in X and $(f^{-1})^{-1}[X_0] = f[X_0]$ is large in Y. Hence $f^{-1}\colon Y \rightharpoonup X$ is a generic bijection.

Next we observe that $(g \circ f)[X_0] = g[f[X_0]]$ is large in Z and $(g \circ f)^{-1}[Z_0] = f^{-1}[g^{-1}[Z_0]]$ is large in X, which shows that $g \circ f\colon Z \rightharpoonup X$ is a generic bijection. □

Lemma 1.8. *Let $(X, \mathcal{F}_X), (Y, \mathcal{F}_Y)$ be filtered spaces and $f\colon X \rightharpoonup Y$ a generic bijection. Then $Y_0 \in \mathcal{F}_Y$ iff $Y_0 \supseteq f[X_0]$ for some $X_0 \in \mathcal{F}_X$.*

Proof. The backward direction immediately follows from $(F3)$ together with the definition of a generic bijection. For the other direction let $Y_0 \in \mathcal{F}_Y$. Then $f^{-1}[Y_0] \in \mathcal{F}_X$, which implies $f[f^{-1}[Y_0]] \in \mathcal{F}_Y$ and yields the desired result with $X_0 := f^{-1}[Y_0]$. □

Next, we define the pushforward of a given filter.

Lemma 1.9. *Let $(X, \mathcal{F})$ be a filtered space, Y a set and let $h\colon X \rightharpoonup Y$ be injective with $\operatorname{dom} h \in \mathcal{F}$. Then for $Y_0 \subseteq Y$, the following are equivalent:*

(i) $h^{-1}[Y_0] \in \mathcal{F}$;

(ii) *there exists $X_0 \in \mathcal{F}$ such that $h[X_0] \subseteq Y_0$.*

Let $\mathcal{H} \subseteq \mathcal{P}(Y)$ denote the set of Y_0 that satisfy (i) *and* (ii). *Then $\mathcal{H}$ is a filter on Y. We call $\mathcal{H}$ the* ***pushforward filter*** *of $\mathcal{F}$ under h and write $\mathcal{H} = h_*(\mathcal{F})$. In this case $(Y, h_*(\mathcal{F}))$ becomes a filtered space and h a generic bijection.*

Proof. (i) ⇒ (ii): Let $Y_0 \subseteq Y$. Then $X_0 = h^{-1}[Y_0]$ does the job.

(ii) ⇒ (i): Let $Y_0 \subseteq Y$ and $X_0 \in \mathcal{F}$ such that $Y_0 \supseteq h[X_0]$. As h is injective and $\operatorname{dom} h$ is large in X,

$$h^{-1}[Y_0] \;\supseteq\; h^{-1}[h[X_0]] \;=\; X_0 \cap \operatorname{dom} h \;\in\; \mathcal{F}$$

and so $h^{-1}[Y_0] \in \mathcal{F}$ by $(F3)$.

To see that $\mathcal{H}$ is a filter, first note that $\varnothing \notin \mathcal{H}$, as $h^{-1}[\varnothing] = \varnothing \notin \mathcal{F}$. Let $Y_0 \in \mathcal{H}$ and $Y_0 \subseteq Y_1 \subseteq Y$. Then $h^{-1}[Y_0] \in \mathcal{F}$ and $h^{-1}[Y_0] \subseteq h^{-1}[Y_1]$, thus, $h^{-1}[Y_1] \in \mathcal{F}$, which gives $Y_1 \in \mathcal{H}$ by (ii). Now let $Y_0, Y_1 \in \mathcal{H}$. Then, since h is injective,

$$h^{-1}[Y_0 \cap Y_1] \;=\; h^{-1}[Y_0] \cap h^{-1}[Y_1] \;\in\; \mathcal{F}.$$

Thus, $Y_0 \cap Y_1 \in \mathcal{H}$ by (i).

Finally, we show that $h\colon X \rightharpoonup Y$ is a generic bijection. Let X_0 be large in X. Since h is injective, $h^{-1}[h[X_0]] = X_0 \cap \operatorname{dom} h$ is large in X and thus $h[X_0] \in h_*(\mathcal{F})$ by (i). If Y_0 is large in Y, by definition of $h_*(\mathcal{F})$, $h^{-1}[Y_0]$ is large in X. $\square$

In Example 1.5 we saw that there can be many generic bijections between to filtered spaces X and Y. However, as we will show next, the filter on Y will always be uniquely determined by the filter on X and the generic bijection.

Lemma 1.10. *Let $(X, \mathcal{F}_X)$ and $(Y, \mathcal{F}_Y)$ be filtered spaces and $h\colon X \rightharpoonup Y$ be a generic bijection. Then $\mathcal{F}_Y = h_*(\mathcal{F}_X)$.*

Proof. Let $Y_0 \in \mathcal{F}_Y$. As h is a generic bijection, $h^{-1}[Y_0] \in \mathcal{F}_X$, hence $Y_0 \in h_*(\mathcal{F}_X)$. Let now $Y_0 \in h_*(\mathcal{F}_X)$. Then there exists $X_0 \in \mathcal{F}_X$ such that $h[X_0] \subseteq Y_0$. But $h[X_0] \in \mathcal{F}_Y$, so by upward closure we get $Y_0 \in \mathcal{F}_Y$. $\square$

We fix a filtered space $X = (X, \mathcal{F})$ for the remainder of this section. Consider the set

$$\mathcal{G} \;:=\; \{f : f \text{ is a generic bijection } X \rightharpoonup X\},$$

and define a binary relation $\sim$ on $\mathcal{G}$ by setting

$$f \sim g \;:\Leftrightarrow\; \text{there exists } X_0 \in \mathcal{F} \text{ such that } f|_{X_0} = g|_{X_0} \qquad (f, g \in \mathcal{G}).$$

In general, $\mathcal{G}$ does not need to be a group under composition, as for example any partial function with a proper subset of X as its domain does not have a compositional inverse. We will show however, that $\mathcal{G}/{\sim}$ can be given a natural group structure. For this we need the following result.

Lemma 1.11. *The relation $\sim$ is an equivalence relation on $\mathcal{G}$. Moreover, if $f_1 \sim f_2$ and $g_1 \sim g_2$, then $(f_1 \circ g_1) \sim (f_2 \circ g_2)$.*

Proof. Clearly $\sim$ is symmetric and reflexive. It remains to show transitivity. For this let $f, g, h \in \mathcal{G}$ where $f \sim g$ and $g \sim h$. Then there are $X_0, X_1 \in \mathcal{F}$ such that $f|_{X_0} = g|_{X_0}$ and $g|_{X_1} = h|_{X_1}$. This gives $f|_{X_0 \cap X_1} = h|_{X_0 \cap X_1}$ and, as $X_0 \cap X_1 \in \mathcal{F}$ by $(F2)$, we have $f \sim h$.

For the second part of the lemma, let $f_1, f_2, g_1, g_2 \in \mathcal{G}$ such that $f_1 \sim f_2$ and $g_1 \sim g_2$. Then there are large sets $X_1, X_2 \subseteq X$ such that $f_1|_{X_1} = f_2|_{X_1}$ and $g_1|_{X_2} = g_2|_{X_2}$. By Lemma 1.7, the compositions $f_1 \circ g_1$ and $f_2 \circ g_2$ are again generic bijections $X \rightharpoonup X$ and agree on the large set $X_2 \cap g_1^{-1}[X_1] \cap g_2^{-1}[X_1]$, hence $f_1 \circ g_1 \sim f_2 \circ g_2$. $\square$

This allows us to introduce the quotient $\tilde{\mathcal{G}} := \mathcal{G}/{\sim}$. We denote by $f^{\sim}$ the equivalence class of $f \in \mathcal{G}$. By Lemma 1.11 we can define a binary operation on $\tilde{\mathcal{G}}$ by setting $f^{\sim} \cdot g^{\sim} := (f \circ g)^{\sim}$ for $f, g \in \mathcal{G}$.

Lemma 1.12. $(\tilde{\mathcal{G}}, \cdot)$ *is a group with neutral element* $\mathrm{id}_X^\sim$.

Proof. Associativity follows immediately from the associativity of composition of functions. By Lemma 1.7, $\mathcal{G}$ is closed under taking inverses. For $f \in \mathcal{G}$ the functions $f \circ f^{-1}$ and $f^{-1} \circ f$ agree with id_X on the large set $\operatorname{dom} f \cap f[X]$. Thus, $(f^{-1})^\sim = (f^\sim)^{-1}$. □

Before we get to the central notion of this paper, the group chunk, we have one more type of sets to define: a weakened version of largeness with respect to a given filter.

1.3 Stationary sets

We return briefly to the setting of arbitrary filtered spaces. Stationary sets come from set theory, where they are also known as co-ideal or positive sets. We give a generalized version for arbitrary filters. While elements in the filter are viewed as large, stationary sets can be thought of as "not small".

Definition. Let $(X, \mathcal{F})$ be a filtered space and $U \subseteq X$. We say that U is **$\mathcal{F}$-stationary** if $U \cap X_0 \neq \varnothing$ for every $X_0 \in \mathcal{F}$. If the filter is clear from context, we may just say U is stationary (in X).

Note that large sets are also stationary by (F2). We first investigate what stationary sets look like in the previously considered examples of filtered spaces.

Example 1.13. Let X be a set and consider the Fréchet filter of cofinite sets (see Example 1.1). Then any infinite set is stationary: Let $U \subseteq X$ be infinite and $X_0 \subseteq X$ be large, i.e., $X \setminus X_0$ finite. Then $(U \setminus X_0) \subseteq (X \setminus X_0) \neq U$, hence $U \cap X_0 \neq \varnothing$.

Example 1.14. Let X be a topological space, $x \in X$, and equip X with the neighborhood filter of x (see Example 1.2). Let $U \subseteq X$. We call x a *limit point* (or *accumulation point*) of U if every neighborhood of x contains at least one point of U different from x. Then every set which has x as a limit point is stationary.

Example 1.15. Let X be a topological space equipped with the filter generated by the dense open subsets (cf. Example 1.3). Then any non-empty open set is stationary. In particular a subset of X is stationary iff its closure has non-empty interior. See Lemma 2.4.

In the remainder of the section we state some results about stationary sets. They are of independent interest and may be skipped by the reader.

Lemma 1.16. *Let $(X, \mathcal{F})$ be a filtered space. If $V \subseteq X$ is not large, then $X \setminus V$ is stationary.*

Proof. If $V \subseteq X$ is not large, then for all large $X_0 \subseteq X$ we have $X_0 \not\subseteq V$, so $X_0 \setminus V \neq \varnothing$. Since

$$X \setminus V \supseteq \bigcup_{X_0 \in \mathcal{F}} X_0 \setminus V,$$

we get $(X \setminus V) \cap X_0 \neq \varnothing$ for all $X_0 \in \mathcal{F}$, i.e., $X \setminus V$ is stationary. □

Lemma 1.17. *Let $(X, \mathcal{F}_X), (Y, \mathcal{F}_Y)$ be filtered spaces and $f\colon X \rightharpoonup Y$ be a generic bijection. Then*

(i) *if $U \subseteq X$ is stationary in X, then $f[U]$ is stationary in Y;*

(ii) *if $V \subseteq Y$ is stationary in Y, then $f^{-1}[V]$ is stationary in X.*

Proof. (i) Let Y_0 be large in Y. Then $f^{-1}[Y_0]$ is large in X. Take an $x \in U \cap f^{-1}[Y_0]$. Then $f(x)$ is defined, and $f(x) \in f[U] \cap Y_0$.

(ii) Let X_0 be large in X. Then $f[X_0]$ is large in Y, and we can take $y \in V \cap f[X_0]$. This yields $x \in X_0$ such that $f(x) = y$, which gives $x \in f^{-1}[V] \cap X_0$. □

We now have have all the tools to define group chunks.

1.4 Group chunks

The axioms that we employ here are a direct generalization of van den Dries' topological group chunks [Dri90]. In Chapter 2 we discuss topological group chunks in more detail, and in Chapters 3 and 4 we consider other variants that can be derived from the results in this section. For more on group chunks defined with different axioms, see Section 0.2.

Convention. For the remainder of the chapter we fix a filtered space $X = (X, \mathcal{F})$ and let X_0, X_1, U, V range over subsets of X.

Definition. A **group chunk** on X is a triple $(X; p, i)$, where

$$p\colon X \times X \rightharpoonup X \quad \text{and} \quad i\colon X \rightharpoonup X$$

satisfy the following conditions:

(I) For every $x \in X$, the partial map

$$\lambda_x := p(x, -)\colon X \rightharpoonup X \qquad \text{(left multiplication by } x\text{)}$$

is a generic bijection;

(II) the inversion map i is a generic bijection;

(III) for every $x \in X$, the set of z for which $p(p(x,z), i(z))$ is defined is stationary;

(IV) for $x, y, z \in X$ the following identities hold whenever both sides are defined:

$$\begin{aligned} p(p(x,y),z) &= p(x,p(y,z)), \\ p(p(x,z),i(z)) &= x, \\ p(i(z),p(z,x)) &= x. \end{aligned}$$

Notation. We will often write xy for $p(x,y)$ and z^{-1} for $i(z)$.

As indicated by the terminology, the partial functions p and i behave like group multiplication and inversion maps on a large part of X. We will show that this is just enough to let us recover a group from the group chunk $(X; p, i)$. Note that the definition does not mention anything about right multiplication. However, in Chapters 2 and 3 we have to impose an extra axiom to get the desired results concerning the topological and measure theoretical aspects.

In the study of groups, a first step is to recover inductively an n-fold version of associativity, cf. [Lan89, p. 4]. We will prove a corresponding result for group chunks using axioms (III) and (IV). Note that in the setting where p is totally defined, the proof does not use anything about i, as it follows immediately from associativity, whereas in the group chunk setting we do need inversion as well.

Convention. For the remaining chapter we fix a group chunk $(X; p, i)$ on X.

First we set some notation for left multiplication by a tuple and define a relation that allows us to talk about various instances of an n-fold product.

Definition. Let $x = (x_1, \dots, x_n) \in X^n$.

(1) Define by recursion on n a generic bijection $\lambda_x \colon X \rightharpoonup X$:

$$\lambda_x := \begin{cases} \mathrm{id}_X & \text{if } n = 0, \\ \lambda_{x_1} \circ \lambda_{(x_2,\dots,x_n)} & \text{if } n \geq 1. \end{cases}$$

(2) For $x \in X^n$, $n \geq 1$ and $y \in X$ we define the relation x **represents** y by recursion on n:

- For $n = 1$, x represents y if $x = y$.
- For $n > 1$, x represents y if there are $m \in \{1, \dots, n-1\}$ and $y_1, y_2 \in X$ such that $(x_1, \dots, x_m)$ represents y_1, $(x_{m+1}, \dots, x_n)$ represents y_2, and $p(y_1, y_2) = y$.

Recall that we denote by $f^{\sim}$ the equivalence class of a generic bijection $f : X \rightharpoonup X$ under the equivalence relation $\sim$, as defined in Section 1.2.

Lemma 1.18 (Associativity Lemma). *For all $(x, y) \in \operatorname{dom} p$ we have $\lambda_{p(x,y)} \sim \lambda_x \circ \lambda_y$. Moreover, for every $x \in X^n$, $n \geq 1$:*

(i) *if x represents $y \in X$, then $\lambda_x \sim \lambda_y$;*

(ii) *x represents at most one element of X.*

Proof. Let $x, y \in X$ and suppose $p(x, y)$ is defined. Then $\lambda_{p(x,y)}$ is a generic bijection, and $\lambda_{p(x,y)} \sim \lambda_x \circ \lambda_y$ follows from (IV), as the two maps agree on the intersection of their domains.

(i) By induction on $n \geq 1$. The case $n = 1$ is trivial. For $n > 1$, take $m \in \{1, \ldots, n-1\}$ and $y_1, y_2 \in X$ such that $p(y_1, y_2) = y$, $x' := (x_1, \ldots, x_m)$ represents y_1, and $x'' := (x_{m+1}, \ldots, x_n)$ represents y_2. The inductive assumption applied twice yields:

$$\lambda_{x'} \sim \lambda_{y_1} \quad \text{and} \quad \lambda_{x''} \sim \lambda_{y_2}.$$

Thus:

$$\lambda_x \sim \lambda_{x'} \circ \lambda_{x''} \sim \lambda_{y_1} \circ \lambda_{y_2} \sim \lambda_{p(y_1,y_2)} \sim \lambda_y.$$

(ii) Suppose x represents both y_1 and y_2. Then $\lambda_{y_1} \sim \lambda_{y_2}$ and by (III), there is $z \in X$ which simultaneously satisfies:

- $p(y_1, z) = p(y_2, z)$ (a large condition), and
- $p(p(y_1, z), i(z))$ is defined (a stationary condition)

By (IV), for such a z:

$$y_1 = p(p(y_1, z), i(z)) = p(p(y_2, z), i(z)) = y_2. \qquad \square$$

Note that left cancellation in the group chunk follows immediately from axiom (I): Let $x, y, y' \in X$ be such that $p(x, y) = p(x, y')$. Then $\lambda_x(y) = \lambda_x(y')$ which gives $y = y'$ by injectivity of λ_x.

For right cancellation we use the Associativity Lemma together with axioms (III) and (IV).

Lemma 1.19 (Right cancellation). *Let $x, x', y \in X$. If $p(x, y) = p(x', y)$, then $x = x'$. In particular, if $\lambda_x \sim \lambda_{x'}$, then $x = x'$.*

Proof. By the Associativity Lemma we get

$$\lambda_x \circ \lambda_y \;\sim\; \lambda_{p(x,y)} \;=\; \lambda_{p(x,'y)} \;\sim\; \lambda_{x'} \circ \lambda_y$$

and thus $\lambda_x^\sim \cdot \lambda_y^\sim = \lambda_{x'}^\sim \cdot \lambda_y^\sim$ in the group $\tilde{\mathcal{G}}$. Canceling yields $\lambda_x^\sim = \lambda_y^\sim$, i.e., $\lambda_x \sim \lambda_{x'}$. Next, choose z such that:

- $p(x,z) = p(x',z)$ (large condition),
- $p(p(x,z),i(z))$ is defined (stationary condition).

Then by (IV):

$$x \;=\; p(p(x,z),i(z)) \;=\; p((x',z),i(z)) \;=\; x'. \qquad \square$$

After having established some basic properties of our group chunk, we are ready to move on to the main theorem.

1.5 Realizing group chunks

Our aim is to embed the group chunk into a unique group in a minimal way, where the embedding behaves like a group morphism whenever the multiplication in the group chunk is defined. Recall that $X = (X;p,i)$ denotes a group chunk and $\mathcal{F}$ our filter on X.

Definition.

(1) Let G be a group. A partial map $h\colon X \rightharpoonup G$ is an **embedding** if it is injective, $\operatorname{dom} h$ is large in X, and $h(p(x,y)) = h(x) \cdot h(y)$ for every $(x,y) \in \operatorname{dom} p$ such that $x, y, p(x,y) \in \operatorname{dom} h$. We say X **embeds** into G if there exists an embedding $X \rightharpoonup G$.

(2) A **realization** of $(X;p,i)$ is a pair (G,h) where G is a group chunk which is also a group, and $h\colon X \rightharpoonup G$ is both an embedding and a generic bijection.

Remark. By Lemma 1.10, any realization of $(X;p,i)$ must be equipped with the filter $h_*(\mathcal{F})$, i.e., the filter on G is uniquely determined by the filter on X and the partial map h.

Note that here we have no requirement of h behaving well with inversion in the group chunk. As we show next, this follows from axioms (I) and (IV).

Lemma 1.20. *Let G be a group and $h\colon X \rightharpoonup G$ be an embedding. Then $h(z)^{-1} = h(i(z))$ for every $z \in i^{-1}[\operatorname{dom} h] \cap \operatorname{dom} h$.*

Proof. Set $X_0 := \operatorname{dom} h$ and let $z \in i^{-1}[X_0] \cap X_0$. Then $i(z)$ is defined and in X_0. We may choose

$$x \;\in\; (\lambda_{i(z)} \circ \lambda_z)^{-1}[X_0] \cap \lambda_z^{-1}[X_0],$$

since the intersection is non-empty, as both sets are large. Then zx and $i(z)(zx)$ are defined and in X_0 and, moreover, by axiom (IV) we have $i(z)(zx) = x$. Applying h, we obtain:

$$h(i(z)) \cdot h(z) \cdot h(x) \;=\; h(x).$$

Cancelling $h(x)$ yields $h(i(z)) \cdot h(z) = e$, i.e., $h(i(z)) = h(z)^{-1}$. □

We arrive at the main result of this chapter. Recall that a realization (G, h) of X always comes equipped with a filter on G. As h is a generic bijection, $h[X]$ is then large in G.

Theorem 1.21. *Let $X = (X; p, i)$ be a group chunk.*

(1) *There exists a realization (G, h) of X with* $\operatorname{dom} h = X$.

