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Combinatorial Games on Cayley Graphs

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Abstract

We study two- and more-player games on Cayley graphs. In the relator games introduced by Gates and Kelvey, two players add to a joint path through the Cayley graph while seeking or avoiding cycles. We present our insights about relator games on Coxeter groups. We also consider the cops and robbers game and upper bounds on the cop number, the minimum number of cops needed to always win the game, for a given graph. We discuss a paper by Frankl on the cop number of Cayley graphs (under some assumptions on the group's generating set), as well as a recent preprint by Biswas and Saha extending Frankl's method to Cayley sum graphs and other algebraic graphs. In both cases, we point out minor issues in the proofs and propose corrections.

Abstract

Wir betrachten Spiele für zwei und mehrere Spieler auf Cayleygraphen. In den Relator Games von Gates und Kelvey verlängern zwei Spieler abwechselnd einen gemeinsamen Pfad in dem Cayleygraphen, wobei Kreise entweder angestrebt oder vermieden werden. Wir präsentieren einige Erkenntnisse über Relator Games für Coxeter-Gruppen. Weiters betrachten wir das Spiel Cops and Robbers sowie obere Schranken für die Cop-Zahl, die minimale Anzahl an Cops, die für einen bestimmten Graphen notwendig ist, um das Spiel immer zu gewinnen. Wir besprechen eine Publikation von Frankl über die Cop-Zahl von Cayleygraphen, sowie ein aktuelles Preprint von Biswas und Saha, in dem Frankls Methode auf Cayley-Summengraphen und andere algebraische Graphen angewendet wird. Für beide Arbeiten weisen wir auf kleinere Probleme in den Beweisen hin und schlagen Möglichkeiten vor, diese zu korrigieren.

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1 Introduction

Many different games have been defined and analyzed that involve the structure of a — sometimes finite, sometimes infinite — graph in their rules. We study some of these games in the case of Cayley graphs of group presentations.

Some graph games fall into the broader category of impartial games: These are deterministic games with perfect information between two players who have identical options and alternate turns. In addition, they must be guaranteed to end after finitely many turns. A prime example of an impartial game is the game of nim (see for example [8, Chapter I]), which is played with a finite number of heaps, each consisting of finitely many tokens. Players take turns removing any number of tokens from a single heap, and the player who can remove the last tokens wins.

Impartial games on graphs include impartial hypergraph games [7] and relator games [6]. The latter are defined exclusively on Cayley graphs, as they are based on building words on the generators of a group presentation, and the goal is either to produce one of the relators as a sub-word or to avoid doing so.

We introduce the relator games REL (the relator achievement game) and RAV (the relator avoidance game) in Chapter 3 of this thesis and repeat some basic observations from Gates and Kelvey [6]. We consider the relator games in the framework of impartial games (for this we refer to [8]) and contrast them with impartial hypergraph games. We focus particularly on relator games of Coxeter groups. For RAV, we easily obtain that the first player can always win for any Coxeter presentation. For REL, the situation is unclear in general. Besides the 2-generator Coxeter groups, which have very simple Cayley graphs, we are only able to give results for a few 3-generator Coxeter groups, which were obtained with a computer-based brute force approach. In all these known cases, the second player wins the game.

There are also many graph games that diverge from the definition of impartial games in different ways: Games can have more or less than two players, a game can be asymmetric in the sense that the different players have different moves available to them or have different winning conditions, there are games that sometimes go on for infinitely many rounds, and so on.

In Chapter 4, we discuss the game of cops and robbers, where N cops play as a team against a single robber. The robber moves around the finite graph trying to evade the cops forever, while the cops try to catch the robber in finitely many rounds. The cop number of a graph is the lowest number N such that N cops can guarantee a win.

1 Introduction

Frankl [5] showed that for a finite abelian group G with generating set S , the cop number of the Cayley graph $C(G, S)$ is at most $\left\lceil \frac{|S|+1}{2} \right\rceil$. We focus on another result by Frankl [4], in the setting where the group need not be abelian, but it is assumed that the generating set S is symmetric and closed under conjugation. Under this assumption, Frankl describes a strategy for $|S| + 1$ many cops. The ideas of this strategy are generalized by Biswas and Saha [1] to three variants of Cayley graphs, namely Cayley sum graphs, twisted Cayley graphs and twisted Cayley sum graphs. Here we only discuss their strategy for Cayley sum graphs, as their treatments of all three graphs follow a very similar structure. We highlight minor issues that we found with the strategies in [4] and [1]. We are able to present an option to correct each strategy, and thus show that the cop number is indeed bounded by $|S|$ and $|S| + 1$ for Cayley graphs and Cayley sum graphs, respectively.

To round out the discussion, we repeat the asymptotic discussion of the cop number of Cayley and Cayley sum graphs given in [1], to show that a weak version of Meyniel's conjecture holds for these graphs: For any group G with n elements, a symmetric generating set S that is closed under conjugation, and any positive constant $c < \frac{1}{2}$, the cop numbers of the Cayley and Cayley sum graphs are both in $\mathcal{O}(n^{1-c})$.

2 Preliminaries

Group presentations and their associated Cayley graphs play a fundamental role in this thesis. We recall some basic concepts about them here.

Definition 2.1. For a group G and a subset S , the *Cayley graph* $C(G, S)$ is a directed graph with vertices $V = G$ and edges $E = \{(u, v) | v = us \text{ for some } s \in S\}$.

Definition 2.2. (i) An undirected graph is *connected* if there exists a path between any two vertices u and v in the graph.
(ii) A directed graph is (*weakly*) *connected* if the underlying undirected graph (obtained by replacing all directed edges with undirected ones) is connected.

All the graph games in this thesis are played on the undirected version of the Cayley graph, so we will never talk about *strong connectivity*, where all vertices must be connected by directed paths.

Definition 2.3. A directed graph is sometimes called *undirected* if every directed edge has an opposite-direction counterpart, i.e. (u, v) is an edge if and only if (v, u) is an edge for all vertices u and v .

Lemma 2.4. (i) For a group G and a subset S , the Cayley graph $C(G, S)$ is connected if and only if S generates G .
(ii) The Cayley graph is undirected if and only if S is symmetric, i.e. $S = S^{-1}$.

Definition 2.5. For a group G , a *presentation* consists of two sets S and R satisfying the following:

- S is a subset of G which generates the whole group.
- The elements of R are words on S (called *relators*).
- The group G is isomorphic to the quotient group of the free group F_S on S by the normal closure of R in F_S .

Such a group presentation is denoted by $G = \langle S | R \rangle$. If $S = \{s_1, \dots, s_n\}$ and $R = \{r_1, \dots, r_m\}$ are both finite sets, the presentation may also be written as $G = \langle s_1, \dots, s_n | r_1, \dots, r_m \rangle$ or $G = \langle s_1, \dots, s_n | r_1 = \dots = r_m = e \rangle$.

For a given group presentation $G = \langle S | R \rangle$, the associated Cayley graph is $C(G, S)$.

Let us also establish some phrases we will use to talk about games.

Notation. Unless other names are specified, we will refer to the player who takes the first turn in a game as Player 1, the player who takes the next turn as Player 2, and so on.

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Definition 2.6 (Winner of a game). For a given game, we say that Player X *has a winning strategy* if this player is able to guarantee a win no matter what choices the other player(s) make. In this case we also say that Player X *can always win* or that Player X *(always) wins the game*. (The last two formulations implicitly assume that Player X chooses optimally at every turn.)

3 Relator games

Gates and Kelvey [6] define two similar two-player games based on forming words from the generators of a given group presentation: Let G be a finite group and S a generating set of G . The two players alternate; at each turn a player chooses an element from $S \cup S^{-1}$ and adds it at the end of the shared word. It is not allowed to play the inverse of the element that has just been played by the other player. The shared word can be viewed as a path in (the undirected version of) the Cayley graph that is being augmented by one edge per turn. We say that a *cycle* is formed (or closed) when the newly added element causes the shared word to contain a sub-word that is equal to the identity in G . On the Cayley graph, this corresponds to the path visiting some node that has already been visited previously, hence the term “cycle”.

In the *relator achievement game* (REL), the first player to close a cycle wins. In the *relator avoidance game* (RAV), the first player to form a cycle loses.

Example 3.1. Consider the cyclic group $C_n = \langle a | a^n = e \rangle$, where the undirected Cayley graph is an n -gon. For $n = 2$, every game trivially ends before there is a winner, because there is no legal move available on the second turn. For $n \geq 3$, Player 1 can choose either a or a^{-1} , resulting in two completely symmetric game states, and on every subsequent turn there is only one legal move. Therefore the outcome of the games is fully determined by the parity of n . If n is odd, Player 1 always wins REL and Player 2 always wins RAV. If n is even, Player 2 always wins REL and Player 1 always wins RAV.

Notice that considering the same group with a different generating set can change the outcome of the relator games:

Example 3.2. Take now $G = C_n$ or any other group G with n elements for $n \geq 3$ with the generating set $S = G$. Then the Cayley graph is the complete graph on n vertices. Notice that on every turn, the active player can move to any vertex except stay at the current one or backtrack to the previous one. We find that for any $n \geq 3$, Player 1 always wins REL on the third turn of the game: First, Player 1 moves from e to some other vertex, then Player 2 makes a move, but is not allowed to go back to e , and on the third turn, Player 1 can always go to e to close the cycle. For the game of RAV, the goal is to avoid repeating a vertex as long as possible. On every turn, if there exists a vertex in the graph that has never been visited, the active player is allowed to move there. Therefore the parity of n decides which player is eventually forced to repeat a vertex. For n odd, Player 2 wins RAV, and for n even, Player 1 wins RAV.

We had a particular interest in understanding the relator games on Cayley graphs of Coxeter presentations. All of the information we mention here regarding Coxeter groups can be found in [3].

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Definition 3.3. A group presentation is called *Coxeter presentation* if it is of the form $\langle S | (st)^{m_{st}} = e \text{ for all } s, t \in S \rangle$, where S is a finite set of generators, $m_{ss} = 1$ for all $s \in S$, $m_{st} = m_{ts} \in \{2, 3, \dots\} \cup \{\infty\}$ for $s \neq t$ and infinitely long relators are meant to be ignored, i.e. $m_{st} = \infty$ is understood as no relation between s and t .

For our analysis of graph games, we are only interested in Coxeter presentations that result in finite groups.

