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On the joint spectra of independence and almost disjointness

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Abstract

This thesis is centered around the combinatorial set theoretic objects of almost disjoint and independent families, the associated cardinal characteristics $\mathfrak{i}$, $\mathfrak{a}$, and in particular the spectra $Spec(\mathfrak{i})$ as well as $Spec(\mathfrak{a})$. Specifically, building upon the work of Vera Fischer and Saharon Shelah in [8] it presents new results (to our best knowledge) about the joint spectra of independence and almost disjointness, with the main finding that consistently $Spec(\mathfrak{i}) = Spec(\mathfrak{a}) = C$ for so-called pre-spectra C. Any such C is a countably infinite set of uncountable cardinals with certain restrictions imposed (see Definition 4.3.10). Towards this aim, ways of adding and removing cardinals with respect to the spectra are examined separately, providing an overview and motivating the choice of methods. This happens with different sets C such as sets consisting of infinitely many uncountable regular cardinals or finite, strictly ordered, measurable cardinals. Regarding this examination, the (as of yet) open problem of a missing analogue of the Hechler mad forcing notion for independent families is discussed.

Zusammenfassung

Diese Arbeit befasst sich mit fast-disjunkten und unabhängigen Familien im Kontext der kombinatorischen Mengenlehre wie auch den dazu assoziierten Kardinalcharakteristiken $\mathfrak{i}$, $\mathfrak{a}$ und insbesondere deren Spektren $Spec(\mathfrak{i})$ und $Spec(\mathfrak{a})$. Aufbauend auf dem Werk von Vera Fischer und Saharon Shelah in [8] werden neue (soweit bekannt) Resultate über das gemeinsame Spektrum von Unabhängigkeit und Fast-Disjunktheit präsentiert, mit dem Hauptergebnis, dass konsistent $Spec(\mathfrak{i}) = Spec(\mathfrak{a}) = C$, wobei C ein sogenanntes Präspektrum ist. Letzteres bezeichnet eine abzählbar unendliche Menge unendlicher Kardinalzahlen, welche bestimmten Restriktionen unterliegt (siehe Definition 4.3.10). In Hinblick auf dieses Ziel werden Möglichkeiten besprochen, wie Kardinalzahlen aus den einzelnen Spektren entfernt und diesen hinzugefügt werden können, wobei die Methodenwahl motiviert und diskutiert wird. Dies geschieht bezüglich unterschiedlicher Mengen C, wie beispielsweise Mengen bestehend aus unendlich vielen, überabzählbaren, regulären, oder aber endlich vielen, streng geordneten, messbaren Kardinalzahlen. Im Hintergrund dieser Untersuchung wird das (zum Zeitpunkt dieser Abfassung) offene Problem des Nichtvorhandenseins eines dem Hechler mad forcing analogen Objektes besprochen.

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Lastly, I would like to dedicate this work to Dustin. You passed away much too early. As the first project that took so many hours to grow, I feel honored to make this dedication. Your peaceful courage, boundless curiosity, and your way to look at things full of so much imagination will forever amaze me and let me grow in turn.

Contents

Introduction

This introduction serves to provide an overview for the thesis. We begin with an explanation of the underlying basic concepts, which will be continued in Chapter 1.

0.1 Almost disjoint and independent families

Collections $\mathcal{A} \subseteq [\omega]^\omega$ with each pair of different elements having finite intersection are known as almost disjoint (a.d.) families. If on the other hand for any pair of subsets of $\mathcal{A}$, $A = \{a_1, a_2, \ldots, a_n\} \neq B = \{b_1, b_2, \ldots, b_m\}$, combinations of the form $\bigcap A \backslash \bigcup B$ are infinite, we call $\mathcal{A}$ an independent family. Both of these types of families are basic objects in combinatorial set theory and in a sense form completely opposite notions: While the former consist of infinite subsets being pairwise spread apart in ω with the exception of perhaps finitely many common elements, the concept of independence captures infinite sets which are mixed up to a considerable degree but remain distinct, as witnessed by infinitely many $n \in \omega$ contained in $\bigcap A \backslash \bigcup B$ as well as in $\bigcap B \backslash \bigcup A$ (and hence in the complement of the former; in contrast, if a and b are almost disjoint, $\omega \subseteq^* \omega \backslash (a \cap b)$). In both cases, $\mathcal{A}$ is maximal if it is not strictly contained in another a.d. or independent family. Finally, the minimal cardinality over all maximal families of the respective kind is $\mathfrak{i}$ in the independence case and $\mathfrak{a}$ for maximal almost disjoint (mad) families.

What is a spectrum?

The cardinals $\mathfrak{a}$ and $\mathfrak{i}$ are examples of so-called cardinal characteristics, of which there exist more than a handful - see e.g. Andreas Blass's work [1]. By spectrum we mean the set of all possible cardinal values the cardinal characteristic in question can take. Let us explicitly state the definitions of the spectra for $\mathfrak{a}$ and $\mathfrak{i}$:

$$Spec(\mathfrak{a}) = \{\kappa : \exists \text{ mad family of cardinality } \kappa\},$$
$$Spec(\mathfrak{i}) = \{\kappa : \exists \text{ maximal independent family of cardinality } \kappa\}.$$

Thereby, the joint spectra of $\mathfrak{i}$ and $\mathfrak{a}$ relate to results about $Spec(\mathfrak{i}) \cap Spec(\mathfrak{a})$.

0.2 The main goal of this work

The main topic of this work is an inspection of the joint spectra of independence and almost disjointness. Herein, the principle task can be seen in proving that consistently both spectra are equal to certain sets of cardinals. Following a chain of results for each spectrum separately we are going to split this task into two subgoals: Positive (consistency) results, where sets of cardinals are added to the spectrum, and negative counterparts, where other unwanted sets are in turn removed.

0.3 Structure of the thesis

Chapters 1, 2, and 3 have a preparatory purpose: The first chapter provides elementary definitions concerning $\mathfrak{i}$ and $\mathfrak{a}$, corresponding *ZFC* results about the spectra as well as required connections to the cardinal characteristics $\mathfrak{d}$ and $\mathfrak{b}$. The next two chapters introduce a basic toolset consisting of diagonalization and finite support iterations, both of which employed together will form the basis of many of the arguments to follow.

Returning to the main goal stated above, the *ZFC* results from the first chapter form a starting point for our endeavor to control the joint spectra which are found to be bounded by $[\aleph_1, \mathfrak{c}]$ (Sections 1.1.2, 1.2.2); in terms of the opposite inclusion we only know that $\mathfrak{c} \in Spec(\mathfrak{i}) \cap Spec(\mathfrak{a})$ (Prop 1.1.2, Cor 1.2.5). In the subsequent parts, we turn our attention to relative consistency results in order to further modulate the spectra. For the moment, let us solely state the main development and refer to the introduction of each chapter for further information: Chapter 4 serves to gradually develop the positive results. We will approach this in a step by step manner, tracing a way to develop techniques of adding single, finitely, and infinitely many regular cardinals. The progression towards any uncountable cardinals will require a modification of the previous methods. The reason for a stepwise approach is that it allows for a structured and accessible presentation of the motivations underlying the choice of methods and their variations. This in part mirrors historical developments, and in part allows us to highlight the techniques used in the respective proofs. The chapter ends with more complicated restatements of the major outcomes of the foregoing sections, as the previous forcing iterations are "embedded" into a tree structure. This addition of seemingly unnecessary structure is important for the isomorphism of names arguments in Chapter 6, where the negative results are considered. As an interlude, we will see in Chapter 5 that the joint spectra of $\mathfrak{i}$ and $\mathfrak{a}$ can be consistently shown to be equal to finite sets of measurable cardinals using the method of taking averages of names, a precursor to the isomorphism of names approach.

A second theme of the thesis is the attempt to analyze differences between the concepts of almost disjointness and independence in connection to the aforementioned inspection of the spectra: While the overview above sketches ways to control the joint spectra, the ways in which the inspection takes place are strongly influenced by these differences. Therefore for us, considerations of specific differences translate to a better understanding of difficulties arising during our inquiry about the joint spectra. These differences are connected to the major longstanding problem of whether $Con(\mathfrak{i} < \mathfrak{a})$ (known as Vaughan's problem), which may be considered to be a long-term motivation for the present inquiries (see e.g. the open questions stated at the end of [7] and [8]). In fact, already in Chapter 2, there appears to be a missing analogue of the a.d. coding poset for independent families, and as we will see in Section 9, a.d. coding can be recovered as the quotient poset of Hechler mad forcing. The latter, introduced in Section 4.2.2, simplifies considerably the positive results for $\mathfrak{a}$, as it provides an alternative to the iterated forcing approach by "coding" multiple sets of mad families into a single forcing. The existence of a comparable forcing poset for $\mathfrak{i}$ is unknown as of now (which may be seen as the reason why the iterative techniques have been developed in the first place, as alluded to in the introduction of [8]). Also in Chapter 2, we will see that Mathias forcing relativized to different filters can be employed to obtain diagonalizing reals for both, independent and almost disjoint families. Whereas both variants of Mathias forcing will play the same role in the course of obtaining the positive results, interestingly, this analogous usage breaks down when transitioning to the negative results, again due to basic differences between almost disjointness and independence (Remark 4.3.7). With this background, a comparison between the employed forcing notions will be provided aiming to shed some light on problems arising in trying to translate the a.d. context to independence. Since such considerations are of a very wide nature, they take a supplementary role outside of the main text, forming the last part of the appendix (Chapter 9). Lastly, Chapter 7 is composed of a collection of the main results from the previous chapters. It ends with an outlook (Section 7.4) aiming at further directions of inquiry connected to this work.

0.4 Summary of results

The results which are to our knowledge new can be summarized as follows:

- About sets of finitely many measurable cardinals $\kappa_1 < \kappa_2 < \ldots < \kappa_n$: It consistently holds that $\{\kappa_1, \kappa_2, \ldots, \kappa_n\} = Spec(\mathfrak{i}) = Spec(\mathfrak{a})$ (see 5.2.3).
- About the joint spectra for any infinite cardinals:
  - For any set C of uncountable cardinals, consistently $C \subseteq Spec(\mathfrak{i}) \cap Spec(\mathfrak{a})$ (see 7.1)
  - For pre-spectra C (Definition 4.3.10), consistently $C = Spec(\mathfrak{i}) \cap Spec(\mathfrak{a})$ (see 7.3).

General remarks and notation

As mentioned, Chapters 8 and 9 form an appendix encompassing all those results which made the impression of stifling the reading flow regarding the main topic. Concerning general notation, $f(A)$ or at times $f[A]$ denotes the image of a set A with respect to a function f, and $\mathcal{P}(A)$ denotes the power set of A. Let us also remark that in order to stay focused on the main arguments, upcoming density results will not contain proofs (and in most cases not even mention) that the dense sets are in the ground model.

1 ZFC Results

In this chapter we will work using solely the axiomatic system of *ZFC*. Unless stated otherwise, all of the results can be found in [11].

1.1 Bounding and almost disjointness

1.1.1 Basics

We define a family $\mathcal{A}$ of infinite subsets of ω to be an almost disjoint family if any two distinct elements of $\mathcal{A}$ have finite intersection. Whenever there is no almost disjoint family strictly containing $\mathcal{A}$ as a subset, we say that $\mathcal{A}$ is a maximal almost disjoint family, also called a mad family. In other words, $\mathcal{A}$ is mad iff any infinite subset of ω not in $\mathcal{A}$ intersects some $a \in \mathcal{A}$ infinitely often. The almost disjointness number $\mathfrak{a}$ is defined as the minimal cardinality of a mad family. Sometimes we will use a.d. to mean almost disjoint or almost disjointness. Let us briefly introduce unbounded families. An unbounded family $\mathcal{B} \subseteq \omega^\omega$ is a family of infinite functions on ω so that there is no $f \in \omega^\omega$ which eventually dominates all functions in $\mathcal{B}$ (f eventually dominates g, in symbols $g <^* f$, if there exists $m \in \omega$ so that for all $n \geq m$, $g(n) < f(n)$). Then the bounding number $\mathfrak{b}$ is defined as the least cardinality of an unbounded family.

1.1.2 Spectrum of almost disjointness and bounding

As it turns out, the possible values of $\mathfrak{a}$ and $\mathfrak{b}$ have the following lower and upper bounds:

Proposition 1.1.1. $\omega_1 \leq \mathfrak{a}, \mathfrak{b} \leq \mathfrak{c}$.

Proof. Indeed, $\mathfrak{a} \leq \mathfrak{c}$ because each mad family $\mathcal{A}$ is a subset of $[\omega]^\omega$, where $|[\omega]^\omega| = \mathfrak{c}$. For $\mathfrak{b} \leq \mathfrak{c}$ it suffices to note that ω^ω is an unbounded family.

To see $\mathfrak{a} \geq \omega_1$, assume that $\mathcal{A}$ is an almost disjoint family of countable cardinality and show that $\mathcal{A}$ cannot be maximal in this case, i.e., that there exists some $b \in [\omega]^\omega \setminus \mathcal{A}$ which is almost disjoint from every $a \in \mathcal{A}$. As $\mathcal{A}$ is countable, we can enumerate it by

$\mathcal{A} = \{a_n : n \in \omega\}$. From $\mathcal{A}$ we construct a set of pairwise disjoint elements by using the fact that for any $n \neq m$, the intersections $a_n \cap a_m$ are finite. Therefore, the sets $b_0 := a_0$, $b_1 := a_1 \backslash b_0$, ..., $b_k := a_k \backslash \bigcup_{j<k} b_j$ are infinite and pairwise disjoint. It follows that $b := \{n \in \omega : n = \min(b_n) \vee (n = 0 \text{ if } b_n = 0)\}$ is as wanted. By construction, $|b \cap b_k| \leq 1$ for each $k \in \omega$ (thus in particular, $b \notin \mathcal{A}$), and so $|b \cap a_k| < \omega$ for all $k \in \omega$.

Finally, to obtain $\mathfrak{b} \geq \omega_1$, assume that $\mathcal{B} \subseteq \omega^\omega$ is a family of countable cardinality, $\mathcal{B} = \{f_k : k \in \omega\}$, and proceed to show that in this case $\mathcal{B}$ cannot be unbounded. To see this, define a function $g \in \omega^\omega$ witnessing $f_k <^* g$ for every $k \in \omega$ via $g : \omega \to \omega$, $g(n) = \sup\{f_k(n) : k \leq n\} + 1$ for every $n \in \omega$. Then, given any $f_k \in \mathcal{B}$, $f_k <^* g$ because for all $n \geq k$, $g(n) \geq f_k(n) + 1 > f_k(n)$. □

Thereby, $Spec(\mathfrak{a}) \subseteq [\aleph_1, \mathfrak{c}]$, to use a suggestive interval notation for cardinal numbers. Note that by the above, there in particular exists an unbounded family of cardinality $\mathfrak{c}$, namely ω^ω itself. What about the a.d. case?

Proposition 1.1.2. *There exists a mad family of cardinality* $\mathfrak{c}$.

Proof. Consider $\mathcal{A} = \{a_x\}_{x \in [\omega]^\omega}$, where $a_x := \{x \cap n : n \in \omega\}$, $x \in [\omega]^\omega$. Then $\mathcal{A}$ is an almost disjoint family of cardinality continuum, where almost disjointness follows from the fact that any two different initial segments $x \cap n$ and $y \cap n$ can be at most equal up until some $N \in \omega$, from which on they are bound to differ. The reason for this is that since $x \neq y$, without loss there exists some $k \in x$ so that $k \notin y$. Therefore, for any $n \geq k+1$, $k \in x \cap n$ but $k \notin y \cap n$, implying that $a_x \cap a_y \subseteq \{x \cap n : n \leq k\}$. To extend $\mathcal{A}$ to a mad family, apply e.g. Zorn's Lemma (with respect to the poset $(\{\mathcal{A} \subseteq [\omega]^\omega : \mathcal{A} \text{ a.d. family of cardinality } \mathfrak{c}\}, \subseteq)$). □

1.1.3 Proof that $\mathfrak{b} \leq \mathfrak{a}$

Let us now establish the relation between $\mathfrak{b}$ and $\mathfrak{a}$. Aside from being interesting in its own right, this result will be of importance in later parts of this work.

Theorem 1.1.3. $\mathfrak{b} \leq \mathfrak{a}$

Proof. In order to show the statement, it suffices to construct for any given mad family $\mathcal{A}$ with $|\mathcal{A}| = \kappa$ an unbounded family $\mathcal{B}$ of at most the same cardinality. If this is the case, $\{|F| : F \text{ mad family}\} \subseteq \{|F| : F \text{ unbounded family}\}$, which leads to the stated inequality for the respective minima of these sets.

So let $\mathcal{A} = \{a_\xi : \xi < \kappa\}$ be a given mad family. The idea is to start by substituting the first countably many elements $a_n \in \mathcal{A}$ by certain modified elements a'_n to produce an "equivalent" mad family $\mathcal{A}'$ so that $\{a'_n : n \in \omega\} \subseteq \mathcal{A}'$ forms a partition of ω into infinite sets. In order to find $\mathcal{B}$, we have to achieve a transition from infinite subsets of ω to functions in ω^ω. This will be achieved, for each $n \in \omega$, via the maps $g_n : a'_n \to \omega$, the inverses of the enumerating functions. The g_n will form the basic building blocks of the elements of $\mathcal{B}$, and the modification of $\mathcal{A}$ will play a crucial role in proving (by contradiction) that $\mathcal{B}$ is unbounded.

Part 1 - Obtain $\mathcal{A}'$: For $n > 0$, define $a'_n = (a_n \cup \{n\}) \setminus \bigcup_{k<n} a'_k$, and for $n = 0$, $a'_0 = a_0 \cup z \cup \{0\}$, where in order not to forget any elements outside of $\mathcal{A}$, we just add them manually in the form of $z := \omega \setminus \bigcup_{\xi<\kappa} a_\xi$, which by maximality of $\mathcal{A}$ must be finite. This construction directly implies that the a'_n are pairwise disjoint. We also make sure to add each $n \in \omega$ separately to obtain a covering of ω. Then define $\mathcal{A}' = (\mathcal{A} \setminus \{a_n : n \in \omega\}) \cup \{a'_n : n \in \omega\}$. As each a_n is modified only up to finitely many elements (i.e., $a_n =^* a'_n$ for all n), $\mathcal{A}'$ remains mad, and this makes concrete what has been stated above as "equivalence" of $\mathcal{A}$ and $\mathcal{A}'$.

Part 2 - Construct $\mathcal{B}$: Fix $n \in \omega$. We can enumerate the elements of a'_n via a strictly increasing bijection $\omega \to a'_n$. Its inverse g_n also forms a strictly increasing bijection. Since $\mathcal{A}'$ is almost disjoint, for every $\xi < \kappa$ with $\xi \geq \omega$, the intersection $a_\xi \cap a'_n$ is finite. So we can define functions $f_\xi \in \omega^\omega$ by setting $f_\xi(n) := \max\{g_n(k) : k \in a_\xi \cap a'_n\}$ (if $a_\xi \cap a'_n = \emptyset$, take $f_\xi(n) := 0$; this case has no importance by the pigeonhole-principle and the definition of $\mathcal{A}'$). Finally, let $\mathcal{B} = \{f_\xi : \omega \leq \xi < \kappa\}$.

Part 3 - Show that $\mathcal{B}$ is an unbounded family: Assume towards a contradiction that this is not so, meaning that there exists some $f \in \omega^\omega$ such that $f_\xi <^* f$ for every ξ. By definition of $<^*$, this says that for any given ξ we can find some $n_\xi \in \omega$ so that for all $m \geq n_\xi$, $f_\xi(m) < f(m)$. Using this, we wish to construct a set $Y \in [\omega]^\omega \setminus \mathcal{A}'$ so that $\mathcal{A}' \cup \{Y\}$ forms an almost disjoint family, in contradiction to the maximality of $\mathcal{A}'$. Define $Y = \{g_n^{-1}(f(n) : n \in \omega)\}$. If we can show that Y intersects each element of $\mathcal{A}'$ at most finitely often, then it automatically follows that $Y \notin \mathcal{A}'$, as Y is infinite. Certainly, as $f \in \omega^\omega$ is a function and the $g_n^{-1} : \omega \to a'_n$ are bijections, for each n, $Y \cap a'_n$ contains exactly one element, namely $g_n^{-1}(f(n))$ itself. It remains to show that the intersections $Y \cap a_\xi$, $\omega \leq \xi < \kappa$, are finite. We can simplify this task by noting that for any such ξ:

$$Y \cap a_\xi = Y \cap a_\xi \cap \omega = \bigcup_{n\in\omega} Y \cap a_\xi \cap a'_n.$$

This union can be split up into two parts, one with $n < n_\xi$ and one where $n \geq n_\xi$. From Part 1 we know that the $a_\xi \cap a'_n$ are finite, so in particular $\bigcup_{n<n_\xi} Y \cap a_\xi \cap a'_n$ is finite. Next, prove that for each $n \geq n_\xi$, $Y \cap a_\xi \cap a'_n = \emptyset$. Since the g_n are increasing bijections, every element of $g_n(a_\xi \cap a'_n)$ is in $f_\xi(n)$, that is, $g_n(a_\xi \cap a'_n) \subseteq f_\xi(n)$, or in other words $a_\xi \cap a'_n \subseteq g_n^{-1}(f_\xi(n))$. On the other hand, we know that $Y \cap a_\xi = \{g_n^{-1}(f(n))\}$. But now $n \geq n_\xi$, so $f_\xi(n) < f(n)$, and since the g_n^{-1} are *strictly* increasing, $g_n^{-1}(f(n)) \notin a_\xi \cap a'_n$. Therefore, $Y \cap a_\xi \cap a'_n = \emptyset$ for all n and for any given ξ. Finally, this implies that $Y \cap a_\xi \subseteq \bigcup_{n<n_\xi} a_\xi \cap a'_n$ is finite, which concludes the proof. □

1.2 Independence and dominating

1.2.1 Basics

As stated in the introduction, a family $\mathcal{A}$ of infinite subsets of ω is called independent if all of its Boolean combinations are infinite, where a Boolean combination with respect to $\mathcal{A}$ is a set of the form $\bigcap X \setminus \bigcup Y$, with any $X \neq Y \in [\mathcal{A}]^{<\omega}$ being disjoint. We will often use the following alternative notation for Boolean combinations. Take $FF(\mathcal{A})$ to be the set of all finite functions from $\mathcal{A}$ to $2 = \{0, 1\}$, i.e., $h \in FF(\mathcal{A})$ is a function $h : \mathcal{A} \to 2$ with finite domain. Then any Boolean combination of $\mathcal{A}$ can be denoted by $\mathcal{A}^h := \bigcap_{Z \in dom(h)} Z^{h(Z)}$, for some $h \in FF(\mathcal{A})$ and $Z^0 := Z$, $Z^1 := \omega \setminus Z$. Write $BC(\mathcal{A}) := \{\mathcal{A}^h : h \in FF(\mathcal{A})\}$ for the set of all Boolean combinations of $\mathcal{A}$. Dominating families are families consisting of infinite functions $f \in \omega^\omega$ with the property that any $g \in \omega^\omega$ is eventually dominated by some f in the dominating family. Finally, the independence number $\mathfrak{i}$ and the dominating number $\mathfrak{d}$ denote the minimal cardinality of all maximal independent families respectively all dominating families.

Remark 1.2.1. Note that $\mathfrak{b} \leq \mathfrak{d}$ because certainly $g <^* f$ implies $\neg(f <^* g)$, so every dominating family is unbounded.

1.2.2 Spectrum of independence and dominating

Similarly to the a.d. case, the following shows that $Spec(\mathfrak{i}) \subseteq [\aleph_1, \mathfrak{c}]$.

Proposition 1.2.2. $\aleph_1 \leq \mathfrak{i}, \mathfrak{d} \leq \mathfrak{c}$.

Proof. Indeed, $\mathfrak{i} \leq \mathfrak{c}$ because each maximal independent family is a subset of $[\omega]^\omega$, while $\mathfrak{d} \leq \mathfrak{c}$, as ω^ω itself is dominating.

To see $\mathfrak{i} \geq \aleph_1$, assume that $\mathcal{A} := \{a_k : k \in \omega\}$ is a countable independent family and show that it cannot be maximal by constructing an element $b \in [\omega]^\omega \setminus \mathcal{A}$ so that $\mathcal{A} \cup \{b\}$ is still independent. One way to obtain such b is to define $b_0 = a_0$, $b_n = a_n \setminus \bigcup_{i \in n} a_i$ for every $n \geq 1$, and $b'_n = \omega \setminus a_n$ for every $n \in \omega$. Each of these sets is infinite by independence of $\mathcal{A}$. Then pick every second element from every b_n and b'_n to construct the set b. As countable unions of countable sets are countable, $b \in [\omega]^\omega$, and by construction b differs from a_k for all $k \in \omega$. Since we made sure to include infinitely many elements from not only every a_k but also the complement of each a_k in b, any Boolean combination with respect to $\mathcal{A} \cup \{b\}$ is infinite, which means that $\mathcal{A} \cup \{b\}$ forms an independent family.

Similarly, for $\mathfrak{d} \geq \aleph_1$, show that given any countable family $\mathcal{A} \subseteq \omega^\omega$ one can construct a function $g \in \omega^\omega$ so that g eventually dominates every $f \in \mathcal{A}$. As $\mathcal{A}$ is countable we can enumerate its elements by $\{f_i : i \in \omega\}$. Define g as follows. Let $g(0) := f_0(0) + 1$, and for $n > 0$ continue by setting $g(n) := \max\{f_k(n) : k < n\} + 1$. It follows that g dominates every member of $\mathcal{A}$. $\square$

Again we can ask the question: Seeing that ω^ω itself is a dominating family of cardinality $\mathfrak{c}$, what about the independence case? The following proof by Fichtenholz and Kantorovich ([4]) uses a different approach than the one given in [11].

Proposition 1.2.3. *There exists an independent family of cardinality $\mathfrak{c}$.*

Proof. Consider the family $C := [\mathbb{Q}]^{<\omega}$ of all finite subsets of the rational numbers. Let $r \in \mathbb{R}$ be a real number. Define the sets $A_r = \{a \in C : |(-\infty, r] \cap a| \text{ even}\}$. Then the family $\{A_r : r \in \mathbb{R}\}$ is an independent family of cardinality $\mathfrak{c}$. $\square$

Discussion 1.2.4. *To make the argument used in the preceding proof a little clearer, note that:*

1. *For the simplest non-trivial kind of Boolean combination* $A_{r_1} \cap \mathbb{R} \setminus A_{r_2}$, $r_1 < r_2 \in \mathbb{R}$*: As* $(r_1, r_2] \subseteq \mathbb{R}$ *is non-empty, there are infinitely many rationals in this interval. So using the pigeonhole principle, there are infinitely many subsets consisting of an odd number of rationals in* $(r_1, r_2]$*. On the other hand, there are infinitely many rationals below* r_1 *(i.e., in* $(-\infty, r_1]$*, and in particular not in* $(r_1, r_2]$*), so if we are interested in the elements which are in both,* A_{r_1} *and* $\mathbb{R} \setminus A_{r_2}$*, we have infinitely many witnesses for this at our disposal. These come in the form of e.g. one fixed even-element rational subset of* $(r_1, r_2]$*, to which we add a single rational from* $(-\infty, r_1]$ *in order to obtain an even intersection with* $(-\infty, r_1]$*.*

2. *Similarly, if we have any finite Boolean combination with respect to a given set of reals, say* $r_1 < r_2 < \cdots < r_n$*, we can in principle repeat the combinatorial argument from above (again by exploiting* $\bar{\mathbb{Q}} = \mathbb{R}$ *and the pigeonhole principle).*

   This case is slightly more complicated in that we have to take care of the right number of intersections for all of the n *sets* A_{r_i} *at once. One way to reuse the argument from (1) is to provide an algorithm starting from the top as follows. Fix an arbitrary* $n \in \omega$*. Then:*

   - *Begin with* r_n*. Either* A_{r_n} *or* $\mathbb{R} \setminus A_{r_n}$ *is in the Boolean combination.*
   - *If* A_{r_n} *appears in the Boolean combination, continue with* r_{n-1}*.*
   - *If* $\mathbb{R} \setminus A_{r_n}$ *is part of the Boolean combination, pick an even-element subset of* $(r_{n-1}, r_n]$ *and add to it a rational from below* r_{n-1} *in case that* $A_{r_{n-1}}$ *is in the Boolean combination; otherwise add nothing. Continue from the top with* r_{n-2}*.*
   - *Repeat this process* $\aleph_0$*-many times, always choosing different rationals to add.*

   In total we add only finitely many elements with one application of this loop. Therefore indeed, by $\bar{\mathbb{Q}} = \mathbb{R}$ *the usage of the pigeonhole principle in each step is allowed and every such loop can be repeated infinitely often.*

As before (see Proposition 1.1.2), we can extend $\mathcal{A}$ to a maximal independent family via Zorn's Lemma.

Corollary 1.2.5. *There exists a maximal independent family of cardinality* $\mathfrak{c}$*.*

1.2.3 Proof that $\mathfrak{d} \leq \mathfrak{i}$

As in the case of $\mathfrak{b}$ and $\mathfrak{a}$, in particular because it will be used later on, we proceed by establishing the connection between $\mathfrak{d}$ and $\mathfrak{i}$, which is the content of the main theorem for this section:

Theorem 1.2.6. $\mathfrak{d} \leq \mathfrak{i}$.

Notation. In the proof we will sometimes use the shorthand notation for Boolean combinations of length n with respect to some *infinite* function $g \in 2^\omega$ given by $C_{n,g} := \bigcap_{k \in n} X_k^{g(k)}$, where the X_k are taken from the given independent family $\mathcal{A}$ or some subfamily of $\mathcal{A}$, and as usual $X_k^0 := X_k$ and $X_k^1 := \omega \setminus X_k$.

Discussion 1.2.7 (Idea of the proof)**.** *Our approach will be to show that any independent family with cardinality $\kappa < \mathfrak{d}$ cannot be maximal. To this end, let $\mathcal{A} = \{X_\xi : \xi \in \kappa\}$ be an independent family of cardinality $\kappa > \omega$ ($\kappa = \aleph_0$ is excluded by $\mathfrak{i} \geq \aleph_1$). In particular, any family of functions of cardinality κ is not dominating.*

It is sufficient to construct a set $Z \in [\omega]^\omega$ outside of $\mathcal{A}$ such that $\mathcal{A} \cup \{Z\}$ is independent. In other words, if we take any disjoint $A, B \in [\mathcal{A}]^{<\omega}$, then Z splits the Boolean combination $\bigcap A \setminus \bigcup B$ in the sense that

$$|Z \cap \bigcap A \setminus \bigcup B| = \omega = |(\omega \setminus Z) \cap \bigcap A \setminus \bigcup B|.$$

How to construct Z? Roughly, as $|BC(\mathcal{A})| = \kappa$, we have to make sure that Z intersects at least uncountably many different Boolean combinations. But we cannot define Z by quantifying over uncountably many elements because Z has to be an infinite subset of ω. In particular, it cannot contain infinitely many elements of ω and at the same time be defined by uncountably many conditions to make sure that it contains enough elements from each Boolean combination.

A more sensible approach is to find an "almost-base" $\mathcal{B}$ of elements which intersect any of the Boolean combinations infinitely often, and construct Z via $\mathcal{B}$. It turns out that to obtain such $\mathcal{B}$ we only require a fixed countable set of Boolean combinations, $\mathcal{A}_\omega$. To address all Boolean combinations outside of $\mathcal{A}_\omega$ consider $\mathcal{A}' := \mathcal{A} \setminus \mathcal{A}_\omega$. Now, in order to access every possible Boolean combination, we make sure to have enough, in fact continuum-many, base elements, which is accomplished by constructing a base element for every function $g \in 2^\omega$. We want to be certain to define the base in a way to be able to obtain Z and its complement $\omega \setminus Z$ at the same time. It suffices to take a subset of $\omega \setminus Z$ consisting of exactly those "almost-base" elements which have associated functions

g mapping to the "removed union parts" of the Boolean combinations, that is, to $\bigcup B$ in $\bigcap A \setminus \bigcup B$. To summarize, we will stick to the following plan:

> *Part I: Set the stage by introducing $\mathcal{A}_\omega, \mathcal{A}'$, and the associated sets of Boolean combinations.*
>
> *Part II: Obtain an "almost-base" $\mathcal{B}$ (the building blocks for Z) and the wanted set Z^- contained in $\omega \setminus Z$ (where in particular we will use the assumption that $\kappa < \mathfrak{d}$, so we will have to associate $[\omega]^\omega$ and ω^ω).*
>
> *Part III: Define Z and Z^- using the first two parts and show that Z indeed splits every Boolean combination of $\mathcal{A}$. Also, show that $Z \in [\omega]^\omega \setminus \mathcal{A}$.*

Proof. Part I: Let $\mathcal{A}_\omega := \{X_n : n \in \omega\}$ be a countable subfamily of $\mathcal{A}$ and let $\mathcal{A}' := \mathcal{A} \setminus \mathcal{A}_\omega$. With respect to each of these two subfamilies of $\mathcal{A}$ introduce the following objects:

- $F := \{\bigcap A' \setminus \bigcup B' : A' \neq B' \in [\mathcal{A}']^{<\omega}$ disjoint$\}$, the set of all Boolean combinations for elements of $\mathcal{A}'$. Note that $|F| \leq \kappa < \mathfrak{d}$.
- $C_g := \{C_{n,g} \in [\mathcal{A}_\omega]^{<\omega} : n \in \omega\}$, the set of all Boolean combinations for elements of $\mathcal{A}_\omega$, defined for any given $g \in 2^\omega$.

Note: For arbitrary g and all $X \in F, Y \in C_g$, it holds that $X \cap Y$ is infinite because it can be written as a Boolean combination of elements of $\mathcal{A}$.

Part II: Any member of the wanted "almost-base" will certainly have to intersect all elements of F and C_g. To construct such members, use the families C_g for different $g \in 2^\omega$, which is sensible because of the following

Claim. *Every C_g has a pseudo-intersection which intersects every element of F infinitely often.*

Proof. A natural candidate for a pseudo-intersection of C_g is given by

$$Y_g^h := \bigcup_{n \in \omega} (C_{n,g} \cap h(n)),$$

for fixed $g \in 2^\omega$ and $h \in \omega^\omega$. In general, Y_g^h can be finite depending on h (e.g. if h is constant or almost constant). In this case it trivially forms a pseudo-intersection of C_g. If Y_g^h is infinite, it must hold that either $Y_g^h \subseteq^* C_{k,g}$ or $Y_g^h \subseteq^* \omega \setminus C_{k,g}$, $k \in \omega$. Can the former, that is, $|Y_g^h \cap C_{k,g}| < \omega$, be true for some $k \in \omega$? Assume towards a contradiction that this is the case. Then $Y_g^h \setminus C_{k,g}$ is infinite. But $C_{m,g} \subseteq C_{k,g}$ whenever

$m \geq k$, and so $Y_g^h \setminus C_{k,g} = \bigcup_{n<k} C_{n,g} \cap h(n)$ must be infinite. Yet this contradicts that $|\bigcup_{n<k} C_{n,g} \cap h(n)| < \omega$.

Next, we wish to find some $h_0 \in \omega^\omega$ to witness that $Y_g^{h_0}$ and $Y_g^{h_0} \cap X$ are infinite for each $X \in F$. This can be accomplished by using our assumption on $\kappa < \mathfrak{d}$. We first need to find a family of functions associated to the $X \in F$. Naturally, for any $X \in F$ pick the enumerating function of $X \cap C_{n,g}$, denoted by $f_X \in \omega^\omega$, where $f_X(n)$ is the n-th element of $X \cap C_{n,g}$, and $f_X(m) < f_X(n)$ if $m < n \in \omega$. As seen at the end of Part I, every $X \cap C_{n,g}$ is infinite. Note that the set of all enumerating functions of sets in F, $\{f_X : X \in F\}$, cannot be dominating, since $|F| < \mathfrak{d}$. This means that there exists $h_0 \in \omega^\omega$ which is not dominated by any f_X, or put differently, the sets $D_X := \{n \in \omega : h_0(n) > f_X(n)\}$ must be infinite for every $X \in F$. But then, for any $X \in F$ and $n \in D_X$, since $h_0(n) \geq f_X(n) + 1$, we have $|X \cap h_0(n)| \geq |X \cap f_X(n) + 1|$. Also, by definition of f_X, $\{f_X(0), f_X(1), \ldots, f_X(n)\} \subseteq \{0, 1, \ldots, f_X(n)\}$. It follows that $|X \cap f_X(n) + 1| \geq n$, i.e., $|X \cap h_0(n)| \geq n$. Yet D_X is infinite and $f_X(n) \in h_0(n)$ for all $n \in D_X$. In conclusion, $X \cap Y_g^{h_0}$ is infinite. Furthermore, as h_0 is strictly increasing, $Y_g^{h_0}$ itself is infinite, where we use that the f_X enumerate $X \cap C_{n,g}$. $\square_{Claim}$

By the claim, for every $g \in 2^\omega$ we can find a set $Y_g \in [\omega]^\omega$ with the following two properties:

1. For every $n \in \omega$, $Y_g \subseteq^* C_{n,g} = \bigcap_{k \in n} X_k^{g(k)}$.
2. $Y_g \cap \bigcap A' \setminus \bigcup B'$ is infinite for all $\bigcap A' \setminus \bigcup B' \in F$.

These Y_g will be the basic building blocks of the "almost-base". To begin with, note that the Y_g are pairwise almost disjoint.

Claim. *If $g \neq g' \in 2^\omega$, then $Y_g \cap Y_{g'}$ is finite.*

Proof. By point (1) above, we know that $Y_g \subseteq^* \bigcap_k X_k^{g(k)}$, and similarly, $Y_{g'} \subseteq^* \bigcap_k X_k^{g'(k)}$. Since $g \neq g'$, there must be some $k \in \omega$ for which $X_k^{g(k)} \cap X_k^{g'(k)} = \emptyset$, meaning that $Y_g \cap Y_{g'}$ is finite. $\square_{Claim}$

Next, in order to be able to differentiate between functions $g \in 2^\omega$ mapping "almost all" (all except finitely many) elements to the respective intersection part ($\bigcap A$ or $g(k) = 0$) or the union part ($\bigcup B$ or $g(k) = 1$) of a Boolean combination associated to g, we define the sets

$$Q_0 = \{g \in 2^\omega : \exists n_0 \in \omega \forall k \geq n_0 (g(k) = 0)\}, Q_1 = \{g \in 2^\omega : \exists n_1 \in \omega \forall k \geq n_1 (g(k) = 1)\}.$$

In other words, $g \in Q_0$ means that almost all of the corresponding X_k will be in the intersection part of the respective Boolean combination and $g \in Q_1$ means that almost all of them will instead be in the removed union part. By simply counting the number of possibilities for values of n_0 respectively n_1, and adding to this the finitely many different possible variations of natural numbers occurring below n_0 and n_1, it follows that $Q_0 \cup Q_1 \subseteq \omega^\omega$ is countable. Thus we can enumerate it by $Q_0 \cup Q_1 = \{g_n : n \in \omega\}$.

As a final step towards our "almost-base" we want to make the Y_{g_n}, $g_n \in Q_0 \cup Q_1$, pairwise disjoint. To this end, define $Y'_{g_n} = Y_{g_n} \setminus \bigcup_{k<n} Y_{g_k}$. Then, by the claim above saying that the Y_{g_n} are pairwise almost disjoint, each Y'_{g_n} is infinite.

In total, each "almost-base" element Y'_{g_n} ...

> ...is almost contained in every Boolean combination with respect to g_n.
>
> ...intersects the Boolean combinations with respect to $\mathcal{A}'$ infinitely often.
>
> ...consists almost entirely of either an intersection part or a union part.
>
> ...is disjoint from any other Y'_{g_m}.

Part III: Finally, define $Z = \bigcup_{g \in Q_0} Y_{g'}$ and $Z^- = \bigcup_{g \in Q_1} Y_{g'}$. Certainly $Z, Z^- \in [\omega]^\omega$, and by the comments above, $Z \cap Z^- = \emptyset$, so in particular $Z^- \subseteq \omega \setminus Z$. We want to show that

- $Z \cap \bigcap A \setminus \bigcup B$ is infinite for all disjoint $A, B \in [\mathcal{A}]^{<\omega}$,
- and $Z^- \cap \bigcap A \setminus \bigcup B$ is infinite for all disjoint $A, B \in [\mathcal{A}]^{<\omega}$.

For the first point, take any such A, B and let $A_0 := A \cap \mathcal{A}_\omega$, $B_0 := B \cap \mathcal{A}_\omega$. Furthermore, let $A' := A \setminus A_0$ and $B' := B \setminus B_0$. Note that we can pick $m \in \omega$ such that $A_0 \cup B_0 \subseteq \{X_n : n \in m\} \subseteq \mathcal{A}_\omega$. Then fix some $g \in Q_0$ so that for any $n \in m$, $X_n \in A_0 \cup B_0 \wedge g(n) = 0$ iff $X_n \in A_0$ (specifying the "decisions" of g). It follows that

$$\bigcap A \setminus \bigcup B \supseteq (\bigcap A' \setminus \bigcup B') \cap \bigcap_{n<m} X_n^{g(n)} \supseteq^* (\bigcap A' \setminus \bigcup B') \cap Y_g,$$

where the first inclusion follows from $\bigcap A \setminus \bigcup B = \bigcap A' \setminus \bigcup B' \cap \bigcup A_0 \setminus \bigcap B_0$ and the choice of m and g. For the second inclusion we use (1) from the definition of Y_g. Point (2) of the definition in turn gives us that the set $Y_g \cap (\bigcap A' \setminus \bigcup B')$ is infinite. Also, $(\bigcap A' \setminus \bigcup B') \cap Y_g$ is almost contained in Z, as $g \in Q_0$ and $Y_{g'} =^* Y_g$ for each g'. Taking the intersection with Z, the above implies that $\bigcap A' \setminus \bigcup B' \cap Y_g \subseteq^* Z \cap \bigcap A \setminus \bigcup B$, and so the latter set is infinite. The second point follows analogously.

Finally, $Z \notin \mathcal{A}$ because otherwise there exists a Boolean combination $\bigcap A \setminus \bigcup B$ with $Z \in B$, so that $Z \cap \bigcap A \setminus \bigcup B = \emptyset$, in contradiction to the above. Overall, we have constructed a set Z which splits every Boolean combination of $\mathcal{A}$, therefore proving that $\mathcal{A}$ is not maximal. $\square$

2 Diagonalization

The ground work laid in Chapter 1 specifically provides a first classification of the cardinal characteristics $\mathfrak{a}$ and $\mathfrak{i}$. In particular, we were able to construct mad and maximal independent families of cardinality $\mathfrak{c}$, thereby showing that their spectra are non-empty. To further inspect possible values of $\mathfrak{a}$ and $\mathfrak{i}$ we now turn to relative consistency results obtained by employing certain forcing methods, most prominently finite support iterations of ccc forcing notions. In this context it is worthwhile to introduce diagonalization, a method to extend a given a.d. or independent family. Roughly, given an a.d. or an independent family $\mathcal{A}$ in the ground model, we can construct a generic object d in the generic extension to witness the following two conditions:

> Extension condition: $\mathcal{A} \cup \{d\}$ is still a.d./independent.
>
> Maximality condition: For any ground model element x such that $\mathcal{A} \cup \{x\}$ is a.d./independent, in the generic extension the family $\mathcal{A} \cup \{x, d\}$ is not a.d./independent anymore.

Throughout this chapter let $\mathcal{M}$ denote our ground model, a countable transitive model (ctm) satisfying *ZFC*. For a given generic filter G, let $\mathcal{M}[G]$ denote the generic extension of $\mathcal{M}$ obtained by forcing with a given forcing notion.

Let us restate the two conditions from above in form of a definition:

Definition 2.0.1 (Diagonalizing set)**.** Let $\mathcal{M}$, $\mathcal{N}$ be ctms of *ZFC*, $d \in [\omega]^\omega$, and $\mathcal{A} \in \mathcal{M}$ a given set. Then d *diagonalizes* $\mathcal{A}$ *over* $\mathcal{M}$ (in $\mathcal{N}$) if:

1. $\mathcal{N}$ is a generic extension of $\mathcal{M}$ and ($\mathcal{A}$ is a.d./independent)$^{\mathcal{M}}$.
2. We can add $d \in ([\omega]^\omega)^{\mathcal{N}} \setminus \mathcal{M}$ to $\mathcal{A}$ and retain almost disjointness/independence, i.e., $\mathcal{A} \cup \{d\}$ is an a.d./independent family. (Extension)
3. We cannot add another element outside of $\mathcal{A}$ (in $\mathcal{M}$) to the resulting family in (2) while retaining independence: For any $x \in ([\omega]^\omega)^{\mathcal{M}} \setminus \mathcal{A}$ with ($\mathcal{A} \cup \{x\}$ independent)$^{\mathcal{M}}$, the family $\mathcal{A} \cup \{x, d\}$ is not independent in $\mathcal{N}$. (Maximality)

In what follows, different ways to achieve diagonalization are going to be considered. Section 2.1 will be concerned with a.d. families. It is based on the lecture notes [5] of V. Fischer, where the corresponding results are given in terms of Martin's axiom. The independence context is discussed in Section 2.2, which follows Fischer's and S. Shelah's work [7].

2.1 Almost disjoint families

We present two approaches to diagonalization in the case of almost disjointness: One via Mathias forcing relativized to a certain filter (2.1.1), the other one via almost disjoint coding (2.1.2), which makes use of Solovay's Lemma. Both notions will play a role in subsequent parts of the thesis, as mentioned in the introduction.

2.1.1 Mathias forcing approach

To begin with, consider the general notion of Mathias forcing relativized to an arbitrary filter $\mathcal{F} \subseteq [\omega]^\omega$.

Definition 2.1.1 (Mathias forcing)**.** $\mathbb{M}(\mathcal{F})$ is a forcing poset with conditions $(s, A) \in [\omega]^{<\omega} \times \mathcal{F}$ so that $max(s) < min(A)$, and $(t, B) \leq (s, A)$ iff

1. $t \supseteq s$. 2. $B \subseteq A$. 3. $t \backslash s \subseteq A$.

Let $G \subseteq \mathbb{M}(\mathcal{F})$ be an $\mathbb{M}(\mathcal{F})$-generic filter over $\mathcal{M}$. Our candidate for d as in Definition 2.0.1 is given by a so-called *Mathias real*

$$d_G := \bigcup \{s : \exists A \in \mathcal{F}((s, A) \in G)\}. \tag{2.1}$$

In order to show that d_G diagonalizes a given almost disjoint family $\mathcal{A}$ over $\mathcal{M}$ (in the generic extension $\mathcal{N} = \mathcal{M}[G]$), we have to:

I. First find a way to associate our forcing notion $\mathbb{M}(\mathcal{F})$ to $\mathcal{A}$.

II. Secondly show that d_G fulfills the Extension and Maximality conditions from 2.0.1.

III. And finally prove that $\mathbb{M}(\mathcal{F})$ has the countable chain condition (ccc), in order to make sure to have cardinal preservation when forcing with $\mathbb{M}(\mathcal{F})$ (Appendix - Proposition 8.2.8). This last point will be of particular importance for our application of diagonalization in the context of ccc finite support iterations.