(2) *For any two realizations (G', h'), (G'', h'') of X there exists a unique group morphism $\gamma\colon G' \to G''$ such that $\gamma \circ h' = h''$ holds on a large set. Moreover, $\gamma \circ h' = h''$ holds on* $\operatorname{dom} h' \cap \operatorname{dom} h''$, *and γ is a group isomorphism and a generic bijection.*

$$\begin{tikzcd} G' \arrow[r, dashed, "\gamma"] & G'' \\ X \arrow[u, "h'"] \arrow[ur, "h''"'] & \end{tikzcd}$$

We first show (1). Then we show (2) for the special realization (G, h) constructed in the proof of (1) in place of an arbitrary realization (G', h'), and deduce the general statements from this. Define the map

$$h : X \to \tilde{\mathcal{G}}, \qquad x \mapsto \; \lambda_x^{\sim},$$

which is well defined by Lemma 1.19, and let $G \subseteq \tilde{\mathcal{G}}$ to be the subgroup generated by $h[X]$. Next equip G with the filter $h_*(\mathcal{F})$. By Lemma 1.9, h then becomes a generic bijection. Lemma 1.18 yields $h(p(x, y)) = h(x) \cdot h(y)$ for all $(x, y) \in \operatorname{dom} p$.

The next fact is a crucial ingredient. It shows that G can be covered by "stationarily many" translates of a large set.

Lemma 1.22. *If U is stationary and X_0 is large, then:*

(i) $h[X] \subseteq h[X_0] \cdot h[U]$;

(ii) $h[X_0] \cdot h[U]$ *is a subgroup of G;*

(iii) $h[X_0] \cdot h[U] = G$.

Proof. We may assume $U \subseteq X_0 \subseteq \mathrm{dom}(i)$ by first replacing X_0 with $X_0 \cap \mathrm{dom}(i)$ and then replacing U with $U \cap X_0$. Then this special case of (iii) will imply the general one and hence (i) and (ii) as well.

(i) Let $x \in X$. By Lemma 1.7, $\lambda_x \circ i$ is a generic bijection, hence $X_1 := (\lambda_x \circ i)^{-1}[X_0]$ is large. For any $v \in X_1$, the expression xv^{-1} is defined and in X_0, and since U is stationary we can pick $v \in U \cap X_1$. By the properties of h, we have

$$h(x) \;=\; h(x) \cdot h(v)^{-1} \cdot h(v) \;=\; h(xv^{-1}) \cdot h(v) \;\in\; h[X_0] \cdot h[U].$$

(ii) Let $x \in X_0$ and $y \in U$. Then x^{-1} and y^{-1} are defined and there is a large set of v such that $y^{-1}(x^{-1}v^{-1})$ is defined and in X_0 (as $\lambda_{y^{-1}} \circ \lambda_{x^{-1}} \circ i$ is a generic bijection). Again we can choose v such that $v \in U$. Then

$$(h(y) \cdot h(x))^{-1} \;=\; h(x)^{-1} \cdot h(y)^{-1} \cdot h(v)^{-1} \cdot h(v) \;=\; h(y^{-1}(x^{-1}v^{-1})) \cdot h(v) \;\in\; h[X_0] \cdot h[U].$$

Similarly, for $x, x' \in X_0$, $y, y' \in U$ there is $v \in U$ such that $x(y(x'(y'v^{-1})))$ is defined and in X_0. Then

$$h(x) \cdot h(y) \cdot h(x') \cdot h(y') \;= h(x(y(x'(y'v^{-1})))) \cdot h(v) \;\in\; h[X_0] \cdot h[U]. \qquad \square$$

We use this representation to show (G, h) is a realization of X. We already observed that $h\colon X \to G$ is a generic bijection and commutes with p whenever defined, i.e., X embeds into G. It remains to prove:

Lemma 1.23. *$(G, h_*(\mathcal{F}))$, together with the inversion and group operation of G, is a group chunk.*

Proof. Axioms (III) and (IV) are immediate, as G is a group. We will show that left multiplication and inversion in G are generic bijections, i.e., (I) and (II) hold. Injectivity is clear for both maps. We start with inversion.

Let G_0 be large in G, then $G_0 \supseteq h[X_0]$ for some $X_0 \in \mathcal{F}$. We may assume $X_0 \subseteq \mathrm{dom}\, i$ by replacing X_0 with $X_0 \cap \mathrm{dom}\, i$. Then $G_0^{-1} \supseteq h[X_0]^{-1} = h[i[X_0]]$, where the last equality follows from Lemma 1.20 since $X_0 \subseteq \mathrm{dom}\, i$. As $h[i[X_0]] \in h_*(\mathcal{F})$, the set G_0^{-1} contains a large set and must thus itself be large in G.

Next we show that for all $g \in G$ left multiplication by g is a generic bijection. Again, let $G_0 \supseteq h[X_0] \in h_*(\mathcal{F})$ for some $X_0 \in \mathcal{F}$. Then by Lemma 1.22,

$$g \cdot G_0 \;\supseteq\; g \cdot h[X_0] \;=\; h(x)h(y)h[X_0],$$

for some $x \in X_0$ and $y \in U$ with U stationary. The following claim finishes the proof, as (I) follows from it, using upward closure of the filter.

Claim. $h(x)h(y)h[X_0] \supseteq h[(\lambda_x \circ \lambda_y)[X_0]]$.

Let $g' \in h[(\lambda_x \circ \lambda_y)[X_0]]$. Then $g' = h(x(yz))$ for some $z \in X_0$. But then $x(yz)$ is defined and hence

$$g' = h(x(yz)) = h(x)h(y)h(z) \in h(x)h(y)h[X_0]. \qquad \square$$

This finishes the proof of part (1) of Theorem 1.21. Next we show part (2) in the case where the initial group is the just constructed group G. We first prove a slightly more general version of uniqueness: Any group morphism $\alpha\colon G \to G'$ such that X embeds into G' via the embedding h' is uniquely determined provided that $\alpha \circ h = h'$ holds on a large subset of X. Then we show the existence of such a group morphism, and that α is an isomorphism whenever (G', h') is a realization of X. Thus, in the remainder of this section we let G' be a group and $h'\colon X \rightharpoonup G'$ be an embedding.

Lemma 1.24. *If there exists a group morphism $\alpha\colon G \to G'$ such that $\alpha \circ h = h'$ holds on a large set, then α is unique and $\alpha \circ h = h'$ holds on all of* $\operatorname{dom} h'$.

Proof. Suppose we have two group morphisms $\alpha_1, \alpha_2\colon G \to G'$ as in the statement of the lemma. Let $X_i \in \mathcal{F}$ be such that $\alpha_i \circ h = h'$ holds on X_i for $i = 1, 2$, and set $X_0 := X_1 \cap X_2$. Let $g \in G$. By Lemma 1.22, we have $G = h[X_0] \cdot h[X_0]$, hence $g = h(x)h(x')$ where $x, x' \in X_0$. Then

$$\begin{aligned}
\alpha_1(g) &= \alpha_1(h(x)h(x')) \\
&= \alpha_1(h(x)) \cdot \alpha_1(h(x')) \\
&= h'(x) \cdot h'(x') \\
&= \alpha_2(h(x)) \cdot \alpha_2(h(x')) \\
&= \alpha_2(h(x)h(x')) \\
&= \alpha_2(g).
\end{aligned}$$

Thus, $\alpha_1 = \alpha_2$, which proves uniqueness of α.

Now assume $\alpha\colon G \to G'$ is a group morphism such that $\alpha \circ h = h'$ holds on the large set X_0 (note that then necessarily $X_0 \subseteq \operatorname{dom} h'$). We show that then $\alpha \circ h = h'$ holds on all of $\operatorname{dom} h'$. Let $x \in \operatorname{dom} h'$ be arbitrary. By axiom (III) there exists a stationary set U such that for all $z \in U$, $(xz)z^{-1}$ is defined and by axiom (IV) this expression must equal x. Pick

$$y \;\in\; U \cap \lambda_x^{-1}[X_0] \cap i^{-1}[X_0] \cap X_0.$$

Then $(xy)y^{-1}$ is defined as $y \in U$ and additionally xy and y^{-1} are defined and in X_0. As $y \in X_0 \subseteq \operatorname{dom} h'$, Lemma 1.20 yields $h'(y^{-1}) = h'(y)^{-1}$. We get

$$\begin{aligned} \alpha(h(x)) &= \alpha(h((xy)y^{-1})) \\ &= \alpha(h(xy)) \cdot \alpha(h(y^{-1})) \\ &= h'(xy) \cdot h'(y^{-1}) \\ &= h'(x) \cdot h'(y) \cdot h'(y)^{-1} \\ &= h'(x). \end{aligned}$$

□

We insert an auxiliary fact before showing the existence of α as in Lemma 1.24.

Lemma 1.25. *Let $x_1, \dots, x_m, y_1, \dots, y_n \in \operatorname{dom} h'$. Then*

$$h(x_1) \cdots h(x_m) = h(y_1) \cdots h(y_n) \implies h'(x_1) \cdots h'(x_m) = h'(y_1) \cdots h'(y_n).$$

Proof. Set $X_0 := \operatorname{dom} h'$, and suppose $h(x_1) \cdots h(x_m) = h(y_1) \cdots h(y_n)$. We want to choose $z \in X_0$ such that the following expressions are defined and in X_0:

- $x_i(x_{i+1}(\dots(x_m z)\dots))$ for $i = 1, \dots, m$.
- $y_j(y_{j+1}(\dots(y_n z)\dots))$ for $j = 1, \dots, n$.

To achieve this we take

$$z \in X_0 \cap \bigcap_{i=1}^{m} (\lambda_{x_i} \circ \lambda_{x_{i+1}} \circ \dots \circ \lambda_{x_m})^{-1}[X_0] \cap \bigcap_{j=1}^{n} (\lambda_{y_j} \circ \lambda_{y_{j+1}} \circ \dots \circ \lambda_{y_n})^{-1}[X_0].$$

Then for such z we obtain

$$h(\lambda_x(z)) = h(x_1) \cdots h(x_m) h(z) = h(y_1) \cdots h(y_n) h(z) = h(\lambda_y(z)).$$

By injectivity of h, we get $\lambda_x(z) = \lambda_y(z)$. Applying h' and using the choice of z yields:

$$h'(x_1) \cdots h'(x_m) \cdot h'(z) = h'(\lambda_x(z)) = h'(\lambda_y(z)) = h'(y_1) \cdots h'(y_n) \cdot h'(z)$$

and so canceling $h'(z)$ yields the desired result. □

Lemma 1.26. *There exists a unique group morphism $\alpha \colon G \to G'$ such that $\alpha \circ h = h'$ holds on a large set. Moreover, $\alpha \circ h = h'$ holds on $\operatorname{dom} h'$, α is injective, and if (G', h') is a realization of X, then α is an isomorphism.*

Proof. Uniqueness follows from Lemma 1.24, hence it suffices to define α. Let $X_0 := \operatorname{dom} h'$. As X_0 is large in X by Lemma 1.22, we know that every $g \in G$ can be written as

$g = h(x_1)h(x_2)$, where $x_1, x_2 \in X_0$. This, together with the case $m = n = 2$ in Lemma 1.25, justifies the following definition:

$$\alpha\colon G \to G', \quad \alpha(h(x_1)h(x_2)) \coloneqq h'(x_1)h'(x_2) \quad (x_1, x_2 \in X_0).$$

To show that α is a group morphism, take $g, \tilde{g} \in G$ and take $x_i, y_i, z_i \in X_0$ (for $i = 1, 2$) such that $g = h(x_1)h(x_2)$, $\tilde{g} = h(y_1)h(y_2)$ and $g \cdot \tilde{g} = h(z_1)h(z_2)$. Then

$$h(z_1) \cdot h(z_2) \;=\; g \cdot \tilde{g} \;=\; h(x_1) \cdot h(x_2) \cdot h(y_1) \cdot h(y_2).$$

By Lemma 1.25 it follows that

$$\alpha(g \cdot g') \;=\; h'(z_1) \cdot h'(z_2) \;=\; h'(x_1) \cdot h'(x_2) \cdot h'(y_1) \cdot h'(y_2) \;=\; \alpha(g) \cdot \alpha(g').$$

Next we show that α is injective. Let $g, \tilde{g} \in G$ and $x_1, x_2, y_1, y_2 \in X_0$ be such that $g = h(x_1)h(x_2)$, $\tilde{g} = h(y_1)h(y_2)$. Suppose that $\alpha(g) = \alpha(\tilde{g})$, i.e., $h'(x_1)h'(x_2) = h'(y_1)h'(y_2)$. Let $x = (x_1, x_2), y = (y_1, y_2)$. Similar to the proof of Lemma 1.25, choose $z \in X_0$ such that $\lambda_x(z)$, $\lambda_y(z)$, $\lambda_{x_2}(z)$, and $\lambda_{y_2}(z)$ are all defined and in X_0. Then $h'(\lambda_x(z)) = h'(\lambda_y(z))$. By injectivity of h' we get $\lambda_x(z) = \lambda_y(z)$. Applying h and canceling $h(z)$ gives $h(x_1)h(x_2) = h(y_1)h(y_2)$, i.e., $g = \tilde{g}$.

Now assume that (G', h') is a realization of $(X; p, i)$. We show that then α is surjective. We have $h'[X] \subseteq \alpha[G] \subseteq G'$, so $\alpha[G]$ is large in G'. As α is a group morphism we also have that $\alpha[G]$ is a subgroup of G' and thus, by Lemma 1.4, $\alpha[G] = G'$. □

Before finishing the proof of Theorem 1.21 we show that $\alpha\colon G \to G'$ from Lemma 1.26 is a generic bijection, where G' is equipped with the pushforward filter $h'_*(\mathcal{F})$. Recall that if (G', h') is a realization of X, then G' must be equipped with the filter $h'_*(\mathcal{F})$ as $h'\colon X \rightharpoonup G'$ is a generic bijection.

Lemma 1.27. *Let $\alpha\colon G \to G'$ be the unique group morphism from the previous lemma. Then, if G' is equipped with the filter $h'_*(\mathcal{F})$, h' and α are generic bijections.*

Proof. It follows immediately from Lemma 1.9 that h' is a generic bijection when G' is equipped with the push forward filter $h'_*(\mathcal{F})$. To show that α is a generic bijection, let G_0 be large in G. Then $G_0 \supseteq h[X_0]$ for some $X_0 \in \mathcal{F}$. We may assume $X_0 \subseteq \operatorname{dom} h'$ by replacing X_0 with $X_0 \cap \operatorname{dom} h'$. Thus,

$$\alpha[G_0] \;\supseteq\; (\alpha \circ h)[X_0] \;=\; h'[X_0],$$

which shows $\alpha[G_0]$ is large in G', as it contains a set from the filter $h'_*(\mathcal{F})$.

Now let G_0' be large in G'. Then $G_0' \supseteq h'[X_0]$ for some X_0 large in X. Again we may assume $X_0 \subseteq \operatorname{dom} h'$. We get

$$\alpha^{-1}[G_0] \;\supseteq\; (\alpha^{-1} \circ h')[X_0] \;=\; h[X_0],$$

i.e., $\alpha^{-1}[G_0']$ is large in G. $\square$

Finally, we show how to derive the general case of (2) as stated in Theorem 1.21 from Lemmas 1.24, 1.26 and 1.27.

Lemma 1.28. *Let (G', h') and (G'', h'') be realizations of X. Then there is a unique group morphism $\gamma\colon G' \to G''$ such that $\gamma \circ h' = h''$ holds on a large set. Moreover, $\gamma \circ h' = h''$ holds on $\operatorname{dom} h' \cap \operatorname{dom} h''$ and γ is a group isomorphism and a generic bijection.*

Proof. Let $\alpha\colon G \to G'$, $\beta\colon G \to G''$ be the unique group morphism from Lemma 1.26. Thus, $\alpha \circ h = h'$ and $\beta \circ h = h''$ holds whenever both sides are defined and α, β are both isomorphisms.

$$\begin{tikzcd}
G \arrow[r, dashed, "\alpha"] \arrow[rr, dashed, bend left, "\beta"] & G' \arrow[r, "\gamma"] & G'' \\
& X \arrow[ul, "h"] \arrow[u, "h'"'] \arrow[ur, "h''"'] &
\end{tikzcd}$$

To show the existence of $\gamma\colon G' \to G''$ such that $\gamma \circ h' = h''$ holds on $\operatorname{dom} h' \cap \operatorname{dom} h''$, it suffices to note that $\gamma = \beta \circ \alpha^{-1}$ has the desired property:

$$h'' \;=\; \beta \circ h \;=\; \beta \circ \alpha^{-1} \circ h'.$$

For uniqueness, suppose $\gamma_1, \gamma_2\colon G' \to G''$ are two group morphisms such that $\gamma_1 \circ h' = h''$ holds on a large set $X_1 \subseteq \operatorname{dom} h'$ and $\gamma_2 \circ h' = h''$ holds on a large set $X_2 \subseteq \operatorname{dom} h'$. Set $X_0 \coloneqq X_1 \cap X_2$. Then, for $i = 1, 2$, we have the map $\gamma_i \circ \alpha : G \to G''$ which satisfies

$$\gamma_i \circ \alpha \circ h \;=\; \gamma_i \circ h' \;=\; h''$$

on the large set X_0. By Lemma 1.24, β is the unique group morphism with this property which implies

$$\beta \;= \gamma_1 \circ \alpha \;=\; \gamma_2 \circ \alpha.$$

Surjectivity of α yields $\gamma_1 = \gamma_2$.

Finally, it is clear that γ bijective and that γ is a generic bijection since α and β are bijective and generic bijections by Lemma 1.27, and generic bijections are preserved under composition and inversion. $\square$

This finishes the proof of Theorem 1.21. For completeness, we show a slightly more general version of (2), where one of the groups is not required to be a realization of our group chunk. We may loose surjectivity of γ, although the remaining arguments go through.

Proposition 1.29. *Let G'' be a group and $h\colon X \rightharpoonup G''$ be an embedding, and suppose (G', h') is a realization of X. Then there is a unique group morphism $\gamma\colon G' \to G''$ such that $\gamma \circ h' = h''$ holds on a large set. Moreover, $\gamma \circ h' = h''$ holds on* $\operatorname{dom} h' \cap \operatorname{dom} h''$*, γ is injective,*

$$\gamma(g') = g'' \quad \Leftrightarrow \quad \textit{there exists } x, y \in X \textit{ such that } g' = h'(x)h'(y) \textit{ and } g'' = h''(x)h''(y),$$

and if G'' is equipped with the pushforward filter $h''_*(\mathcal{F})$, then γ, h'' are generic bijections.

Proof. Lemma 1.26 gives a unique group isomorphism $\alpha\colon G \to G'$ and a unique injective group morphism $\beta\colon G \to G''$.

$$\begin{tikzcd}
G \arrow[r, dashed, "\alpha"] \arrow[rr, dashed, bend left, "\beta"] & G' \arrow[r, hook, "\gamma"] & G'' \\
& X \arrow[ul, "h"] \arrow[u, "h'"'] \arrow[ur, "h''"'] &
\end{tikzcd}$$

As in the proof of Lemma 1.28 we get a unique group morphism $\gamma\colon G' \to G''$ where $\gamma = \beta \circ \alpha^{-1}$. Clearly γ is injective. By Lemma 1.27, γ and h'' are generic bijections when G'' is equipped with the filter $h''_*(\mathcal{F})$.

Note that by construction of α, β in Lemma 1.26 we have

$$\begin{aligned}
\alpha(g) = g' &\iff \exists x, y \in X : g = h(x)h(y) \wedge g' = h'(x)h'(y), \\
\beta(g) = g'' &\iff \exists x, y \in X : g = h(x)h(y) \wedge g'' = h''(x)h''(y).
\end{aligned}$$

We can thus give a concrete description of $\gamma = \beta \circ \alpha^{-1}$ via

$$\gamma(g') = g'' \iff \exists x, y \in X : g' = h'(x)h'(y) \wedge g'' = h''(x)h''(y). \qquad \square$$

To conclude the chapter we give an equivalent way of defining the relation $\sim$ on the set of left multiplication maps $X \rightharpoonup X$. This is a consequence of Theorem 1.21, as we know that (G, h) is a realization of X.

Corollary 1.30. *Let $x, y \in X$. Then*

$$\lambda_x \;\sim\; \lambda_y \quad \Leftrightarrow \quad \textit{there exists at least one } z \textit{ such that } p(x,z) = p(y,z)$$

Proof. Applying h yields $h(x)h(z) = h(y)h(z)$ in G, and canceling $h(z)$ yields $h(x) = h(y)$, i.e., $\lambda_x \sim \lambda_y$. $\square$

2 Topological Group Chunks

In this chapter we use the results from Chapter 1 to derive van den Dries' topological Group Chunk Theorem. For this we translate the concepts and definitions from [Dri90] to the filter theoretic setting. We then apply Theorem 1.21 to obtain an abstract group and only need to prove the required results about the topology on the group.

Convention. For this chapter we let X, Y, Z denote non-empty topological spaces.