Example 3.4. Consider any Coxeter presentation with 2 generators. If $m_{12} = \infty$, then the resulting group is not finite. Otherwise, the presentation is of the form $\langle s, t | s^2 = t^2 = (st)^n = e \rangle$ for some $n \in \mathbb{N}$ and the group is isomorphic to a dihedral group. The undirected Cayley graph of this presentation is isomorphic to the one of $C_{2n} = \langle a | a^{2n} = e \rangle$. Hence Example 3.1 shows that Player 2 always wins REL and Player 1 always wins RAV.

Remark 3.5. For the other canonical presentation of the dihedral group (which is not a Coxeter presentation), the Cayley graph has a slightly more complex structure. Hence the analysis of the relator games is also less trivial. The results can be found in [6].

The following result can be found as Theorem 3.9 in [6].

Theorem 3.6. *Suppose that a group presentation has at least one generator of order 2. Then Player 1 has a winning strategy for the relator avoidance game.*

Proof. Let $s \in S$ be a generator of order 2. Then the winning strategy for Player 1 is to always play s . (This is always legal: For the first move Player 1 can choose any element of $S \cup S^{-1}$, and from then on, Player 2 is never allowed to play $s^{-1} = s$, so s is always legal on Player 1's turn.) Suppose that on some turn n (with n odd), Player 1 completes a cycle with this strategy, i.e. Player 1 enters a vertex g on the undirected Cayley graph that has been visited by the shared path once before. We know that Player 1 goes from gs to $gs^2 = g$, so on turn $n - 1$, Player 2 has entered vertex gs .

Consider now the first time that the shared path has visited vertex g , at some turn $m < n$. There are two cases: If m is odd, i.e. Player 1 has caused the first visit to g , then by the same argument as above, the previous end point of the shared path must have been gs . (I.e. Player 2 must have entered gs on turn $m - 1$ if $m > 1$, or gs is equivalent to the starting point e of the game if $m = 1$.) Hence Player 2 has closed a cycle on turn $n - 1$, losing the game and ending it before Player 1 could lose it on turn n .

Suppose now that m is even, i.e. Player 2 was responsible for the first visit to g . Then Player 1, as a consequence of always playing s , must have gone to gs on turn $m + 1$. This again means that the move of Player 2 on turn $n - 1$ closes a cycle, ending the game.

There is also the case that $g = e$ and no player has previously returned to the starting vertex e . We know that Player 1 moves to s on the first turn. So when Player 2 moves to $gs = s$ on turn $n - 1$, it closes a cycle. Hence it is not possible that Player 1 is the first one to close a cycle. $\square$

Remark 3.7. Consider what happens if the group presentation involves multiple generators of order 2 that are not inverses of each other. Then Player 2 can also pick one of them and use the same strategy. However, one cannot modify the proof above to also show that Player 2 is never the first to close a cycle. If one were to swap Player 1 and Player 2 in the argument from the proof and try to reach a contradiction, this would fail in the case where Player 2 closes the cycle at vertex e .

Corollary 3.8. *In the relator avoidance game on any Coxeter presentation of a finite group, Player 1 always has a winning strategy.*

Proof. By definition, every generator in a Coxeter presentation has order 2. Therefore by Theorem 3.6, Player 1 has a winning strategy. $\square$

For the relator achievement game on Coxeter presentations, no such general result is known. In the following we describe our approaches to analyzing the relator achievement game and the results we were able to obtain — with a focus on Coxeter presentations.

3.1 The theory of impartial games

The relator games can be categorized as examples of *impartial games*: An impartial game is a deterministic two-player game with perfect information where both players alternate turns, and have the same options whenever it is their turn, and the same winning conditions. This definition is clearly satisfied by the relator games.

In [7], the following formal description is given:

Definition 3.9. An *impartial game* consists of a finite set $\mathcal{P}$ of *positions*, with one of them being the starting position, and a possibly empty set of *options* $\text{Opt}(P) \subseteq \mathcal{P}$ for each position $P \in \mathcal{P}$. On each turn, the active player picks one of the options of the current position P to be the next position. If a position has no options, then it is called a *terminal position*, and the player who is tasked with picking the next position loses the game. It is required that every game must finish (i.e. have a loser) after finitely many turns.

An impartial game can be represented as a directed acyclic graph called the *game digraph*, where the vertices are the positions of the game, and the successors of a vertex are exactly its options. We can imagine that the two players take turns moving a token along a directed edge to a new position, until it reaches a vertex without outgoing edges. Examples of game digraphs can be found in the next section. (see Figure 3.1 and Figure 3.2)

An important result in the theory of combinatorial games is the *Sprague-Grundy theorem*. It states that every impartial game in the sense of the definition above is equal to a single-heap game of nim with some number of tokens. This motivates the following definition.

Definition 3.10. The *nim-value* $\text{nim}(P)$ of a position P is defined recursively as the *minimum excludent* of the set of nim-values of the options of P . The *minimum*

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excludent of a set of natural numbers is the smallest natural number missing from the set.

Nim-values are an important tool for analyzing the positions of an impartial game, and for determining the overall winner of the game given that both players play optimally.

A terminal position P has no options, so $\text{nim}(P) = 0$, and this is clearly a losing position for the player who is about to move. If $\text{nim}(P) > 0$, then it is a winning position for the player who is about to move, and their optimal strategy is to move the token in the game digraph to one of the options (i.e. children) with nim-value 0 (which always exists). Any position P with $\text{nim}(P) = 0$ has no options that also have nim-value 0, and P is a losing position for the player who is about to move. Hence Player 1 has a winning strategy for the entire game if the nim-value of the starting position is non-zero, and Player 2 has a winning strategy if it is zero.

Impartial games and nim-values are discussed in further detail in [8, Chapter IV].

3.2 Comparison to impartial hypergraph games

Impartial hypergraph games are discussed by Sieben [7] as a sub-family of impartial games. We explored this theory in the hope of applying it to the relator achievement game. However, it soon became apparent that the relator games cannot be equivalent to any impartial hypergraph game. This is essentially because the number of distinct positions after a given number of turns grows very fast for relator games.

Definition 3.11. A *hypergraph* $H = (V, \mathcal{H})$ consists of a finite set V of vertices, and a set $\mathcal{H}$ of edges, where an edge can be any subset of V (whereas an edge in a regular graph is a set of exactly 2 vertices).

For each of the impartial hypergraph games in [7], the current position of the game is the set P of selected vertices, and each turn consists of the active player adding one previously unselected edge to P .

In the *achieve game* $\text{ACV}(H)$, the game ends as soon as P contains one of the edges of H as a subset, i.e. any position that contains an edge is terminal. The player who added the last vertex to complete the edge wins.

In the *avoid game* $\text{AVD}(H)$, the set P may never contain one of the edges of H as a subset, so any position that would force the next player to complete an edge is terminal.

Let us take an example from [7] to make some elementary observations about the game digraph of hypergraph games. We consider $\text{ACV}(H)$ and $\text{AVD}(H)$ for the hypergraph H with vertices $V = \{a, b, c\}$ and edges $\mathcal{H} = \{\{a, b\}, \{b, c\}\}$ (see Figure 3.1).

It is clear that a position in the n -th layer of the game digraph always contains n vertices. In Figure 3.1a we notice that the set $\{a, b\}$ describes a single position, meaning it does not matter for subsequent rounds of the game whether a or b was

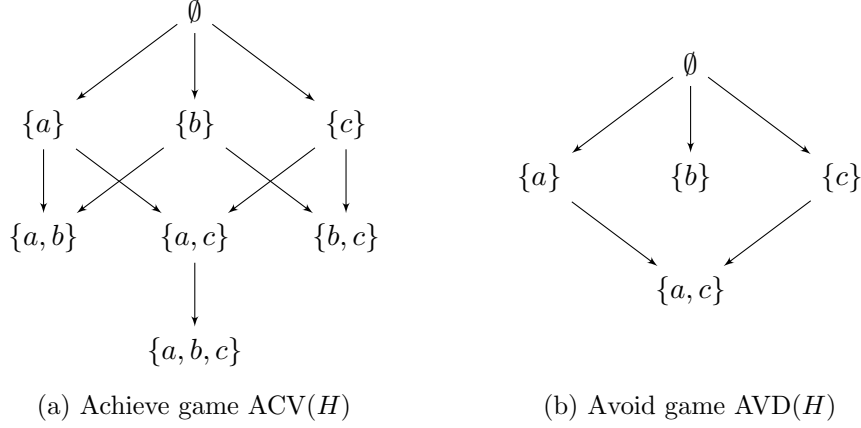


Figure 3.1: Game digraphs of impartial hypergraph games for the hypergraph $H = (V, \mathcal{H})$ with $V = \{a, b, c\}$ and $\mathcal{H} = \{\{a, b\}, \{b, c\}\}$ ([7, Figure 3.1])

picked first. All positions in the hypergraph games can be described as unordered sets, and as a result it is very common that the game digraph contains such “diamond” shapes, where a vertex has more than one parent. It follows that the k -th layer contains at most $\binom{|V|}{k}$ positions, that being all the k -element subsets of V . A subset of V can also be absent from the game digraph, meaning that it can never be completed before terminating the game. For example, in the game in Figure 3.1b, one cannot make the set $\{a, b, c\}$ without previously reaching a set that contains either $\{b\}$ or $\{a, c\}$ as a subset, both of which terminate the game. Hence the position $\{a, b, c\}$ is not included in the game digraph.

Let us now consider the game digraph of a relator game, and compare it to those of impartial hypergraph games.

Example 3.12. Take the group presentation $D_3 = \langle s, r \mid r^3 = s^2 = rsrs = e \rangle$ of the dihedral group with six elements, and consider the relator achievement game. At every turn, it is clear that the current position of the game is described by the shared path through the Cayley graph that the players have taken so far. It is not enough to characterize the path by the unordered set of all vertices visited so far. For instance, the paths (e, r, r^2) and (e, r^2, r) cannot be considered as the same game state, because they have different sets of options: From (e, r, r^2) one can move to r^2s , but from (e, r^2, r) one cannot.

Hence we see that the “current” (most recent) vertex in the path is necessary to determine the set of options; but this vertex alone cannot describe the position either: The paths (e, r, rs) and (e, s, rs) have the same most recent vertex, but from the first path the players are allowed to go to s next, and from the second path they are not. For a different example, consider the paths (e, s, rs, r^2s) and (e, r^2, r^2s) . Both are currently at vertex r^2s , and in both cases the next player is allowed to add vertex s to the path. But in the first case this ends the game, and in the second case it does not.

