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DAHA and Dyck path algebras

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Abstract

This thesis is composed of two parts. The first one is based on our work on DAHA representations at roots of unity and doubly periodic tableaux [BCMN23]. We give a combinatorial description of X-semisimple graded representations of the DAHA, showing that these can also be constructed using certain spaces of intertwiners of quantum group representations. As an application, we prove a conjecture of Morton and Samuelson [MS21] on the tangle algebra presentation of the DAHA. In the second half of this work, which is based on [BMN], we begin by defining in a skein theoretic fashion the $\mathbb{A}_q$ algebra of a given surface. We then focus on the $\mathbb{A}_q$ of the disc. We give a basis for this algebra by proving that it is isomorphic to the space of endomorphisms of a certain quantum group representation. Finally, we construct representations for the $\mathbb{A}_q$ of the annulus and $\mathbb{A}_{q,t}$ of the torus from spaces of intertwiners.

Zusammenfassung

Diese Arbeit besteht aus zwei Teilen. Der erste Teil basiert auf unserer Arbeit über DAHA-Darstellungen an Einheitswurzeln und doubly periodic tableaux [BCMN23]. Wir geben eine kombinatorische Beschreibung von X-halbeinfachen graduierten Darstellungen von DAHA und zeigen, dass diese auch mithilfe bestimmter Räume von Intertwinern von Quantengruppendarstellungen konstruiert werden können. Als Anwendung beweisen wir eine Vermutung von Morton und Samuelson [MS21] über die Tangle-Algebra-Beschreibung von DAHA. In der zweiten Hälfte dieser Arbeit, die auf [BMN] basiert, definieren wir zunächst die $\mathbb{A}_q$-Algebra einer gegebenen Oberfläche auf skeintheoretische Weise. Dann konzentrieren wir uns auf die $\mathbb{A}_q$ der Kreisscheibe. Wir geben eine Basis an diese Algebra, indem wir beweisen, dass sie isomorph zum Raum der Endomorphismen einer bestimmten Quantengruppendarstellung ist. Schließlich zeigen wir, wie man Intertwinern verwenden kann, um Darstellungen für $\mathbb{A}_q$ des Kreisrings und für $\mathbb{A}_{q,t}$ des Torus zu konstruieren.

A mio nonno Anno

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Contents

Introduction

The first chapter of this thesis is based on [BCMN23]. The paper contains two main results: the classification of X-semisimple representations at roots of one and their reincarnation as spaces of intertwiners, which are then combined to prove the tangle algebra presentation of the DAHA proposed in [MS21]. The crucial objects in both results turn out to be *doubly periodic tableaux*. In [BCMN23], we explore in detail the combinatorial properties of these tableaux, however, in this work, we choose to take the perspective of DAHA representations, therefore, we limit ourselves to the facts that are relevant to prove the results above. Furthermore, we would like to explain the general context in which the representations defined in the paper arise.

The second chapter is based on joint work with Léa Bittmann and Anton Mellit, that will converge into the upcoming paper [BMN]. The aim of this project is to approach similar questions to those we answered for the DAHA for the double Dyck path algebra $\mathbb{A}_{q,t}$. We take a reverse approach, as we begin by defining an $\mathbb{A}_{q,t}$ algebra attached to a surface, to then focus on some specific examples that should recover the usual $\mathbb{A}_q$ and $\mathbb{A}_{q,t}$ algebras (with inverses of the variables). We prove then some Schur-Weyl duality-like results for these algebras.

0.1 Representation theory and skein theory of the GL_n DAHA

The double affine Hecke algebra (DAHA) was introduced by Cherednik and gained its popularity when he gave a complete proof of Macdonald's constant term conjecture via DAHA [Che95]. Being related to Macdonald polynomials, the DAHA shares with this special family of polynomials its ubiquity in several fields of Mathematics, such as Algebraic Geometry, Representation Theory, Quantum Algebra and Knot Theory. The DAHA and its degenerations, the trigonometric and rational Cherednik algebras, were indeed connected to many geometric objects, such as the Hilbert scheme (see e.g. [GN15], [GS05], [GS06]) and the affine Springer fiber ([OY16]), and to torus knot invariants ([GORS14], [GN15], [CD16]). This work will be focused on three main aspects of DAHA theory. The first is the study of its X-semisimple representations. This class of representations is quite natural to consider, as even the very first example of DAHA representation, the polynomial representation, falls into this category. Moreover, as we shall see in detail, this class of representations can be fully understood by means of some very interesting combinatorial tools. When considering X-semisimple representations away from roots of unity, the combinatorial objects we end up dealing with are periodic skew tableaux [SV05], whereas, when the parameters are roots of unity, we have to use doubly periodic tableux [BCMN23]. The second element of DAHA theory that we want to consider is Schur-Weyl duality-like results. These results (in [JV18] away from roots of unity and [BCMN23] for the root of unity case) give a construction of some DAHA module using intertwiners of quantum group representations. As it turns out, this type of Schur-Weyl duality is in fact linked to our first point of interest. In both cases, the authors are indeed able to identify these DAHA modules with a certain X-semisimple representations, reconstructing the corresponding tableaux. Finally, we will be interested in the skein description of the DAHA. The DAHA is known to be isomorphic to a braid skein algebra ([MS21]). We will show, following [BCMN23], that the tangle skein algebra description proposed in [MS21] is also isomorphic to the DAHA.