2.1 Dense open sets

We begin by introducing the filter generated by dense open sets. Recall that a subset $X_0 \subseteq X$ is **dense** iff $X_0 \cap U \neq \varnothing$ for all non-empty open sets $U \subseteq X$.

Lemma 2.1. *The intersection of two dense open subsets of X is dense open in X.*

Proof. Let $U \subseteq X$ be non-empty open and $X_0, X_1 \subseteq X$ be dense open. Then $X_0 \cap U$ is non-empty and open, hence $X_1 \cap (X_0 \cap U) = (X_1 \cap X_0) \cap U \neq \varnothing$. $\square$

Lemma 2.2. *Let $\mathcal{F}_X \subseteq \mathcal{P}(X)$ denote the collection of all subets of X that contain a dense open set. Then $\mathcal{F}_X$ is a filter on X.*

Proof. Clearly $\varnothing \notin \mathcal{F}_X$ and if $X_0 \in \mathcal{F}_X, X_0 \subseteq X_0'$ then $X_0' \in \mathcal{F}_X$. Now let $X_0, X_0' \in \mathcal{F}_X$. Then each of those sets contains a dense open set and hence $X_0 \cap X_0'$ contains the intersection of two dense open sets, which is again dense open by the previous lemma. Thus, $X_0 \cap X_0' \in \mathcal{F}_X$. $\square$

Recall that for $X_0 \subseteq X$ we define the **subspace topology** on X_0 by letting U be open in X_0 if and only if $U = X_0 \cap V$ for some V that is open in X.

Lemma 2.3. *Let X_0 be dense open in X and equip it with the subspace topology. If X_0' is dense open in X_0, then X_0' is dense open in X.*

Proof. Let X_0' be dense open in X_0. Then $X_0' = X_0 \cap U$ for some open set $U \subseteq X$. Hence X_0' is open in X as the intersection of two open sets. Now let $V \subseteq X$ be non-empty open in X. Then $X_0 \cap V$ is non-empty and open in X_0. Hence $X_0' \cap (X_0 \cap V) \neq \varnothing$ since X_0' is dense in X_0. But $X_0' \cap (X_0 \cap V) = X_0' \cap V$ and thus X_0' is dense in X. $\square$

Let $\mathcal{F}_X$ denote the filter on X generated by the dense open sets from Lemma 2.2. We give a characterization of stationary sets with respect to $\mathcal{F}_X$.

Lemma 2.4. *Let $U \subseteq X$. Then U is stationary with respect to $\mathcal{F}_X$ iff* $\operatorname{int}(\overline{U}) \neq \varnothing$*.*

Proof. Let U be such that $\operatorname{int}(\overline{U}) \neq \varnothing$ and let $X_0 \in \mathcal{F}_X$. We may assume X_0 is dense open. Then there is a non-empty open set W such that $W \subseteq \overline{U}$ and $W \cap X_0 \neq \varnothing$. Let $x \in W \cap X_0$. Then X_0 is a neighborhood of x and $x \in \overline{U}$, hence $X_0 \cap U \neq \varnothing$.

For the other direction, assume $\operatorname{int}(\overline{U}) = \varnothing$. We show that then there exists a dense open set X_0 such that $X_0 \cap U = \varnothing$. Consider $X_0 \coloneqq X \setminus \overline{U}$. Then X_0 is open, as the complement of a closed set, so it remains to show that X_0 is dense. This can be seen easily, as any non-empty open set W with $W \cap X_0 = \varnothing$ must be contained in $\overline{U}$, which contradicts $\operatorname{int}(\overline{U}) = \varnothing$. □

In particular, any non-empty open subset of X is stationary with respect to $\mathcal{F}_X$. Next we consider generic bijections with respect to $\mathcal{F}_X$.

2.2 Dense homeomorphisms

In order to relate topological group chunks to the filter theoretic setting we need to specify the generic bijections. We begin by showing that homeomorphisms preserve dense open sets.

Lemma 2.5. *Let $f\colon X \to Y$ be a homeomorphism.*

(i) *If X_0 is dense open in X, then $f[X_0]$ is dense open in Y.*

(ii) *If Y_0 is dense open in Y, then $f^{-1}[Y_0]$ is dense open in X.*

Proof. Let X_0 be dense open in X and Y_0 be dense open in Y. As f is a homeomorphism it is bijective and f, f^{-1} are continous, hence $f[X_0]$ and $f^{-1}[Y_0]$ are open in X and Y, respectively.

(i) Let $V \subseteq Y$ be non-empty open. Then $V \cap f[X_0] = f[f^{-1}[V] \cap X_0] \neq \varnothing$ since $f^{-1}[V]$ is non-empty open in X.

(ii) Let $U \subseteq X$ be non-empty open. Then $U \cap f^{-1}[Y_0] = f^{-1}[f[U] \cap Y_0] \neq \varnothing$ since $f[U]$ is non-empty open in Y. □

Definition. A **dense homeomorphism** from X to Y is a homeomorphism $f\colon X_0 \to Y_0$ where X_0, Y_0 are dense open in X, Y, respectively. If $X = Y$ we say f is a **dense X-homeomorphism**.

Convention. From now on, let X, Y, Z be equipped with the corresponding filters generated by the dense open sets, which we denote by $\mathcal{F}_X$, $\mathcal{F}_Y$, $\mathcal{F}_Z$, respectively.

Lemma 2.6. *Let X_0, Y_0 be dense open in X, Y respectively, and let $f\colon X_0 \to Y_0$ be a dense homeomorphism. Then f is a generic bijection.*

Proof. Since f is a homeomorphism, it is injective. Let $X_1 \in \mathcal{F}_X$. Then X_1 contains a dense open set $U \subseteq X$. We have

$$f[X_1] \supseteq f[X_0 \cap X_1] \supseteq f[X_0 \cap U].$$

Since U is dense open in X, $X_0 \cap U$ is dense open in X_0. Then $f[X_0 \cap U]$ is dense open in Y_0 and, by Lemma 2.3, $f[X_0 \cap U]$ is dense open in Y. Thus, $f[X_1]$ contains a dense open set and is therefore in $\mathcal{F}_Y$.

Now let $Y_1 \in \mathcal{F}_Y$. Then Y_1 contains a dense open set $V \subseteq Y$ and

$$f^{-1}[Y_1] \supseteq f^{-1}[Y_0 \cap Y_1] \supseteq f^{-1}[Y_0 \cap V].$$

As above, since V is dense open in Y, $Y_0 \cap V$ is dense open in Y_0, hence $f^{-1}[Y_0 \cap V]$ is dense open in X and we get $f^{-1}[Y_1] \in \mathcal{F}_X$. $\square$

Lemma 2.7. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be dense homeomorphisms. Then so is $g \circ f\colon X \to Z$.*

Proof. Let $Y_1 := \operatorname{dom} g$ and $Y_0 := \operatorname{im} f$. Then Y_0, Y_1 are dense open in Y, hence $Y_0 \cap Y_1$ is dense open in Y. As g is a homeomorphism, $Z_0 := g[Y_0 \cap Y_1]$ is dense open in Z and $g \circ f\colon \operatorname{dom}(f) \to Z_0$ is a homeomorphism, i.e., $g \circ f$ is a dense homeomorphism. $\square$

2.3 A topological Group Chunk Theorem

We state the definition of a topological group chunk from [Dri90]. By what we showed so far, this is a group chunk in the sense of the filter theoretic definition in Chapter 1, if we equip X with the filter generated by its dense open subsets. Note that there is an extra axiom which imposes a condition on the right multiplication maps as well. Although this was not needed in Chapter 1, it is necessary to prove results about the topology on the realization G.

Definition. A **group chunk** on a topological space X is a triple $(X; p, i)$, where the partial maps $p\colon X \times X \rightharpoonup X$ and $i\colon X \rightharpoonup X$ satisfy the following conditions:

(Ia) For every $x \in X$ left multiplication by x ($\lambda_x\colon X \rightharpoonup X$) is a dense X-homeomorphism, (Ib) For $y \in \operatorname{dom} i$ right multiplication by y ($\rho_y \colon X \rightharpoonup X$) is a dense X-homeomorphism,

(II) The inversion map i is a dense X-homeomorphism,

(III) For every $x \in X$, there is a non-empty open set of z's for which $p(p(x,z), i(z))$ is defined,

(IV) For $x, y, z \in X$, the following identities hold whenever both sides are defined:

$$\begin{aligned} p(p(x,y),z) &= p(x,p(y,z)), \\ p(p(x,z),i(z)) &= x, \\ p(i(z),p(z,x)) &= x. \end{aligned}$$

Now we make precise what we consider a realization of a topological group chunk.

Definition. A **homogeneous group** is a group carrying a topology for which inversion and all left multiplication maps are continuous.

Note that then right multiplication is also continuous, as for $g, h \in G$ we have

$$\rho_g(h) = hg = (g^{-1}h^{-1})^{-1}.$$

Remark. Any homogeneous group equipped with the filter generated by its large sets is a group chunk. This is immediate, as every homeomorphism is a dense homeomorphism, hence a genric bijection.

Definition. A **realization** of a group chunk $X = (X; p, i)$ is a realization (G', h') of X viewed as a group chunk with respect to the filter generated by its dense open sets, together with a topology on G' making G' a homogeneous group and h' a dense homeomorphism.

We arrive at the topological Group Chunk Theorem. Let $X = (X; p, i)$ be a group chunk on the topological space X and let (G, h) be the realization of X (in the sense of Chapter 1) from Theorem 1.21.

Theorem 2.8.

(1) *There exists a topology on (G, h) such that it becomes a realization of X.*

(2) *Let (G', h') and (G'', h'') be realizations of X. Then the unique group isomorphism $\gamma \colon G' \to G''$ from Theorem 1.21 is a homeomorphism.*

We begin by proving part (1) of the theorem. Let $V := \operatorname{dom} i$. By Lemma 1.22 we have

$$G \;=\; h[V] \cdot h[V].$$

We proceed as in [Dri90]. For $x \in V$ define $\phi_x \colon V \to G$ by $\phi_x(y) = h(x)h(y)$. A set $U \subseteq G$ will be taken to be open if and only if its preimages under all the ϕ_x are.

Remark. This is the finest topology such that all ϕ_x are continuous. It is called the **final topology** induced by $\{\phi_x : x \in V\}$. See [Bou89a, Chapter 1.2.4].

Lemma 2.9. *G is a homogeneous group.*

Proof. We first check continuity of inversion. For this, let U be open in G, $x \in V$, and $y \in \phi_x^{-1}[U^{-1}]$, that is $h(y)^{-1}h(x)^{-1} = h(y^{-1})h(x^{-1}) \in U$. We show that $\phi_x^{-1}[U^{-1}]$ contains a neighborhood of y. Since $\rho_{x^{-1}} \circ \rho_{y^{-1}} \circ i$ is a dense X-homeomorphism by axioms (Ib) and (II), we can choose $v \in V$ such that

$$(v^{-1}y^{-1})x^{-1} \;=\; (\rho_{x^{-1}} \circ \rho_{y^{-1}} \circ i)(v)$$

is defined and in V. Then

$$h(y^{-1})h(x^{-1}) \;=\; \phi_v((v^{-1}y^{-1})x^{-1}) \;=\; (\phi_v \circ \rho_{x^{-1}} \circ \lambda_{v^{-1}} \circ i)(y) \;\in\; U.$$

Since $(\phi_v \circ \rho_{x^{-1}} \circ \lambda_{v^{-1}} \circ i)$ is continuous and U is open, we can find a neighborhood of y that gets mapped into U. For y_1 in this neighborhood we get

$$(\phi_v \circ \rho_{x^{-1}} \circ \lambda_{v^{-1}} \circ i)(y_1) \;=\; \phi_v((v^{-1}y_1^{-1})x^{-1}) \;=\; h(y_1^{-1})h(x^{-1}) \;=\; h(y_1)^{-1}h(x)^{-1} \;\in\; U,$$

i.e., $y_1 \in \phi_x^{-1}[U^{-1}]$ for y_1 in a neighborhood of y as required.

Similarly we check that $g^{-1}U$ is open for $g \in G$, U open. As $h[X]$ generates G we may assume $g = h(a)$ for some $a \in X$. Then for $x \in V$ we have

$$y \in \phi_x^{-1}[g^{-1}U] \iff h(a)h(x)h(y) \in U.$$

Fix x, y with $y \in \phi_x^{-1}[g^{-1}U]$. Then $x, y \in V$ and, by Axiom (Ib), $\rho_y \circ \rho_x$ is a dense X-homeomorphism. We can pick $v \in V$ such that av^{-1} and $(vx)y$ are defined and in V. Then

$$\begin{aligned} h(a)h(x)h(y) &= h(a)h(v)^{-1}h(v)h(x)h(y) \\ &= h(av^{-1})h((vx)y) \\ &= \phi_{av^{-1}}((vx)y) \\ &= (\phi_{av^{-1}} \circ \lambda_v \circ \lambda_x)(y) \in U. \end{aligned}$$

Again, since $(\phi_{av^{-1}} \circ \lambda_v \circ \lambda_x)$ is continuous, we get a neighborhood of y that is mapped into U. For y_1 in this neighborhood we have $(vx)y_1 \in \phi_{av^{-1}}^{-1}[U]$, i.e., $y_1 \in \phi_x^{-1}[g^{-1}U]$. $\square$

Lemma 2.10. *The map $h\colon X \to G$ is a dense homeomorphism.*

Proof. Let $x \in X$. By axioms (III) and (IV) we have $x = uv$ for some $u, v \in V$. Then $uV = \lambda_u[V]$ is a neighborhood of x on which $h = \phi_u \circ \lambda_u^{-1}$. Thus, h is an injective map which is locally a homeomorphism with open image.

Now we check that $h[X]$ is dense in G. Let U be non-empty and open in G, and take $u, v \in V$ such that $h(u)h(v) \in U$. Then $\phi_u^{-1}[U] \subseteq X$ is non-empty and open. For a large set of z, $\lambda_u(z) = uz$ is defined, hence also for some $z \in \phi_u^{-1}[U]$, uz is defined, and $h(uz) = h(u)h(z) \in U$. $\square$

This finishes the proof of part (1) of Theorem 2.8. For part (2), let (G', h') be a realization of X. Note that it suffices to show that the unique isomorphism $\alpha\colon G \to G'$ from Lemma 1.26 is a homeomorphism. The general statement will then follow by composing two such maps as in the proof of Lemma 1.28.

Lemma 2.11. *Let (G', h') be a realization of X. Then the unique group isomorphism $\alpha\colon G \to G'$ from Lemma 1.26 is a homeomorphism.*

Proof. It remains to prove continuity of α and α^{-1}. Let $U \subseteq G'$ be open. We need to show that $\alpha^{-1}[U]$ is open in G, i.e., for all $x \in V$, $\phi_x^{-1}[\alpha^{-1}[U]]$ is open in V. Let $x \in V$. Then

$$\begin{aligned}\phi_x^{-1}[\alpha^{-1}[U]] &= \{v \in V : h'(x)h'(v) \in U\}\\ &= \{v \in V : h'(v) \in h'(x^{-1})U\}\\ &= V \cap h'^{-1}[h'(x^{-1})U],\end{aligned}$$

which is open.

To see that α^{-1} is open we note that for open $U \subseteq G$ we have $\alpha[U] = h'[h^{-1}[U]]$. $\square$

This concludes the proof of Theorem 2.8. As noted in Chapter 1, the filter on G must equal the push forward filter $h_*(\mathcal{F})$ of the filter $\mathcal{F}$ on X, generated by its dense open sets. We conclude the chapter by showing that this coincides with the filter on G which is generated by its own dense open sets.

Lemma 2.12. *Let $G_0 \subseteq G$. Then*

$$G_0 \in h_*(\mathcal{F}) \iff U \subseteq G_0 \text{ for some dense open } U \subseteq G.$$

Proof. "$\Rightarrow$": Let $G_0 \in h_*(\mathcal{F})$. Then $G_0 \supseteq h[X_0]$ for some $X_0 \in \mathfrak{f}$. We can thus take $X_0' \subseteq X_0$ dense open in X such that $G_0 \supseteq h[X_0']$. But since h is a homeomorphism, $h[X_0']$ is dense open in G.

"$\Leftarrow$": Suppose $U \subseteq G_0$ for some dense open $U \subseteq G$. Let $G_0' = h[X] \cap U$. Then $G_0' \subseteq G_0$ is also dense open in G and thus $h^{-1}[G_0']$ is dense open in X, as h is a homeomorphism. Hence, $h^{-1}[G_0'] \in \mathcal{F}$. This is equivalent to $h[h^{-1}[G_0']] = G_0' \in h_*(\mathcal{F})$ and thus, by upwards closure, $G_0 \in h_*(\mathcal{F})$. $\square$

3 Measurable Group Chunks

In this chapter we turn to Adam Lewenberg's thesis [Lew95], where he proves a measurable version of the Group Chunk Theorem, using the same axioms as in [Dri90]. We first translate the measurable setting into the filter theoretic language of Chapter 1. Then we give an exposition of the additional work needed, to obtain the appropriate version of the Group Chunk Theorem in this setting.

3.1 Some measure theory

We begin with some measure theoretic definitions and basic properties needed to state the measurable Group Chunk Theorem.

Fix a set X. An **algebra of subsets of** X is a collection of subsets of X, closed under finite unions and complements. Note that an algebra is also closed under finite intersections and must contain both X and $\varnothing$. Fix an algebra S of X. The sets in S are called **measurable** subsets of X and we call (X, S) a **measurable space**.

A **finitely additive measure** on (X, S) is a map $\mu\colon S \to [0, \infty] \subseteq \mathbb{R}_{\pm\infty}$ such that $\mu(\varnothing) = 0$ and for all disjoint $U, V \in S$ we have $\mu(U \cup V) = \mu(U) + \mu(V)$. We call μ the **zero measure** if it assigns each $U \in S$ the value 0.

A **measure space** is a triple (X, S, μ), where (X, S) is a measurable space and μ is a finitely additive non-zero measure on (X, S). We call a measurable subset of X with measure zero a **null set** and a non-null measurable set **visible**. If every subset of every null set is also measurable, we call (X, S, μ) a **complete** measure space. If for two measurable sets A, B the symmetric difference $(A \setminus B) \cup (B \setminus A)$ is a null set, then we say A and B are **almost equal** and write $A \approx B$.

Let (X, S, μ), (X, S', μ') be a measure spaces. We say that (X, S', μ') **extends** (X, S, μ) if $S' \supseteq S$ and $\mu'|_S = \mu$.

Convention. For the remainder of the chapter, $X = (X, S, \mu)$ denotes a measure space.

Note that for each $n \geq 1$ and pairwise disjoint $U_1, \dots, U_n \in S$ we have

$$\mu(U_1 \cup \dots \cup U_n) \;=\; \mu(U_1) + \dots + \mu(U_n).$$

Lemma 3.1 (Monotonicity). *Let $U, V \in S$, $U \subseteq V$. Then $\mu(U) \leq \mu(V)$.*

Proof. We have
$$\mu(V) \;=\; \mu(U \cup (V \setminus U)) \;=\; \mu(U) + \mu(V \setminus U).$$
Hence, $\mu(U) \leq \mu(V)$. $\square$

We move towards defining the filter of large sets in the measure theoretic setting.

3.2 Thick sets

Definition. A measurable set $X_0 \subseteq X$ is **thick** (in X) if $\mu(X \setminus X_0) = 0$.

Lemma 3.2. *$X_0 \in S$ is thick in X if and only if for all visible $U \subseteq X$ the set $U \cap X_0$ is visible.*

Proof. Let $X_0 \subseteq X$ be thick in X and $U \subseteq X$ be visible. Then

$$0 \;<\; \mu(U) \;=\; \mu(U \cap X_0) + \mu(U \setminus X_0) \;\leq\; \mu(U \cap X_0) + \mu(X \setminus X_0) \;=\; \mu(U \cap X_0).$$

Thus, $U \cap X_0$ is visible.

Now assume $X_0 \subseteq X$ is not thick in X. Then $\mu(X \setminus X_0) > 0$. But then $X \setminus X_0$ is visible and
$$\mu(X_0 \cap (X \setminus X_0)) \;=\; \mu(\varnothing) \;=\; 0.$$
$\square$

Lemma 3.3. *The intersection of two thick sets in X is thick in X.*

Proof. Let $U \subseteq X$ be visible and $X_0, X_1 \subseteq X$ be thick in X. Then $X_0 \cap U$ is visible by Lemma 3.2, hence $X_1 \cap (X_0 \cap U) = (X_1 \cap X_0) \cap U$ is visible. $\square$

Lemma 3.4. *Let X_0 be thick in X and $U \in S$. Then $\mu(X_0 \cap U) = \mu(U)$.*

Proof. As X_0 is thick in X we have $0 = \mu(X \setminus X_0) \geq \mu(U \setminus X_0)$. This gives

$$\mu(U) \;=\; \mu(U \cap X_0) + \mu(U \setminus X_0) \;=\; \mu(U \cap X_0).$$
$\square$

We show that the thick subsets of X generate a filter on X.