We deduce that at the least, two paths must be considered distinct positions if they

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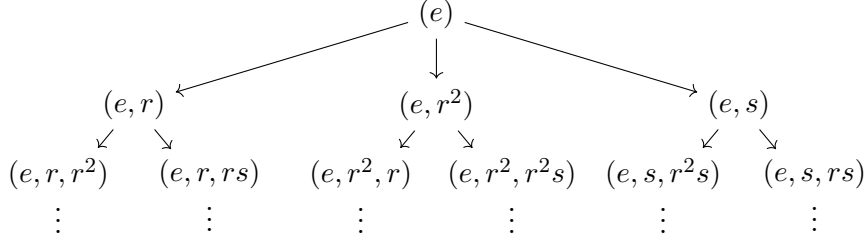


Figure 3.2: Part of the game digraph of the relator achievement game on D_3 with generators r and s

either do not have the same set of previously visited vertices or do not have the same most recent vertex. (This suggests that the game digraphs of relator games are quite large relative to the size of the underlying group.)

It follows that the first few layers of the game digraph must look as depicted in Figure 3.2.

Lemma 3.13. (i) *There is no hypergraph H such that the achieve game $ACV(H)$ has the same game digraph as the relator achievement game from Example 3.12.*
(ii) *There is also no hypergraph H such that the avoid game $AVD(H)$ has the same game digraph as the relator achievement game from Example 3.12.*

Proof. (i) Suppose there is such a hypergraph $H = (V, \mathcal{H})$. Then H has exactly 3 vertices, since layer 1 of the game digraph has 3 positions. Hence layer 2 can have at most $\binom{|V|}{2} = 3$ positions. But in the game digraph in Figure 3.2, there are 6 positions in layer 2, a contradiction.

(ii) Suppose there is such a hypergraph $H = (V, \mathcal{H})$. Then H must have 3 vertices, call them a, b, c , corresponding to the 3 positions in layer 1. For the avoid game, it is also possible that V contains one or more additional vertices which do not correspond to positions: There may be a vertex $d \in V$ such that the singleton $\{d\}$ is an edge in $\mathcal{H}$, and therefore $\{d\}$ is not an option at the beginning of the game and thus does not appear in the game digraph. In this case, we note that any set containing d , such as $\{a, d\}$, is also not an option in the avoid game, as it contains the edge $\{d\}$ as a subset. Hence the potential existence of such an additional vertex d does not increase the upper bound on the number of options that can appear in subsequent layers of the game digraph. So we can argue as in the previous proof that layer 2 contains at most $\binom{3}{2} = 3$ positions, contradicting the 6 positions in Figure 3.2. $\square$

This shows that in general, relator achievement games of group presentations cannot be viewed as instances of the impartial hypergraph games as described in [7]. The next example shows that it also cannot hold if we limit the scope to Coxeter presentations.

Example 3.14. Consider the Coxeter presentation of the dihedral group $D_3 = \langle s, t \mid s^2 = t^2 = (st)^3 = e \rangle$. We recall from Example 3.1 and Example 3.4: The undirected Cayley graph is just a cycle with 6 vertices. For the first move of REL,

there are two choices, and hence the game digraph has two positions in layer 1. Every position apart from the starting position has at most 1 option. As the Cayley graph contains exactly one cycle, the game always ends after exactly 6 moves, and hence the game digraph consists of layers 0 to 6.

Lemma 3.15. (i) *There is no hypergraph H such that the achieve game $ACV(H)$ has the same game digraph as the relator achievement game from Example 3.14.*
(ii) *There is also no hypergraph H such that the avoid game $AVD(H)$ has the same game digraph as the relator achievement game from Example 3.14.*

Proof. (i) Suppose that there is a hypergraph $H = (V, \mathcal{H})$ s.t. $ACV(H)$ has the same game digraph. As there are 2 positions in the first layer, H must have exactly 2 vertices. But this means that every edge in $\mathcal{H}$ contains at most 2 vertices, and will therefore be achieved after at most 2 rounds of the game. So the game digraph cannot have more layers beyond layer 2, a contradiction.

(ii) We can argue very similarly that the game digraph cannot belong to an avoid game $AVD(H)$ for any hypergraph H . $\square$

3.3 Some numerical results

We also explored the relator achievement game with a brute force approach. For a particular group presentation (we considered mainly Coxeter presentations), the idea was to generate the entire game digraph using a computer program.

In Example 3.12, we observed that paths in the relator games must be considered as distinct game positions if they have a different set of previously visited vertices or a different most recent vertex. In our implementation, we even considered every path as a distinct position. This potentially increases the number of positions, but it also conveniently makes the game digraph a tree when viewed as an undirected graph, i.e. no vertex has more than one parent.

Once the game digraph is established, one can analyze each position from the bottom up. For our purposes, a slight simplification of the nim-values is sufficient, where instead of distinguishing all natural numbers, each position just has a value zero (losing) or non-zero (winning). Terminal positions are assigned zero, and any other position is assigned zero if and only if all of its children are non-zero.

We soon faced the issue that the number of positions in the game digraphs is increasing extremely fast relative to the size of the groups, to the point of being computationally infeasible. To illustrate the problem, consider once more the group presentation $D_3 = \langle s, r \mid r^3 = s^2 = rsrs = e \rangle$ with 6 group elements. In the relator achievement game on this group presentation, which is partially shown in Figure 3.2, already 90 distinct paths (of length at most 6) can appear.

We used some tricks to decrease the running time and space requirements of the program: The nim-values are computed from the bottom up, and parent nodes aggregate all the necessary information about their children. Therefore we are able

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to delete each sub-tree of the game digraph as soon as the nim-value of the sub-tree's root has been computed. Additionally, we check for each position whether the latest visited vertex is a neighbor of any previous vertex in the path. Whenever this is the case, we can immediately mark the position as losing, since the next player has the option of closing a cycle. In this way we avoid checking any other possible moves of the next player and the sub-trees they might have spawned. With these changes the program did run significantly faster, but we were only able to analyze a few more groups, as the size of the game digraphs still increased too much.

First, we executed the program for some Coxeter presentations with 2 generators, namely $D_n = \langle s, t | s^2 = t^2 = (st)^n = e \rangle$ for $n \in \{2, 3, 4, 5\}$. We already observed in Example 3.4 that the Cayley graphs of these groups are quite trivial and that Player 2 wins REL for all of them. The program produced the same results and the running times were extremely short due to the game digraph only branching a single time.

We were able to obtain results for three additional Coxeter presentations — all with 3 generators — which had not been known beforehand. To describe the groups, we recall the general form of a 3-generator Coxeter presentation:

$$\langle s_1, s_2, s_3 | (s_i s_j)^{m_{ij}} = e \rangle \text{ with } m_{ii} = 1 \text{ for all } i \in \{1, 2, 3\} \\ \text{and } m_{ij} = m_{ji} \text{ for } i, j \in \{1, 2, 3\}$$

The finite Coxeter groups are also commonly described based on their *Coxeter diagrams*. We will not explain the details of this classification here, but mention the corresponding group symbols of the three analyzed groups.

We analyzed the following groups:

- $m_{12} = m_{13} = m_{23} = 2$, resulting in the group $A_1 \times A_1 \times A_1$ in the classification of finite Coxeter groups
- $m_{12} = m_{23} = 3, m_{13} = 2$, resulting in the group A_3 in the classification of finite Coxeter groups
- $m_{12} = 4, m_{23} = 3, m_{13} = 2$, resulting in the group B_3 in the classification of finite Coxeter groups

For all three, the result of the brute force analysis was that Player 2 has a winning strategy.

Hence all of the tested Coxeter presentations have the same outcome for the game of REL, but we were only able to collect very few relevant data points due to the computational complexity. Still, the results suggest the following conjecture:

Conjecture 3.16. *For any Coxeter presentation $G = \langle S | R \rangle$, Player 2 can always win REL.*

Together with Corollary 3.8, this would mean that Player 1 always wins RAV and Player 2 always wins REL for any Coxeter presentation.

4 The cops and robbers game

For any finite, connected and undirected graph $G = (V, E)$ and a natural number N , N cops and one robber can play the game of cops and robbers: Every player chooses a vertex in the graph as their starting position, first the cops, then the robber. The game consists of the cops and the robber taking alternating turns. First, every cop chooses to either stay at their vertex, or move along an edge to one of the adjacent vertices. Then, the robber has the same choice of either staying or moving to one of the neighboring vertices. The game has perfect information, so every player knows the position of all others at all times. If it happens at any point that the robber occupies the same vertex as at least one cop, we say that the robber is “caught” and the game ends. (It does not matter if the cop moves to a vertex already occupied by the robber, or the other way around.) The cops play as a team and they win the game if one of them catches the robber after finitely many turns. The robber wins if the game continues indefinitely without a capture.

Definition 4.1. The *cop number* of a graph $G = (V, E)$ is the lowest natural number c such that in a cops and robbers game on G , c many cops can always capture the robber in finitely many steps, no matter which moves the robber chooses.

4.1 Cops and robbers on the Cayley graph

Frankl [4] claims the following bound for the cop number of a Cayley graph given some assumptions on the generating set:

Theorem 4.2. *Suppose S is a symmetric generating set for the group G and closed under conjugation. Then the cop number of the Cayley graph $C(G, S)$ is at most $|S|$.*

Frankl argues this by giving a strategy that can be used by $|S|$ many cops to guarantee catching the robber, which we will discuss in the following. We have found a minor issue with Frankl’s proof and will also describe how it can be fixed. At first, we will ignore that both the robber and the cops can choose to skip a turn at any time, i.e. not move for a turn. This case is discussed separately at the end of the chapter.

4.1.1 The strategy of Frankl

All cops choose the identity element e as their initial position, then the robber picks an initial position. Although the rules state that the cops move first, we want to consider the game as consisting of individual rounds that all start with a move of the robber, followed by a move of each cop. We can easily switch to this perspective by making the cops all move from a neighboring node to e on their first turn. We say that the subsequent move of the robber starts round 1.

4 The cops and robbers game

At the beginning of the game, we label each cop by a different generator $s \in S$ arbitrarily. These labels will be updated at the end of each round, so the same cop may have different labels throughout the game.