0.2 Representation theory and skein theory of $\mathbb{A}_{q,t}$

The $\mathbb{A}_{q,t}$ algebra (or double Dyck path algebra or Carlsson-Mellit algebra) was first introduced in [CM18] by Carlsson and Mellit to prove the shuffle conjecture. Its connections to Geometry were later explored in [CGM20], where the authors define an $\mathbb{A}_{q,t}$-action on the parabolic flag Hilbert scheme. In some sense, the $\mathbb{A}_{q,t}$ algebra can be considered as a relative of the DAHA: already in [CM18], we can find a comparison between the two, as, at a first glance, one might be tempted to hope that the (positive half of the) DAHA embeds into $\mathbb{A}_{q,t}$. This is not the case, however, one can make sense of the relation between the two algebras by consulting for example [IW24] and [Wei24]. As a matter of fact, in [IW24], we can find a way to interpret the subalgebra generated by some elements of $\mathbb{A}_{q,t}$ as a stable limit of the DAHA and the polynomial representation restricted to this subalgebra is constructed from polynomial representation of the DAHA. More recently, in [Wei24], the author constructs representations for certain subalgebras of $\mathbb{B}_{q,t}$, an algebra introduced and proved to be equivalent to $\mathbb{A}_{q,t}$ in [CGM20], from representations of DAHA. In conclusion, it is natural to investigate whether we can have $\mathbb{A}_{q,t}$ analogues for our DAHA results. We will start from giving a skein theoretic version of $\mathbb{A}_{q,t}$. We would like to remark that [Mel21] already showed that the $\mathbb{A}_{q,t}$ has interesting connections to skein theory and is somewhat expected to admit a skein theoretic interpretation. We want to emphasize that the construction we present here is not original and is based on unpublished notes by Anton Mellit [Mel19]. From here, it will be natural to consider certain subalgebras of the $\mathbb{A}_{q,t}$ of the torus and we will prove certain Schur-Weyl duality results for them. A very nice byproduct of these proofs will be a description of a basis for one of these subalgebras. It is worth remarking at this point that, even if the results appear to be very similar to the ones for the DAHA, they cannot be easily derived from those; on the contrary, the techniques used in the proofs turn out to be quite different and only occasionally rely on some classic facts about AHA representations.

Conventions

Throughout the text we will use English notation for tableaux. We will use reading conventions, so, when writing (i, j) for a cell, we mean that i goes from left to right and j from top to bottom. We depict tableaux as follows: we embed $\mathbb{Z}^2 \subseteq \mathbb{R}^2$ and draw a tableau σ by writing the label $\sigma(x, y)$ in the center of the square cell with top left corner (x, y). In Chapter 1, we will follow the notations of [BCMN23] with the sole difference that we replace the m of the DAHA $\ddot{H}_{q,t}(m)$ in [BCMN23] by n to try to make our conventions as uniform as possible.

Chapter 1

Doubly periodic tableaux, DAHA and quantum group representations

1.1 Introduction

The goal of this chapter is to explain the content of [BCMN23] focusing on its representation theoretic aspects. More precisely, we will take the perspective of DAHA theory, avoiding diving into the combinatorial results concerning doubly periodic tableaux. We will also offer an overview of the context in which the original results therein arise. Before presenting the results of the paper, we recall the main facts concerning the GL_n DAHA that are connected to this work. We begin by giving an introduction to affine root systems and, in particular, to the extended affine Weyl group of type GL_n, the extended affine permutation group. We will then give a general introduction to the GL_n DAHA and its polynomial representation to then proceed to focus on X-semisimple representations. We will give a brief explanation of how these were classifed in [SV05] in the generic case. Afterwards, we start presenting the actual content of [BCMN23]. Here we focus on the root of unity case and give a combinatorial description of graded X-semisimple representations. We will use these representations to construct a faithful representation, which will be later used to prove the tangle algebra presentation of the DAHA. Graded X-semisimple representations have an alternative construction based on quantum group representations. This will be the main topic of Section 1.4. This is the key to link the results obtained from our DPT construction to the skein presentation of the DAHA, which will be presented in Section 1.5.

1.2 The DAHA of type GL_n

In this work, we will be interested in the DAHA of type GL_n (we prefer this terminology to "type A" to avoid confusion with the DAHA of type SL_n). The recipe to construct a DAHA of a given type goes as follows. We consider the Weyl group of this type and take its extended affine version. At this point, we build its corresponding affine Hecke algebra. Finally, we glue two copies of this AHA along a common representation. As we will be solely dealing with the DAHA of type GL_n, after giving some general preliminaries on root systems and Weyl groups, we will soon specialize to type A and define the DAHA and its polynomial representation for type GL_n.

