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When the Bellman Equation Cannot Be Solved Analytically

Verfasser

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I hereby certify that the work reported in this thesis is my own, and the work done by other authors is appropriately cited.

Caban Sebastian

Sebastian Caban
Vienna, January 28, 2010

Abstract

Many questions in managerial decision making imply —if uncertainty is involved— stochastic optimization problems of the form

$$V(S) = \max_x \mathrm{E}\left\{\int_0^\infty e^{-\rho t} f(S,x)\,\mathrm{d}t\right\}$$

where the state transition $\mathrm{d}S = g(S,x)\,\mathrm{d}t + \sigma(S)\,\mathrm{d}z$ describes the underlying stochastic process S as an Itō process and $\mathrm{d}z$ is a Brownian motion.

For very simple problems, results can be obtained analytically by solving the corresponding Bellman equation. In all other cases, $V(S)$ has to be computed numerically. Unfortunately, conventional numerical methods are either slow, or completely fail as the solution is a saddlepoint path so that all neighboring solution paths diverge.

This thesis applies methods developed in “Applied Computational Economics and Finance” by M.J. Miranda and P.L. Fackler to solve selected problems that arise in modeling managerial decision making, optimal pricing of nondurables, modeling the ups and downs in oil prices, and modeling the impact of CO_2 emission into the atmosphere.

Kurzfassung

Viele Fragen der betrieblichen Entscheidungsfindung führen —wenn Unsicherheit involviert ist— zu stochastischen Optimierungsproblemen der Form

$$V(S) = \max_x \mathrm{E}\left\{\int_0^\infty e^{-\rho t} f(S,x)\,\mathrm{d}t\right\}$$

wobei der Zustandsübergang $\mathrm{d}S = g(S,x)\,\mathrm{d}t + \sigma(S)\,\mathrm{d}z$ den zugrunde liegenden stochastischen Prozess S als Itō Prozess beschreibt und $\mathrm{d}z$ eine Brownsche Bewegung ist.

Für sehr einfache Probleme kann durch Lösung der dazugehörigen Bellman Gleichung eine analytische Lösung berechnet werden. In allen anderen Fällen ist dies jedoch mittels numerischer Methoden möglich. Konventionelle numerische Lösungsmethoden sind allerdings langsam oder teilweise unbrauchbar, weil die Lösung einen Sattelpunktpfad beschreibt und alle benachbarten Pfade divergieren.

Diese Magisterarbeit verwendet die in “Applied Computational Economics and Finance” von M.J. Miranda and P.L. Fackler entwickelten numerischen Methoden, um ausgewählte Probleme der Modellierung der betrieblichen Entscheidungsfindung, der optimalen Preisfestlegung von kurzlebigen Konsumgütern, der Modellierung von Schwankungen im Ölpreis und der Modellierung der Auswirkungen von CO_2 in der Atmosphäre zu lösen.

Contents

1 Introduction

Continuous time optimization problems of the form

$$V(S) = \max_x \mathrm{E}\left\{\int_0^\infty e^{-\rho t} f(S,x)\,\mathrm{d}t\right\} \tag{1.1}$$

occur very often in modern economics. If in addition a state transition as for example

$$\mathrm{d}S = g(S,x)\,\mathrm{d}t + \sigma(S)\,\mathrm{d}z \tag{1.2}$$

describes the underlying stochastic process S as an Itō process[1] (where $\mathrm{d}z$ is a Brownian motion[2]), we obtain a "continuous time stochastic control problem".

In simple cases, an analytical solution of $V(S)$ can be obtained by solving the corresponding Bellman equation as shown in the next section. Unfortunately, not many problems admit a closed form solution. Especially if the optimal strategy is nonlinear, $V(S)$ has to be computed numerically as shown in the next but one section. The remaining two sections of this introduction are devoted to simple examples that illustrate the power of the described analytical and numerical solution methods.

[1] The Itō process, a generalized Wiener process, is named after the Japanese mathematician Itō Kiyoshi (伊藤 清, 1915–2008). See e.g. [1] and [2] for more information.

[2] The Brownian motion, often referred to as Wiener process, is named after Scottish botanist Robert Brown (1773–1858). See e.g. [3] for further information.

1.1 How to Obtain An Analytical Solution?

The first step is to convert[3] the stochastic optimization problem

$$V(S) = \max_x \mathrm{E}\left\{\int_0^\infty e^{-\rho t} f(S,x)\,\mathrm{d}t\right\} \tag{1.1}$$

$$\mathrm{d}S = g(S,x)\,\mathrm{d}t + \sigma(S)\,\mathrm{d}z \tag{1.2}$$

into a deterministic differential equation, the so called "Bellman equation"[4]

$$\rho V(S) = \max_x \left\{ f(S,x) + g(S,x)V_S(S) + \frac{1}{2}\sigma^2(S)V_{SS}(S) \right\}. \tag{1.3}$$

Please note that the Bellman equation is not only non-stochastic, but also time invariant as we deal here with an infinite time-horizon problem.

The second step is solve the fist order condition of the maximization problem in Equation (1.3)

$$f_x(S,x) + g_x(S,x)V_S(S) = 0 \tag{1.4}$$

for the "optimal control" $x(V_S, S)$ to insert it again into Equation (1.3). The resulting differential equation is called the "concentrated Bellman equation":

$$\rho V(S) = f(S, x(V_S,S)) + g(S, x(V_S,S))V_S(S) + \frac{1}{2}\sigma^2(S)V_{SS}(S). \tag{1.5}$$

The final step is to obtain a particular solution for the differential equation Equation (1.5) using standard calculus. Unfortunately, a closed form solution can be found only for very simple problems. For more complex cases, numerical methods must be used as shown in the next subsection.

[3] By using Itō's Lemma we remove the the randomness and therefore also the expectation (see [2, p. 321] for more details on the derivation).

[4] The "Bellman equation," or Hamilton-Jacobi-Bellman equation, often referred to as dynamic programming equation, is named after the American mathematician Richard Bellman (1920–1984).

1.2 How to Obtain A Numerical Solution?

As standard shooting algorithms[5] are likely to fail to numerically solve

$$\rho V(S) = \max_x \left\{ f(S,x) + g(S,x)V_S(S) + \frac{1}{2}\sigma^2(S)V_{SS}(S) \right\} \tag{1.3}$$

and finite difference methods suffer from show convergence [4], we use the algorithm developed in [5, Section 11.2] to numerically solve the differential equation. It works as follows:

Firstly, the value function $V(S)$ is approximated in the interval $[a,b]$ as

$$V(S) \approx \sum_{j=1}^{n} c_j \Phi_j(S) = \Phi(S)c \tag{1.6}$$

where $\Phi(S)$ is a set of n orthogonal basis functions and c is a vector of the n basis coefficients[5, Section 6.1]. For this interpolation of $V(S)$ at the nodal points s_i we use use

- Chebychev basis polynomials of the first kind

  $$\Phi_j(S) = T_{j-1}(z) \tag{1.7}$$

  recursively defined as

  $$T_0(z) = 1, \quad T_1(z) = z, \quad T_j(z) = 2zT_{j-1}(z) - T_{j-2}(z) \tag{1.8}$$

  in combination with Chebychev interpolation nodes

  $$s_i = \frac{a+b}{2} + \frac{b-a}{2}\cos\left(\frac{n-i+0.5}{n}\pi\right), \quad \forall i = 1,2,\ldots,n. \tag{1.9}$$

  The advantage of Chebychev node interpolation is that it results in nearly optimal polynomial approximants (Rivlin's theorem [6]) satisfying the equi-oscillation-property [7]. Therefore, at least in theory, the interpolation error can be made arbitrarily small by increasing the order n. But even if the Chebychev polynomials used in combination with Chebychev interpolation nodes result in extremely-well conditioned interpolation equations, numeri-

[5] using, for example, an Euler or an Runge-Kutta algorithm

cal instabilities prevent the use of too high orders. If required (for example, to calculate the optimal stopping point $\hat{S}$ in Figure 1.3 with arbitrary precission) these instabilities can be overcome by using cubic splines:

- cubic spline basis functions (see [8, Chapter 3.]) in combination with evenly distributed interpolation nodes (chosen to be also the breakpoints of the splines)

$$s_i = a + \frac{i-1}{n-1}(b-a), \quad \forall i = 1, 2, \ldots, n \tag{1.10}$$

  or in combination with to the specific problem adapted interpolation nodes. Splines are advantageous if functions experience a wide range of curvature.

  It is to be noted that spline interpolation matrices are sparse[6]. In addition, their values are are bounded. This not only makes storage and matrix manipulation algorithms very efficient, but also reduces numerical difficulties due to the very well conditioned structure.

Secondly, after choosing a set of basis functions of order n in the interval $[a, b]$, we make an initial guess for the coefficient vector c. To do so, we either use a previously calculated result or simply set $c_j = j \quad \forall j = 1, 2, \ldots, n$. Applied to the examples in presented in the next chapters, it turned out that using the previous results leads to fast convergence but sometimes also numerical instabilities. On the other hand, $c_j = j$ worked very efficient in all investigated examples.

The resulting Bellman equation looks as follows

$$\rho \Phi(s_i)c = \max_x \left\{ f(s_i, x) + g(s_i, x)\Phi'(s_i)c + \frac{1}{2}\sigma^2(s_i)\Phi''(s_i)c \right\}. \tag{1.11}$$

Thirdly, we calculate the optimal control $x^*(s_i, c)$ by solving the first order condition of the maximization problem of Equation (1.11)

$$f_x(s_i, x^*) + g_x(s_i, x^*)\Phi'(s_i)c = 0. \tag{1.12}$$

[6] A sparse matrix is a matrix that consists primarily of "zeros."

Fifthly, after obtaining the optimal control, we update the value function, that is the vector c, by using Newton's method

$$c = (\rho\Phi_0 - m\Phi_1 - v\Phi_2)^{-1} f \tag{1.13}$$

where Φ_0, Φ_1, and Φ_0 are the $n \times n$ basis matrices formed by the n values of s_i, ρ is the discount rate, and m, v, and f are vectors defined by $m_i = g(s_i, x_i)$, $v_i = 0.5\sigma^2(s_i)$, and $f_i = f(s_i, x_i)$.

Sixthly, we continue with step three, until the result is accurate enough or a predefined number of iterations (for example, 300) is exceeded.

Finally, if the algorithm did not converge, we reduce the order n or choose different initial values for the value function at the collocation nodes s_i.

1.3 A First Example: "No CO_2 Emissions"

Assume the following very simple example that admits a closed form solution.