For every $n \geq 1$, the n -th round begins with the robber moving from their current position y to the vertex yt for some $t \in S$. For a cop with label $s \in S$ who starts the n -th round at a vertex z , we consider $z^{-1}y$ as a measure of the distance to the robber. We write $z^{-1}y = w_{z,y,k}s^i$, where $w_{z,y,k}$ is the (not necessarily unique) product of k many elements of S for k as small as possible, i.e. $z^{-1}y = g_1g_2 \cdots g_k s^i$ for some $g_1, g_2, \dots, g_k \in S$. We call the number k the *tail* of the cop. In the case that $k = 0$, the *power* of the cop is defined as the smallest natural number s.t. $z^{-1}y = s^i$.

The goal of each cop is at first to decrease their tail to zero. Then, once a cop is only a power s^i away from the robber, we can consider the cycle consisting of vertices $z, zs, zs^2, \dots, zs^{\text{ord}(s)-1}, z$, which both the cop and the robber are currently on. The cop is going to chase the robber around this cycle. Whenever the robber leaves the cycle by choosing a t that is not a power of s , the cop also traverses an edge labelled t . Notice if we shift all vertices on the above cycle by t , the result is of the form $zt, zt(t^{-1}st), zt(t^{-1}st)^2, \dots, zt(t^{-1}st)^{\text{ord}(s)-1}, zt$, and $t^{-1}st \in S$. Thus cop and robber are again on a cycle together, but one that can be traversed by edges corresponding to the generator $t^{-1}st$ instead of s , and their relative position has not changed. If two cops with inverse labels both reach tail zero, they can chase the robber around the same cycle in opposite directions.

Frankl [4] gives the following strategy for the cops: In the n -th round, the cop currently labelled by s moves from z to

$$\begin{cases} zt & \text{if } t \text{ is not a power of } s \\ zs & \text{if } t \text{ is a power of } s \text{ and } k = 0 \\ zg_1 & \text{if } t \text{ is a power of } s \text{ and } k \geq 1 \text{ with } w_{z,y,k} = g_1g_2 \cdots g_k \end{cases}$$

and changes their label from s to $t^{-1}st$ for the next round. Note that the new label of each cop is obtained by conjugating the old label by the same element, which is a bijection. So at the beginning of the next round, there will again be a 1-to-1 mapping between the labels of the cops and the generators in S .

Notation. We call the first case a “keeping up” move, since the cop is merely ensuring that the distance does not get larger. The second case is a “chasing” move, because this is when the cop chases the robber around a circle, and the third case is an “approaching” move where the cop can reduce the tail.

The idea is that each round will have either some cops that are able to “approach” and thus reduce their tail, or some cops with inverse labels “chasing” the robber, which should inevitably lead to capture if it happens often enough. Meanwhile, all other cops are “keeping up” and ensuring that their tail or power is never increasing.

However, the strategy above does not take into account that there might be two distinct generators $s_1, s_2 \in S$ such that s_2 is a power of s_1 other than s_1^{-1} . If

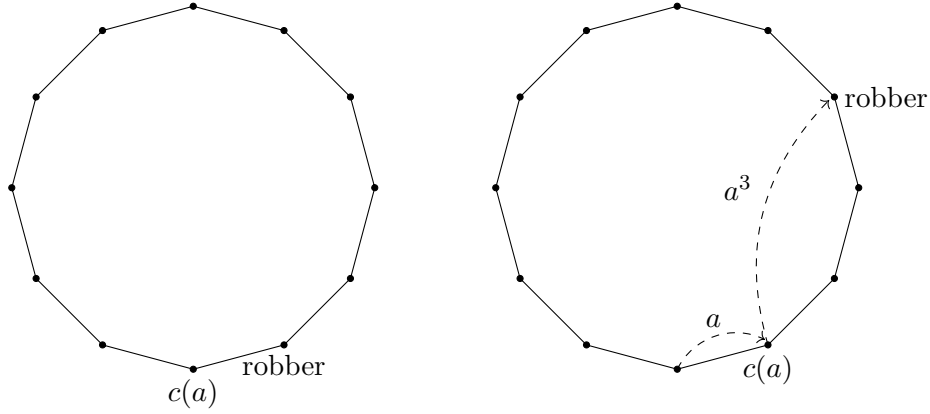


Figure 4.1: One round in a cops and robbers game on C_{12} . The power of the cop $c(a)$ goes from $i = 1$ to $i = 3$.

only one of these generators is a power of the other, this leads to some unexpected behaviour in the “chasing” case, but still results in the cops eventually winning (see Example 4.3). If both generators are a power of the other, this gives the robber a winning strategy in some cases (see Example 4.4).

Example 4.3. The power of cops with tail zero can behave unexpectedly in games where some generators are powers of other generators. Consider the cyclic group $C_{12} = \langle a | a^{12} = e \rangle$ with the generating set $S = \{a, a^{-1}, a^3, a^{-3}\}$. Notice that the labels of the cops never change in an abelian group. In this group, a^3 is a power of a , but a is not a power of a^3 . It follows that according to the strategy above, if the robber uses $t = a$, the cop with label a^3 will make a “keeping up” move $z \rightarrow za$. On the other hand, if the robber uses $t = a^3$, the cop with label a will make a “chasing” move $z \rightarrow za$ since t is a power of the label.

Consider the situation depicted in Figure 4.1, where the cop with label a has power $i = 1$, and the robber’s next move uses $t = a^3$. After the round, the cop has power $i = 3$, larger than before.

Notice that the cop with label a^3 , on the other hand, is chasing the robber across their cycle (which has only four nodes, and edges corresponding to a^3) as intended, and never increases their power. This ensures that the robber will be caught eventually: If the robber chooses $t = a^{\pm 3}$ enough times, $c(a^3)$ and $c(a^{-3})$ will catch the robber. Otherwise, the robber stops using $t = a^{\pm 3}$ at some point, and from then on $c(a)$ and $c(a^{-1})$ will only move around their cycle in steps of length 1, and catch the robber. The same argument holds whenever one generator is a power of another, but not vice versa. Therefore the strategy above wins for any such presentation.

Consider now an example where two generators are both powers of the other, and Frankl’s strategy is not winning.

Example 4.4. Take the cyclic group $C_{14} = \langle a | a^{14} = e \rangle$, and choose the generating set $S = \{a, a^{-1}, a^3, a^{-3}\}$. Any generator $t \in S$ which the robber can pick to go $y \rightarrow yt$ is a power of any other generator $s \in S$ which a cop can be labelled by. So every cop will always make a “chasing” move $z \rightarrow zs$ for their own label s .

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round	new position of ...				
	the robber	$c(a)$	$c(a^{-1})$	$c(a^3)$	$c(a^{-3})$
$n = 1$	a^5	a	a^{13}	a^3	a^{11}
$n = 2$	a^4	a^2	a^{12}	a^6	a^8
$n = 3$	a^1	a^3	a^{11}	a^9	a^5
$n = 4$	a^0	a^4	a^{10}	a^{12}	a^2
$n = 5$	a^{11}	a^5	a^9	a^1	a^{13}
$n = 6$	a^{12}	a^6	a^8	a^4	a^{10}
$n = 7$	a^{13}	a^7	a^7	a^7	a^7
$n = 8$	a^2	a^8	a^6	a^{10}	a^4
$n = 9$	a^3	a^9	a^5	a^{13}	a^1
$n = 10$	a^6	a^{10}	a^4	a^2	a^{12}
$n = 11$	a^7	a^{11}	a^3	a^5	a^9
$n = 12$	a^{10}	a^{12}	a^2	a^8	a^6
$n = 13$	a^9	a^{13}	a^1	a^{11}	a^3
$n = 14$	a^8	a^0	a^0	a^0	a^0

Table 4.1: A cops and robbers game on C_{14} . All cops start at a^0 , the robber starts at a^8 .

Recall that the cops will never change labels because the group is abelian. With this strategy of the cops, it is possible that the robber will escape them indefinitely. We describe one such game:

All cops start at $e = a^0$, the robber starts at a^8 . The subsequent moves can be found in Table 4.1. Note that in order to evade capture, the new position of the robber must be different from all positions of cops in the previous row as well as the current row of the table.

At the end of the fourteenth round, the robber and all cops are in the same position as they were at the beginning of the first round. Hence the robber can repeat the same moves indefinitely and never be caught.

4.1.2 A small correction

We propose a correction to the strategy given in [4]. In response to the problem described above, we split apart the case where a cop has tail zero and t is a power of their label.

In the n -th round, the cop currently labelled by s moves from z to

$$\begin{cases} zt & \text{if } t \text{ is not a power of } s \\ zs & \text{if } t \text{ is } s \text{ or } s^{-1}, \text{ and } k = 0 \\ zt & \text{if } t \text{ is a power of } s \text{ other than } s \text{ or } s^{-1}, \text{ and } k = 0 \\ zg_1 & \text{if } t \text{ is a power of } s \text{ and } k \geq 1 \text{ with } w_{z,y,k} = g_1 g_2 \cdots g_k \end{cases}$$

and changes their label from s to $t^{-1}st$ for the next round.

4.1 Cops and robbers on the Cayley graph

Notation. Similarly to before, we call the first and third case “keeping up” moves, the second case a “chasing” move, and the fourth case an “approaching” move.

Lemma 4.5. *(Tail and power of each cop are monotonically decreasing) Let n be a natural number, and fix any cop C . The tail of C at the beginning of the $(n+1)$ -st round is either equal to the tail of the same cop at the beginning of the n -th round, or lower. (It is lower exactly if the cop makes an approaching move in the n -th round.) If C starts the n -th round with tail 0 and power i , and $i \geq 2$, then the power of C at the start of the $(n+1)$ -st round is either i or $i-2$. (It is $i-2$ exactly if the robber chose $t = s^{-1}$.)*

If C starts the n -th round with tail 0 and power 1, then either the robber runs into C in the n -th round, or the power of C at the start of the $(n+1)$ -st round is again 1. (The robber runs into C exactly if $t = s^{-1}$.)

Proof. Suppose that at the beginning of the n -th round, cop C has some label $s \in S$ and is at a vertex z . Say that the robber moves from y to yt during this round. Let $z^{-1}y = g_1g_2 \cdots g_k s^i$ as above.

(1) Suppose that t is not a power of s . Then C moves to zt , and to determine the tail of C in the next round, we consider

$$\begin{aligned} (zt)^{-1}(yt) &= t^{-1}z^{-1}yt \\ &= t^{-1}g_1g_2 \cdots g_k s^i t \\ &= (t^{-1}g_1t)(t^{-1}g_2t) \cdots (t^{-1}g_kt)(t^{-1}st)^i. \end{aligned}$$

Notice that $t^{-1}st$ is the label of C in the $(n+1)$ -st round, and that $t^{-1}g_jt$ are in the generating set S for $j = 1, \dots, k$, as we assume it is closed under conjugation. Therefore, the tail of C at the beginning of the $(n+1)$ -st round is less or equal to k . If $k = 0$, we also notice that the power remains i .