1.2.1 Affine root systems and extended affine Weyl group

Let us quickly recall the general definitions of affine root system and corresponding (extended) affine Weyl group. We follow [Mac03] in our exposition.

Let R be a reduced irreducible root system, spanning an n dimensional Euclidean vector space E. The space E is endowed with a scalar product $(\cdot,\cdot)$. We fix simple roots $\alpha_1,\dots,\alpha_n$ and denote by R_+ and R_- the positive and the negative roots respectively. For a root $\alpha\in R$, we denote $\alpha^\vee=\frac{2\alpha}{(\alpha,\alpha)}$ the corresponding coroot and $R^\vee$ the set of the coroots. Let Q be the root lattice $\sum_{\alpha\in R}\mathbb{Z}\alpha$ and let P be the weight lattice, i.e. the elements of E having integral pairing with the coroots. Similarly, we set $Q^\vee=\sum_{\alpha^\vee\in R^\vee}\mathbb{Z}\alpha^\vee$ the coroot lattice and $P^\vee$ to be the coweight lattice, i.e. the elements of E with integral pairing with the roots. Recall the Weyl group W is the group generated by reflections $s_\alpha(x)=x-\frac{2(\alpha,x)}{(\alpha,\alpha)}\alpha$.

Definition 1.2.1. The affine root system is the subset of $\mathbb{R}^{n+1}$ given by

$$R^a=\{[\alpha,k]:\alpha\in R,k\in\mathbb{Z}\}.$$

For $a=[\alpha,k]\in R^a$, the corresponding reflection is given by $s_a(x)=x-\frac{2((x,\alpha)+k)}{(\alpha,\alpha)}\alpha$.

Definition 1.2.2. The *affine Weyl group* W^a is the group generated by the reflections s_a, for $a\in R^a$.

Remark 1.2.3. One can verify that $W^a=t(Q^\vee)\rtimes W$, where $t(Q^\vee)$ denotes the translations by the elements of the coroot lattice.

Remark 1.2.4. The affine Weyl group is generated by the reflections s_{α_i}, for simple roots α_i, and $s_{-\theta,1}$, where θ is the highest root.

The simple affine roots are give by the (finite) simple roots $\alpha_1,\dots,\alpha_n$ and $\alpha_0=[-\theta,1]$.

Definition 1.2.5. The *alcoves* are the connected components of $E\setminus\cup H_{\alpha,k}$, where $H_{\alpha,k}\{x\in E:(x,\alpha)=k\}$. The *fundamental alcove* is the alcove defined by $\mathcal{A}=\{x\in E:\alpha_i(x),0\le i\le n\}$.

Definition 1.2.6. The *extended affine Weyl group* W^{ae} is defined as $t(P^\vee) \rtimes W$.

Let us recall the definition of length for an element of the Weyl group. The length of w is the length $l(w)$ of a minimal expressions in the generators s_{α_i}. We know the length is also equal to $|\{R_+ \cap w^{-1}(R_-)\}|$. This definition can actually be extended to elements of the extended affine Weyl group by replacing $R_\pm$ with $R^a_\pm$.

Let

$$\Omega = \{w \in W^{ae} : l(w) = 0\}.$$

Then $\Omega \simeq P^\vee/Q^\vee$. One can prove that $W^{ae} \simeq \Omega \ltimes W$

1.2.2 The extended affine permutation group

We now want to specialize the notion of extended affine Weyl group to type GL_n and study some of its properties. In this case the (finite) Weyl group is the permutation group, whereas the extended affine Weyl group is the *extended affine permutation group* $\widehat{\mathfrak{S}}^e_n$.

We recall here its definition:

$$\widehat{\mathfrak{S}}^e_n = \left\langle \pi, s_i, i \in \mathbb{Z}/n\mathbb{Z} \;\middle|\; \begin{array}{ll} s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, & i \in \mathbb{Z}/n\mathbb{Z} \\ s_i s_j = s_j s_i, & j \neq i \pm 1 \mod n \\ s_i^2 = 1, & i \in \mathbb{Z}/n\mathbb{Z} \\ \pi s_i = s_{i+1}\pi, & i \in \mathbb{Z}/n\mathbb{Z}. \end{array} \right\rangle.$$

Remark 1.2.7. In general, the extended affine Weyl group contains the Weyl group and the affine Weyl group. In this case, we can see that $\widehat{\mathfrak{S}}^e_n$ contains the symmetric group $\mathfrak{S}_n$, generated by the reflections s_i for $1 \leq i < n$, as well as the affine permutation group $\widehat{\mathfrak{S}}_n$ as the subgroup generated by the reflections $s_i, i \in \mathbb{Z}/n\mathbb{Z}$.

An element of the extended affine permutation group can be interpreted as bijection $f : \mathbb{Z} \to \mathbb{Z}$ satisfying $f(i+n) = f(i)+n$. The generators act as follows:

$$\begin{array}{ll} s_i(j) = j+1 & \text{for } j \equiv i \mod n \\ s_i(j) = j-1 & \text{for } j \equiv i+1 \mod n \\ s_i(j) = j & \text{for } j \not\equiv i, i+1 \mod n \\ \pi(j) = j+1 & \text{for all } j. \end{array}$$

Remark 1.2.8. The elements of the affine permutation group $\widehat{\mathfrak{S}}_n$ are the affine permutations f satisfying $\sum_{i=1}^n f(i) = \frac{n(n+1)}{2}$.