Lemma 3.5. *Let $\mathcal{F} \subseteq \mathcal{P}(X)$ denote the collection of all subsets of X that contain a thick set. Then $\mathcal{F}$ is a filter on X.*

Proof. Clearly $\varnothing \notin \mathcal{F}$ (as μ is non-zero) and if $X_0 \in \mathcal{F}, X_0 \subseteq X_0'$ then $X_0' \in \mathcal{F}$. Now let $X_0, X_0' \in \mathcal{F}$. Each of those sets contains a thick set, hence $X_0 \cap X_0'$ contains the intersection of two thick sets, which is again thick by Lemma 3.3. Thus $X_0 \cap X_0' \in \mathcal{F}$. □

Remark. If (X, S, μ) is complete, then $X_0 \in \mathcal{F}$ iff X_0 is thick in X. Indeed, suppose X_0 contains a thick set X_0'. Then $\mu(X \setminus X_0') = 0$ and $X_0 = X_0' \cup V$ for some $V \subseteq X \setminus X_0'$. By completeness, $V \in S$ and $\mu(V) = 0$, hence $X_0 \in S$ and $\mu(X \setminus X_0) \leq \mu(X \setminus X_0') = 0$.

The following two lemmas will be used in the next section. The second result is the corresponding variant of Lemma 2.3.

Lemma 3.6. *Let $U \in S$ be visible. Set $S_U := \{V \cap U : V \in S\}$. Then $(U, S_U, \mu|_{S_U})$ is a measure space.*

Proof. Clearly S_U is an algebra, as for $V, W \in S$,

$$(U \cap V) \cup (U \cap W) \;=\; U \cap (V \cup W)) \;\in\; S_U$$

and

$$(U \cap V) \setminus (U \cap W) \;=\; U \cap (V \setminus W) \;\in\; S_U.$$

To see that $\mu|_{S_U}$ is a measure, note that $\mu(U \cap \varnothing) = \mu(\varnothing) = 0$ and finite additivity follows directly from finite additivity of μ. Finally, $\mu|_{S_U}$ is non-zero because of the choice of U. □

Lemma 3.7. *Let $X_0' \subseteq X_0 \subseteq X$. If X_0 is thick in X, then X_0' is thick in the measure space $X_0 = (X_0, S_{X_0}, \mu|_{X_0})$ if and only if X_0' is thick in X.*

Proof. Let X_0 be thick in X. For the forward direction suppose X_0' is thick in X_0. Then $\mu(X \setminus X_0) = 0$ and $\mu(X_0 \setminus X_0') = 0$. Hence,

$$\mu(X \setminus X_0') \;=\; \mu((X \setminus X_0) \cup (X_0 \setminus X_0')) \;=\; \mu(X \setminus X_0) + \mu(X_0 \setminus X_0') \;=\; 0.$$

Now suppose X_0' is thick in X. Then $\mu(X \setminus X_0') = 0$ and thus,

$$\mu(X_0 \setminus X_0') \;\leq\; \mu(X \setminus X_0') \;=\; 0.$$ □

Before we move on to generic bijections between to measure spaces we give a sufficient condition for being a stationary set.

Lemma 3.8. *If $U \subseteq X$ contains a visible subset of X, then U is stationary with respect to $\mathcal{F}$.*

Proof. Suppose U contains a visible set and let $X_0 \in \mathcal{F}$. Then X_0 contains a thick set which means it has visible intersection with any visible set. In particular, $X_0 \cap U \neq \varnothing$. □

3.3 Thick automorphisms

Definition. Let (X, S_X, μ_X) and (Y, S_Y, μ_Y) be two measure spaces and $f\colon X \rightharpoonup Y$.

(1) f is **measurable** if for all measurable sets $V \subseteq Y$ the preimage $f^{-1}[V]$ is also measurable;

(2) f is **measurability preserving** if f is injective, measurable and $f^{-1}\colon Y \rightharpoonup X$ is also measurable;

(3) f is **measure preserving** if f is measurability preserving and for all $V \in S_Y$ such that $V \subseteq f[X]$ we have $\mu_X(f^{-1}[V]) = \mu_Y(V)$;

(4) If f is measure preserving, its inverse $f^{-1}\colon Y \rightharpoonup X$ is measurability preserving, $f[X]$ is thick in Y, and $f^{-1}[Y] = \operatorname{dom} f$ is thick in X, we call f a **thick measure embedding** of X into Y. A **thick X-automorphism** is a thick measure embedding of X into X.

Remark. Note that if $f\colon X \rightharpoonup Y$ is measure preserving and $f[X]$ is thick in Y, then the condition $V \subseteq f[X]$ in (3) can be dropped, i.e., $\mu_X(f^{-1}[V]) = \mu_Y(V)$ holds for all $V \in S_Y$. Indeed, let $V \in S_Y$, then

$$\mu_X(f^{-1}[V]) \;=\; \mu_X(f^{-1}[V \cap f[X]]) \;=\; \mu_Y(V \cap f[X]) = \mu_Y(V) - \mu_Y(V \setminus f[X]) \;=\; \mu_Y(V).$$

We prove some properties of thick measure embeddings, before we show that any thick measure embedding between to measure spaces equipped with the corresponding filter of thick sets is a generic bijection.

Lemma 3.9. *Let (X, S_X, μ_X) and (Y, S_Y, μ_Y) be two measure spaces and $f\colon X \rightharpoonup Y$ be a thick measure embedding. Then $f^{-1}\colon Y \to X$ is measure preserving.*

Proof. Let $U \in S_X$, $U \subseteq f^{-1}[Y]$. Then

$$\mu_X(U) \;=\; \mu_X(f^{-1}[f[U]]) \;=\; \mu_Y(f[U]). \qquad \square$$

Lemma 3.10. *Let (X, S_X, μ_X), (Y, S_Y, μ_Y), and (Z, S_Z, μ_Z) be measure spaces, and let $f\colon X \rightharpoonup Y$ and $g\colon Z \rightharpoonup Y$ be thick measure embeddings. Then*

(i) *$g \circ f$ and $(g \circ f)^{-1}$ are measurability preserving;*

(ii) *$g \circ f$ and $(g \circ f)^{-1}$ are measure preserving;*

(iii) *$(g \circ f)[X]$ is thick in Z;*

(iv) *$(g \circ f)^{-1}[Z]$ is thick in X; and*

(v) *$g \circ f$ is a thick measure embedding.*

Proof. We prove (i)–(iv), then (v) is immediate.

(i) Clearly $g \circ f$ and $(g \circ f)^{-1}$ are injective. If $Z_0 \in S_Z$, then $g^{-1}[Z_0] \in S_Y$ and hence

$$(g \circ f)^{-1}[Z_0] \;=\; f^{-1}[g^{-1}[Z_0]] \;\in\; S_X.$$

Thus, $g \circ f$ is measurable. For $X_0 \in S_X$ we have $g[f[X_0]] \in S_Z$, so $(g \circ f)^{-1}$ is measurable as well. It follows that $g \circ f$ and $(g \circ f)^{-1}$ are measurability preserving.

(ii) To show that $g \circ f$ and $(g \circ f)^{-1}$ are measure preserving, let first $V \in S_Z$ be a subset of $(g \circ f)[X]$. Then $g^{-1}[V] \in S_Y, g^{-1}[V] \subseteq f[X]$ and $V \subseteq g[Y]$. Hence we can apply that f and g are measure preserving to obtain

$$\mu_X((g \circ f)^{-1}[V]) \;=\; \mu_X(f^{-1}[g^{-1}[V]]) \;=\; \mu_Y(g^{-1}[V]) \;=\; \mu_Z[V].$$

Now let $U \in S_X, U \subseteq (g \circ f)^{-1}[Z]$. Then $f[U] \in S_Y, f[U] \subseteq g^{-1}[Z]$ and $U \subseteq f^{-1}[Y]$. As f^{-1} and g^{-1} are measure preserving we get

$$\mu_Z((g \circ f)[U]) \;=\; \mu_Z(g[f[U]]) \;=\; \mu_Y(g[U]) \;=\; \mu_X[U].$$

(iii) Note that $f[X]$ is thick in Y and $g^{-1}[Z], f[X] \in S_Y$. Hence

$$0 \;=\; \mu_Y(Y \backslash f[X]) \;\geq\; \mu_Y(g^{-1}[Z] \backslash f[X]) \;=\; \mu_Z(g[g^{-1}[Z] \backslash f[X]]) \;=\; \mu_Z(g[Y] \backslash (g \circ f)[X]),$$

where we used that g^{-1} is measure preserving. This shows that $(g \circ f)[X]$ is thick in $g[Y]$, which is thick in Z, so $(g \circ f)[X]$ is thick in Z by Lemma 3.7.

(iv) We have that $g^{-1}[Z]$ is thick in Y and $g^{-1}[Z], f[X] \in S_Y$. This time we use that f is measure preserving to obtain

$$\begin{aligned} 0 \;&=\; \mu_Y(Y \setminus g^{-1}[Z]) \\ &\geq\; \mu_Y(f[X] \setminus g^{-1}[Z]) \\ &=\; \mu_X(f^{-1}[f[X] \setminus g^{-1}[Z]]) \\ &=\; \mu_X(f^{-1}[Y] \setminus (g \circ f)^{-1}[Z]), \end{aligned}$$

i.e., $(g \circ f)^{-1}[Z]$ is thick in $f^{-1}[Y]$, hence also in X. $\square$

Lemma 3.11. *Let (X, S_X, μ_X) and (Y, S_Y, μ_Y) be two measure spaces equipped with the corresponding filter of thick sets from Lemma 3.5 and let $f \colon X \rightharpoonup Y$ be a thick measure embedding. Then f is a generic bijection.*

Proof. Let $X_1 \in \mathcal{F}_X$. Then $X_1 \supseteq X_0$ for a thick set $X_0 \subseteq X$. In particular $X_0 \in S_X$, which implies $f[X_0] \in S_Y$, by measurability of f^{-1}. We get

$$\begin{aligned} 0 &= \mu_X(X \setminus X_0) \\ &\geq \mu_X((X \cap f^{-1}[Y]) \setminus X_0) \\ &= \mu_X(f^{-1}[(f[X] \cap Y) \setminus f[X_0]]) \\ &= \mu_Y(f[X] \setminus f[X_0]). \end{aligned}$$

Thus $f[X_0]$ is thick in $f[X]$, which is thick in Y, which means $f[X_0]$ is thick in Y by Lemma 3.7. As $f[X_1] \supseteq f[X_0]$, this implies $f[X_1] \in \mathcal{F}_Y$.

Let $Y_1 \in \mathcal{F}_Y$. Then $Y_1 \supseteq Y_0$ for a thick set $Y_0 \subseteq Y$. In particular $Y_0 \in S_Y$, which implies $f^{-1}[Y_0] \in S_X$, by measurability of f. We get

$$\begin{aligned} 0 &= \mu_Y(Y \setminus Y_0) \\ &\geq \mu_Y((Y \cap f[X]) \setminus Y_0) \\ &= \mu_X(f^{-1}[(Y \cap f[X]) \setminus Y_0]) \\ &= \mu_X(f^{-1}[Y] \setminus f^{-1}[Y_0]). \end{aligned}$$

Thus, $f^{-1}[Y_0]$ is thick in $f^{-1}[Y]$ and, by Lemma 3.7, $f^{-1}[Y_0]$ is thick in X, as $f^{-1}[Y]$ is thick in X. As $f^{-1}[Y_1] \supseteq f^{-1}[Y_0]$, this implies $f^{-1}[Y_1] \in \mathcal{F}_X$. □

As a special case we get the following corollary.

Corollary 3.12. *Each thick X-automorphism $X \rightharpoonup X$ is a generic bijection.*

This allows us to define a group chunk on the measure space X.

3.4 A measurable Group Chunk Theorem

We now have everything in place to define group chunks on measure spaces as in [Lew95]. As in the topological setting, an extra axiom regarding right multiplication is needed, to prove the additional results about the measure on the realization.

Definition. A **group chunk** on a measure space $X = (X, S, \mu)$ is a triple $(X; p, i)$, where $p\colon X \times X \rightharpoonup X$ and $i\colon X \rightharpoonup X$ satisfy the following conditions:

(Ia) For every $x \in X$ left multiplication by x ($\lambda_x\colon X \rightharpoonup X$) is a thick X-automorphism,

(Ib) For $y \in \operatorname{dom} i$ right multiplication by y ($\rho_y\colon X \rightharpoonup X$) is a thick X-automorphism.

(II) The inversion map i is a thick X-automorphism,

(III) For every $x \in X$, there is a visible set of z's for which $p(p(x,z), i(z))$ is defined.

(IV) For $x, y, z \in X$, the following identities hold whenever both sides are defined:

$$\begin{aligned} p(p(x,y),z) &= p(x,p(y,z)), \\ p(p(x,z),i(z)) &= x, \\ p(i(z),p(z,x)) &= x. \end{aligned}$$

By what we have showed so far, this is a group chunk on X viewed as a filtered space with the filter generated by its thick sets.

Definition. Let (G', S', μ') be a measure space, where G' is the underlying set of a group. If inversion and left multiplication in G' are measure preserving, then (G', S', μ') is called a **homogeneous measurable group**.

Definition. A **realization** of a group chunk $X = (X; p, i)$ is a realization (G', h') of X viewed as a group chunk with respect to the filter generated by its thick sets, together with an algebra S' and a measure μ' on G' such that (G', S', μ') becomes a homogeneous measurable group and h' a thick measure embedding.

We now state the measurable Group Chunk Theorem. Let $X = (X; p, i)$ be a group chunk on the measure space (X, S, μ) and let (G, h) be as defined in Chapter 1.

Theorem 3.13.

(1) *There is an algebra S_G and a measure μ_G on G such that (G, h) becomes a realization of X.*

(2) *Let (G', h') and (G'', h'') be realizations of X. Then the unique group isomorphism $\gamma\colon G' \to G''$ from Theorem 1.21 is measure preserving.*

To prove part (1) of Theorem 3.13, we start by defining S_G. Let $V = \operatorname{dom} i$; then, by Lemma 1.22,

$$G = h[V] \cdot h[V].$$

Define $\phi_x\colon V \to G$ as in Chapter 2 by $\phi_x(y) = h(x)h(y)$. A set $U \subseteq G$ will be taken to be measurable iff

$$\phi_x^{-1}[U] = \{v \in V : h(x)h(v) \in U\}$$

is measurable for all $x \in V$. Let S_G be the collection of all measurable subsets of G.

Lemma 3.14. *S_G is an algebra on G.*

Proof. Let $U, U' \in S_G$. Then for all $x \in V$ we have

$$\phi_x^{-1}[U \cap U'] \;=\; \phi_x^{-1}[U] \cap \phi_x^{-1}[U'] \;\in\; S.$$

and

$$\phi_x^{-1}[U \setminus U'] \;=\; \phi_x^{-1}[U] \setminus \phi_x^{-1}[U'] \;\in\; S.$$

Thus, $U \cap U' \in S_G$ and $U \setminus U' \in S_G$ for all $U, U' \in S_G$. □

Next we show that the measure of a set $\phi_x^{-1}[U]$ for $U \in S_G$ and $x \in V$ does not depend on the choice of x.

Lemma 3.15. *If U is measurable, then $\mu(\phi_x^{-1}[U]) = \mu(\phi_y^{-1}[U])$ for all $x, y \in V$.*

Proof. Let $A \coloneqq \phi_x^{-1}[U]$ and $B \coloneqq \phi_y^{-1}[U]$. Let $f \coloneqq \lambda_{y^{-1}} \circ \lambda_x$. Then f is a thick X-automorphism, hence $\operatorname{dom} f$ is thick in X. Let $A_1 \coloneqq A \cap \operatorname{dom} f$. Then

$$\mu(A_1) \;=\; \mu(A \cap \operatorname{dom} f) \;=\; \mu(A).$$

We first show that $f[A_1] \subseteq B$. Let $a \in A_1$. Then $h(x)h(a) \in U$ and $f(a) = y^{-1}(xa)$ is defined. To show $f(a) \in B$ we need to show $h(y)h(y^{-1}(xa)) \in U$. But

$$h(y)h(y^{-1}(xa)) \;=\; h(y)h(y)^{-1}h(x)h(a) \;=\; h(x)h(a) \;\in\; U.$$

As f is a thick X-automorphism we have $\mu(f[A_1]) = \mu(A_1) = \mu(A)$. Together with $f[A_1] \subseteq B$ this yields $\mu(A) \leq \mu(B)$.

For the other inequality, let $g = \lambda_{x^{-1}} \circ \lambda_y$ and set $B_1 \coloneqq B \cap \operatorname{dom} g$. With the same reasoning as above we get $\mu(B_1) = \mu(B)$. We show $g[B_1] \subseteq A$. Let $b \in B_1$. Then $h(y)h(b) \in U$ and $g(b) = x^{-1}(yb)$ is defined. We get

$$h(x)h(x^{-1}(yb)) \;=\; h(x)h(x)^{-1}h(y)h(b) \;=\; h(y)h(b) \;\in\; U.$$

Thus $g[B_1] \subseteq A$, and as g is a thick X-automorphism, $\mu(B) \leq \mu(A)$, completing the proof. □

We can now define μ_G by setting $\mu_G(U) = \mu(\phi_x^{-1}[U])$. By Lemma 3.15, this is well-defined, as the choice of $x \in V$ does not matter. To see that μ_G is indeed a measure, note that for disjoint $U_1, U_2 \in S_G$ and $x \in V$ the sets $\phi_x^{-1}[U_1]$, $\phi_x^{-1}[U_2]$ are disjoint, so

$$\mu_G(U_1 \cup U_2) \;=\; \mu(\phi_x^{-1}[U_1] \cup \phi_x^{-1}[U_2]) \;=\; \mu(\phi_x^{-1}[U_1]) + \mu(\phi_x^{-1}[U_2]) \;=\; \mu_G(U_1) + \mu_G(U_2).$$

Lemma 3.16. *Let $x \in V$. Then the set $\phi_x[V]$ is thick in G.*

Proof. $\mu_G(G \setminus \phi_x[V]) = \mu(\phi_x^{-1}[G \setminus \phi_x[V]]) = \mu(\varnothing) = 0$. □

The next step is to show that (G, S_G, μ_G) is a homogeneous measurable group. For this we need a useful auxiliary result.

Lemma 3.17. *Let $x \in V$. Let also $P \subseteq X$ be a measurable property and*

$$A = \{v \in V : P(v)\} = P \cap V.$$

Then the following hold:

(i) $\lambda_x[A] \approx \{v \in V : P(x^{-1}v)\}$,

(ii) $\rho_x[A] \approx \{v \in V : P(vx^{-1})\}$,

(iii) $i[A] = \{v \in V : P(v^{-1})\}$.

Proof. It suffices to prove (i), as (ii) is analogous to (i) and (iii) is clear.

First note that $(V \cap P) \approx P$ and $\lambda_x[V \cap P] \approx \lambda_x[P]$, as V is thick and λ_x is a thick X-automorphism. Furthermore, we claim that $\lambda_x[P] \approx \lambda_{x^{-1}}^{-1}[P]$. Indeed, we have

$$(\lambda_{x^{-1}}^{-1})^{\sim} \;=\; (\lambda_{x^{-1}}^{\sim})^{-1} \;=\; h(x^{-1})^{-1} \;=\; h(x) \;=\; \lambda_x^{\sim}$$

by Lemma 1.20, i.e., $\lambda_{x^{-1}}^{-1}$ and λ_x agree on a large set. This implies that the symmetric difference of $\lambda_x[P]$ and $\lambda_{x^{-1}}^{-1}[P]$ must have measure 0. Thus,

$$\lambda_x[A] \;=\; \lambda_x[V \cap P] \;\approx\; \lambda_x[P] \;\approx\; \lambda_{x^{-1}}^{-1}[P] \;\approx\; V \cap \lambda_{x^{-1}}^{-1}[P] \;=\; \{v \in V : P(x^{-1}v)\}.$$ □

Remark. The relation $\approx$ could be stated in Chapter 1 for filtered spaces but we chose to state it here as it appears in [Lew95], since it is not clear how it applies in Chapter 2.

Lemma 3.18. *(G, S_G, μ_G) is a homogeneous measurable group.*

Proof. We start by showing that inversion in G is measure preserving. For this, we have to prove that for any $U \in S_G$ and $x \in V$ we have $\phi_x^{-1}[U^{-1}] \in S$ and $\mu_G(U^{-1}) = \mu_G(U)$.

So let $x \in V$ and $U \in S_G$. Then

$$\phi_x^{-1}[U] \;=\; \{v \in V : h(x)h(v) \in U\} \;\in\; S$$

and by Lemma 3.17,

$$\begin{aligned}(\rho_{x^{-1}} \circ \lambda_{x^{-1}} \circ i)[\phi_x^{-1}[U]] &\approx \{v \in V : h(x)h(((xv)x)^{-1}) \in U\} \\ &= \{v \in V : h(v)^{-1}h(x)^{-1} \in U\} \\ &= \{v \in V : h(x)h(v) \in U^{-1}\} \\ &= \phi_x^{-1}[U^{-1}].\end{aligned}$$

As $\rho_{x^{-1}} \circ \lambda_{x^{-1}} \circ i$ is a thick X-automorphism, the set $\phi_x^{-1}[U^{-1}]$ is measurable and has the same measure as $\phi_x^{-1}[U]$. Thus, by definition, $U^{-1} \in S_G$ and U^{-1} has the same measure in G as U.