We begin with point (I) by introducing special filters $\mathcal{F}_{\mathcal{A}}$:

Remark 2.1.2 (Induced filters $\mathcal{F}_{\mathcal{A}}$). Any almost disjoint family $\mathcal{A}$ induces an ideal

$$I(\mathcal{A}) := \{X \subseteq \omega : \exists Y \in [\mathcal{A}]^{<\omega}(X \subseteq^* \bigcup Y)\},$$

with respect to which one can define the dual filter

$$\mathcal{F}_{\mathcal{A}} := \mathcal{F}(\mathcal{A}) := \{X \subseteq \omega : \omega \backslash X \in I(\mathcal{A})\}.$$

Note that $I(\mathcal{A})$ contains all finite subsets of ω, so each cofinite $X \subseteq \omega$ must be in $\mathcal{F}_{\mathcal{A}}$. Also, $\mathcal{A} \subseteq I(\mathcal{A})$ (and so no element of $\mathcal{A}$ can be in $\mathcal{F}_{\mathcal{A}}$). To summarize, $I(\mathcal{A}) \supseteq \mathcal{A} \cup [\omega]^{<\omega}$.

From the definition of $\mathbb{M}(\mathcal{F})$ it is clear that the generic object d_G will be almost contained in some element $A \in \mathcal{F}$ (which formally follows from the next lemma). Using $\mathcal{F}_{\mathcal{A}}$ instead of $\mathcal{F}$ ensures that d_G does not approximate any of the elements of $\mathcal{A}$.

Before presenting the density arguments for (II), we give a similar proof to show that d_G is an infinite subset of ω. Note that the only part where we will require to use $\mathcal{F}_{\mathcal{A}}$ is the (crucial) maximality condition (Lemma 2.1.4 part (2)). Everything else could be stated in terms of an arbitrary filter $\mathcal{F}$.

Lemma 2.1.3. *Let d_G be as in (2.1). Then $d_G \in [\omega]^\omega \cap \mathcal{M}[G]$.*

Proof. To see that $d_G \in [\omega]^\omega \cap \mathcal{M}[G]$, for any $n \in \omega$ define the sets

$$D_n = \{(s, X) \in \mathbb{M}(\mathcal{F}_{\mathcal{A}}) : \exists m > n(m \in s)\}.$$

Each D_n is dense, as for any $(s, A) \in \mathbb{M}(\mathcal{F}_{\mathcal{A}})$ with $m \notin s$ for all $m > n$ we can define an extension in D_n given by $(t, B) := (s \cup \{m\}, A \setminus m + 1)$, where $m \in A$ such that $m > n$ exists because s is finite, $\max(s) < \min(A)$, and A is infinite. Therefore, for each $n \in \omega$, $G \cap D_n \neq \emptyset$, and so d_G is infinite in $\mathcal{M}[G]$. □

Next, let us prove that d_G indeed diagonalizes $\mathcal{A}$.

Lemma 2.1.4. *Let κ be a cardinal such that $\omega \leq \kappa < \mathfrak{c}$, and let $\mathcal{A}$ be an almost disjoint family of cardinality κ with induced filter $\mathcal{F}_{\mathcal{A}} \subseteq [\omega]^\omega$ as in Remark 2.1.2. Then for an $\mathbb{M}(\mathcal{F}_{\mathcal{A}})$-generic filter G (over $\mathcal{M}$) and d_G defined as in (2.1) it holds that in $\mathcal{M}[G]$:*

1. *$\mathcal{A} \cup \{d_G\}$ is an almost disjoint family.*
2. *$\forall x \in ([\omega]^\omega \setminus \mathcal{A}) \cap \mathcal{M}$ so that $\mathcal{A} \cup \{x\}$ forms an almost disjoint family, $\mathcal{A} \cup \{x, d_G\}$ is not almost disjoint.*

Proof. We prove both statements using density arguments. To see part (1), for any $a \in \mathcal{A}$ define the set

$$D_{\omega\backslash a} = \{(s, A) \in \mathbb{M}(\mathcal{F}_{\mathcal{A}}) : A \subseteq \omega\backslash a\}.$$

Fix such a and show that $D_{\omega\backslash a}$ is dense by taking any $(s, A) \in \mathbb{M}(\mathcal{F}_{\mathcal{A}})$ with $A \nsubseteq \omega\backslash a$, and providing a suitable extension in $D_{\omega\backslash a}$. By assumption, there exists $n \in A$ with $n \notin \omega\backslash a$, i.e., $n \in a$. Thus we can define an extension of (s, A) by $(s, A\backslash a)$, where $(s, A\backslash a) \in D_{\omega\backslash a}$ because $A \backslash a = A \cap (\omega \backslash a)$ is again an element of $\mathcal{F}_{\mathcal{A}}$ (using that $\mathcal{F}_{\mathcal{A}}$ is a filter) and $A\backslash a \subseteq \omega\backslash a$. Therefore, for any $a \in \mathcal{A}$, $D_{\omega\backslash a}$ is dense, and so the generic filter G intersects all of the $D_{\omega\backslash a}$. This implies that $d_G \subseteq^* \omega \backslash a$, and by $a \in \mathcal{A}$ being arbitrary, that (1) holds.

For (2), it suffices to find dense sets ensuring that $|d_G \cap x| = \aleph_0$ for any $x \in [\omega]^\omega \backslash \mathcal{A} \cap \mathcal{M}$ so that $\mathcal{A} \cup \{x\}$ forms an a.d. family. To this end, for $n \in \omega$ and $x \in [\omega]^\omega \backslash \mathcal{A}$, let

$$D_{n,x} := \{(s, A) : \exists m > n(m \in s \cap x)\}.$$

There are two cases to consider. If $x \in \mathcal{F}_{\mathcal{A}}$, repeat the argument from Lemma 2.1.3. Otherwise, $x \in I(\mathcal{A}) \setminus \mathcal{A}$, where x must be infinite and coinfinite. Since $x \cap A$ does not have to be infinite in general, $D_{n,x}$ is not dense in $\mathbb{M}(\mathcal{F}_{\mathcal{A}})$. Yet $\omega\backslash x \in \mathcal{F}_{\mathcal{A}}$, and thus, by employing $D_{n,\omega\backslash x}$ instead, we can now prove that these sets are dense (again as in 2.1.3). It follows that $d_G \subseteq^* \omega \backslash x$. Let us conclude by showing that this implies (2). Indeed, take an arbitrary $a \in \mathcal{A}$. As $|x \cap a| < \omega$, $|\omega \backslash x \cap a| = \omega$. Thus, $d_G \subseteq^* a$, implying that $\mathcal{A} \cup \{d_G, x\}$ is not almost disjoint. $\square$

Finally, we show that $\mathbb{M}(\mathcal{F}_{\mathcal{A}})$ has the ccc by proving that it fulfills the stronger notion of being σ-centered (Appendix, Definition 8.2.10).

Remark 2.1.5. $|[\omega]^{<\omega}| = \aleph_0$.

Lemma 2.1.6. *$\mathbb{M}(\mathcal{F}_{\mathcal{A}})$ is σ-centered.*

Proof. We can express $\mathbb{M}(\mathcal{F}_{\mathcal{A}})$ as the countable union $\bigcup_{s\in[\omega]^{<\omega}} M_s$, where each M_s is given by $\{(s, A) : A \in \mathcal{F}_{\mathcal{A}}\}$. Furthermore, each M_s is centered, as for any two $(s, A_1), (s, A_2) \in M_s$, the extending condition $(s, A_1 \cap A_2)$ is in M_s, where $A_1 \cap A_2 \in \mathcal{F}_{\mathcal{A}}$ because $\mathcal{F}_{\mathcal{A}}$ is a filter and therefore closed under intersections. $\square$

2.1.2 The almost disjoint coding approach

Definition 2.1.7 (Almost disjoint coding)**.** Let $\mathcal{A} \subseteq [\omega]^\omega$. Then the a.d. coding forcing poset $\mathbb{P}_\mathcal{A}$ is given by

- conditions $(s, F) \in [\omega]^{<\omega} \times [\mathcal{A}]^{<\omega}$
- the relation $(s', F') \leq (s, F)$ iff $s' \supseteq s$, $F' \supseteq F$, $\forall x \in F(x \cap s' \subseteq s)$.

For the same reasons as in the case of Mathias forcing we want to make sure that $\mathbb{P}_\mathcal{A}$ has the ccc:

Lemma 2.1.8. *$\mathbb{P}_\mathcal{A}$ is σ-centered.*

Proof. For $a \in [\omega]^{<\omega}$ set $P_a := \{(s, F) \in \mathbb{P}_\mathcal{A} : s = a\} = \{(a, F) : F \in [\mathcal{A}]^{<\omega}\}$. Then certainly $\bigcup_{a \in [\omega]^{<\omega}} P_a = \mathbb{P}_\mathcal{A}$ is countable. Furthermore, P_a is centered, as for any two conditions $(a, F_1), (a, F_2) \in P_a$ we can always find an extension $(a, F_1 \cup F_2) \in P_a$. □

For G an $(\mathcal{M}, \mathbb{P}_\mathcal{A})$-generic filter consider

$$d_G := \bigcup \{s : \exists F((s, F) \in G)\}. \tag{2.2}$$

We want to show that d_G fulfills the second of the diagonalization properties stated in Definition 2.0.1. To this end begin by noting:

Lemma 2.1.9. *If $G \subseteq \mathbb{P}_\mathcal{A}$ is a filter and $(s, F) \in G$, then $d_G \cap x \subseteq s$ for every $x \in F$.*

Proof. Let $(s, F) \in G$. It suffices to show that for any fixed $x \in F$ the intersection $(d_G \setminus s) \cap x$ is empty. To see this, we will prove that any $n \in d_G \setminus s$ is not in x, and conversely any $n \in x$ cannot be in $d_G \setminus s$. So let $n \in d_G \setminus s$. Then by definition of d_G there must be some $(s', F') \in G$ so that $n \in s'$. Therefore, $n \in s' \setminus s$. As G is a filter and thus in particular downwards directed, we can find an extension in G for both (s, F) and (s', F'). To simplify notation it is without loss possible to assume that $(s', F') \leq (s, F)$ (the same proof works for the common extension). Using the definition of $\leq$, it holds that $x \cap s' \subseteq s$, which implies $(s' \setminus s) \cap x = \emptyset$. Thus $n \notin x$.

For the converse, take any $n \in x$ and assume towards a contradiction that $n \in d_G \setminus s$. Again, this implies that there exists some $(s', F') \in G$ with $n \in s'$. Using that G is upwards closed and therefore its elements are pairwise compatible, (s, F) and (s', F') are compatible. As before, assume without loss of generality that $(s', F') \leq (s, F)$. In particular, for all $x \in F$, $x \cap s' \subseteq s$. It follows that for all $x \in F$ and any $m \in x \setminus s$, $m \notin s'$ (since $m \in s'$ would mean that $m \in s$). But then $n \notin s'$, a contradiction. □

Next, assume that G is $(\mathcal{M}, \mathbb{P}_{\mathcal{A}})$-generic. Then indeed via a density argument and Lemma 2.1.9 we obtain:

Lemma 2.1.10. *Given any family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ and $x \in \mathcal{A}$, the intersection $d_G \cap x$ is finite.*

Proof. Fix any $x \in \mathcal{A}$ and define

$$D_x = \{(s, F) \in \mathbb{P}_{\mathcal{A}} : x \in F\}.$$

To see that D_x is dense in $\mathbb{P}_{\mathcal{A}}$, for any $(s, F) \in \mathbb{P}_{\mathcal{A}}$ with $x \notin F$, set $F' := F \cup \{x\}$. Then $(s, F') \in D_x$ extends (s, F). Therefore, as $D_x \cap G \neq \emptyset$, we can apply Lemma 2.1.9 to conclude the proof. $\square$

In particular, the previous lemma states that for $\mathcal{A} \subseteq [\omega]^\omega$ an almost disjoint family, d_G fulfills property (2) of Definition 2.0.1. Property (3) of 2.0.1 is taken care of in the next result, also known as Solovay's Lemma.

Proposition 2.1.11. *Let $\kappa \geq \omega$ be a cardinal. Consider the families $\mathcal{A}, \mathcal{C} \subseteq \mathcal{P}(\omega)$ with $|\mathcal{A}|, |\mathcal{C}| \leq \kappa$. Furthermore, assume that for any $y \in \mathcal{C}$ and any $F \in [\mathcal{A}]^{<\omega}$, the set $y \setminus \bigcup F$ is infinite. Then there exists $d \in [\omega]^\omega \cap \mathcal{M}[G]$ so that in $\mathcal{M}[G]$ the following holds:*

1. *For any $x \in \mathcal{A}$, $|x \cap d| < \omega$,*
2. *For any $y \in \mathcal{C}$, $|y \cap c| = \omega$.*

Proof. Let $d := d_G$ be as in (2.2). Proceed by providing a density argument for each of the statements. To see that d is infinite, define for every $n \in \omega$ the sets

$$D_n = \{(s, F) \in \mathbb{P}_{\mathcal{A}} : \exists m > n(m \in s \land m \notin \bigcup F)\}.$$

For the density of D_n, let $(s, F) \in \mathbb{P}_{\mathcal{A}}$ be such that $\forall m > n(m \notin s \ \lor \ m \in \bigcup F)$. Note that we cannot assume $m \in \bigcup F$ for all $m > n$, because otherwise $y \setminus \bigcup F$ would be finite for all $y \in \mathcal{C}$, in contradiction to our assumption. Obtain $(s', F') \leq (s, F)$ by setting $s' := s \cup \{n\}$ and $F' := F$, where by the above we may choose $m > n$ such that $m \notin \bigcup F$. Then $(s', F') \in D_n$, and so $G \cap D_n \neq \emptyset$ for any $n \in \omega$, proving that $d \in [\omega]^\omega \cap \mathcal{M}[G]$.

Too see (1) we can use Lemma 2.1.10. Finally, for (2) define the sets

$$E_n^y = \{(s, F) \in \mathbb{P}_{\mathcal{A}} : s \cap y \not\subseteq n\},$$

where $n \in \omega$ and $y \in \mathcal{C}$. To prove density, for arbitrary $(s, F) \in \mathbb{P}_{\mathcal{A}}$ an extension is given by $(s \cup m, F) \in E_n^y$, where $m > n$ has been picked such that $m \in y$ but $m \notin \bigcup F$. This choice of m is possible due to the pigeonhole principle, because $|y \setminus \bigcup F| = \omega$. Indeed, $(s \cup m, F) \leq (s, F)$: We must show that for all $x \in F$, $x \cap (s \cup \{m\}) \subseteq s$, which is true by our choice of m, as $m \notin \bigcup F$ results in $m \notin x$, implying that $x \cap (s \cup \{m\}) = x \cap s \subseteq s$. The fact that $(s \cup m, F) \in E_n^y$ follows from $m > n$ and $m \in y$, as therefore $(s \cup \{m\}) \cap y \not\subseteq n$. In total, G intersects every E_n^y, which means that for fixed $y \in \mathcal{C}$ the finite approximations $s \cap y$ cannot be contained by any finite $n \in \omega$. As d contains all such s, $|d \cap y| = \aleph_0$. $\square$

Note that up until now we have not made any further assumptions on $\mathcal{A}$ other than it being a subfamily of $\mathcal{P}(\omega)$, with the additional requirement for Solovay's Lemma that $|\mathcal{A}| \leq \kappa$. Next, we wish to employ Solovay's Lemma with respect to an almost disjoint family $\mathcal{A}$. This makes property (3) mentioned above explicit.

Corollary 2.1.12. *Let $\mathcal{A} \subseteq [\omega]^\omega$ be an almost disjoint family and $\kappa < 2^\omega$ an infinite cardinal such that $|\mathcal{A}| = \kappa$. Then $\mathcal{A}$ is not maximal.*

Proof. As $\mathcal{A}$ is infinite and almost disjoint, for any $F \in [\mathcal{A}]^{<\omega}$, $\omega \setminus \bigcup F$ is infinite. Indeed, assume towards a contradiction that there exists $F \in [\mathcal{A}]^{<\omega}$ so that $\omega \setminus \bigcup F$ is finite. Let $B \in \mathcal{A} \setminus F$. Then there must be $B_0 \in F$ with $|B \cap B_0| = \aleph_0$ since otherwise $|B| < \omega$, a contradiction. Thus, by applying Solovay's Lemma to $\mathcal{A}$ and $\mathcal{C} = \{\omega\}$, find $d \in [\omega]^\omega \cap \mathcal{M}[G]$ with $|d \cap x| < \omega$, for any $x \in \mathcal{A}$. In particular, $d \notin \mathcal{A}$. So d witnesses that $\mathcal{A}$ is not maximal because $\mathcal{A} \cup \{d\}$ forms an almost disjoint family. $\square$

2.2 Independent families - Mathias forcing with diagonalization filters

In order to obtain a diagonalizing element as in Definition 2.0.1, proceed similarly as in the case of almost disjointness, that is, by defining Mathias forcing relativized to a certain filter. Let $\mathcal{A}$ be an independent family. Denote the Fréchet filter (the filter consisting of all cofinite sets) by $\mathcal{F}_0$. We first note that $\mathcal{F}_0$ has the following two properties.

Claim 2.2.1.

1. *$\forall A \in \mathcal{F}_0 \forall B \in BC(\mathcal{A})(A \cap B$ is infinite).*

2. *$\mathcal{F}_0 \cap BC(\mathcal{A}) = \emptyset$.*

Proof. To see the first statement, assume towards a contradiction that $A \cap B$ is finite. As $|B| = \aleph_0$, infinitely many $b \in B$ must be outside of A, i.e., $(\omega \setminus A) \cap B$ must be infinite, contradicting A being cofinite. For the second point assume towards a contradiction that

A is an element of both, $\mathcal{F}_0$ and $BC(\mathcal{A})$. Then in particular $\omega \setminus A \in BC(A)$, so $\omega \setminus A$ is infinite leading to the same contradiction. $\square$

Using the above, define the following type of filters to which we will relativize our Mathias forcing notion.

Definition 2.2.2 (Diagonalization filter)**.** Call $\mathcal{U}$ an $\mathcal{A}$-diagonalization filter if $\mathcal{U}$ is an extension of $\mathcal{F}_0$ which is maximal with respect to (1) and (2) from Claim 2.2.1.

Remark 2.2.3 (Comparison to $\mathcal{F}_\mathcal{A}$). Knowing that the Mathias forcing notion relativized to a filter $\mathcal{F}$ aims to provide a generic object which approximates an element of $\mathcal{F}$, generally speaking, the usage of the notion of a filter allows for a reduction of complexity in addressing all objects of the given family $\mathcal{A}$ in an indirect fashion. That is to say, we do not have to add or remove (i.e., encode) the values of d_G by hand, and instead we say that any potential d_G must have the wanted properties (e.g. intersect every Boolean combination $\mathcal{A}^h$ infinitely often) because it will be "big" with respect to the filter $\mathcal{U}$. In the case of $\mathcal{F}_\mathcal{A}$, it seems natural to start with the ideal induced by $\mathcal{A}$ since the definition of almost disjointness itself guarantees to have sets which are "spread apart" from one another, and therefore must be "small" in the sense of the ideal $I(\mathcal{A})$. The more complex notion of independence enforces another, more indirect approach, as above. So (1) makes sure that the generic object will intersect every $\mathcal{A}^h$ "often enough", whereas (2) guarantees that d_G will be distinct from all of the $\mathcal{A}^h$.

We now employ $\mathbb{M}(\mathcal{F})$ from Definition 2.1.1 with $\mathcal{F} = \mathcal{U}$. The following lemma is the counterpart to Lemma 2.1.4 from the a.d. context.

Lemma 2.2.4. *In the ground model $\mathcal{M}$, let $\mathcal{A}$ be an independent family, $\mathcal{U}$ an $\mathcal{A}$-diagonalization filter, and $\mathbb{M}(\mathcal{U})$ the Mathias forcing poset relativized to $\mathcal{U}$. Let G be an $\mathbb{M}(\mathcal{U})$-generic filter over $\mathcal{M}$. In $\mathcal{M}[G]$, define $d_G = \bigcup\{s : \exists(s, A) \in G\}$. Then:*

1. *$\mathcal{A} \cup \{d_G\}$ is an independent family in $\mathcal{M}[G]$.*
2. *$\forall x \in ([\omega]^\omega \setminus \mathcal{A}) \cap \mathcal{M}$ so that $\mathcal{A} \cup \{x\}$ forms an independent family in $\mathcal{M}$, the family $\mathcal{A} \cup \{x, d_G\}$ is not independent in $\mathcal{M}[G]$.*

Proof. Note that by Lemma 2.1.3 d_G is infinite. To see (1), it suffices to show that d_G splits every Boolean combination of $\mathcal{A}$, so we require two families of dense sets, one to witness that d_G intersects every Boolean combination infinitely often, the other to witness the same for $\omega \setminus d_G$. For the former, consider for any finite function $h \in FF(\mathcal{A})$ and $n \in \omega$ the sets

$$D_{n,h} := \{(s, A) \in \mathbb{M}(\mathcal{U}) : |s \cap \mathcal{A}^h| > n\}.$$

To show that these sets are dense in $\mathbb{M}(\mathcal{U})$ take any $(s, A) \in \mathbb{M}(\mathcal{U})$ so that $|s \cap \mathcal{A}^h| \leq n$. Now $A \in \mathcal{U}$ has property (1) from the definition of $\mathcal{U}$, that is, $A \cap \mathcal{A}^h$ is infinite. But this means that we can surely find a finite $t \subseteq A \cap \mathcal{A}^h$ such that $min(t) > max(s)$ and $|t| > n$, which allows us to define an extension $(s \cup t, X \setminus (max(t) + 1)) \in D_{h,n}$ of (s, A). Thus $d_G \cap \mathcal{A}^h$ is infinite for any $h \in FF(\mathcal{A})$.

We still have to show that $(\omega \setminus d_G) \cap \mathcal{A}^h$ is likewise infinite. To this end, introduce for n and h as above the sets

$$E_{n,h} := \{(s, A) \in \mathbb{M}(\mathcal{U}) : |(min(A) \setminus max(s)) \cap \mathcal{A}^h| > n\},$$

which aim to add elements to the complement of d_G by making the space in between s and A arbitrarily big. Let $(s, A) \in \mathbb{M}(\mathcal{U})$ be such that $|(min(A) \setminus max(s)) \cap \mathcal{A}^h| \leq n$. To find an extension in $E_{n,h}$, take an initial segment t of $A \cap \mathcal{A}^h \setminus \min(A)$ with $|t| > n$, which is possible due to the definition of $\mathcal{U}$. Then it follows that $(s \cup t, A \setminus (max(t) + 1))$ is a condition extending (s, A). It is also the case that $(s \cup t, A \setminus (max(t) + 1)) \in E_{n,h}$: Indeed, note that by the construction above, $\min(A \setminus \max(t) + 1) = \max(t) + 1$ as well as $\max(s \cup t) = \max(t)$. Hence

$$|(\min(A \setminus \max(t) + 1) \setminus \max(s \cup t)) \cap \mathcal{A}^h| = |\max(t) + 1 \setminus \max(t)| = |\max(t)| = |t| > n.$$

Since n, h were chosen arbitrarily, $\mathcal{A}^h \setminus d_G = \mathcal{A}^h \cap (\omega \setminus d_G)$ is infinite for any $h \in FF(\mathcal{A})$.

For (2), proceed by considering the following cases. If $x \in \mathcal{U}$, then $d_G \subseteq^* x$, which means that $d_G \cap (\omega \setminus x)$ is finite. Say $x \notin \mathcal{U}$. To not contradict maximality of $\mathcal{U}$, either one of the following two cases must hold:

(a) There exists $h \in FF(\mathcal{A})$ witnessing $|\mathcal{A}^h \cap x| < \omega$. (Negation of 2.2.1 (1))

(b) $(\mathcal{U} \cup \{x\}) \cap BC(\mathcal{A}) \neq \emptyset$. (Negation of 2.2.1 (2))

As by assumption, case (a) is not possible, (b) must be true. By $\mathcal{U} \cap BC(\mathcal{A}) = \emptyset$, there has to be some $h \in FF(\mathcal{A})$ so that $x = \mathcal{A}^h$. Hence there exists $A \in \mathcal{U}$ with $x \cap A \subseteq \mathcal{A}^h$. Without loss, pick $C \in dom(h)$ such that $h(C) = 1$. Then $A \cap x \subseteq \mathcal{A}^h \subseteq C$ implies that $A \cap x \cap \omega \setminus C$ is finite (in case that $h(C) = 0$ for all $C \in dom(h)$, obtain $A \cap x \subseteq \omega \setminus C$, and so $|A \cap x \cap C| < \omega$). Now $d_G \subseteq^* A$ implies the statement. $\square$

3 Finite support iterations

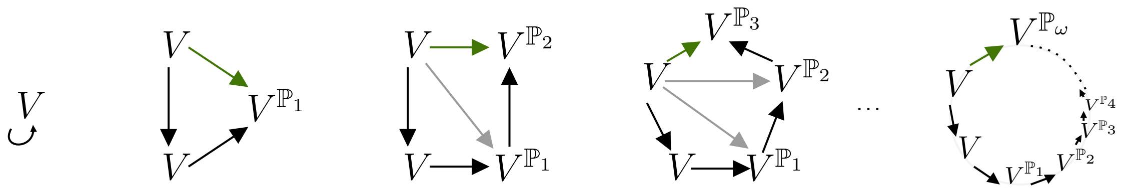
As finite supports iterations play a prominent role in this work, the aim of this chapter is to provide an introduction focusing on the application of the method. We first wish to highlight the basic idea of iterated forcing (3.1), and continue discussing finite (3.2) and infinite (3.3) iterations. As this subject is very rich in technical details and subtleties, our discussion will be centered around the solution of an example problem (3.1.1), where results from the theory will be introduced whenever necessary. This example, while not being of importance for the rest of the thesis, illustrates (and was in fact influenced by) the usage of finite support iterations in applications concerning the positive results from Chapter 4. It will be central to discuss an inherent property of finite support iterations which arises in the case that the iteration length is of countable cofinality (3.4).

The expressions "catching the tail" as well as "going full n-gon/circle" in Section 3.1 have been inspired by Shelah's phrasing that we "catch our tail" in [17]. The illustration from the next section in turn was inspired by this expression. The sources for the theoretical results in Sections 3.2 and 3.3 are two books from K. Kunen: [15] and [16]. Concerning these results specifically, the corresponding proofs have been significantly shortened (or left out and referred to) compared to much of the rest of the thesis for reasons of readability and limitation of scope. Finally, Fact 3.4.1 is Example 2.0.2 from M. Goldstern's work [10].

3.1 A very short introduction

Roughly speaking, iterated forcing gives us a way to split a task, the wished-for forcing procedure, into finitely or infinitely many sub-tasks, each of which takes care of a certain part of the wanted construction. It is important to realize that this method allows us to work with forcing posets which in general are not available in the ground model, in contrast to product forcing (see for example [16], Cha. V.1).

Consider the following illustration:



Here we see depictions of iterated forcing constructions with iteration lengths $\delta = 0, 1, 2, 3$ and ω, from left to right. We use V to denote the ground model. An arrow $A \to B$ means that there exists a forcing notion in A leading to the extension B. Each $V^{\mathbb{P}_\alpha}$ is the generic extension of V when forcing with $\mathbb{P}_\alpha$. In order to shorten the definition of iterated forcing, we allow for the cases $\delta = 0$ and $\delta = 1$, which correspond to not forcing at all and "simple", that is, non-iterated forcing, respectively.

$\delta = 0, 1$: In the first picture one can use the trivial forcing poset $\mathbb{P}_0 := \{\emptyset\}$ to obtain the curved arrow. Thus, $V = V^{\mathbb{P}_0}$ leads to itself. The second picture shows simple forcing using a forcing poset $\mathbb{P}_1$. Basically, the green arrow would suffice here. We include the repetition of the trivial forcing $V \to V$ via $\mathbb{P}_0$ in order to reflect the definition of iterated forcing. This will become clearer in the following discussion for the case $\delta \geq 2$.

$\delta \geq 2$: Looking at the illustration, one thing to notice is that all arrows from the previous steps are reproduced in the next step, where an eventual previously green arrow becomes a grey arrow in the next picture (the grey arrows have been excluded in the picture for $\delta = \omega$). The outer (black) arrows can be thought of as a "tail" of forcings corresponding to taking care of the given subtasks one step at a time. Certainly, this is easier to imagine for finite iteration lenghts, yet as we will see, properties inherent to the method emerge only after transitioning to infinite iterations. The green arrows correspond to "catching the tail" (or "going full circle" respectively "going full n-gon"), i.e., obtaining a forcing poset in V which conveniently fulfills all subtasks at once. Therefore crucially, the green arrows stand for the actual iterated forcing constructions.

Problem 3.1.1. Back to the practicality of the method. Let us state two example problems we would like to solve in the course of the next two sections:

1. Say we force the existence of a mad family $\mathcal{A}$ with $|\mathcal{A}| = \aleph_1$ (see Appendix - Section 8.3.5), and for some reason we wish to force that $\mathcal{A}$ is not maximal, that is, in the generic extension we wish to construct a witness for the non-maximality of $\mathcal{A}$. One way to achieve this is to employ almost disjoint coding (Section 2.1.2).

2. Assume CH. Then $\aleph_1 \in Spec(\mathfrak{a})$.

3.2 Finite iterations or "going full n-gon"

To approach (1) we use a so-called two-step iteration ($\delta = 2$). As such iterations are in part instructive for all longer iterations, we will begin with this example, noting that any iteration length of finite cofinality $n < \omega$ can be split up into a successive series of two-step iterations. Interestingly, the results from (1) will be of importance for (2).

Let $V \models ZFC + CH$. To obtain $\mathcal{A}$, define $\mathbb{P}_1 = \mathbb{C}$ (in V) to be the Cohen forcing poset from Proposition 8.3.5. Here we see the necessity, as already suggested by the illustration above, of finding names for forcing posets.

Definition 3.2.1 (Name for a forcing notion)**.** For a given forcing poset $\mathbb{P} \in V$ and G a $(V, \mathbb{P})$-generic filter, let $\dot{\mathbb{Q}}$ be an abbreviation for $(\dot{\mathbb{Q}}, \leq_{\dot{\mathbb{Q}}}, 1_{\dot{\mathbb{Q}}})$, where in $V[G]$:

- $\dot{\mathbb{Q}}_G = \mathbb{Q}$ is a set preordered by $(\leq_{\dot{\mathbb{Q}}})_G = \leq_{\mathbb{Q}}$,
- $(1_{\dot{\mathbb{Q}}})_G = 1_{\mathbb{Q}} \in \mathbb{Q}$ forms the weakest element with respect to $\leq_{\mathbb{Q}}$.

Definition 3.2.2 (Two-step iteration)**.** Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a forcing poset. Define $\mathbb{P} * \dot{\mathbb{Q}} := \{\langle p, \dot{q}\rangle : p \in \mathbb{P} \wedge 1_{\mathbb{P}} \Vdash_{\mathbb{P}} "\dot{q} \in \dot{\mathbb{Q}}"\}$, ordered by $\langle p_1, \dot{q}_1\rangle \leq \langle p_0, \dot{q}_0\rangle$ iff $p_1 \leq_{\mathbb{P}} p_0$ and $p_1 \Vdash_{\mathbb{P}} "\dot{q}_1 \leq_{\dot{\mathbb{Q}}} \dot{q}_0"$. Furthermore, set $1_{\mathbb{P}*\dot{\mathbb{Q}}} := \langle 1_{\mathbb{P}}, 1_{\dot{\mathbb{Q}}}\rangle$.

Remark 3.2.3. $\mathbb{P} * \dot{\mathbb{Q}}$ may not be well-defined as a set: If we do not specify a way to restrict the possibilities for the number of $\mathbb{P}$-names $\dot{q}$ for $q \in \mathbb{Q}$ such that $1_{\mathbb{P}} \Vdash_{\mathbb{P}} "\dot{q} \in \dot{\mathbb{Q}}"$, it might be that all of such names together form a proper class. One way to deal with this is by implicitly assuming that $\dot{\mathbb{Q}} \cong (\alpha, \beta, \emptyset)$ for α, β appropriate ordinals, which can be achieved using the Forcing Theorem and the Mixing Lemma (Appendix, Section 8.1). How this is done in detail can be found in [15].

So let $\dot{\mathbb{Q}}_1 = \dot{\mathbb{P}}_{\mathcal{A}}$ be a $\mathbb{P}_1$-name for the almost disjoint coding poset $\mathbb{P}_{\mathcal{A}}$. In order to show that forcing with $\mathbb{P}_1 * \dot{\mathbb{Q}}_1$ leads to the same results as forcing first with $\mathbb{P}_1$ and then with $\mathbb{Q}_1$, it is enough to show that the corresponding generic extensions are the same. For this we need to introduce the notion of a $\mathbb{P}_1 * \dot{\mathbb{Q}}_1$-generic filter over V:

Definition 3.2.4 (Generic filter on $\mathbb{P} * \dot{\mathbb{Q}}$)**.** Let G be a $(V, \mathbb{P})$-generic filter and $H \subseteq \dot{\mathbb{Q}}_G$ be a set. Then define $G * H = \{(p, \tau) \in \mathbb{P} * \dot{\mathbb{Q}} : p \in G \wedge \tau_G \in H\}$.

First, let us note that $\mathbb{P}$ completely embeds into $\mathbb{P} * \dot{\mathbb{Q}}$. As the next lemma shows, this entails that what can be done in $\mathbb{P}$ is also possible in $\mathbb{P} * \dot{Q}$. For further details, see Appendix - Section 8.4, where complete embeddings and related concepts have been introduced. In particular, the notion of equivalent forcing posets has been provided there, together with corresponding results which are reminiscent of what is to follow.

Lemma 3.2.5. *For $\mathbb{P}, \dot{\mathbb{Q}}$ as above, define $i : \mathbb{P} \to \mathbb{P} * \dot{\mathbb{Q}}$ via $i(p) = (p, 1_{\dot{\mathbb{Q}}})$. Then i is a complete embedding.*

Proof. This can be shown by proving:

1. $\forall p, p' \in \mathbb{P}$: $p \leq_{\mathbb{P}} p'$ iff $i(p) \leq_{\mathbb{P}*\dot{\mathbb{Q}}} i(p')$.
2. $i(1_{\mathbb{P}}) = 1_{\mathbb{P}*\dot{\mathbb{Q}}}$.
3. $\forall (p, \dot{q}), (p', \dot{q}') \in \mathbb{P} * \dot{\mathbb{Q}}$: $p \perp_{\mathbb{P}} p' \Rightarrow i(p) \perp_{\mathbb{P}*\dot{\mathbb{Q}}} i(p')$.
4. $\forall (p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}}, \forall p' \in \mathbb{P}$: $p \perp_{\mathbb{P}} p'$ iff $i(p) \perp_{\mathbb{P}*\dot{\mathbb{Q}}} 1_{\dot{\mathbb{Q}}}$.
5. $\forall p, p' \in \mathbb{P}$: $p \perp_{\mathbb{P}} p'$ iff $i(p) \perp_{\mathbb{P}*\dot{\mathbb{Q}}} i(p')$.

Then $(1), (2), (5)$ imply the statement of the lemma. □

Next, let K be $\mathbb{P} * \dot{\mathbb{Q}}$-generic over V, and define

$$G = i^{-1}(K),\ H = \{\tau_G : \tau \in dom(\dot{\mathbb{Q}}) \wedge \exists q \in \mathbb{P}((q, \tau) \in K)\}.$$

We are now ready to formulate the following important result, which may be viewed as catching the tail of the two-step iteration.

Theorem 3.2.6.

(a) G is $\mathbb{P}$-generic over V and H is $\mathbb{Q}$-generic over $V[G]$.

(b) $K = G * H$ and $V[K] = V[G][H]$.

Proof. This is similar to the proof of Proposition 8.4.10. See also Theorem V.3.6 in [16]. □

All of these results hold for $\mathbb{P}_2 = \mathbb{P}_1 * \dot{\mathbb{Q}}_1 = \mathbb{C} * \dot{\mathbb{P}}_{\mathcal{A}}$, which thereby leads to the generic extension $V^{\mathbb{P}_2} = V[\mathcal{A}][d]$, $\mathcal{A}$ being the almost disjoint family added by $\mathbb{C}$, and d the diagonalizing real added by $\dot{\mathbb{P}}_{\mathcal{A}}$. Using Corollary 2.1.12 it follows that $\mathcal{A}$ is not maximal in $V^{\mathbb{P}_2}$. We can even make sure that $|\mathcal{A}| = \aleph_1$ in $V^{\mathbb{P}_2}$. To do so, use the following general result:

Lemma 3.2.7. *The two-step iteration of ccc forcing posets is ccc.*

Proof. This is a special case of Lemma V.3.9 in [16]. □

Knowing that $\mathbb{P}_1$ and $\mathbb{Q}_1$ are ccc, we are done because ccc forcing notions preserve cardinalities (Appendix - Section 8.2.2).

3.3 Infinite iterations or "going full circle"

3.3.1 Definition of finite support iterations

For iterated forcing of length $\delta > 2$, especially $\delta \geq \omega$, it makes sense to introduce a change in notation:

Discussion 3.3.1. *Already for $\delta = 3$, simple repetition of the two-step procedure leads to elements $p \in \mathbb{P}_3$ having the cumbersome form $p = \langle\langle \dot{q}_0, \dot{q}_1\rangle, \dot{q}_2\rangle$, with $\dot{q}_0 \in \dot{\mathbb{Q}}_0$, $\dot{q}_0 \Vdash_{\mathbb{P}_1}$ "$\dot{q}_1 \in \dot{\mathbb{Q}}_1$", and $\langle \dot{q}_0, \dot{q}_1\rangle \Vdash_{\mathbb{P}_2}$ "$\dot{q}_2 \in \dot{\mathbb{Q}}_2$". The definition of general iterations will be formulated in a way to simplify the notation allowing us to write e.g. $p = \langle \dot{q}_0, \dot{q}_1, \dot{q}_2\rangle$.*

We now define general forcing iterations tailored to the finite support case.

Definition 3.3.2 (Finite support iteration)**.** Let δ be a given ordinal in the ground model V. Define recursively on $\alpha < \delta$ the finite support iteration (fsi) of length δ to be the sequence $\mathbb{P}_\delta = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \delta, \beta < \delta\rangle$, where for each α and β, $\mathbb{P}_\alpha := (\mathbb{P}_\alpha, \leq_{\mathbb{P}_\alpha}, 1_{\mathbb{P}_\alpha})$ is a forcing poset, and $\dot{\mathbb{Q}}_\beta$ abbreviates $(\dot{\mathbb{Q}}_\beta, \leq_{\dot{\mathbb{Q}}_\beta}, 1_{\dot{\mathbb{Q}}_\beta})$, with the properties that:

1. For all α: $\mathbb{P}_\alpha$ is the set of all α-length sequences $p = \langle \dot{q}_\mu : \mu < \alpha\rangle$, where $\dot{q}_\mu \in dom(\dot{\mathbb{Q}}_\mu)$ for all $\mu < \alpha$. Also, if $p \in \mathbb{P}_\alpha$, then $p\restriction\beta \in \mathbb{P}_\beta$ for each $\beta < \alpha$. In addition, introduce the notation $p(\mu) := \dot{q}_\mu$.

2. For all β: $\dot{\mathbb{Q}}_\beta$ is a $\mathbb{P}_\beta$-name for a forcing notion (i.e., $1_{\mathbb{P}_\beta} \Vdash_{\mathbb{P}_\beta}$ "$\dot{\mathbb{Q}}_\beta$ is a forcing poset with ordering $\dot{\leq}_{\dot{\mathbb{Q}}_\beta}$ and weakest element $\dot{1}_{\dot{\mathbb{Q}}_\beta}$").

3. Complete embeddings: Say $\alpha_1 \leq \alpha_2 \leq \delta$. Then $\mathbb{P}_{\alpha_1}$ may be embedded in $\mathbb{P}_{\alpha_2}$ by filling up all spaces in $\alpha_2 \setminus \alpha_1$ with the respective 1-elements. This means that $\mathbb{P}_{\alpha_1} \lessdot \mathbb{P}_{\alpha_2}$, as in Definition 8.4.1. In what follows, we denote every such mapping by i and note that i fulfills all of the properties stated in the proof of Lemma 3.2.5.

4. For each $\alpha \leq \delta$, $1_{\mathbb{P}_\alpha}$ is the sequence $\langle 1_{\dot{\mathbb{Q}}_\gamma} : \gamma < \alpha\rangle$.

5. For any $p, q \in \mathbb{P}_\alpha$, $p \leq_{\mathbb{P}_\alpha} q$ iff for all $\beta < \alpha$: $p\restriction\beta \Vdash_{\mathbb{P}_\beta}$ "$p(\beta) \leq_{\dot{\mathbb{Q}}_\beta} q(\beta)$".

6. Successor cases: For any successor $\alpha + 1 < \delta$, $\mathbb{P}_{\alpha+1}$ consists of all sequences $p \in \mathbb{P}_\alpha$ end-extended by $\dot{q}_\alpha \in \dot{\mathbb{Q}}_\alpha$, where $p \Vdash_{\mathbb{P}_\alpha}$ "$\dot{q}_\alpha \in \dot{\mathbb{Q}}_\alpha$".

7. Supports: Let $\alpha \leq \delta$ be a limit ordinal and p be *any* sequence of length α. Call $supt(p) := \{\beta < \alpha : p(\beta) \neq 1_{\dot{\mathbb{Q}}_\beta}\}$ the support of p.

8. Limit cases/Finite supports: For any α and p as in (7), $p \in \mathbb{P}_\alpha$ iff $p\restriction\beta \in \mathbb{P}_\beta$ for all $\beta < \alpha$ and $supt(p)$ is finite.

So we see that an iteration of infinite length differs from the previous ones in that now, in addition to the successor steps, we also have to specify what happens at limit steps. We will often make use of the following notational conveniences:

Remark 3.3.3 (Implicit identifications). Let $\mathbb{P}_\delta$ be a finite support iteration of length δ.

- The statement '$\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$' shall abbreviate the identification $\mathbb{P}_{\alpha+1} \sim \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$, given by $(p, \dot{q}) \mapsto p^\frown \dot{q}$, the end-extension of the α-sequence p by the element $\dot{q}$.

- Say $\alpha_1 \leq \alpha_2 \leq \delta$. Then by writing $\mathbb{P}_{\alpha_1} \subseteq \mathbb{P}_{\alpha_2}$ (set inclusion, with the former being a complete suborder of the latter) we are implicitly identifying $\mathbb{P}_{\alpha_1}$ with $i(\mathbb{P}_{\alpha_1})$, where $i : \mathbb{P}_{\alpha_1} \to \mathbb{P}_{\alpha_2}$ is the complete embedding from point (3) above.

- As a special case: If $\lambda \leq \delta$ is a limit ordinal, then we write $\mathbb{P}_\lambda = \bigcup_{\alpha<\lambda} \mathbb{P}_\alpha$ with the above identifications in place for every $\mathbb{P}_\alpha$ and $\mathbb{P}_\lambda$ itself. In particular, as a union of a chain of posets, $\mathbb{P}_\lambda$ is itself a poset.

3.3.2 Some important properties of finite support iterations

Before returning to point (2) of Problem 3.1.1, let us first note that the following crucial properties, which were required to construct a ccc two-step iteration for 3.1.1(1), also hold for finite support iterations:

I. "Going full circle": Forcing with $\mathbb{P}_\delta$ in V is equivalent to successively forcing with all of the forcing notions $\mathbb{Q}_\alpha$, $\alpha < \delta$.

II. Preservation of the ccc.

Finally, there is one more property we have to discuss at this point:

III. In general, it is not the case that $V^{\mathbb{P}_\delta} = \bigcup_{\alpha<\delta} V^{\mathbb{P}_\alpha}$.

For (I), let G be a $\mathbb{P}_\delta$-generic filter over V. Then for arbitrary $\alpha \leq \delta$, $G_\alpha := i^{-1}(G)$, where $i : \mathbb{P}_\alpha \to \mathbb{P}_\delta$ is a complete embedding as in Definition 3.3.2 (3). Furthermore, let $H_\alpha := \{\dot{q}_{G_\alpha} : \dot{q} \in dom(\dot{\mathbb{Q}}_\alpha) \wedge \exists p\ (p^\frown \dot{q} \in G_{\alpha+1})\}$. Using these concrete generic filters, $V^{\mathbb{P}_\alpha} = V[G_\alpha]$. With this data, obtain:

Lemma 3.3.4.

(a) G_α *is a* $\mathbb{P}_\alpha$*-generic filter over* V*.*

(b) For any $\alpha_1 \leq \alpha_2 \leq \delta$, $V[G_{\alpha_1}] \subseteq V[G_{\alpha_2}]$*.*

(c) For $\alpha < \delta$, $H_\alpha \in V[G_{\alpha+1}]$*, and* H_α *forms a* $\mathbb{Q}_\alpha$*-generic filter over* $V[G_\alpha]$*.*

Proof. This is Lemma 5.13 (Chapter VIII) in [15]. □

Now we can formulate (I) as the following statement for $\alpha = \delta$:

Fact 3.3.5. For any $\alpha \leq \delta$, $V[G_\alpha] = V[\langle H_\beta : \beta < \alpha \rangle]$.

Proof. Since the successor case is just as in the two-step iteration context, assume α is a limit ordinal. Without loss take $\alpha = \delta$. Use Proposition 8.1.2 for both inclusions separately: For "$\supseteq$": $V \subseteq V[G_\delta]$ and $V[G_\delta] \models ZFC$ because $V[G_\delta]$ is a generic extension of a countable transitive model V, and $\langle H_\beta : \beta < \delta \rangle \in V[G_\delta]$ by the previous lemma. Similarly for "$\subseteq$", in $V[\langle H_\beta : \beta < \alpha \rangle]$, construct G_δ using the H_β and i from above. □

Continue with (II): As seen in the definition, finite support iterations make sure that the conditions are non-trivial only in finitely many instances of the iteration. Why are we interested in iterations with finite supports as opposed to, say, countable supports? The main reason is that finite supports allow for suitable preservation of cardinals (and cofinalities), in the sense that the ccc is preserved.

Lemma 3.3.6. *Let $\mathbb{P}_\delta = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \delta, \beta < \delta \rangle$ be a finite support iteration of length δ. Then $\mathbb{P}_\delta$ is ccc if for all $\alpha < \delta$, $1_{\dot{\mathbb{Q}}_\alpha} \Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ ccc".*

Proof. This is Lemma V.3.17 in [16]. □

Finally, for (III): We will see in Section 3.4 that finite support iterations add Cohen reals in limit stages of countable cofinality.

3.3.3 Back to the example

Returning to point (2) of Problem 3.1.1, we wish to show:

Claim 3.3.7. *Assume CH. Then consistently, $\aleph_1 \in Spec(\mathfrak{a})$.*

Proof. Recursively define a finite support iteration $\mathbb{P} := \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \aleph_1, \beta < \aleph_1 \rangle$ of length $\aleph_1$ as follows.