Remark 1.2.9. We can give a convenient description of the minimal length left coset of representatives $\widehat{\mathfrak{S}}_n/\mathfrak{S}_n$ as the permutations satisfying $f(1) < f(2) < \cdots < f(n)$ (see e.g. [Hum90]).

1.2.3 The DAHA and its polynomial representation

Let us begin this section by defining the affine Hecke algebra of type GL_n.

Definition 1.2.10. The affine Hecke algebra is the algebra over $\mathbb{K}$ generated by $\pi^{\pm 1}, T_0, \dots, T_{n-1}$ subject to the relations

(1) $(T_i - q)(T_i + 1) = 0$, for $i = 0, \dots, n-1$;

(2) $T_i T_j = T_j T_i$ if $i \not\equiv j \pm 1 \mod n$;

(3) $T_i T_j T_i = T_j T_i T_j$ if $i \equiv j \pm 1 \mod n$.

(4) $\pi T_i \pi^{-1} = T_{i+1}$, for $i = 0, \dots, n-2$, $\pi T_{n-1} \pi^{-1} = T_0$.

The affine Hecke algebra admits the induced representation and the so-called Cherednik's basic representation on the same polynomial ring (see e.g. [Sim17]). The DAHA is obtained by gluing two copies of the AHA along this common representation.

Definition 1.2.11. Let $n \in \mathbb{Z}_{>2}$. The double affine Hecke algebra $\ddot{H}_{q,t}(n)$ is the algebra over $\mathbb{C}[q^{\pm 1}, t^{\pm 1}]$ generated by

$$T_0, T_1, \dots, T_{n-1}, \pi^{\pm 1}, X_1^{\pm 1}, X_2^{\pm 1}, \dots, X_n^{\pm 1},$$

subject to the relations

(1) $(T_i - q)(T_i + 1) = 0$, for $i = 0, \dots, n-1$;

(2) $T_i T_j T_i = T_j T_i T_j$, for $j = i \pm 1 \mod n$;

(3) $T_i T_j = T_j T_i$ if $j \neq i \pm 1 \mod n$;

(4) $T_i X_i T_i = q X_{i+1}$, for $i = 1, \dots n-1$, $T_0 X_n T_0 = t^{-1} q X_1$;

(5) $T_i X_j = X_j T_i$, for $j \neq i, i+1$;

(6) $\pi X_i \pi^{-1} = X_{i+1}$, for $i = 1, \dots, n-1$, $\pi X_n \pi^{-1} = t^{-1} X_1$;

(7) $\pi T_i \pi^{-1} = T_{i+1}$, for $i = 0, \dots, n-2$, $\pi T_{n-1} \pi^{-1} = T_0$;

(8) $X_i X_j = X_j X_i$, for $i, j \in \{1, \dots, n\}$.

For $n = 2$, the double affine Hecke algebra $\ddot{H}_{q,t}(2)$ is the algebra defined by the same generators and relations (1),(4)-(8).

We call the *small DAHA* $\ddot{H}_{q,t}(n)^s$ the subalgebra of $\ddot{H}_{q,t}(n)$ generated by $T_0, T_1, \dots, T_{n-1}$ and $X_1^{\pm 1}, X_2^{\pm 1}, \dots, X_n^{\pm 1}$.

Remark 1.2.12. The generators $T_1, \dots, T_{n-1}, \pi^{\pm 1}$ form an affine Hecke algebra (according to our definition) and so do the generators $T_0, \dots, T_{n-1}, X_1^{\pm 1}, \dots, X_n^{\pm 1}$ (see for example section 2.4 of [Sim17]). These two presentations reflect the two ways to express the Weyl group as a semidirect product we saw in Section 1.2.1.

Remark 1.2.13. The DAHA admits a grading given by

$$\deg(\pi^{\pm 1}) = \pm 1,$$
$$\deg(X_i^{\pm 1}) = \deg(T_i) = 0, \quad (i \in \mathbb{Z}).$$

The DAHA admits a so-called polynomial representations. In fact, this is precisely the common representation of the two copies of the AHA glued together to form the DAHA. This was used by Cherednik in [Che95] to prove Macdonald constant term conjecture. We refer the reader to Cherednik's book [Che05] for details on the construction. The fact that will be crucial later on is that this representation is faithful. To match the notation in [BCMN23], we use $\mathbb{C}(q,t)[Y_1^{\pm},\dots,Y_n^{\pm}]$ as the base space of the polynomial representation.

The polynomial representation, which we denote from now on by $\mathcal{P}$, admits a basis of weight vectors indexed by the the minimal length left coset representatives in $\widehat{\mathfrak{S}}_n^e/\mathfrak{S}_n$. It will be important to us that 1 is an element of this basis with weight $(1,q,\dots,q^{n-1})$. Moreover, as f varies in the set of minimal length coset representatives $S=\{f\in\widehat{\mathfrak{S}}_n^e|f(1)<f(2)<\dots<f(n)\}$, the set $T_f(1)$ forms a basis.