Now we show that for any $g \in G$ the set $g^{-1}U$ is measurable and has the same measure as U. For this let $g \in G$. As $h[V]$ generates X, we may assume $g = h(a)$ for some $a \in V$. Then, again by Lemma 3.17, for $x \in V$ we have

$$\begin{aligned}(\lambda_{x^{-1}} \circ \lambda_{a^{-1}} \circ \lambda_x)[\phi_x^{-1}[U]] &\approx \{v \in V : h(x)h(x^{-1}(a(xv))) \in U\} \\ &= \{v \in V : h(a)h(x)h(v) \in U\} \\ &= \{v \in V : h(x)h(v) \in h(a)^{-1}U\} \\ &= \{v \in V : h(x)h(v) \in g^{-1}U\} \\ &= \phi_x^{-1}[g^{-1}U].\end{aligned}$$

As $\lambda_{x^{-1}} \circ \lambda_a \circ \lambda_x$ is a thick X-automorphism, $\phi_x^{-1}[g^{-1}U]$ is measurable in X, hence $g^{-1}U$ is measurable in G with the same measure as U. □

To finish the proof of part (1) of Theorem 3.13 it remains to show that $h\colon X \to G$ is a thick measure embedding. For this we need to first prove an intermediate result.

Lemma 3.19. *Let $x \in V$. Then $\phi_x\colon V \to G$ is a thick measure embedding.*

Proof. By definition, ϕ_x is injective, measurable and measure preserving. We have to show that $\phi_x^{-1}\colon \phi_x[V] \to V$ is measurable and measure preserving. Let $A \subseteq V$ be measurable. Then we show that for each $y \in V$ the set

$$\begin{aligned}\phi_y^{-1}[\phi_x[A]] &= \{v \in V : h(y)h(v) \in \phi_x[A]\} \\ &= \{v \in V : h(y)h(v) \in h(x)h[A]\}\end{aligned}$$

is measurable and $\mu_G(\phi_x[A]) = \mu(A)$. By Lemma 3.17 we have

$$\begin{aligned}(\lambda_{y^{-1}} \circ \lambda_x)[A] &\approx \{v \in V : x^{-1}(yv) \in A\} \\ &= \{v \in V : h(x^{-1}(yv)) \in h[A]\} \\ &= \{v \in V : h(y)h(v) \in h(x)h[A]\} \\ &= \phi_y^{-1}[\phi_x[A]].\end{aligned}$$

But then, by definition of μ_G,

$$\mu(A) \;=\; \mu((\lambda_{y^{-1}} \circ \lambda_x)[A]) \;=\; \mu(\phi_y^{-1}[\phi_x[A]]) \;=\; \mu_G(\phi_x[A]).$$

This finishes the proof as, by Lemma 3.16, $\phi_x[V]$ is thick in G. □

Lemma 3.20. *The map $h\colon X \to G$ is a thick measure embedding.*

Proof. Let $x \in V$ and let $A := V \cap \lambda_x[V]$. Then for any $a \in A$ there is $v \in V$ such that $a = xv$. We see that

$$h(a) \;=\; h(xv) \;=\; h(x)h(v) \;=\; \phi_x(v) \;=\; (\phi_x \circ \lambda_x^{-1})(a),$$

hence $h|_A = \phi_x \circ \lambda_x^{-1}$ is a thick measure embedding. As A is thick in X, it follows that $h\colon X \to G$ is a thick measure embedding. □

This finishes the proof of Theorem 3.13 (1). For part (2), let (G', h') be an arbitrary realization of X. Note that it suffices to show that the unique isomorphism $\alpha\colon G \to G'$ from Lemma 1.26 is measure preserving. The general statement will then follow by composing two such maps as in the proof of Lemma 1.28.

Lemma 3.21. *Let (G', h') be a realizations of X. Then the unique group isomorphism $\alpha\colon G \to G'$ from Lemma 1.26 is a measure isomorphism.*

Proof. Let $U \subseteq G''$ be measurable. We must check that for $x \in X$ the set $\phi_x^{-1}[\alpha^{-1}[U]]$ is measurable in X. We have

$$\begin{aligned}\phi_x^{-1}[\alpha^{-1}[U]] &= \{v \in V : h(x)h(v) \in \alpha^{-1}[U]\} \\ &= \{v \in V : h'(x)h'(v) \in U\} \\ &= V \cap (h')^{-1}[h'(x^{-1})[U]].\end{aligned}$$

As $h'\colon X \rightharpoonup G'$ is a thick measure embedding it follows that the set $\phi_x^{-1}[\alpha^{-1}[U]]$ is measurable. Denote by $\mu_{G'}$ the measure on G'. Then we have

$$\mu_G(\alpha^{-1}[U]) \;=\; \mu(h^{-1}[\alpha^{-1}[U]]) \;=\; \mu[(h')^{-1}[U]] \;=\; \mu_{G'}(U),$$

hence α is measure preserving. Similarly, it follows that α^{-1} is measure preserving, which shows that $\alpha\colon G \to G'$ is a measure isomorphism. $\square$

4 Existential Matroids

We turn to a model theoretic setting. In various applications, a version of the Group Chunk Theorem gets used to characterize definable groups in a given structure (see Section 0.2). In this chapter we consider definable group chunks with regard to a filter coming from a notion of dimension on a model. A natural framework for this is a fixed monster model with an existential matroid, as introduced in [For11].

We begin by introducing basic properties of pregeometries and existential matroids. Then we define a notion of dimension induced by existential matroids and show how they give rise to a filter on a fixed model. Finally, we state and prove a corresponding Group Chunk Theorem using the results from Chapter 1 as well as ideas from [PPP23]. A key result to show definability of the abstract group will be Corollary 1.30.

4.1 Pregeometries

Before specifying the model theoretic framework, we give a brief introduction to pregeometries on arbitrary sets. We follow [For11], with occasional results from [Elb25], [GL00] and [TZ12, Appendix C.1]. Additional results can be found in these sources.

Convention. In what follows we let M denote a set, let A, B, C, D range over subsets of M and a, b, c, d over elements of M. We will write AB for $A \cup B$, and similarly Aa for $A \cup \{a\}$.

Definition. A map $\mathrm{cl}\colon \mathcal{P}(M) \to \mathcal{P}(M)$ is called a **finitary closure operator** on M, if for all A, B, the following holds:

- **Extension**: $A \subseteq \mathrm{cl}(A)$;
- **Monotonicity**: $A \subseteq B \Rightarrow \mathrm{cl}(A) \subseteq \mathrm{cl}(B)$;
- **Idempotency**: $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$;
- **Finite Character**: $\mathrm{cl}(A) = \bigcup\{\mathrm{cl}(A_0) : A_0 \subseteq A \text{ finite}\}$.

Such a closure operator cl is called a **pregeometry** (or finitary matroid) if it additionally satisfies, for all a, b, A:

- **Exchange Property**: $a \in \mathrm{cl}(Ab) \setminus \mathrm{cl}(A) \Rightarrow b \in \mathrm{cl}(Aa)$.

Remark. Note that finite character implies monotonicity: if $A \subseteq B$, then

$$\mathrm{cl}(A) = \bigcup\{\mathrm{cl}(A_0) : A_0 \subseteq A \text{ finite}\} \subseteq \bigcup\{\mathrm{cl}(B_0) : B_0 \subseteq B \text{ finite}\} = \mathrm{cl}(B).$$

Example 4.1. On any set M we can consider the *trivial pregeometry* cl_0, given by

$$\mathrm{cl}_0(A) := M \quad \text{for all } A.$$

Example 4.2. Let V be a vector space over a division ring K. Then the K-linear span in V defines a pregeometry on V.

Example 4.3. Let K be a field and denote by $\langle A\rangle_K$ the subfield of K generated by A. Then

$$\mathrm{cl}(A) \ := \ \big\{a \in K : a \text{ is algebraic over } \langle A\rangle_K\big\}$$

is a pregeometry on K: Extension, monotonicity and idempotency are clear. To show that cl is finitary, it suffices to note that any $a \in \mathrm{cl}(A)$ is the zero of a non-zero polynomial with finitely many coefficients $a_1, \dots, a_n \in \langle A\rangle_K$ and is therefore algebraic over $\langle\{a_1, \dots, a_n\}\rangle_K$, i.e., in $\mathrm{cl}(\{a_1, \dots, a_n\})$.

To prove exchange, let a be algebraic over $\langle Ab\rangle_K$ but transcendental over $\langle A\rangle_K$. Then there exists a polynomial p with coefficients in $\langle Ab\rangle_K$ such that $p(a) = 0$, where at least one of the coefficients must be in $\langle Ab\rangle_K \setminus \langle A\rangle_K$. This yields a polynomial q with coefficients in $\langle Aa\rangle_K$ such that $q(b) = p(a) = 0$.

Example 4.4. The (model theoretic) algebraic closure acl on a structure $\boldsymbol{M} = (M; \dots)$ is a finitary closure operator on M. Pillay and Steinhorn showed in [PS86] that if $\boldsymbol{M}$ is o-minimal, then acl satisfies exchange, hence is a pregeometry.

In general however, acl must not satisfy exchange: Consider the language $\mathcal{L} = \{E, P\}$ consisting of a binary predicate E and a unary predicate P, and the theory T that states that E is an equivalence relation with infinitely many infinite equivalence classes and $P(x)$ holds for a unique element in each equivalence class. In any model of T, take an arbitrary element b in one of the classes, and let a be the unique element such that $E(a, b)$ and $P(a)$ hold. Then $a \in \mathrm{acl}(b) \setminus \mathrm{acl}(\varnothing)$, but $b \notin \mathrm{acl}(a)$.

Convention. From now on we let cl denote a pregeometry on the set M.

There are two natural ways to obtain a new pregeometry on M with regard to a fixed subset of M. For completeness we state both in the next definition, although only the relativization is needed for later proofs.

Definition. Let $C \subseteq M$ and define

$$\begin{aligned}
&\mathrm{cl}_C \colon \mathcal{P}(M) \to \mathcal{P}(M), && \mathrm{cl}_C(A) = \mathrm{cl}(AC),\\
&\mathrm{cl}^C \colon \mathcal{P}(C) \to \mathcal{P}(C), && \mathrm{cl}^C(B) = \mathrm{cl}(B) \cap C.
\end{aligned}$$

The operator cl_C is called the **relativization** of cl with respect to C and cl^C is called the **restriction** of cl with respect to C.

Lemma 4.5. *Let $C \subseteq M$. Then cl_C is a pregeometry on M.*

Proof. We check that cl_C satisfies the axioms of a pregeometry. We start with extension:

$$A \ \subseteq \ AC \ \subseteq \ \mathrm{cl}(AC) \ = \ \mathrm{cl}_C(A).$$

Next we show monotonicity. Let $A \subseteq B$. Then $AC \subseteq BC$, which gives $\mathrm{cl}(AC) \subseteq \mathrm{cl}(BC)$, i.e., $\mathrm{cl}_C(A) \subseteq \mathrm{cl}_C(B)$.

Idempotency: We have

$$\mathrm{cl}_C(\mathrm{cl}_C(A)) \ = \ \mathrm{cl}_C(\mathrm{cl}(AC)) \ = \ \mathrm{cl}(\mathrm{cl}(AC) \cup C) \ = \ \mathrm{cl}(\mathrm{cl}(AC)) \ = \ \mathrm{cl}(AC) \ = \ \mathrm{cl}_C(A).$$

Finite Character: We see

$$\begin{aligned}
\mathrm{cl}_C(A) \ &= \ \bigcup\{\mathrm{cl}(B_0) : B_0 \subseteq AC \text{ finite}\}\\
&= \ \bigcup\{\mathrm{cl}(A_0C_0) : A_0 \subseteq A \text{ finite}, C_0 \subseteq C \text{ finite}\}\\
&= \ \bigcup\{\mathrm{cl}(A_0C) : A_0 \subseteq A \text{ finite}\}\\
&= \ \bigcup\{\mathrm{cl}_C(A_0) : A_0 \subseteq A \text{ finite}\}.
\end{aligned}$$

Exchange Property: Let $a \in \mathrm{cl}_C(Ab) \setminus \mathrm{cl}_C(A)$. Then $a \in \mathrm{cl}(ACb) \setminus \mathrm{cl}(AC)$. By the exchange property of cl we get $b \in \mathrm{cl}(ACa)$, i.e., $b \in \mathrm{cl}_C(Aa)$. $\square$

An advantage of working with pregeometries is that it allows us to define a notion of independence, similar to the case of vector spaces.

Definition.

(1) A **generates** B over C if $\mathrm{cl}(AC) = \mathrm{cl}(BC)$.

(2) A is **independent** over C if, for every $a \in A, a \notin \mathrm{cl}(C \cup (A \setminus \{a\}))$.

(3) $A_0 \subseteq A$ is a **basis** of A over C if A_0 is independent over C and A_0 generates A.

If $C = \varnothing$, then we say A generates B, A is independent, or A_0 is a basis of A.

Example 4.6. Consider, as in Example 4.2, a vector space over a division ring K and the pregeometry mapping a set of vectors to its K-linear span. Then the notion of independence coming from the pregeometry is precisely the notion of linear independence over K.

Example 4.7. Let K be a field and consider the pregeometry cl from Example 4.3. Then $A \subseteq K$ is independent if it is algebraically independent (over the prime subfield of K): a is transcendental over $\langle A \setminus \{a\}\rangle_K$ for each $a \in A$.

The following two results show that the notion of independence behaves as expected: We can extend independent sets to a basis and moreover, any to bases have the same cardinality. The proofs of these results are from [TZ12, Appendix C.1].

Proposition 4.8. *Let $A' \subseteq A$ be independent over C. Then A' can be extended to a basis of A over C. In particular, any set A has a basis over C.*

Proof. By passing to cl_C if necessary, we may assume $C = \varnothing$. First we show that each set A contains a maximal independent subset. Let $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$ be a chain of independent subsets of A. We claim that $X = \bigcup_{i\in\mathbb{N}} X_i$ is independent. To see this, let $x \in X$. Then there is n_1 such that $x \in X_i$ for all $i \geq n_1$. Assume $x \in \mathrm{cl}(X \setminus \{x\})$. By finite character there exists n_2 such that $x \in \mathrm{cl}(X_j \setminus \{x\})$ for all $j \geq n_2$. Then for $N := \max\{n_1, n_2\}$ we have $x \in X_N$ and $x \in \mathrm{cl}(X_N \setminus \{x\})$, a contradiction to X_N being independent. Thus, there exists a maximal independent subset of A by Zorn's Lemma.

Now let $A' \subseteq A$ be independent. If $a \in A\setminus\mathrm{cl}(A')$, then $A'a$ is again independent. To see this, consider $b \in A'$ arbitrary. Then $b \notin \mathrm{cl}(A'\setminus\{b\})$ and by exchange $b \notin \mathrm{cl}((A'\setminus\{b\})\cup a)$. This implies that for a maximal independent subset $A_0 \subseteq A$ we have $A_0 \subseteq A \subseteq \mathrm{cl}(A_0)$ and therefore $\mathrm{cl}(A) = \mathrm{cl}(A_0)$. □

It follows that if A_0 is a basis of A over C, then A_0 is a maximal independent subset of A over C and a quick argument shows that A_0 is also a minimal generating set of A over C: If A_0 is a basis, then for all $a \in A_0$ we have $a \notin \mathrm{cl}(C \cup (A_0 \setminus \{a\}))$. Hence $\mathrm{cl}(C \cup (A_0 \setminus \{a\})) \neq \mathrm{cl}(A)$.

Lemma 4.9. *All bases of A over C have the same cardinality.*

Proof. Again we may assume $C = \varnothing$. Let $B, D \subseteq A$. We show that if B is independent and D generates A, then $|B| \leq |D|$.

First assume B is infinite. Then we extend B to a basis B' of A. By finite character we can choose for every $d \in D$ a finite subset $B_d \subseteq B'$ with $d \in \mathrm{cl}(B_d)$. We claim that $\bigcup_{d\in D} B_d$ generates A. Indeed, $D \subseteq \bigcup_{d\in D} \mathrm{cl}(B_d) \subseteq \mathrm{cl}(\bigcup_{d\in D} B_d)$ by finite character. Then, by idempotency and monotonicity, we get

$$\mathrm{cl}(\bigcup_{d\in D} B_d) \;\subseteq\; \mathrm{cl}(B') \;=\; \mathrm{cl}(A) \;=\; \mathrm{cl}(D) \;\subseteq\; \mathrm{cl}(\bigcup_{d\in D} B_d).$$

As B' generates A and contains each B_d we get $B' = \bigcup_{d\in D} B_d$. This implies that D is infinite and $|B| \leq |B'| \leq |D|$.

Now assume B is finite. We claim that given $b \in B \setminus D$ there is some $d \in D \setminus B$ such that $B' = (B \setminus \{b\}) \cup \{d\}$ is independent. To see this, let $b \in B \setminus D$. Then $b \in \mathrm{cl}(A) = \mathrm{cl}(D)$, hence $D \not\subseteq \mathrm{cl}(B \setminus \{b\})$. (Otherwise by monotonicity and idempotency $b \in \mathrm{cl}(D) \subseteq \mathrm{cl}(B \setminus \{b\})$, which contradicts independence of B.) Thus, we can choose $d \in D \setminus \mathrm{cl}(B \setminus \{b\})$. Then B' is independent by the exchange property. □

This justifies the following definition of rank of a pregeometry, which later gives rise to a notion of dimension.

Definition. Define the **rank** of A over C by

$$\begin{aligned}\mathrm{rk}^{\mathrm{cl}}(A/C) \;&:=\; \min\{|A_0| : A_0 \subseteq A, \mathrm{cl}(A_0C) = \mathrm{cl}(AC)\}\\ &=\; \text{cardinality of a basis of } A \text{ over } C.\end{aligned}$$

Example 4.10. Let V be a vector space over a division ring K. We have noted in Example 4.6 that the K-linear span is a pregeometry on V. Then for $A \subseteq V$ we have

$$\mathrm{rk}^{\mathrm{span}_K}(A) \;=\; \dim_K(\mathrm{span}_K A).$$

Before we move on and prove the rank formula, an important tool for the remaining chapter, we give some auxiliary results. The first lemma shows how the rank of a pregeometry behaves when passing to the relativization.

Lemma 4.11. *We have* $\mathrm{rk}^{\mathrm{cl}}(A/C) = \mathrm{rk}^{\mathrm{cl}_C}(A)$.

Proof. If A_0 is a basis of A over C with respect to cl, then for all $a \in A_0$,

$$a \;\notin\; \mathrm{cl}(C \cup (A_0 \setminus \{a\})) \;=\; \mathrm{cl}_C(A_0 \setminus \{a\})$$

and

$$\mathrm{cl}_C(A_0) \;=\; \mathrm{cl}(A_0C) \;=\; \mathrm{cl}(AC) \;=\; \mathrm{cl}_C(A).$$

Thus, A_0 is a basis of A with respect to cl_C. □

Lemma 4.12. *Let A_0 be a basis of A over B, and B_0 be a basis of B. Then $A_0 \cap B_0 = \varnothing$ and A_0B_0 is a basis of AB.*

Proof. If $a \in A_0$, then

$$a \ \notin \ \mathrm{cl}(B \cup (A_0 \setminus \{a\})) \ \supseteq \ \mathrm{cl}(B) \ \supseteq \ B \ \supseteq \ B_0.$$

Thus, $A_0 \cap B_0 = \varnothing$.

Next we show that A_0B_0 is independent. Let $a \in A_0B_0$. If $a \in A_0$, then $a \notin B_0$ and, by independence of A_0, we have $a \notin \mathrm{cl}(B \cup (A_0 \setminus \{a\})) = \mathrm{cl}(A_0B_0 \setminus \{a\})$. Let $a \in B_0 \setminus A_0$ and suppose $a \in \mathrm{cl}(A_0 \cup (B_0 \setminus \{a\}))$. As $a \notin \mathrm{cl}(B_0 \setminus \{a\})$, there is a minimal subset $\{a_1, \dots a_n\} \subseteq A_0$ such that $a \in \mathrm{cl}(\{a_1, \dots a_n\} \cup (B_0 \setminus \{a\}))$. Then, by the exchange property, we have $a_1 \in \mathrm{cl}(\{a_2, \dots, a_n\} \cup B_0) \subseteq \mathrm{cl}(B \cup (A_0 \setminus \{a_1\}))$, which contradicts independence of A_0 over B. Thus, A_0B_0 is independent.