The example is taken from [9, Section 2.1.]. The meaning of the model is described in [9, p. 107]. As the purpose of this introduction is to present the method used to obtain an analytical/numerical solution to the stochastic control problem we refrain from explaining the model in detail here.

$$f(S,x) = -\frac{\gamma}{2}S^2 \tag{1.14}$$

$$g(S,x) = 0 \tag{1.15}$$

$$\sigma(S) = \sigma \tag{1.16}$$

and

$$\gamma = 0.868187$$
$$\sigma = 0.2$$
$$\rho = 0.03,$$

that is,

$$V(S) = \max_{x} \mathrm{E}\left\{\int_0^{\infty} e^{-\rho t}\left(-\frac{\gamma}{2}S^2\right)\,\mathrm{d}t\right\}, \tag{1.17}$$

$$\mathrm{d}S = \sigma\,\mathrm{d}z. \tag{1.18}$$

1.3.1 Analytical Solution

To do so, we insert Equations (1.14) and (1.15) into Equation (1.3) to obtain the deterministic problem

$$\rho V(S) = -\frac{\gamma}{2}S^2 + \frac{1}{2}\sigma^2 V_{SS}(S). \tag{1.19}$$

As there is no maximization problem left, we now only have to obtain a particular solution for the differential equation Equation (1.19) (see also [9, Eq. (12)]):

$$V(S) = -\frac{1}{2}\frac{\gamma S^2}{\rho} - \frac{1}{2}\frac{\gamma\sigma^2}{\rho^2}. \tag{1.20}$$

1.3.2 Numerical Solution

We apply the method described in Section 1.2 (see also [5, Section 11.2]) using the MATLAB code shown below:

wirl1fun.m:

```
function out = wirl1fun(flag,s,x,Vs,xrho,xsig,xgamma)
switch flag
  case 'x';     out = NaN   + zeros(size(s,1),1);      % optimal control
  case 'f';     out = -xgamma/2*s.^2;                  % reward function
  case 'g';     out = 0     + zeros(size(s,1),1);      % drift function
  case 'sigma'; out = xsig + zeros(size(s,1),1);       % diffusion function
  case 'rho';   out = xrho + zeros(size(s,1),1);       % discount function
end
```

main file:

```
  clc; clear; optset('scsolve','defaults');
% Define problem parameters
  xgamma = 0.868187; xsig = 0.2; xrho = 0.03;
% Pack model structure
  model.func='wirl1fun'; model.params={xrho,xsig,xgamma};
% Define nodes and basis matrices
  fspace=fundefn('cheb',5,-1,3); snodes=funnode(fspace);
% Call solver
  [cv,s,v,x,resid]=scsolve(model,fspace,snodes,snodes);
```

Figure 1.1 shows the the numerically calculated value function as well as the analytically calculated value function on top of each other. They turn out to be equal.

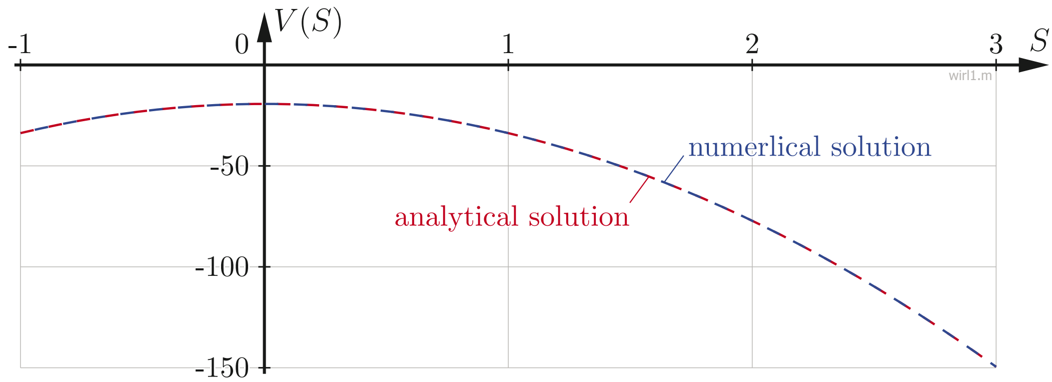

Figure 1.1: The numerical and analytical solution —both plotted on top of each other— turn out to be equal.

1.4 A Second Example: "Constant CO_2 Emissions"

For the model described in [9, Section 3.], we obtain the functions

$$f(S,x) = \bar{u} - \frac{\gamma}{2} S^2 \tag{1.21}$$

$$g(S,x) = \bar{x} \tag{1.22}$$

$$\sigma(S) = \sigma \tag{1.23}$$

using the constants

$$\gamma = 0.868187$$
$$\sigma = 0.2$$
$$\rho = 0.03$$
$$\bar{u} = 1$$
$$\bar{x} = 0.01446.$$

1.4.1 Analytical Solution

The analytical solution to the problem is obtained by solving the Bellman equation

$$\rho V(S) = \bar{u} - \frac{\gamma}{2} S^2 + \bar{x} V_S(S) + \frac{1}{2} \sigma^2 V_{SS}(S) \tag{1.24}$$

leading to the particular solution (equal to [9, Eq. (17)])

$$V(S) = -\frac{1}{2} \frac{\gamma S^2}{\rho} - \frac{1}{2} \frac{\gamma \sigma^2}{\rho^2} + \frac{\bar{u}}{\rho} - \frac{\gamma \bar{x} S}{\rho^2} - \frac{\gamma \bar{x}^2}{\rho^3}. \tag{1.25}$$

1.4.2 Numerical Solution

We again apply the algorithm described in [5, Section 11.2] to obtain the numerical solution shown in Figure 1.2.

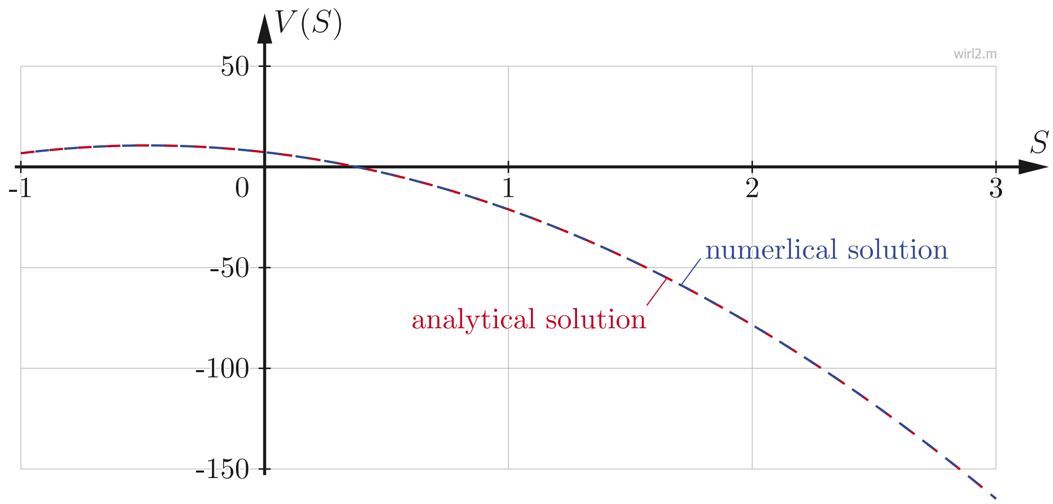

Figure 1.2: The numerical and analytical solution —both plotted on top of each other— turn out to be equal.

1.5 A Third Example: "Irreversible CO_2 Emissions"

For the model described in [9, Section 4.] we obtain the functions

$$
\begin{aligned}
f(S,x) &= \alpha x - \frac{\beta}{2}x^2 - \frac{\gamma}{2}S^2 && (1.26)\\
g(S,x) &= x && (1.27)\\
\sigma(S) &= \sigma && (1.28)\\
x &> 0 && (1.29)
\end{aligned}
$$

using the constants

$$
\begin{aligned}
\gamma &= 0.868187\\
\sigma &= 0.2\\
\rho &= 0.03\\
\alpha &= 70.142\\
\beta &= 136.341.
\end{aligned}
$$

The associated first order condition is

$$f_x(S,x) + g_x(S,x)V_S(S) = 0. \tag{1.30}$$

Using Equations (1.26) and (1.27) we obtain

$$\alpha - \beta x + 1V_S(S) = 0 \tag{1.31}$$

to calculate $x(V_S)$ as

$$x(V_S) = \frac{\alpha + V_S(S)}{\beta} > 0. \tag{1.32}$$

This time, the analytical solution to the problem cannot be obtained by solving the Bellman equation

$$\rho V(S) = \alpha x(V_S) - \frac{\beta}{2}x(V_S)^2 - \frac{\gamma}{2}S^2 + x(V_S)V_S(S) + \frac{1}{2}\sigma^2 V_{SS}(S). \tag{1.33}$$

1.5.1 Numerical Solution

We again apply the algorithm described in [5, Section 11.2] to obtain the solution shown in Figure 1.3. We use the MATLAB code shown below.

wirl3fun.m:

```
function out = wirl3fun(flag,s,x,Vs,xgamma,xsig,xrho,xalpha,xbeta)
switch flag
  case 'x';     out = max((xalpha+Vs)/xbeta,zeros(size(s,1),1)); % optimal control
  case 'f';     out = xalpha*x-xbeta*x.^2/2-xgamma/2*s.^2;       % reward function
  case 'g';     out = x;                                         % drift function
  case 'sigma'; out = xsig  + zeros(size(s,1),1);                % diffusion function
  case 'rho';   out = xrho  + zeros(size(s,1),1);                % discount function
end
```

wirl3.m:

```
  clc; clear;
  optset('scsolve','defaults');

% Define problem parameters
  xgamma = 0.868187; xsig = 0.2; xrho = 0.03; xalpha=70.142; xbeta=136.341;

% Pack model structure
  model.func='wirl3fun'; model.params={xgamma,xsig,xrho,xalpha,xbeta};

% Define nodes and basis matrices
  fspace=fundefn('spli',2000,-30,30); snodes=funnode(fspace);

% Call solver
  [cv,s,v,x,resid]=scsolve(model,fspace,snodes,snodes);
```

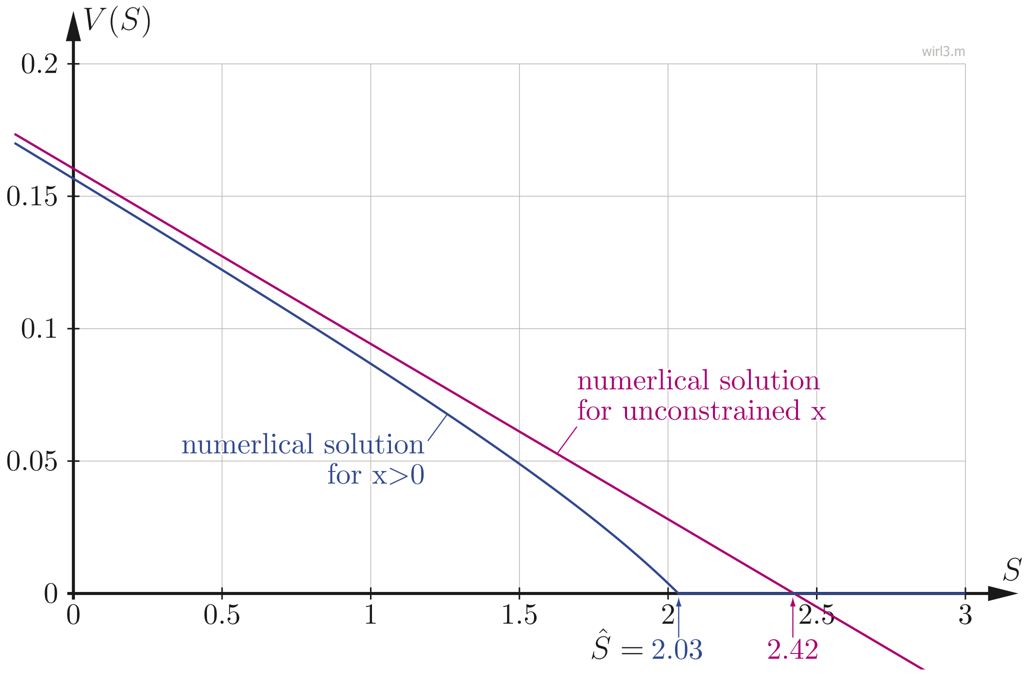

Figure 1.3: The numerical solution, same as [10, Fig. 3].