(2) Suppose that $t \in \{s, s^{-1}\}$ and $k = 0$. Then $z^{-1}y = s^i$, and C moves to zs . At the beginning of the next round, we have that

$$\begin{aligned} (zs)^{-1}(yt) &= s^{-1}z^{-1}yt \\ &= s^{-1}s^i t \\ &= \begin{cases} s^i & \text{if } t = s \\ s^{i-2} & \text{if } t = s^{-1}. \end{cases} \end{aligned}$$

In this case, the label of C remains s after the n -th round. The equation above shows that the tail of C remains zero, and the new power of C is either i or $i-2$.

(3) Suppose that t is a power of s other than s, s^{-1} and $k = 0$. Then $z^{-1}y = s^i$, and C moves to zt . At the beginning of the next round, we have that

$$\begin{aligned} (zt)^{-1}(yt) &= t^{-1}z^{-1}yt \\ &= t^{-1}s^i t \\ &= s^i, \end{aligned}$$

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since t is a power of s . In this case, the label of C remains s after the n -th round. The equation shows that the tail of C remains zero and the power remains i .

(4) Suppose that t is a power of s and $k \geq 1$. Then C moves to zg_1 , and regarding the next round, we note that

$$\begin{aligned} (zg_1)^{-1}(yt) &= g_1^{-1}z^{-1}yt \\ &= g_1^{-1}g_1g_2 \cdots g_k s^i t \\ &= g_2 \cdots g_k s^i t. \end{aligned}$$

Note that $s^i t$ is a power of S , and that the label of C remains s after this round. Thus we see that the tail of C at the beginning of the next round is at most $k - 1$, so it has been reduced. $\square$

Now we show that $|S|$ many cops using the modified strategy we gave above will always catch the robber after finitely many turns. This will show that the cop number is at most $|S|$.

Proof of Theorem 4.2. Let $x \in G$ be the initial position of the robber. This x can be written as a product of m many elements of S , for some $m \leq \frac{|G|}{2}$. Let ℓ be the largest order of any group element, with $\ell \leq |G|$.

In each round, the robber chooses some $t \in S$ to move from y to yt . We say the round is a special round for a cop C with label s if t is either s or s^{-1} . All of the cops who are experiencing a special round will make an “approaching” move to decrease their tail by 1 if they do not have a tail of zero yet.

Notice that if two cops C and D were labelled with inverse elements at the start of the game, they keep this property after each relabelling, and they always have special rounds at the same time.

By the pigeonhole principle, during the first $(m + \ell) \cdot |S|$ rounds of the game, at least one cop C must have at least $(m + \ell)$ many special rounds. After m many special rounds, C is guaranteed to have a tail of zero. The same is true for the cop D who's label is the inverse of C 's label. (If the label of C has order 2 in G , then D is not a separate cop, and C can catch the robber on their own in the next special round. So from now on we assume C and D are different.)

After C and D have reached tail zero, we can consider the relative positions of the robber and the cops C and D to be on a cycle whose edges all correspond the label of C . In non-special rounds, all three travel in the same direction t , and we can take the viewpoint that the whole cycle is shifted by t alongside them, as it is described in Section 4.1.1. In every special round, C and D make a “chasing” move to pursue the robber around this cycle from opposite directions in steps of length 1. Hence it is clear that one of them will catch the robber, or the robber has to run into one of them, after at most ℓ more special rounds. $\square$

4.1.3 Skipping turns

We have not yet discussed what action the cops should take in a round where the robber chooses to remain at the same vertex. It is clear that the cops cannot simply ignore all such rounds by copying the robber and also remaining at their positions. If they did, then the robber could ensure a win by skipping all their turns, causing the game to continue for infinitely many rounds (in which no one is ever moving).

For any finite graph, it is always possible to deal with this by increasing the number of cops by one. The additional cop finds a finite path from their starting position to that of the robber, and walks along this path. This ensures that a robber who never moves is caught in finitely many turns. Hence, all other cops can skip all turns that the robber skips, and follow the strategy that they have for games without skipping. In fact, this method is considered so standard that most papers on the cops and robbers game do not mention it at all in the descriptions of strategies, and sometimes do not even add a corresponding $+1$ in their results. For example Frankl [4] states that the cop number is bounded by $|S|$ instead of $|S| + 1$.

For the particular strategy on the Cayley graph presented above, there is also a way to actually avoid using an additional cop, by letting the $|S|$ many cops from before make the following moves if the robber stays in place: Every cop with tail $k \geq 1$ will make an “approaching” move, and every cop with tail $k = 0$ will make a “chasing” move. More precisely, the cop with label s will go from z to

$$\begin{cases} zs & \text{if the robber is skipping a turn and } k = 0 \\ zg_1 & \text{if the robber is skipping a turn and } k \geq 1 \text{ with } w_{z,y,k} = g_1g_2 \cdots g_k \end{cases}$$

and the label will not be changed.

Lemma 4.6. *If the robber remains at the same vertex y during the n -th round, then the cop C with label s will either decrease their tail from k at the start of the n -th round to $k - 1$ the start of the $(n + 1)$ -st round, or decrease their power from i to $i - 1$ within the same time.*

Proof. (1) Suppose that at the beginning of the n -th round we have $z^{-1}y = s^i$. Then C moves from z to zs , and for the next round, we consider

$$\begin{aligned} (zs)^{-1}y &= s^{-1}z^{-1}y \\ &= s^{-1}s^i \\ &= s^{i-1}. \end{aligned}$$

This shows that the new power of C is $i - 1$, since the label of the cop is still s .

(2) Suppose that at the beginning of the n -th round, the tail of C is $k \geq 1$. Then C moves to zg_1 , and for the next round, we note that

$$\begin{aligned} (zg_1)^{-1}y &= g_1^{-1}z^{-1}y \\ &= g_1^{-1}g_1g_2 \cdots g_k s^i \\ &= g_2 \cdots g_k s^i. \end{aligned}$$

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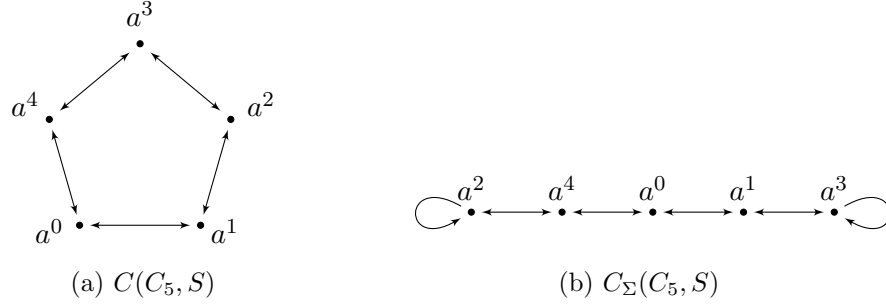


Figure 4.2: Cayley and Cayley sum graph for C_5 with the generating set $S = \{a, a^{-1}\}$

Since the label of C remains s , we see that the tail of C at the beginning of the next round is $k - 1$. $\square$

4.2 Cops and robbers on the Cayley sum graph

We introduce Cayley sum graphs and some of their basic properties. Compare this to the description of Cayley graphs in Chapter 2.

Definition 4.7. For a group G and a subset S , the *Cayley sum graph* $C_\Sigma(G, S)$ is a directed graph with vertices $V = G$ and edges $E = \{(u, v) | v = u^{-1}s \text{ for some } s \in S\}$.

Cayley sum graphs are closely related to Cayley graphs. However, Cayley sum graphs do not generally have such a high degree of symmetry, and can therefore be more difficult to analyze.

Example 4.8. Consider the cyclic group $C_5 = \langle a | a^5 = e \rangle$ with the generating set $S = \{a, a^{-1}\}$. Clearly, the Cayley graph $C(C_5, S)$ is a cycle graph. On the other hand, the Cayley sum graph $C_\Sigma(C_5, S)$ is a path graph with loops at the endpoints (see Figure 4.2).

Lemma 4.9. (i) For a group G and a symmetric subset S , the Cayley sum graph $C_\Sigma(G, S)$ is connected if and only if S generates G .
(ii) The Cayley sum graph is undirected if and only if S is closed under conjugation.

Proof. (i) Suppose that $C_\Sigma(G, S)$ is connected and take an arbitrary element $g \in G$. There exists a path from e to g in the underlying undirected version of the Cayley sum graph, say of length n . This path is of the form $e \rightarrow s_1 \rightarrow s_1^{-1}s_2 \rightarrow s_2^{-1}s_1s_3 \rightarrow \dots \rightarrow \bar{s}(n) = g$ where

$$\bar{s}(n) = \begin{cases} s_{n-1}^{-1} \dots s_4^{-1} s_2^{-1} s_1 s_3 \dots s_n & \text{if } n \text{ odd} \\ s_{n-1}^{-1} \dots s_3^{-1} s_1^{-1} s_2 s_4 \dots s_n & \text{if } n \text{ even} \end{cases}$$

for some $s_i \in S \cup S^{-1}$ for $i \in \{1, \dots, n\}$. This shows that g can be written as a finite product of elements in S and their inverses. Since g was arbitrary, this means that S is a generating set.

Conversely, assume that S generates G . It suffices to show that there is an undirected path in the Cayley sum graph from e to an arbitrary element $g \in G$. The element g

4.2 Cops and robbers on the Cayley sum graph

is equal to a product of n elements of $S \cup S^{-1}$ for some n . Hence g can also be written in the form of $\bar{s}(n)$ from above, by replacing some s_i by $(s_i^{-1})^{-1}$ if necessary. This shows that the Cayley sum graph contains an undirected path from e to $\bar{s}(n) = g$. Hence any two elements $g, h \in G$ are connected by an undirected path going through the vertex e .

(ii) For a single edge $(g, h) \in E$, where $h = g^{-1}s$ for some $s \in S$, the inverse edge (h, g) exists if and only if $hsh^{-1} \in S$, because g can be written as $g = sh^{-1} = h^{-1}(hsh^{-1})$. Consequently, for a single node $g \in G$ with outgoing edges $(g, g^{-1}s)$, all edges $(g^{-1}s, g)$ exist if and only if $(g^{-1}s)s(g^{-1}s)^{-1} = g^{-1}sg \in S$ for all $s \in S$. Therefore, all edges are undirected if and only if S is closed under conjugation by all elements $g \in G$. $\square$

The following fact gives some intuition for the strategies that will be described in this section.