- $\alpha = 0$: Set $\mathbb{P}_0 = \{\emptyset\}$, and take $\mathcal{A}_0$ to be the empty almost disjoint family $\emptyset$. Let $\dot{\mathbb{Q}}_0$ be a $\mathbb{P}_0$-name for the a.d. coding poset $\mathbb{P}_{\mathcal{A}_0}$ used to force the existence of a real which diagonalizes $\mathcal{A}_0$ (Definition 2.1.7 and Proposition 2.1.11).

- $\alpha = 1$: Working in $V^{\mathbb{P}_1}$, set $\mathcal{A}_1 = \mathcal{A}_0 \cup \{d_0\}$, where d_0 is the real forced in the previous step. Then let $\dot{\mathbb{Q}}_1$ be a $\mathbb{P}_1$-name for the almost disjoint coding forcing notion $\mathbb{P}_{\mathcal{A}_1}$. In $V^{\mathbb{P}_2}$ there exists a $\mathbb{P}_{\mathcal{A}_1}$-generic real d_1 diagonalizing $\mathcal{A}_1$. Thus in $V^{\mathbb{P}_2}$ define $\mathcal{A}_1 = \mathcal{A}_0 \cup \{d_1\}$, which trivially forms an almost disjoint family.

- Successor case: Assume that for $\alpha \geq 2$, $\mathcal{A}_{\alpha-1}$ is an almost disjoint family, and d_α is a given $\mathbb{P}_{\mathcal{A}_\alpha}$-generic real. In $V^{\mathbb{P}_{\alpha+1}}$, define $\mathcal{A}_\alpha = \mathcal{A}_{\alpha-1} \cup \{d_\alpha\}$. Then set $\dot{\mathbb{Q}}_{\alpha+1}$ to be a $\mathbb{P}_{\alpha+1}$-name for $\mathbb{P}_{\mathcal{A}_\alpha}$; so in $V^{\mathbb{P}_{\alpha+2}}$ we obtain the generic real $d_{\alpha+1}$.

- Limit case: For $\lambda < \aleph_1$ a limit ordinal assume that the finite support iteration $\mathbb{P}_\lambda = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \lambda, \beta < \lambda \rangle$ is defined. Then in $V^{\mathbb{P}_\lambda}$, let $\dot{\mathbb{Q}}_\lambda := \bigcup_{\beta<\lambda} \dot{\mathbb{Q}}_\beta$, which is a $\mathbb{P}_\lambda$-name for $\mathbb{P}_{\mathcal{A}_\lambda}$, where $\mathcal{A}_\lambda := \bigcup_{\beta<\lambda} \mathcal{A}_\beta$, and proceed as in the successor case.

Finally, in $V^{\mathbb{P}}$ set $\mathcal{A} := \mathcal{A}_{\aleph_1} := \bigcup_{\beta<\aleph_1} \mathcal{A}_\beta$. Note that $\mathcal{A}$ is almost disjoint by construction. Furthermore, $\mathcal{A}$ is maximal: Assume towards a contradiction that this is not so. Then there exists some $x \in V^{\mathbb{P}} \cap [\omega]^\omega \setminus \mathcal{A}$ such that $|x \cap y| < \aleph_0$ for all $y \in \mathcal{A}$. Employing Lemma 3.4.2 find $\alpha < \aleph_1$ with $x \in [\omega]^\omega \cap V^{\mathbb{P}_\alpha}$. Let $d_{\alpha+1} \in V^{\mathbb{P}_{\alpha+1}}$ be the $\mathbb{P}_{\mathcal{A}_\alpha}$-generic real. Thus in particular, $\mathcal{A}_\alpha \cup \{x, d_{\alpha+1}\}$ is not almost disjoint, implying that there must be some $y \in \mathcal{A}$ with $y \cap x$ infinite, a contradiction. $\square$

Remark 3.3.8.

a. In the last part of the proof we see the main application of Lemma 3.4.2 from the upcoming section to obtain maximality. This usage is exemplary for many finite support constructions to follow.

b. One way to think about the last part is to note that one cannot find a witness for non-maximality, as for every potential witness there is a diagonalizing real in the successive model constructed during the iteration. See e.g. Remark 4.1.2 for a discussion of how this fails in the independence case when $cf(\kappa) = \omega$.

c. It has not been stated explicitly, but *supt*(p) is as in 3.3.2(8) as per assumption, not because of properties of any of the occurring forcing posets. On the other hand, every $\mathbb{Q}_\beta$ has the ccc (Lemma 2.1.8).

d. For a connection between the a.d. coding poset and $\mathbb{H}$ (Def. 4.2.3) see Cha. 9.

3.4 About countable cofinality iteration lengths

Next, we wish to discuss certain aspects inherent to the fsi method. Let κ be a cardinal and $\mathbb{P}_\kappa$ a forcing iteration of length κ.

Fact 3.4.1. In any finite support iteration, any limit stage of countable cofinality adds Cohen reals.

Proof. To see this, we can without loss take $\kappa = \omega$, so the finite support iteration is given by $\mathbb{P} := \langle \mathbb{P}_n, \dot{\mathbb{Q}}_m : n \leq \omega, m < \omega \rangle$. Such choice is possible by $\mathbb{P}$ being an fsi. Fix an arbitrary $n \in \omega$. As per our usual assumptions on forcing posets, $1_{\mathbb{P}_n} \Vdash_{\mathbb{P}_n}$ "$\exists \dot{q}_n^0 \neq \dot{q}_n^1 \in \dot{\mathbb{Q}}_n(\dot{q}_n^0 \perp_{\dot{\mathbb{Q}}_n} \dot{q}_n^1)$", which means that $\dot{\mathbb{Q}}_n$ is non-trivial. Let G_n be a $\dot{\mathbb{Q}}_n$-generic filter over $V^{\mathbb{P}_n}$. Then in $V^{\mathbb{P}}$, define a function

$$f : \omega \to 2, \ f(n) = 1 \text{ iff } q_n \in G_n.$$

To see that f is a Cohen real, it is enough to show that there exists a $(V^{\mathbb{P}}, \mathbb{C})$-generic filter H such that $f = \bigcup H$. It is possible to reconstruct H using f. Indeed, take $H = [f]^{<\omega}$. Certainly, H forms a filter on $\mathbb{C}$. Employing the characterization of genericity from Lemma 8.2.5, take any maximal antichain $A \subseteq \mathbb{C}$ and assume towards a contradiction that $H \cap A = \emptyset$. Thereby, there is $n \in \omega$ with $\{(n, f(n))\} \in H$ but $(n, f(n)) \neq (n, p(n))$ for all $p \in A$, and so $A \cup \{(n, f(n))\}$ forms an antichain in $\mathbb{C}$. $\square$

On the other hand, consider $V^{\mathbb{P}_\kappa}$ with $cf(\kappa) > \omega$. Then any Cohen reals added in countable cofinality limit steps do "not affect" the final model obtained through forcing with $\mathbb{P}_\kappa$ or put differently, essentially no new Cohen reals are added in $V^{\mathbb{P}_\kappa}$. To make this more exact, consider the following lemma which will find its use in many finite support iterations occurring in this thesis.

Lemma 3.4.2. *For κ a cardinal with $cf(\kappa) > \omega$, let $\mathbb{P}_\kappa$ be a finite support iteration of ccc forcing notions. Then in the final model $V^{\mathbb{P}_\kappa}$ it holds that: For every function $f \in \omega^\omega \cap V^{\mathbb{P}_\kappa}$ there exists a limit ordinal $\lambda < \kappa$ such that $f \in \omega^\omega \cap V^{\mathbb{P}_\lambda}$.*

Proof. Let $f \in \omega^\omega \cap V^{\mathbb{P}_\kappa}$. Any given $\mathbb{P}_\kappa$-name $\dot{f}$ can be viewed without loss as a nice name (Appendix - Section 8.2.1). Then by Lemma 8.2.15, there exist sets $\{p_{n,i} \in \mathbb{P}_\kappa : n, i \in \omega\}$, $\{k_{n,i} \in \omega : n, i \in \omega\}$, such that

$$A_n := \{p_{n,i} \in \mathbb{P}_\kappa : p_{n,i} \Vdash \text{"}\dot{f}(n) = \check{k}_m\text{"}, i \in \omega\}$$

forms a maximal antichain for all $n \in \omega$. Therefore, $\dot{f}$ is completely determined by the countable sets $\{A_n : n \in \omega\}$ respectively $\{p_{n,i} \in A_n : n, i \in \omega\}$.

As $cf(\kappa)$ is uncountable, there exists a countable set of cardinals $\{\kappa_{n,i} : n, i < \omega\}$ so that each $\kappa_{n,i} < \kappa$ is countable, and additionally $\lambda := \sup\{\kappa_{n,i} : n, i < \omega\}$ is countable. We wish to use λ as an upper bound for all of the information about $\dot{f}$ as given by $\{p_{n,i} : n, i \in \omega\}$. To do so, use that $\mathbb{P}_\kappa$ forms a finite support iteration: Since $supt(p_{n,i})$ is finite for all $n, m \in \omega$, without loss we can assume that $supt(p_{n,i}) \subseteq \kappa_{n,i} < \lambda$ for all n, i. Hence $\{p_{n,i} : n, i \in \omega\}$ is contained in $\mathbb{P}_\lambda$. But by the above this means that $\dot{f}$ is "effectively" a $\mathbb{P}_\lambda$-name, and therefore $f = \dot{f}_G$ is in $V^{\mathbb{P}_\lambda} \cap \omega^\omega$. $\square$

Remark 3.4.3. Notice the usage of identifications as mentioned in Remark 3.3.3; if we do not suppress the underlying complete embeddings, the above may be stated in more exact terms. As an example of this, see e.g. Lemma 5.14 in [15], p.276.

4 Positive results aka. adding cardinals to the spectra

Let $\mathcal{A}$ be a maximal independent or a mad family. Now that the main preparations are accomplished, we turn to the possible sizes of $|\mathcal{A}|$, that is, the spectra as defined in 0.1. Let us for now write $Spec(\mathcal{A})$ to address both spectra at once. At this point we have verified that $Spec(\mathcal{A}) \subseteq [\aleph_1, \mathfrak{c}]$ and $\mathfrak{c} \in Spec(\mathcal{A})$. From Section 3.3.3 we also know that $Con(\aleph_1 \in Spec(\mathfrak{a}))$. Yet there are many more questions to be investigated in connection to the "flexibility" of $Spec(\mathcal{A})$. To state some examples: Under which assumptions is $Spec(\mathcal{A})$ small? Certainly, if we assume $ZFC + CH \vee MA$, it follows that $Spec(\mathcal{A}) = \{\mathfrak{c}\}$. How small can we force it to be to be without such assumptions? (In the case of $\mathfrak{i}$ see e.g. [7], part (3)). The investigation of such inquiries proves to be a rich source for the development of various forcing methods. As mentioned in the introduction, the main focus of this work lies in the addition (positive results) and exclusion (negative results) of cardinals concerning the spectra. For certain sets of uncountable cardinals C, the combination of both will result in $Spec(\mathcal{A}) = C$. In the following we are interested in the positive results, with the aim to prove that consistently, any such C can be included in $Spec(\mathcal{A})$.

4.1 Spectrum of independence

Sections 4.1.1 and 4.1.2, the source of which is [7] from Fischer and Shelah, address the finite regular case. More general sets C of arbitrary (uncountable) cardinals will be approached in Section 4.1.4, based on [8], the continuation of the aforementioned work. Before getting to this however, as a preparation, finite support products and the Knaster property will be introduced in Section 4.1.3.

4.1.1 How to add a single regular cardinal

The proof of the upcoming lemma is very similar to the approach taken in 3.3.3.

Lemma 4.1.1. *For any cardinal κ with $cf(\kappa) > \omega$, it is consistent that $\kappa \in Spec(\mathfrak{i})$.*

Proof. We prove the statement by constructing a maximal independent family $\mathcal{A}$ using a finite support iteration $\mathbb{P}_\kappa = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$, where κ is the wanted size of the family. Define it recursively as follows:

- $\alpha = 0$: Take $\mathcal{A}_0$ to be a given independent family with $\mathcal{U}_0$ the $\mathcal{A}_0$-diagonalization filter as in Definition 2.2.2 (e.g. $\mathcal{A}_0 = \emptyset$ and $\mathcal{U}_0$ any ultrafilter extending the Fréchet filter). Set $\mathbb{P}_0 = \{\emptyset\}$, and let $\dot{\mathbb{Q}}_0$ be a $\mathbb{P}_0$-name for the Mathias forcing notion $\mathbb{M}(\mathcal{U}_0)$ relativized to $\mathcal{U}_0$.

- $\alpha = 1$: Then in $V^{\mathbb{P}_1}$, d_0 is an $\mathbb{M}(\mathcal{U}_0)$-generic Mathias real diagonalizing $\mathcal{A}_0$ (Definition 2.0.1 + Lemma 2.2.4). Form the independent family $\mathcal{A}_1 := \mathcal{A}_0 \cup \{d_0\}$ and set $\dot{\mathbb{Q}}_1$ to be a $\mathbb{P}_1$-name for $\mathbb{M}(\mathcal{U}_1)$, where $\mathcal{U}_1$ is an $\mathcal{A}_1$-diagonalization filter.

- Successor case: Assume that for $\alpha \geq 2$, $\mathcal{A}_\alpha$ is independent, and d_α is a given $\mathbb{M}(\mathcal{U}_\alpha)$-generic real. In $V^{\mathbb{P}_{\alpha+1}}$, define $\mathcal{A}_{\alpha+1} = \mathcal{A}_\alpha \cup \{d_\alpha\}$ and take $\mathcal{U}_{\alpha+1}$ to be a $\mathbb{P}_{\alpha+1}$-name for a corresponding diagonalization filter. Then set $\dot{\mathbb{Q}}_{\alpha+1}$ to be a $\mathbb{P}_{\alpha+1}$-name for $\mathbb{M}(\mathcal{U}_{\alpha+1})$.

- Limit case: For $\lambda < \aleph_1$ a limit ordinal assume that the finite support iteration $\mathbb{P}_\lambda = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \lambda, \beta < \lambda \rangle$ is defined. Then in $V^{\mathbb{P}_\lambda}$, let $\dot{\mathbb{Q}}_\lambda := \bigcup_{\beta<\lambda} \dot{\mathbb{Q}}_\beta$, which is a $\mathbb{P}_\lambda$-name for $\mathbb{M}(\mathcal{U}_\lambda)$, where $\mathcal{U}_\lambda$ is defined as the $\mathcal{A}_\lambda$-diagonalization filter, with $\mathcal{A}_\lambda := \bigcup_{\beta<\lambda} \mathcal{A}_\beta$.

In the final model $V^{\mathbb{P}_\kappa}$, set $\mathcal{A} := \bigcup_{\beta<\kappa} \mathcal{A}_\beta$. Since we add a new element κ-many times, $|\mathcal{A}_\kappa| = \kappa$. Independence follows from the first diagonalizing property (Lemma 2.2.4 (1)), as each newly added generic real retains independence of $\mathcal{A}_0$ respectively the newly obtained larger family. We proceed by showing:

Claim. *$\mathcal{A}$ is maximal in $V^{\mathbb{P}_\kappa}$.*

Proof. Assume otherwise, i.e., that in $V^{\mathbb{P}_\kappa}$ there is $x \in [\omega]^\omega \setminus \mathcal{A}$ such that $\mathcal{A} \cup \{x\}$ forms an independent family. By κ being uncountable and $\mathbb{P}^\kappa$ a ccc finite support iteration, find $\alpha < \kappa$ with $x \in [\omega]^\omega \cap V^{\mathbb{P}_\alpha}$ (Lemma 3.4.2). Now $\mathcal{A}_\alpha \cup \{x\}$ must be independent. Let $d_{\alpha+1}$ be the diagonalization real obtained in the next iteration step. Thereby, $V^{\mathbb{P}_{\alpha+1}} \models$ "$\mathcal{A}_{\alpha+1} \cup \{x, d_{\alpha+1}\}$ is not independent", and so $\mathcal{A} \cup \{x\}$ is not independent. $\square_{Claim}$

$\square$

Remark 4.1.2 (Assumption of uncountable cofinality). By Fact 3.4.1, the assumption that $cf(\kappa) > \omega$ is indeed necessary. Say we have a countable cofinality limit ordinal λ. If we take λ to be the length of our fsi, then in the final generic extension a Cohen real h is added. Using density arguments one can show that any given Cohen real splits all $x \in [\omega]^\omega$ in the preceding model of the iteration (Appendix - 8.3.3). Now assume that in some iteration step $\alpha < \lambda$ we have constructed an independent family $\mathcal{A}_\alpha$. Then h splits all elements of $\mathcal{A}_\alpha$. Hence $h \notin \mathcal{A}_\alpha$. Furthermore, $\mathcal{A}_\alpha \cup \{h\}$ is an independent family: If we add $h^k \in \{h, \omega \setminus h\}$ (with $k \in \{0,1\}$ and the convention for Boolean combinations from Section 1.2) to some Boolean combination $Y \in BC(\mathcal{A}_\alpha)$, since h splits every element in Y, the resulting Boolean combination will still be an infinite subset of ω. But this means that $\mathcal{A}_\alpha$ is not maximal for any $\alpha < \lambda$. As in this case we do not add any Mathias reals to witness maximality at a later stage of the iteration, the method fails to show whether $\lambda \in Spec(\mathfrak{i})$ or not. Therefore, by necessitating $cf(\kappa) > \omega$ we omit this problem.

4.1.2 Finitely many regular uncountable cardinals

It is not possible to add to $Spec(\mathfrak{i})$ uncountable regular cardinals $\kappa_1 < \kappa_2 < \cdots < \kappa_n$ for any $n \geq 2$ by naively repeating the proof of Lemma 4.1.1, that is, by extending the length of the finite support iteration to κ_n and doing everything else as before:

Discussion 4.1.3 (Necessity to change previous approach). *Let $n = 2$ and assume that we have constructed a maximal independent family A_{κ_1} of cardinality κ_1 as in the proof of 4.1.1. Since we have to continue the iteration to obtain a maximal independent family A_{κ_2} of cardinality $\kappa_2 > \kappa_1$, each consecutive iteration step will add a Mathias real witnessing that A_{κ_1} is not maximal. In other words, we cannot repeat the previous proof because the destruction of maximality in earlier models of the iteration is necessary for the procedure to work in the first place.*

To resolve this issue while still making use of the approach taken in Lemma 4.1.1, we will force the existence of the wanted n maximal independent families "at once" with the help of ordinal products and appropriate index sets for the given families. This idea is captured already in the case where $n = 2$.

Lemma 4.1.4. *Let $\kappa_1 < \kappa_2$ be uncountable regular cardinals. Assume that GCH holds in the ground model. Then consistently there are maximal independent families A_{κ_1} and A_{κ_2} of cardinality κ_1 and κ_2, respectively.*

Proof. In order to simultaneously force the existence of A_{κ_1} and A_{κ_2} using a finite support iteration as before, we require two things: (1) a suitable length for the iteration to provide enough iteration steps to construct both families, and (2), two disjoint sets of indices to address the members of A_{κ_1} and A_{κ_2}. We solve both requirements at once by considering the ordinal product $\gamma^* := \kappa_2 \cdot \kappa_1 = type(\kappa_1 \times \kappa_2, <_{lex})$ (ordered lexicographically). Thereby (1) is taken care of because $|\gamma^*| = \max\{\kappa_1, \kappa_2\} = \kappa_2$, and so γ^* will work as iteration length. Meanwhile (2) is obtained by taking disjoint subsets $I_1, I_2 \subseteq \gamma^*$ so that $|I_1| = \kappa_1$, $|I_2| = \kappa_2$, and both are unbounded in γ^* (for an example see the upcoming remark).

Next, recursively define a finite support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \gamma^*, \beta < \gamma^* \rangle$ using the following case distinction:

- If $\alpha \in I_1$ or $\alpha \in I_2$, let $\mathbb{P}_\alpha$ and $\dot{\mathbb{Q}}_\alpha$ be as in the proof of Lemma 4.1.1. Without loss, fix $\alpha \in I_1$. Then we obtain an independent family $\mathcal{A}^\alpha_{\kappa_1} := \{a_\delta : \delta \in I_1 \cap \alpha\}$, where each a_δ is the Mathias real obtained in the respective iteration step as before. Thus, for any $\delta \in I_1$ we have that a_δ diagonalizes $\mathcal{A}^\delta_{\kappa_1}$ over the model $V^{\mathbb{P}_\delta}$, resulting in an independent family $\mathcal{A}_{\kappa_1}$ in the final model. Proceed analogously for $\mathcal{A}_{\kappa_2}$. By unboundedness of I_1 and I_2 in γ^* this means that there are no witnesses to destroy the maximality of the obtained maximal independent families $\mathcal{A}_{\kappa_1}$ and $\mathcal{A}_{\kappa_2}$ in the final model, as there cannot be a witness in both separate limit cases.
- We still need to specify what happens if $\alpha \notin I_1 \cup I_2$, or else the finite support iteration is not well-defined. In this case, let $\dot{\mathbb{Q}}_\alpha$ name a forcing poset to fill up the vacancies so that there will be no interference with the rest of the forcing construction. Along these lines, let the place holders be given by Cohen forcing.

$\square$

Remark 4.1.5 (Disjoint unbounded subsets as in the proof).

1. As mentioned in the proof above, one can find disjoint subsets, each of which is unbounded in $\gamma^* = \kappa_2 \cdot \kappa_1$. Consider for example $I_1 := \{(\delta, 0) : \delta \in \kappa_1\}, I_2 := \gamma^* \setminus I_1 \subseteq \kappa_1 \times \kappa_2$. Then $I_1 \cap I_2 = \emptyset$, $|I_1| = \kappa_1$, and I_1 (is order-isomorphic to a set, which) is unbounded in γ^*: For any fixed $\alpha \in \gamma^*$, there exists a unique associated $(\alpha_1, \alpha_2) \in \kappa_1 \times \kappa_2$. Setting $(\beta_1, \beta_2) := (\alpha_1 + 1, 0)$, $(\alpha_1, \alpha_2) <_{lex} (\beta_1, \beta_2)$. As $(\beta_1, \beta_2) \in I_1$, we are done. Since $|I_2| = \kappa_2$, I_2 is also unbounded in γ^*.
2. Generalization to $n > 2$ strictly ordered cardinals $\kappa_1 < \kappa_2 < \cdots < \kappa_n$: Consider for example $I_1 := \{(\delta_1, \delta_2, \ldots, \delta_n) : \delta_1 < \kappa_1 \wedge \forall k \in n+1 \setminus \{0,1\} : \delta_k = 0\}$. Then define recursively, given $j - 1 \in \{1, 2, 3, \ldots, n-1\}$:

$$I_j = \{(\delta_1, \delta_2, \ldots, \delta_n) : \delta_1 < \kappa_1 \wedge \forall k \in n+1 \setminus \{0,1,j\} : \delta_k = 0 \wedge \delta_j < \kappa_j\} \setminus \bigcup_{i<j} I_i.$$

By construction it follows that all of the I_j are pairwise disjoint and unbounded in γ^*, and $|I_j| = \kappa_j$.

Theorem 4.1.6. *Assume GCH. Let $\kappa_1 < \kappa_2 < \cdots < \kappa_n$, $n \in \omega$, be uncountable regular cardinals. Then consistently $\{\kappa_i : 1 \leq i \leq n\} \subseteq Spec(\mathfrak{i})$.*

Proof. Repeat the previous proof with $\gamma^* = \kappa_n \cdot \kappa_{n-1} \cdot \ldots \cdot \kappa_1$. □

Remark 4.1.7 (Further generalization)*.* As stated in [7], it is possible to generalize this construction using an arbitrary cardinal ϑ instead of n to obtain $Con(\{\vartheta_\alpha : \alpha < \vartheta\} \subseteq Spec(\mathfrak{i}))$. Here one forces that $\mathfrak{c} = \lambda$ for any given cardinal λ of uncountable cofinality. The finite support iteration in this case uses a standard bookkeeping argument. In the context of this work, the upcoming results will accomplish the same as this generalization while also allowing cardinals of singular cofinality.

4.1.3 Finite support products and the Knaster property

We begin with preliminaries, the source of which is T. Jech's work [13].

Definition 4.1.8 (Finite support products, Knaster)**.**

1. For a given collection of posets $\{(\mathbb{P}_i, \leq_{\mathbb{P}_i}, 1_{\mathbb{P}_i}) : i \in I\}$ we can define the finite support product as $\mathbb{P} := \prod_{i \in I} \mathbb{P}_i$, where each $p \in \mathbb{P}$ is a function on I with $p(i) \in \mathbb{P}_i$, and any such p takes the value $1_{\mathbb{P}_i}$ cofinitely often. $(\mathbb{P}, \leq)$ is a poset where for any $p, q \in \mathbb{P}$, $p \leq q$ iff $p(i) \leq_{\mathbb{P}_i} q(i)$ for all $i \in I$. Set $supt(p) := \{i \in I : p(i) \neq 1_{\mathbb{P}_i}\}$ to be the support of $p \in \mathbb{P}$. Finally, note that any generic filter G on $\mathbb{P}$ induces $(V, \mathbb{P}_i)$-generic filters $G_i := \{p(i) : p \in G\}$.

2. Let $\mathbb{P}$ be a forcing notion. Then $\mathbb{P}$ has the Knaster property (or is Knaster) if for every uncountable set of conditions in $\mathbb{P}$ one can find an uncountable subset consisting only of pairwise compatible elements.

Remark 4.1.9 (G_i is a generic filter)*.* Each G_i is indeed a $\mathbb{P}_i$-generic filter (over some ground model V): Fix $i \in I$ and begin by proving that G_i is a filter on $\mathbb{P}_i$. Given any $p(i), q(i) \in G_i$, the existence of a common extension follows directly from G being a filter. Furthermore, given any $p(i) \in G_i$ and $q_i \in \mathbb{P}_i$ with $p(i) \leq q_i$, one can define $q \in \mathbb{P}$ via $q(j) = q_i$ if $i = j$, while $q(j) = 1_{\mathbb{P}_j}$ otherwise, and again use that G is a filter to find that G_i is upwards-closed. Similarly, to show genericity of G_i, for any $i \in I$ take sets $D_i \subseteq \mathbb{P}_i$ which are dense in $\mathbb{P}_i$, and define the dense set $D = \prod_{i \in I} D_i$ (this is again a finite support product; it is dense since for any $p \in \mathbb{P}$, each of the finitely many conditions $p(i) \neq 1_{\mathbb{P}_i}$

will be extended by $q_i \in D_i$, and otherwise one can define $q(j) = 1_{\mathbb{P}_j}$ as before). Using that G is $\mathbb{P}$-generic, $G \cap D \neq \emptyset$, which implies that $D_i \neq \emptyset$. With this definition of G_i we can write $G = \prod_{i \in I} G_i$.

Example 4.1.10 (Cohen forcing, Mathias forcing)**.** Let $\mathbb{C}_I$ denote Cohen forcing adding $|I|$-many side by side Cohen reals (Appendix - 8.3.2). Then $\mathbb{C}_I = \prod_{i \in I} \mathbb{C}_i$, where each $\mathbb{C}_i := Fn(\omega, 2)$ adds a single Cohen real.

- $\mathbb{C}_I$ is Knaster: Indeed, for any uncountable $A \subseteq \mathbb{C}_I$ one can find a subfamily consisting of pairwise compatible elements using the pigeon-hole principle and the Δ-system Lemma. We omit the proof here, as it is very similar to the Hechler mad case (Lemma 4.2.5) below.

As another example, consider Mathias forcing $\mathbb{M}$ as introduced in Sections 2.1.1 and 2.2:

- $\mathbb{M}(\mathcal{F})$ is Knaster: By Lemma 2.1.6, Mathias forcing relativized to a filter is σ-centered, which implies the Knaster property. To see this, take any uncountable subset $A \subseteq \mathbb{M}(\mathcal{F})$. Then all conditions $p \in A$ are distributed over at most countably many pairwise disjoint centered subsets $M_i \subseteq \mathbb{M}(\mathcal{F})$, which implies that all except possibly countably many of the conditions in A must be pairwise compatible.

Remark 4.1.11 (Knaster implies ccc). From the assumption that $\mathbb{P}$ is not ccc it follows that there exists an uncountable antichain $A \subseteq \mathbb{P}$. In this case, there is no uncountable subset $B \subseteq A$ of pairwise compatible conditions meaning that $\mathbb{P}$ is not Knaster.

The reason we introduce the Knaster property is that finite support products (or finite products, for that matter) of ccc forcing notions are in general not ccc, whereas products of Knaster forcing notions preserve the ccc:

Lemma 4.1.12. *If* $\mathbb{P}_1, \mathbb{P}_2$ *are Knaster, then* $\mathbb{P}_1 \times \mathbb{P}_2$ *is Knaster.*

Proof. Without loss of generality assume that $\mathbb{P}_1$ and $\mathbb{P}_2$ are uncountable. Take any uncountable subset $A \subseteq \mathbb{P}_1 \times \mathbb{P}_2$. Then at least one of the following holds:

1. The set of conditions $A_1 := \{p_1 \in \mathbb{P}_1 : (p_1, p_2) \in A\}$ is uncountable for some fixed $p_2 \in \mathbb{P}_2$.

2. The set $A_2 := \{p_2 \in \mathbb{P}_2 : (p_1, p_2) \in A\}$ is uncountable for some fixed $p_1 \in \mathbb{P}_1$.

3. None of the above is true.

In the first case use that $\mathbb{P}_1$ is Knaster, and so there exists $B \subseteq A_1$ uncountable with pairwise compatible conditions. This results in a subset of A of uncountable cardinality with pairwise compatible conditions. The second case is similar. In the third case, since any sets of the form A_1, A_2 are countable, there must be an uncountable set $C \subseteq A$ such that $\forall (c_1, c_2), (d_1, d_2) \in C : c_1 \neq d_1$ and $c_2 \neq d_2$. As C is uncountable, we can form the uncountable set

$$C_1 := \{c_1 \in \mathbb{P}_1 : \exists d_1 \in \mathbb{P}_2((c_1, d_1) \in C)\}.$$

Since $\mathbb{P}_1$ is Knaster, we can without loss assume that all conditions in C_1 are pairwise compatible. Similarly define an uncountable set $C_2 := \{d_2 \in \mathbb{P}_2 : \exists c_2 \in C_1((c_2, d_2) \in C)\}$ of pairwise compatible conditions. This construction guarantees that $C_1 \times C_2 \subseteq C$ consists of pairwise compatible conditions, which concludes the proof. □

Remark 4.1.13. Using natural induction, a direct consequence of the lemma above is that the Knaster property is preserved for any finite product of Knaster forcing posets.

Lemma 4.1.14. *The finite support product $\mathbb{P}$ of Knaster posets is again Knaster.*

Proof. Let $X \subseteq \mathbb{P} := \prod_{i \in I} \mathbb{P}_i$ be uncountable and define the set of supports for elements in X by $Y = \{supt(p) : p \in X\}$. Per definition, each $supt(p)$ is finite. If Y is countable, using the pigeonhole-principle one can find a finite subset $J \subseteq I$ such that $supt(p) = J$ for uncountably many $p \in X$, implying that $\prod_{i \in J} \mathbb{P}_i$ consists of uncountably many conditions $p \restriction J$, where $p \in X$. Since the latter product is finite, it is Knaster by the previous remark. Therefore, we can find an uncountable subset of pairwise compatible conditions of X by defining an injection $\prod_{i \in J} \mathbb{P}_i \to X \cap \prod_{i \in I} \mathbb{P}_i$, which is the identity on J and returns $1_{\mathbb{P}_i}$ for every $i \in I \setminus J$. If on the other hand Y is uncountable, we can use the Δ-system Lemma (Appendix, 8.2.3) to obtain an uncountable subset $Z \subseteq Y$ and a finite root $J \subseteq I$ such that $supt(p) \cap supt(q) = J$ for all $p \neq q \in Y$. Using J, repeat the above to find an uncountable subset of X of pairwise compatible conditions. □

4.1.4 Adding arbitrary sets of any uncountable cardinals

As we will see in Section 4.2.2, by defining the "right" kind of forcing notion, it is possible to force the existence of a mad family in one step instead of using iterated forcing. An analogous poset for the independence case is currently not known. In Chapter 9 we shortly address one of the difficulties in trying to define such a poset. Nevertheless, in this section we will use a different idea to prove that given any set C of uncountable cardinals, consistently $C \subseteq Spec(\mathfrak{i})$.

Remark 4.1.15 (Adding arbitrary sets C?). Let us first specify what "arbitrary" refers to. Certainly, from the *ZFC* results we know that $Spec(\mathfrak{i})$ and $Spec(\mathfrak{a})$ are both bounded above by $\mathfrak{c}$. Therefore, if we prove that $Con(C \subseteq Spec(\mathfrak{i}))$, this implicitly means that we add $\lambda := \sup C$ to $Spec(\mathfrak{i})$, where consistently $\lambda \leq \mathfrak{c}$. The same holds for $\mathfrak{a}$. This point will be discussed further in the course of preparing the negative results (Section 4.3.3).

For the proof, besides the preliminaries from Section 4.1.3, we will make use of the fact that $\mathbb{C}_I = Fn(I \times \omega, \supseteq)$ adds an independent family of size $|I|$ (Appendix, Lemma 8.3.2). As seen in Remark 8.3.3, the added family is not maximal.

Discussion 4.1.16. *We are now ready to prove $Con(C \subseteq Spec(\mathfrak{i}))$, where C consists of arbitrarily many uncountable cardinals. Again, our approach will be to first consider finite C, and repeat the same idea with $|C| \geq \aleph_0$ afterwards. Let us see how the previous results fit into the following proof:*

1. *Even if $|C| = 1$, $\mathbb{C}_I$ by itself will add no maximal independent family.*

2. *Mathias forcing by itself, using the previous approach, does not work either: As seen before, this fails when we try to construct maximal independent families of cardinality with countable cofinality.*

3. *New approach: The following proof employs a combination of both, Cohen and Mathias forcing. Say we wish to add $C = \{\kappa_1, \kappa_2, \dots, \kappa_n\}$ to $Spec(\mathfrak{i})$.*

    - *To take care of independence: First force the existence of n "separated" independent (not necessarily maximal) families $\mathcal{A}_1^1, \mathcal{A}_2^1, \dots, \mathcal{A}_n^1$ at once by using Cohen forcing. These will already have the right sizes ($|\mathcal{A}_j^1| = \kappa_j$ for every j). The separation will be achieved using finite support products and certain disjoint index sets.*

    - *For maximality: We will proceed as before, that is, by adding "enough" Mathias reals (so we can in particular use their diagonalizing properties - see Definition 2.0.1 and Lemma 2.2.4) to each of the n parts of the product, which we can think of as branches. In this case it is sufficient to consider an iteration of length $\aleph_1$, since adding this many Mathias reals witnesses maximality, ultimately because no new reals are added in uncountable limit steps.*

Theorem 4.1.17. *Assume GCH. Let C be a finite set consisting of arbitrary uncountable cardinals. Then consistently $C \subseteq Spec(\mathfrak{i})$.*

Proof. Consider $C = \{\kappa_1, \kappa_2, \dots, \kappa_n\}$. For $j \in \{1, 2, \dots, n\}$ find pairwise disjoint sets I_j so that $|I_j| = \kappa_j$. We define the finite support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_1, \beta < \omega_1 \rangle$ of length $\aleph_1$ recursively as follows: Let $\mathbb{P}_0$ be the trivial forcing and $\dot{\mathbb{Q}}_0$ a $\mathbb{P}_0$-name for the finite support product $\prod_{j=1}^n \mathbb{C}_{I_j}$ of Cohen posets $\mathbb{C}_{I_j}$.

- $\alpha = 1$: In $V^{\mathbb{P}_1}$, each $\mathbb{C}_{I_j}$ adds a set of Cohen reals $\mathcal{A}_j^1 = \{c_i : i \in I_j\}$, which forms an independent family of size κ_j. To begin adding n Mathias reals separately to each of the n independent families, let $\dot{\mathbb{Q}}_1$ be a $\mathbb{P}_1$-name for the finite support product $\prod_{j=1}^n \mathbb{M}(\mathcal{U}_j^1)$, $\mathcal{U}_j^1$ being the respective diagonalization filters.
- $\alpha = 2$: Work in $V^{\mathbb{P}_2}$. Fix $j \in \{1, 2, \ldots, n\}$, and let d_j^1 denote the Mathias real added by $\mathbb{M}(\mathcal{U}_j^1)$. Then define $\mathcal{A}_j^2 = \mathcal{A}_j^1 \cup \{d_j^1\}$, and proceed by taking the $\mathbb{P}_2$-name $\dot{\mathbb{Q}}_2$ for $\prod_{j=1}^n \mathbb{M}(\mathcal{U}_j^2)$ as before.
- Let $\alpha > 2$. For the recursive definition assume that the $\dot{\mathbb{Q}}_\beta$, $\{\mathcal{A}_j^\beta : j \in \{1, 2, \ldots, n\}\}$ together with the corresponding diagonalizing filters and $\mathbb{M}(\mathcal{U}_j^\beta)$-reals d_j^β are defined for all $\beta < \alpha$.
  - If $\alpha = \gamma + 1$, let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for the finite support product $\prod_{j=1}^n \mathbb{M}(\mathcal{U}_j^\gamma)$. In $V^{\mathbb{P}_\alpha}$, proceed as above, setting $\mathcal{A}_j^\alpha := \mathcal{A}_j^\gamma \cup \{d_j^\gamma\}$ for every j.
  - If α is a limit ordinal, let $\mathbb{P}_\alpha = \langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\gamma : \beta \leq \alpha, \gamma < \alpha \rangle$ be a finite support iteration. Then in $V^{\mathbb{P}_\alpha}$, for any j let $\mathcal{A}_j^\alpha := \bigcup_{\beta<\alpha} \mathcal{A}_j^\beta$ and $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for $\prod_{j=1}^n \mathbb{M}(\mathcal{U}_j^\alpha)$.

Finally, in $V^{\mathbb{P}_{\omega_1}}$ define for every j the families $\mathcal{A}_j := \mathcal{A}_j^{\omega_1} := \bigcup_{\beta<\omega_1} \mathcal{A}_j^\beta$, where we note that $|\mathcal{A}_j| = \kappa_j$ since each κ_j is at least uncountable and we have added to every $\mathcal{A}_j^1$ only $\aleph_1$-many elements. Furthermore, each $\mathcal{A}_j$ is independent. It remains to be shown that:

Claim. *For every $j \in \{1, 2, \ldots, n\}$, $\mathcal{A}_j$ is maximal.*

Proof. All of the products from the construction are ccc forcing posets by Remark 4.1.13. Thus $\mathbb{P}_{\omega_1}$ is ccc, and we can proceed as in the proof of Lemma 4.1.1: Fix j and let $x \in [\omega]^\omega \cap V^{\mathbb{P}_{\omega_1}}$ be such that $x \notin \mathcal{A}_j$. Hence there is $\alpha < \omega_1$ with $x \in [\omega]^\omega \cap V^{\mathbb{P}_\alpha}$. As $\mathbb{Q}_\alpha = \prod_{i=1}^n \mathbb{M}(\mathcal{A}_i^\alpha)$ in particular adds a Mathias real d_j^α which diagonalizes $\mathcal{A}_j^\alpha$, the assumption that $\mathcal{A}_j \cup \{x\}$ is independent leads to a contradiction. $\square_{Claim}$

Thus every $\mathcal{A}_j$ is indeed a maximal independent family, which concludes the proof. $\square$

Remark 4.1.18 (GCH assumption). This comparatively strong assumption will be of importance for the negative results.

Corollary 4.1.19. *Assume GCH. Let C be an arbitrary set consisting of uncountable cardinals. Then consistently $C \subseteq Spec(\mathfrak{i})$.*

Proof. Say $|C| = \Theta$. Then we can find pairwise disjoint sets I_α, $\alpha < \Theta$, each of which has cardinality $\kappa_\alpha := |I_\alpha|$. Using finite support products indexed by Θ we can repeat the steps of the proof of the theorem above, employing that any finite support product of Knaster posets is again Knaster (Lemma 4.1.14), and so the fsi is ccc. $\square$

4.2 Spectrum of almost disjointness

In the finite regular case (4.2.1) we can mostly mirror the results for independent families, so the accompanying proofs will be considerably shortened. For arbitrary uncountable cardinals however (4.2.3), it will be necessary to find an alternative to the forcing posets $\mathbb{C}_\lambda$. This will be approached in Section 4.2.2, based on Blass's work [2].

4.2.1 Finitely many regular uncountable cardinals

Let us consider the situation analogous to the one in Lemma 4.1.1. See its proof for the details missing in the following one.

Lemma 4.2.1. *For any cardinal κ with $cf(\kappa) > \omega$, $\kappa \in Spec(\mathfrak{a})$.*

Proof. Define recursively a finite support iteration $\mathbb{P}_\kappa = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$: Let $\mathbb{P}_0 = \{\emptyset\}$, and let $\mathcal{A}_0 = \emptyset$. Set $\mathcal{F}_{\mathcal{A}_0}$ to be the dual filter with respect to $\mathcal{A}_0$ (see Remark 2.1.2). Further let $\dot{\mathbb{Q}}_0$ be a $\mathbb{P}_0$-name for $\mathbb{M}(\mathcal{F}_{\mathcal{A}_0})$, the Mathias forcing notion relativized to $\mathcal{F}_{\mathcal{A}_0}$. In $V^{\mathbb{P}_1}$, define $\mathcal{A}_1 = \mathcal{A}_0 \cup \{d_0\}$, where d_0 is the $\mathbb{M}(\mathcal{F}_{\mathcal{A}_0})$-generic Mathias real diagonalizing $\mathcal{A}_0$ (Lemma 2.1.4). With $\mathcal{F}_{\mathcal{A}_1}$ the induced filter for $\mathcal{A}_1$, define $\dot{\mathbb{Q}}_1$ to be a $\mathbb{P}_1$-name for $\mathbb{M}(\mathcal{F}_{\mathcal{A}_1})$. Next, let $\alpha \geq 2$ and assume that the above objects are defined for all $\beta \leq \alpha$. For successors $\alpha + 1$ proceed by letting $\mathcal{A}_{\alpha+1} := \mathcal{A}_\alpha \cup \{d_\alpha\}$, and $\dot{\mathbb{Q}}_{\alpha+1}$ name $\mathbb{M}(\mathcal{F}_{\mathcal{A}_{\alpha+1}})$ in $V^{\mathbb{P}_{\alpha+1}}$. For limit steps λ let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \lambda, \beta < \lambda \rangle$ be a given finite support iteration, and define $\dot{\mathbb{Q}}_\lambda$ as a $\mathbb{P}_\lambda$-name for $\bigcup_{\alpha<\lambda} \mathbb{M}(\mathcal{F}_{\mathcal{A}_\alpha})$, together with $\dot{\mathcal{A}}_\lambda$ a $\mathbb{P}_\lambda$-name for the almost disjoint family $\bigcup_{\alpha<\lambda} \mathcal{A}_\alpha$. This concludes the construction of $\mathbb{P}_\kappa$. It follows that $V^{\mathbb{P}_\kappa} \models$ "$\mathcal{A}_\kappa := \bigcup_{\alpha<\kappa}$ is a mad family of cardinality κ". $\square$

It is possible to generalize the above in the same manner as in Theorem 4.1.6:

Theorem 4.2.2. *Assume GCH. Let $\kappa_1 < \kappa_2 < \cdots < \kappa_n$, $n \in \omega$, be uncountable regular cardinals. Then consistently $\{\kappa_i : 1 \leq i \leq n\} \subseteq Spec(\mathfrak{a})$.*

4.2.2 Hechler mad forcing

In his work "Short complete nested sequences in $\beta N \setminus N$ and small maximal almost-disjoint families" ([12]), S. Hechler provides a forcing notion which in part aims to prove the following theorem.

Theorem. *Let $\kappa \leq 2^{\aleph_0}$ be an uncountable cardinal. Then there exists a mad family of cardinality κ.*

The original approach of Hechler encodes into the forcing poset a way to simultaneously prove the theorem above as well as certain claims concerning towers. In Blass's "Simple Cardinal Characteristics of the Continuum" ([2]), a slightly different but equivalent reformulation of Hechler's original approach concerning the a.d. part is given. The following will be along those lines, and the forcing notion we are about to define is known as Hechler mad forcing. For our purposes a simplified version of Blass's definition will suffice in that our poset shall add only a single cardinal.

Definition 4.2.3 (Hechler mad forcing)**.** Let κ be an uncountable cardinal. Then $\mathbb{H}_\kappa$ is a forcing notion consisting of ...

- ... conditions being finite partial functions $p : F_p \times n_p \to 2$, where $F_p \in [\kappa]^{<\omega}$ and $n_p \in \omega$,
- ... an extension relation $\leq$ so that for any $p : F_p \times n_p \to 2$ and $p' : F_{p'} \times n_{p'} \to 2$ in $\mathbb{H}_\kappa$, $p' \leq p$ iff p' extends p as a function, i.e., $p' \supseteq p$ (in particular, $F'_p \supseteq F_p, n_{p'} \geq n_p$) and for all $\xi \neq \eta \in F_p$ and any $m \in n_{p'} \setminus n_p$: $\neg(p'(\xi, m) = 1 \wedge p'(\eta, m) = 1)$.

Remark 4.2.4. Using $\mathbb{H}_\kappa$ we are planning to adjoin generic sets a_ξ which shall fulfill the following: First we want $p(\xi, k) = 1$ iff $k \in a_\xi$. Secondly, $\mathbb{H}_\kappa$ shall make sure that if $\xi \neq \eta \in F_p$, then the associated intersections are bounded in the sense that $a_\xi \cap a_\eta \subseteq n$ for some $n \in \omega$. This in particular implies almost disjointness of the different a_ξ.

Let κ be a fixed uncountable cardinal. For cardinal preservation and our iterated forcing constructions we must make sure that $\mathbb{H}_\kappa$ has the ccc. We show that it has the stronger property of being Knaster, which will also be useful for the wanted positive result.