Finally, we show that $\mathrm{cl}(A_0B_0) = \mathrm{cl}(AB)$. As B_0 generates B, we have $\mathrm{cl}(B) = \mathrm{cl}(B_0)$. As $A_0 \subseteq \mathrm{cl}(A_0B_0)$ and $\mathrm{cl}(B_0) \subseteq \mathrm{cl}(A_0B_0)$ we get $A_0 \,\mathrm{cl}(B_0) \subseteq \mathrm{cl}(A_0B_0)$, hence, by monotonicity and idempotency, $\mathrm{cl}(A_0 \,\mathrm{cl}\, B_0) \subseteq \mathrm{cl}(A_0B_0)$. Together this yields

$$\mathrm{cl}(AB) \ = \ \mathrm{cl}(A_0B) \ = \ \mathrm{cl}(A_0 \,\mathrm{cl}(B_0)) \ \subseteq \ \mathrm{cl}(A_0B_0).$$

The other direction follows directly from monotonicity. □

We finish this section with the rank formula.

Corollary 4.13. *The following hold for all A, B, C:*

(i) $\mathrm{rk}^{\mathrm{cl}}(AB/C) = \mathrm{rk}^{\mathrm{cl}}(A/BC) + \mathrm{rk}^{\mathrm{cl}}(B/C)$; (Rank Formula)

(ii) *if $C \subseteq B$, then* $\mathrm{rk}^{\mathrm{cl}}(A/B) \leq \mathrm{rk}^{\mathrm{cl}}(A/C)$;

(iii) *if $A \subseteq \mathrm{cl}(BC)$, then* $\mathrm{rk}^{\mathrm{cl}}(A/C) \leq \mathrm{rk}^{\mathrm{cl}}(B/C)$.

Proof. (i) Passing to cl_C if necessary we may assume $C = \varnothing$. Then we have to show for all A, B the equality $\mathrm{rk}^{\mathrm{cl}}(AB) = \mathrm{rk}^{\mathrm{cl}}(A/B) + \mathrm{rk}^{\mathrm{cl}}(B)$. Let A_0 be a basis of A over B and B_0 be a basis of B. By Lemma 4.12,

$$\mathrm{rk}^{\mathrm{cl}}(AB) = |A_0B_0| = |A_0| + |B_0| = \mathrm{rk}^{\mathrm{cl}}(A/B) + \mathrm{rk}^{\mathrm{cl}}(B).$$

(ii) Let $C \subseteq B$ and let $A_0 \subseteq A$ be independent over B. Then A_0 remains independent over C. Thus, $\mathrm{rk}^{\mathrm{cl}}(A/B) \leq \mathrm{rk}^{\mathrm{cl}}(A/C)$.

(iii) Let $A \subseteq \mathrm{cl}(BC)$. Then also $\mathrm{cl}(AC) \subseteq \mathrm{cl}(BC)$, hence $\mathrm{rk}^{\mathrm{cl}}(A/C) \leq \mathrm{rk}^{\mathrm{cl}}(B/C)$. □

Now that we have introduced the relevant background and acquired the necessary tools, we advance to a model theoretic setting.

4.2 Existential matroids

We begin by making our setting precise, and then introduce the notion of existential matroid. The results and definitions in this section are due to Fornasiero [For11]. For an in depth background in model theory we refer to [TZ12] or [ADH17a, Appendix B].

In what follows we fix a one-sorted language $\mathcal{L}$, a complete theory T that admits only infinite models and a monster model $\boldsymbol{M} = (M; \dots)$ of T with respect to a sufficiently large cardinal $\kappa > |\mathcal{L}|$, i.e., $\boldsymbol{M}$ is κ-saturated and every reduct of $\boldsymbol{M}$ is strongly κ-homogeneous.

We let A, B, C, D range over small subsets of M (i.e., of cardinality less than κ) and let a, b, c, d range over finite tuples from M. We write AB for $A \cup B$. If $a = (a_1, \dots, a_n)$ we write Aa for $A \cup \{a_1, \dots a_n\}$ and $a \in \operatorname{acl}(A)$ for $\{a_1, \dots, a_n\} \subseteq \operatorname{acl}(A)$. Furthermore, we let X, Y denote subsets of M_x and M_y respectively, with finite multivariables, $x = (x_1, \dots, x_n)$ and y.

Denote by $\operatorname{Aut}(\boldsymbol{M}/B)$ the set of automorphisms of $\boldsymbol{M}$ that fix B pointwise. We say that $X \subseteq M^n$ is B-invariant if X is $\operatorname{Aut}(\boldsymbol{M}/B)$ invariant. We write $A \equiv_B A'$ if there is some $h \in \operatorname{Aut}(\boldsymbol{M}/B)$ such that $h(A) = A'$. Note that as $\boldsymbol{M}$ is κ-saturated and $|B| < \kappa$ we have $a \equiv_B a'$ if and only if $\operatorname{tp}(a/B) = \operatorname{tp}(a'/B)$.

Definition. Let $\varphi(x, y)$ be an $\mathcal{L}$-formula, where x is a single variable and $y = (y_1, \dots, y_n)$. We say that $\varphi(x, y)$ is **x-narrow** if for every a, b, if $\boldsymbol{M} \models \varphi(a, b)$, then $a \in \operatorname{cl}(b)$. We say that cl is **definable** if for every B,

$$\operatorname{cl}(B) \;=\; \{a \in M : \boldsymbol{M} \models \varphi(a, b) \text{ for some } x\text{-narrow formula } \varphi(x, y), b \in B^n\}.$$

Convention. From now on, fix a definable pregeometry $\operatorname{cl}\colon \mathcal{P}(M) \to \mathcal{P}(M)$ on M.

We show next that definability of the pregeometry allows for compatibility with automorphisms of the monster model.

Lemma 4.14. *Let $h \in \operatorname{Aut}(\boldsymbol{M})$. Then for any B we have $h(\operatorname{cl}(B)) = \operatorname{cl}(h(B))$.*

Proof. First let $a \in \operatorname{cl}(B)$. Then there is an x-narrow formula $\varphi(x, y)$ and $b \in B^n$ such that $\boldsymbol{M} \models \varphi(a, b)$. As $h \in \operatorname{Aut}(\boldsymbol{M})$ we get $\boldsymbol{M} \models \varphi(h(a), h(b))$, hence $h(a) \in \operatorname{cl}(h(B))$ which proves $h(\operatorname{cl}(B)) \subseteq \operatorname{cl}(h(B))$.

For the other direction, let $a \in \mathrm{cl}(h(B))$. Then there is an x-narrow formula $\varphi(x, y)$ and $b \in B^n$ with $\boldsymbol{M} \models \varphi(a, h(b))$. Again, as $h \in \mathrm{Aut}(\boldsymbol{M})$, we get $\boldsymbol{M} \models \varphi(h^{-1}(a), b)$, hence $a = h(h^{-1}(a)) \in h(\mathrm{cl}(B))$. □

Lemma 4.15. *Let* $h \in \mathrm{Aut}(\boldsymbol{M})$. *Then* $\mathrm{rk}^{\mathrm{cl}}(A/C) = \mathrm{rk}^{\mathrm{cl}}(h(A)/h(C))$

Proof. let A_0 be a basis of A over C. We show that then $h(A_0)$ is a basis of $h(A)$ over $h(C)$. We have $\mathrm{cl}(A_0C) = \mathrm{cl}(AC)$. Applying h to both sides yields $\mathrm{cl}(h(A_0)h(C)) = \mathrm{cl}(h(A)h(C))$ by Lemma 4.14, so $h(A_0)$ generates $h(A)$ over $h(C)$.

For $a \in A_0$ we have $a \notin \mathrm{cl}(C \cup (A_0 \setminus \{a\}))$. Then

$$h(a) \ \notin \ h(\mathrm{cl}(C \cup (A_0 \setminus \{a\}))) \ = \ \mathrm{cl}(h(C) \cup (h(A_0) \setminus h(a))),$$

i.e., $h(A)$ is independent over $h(C)$. □

Definition. The pregeometry cl satisfies **existence** if for every a, B, C, if $a \notin \mathrm{cl}(C)$, then there exists $a' \equiv_C a$ such that $a' \notin \mathrm{cl}(BC)$.

A crucial observation, needed for later, is that any definable pregeometry that satisfies existence is an extension of the algebraic closure acl.

Lemma 4.16. *If* cl *satisfies existence, then* $\mathrm{acl}\, B \subseteq \mathrm{cl}\, B$ *for all* $B \subseteq M$.

Proof. If $a \notin \mathrm{cl}(B)$, then, by existence, there is $a' \equiv_B a$ such that $a' \notin \mathrm{cl}(\mathrm{acl}\, B) \supseteq \mathrm{acl}\, B$. Hence, $a \notin \mathrm{acl}\, B$. □

Definition. The **trivial pregeometry** cl^0 is given by $\mathrm{cl}^0(X) = M$ for every $X \subseteq M$.

Definition. We say that cl is an **existential matroid** if cl is a definable pregeometry, satisfies exstence and is non-trivial.

Example 4.17. In monster models of the theory of transseries (see [ADH17a]) the differential algebraic closure is the unique existential matroid. See Proposition 4.45 for details.

Example 4.18. In the asymptotic couple of models of the theory of logarithmic transseries $\mathbb{T}_{\log}$, acl is not a pregeometry, but there are many existential matroids. See [GKP25, Lemma 3.10].

Example 4.19. In general, if acl is a pregeometry, then it is an existential matroid. To see this, let $\boldsymbol{M} = (M; \dots)$ be a monster model of a complete theory T such that acl is a pregeometry on M. We first show that acl is definable. Let $a \in \mathrm{acl}(B)$. Then there exists a formula $\varphi(x, y)$ with $y = (y_1, \dots, y_n)$, $b \in B^n$ and $k \in \mathbb{N}$ such that $\boldsymbol{M} \models \varphi(a, b)$ and $|\varphi(\boldsymbol{M}, b)| < k$. We can then define a new formula

$$\psi(x, y) \coloneqq \begin{cases} \varphi(x, y) \text{ if } |\varphi(\boldsymbol{M}, b)| < k, \\ x \neq x \text{ otherwise.} \end{cases}$$

Then $\psi(x, y)$ is clearly x-narrow and $\boldsymbol{M} \models \psi(a, b)$. As the other inclusion is clearly true we have shown that acl is a definable pregeometry.

To show that acl satisfies existence, suppose $a \notin \operatorname{acl}(C)$. As $|C| < \kappa$ and $\kappa > |\mathcal{L}|$ we have $\operatorname{acl}(C) < \kappa$. As $a \notin \operatorname{acl}(C)$ the type $\operatorname{tp}(a/C)$ has infinitely many, hence, by saturation of $\boldsymbol{M}$, it has at least κ many realizations. We can thus pick $a' \equiv_C a$ avoiding the small set $\operatorname{acl}(BC)$ for any B with $|B| < \kappa$.

4.3 Dimension

We define a notion of dimension coming from an existential matroid. The definition used in this thesis is slightly more general than the one given in [For11], as it doesn't restrict to definable sets. For this wee need an intermediate step, namely dimension with regard to a parameter set. While this definition is internal to the monster model, there is another approach by Ángel and van den Dries, where dimension is defined by computing the rank in a suitably saturated elementary extension [ÁD16]. The two constructions turn out to yield the same dimension in this setting; for a more detailed discussion see Section 4.7.

From now on, we fix an existential matroid cl on M and set $\operatorname{rk} = \operatorname{rk}^{\mathrm{cl}}$. For $X \neq \varnothing$ and a small set A we define the **A-dimension** of X by

$$\dim_A X \ \coloneqq \ \max_{a \in X} \operatorname{rk}(a/A).$$

We set $\dim_A(\varnothing) \coloneqq -\infty$ for all A. Note that for every $A \subseteq B$ we have $\dim_A X \geq \dim_B X$. Now we define the (absolute) **dimension** of X to be

$$\dim X \ \coloneqq \ \min_{\substack{A \subseteq M \\ |A| < \kappa}} \dim_A X.$$

Definition. We call A **sufficient** for X if $\dim_A X = \dim X$.

Remark. Note that if A is sufficient for X and $B \supseteq A$, then B is sufficient for X as well. This follows from $\dim X = \dim_A X \geq \dim_B X \geq \dim X$.

A first goal of this section is to show that if $X \subseteq M_x$ is A-invariant, then A is sufficient for X. The next proposition shows how the dimension of a type relates to its realizations. This in turn yields the Refinement Lemma, which is used to prove the desired result.

Proposition 4.20. *Let $p \in \mathrm{St}_x(A)$. Then $\dim p = \mathrm{rk}(a/A)$ for every realization a of p. In particular, for all a, A, we have $\mathrm{rk}(a/A) = \dim(\mathrm{tp}(a/A))$.*

Proof. Let $a, b \in M_x$ realize p. Then there is an $h \in \mathrm{Aut}(\boldsymbol{M}/A)$ such that $h(a) = b$, hence, by Lemma 4.15, $\mathrm{rk}(a/A) = \mathrm{rk}(h(a)/h(A)) = \mathrm{rk}(b/A)$. It remains to show that A is sufficient for p. For this, let $a = (a_1, \ldots, a_n)$ realize p and let B be an arbitrary small parameter set. We may assume $\dim_A p = \mathrm{rk}(a/A) = n$. Then $a_1, \ldots, a_n$ are independent over A, hence $a_1 \notin \mathrm{cl}(A \cup \{a_2, \ldots, a_n\})$. By existence there is $b_1 \equiv_{A\cup\{a_2,\ldots,a_n\}} a_1$ with

$$b_1 \;\notin\; \mathrm{cl}(AB \cup \{a_2, \ldots a_n\}) \;\supseteq\; \mathrm{cl}(B \cup \{a_2, \ldots a_n\}).$$

Continue by recursively replacing each a_i by b_i such that

- $b_i \equiv_{A\cup\{b_1,\ldots,b_{i-1}\}\cup\{a_{i+1},\ldots,a_n\}} a_i$,
- $b_i \notin \mathrm{cl}(B \cup \{b_1, \ldots, b_{i-1}\} \cup \{a_{i+1}, \ldots, a_n\})$.

Then $b = (b_1, \ldots, b_n) \models p$ and $\dim_B p = \mathrm{rk}(b/B) = n$. As B was arbitrary, we obtain $\dim p = \min_{|B|<\kappa} \dim_B p = \dim_A p$, i.e., A is sufficient for p. $\square$

Corollary 4.21. *Let $p \in \mathrm{St}_x(A)$ and $B \supseteq A$. Then there exists $p' \in \mathrm{St}_x(B)$ such that $\dim p = \dim p'$ and $p \subseteq p'$.*

Proof. Let $a = (a_1, \ldots, a_n)$ be a realization of p. We can assume $\dim p = n$. Replace $a_1, \ldots, a_n$ recursively by $(b_1, \ldots, b_n)$ as in the proof of the proposition above. Then $p = \mathrm{tp}(a/A) = \mathrm{tp}(b/A)$ and for $p' := \mathrm{tp}(b/B)$ we have $\dim p' = n = \dim p$. $\square$

Lemma 4.22. *If X is A-invariant, then A is sufficient for X.*

Proof. We may assume X is nonempty. Choose $a \in X$ such that $\mathrm{rk}(a/A) = \dim_A X$. Set $p := \mathrm{tp}(a/A) \in \mathrm{St}_x(A)$. Let $B \supseteq A$ be arbitrary. Lemma 4.21 yields $p' \in \mathrm{St}_x(B)$ such that $\dim p' = \dim p$ and $p \subseteq p'$. Let $b \in M_x$ realize p'. Then $\mathrm{rk}(b/B) = \mathrm{rk}(a/A)$ and since X is A-invariant, we have $b \in X$ as $\mathrm{tp}(a/A) = \mathrm{tp}(b/A)$. Thus

$$\dim_A X \;=\; \mathrm{rk}(a/A) \;=\; \mathrm{rk}(b/B) \;\leq\; \dim_B X \;\leq\; \dim_A X,$$

so $\dim_A X = \dim_B X$. Since $A^* \mapsto \dim_{A^*} X$ is decreasing, we get $\dim X = \dim_A X$. $\square$

In particular this yields that if X is A-definable, then A is sufficient for X. We finish this section with some basic properties of dimension.

Lemma 4.23. *For any X, Y:*

(i) $\dim M = 1$*;*

(ii) *if* $\dim X > 0$*, then* $|X| \geq \kappa$*;*

(iii) $\dim(X \times Y) = \dim X + \dim Y$*;*

(iv) *if* $X = \bigcup_{i=1}^{n} U_i$ *with* $n \geq 1$*, then* $\dim X = \max \dim U_i$*;*

(v) *if* $f \colon X \to Y$ *is definable, then:*

   (a) *if f is surjective, then* $\dim X \geq \dim Y$*;*

   (b) *if f is injective, then* $\dim X \leq \dim Y$*;*

Proof. (i) Let B be sufficient for M, i.e., $\dim_B M = \dim M$. As cl is non-trivial, there exists some $C \subseteq M$ such that $\mathrm{cl}(C) \neq M$. Let $a \in M \setminus \mathrm{cl}(C)$. Existence yields an $a' \equiv_C a$ such that $a' \notin \mathrm{cl}(BC) \supseteq \mathrm{cl}(B)$. But then

$$\dim M \;=\; \dim_B M \;=\; \sup_{a \in M} \mathrm{rk}(a/B) \;=\; \mathrm{rk}(a'/B) \;=\; 1.$$

(ii) Suppose $|X| < \kappa$. Then there is a small set $A \subseteq M$ such that for every tuple $a \in X$, each coefficient of a is in A. For such an A we have $\dim_A X \leq 0$, thus $\dim X \leq 0$.

(iii) We may assume X and Y are both non-empty. Take A large enough so that it is sufficient for $X \times Y$, X and Y. First, take $(a, b) \in X \times Y$ with $\mathrm{rk}(ab/A) = \dim(X \times Y)$. By the dimension formula,

$$\dim(X \times Y) \;=\; \mathrm{rk}(ab/A) \;=\; \mathrm{rk}(a/Ab) + \mathrm{rk}(b/A) \;\leq\; \dim_{Ab} X + \dim_A Y \;=\; \dim X + \dim Y.$$

Next, take $b \in Y$ with $\dim(b/A) = \dim X$, and then $a \in X$ with $\dim(a/Ab) = \dim X$. Then

$$\dim X + \dim Y \;=\; \mathrm{rk}(a/Ab) + \mathrm{rk}(b/A) \;=\; \mathrm{rk}(ab/A) \;\leq\; \dim_A(X \times Y) \;=\; \dim(X \times Y).$$

(iv) Let A be sufficient for X. Then $\dim X = \dim_A X = \sup_{a \in X} \mathrm{rk}(a/A)$. Any point attaining the maximum in the definition of dimension of X must be contained in one of the U_i, thus the equality holds.

(v) We may assume $X \neq \emptyset$ and take A such that X, Y and $f \colon X \to Y$ are all A-definable. Note that then A is sufficient for X and Y by Lemma 4.22.

For (a), let f be surjective. Take $b \in Y$ such that $\mathrm{rk}(b/A) = \dim Y$, and fix $a \in X$ with $b = f(a)$. Then, by Lemma 4.16, $b \in \mathrm{acl}(Aa) \subseteq \mathrm{cl}(Aa)$, and so by Lemma 4.13 (iii):

$$\dim Y \;=\; \mathrm{rk}(b/A) \;\leq\; \mathrm{rk}(a/A) \;\leq\; \dim_A X \;=\; \dim X.$$

For (b), let f be injective. Take $a \in X$ with $\mathrm{rk}(a/A) = \dim X$. Then $a \in \mathrm{acl}(Ab) \subseteq \mathrm{cl}(Ab)$ and $b \in \mathrm{acl}(Aa) \subseteq \mathrm{cl}(Aa)$. Thus,

$$\dim X \;=\; \mathrm{rk}(a/A) \;=\; \mathrm{rk}(b/A) \;\leq\; \dim_A Y \;=\; \dim Y. \qquad \square$$

Lemma 4.24. *Let A be sufficient for X. Then A is sufficient for X^n for any n.*

Proof. We proceed by induction. Suppose the statement holds for a given $n \geq 1$. We show that then $\dim_A X^{n+1} = \dim X^{n+1}$. As the other direction holds by definition, it suffices to show $\dim_A X^{n+1} \leq \dim X^{n+1}$. By the dimension formula, with x, y ranging over X^n, X, respectively, we have

$$\dim_A X^{n+1} \;=\; \dim_A(X^n \times X) \;=\; \max \mathrm{rk}(xy/A) \;=\; \max\,[\mathrm{rk}(x/A) + \mathrm{rk}(y/Ax)].$$

By the inductive assumption A is sufficient for X^n, hence

$$\dim_A X^{n+1} \;=\; \max\,[\mathrm{rk}(x/A) + \mathrm{rk}(y/Ax)] \;\leq\; \dim X^n + \dim X \;=\; \dim X^{n+1},$$

where the last equality holds by Lemma 4.23 (iii). $\square$

4.4 Generic points

In order to apply the results from Chapter 1 and prove a definable Group Chunk Theorem we need to define a notion of largeness, i.e., a filter associated to the notion of dimension coming from the existential matroid cl. We begin by associating to each set $X \subseteq M_x$ a set of generic points.