2 Ups and Downs of (Oil) Prices

Note that the problem statement below is quoted from [11].

"The risk neutral monopoly sets its price strategy such that the expected net present value of profits is maximized

$$V(x) = \max_{p(t)} \mathrm{E}\left\{ \int_0^\infty e^{-rt} \big(p(t) - c\big) x(t)\, \mathrm{d}t \right\}. \tag{2.1}$$

The function $x(t)$ denotes the demand in period t and c denotes the unit cost of production that can be ignored without loss in generality by interpreting p as the price above the constant marginal costs c.

Sluggishness is an important characteristic of world energy markets. Demand depends on the energy efficiencies of durables with an average life time of, for example, above ten years for cars and of many decades for buildings. A particularly convenient and thus often used specification of dynamic demand (and also of supply) is the one implied by the Eisner-Strotz model [12, 13]. Therefore, define with $D(p)$ as the (expected) equilibrium or long run demand and assume that the static (and also stationary) profit (or revenue) function $\pi(p) = pD(p)$ is concave with a unique, interior profit maximizing price denoted with p^*. The consumers trade off (quadratic) costs for being out of equilibrium $(\frac{1}{2}\left(D(p) - x\right)^2)$ against the costs of adjustment $\left(\frac{\gamma}{2}\, \mathrm{d}x^2\right)$. Assuming myopic behavior (that is, all present adjustments are made from a plan for future adjustments based on the assumption that the current price prevails in all future) implies the dynamic demand pattern

$$\mathrm{d}x(t) = \eta\,[D(p(t)) - x(t)]\;\mathrm{d}t + \sigma(x(t))\,\mathrm{d}z \tag{2.2}$$

in which $\mathrm{d}z$ denotes the increment of a standardized Wiener process z and σ the corresponding standard error. The adjustment parameter η determines in the deterministic model the time constant of adjustment $\tau = 1/\eta$ which increases with respect to (subjective) discounting and adjustment costs γ.

Consumers are myopic, or at least were myopic in this sense according to empirical investigations in [14, 15]. This hypothesis seems in line also with recent evidence because anticipation of higher prices should trigger conservation prior to a price jump. However, consumers were instead on a spending spree on energy intensive services such as gas guzzling SUVs until very recently (following a period of caution after the twelve years of high energy prices from 1973 to 1985). Although different specifications of the random term in the stochastic process[1] described in Equation (2.2) are conceivable, the following analysis is restricted to the square root

$$\sigma(x) = \sigma\sqrt{x} \tag{2.3}$$

because it introduces level dependent uncertainty while still allowing for explicit, analytical solutions."

2.1 The Bellman Equation

A price strategy maximizing the expected net present value defined in Equation (2.1) that is subject to the state transition defined in Equation (2.2) must satisfy the Hamilton-Jacobi-Bellman equation (see [2])

$$rV(x) = \max_p \left\{ (p-c)x + \eta\,[D(p) - x]\,V'(x) + \frac{1}{2}\sigma^2 x V''(x) \right\}. \tag{2.4}$$

The interior first order condition is

$$x + \eta D'(p)V'(x) = 0. \tag{2.5}$$

[1] Pindyck introduces stochastic processes (of prices and discoveries) into the analysis of resource markets but with little explanatory power for the recent price jump [16–18].

Assuming a quadratic and concave demand[2]

$$D(p) = A - Bp - \frac{\varepsilon}{2}p^2 \tag{2.6}$$

yields

$$x - (B + \varepsilon p)\,\eta V' = 0 \tag{2.7}$$

$$p^* = \frac{x}{\varepsilon \eta V'} - \frac{B}{\varepsilon}. \tag{2.8}$$

Inserting Equation (2.8) into Equation (2.6) and then into Equation (2.4) results in the the differential equation

$$rV = -cx - \frac{Bx}{\varepsilon} + \frac{x^2}{2\varepsilon\eta}\frac{1}{V'} + (A\eta - x\eta + \frac{B^2\eta}{2\varepsilon})V' + \frac{x\sigma^2}{2}V'' \tag{2.9}$$

which we will solve in the following sections numerically.

2.2 Concave Demand

Using the variables

$$\begin{aligned} A &= 30 \\ B &= 0.2 \\ \varepsilon &= 0.05 \\ r &= 0.05 \\ \eta &= 0.20 \\ \sigma &= 0.5 \\ c &= 0 \end{aligned}$$

[2] linear or convex demand yields bang-bang policies, see Section 2.3.

we obtain for the framework introduced in Equations (1.1) to (1.3) the following functions[3]

$$f(S,x) = (x-c)S \tag{2.10}$$
$$g(S,x) = \eta\left(A - Bx - \frac{\varepsilon}{2}x^2 - S\right) \tag{2.11}$$
$$\sigma(S) = \sigma\sqrt{S} \tag{2.12}$$
$$x(S,V_s) = \frac{S}{\varepsilon\eta V_s} - \frac{B}{\varepsilon}. \tag{2.13}$$

Figure 2.1 shows the expected steady state solution x_∞ of the demand for different values of ε and $\sigma(x)$. Note that the case of $\varepsilon \to -\infty$, $\sigma(x) = 0$ corresponds to the steady state solution of the linear, deterministic problem $x_\infty = 13.3$ (compare [11]).

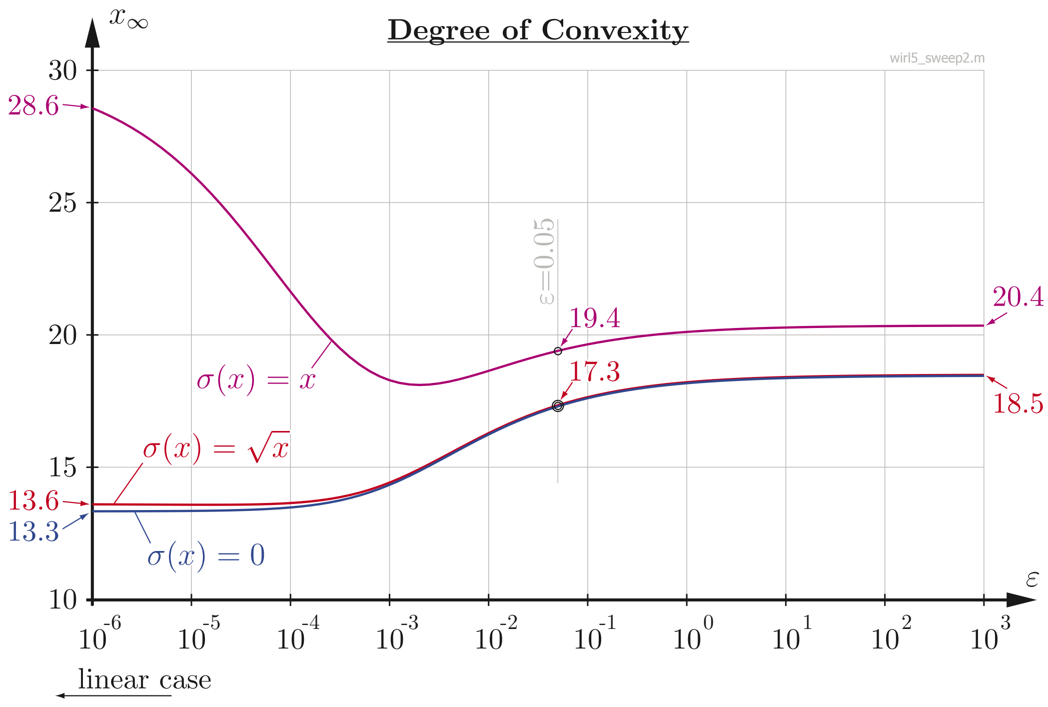

Figure 2.1: Expected steady state solution for different levels of ε and $\sigma(x)$.

[3] Note that the variable names change: $r \to \rho$, $x \to S$, and $p \to x$.

Figure 2.1 shows the same results again, but now presenting $\sigma(x)$ on the abscissa.

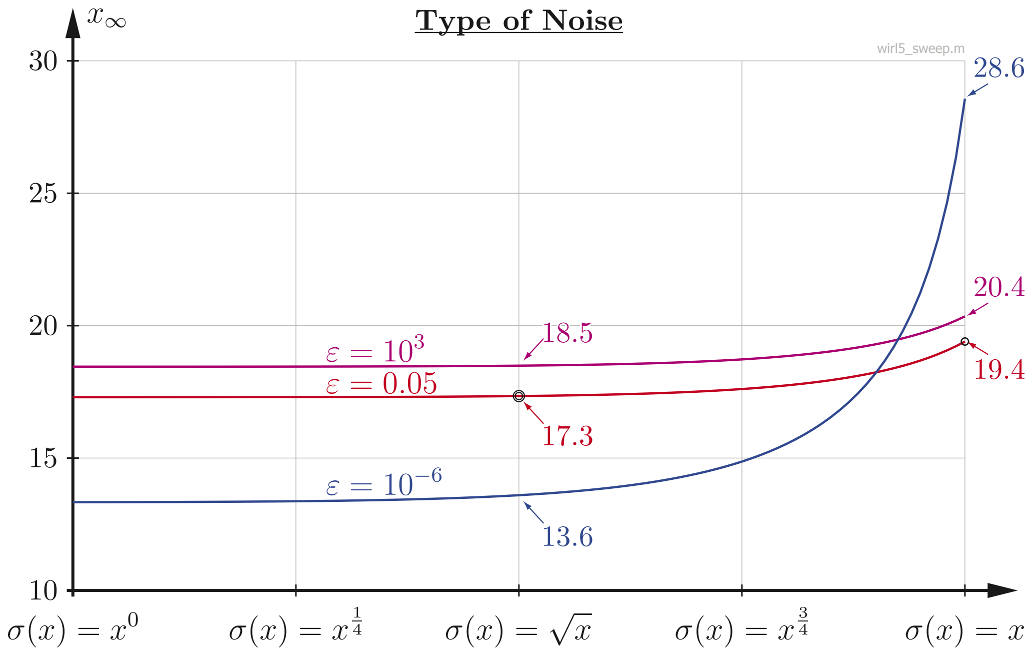

Figure 2.2: Expected steady state solution for different levels of $\sigma(x)$ and ε.