Lemma 4.2.5. *Hechler mad forcing is Knaster.*

Proof. Consider an arbitrary uncountable subset $A \subseteq \mathbb{H}_\kappa$ given by

$$A := \{p_\alpha : F_{p_\alpha} \times n_{p_\alpha} \to 2 \mid \alpha < \aleph_1\}.$$

We wish to construct an uncountable subfamily with pairwise compatible elements. Define $A' \subseteq A$ uncountable via

$$A' := \{p_\alpha : n_{p_\alpha} = n \text{ for } F_{p_\alpha} \times n_{p_\alpha} = dom(p_\alpha))\},$$

where such $n \in \omega$ exists by the pigeonhole principle. Next, apply the Δ-System Lemma to the set $\{F_{p_\alpha} \in [\kappa]^{<\omega} : \exists p_\alpha \in A'(F_{p_\alpha} = pr_1(dom(p_\alpha)))\}$ (pr_1 denotes the natural projection onto the first coordinate), obtaining a Δ-system with root F, such that for any $p_{\alpha_1} \neq p_{\alpha_2}$, $F_{p_{\alpha_1}} \cap F_{p_{\alpha_2}} = F$. This allows for the definition of the uncountable subfamily

$$A'' = \{p_\alpha \in A' : \forall q \in A' \ (q \neq p_\alpha \to dom(p_\alpha) \cap dom(q) = F \times n)\}.$$

Since for any $p_\alpha \in A''$, $range(p_\alpha)$ is finite, employ the pigeon-hole principle to find an uncountable subfamily $B \subseteq A''$ such that any two elements of B coincide on their common domain. Consequently, $B \subseteq A$ forms an uncountable family of pairwise compatible elements. $\square$

Let G be an $\mathbb{H}_\kappa$-generic filter over the given ground model $\mathcal{M}$. In order to formalize the intuitive meaning of $\mathbb{H}_\kappa$, define $\mathcal{A}_\kappa = \{a_\xi : \xi < \kappa\}$ with

$$a_\xi := \{k \in \omega : \exists p \in G \ (p(\xi, k) = 1)\}.$$

Theorem 4.2.6. *For any uncountable cardinal κ, $\mathcal{A}_\kappa$ is a mad family of cardinality κ.*

Proof. Fix κ uncountable. We first prove that:

Claim. *$\mathcal{A}_\kappa$ is an almost disjoint family.*

Proof. Employ a density argument. As mentioned above, almost disjointness is basically directly included in the definition of $\mathbb{H}_\kappa$, and so, considering $\xi, \eta < \kappa$ with $\xi < \eta$, define

$$D_{\xi\eta} = \{p : F_p \times n_p \to 2 \mid \xi, \eta \in F_p\} \subseteq \mathbb{H}_\kappa.$$

To see that these sets are dense in $\mathbb{H}_\kappa$, pick any $p \in \mathbb{H}_\kappa$ with $\xi \notin F_p$ or $\eta \notin F_p$. Assume that both parts of the disjunction hold, and define an extension $p' : F_{p'} \times n_{p'} \to 2$ by setting $n_{p'} = n_p$ and $F_{p'} = F_p \cup \{\xi, \eta\}$. Thus $p' \in \mathbb{H}_\kappa$, $p' \leq p$ and $p' \in D_{\xi\eta}$.

Having this, indeed, for any two distinct elements in F_p the associated sets are almost disjoint: Take $p \in G \cap D_{\xi\eta}$. Fix an arbitrary $q \in G$. Since G is a filter, by downwards directedness, there is $r \in G$ extending both p and q, in particular $r \supseteq q$. As $r \leq p$, ξ and η are in F_r, and additionally $\neg(r(\xi, k) = r(\eta, k) = 1)$ for any $k \geq n_p$. But then the same must be true for q, and since q is arbitrary, for any element of G. So by definition of a_ξ respectively a_η, these two sets can only have at most n_p elements in common; in other words $a_\xi \cap a_\eta \subseteq n_p$. $\square_{Claim}$

Proceed by proving that $\mathcal{A}_\kappa$ is maximal: Assume towards a contradiction that this is not the case. Then there exists a witness for non-maximality, that is, a set $x \in ([\omega]^\omega \setminus \mathcal{A}_\kappa) \cap V[G]$ so that x is almost disjoint from every a_ξ. Our plan is to use combinatorial arguments such as the ccc to reduce the number of conditions associated to a name for x in order to be sure to find a condition not associated to that name. This will allow us to define a condition providing the wanted contradiction.

By definition of $V[G]$, x has an $\mathbb{H}_\kappa$-name $\dot{x}$ in V. As $\mathbb{H}_\kappa$ is ccc, we can assume that $\dot{x}$ is determined by at most countably many conditions (as in 8.2.4). Therefore, we can fix a countable set $J \subseteq \kappa$ so that all conditions occurring in $\dot{x}$ have domains which are subsets of $J \times \omega$. By the Forcing Theorem there is a condition $p : F_p \times n_p \to 2$ so that

$$p \Vdash \text{"}\dot{x} \text{ is an infinite subset of } \omega \wedge \dot{x} \text{ is almost disjoint from } \dot{a}_\xi \text{ for every } \xi < \kappa\text{"}.$$

We can assume that $F_p \subseteq J$ (otherwise enlarge J, which is possible because F_p is finite). By J being countable yet κ uncountable, there must be some $\xi < \kappa$ so that $\xi \notin J$. Fixing this ξ we see that $p \Vdash$ "$\dot{x} \cap \dot{a}_\xi$ is finite". It is possible to find an extending condition $p' \leq p$ to obtain $p' \Vdash$ "$\dot{x} \cap \dot{a}_\xi \subseteq m$" for some $m \in \omega$. Without loss (else change up the choice of $n_{p'}$) we can even assume that

$$p' \Vdash \text{"}\dot{x} \cap \dot{a}_\xi \subseteq n_{p'}\text{"}.$$

Furthermore, by coherence of "$\Vdash$" (Appendix - 8.1),

$$p' \Vdash \text{"}\dot{x} \text{ is an infinite subset of } \omega \wedge \dot{x} \text{ is almost disjoint from } \dot{a}_\xi \text{ for every } \xi < \kappa\text{"}.$$

Define the set of ordinals $X = \{\eta : \eta \in F_{p'} \cap J\}$ and note that $X \neq \emptyset$ because $F_p \subseteq F_{p'} \cap J$, where $F_p \neq \emptyset$ by choice of p respectively J. Since $F_{p'}$ is finite and as we can, for any $\eta \in X$, find some element in $\dot{x}$ which is in none of the $\dot{a}_\eta$,

$$p' \Vdash \text{"}\exists k \geq n_{p'} (k \in \dot{x} \text{ and } \forall \eta \in X (k \notin \dot{a}_\eta))\text{"}. \tag{4.1}$$

Next, extend p' to a condition $q : F_q \times n_q \to 2$ such that q forces the statement in (4.1) for a single fixed $k \geq n_{p'}$. Setting $F'' = F_q \cap J$, let $p'' = q \restriction (F'' \times n_q)$. Note that $F'' \neq \emptyset$ because $F_p \subseteq F_q \cap J$. We want to show the following

Claim. *p'' forces the statement from* (4.1) *for the same k as in the definition of q.*

Proof. Assume towards a contradiction that the claim is false. Extend p'' to a condition r which forces the negation of the statement from (4.1), where we choose r so that $dom(r) \subseteq J \times \omega$. Such choice is possible, as p'' and all conditions involved in the statement of (4.1) have domain contained in $J \times \omega$. Now consider the function $q \cup r$, which by itself is not in general stronger than both q and r. Yet this can be accomplished as long as we define $q \cup r$ to be 0 for any newly added m as in the second point of the definition of the extension relation $\leq$ (Definition 4.2.3). But then $q \cup r$ forces both the statement of (4.1) as well as its negation, a contradiction. $\square_{Claim}$

By construction of the extensions, $dom(p') \cap dom(p'') = (F_{p'} \cap J) \times n_{p'}$. Furthermore, since $q \leq p''$ and $q \leq p'$, $p'' \upharpoonright (F_{p'} \cap J) \times n_{p'} = p' \upharpoonright (F_{p'} \cap J) \times n_{p'}$. Also, we can if necessary extend p'' in order to assume $n_q > k$. Then overall, $k \in n_q \setminus n_{p'}$, and so, by the claim above and the definition of $a_\eta := \{m \in \omega : \exists p \in G\ (p(\eta, m) = 1)\}$,

$$p''(\eta, k) = 0 \text{ for any } \eta \in X. \tag{4.2}$$

As a final building block to obtain a contradiction from our assumption that $\mathcal{A}_\kappa$ is not maximal, define the function $s : F_{p'} \times n_q \to 2$ by setting

a. $s \upharpoonright F_{p'} \times n_{p'} = p'$,

b. $s \upharpoonright (F_{p'} \cap J) \times n_q = p''$,

c. $s(\xi, k) = 1$ for some fixed $\xi \in F_{p'}$, and $s = 0$ everywhere else.

Certainly $s \supseteq p'$ as a function. But more than that, we have that

Claim. *$s \leq p'$.*

Proof. Assume towards a contradiction that this is not the case, i.e., that there exists a witness $j \in n_q \setminus n_{p'}$ for

$$s(\alpha, j) = s(\beta, j) = 1, \text{ where } \alpha \neq \beta \in F_{p'}.$$

In the following we show that no part of the definition of s works out with this assumption: First of all, s as in (a) does not witness the equation because $j \geq n_{p'}$, and so $s \upharpoonright F_{p'} \times n_{p'}$ is not defined at j. Secondly, part (c) means that $s(\alpha, j) = 0$ or $s(\beta, j) = 0$, as by assumption $\alpha \neq \beta$. Without loss take $s(\alpha, j) = 1$. Then necessarily $(\alpha, j) = (\xi, k)$. Finally, (b) cannot provide some $\beta \neq \xi$ in $F_{p'} \cap J$ with $s(\beta, k) = 1$ by equation (4.2).

$\square_{Claim}$

Now fix $\xi < \kappa$ as in the definition of s. The above claim implies in particular that $s \Vdash$ "$k \in \dot{x}$ and $\dot{x} \cap \dot{a}_\xi \subseteq n_{p'}$", as the same statement is forced by p'. Since $k \geq n_{p'}$ the latter implies that $s \Vdash$ "$k \notin \dot{a}_\xi$", in contradiction to part (c) of the definition of s. $\square$

4.2.3 Arbitrary sets of any uncountable cardinals

As shown in the proof of Theorem 4.2.6, Hechler mad forcing does indeed add an arbitrary cardinal to $Spec(\mathfrak{a})$:

Corollary 4.2.7. *Consistently for any uncountable cardinal κ, $\kappa \in Spec(\mathfrak{a})$.*

Using this corollary and the fsi approach we can make sure that for any given finite set of uncountable cardinals C, consistently $C \subseteq Spec(\mathfrak{a})$:

Theorem 4.2.8. *Assume GCH. Let C be a finite set of uncountable cardinals. Then there exists a generic extension in which $C \subseteq Spec(\mathfrak{a})$.*

Proof. Proceed as in the proof of Theorem 4.1.17, but instead of the $\prod_j \mathbb{C}_{I_j}$ use finite support products of Hechler mad forcing notions, and instead of $\prod_j \mathbb{M}(\mathcal{A}_j)$ use the corresponding relativization of Mathias forcing for the a.d. case (as introduced in Section 2.1.1). Since Hechler mad forcing is Knaster, the finite support product of such forcing notions is ccc. $\square$

Thus, as in the case of independence (Corollary 4.1.19) we obtain:

Corollary 4.2.9. *Assume GCH. Let C be an arbitrary set consisting of uncountable cardinals. Then there exists a generic extension in which $C \subseteq Spec(\mathfrak{a})$.*

4.3 Approach using tree forcing

Although we already employed in an intuitive sense some terminology from the subject of trees, we now wish to formally introduce trees (Section 4.3.1). After that, we will see an example of how one can combine some of the tools used from previous sections with trees to find different proofs of the positive results (4.3.2). The last part (4.3.3) is concerned with the introduction of suitable families C called pre-spectra, for which the negative results will be formulated in Chapter 6.

All of this builds on results given in [8].

4.3.1 Tree basics

A characterizing property of trees can be seen as their closure under initial segments, meaning that if we consider any branch of a given tree, then it is possible to access every sub-branch.

Definition 4.3.1 (Tree)**.** A partially ordered set is called a tree if for any of its nodes the set of all of its predecessors is well-ordered by the given order.

The following example introduces the kind of trees we are going to be interested in.

Example 4.3.2. For cardinals σ and θ, let $\theta^{<\sigma} := \{\tau : \alpha \to \theta \mid \alpha < \sigma\}$. Let $\trianglelefteq$ denote the relation on $\theta^{<\sigma}$ given by $\eta \trianglelefteq \tau$ iff $dom(\eta) \subseteq dom(\tau)$ and $\eta = \tau \restriction dom(\eta)$. In other words, τ end-extends η. Then $(\theta^{<\sigma}, \trianglelefteq)$ forms a tree. Note that any subset $T \subseteq \theta^{<\sigma}$ closed under initial segments is itself a tree.

Some more vocabulary concerning trees:

Notation. Let S be a tree. Then:

- $\tau{\downarrow}_S$ respectively $\tau{\downarrow}$ denotes the set of all proper initial segments of $\tau \in S$
- The height of a node $\tau \in S$ is given by $ht_S(\tau) = type(\tau{\downarrow}_S)$.
- The α-th level of S is defined as $S_\alpha = \{\tau \in S : ht_S(\tau) = \alpha\}$.
- Using the levels of S one can define $ht(S) = min\{\alpha : S_\alpha = \emptyset\}$ to be the height of the whole tree.
- We say that μ is an immediate successor of τ if $\exists \epsilon\ (\mu = \tau^\frown \langle \epsilon \rangle)$.
- $Succ_S(\tau) = \{\mu \in S : \mu$ immediate successor of $\tau\}$ is called the successor set of τ in S.
- S is called θ-splitting if $\forall \tau \in S$: $|Succ_S(\tau)| = \theta$.
- Finally, a path through S is a chain $P \subseteq S$, so that $\forall \alpha < ht(S) : P \cap S_\alpha \neq \emptyset$.

4.3.2 Adding any uncountable cardinals to the spectra via trees

In this section we provide another way to prove Corollaries 4.1.19 and 4.2.9, both being restated as theorems below. Here we will see that it is not possible to directly transfer the Mathias forcing approach for $\mathfrak{i}$ to the a.d. case due to differences inherent to the underlying concepts.

Independence case

We first show that it is possible to add multiple mutually generic Mathias reals at once.

Lemma 4.3.3. *Let $I \neq \emptyset$ be an index set. For any given independent family $\mathcal{A}$ (in the ground model V) and $\mathcal{U}$ an $\mathcal{A}$-diagonalization filter, consider the finite support product $\mathbb{Q} := \prod_{i \in I} \mathbb{M}(\mathcal{U}_i)$, where $\mathcal{U}_i = \mathcal{U}$ for all $i \in I$. Let $G := \prod_{i \in I} G_i$ be a $(V, \mathbb{Q})$-generic filter. Then in $V[G]$, with the Mathias reals denoted as $x_i := x_{G_i}$, obtain:*

- *$\mathcal{A} \cup \{x_i\}_{i \in I}$ is still an independent family,*
- *but for any $y \in V \cap ([\omega]^\omega \setminus \mathcal{A})$, the family $\mathcal{A} \cup \{x_i, y\}$ ceases to be independent, for any $i \in I$.*

Proof. Knowing that each x_i is diagonalizing, we only have to show that $\mathcal{A} \cup \{x_i\}_{i \in I}$ forms an independent family, which is achieved by a density argument. Fix an arbitrary finite function $h \in FF(\mathcal{A})$, a finite map $j : I \to 2$, and $n \in \omega$. Then define

$$D_{h,j,n} = \{\bar{q} \in \mathbb{Q} : \bar{q} \Vdash \text{"}\exists \check{k} > \check{n}(\check{k} \in \bigcap_{i \in dom(j)} \dot{x}_i^{j(i)} \cap \check{\mathcal{A}}^h)\text{"}\}.$$

Claim. *$D_{h,j,n}$ is dense in $\mathbb{Q}$.*

Proof. Take any $\bar{p} = \prod_{i \in dom(\bar{p})} (s_i, F_i) \in \mathbb{Q}$, where $(s_i, F_i) \in [\omega]^{<\omega} \times \mathcal{U}$ as in the definition of Mathias forcing (Definition 2.1.1). Without loss assume that $dom(\bar{p}) = dom(j) = I$ (We know that by definition $\bar{p}$ is a function on I with $\bar{p}(i) \in \mathbb{M}(\mathcal{U})$ for all $i \in I$, and $\bar{p}(i) = (\emptyset, \omega)$ cofinitely often. So only a finite part of the domain of certain functions in $\mathbb{Q}$, namely $dom(j)$, is of interest to us in the context of the density argument. In particular, for the wanted extension we could simply copy the entries of $\bar{p}$ for any index in $I \setminus dom(j)$). We wish to construct an extension of $\bar{p}$ in $D_{h,j,n}$. Split $dom(j)$ into $I_0 := j^{-1}(0)$ and $I_1 := j^{-1}(1)$. The key object which allows us to conclude the proof is $\mathcal{U}$, by the definition of which (Definition 2.2.2) the intersection $\mathcal{A}^h \cap \bigcap_{i \in dom(j)} F_i$ is an infinite subset of ω. As n and $\max_{i \in I_0} s_i$ are finite, there exists $k \in \mathcal{A}^h \cap \bigcap_{i \in dom(j)} F_i$ with $k > \max\{n, \max_{i \in I_0} s_i\}$. Thereby, the wanted extension is given by $\bar{q} \in \mathbb{Q}$ with

$$\bar{q}(i) := \begin{cases} (s_i \cup \{k\}, F_i \setminus (k+1)), & \text{if } i \in I_0, \\ (s_i, F_i \setminus (k+1)), & \text{if } i \in I_1. \end{cases}$$

Indeed, $\bar{q} \leq \bar{p}$, as in both cases $\bar{q}(i) \leq_{\mathbb{M}(\mathcal{U})} (s_i, F_i)$. Furthermore,

$$\forall i \in I_0 : \bar{q}(i) \Vdash_{\mathbb{M}(\mathcal{U})} \text{"}\check{k} \in \dot{x}_i \cap \check{\mathcal{A}}^h\text{"} \text{ as well as } \forall i \in I_1 : \bar{q}(i) \Vdash_{\mathbb{M}(\mathcal{U})} \text{"}\check{k} \in \omega \setminus \dot{x}_i \cap \check{\mathcal{A}}^h\text{"}.$$

The former can be seen by noting that if $\bar{q}(i) \in G_i$ for some $\mathbb{M}(\mathcal{U})$-generic filter G_i, then the fact that $x_i = \bigcup\{s : \exists F \in \mathcal{U}\ ((s,F) \in G_i)\}$ provides $s_i \cup \{k\} \subseteq x_i$. The latter case holds, as we know that $k \notin s_i$, and any condition in G_i compatible to $\bar{q}(i)$ cannot contain k because its infinite part specifically excludes k, and $max(s_i) < k$. $\square_{Claim}$

By the Forcing Theorem, $D_{h,j,n} \in V$. Therefore, $G \cap D_{h,j,n} \neq \emptyset$ for any h, j, n as above. But this means that any Boolean combination with respect to the elements of $\mathcal{A} \cup \{x_i\}_{i \in I}$ is infinite, and hence the statement of the lemma follows. $\square$

Discussion 4.3.4 (Mutual genericity)**.** *Although each x_i is a Mathias real stemming from the same forcing poset $\mathbb{M}(\mathcal{U}_i) = \mathbb{M}(\mathcal{U})$, where the only way to differentiate between the $\mathbb{M}(\mathcal{U}_i)$ is their position in the product, it is not the case that $(\{x_i : i \in I\} = \{x_i\})^{V[G]}$. The reason for this is that each Mathias real must form a different generic object, as exhibited by the dense sets $D_{h,j,n}$.*

Theorem 4.3.5 (Restatement of Corollary 4.1.19)**.** *Assume GCH. Let C be an arbitrary set consisting of uncountable cardinals. Then consistently $C \subseteq Spec(\mathfrak{i})$.*

Proof. Fix $\theta \in C$. Consider the tree $S_\theta = (\theta \times \{\theta\})^{<\omega_1}$. Then S_θ is θ-splitting and has height ω_1. Defining a finite support iteration using the tree structure of S_θ will take care of forcing the existence of a maximal independent family of cardinality θ. For any $\alpha < \omega_1$ the αth level of the tree for θ is given by $S_{\theta,\alpha} = S_\theta \cap \theta^{\alpha+1}$. To consider all of the trees at once define $\bar{S} = (S_\theta : \theta \in C)$. Note that the definition of the S_θ makes sure that $\bar{S}$ consists of pairwise disjoint trees. Call $S = \bigcup \bar{S}$. Finally, for any $\alpha < \theta$ let $\bar{S}_\alpha = \bigcup_{\theta \in C} S_{\theta,\alpha}$, the "global" αth level of the tree sequence.

We define a finite support iteration $\mathbb{P}_{\bar{S}} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_1, \beta < \omega_1 \rangle$ recursively as follows (in terms of levels of $\bar{S}$):

- For $\bar{S}_0$: Let $\mathbb{P}_0 = \{\emptyset\}$, and $\dot{\mathbb{Q}}_0$ a $\mathbb{P}_0$-name for the finite support product $\prod_{\eta \in \bar{S}_0} \mathbb{C}_\eta$, where $\mathbb{C}_\eta := \mathbb{C}_\theta$ whenever $\eta \in S_{\theta,0}$, i.e., $\prod_{\eta \in \bar{S}_0} \mathbb{C}_\eta = \prod_{\theta \in C} \mathbb{C}_\theta$.
- For $\bar{S}_1$: Work in $V^{\mathbb{P}_1}$. For each $\eta \in \bar{S}_0$ associate to every $\nu \in Succ_S(\eta)$ a copy $\mathcal{A}^1_\eta$ of the independent family added by $\mathbb{C}_\eta$. In other words, in every tree S_θ we add θ-many copies of an independent family of cardinality θ. Furthermore, for each such ν define an $\mathcal{A}^1_\eta$-diagonalization filter $\mathcal{U}_\eta$. Let $\dot{\mathbb{Q}}_1$ be a $\mathbb{P}_1$-name for the finite support product $\prod_{\eta \in \bar{S}_1} \mathbb{M}(\mathcal{U}_\eta)$. Then in $V^{\mathbb{P}_2}$, define the $\mathbb{M}(\mathcal{U}_\eta)$-generic reals a_η for all $\eta \in \bar{S}_1$, and set $\mathcal{A}^2_\eta := \mathcal{A}^1_\eta \cup \{a_\eta : \eta \in \bar{S}_1\}$. Note that each $\mathcal{A}^2_\eta$ forms an independent family by the previous lemma. We can rewrite it via $\mathcal{A}^2_\eta = \{a_\nu : \nu \in Succ_S(\eta{\restriction}\xi), \xi < 2\}$, setting $Succ_S(\eta{\restriction}0) = Succ_S(\emptyset)$ to be an index set for $\mathcal{A}^1_\eta$.

- For $\bar{S}_\alpha$, $\alpha \geq 2$: Assume that for all $\eta \in \bar{S}_{\alpha-1}$, $\mathcal{A}^\alpha_\eta$ is defined. Let us again produce multiple copies of each of those independent families: To every $\nu \in Succ_S(\eta)$ assign $\mathcal{A}^\alpha_\eta$ and $\mathcal{U}_\eta$, a copy of the $\mathcal{A}^\alpha_\eta$-diagonalization filter. Working in $V^{\mathbb{P}_\alpha}$, for all $\eta \in \bar{S}_\alpha$, let a_η be the corresponding $\mathbb{M}(\mathcal{U}_\eta)$-generic Mathias real, and set

$$\mathcal{A}^{\alpha+1}_\eta := \{a_\nu : \nu \in Succ_S(\eta \restriction \xi), \xi < \alpha + 1\}.$$

  Finally, define $\dot{\mathbb{Q}}_\alpha$ to be a $\mathbb{P}_\alpha$-name for the finite support product $\prod_{\eta \in \bar{S}_\alpha} \mathbb{M}(\mathcal{U}_\eta)$.

In the resulting model $V^{\mathbb{P}_{\bar{S}}}$, for an arbitrary fixed $\theta \in C$ consider any path g through S_θ, that is, g is a linearly ordered subset of S_θ going through every level $S_{\theta,\alpha}$, $\alpha < \omega_1$. Then define

$$\mathcal{A}_g = \{a_\nu : \nu \in Succ_S(g \restriction \xi), \xi < \omega_1\}.$$

Claim. *$\mathcal{A}_g$ forms a maximal independent family in $V^{\mathbb{P}_{\bar{S}}}$ of cardinality θ.*

Proof. Certainly, $\mathcal{A}_g$ has cardinality θ, as for any given path g, $\theta \geq \omega_1$ implies that

$$|\mathcal{A}_g| = \bigcup_{\xi < \omega_1} |Succ_S(g \restriction \xi)| = \theta.$$

Furthermore:

- $\mathcal{A}_g$ is an independent family: This basically follows directly from the construction, since along any such path g the collected reals form an independent family in each step by the previous lemma.

- $\mathcal{A}_g$ is maximal: As in the proof of Corollary 4.1.19, the finite support products preserve the ccc in the final model. Thus we can proceed as we did there, assuming towards a contradiction that $\mathcal{A}_g$ is not maximal, and using the second part of the previous lemma together with a witness for the non-maximality stemming from one of the previous iteration steps, to obtain a contradiction as in Lemma 4.1.1.

$\square_{Claim}$

As $\theta \in C$ was arbitrary, we conclude that in $V^{\mathbb{P}_{\bar{S}}}$, $C \subseteq Spec(\mathfrak{i})$. $\square$

In the above proof we could have used only the Mathias poset and not add Cohen reals in the first step, see [8].

Remark 4.3.6 (Comparison of the proofs from before vs. tree context).

- In the proof of Corollary 4.1.19 we technically also worked with trees, that is, one can view the situation as having $|C|$-many disjoint trees, each consisting of only a single branch of size $\aleph_1$. Here we are adding much more structure than apparently needed, reason being the wanted negative results. More concretely, we want to have a certain amount of names for reals to work with in order to enable an isomorphism of names argument. For details see Chapter 6 and particularly Section 6.2.2.

- Another point to mention is that in the tree context we do not need to specify successor and limit cases as before. The reason is that the finite support iteration has been defined along the trees.

Almost disjointness case

In transitioning to the a.d. case, it is important to note that the diagonalization filters used in the previous proof are a tool introduced for independent families. Is it possible to define a similar notion for almost disjoint families, or maybe employ the induced filters from Remark 2.1.2?

Remark 4.3.7 (Transitioning to almost disjoint families). It is not possible to obtain a version of Lemma 4.3.3 with $\mathbb{M}(\mathcal{F}_A)$ instead of $\mathbb{M}(\mathcal{U})$. The reason lies not in the change of filters, but in the concept of almost disjointness itself. Indeed, the mentioned proof shows in particular that the added mutually generic Mathias reals have infinite intersections with one another. Hence, they are not almost disjoint.

Nevertheless, for the upcoming negative results it will be important to have an analogue of Theorem 4.3.5 corresponding to the a.d. case. At this point, we can use that the Hechler mad forcing notion is product-like (Appendix - Proposition 9.0.2), and employ the quotients $\mathbb{H}_{[\theta,\lambda)}$ as introduced in Appendix - Discussion 9.0.3. For our context, consider a tree version of these posets:

Discussion 4.3.8 (Variation of $\mathbb{H}_{[\theta,\lambda)}$)**.** *As we have seen, $\mathbb{H}_\lambda \sim \mathbb{H}_\theta * \mathbb{H}_{[\theta,\lambda)}$ $(\theta < \lambda)$, where the mad family $\mathcal{A}_\theta$ added by $\mathbb{H}_\theta$ can be expanded by the reals added via $\mathbb{H}_{[\theta,\lambda)}$ to obtain the mad family forced by $\mathbb{H}_\lambda$. We wish to directly associate the added reals to the nodes of the tree. To do so, change the definition of $\mathbb{H}_{[\theta,\lambda)}$ to obtain a forcing poset $\mathbb{H}_{\mathcal{A}^\alpha_\eta, Succ_S(\eta)}$ as follows:*

- *As before consider $S := \bigcup \bar{S}$, the set of all nodes appearing in the tree sequence $\bar{S} := \langle S_\theta : \theta \in C \rangle$. Fix $\eta \in S$. Say $\mathcal{A}^\alpha_\eta$ is a given mad family associated to η. To any $\nu \in Succ_S(\eta)$ allocate a copy of $\mathcal{A}^\alpha_\eta$.*

- *Let* $(p, H) \in \mathbb{H}_{\mathcal{A}^\alpha_\eta, Succ_S(\eta)}$ *be such that* $p : Succ_S(\eta) \times \omega \to 2$, $dom(p) = A_p \times n_p \in [Succ_S(\eta)]^{<\omega} \times \omega$, *and* $H \in [|\mathcal{A}^\alpha_\eta|]^{<\omega}$.

- *Then define* $(q, K) \leq (p, H)$ *iff* $q \leq_{\mathbb{H}_{|\mathcal{A}^\alpha_\eta|+|Succ_S(\eta)|}} p$, $K \supseteq H$, *and furthermore*

$$\forall \alpha \in A_p \forall \beta \in H \ \forall m \in n_q \backslash n_p \ (m \in a_\beta \Rightarrow q(\alpha, m) = 0),$$

where $a_\beta \in \mathcal{A}^\alpha_\eta$, $a_\beta := \{k \in \omega : \exists p \in G(p(\beta, k) = 1)\}$ *as before.*

Theorem 4.3.9 (Restatement of Corollary 4.2.9)**.** *Assume GCH. Let C be a set consisting of uncountable cardinals. Then there exists a generic extension in which $C \subseteq Spec(\mathfrak{a})$.*

Proof. With the same notation as in the proof for the independence case, define the finite support iteration $\mathbb{P}_{\bar{S}} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_1, \beta < \omega_1 \rangle$ recursively as follows:

- For $\bar{S}_0$: Let $\mathbb{P}_0 = \{\emptyset\}$. Now define $\dot{\mathbb{Q}}_0$ to be a $\mathbb{P}_0$-name for the finite support product $\prod_{\eta \in \bar{S}_0} \mathbb{H}_\eta$, where $\mathbb{H}_\eta := \mathbb{H}_\theta$ whenever $\eta \in S_{\theta,0}$.

- For $\bar{S}_1$: Work in $V^{\mathbb{P}_1}$. For all $\eta \in \bar{S}_0$, to each $\nu \in Succ_S(\eta)$ assign a copy of $\mathcal{A}^1_\eta := \{a_\nu : \nu \in Succ_S(\eta)\}$, the mad family forced by $\mathbb{H}_\eta$, where the a_ν are $\mathbb{H}_\eta$-reals. Let $\dot{\mathbb{Q}}_1$ be a $\mathbb{P}_1$-name for the finite support product $\prod_{\eta \in \bar{S}_1} \mathbb{H}_{\mathcal{A}^1_\eta, Succ_S(\eta)}$. Every such $\mathbb{H}_{\mathcal{A}^1_\eta, Succ_S(\eta)}$ forces the existence (in $V^{\mathbb{P}_2}$) of a set of θ-many reals $\{a_\nu : \nu \in Succ_S(\eta)\}$, whenever $\eta \in S_{\theta,1}$. Thus, continuing in $V^{\mathbb{P}_2}$, for any $\eta \in \bar{S}_1$, to each node $\nu \in Succ_S(\eta)$ assign $\mathcal{A}^2_\eta := \{a_\nu : \nu \in Succ_S(\eta \restriction \xi), \xi < 2\}$.

- For $\bar{S}_\alpha$, $\alpha \geq 2$, we can basically proceed as before: In $V^{\mathbb{P}_\alpha}$, assume that for all $\eta \in \bar{S}_{\alpha-1}$ the families $\mathcal{A}^\alpha_\eta$ are given. Define $\dot{\mathbb{Q}}_\alpha$ as the $\mathbb{P}_\alpha$-name for the finite support product $\prod_{\eta \in \bar{S}_\alpha} \mathbb{H}_{\mathcal{A}^\alpha_\eta, Succ_S(\eta)}$. In $V^{\mathbb{P}_{\alpha+1}}$, let $\mathcal{A}^{\alpha+1}_\eta := \{a_\nu : \nu \in Succ_S(\eta \restriction \xi), \xi < \alpha + 1\}$, $\forall \eta \in \bar{S}_\alpha$.

Fix an arbitrary $\theta \in C$ and continue working in the final model $V^{\mathbb{P}_{\bar{S}}}$. Let g be any path in S_θ, and define $\mathcal{A}_g = \{a_\nu : \nu \in Succ_S(g \restriction \xi), \xi < \omega_1\}$. As before, if we are able to prove that $\mathcal{A}_g$ is a mad family of cardinality θ, then we are done. Knowing that $|\mathcal{A}_g| = \theta$ from the independence case, it remains to show that $\mathcal{A}_g$ is almost disjoint and maximal:

Almost disjointness: We can write $\mathcal{A}_g = \bigcup_{\xi < \omega_1} \mathcal{A}_{g \restriction \xi}$, a union of a chain of a.d. (even mad) families $\mathcal{A}_{g \restriction \xi} := \{a_\nu : \nu \in Succ_S(g \restriction \xi)\}$, where in each step, for growing ξ, the newly added reals are obtained by the quotient posets in the construction above.

Maximality: Assume otherwise. Then there is some $x \in [\omega]^\omega \setminus \mathcal{A}_g$ with $|x \cap a_\nu| < \omega$ for all ν. Again, there exists $\alpha < \omega_1$ such that $x \in V^{\mathbb{P}_\alpha}$. So in $V^{\mathbb{P}_{\alpha+1}}$, $\mathcal{A}_{g \restriction \alpha+1} \cup \{x\}$ forms an a.d. family, in contradiction to the maximality of $\mathcal{A}_{g \restriction \alpha+1}$. $\square$

4.3.3 Pre-spectra

Let us return to Remark 4.1.15 from the beginning of this section, where it was mentioned that adding arbitrary C to the spectra implies that in the generic extension, $\lambda := \sup C \leq \mathfrak{c}$ is included in the spectrum. This will be made more explicit in the following, and to conclude the positive results, we put some additional limitations on C, which will allow for the formulation of corresponding negative results. In the case of independent families the restricted sets C are known as so-called pre-independence spectra ([8]). Since we are working towards results for the joint spectra of $\mathfrak{i}$ and $\mathfrak{a}$, let us speak of 'pre-spectra' instead.

Definition 4.3.10 (Pre-spectrum)**.** Let C be a set of uncountable cardinals. Call C a pre-spectrum (with parameters σ, λ) if:

1. C has minimum σ which is regular.
2. C is bounded above by λ, where $cf(\lambda) > \omega$ and $\lambda = \max C$.
3. If $|C| \geq \omega$, C is closed with respect to singular limits of countable cofinality.

Discussion 4.3.11. *In order to prove equality with the spectra, we will further impose the limitation on* C *that* $|C| \leq \aleph_0$*. Point (2) of the above definition reflects the fact that in ZFC the spectra are bounded by* $\mathfrak{c}$*, which has uncountable cofinality (and this is preserved in our case because of the ccc forcing notions we are using). Point (3) specifies that the upper bound of possibly infinitely many sets has to be included in the spectrum, as the fact that Spec*$(\mathfrak{i})$ *is bounded is a ZFC result (for example,* $\{\aleph_n : 1 \leq n < \omega\}$ *does not form a pre-spectrum because* $\aleph_\omega$ *is not included in the set, but even this is not enough, as* $cf(\aleph_\omega)$ *is countable and so (2) fails). Finally, point (1) is a restriction coming from our choice of proof method, where the minimal iteration length* σ *is being used as before to make the diagonalization arguments work. The question whether conditions (1) and (3) on such sets may be weakened or not is still open.*

Similarly as before one can show that a given pre-spectrum may be added to *Spec*$(\mathfrak{i})$ or *Spec*$(\mathfrak{a})$. In addition, it is now necessary to prove that σ indeed takes the value $\mathfrak{i}$ or $\mathfrak{a}$ in the generic extension to ensure that we do not add any cardinal below $\min C$.

Theorem 4.3.12. *Assume GCH. Let* C *be a pre-spectrum with parameters* σ *and* λ*. Then consistently,* $C \subseteq Spec(\mathfrak{i})$*, where* $\sigma = \mathfrak{i}$ *and* $\lambda = \mathfrak{c}$*.*

Proof. Again consider $\bar{S} = \{S_\theta : \theta \in C\}$. To show that $Con(C \subseteq Spec(\mathfrak{i}))$, proceed as in the proof of Theorem 4.3.5, with the finite support iteration $\mathbb{P}_{\bar{S}} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \sigma, \beta < \sigma \rangle$

defined analogously, except that now the length of the iteration is $\sigma \geq \aleph_1$. As $\mathbb{P}_{\bar{S}}$ is ccc, it follows that $V^{\mathbb{P}_{\bar{S}}} \models$ "$\lambda = \mathfrak{c}$".

Finally, to see that $V^{\mathbb{P}_{\bar{S}}} \models$ "$\sigma = \mathfrak{i}$", consider the Cohen reals added along the iteration:

Claim 4.3.13. *The number of Cohen reals adjoined via $\mathbb{P}_{\bar{S}}$ is unbounded in σ.*

Proof. Since σ is regular uncountable, it is of uncountable cofinality. Therefore, for any $\alpha \in \sigma$ there exists a countable cofinality limit stage of the iteration where a Cohen real c_α is added (Fact 3.4.1). $\square_{Claim}$

Fix an arbitrary $\alpha < \sigma$. The real c_α is not eventually dominated by any $f \in \omega^\omega \cap V^{\mathbb{P}_\gamma}$, $\gamma < \alpha$ (Appendix - 8.3.4). Now any dominating family $\mathbb{D}$ in $V^{\mathbb{P}_\alpha}$ must have cardinality $|\mathbb{D}| \geq \sigma$ because in case that $|\mathbb{D}| < \sigma$, there exists a Cohen real not dominated by any of the elements of $\mathbb{D}$, contradicting $\mathbb{D}$ being dominating. But then $\mathfrak{d} \geq \sigma$ and therefore $\mathfrak{i} = \sigma$, since $\mathfrak{d} \leq \mathfrak{i}$ (Section 1.2.3) and $\mathfrak{i} \leq \sigma$ (by $Con(\sigma \in Spec(\mathfrak{i}))$ and the definition of $\mathfrak{i}$). $\square$

Except for the last part, we can copy the proof for a corresponding result for $\mathfrak{a}$:

Theorem 4.3.14. *Assume GCH. Let C be a pre-spectrum with parameters σ and λ. Then consistently, $C \subseteq Spec(\mathfrak{a})$, where $\sigma = \mathfrak{a}$ and $\lambda = \mathfrak{c}$.*

Proof. Define the fsi $\mathbb{P}_{\bar{S}} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \sigma, \beta < \sigma \rangle$ as in the proof of Theorem 4.3.9 and proceed as before. To see that $\sigma = \mathfrak{a}$: Again, $\sigma \in Spec(\mathfrak{a})$ implies $\mathfrak{a} \leq \sigma$. For the converse, use Claim 4.3.13 to find that any unbounded family $\mathcal{B}$ in $V^{\mathbb{P}_\alpha}$ with $|\mathcal{B}| < \sigma$ is not unbounded in $V^{\mathbb{P}_{\bar{S}}}$. Indeed, take such family. Then by the claim there exists a Cohen real c_γ, $\gamma \geq \sigma$, dominating all elements of $\mathcal{B}$. Therefore, $\mathfrak{a} \geq \sigma$ because $\mathfrak{b} \leq \mathfrak{a}$ (Theorem 1.1.3) and $\mathfrak{b} \geq \sigma$.

$\square$

5 Interlude - measurable cardinals

Our next aim is to find out when the spectra of $\mathfrak{a}$ and $\mathfrak{i}$ are equal. Before discussing this question as a continuation of the previous chapter, let us consider the case where $C := \{\kappa_i\}_{i=1}^n$ is a finite set of strictly ordered measurable cardinals. We will see that in this case it is consistent that both spectra coincide with C. In a rough sense, the upcoming results minus the measurability assumption can be seen as a blue print for the next two chapters.

Following [13], in Section 5.1 we start by stating a few definitions leading to the concept of an ultrapower. Section 5.2, after introducing the method of taking averages (5.2.1) and the dominating forcing notion (5.2.2), delivers the proof of the above statement (5.2.3). It is based on [7] and J. Brendle's work [3].

5.1 Basic definitions

Let us begin with a few necessary preparations:

Definition 5.1.1 (Ultrafilter, measurable cardinal)**.** Let $\mathcal{U}$ be a filter over some set S, θ a regular uncountable cardinal and κ some uncountable cardinal. Then:

1. $\mathcal{U}$ is called an ultrafilter, if for any $A \subseteq S$ either $A \in \mathcal{U}$ or $S \setminus A \in \mathcal{U}$. If furthermore every set in $\mathcal{U}$ is infinite, then $\mathcal{U}$ is called a non-principal ultrafilter.

2. $\mathcal{U}$ is θ-complete, if for any set $\{A_\alpha \in \mathcal{U} : \alpha < \theta\}$, $\bigcap_{\alpha<\theta} A_\alpha \in \mathcal{U}$.

3. κ is measurable, if there exists a κ-complete non-principal ultrafilter $\mathcal{U}$ over κ.

We will further require the notion of an ultrapower. In terms of notation, the following is already stated in the set-theoretic context used later on instead of a general model-theoretic treatment.

Discussion 5.1.2. *Ultrapowers are special cases of ultraproducts, which we define as follows:*

- *Given a set $S \neq \emptyset$, $\mathcal{U}$ an ultrafilter on S, and $\{\in\}$-structures $\mathcal{A}_x$ indexed by $x \in S$, consider the Cartesian product $\prod_{x\in S} \boldsymbol{A}_x$ of the respective universes.*

- *Introduce an equivalence relation $\equiv$ on $\prod_{x\in S} \boldsymbol{A}_x$ by saying that*

$$f \equiv g \text{ iff } \{x \in S : f(x) = g(x)\} \in \mathcal{U}.$$

- *It can be shown that the quotient set $\boldsymbol{A} = \prod_{x\in S} \boldsymbol{A}_x / \equiv$ forms the universe of a model, which we call the ultraproduct of the $\boldsymbol{A}_x$ with respect to $\mathcal{U}$. Let us denote this model by $Ult_{\mathcal{U}}\{\mathcal{A}_x : x \in S\}$.*

- *An important result in this context known as Łoś's Theorem says that for any given formula (resp. sentence) ϕ, for all $f_1, f_2, \ldots, f_n \in \prod_{x\in S} \boldsymbol{A}_x$, $Ult_{\mathcal{U}}\{\boldsymbol{A}_x : x \in S\} \models \phi([f_1], [f_2], \ldots, [f_n])$ is equivalent to $\mathcal{A}_x \models \phi[f_1(x), f_2(x), \ldots, f_n(x)]$ almost everywhere (relative to $\mathcal{U}$), i.e., $\{x \in S : \boldsymbol{A}_x \models \phi[f_1(x), f_2(x), \ldots, f_n(x)]\} \in \mathcal{U}$.*

Finally, $Ult_{\mathcal{U}}\{\mathcal{A}_x : x \in S\}$ is called an ultrapower if all of the $\mathcal{A}_x$ coincide, that is, $\mathcal{A}_x = \mathcal{A}$ for all $x \in S$. In this case we write $Ult_{\mathcal{U}}\mathcal{A} := Ult_{\mathcal{U}}\{\mathcal{A}_x : x \in S\}$. A consequence of Łoś's Theorem is that any model is elementarily equivalent to its ultrapower. Therefore, even though we can construct a potentially much more complicated model from given ones, it is not possible to differentiate between them using first-order sentences in the given language. For proofs of the above statements see [13], pp.158.

5.2 Measurable cardinals and the joint spectra of $\mathfrak{i}$ and $\mathfrak{a}$

As mentioned in the introduction to this section we wish to prove the following theorem:

Theorem 5.2.1. *Assume GCH. Let $\kappa_1 < \kappa_2 < \cdots < \kappa_n$, $n \in \omega$, be measurable cardinals. Then consistently $Spec(\mathfrak{i}) = Spec(\mathfrak{a}) = \{\kappa_i\}_{i=1}^n$.*

Discussion 5.2.2. *The proof (see Section 5.2.3) consists of three parts: We have to make sure to force that ...*

I. *... $\{\kappa_i\}_{i=1}^n \subseteq Spec(\mathfrak{i}) \cap Spec(\mathfrak{a})$,*

II. *... there are no cardinals in between the given ones and there are none above $\max_{i=1,\ldots,n} \kappa_i = \kappa_n$ in $Spec(\mathfrak{i}) \cap Spec(\mathfrak{a})$,*

III. *... no cardinal below $\min_{i=1,\ldots,n} \kappa_i = \kappa_1$ is in $Spec(\mathfrak{i}) \cap Spec(\mathfrak{a})$.*

Whereas (I) basically follows from the chapter on positive results, for (II) we can use the measurability assumption, which ultimately provides a way to define a witness for non-maximality of the given independent and a.d. families, respectively. For this we will employ the method of taking the average of $\mathbb{P}$-names, which will be introduced in the upcoming section. Lastly, (III) will require special attention, as we will have to make sure to show that $\mathfrak{i}, \mathfrak{a} \geq \kappa_1$ in the generic extension. This will actually entail that consistently $\mathfrak{i} = \mathfrak{a} = \kappa_1$.

5.2.1 Taking the average of $\mathbb{P}$-names

Let $\mathbb{P}$ be a given forcing poset and κ a measurable cardinal with $\mathcal{D} \subseteq \kappa$ the associated κ-complete non-principal ultrafilter as in Definition 5.1.1. For any function $f \in \mathbb{P}^\kappa$ define $[f] = \{g \in \mathbb{P}^\kappa : \{\alpha \in \kappa : f(\alpha) = g(\alpha)\} \in \mathcal{D}\}$. Consider the ultrapower

$$\mathbb{P}^\kappa/\mathcal{D} := \{[f] : f \in \mathbb{P}^\kappa\}.$$

It then follows that:

Fact 5.2.3.

1. The relation $\leq$ on $\mathbb{P}^\kappa/\mathcal{D} \times \mathbb{P}^\kappa/\mathcal{D}$ given by $[f] \leq [g]$ iff $\{\alpha \in \kappa : f(\alpha) \leq g(\alpha)\} \in \mathcal{D}$ forms a partial order on $\mathbb{P}^\kappa/\mathcal{D}$.

2. $\mathbb{P}$ can be considered as a subset of $\mathbb{P}^\kappa/\mathcal{D}$.

Proof.

1. Use that $\mathcal{D}$ is a filter to find that $\{\alpha \in \kappa : f(\alpha) \leq f(\alpha)\} = \kappa \in \mathcal{D}$ for reflexivity. Transitivity and anti-symmetry follow from $\mathcal{D}$ being closed under intersections.

2. Identify each $p \in \mathbb{P}$ with the function $f \in \mathbb{P}^\kappa$, $f(\alpha) := p$ for all $\alpha \in \kappa$.

□

From now on we will frequently employ this fact without mention. Denote incompatibility of two elements of $\mathbb{P}^\kappa/\mathcal{D}$ with respect to $\leq$ by $\perp$. Using the generalization of the ccc to the κ-chain condition (or κ-cc - Definition 8.2.7) and the notion of $\lessdot$ from Definition 8.4.1, we find that whether $\mathbb{P}$ embeds completely into $\mathbb{P}^\kappa/\mathcal{D}$ or not can be characterized by a combinatorial property of $\mathbb{P}$ itself.