We state here the faithfulness result for the polynomial representation, whose proof can be found in [Che05].

Theorem 1.2.14. *The polynomial representation of the DAHA is faithful when q and t are treated as formal parameters or specialized to complex values which are not roots of unity.*

1.3 X-semisimple representations of DAHA

1.3.1 X-semisimple representations and their properties

Definition 1.3.1. A representation M of $\ddot{H}_{q,t}$ is called X-semisimple if it is finitely generated and admits a decomposition $M=\oplus_{\underline{w}}M_{\underline{w}}$ with $\dim M_{\underline{w}}<\infty$ for all weights $\underline{w}$.

In Section 4.2 of [SV05], the authors study the general structure of X-semisimple representations making use of intertwiners. DAHA intertwiners are elements ϕ_f, for f in the extended affine permutation group, that send an element of the weight space M_w to the space $M_{f(w)}$. In that section, it shown that, when q is not a root of unity, the nonzero weight spaces are always one-dimensional. As we shall see, this will no longer be the case when the parameters are specialized to roots of unity.

1.3.2 Periodic skew tableaux

We will give a short overview of the content of [SV05], where the authors classify X-semisimple representations via periodic skew tableaux. It was proved that there is an X-semisimple DAHA action on the space whose basis is indexed by standard tableaux with a certain periodic skew shape. In fact, any irreducible X-semisimple representation of the DAHA away from roots of unity is isomorphic to one constructed in this fashion.

Let us begin by introducing the necessary combinatorial notions. We want to consider partitions in

$$\mathcal{A}_{K,N} := \{\lambda = (\lambda_1 \geq \cdots \geq \lambda_N) \in \mathbb{Z}^N : \lambda_1 - \lambda_N \leq K\}.$$

Recall that, for partitions λ, μ such that $\lambda_i \geq \mu_i$, we can define the skew diagram λ/μ. More precisely, we are interested in pairs of diagrams in the sets

$$J^n_{(K,N)} = \{(\lambda, \mu) \in \mathcal{A}_{K,N} \times \mathcal{A}_{K,N} : \lambda_i \geq \mu_i, \sum_{i=1}^{N} (\lambda_i - \mu_i) = n\}$$

and

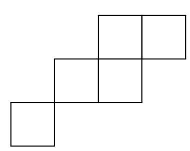
$$J^{n*}_{(K,N)} = \{(\lambda, \mu) \in \mathcal{A}_{K,N} \times \mathcal{A}_{K,N} : \lambda_i > \mu_i, \sum_{i=1}^{N} (\lambda_i - \mu_i) = n\}$$

For $\lambda, \mu \in J^n_{(K,N)}$, we can define λ/μ as

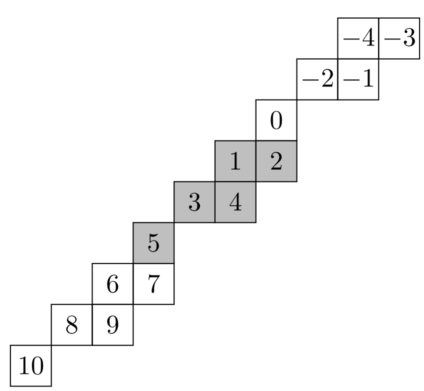
$$\lambda/\mu = \{(x, y) \in \mathbb{Z}^2 : x \in \{\mu_y + 1, \ldots, \lambda_y\}, y \in \{1, \ldots, N\}\}.$$

Here and throughout this work, we are using *reading directions*, meaning that the x-axis points towards the right and the y-axis points downwards. We can define the corresponding periodic skew diagram as $\widehat{\lambda/\mu}$ as the union of translates $\lambda/\mu + m(K, -N)$. A *periodic skew tableaux* is a function $T : \lambda/\mu \to \mathbb{Z}$ such that $T(u + (K, -N)) = T(u) + n$. A periodic skew tableaux is said to be *standard* if it is increasing from left to right and top to bottom. We denote by $\mathrm{PST}(\lambda, \mu)$ the set of standard periodic skew tableaux of shape λ/μ.

Example 1.3.2. $\lambda = (4, 3, 1)$ and $\mu = (2, 1, 0)$ belong to $J^5_{3,3}$ and the skew diagram λ/μ is



Here an example of (part of) a standard periodic skew tableau of shape $\widehat{\lambda/\mu}$ with fundamental domain colored in gray:



The extended affine permutation group clearly acts on periodic skew diagrams by permuting the labels, however, in general, it does not send a standard tableaux to a standard tableaux.

We define the content of a periodic skew tableaux to be the function $C_T^{\widehat{\lambda/\mu}}$: $\mathbb{Z} \to \mathbb{Z}$ given by $C_T^{\widehat{\lambda/\mu}}(i) = x - y$, where $T(x,y) = i$.