Figure 2.3 shows the same results for different levels of the discount rate r. Note the solution for $\varepsilon \to -\infty$, $\sigma(x) = 0$, $r \to -\infty$ which corresponds to the steady state demand of $x = 15$ (compare [11]).

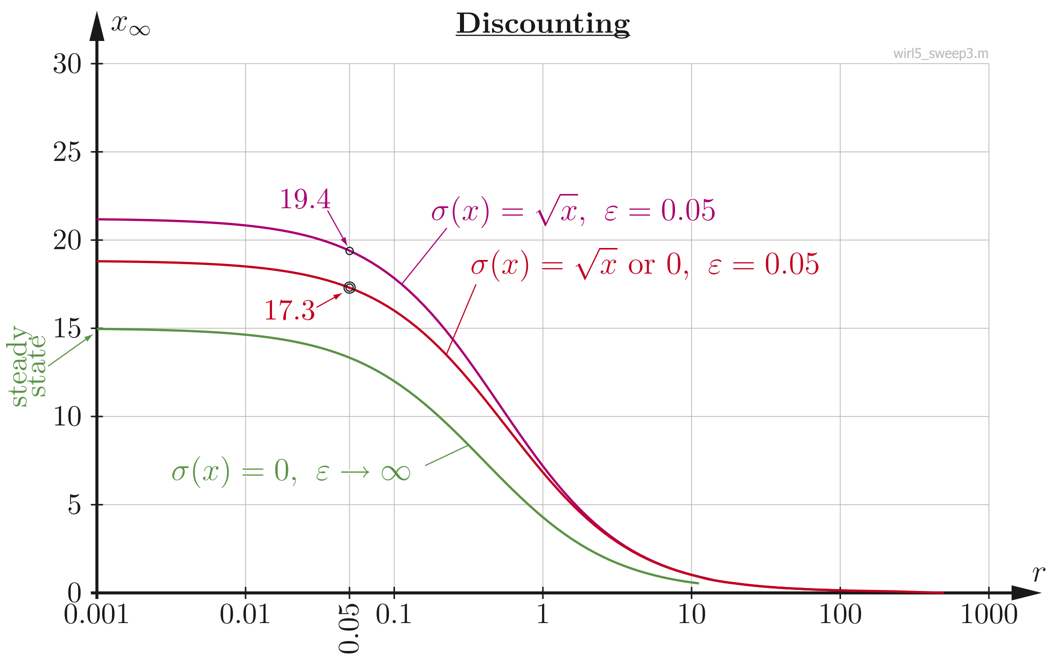

Figure 2.3: Expected steady state solution for different levels of the discount rate r.

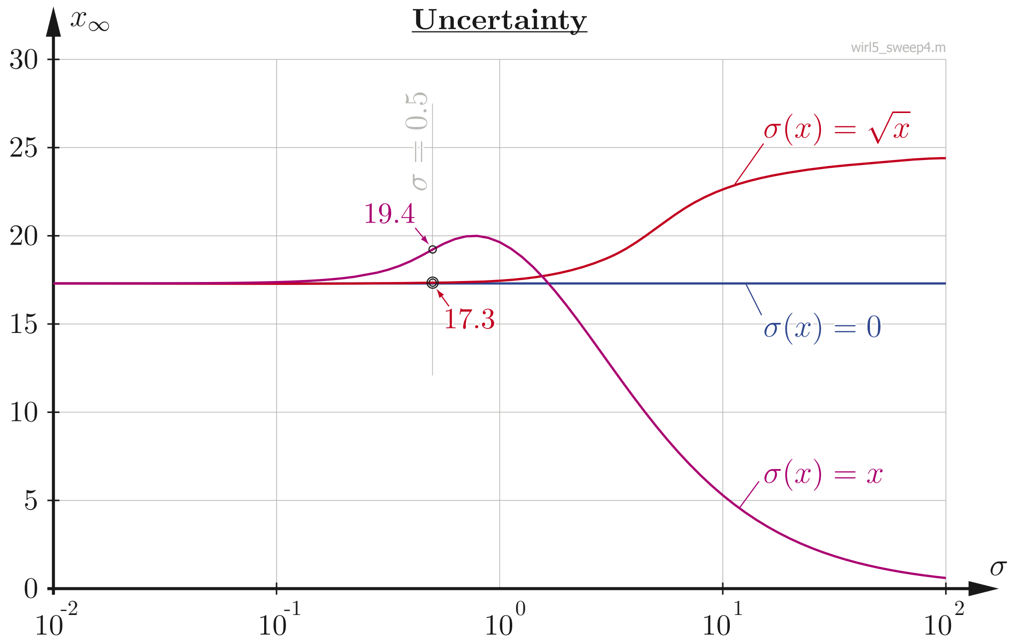

Figure 2.4: Expected steady state solution for different levels of the uncertainty σ.

Figure 2.5 shows the expected steady state solution for different levels of the time constant η. Note that for high and low time constants, the steady state solution does not depend on the type of noise anymore.

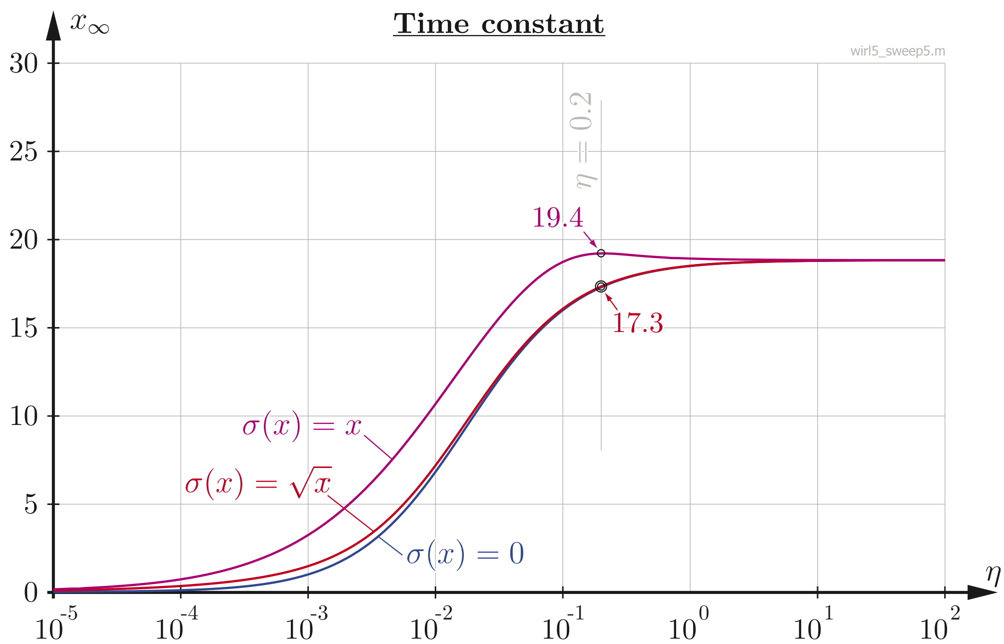

Figure 2.5: Expected steady state solution for different levels of the time constant η.

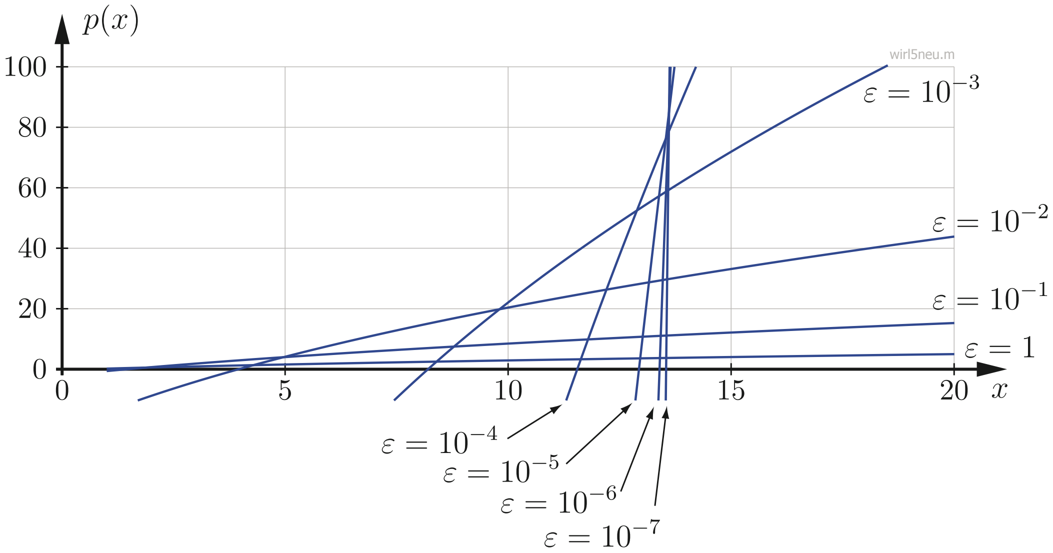

Figure 2.6: For decreasing ε, the concave problem converges to the non-concave demand problem solved in the next section.