Lemma 5.2.4. *Let κ and $\mathbb{P}$ be as above. Then $\mathbb{P} \lessdot \mathbb{P}^\kappa/\mathcal{D}$ iff $\mathbb{P}$ has the κ-cc.*

Proof. For "$\Rightarrow$" prove the contrapositive. So assume that $\mathbb{P}$ is not κ-cc. In order to show that $\mathbb{P} \not\lessdot \mathbb{P}^\kappa/\mathcal{D}$, employ the equivalent characterization of $\lessdot$ from Lemma 8.4.2 ($\mathbb{P}$ embeds into $\mathbb{P}^\kappa/\mathcal{D}$ and the identity forms a complete embedding), and show that any maximal antichain in $\mathbb{P}$ is not maximal with respect to $\mathbb{P}^\kappa/\mathcal{D}$. Let $A := \{p_\alpha : \alpha < \lambda\}$, $\lambda \geq \kappa$, be a maximal antichain in $\mathbb{P}$. Define $f \in \mathbb{P}^\kappa$ by $f(\alpha) = p_\alpha, \forall \alpha \in \kappa$. By pairwise incompatibility of the elements of A, for any fixed $\alpha \in \lambda$ and any $g \in \mathbb{P}^\kappa$, the sets $\{\beta \in \kappa : g(\beta) \leq f(\beta)\}$ and $\{\beta \in \kappa : g(\beta) \leq p_\alpha\}$ cannot be both in $\mathcal{D}$. Thus $f \perp p_\alpha$ for all $\alpha < \lambda$. Thereby $A \cup \{f\} \supsetneq A$ forms an antichain in $\mathbb{P}^\kappa/\mathcal{D}$.

For "$\Leftarrow$", first note that for all $p, q \in \mathbb{P}$, $p \leq_\mathbb{P} q$ iff $p \leq q$ by (2) of the previous fact. This implies that $p \perp_\mathbb{P} q \Rightarrow p \perp q$ for any $p, q \in \mathbb{P}$. To see that maximality of antichains is preserved use that κ is measurable. Take any maximal antichain A of $\mathbb{P}$ with some index set $\mu < \kappa$, which exists by $\mathbb{P}$ having the κ-cc. Now assume towards a contradiction that A loses maximality in $\mathbb{P}^\kappa/\mathcal{D}$, as witnessed by some $[f] \in \mathbb{P}^\kappa/\mathcal{D}$. Then $\{\beta \in \kappa : f(\beta) \perp p_\alpha\} \in \mathcal{D}$ for any $\alpha \in \mu$, and so by $\mathcal{D}$ being κ-closed, $\bigcap_{\alpha\in\kappa}\{\beta \in \kappa : f(\beta) \perp p_\alpha\} \in \mathcal{D}$. Since there exists $\beta \in \kappa\backslash\mu$ with $f(\beta) \in \mathbb{P}$, $f(\beta) \neq p_\alpha$, and $f(\beta) \perp_\mathbb{P} p_\alpha$ for all $\alpha < \mu$, $f(\beta)$ witnesses that A is not maximal with respect to $\mathbb{P}$, a contradiction. $\square$

Next, let us establish that the ccc is preserved when taking the ultrapower $\mathbb{P}^\kappa/\mathcal{D}$. More generally:

Lemma 5.2.5. *If $\mathbb{P}$ has the λ-cc, then so does $\mathbb{P}^\kappa/\mathcal{D}$.*

Proof. To see that any maximal antichain of $\mathbb{P}^\kappa/\mathcal{D}$ has cardinality less than λ again employ measurability of κ. Assume towards a contradiction that $\{[f_\alpha] : \alpha < \lambda\}$ forms an antichain. Then for any $\alpha \neq \beta \in \lambda$, $\{\gamma \in \kappa : f_\alpha(\gamma) \perp f_\beta(\gamma)\} \in \mathcal{D}$, and so $\bigcap_{\alpha\neq\beta\in\lambda}\{\gamma \in \kappa : f_\alpha(\gamma) \perp f_\beta(\gamma)\} \in \mathcal{D}$. But this results in $\{f_\alpha(\gamma) : \gamma \in \kappa \wedge \alpha \in \lambda\}$ being an antichain in $\mathbb{P}$ of cardinality $\lambda \cdot \kappa$, a contradiction. $\square$

Now we are ready to introduce the average of $\mathbb{P}$-names.

Discussion 5.2.6 (How to take the average of $\mathbb{P}$-names)**.** *Assume that $\mathbb{P}$ is ccc. Taking as motivation the construction of a witness to non-maximality of a given independent or a.d. family, let us construct the average for any given $\dot{\mathcal{A}}$, which forms a $\mathbb{P}$-name for a family $\mathcal{A} \subseteq \omega^\omega$.*

- *Full description of $\dot{\mathcal{A}}$: Writing $\dot{\mathcal{A}} = \{\dot{f}^\alpha : \alpha < \kappa\}$, describe each $\dot{f}^\alpha \in \dot{\mathcal{A}}$ completely using maximal antichains $\{p^\alpha_{n,i} : i, n \in \omega\} \subseteq \mathbb{P}_\kappa$ and values $\{k^\alpha_{n,i} : i, n \in \omega\} \subseteq \omega$, where $p^\alpha_{n,i} \Vdash_\mathbb{P}$ "$\dot{f}(n) = k^\alpha_{n,i}$" for all $n \in \omega$ and $\alpha < \kappa$ (see Appendix - Section 8.2.4). Therefore, the set $\{p^\alpha_{n,i}, k^\alpha_{n,i} : i, n \in \omega, \alpha < \kappa\}$ serves to characterize $\dot{\mathcal{A}}$.*

- *Reduction steps: Form* $[p_{n,i}] := \langle p^\alpha_{n,i} : \alpha < \kappa\rangle/\mathcal{D} \in \mathbb{P}^\kappa/\mathcal{D}$. *Notice that by the pigeon-hole principle and* $\kappa > \aleph_0$ *it is possible to identify* $\omega^\kappa/\mathcal{D} = \omega$. *Then for each* $[k_{n,i}] = \langle k^\alpha_{n,i} : \alpha < \kappa\rangle/\mathcal{D}$, $[k_{n,i}] = k_{n,i}$. *As* $\mathcal{D}$ *is in particular* $\aleph_1$*-complete, we get the following*

  Claim. *For any* $n \in \omega$, $\{[p_{n,i}] : i \in \omega\}$ *forms a maximal antichain in* $\mathbb{P}^\kappa/\mathcal{D}$.

  Proof. *Fix* n *and take a maximal antichain* $\{p^\alpha_{n,i} : i \in \omega\}$ *in* $\mathbb{P}$. *Assume that* $\{[p_{n,i}] : i \in \omega\}$ *is not maximal in* $\mathbb{P}^\kappa/\mathcal{D}$, *as witnessed by some* $[f]$. *Thereby,* $\{\beta \in \kappa : f(\beta) \perp_{\mathbb{P}} p^\alpha_{n,i}\} \in \mathcal{D}$ *for every* $i \in \omega$. *Since* $\mathcal{D}$ *is* $\aleph_1$*-complete,* $\bigcap_{i\in\omega}\{\beta \in \kappa : f(\beta) \perp_P p^\alpha_{n,i}\} \in \mathcal{D}$. *Certainly we can find* $\beta \in \kappa \setminus \aleph_0$ *such that* $f(\beta) \in \mathbb{P}$, $f(\beta) \neq p^\alpha_{n,i}$, *and* $f(\beta) \perp_{\mathbb{P}} p^\alpha_{n,i}$ *for all* $i \in \omega$. *Therefore,* $\{p^\alpha_{n,i} : i \in \omega\} \cup \{f(\beta)\}$ *forms a maximal antichain in* $\mathbb{P}$, *a contradiction.* $\Box_{Claim}$

- *Construction of a witness: As mentioned in Remark 8.2.16, it is also possible to construct a name out of the* $\{[p_{n,i}] : i \in \omega\}$ *and the corresponding* $k_{n,i}$*-values: Define* $\dot{f}^\kappa = \langle \dot{f}^\alpha : \alpha < \kappa\rangle/\mathcal{D}$ *to be such* $\mathbb{P}^\kappa/\mathcal{D}$*-name, with*

$$[p_{n,i}] \Vdash_{\mathbb{P}^\kappa/\mathcal{D}} \text{"}\dot{f}^\kappa(n) = k_{n,i}\text{"} \ \forall i, n \in \omega.$$

  This $\dot{f}^\kappa$ *we call the average of the* $\dot{f}^\alpha$.

All of the above steps can be repeated for families of infinite subsets of ω, employing the description in Lemma 8.2.15(b). Here is how the method can be used for independent or almost disjoint families $\mathcal{A}$. Begin with the a.d. case:

Lemma 5.2.7. *Let* $\dot{\mathcal{A}}$ *be a* $\mathbb{P}$*-name for an almost disjoint family of cardinality* $\lambda \geq \kappa$, κ *a measurable cardinal. Then* $1_{\mathbb{P}^\kappa/\mathcal{D}} \Vdash_{\mathbb{P}^\kappa/\mathcal{D}}$ *"*$\dot{\mathcal{A}}$ *is not maximal".*

Proof. Given the name $\dot{\mathcal{A}} = \{\dot{a}^\alpha : \alpha < \lambda\}$, where each $\dot{a}^\alpha$ is a $\mathbb{P}$-name for a real $a \in [\omega]^\omega$, proceed by taking the average $\dot{a}^\kappa$ of the $\dot{a}^\alpha$, which is going to witness the non-maximality of $\dot{\mathcal{A}}$. For every $\alpha \in \lambda$ obtain a description of $\dot{a}^\alpha$ via $p^\alpha_{n,i} \in \mathbb{P}$ and $k^\alpha_{n,i} \in 2$ such that

- $p^\alpha_{n,i} \Vdash_{\mathbb{P}}$ "$n \in \dot{a}^\alpha$ iff $k^\alpha_{n,i} = 1$",
- $\{p^\alpha_{n,i} : i \in \omega\}$ is a maximal antichain for all $\alpha \in \kappa$, $n \in \omega$.

Along the lines of Discussion 5.2.6 proceed as follows: Again form the objects over $\mathcal{D}$, namely $[p_{n,i}] = \langle p^\alpha_{n,i}\rangle_{\alpha\in\kappa}/\mathcal{D} \in \mathbb{P}^\kappa/\mathcal{D}$ and $k_{n,i} = \langle k^\alpha_{n,i}\rangle_{\alpha\in\kappa}/\mathcal{D} \in 2$, where for any $n \in \omega$,

$\{[p_{n,i}] : i \in \omega\}$ is a maximal antichain. This data characterizes completely a corresponding $\mathbb{P}^\kappa/\mathcal{D}$-name $\dot{a}^\kappa = \langle \dot{a}^\alpha \rangle_{\alpha\in\kappa}/\mathcal{D}$, the wanted average over the $\dot{a}^\alpha$ $(\alpha < \kappa)$, where

$$\forall i, n \in \omega \ ([p_{n,i}] \Vdash_{\mathbb{P}^\kappa/\mathcal{D}} \text{"} n \in \dot{a}^\kappa \text{ iff } k_{n,i} = 1\text{"}).$$

Claim 5.2.8. $1_{\mathbb{P}^\kappa/\mathcal{D}} \Vdash_{\mathbb{P}^\kappa/\mathcal{D}}$ "$\forall \beta < \lambda \ (|\dot{a}^\kappa \cap \dot{a}^\beta| < \infty)$".

Proof. For $\beta \in \lambda$ fixed, by $\dot{\mathcal{A}}$ naming an a.d. family, for almost all $\alpha \in \kappa$ (all except perhaps $\alpha = \beta$, if $\beta < \kappa$), there exist values $n_i^\alpha \in \omega$ and conditions $q_i^\alpha \in \mathbb{P}$ such that $\{q_i^\alpha : i \in \omega\}$ forms a maximal antichain and $q_i^\alpha \Vdash_{\mathbb{P}}$ "$\dot{a}^\alpha \cap \dot{a}^\beta \subseteq n_i^\alpha$". As before, proceed by considering $[q_i] = \langle q_i^\alpha : \alpha < \kappa \rangle/\mathcal{D} \in \mathbb{P}^\kappa/\mathcal{D}$ and $[n_i] = \langle n_i^\alpha : \alpha < \kappa \rangle/\mathcal{D}$. Again $[n_i] = n_i$, and furthermore, by assumption on the q_i^α, $\{[q_i] : i \in \omega\}$ is a maximal antichain. To prove the claim, it suffices to show that $[q_i] \Vdash_{\mathbb{P}^\kappa/\mathcal{D}}$ "$\dot{a}^\beta \cap \dot{a}^\kappa \subseteq n_i$" for all $i \in \omega$. Assume towards a contradiction that this is not the case. Then there must be $i, m \in \omega$, $m \geq n_i$, and $[r] = \langle r^\alpha \rangle_{\alpha\in\kappa}/\mathcal{D} \in \mathbb{P}^\kappa/\mathcal{D}$ so that $[r] \leq [q_i]$, and $[r] \Vdash_{\mathbb{P}^\kappa/\mathcal{D}}$ "$m \in \dot{a}^\beta \cap \dot{a}^\kappa$". Let $j \in \omega$ be such that $[r] \leq [p_{m,j}]$ and $k_{m,j} = 1$. Combining this information, find that the set of all $\alpha \in \kappa$ such that $r^\alpha \leq q_i^\alpha, r^\alpha \leq p_{m,j}^\alpha, k_{m,j}^\alpha = 1, n_i^\alpha = n_i$ and $r^\alpha \Vdash_{\mathbb{P}}$ "$m \in \dot{a}^\beta$" is big with respect to $\mathcal{D}$:

$$S := \{\alpha \in \kappa : r^\alpha \leq q_i^\alpha \wedge r^\alpha \leq p_{m,j}^\alpha \wedge k_{m,j}^\alpha = 1 \wedge n_i^\alpha = n_i \wedge r^\alpha \Vdash_{\mathbb{P}} \text{"} m \in \dot{a}^\beta \text{"}\} \in \mathcal{D}.$$

Take any $\alpha \in S$. Then $r^\alpha \leq p_{m,j}^\alpha$ together with $k_{m,j}^\alpha = 1$ imply that $r^\alpha \Vdash_{\mathbb{P}}$ "$m \in \dot{a}^\alpha \cap \dot{a}^\beta$". On the other hand, we know that $q_i^\alpha \Vdash_{\mathbb{P}}$ "$\dot{a}^\alpha \cap \dot{a}^\beta \subseteq n_i^\alpha$" and $n_i^\alpha \leq m$. Since $r^\alpha \leq_{\mathbb{P}} q_i^\alpha$, $r^\alpha \Vdash_{\mathbb{P}}$ "$m \notin \dot{a}^\alpha$", a contradiction. $\square_{Claim}$

$\square$

In the case of independence proceed similarly:

Lemma 5.2.9. *Let $\dot{\mathcal{A}}$ be a $\mathbb{P}$-name for an independent family of cardinality $\lambda \geq \kappa$, κ measurable. Then $1_{\mathbb{P}^\kappa/\mathcal{D}} \Vdash_{\mathbb{P}^\kappa/\mathcal{D}}$ "$\dot{\mathcal{A}}$ is not maximal".*

Proof. Without loss assume that $1_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "$\dot{BC}(\mathcal{A}) = \{\dot{a}^\alpha : \alpha < \lambda\}$", where $BC(\mathcal{A})$ denotes the set of all Boolean combinations of $\mathcal{A}$, as introduced in Section 1.2.1. This is possible due to $|BC| = |\mathcal{A}|$ and every $a^\alpha \in BC(\mathcal{A})$ being a real. So each $\dot{a}^\alpha$ is a $\mathbb{P}$-name for a Boolean combination with respect to the independent family $\mathcal{A}$. As in the previous lemma, take the average over all the $\dot{a}^\alpha$, likewise introducing $p_{n,i}^\alpha$ and $k_{n,i}^\alpha$. Thereby, obtain $\dot{a}^\kappa := \langle \dot{a}^\alpha : \alpha < \kappa \rangle/\mathcal{D}$. In analogy to Claim 5.2.8 we wish to show that

$$1_{\mathbb{P}^\kappa/\mathcal{D}} \Vdash_{\mathbb{P}^\kappa/\mathcal{D}} \text{"}\forall \beta < \lambda \ (|\dot{a}^\kappa \cap \dot{a}^\beta| = \aleph_0)\text{"},$$

that is, we want $|\dot{a}^\kappa \cap \dot{a}^\beta| > n$ for any given $n \in \omega$. For $\beta \in \lambda$ fixed, since $\dot{\mathcal{A}}$ names an independent family, for all $\alpha < \kappa$ there exist values $n_i^\alpha \in \omega$ and conditions $q_i^\alpha \in \mathbb{P}$ such that $\{q_i^\alpha : i \in \omega\}$ forms a maximal antichain and

$$q_i^\alpha \Vdash_{\mathbb{P}} \text{"}\exists m > n_i^\alpha (m \in \dot{a}^\alpha \cap \dot{a}^\beta)\text{"}. \tag{5.1}$$

As before, proceed by considering $[q_i] = \langle q_i^\alpha \rangle_{\alpha \in \kappa}/\mathcal{D} \in \mathbb{P}^\kappa/\mathcal{D}$ and $n_i = [n_i] = \langle n_i^\alpha : \alpha < \kappa \rangle/\mathcal{D} \in \omega$, where $\{[q_i] : i \in \omega\}$ forms a maximal antichain. It suffices to prove the following claim:

Claim. $[q_i] \Vdash_{\mathbb{P}^\kappa/\mathcal{D}} \text{"}\exists m > n_i \ (m \in \dot{a}^\kappa \cap \dot{a}^\beta)\text{"}$.

Proof. Assume otherwise, that is, there exist $i \in \omega$, $[r] = \langle r^\alpha : \alpha < \kappa \rangle/\mathcal{D} \in \mathbb{P}^\kappa/\mathcal{D}$ with $[r] \leq [q_i]$ such that

$$\forall l > n_i : \ [r] \Vdash_{\mathbb{P}^\kappa/\mathcal{D}} \text{"}l \notin \dot{a}^\beta \cap \dot{a}^\kappa\text{"}.$$

Without loss of generality take $j \in \omega$ so that $[r] \leq [p_{l,j}]$ and $k_{l,j} = 0$. As before, for $l > n_i$ arbitrary but fixed, it follows that

$$S := \{\alpha \in \kappa : r^\alpha \leq q_i^\alpha \wedge r^\alpha \leq p_{l,j}^\alpha \wedge k_{l,j}^\alpha = 0 \wedge n_i^\alpha = n_i \wedge r^\alpha \Vdash_{\mathbb{P}} \text{"}l \notin \dot{a}^\beta\text{"}\} \in \mathcal{D}.$$

Let $\alpha \in S$. Then $r^\alpha \Vdash_{\mathbb{P}} \text{"}l \notin \dot{a}^\alpha \cup \dot{a}^\beta\text{"}$, in particular, $r^\alpha \Vdash_{\mathbb{P}} \text{"}l \notin \dot{a}^\alpha \cap \dot{a}^\beta\text{"}$. As $l > n_i$ is chosen to be arbitrary, this holds for any $l > n_i = n_i^\alpha$, or equivalently:

$$r^\alpha \Vdash_{\mathbb{P}} \text{"}\forall l > n_i \ (l \notin \dot{a}^\alpha \cap \dot{a}^\beta)\text{"}.$$

But $q_i^\alpha \Vdash_{\mathbb{P}} \text{"}\exists m > n_i^\alpha (m \in \dot{a}^\alpha \cap \dot{a}^\beta)\text{"}$ by equation (5.1). So since $r^\alpha \leq_{\mathbb{P}} q_i^\alpha$, we obtain the contradiction that $r^\alpha \Vdash_{\mathbb{P}} \text{"}\exists m > n_i \ (m \in \dot{a}^\alpha \cap \dot{a}^\beta \wedge m \notin \dot{a}^\alpha \cap \dot{a}^\beta)\text{"}$. $\square_{Claim}$

$\square$

5.2.2 Dominating forcing

As another preparation for the proof of Theorem 5.2.1, let us introduce the dominating forcing notion (also known as Hechler forcing), which is defined as follows.

Definition 5.2.10 (Dominating forcing)**.** Let $\mathbb{D}$ be the forcing poset given by ...

... conditions $(s, f) \in \omega^{<\omega} \times \omega^\omega$,

... an extension relation defined via $(s, f) \leq_{\mathbb{D}} (t, g)$ iff $s \supseteq t$, $f(n) \geq g(n)$ $\forall n \in \omega$, and $\forall k \in dom(s) \setminus dom(t)$ $(s(k) > g(k))$.

The following proposition gathers all of the important properties we require from $\mathbb{D}$.

Proposition 5.2.11. *Let $\mathbb{D}$ be the dominating forcing poset. Then*

1. *$\mathbb{D}$ is ccc.*
2. *$\mathbb{D}$ adds a dominating real d over the ground model, that is, for $G \subseteq \mathbb{D}$ a (V, G)-generic filter, $V[G] \models$ "$d <^* f$ for any ground model function $f \in \omega^\omega$".*

Proof. Begin with (1): This essentially follows from the definition of $(\mathbb{D}, \leq)$, since for any two conditions $(s, f), (t, g) \in \mathbb{D}$, $s = t$ implies that the conditions must be compatible. Define an extension $(u, h) \in \mathbb{D}$ by setting $u = s$ and h such that $h(n) = max(f(n), g(n))$ for every $n \in \omega$; then the last part of the definition of $\leq$ holds vacuously. So take any antichain $A \subseteq \mathbb{D}$. If we want to count all possible incompatible elements, the second coordinates of the elements of $\mathbb{D}$ essentially "do not count". More precisely, for any $(s, f) \neq (t, g) \in A$, since $(s, f) \perp (t, g)$, we must have $s \neq t$. But there are only $|\omega^{<\omega}| = \omega$ many different possibilities for such elements, so A must also be countable.

Part (2): Let $d = f_G = \bigcup\{s : \exists f((s, f) \in G)\}$. Let us begin with the following

Claim. *f_G is a $\mathbb{D}$-generic real.*

Proof. We need to show that $(f_G \in \omega^\omega)^{V[G]}$. To see this, define for each $n \in \omega$,

$$D_n = \{(s, f) \in \mathbb{D} : n \in dom(s)\}.$$

Fix $n \in \omega$. We show that D_n is dense in $\mathbb{D}$. Pick any $(s, f) \in \mathbb{D}$ with $n \notin dom(s)$. Define $t \in \omega^{<\omega}$ by $t = f$ on $dom(s)$ and set

$$dom(t) = dom(s) \cup \{(k, f(k)) : |dom(s)| + 1 \leq k \leq n\}.$$

Thus $t \in \omega^{<\omega}$, $t \supseteq s$, and $t(k) = f(k)\ \forall k \in dom(t) \backslash dom(s)$. Letting $g := f$, $(t, g) \leq (s, f)$, and since $n \in dom(s)$, $(t, g) \in D_n$.

To see that f_G is defined on all of ω note that $f_G : \omega \to \omega$ is a partial function by downwards-directedness of G. But since G is generic, it furthermore intersects every D_n for $n \in \omega$, meaning that every $n \in \omega$ must be in $dom(f_G)$ by definition of f_G. $\square_{Claim}$

Next, we want to make sure that, working in $V[G]$, f_G eventually dominates any given $f \in \omega^\omega \cap V$. To this end, for any $f \in \omega^\omega \cap V$ define the sets

$$D_f = \{(t, g) \in \mathbb{D} : \forall n \in \omega(g(n) > f(n))\}.$$

Claim. *For all $f \in \omega^\omega \cap V$, D_f is dense in $\mathbb{D}$.*

Proof. Fix $f \in \omega^\omega \cap V$ and let $(u, h) \in D$ be arbitrary such that $\exists n \in \omega(h(n) \leq f(n))$. We define $(t, g) \in D$ as follows: Let $g \in \omega^\omega$ be given by $g := \max\{f, h\} + 1$, and $t := u$. Then for all $n \in \omega$ it follows that $g(n) = max(f(n), h(n)) + 1 > f(n)$. This and the definition of t guarantee that $(t, g) \in D_f$ and $(t, g) \leq (u, h)$. $\square_{Claim}$

Why does f_G eventually dominate any $f \in \omega^\omega \cap V$? For $f \in \omega^\omega \cap V$ arbitrary, by genericity of G, $D_f \cap G \neq \emptyset$, so there exists some $(t, g) \in D_f \cap G$ such that $\forall n \in \omega(g(n) > f(n))$. Pick an arbitrary $(u, h) \in G$. By downwards-directedness of G there exists some $(v, k) \leq (u, h), (t, g)$. Without loss of generality let $(v, k) = (u, h)$, i.e., $(u, h) \leq (t, g)$ (we could just keep on using (v, k) and do all the following steps with (v, k) instead of (u, h)). Then in particular $u \supseteq t$, $h \geq g$ and for any $k \in dom(u) \setminus dom(t)$, $u(k) \geq g(k) > f(k)$. So $f_G(k) > f(k)$ for each $k \in dom(u) \setminus dom(t)$. Now as things are we have no guarantee that $f_G > f$ on $dom(t)$, but this poses no problem because we are only interested in $<^*$ and $dom(t)$ is finite. On the other hand, to make sure that f_G dominates f above $dom(t)$, collect every u with $(u, h) \in G$ into one function, which per definition is f_G, noting that for any $(u, h) \in G$ we can always pick a better approximation (u', h), where $u' \supseteq u$, and for any $n \in dom(u') \setminus dom(u)$, $u'(n) := h(n) + 1$. Since $h(n) \geq g(n) > f(n)$, such finite approximations suffice. Finally in $V[G]$, pick $m := |dom(t)| + 1$. Therefore, for any $n \geq m$, $f_G(n) > h(n) > f(n)$. $\square$

Remark 5.2.12 (Dominating reals are diagonalizing). As with independence and almost disjointness one can show that if $\mathcal{A}$ is a dominating family in the ground model V, forcing with $\mathbb{D}$ adds a dominating real d witnessing that $\mathcal{A}$ is not dominating in the generic extension of V, whereas $\mathcal{A} \cup \{d\}$ forms a dominating family; in short, d diagonalizes any given ground model dominating family. In this case we do not need a maximality clause by definition of the dominating number $\mathfrak{d}$. Similarly, for λ a regular cardinal and $\mathbb{P}$ a finite support iteration of length λ, we can use $\mathbb{D}$ as above to add a dominating family $\mathcal{A}$ of cardinality λ. In this case additionally, $\mathfrak{d} = \lambda$ because any element in $\mathcal{A}$ witnesses that any family of reals with cardinality $< \lambda$ cannot be dominating.

5.2.3 Proof of the theorem

Let us restate Theorem 5.2.1 and repeat the motivating steps mentioned at the beginning, which will make up the proof:

Theorem. *Assume GCH. Let $\kappa_1 < \kappa_2 < \cdots < \kappa_n$, $n \in \omega$, be measurable cardinals. Then consistently $Spec(\mathfrak{i}) = Spec(\mathfrak{a}) = \{\kappa_i\}_{i=1}^n$.*

Remark. We will show that:

I. $\ldots \{\kappa_i\}_{i=1}^n \subseteq Spec(\mathfrak{i}) \cap Spec(\mathfrak{a})$,

II. ...there are no cardinals in between the given ones and there are none above $\max_{i=1,\ldots,n} \kappa_i = \kappa_n$ in $Spec(\mathfrak{i}) \cap Spec(\mathfrak{a})$,

III. ...no cardinal below $\min_{i=1,\ldots,n} \kappa_i = \kappa_1$ is in $Spec(\mathfrak{i}) \cap Spec(\mathfrak{a})$.

To take care of (I), (II) and (III), we are going to proceed similarly as in the proof of Lemma 4.1.4 and related statements from the previous sections. The main difference is that in each of the n "branches" we will now have to distinguish more cases. In order to do so, introduce the sets

$$Even = \{\alpha : \alpha = \beta + 2k \text{ for some limit ordinal } \beta \text{ and } k \in \omega\},$$
$$Odd = \{\alpha : \alpha = \beta + 2k + 1 \text{ for some limit ordinal } \beta \text{ and } k \in \omega\}.$$

Proof. Let $\gamma^* = \kappa_n \cdot \kappa_{n-1} \cdot \ldots \cdot \kappa_1$. Each κ_i is measurable and comes with an associated κ_i-complete non-principal ultrafilter $\mathcal{D}_i$. As before, for every $j \in \{1, 2, \ldots, n\}$ take $I_j \subseteq \gamma^*$ unbounded, such that $I_j \cap I_k = \emptyset$, $j \neq k$. As we will have to distinguish between more cases this time, impose the additional requirements that

- each I_j consists of successor cardinals only,
- $\gamma^* = sup_j\{\gamma \in I_j : \gamma \in Even\} = sup_j\{\gamma \in I_j : \gamma \in Odd\}$,
- for every $k \in \{1, 2, \ldots, n\}$, $I_k \cap Even$ is split into two disjoint unbounded subsets $I^{\mathfrak{i}}_{k,even}$ and $I^{\mathfrak{a}}_{k,even}$,
- if $k = 1$, split $I_k \cap Odd$ into three disjoint unbounded subsets $I^{\mathfrak{i}}_{1,odd}$, $I^{\mathfrak{a}}_{1,odd}$ and $I^{\mathfrak{d}}_{1,odd}$, where in particular $|I^{\mathfrak{d}}_{1,odd}| = \kappa_1$; for every $k \in \{2, 3, \ldots, n\}$, $I_k \cap Odd$ is split into two disjoint unbounded subsets $I^{\mathfrak{i}}_{k,odd}$ and $I^{\mathfrak{a}}_{k,odd}$.

Proceed by recursively defining the ccc finite support iteration $\mathbb{P}_{\gamma^*} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \gamma^* \rangle$: Fix $\alpha \in \gamma^*$ and assume that for all $k \in \{1, 2, \ldots, n\}$ we have already defined ...

... sequences $\langle r^k_\gamma : \gamma \in I^{\mathfrak{i}}_{k,even}, \gamma < \alpha \rangle$ of Mathias generic reals as in the independence case (e.g. Lemma 4.1.1), such that $J^{\mathfrak{i}}_{\alpha,k} := \{r^k_\gamma : \gamma \in I^{\mathfrak{i}}_{k,even} \cap \alpha\}$ forms an independent family, and for all $\gamma \in I^{\mathfrak{i}}_{k,even} \cap \alpha$, r^k_γ diagonalizes $J^{\mathfrak{i}}_{\gamma,k}$.

... sequences of Mathias generic reals as in the a.d. case (e.g. Lemma 4.2.1), such that $J^{\mathfrak{a}}_{\alpha,k} := \{r^k_\gamma : \gamma \in I^{\mathfrak{a}}_{k,even} \cap \alpha\}$ forms an almost disjoint family, and for all $\gamma \in I^{\mathfrak{a}}_{k,even} \cap \alpha$, r^k_γ diagonalizes $J^{\mathfrak{a}}_{\gamma,k}$.

... and furthermore: for $k = 1$, sequences $\langle d_\gamma : \gamma \in I^{\mathfrak{d}}_{1,odd}, \gamma < \alpha\rangle$ of dominating reals as in Proposition 5.2.11 such that $J^{\mathfrak{d}}_{\alpha,1} := \{d_\gamma : \gamma \in I^{\mathfrak{d}}_{1,odd} \cap \alpha\}$ forms a dominating family, and for all $\gamma \in I^{\mathfrak{d}}_{1,odd} \cap \alpha$, d_γ diagonalizes $J^{\mathfrak{d}}_{\gamma,1}$.

For the next step we have the following case distinction:

(a_1) If there is $k \in \{1, 2, \ldots, n\}$ with $\alpha \in I^{\mathfrak{i}}_{k,even}$, let $\mathcal{U}^{\mathfrak{i}}_\alpha$ be a $J^{\mathfrak{i}}_{\alpha,k}$-diagonalization filter in $V^{\mathbb{P}_\alpha}$, $\dot{\mathbb{Q}}_\alpha$ a $\mathbb{P}_\alpha$-name for the Mathias forcing poset $\mathbb{M}(\mathcal{U}^{\mathfrak{i}}_\alpha)$, and r^k_α the associated Mathias generic real (all with respect to independent families).

(a_2) If there is $k \in \{1, 2, \ldots, n\}$ with $\alpha \in I^{\mathfrak{a}}_{k,even}$, let $\mathcal{F}_{J^{\mathfrak{a}}_{\alpha,k}}$ be an induced filter in $V^{\mathbb{P}_\alpha}$, $\dot{\mathbb{Q}}_\alpha$ a $\mathbb{P}_\alpha$-name for $\mathbb{M}(\mathcal{F}_{J^{\mathfrak{a}}_{\alpha,k}})$, and r^k_α the associated Mathias generic real (all with respect to almost disjoint families).

(b_1) If there is $k \in \{2, 3, \ldots, n\}$ with $\alpha \in I^{\mathfrak{i}}_{k,odd}$, then there exists β such that $\alpha = \beta + 1$. Thereby, take $\dot{\mathbb{Q}}_\alpha$ to be a $\mathbb{P}_\alpha$-name for the quotient poset $\mathbb{R}_\alpha$ such that $\mathbb{P}^{\kappa_k^+}/\mathcal{D}_k = \mathbb{P}_\beta * \mathbb{R}_\alpha$. Here κ_k^+ denotes the cardinal successor of κ_k.

(b_2) If there is $k \in \{2, 3, \ldots, n\}$ with $\alpha \in I^{\mathfrak{a}}_{k,odd}$, then there exists β such that $\alpha = \beta + 1$. Take $\dot{\mathbb{Q}}_\alpha$ to name $\mathbb{R}_\alpha$ as in (b_1).

(b_3) For $k = 1$, if $\alpha \in I^{\mathfrak{a}}_{k,odd} \cup I^{\mathfrak{i}}_{k,odd}$, proceed as in (b_1) and (b_2), respectively. If on the other hand $\alpha \in I^{\mathfrak{d}}_{k,odd}$, let $\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{D}}$, where $\dot{\mathbb{D}}$ denotes a $\mathbb{P}_\alpha$-name for the dominating forcing notion as given in Definition 5.2.10.

(c) If $\alpha \notin \bigcup_{k=1}^n I_k$, let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for Cohen forcing.

Note that (a_1) and (a_2) together guarantee $\{\kappa_1, \kappa_2, \ldots, \kappa_n\} \subseteq Spec(\mathfrak{i}) \cap Spec(\mathfrak{a})$. Yet we need to include the cases (b_1) and (b_2) in order to make sure that all cardinals in between each κ_i and κ_{i+1} are excluded from the spectra. More concretely, (b_1) and in part (b_3) utilize Lemma 5.2.9 so that for each $k = 1, 2, \ldots, n$, maximal independent families with cardinality above κ_k do not exist. This works again as in the previous results because of the unboundedness assumptions on the sets $I^{\mathfrak{i}}_{k,odd}$. In the same fashion, (b_2) and (b_3) take care of the a.d. case using Lemma 5.2.7. Finally, (c) acts as a placeholder for the finite support iteration to be well-defined.

It remains to show that:

Claim. *The finite support iteration as defined above does not add any cardinals below κ_1 to the spectra.*

Proof. Proceed in two parts:

(i) $V^{\mathbb{P}_{\gamma^*}} \models$ "$\mathfrak{d} = \kappa_1$".

(ii) $V^{\mathbb{P}_{\gamma^*}} \models$ "$\mathfrak{d} = \mathfrak{b}$".

(i): By clause (b_3), Proposition 5.2.11, and Remark 5.2.12, as $I^{\mathfrak{d}}_{k,odd} \subseteq \gamma^*$ is unbounded with $|I^{\mathfrak{d}}_{k,odd}| = \kappa_1$, obtain a dominating family $\mathcal{A} = \{d_\gamma : \gamma \in I^{\mathfrak{d}}_{k,odd}\}$ with $|\mathcal{A}| = \kappa_1$. Since $I^{\mathfrak{d}}_{k,odd}$ is in particular unbounded in γ^*, $\mathcal{A}$ is dominating with respect to all reals in the final model $V^{\mathbb{P}_{\gamma^*}} \cap \omega^\omega$. Now by construction, $cf(\gamma^*) = \kappa_1$. Thus $V^{\mathbb{P}_{\gamma^*}} \models$ "$\mathfrak{d} = \kappa_1$".

(ii): It is enough to prove that any family $\mathcal{B} \subseteq [\omega]^\omega$ with $\delta := |\mathcal{B}| < \kappa_1$ cannot be unbounded. Indeed, in this case $\mathfrak{b} \geq \kappa_1 = \mathfrak{d}$, and together with the *ZFC* result $\mathfrak{d} \leq \mathfrak{b}$ (Remark 1.2.1) the statement follows. Hence let $\mathcal{B}$ be as above. We can basically proceed as in the proof of Theorem 4.3.14. Without loss assume that δ is a limit ordinal. Since the finite support iteration adds a Cohen real $c_\delta \in [\omega]^\omega \cap V^{\mathbb{P}_\delta}$, c_δ splits each $x \in \mathcal{B}$. $\square_{Claim}$

Finally, from Chapter 1 we know that $\mathfrak{b} \leq \mathfrak{a}$ and $\mathfrak{d} \leq \mathfrak{i}$ (Theorems 1.1.3 and 1.2.6). So in total, $\kappa_1 \leq \mathfrak{a}, \mathfrak{i}$, which concludes the proof.

$\square$

6 Negative results aka. excluding cardinals from the spectra

The final positive results from Chapter 4 can be summarized as follows:

For any set of uncountable cardinals C, $Con(C \subseteq Spec(\mathfrak{i}))$ and $Con(C \subseteq Spec(\mathfrak{a}))$.

With the aim to achieve what has been accomplished in the interlude (Chapter 5), namely equality concerning the spectra, we wish to prove the opposite inclusions. Yet here we work with any uncountable cardinals, not measurable ones in particular, so essential tools we had at our disposal to construct a witness for non-maximality, namely κ-complete non-principal ultrafilters, have to be replaced in some way. To this end, Section 6.1 will, with the help of an example, introduce the so-called isomorphism of names method and, extracting the key steps of the procedure (Discussion 6.1.2), pave the way towards the tree context from Section 4.3.2. The latter is taken up in Section 6.2. Specifically, 6.2.1 serves to refine and add to previous notations and definitions introduced for pre-spectra (4.3.3). Following this, isomorphism of names arguments are given for the independence (6.2.2) and a.d. case (6.2.3).

The example in the upcoming section is taken from [3], whereas 6.2.1 and 6.2.2 are based on [8].

6.1 Isomorphism of names arguments

There are many similarities between taking the average of $\mathbb{P}$-names (5.2.1) and what is to be discussed in the following. Both methods contain combinatorial arguments to allow a reduction of given information, which keeps intact essential properties associated to each element of the given family of names. This is exactly what happens when taking an average over a base set. Yet in a sense, the measurability assumption allows for this to happen in a very direct but somewhat crude way: It simply states that a reduction preserving the wished-for properties exists. Naturally, one way to make precise how similar mathematical objects are, is by employing morphisms. As we will see, an interplay of such algebraic objects will allow us to reproduce the previous results.

6.1.1 An introductory example

We begin with an example in the a.d. context which makes use of the method in a relatively transparent way.

Proposition 6.1.1. *Assume CH in the ground model $\mathcal{M}$, and let $\mathfrak{c} = \lambda$ hold in the generic extension $\mathcal{M}[G]$, where $\lambda \geq \aleph_2$ and G is an $(\mathcal{M}, \mathbb{C}_\lambda)$-generic filter (that is, using Cohen forcing $\mathbb{C}_\lambda$ force the continuum to be λ, which requires the additional assumption that $\lambda^\omega = \lambda$). Then in $\mathcal{M}[G]$, every mad family has size equal to either $\aleph_1$ or λ.*

Proof. For the existence of a mad family of cardinality $\aleph_1$ see Appendix - Prop. 8.3.5.

We are planning to show that any almost disjoint family of cardinality κ with $\aleph_1 < \kappa < \lambda$ cannot be maximal. Knowing that $\aleph_1 \leq \mathfrak{a} \leq \mathfrak{c}$ holds in *ZFC* (see Chapter 1), this implies the statement of the proposition.

Part 1: First of all take any family of infinite subsets of ω, say $\mathcal{A} = \{a^\alpha : \alpha < \kappa\}$. Our aim in this part is to describe the elements of $\mathcal{A}$ completely using countably many maximal antichains with respect to $\mathbb{C}_\lambda$, and associated values in $\{0, 1\}$.

Claim. *For each $\alpha < \kappa$ and $n \in \omega$ there exists a maximal antichain $A^\alpha_n = \{p^\alpha_{n,i} : i \in \omega\} \subseteq \mathbb{C}_\lambda$ and a set $\{k^\alpha_{n,i} : i \in \omega\} \subseteq \{0, 1\}$ so that every $p^\alpha_{n,i}$ decides whether n is in a^α or not, i.e., $p^\alpha_{n,i} \Vdash$ "$\check{n} \in \dot{a}^\alpha$ iff $k^\alpha_{n,i} = 1$".*

Proof. See Section 8.2.4 of the appendix. For convenience, we restate the functions f^α_n characterizing the A^α_n to obtain the $k^\alpha_{n,i}$: Define f^α_n via $p^\alpha_{n,i} \mapsto 1$ iff $p^\alpha_{n,i} \Vdash$ "$\check{n} \in \dot{a}^\alpha$". Then set $k^\alpha_{n,i} := f^\alpha_n(p^\alpha_{n,i})$. $\square_{Claim}$

Part 2: With the ultimate aim to construct a witness for the non-maximality of $\mathcal{A}$, in this part we wish to extract enough information from the $p^\alpha_{n,i}$, $k^\alpha_{n,i}$ to allow such construction. This will happen in a couple of reduction steps. Let us first of all collect all of the information which one may use to characterize any fixed a^α. To do so, define sets B^α as follows: For all $\alpha < \kappa$, let

$$B^\alpha := \bigcup \{dom(p^\alpha_{n,i}) : i, n \in \omega\}.$$

Clearly, each B^α is countable, and so $|\bigcup_{\alpha<\kappa} B^\alpha| \leq \kappa < \lambda$. Using the generalized version of the Δ-system Lemma (Theorem 8.2.14 applied to $\{B^\alpha : \alpha < \omega_2 \cdot 2\}$), there exists a family $\mathcal{B} = \{B^\alpha : \alpha < \omega_2\}$ (where we reindexed the sets B^α to enumerate them from 0 onwards) forming a Δ-system with root $R \in [\lambda \times \omega]^{\leq \omega}$. Note that it suffices to use bounds of cardinality $\aleph_2$ because this is the smallest one of the cardinals to be removed, and for any $\kappa > \aleph_2$ the following can be repeated. Also, the reason we are considering $\omega_2 \cdot 2$ is "space-allocation", which is a more exact way to employ the Δ-system Lemma, the latter being stated in terms of $\aleph_2 = |\omega_2| = |\omega_2 \cdot 2|$.

The upcoming reduction step is crucial to the argument in that we are about to construct the wanted isomorphism of names using the Δ-system $\mathcal{B}$: As each of the elements in $\mathcal{B}$ is countably infinite, for any $\alpha, \beta < \omega_2$ there exists a bijection $\phi := \phi_{\alpha,\beta} : B^\alpha \to B^\beta$. Not only can we choose ϕ so that the root R (i.e., the common domain) gets mapped to itself, we can even make sure that $\phi \restriction R = id_R$. Indeed, there are $\aleph_0^{\aleph_0}$ ways to permute ω. Out of those ways pick one which leaves R as it is, while bijectively assigning the rest of the elements to one another. The latter is possible due to all B^α having equal cardinality, implying the same for the $B^\alpha \backslash R$.

Next, for any $\gamma < \omega_2$ consider $\mathbb{C}_{B^\gamma} := \{p \in \mathbb{C}_\lambda : dom(p) \subseteq B^\gamma\}$, the restriction of $\mathbb{C}_\lambda$ to B^γ. Fix $\alpha, \beta < \omega_2$. Then:

Claim.

1. *ϕ induces an order-isomorphism $\psi := \psi_{\alpha,\beta} : \mathbb{C}_{B^\alpha} \to \mathbb{C}_{B^\beta}$.*

2. *Furthermore, we can identify the corresponding $\mathbb{C}_\lambda$-names $\dot{a}^\alpha$ and $\dot{a}^\beta$.*

Proof. Fix any $p \in \mathbb{C}_{B^\alpha}$. For (1), a natural way to define ψ is to let $\psi(p) \in \mathbb{C}_{B^\beta}$ such that $dom(\psi(p)) := \phi(dom(p))$, and

$$\psi(p)(x) := \begin{cases} p(x), & \text{if } x \in \phi(dom(p)) \cap R, \\ p(\phi^{-1}(x)), & \text{if } x \in \phi(dom(p)) \backslash R. \end{cases}$$

The reason we can define ψ in this way is that we ensured beforehand for any function in $\mathbb{C}_{B^\alpha}$ to have common domain with any function in $\mathbb{C}_{B^\beta}$. To see that ψ is surjective, fix $q \in \mathbb{C}_{B^\beta}$. Note that then $\phi^{-1}(dom(q)) \subseteq B^\alpha$ defines a function $p \in \mathbb{C}_{B^\alpha}$ with $dom(p) := \phi^{-1}(dom(q))$, where $p(x) := q(\phi(x))$ for all $x \in dom(p)$. Now $dom(\psi(p)) = \phi(dom(p)) = dom(q)$. It remains to show that for all $x \in dom(q)$, $\psi(p)(x) = q(x)$. Indeed, whenever $x \in dom(q) \cap R$, $\psi(p)(x) = p(x) = q(x)$, as $\phi(x) = x$. If on the other hand $x \in dom(q) \setminus R = \phi(dom(p)) \setminus R$, then $\psi(p)(x) = p(\phi^{-1}(x)) = q(x)$. The fact that ψ is an injection can be seen directly from the definition. Finally, to see that ψ is order-preserving, let $p_2 \supseteq p_1$ for arbitrary $p_1, p_2 \in \mathbb{C}_{B^\alpha}$, and show $\psi(p_1) \subseteq \psi(p_2)$.

In case that $x \in \phi(dom(p_1)) \cap R$, we are done. So say $x \in \phi(dom(p_1)) \setminus R$. Hence $\psi(p_1)(x) = p_1(\phi^{-1}(x)) \subseteq p_2(\phi^{-1}(x))$, where $\phi^{-1}(x) \in dom(p_1)$ by $x \in \phi(dom(p_1))$.