Lemma 4.10. *Let G be a group, and S be a symmetric (generating) set that is closed under conjugation. Let u and v be two elements of G .*

- (i) *There exists a path of length 2 from u to v in the Cayley graph $C(G, S)$ if and only if there exists a path of length 2 from u to v in the Cayley sum graph $C_\Sigma(G, S)$.*
- (ii) *For any natural number n , there exists a path of length $2n$ from u to v in the Cayley graph $C(G, S)$ if and only if there exists a path of length $2n$ from u to v in the Cayley sum graph $C_\Sigma(G, S)$.*

Proof. (i) Suppose that there is a path of length 2 from u to v in the Cayley graph. The edges on this path must be of the form (u, us_1) and (us_1, us_1s_2) for some generators $s_1, s_2 \in S$, and $us_1s_2 = v$. Since S is symmetric and closed under conjugation, we have that $s = us_1^{-1}u^{-1} \in S$. Therefore, the Cayley sum graph contains the edges $(u, u^{-1}s)$ and $(u^{-1}s, (u^{-1}s)^{-1}s_2)$, and we see that $(u^{-1}s)^{-1}s_2 = us_1u^{-1}us_2 = v$, i.e. these edges form a path from u to v in the Cayley sum graph. Conversely, suppose that the Cayley sum graph contains a path of length 2 from u to v . The edges on this path must be of the form $(u, u^{-1}s_1)$ and $(u^{-1}s_1, s_1^{-1}us_2)$ for some generators $s_1, s_2 \in S$, and $s_1^{-1}us_2 = v$. It follows from the assumptions that $s = u^{-1}s_1^{-1}u \in S$. Therefore, the Cayley graph contains edges (u, us) and (us, uss_2) , and it holds that $uss_2 = uu^{-1}s_1^{-1}us_2 = v$, so there is a path of length 2 from u to v in the Cayley graph.

(ii) This follows immediately by induction. $\square$

Biswas and Saha [1] state that for an undirected, connected Cayley sum graph $C_\Sigma(G, S)$ with S symmetric, the cop number is at most $|S| + 1$. (The $+1$ is implicitly assumed to deal with the skipping case.) However, some issues were found in this version of the paper ([v1] on arXiv). We will describe these issues, and then propose a way to fix the strategy.

4.2.1 The strategy of Biswas and Saha

For now, we describe the strategy suggested by Biswas and Saha [1] for $|S| + 1$ many cops. It is derived from the strategy for Cayley graphs given by Frankl [4] (discussed

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in Section 4.1).

All cops choose the identity element e as their initial position, then the robber picks an initial position. Like before, we consider the game as consisting of individual rounds that all start with a move of the robber, followed by a move of each cop. Again each cop is labelled by a different element $s \in S$, which will be updated at the end of each round.

Like in the discussion on the strategy for Cayley graphs, we will ignore at first that turns may be skipped, so let us say that every round begins with the robber moving from his position at some vertex y to $y^{-1}t$ for some $t \in S$. We define the following notion:

Definition 4.11. The *connection* of the robber during the n -th round is $\hat{t} := y^{-1}t^{-1}y \in S$ if n is odd, and $\hat{t} := t \in S$ if n is even.

For a cop who has some label $s \in S$ and is at some vertex $z \in G$ at the beginning of the n -th round, we define the *tail* and *power* as a measure of how close this cop is to catching the robber. At first glance, these following definitions measure the distance of a cop to the robber only in the Cayley graph. Lemma 4.10 gives some motivation for why this might be of interest also when dealing with Cayley sum graphs.

Definition 4.12. (a) If n is odd, we write $z^{-1}y = w_{z,y,k}s^i$, where i is an integer, and $w_{z,y,k}$ is a product of k many elements of S with k as small as possible. We call k the *tail* of the cop. For $k = 0$, we say the *power* of the cop is the smallest positive integer i s.t. $z^{-1}y = s^i$.

(b) If n is even, we instead write $zy^{-1} = w_{z,y,k}s^i$, where i is an integer, and $w_{z,y,k}$ is a product of k many elements of S with k as small as possible. We call k the *tail* of the cop. For $k = 0$, we say the *power* of the cop is the smallest positive integer i s.t. $zy^{-1} = s^i$.

Clearly a cop catches the robber exactly if and when their tail and power are both 0.

Remark 4.13. Note that there can be multiple such representations of the distance of a cop to the robber (i.e. $z^{-1}y$ or zy^{-1} respectively) that all have minimal tail k , but use a different sequence of generators $g_1, \dots, g_k$. The following strategy, given in [1], refers to $g_1, \dots, g_k$ in its definition. So the move of a cop depends on which representation of the distance is arbitrarily chosen (among those with k minimal). Hence the strategy can be thought of as being non-deterministic.

In the setting described above, in the n -th round, the strategy of the cop with label s is the following:

If n is odd, move from z to

$$\begin{cases} \hat{t}^{-1}z^{-1} & \text{if } \hat{t} \text{ is not a power of } s \\ s^{-1}z^{-1} & \text{if } \hat{t} \in \{s, s^{-1}\} \text{ and } k = 0 \\ \hat{t}^{-1}z^{-1} & \text{if } \hat{t} \text{ is a power of } s \text{ other than } s \text{ or } s^{-1}, \text{ and } k = 0 \\ g_1^{-1}z^{-1} & \text{if } \hat{t} \text{ is a power of } s \text{ and } k \geq 1 \text{ with } w_{z,y,k} = g_1g_2 \cdots g_k \end{cases}$$

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and change the label from s to $\hat{t}^{-1}s\hat{t}$ for the next round.

(Note that the above moves are all of the form $z \rightarrow x^{-1}z^{-1}$ for $x \in S$. Thus they are allowed in the game on $C_\Sigma(G, S)$ because we can rewrite $x^{-1}z^{-1} = z^{-1}(zx^{-1}z^{-1})$ and S is symmetric and closed under conjugation.)

If n is even, move from z to

$$\begin{cases} z^{-1}\hat{t} & \text{if } \hat{t} \text{ is not a power of } s \\ z^{-1}s & \text{if } \hat{t} \in \{s, s^{-1}\} \text{ and } k = 0 \\ z^{-1}\hat{t} & \text{if } \hat{t} \text{ is a power of } s \text{ other than } s, s^{-1} \text{ and } k = 0 \\ z^{-1}g_1 & \text{if } \hat{t} \text{ is a power of } s \text{ and } k \geq 1 \text{ with } w_{z,y,k} = g_1g_2 \cdots g_k \end{cases}$$

and change the label from s to $\hat{t}^{-1}s\hat{t}$ for the next round.

We give names to the moves like in the previous section. For n odd or even, the first and third of the respective cases will be called “keeping up” moves, the second case is a “chasing” move, and the fourth case an “approaching” move.

We show that $|S|$ many cops starting at e are not guaranteed to catch the robber with this strategy, by giving a counter-example:

Example 4.14. Consider the symmetric group $G = S_4$ with the generating set S consisting of the six 4-cycles. (The 4-cycles really generate S_4 because the set $\{(12), (1234)\}$ generates S_4 , and the transposition can be written as $(12) = (1423)(1243)^2$. Notice moreover that S is symmetric and closed under conjugation.)

The corresponding cops and robbers game has six cops. We will use capital letters to give the cops names that do not change throughout the game, in addition to the labelling by generators which is updated at the end of every round. We start with cops $A(1234)$, $B(1432)$, $C(1243)$, $D(1342)$, $E(1324)$ and $F(1423)$ all positioned at vertex e . (The pairs A - B , C - D and E - F correspond to inverse generators.)

We will choose the moves of the robber in such a way that the connection is always (1234) or (1432) , so only cops A and B will make “interesting” moves. This also means that the label of A and B will never change. The cops C - F might change labels between each other, but in each round, all four will make the same “keeping up” move, so we can treat them collectively.

Suppose that the robber chooses the starting position (12) , and all cops start at e as the strategy above prescribes. The table below lists the robber’s choice of t in each round, and the moves of all players that follow from this.

Observe that the robber is not caught in any of these rounds, as they always choose a new position which is different from the position of all cops in the previous round as well as the current round. Notice that the outcome of round $n = 6$ is exactly the same as for $n = 2$, and they are both even. So the robber can repeatedly perform their moves from rounds 3–6 and never be caught, if the cops adhere to the strategy given above.

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round	t	$\hat{t}$	new position of ...			
			the robber	A	B	$C-F$
$n = 1$	(1243)	(1234)	(243)	(1423)	(1423)	(1432)
$n = 2$	(1234)	(1234)	(1324)	(143)	(234)	(13)(24)
$n = 3$	(1243)	(1234)	(134)	(12)	(12)	(1234)
$n = 4$	(1432)	(1432)	(1324)	(234)	(143)	(13)(24)
$n = 5$	(1243)	(1234)	(134)	(1423)	(1423)	(1234)
$n = 6$	(1432)	(1432)	(1324)	(143)	(234)	(13)(24)

Table 4.2: A cops and robbers game on $C_\Sigma(S_4)$. Cops start at e , the robber starts at (12).

This shows that the given strategy is not sufficient to catch the robber. However, the following lemma shows that it is “mostly” the correct idea and “mostly” successful at getting each cop closer and closer to the robber.

Lemma 4.15 (Tail and power of each cop are “mostly” decreasing). *Let n be a positive integer and fix some cop C (the label of C is not fixed) who is following the above strategy from any starting position. Then the tail of C at the beginning of the $(n + 1)$ -st round is less or equal to the tail of the same cop at the beginning of the n -th round.*

Moreover, if C has tail zero and power α at the beginning of the n -th round, then the power of C at the beginning of the $(n + 1)$ -st round is

$$\begin{cases} \alpha - 2 & \text{if in the } n\text{-th round, } \hat{t} \text{ is the inverse of the label of } C, \text{ and } \alpha \geq 2 \\ \text{ord}_G(s) - 1 & \text{if in the } n\text{-th round, } \hat{t} \text{ is the inverse of the label of } C, \text{ and } \alpha = 1 \\ \alpha & \text{if in the } n\text{-th round, } \hat{t} \text{ is some other power of the label of } C, \end{cases}$$

where $\text{ord}_G(s)$ is the order of the current label s of C . We remark that if $\alpha = 1$, then $\alpha - 2 = -1$, which is not a valid value for the power, as it is defined as the smallest non-negative exponent.