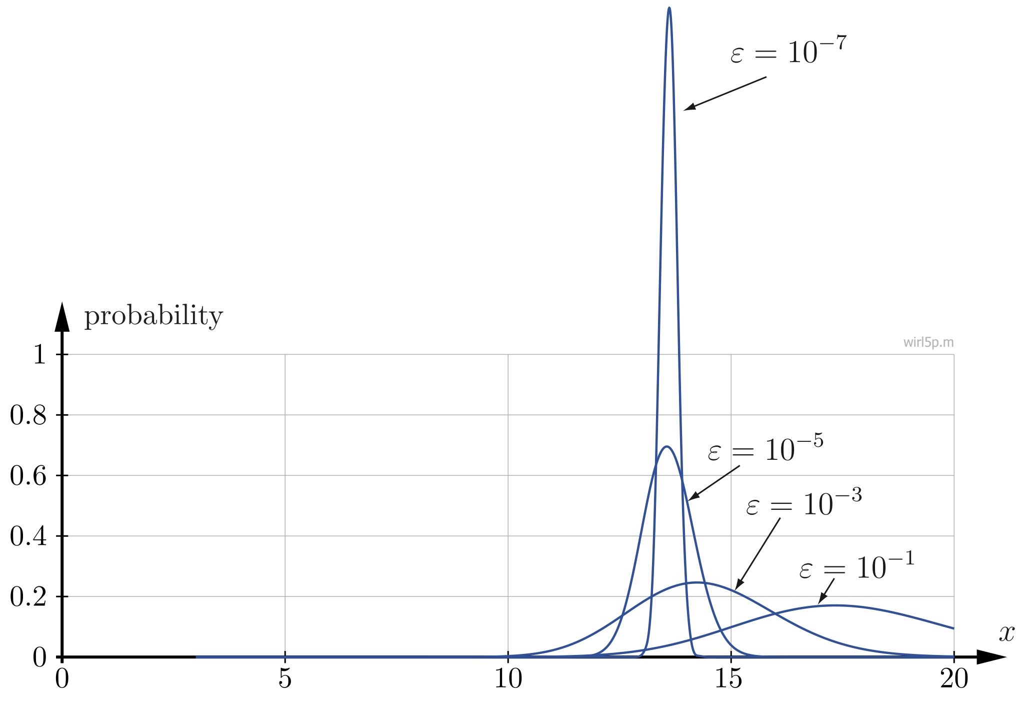

Figure 2.7: For decreasing ε, the long-run state density converges do a dirac.

2.3 Non-Concave Demand (Big Bang Problem)

Let us now assume the non-concave ($D''(p) \geq 0$) demand

$$D(p) = 1 - p + \frac{\varepsilon}{2} p^2 \tag{2.14}$$

and a price that is bounded by $[\underline{p}, \bar{p}]$

$$V(x) = \max_{p(t) \in [\underline{p}, \bar{p}]} \mathrm{E} \left\{ \int_0^\infty e^{-rt} \big(p(t) - c\big) x(t) \, \mathrm{d}t \right\} \tag{2.15}$$

where $\underline{p}$ denotes the minimum and $\bar{p}$ denotes the maximum price the firm can charge the customer. Using the constants

$$\begin{aligned} \varepsilon &= 0.20 \\ r &= 0.10 \\ \eta &= 0.20 \\ \sigma &= 0.10 \end{aligned}$$

$$\begin{aligned} \underline{p} &= 0 & \bar{p} &= 1 \\ \underline{D} &= D(\underline{p}) = 1 & \bar{D} &= D(\bar{p}) = 0.10 \end{aligned}$$

the settings of low and high demands in Equation (2.14) correspond in a convex set up for $D(p)$ with the static profit maximizing price of $p^s = 0.544$ and the corresponding demand $x^s = D(p^s) = 0.485$.

In the dynamic case, applying continuity in V' and V'' implies that the firm charges the upper price limit

$$p(x) = \bar{p} \qquad \text{iif} \qquad (\bar{p} - c)x + \eta \left[\bar{D} - x\right] V' - \eta \left[\underline{D} - x\right] V' > 0 \tag{2.16}$$

and the lower price limit

$$p(x) = \underline{p} \qquad \text{iif} \qquad (\bar{p} - c)x + \eta \left[\bar{D} - x\right] V' - \eta \left[\underline{D} - x\right] V' < 0. \tag{2.17}$$

In other words, the optimal policy is always at the boundary as also shown in the bottom of Figure 2.8. The threshold at which the price jumps from $\underline{p}$ to $\bar{p}$ is denoted as $\hat{x}$.

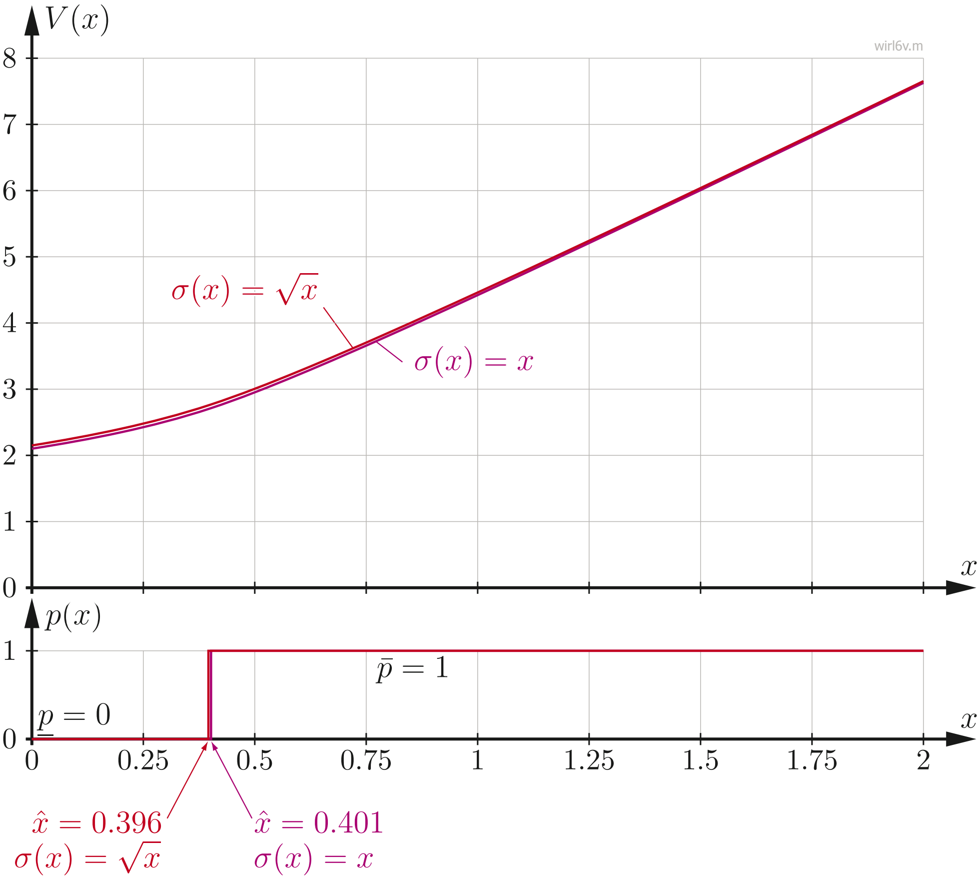

Figure 2.8: Value function $V(x)$ and optimal pricing strategy $p(x)$.

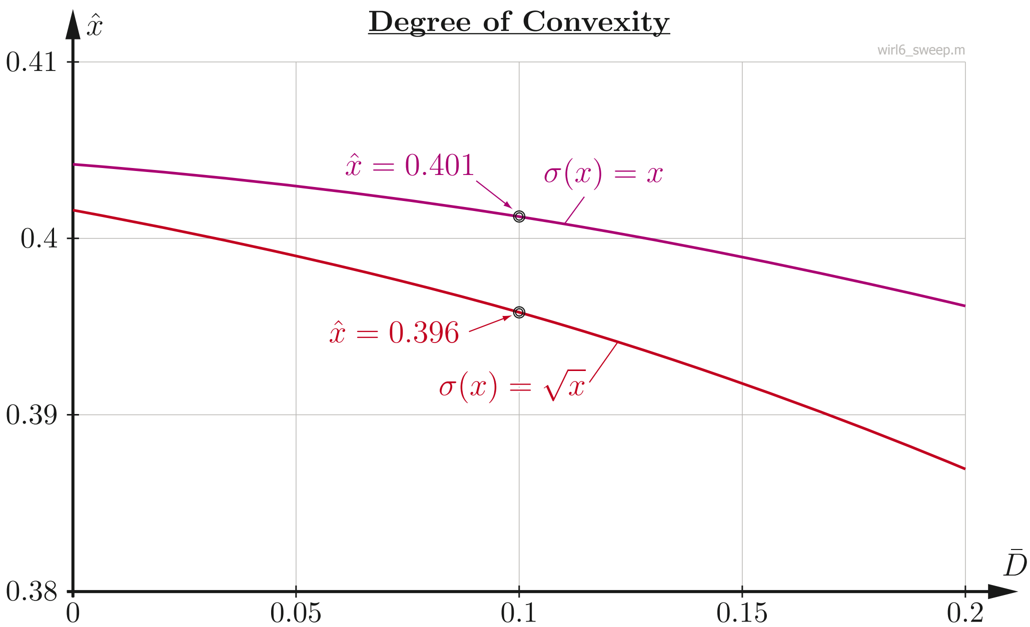

Figure 2.9: Threshold $\hat{x}$ for different levels of $\bar{D}$. ($\varepsilon = 2\bar{D}$)

3 Global Warming

In a world subject to global warming, ones goal should be to maximize the global welfare[1] (the social optimum $\widehat{=}$ first best)

$$W(T) = \max_{y(t)} \mathrm{E}\left\{\int_0^\infty e^{-rt}\big(u(y) - C(T)\big)\,\mathrm{d}t\right\} \tag{3.1}$$

where $u(y)$ is the global benefit due to the emissions y (a and b are constants)

$$u(y) = ay - \frac{b}{2}y^2, \tag{3.2}$$

$C(T)$ are the global external costs due to an increase in temperature T (c is a constant)

$$C(T) = \frac{c}{2}T^2, \tag{3.3}$$

the temperature T follows the Itō process (δ and σ are a constants)

$$\mathrm{d}T = (y - \delta T)\,\mathrm{d}t + \sigma T\,\mathrm{d}z, \tag{3.4}$$

and $\mathrm{d}z$ is a Brownian motion.

[1] This example is taken from [19] and extended by missing numerical simulations. For more detailed information on the model and the parameter sets used, see [19].