For (2), we can take the disjoint union $\bigcup_{n\in\omega} A_n^\alpha$ to define a function $f^\alpha : \bigcup_{n\in\omega} A_n^\alpha \to 2$. By construction, f^α uniquely determines $\dot{a}^\alpha$. As $|\bigcup_{n\in\omega} A_n^\alpha| = \aleph_0$, there are at most $2^{\aleph_0} = \aleph_1$ (via CH) of such functions, and thus, considering the equivalence relation

$$\dot{a}^\alpha \sim \dot{a}^\beta \text{ iff } f^\alpha = f^\beta,$$

there can be at most $\aleph_1$-many different equivalence classes for names. There are at least $\aleph_2$-many names to choose from, so without loss of generality, we can identify $\dot{a}^\alpha$ and $\dot{a}^\beta$ using the pigeonhole principle. Therefore, $\psi(p_{n,i}^\alpha) = p_{n,i}^\beta$. Similarly, employing $k_{n,i}^\alpha = f_n^\alpha(p_{n,i}^\alpha)$ and $k_{n,i}^\beta = f_n^\beta(p_{n,i}^\beta)$ as in part 1, it follows that $k_{n,i}^\alpha = k_{n,i}^\beta$, since we have $> \aleph_0$-many such functions but only countably many possible function values. Hence one can set $k_{n,i} := k_{n,i}^\alpha = k_{n,i}^\beta$. $\square_{Claim}$

Part 3: Let us now construct the object mentioned in part 2. More concretely, we wish to obtain the name $\dot{a}^\kappa$ of an infinite subset $a^\kappa \subseteq \omega$ so that $1 \Vdash$ "$\dot{a}^\kappa \neq \dot{a}^\alpha$" for all $\alpha < \omega_2$, and the evaluation a^κ witnesses non-maximality of $\mathcal{A}$ in the generic extension (in case that the latter is an almost disjoint family). Start off by noting that:

Claim. *There exists a set $B^\kappa \in [\lambda \times \omega]^\omega$ such that $\bigcup_{\alpha<\kappa} B^\alpha \cap B^\kappa = R$, (and thus, $R \subseteq B^\kappa$).*

Proof. Fix any B^α, $\alpha < \omega_2$. Consider $Y := (\lambda \times \omega) \setminus \bigcup_{\alpha<\kappa} B^\alpha$. Since $|\bigcup_{\alpha<\kappa} B^\alpha| \leq \kappa$, $|Y| = \lambda$. Next, let $j : B^\alpha \setminus R \to Y$ be some injection, which we can be certain to find because of $|B^\alpha| < |Y|$. Then define

$$B^\kappa = R \cup j[B^\alpha \setminus R].$$

By R being the root of the B^α for $\alpha < \omega_2$, the claim follows. Note: If $B^\alpha \setminus R = \emptyset$ for the chosen α, there are still $\aleph_2$-many indices $\beta \neq \alpha$ left with $B^\beta \setminus R \neq \emptyset$, and this is sufficient for our purposes. Thereby, without loss of generality let the $B^\alpha \setminus R \neq \emptyset$. Also, B^κ is infinite by j being an injection with infinite domain. $\square_{Claim}$

The resulting set B^κ contains at least one element which is not in any of the B^α, as guaranteed by the definition of j. Yet in another sense, with respect to the isomorphism discussed above, the elements of B^κ are indistinguishable from the ones in B^α, as we will see soon. The whole reason why so much effort was put into the reduction steps is to be able to find such a B^κ through which we can now define $\dot{a}^\kappa$ as follows: Take an arbitrary

$\alpha < \omega_2$. Since $|B^\kappa| = \aleph_0$ and $R \subseteq B^\kappa$, as before we can find a bijection $\phi_{\alpha,\kappa} : B^\alpha \to B^\kappa$, so that $\phi_{\alpha,\kappa}$ fixes R. Again there exists a naturally induced isomorphism $\psi_{\alpha,\kappa} : \mathbb{C}_{B^\alpha} \to \mathbb{C}_{B^\kappa}$. For each $i \in \omega$, define $p^\kappa_{n,i} = \psi_{\alpha,\kappa}(p^\alpha_{n,i})$, and set $k^\kappa_{n,i} = k_{n,i}$. Then $\{p^\kappa_{n,i} : i \in \omega\}$ forms a maximal antichain because $\psi_{\alpha,\kappa}$ is an order isomorphism and each $\{p^\alpha_{n,i} : i \in \omega\}$ is a maximal antichain. This data completely determines $\dot{a}^\kappa$, as seen in part 1 and Remark 8.2.16.

Part 4: Now without loss assume that $1 \Vdash$ "$\dot{\mathcal{A}} = \{\dot{a}^\alpha : \alpha < \kappa\}$ is an almost disjoint family" (If instead of 1 a stronger condition p forces the statement, the following steps can be repeated with the addition that the statements hold below p).

Claim. *For all $\beta < \kappa$, $\dot{a}^\kappa \cap \dot{a}^\beta$ is finite. That is: $1 \Vdash$ "$\dot{\mathcal{A}} \cup \{\dot{a}^\kappa\}$ is almost disjoint".*

Proof. Fix $\beta < \kappa$ and find $\alpha < \omega_2$ such that $B^\alpha \cap B^\beta \subseteq R$: There exist $\alpha_1, \alpha_2 \in \omega_2$ such that $B^\beta \cap B^{\alpha_1} = B^\beta \cap B^{\alpha_2}$. Thus $B^\beta \cap B^{\alpha_1} \subseteq B^{\alpha_1} \cap B^{\alpha_2} = R$. So $\alpha = \alpha_1$ is as wanted. As $B^\kappa \cap B^\beta \subseteq R$, by definition of B^κ it holds that $B^\kappa \cap B^\beta = B^\alpha \cap B^\beta$. Consequently, there exists a bijection $\phi' : B^\alpha \cup B^\beta \to B^\kappa \cup B^\beta$ with $\phi' \restriction R = id_R$. As before, ϕ' induces an isomorphism $\psi' : \mathbb{C}_{B^\alpha \cup B^\beta} \to \mathbb{C}_{B^\kappa \cup B^\beta}$. From $1 \Vdash_{\mathbb{C}_{B^\alpha \cup B^\beta}}$ "$\dot{a}^\alpha \cap \dot{a}^\beta$ is finite", it follows that $1 \Vdash_{\mathbb{C}_{B^\kappa \cup B^\beta}}$ "$\psi'(\dot{a}^\alpha) \cap \psi'(\dot{a}^\beta)$ is finite". But $\psi'(\dot{a}^\alpha) = \dot{a}^\kappa$ and $\psi'(\dot{a}^\beta) = \dot{a}^\beta$. Thus $1 \Vdash_{\mathbb{C}_{B^\kappa \cup B^\beta}}$ "$\dot{a}^\kappa \cap \dot{a}^\beta$ is finite". As $\beta < \kappa$ was arbitrary, the statement follows. $\square_{Claim}$

To conclude, $\dot{a}^\kappa$ witnesses that there cannot be a mad family of cardinality κ. $\square$

Discussion 6.1.2 (Isomorphism of names vs. Taking averages)**.** *Let us continue from where we left off in the introduction to this section. To underline the connection between the methods, we refer to the previous analogy of taking general "averages". During this discussion, we understand an average $\mathfrak{A}$ taken over a base set $\mathfrak{B}$ to have the following two properties:*

1. *Descriptivity: $\mathfrak{A}$ can be associated to each member of $\mathfrak{B}$ via some characteristic inherent to all $a \in \mathfrak{B}$.*
2. *Reductivity: $\mathfrak{A} \notin \mathfrak{B}$.*

In a sense, already the theorem of Łoś (5.1.2) (respectively its application to the case where $\mathcal{A}_x = \mathcal{A}\ \forall x \in S$) can be seen as taking the average of a given family of structures. Indeed, the ultrapower $Ult_{\mathcal{U}}\mathcal{A}$ of a given structure $\mathcal{A}$ fulfills (1) because $Ult_{\mathcal{U}}\{\mathcal{A}_x : x \in S\} \equiv \mathcal{A}$, whereas (2) is guaranteed by the ultraproduct containing elements which are not in $\mathcal{A}$ (take e.g. $\mathcal{A} = \mathbb{N}$, $\mathcal{U}$ any ultrafilter on $\mathbb{N}$, where two sequences $f, g \in \mathbb{N}^{\mathbb{N}}$ are called equivalent iff $\{n \in \mathbb{N} : f(n) = g(n)\} \in \mathcal{U}$, i.e., $\mathfrak{A} = \mathbb{N}^{\mathbb{N}}/\mathcal{U}$ forms the hypernatural numbers. So in particular, the ultraproduct in this case contains non-standard natural numbers).

Similarly, the methods of taking averages as well as isomorphisms of names employ the *following data: $\mathfrak{B} = \dot{A}$, and $\mathfrak{A} = \dot{a}^{\kappa}$, where (1) is satisfied by $\mathfrak{A}$ retaining independence/almost disjointness, while (2) follows by construction (e.g., the injection j from part 3 of the previous proof guarantees this).*

Finally, there are some obvious parallels between the methods such as the characterization of the names for elements of $\dot{A}$, and various reductions in order to access surplus information (such as the set Y from part 3 in the above proof) which refers "outside of $\dot{A}$" guaranteeing (2).

6.2 Negative results

6.2.1 Preparing an isomorphism of names argument

First note that given a pre-spectrum C, there can be multiple sequences of pairwise disjoint trees $\bar{S}$, depending on the order of the trees in each sequence. To address such differences let us introduce the following notation.

Definition 6.2.1 ($\mathfrak{m}$, spectrum-parameters, $S_{\theta,<\alpha}$, $S_{\mathfrak{m},\alpha}$, $S_{\mathfrak{m},<\alpha}$, $Q_{\mathfrak{m}}$).
For C a pre-spectrum with parameters σ and λ define:

- $\mathfrak{m}(C)$ (or just $\mathfrak{m}$ if C is known from the context) as the set of all possible different sequences $S_{\mathfrak{m}} := (S_\theta : \theta \in C)$, so-called spectrum-parameters for C. Note that $S_{\mathfrak{m}}$ is just $\bar{S}$ in the previous notation. Talking about $\mathfrak{i}$ specifically, we also use the terminology *Spec*($\mathfrak{i}$)-parameters and write $S^{\mathfrak{i}}_{\mathfrak{m}}$. The same holds for $\mathfrak{a}$, where we write $S^{\mathfrak{a}}_{\mathfrak{m}}$.
- $S_{\theta,<\alpha} := \bigcup_{\beta<\alpha} S_{\theta,\beta}$, the set of all levels of the tree for $\theta \in C$ below the αth level.
- In the context of $\mathfrak{m}(C)$, let $S_{\mathfrak{m},\alpha} := \bar{S}_\alpha$ from before (that is, all αth levels taken together), and $S_{\mathfrak{m},<\alpha} := \bigcup_{\theta\in C} S_{\theta,<\alpha}$.
- Given a spectrum-parameter $S_{\mathfrak{m}}$, let $Q_{\mathfrak{m}}$ be the set of all possible finite support iterations $\mathbb{P}(S_{\mathfrak{m}})$ as defined earlier (but with trees of height σ instead of ω_1) with respect to $S_{\mathfrak{m}}$. For details see the upcoming remark.

Remark 6.2.2 ($\mathbb{P}(S_{\mathfrak{m}}) \in Q_{\mathfrak{m}}$). For the convenience of the reader we restate the main properties of the finite support iteration $\mathbb{P}(S_{\mathfrak{m}})$, as in Theorem 4.3.5 and Theorem 4.3.9:

- $S_{\mathfrak{m}} = \bar{S} = \{S_\theta : \theta \in C\}$, $\sigma, \lambda \in C)$.
- $\mathbb{P}(S_{\mathfrak{m}}) := \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \sigma, \beta < \sigma\rangle$.

- $S_{\mathfrak{m},0}$: If $S_\mathfrak{m} = S^\mathfrak{i}_\mathfrak{m}$, take $\dot{\mathbb{Q}}_0$ to be a $\mathbb{P}_0$-name for $\prod_{\eta\in\bar{S}_0} \mathbb{C}_\eta$ ($\mathbb{C}_\eta := \mathbb{C}_\theta$ whenever $\eta \in S_{\theta,0}$). If $S_\mathfrak{m} = S^\mathfrak{a}_\mathfrak{m}$, let $\dot{\mathbb{Q}}_0$ be a $\mathbb{P}_0$-name for $\prod_{\eta\in\bar{S}_0} \mathbb{H}_\eta$ ($\mathbb{H}_\eta := \mathbb{H}_\theta$ whenever $\eta \in S_{\theta,0}$). To each node $\eta \in S_{\mathfrak{m},0}$, assign a copy of the family $\mathcal{A}^1_\eta$ (either a mad family or an independent family of cardinality θ, depending on whether $\eta \in S^\mathfrak{a}_{\theta,0}$ or $\eta \in S^\mathfrak{i}_{\theta,0}$).

- $S_{\mathfrak{m},\alpha}$, $\alpha \geq 1$: If $S_\mathfrak{m} = S^\mathfrak{i}_\mathfrak{m}$, let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for $\prod_{\eta\in\bar{S}_\alpha} \mathbb{M}(\mathcal{U}_\eta)$, where each $\mathcal{U}_\eta$ is a fitting diagonalization filter. Else, if $S_\mathfrak{m} = S^\mathfrak{a}_\mathfrak{m}$, let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for $\prod_{\eta\in\bar{S}_\alpha} \mathbb{H}_{\mathcal{A}^\alpha_\eta, Succ_S(\eta)}$. Define in any case $\mathcal{A}^{\alpha+1}_\eta := \{a_\nu : \nu \in Succ_S(\eta\restriction\xi), \xi < \alpha+1\}$.

Due to the tree structure, there are several complexity levels concerning the forcing conditions:

I. Conditions of the fsi are σ-sequences. Those may span over all trees.

II. Each element of such a sequence is an element of a finite support product, which can occur at different levels of the trees.

III. Mathias, Cohen or Hechler mad conditions occur inside the products.

More concretely, by definition, $p(\alpha)$ is the αth entry of the σ-sequence $p \in \mathbb{P}(S_\mathfrak{m})$, which is of the form $\dot{q}_\alpha \in dom(\dot{\mathbb{Q}}_\alpha)$. Before we can even start thinking about possible isomorphisms, it is necessary to associate the forcing conditions to the nodes of the trees. For this we will, using the above complexity levels, start distinguishing different supports:

Definition 6.2.3 (Supports, generic real part, trunk)**.** Let $p \in \mathbb{P}(S_\mathfrak{m})$. Then:

1. $supt(p) := \{\alpha : 1_{\mathbb{P}(S_\mathfrak{m})} \Vdash_{\mathbb{P}(S_\mathfrak{m})} "\check{p}(\alpha) \neq \dot{1}_{\dot{\mathbb{Q}}_\alpha}"\}$ is the support of p as defined for finite support iterations (Definition 3.3.2). So here we are picking out levels of the tree sequence with at least one condition $\dot{q}_\alpha \neq 1_{\dot{\mathbb{Q}}_\alpha}$. (I)

2. Next, take $\alpha \in supt(p)$. Depending on whether $\mathbb{P}(S_\mathfrak{m})$ is $\mathbb{P}(S^\mathfrak{i}_\mathfrak{m})$ or $\mathbb{P}(S^\mathfrak{a}_\mathfrak{m})$ as well as $\alpha = 0$ or $\alpha \geq 1$, the trivial condition 1 equals $1_{\mathbb{M}(\mathcal{U}_\xi)} = (\emptyset, \omega)$ or one of $1_{\mathbb{C}_\eta} = 1_{\mathbb{H}_\eta} = \emptyset$ (the empty function). Let $supt(p(\alpha)) := \{\xi : p(\alpha)(\xi) \neq 1\}$ be concerned with the non-trivial conditions of the posets in the finite support product. (II)

3. Note: It is always the case that $supt(p(\alpha)) \in [S_{\mathfrak{m},\alpha}]^{<\omega}$. Indeed, by definition of finite support products, $p(\alpha)(\xi) = 1$ cofinitely often.

4. Generic real parts and trunks: This takes care of (III). Let $\eta \in supt(p(\alpha))$. Then:

a) If $p(\alpha)(\eta) \in \mathbb{M}(\mathcal{U}_\eta)$, then $p(\alpha)(\eta) = (s, A)$, with finite part $s \in [\omega]^{<\omega}$ and infinite part $A \in \mathcal{U}_\eta \subseteq [\omega]^\omega$ (Definitions 2.1.1 and 2.2.2). Call the former the trunk of $p(\alpha)(\eta)$, and the latter the (generic) real part of $p(\alpha)(\eta)$.

b) Let $\dot{a}_\alpha$ be a given $\mathbb{P}(S_\mathfrak{m})$-name for a real. If $p(\alpha)(\eta) \in \mathbb{H}_\eta \cup \mathbb{C}_\eta$ appears as some q in $a_\alpha = \{k \in \omega : \exists q \in G(q(\alpha, k) = 1)\}$ (see Sections 4.2.2 and 8.3.2), call a_α itself the real part of $p(\alpha)(\eta)$. The trunks on the other hand are provided by the conditions themselves.

c) Similarly, in case that $p(\alpha)(\eta) \in \mathbb{H}_{\mathcal{A}^\alpha_\eta, Succ_S(\eta)}$ (see Discussion 4.3.8), $p(\alpha)(\eta) = (p, K)$. Here, the real part is as in b), while the trunk is given by p.

5. A connection between $\bigcup S_\mathfrak{m}$ and $\mathbb{P}(S_\mathfrak{m})$: We wish to identify the real parts of the respective non-trivial conditions with certain nodes of the trees. For the identification, let ι be a function which associates to the parameters α, η, p an index set $\iota(\alpha, \eta, p) < \omega_1$. Then let $\{\eta_i : i < \iota(\alpha, \eta, p)\} \subseteq S_{\mathfrak{m},<\alpha}$ be the nodes from $\bigcup S_\mathfrak{m}$ which shall be identified with $A \in \mathcal{U}_\eta$ or a_η, depending on the cases given in (4). This happens by setting $B(\{\eta_i : i \in \iota(\alpha, \eta, p)\}) = A$ or $B(\{\eta_i : i \in \iota(\alpha, \eta, p)\}) = a_\eta$, where B is a Borel function.

6. Now define the actual support of $p(\alpha)(\eta)$ via

$$asupt(p(\alpha)(\eta)) := \{\eta_i : i < \iota(\alpha, \eta, p)\},$$

and let $asupt(p(\alpha)) := \bigcup\{asupt(p(\alpha)(\eta)) : \eta \in supt(p(\alpha))\}$. Finally, collect all actual supports for p itself to obtain the full support

$$fsupt(p) := \bigcup\{asupt(p(\alpha)) : \alpha \in supt(p)\}$$

7. To conclude, let $dom(p) := \bigcup\{supt(p(\alpha)) : \alpha \in supt(p)\}$.

Crucially, it will be necessary for us to define isomorphisms of some kind, and to do so, we next introduce certain permutation groups of the nodes of $S_\mathfrak{m}$:

Definition 6.2.4 ($S_\mathfrak{m}$-group)**.** Given $S_\mathfrak{m} \in \mathfrak{m}(C)$, a permutation group $K = K(S_\mathfrak{m})$ on $\bigcup S_\mathfrak{m}$ is called an $S_\mathfrak{m}$-group if for any permutations $\pi, \pi_1, \pi_2 \in K$ the following properties hold:

1. If $\eta \in S_{\theta,\alpha}$, then $\pi(\eta) \in S_{\theta,\alpha}$ (closure under specific tree levels).

2. If $\eta, \nu \in S_\theta$ with η an initial segment of ν ($\eta \trianglelefteq \nu$), then $\pi(\eta) \trianglelefteq \pi(\nu)$ (preservation of $\trianglelefteq$).

3. If for some $\alpha < \sigma$ and $\eta \in S_{\mathfrak{m},\alpha}$, $\pi_1(\eta) = \pi_2(\eta)$, then for any $\nu \in S_{\mathfrak{m},<\alpha}$, $\pi_1(\nu) = \pi_2(\nu)$ (equality of permutations with common value below a given level).

In this context, for any $\alpha \leq \sigma$ define $K_\alpha = \{\pi \restriction S_{\mathfrak{m},<\alpha} : \pi \in K\}$.

Now we can construct the wanted isomorphisms as automorphisms of the $\mathbb{P}_\alpha$ occurring in $\mathbb{P}(S_\mathfrak{m})$:

Definition 6.2.5 ($Q_{\mathfrak{m},K}$)**.** For an $S_\mathfrak{m}$-group K let

$$Q_{\mathfrak{m},K} := \{\mathbb{P}(S_\mathfrak{m}) \in Q_\mathfrak{m} : \mathbb{P}(S_\mathfrak{m}) \text{ fulfills (a) and (b)}\},$$

where:

(a) For any $\alpha \leq \sigma$, $\pi \in K_\alpha$ induces an automorphism $\hat{\pi}$ on $\mathbb{P}_\alpha$.

(b) Each $\hat{\pi}$ as in (a) additionally fulfills that for any $\theta \in C$ and $\eta \in S^{\mathfrak{i}}_{\mathfrak{m},\theta}$, $\hat{\pi}(\mathcal{U}_\eta) = \mathcal{U}_{\pi(\eta)}$. In case that $\eta \in S^{\mathfrak{a}}_\mathfrak{m}$, let $\hat{\pi}$ be any bijection between the real parts of conditions $p(\alpha)(\eta)$ corresponding to the respective nodes of $\bigcup S^{\mathfrak{a}}_\mathfrak{m}$.

Notice that the discrepancy stated in (b) is an artifact of Remark 4.3.7: By mutual genericity of the Mathias reals, their order is crucial in distinguishing between them. Therefore, the corresponding real parts (as elements of $\mathcal{U}_\eta$) have to be preserved by $\hat{\pi}$. On the contrary, the Hechler mad reals added in each new step of the iteration are completely new generic reals, and so we only require some kind of bijective mapping between their corresponding real parts. The isomorphisms are meant to consist of mappings between the respective generic real parts (level III. from above) of the elements occuring inside the sequences on complexity level I. from above.

Lemma 6.2.6. *If K is an $S_\mathfrak{m}$-group, then $Q_{\mathfrak{m},K} \neq \emptyset$.*

Proof. This is Lemma 4.4 from [8], which we just accept without proof. □

6.2.2 Independence case

The following lemma is proved by introducing a suitable isomorphism of names argument:

Lemma 6.2.7. *Assume GCH. Let C be a pre-spectrum with parameters σ and λ, $S_\mathfrak{m} = S^{\mathfrak{i}}_\mathfrak{m} \in \mathfrak{m}(C)$, $K = K(S_\mathfrak{m})$, and θ an arbitrary cardinal with $\sigma < \theta < \lambda$ but $\theta \notin C$. Fix $\mathbb{P}(S_\mathfrak{m}) \in Q_{\mathfrak{m},K}$. Consider the following condition $\star$:*

$$\exists \theta_* \in [\sigma, \theta) : \ \theta_* \geq \sup(C \cap \theta)$$

Then $\star$ implies that $\Vdash_{\mathbb{P}(S_{\mathfrak{m}})}$ "$\theta \notin Spec(\mathfrak{i})$".

Proof. Assume towards a contradiction that the conclusion does not hold for some $p_1 \in \mathbb{P}(S_{\mathfrak{m}})$, i.e., $p_1 \Vdash$ "$\theta \in Spec(\mathfrak{i})$". Thereby, there exists a family of names $\{\dot{a}_\alpha : \alpha < \theta\}$, which we shall call by the $\mathbb{P}(S_{\mathfrak{m}})$-name $\dot{\mathcal{A}}$, such that

$$p_1 \Vdash \text{"}\dot{\mathcal{A}} \text{ is a maximal independent family"}.$$

Here, each $\dot{a}_\alpha$ is a $\mathbb{P}(S_{\mathfrak{m}})$-name for a set in $[\omega]^\omega$. As before, we can characterize the elements of $\dot{\mathcal{A}}$ using maximal antichains and truth values: For any $\alpha \in \theta$, consider the maximal antichain $\{p^\alpha_{n,l} \in \mathbb{P}(S_{\mathfrak{m}}) : n, l \in \omega\}$ and the set $\{k^\alpha_{n,l} : n, l \in \omega\} \subseteq \{0, 1\}$, with $p^\alpha_{n,l} \Vdash$ "$\check{n} \in \dot{a}_\alpha$ iff $\check{k}^\alpha_{n,l} = \check{1}$". Up until now, the procedure has been as in Proposition 6.1.1.

The reduction part in this case turns out to be more complicated in that we have to take into account the additional tree structure. For fixed $\alpha < \theta$, consider the non-repeating sequence $\bar{t}_\alpha := \langle t_{\alpha,\varepsilon} : \varepsilon \in \varepsilon_\bullet \rangle \subseteq \bigcup S_{\mathfrak{m}}$, where $\varepsilon_\bullet$ is a countable ordinal chosen so that $\bar{t}_\alpha$ includes "all of the information needed to determine $\dot{a}_\alpha$ completely"; or more concretely, we choose $\varepsilon_\bullet$ so that $\bar{t}_\alpha$ contains $fsupt(\dot{a}_\alpha)$, which can be written out as

$$fsupt(\dot{a}_\alpha) := \bigcup_{n,l\in\omega} fsupt(p^\alpha_{n,l}) = \bigcup_{n,l\in\omega} \bigcup_{\beta \in supt(p^\alpha_{n,l})} \bigcup \{asupt(p^\alpha_{n,l}(\beta)(\eta)) : \eta \in supt(p^\alpha_{n,l}(\beta))\},$$

where $|fsupt(\dot{a}_\alpha)| = \aleph_0$. The reason this works is because of the definition of *asupt* and $\iota(\beta, \eta, p^\alpha_{n,l})$ (Definition 6.2.3). At this point, $\bar{t}_\alpha$ may consist of nodes in $\bigcup S_{\mathfrak{m}}$ with no assumptions on how the nodes relate to one another other than the association to $\dot{a}_\alpha$, and that the order of the sequence elements matters with respect to the referred-to tree nodes. Using combinatorial arguments, we can extract subsets of nodes with special relations. This happens in a couple of reduction steps, starting with:

Claim.

1. *There exists some $W_1 \in [\theta]^{\theta_*^+}$ (θ_*^+ being the cardinal successor of θ_*), such that for any $\theta' \in C \cap \theta$ and any $h \in \varepsilon_\bullet$*

$$\alpha, \beta \in W_1 \textit{ iff } t_{\alpha,h}, t_{\beta,h} \in S_{\theta'} \Rightarrow t_{\alpha,h} = t_{\beta,h}.$$

2. *There exists some $W_2 \in [W_1]^{\theta_*^+}$, such that*

   - $\forall \alpha, \beta \in W_2 \forall h \in \varepsilon_\bullet : ht(t_{\alpha,h}) = ht(t_{\beta,h})$. *(2.1)*

- $\forall \alpha, \beta \in W_2 \forall h, l \in \varepsilon_\bullet$: *There is* $\theta' \in C$ *with* $t_{\alpha,h}, t_{\alpha,l} \in S_{\theta'}$ *and* $t_{\alpha,h} <_{S_{\theta'}} t_{\alpha,l}$ *iff there is* $\theta' \in C$ *with* $t_{\beta,h}, t_{\beta,l} \in S_{\theta'}$ *and* $t_{\beta,h} <_{S_{\theta'}} t_{\beta,l}$. *(2.2)*

Proof. We use the infinite pigeonhole principle and assumption $\star$:

For (1), whereas α, β range through θ, $S_{\theta'}$ is θ'-splitting and hence $|S_{\theta'}| = \theta'$ (by $\theta' \geq \sigma$). Since $\theta' < \theta$, $S_{\theta'}$ can contain up to $\theta \geq \theta_*^+$-many equal $t_{\alpha,h}$ elements by the pigeonhole principle. So in particular, there must be some W_1 as claimed in (1).

Point (2) can be seen in a similar fashion: In (2.1), α, β again range through θ, whereas there are at most $\sigma < \theta$ many possible heights $ht(t_{\alpha,l})$. The result then follows from $\theta \geq \theta_*^+$. In (2.2), it is possible that $\theta' > \theta$, as we take arbitrary $\theta' \in C$. This is of no concern, as for each fixed α, only $h \in \varepsilon_\bullet$ is varied in $t_{\alpha,h}$, and $|\varepsilon_\bullet| = \aleph_0$. So $<_{S_{\theta'}}$ seen as a subset of $\{(t_{\alpha,h}, t_{\alpha,l}) : h \neq l \in \varepsilon_\bullet\}$ has cardinality below $\aleph_1$. Now as $\theta > \sigma \geq \aleph_1$, we can again use the pigeonhole principle as above to find the wanted subfamily of θ. Yet there is one more case to consider, namely that the left-hand side and right-hand side of the stated equivalence refer to different trees, i.e., to $S_{\theta_\alpha} \neq S_{\theta_\beta}$. But since $|C| = \aleph_0$, again the total number of possibilities is bounded by $\aleph_0 \cdot \aleph_1 = \aleph_1$. $\square_{Claim}$

For arbitrary $\varepsilon \in \varepsilon_\bullet$ and $\alpha \in W_2$ consider $\theta_{\alpha,\varepsilon} < \theta_*^+$. Although the $\theta_{\alpha,\varepsilon}$ are not necessarily elements of C, we can still consider associated trees $S_{\theta_{\alpha,\varepsilon}}$. Then:

Claim. *Fix* $\varepsilon \in \varepsilon_\bullet$. *There exists some* $W_3 \in [W_2]^{\theta_*^+}$ *such that* $\alpha \in W_3$ *iff* $\theta_{\alpha,\varepsilon} = \theta_\varepsilon$.

Proof. Again use the pigeonhole principle and $|W_2| = \theta_*^+ > \aleph_0 = |\varepsilon_\bullet|$. $\square_{Claim}$

By the previous claim we can form a sequence $\langle \theta_{\alpha,\varepsilon} : \alpha \in W_3 \rangle$ with $\varepsilon \in \varepsilon_\bullet$ fixed and $\theta_{\alpha,\varepsilon} = \theta_\varepsilon$ for θ_*^+-many cardinals.

We have a few more reduction steps to discuss. For all $\alpha, \beta \in W_3$ (without loss again use W_3):

1. $\forall k, n, l \in \omega$:
   - $t_{\alpha,k} \in fsupt(p^\alpha_{n,l})$ iff $t_{\beta,k} \in fsupt(p^\beta_{n,l})$.
   - $t_{\alpha,k} \in dom(p^\alpha_{n,l})$ iff $t_{\beta,k} \in dom(p^\beta_{n,l})$.
   - For any $t_{\alpha,k} \in dom(p^\alpha_{n,l})$ (hence by the previous point, $t_{\beta,k} \in dom(p^\beta_{n,l})$), it holds that $trunk(p^\alpha_{n,l}(t_{\alpha,k})) = trunk(p^\beta_{n,l}(t_{\beta,k}))$, where $trunk$ refers to the finite part of the given condition (where the γ in $p(\gamma)(\eta)$ is suppressed).

2. $\bar{t}_\alpha, \bar{t}_\beta$ realize the same quantifier free type in $S_{\mathfrak{m}}$.

To prove the above points, use the pigeonhole principle as before. Here, (1) guarantees that if one considers nodes referring to different elements $\dot{a}_\alpha$ and $\dot{a}_\beta$ which occur at the same point of the respective sequence, $\bar{t}_\alpha$ or $\bar{t}_\beta$, then this fact alone means that $fsupt$ and dom contain such a node each, and even that the nodes are associated to the same trunk. Point (2) makes sure that the same decisions are possible using the nodes from $\bar{t}_\alpha$ and the ones from $\bar{t}_\beta$.

Next, introduce an equivalence relation E on $\varepsilon_\bullet$ as follows: For all $\varepsilon_1, \varepsilon_2 \in \varepsilon_\bullet$ let $\varepsilon_1 E \varepsilon_2$ iff $\theta_{\varepsilon_1} = \theta_{\varepsilon_2}$, where $\theta_{\varepsilon_1}, \theta_{\varepsilon_2} < \theta_*^+$. Call $v_\varepsilon := [\varepsilon]_E$, the equivalence class of ε with respect to E. Let $\sigma_\bullet \leq \varepsilon_\bullet < \sigma$ denote the number of such equivalence classes. Furthermore, for any $l \in \sigma_\bullet$ there exists some $\varepsilon \in v_l$ such that $\theta_l = \theta_{\alpha,\varepsilon}$ for some $\alpha < \theta$. This actually holds for θ_*^+-many such α. Thus, by considering θ_l, v_l allows us to address elements using indices which are not in any of the other classes.

Now consider for any given $\bar{t}_\alpha \subseteq \bigcup S_\mathfrak{m}$ the downwards closure $\bar{t}_\alpha^+$, which necessarily has length less than σ. For fixed $l < \sigma_\bullet$, take $\bar{t}_\alpha^+ \restriction v_l$, which is again a sequence of countable length. Then applying the Δ-system Lemma (8.2.14) to the family $\{\bar{t}_\alpha^+ \restriction v_l : \alpha < \theta_*^+\}$, obtain that the sequence $\langle \bar{t}_\alpha^+ \restriction v_l : \alpha \in W_3 \rangle$ forms a Δ-system. Furthermore, we can make sure to choose the elements of the sequence so that they realize the same quantifier-free type in S_{θ_l}. Then define

$$A^{\theta_l} := \bigcap_{\alpha \in W_3} Range(\bar{t}_\alpha^+ \restriction v_l) \subseteq \bigcup S_\mathfrak{m}.$$

First note that $|A^{\theta_l}| \leq \aleph_0$. Set $w_l := \{\varepsilon \in v_l : t_{\alpha,\varepsilon} \in A^{\theta_l}\}$. With $t_\varepsilon^l := t_{\alpha,\varepsilon}$ (using some $\alpha \in W_3$), we can rewrite $A^{\theta_l} = \langle t_\varepsilon^l : \varepsilon \in w_l \rangle$.

After these preparations, we wish to construct a witness to destroy the maximality of $\mathcal{A}$. For this, define a condition $p_2 \leq p_1$ and a sequence $\bar{t} \subseteq \bigcup S_\mathfrak{m}$ of length ω such that $\bar{t} \supseteq fsupt(p_2)$, i.e., $\bar{t}^+$ is a sequence consisting of potentially new tree nodes. Furthermore, define the sequence $\bar{t}^\circ := \langle t_\varepsilon : \varepsilon < \varepsilon_\bullet \rangle$ so that for all $l < \sigma_\bullet$:

1. $\bar{t}^\circ \restriction v_l$ forms a subtree of S_{θ_l} and $\bar{t}^\circ \restriction v_l$ realizes the same type as $\bar{t}_\alpha^+ \restriction v_l$ for all $\alpha \in W_3$.
2. $\bar{t}^\circ \restriction w_l = \langle t_\varepsilon^l : \varepsilon \in w_l \rangle = A^{\theta_l}$.
3. If $\varepsilon \in v_l \backslash w_l$, then $t_\varepsilon \notin \bigcup\{Range(\bar{t}_\alpha^+) : \alpha < \theta\} \cup Range(\bar{t}^+)$.

Thereby, $\bar{t}^\circ$ is "comparable" to the $\bar{t}_\alpha^+ \restriction v_l$, incorporates the common intersection A^{θ_l} (wrt. all $\alpha \in W_3$), and yet also contains indices for nodes outside of any elements of the $\bar{t}_\alpha^+$ (for any $\alpha < \theta$) or the $\bar{t}^+$, i.e., indices referring to a "new" element. We proceed by defining

$$bad_{\bar{t}} := \{\alpha \in W_3 : \exists l < \varepsilon_\bullet \; (\langle t_{\beta,l} : \beta \in W_3\rangle \text{ has no repetitions} \wedge t_{\alpha,l} \in Range(\bar{t}^+))\}.$$

As $\bar{t}^+$ a priori consists of arbitrary tree nodes, with this definition we want to make sure that there are enough $\bar{t}^+$-nodes different from any of the $\bar{t}_\alpha$-nodes. By $|bad_{\bar{t}}| < \sigma$ and thereby $|W_3 \setminus bad_{\bar{t}}| = \theta_*^+$, it is certainly possible to find such nodes. Indeed, $|bad_{\bar{t}}| < \sigma$ holds, as on the one hand, $\varepsilon_\bullet < \sigma$, whereas on the other hand, $|Range(\bar{t}^+)| < \sigma$, and so for any $l \in \varepsilon_\bullet$, $|\{\alpha \in W_3 : t_{\alpha,l} \in Range(\bar{t}^+)\}| < \sigma$.

Now we can introduce the wanted isomorphisms, which are automorphisms on $\mathbb{P}(S_\mathfrak{m})$ in this case. First, for any $\alpha \in W_3 \setminus bad_{\bar{t}}$ there exists an automorphism $\pi_\alpha : \bigcup S_\mathfrak{m} \to \bigcup S_\mathfrak{m}$ with $\pi_\alpha(\bar{t}^+) = \bar{t}^+$ and $\pi_\alpha(\bar{t}_\alpha^+) = \bar{t}^\circ$. Fix an arbitrary $\alpha \in W_3 \setminus bad_{\bar{t}}$ and set $\dot{a}^\circ := \hat{\pi}_\alpha(\dot{a}_\alpha)$, where $\hat{\pi}_\alpha$ is the extended automorphism on $\mathbb{P}(S_\mathfrak{m})$ induced by π_α, which exists by Lemma 6.2.6. To see that $\dot{a}^\circ$ is the wanted witness, show the following:

Claim. *For any finite subfamily $\{\dot{a}_{\alpha_l} : l < k\} \subseteq \dot{\mathcal{A}}$ it holds that*

$$p_2 \Vdash \text{"}\{\dot{a}_{\alpha_l}\}_{l<k} \cup \{\dot{a}^\circ\} \textit{ is independent"}.$$

Proof. For any $k \in \omega$ there exists $\{\alpha_l : l < k\} \subseteq W_3$, $\gamma \in W_3 \setminus \{\alpha_l : l < k\}$, $\pi \in K$ so that:

(0) $\{\dot{a}_{\alpha_l} : l < k\} \subseteq \dot{\mathcal{A}}$ consists of pairwise different elements.

(1) $\pi \restriction \bar{t}^+ = id_{\bar{t}^+}$ and thus $\hat{\pi}(p_2) = p_2$.

(2) $\hat{\pi}(\dot{a}_{\alpha_l}) = \dot{a}_{\alpha_l}$ for any $l < k$.

(3) $\hat{\pi}(\dot{a}_\gamma) = \dot{a}^\circ$.

Certainly, $p_2 \Vdash$ "$\{\dot{a}_{\alpha_l} : l < k\} \cup \{\dot{a}_\gamma\}$ forms an independent family". By employing that $\hat{\pi}$ is an order isomorphism on $\mathbb{P}(S_\mathfrak{m})$, it follows that $\hat{\pi}(p_2) \Vdash$ "$\{\hat{\pi}(\dot{a}_{\alpha_l}) : l < k\} \cup \{\hat{\pi}(\dot{a}_\gamma)\}$ is independent". Finally, using points (1),(2) and (3) from above, we obtain that

$$p_2 \Vdash \text{"}\{\dot{a}_{\alpha_l} : l < k\} \cup \{\dot{a}^\circ\} \text{ is independent"}.$$

$\square_{Claim}$

So $p_2 \Vdash$ "$\mathcal{A}$ is not maximal". But by coherence (as $p_2 \leq p_1$), p_2 must force that $\mathcal{A}$ is maximal, a contradiction. $\square$

6.2.3 Almost disjointness case

In this section we provide a version of Lemma 6.2.7 for the a.d. context. Note that most of the argument for $\mathfrak{i}$ can be repeated here.

Lemma 6.2.8. *Assume GCH. Let C be a pre-spectrum with parameters σ, λ, $S_\mathfrak{m} = S_\mathfrak{m}^\mathfrak{a}$, $K = K(S_\mathfrak{m})$, $\theta \in (\sigma, \lambda) \setminus C$, $\mathbb{P}(S_\mathfrak{m}) \in Q_{\mathfrak{m},K}$, and assume that*

$$\exists \theta_* \in [\sigma, \theta) : \ \theta_* \geq \sup(C \cap \theta). \quad (\star)$$

Then $\star \Rightarrow 1 \Vdash$ "$\theta \notin Spec(\mathfrak{a})$".

Proof. Again we assume towards a contradiction that there is some $p_1 \in \mathbb{P}(S_\mathfrak{m}^\mathfrak{a})$ such that

$$p_1 \Vdash \text{"}\dot{\mathcal{A}} := \{\dot{a}_\alpha : \alpha < \theta\} \text{ forms a mad family"}.$$

As before, obtain maximal antichains $\{p_{n,l}^\alpha \in \mathbb{P}(S_\mathfrak{m}^\mathfrak{a}) : n, l \in \omega\}$ to characterize each $\dot{a}_\alpha$. Repeat the reduction steps, and obtain $v_l, w_l, \langle \bar{t}_\alpha^+ \restriction v_l : \alpha < \theta_*^+ \rangle$, A^{θ_l}, p_2, $\bar{t}$, $\bar{t}^\circ, bad(\bar{t})$, π_α, and $\dot{a}^\circ$, where a° is the wanted witness for the non-maximality of $\mathcal{A}$. Proceed by proving:

Claim. *For all $\dot{a}_\alpha \in \dot{\mathcal{A}}$, $p_2 \Vdash$ "$\dot{a}_\alpha \cap \dot{a}^\circ$ is finite".*

Proof. Fix α and let $\gamma \in W_3 \setminus \{\alpha\}$. Knowing that $\dot{a}_\gamma, \dot{a}_\alpha \in \dot{\mathcal{A}}$, $p_2 \Vdash$ "$\dot{a}_\alpha \cap \dot{a}_\gamma$ is finite". By $\hat{\pi}$ being an order-isomorphism on $\mathbb{P}_{S_\mathfrak{m}}$, $\hat{\pi}(p_2) \Vdash$ "$\hat{\pi}(\dot{a}_\alpha) \cap \hat{\pi}(\dot{a}_\gamma)$ is finite", and so $p_2 \Vdash$ "$\dot{a}_\alpha \cap \dot{a}^\circ$ is finite". Therefore, since $\alpha < \theta$ was arbitrary, p_2 forces that there exists some $\dot{x} \in [\omega]^\omega \setminus \dot{\mathcal{A}}$ so that for all $\dot{a} \in \dot{\mathcal{A}}$, $\dot{x} \cap \dot{a}$ is finite, meaning that $\dot{\mathcal{A}}$ is not maximal. But $p_2 \leq p_1 \Vdash$ "$\dot{\mathcal{A}}$ is maximal", a contradiction. $\square_{Claim}$

$\square$

7 On the joint spectra of $\mathfrak{i}$ and $\mathfrak{a}$

Already in Section 5.2.3 we have seen a first result for the joint spectra in case that $\kappa_1 < \kappa_2 < \cdots < \kappa_n$ are measurable cardinals. Most of the work in this section will consist of combining what we obtained for $Spec(\mathfrak{a})$ and $Spec(\mathfrak{i})$ separately into statements about the joint spectra $Spec(\mathfrak{a}) \cap Spec(\mathfrak{i})$.

7.1 Positive result

Combining the proofs which lead to Corollaries 4.1.19 and 4.2.9 provides us with the following additional result:

Corollary 7.1.1. *Assume GCH and let C be an arbitrary set of uncountable cardinals. Then consistently $C \subseteq Spec(\mathfrak{i}) \cap Spec(\mathfrak{a})$.*

Proof. We proceed as in the proofs of the stated results, where for $\theta := |C|$ we find pairwise disjoint index sets I^0_α and I^1_α, $\alpha < \theta$. Setting $\kappa^0_\alpha = |I^0_\alpha|$ and $\kappa^1_\alpha = |I^1_\alpha|$ we can use finite support products as before, but this time with "double" the amount of branches. $\square$

Certainly, it is also possible to reproduce this in the tree context. We are presenting here a combination of Theorems 4.3.5 and 4.3.9.

Theorem (Restatement of Corollary 7.1.1)**.** *Assume GCH and let C be an arbitrary set of uncountable cardinals. Then consistently $C \subseteq Spec(\mathfrak{i}) \cap Spec(\mathfrak{a})$.*

Proof. For any $\theta \in C$, consider $S^{\mathfrak{i}}_\theta := (\theta \times \{(\theta, 0)\})^{<\omega_1}$ and $S^{\mathfrak{a}}_\theta := (\theta \times \{(\theta, 1)\})^{<\omega_1}$, together with $\bar{S} = \langle S^{\mathfrak{i}}_\theta, S^{\mathfrak{a}}_\theta : \theta \in C\rangle$, where the independence and a.d. cases are to be taken care of for every θ by the disjoint copies $S^{\mathfrak{i}}_\theta$ and $S^{\mathfrak{a}}_\theta$, respectively. In addition to the previous notation, define the sets $\bar{S}^{\mathfrak{i}}_\alpha = \bigcup_{\theta \in C} S^{\mathfrak{i}}_{\theta,\alpha}$ and $\bar{S}^{\mathfrak{a}}_\alpha = \bigcup_{\theta \in C} S^{\mathfrak{a}}_{\theta,\alpha}$, to be able to address any level for all trees with respect to the independence case or the a.d. case separately. Then define the finite support iteration $\mathbb{P}_{\bar{S}} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_1, \beta < \omega_1\rangle$ by recursion as in the previous proofs, with the following changes to the finite support products incorporated:

- For $\bar{S}_0$: Define $\dot{\mathbb{Q}}_0$ to be a $\mathbb{P}_0$-name for the finite support product $\prod_{\eta\in\bar{S}^{\mathfrak{i}}_0}\mathbb{C}_\eta \times \prod_{\eta\in\bar{S}^{\mathfrak{a}}_0}\mathbb{H}_\eta$, where $\mathbb{C}_\eta=\mathbb{C}_\theta$ whenever $\eta\in S^{\mathfrak{i}}_{\theta,0}$, and $\mathbb{H}_\eta=\mathbb{H}_\theta$ whenever $\eta\in S^{\mathfrak{a}}_{\theta,0}$.