Definition. Let A be sufficient for X. Then define the subset of **A-generic points** of X to be:

$$X^A \;:=\; \big\{a \in X : \mathrm{rk}(a/A) = \dim X\big\}.$$

We say a is **A-generic** in X iff $a \in X^A$.

Lemma 4.25. *Let A be sufficient for X.*

(i) *If X is nonempty, then so is X^A.*

(ii) *If $A \subseteq B$, then $X^B \subseteq X^A$.*

Proof. (i) Choose $a \in X$ such that $\mathrm{rk}(a/A) = \dim_A X$. Then $a \in X^A$.

(ii) Let $a \in X^B$. Then

$$\dim X \;=\; \mathrm{rk}(a/B) \;\leq\; \mathrm{rk}(a/A) \;\leq\; \dim_A X \;=\; \dim X,$$

hence $\mathrm{rk}(a/A) = \dim X$ and $a \in X^A$. □

Before we define the filter generated by generic sets we prove two results concerning finite sequences of generic points. Such sequences of sufficient length will be used later as a kind of approximation to stationary sets, to obtain a similar result as in Lemma 1.22.

Lemma 4.26. *Let A be sufficient for X and $t = (t_i)_{i\in I}$ be a sequence from X where I is a finite index set. Then the following are equivalent:*

(i) *t is A-generic in $X^{|I|}$;*

(ii) $\mathrm{rk}(t|_{I_0}/A) = |I_0| \cdot \dim X$ *for every $I_0 \subseteq I$.*

Proof. (ii) $\Rightarrow$ (i) is clear. For (i) $\Rightarrow$ (ii), let t be A-generic in $X^{|I|}$ and let $I_0 \subseteq I$. By the dimension formula and Lemma 4.23 (iii) we get

$$|I| \cdot \dim X \;=\; \dim X^{|I|} \;=\; \mathrm{rk}(t/A) \;=\; \mathrm{rk}(t|_{I_0}/A) + \mathrm{rk}(t|_{(I\setminus I_0)}/At|_{I_0}).$$

Since

$$\mathrm{rk}(t|_{(I\setminus I_0)}/At|_{I_0}) \;\leq\; \mathrm{rk}(t|_{(I\setminus I_0)}/A) \;\leq\; |I \setminus I_0| \cdot \dim_A X \;=\; |I \setminus I_0| \cdot \dim X,$$

we get

$$\mathrm{rk}(t|_{I_0}/A) \geq |I_0| \cdot \dim X = \dim X^{|I_0|}.$$

But as A is sufficient for $X^{|I_0|}$ by Lemma 4.24, this implies $\mathrm{rk}(t|_{I_0}/A) = |I_0| \cdot \dim X$. □

Definition. Let A be sufficient for X. A finite sequence $(t_i)_{i\in I}$ in X which is A-generic as an element of $X^{|I|}$ is called A-**generic**.

Lemma 4.27. *Let A be sufficient for X, $(t_i)_{i\in I}$ be an A-generic sequence with $|I| > \dim X$, and $x \in X$. Then there is $I_0 \subseteq I$ with $|I_0| \leq \dim X$ such that $(t_i)_{i\in I\setminus I_0}$ is Ax-generic.*

Proof. We may assume $\dim X = \dim_A X = n$ and $I = \{1, \ldots, m\}$ for $m > n$. Towards a contradiction, assume the conclusion of the lemma fails. Then there must be at least $n+1$ many components of $(t_i)_{i\in I}$ that are not Ax-generic. Since the order is irrelevant we may suppose $\mathrm{rk}(t_i/Ax) < n$ for $i \in \{1, \ldots, n+1\}$. By Lemma 4.26 we have

$$\begin{aligned} n(n+1) &= \mathrm{rk}(t_1 \cdots t_{n+1}/A) \\ &\leq \mathrm{rk}(t_1 \cdots t_{n+1}x/A) \\ &= \mathrm{rk}(t_1 \cdots t_{n+1}/Ax) \ + \ \mathrm{rk}(x/A) \\ &= \mathrm{rk}(t_1 \cdots t_n/Axt_{n+1}) \ + \ \mathrm{rk}(t_{n+1}/Ax) \ + \ \mathrm{rk}(x/A). \end{aligned}$$

Since $\mathrm{rk}(x/A) \leq n$ and $\mathrm{rk}(t_{n+1}/Ax) < n$, we can use the upper equation to obtain

$$\begin{aligned} n^2 - n + 1 &\leq \mathrm{rk}(t_1 \cdots t_n/Axt_{n+1}) \\ &= \mathrm{rk}(t_1 \cdots t_{n-1}/Axt_nt_{n+1}) \ + \ \mathrm{rk}(t_n/Axt_{n+1}) \\ &\quad \vdots \\ &= \mathrm{rk}(t_1/Axt_2 \cdots t_{n+1}) \ + \ \mathrm{rk}(t_2/Axt_3 \cdots t_{n+1}) \ + \ \cdots \ + \ \mathrm{rk}(t_n/Axt_{n+1}) \\ &\leq \mathrm{rk}(t_1/Ax) \ + \ \cdots \ + \ \mathrm{rk}(t_n/Ax) \\ &\leq n^2 - n, \end{aligned}$$

a contradiction. □

Convention. For the remainder of this section, let X be non-empty.

Lemma 4.28. *Let $X_0 \subseteq X$. Then the following are equivalent:*

(i) *There exists A sufficient for X such that $X^A \subseteq X_0$;*

(ii) *for every A which is sufficient for X and $X \setminus X_0$, $X^A \subseteq X_0$;*

(iii) $\dim(X \setminus X_0) < \dim X$.

Proof. (ii) ⇒ (i) is clear.

(i) ⇒ (iii): Let A be sufficient for X such that $X^A \subseteq X_0$. By enlarging A if necessary, we may assume A is sufficient for $X \setminus X_0$ as well. Then there is $a \in X \setminus X_0$ such that $\mathrm{rk}(a/A) = \dim(X \setminus X_0)$. But then $a \notin X^A$, hence $\dim(X \setminus X_0) = \mathrm{rk}(a/A) < \dim X$.

(iii) ⇒ (ii): Suppose A is sufficient for $X, X \setminus X_0$ and let $a \in X^A$, i.e., $\mathrm{rk}(a/A) = \dim X$. Then $\mathrm{rk}(a/A) > \dim(X \setminus X_0) = \dim_A(X \setminus X_0)$ and so $a \notin X \setminus X_0$. Thus $X^A \subseteq X_0$, i.e., X_0 is large. □

Definition. We say $X_0 \subseteq X$ is **large** (in X) if X_0 satisfies one of the conditions of Lemma 4.28.

Lemma 4.29. *The family of large subsets of X is a filter on X.*

Proof. Clearly the filter axioms (F1) and (F3) hold. It remains to show (F2). Let X_0, X_1 be large in X. Then there are A, B such that $X^A \subseteq X_0$ and $X^B \subseteq X_1$. Then

$$\varnothing \neq X^{AB} \subseteq X^A \cap X^B \subseteq X_0 \cap X_1. \qquad \square$$

The next lemma plays a crucial role in the proof of the Group Chunk Theorem in the existential matroid setting. It characterizes generic bijections with respect to the filter generated by the sets of generic points.

Lemma 4.30. *Let $f\colon X \rightharpoonup X$ be A-definable and injective, such that $f[X]$ and $f^{-1}[X]$ are large. Then f is a generic bijection. Moreover, for A sufficient for X, f restricts to a bijection $X^A \to X^A$.*

Proof. The first statement follows from the second, which we will prove. Choose A sufficiently big such that f is A-definable, $X^A \subseteq f[X]$, and $X^A \subseteq f^{-1}[X]$. Since $X^A \subseteq f^{-1}[X]$, the restriction of the domain of f to X^A is a total function $X^A \to X$ which is injective.

Next, note that for $a \in X^A$, we have $\mathrm{rk}(f(a)/A) = \mathrm{rk}(a/A) = \dim X$, thus $f(a) \in X^A$. In particular, we can further restrict the codomain to X^A.

For surjectivity, choose $b \in X^A$. Since $X^A \subseteq f[X]$, there is $a \in X$ with $f(a) = b$. As f is injective and A-definable, $\mathrm{rk}(a/A) = \mathrm{rk}(f(a)/A) = \dim X$, i.e., $a \in X^A$. $\square$

4.5 A definable Group Chunk Theorem

Let $X \subseteq M_x$. We call $(X; p, i)$ an **A-definable group chunk** if $(X; p, i)$ is a group chunk on X viewed as a filtered space with respect to the filter coming from the dimension (see Lemma 4.29) and the set X as well as the maps $p\colon X \times X \rightharpoonup X$ and $i\colon X \rightharpoonup X$ are all A-definable. Similarly we say a realization (G', h') of $(X; p, i)$ is **A-definable** if the set G', the group operations on G', and the map $h'\colon X \rightharpoonup G'$ are all A-definable.

Fix an A-definable group chunk $X = (X; p, i)$. Note that then, by Lemma 4.22, A is sufficient for X. Furthermore, fix an A-generic sequence $(t_i)_{i \in I}$ with $|I| > 4 \dim X$ and set $T = \{t_i : i \in I\}$.

Theorem 4.31.

(1) *There exists an AT-definable realization $(\tilde{G}, \tilde{h})$ of X with $\operatorname{dom} \tilde{h} = X$.*

(2) *Let (G', h') be a B-definable and (G'', h'') a C-definable realization of X. Then the unique isomorphism $\gamma\colon G' \to G''$ from Theorem 1.21 is ABC-definable.*

We begin by showing part (1) of the theorem. Let (G, h) denote the realization defined in the proof of Theorem 1.21. As a first step we prove a similar result as Lemma 1.22. The idea is that the A-generic sequence approximates a stationary set well enough to get a representation $G = h[X^A] \cdot h[T]$. As T is not actually stationary the proof has some subtle differences, hence it is not clear if it can be directly derived from Lemma 1.22.

Lemma 4.32.

(i) $h[X] \subseteq h[X^A] \cdot h[T]$.

(ii) $h[X^A] \cdot h[T]$ *is a subgroup of* G.

(iii) $h[X^A] \cdot h[T] = G$.

Proof. We use the same argument as in Lemma 1.22. Note that we are in the case $T \subseteq X^A \subseteq \mathrm{dom}(i)$, since i is an A-definable generic bijection and thus restricts to a bijection $X^A \to X^A$ by Lemma 4.30. As in the proof of Lemma 1.22 (iii) follows from (i) and (ii).

(i) Let $x \in X$. Since $\lambda_x \circ i$ is an Ax-definable generic bijection it restricts to a bijection $X^{Ax} \to X^{Ax}$ by Lemma 4.30. By Lemma 4.27 we can pick $t \in (t_i)_{i \in I}$ such that t is Ax-generic. Then

$$h(x) \;=\; h(xt^{-1}) \cdot h(t) \;\in\; h[X^A] \cdot h[T].$$

(ii) Let $x \in X^A$, $t \in (t_i)_{i \in I}$. Pick $s \in (t_i)_{i \in I}$ such that s is Axt-generic. Since the map $\lambda_{t^{-1}} \circ \lambda_{x^{-1}} \circ i$ restricts to a bijection $X^{Axt} \to X^{Axt}$ we have

$$(h(x) \cdot h(t))^{-1} \;=\; h(t^{-1}x^{-1}s^{-1}) \cdot h(s) \;\in\; h[X^A] \cdot h[T].$$

Now let $x, y \in X^A$, $t, s \in (t_i)_{i \in I}$. Choose $r \in (t_i)_{i \in I}$ such that r is $Axyts$-generic. Then, as above, $xtysr^{-1}$ is defined and in X^{Axyts}. Hence

$$h(x) \cdot h(t) \cdot h(y) \cdot h(s) \;=\; h(xtysr^{-1}) \cdot h(r) \;\in\; h[X^A] \cdot h[T]. \qquad \square$$

Next we introduce a definable variant of the relation $\sim$. For $x_1, x_2, y_1, y_2 \in X$, set

$$\begin{aligned}(x_1, x_2) \equiv (y_1, y_2) \;:\iff\; & \exists z \in X \text{ such that } p(x_1, p(x_2, z)) = p(y_1, p(y_2, z)) \\ \iff\; & \lambda_{x_1} \circ \lambda_{x_2} \sim \lambda_{y_1} \circ \lambda_{y_2},\end{aligned}$$

where the last equivalence follows from Corollary 1.30. Now we define the underlying set of the AT-definable group $\tilde{G}$ using a strategy due to Peterzil, Pillay and Point [PPP23]. We first impose an ordering on the index set I. By Lemma 4.32 we can choose for each

$g \in G$ a pair $(x, t_i) \in X^A \times T$ with minimal $i \in I$ such that $g = h(x)h(t_i)$. This allows us to define a map

$$\tilde{\alpha} \colon G \to X \times T, \quad \tilde{\alpha}(g) := (x, t_i) \text{ where } i \in I \text{ is minimal such that } g = h(x)h(t_i).$$

We set $\tilde{G} := \tilde{\alpha}[G]$ and define an operation $\star$ on $\tilde{G}$ via

$$\begin{aligned} \star \colon \tilde{G} \times \tilde{G} \to \tilde{G}, \quad (x, t_i) \star (y, t_j) &:= (z, t_k), \text{ where } (z, t_k) \in \tilde{G} \text{ such that } \exists w \in X \\ &\qquad \text{with } (\lambda_x \circ \lambda_{t_i} \circ \lambda_y \circ \lambda_{t_j})(w) = (\lambda_z \circ \lambda_{t_k})(w). \end{aligned}$$

Again, by Corollary 1.30 we have

$$\begin{aligned} (x, t_i) \star (y, t_j) = (z, t_k) &\iff \lambda_x \circ \lambda_{t_i} \circ \lambda_y \circ \lambda_{t_j} \sim \lambda_z \circ \lambda_{t_k} \\ &\iff h(x)h(t_i)h(y)h(t_j) = h(z)h(t_k). \end{aligned}$$

Thus, $\operatorname{dom} \star = \tilde{G} \times \tilde{G}$ by Lemma 4.32. Furthermore, for each pair $(x, t_i), (y, t_j) \in \tilde{G}$ the expression $(x, t_i) \star (y, t_j)$ is unique: To see this, let $(z, t_k), (z', t_l) \in \tilde{G}$ with $h(z)h(t_k) = h(z')h(t_l)$. By definition of $\tilde{G}$, k and l are minimal in I. Hence $k = l$. But then cancellation in G yields $h(z) = h(z')$ and as h is injective we get $z = z'$.

Lemma 4.33. *$(\tilde{G}, \star)$ is an AT-definable group.*

Proof. We prove that $(\tilde{G}, \star)$ is a group we show that $\tilde{\alpha} \colon G \to \tilde{G}$ is bijective and $\tilde{\alpha}(g \cdot g') = \tilde{\alpha}(g) \star \tilde{\alpha}(g')$ for all $g, g' \in G$. Surjectivity of $\tilde{\alpha}$ is clear. For injectivity let $g, g' \in G$ be such that $\tilde{\alpha}(g) = \tilde{\alpha}(g') = (x, t_i)$. Then by definition we have $g = h(x)h(t_i) = g'$. Next, let $g, g' \in G$. Then tracing the definitions clearly yields $\tilde{\alpha}(g \cdot g') = \tilde{\alpha}(g) \star \tilde{\alpha}(g')$.

To see that the set $\tilde{G}$ is AT-definable, note that, as X and $p \colon X \times X \rightharpoonup X$ are A-definable, the relation $\equiv$ is A-definable and

$$(x, t_i) \in \tilde{G} \iff x \in X \wedge t_i \in T \wedge \forall j < i \, \neg(\exists y \in X : (y, t_j) \equiv (x, t_i)).$$

For the group operation $\star$ we see

$$(x, t_i) \star (y, t_j) = (z, t_k) \iff \exists z \in X : p(x, p(t_i, p(y, p(t_j, z)))) = p(z, p(t_k, z)). \qquad \square$$

Next we define the map $\tilde{h}$ by

$$\tilde{h} \colon X \to \tilde{G}, \quad x \mapsto \tilde{h}(x) := (y, t_j), \text{ where } (y, t_j) \in \tilde{G} \text{ such that } h(x) = h(y)h(t_j)$$

and equip $\tilde{G}$ with the filter induced by the dimension (see Lemma 4.29).

Lemma 4.34. *The map $\tilde{h} \colon X \to \tilde{G}$ is an AT-definable generic bijection.*

Proof. We have $\tilde{h} = \tilde{\alpha} \circ h$, hence $\tilde{h}$ is injective. To see that $\tilde{h}$ is AT-definable note that, by Corollary 1.30, we have

$$\tilde{h}(x) = (y, t_j) \iff \exists z \in X : p(x, z) = p(y, p(t_j, z)).$$

It remains to show that the image and preimage of large sets under $\tilde{h}$ remain large. First note that $\dim \tilde{G} = \dim X$ by Lemma 4.23 (iv), as $\tilde{G}$ is a finite union of sets of the form $X \times \{t_i\}$.

Let $B \supseteq AT$ be sufficient for $\tilde{G}$. We show that then $\tilde{h}^{-1}[\tilde{G}^B] \supseteq X^B$. Let $x \in X^B$. As $\tilde{h}$ is B-definable and injective we get $\dim(X) = \operatorname{rk}(x/B) = \operatorname{rk}(\tilde{h}(x)/B)$, i.e., $\tilde{h}(x) \in \tilde{G}^B$.

Now let $B \supseteq AT$ be sufficient for X. We show that $\tilde{h}[X^B] \supseteq \tilde{G}^B$. Let $(x, t_i) \in \tilde{G}^B$. Then $\operatorname{rk}(x/B) = \dim(X)$, i.e., $x \in X^B$. In particular, x is At_i generic. Applying the dimension formula yields

$$\operatorname{rk}(xt_i/A) \;=\; \operatorname{rk}(x/At_i) + \operatorname{rk}(t_i/A) \;=\; 2 \dim X$$

and as x is also A-generic we get

$$2 \dim X \;=\; \operatorname{rk}(xt_i/A) \;=\; \operatorname{rk}(t_i/Ax) + \operatorname{rk}(x/A) \iff \operatorname{rk}(t_i/Ax) \;=\; \dim X,$$

i.e., $t_i \in X^{Ax}$. By Lemma 4.30 λ_x restricts to a bijection $X^{Ax} \to X^{Ax}$, hence $p(x, t_i)$ is defined and by definition of $\tilde{h}$ we get $\tilde{h}(p(x, t_i)) = (x, t_i)$, i.e., $(x, t_i) \in \tilde{h}[X^B]$. □

Denote by $\operatorname{inv} \colon \tilde{G} \to \tilde{G}$ the inversion in the group $(\tilde{G}, \star)$.

Lemma 4.35.

(i) *For all* $(x, y) \in \operatorname{dom} p$, $\tilde{h}(p(x, y)) = \tilde{h}(x) \star \tilde{h}(y)$.

(ii) *For all* $x \in \operatorname{dom} i$, $\tilde{h}(i(x)) = \operatorname{inv}(\tilde{h}(x))$.

Proof. (i) can be easily seen by tracing the definitions and (ii) is a consequence of (i) and Lemma 1.20. □

Lemma 4.36. $(\tilde{G}; \star, \operatorname{inv})$ *is a group chunk.*

Proof. We need to show that inversion and left multiplication for any $g \in \tilde{G}$ are generic bijections. Injectivity is clear for both maps. We start with inversion. Let G_0 be large in $\tilde{G}$. Then $G_0 \supseteq \tilde{h}[X_0]$ for some X_0 large in X, hence $\operatorname{inv}[G_0] \supseteq \operatorname{inv}[\tilde{h}[X_0]]$. By further restricting X_0 if necessary we may assume $X_0 \subseteq \operatorname{dom} i$. We show that $\operatorname{inv}[\tilde{h}[X_0]] \supseteq \tilde{h}[i[X_0]]$, then we are done by upward closedness. Let $(y, t_j) \in \tilde{h}[i[X_0]]$. Then $(y, t_j) =$

$\tilde{h}(i(x))$ for some $x \in X_0$. Thus,

$$(y, t_j) \;=\; \tilde{h}(i(x)) \;=\; \operatorname{inv}(\tilde{h}(x)) \;\in\; \operatorname{inv}[\tilde{h}[X_0]].$$

Next we show that for all $(y, t_j) \in \tilde{G}$ left multiplication by (y, t_j) is a generic bijection. Again, let $G_0 \supseteq \tilde{h}[X_0]$ be large in $\tilde{G}$ for some X_0 large in X. Then

$$(y, t_j) \star G_0 \;\supseteq\; (y, t_j) \star \tilde{h}[X_0].$$

We show

$$(y, t_j) \star \tilde{h}[X_0] \;\supseteq\; \tilde{h}[(\lambda_y \circ \lambda_{t_j})[X_0]].$$

Let $(z, t_k) \in \tilde{h}[(\lambda_y \circ \lambda_{t_j})[X_0]]$. Then there is $x \in X_0$ such that

$$(z, t_k) \;=\; \tilde{h}((\lambda_y \circ \lambda_{t_j})(x)) \;=\; \tilde{h}(p(y, p(t_j, x))) \;=\; \tilde{h}(y) \star \tilde{h}(t_j) \star \tilde{h}(x) \;=\; (y, t_j) \star \tilde{h}(x). \quad \square$$

The previous results put together show that $(\tilde{G}, \tilde{h})$ is an AT-definable realization of X and as $\operatorname{dom} \tilde{h} = X$ we have finished the proof of part (1) of Theorem 4.31. It remains to prove part (2) of the Theorem. This is established in the following lemma.