Attention should be paid to the last part of the lemma, which implies that it is possible for the power of a cop to increase in some rounds. This seems to have been overlooked in [1] (where the lemma was presented without this caveat). The following example disproves the claim that the power of each cop is monotonically decreasing in all games.

Example 4.16. Consider again the game described in Example 4.14. At the beginning of the second round, the robber is at vertex $y = (243)$, and cop B with label $s = (1432)$ is at vertex $z = (1423)$. Writing $zy^{-1} = (1423)(234) = (1432)$ shows that B has tail $k = 0$ and power $i = 1$. At the beginning of the third round, we find the robber at $y = (1324)$ and cop B at $z = (234)$. We have $z^{-1}y = (243)(1324) = (1234) = (1432)^3$, so cop B now has tail $k = 0$ and power $i = 3$, higher than before.

In [1] it is also stated at the end of this lemma that (assuming n is odd), a cop C who starts the round with tail zero and power 1 and whose label is equal to the connection, will always capture the robber during the n -th round. This is disproved by the following observation:

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Example 4.17. Consider once more the game described in Example 4.14. Let $n = 5$ and consider cop A , who always has the label (1234) . At the beginning of the n -th round, the distance between cop A and the robber is $z^{-1}y = (234)^{-1}(1324) = (1234)$, hence the cop has $k = 0$ and $i = 1$. Further, the connection of the robber in the n -th round is $\hat{t} = (1234)$, hence all the stated conditions are satisfied. However, Table 4.2 shows that cop A does not catch the robber during the n -th round.

Proof of Lemma 4.15. We always say that at the beginning of the n -th round, the cop C has some label $s \in S$ and is at some vertex z , the tail of this cop is k and if $k = 0$, the power is i . We suppose further that during the n -th round, the robber moves from a vertex y to $y^{-1}t$.

In the cases where n is odd, we write $z^{-1}y = w_{z,y,k}s^i$ as in Def. 4.11, in the cases where n is even, we write $zy^{-1} = w_{z,y,k}s^i$.

(1) Suppose that the connection $\hat{t}$ is not a power of s . If n is odd, C moves to $\hat{t}^{-1}z^{-1} = (y^{-1}t^{-1}y)^{-1}z^{-1}$. To determine the tail of C in the $(n+1)$ -st round, we write

$$\begin{aligned} ((y^{-1}t^{-1}y)^{-1}z^{-1})(y^{-1}t)^{-1} &= (y^{-1}t^{-1}y)^{-1}z^{-1}y(y^{-1}t^{-1}y) \\ &= (y^{-1}t^{-1}y)^{-1}w_{z,y,k}s^i(y^{-1}t^{-1}y) \\ &= (y^{-1}t^{-1}y)^{-1}w_{z,y,k}(y^{-1}t^{-1}y)((y^{-1}t^{-1}y)^{-1}s(y^{-1}t^{-1}y))^i. \end{aligned}$$

Note that $(y^{-1}t^{-1}y)^{-1}w_{z,y,k}(y^{-1}t^{-1}y)$ can be rewritten as a product of elements $\tilde{g}_j := (y^{-1}t^{-1}y)^{-1}g_j(y^{-1}t^{-1}y)$ for $j = 1 \dots k$, each of which is again in S because S is closed under conjugation. Further, $((y^{-1}t^{-1}y)^{-1}s(y^{-1}t^{-1}y))^i$ is the label of C at the beginning of the $(n+1)$ -st round. Therefore the new tail of C is less or equal to k .

Let us point out the case where the tail of C was already zero at the beginning of the n -th round, so $w_{z,y,k} = e$ and the power of C was i . Then the last line in the previous equation simplifies to $((y^{-1}t^{-1}y)^{-1}s(y^{-1}t^{-1}y))^i$, so the power of C at the beginning of the $(n+1)$ -st round is still i , as stated in the lemma.

If n is even, C moves to $z^{-1}t$. Regarding the $(n+1)$ -st round, we note that

$$\begin{aligned} (z^{-1}t)^{-1}(y^{-1}t) &= t^{-1}zy^{-1}t \\ &= t^{-1}w_{z,y,k}s^i t \\ &= t^{-1}w_{z,y,k}t(t^{-1}st)^i. \end{aligned}$$

Similarly to the odd case, we note that $t^{-1}w_{z,y,k}t$ can be written as a product of k many elements of S , and that $(t^{-1}st)$ is the new label of the cop C . Therefore the tail in the $(n+1)$ -st round is less than or equal to k .

If the tail of C was already zero, we again notice that the power of C does not change from the n -th to the $(n+1)$ -st round, but remains i .

(2) Suppose that $\hat{t}$ is s or s^{-1} and $k = 0$. If n is odd, C moves to $s^{-1}z^{-1}$. For the

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$(n+1)$ -st round, we note that

$$\begin{aligned} (s^{-1}z^{-1})(y^{-1}t)^{-1} &= s^{-1}z^{-1}yy^{-1}t^{-1}y \\ &= s^{-1}s^i y^{-1}t^{-1}y \\ &= \begin{cases} s^i & \text{if } y^{-1}t^{-1}y = s \\ s^{i-2} & \text{if } y^{-1}t^{-1}y = s^{-1}. \end{cases} \end{aligned}$$

From this we see that the tail of the cop remains zero, and the power stays the same if the connection of the robber was s and decreases by two if the connection was s^{-1} .

If n is even, C moves to $z^{-1}s$. So we note that

$$\begin{aligned} (z^{-1}s)^{-1}(y^{-1}t) &= s^{-1}zy^{-1}t \\ &= s^{-1}s^i t \\ &= \begin{cases} s^i & \text{if } t = s \\ s^{i-2} & \text{if } t = s^{-1}. \end{cases} \end{aligned}$$

Again we see that the tail of the cop is still zero at the beginning of the $(n+1)$ -st round and the power behaves as stated in the lemma.

(3) Suppose that $\hat{t}$ is a power of s other than s, s^{-1} and $k = 0$. If n is odd, C moves to $(y^{-1}t^{-1}y)^{-1}z^{-1}$. Note that

$$\begin{aligned} ((y^{-1}t^{-1}y)^{-1}z^{-1})(y^{-1}t)^{-1} &= (y^{-1}t^{-1}y)^{-1}z^{-1}y(y^{-1}t^{-1}y) \\ &= (y^{-1}t^{-1}y)^{-1}s^i(y^{-1}t^{-1}y) \\ &= s^i, \end{aligned}$$

where the last equality follows because $(y^{-1}t^{-1}y)$ is a power of s . Therefore, the tail of the cop remains zero and the power remains i .

If n is even, C moves to $z^{-1}t$, and we have for the $(n+1)$ -st round that

$$\begin{aligned} (z^{-1}t)^{-1}(y^{-1}t) &= t^{-1}zy^{-1}t \\ &= t^{-1}s^i t \\ &= s^i, \end{aligned}$$

which concludes the proof in this case.

(4) Suppose that $\hat{t}$ is a power of s and $k \geq 1$. Let $w_{z,y,k} = g_1g_2 \cdots g_k$ for some $g_1, g_2, \dots, g_k \in S$.

If n is odd, C moves to $g_1^{-1}z^{-1}$, where $z^{-1}y = g_1g_2 \cdots g_k s^i$. Note that

$$\begin{aligned} (g_1^{-1}z^{-1})(y^{-1}t)^{-1} &= g_1^{-1}z^{-1}y(y^{-1}t^{-1}y) \\ &= g_2 \cdots g_k s^i (y^{-1}t^{-1}y). \end{aligned}$$

Since $y^{-1}t^{-1}y$ is a power of s , this shows that at the beginning of round $(n+1)$, the tail of this cop is at most $k-1$.

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If n is even, C moves to $z^{-1}g_1$, where $zy^{-1} = g_1g_2 \cdots g_k s^i$. In this case we have for the $(n+1)$ -st round

$$\begin{aligned} (z^{-1}g_1)^{-1}(y^{-1}t) &= g_1^{-1}zy^{-1}t \\ &= g_2 \cdots g_k s^i t. \end{aligned}$$

Since t is a power of s , this again shows that the tail of this cop is at most $k-1$. $\square$

We have been told in private conversation that the authors are aware of these issues and have resubmitted their paper with a corrected version of the strategy (with $2 \cdot |S| + 1$ many cops). To date, they have not publically released the new version of the preprint. We have independently found a different method of adjusting the strategy, which we will discuss in the following. Our approach requires $|S| + 1$ many cops.

4.2.2 The adjusted strategy

The main issue with the original strategy is that a cop needs to have even power to catch the robber. We just showed that the power can only decrease in steps of 2 when viewed over an entire round. A cop whose power is odd at the beginning of some round has no reliable way of changing it to even. Such a cop can only hope to be run into by the robber in the middle of the round, but this can clearly not be guaranteed either.

At first, we propose to amend this by adding a second “copy” of each cop who always has a power with the opposite parity. This gives a weaker upper bound for the cop number of Cayley sum graphs:

Theorem 4.18. *Suppose the Cayley sum graph $C_\Sigma(G, S)$ is undirected and connected, and that S is symmetric. Then the cop number of the graph is at most $2 \cdot |S| + 1$.*

Definition 4.19. To be consistent with the notation in [1], we have previously defined the power i of a cop only if $k = 0$. But in this section, we need to keep track of the value of i even before the tail becomes zero, so let us call it the *exponent* of the cop.

For every generator $s \in S$, let there be two cops $c(s)$ and $c'(s)$. At the start of the game, we place $c(s)$ at e and $c'(s)$ at s .

For the proof below to work out, we must set a few new rules concerning how cops compute their tail and exponent throughout the game. It is not enough that the cop finds an arbitrary representation of $z^{-1}y$ or zy^{-1} with minimal k and i in every round, like it was in the case of Cayley graphs.

At the beginning of the first round, all cops of the form $c(s)$ arbitrarily choose a representation with minimal tail and exponent. Suppose that cop $c(s)$ at position $z = e$ finds the following representation with tail k and exponent i :

$$z^{-1}y = y = g_1 \cdots g_k s^i$$

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From this we obtain for the cop $c'(s)$ at position $z = s$:

$$z^{-1}y = s^{-1}y = s^{-1}g_1 \cdots g_k s^i = (s^{-1}g_1 s) \cdots (s^{-1}g_k s) s^{i-1}$$

In other words, we have found one representation of the distance of $c'(s)$ to the robber with tail k and exponent $i - 1$. It is easy to see that $c'(s)$ cannot have another representation with shorter tail — this would contradict the minimality of k in the representation of $c(s)$. But there might be other representations with a different sequence of generators $\tilde{g}_1, \dots, \tilde{g}_k$. However, we require that $c'(s)$ will use exactly the representation that is derived from $c(s)$ as above.