- For $\bar{S}_1$: Work in $V^{\mathbb{P}_1}$. Then for every $\eta\in\bar{S}_0$,

$$\begin{cases}\text{if } \eta\in\bar{S}^{\mathfrak{i}}_0, \text{ assign to each } \nu\in Succ_S(\eta) \text{ copies } \mathcal{A}^{1,\mathfrak{i}}_\eta \text{ and } \mathcal{U}_\eta,\\ \text{if } \eta\in\bar{S}^{\mathfrak{a}}_0, \text{ assign to each } \nu\in Succ_S(\eta) \text{ a copy } \mathcal{A}^{1,\mathfrak{a}}_\eta,\end{cases}$$

  where $\mathcal{A}^{1,\mathfrak{i}}_\eta$ is the independent family added by $\mathbb{C}_\eta$, $\mathcal{U}_\eta$ is an $\mathcal{A}^{1,\mathfrak{i}}_\eta$-diagonalization filter, and $\mathcal{A}^{1,\mathfrak{a}}_\eta$ the mad family added by $\mathbb{H}_\eta$. Next, let $\dot{\mathbb{Q}}_1$ be a $\mathbb{P}_1$-name for the finite support product $\prod_{\eta\in\bar{S}^{\mathfrak{i}}_1}\mathbb{M}(\mathcal{U}_\eta)\times\prod_{\eta\in\bar{S}^{\mathfrak{a}}_1}\mathbb{H}_{\mathcal{A}^{1,\mathfrak{a}}_\eta,Succ_S(\eta)}$. Then in $V^{\mathbb{P}_2}$, for all $\eta\in\bar{S}_1$ define the family

$$\mathcal{A}^2_\eta:=\begin{cases}\mathcal{A}^{2,\mathfrak{i}}_\eta:=\{a_\nu:\nu\in Succ_S(\eta{\restriction}\xi),\xi<1\}, \text{ if } \eta\in\bar{S}^{\mathfrak{i}}_1,\\ \mathcal{A}^{2,\mathfrak{a}}_\eta:=\{a_\nu:\nu\in Succ_S(\eta{\restriction}\xi),\xi<1\}, \text{ if } \eta\in\bar{S}^{\mathfrak{a}}_1.\end{cases}$$

- Similarly, for $\alpha\geq 2$, assume that for any $\eta\in\bar{S}_{\alpha-1}$ the families $\mathcal{A}^{\alpha,\mathfrak{i}}_\eta$ (plus the corresponding $\mathcal{U}_\eta$) and $\mathcal{A}^{\alpha,\mathfrak{a}}_\eta$ are defined. Work in $V^{\mathbb{P}_\alpha}$. Set $\dot{\mathbb{Q}}_\alpha$ to be a $\mathbb{P}_\alpha$-name for $\prod_{\eta\in\bar{S}^{\mathfrak{i}}_\alpha}\mathbb{M}(\mathcal{U}_\eta)\times\prod_{\eta\in\bar{S}^{\mathfrak{a}}_\alpha}\mathbb{H}_{\mathcal{A}^{\alpha,\mathfrak{a}}_\eta,Succ_S(\eta)}$. In $V^{\mathbb{P}_{\alpha+1}}$, set

$$\mathcal{A}^{\alpha+1}_\eta:=\begin{cases}\mathcal{A}^{\alpha+1,\mathfrak{i}}_\eta:=\{a_\nu:\nu\in Succ_S(\eta{\restriction}\xi),\xi<\alpha\}, \text{ if } \eta\in\bar{S}^{\mathfrak{i}}_\alpha,\\ \mathcal{A}^{\alpha+1,\mathfrak{a}}_\eta:=\{a_\nu:\nu\in Succ_S(\eta{\restriction}\xi),\xi<\alpha\}, \text{ if } \eta\in\bar{S}^{\mathfrak{a}}_\alpha.\end{cases}$$

□

To incorporate the notational changes from Section 6.2.1, as before substitute $\bar{S}=S_{\mathfrak{m}}$, where $\bar{S}^{\mathfrak{i}}=S^{\mathfrak{i}}_{\mathfrak{m}}$ and $\bar{S}^{\mathfrak{a}}=S^{\mathfrak{a}}_{\mathfrak{m}}$, and consider trees of length $\sigma\geq\aleph_1$. Finally, to take into account pre-spectra as in the case of Theorems 4.3.12 and 4.3.14, repeating the above and the additional arguments as in the corresponding proofs, obtain:

Theorem 7.1.2. *Assume GCH and let C be a pre-spectrum with parameters σ and λ. Then consistently $C\subseteq Spec(\mathfrak{i})\cap Spec(\mathfrak{a})$, $\sigma=\mathfrak{i}=\mathfrak{a}$ as well as $\lambda=\mathfrak{c}$.*

7.2 Negative result

Combining Lemmas 6.2.7 and 6.2.8, the joint result also holds:

Lemma 7.2.1. *Assume GCH. Let C be a pre-spectrum with parameters σ and λ, $S_{\mathfrak{m}} = S^{\mathfrak{i}}_{\mathfrak{m}} \cup S^{\mathfrak{a}}_{\mathfrak{m}} \in \mathfrak{m}(C)$, $K = K(S_{\mathfrak{m}})$, and θ an arbitrary cardinal with $\sigma < \theta < \lambda$ but $\theta \notin C$. Fix $\mathbb{P}(S_{\mathfrak{m}}) \in Q_{\mathfrak{m},K}$. As before, consider condition $\star$:*

$$\exists \theta_* \in [\sigma, \theta) : \ \theta_* \geq \sup(C \cap \theta).$$

Then $\star$ implies that $\Vdash_{\mathbb{P}(S_{\mathfrak{m}})}$ "$\theta \notin Spec(\mathfrak{i}) \cap Spec(\mathfrak{a})$".

7.3 Main result

Assume GCH and let C be a pre-spectrum with $|C| = \aleph_0$. Separately using Lemmas 6.2.7 and 6.2.8, it follows that

$$Con(Spec(\mathfrak{i}) = C) \ \text{ and } \ Con(Spec(\mathfrak{a}) = C).$$

Employing Theorem 7.1.2 together with Lemma 7.2.1 in turn yields

Theorem 7.3.1. *Assume GCH. Consider a countable pre-spectrum C with condition $\star$ as above. Then consistently $Spec(\mathfrak{i}) = C = Spec(\mathfrak{a})$.*

Proof. Similarly to the proof of Theorem 5.2.1, let us take care of the following three points:

I. Positive results: Consistently, $C \subseteq Spec(\mathfrak{i}) \cap Spec(\mathfrak{a})$.

II. There are no cardinals in between the given ones in C.

III. There are no cardinals below $\min(C)$ and above $\max(C)$.

For a spectrum-parameter $S_{\mathfrak{m}}$ take $\mathbb{P}(S_{\mathfrak{m}}) \in Q_{\mathfrak{m},K}$ to be a finite support iteration as in Theorem 7.1.2, and continue working in $V^{\mathbb{P}(S_{\mathfrak{m}})}$. Consequently (I) is fulfilled. Additionally, (III) holds because the theorem guarantees that $\min(C) = \sigma = \mathfrak{i} = \mathfrak{a}$, and $\max(C) = \lambda = \mathfrak{c}$, where there cannot exist cardinals below and above the minimum and maximum by the corresponding *ZFC* results for the spectra. Finally, (II) follows from Lemma 7.2.1. $\square$

7.4 Outlook

As a follow up to the main result of the last section, a natural question is to inquire about weakenings of the restrictions imposed on the pre-spectra (Definition 4.3.10). In the course of constructing the model realizing the main result it became apparent that the employed methods (occurring in the positive and negative results) are intimately connected to these restrictions: The regularity of σ is required for successful application of the maximality condition of the diagonalizing reals (Definition 2.0.1) in the context of ccc finite support iterations, as witnessed by most of the positive results (e.g. Thm 4.3.12). Therefore, modifications of 4.3.10(1) seem to involve changes to the basic approach. Point (2) is not truly restrictive, as it stems from the *ZFC* limitations on the spectra. It might be lucrative to work on ways of weakening (3), perhaps by modifying the isomorphism of names part of the argument.

It should be mentioned that for the a.d. case, there is considerable work done in weakening the analogous Blass spectrum ([2]) by Shelah and Spinas in [18]. To the knowledge of the author, there exists new work yet to be published on weakenings for pre-independence spectra. It would be interesting to compare these results, and take into account the peculiarity for the joint spectra as described in Remark 4.3.7.

As stated frequently in this work, a Hechler mad analogue for $\mathfrak{i}$ is yet to be found. If we were able to obtain maximal independent families in a single forcing step, this would allow for an alternative to the iterated approach exhibited in this thesis and would in turn be a valuable tool for the inspection of the weakenings mentioned above. Already an "independent coding" poset would be considered great progress, though as we have seen in Chapter 8.4, the natural candidate $\mathbb{C}$ does not work out. Finally, the question whether $Con(cf(\mathfrak{i}) = \omega)$ is yet to be decided ([8]), whereas the a.d. case has been worked out, as mentioned in [18].

Turning to the longstanding open problem of $\mathfrak{i} < \mathfrak{a}$ (for $\mathfrak{a} < \mathfrak{i}$ see Appendix, 8.3.6) let us note that $\mathfrak{d} \leq \mathfrak{i}$ (Thm 1.2.6) implies that in case $\mathfrak{i} < \mathfrak{a}$ is true, $\mathfrak{d} < \mathfrak{a}$ must hold. So might models of the latter indicate ways to approach the former? In our case, we obtain a model of $\mathfrak{d} \leq \sigma = \mathfrak{a}$, where it also holds that $\mathfrak{i} = \mathfrak{a}$ due to the construction of the finite support iteration (Thm 7.3.1). In this sense, $\mathfrak{i} = \mathfrak{a}$ may be viewed as an artifact of our choice of methods. This seems to be the norm: As mentioned in e.g. [8], all known models of $\mathfrak{d} < \mathfrak{a}$ result in such equality. Again, a Hechler mad analogue would help to bypass iterative methods. Recent work towards such model has been done by Fischer in [6] and Fischer together with C. Switzer in [9], where the concepts of selective and generic selective independence are introduced, respectively. Such families come with their own set of interesting questions such as the necessary assumptions under which their existence is guaranteed.

8 Appendix: Some forcing theory and other preliminaries

This chapter does not serve as an introduction to forcing but rather as an appendix for the rest of the thesis. In this sense, parts of it, especially the applications in Section 8.3, will often be referred to in the preceding chapters. Nevertheless the aim was to write the appendix in an introductory style. Many of the given results can be found in K. Kunen's work [16]. The exact sources will be stated at the beginning of each section.

On the employed forcing approach

Forcing can be accessed through different means, one of which uses Boolean-valued models (as introduced for example in T. Jech's [13]). Another standard-approach, the one taken in this thesis, can be found in e.g. [16].

With the aim of presenting some underlying conventions and notations, let us give a short overview of this approach:

Ground model: Any forcing extension builds upon the notion of a ground model, denoted by $\mathcal{M}$ or V. Such a ground model takes the form of a countable transitive model (ctm) of *ZFC*. More concretely, $\mathcal{M}$ models a finite fragment of *ZFC* found using the Reflection Principle, and this is sufficient for our aim, namely to prove any given consistency result.

Forcing notion: Any forcing procedure is encoded in a forcing poset (or forcing notion) $\mathbb{P} := (\mathbb{P}, \leq_{\mathbb{P}}, 1_{\mathbb{P}})$, a set $\mathbb{P}$ pre-ordered by $\leq_{\mathbb{P}}$ with maximal element $1_{\mathbb{P}} \in \mathbb{P}$. If no ambiguities arise, we will omit the subscripts and write $\leq$ and 1. In standard forcing terminology, the elements of $\mathbb{P}$ are called (forcing) conditions, and for any $p, q \in \mathbb{P}$, $p \leq q$ means that p is stronger than q. Thus 1 is the weakest condition in $\mathbb{P}$. Incompatibility of p and q is denoted by $p \perp q$.

Implicit assumptions on $\mathbb{P}$: Unless stated otherwise, $\mathbb{P}$ is a separative preorder (that is, $\forall p, q \in \mathbb{P}$: either $q < p$ and $p \not\leq q$, or $\exists r \leq q$: $r \perp p$. This turns $\mathbb{P}$ into a partial order.

Furthermore, $\mathbb{P}$ is assumed to be atomless (i.e., it contains no element $r \in \mathbb{P}$ so that for $p, q \in \mathbb{P}$ with $p, q \leq r$, p and q are incompatible). Finally, in general we assume that $(\mathbb{P}, \leq, 1) \in \mathcal{M}$.

Generic filter: Let $\mathbb{P}$ be a given forcing poset. Usually, $G \subseteq \mathbb{P}$ (and sometimes H) will denote a $\mathbb{P}$-generic filter over $\mathcal{M}$, where G is defined to be a filter intersecting every dense set in $\mathcal{M}$. At times we will call G an $(\mathcal{M}, \mathbb{P})$-generic filter. Such G exists by $\mathcal{M}$ being countable. Furthermore, the assumption that $\mathbb{P}$ is atomless implies that $G \notin \mathcal{M}$.

Generic extension: In applications of forcing, the definition of $\mathbb{P}$ is motivated by the generic object we wish to add to the ground model. This object is without loss given by G itself, and the notation $\mathcal{M}[G]$ for *the* (by Proposition 8.1.2) generic extension reflects that we are "adjoining" G to $\mathcal{M}$. Oftentimes we will derive the wanted generic object from G, such as in the case of adding a Cohen real $c := \bigcup G$ or a certain family of sets $\mathcal{A}$; to emphasize which generic object is added one can write $\mathcal{M}[c]$ and $\mathcal{M}[\mathcal{A}]$, though we will seldom employ this notation. In the context of finite support iterations we will often write $V^{\mathbb{P}}$ for the generic extension obtained by forcing with $\mathbb{P} \in V$.

All concepts above, as well as omitted ones such as absoluteness, the forcing language or the definition and properties of $\Vdash$ ($\Vdash_{\mathcal{M},\mathbb{P}}$) can be found in [16]. Aside from a generic extension, $V^{\mathbb{P}}$ can also stand for the class of all $\mathbb{P}$-names, although this should be clear from the context. In this case, $\mathcal{M}^{\mathbb{P}}$ means the class of all $\mathbb{P}$-names relativized to $\mathcal{M}$. Finally, some of the following results do not have to use all of *ZFC*, though we will not be concerned with such subtleties here; Proposition 8.1.2 for example requires $\mathcal{M}$ to be a model of *ZF-P* only.

8.1 Miscellaneous forcing results

The proof of Lemma 8.1.4 is based on the proof of Theorem 2.10 in A. Karagila's script [14]. The other results are taken from [16].

Forcing Theorem: Truth and Definability

The Forcing Theorem refers to the Truth Lemma and the Definability Lemma from [16]:

Theorem 8.1.1 (Forcing Theorem)**.** *Let $\mathcal{M}$ be a ctm. Then it holds that:*

- ***Truth**: For $\mathbb{P} \in \mathcal{M}$ a forcing poset, ψ a sentence in the forcing language relativized to $\mathcal{M}$ and G a $\mathbb{P}$-generic filter over $\mathcal{M}$, it holds that:*

$$\mathcal{M}[G] \models \psi \textit{ iff } \exists p \in G \textit{ with } p \Vdash \psi.$$

- ***Definability***: *In short, a statement of the form $p \Vdash \phi$, in words, "a formula in the forcing language is forced by some forcing condition" is definable in the ground model. To be more concrete, for $\phi(\bar{x})$ a formula in the language of set theory with $\bar{x} = (x_1, x_2, \ldots, x_n)$ the only free variables of ϕ, it holds that the set $\{(p, \mathbb{P}, \leq, 1, \vartheta_1, \vartheta_2, \ldots, \vartheta_n) : (\mathbb{P}, \leq, 1)$ forcing notion $, p \in \mathbb{P}, \mathbb{P} \in \mathcal{M}, \vartheta_i \in \mathcal{M}^{\mathbb{P}} \forall i = 1, \ldots, n,\ p \Vdash_{\mathbb{P}} \phi(\vartheta_1, \ldots, \vartheta_n)\}$ is definable over $(\mathcal{M}, \in)$ without parameters.*

Minimality of the generic extension

To begin with, the generic extension $\mathcal{M}[G]$ is the minimal extension of $\mathcal{M}$ in the following sense:

Proposition 8.1.2. *Let $G \subseteq \mathbb{P}$ be a filter on $\mathbb{P}$. Then for any transitive model $\mathcal{N} \models ZFC$ with $\mathcal{M} \subseteq \mathcal{N}$ and $G \in \mathcal{N}$, $\mathcal{M}[G] \subseteq \mathcal{N}$.*

Proof. Let $x \in \mathcal{M}[G]$, i.e., $x = \tau_G$ for some $\tau \in \mathcal{M}^{\mathbb{P}}$. As the evaluation $\tau_G := val(\tau, G)$ of a $\mathbb{P}$-name τ is recursively defined, it is absolute for transitive models. Then $\tau_G \in \mathcal{N}$, because τ and G are in $\mathcal{N}$ by assumption. Thus the statement follows. $\square$

Coherence

The forcing relation $\Vdash$ has many properties which specify how it behaves in the context of complex sentences in the forcing language. One of the properties will be particularly useful, as it will enable us to find stronger conditions in the same "branch" of the forcing:

Proposition 8.1.3. *Let $p, q \in \mathbb{P}$ be such that $q \leq p$, and let ψ be a sentence in the forcing language relativized to $\mathcal{M}$. Further assume that $p \Vdash \psi$. Then also $q \Vdash \psi$.*

Proof. Let G be a generic filter with $q \in G$, and assume towards a contradiction that $q \nVdash \psi$. This means that $q \notin G$ or $\mathcal{M}[G] \not\models \psi$, and so the latter must be true. But as $q \leq p$, $p \in G$. Hence by assumption on p, $\mathcal{M}[G] \models \psi$, a contradiction to $\mathcal{M}[G]$ being a model. $\square$

Mixing of P-names

In the context of quotient forcing, it will make life easier to "mix together" certain names. The basic tool for this is the following lemma.

Lemma 8.1.4. *Let $\mathbb{P}$ be a forcing notion and $p \in \mathbb{P}$ a condition such that $p \Vdash$ "$\exists x \varphi(x)$". Then there exists a name $\tau \in \mathcal{M}^{\mathbb{P}}$ with $p \Vdash \varphi(\tau)$.*

Proof. Fix $p \in \mathbb{P}$ as above. We begin with the following

Claim. *The set* $D_p := \{q \leq p : \exists \tau \in \mathcal{M}^{\mathbb{P}} \ (q \Vdash \varphi(\tau))\}$ *is dense below* p.

Proof. By assumption, $p \Vdash$ "$\exists x \varphi(x)$". We show that this is equivalent to the claim. Indeed, we can equivalently write that $p \Vdash$ "$\neg \forall x \neg \varphi(x)$". Call $\phi = \forall x \neg \varphi(x)$. Then

$$p \Vdash \neg \phi \text{ iff } \neg \exists r \leq p (r \Vdash \phi) \text{ iff } \forall r \leq p (r \not\Vdash \phi),$$

where the first equivalence uses basic forcing properties (that is, coherence, the inability of a condition to force a statement and its negation at the same time (the $0,1$-law), the Truth Lemma and downwards-directedness of the underlying generic filter). Now the latter is equivalent to the existence of a witness $\tau \in \mathcal{M}^{\mathbb{P}}$ so that $r \not\Vdash \neg\varphi(\tau)$ whenever $r \leq p$, and thus as before, for any $r \leq p$ there is $\tau \in \mathcal{M}^{\mathbb{P}}$ such that $\exists q \leq r (q \Vdash \varphi(\tau))$; in other words, D_p is dense below p. $\square_{Claim}$

Next we show that D_p contains a maximal antichain A. By the claim, there exists $d \in D_p$, and so $\{d\} \subseteq D_p$ forms a trivial antichain. Using Zorn's Lemma one can extend $\{d\}$ to a maximal antichain $A \subseteq D_p$ (We can, for instance, take $X = \{A' \subseteq D_p : A' \text{ is an antichain}\}$, which is partially ordered by $\subseteq$. Then any chain $Y \subseteq X$ has an upper bound given by $\bigcup Y$; application of Zorn's Lemma yields a maximal element of X.) For all $a \in A$ let $\tau_a \in \mathcal{M}^{\mathbb{P}}$ be a name such that $a \Vdash \varphi(\tau_a)$. Change up those names as follows. For $a \in A$ fixed, consider all elements $(\sigma, b) \in \tau_a$ so that if $a \perp b$, remove (σ, b) from τ, whereas if $a \not\perp b$, the set $\{c : c \leq a, b\}$ is non-empty, in which case we replace (σ, b) by all of the (σ, c). Again we use τ_a to denote the resulting name. Then let $\tau := \bigcup \{\tau_a : a \in A\}$. It follows that:

Claim. *For any* $a \in A$, $a \Vdash$ "$\tau = \tau_a$".

Proof. Indeed, whenever $a \in G$, where G is an $(\mathcal{M}, \mathbb{P})$-generic filter, assume towards a contradiction that $a \Vdash$ "$\exists (\sigma, b) \in \tau : \ (\sigma, b) \notin \tau_a$". Now $(\sigma, b) \in \tau_{a'}$ for some $a' \in A \setminus \{a\}$. But since A is an antichain, $a \perp a'$. Thereby, $a \Vdash$ "$\tau_{a'} = \check{\emptyset}$" by the modified definition of the τ_a. $\square_{Claim}$

Finally, we show that $p \Vdash \varphi(\tau)$. Assume otherwise. Then there exists some $r \leq p$ with $r \Vdash \neg\varphi(\tau)$. Consequently, by maximality of A, there is $a \in A$ such that $a \not\perp r$, and hence $a \Vdash \varphi(\tau)$. $\square$

The proof above exemplifies how the mixing takes place, namely, by specifying incompatible conditions deciding which part of τ gets realized in the forcing extension, whereas the rest of the conditions get discarded by the generic filter. In this sense, we can think of τ as a "superposition" of names. We restate the mixing part of the proof as follows:

Corollary 8.1.5. *Let $A \subseteq \mathbb{P}$ be an antichain. For each $p \in A$, let τ_p be an associated $\mathbb{P}$-name. Then there exists a $\mathbb{P}$-name τ so that $p \Vdash$ "$\tau = \tau_p$" for each $p \in A$.*

8.2 Combinatorial properties and nice names

There is a useful connection between combinatorial properties of a given forcing poset and certain properties of the generic extension, in particular concerning cardinal preservation. Section 8.2.1 will motivate this connection by establishing equivalent ways to define a generic filter. Additionally, so-called nice names will be introduced, which will prove to be an invaluable tool for various parts of the thesis. In 8.2.2 the combinatorial properties are provided, starting with the countable chain condition, and continuing with a stronger notion playing a role in the context of finite support products and iterations. Afterwards, in 8.2.3 we will consider another valuable combinatorial tool known as the Delta System Lemma and prove a generalized version of it. All of these results are taken from [16] except the last, which stems from [15] (Thm II.1.5). Lastly, Section 8.2.4 provides a nice name construction for a given family $\mathcal{A} \subseteq [\omega]^\omega$. This will be of central use, for example when taking averages (5.2.1) and constructing isomorphisms of names (6.1.1).

8.2.1 Genericity, antichains and nice names

Let $\mathcal{M}, \mathbb{P}$, etc. be as before. In the following, fix an arbitrary $\mathbb{P}$-name $\tau \in V^{\mathbb{P}}$, i.e., a recursively defined family of tuples $\tau = \{\langle \sigma, p \rangle : \sigma \text{ a } \mathbb{P}\text{-name } \wedge p \in \mathbb{P}\}$. Nice names are $\mathbb{P}$-names of a special form. In this section we want to discuss what is "nice" about nice names. We begin with the definition.

Definition 8.2.1 (Nice name)**.** Let $X \subseteq \tau$. A nice name for X is a $\mathbb{P}$-name of the form

$$\bigcup \{\{\sigma\} \times A_\sigma : \sigma \in dom(\tau)\},$$

where each $A_\sigma \subseteq \mathbb{P}$ forms an antichain.

It is possible to associate *in the ground model* a nice name to every subset $X \subseteq \tau$ in case that $\tau \in \mathcal{M}^{\mathbb{P}}$:

Lemma 8.2.2. *Let $\tau \in \mathcal{M}^{\mathbb{P}}$. Assume that $1 \Vdash$ "$\mu \subseteq \tau$" for some $\mu \in \mathcal{M}^{\mathbb{P}}$. Then there exists a nice name $\vartheta \in \mathcal{M}^{\mathbb{P}}$ for a subset of τ such that $1 \Vdash$ "$\mu = \vartheta$".*

Proof. Proceed in two parts: First, find a suitable nice name ϑ. Then show that $\mu_G = \vartheta_G$. Assume that $\mu \neq \emptyset$ (otherwise, there is nothing to show). Define ϑ as follows: Knowing that $\mathbb{P} \neq \emptyset$ per assumption there must be some non-empty antichain in $\mathbb{P}$. Thus, we may use AC (in $\mathcal{M}$) to choose, for any $\sigma \in dom(\tau)$, an antichain A_σ. By our assumption on μ, it is further possible to guarantee that for each $\sigma \in dom(\tau)$ the chosen A_σ fulfills

$$\forall p \in A_\sigma : \; p \Vdash \text{"}\sigma \in \mu\text{"},$$

where we used the Definability Lemma (see Forcing Theorem - 8.1) to make sure that the stated formula containing $\Vdash$ is definable in $\mathcal{M}$ (thereby the sequence of all A_σ is in $\mathcal{M}$, and the following is possible in $\mathcal{M}$). Again using AC (respectively Zorn's Lemma) each A_σ fulfilling the stated properties can be assumed to be maximal. Having this, define

$$\vartheta = \bigcup \{\{\sigma\} \times A_\sigma : \sigma \in dom(\tau)\}.$$

To see that $\mu_G = \vartheta_G$, show both inclusions separately:
For "$\subseteq$", let $x \in \mu_G$. Then $x = \sigma_G$ for some $\sigma \in dom(\tau)$, where because of $\mu_G \subseteq \tau_G$, $x \in \tau_G$. Assume towards a contradiction that $x \notin \vartheta_G$. Using the Truth Lemma, find a condition $p \in G$ with $p \Vdash \text{"}\check{x} \notin \vartheta \wedge \check{x} \in \mu\text{"}$. Say there exists $\hat{p} \in A_\sigma$ with $\hat{p} \not\perp p$. We show that this assumption leads to a contradiction (and therefore $p \perp \hat{p}$ for any $\hat{p} \in A_\sigma$, contradicting in turn the maximality of A_σ). Indeed, our assumption implies that $\hat{p} \in G$ and $\langle \sigma, \hat{p} \rangle \in \vartheta$. Now since $\sigma_G \notin \vartheta_G$ (in $\mathcal{M}[G]$), by definition of ϑ_G it must hold that $\forall p \in G$: $\langle \sigma, p \rangle \notin \vartheta$, a contradiction.
To see "$\supseteq$", let $x \in \vartheta_G$. Then there exists $\sigma \in dom(\tau)$ such that $x = \sigma_G$ and $\langle \sigma, p \rangle \in \vartheta$ for some $p \in A_\sigma$. Furthermore, $p \Vdash \text{"}\sigma \in \mu\text{"}$, and so $\mathcal{M}[G] \models \text{"}x = \sigma_G \in \mu_G\text{"}$. □

What is nice about nice names

To answer this question, we refer to some results where nice names are being employed:

1. We can completely characterize any real using nice names, see e.g. 8.2.4.

2. More generally, any subset of any given $\mathbb{P}$-name can be associated to a nice name, as proved in Lemma 8.2.2.

3. Furthermore, if the underlying forcing poset has the countable chain condition, then not only is the size of a maximal antichain bounded, but also the number of maximal antichains and hence of nice names. This is shown in Lemma 8.2.9. In

particular this means that we can find an upper bound on the "information" used to characterize an object such as a given real.

Density and maximal antichains

Let us establish how the A_σ from Definition 8.2.1 and a given generic filter G are connected. Start by providing a few notions related to density.

Lemma 8.2.3 (Weaker and stronger notions wrt. density)**.** *Let $X \subseteq \mathbb{P}$. Consider the following properties of X:*

1. *X is dense.*

2. *X is open dense: X is open (downwards closed) and dense.*

3. *X is a maximal antichain.*

4. *X is predense: $\forall p \in \mathbb{P}\ \exists q \in X : p \not\perp q$.*

Then it holds that $(2) \Rightarrow (1) \Rightarrow (4)$, $(3) \Rightarrow (4)$, $(1) \Rightarrow \exists Y \subseteq X$ *with Y being a maximal antichain.*

Proof. See e.g. [16], Theorem III.3.60. □

Remark 8.2.4. The "vertical" notions (1),(2),(4) can be defined to hold *below* a given $p \in \mathbb{P}$ by relativizing the statements to p. For example, X is dense below some $q \in \mathbb{P}$ iff $\forall p \in \mathbb{P}$ with $p \leq q$: there exists $r \in X$ such that $r \leq p$.

We next provide a characterization of genericity using the "horizontal" notion (3). The connection between a maximal antichain A and a dense set D of $\mathbb{P}$ may be seen as follows: the maximality of A implies that any given condition $p \in \mathbb{P}$ must be compatible to some element of A.

Lemma 8.2.5. *Let $\mathbb{P} \in \mathcal{M}$ be a forcing poset, $G \subseteq \mathbb{P}$ a filter. The following are equivalent:*

1. *G is an $(\mathcal{M}, \mathbb{P})$-generic filter.*

2. *For any maximal antichain in $A \in \mathcal{M}$: $G \cap A \neq \emptyset$.*

Proof. $(1) \Rightarrow (2)$: Fix a maximal antichain A and show $G \cap A \neq \emptyset$. For this define $D_A = \{q \in \mathbb{P} : \exists r \in A\ (q \leq r)\}$. It holds that D_A dense in $\mathbb{P}$. Indeed, let $p \in \mathbb{P}$ such that $p \notin A$. Then $\exists r \in A\ (p \not\perp r)$ (by maximality of A). So $\exists q \in \mathbb{P}$ with $q \leq p, r$. But this means that $q \in D_A$. Next, show $G \cap A \neq \emptyset$: Since there exists $q \in G \cap D_A$, there is

$r \in A$ with $q \leq r$. As G is upwards closed, $r \in G \cap A$.

(2) $\Rightarrow$ (1): Take $D \subseteq \mathbb{P}$ to be a dense set in $\mathcal{M}$. Show that $G \cap D \neq \emptyset$. Consider the set $A_D \subseteq D$ of pairwise incompatible elements of D such that A_D is maximal with this property. If necessary, extend A_D to a maximal antichain $\hat{A}_D$ in $\mathcal{M}$. Then $\hat{A}_D \cap G \neq \emptyset$ per assumption. Let $p \in G \cap \hat{A}_D$. If $p \in D$, we are done, so assume that this is not the case, meaning that $p \in \hat{A}_D \backslash A_D$. By density of D, there exists $q \in D$ such that $q \leq p$. Consequently, $q \perp a$ for every $a \in \hat{A}_D \backslash \{p\}$, and so q cannot be an element of $\hat{A}_D$ (if it were, $q \perp p$, a contradiction). But this means that $\hat{A}_D \cup \{q\}$ is an antichain properly containing $\hat{A}_D$, in contradiction to the maximality of $\hat{A}_D$. $\square$

8.2.2 Countable chain condition and related notions

In the following, let $\mathbb{P}$ be a forcing poset in some ground model $\mathcal{M}$ as usual.

The countable chain condition

Definition 8.2.6 (Countable chain condition)**.** $\mathbb{P}$ is said to have the countable chain condition (ccc), if for any antichain $A \subseteq \mathbb{P}$, $|A| \leq \aleph_0$. Frequently we will also say that $\mathbb{P}$ is ccc for the sake of brevity.

In this work we will (albeit seldom) require a generalization of the ccc:

Definition 8.2.7 (κ-cc)**.** For a given cardinal κ, $\mathbb{P}$ has the κ-cc if for every antichain $A \subseteq \mathbb{P}$, $|A| < \kappa$ (e.g. *ccc* $= \aleph_1$*-cc*).

Next, consider two results which depend on the ccc.

Cardinal preservation

Proposition 8.2.8 (Cardinal preservation)**.** *Any ccc forcing poset $\mathbb{P}$ (in $\mathcal{M}$) preserves cofinalities and therefore cardinals.*

Proof. This is Theorem IV.3.4 from [16]. $\square$

Nice names and the ccc

Given a ccc forcing poset it is possible to find an upper bound for the number of nice names.

Lemma 8.2.9. *Let $\tau \in V^{\mathbb{P}}$ and fix $X \subseteq \tau$. Assume that κ, λ are infinite cardinals such that $|\mathbb{P}| = \kappa$ and $|dom(\tau)| = \lambda$. If $\mathbb{P}$ has the ccc, then*

$$|\{\tau_X : \tau_X \text{ nice name for } X\}| \leq \kappa^{\lambda}.$$

Proof. Since $\mathbb{P}$ is ccc, for every $\sigma \in dom(\tau)$ with associated antichain $A_\sigma \subseteq \mathbb{P}$, $|A_\sigma| \leq \aleph_0$. Thereby the number of possible antichains in $\mathbb{P}$ is $\leq |\mathbb{P}|^{\aleph_0} = \kappa^{\aleph_0}$. Now fix $X \subseteq \tau$. Then the number of possibilities to assign to X a nice name is $\leq (\kappa^{\aleph_0})^{|dom(\tau)|} = (\kappa^{\aleph_0})^{\lambda}$ because we assign to each $\sigma \in dom(X)$ some antichain A_σ by definition of a nice name. It follows that the total number of possibilities to do is given by at most $(\kappa^{\aleph_0})^{\lambda} = \kappa^{\lambda}$, as $\lambda \geq \aleph_0$. $\square$

Notions stronger than the ccc

Definition 8.2.10 (Centered, σ-centered)**.** Let $\mathbb{P}$ be a forcing poset. $\mathbb{P}$ is σ-centered if $\mathbb{P} = \bigcup_{i<\omega} C_i$, where every $C_i \subseteq \mathbb{P}$ is centered, that is, $\forall n \in \omega, \forall p_1, p_2, \ldots, p_n \in C_i \exists q \in \mathbb{P}$ with $q \leq p_i$ for all $i = 1, \ldots, n$.

Remark 8.2.11 (Antichains and centeredness). Let $A \subseteq \mathbb{P}$ be an antichain. Then $|A \cap C| \leq 1$ for any $C \subseteq \mathbb{P}$ which is centered.

Lemma 8.2.12. $\mathbb{P}$ *σ-centered* $\Rightarrow \mathbb{P}$ *ccc.*

Proof. Let $A \subseteq \mathbb{P}$ be an antichain and assume towards a contradiction that A is uncountable. As $A \subseteq \bigcup_{i \in \omega} C_i$, there exist $a_1 \neq a_2 \in A$ so that $a_1, a_2 \in C_i$ for some $i \in \omega$, in contradiction to the previous remark. $\square$

In Definition 4.1.8 another related notion, the so-called Knaster property, has been introduced, where we find that $\mathbb{P}$ σ-centered $\Rightarrow \mathbb{P}$ Knaster $\Rightarrow \mathbb{P}$ ccc.

8.2.3 The Delta System Lemma

Proposition 8.2.13 (Δ-system Lemma)**.** *Every uncountable collection of finite sets contains an uncountable Δ-system, i.e., a subfamily of finite sets with common intersection (or root) R.*

For us, obtaining a Δ-system is useful for "thinning out purposes" e.g. when trying to prove that certain forcing notions are Knaster (such as in Lemma 4.2.5) or in isomorphism of names arguments (Chapter 6). Let us state and prove a generalized version of the above:

Theorem 8.2.14 (Generalized Δ-system Lemma)**.** *Let κ, θ be cardinals with $\omega \leq \kappa < \theta$, θ regular, and the property that for all $\alpha < \theta$ ($|\alpha^{<\kappa}| < \theta$). Consider a family $\mathcal{A}$ of cardinality $|\mathcal{A}| = \theta$ such that $|a| < \kappa$ for all $a \in \mathcal{A}$. Then there exists a family $B \in [\mathcal{A}]^\theta$ which forms a Δ-system.*

Proof. As $|\mathcal{A}| = \theta$, $|\bigcup \mathcal{A}| \leq \theta$, $\bigcup \mathcal{A}$ being a union of θ-many elements, each of cardinality less than θ. Therefore, there exists an injection $\bigcup \mathcal{A} \to \theta$ which can be used to identify $\bigcup \mathcal{A} \subseteq \theta$ (without loss, as we are not interested in the particular form of the elements of $\mathcal{A}$). With this identification in place, for any $x \in \mathcal{A}$, $x \subseteq \theta$, and in particular $type(x) < \kappa$ because $|x| < \kappa$ by assumption of the theorem. Using the infinite pigeonhole principle it follows that since $\theta = cf(\theta)$ and $\theta > \kappa$, there must be some $\rho < \kappa$ such that $type(x) = \rho$ for θ-many elements $x \in \mathcal{A}$. In other words $|\mathcal{A}_1| = \theta$, where

$$\mathcal{A}_1 := \{x \in \mathcal{A} : type(x) = \rho\}.$$

Fix such ρ and consider an arbitrary $\alpha < \theta$. Then our assumption that $|\alpha^{<\kappa}| < \theta$ implies that the set $\{x \in \mathcal{A}_1 : x \subseteq \alpha\}$ must have cardinality less than θ. Indeed, any element of $\{x \in \mathcal{A}_1 : x \subseteq \alpha\}$ can be thought of as a $< \kappa$-sequence of elements in α, by $|x| < \kappa$ and $x \subseteq \alpha$. Therefore, $|\{x \in \mathcal{A}_1 : x \subseteq \alpha\}| \leq |\alpha^{<\kappa}| < \theta$. Now $\bigcup \mathcal{A}_1$ is unbounded in θ, for if we take any $\alpha \in \theta$, it must follow that $|\{x \in \mathcal{A}_1 : x \subseteq \alpha\}| < \theta$ by the above, while at the same time $|\mathcal{A}_1| = \theta$. So in order to "fill up" all of θ using the elements of $\mathcal{A}_1$, there has to be an element $x \in \mathcal{A}_1$ such that $x \supsetneq \alpha$, i.e., there exists $y \in \bigcup \mathcal{A}_1$ with $y > \alpha$ (note that $y \in x$ and $y \notin \alpha$ means that since $\mathcal{A} \subseteq \theta$, $x \in \theta$ implies that $y \in \theta$. In conclusion, $y \notin \alpha \Rightarrow \alpha \in y$).

Next, in order to be able to address the elements of $\bigcup \mathcal{A}_1$, define $x(\xi)$ to be the ξth element of $x \in \mathcal{A}_1$, for any $\xi < \rho$. By regularity of θ, there is $\xi \in \rho$ such that $\{x(\xi) \in \bigcup \mathcal{A}_1 : x \in \mathcal{A}_1\}$ is unbounded in θ. Take ξ_0 to be the least such ξ and define

$$\alpha_0 = \sup\{x(\eta) + 1 : x \in \mathcal{A}_1,\ \eta < \xi_0\}.$$

It follows that $\alpha_0 < \theta$ and $x(\eta) < \alpha_0$ for any $x \in \mathcal{A}_1, \eta < \xi_0$.

Using transfinite recursion on $\mu < \theta$, define $x_\mu \in \mathcal{A}_1$ so that $x_\mu(\xi_0) > \max\{\alpha_0, \sup\{x_\xi(\nu) : \eta < \rho, \nu < \mu\}\}$. To collect all such x_μ, define $\mathcal{A}_2 = \{x_\mu : \mu < \theta\}$. We note that $|\mathcal{A}_2| = \theta$, and for any $x \neq y \in \mathcal{A}_2$, $x \cap y \subseteq \alpha_0$. Now by assumption, $|\alpha_0^{<\kappa}| < \theta$, so there exists $R \subseteq \alpha_0$ and $B \subseteq \mathcal{A}_2$ with $|B| = \theta$, and $x \cap \alpha_0 = R$ for any $x \in B$, as wanted. $\square$

8.2.4 Describe sets of reals via maximal antichains

For us, a real can be a function in the Baire space ω^ω or a set in $[\omega]^\omega$.

Providing a full description via maximal antichains

For $\mathbb{P}$ a given ccc forcing poset consider a $\mathbb{P}$-name $\dot{\mathcal{A}}$ for a family of reals $\mathcal{A}$ with $|\mathcal{A}| = \kappa$. It is possible to write:

(a) $\dot{\mathcal{A}} = \{\dot{f}^\alpha : \alpha < \kappa\}$, where $\dot{f}^\alpha$ is a $\mathbb{P}$-name for a function $f \in \omega^\omega$.

(b) $\dot{\mathcal{A}} = \{\dot{a}_\alpha : \alpha < \kappa\}$, with $\dot{a}_\alpha$ a $\mathbb{P}$-name for a set $a \in [\omega]^\omega$.

In both cases we can basically use the same nice name approach to obtain:

Lemma 8.2.15. *For all $\alpha < \kappa$ we may find maximal antichains $A_n^\alpha = \{p_{n,i}^\alpha : i \in \omega\} \subseteq \mathbb{P}$, sets of values $\{k_{n,i}^\alpha : i \in \omega\} \subseteq \{0,1\}$ for (a), and $\{k_{n,i}^\alpha : i \in \omega\} \subseteq \omega$ for (b), so that each $p_{n,i}^\alpha$ decides whether $(n, k_{n,i}^\alpha) \in f^\alpha$ respectively $n \in a_\alpha$:*

$$p_{n,i}^\alpha \Vdash \text{"}\dot{f}^\alpha(n) = k_{n,i}^\alpha\text{"}. \qquad \textit{Case (a)}$$
$$p_{n,i}^\alpha \Vdash \text{"}n \in \dot{a}_\alpha \textit{ iff } k_{n,i}^\alpha = 1\text{"}. \qquad \textit{Case (b)}$$

Notice that in (a), $\dot{f}^\alpha(n)$ is a $\mathbb{P}$-name for the value $f^\alpha(n) \in \omega$. Also, we are taking great liberty in omitting check names for the sake of readability.

Proof. Fix arbitrary $\alpha < \kappa$.

For (a): The $\mathbb{P}$-name $\dot{f}^\alpha$ can be viewed without loss as a nice name (Section 8.2.1). Indeed, a nice name for $\dot{f}^\alpha$ is a $\mathbb{P}$-name of the form

$$\bigcup \{\{\sigma\} \times A_\sigma^\alpha : \sigma \in dom(\dot{f}^\alpha)\},$$

where each $A_\sigma^\alpha \subseteq \mathbb{P}$ forms an antichain. As in the proof of Lemma 8.2.2 it is possible to assume that every A_σ^α is maximal such that

$$\forall p \in A_\sigma^\alpha : \ p \Vdash_{\mathbb{P}} \text{"}\sigma \in \dot{f}^\alpha\text{" iff } p \Vdash_{\mathbb{P}} \text{"}\sigma = \langle n, k^\alpha \rangle = \dot{f}^\alpha(n)\text{"},$$

for fitting values $k^\alpha \in \omega$. By $\mathbb{P}$ having the ccc, A_σ^α is countable. Therefore, associating σ to n we can write

$$A_n^\alpha := A_\sigma^\alpha = \{p_{n,i}^\alpha \in \mathbb{P} : p_{n,i}^\alpha \Vdash \text{"}\dot{f}^\alpha(n) = k_{n,i}\text{"}, i \in \omega\}.$$

Thus A_n^α encodes the value $f^\alpha(n)$ for every $n \in \omega$. So overall, the sequence of maximal antichains $\{A_n^\alpha\}_{n\in\omega}$, and therefore the sequence of incompatible conditions $\{p_{n,i}^\alpha : n, i \in \omega\}$, determines $\dot{f}^\alpha$ completely. As $\alpha < \kappa$ was arbitrary, $\dot{\mathcal{A}}$ is completely determined.

For (b) we provide a slightly different approach: Fix $n \in \omega$. Then either $n \in a_\alpha$ or not. Thus, we can find $p_n^\alpha, q_n^\alpha \in \mathbb{P}$ such that $p_n^\alpha \Vdash$ "$\check{n} \in \dot{a}_\alpha$" and $q_n^\alpha \Vdash$ "$\check{n} \notin \dot{a}_\alpha$", where $p_n^\alpha \perp q_n^\alpha$. Using Zorn's Lemma, extend the antichain $\{p_n^\alpha, q_n^\alpha\}$ to a maximal antichain A_n^α. By the ccc, we can write $A_n^\alpha = \{p_{n,i}^\alpha : i \in \omega\}$. To obtain the $k_{n,i}^\alpha$, consider a function $g : A_n^\alpha \to \{0, 1\}$ such that

$$g(p_{n,i}^\alpha) = 1 \text{ iff } p_{n,i}^\alpha \Vdash \text{"}\check{n} \in \dot{a}_\alpha\text{"}.$$

By definition of A_n^α, g is well-defined. Then set $k_{n,i}^\alpha := g(p_{n,i}^\alpha)$. $\square$

Remark 8.2.16 (On the converse direction). Given $\{p_{n,i}^\alpha \in \mathbb{P} : i, n \in \omega, \alpha \in \kappa\}$ and $\{k_{n,i}^\alpha \in \omega : i, n \in \omega, \alpha \in \kappa\}$ with the property that

$$\forall \alpha < \kappa \; \forall n \in \omega, \{p_{n,i}^\alpha : i \in \omega \wedge p_{n,i}^\alpha \Vdash \text{"}\dot{f}^\alpha(n) = k_{n,i}^\alpha\text{"}\} \text{ is a maximal antichain,}$$

there exist $\mathbb{P}$-names $\dot{f}^\alpha$ for functions $f^\alpha \in \omega^\omega$.

8.3 Cohen forcing and its applications

After shortly introducing Cohen forcing $\mathbb{C}$ (8.3.1), we proceed by defining the poset $\mathbb{C}_I$ to add multiple Cohen reals at once, and use $\mathbb{C}_I$ to provide a proof that consistently there exists an independent family of any given cardinality (8.3.2). This is followed up by a discussion of the splitting properties of a Cohen real in Section 8.3.3, which shows that the independent family obtained in the previous section is not maximal (Remark 8.3.3). Both of these facts will play a role in motivating the approach to consistently prove that $C \subseteq Spec(\mathfrak{i})$ for arbitrary C (see Section 4.1.4 and in particular Discussion 4.1.16). Furthermore, the splitting properties will be helpful in a discussion of the problem of proving that $\kappa \in Spec(\mathfrak{i})$ for $cf(\kappa) = \aleph_0$ with a certain finite support iteration construction (Remark 4.1.2). After shortly considering the unboundedness of Cohen reals in 8.3.4, Section 8.3.5 provides a side fact appearing during the introduction to isomorphism of names arguments (6.1.1), namely that consistently, there exists an almost disjoint family of cardinality $\aleph_1$. Finally, in Section 8.3.6 we collect all previous results to provide $Con(\mathfrak{a} < \mathfrak{i})$.

For the statements mentioned in 8.3.1 and Lemma 8.3.1, see [16]. Section 8.2.4 is based on the first part of the proof of Proposition 3.1 in [3]. The results from 8.3.5 are taken from [15]. The other statements may be considered as folklore.

8.3.1 Simple Cohen forcing

Cohen forcing can be thought of as a forcing notion to consistently prove the existence of a "new" infinite subset of ω. For any given sets A and B, let $Fn(A, B)$ denote the set of all finite partial functions $p : A \to B$. Then define $\mathbb{C}$ to be the Cohen forcing poset given by $(Fn(\omega, 2), \leq)$, where for any $p_1, p_2 \in \mathbb{C}$, $p_1 \leq p_2$ iff $p_1 \supseteq p_2$, that is, p_1 extends p_2 as a function, and $1_{\mathbb{C}} := \emptyset$. $\mathbb{C}$ may be considered to be the simplest variant of Cohen forcing, and it can be used to illustrate the meaning of the forcing terminology in a relatively straightforward way. Furthermore, the construction of a generic object using $\mathbb{C}$ already presents some key ideas which will be reused in more complicated forcing posets. One of these ideas is to define the posets so as to make sure that the generic object will be an infinite subset of ω. Instead of further discussing these aspects of $\mathbb{C}$, we wish to introduce a variant of this simple kind of Cohen forcing which will be of more use to us.