If n identical but individual countries maximize their own welfare, each country is emitting exactly one nth of the share of the global emissions y

$$x = \frac{y}{n}. \tag{3.5}$$

In order to let the aggregate correspond to Equations (3.2) and (3.3), each country has to maximize its own welfare

$$V_i(T) = \max_{x(t)} \mathrm{E}\left\{\int_0^\infty e^{-rt}\big(v(x_i) - K(T)\big)\,\mathrm{d}t\right\} \tag{3.6}$$

where $v(x_i)$ is the individual benefit due to the emissions x

$$u(y) = ay - \frac{bn}{2}y^2 \tag{3.7}$$

and $K(T)$ is the cost due to an increase in temperature

$$K(T) = \frac{c}{2n}T^2. \tag{3.8}$$

3.1 The Bellman Equation

The optimal, welfare maximizing strategy must satisfy the Hamilton-Jacobi-Bellman equation

$$rW(T) = \max_y \left\{[u(y) - C(T)] + [y - \delta T]\,W'(T) + \frac{1}{2}\sigma^2 T^2 W''(T)\right\} \tag{3.9}$$

for the first best case and

$$rV_i(T) = \max_x \left\{[v(x_i) - K(T)] + [nx - \delta T]\,V_i'(T) + \frac{1}{2}\sigma^2 T^2 V_i''(T)\right\}. \tag{3.10}$$

for the second best case. The corresponding interior first order conditions are

$$a - by + W' = 0 \tag{3.11}$$

$$y = \frac{a + W'}{b} \tag{3.12}$$

and

$$a - bnx + V' = 0 \tag{3.13}$$
$$x = \frac{a + V'}{bn}. \tag{3.14}$$

3.2 Parameter Set 1

Using the variables

$$r = 0.03$$
$$a = 871$$
$$b = 50723$$
$$c = 11.425$$
$$n = 1 \text{ to } 3$$
$$\delta = 0.005$$
$$\sigma = 0.1$$

we obtain for the framework introduced in Equations (1.1) to (1.3) the following functions[2]

$$f(S, x) = ax - \frac{bn}{2}x^2 - \frac{c}{2n}S^2 \tag{3.15}$$
$$g(S, x) = c - \delta S \tag{3.16}$$
$$\sigma(S) = \sigma S \tag{3.17}$$
$$x(S, V_s) = \frac{a + V_s}{bn}. \tag{3.18}$$

[2] Note that the variable names change: $r \to \rho$, $T \to S$, and $y \to x$.

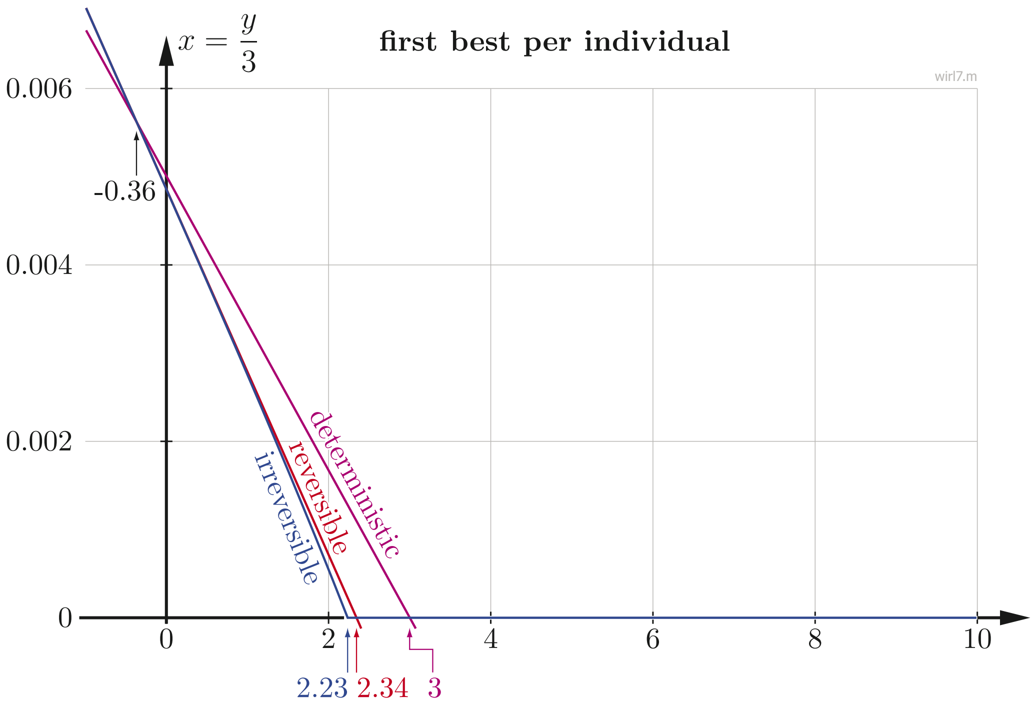

Figure 3.1: Numerical simulation, first best.

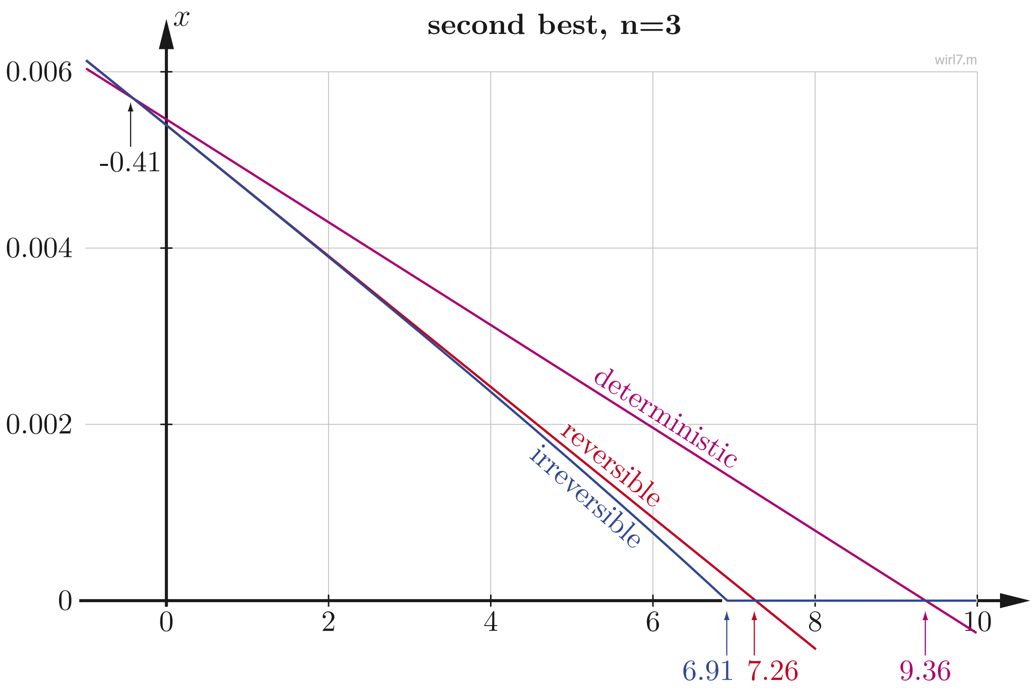

Figure 3.2: Numerical simulation, second best.

3.3 Parameter Set 2

$$r = 0.05$$
$$a = 1$$
$$b = 1$$
$$c = 0.1$$
$$\delta = 0.1$$
$$\sigma = 0.2$$
$$n = 2$$

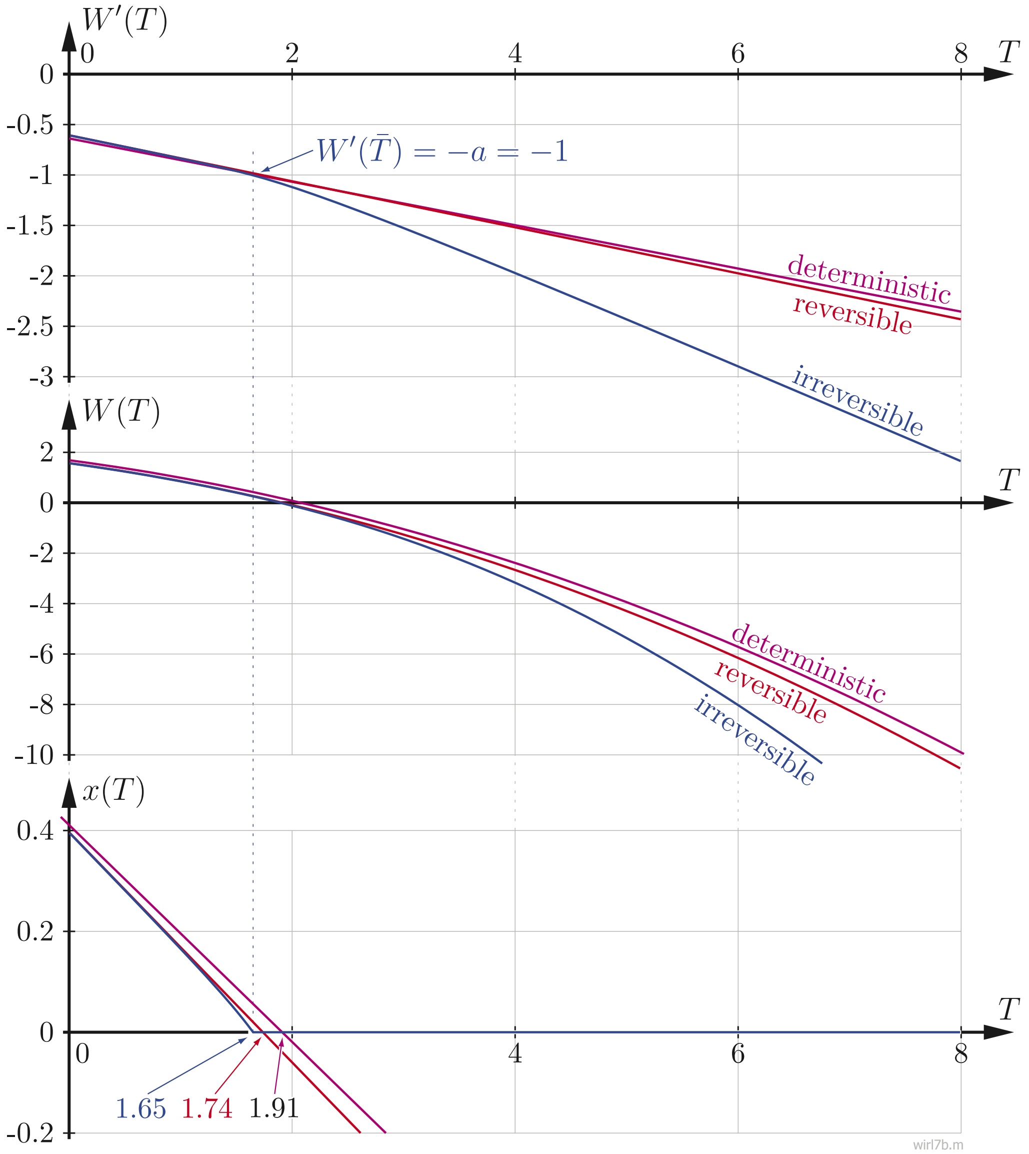

Figure 3.3: Numerical simulation, first best.