We can now define the DAHA action of periodic skew tableaux. Let $V(\lambda,\mu) = \oplus_{T\in \mathrm{PST}(\lambda/\mu)} \mathbb{C}\nu_T$. We assume that q is not a root of unity and $t = q^{K+N}$.

Theorem 1.3.3 (Theorem 4.17 [SV05])**.** *The DAHA $\ddot{H}_{q,t}(n)$ acts $V(\lambda,\mu)$ by*

$$X_i\nu_T = q^{C_T(i)}\nu_T, \qquad \pi\nu_T = \nu_{\pi T},$$

$$T_i\nu_T = \begin{cases} q\nu_T & \text{if } i \text{ is to the left of } i+1,\\ -\nu_T & \text{if } i \text{ is on top of } i+1,\\ -\frac{1-q}{1-q^{C_T(i)-C_T(i+1)}}\nu_T + \frac{1-q^{1+C_T(i)-C_T(i+1)}}{1-q^{C_T(i)-C_T(i+1)}}\nu_{s_i(T)} & \text{otherwise.}\end{cases}$$

The representation $V(\lambda,\mu)$ is irreducible. Moreover, any irreducible X-semisimple representation is isomorphic to $V(\lambda,\mu)$ for some choice of λ and μ.

Theorem 1.3.4 (Theorem 4.20 [SV05])**.** *Let $n \in \mathbb{Z}_{\geq 2}$ and $\kappa \in \mathbb{Z}_{\geq 1}$. Let L be an irreducible X-semisimple $\ddot{H}_{q,t}$-module where t acts as q^κ, then L is isomorphic to $V(\lambda,\mu)$ for some $(\lambda,\mu) \in J^{n*}_{(m,m-\kappa)}$ for some $m \in [1,\kappa]$.*

We give a sketch of the strategy of the proof, as we will use the same method to show our classification result for the root of unity case (Theorem 1.3.31). The first step of the proof is to show that from a weight of the representation one can construct a content function, which in turn gives us a skew periodic tableau. Once this tableau has been determined, the final step consists in showing that one can obtain an isomorphism by sending the chosen weight vector to the vector attached to the tableau.

1.3.3 Graded X-semisimple representations

We now want to introduce the class of representations that we will be dealing with in the root of unity case. Recall from Remark 1.2.13 that the DAHA has a grading

$$\deg(\pi^{\pm 1}) = \pm 1,$$
$$\deg(X_i^{\pm 1}) = \deg(T_i) = 0, \quad (i \in \mathbb{Z}).$$

Definition 1.3.5. A representation M of a graded algebra $\mathcal{A} = \bigoplus_{p\in\mathbb{Z}} A_p$ is a *graded representation* if it admits a decomposition $M = \bigoplus_{i\in\mathbb{Z}} M_i$ such that, for all $(p,i) \in \mathbb{Z}^2$,

$$A_p \cdot M_i \subset M_{p+i}.$$

Remark 1.3.6. A graded representation of the DAHA decomposes into $M = \bigoplus_{i\in\mathbb{Z}} M_i$ such that for all $(p,i) \in \mathbb{Z}^2$, $\pi^p(M_i) \subset M_{p+i}$.

From the definition of the grading, we immediately have the following.

Lemma 1.3.7. *A graded representation $M = \bigoplus_{i\in\mathbb{Z}} M_i$ of $\ddot{H}_{q,t}(n)$ is irreducible as a graded representation if and only if each graded piece M_i is an irreducible representation of the small DAHA $\ddot{H}_{q,t}(n)^s$.*

In that case, we say that M is a *graded irreducible* representation. In the next section, we will give a combinatorial classification of all such representations.

1.3.4 DPT and representations at roots of unity

From this section until the end of the chapter, we will be focusing on the content of [BCMN23]. However, our exposition will be slightly different as we will try to assume as much as possible the perspective of DAHA representation theory.

Fix integers K, N, a, b, let $n = aN - bK$, and assume that $n > 0$.

Definition 1.3.8. A *doubly periodic tableau* with respect to (K, N, a, b) is a surjective function $\sigma : \mathbb{Z}^2 \to \mathbb{Z}$ satisfying

(1) $\sigma(x + K, y - N) = \sigma(x, y)$,

(2) $\sigma(x + a, y - b) = \sigma(x, y) + n$,

for all $x, y \in \mathbb{Z}$.

Let $\mathrm{DPT}(K, N, a, b)$ denote the set of all standard doubly period tableaux with respect to (K, N, a, b).

Also here, we will always use reading coordinates for $\mathbb{Z}^2$, i.e. the x-axis will be oriented from left to right and the y-axis from top to bottom.

Remark 1.3.9. We note that, for any $r \in \mathbb{Z}$, it follows immediately from the definition that

$$\mathrm{DPT}(K, N, a, b) = \mathrm{DPT}(K, N, a + rK, b + rN).$$

We state here a few facts concerning DPT that will be used later; the complete proofs can be found in [BCMN23].