Further, the representation of the distance between each cop and the robber should remain consistent from round to round: In the proof of Lemma 4.15 we derived equations bounding the tail and power of a cop in terms of the tail and power in the previous round. Recall for example the case when n is even, $c(s)$ has tail $k > 0$, and $t = s$. At the beginning of the n -th round, $c(s)$ at position z represents the distance from the robber as:

$$zy^{-1} = g_1 g_2 \cdots g_k s^i$$

In the proof, we have obtained for the beginning of round $n + 1$:

$$(z^{-1}g_1)^{-1}(y^{-1}t) = g_2 \cdots g_k s^{i+1}$$

The cop $c(s)$ should use this exact representation for the $(n + 1)$ -st round. The same principle applies to all other cases described in the proof.

However, we do not want to allow an exponent to become -1 (which could happen in a “chasing” or “approaching” move). Representing the exponent -1 as $\text{ord}_G(s) - 1$ can also lead to some issues, in particular because $\text{ord}_G(s)$ can be an odd number and therefore mess up some claims regarding exactly when the exponent changes parity. Instead we replace the exponent -1 by $2 \cdot \text{ord}_G(s) - 1$, which does not change the parity (and maintains the correctness of the formula).

Finally, we make one change concerning the movement of the cops: Only make an “approaching” move if t is exactly s or s^{-1} . If t is any other power of s , do a “keeping up” move. The reason for this change is to ensure that the exponent i of a cop changes parity in every “approaching” move.

Each cop follows this strategy independently of all other cops, including the one sharing the same label s . At the end of each round, cops of the form $c(s)$ are relabelled to $c(\hat{t}^{-1} s \hat{t})$, and cops of the form $c'(s)$ are relabelled to $c'(\hat{t}^{-1} s \hat{t})$.

Proof of Theorem 4.18. First, let us show that, assuming the robber never skips a turn, the strategy with $2 \cdot |S|$ many cops which we just described always catches the robber (within at most $\frac{5}{2}|G| \cdot |S|$ many turns).

Say that the initial position of the robber is some $x \in G$. For every cop $c(s)$ with tail k and exponent i , we have ensured that $c'(s)$ has tail k and exponent $i - 1$. Since S generates the whole group, we can express any element as a product of at most

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$\frac{1}{2}|G|$ many elements of S . In particular, the tails of all cops are at most $\frac{1}{2}|G|$, since the cops $c(s)$ always choose a minimal representation in the beginning.

We say that a pair of cops C and C' has a “special round” if the label of these cops at the beginning of the round is equal to or the inverse of the connection of the robber. (This is equivalent to the cops doing an “approaching” or a “chasing” move.) By the pigeonhole principle, there must exist at least one pair of cops who experiences at least $\frac{5}{2}|G|$ many special rounds within the first $\frac{5}{2}|G| \cdot |S|$ many rounds of the game.

Let C and C' be these cops. From now on, we will say “special rounds” only for the special rounds of C and C' . Let D be the cop whose label is the inverse of the label of C at the beginning of the first round. Notice that after changing labels at the end of each round, the labels of C and D will always remain inverses of each other. This means C , C' , D and D' always have special rounds together. (It could happen that C and D are not distinct, if the label has order 2.)

Notice that C and D start out with the same tail k and have the same special rounds, therefore they reach a tail of zero during the same round. C' also makes “approaching” moves in the same rounds as C , albeit possibly at different vertices. The same holds for D' . Therefore all four cops reach tail zero during the same round. This happens after at most $\frac{|G|}{2}$ many special rounds, and the tails will then remain zero for the rest of the game.

With the modified strategy, after each “approaching” move, the tail of a cop has decreased by exactly 1, and the exponent has changed parity. This means if C has odd exponent upon reaching tail zero, then C' has even exponent, and vice versa.

Once the tail of all cops in $\{C, C', D, D'\}$ is zero, in each subsequent special round, either the powers of C and C' will each go down by exactly 2 and the powers of D and D' will remain the same, or the other way around. Since the cops with the same label always have powers with opposite parity, and any power can be at most $2|G|$, within at most $2|G|$ more special rounds, one of the four cops will reach power zero and thus catch the robber.

Therefore the robber will always be captured within at most $\frac{5}{2}|G| \cdot |S|$ many rounds.

To deal with the robber skipping turns, we can dedicate one additional cop and use the general strategy explained in Section 4.1.3.

□

By looking at the proof in detail, we can improve the strategy: Let C and C' be the pair of cops such that one of them catches the robber at the end of the game. Then we can deduce already at the start of the game which one of them will do so. If C starts the game with tail k and exponent i (hence C' starts with tail k and exponent $i - 1$), then the following holds: If $k + i$ is even, then C will catch the robber, and if $k + i$ is odd, then C' will catch the robber. Therefore we can use only one cop for each generator s , if we ensure that this cop can at some point achieve an even value for $k + i$.

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Corollary 4.20. *Suppose that the Cayley sum graph $C_\Sigma(G, S)$ is undirected and connected, and that S is symmetric. Then the cop number of the graph is at most $|S| + 1$.*

Proof. We deploy one cop to deal with the robber skipping turns; the details are discussed in Section 4.1.3. For the strategy of the remaining cops, any round that the robber skips will also be skipped by these cops and will not be counted.

We have one cop $c(s)$ for every generator s . Recall the original definition of the cops and robbers game: The cops pick their positions, then the robber picks a position, then the cops are the first to make a move. We place all cops at starting position e . After the robber has chosen a starting position, we use the first move of the cops to adjust their position: Each cop $c(s)$ either stays at e , or moves to $e^{-1}s = s$, depending on which choice results in $k + i$ being even. The subsequent move of the robber is considered as the start of the first round. This is exactly equivalent to the situation of the previous theorem, but out of each pair of cops C and C' , only the one who starts out with an even value for $k + i$ exists.

Let us go through all possible moves that $c(s)$ can make during the game, recalling the proof of Lemma 4.15. In both cases of “keeping up” moves, the tail and exponent are unchanged. In a “chasing” move, the exponent either stays the same or decreases by 2. Finally, in an “approaching” move, the tail decreases by 1 and the power changes by ± 1 . In all cases, the sum of tail and exponent has the same parity at the start of the current and of the next round. So it is ensured that any cop C who has enough special rounds will end up with an even power, and hence C or the cop who has the inverse label to C can subsequently catch the robber. $\square$

4.3 Asymptotic discussion

At the end of this discussion, we want to mention an important open conjecture about the asymptotic behaviour of the cop number.

Conjecture 4.21 (Meyniel’s Conjecture). *Let G be any graph with n vertices. Then the cop number of G is in $\mathcal{O}(\sqrt{n})$.*

Conjecture 4.22 (Weak Meyniel’s Conjecture). *There exists some fixed constant $c > 0$ such that for any graph G on n vertices, the cop number is in $\mathcal{O}(n^{1-c})$.*

It is shown in [1] that the weak Meyniel’s conjecture holds for Cayley graphs and Cayley sum graphs for any real number $c < \frac{1}{2}$, under the assumptions on S we have made above. In this section we give an expanded exposition of their proof, which relies mostly on the main results from Sections 4.1 and 4.2. The proof additionally requires the following result, which can be found as Corollary 3.2 in [2].

Lemma 4.23. *Let G be a finite group and S be any subset. Let $n = |G|$ and $k = |S|$. Let $\tau(S, G)$ be the minimum number of elements $t \in G$ needed so that the left-translates tS cover the entire group G . Let $H_k = \sum_{j=1}^k \frac{1}{j}$ be the k -th harmonic number, which is well-known to satisfy $H_k \leq \log k + 1$. Then*

$$\tau(S, G) \leq \frac{n}{k} H_k \leq \frac{n}{k} (\log k + 1).$$

This allows us to prove the following results:

Theorem 4.24. *Let G be a group with n elements and S a symmetric generating set that is closed under conjugation. Then the cop number of the Cayley graph $C(G, S)$ is at most $2\sqrt{n} \log n$.*

Proof. We distinguish two cases depending on the size of S .

Suppose that $|S| < \sqrt{n} \log n$. Then by Theorem 4.2, the cop number is at most $|S| < 2\sqrt{n} \log n$.

Suppose that $k = |S| \geq \sqrt{n} \log n$. Then by Lemma 4.23, we have

$$\begin{aligned} \tau(S, G) &\leq \frac{n}{k}(\log k + 1) \\ &\leq \frac{n}{\sqrt{n} \log n}(\log k + 1) \\ &\leq \frac{n}{\sqrt{n} \log n}(\log n + 1) \\ &\leq \sqrt{n} \left(1 + \frac{1}{\log n}\right) \\ &\leq 2\sqrt{n} \text{ for } n \geq 3. \end{aligned}$$

Let $T \subseteq G$ be the set with at most $2\sqrt{n}$ elements such that $TS = G$. At the start of the cops and robbers game on G , place one cop at each of the elements of T . Let x be the starting position of the robber. Then there exist elements $t \in T$ and $s \in S$ such that $ts = x$. Hence the cop placed at t can catch the robber on the first turn of the game. This strategy requires $|T| \leq 2\sqrt{n} \leq 2\sqrt{n} \log n$ cops. $\square$

Theorem 4.25. *Let G be a group with n elements and S a symmetric generating set that is closed under conjugation. Then the cop number of the Cayley sum graph $C_\Sigma(G, S)$ is at most $2\sqrt{n} \log n$.*

Proof. Suppose $|S| < \sqrt{n} \log n$. Then by Corollary 4.20, the cop number of the graph is at most $|S| + 1 < 2\sqrt{n} \log n$.

Suppose $|S| \geq \sqrt{n} \log n$. As in the previous proof, Lemma 4.23 implies that there is a subset T of G with size at most $2\sqrt{n}$ such that $TS = G$. Place one cop at each element of T^{-1} . Let x be the starting position of the robber. There exist elements $t \in T$ and $s \in S$ such that $ts = x$. Consider the cop placed at t^{-1} . On the first turn, this cop can move to $(t^{-1})^{-1}s = x$ and catch the robber. $\square$

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