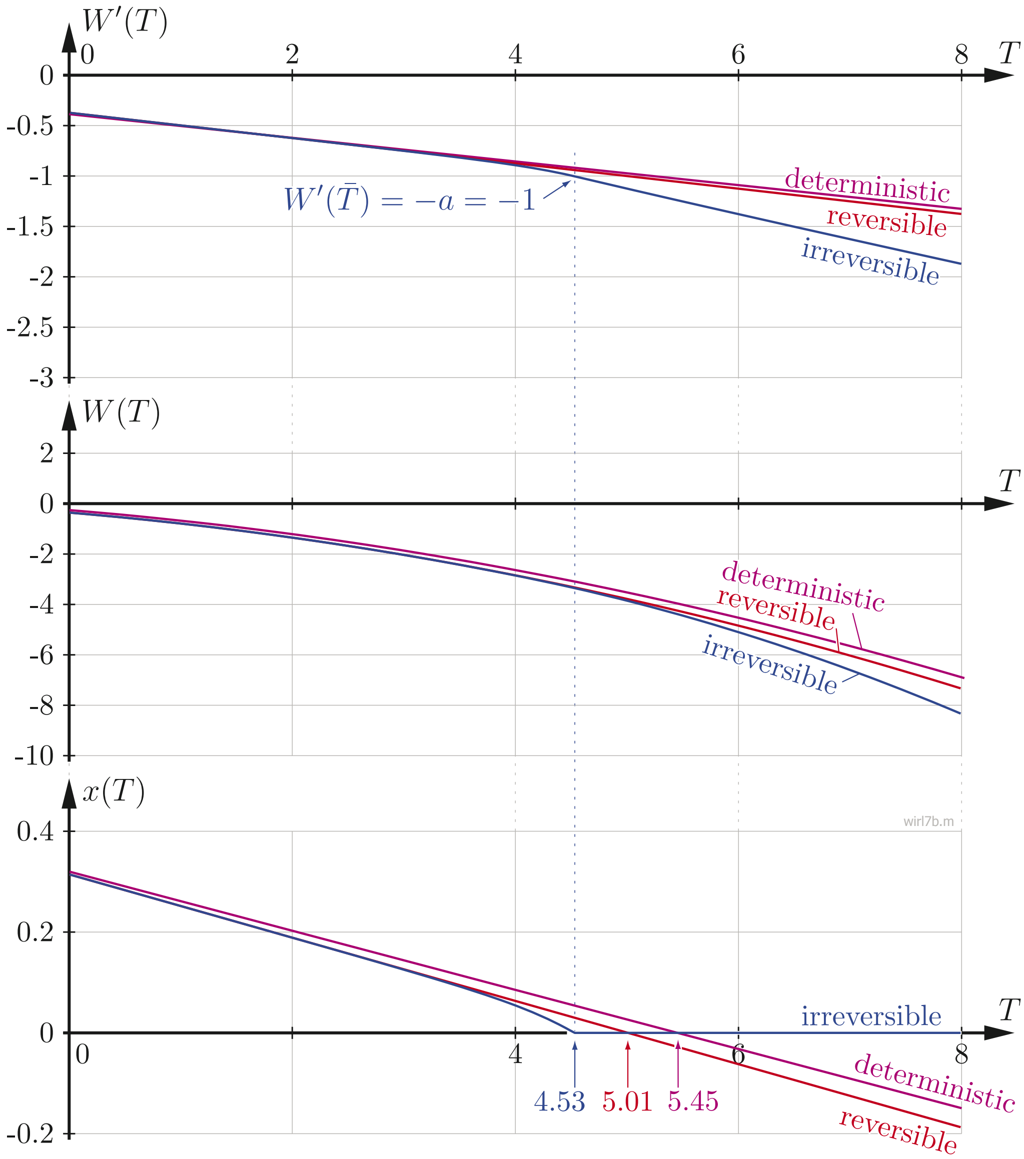

Figure 3.4: Numerical simulation, second best.

4 Pricing of Nondurables

Assume that n equal players choose their emissions x at time t such that the net present value of the total benefits minus costs is maximized[1] (first best)

$$W(X) = \max_{x} \int_0^\infty e^{-rt} n \big[u(x) - \frac{c}{2} X^2\big] \,\mathrm{d}t \tag{4.1}$$

or that each of the n identical players chooses his emissions x_i such that the net present value of his own benefits minus costs is maximized (second best)

$$V_i(X) = \max_{x_i} \int_0^\infty e^{-rt} \big[u(x_i) - \frac{c}{2} X^2\big] \,\mathrm{d}t \tag{4.2}$$

where $u(x_i)$ is the individual benefit due to the emissions x_i ($0 < a < 1$ and $\underline{x} > 0$ are constants)

$$u(x_i) = \frac{1}{a}(x_i - \underline{x})^a, \tag{4.3}$$

and X is the quadratic external cost due to the pollution stock X ($c > 0$ and $\delta > 0$ are constants)

$$\mathrm{d}X = \left[x_i(t) + \sum_{i \neq j} x_j(t) - \delta X) \right] \mathrm{d}t. \tag{4.4}$$

[1] This example is taken from [20] and extended by missing numerical simulations. For more detailed information on the model and the parameter sets used, see [20].

4.1 The Bellman Equation

To solve the first best dynamic optimization problem (Equation (4.1)), we calculate the Hamilton-Jacobi-Bellman equation

$$rW(X) = \max_{x} \left[n \left(\frac{1}{a}(x - \underline{x})^a - \frac{c}{2}X^2 \right) + (nx - \delta X)\, W'(X) \right] \tag{4.5}$$

and maximize the right hand side

$$x^* = \underline{x} + (-W')^{\frac{1}{a-1}}. \tag{4.6}$$

The second best dynamic optimization problem (Equation (4.2)) leads the Hamilton-Jacobi-Bellman equation

$$rV_i(X) = \max_{x_i} \left[\frac{1}{a}(x_i - \underline{x})^a - \frac{c}{2}X^2 + \left(x_i + \sum_{i \neq j} x_j - \delta X \right) V'(X) \right] \tag{4.7}$$

with the corresponding interior first order condition

$$x^* = \underline{x} + (-V')^{\frac{1}{a-1}}. \tag{4.8}$$

4.2 A Numerical Example

Using the variables

$$\begin{aligned} a &= 0.6 & \delta &= 0.2 \\ c &= 1500 & r &= 10 \\ \underline{x} &= 0.01 & n &= 2 \end{aligned}$$

we obtain for the framework introduced in Equations (1.1) to (1.3) the following functions ($n^* = n$ for first best, $n^* = 1$ for second best)[2]

$$f(S, x) = n^* \frac{1}{a}(x - \underline{x})^a - \frac{c}{2}S^2 \tag{4.9}$$

$$g(S, x) = nx + \delta S \tag{4.10}$$

$$\sigma(S) = [\,] \tag{4.11}$$

$$x(S, V_s) = \underline{x} + (-V_s)^{\frac{1}{a-1}}. \tag{4.12}$$

[2] Note that the variable names change: $r \to \rho$, $X \to S$, $W \to V$.

The optimal level of pollution X^*_∞ can be calculated for the first best case as

$$\frac{ncX^*_\infty}{r+\delta} = \left(\frac{\delta X^*_\infty - n\underline{x}}{n}\right)^{a-1} \tag{4.13}$$

$$X^*_\infty = 0.102. \tag{4.14}$$

The numerical simulation shown in Figure 4.1 leads to the same solution:

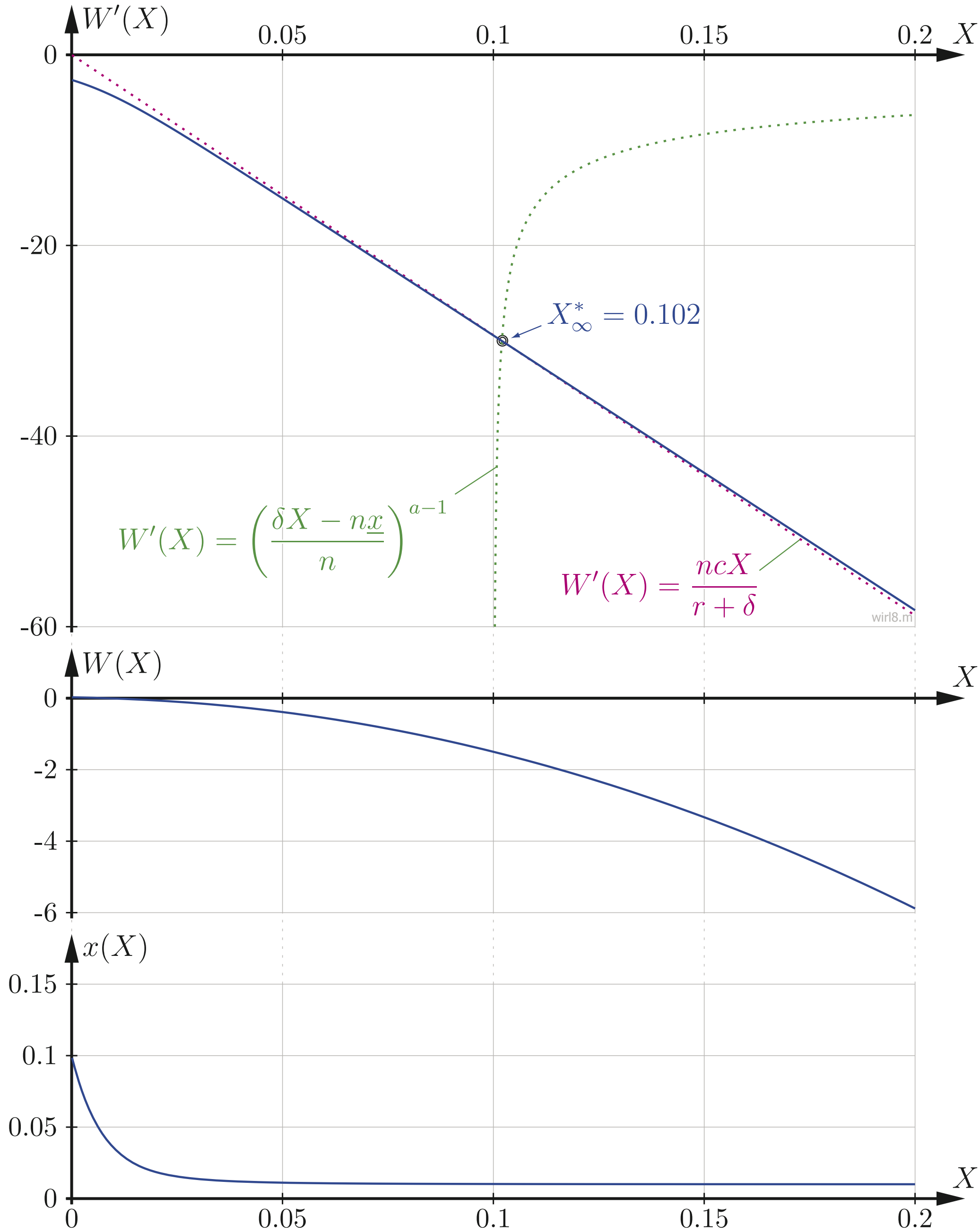

Figure 4.1: First best solution.