Lemma 1.3.10. *The set $\mathrm{DPT}(K, N, a, b)$ is nonempty if and only if $K, N > 0$.*

Lemma 1.3.11. *Let $\sigma \in \mathrm{DPT}(K, N, a, b)$, $r \in \mathbb{Z}$, and $(x, y), (x', y') \in \mathbb{Z}^2$. Then $\sigma(x, y) = \sigma(x', y') + rn$ if and only if $(x, y) = (x', y') + s(K, -N) + r(a, -b)$ for some $s \in \mathbb{Z}$.*

Let us introduce a few definitions that will help us encode the information contained in a DPT.

Definition 1.3.12. Let $\sigma \in \mathrm{DPT}(K, N, a, b)$. The *extended fundamental domain* of σ is

$$\Delta'(\sigma) = \{(x, y) \in \mathbb{Z}^2 \mid 1 \leq \sigma(x, y) \leq n\}.$$

The *fundamental domain* of σ is

$$\Delta(\sigma) = \{(x, y) \in \Delta'(\sigma) \mid 0 \leq y < N\}.$$

Example 1.3.13. Here an example of a (portion of a) standard doubly periodic tableau with $K = 3, N = 2, a = 1, b = -1, n = 1 \cdot 2 - (-1) \cdot 3 = 5$.

-14	-12	-10	-8	-6	-4	-2	0	2	4
-11	-9	-7	-5	-3	-1	1	3	5	7
-8	-6	-4	-2	0	2	4	6	8	10
-5	-3	-1	1	3	5	7	9	11	13
-2	0	2	4	6	8	10	12	14	16
1	3	5	7	9	11	13	15	17	19

We highlight the fundamental domain in gray, in pink its translates by $\pm(a, -b)$ and in teal by $\pm(K, -N)$. The portion of the extended fundamental domain which appears in the picture is therefore made of the boxes in gray and teal. We write in purple the filling of the $(0, 0)$ box.

It will be useful to introduce this notation. To a DPT σ we associate a partition $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{Z}^N$ such that $\lambda_1 \geq \cdots \geq \lambda_N$ as follows

$$\lambda_i = \min\{x \in \mathbb{Z} \mid \sigma(x, i - 1) \geq 1\}.$$

We will see later that these partitions end up belonging to the fundamental alcove $\mathcal{A}_{K,N}$. Clearly, we can let the extended affine permutation group act on the set of DPT, however, it is also immediate to see that the image of a standard DPT under this action might not still be standard. It is therefore worth taking a look at what happens when "acting" by permutations to then be able to define a(n actual) affine Hecke algebra action on $\mathrm{DPT}(K, N, a, b)$.

Definition 1.3.14. We say a permutation is *allowed to act* on a standard DPT σ if the image of σ is again standard.

Remark 1.3.15. For any tableau $\sigma \in \mathrm{DPT}(K, N, a, b)$, the action of the Weyl group $\mathfrak{S}_n$ (when it is allowed to act) preserves the extended fundamental domain and the fundamental domain.

Proposition 1.3.16. *For $\sigma_1, \sigma_2 \in \mathrm{DPT}(K, N, a, b)$ there exists a unique n-affine permutation $f \in \widehat{\mathfrak{S}}_n^e$ such that $\sigma_2(x, y) = f(\sigma_1(x, y))$ for all $x, y \in \mathbb{Z}^2$. We denote this affine permutation by $\sigma_2\sigma_1^{-1}$.*

Proof. Given an affine permutation f with $\sigma_2 = f\sigma_1$, then for $0 \leq i < n$, pick $(x, y) \in \mathbb{Z}^2$ with $\sigma_1(x, y) = i$. Then

$$f(i) = \sigma_2(x, y). \tag{1.3.1}$$

This proves uniqueness.

Moreover, (1.3.1) gives a well defined affine permutation. Indeed, if $\sigma_1(x, y) = \sigma_1(x', y')$, then $(x', y') = (x, y) + s(K, -N)$ for some $s \in \mathbb{Z}$ by Lemma 1.3.11, and it follows that $\sigma_2(x, y) = \sigma_2(x', y')$. We also have to show the values $f(0), \dots, f(n-1)$ are distinct modulo n. Suppose this is not the case and let $0 \leq i < j < n$ with $f(i) = f(j) + rn$ for some n. There exists $(x, y) \in \mathbb{Z}^2$ and $(x', y') \in \mathbb{Z}^2$ such that $\sigma_1(x, y) = i$, and $\sigma_1(x', y') = j$. Thus $\sigma_2(x, y) = f(i) = \sigma_2(x', y') + rn$. By Lemma 1.3.11, we have $(x, y) = (x', y') + r(a, -b) + s(K, -N)$ for some $s \in \mathbb{Z}$. We can conclude that $i = j + rn$, from which it follows that $r = 0$ and $i = j$, a contradiction. □

Proposition 1.3.17. *If an n-affine permutation f is allowed to act on σ and $f = \pi^r s_{j_1} \dots s_{j_\ell}$ is a minimal length expression of f, thenm, for each $i = 1 \dots \ell$, the permutation s_{j_i} is allowed to act on $s_{j_{i+1}} \dots s_{j_\ell}\sigma$.*