Lemma 4.37. *Let (G', h') be a B-definable and (G'', h'') a C-definable realization of X. Then the unique isomorphism $\gamma\colon G' \to G''$ from Theorem 1.21 is ABC-definable.*

Proof. This is an immediate consequence of Proposition 1.29. We can define γ by

$$\gamma(g') = g'' \iff \exists x, y \in X : g' = h'(x)h'(y) \wedge g'' = h''(x)h''(y). \qquad \square$$

4.6 From dimension to matroid

In this section we show how to obtain an existential matroid from a dimension function on $\boldsymbol{M}$. This can be useful in applications of the definable Group Chunk Theorem, as in certain structures there might already be a known dimension function, whereas proving that the pregeometry in question satisfies exchange might not be straight forward. As an example we show that in monster models of the theory of transseries there is a unique existential matroid.

The result of passing from a dimension function to an existential matroid is essentially [For11, Theorem 4.1]. However, to obtain wider applicability, we use the slightly more general notion of dimension function from [Dri89]. For a set $T \subseteq M^{m+n}$

and $a \in M^m$, $b \in M^n$ we introduce the notations

$$\begin{aligned} T_a &= \{y \in M^n : (a, y) \in T\}, \\ T^b &= \{x \in M^m : (x, b) \in T\}. \end{aligned}$$

Definition. A map d that associates to any $S \subseteq M^m$ a value $d(S) \in \{-\infty, 0, 1, \ldots, m\}$ is called a **dimension function** on $\boldsymbol{M}$, if the following hold for all m and $S, S_1, S_2 \subseteq M^m$:

(D1) $d(S) = -\infty$ iff $S = \varnothing$, $d(\{a\}) = 0$ for every $a \in M$ and $d(M) = 1$;

(D2) $d(S_1 \cup S_2) = \max\{d(S_1), d(S_2)\}$;

(D3) $d(S^\sigma) = d(S)$ for every permutation σ of the coordinates of M^m;

(D4) Let $T \subseteq M^{m+1}$ be definable and for $i = 0, 1$ let $T(i) := \{x \in M^m : d(T_x) = i\}$. Then $T(i)$ is definable and

$$d(\{(x, y) : x \in T(i)\}) = d(T(i)) + i,$$

for $i = 0, 1$.

The difference to Definition 4.1 in [For11] is axiom (D4), where it is required that the sets $T(i)$ are definable with the same parameters as T. We will call this a reasonable dimension function and prove that one can make a dimension function reasonable by adding only a small amount of constant symbols to the language.

Definition. A dimension function d on $\mathbf{M}$ is called a **reasonable dimension function** on $\boldsymbol{M}$, if for every definable $T \subseteq M^{m+1}$ and $i = 1, 2$, the set $T(i)$ is definable *with the same parameters* as T.

Lemma 4.38. *Suppose d is a dimension function on $\boldsymbol{M}$. The following are equivalent for $i = 1, 2$:*

(i) *Given A-definable $T \subseteq M^{m+1}$, each $T(i)$ is A-definable;*

(ii) *Given $\varnothing$-definable $T \subseteq M^{m+1}$, each $T(i)$ is $\varnothing$-definable.*

Proof. (i) $\Rightarrow$ (ii) is clear, it remains to prove (ii) $\Rightarrow$ (i). Let $T \subseteq M^{m+1}$ be A-definable, defined by the $\mathcal{L}$-formula $\varphi(x, y, z)$, where $|x| = l$, $|y| = m$ and $|z| = 1$, i.e., $T = \varphi(a, \boldsymbol{M})$ for some $a \in A^l$. Then the set $S = \varphi^{\boldsymbol{M}}$ is $\varnothing$-definable, hence $S(i)$ is $\varnothing$-definable for $i = 1, 2$ and $T = S_a$. But then for $i = 1, 2$ we have

$$T(i) = S_a(i) = \{b \in M^m : d(S_{(a,b)}) = i\} = \{b \in M^m : (a, b) \in S(i)\} = S(i)_a.$$

Thus, as $S(i) = \psi_i^{\boldsymbol{M}}$ for an $\mathcal{L}$-formula $\psi_i(x, y, z)$, we have $T(i) = \psi_i(a, \boldsymbol{M})$, i.e., $T(i)$ is A-definable. $\square$

Lemma 4.39. *If d is a dimension function on $\boldsymbol{M}$, then there exists $A \subseteq M$ with $|A| \leq \mathcal{L}$ such that d is a reasonable dimension function on $\boldsymbol{M}_A$.*

Proof. For each $\mathcal{L}$-formula $\varphi(x, y)$ with x a finite variable and y a single variable, let $a_{\varphi(x,y)}$ be a finite tuple of parameters which defines $\varphi^{\boldsymbol{M}}(0)$ and $\varphi^{\boldsymbol{M}}(1)$. Let A be the collection of all these parameters. Then this set A works by Lemma 4.38. $\square$

We now define the existential matroid coming from a reasonable dimension as in [For11] and prove that it satisfies the desired properties.

Definition. Given a dimension function d on $\boldsymbol{M}$ we define an operator $\mathrm{cl}^d \colon M \to \mathcal{P}(M)$ as follows:

$$\mathrm{cl}^d(A) := \bigcup \big\{X : X \subseteq M \text{ is } A\text{-definable with } d(X) = 0\big\}.$$

Theorem 4.40. *Let d be a reasonable dimension function on $\boldsymbol{M}$. Then cl^d is an existential matroid and the dimension induced by cl^d as in Section 4.3 agrees with d on the definable subsets of M^m for each m.*

To prove the theorem we start with an auxiliary lemma characterizing the x-narrow formulas via their dimension. For what follows, we fix a reasonable dimension function d on $\boldsymbol{M}$.

Lemma 4.41. *Let $\varphi(x, y)$ be an $\mathcal{L}$-formula, x a single variable, y finite. Then $\varphi(x, y)$ is x-narrow iff $d(\varphi(\boldsymbol{M}, b)) = 0$ for all $b \in M^{|y|}$.*

Proof. Let $\varphi(x, y)$ be x-narrow and assume that there is a $b \in M^{|y|}$ with $d(\varphi(\boldsymbol{M}, b)) = 1$. Set

$$\Psi(x) = \{\psi(x, b) : \psi(x, y) \text{ is an } \mathcal{L}\text{-formula such that } d(\psi(\boldsymbol{M}, b)) = 0\}.$$

Then for all $\psi(x, b) \in \Psi(x)$ we have $\varphi(\boldsymbol{M}, b) \cap M \setminus \psi(\boldsymbol{M}, b) \neq \varnothing$, hence there exists $a \in M$ such that $a \notin \mathrm{cl}^d(b)$ but $\boldsymbol{M} \models \varphi(a, b)$, a contradiction to $\varphi(x, y)$ being x-narrow.

For the other direction, assume $\varphi(x, y)$ is as in the statement of the lemma such that $d(\varphi(\boldsymbol{M}, b)) = 0$ for every $b \in M^{|y|}$. Then $\varphi(x, y)$ is clearly x-narrow: If $\boldsymbol{M} \models \varphi(a, b)$ for some $a \in M, b \in M^{|y|}$, then a is contained in the b-definable set $\varphi(\boldsymbol{M}, b)$ of dimension 0, hence $a \in \mathrm{cl}^d(b)$ by definition of cl^d. $\square$

Lemma 4.42. *The operator cl^d is an existential matroid.*

Proof. Extension, monotonicity, idempotency and finite character are clear. It remains to prove that cl^d satisfies the exchange property and existence and is definable.

To prove that cl^d satisfies exchange, let $a \in \mathrm{cl}(Ab) \setminus \mathrm{cl}(A)$. Suppose $b \notin \mathrm{cl}(Aa)$. Let $X \subseteq M^2$ be A-definable such that $a \in X_b$ and $d(X_b) = 0$. Let $X' = X \cap (X(0) \times M)$.

By assumption, $(b, a) \in X'$ and, by (D4), we have $d(X') = d(X(0)) \leq 1$. W.l.o.g. we may assume $X = X'$. Let $Z := \{c \in M : d(X^c) = 1\}$. Then Z is A-definable. As $b \in X^a$ and $b \notin \mathrm{cl}(Aa)$ we have $a \in Z$ and as $a \notin \mathrm{cl}(A)$ we have $d(Z) = 1$. Thus, by (D3) (applied to the permutation $\sigma = (12)$), (D2) and (D4), we have

$$2 \;=\; 1 + d(Z) \;=\; 1 + d(X^{(12)}(1)) \;=\; d(X^{(12)} \cap (X^{(12)}(1) \times M)) \;\leq\; d(X^{(12)}) \;=\; d(X),$$

a contradiction.

Now we show that cl^d is definable. Let $a \in \mathrm{cl}^d(B)$ and $X \subseteq M$ be B-definable such that $d(X) = 0$ and $a \in X$. Let $\varphi(x, y)$ define X, i.e., $\varphi(\boldsymbol{M}, b) = X$ for some $b \in B^{|y|}$. By (D4) we may assume w.l.o.g. that $d(\varphi(\boldsymbol{M}, c)) \leq 0$ for all $c \in M^{|y|}$: Indeed, if this is not already the case, we may replace φ by the $\mathcal{L}$-formula

$$\tilde{\varphi}(x, y) := \begin{cases} \varphi(x, y), & \text{if } c \notin \varphi^{\boldsymbol{M}}(1), \\ \varnothing, & \text{if } c \in \varphi^{\boldsymbol{M}}(1). \end{cases}$$

But then $\varphi(x, y)$ is x-narrow. It follows that cl^d is definable.

Next we show that cl^d satisfies existence. Let B, C, a be given such that $a \notin \mathrm{cl}^d(C)$. We may assume $B \supseteq C$ and have to show there exists $a' \equiv_C a$ such that $a' \notin \mathrm{cl}^d(B)$, i.e., $\mathrm{tp}(a'/C) \;=\; \mathrm{tp}(a/C) \cup \Phi_B$, where

$$\Phi_B := \{\neg\varphi(x, b) : \varphi(x, y) \text{ is } x\text{-narrow and } b \in B^{|y|}\}.$$

As both of these sets are closed under finite intersections, it is enough to show that for each $\varphi_1 \in \mathrm{tp}(a/C)$ and $\varphi_2 \in \Phi_B$ we have $\varphi_1^{\boldsymbol{M}} \cap \varphi_2^{\boldsymbol{M}} \neq \varnothing$, then the statement follows by compactness. So let $\varphi_1 \in \mathrm{tp}(a/C)$ and $\varphi_2 \in \Phi_B$. As $a \notin \mathrm{cl}^d(C)$ we have $d(\varphi_1^{\boldsymbol{M}}) = 1$. Furthermore, as $\varphi_2(x) = \neg\varphi(x, b)$ for some x-narrow $\varphi(x, y)$ and $b \in B^{|y|}$ we have

$$d(M \setminus \varphi_2^{\boldsymbol{M}}) = d(\varphi(\boldsymbol{M}, b)) = 0$$

by Lemma 4.41. Thus, we have $\varphi_1^{\boldsymbol{M}} \cap \varphi_2^{\boldsymbol{M}} \neq \varnothing$, as otherwise we have $\varphi_1^{\boldsymbol{M}} \subseteq M \setminus \varphi_2^{\boldsymbol{M}}$ which would contradict $d(\varphi_1^{\boldsymbol{M}}) = 1$. □

To finish the proof of Theorem 4.40 we show that d agrees with the dimension coming from cl^d, defined as in Section 4.3. We shall denote this dimension by dim.

Lemma 4.43. *For all m and definable $X \subseteq M^m$ we have* $\dim X = d(X)$.

Proof. We prove the result for $m = 1$ as the general case easily follows. It furthermore suffices to consider the case of dimension 0. Let $X \subseteq M$ be B-definable with $d(X) = 0$.

Then, by Lemma 4.41, we can assume $X = \varphi(\boldsymbol{M}, b)$ for some x-narrow $\mathcal{L}$-formula $\varphi(x, y)$ and $b \in B^{|y|}$. By definition, we have for all $a \in \mathrm{cl}^d(B)$ for all $a \in X$. Hence, $\mathrm{rk}(a/B) = 0$ for all $a \in X$, which gives

$$\dim X \;=\; \sup_{a \in X} \mathrm{rk}(a/B) \;=\; 0.$$

Now let $X \subseteq M$ be B-definable such that $\dim X = 0$. Then for all $a \in X$ we have $\mathrm{rk}(a/B) = 0$, i.e., $a \in \mathrm{cl}^d(B)$. But then, by Lemma 4.41, we can cover X using all x-arrow formulas,

$$X \;\subseteq\; \bigcup_{\substack{\psi(x,y)\ x\text{-narrow} \\ b \in B^{|y|}}} \psi(\boldsymbol{M}, b).$$

Using compactness we obtain a finite amount of x-narrow $\mathcal{L}$-formulas $\psi_1(x, y_1), \dots, \psi_n(x, y_n)$ and $b_i \in B^{|y_i|}$ for $i = 1, \dots, n$ such that

$$X \;\subseteq\; \bigcup_{i=1}^{n} \psi_i(\boldsymbol{M}, b_i),$$

which gives $d(X) = 0$ by inductively applying (D2). $\square$

This concludes the proof of the theorem. We can now use the results from [ADH25], [ADH17b] and [Dri89], together with [For11], to obtain an existential matroid in models of the complete theory of the field of transseries $\mathbb{T}$, which is shown in [ADH17a] to be the theory of Liouville closed ω-free newtonian H-fields.

Proposition 4.44. *In monster models of the complete theory of $\mathbb{T}$ the differential algebraic dimension is a reasonable dimension function.*

Proof. Let K be a Liouville closed ω-free newtonian H-field. It follows from [Dri89, 2.25] and [ADH17b, §3] that the differential algebraic dimension dim on K is a dimension function. It is also shown in [ADH17b, §4] that for any definable $S \subseteq K^n$,

$$\dim S = 0 \iff S \text{ is discrete.}$$

This gives a definable characterization of dimension 0 sets, hence the dimension function is reasonable. $\square$

Proposition 4.45. *In monster models of the complete theory of of $\mathbb{T}$ the differential algebraic closure is the unique existential matroid.*

Proof. It follows from [ADH25, Proposition 5.2] together with [For11, Lemma 3.23] that the differenital algebraic closure is an existential matroid. As the theory of Liouville closed ω-free newtonian H-fields extends the theory of rings without zero divisors we can apply [For11, Theorem 3.48] to get uniqueness. $\square$

4.7 Comparing dimensions

In [ÁD16], Ángel and van den Dries use a different approach to define dimension coming from a pregeometry. Instead of taking the minimum over all small parameter sets they pass to a sufficiently saturated elementary extension and compute the rank there. We show that the two notions of dimension align. We will refer to results from [ÁD16] as indicated without repeating the proofs.

We begin by defining dimension as in [ÁD16]. Throughout we fix a one-sorted language $\mathcal{L}$. Let $x, x_1, x_2, x_3, \dots, y_1, y_2, \dots$ be distinct variables and let

$$\Phi \ \coloneqq \ (\varphi_i(x, y_1, \dots, y_{n_i}))_{i \in I}$$

be a family of $\mathcal{L}$-formulas. For a given $\mathcal{L}$-structure $\boldsymbol{M} = (M; \dots)$, we associate to each parameter set $B \subseteq M$ its **Φ-image**

$$\Phi(B) \ \coloneqq \ \{a \in M : \boldsymbol{M} \models \varphi_i(a, b) \text{ for some } i \in I \text{ and } b \in B^{n_i}\}.$$

Next, let Σ be a set of $\mathcal{L}$-sentences. We say that (Φ, Σ) **defines a pregeometry** if for every model $\boldsymbol{M} = (M; \dots)$ of Σ, the operation $A \mapsto \Phi(A) \colon \mathcal{P}(M) \to \mathcal{P}(M)$ is a pregeometry on M, to be called the (Φ, Σ)-pregeometry of $\boldsymbol{M}$.

Assume that (Φ, Σ) defines a pregeometry. For $\boldsymbol{M} = (M; \dots) \models \Sigma$ and $A, B \subseteq M$ set

$$\operatorname{rk}(A/B) \ \coloneqq \ \operatorname{rk}^{(\Phi, \Sigma)}(A/B).$$

As mentioned in [ÁD16] the following result can be shown using a routine induction.

Lemma 4.46. *Let (Φ, Σ) define a pregeometry and $\boldsymbol{M} = (M; \dots) \models \Sigma$. Then for each $B \subseteq M$ and $m \geq 1$ there exists a set*

$$I_m \ = \ I_m(x_1, \dots, x_m, y_1, y_2 \dots)$$

of $\mathcal{L}$-formulas $\psi(x_1, \dots, x_m, y_1, y_2 \dots)$ such that for all $a_1, \dots, a_m \in M$,

$$a_1, \dots, a_m \text{ are } (\Phi, \Sigma)\text{-independent over } B \iff (a_1, \dots, a_m) \text{ realizes the partial type } I_m(x_1, \dots, x_m, B),$$

where $I_m(x_1, \dots, x_m, B)$ consists of the formulas $\psi(x_1, \dots, x_m, b_1, b_2, \dots)$ such that

$\psi(x_1, \dots, x_m, y_1, y_2, \dots) \in I_m$ and $b_1, b_2, \dots \in B$. Moreover, I_m can be taken to depend only on $(\mathcal{L}, \Phi, m)$, not on $(\Sigma, \boldsymbol{M}, B)$.

Let now $\boldsymbol{M} = (M; \dots) \models \Sigma$ and $X \subseteq M^n$. We define the **external dimension of** X, $\dim' X \in \{-\infty, 0, 1, \dots, n\}$ as follows. If $X \neq \varnothing$, take a $\kappa > |M|$ and a κ-saturated elementary extension $(\boldsymbol{M}^*, X^*) = (M^*; \dots, X^*)$ of the $\mathcal{L}_n$-structure $(\boldsymbol{M}, X)$, where $\mathcal{L}_n$ is $\mathcal{L}$, augmented by an n-ary relation symbol as a name for X. We set

$$\dim' X \;\coloneqq\; \max\{\mathrm{rk}(Ma/M) : a \in X^*\},$$

where the rank is computed with respect to the (Φ, Σ)-pregeometry of $\boldsymbol{M}^*$. If $X = \varnothing$ we set $\dim' X \coloneqq -\infty$.

Remark. By Lemma 4.46, $\dim' X$ does not depend on the choice of the (sufficiently saturated) elementary extension $(\boldsymbol{M}^*, X^*)$ of $(\boldsymbol{M}, X)$.

The next result is [ÁD16, Lemma 2.2].

Lemma 4.47. *Suppose that $X \subseteq M^n$ and $(\boldsymbol{M}, X) \preceq (\boldsymbol{M}^*, X^*)$. Then*

$$\dim'_{\boldsymbol{M}} X \;=\; \dim'_{\boldsymbol{M}^*}(X^*).$$

Finally, we show that the two definitions of dimension coming from a pregeometry are equivalent.

Proposition 4.48. *Let $\boldsymbol{M} \models \Sigma$ be κ-saturated for some regular cardinal $\kappa > |\mathcal{L}|$ and $X \subseteq M$. Then*

$$\dim' X \;=\; \min_{\substack{A \subseteq M \\ |A| < \kappa}} \dim_A \mathrm{rk}(a/A) \;=\; \dim X.$$

Proof. Let A be sufficient for X, i.e.,

$$\dim_A X \;=\; \min_{\substack{B \subseteq M \\ |B| < \kappa}} \dim_B X \;=\; \dim X.$$

Extend $\boldsymbol{M}$ to an $\mathcal{L}_n$-structure $(\boldsymbol{M}, X)$, where $\mathcal{L}_n$ is $\mathcal{L}$, augmented by an n-ary relation symbol P as a name for X, i.e., $P^{\boldsymbol{M}} = X$.

Next, apply the downward Löwenheim-Skolem Theorem to obtain an elementary substructure $(\boldsymbol{N}, P^{\boldsymbol{N}})$ of $(\boldsymbol{M}, X)$ containing A such that $|N| < \kappa$. By Lemma 4.47:

$$\dim' X \;=\; \dim'_{\boldsymbol{M}} X \;=\; \dim'_{\boldsymbol{N}} P^{\boldsymbol{N}} \;=\; \max\{\mathrm{rk}(Na/N) : a \in X\} \;=\; \dim_N X \;=\; \dim_A X,$$

where the last equality holds because $|N| < \kappa$ and $A \subseteq N$ implies $\dim_N X \leq \dim_A X$. $\square$

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