For the second best case, the optimal level of pollution X^*_∞ can be calculated again using Equation (4.13) as

$$X^*_\infty = 0.110. \tag{4.15}$$

The corresponding numerical result is shown in Figure 4.2:

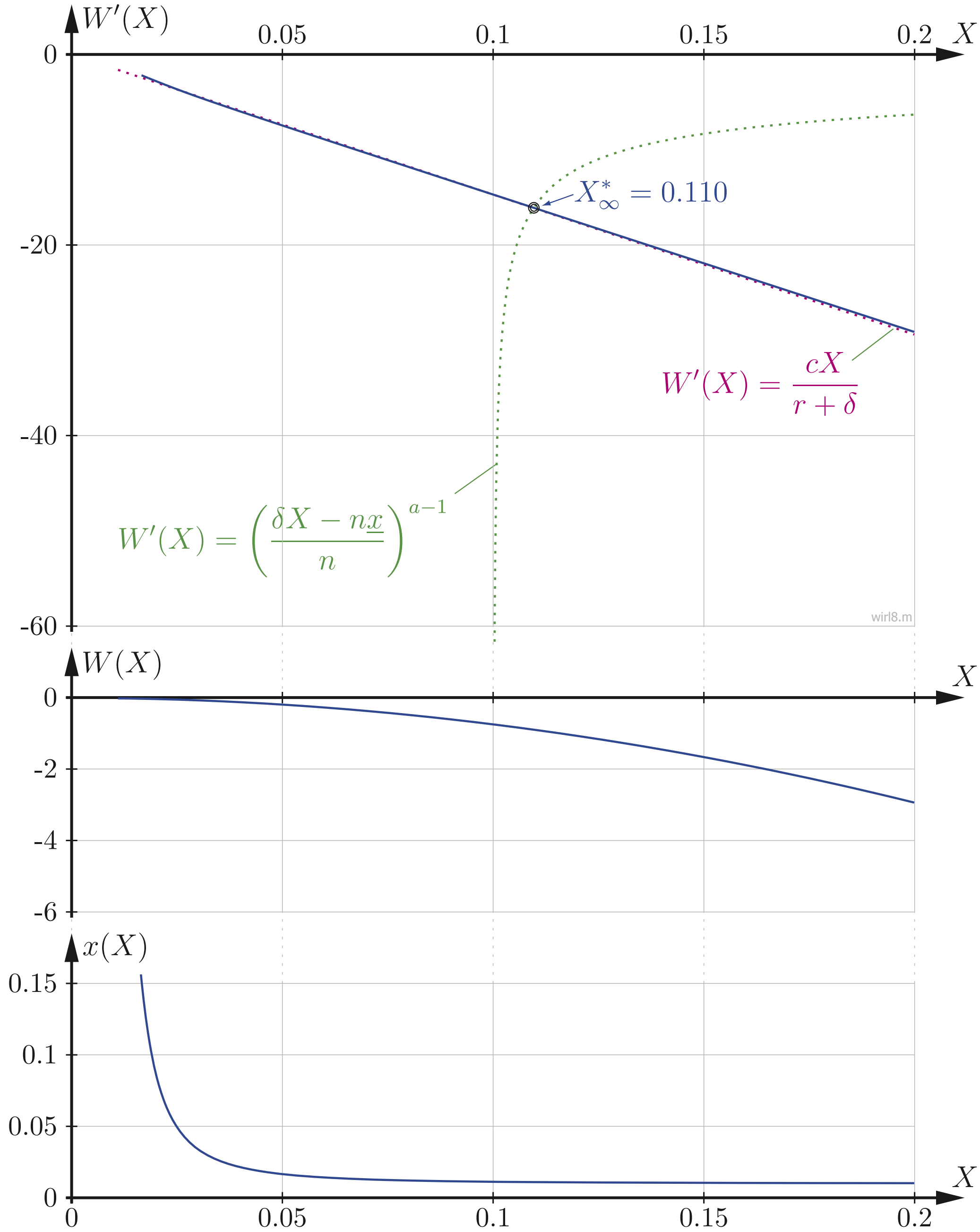

Figure 4.2: Second best solution, $a = 0.6 > \frac{1}{n}$.

To simulate the second best case where $a < \frac{1}{n}$ we set a to 0.4 and obtain the optimal level of pollution (using Equation (4.13)) as

$$X^*_\infty = 0.155. \tag{4.16}$$

The corresponding numerical result is shown in Figure 4.3:

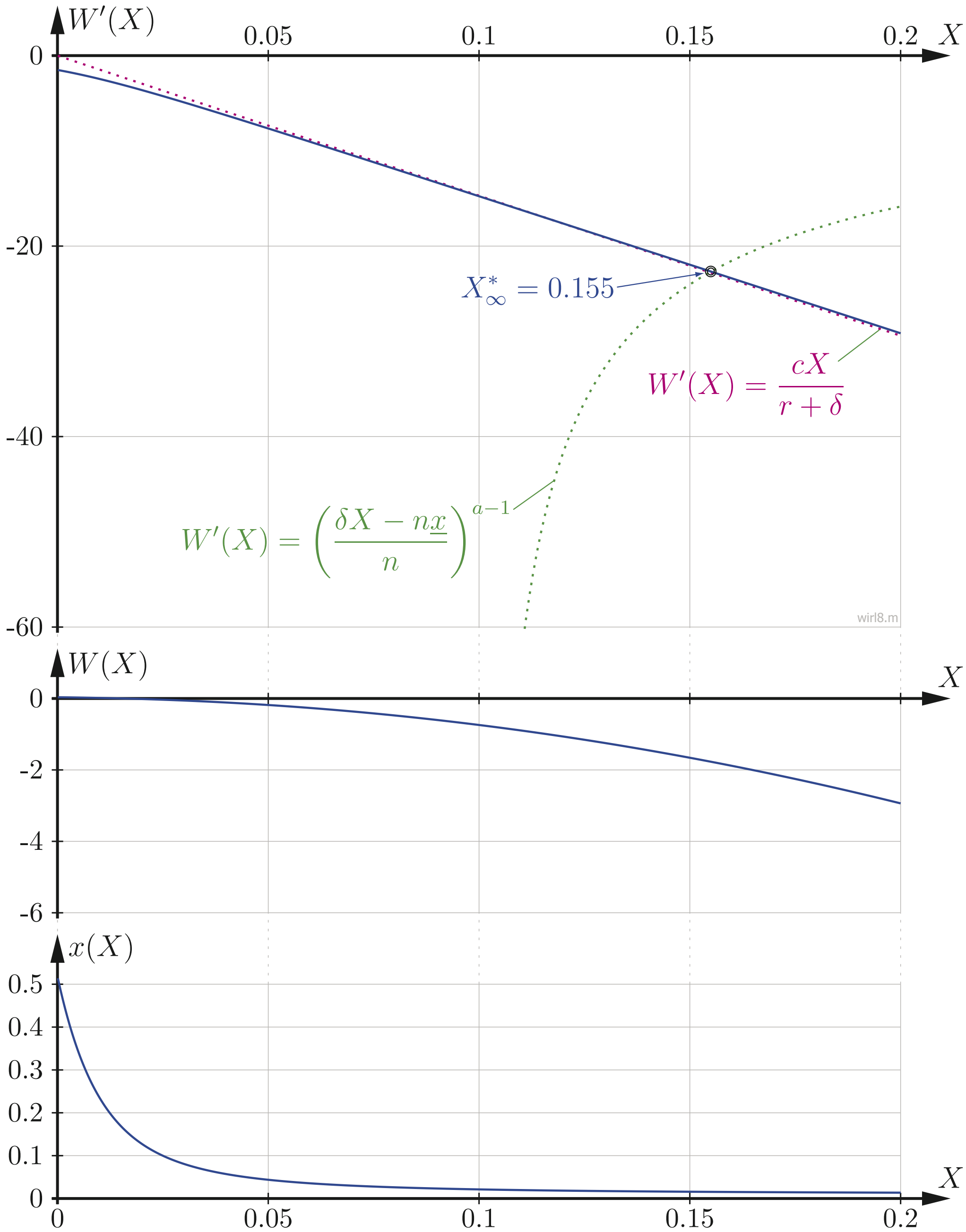

Figure 4.3: Second best solution, $a = 0.4 < \frac{1}{n}$.

5 Conclusion

Solving differential equations of the form

$$\rho V(S) = \max_x \left\{ f(S,x) + g(S,x) V_S(S) + \frac{1}{2}\sigma^2(S) V_{SS}(S) \right\}. \tag{1.3}$$

that is, continuous time stochastic control problems of the form

$$V(S) = \max_x \mathrm{E} \left\{ \int_0^\infty e^{-\rho t} f(S,x)\,\mathrm{d}t \right\} \tag{1.1}$$

$$\mathrm{d}S = g(S,x)\,\mathrm{d}t + \sigma(S)\,\mathrm{d}z, \tag{1.2}$$

using the algorithm described in Section 1.2 (taken from [5, Section 11.2]) turns out to be straight forward. Numerical instabilities only arose if

- the order of the basis functions was chosen too high (then the order had to be decreased iteratively until the numerical instabilities resolved)
- or if optimal stopping problems were solved using Chebychev polynomials as basis functions (cubic splines resolved the numerical problems in these cases).

Summarizing, despite the small difficulties pointed out above, the algorithm of [5, Section 11.2] shows excellent convergence for the examples presented in Chapters 1 to 4.

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Work Experience and Teaching

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July 2005 – Sep. 2009	**Teaching Assistant**, *Vienna University of Technology*, Austria. full time, responsible for the exercises in "Mobile Communication"
Mar. 2005 – Nov. 2008	**Erasmus Coordinator**, *Vienna University of Technology*, Austria. part time, responsible for most Erasmus related activities at the faculty
Oct. 2002 – June 2004	**Tutor**, *Vienna University of Technology*, Vienna. part time, for "Wave Propagation" and "Electrical Engineering Laboratory"
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Publications

(see Page 3 to 5 for details)

Ph.D. thesis	**one**, on measuring wireless communication systems
master thesis	**one**, on developing a 4x4 multiple antenna testbed
journal papers	**four**
conference papers	**nineteen**
invited talks	**six**, all above in the field of "testbedding" wireless communication systems

Languages

German	**native**
English	**business fluent**

Computer Skills

OS	all Windows versions
programs	Office (excellent), Adobe Illustrator (excellent), Photoshop
programming	VB (excellent), VBA, Matlab (excellent), Mathematica, C, C++, HTML
miscellaneous	computer hardware (excellent), LaTeX, SAP (basic)

Interests

sports	volleyball, cycling, orienteering
outdoors	hiking, usually with a tent and friends in the Austrian mountains
traveling	backpacking around the world, e.g. the whole February 2009 in Peru

List of Publications

Journal Papers

[1] C. Mehlführer, S. Caban, and M. Rupp, "**Experimental evaluation of adaptive modulation and coding in MIMO WiMAX with limited feedback**," *EURASIP Journal on Advances in Signal Processing*, vol. 2008, Article ID 837102, 2008, doi: 10.1155/2008/837102.
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