8.3.2 Application 1 - Side-by-side Cohen reals

For any index set I, define the Cohen forcing poset $\mathbb{C}_I := Fn(I \times \omega, 2)$ adding $|I|$-many side-by-side Cohen reals, which forces the existence of an independent family $\{c_i : i \in I\}$ of Cohen reals c_i. Formally, any such c_i can be obtained as follows. Fix a $\mathbb{C}_I$-generic filter G over the ground model. Then for each $i \in I$, let

$$c_i := \{n \in \omega : \exists p \in G(p(i, n) = 1)\}.$$

Lemma 8.3.1 (Cohen forcing and the ccc)**.** *Let I, J be index sets. Then $I \neq \emptyset$ or $|J| \leq \aleph_0$ is equivalent to $Fn(I, J)$ having the ccc.*

Proof. This is Lemma III.3.7 in [16]. □

Lemma 8.3.2. *Let I be an arbitrary index set with $|I| \geq \omega$, and G a $\mathbb{C}_I$-generic filter over the ground model V. Then consistently $\mathcal{A} := \{c_i : i \in I\}$ forms an independent family.*

Proof. We proceed with a density argument. For each $n \in \omega$ and any disjoint, non-empty $I_1, I_2 \in [I]^{<\omega}$ define the sets $D^n_{I_1,I_2} \subseteq \mathbb{C}_I$ by

$$D^n_{I_1,I_2} = \{p : \exists m > n \forall i \in I_1 \forall j \in I_2((i, m), (j, m) \in dom(p) \wedge p(i, m) = 1 \wedge p(j, m) = 0)\}.$$

Claim. *Any such $D^n_{I_1,I_2}$ is dense in $\mathbb{C}_I$.*

Proof. For any $p \in \mathbb{C}_I$ with $(i,m),(j,m) \notin dom(p)$ we can find an extension in $D^n_{I_1,I_2}$ by considering the union of p with the missing elements and approriate values. Pick any $p \in \mathbb{C}_I$ such that for all $m > n$ there are $i \in I_1$ and $j \in I_2$ so that $(i,m),(j,m) \in dom(p)$ and $p(i,m) = 0$ or $p(j,m) = 1$. Assume without loss that both is the case. Then we can define $p' \in \mathbb{C}_I$ extending p such that for all $i \in I_1$, $j \in I_2$ with $(i,m'),(j,m') \in dom(p')$, $p'(i,m') = 1$ and $p'(j,m') = 0$, since $dom(p)$ is finite (and so there must be some $m' > n$ as in the definition of $D^n_{I_1,I_2}$). As $p' \in D^n_{I_1,I_2}$, the statement follows. $\square_{Claim}$

Since the $D^n_{I_1,I_2}$ are in the ground model, we have $G \cap D^n_{I_1,I_2} \neq \emptyset$. In other words, for any $n \in \omega$ and any I_1, I_2 as above we have some $p \in G$ witnessing that there is $m > n$ so that $p(i,m) = 1$ ($m \in c_i$), and at the same time $p(j,m) = 0$ respectively $m \in \omega \setminus c_j$ ($m \notin c_j$).

With this we can simultaneously make sure that any $c_i \in \mathcal{A}$ is infinite and that $\mathcal{A}$ is independent. To see this, take any Boolean combination of Cohen reals $c_{i_1} \cap \ldots \cap c_{i_k} \cap (\omega \setminus c_{j_1}) \cap \ldots \cap (\omega \setminus c_{j_l})$ indexed by I, and set $I_1 := \{i_x : x = 1,2,\ldots,k\}$ and $I_2 := \{j_y : y = 1,2,\ldots,l\}$. Fix $n \in \omega$ and take $p_1 \in G \cap D^n_{I_1,I_2}$. Then there exists $m_1 > n$ with $p_1(i_x, m_1) = 1$ and $p_1(j_y, m_1) = 0$ for any x, y. Thereby, $m_1 \in c_{i_x}$ and $m_1 \notin c_{j_y}$ for arbitrary fixed c_{i_x}, c_{j_y}. In this manner, as n is arbitrary, we make sure that infinitely many elements are contained in the Boolean combination above, and since this holds for any I_1, I_2, the statement follows. Finally, any c_i is infinite, as it is possible to find a Boolean combination in the above manner where $i = i_1$. $\square$

Remark 8.3.3 (The independent family forced by $\mathbb{C}_I$ is not maximal.). Consider an independent family $\mathcal{A} := \{c_\alpha : \alpha \in I\}$ as above. Assume towards a contradiction that there exists $x \in [\omega]^\omega \setminus \mathcal{A}$ such that $|x \cap \mathcal{A}^h| < \omega$ for some Boolean combination $\mathcal{A}^h$ of $\mathcal{A}$ (that is, $h \in FF(\mathcal{A})$ with $\mathcal{A}^h = \bigcap_{c \in h^{-1}(1)} c \;\cap\; \bigcap_{c \in h^{-1}(0)} \omega \setminus c$). As shown in the upcoming section, for any $\alpha \in I$, c_α splits all ground model reals: $\forall x \in [\omega]^\omega \cap V$, $|c_\alpha \cap x| = \aleph_0 = |c_\alpha \cap (\omega \setminus x)|$. Therefore $x \cap \bigcap_{c \in h^{-1}(1)} c$ is infinite, and so by our assumption, $x \cap \bigcap_{c \in h^{-1}(0)} \omega \setminus c$ must be finite. This implies that $\omega \setminus c_\beta \cap x$ is finite for some $\beta \in I$, or in other words, that $\omega \subseteq^* c_\beta \cup \omega \setminus x$. But $\omega \not\subseteq^* c_\beta$, and so $\omega \subseteq^* \omega \setminus x$, a contradiction to $|x| = \omega$.

8.3.3 Application 2 - Cohen real splits all ground model reals

We want to show that in $\mathcal{M}[G]$, the set a_G splits every given $b \in [\omega]^\omega \cap \mathcal{M}$, meaning that in $\mathcal{M}[G]$, $a_G \cap b$ and $b \setminus a_G$ are infinite. The dense sets $D_n := \{p \in \mathbb{P} : \exists m \in dom(p) : m \geq n \wedge p(m) = 1\}$ ensure that $a_G = \{n \in \omega : f_G(n) = 1\} \in [\omega]^\omega$. Next, fix arbitrary $b \in [\omega]^\omega \cap \mathcal{M}$. In order to guarantee that each approximation adds elements of b to f_G

while also leaving out elements of b, find suitable dense sets. For any $n \in \omega$, let

$$D_n^b := \{p \in \mathbb{C} : \exists m \in dom(p)(m \geq n \wedge p(m) = 1 \wedge m \in b)\}.$$

Fix n, b as above. To see that D_n^b is dense, take any $p \in \mathbb{C}$, and assume that $\forall m \in dom(p)$, $m < n \vee p(m) = 0 \vee m \notin b$. The first two cases of the disjunction can be taken care of by a standard argument (the same used for the D_n above). Consider the case where $m \notin b$. Since $b \in [\omega]^\omega$, using the infinite pigeonhole principle find $m' \in b$ with $m' > m$, and add it to p via $p' := p \cup \{(m', 1)\}$. Then $p' \leq p$ and $p' \in D_n^b$ as we wanted.
The sets D_n^b guarantee that $f_G(m) = 1$ for infinitely many $m \in \omega$, that is, $|a_G \cap b| = \aleph_0$. Since $G \cap D_n^b \neq \emptyset$, in $\mathcal{M}[G]$ find for any given $n \in \omega$ some $p \in G \cap D_n^b$, and $m \geq n$ so that $f_G(m) = p(m) = 1$. Thus, $m \in a_G$. As $m \in b$, $m \in a_G \cap b$.

We still need to make sure that a_G does not contain infinitely many elements of b. To this end, for $n \in \omega$ and $b \in [\omega]^\omega \cap \mathcal{M}$ define the sets

$$\tilde{D}_n^b = \{p \in \mathbb{C} : \exists m \in dom(p)(m \geq n \wedge p(m) = 0 \wedge m \in b)\}.$$

To see that these sets are dense in $\mathbb{C}$, proceed as above. It follows that for any $n \in \omega$ there is $m \geq n$ with $m \notin a_G$ (as $f_G(m) = 0$) but $m \in b$.

8.3.4 Application 3 - Unboundedness of Cohen reals

Consider the $(\mathcal{M}, \mathbb{C})$-generic filter G, and let $f_G = \bigcup G \in \omega^\omega$ be a Cohen real in $\mathcal{M}[G]$. Then f_G is unbounded with respect to all ground model functions $f \in \omega^\omega \cap \mathcal{M}$. For the proof, use density arguments, sketched as follows:

- $dom(f_G) = \omega$ can be seen by employing the dense sets $D_n = \{p \in \mathbb{C} : n \in dom(p)\}$ for every $n \in \omega$.

- Define $D_{f,k} = \{p \in \mathbb{C} : k \in dom(p) \wedge p(k) > f(k)\}$, where $k \in \omega$ and $f \in \omega^\omega \cap \mathcal{M}$. These sets are dense, as witnessed by the extension $q \supseteq p$ of any given p with $q := p \cap \{(k+1, m)\}$, $m > f(k+1)$. Finally, show that for arbitrary $n \in \omega$, $f \in \omega^\omega \cap \mathcal{M}$ there is $m \in \omega$ such that $m \geq n$ and $(f_G(m) > f(m))^{\mathcal{M}[G]}$. Fix $m \geq n$. Then there exists $p \in D_{f,m} \cap G$, and in $\mathcal{M}[G]$ it holds that $f_G(m) = p(m) > f(m)$.

8.3.5 Application 4 - A mad family of cardinality $\aleph_1$

In the following we wish to prove the statement that consistently, $\aleph_1 \in Spec(\mathfrak{a})$. As a preparation, consider the next lemma providing us with a way to reduce the amount of information necessary to fully describe a (potentially) generic object in the extension obtained via Cohen forcing, as long as it is possible to find a bound in the ground model for this object.

Lemma 8.3.4. *Let $I, S \in \mathcal{M}$ and G be an $(\mathcal{M}, Fn(I,2))$-generic filter. Consider $X \in \mathcal{M}[G]$ and assume that $X \subseteq S$. Then there exists $I_0 \subseteq I$ with $I_0 \in \mathcal{M}$ and $|I_0| \leq |S|$ in $\mathcal{M}$, such that $X \in \mathcal{M}[G \cap Fn(I_0, 2)]$.*

Proof. Say $|S| < \aleph_0$. Then certainly $X \in \mathcal{M}$, as we can construct any finite set in $\mathcal{M}$. Therefore the statement holds for X. So assume that $|S| \geq \aleph_0$. Consider the name $\check{S} = \{(\check{x}, \emptyset) : x \in S\}$. As $X \subseteq S$, it is possible to find a nice name τ for a subset of $\check{S}$ with $\tau_G = X$ (see Lemma 8.2.2). Per definition, $\tau = \bigcup_{x \in S}(\{\check{x}\} \times A_x)$, where A_x is an antichain in $Fn(I,2)$. Now we are ready to define

$$I_0 = \bigcup\{dom(p) : \exists x \in S(p \in A_x),\ p \in Fn(I,2)\}.$$

As $I \in \mathcal{M}$, the same holds for its power set $\mathcal{P}(I)$, and thus $I_0 \in \mathcal{M}$. Clearly, $I_0 \subseteq I$, and since each $dom(p) \subseteq I$ is finite and every A_x countable (by $Fn(I,2)$ having the ccc), I_0 is a union of at most $|S|$-many finite sets, implying $|I_0| \leq |S|$. Define $G_0 = G \cap Fn(I_0, 2)$. It remains to show that $X \in \mathcal{M}[G_0]$. By definition of I_0, $A_x \subseteq Fn(I_0, 2)$ for any $x \in S$. Thereby τ, consisting only of conditions in A_x for $x \in S$, is an $Fn(I_0, 2)$-name. It follows that evaluating τ with respect to G leads to the same result as evaluating it via G_0:

$$\tau_G = \{\sigma_G : \exists p \in G((\sigma, p) \in \tau)\} = \{x \in S : \exists p \in G_0(p \in A_x)\} = \tau_{G_0}.$$

□

Proposition 8.3.5. *Assume CH in the ground model $\mathcal{M}$ and let $I \in \mathcal{M}$. Then $(\exists \mathcal{A} : \mathcal{A}$ is a mad family of cardinality $\aleph_1)^{\mathcal{M}[G]}$, where G is an $(\mathcal{M}, Fn(I,2))$-generic filter.*

Proof. We want to construct a mad family in $\mathcal{M}$ already of size $\aleph_1$ and show that it is Cohen indestructible, meaning that for any $I \in \mathcal{M}$ and G an $(\mathcal{M}, Fn(I,2))$-generic filter, $\mathcal{M}[G] \models$ "$\mathcal{A}$ is maximal".

Part 1 - Simplification: Employing the lemma above, we may work with countable $I_0 \in \mathcal{M}$ instead of I, together with corresponding generic extensions. Indeed, for all $X \in \mathcal{M}[G]$, $X \subseteq \omega$, there is a countable $I_0 \in \mathcal{M}$ such that $X \in \mathcal{M}[G \cap Fn(I_0, 2)]$. Because of the following claim, from now on consider only the forcing $Fn(\omega, 2)$.

Claim. *It suffices to define $\mathcal{A}$ so that if G is $(\mathcal{M}, Fn(\omega, 2))$-generic, then there cannot exist $x \in [\omega]^\omega \cap \mathcal{M}[G]$ almost disjoint from all $a \in \mathcal{A}$.*

Proof. Knowing that I_0 is countable, we have the following cases. With the restriction $G_0 := G \cap Fn(I_0, 2)$ as before, note that if $(|I_0| < \omega)^{\mathcal{M}}$, $\mathcal{M}[G_0] = \mathcal{M}$. If on the other hand $(|I_0| = \omega)^{\mathcal{M}}$, then in $\mathcal{M}$, $Fn(I_0, 2) \cong Fn(\omega, 2)$, and so $\mathcal{M}[G] = \mathcal{M}[G_0]$ for any $(\mathcal{M}, Fn(\omega, 2))$-generic filter G (this can be seen e.g. by employing our definition of forcing equivalence (8.4.9), and noting that having an order-isomorphism suffices for the definition to hold.). $\square_{Claim}$

Part 2 - Construction of the wanted mad family $\mathcal{A} \subseteq [\omega]^\omega$: Since $|Fn(\omega, 2)| = \aleph_0$, and the number of nice names for subsets of ω is $2^{\aleph_0} = \aleph_1$ (CH), we can enumerate the set of such tuples as $\{(p_\xi, \tau_\xi) : \xi < \omega_1\}$, where $p_\xi \in Fn(\omega, 2)$, and τ_ξ denotes such a nice name. We can even start counting at $\aleph_0$ leading to the enumeration $\{(p_\xi, \tau_\xi) : \omega \leq \xi < \omega_1\}$.

Construct $\mathcal{A}$ recursively as follows. To take care of the first ω-many elements of $\mathcal{A}$, it is always possible to find a family of pairwise disjoint sets $a_n \in [\omega]^\omega$. If on the other hand, ξ is such that $\omega \leq \xi < \omega_1$, assume that the sets a_η are defined for all $\eta < \xi$. In particular, as $\xi < \aleph_1$, the number of such sets is countable.

Then obtain a_ξ with the properties:

1. $\forall \eta < \xi \ (|a_\eta \cap a_\xi| < \aleph_0)$,

2. If $p_\xi \Vdash$ "$|\tau_\xi| = \aleph_0$" and for all $\eta < \xi$, $p_\xi \Vdash$ "$|\tau_\xi \cap \check{a}_\eta| < \aleph_0$", then for any $n \in \omega$ and any $q \leq p_\xi$ there is $m \geq n$ such that $m \in a_\xi$ and $r \Vdash$ "$\check{m} \in \tau_\xi$".

Claim. *Such a set a_ξ exists.*

Proof. Assuming that the antecedent in (2) is false, the whole implication is true, so in this case it remains to find a_ξ fulfilling (1). But since there exists no mad family of cardinality $\aleph_0$, given the almost disjoint family $\{a_\eta : \eta < \xi\}$, there must be some $a_\xi \in [\omega]^\omega \setminus \{a_\eta : \eta < \xi\}$ as in (1). Therefore, assume that the antecedent in (2) holds. Enumerate the sets $\{a_\eta : \eta < \xi\}$ and $\omega \times \{q : q \leq p_\xi\}$ by $\{b_k : k \in \omega\}$ and $\{(n_i, q_i) : i \in \omega\}$, respectively. The latter is possible because $\omega \times Fn(\omega, 2)$ is countably infinite. By the antecedent in (2), it holds that for any $i \in \omega$, $q_i \Vdash$ "$|\tau_\xi \setminus (\bigcup_{k=0}^{i} \check{b}_k)| = \aleph_0$". Thus, there exists some $r_i \leq q_i$ and $m_i \geq n_i$ such that $m_i \notin \bigcup_{k=0}^{i} b_k$, but $r_i \Vdash$ "$m_i \in \tau_\xi$". Finally, define $a_\xi = \{m_i : i \in \omega\}$, with m_i as above. $\square_{Claim}$

Part 3 - Show that $\mathcal{A} := \{a_\xi : \xi < \omega_1^{\mathcal{M}}\}$ is as stated in the proposition:

Claim. *$\mathcal{A}$ forms an almost disjoint family of cardinality $\aleph_1$ in $\mathcal{M}$. Furthermore, $\mathcal{A}$ is maximal in $\mathcal{M}[G]$.*

Proof. Certainly $\mathcal{A}$ has cardinality $\aleph_1$ in $\mathcal{M}$. The fact that $\mathcal{M} \models$ "$\mathcal{A}$ almost disjoint family" follows from the definition of the a_η and the a_ξ (points (1) and (2) above).

We wish to show that $\mathcal{A}$ is maximal. Assume this is not the case. Then there must be some (p_ξ, τ_ξ) with $p_\xi \in G$ so that $p_\xi \Vdash$ "$|\tau_\xi| = \omega$" and $p_\xi \Vdash$ "$\forall \check{a} \in \check{\mathcal{A}}$ $(|\tau_\xi \cap \check{a}| < \aleph_0)$". This means that the antecedent from (2) also holds for this ξ. Yet $p_\xi \Vdash$ "$\tau_\xi \cap \check{a}_\xi$ is finite". Hence there is $q \leq p$ and $n \in \omega$ such that $q \Vdash$ "$\tau_\xi \cap \check{a}_\xi \subseteq \check{n}$", contradicting (2). $\square_{Claim}$

Using the simplification from part 1, the family $\mathcal{A}$ is as we wanted. $\square$

8.3.6 Application 5 - $\mathfrak{a} < \mathfrak{i}$

In Section 8.3.5 we used Cohen forcing $\mathbb{C}_I$ to find that $Con(\mathfrak{a} = \aleph_1)$. Take I such that $|I| = \lambda$ with $\sigma < \lambda$ being the pre-spectrum parameters (Definition 4.3.10), and consider the model $V^{\mathbb{P}_{\bar{s}}}$ from Theorem 4.3.12. On the other hand, by Section 8.3.4, every Cohen real is unbounded. Thus in the same model as above we can realize $\mathfrak{d} = \mathfrak{c} = \lambda$. Finally, use that $\mathfrak{i} \geq \mathfrak{d}$ (Theorem 1.2.6). Then in total $\mathfrak{a} = \sigma < \lambda = \mathfrak{d} \leq \mathfrak{i}$.

8.4 Comparing forcing notions

We want to introduce basic tools to be able to compare different forcing notions.

Definition 8.4.1 (Complete and dense embeddings, $\lessdot$)**.** Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing posets.

- Let $i : \mathbb{P} \to \mathbb{Q}$ be a function. Call i a complete embedding if $i(1_{\mathbb{P}}) = 1_{\mathbb{Q}}$, $p \leq_{\mathbb{P}} q$ implies $i(p) \leq_{\mathbb{Q}} i(q)$ and $p \perp_{\mathbb{P}} q$ iff $i(p) \perp_{\mathbb{Q}} i(q)$ for all conditions $p \in \mathbb{P}$, $q \in \mathbb{Q}$, and i maps any maximal antichain A in $\mathbb{P}$ to a maximal antichain $i(A)$ in $\mathbb{Q}$.
- Call $i : \mathbb{P} \to \mathbb{Q}$ a dense embedding if $i(1_{\mathbb{P}}) = 1_{\mathbb{Q}}$, $p \leq_{\mathbb{P}} q$ implies $i(p) \leq_{\mathbb{Q}} i(q)$, $p \perp_{\mathbb{P}} q$ iff $i(p) \perp_{\mathbb{Q}} i(q)$ for all conditions $p \in \mathbb{P}$, $q \in \mathbb{Q}$, and $i(\mathbb{P}) \subseteq \mathbb{Q}$ is dense in $\mathbb{Q}$.
- We say that $\mathbb{P}$ is completely contained in $\mathbb{Q}$, written $\mathbb{P} \lessdot \mathbb{Q}$, if $\mathbb{P} \subseteq \mathbb{Q}$, and
  1. $p \leq_{\mathbb{P}} p'$ implies $p \leq_{\mathbb{Q}} p'$ for all $p, p' \in \mathbb{P}$,
  2. $p \perp_{\mathbb{P}} p'$ implies $p \perp_{\mathbb{Q}} p'$ for all $p, p' \in \mathbb{P}$,
  3. For every $q \in \mathbb{Q}$ there is a so-called reduction of q to $\mathbb{P}$, $r \in \mathbb{P}$, such that for all $p \in \mathbb{P}$ with $p \leq_{\mathbb{P}} r$ it holds that $p \not\perp_{\mathbb{Q}} q$. (Reduction property)

Let us now consider a few first corollaries stemming from these definitions.

Lemma 8.4.2. *The following are equivalent:*

1. $\mathbb{P} \lessdot \mathbb{Q}$,
2. $\mathbb{P}$ *embeds into* $\mathbb{Q}$ ($\mathbb{P} \subseteq \mathbb{Q}$ *and for all* $p, q \in \mathbb{P}$*:* $p \leq_{\mathbb{P}} q$ *iff* $p \leq_{\mathbb{Q}} q$)*, and additionally the identity mapping is a complete embedding* $\mathbb{P} \to \mathbb{Q}$.

Proof. (1) $\Rightarrow$ (2): First one can see that $p \perp_{\mathbb{Q}} q$ implies $p \perp_{\mathbb{P}} q$ by considering the contrapositive. So let p and q be compatible in $\mathbb{P}$. Then there exists $r \in \mathbb{P}$ with $r \leq_{\mathbb{P}} p, q$. As $p, q, r \in \mathbb{Q}$, it follows that p and q are compatible in $\mathbb{Q}$ as witnessed by r. Secondly, show that maximal antichains are preserved: Let $A \subseteq \mathbb{P}$ be a maximal antichain in $\mathbb{P}$. The only way that A is not maximal in $\mathbb{Q}$ is that $\exists q \in \mathbb{Q} \setminus \mathbb{P}$ $(q \perp_{\mathbb{Q}} a)$ for all $a \in A$. Indeed, assume towards a contradiction that there is such $q \in \mathbb{Q} \setminus \mathbb{P}$. By assumption, q has a reduction $r \in \mathbb{P}$, that is, for all $p \in \mathbb{P}$ with $p \leq_{\mathbb{P}} r$, $p \not\perp_{\mathbb{Q}} q$. Now either $r \in A$ or $r \in \mathbb{P} \setminus A$. In the former case, $q \perp_{\mathbb{Q}} r$ contradicting that r is a reduction of q. So assume that $r \in \mathbb{P} \setminus A$. By maximality of A, there is $a \in A$ with $r \not\perp_{\mathbb{P}} a$, and so $r \not\perp_{\mathbb{Q}} a$. So overall, there are extensions $s_1, s_2 \in \mathbb{Q}$ such that $s_1 \leq r, a$ and $s_2 \leq r, q$. In particular, $s_1, s_2 \leq_{\mathbb{Q}} r$, implying that $s_1 \not\perp_{\mathbb{Q}} s_2$ (by assumption $\mathbb{Q}$ is atomless - see the intro to Chapter 8). But then $a \not\perp_{\mathbb{Q}} q$, in contradiction to the choice of q.

(2) $\Rightarrow$ (1): Let $q \in \mathbb{Q}$ be arbitrary. We wish to find a reduction of q to $\mathbb{P}$. Assume towards a contradiction that there is no such reduction. Thus for any $r \in \mathbb{P}$ there must be $p \in \mathbb{P}$ with $p \leq_{\mathbb{P}} r$ and $p \perp_{\mathbb{Q}} q$. But in this case, $q \in \mathbb{Q} \setminus \mathbb{P}$ because otherwise there is a reduction of r of q, namely $r = q$. Now let $A \subseteq \mathbb{P}$ be an arbitrary maximal antichain in $\mathbb{P}$. Then $A \cup \{q\} \supsetneq A$ forms an antichain in $\mathbb{Q}$: By assumption, for any $a \in A$ there exists $p_a \in \mathbb{P}$ so that $p_a \leq_{\mathbb{P}} a$ and $p_a \perp_{\mathbb{Q}} q$. Therefore, if $q \not\perp_{\mathbb{Q}} a$ for some $a \in A$, then the common extension $s \leq_{\mathbb{Q}} q, a$ must fulfill $s \not\perp_{\mathbb{Q}} p_a$, or else a is an atom of $\mathbb{Q}$. It follows that there exists $r \leq s, p_a$, and hence $r \leq q, p_a$ contradicting $q \perp_{\mathbb{Q}} p_a$. Therefore, A is not maximal in $\mathbb{Q}$, in contradiction to (1). $\square$

Lemma 8.4.3. *Every dense embedding is a complete embedding.*

Proof. It remains to be shown that for any maximal antichain $A \subseteq \mathbb{P}$ in $\mathbb{P}$, the image $i(A) \subseteq \mathbb{Q}$ is a maximal antichain in $\mathbb{Q}$. Since for every $a_1 \neq a_2 \in A$ it holds that $a_1 \perp_{\mathbb{P}} a_2$ iff $i(a_1) \perp_{\mathbb{Q}} i(a_2)$, $i(A)$ is an antichain in $\mathbb{Q}$. To see that $i(A)$ is maximal, assume otherwise. Then there exists some $b \in \mathbb{Q} \setminus i(A)$ with $b \perp_{\mathbb{Q}} i(a)$ for all $a \in A$. But since $i(\mathbb{P})$ is dense in $\mathbb{Q}$, there exists some $p \in \mathbb{P}$ so that $i(p) \leq_{\mathbb{Q}} b$, and so $i(p) \perp_{\mathbb{Q}} i(a)$ for all $a \in A$. This implies that $p \perp_{\mathbb{P}} a$ while $p \neq a$ for all $a \in A$, a contradiction to the maximality of A. $\square$

For an even stronger notion:

Lemma 8.4.4. *Every order-isomorphism is a dense embedding.*

Proof. Let $f : \mathbb{P} \to \mathbb{Q}$ be an order-isomorphism. Then $f(\mathbb{P})$ is dense in $\mathbb{Q}$ by surjectivity of f. $\square$

What is the relation between generic extensions of two forcing notions $\mathbb{P}, \mathbb{Q}$ for which there exists a complete embedding $i : \mathbb{P} \to \mathbb{Q}$?

Lemma 8.4.5. *For $\mathcal{M}$ a ctm of ZFC, let $\mathbb{P}, \mathbb{Q}, i \in \mathcal{M}$ be as above and H a $\mathbb{Q}$-generic filter over $\mathcal{M}$. Then $G := i^{-1}(H)$ forms a $\mathbb{P}$-generic filter over $\mathcal{M}$.*

Proof. G is a filter: The notion of a complete embedding is absolute, as the quantifiers occuring in its definition are bounded by $\mathbb{P}$ and $\mathbb{Q}$, respectively. To see that G is downwards directed take any $p, q \in G$. Then $i(p), i(q) \in H$, and since H is downwards directed, $i(p) \not\perp_{\mathbb{Q}} i(q)$. Thus by the above $p \not\perp_{\mathbb{P}} q$. Similarly, using that H is upwards closed we can show that the same holds for G.

G is generic: Use Lemma 8.2.5. Let A be a maximal antichain in $\mathbb{P}$. By i being a complete embedding we know that $i(A)$ is a maximal antichain in $\mathbb{Q}$. Now $i(A) \cap H \neq \emptyset$, and thus $i^{-1}(i(A) \cap H) = A \cap G \neq \emptyset$. $\square$

Instead of complete embeddings it is possible to consider so-called projections:

Definition 8.4.6. Call $\pi : \mathbb{Q} \to \mathbb{P}$ a projection if

1. $\pi(1_{\mathbb{Q}}) = 1_{\mathbb{P}}$,
2. whenever $q_1 \leq_{\mathbb{Q}} q_2$, $\pi(q_1) \leq_{\mathbb{P}} \pi(q_2)$,
3. for any $p \in \mathbb{P}$ and $q \in \mathbb{Q}$ it holds that if $p \leq_{\mathbb{P}} \pi(q)$, then there exists $q' \leq_{\mathbb{Q}} q$ such that $\pi(q') \leq_{\mathbb{P}} p$.

Lemma 8.4.7. $\mathbb{P} \lessdot \mathbb{Q}$ *implies that there exists a projection* $\mathbb{Q} \to \mathbb{P}$*.*

Proof. By $\mathbb{P} \lessdot \mathbb{Q}$, $1_{\mathbb{P}} = 1_{\mathbb{Q}} =: 1$. Let $\pi : \mathbb{Q} \to \mathbb{P}$ be given by

$$\pi(q) := \begin{cases} q, & \text{if } q \in \mathbb{P}, \\ 1, & \text{if } q \in \mathbb{Q} \backslash \mathbb{P}. \end{cases}$$

Then π is a projection. Indeed, (1) and (2) of the definition above are clear. For (3), take arbitrary $p \in \mathbb{P}$ and $q \in \mathbb{Q}$ and assume that $p \leq_{\mathbb{P}} \pi(q)$. As $\mathbb{Q}$ is separative (see the intro to this chapter) there must be some $q' \in \mathbb{Q}$ with $\pi(q') \leq_{\mathbb{P}} p$. □

Remark 8.4.8 (On the converse). The converse direction is also true, where we may assume $\mathbb{P} \subseteq \mathbb{Q}$ to hold. Concerning the reduction property: Let $q \in \mathbb{Q}$. Then $r = \pi(q)$ forms a reduction of q to $\mathbb{P}$, which easily follows from (3) of the definition of π.

Intuitively, forcing notions can be viewed as "equivalent" if they lead to the same generic extensions. Formalizing this idea we obtain the following definition:

Definition 8.4.9 (Forcing equivalence)**.** Two forcing notions $\mathbb{P}$ and $\mathbb{Q}$ are called (forcing) equivalent, in symbols $\mathbb{P} \sim \mathbb{Q}$, if ...

- ... for any $(\mathcal{M}, \mathbb{P})$-generic filter G there exists an $(\mathcal{M}, \mathbb{Q})$-generic filter H such that $\mathcal{M}[G] = \mathcal{M}[H]$.
- ... for any $(\mathcal{M}, \mathbb{Q})$-generic filter H there exists an $(\mathcal{M}, \mathbb{P})$-generic filter G such that $\mathcal{M}[G] = \mathcal{M}[H]$.

Note that $\sim$ indeed forms an equivalence relation. We next show that the existence of a dense embedding $i : \mathbb{P} \to \mathbb{Q}$ implies that $\mathbb{P} \sim \mathbb{Q}$:

Proposition 8.4.10. *Let* $\mathbb{P}, \mathbb{Q}, i \in \mathcal{M}$ *with* $\mathbb{P}, \mathbb{Q}$ *forcing posets and* $i : \mathbb{P} \to \mathbb{Q}$ *a dense embedding. Then:*

1. *Any given $\mathbb{P}$-generic filter G induces a $\mathbb{Q}$-generic filter $H = \{q \in \mathbb{Q} : \exists p \in G(i(p) \leq_{\mathbb{Q}} q)\}$, and we can characterize G using H via $G = i^{-1}(H)$.*

2. *Any given $\mathbb{Q}$-generic filter H induces a $\mathbb{P}$-generic filter $G = i^{-1}(H)$, and we can characterize H using G via $H = \{q \in \mathbb{Q} : \exists p \in G(i(p) \leq_{\mathbb{Q}} q)\}$.*

3. *In either case, (1) or (2), it holds that $\mathcal{M}[G] = \mathcal{M}[H]$.*

Proof. Begin with (1): Let $G \subseteq \mathbb{P}$ be given. First show that H is a filter. H is downwards directed: Take any $q_1, q_2 \in H$. By definition there exist $p_1, p_2 \in G$ with $i(p_1) \leq_{\mathbb{Q}} q_1$ and $i(p_2) \leq_{\mathbb{Q}} q_2$. Thus, as G is downwards directed, there is $p_3 \leq_{\mathbb{P}} p_1, p_2$ so that in particular $i(p_3) \leq_{\mathbb{Q}} q_1, q_2$. H is upwards closed: Let $q_1, q_2 \in \mathbb{Q}$ be such that $q_1 \in H$ and $q_1 \leq_{\mathbb{Q}} q_2$. Then there is $p_1 \in G$ with $i(p_1) \leq_{\mathbb{Q}} q_1$, so $i(p_1) \leq_{\mathbb{Q}} q_2$. Thus, $q_2 \in H$ by definition of H.

H is $(\mathbb{Q}, \mathcal{M})$-generic: Let $D \subseteq \mathbb{Q}$ be dense in $\mathbb{Q}$ with $D \in \mathcal{M}$. We show that $D \cap H \neq \emptyset$ by proving that $\tilde{D} := \{p \in \mathbb{P} : \exists q \in D\ (i(p) \leq q)\} \cap G \neq \emptyset$. By genericity of G it suffices to prove that $\tilde{D}$ is dense in $\mathbb{P}$. Consider arbitrary $p \in \mathbb{P}$ and find $q \in D$ with $q \leq i(p)$ (using that D is dense in $\mathbb{Q}$). By density of $i(\mathbb{P})$ in $\mathbb{Q}$, there exists $p' \in \mathbb{P}$ so that $i(p') \leq q$, implying that $i(p') \not\perp i(p)$. Consequently, $p' \not\perp p$ have common extension $p'' \in \mathbb{P}$. Now $p'' \in \tilde{D}$ and $p'' \leq p$.

Finally, we show that $G = i^{-1}(H)$. Indeed, if $p \in G$, then by $i(p) \leq i(p)$, $i(p) \in H$. So $p \in i^{-1}(H)$. On the other hand, $p \in i^{-1}(H)$ implies that $i(p) \in H$, and hence $p \in G$.

Continue with (2): We already know that G is a generic filter by Lemma 8.4.5. To see that $H = \{q \in \mathbb{Q} : \exists p \in G(i(p) \leq_{\mathbb{Q}} q)\}$, assume first that $q \in H$. Then by $G = i^{-1}(H)$, $p \in G$ with $i(p) = q$. Thus $q \in \{q \in \mathbb{Q} : \exists p \in G(i(p) \leq_{\mathbb{Q}} q)\}$. Next, say $q \in \{q \in \mathbb{Q} : \exists p \in G(i(p) \leq_{\mathbb{Q}} q)\}$. Then there exists some $p \in G$ such that $i(p) \leq_{\mathbb{Q}} q$. By $G = i^{-1}(H)$, $i(p) \in H$. Then, using that H is upwards closed, $q \in H$.

Finally, show (3): We start with the proof that $\mathcal{M}[G] \subseteq \mathcal{M}[H]$, noting that this actually holds for any function $i : \mathbb{P} \to \mathbb{Q}$ (whereas we require i to be a complete embedding for H to be $\mathbb{Q}$-generic over $\mathcal{M}$, as seen in Lemma 8.4.5). Indeed, begin by defining recursively

$$i_* : V^{\mathbb{P}} \to V^{\mathbb{Q}},\ i_*(\tau) = \{(i_*(\sigma), i(q)) : (\sigma, q) \in \tau\}.$$

We have to show that for any $\tau \in \mathcal{M}^{\mathbb{P}}$, $i_*(\tau) \in \mathcal{M}^{\mathbb{Q}}$ and $val(i_*(\tau), H) = val(\tau, G)$. To see this, proceed by induction on the rank $rk(\tau)$ of $\tau \in \mathcal{M}^{\mathbb{P}}$ If $rk(\tau) = 0$, then $\tau = \emptyset$, and so it follows that $i_*(\tau) = \emptyset \in \mathcal{M}^{\mathbb{Q}}$, as well as $val(i_*(\tau), H) = \emptyset = val(\tau, G)$. Next, assume that the statement holds for names σ with $rk(\sigma) < \alpha$ and show that it also holds for τ with $rk(\tau) = \alpha$. But this is true, since on the one hand, $i_*(\tau) = \{(i_*(\sigma), i(q)) : (\sigma, q) \in \tau\}$, and since $rk(\sigma) < \alpha$ for all $(\sigma, q) \in \tau$, it follows that $i_*(\sigma) \in \mathcal{M}^{\mathbb{Q}}$ for any $(\sigma, q) \in \tau$. Thereby, as $\mathcal{M}$ is a model of *ZFC* (respectively ZF-P), $i_*(\tau) \in \mathcal{M}^{\mathbb{Q}}$. On the other hand,

$$val(i_*(\tau), H) = \{val(i_*(\sigma), H) : \exists(i, q) \in H((\sigma, q) \in \tau)\} = val(\tau, G)$$

because $val(i_*(\sigma), H) = val(\sigma, G)$ per induction hypothesis, and $\exists i(q) \in H((\sigma, q) \in \tau)$ iff $\exists q \in G((\sigma, q) \in \tau)$ by definition of $G = i^{-1}(H)$ as in case (1) or (2).

To see that $\mathcal{M}[H] \subseteq \mathcal{M}[G]$, first note that we can reconstruct G in $\mathcal{M}[H]$, as $H, i \in \mathcal{M}[H]$ and $G = i^{-1}(H)$ holds in both (1) and (2). Therefore, $G \in \mathcal{M}[H]$. Using minimality of $\mathcal{M}[G]$ (see Proposition 8.1.2), the statement follows. □

8.4.1 Quotient forcing

Given two forcing notions $\mathbb{P}, \mathbb{Q}$ we wish to introduce the quotient poset $\mathbb{Q}/\mathbb{P}$. To do so we are going to employ the projection $\mathbb{Q} \to \mathbb{P}$ induced by $\mathbb{P} \lessdot \mathbb{Q}$ as seen in Lemma 8.4.7.

Definition 8.4.11. Let $\mathbb{P} \lessdot \mathbb{Q}$ and $\pi : \mathbb{Q} \to \mathbb{P}$ be the associated projection. Then the quotient poset is defined as the $\mathbb{P}$-name $\mathbb{Q}/\mathbb{P} = \{(\check{q}, \pi(q)) : q \in \mathbb{Q}\}$, ordered by $(\check{q}_1, \pi(q_1)) \leq_{\mathbb{Q}/\mathbb{P}} (\check{q}_2, \pi(q_2))$ iff $q_1 \leq_{\mathbb{Q}} q_2$.

Notice that since $\mathbb{Q}/\mathbb{P}$ is a $\mathbb{P}$-name for a forcing poset, $\check{q} := \{(\check{x}, 1_{\mathbb{P}}) : x \in q\} \in \mathcal{M}^{\mathbb{P}}$. Let $*$ be as defined in Chapter 3. We obtain the following lemma.

Lemma 8.4.12. $\mathbb{P} \lessdot \mathbb{Q}$ *implies that* $\mathbb{Q} \sim \mathbb{P} * \mathbb{Q}/\mathbb{P}$.

Proof. By Corollary 8.4.10 it suffices to define a dense embedding $\mathbb{Q} \to \mathbb{P} * \mathbb{Q}/\mathbb{P}$. Let $\pi : \mathbb{Q} \to \mathbb{P}$ be a projection as above. In order to find such a function, given $q \in \mathbb{Q}$, construct a $\mathbb{P}$-name $\dot{q}$ such that $\pi(q) \Vdash$ "$\dot{q} \in \mathbb{Q}/\mathbb{P}$". In this case, $\pi(q) \Vdash$ "$\dot{q}$ is of the form $(\check{q}, \pi(q))$", as in the definition of the quotient poset. To find $\dot{q}$, employ Corollary 8.1.5 about mixing names. For details see e.g. [14], Theorem 6.5. □

Definition 8.4.13 (Product-like forcing poset)**.** Given forcing posets $\mathbb{P}$ and $\mathbb{Q}$, call $\mathbb{Q}$ product-like if there exists a quotient poset $\mathbb{R}$ such that $\mathbb{Q} \sim \mathbb{P} * \mathbb{R}$.

9 Appendix: Putting some results in perspective

The tools to compare forcing posets from Chapter 8.4 allow us to shine some light on the connection between previously employed forcing notions. In Section 2.1.2 we already got to know the almost disjoint coding poset, which adds a single diagonalizing real to a given almost disjoint family. In Section 4.2.2 we introduced Hechler mad forcing to obtain a mad family of arbitrary uncountable cardinality. We begin by comparing these two posets. After that we turn our attention to Cohen forcing $\mathbb{C}_I$ as introduced in Section 8.3.2 relating it to $\mathbb{H}_\kappa$.

Lemma 9.0.1. *Denote by $\mathbb{H}_\lambda$ the Hechler mad forcing notion adding a mad family $\mathcal{A}$ with $|\mathcal{A}| = \lambda$. Let $\kappa < \lambda$. Then $\mathbb{H}_\kappa \lessdot \mathbb{H}_\lambda$.*

Proof. Without loss assume that $\kappa > 0$. By Remark 8.4.8 it suffices to find a projection $\pi : \mathbb{H}_\lambda \to \mathbb{H}_\kappa$. Define $\pi(p) = p{\restriction}\kappa \times \omega$. Let $\pi(p) = p$ whenever $p \in \mathbb{H}_\kappa$. Then indeed:

- $\pi(1_{\mathbb{H}_\lambda}) = 1_{\mathbb{H}_\kappa}$, as the empty function restricted to anything is still the empty function. Furthermore, if $p_1 \leq_{\mathbb{H}_\lambda} p_2$, then $\pi(p_1) = p_1{\restriction}\kappa \times \omega \leq_{\mathbb{H}_\kappa} p_2{\restriction}\kappa \times \omega$ follows immediately.

- Take $p \in \mathbb{H}_\kappa$ and $q \in \mathbb{H}_\lambda$ with $p \leq_{\mathbb{H}_\kappa} \pi(q)$. Then $q' := p \cup q$ is such that $q' \leq_{\mathbb{H}_\lambda} q$ and $\pi(q') \leq_{\mathbb{H}_\kappa} p$.

□

Proposition 9.0.2. *Let $\kappa < \lambda$ be as above. Then $\mathbb{H}_\lambda$ is product-like, that is, there exists a quotient poset $\mathbb{R}$ such that $\mathbb{H}_\lambda \sim \mathbb{H}_\kappa * \mathbb{R}$.*

Proof. Using Lemma 8.4.12, the previous result implies that $\mathbb{R} = \mathbb{H}_\lambda / \mathbb{H}_\kappa$ is as in the proposition. □

To further inspect $\mathbb{H}_\lambda / \mathbb{H}_\kappa$ we introduce a forcing poset which adds exactly those reals added by $\mathbb{H}_\lambda$ and not by $\mathbb{H}_\kappa$.

Discussion 9.0.3 (Decomposition of Hechler mad forcing). *Let $\lambda > \kappa$. As a reminder, note that $\mathbb{H}_\kappa$ (Definition 4.2.3) forces the existence of a mad family $\mathcal{A}_\kappa := \{a_\alpha : \alpha < \kappa\}$, with generic reals*

$$a_\alpha := \{k \in \omega : \exists p \in G(p(\alpha, k) = 1)\},$$

where G is a $(V, \mathbb{H}_\kappa)$-generic filter. Then in $V[G]$, define the poset $(\mathbb{H}_{[\kappa,\lambda)}, \leq)$ by

$$\mathbb{H}_{[\kappa,\lambda)} = \{(p, H) \mid p \in {}^{\lambda\backslash\kappa\times\omega}2 \wedge dom(p) = F_p \times n_p \in [\lambda\backslash\kappa]^{<\omega} \times \omega \ \wedge \ H \in [\kappa]^{<\omega}\},$$

with ordering $(q, K) \leq (p, H)$ iff

- *$q \leq_{\mathbb{H}_\lambda} p$ and $K \supseteq H$,*
- *$\forall\alpha \in F_p, \forall\beta \in H, \forall k \in n_q\backslash n_p$: If $k \in a_\beta$, then $q(\alpha, k) = 0$.*

Thereby, the newly added values k are not in $a_\beta \cap a_\alpha$, where β marks one of the old reals (in $\mathcal{A}_\kappa$), and α is the index of a new real added via $\mathbb{H}_{[\kappa,\lambda)}$. Note that $\mathbb{H}_{[\kappa,\lambda)}$ has the ccc. To see the connection to the quotient $\mathbb{H}_\lambda/\mathbb{H}_\kappa$, show that:

Claim. $\mathbb{H}_\lambda/\mathbb{H}_\kappa \sim \mathbb{H}_{[\kappa,\lambda)}$.

Proof. Say $\mathcal{A}_\kappa = \{a_\beta : \beta < \kappa\}$ is the mad family forced by $\mathbb{H}_\kappa$. Let $(p, K) \in \mathbb{H}_{[\kappa,\lambda)}$. Define $g : \mathbb{H}_\lambda/\mathbb{H}_\kappa \to \mathbb{H}_{[\kappa,\lambda)}$ via $g((\check{q}, \pi(q))) = (p, K)$ with $p := q\restriction\lambda\backslash\kappa \times \omega$ and $K := F_{\pi(q)}$ such that whenever $\beta \in K$, $\alpha \in F_p$ and $p(\alpha, k) := 1 - \pi(q)(\beta, k)$, for any $k \in n_q$. Then g forms a dense embedding. $\square_{Claim}$

By the claim it holds that $\mathbb{H}_\lambda \sim \mathbb{H}_\kappa * \dot{\mathbb{H}}_{[\kappa,\lambda)}$ in $V^{\mathbb{H}_\kappa * \dot{\mathbb{H}}_{[\kappa,\lambda)}}$, and so $\mathcal{A}_\lambda = \mathcal{A}_\kappa \cup \{a_\alpha : \alpha \in \lambda\backslash\kappa\}$ forms a mad family.

Claim. *For $\lambda = \kappa \cup \{\kappa\}$, $\mathbb{H}_\lambda/\mathbb{H}_\kappa \sim \mathbb{P}_\kappa$, the a.d. coding poset.*

Proof. Employ the previous claim to find that $\mathbb{H}_\lambda/\mathbb{H}_\kappa \sim \mathbb{H}_{[\kappa,\kappa\cup\{\kappa\})}$, where the latter adds a single diagonalizing real d_κ for the family $\mathcal{A}$. Thereby, $V^{\mathbb{H}_{[\kappa,\kappa\cup\{\kappa\})}} = V^{\mathbb{P}_\mathcal{A}}$. $\square_{Claim}$

Trouble in Finding an analogue for the independence case Per definition, every Hechler diagonalizing real fulfills the maximality property as in 2.0.1. The first and simplest candidate for a Hechler mad poset may be considered to be Cohen forcing $\mathbb{C}_I$. As shown in Lemma 8.3.2, $\mathbb{C}_I$ does indeed add an independent family of a wanted size. Furthermore, $\mathbb{C}_I$ can be written as a product $\prod_{i\in I} \mathbb{C}_i$, where $\mathbb{C}_i = \mathbb{C}$ for all $i \in I$. Thus, it can be used in finite support iterations. Despite this and its structural similarities to $\mathbb{H}_\theta$, the added family is not maximal (Remark 8.3.3).

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