Proof. Assume now that there is a value of ℓ for which s_{j_i} is not allowed to act on $s_{j_{i+1}} \dots s_{j_\ell}\sigma$. Set $h = \pi^r s_{j_1} \dots s_{j_{i-1}}$ and $g = s_{j_{i+1}} \dots s_{j_\ell}$. Since $s_{j_i}(g\sigma)$ is not standard, we have $s_{j_i}(g\sigma)(x+1, y) = k$ and $s_{j_i}(g\sigma)(x, y) = k+1$ or $s_{j_i}(g\sigma)(x, y+1) = k$ and $s_{j_i}(g\sigma)(x, y) = k+1$, for some choice of $(x, y) \in \mathbb{Z}^2$. By the minimality of the length of the expression for f, we have that $l(hs_{j_i}) > l(h)$, which implies (see e.g. [Hum90]) that $h(k) < h(k+1)$. This means, in the first case, that $hs_{j_i}(g\sigma)(x+1, y) < hs_{j_i}(g\sigma)(x, y)$. Similarly, in the second case, we see that $hs_{j_i}(g\sigma)(x, y+1) < hs_{j_i}(g\sigma)(x, y)$. In both cases this implies that $f\sigma$ is not standard, contradicting our hypothesis. □

In order to understand which permutations are allowed to act on a given DPT, we introduce the concept of content.

Definition 1.3.18. The *content* function of $\sigma \in \mathrm{DPT}(K, N, a, b)$ is the function $C_\sigma : \mathbb{Z} \to \mathbb{Z}/(N+K)\mathbb{Z}$ sending $i \in \mathbb{Z}$ to the difference $x - y \mod (N+K)$, where $\sigma(x, y) = i$.

We can see that, if two cells $(x, y), (x', y') \in \mathbb{Z}^2$ are both labelled with i, then $x - y - (x' - y')$ is a multiple of $N + K$, so the content function is well defined.

Remark 1.3.19. Recall that a box (x, y) is situated along the j-th diagonal if $x - y = j$. So we note that the content function of i determines along which diagonals we can find the box labelled by i.

The following proposition is immediate to check.

Proposition 1.3.20. *The generator s_i of $\widehat{\mathfrak{S}}_n^e$ is allowed to act on $\sigma \in \mathrm{DPT}(K, N, a, b)$ if and only if $C_\sigma(i) - C_\sigma(i+1) \not\equiv \pm 1 \mod (N+K)$.*

The content function associated to a tableau uniquely determines it in the sense of the following proposition.

Proposition 1.3.21. *Let (K, N, a, b) and (K', N', a', b') be such that $n = aN - bK = a'N' - b'K'$. The following are equivalent:*

(1) $C_\sigma = C_{\sigma'}$ for some $\sigma \in \mathrm{DPT}(K, N, a, b)$ and $\sigma' \in \mathrm{DPT}(K', N', a', b')$,

(2) $K = K'$, $N = N'$ and there exists a pair $(r, s) \in \mathbb{Z}^2$ such that $a' = a + rK$, $b' = b + rN$ and $\sigma'(x, y) = \sigma(x + s, y + s)$, for all $(x, y) \in \mathbb{Z}^2$.

For the proof we rely on Proposition 1.3.30, which we decided to state and prove next to its direct application, however, the reader interested in the proof of this proposition might want to take a look at it before going further.

Proof. We know from Remark 1.3.9 that any $\sigma \in \mathrm{DPT}(K, N, a, b)$ is also an element of $\mathrm{DPT}(K, N, (a+rK, b+rN)$, for all $r \in \mathbb{Z}$. Moreover, it is immediate to see that σ, σ' satisfying $\sigma'(x, y) = \sigma(x + s, y + s)$, for some $s \in \mathbb{Z}$, will have the same content function. So we have proved that (2) $\Rightarrow$ (1).

The proof of (1) $\Rightarrow$ (2) is a consequence of the proof of Proposition 1.3.30. If $C_\sigma = C_{\sigma'}$, then we have $N + K = N' + K'$. Using the notations of the proof of Proposition 1.3.30, $i_0^{(1)}$ is the smallest positive value in the 0-diagonal of both σ and σ'. Thus there exists $(s_1, s_2) \in \mathbb{Z}^2$ such that $\sigma(s_1, s_1) = \sigma'(s_2, s_2) = i_0^{(1)}$. Then, again from the same proof, a (K, N)-periodic lattice path is uniquely constructed from the content function, up to this diagonal shift. This implies that $K = K'$ and $N = N'$.

The filling is then uniquely extended to a (K, N)-periodic standard filling of $\mathbb{Z}^2$, using the content function. We see that $\sigma'(x, y) = \sigma(x + s, y + s)$, with $s = s_1 - s_2$, for all $(x, y) \in \mathbb{Z}^2$.

Finally, $(a, -b)$ (resp. $(a', -b')$) is the n-periodicity of σ (resp. σ'), defined modulo (K, N). Since σ and σ' have the same fillings up to diagonal shift (see again the proof of Proposition 1.3.30), we deduce that there exists $r \in \mathbb{Z}$ such that $(a', b') = (a, b) + r(K, N)$.

□

Proposition 1.3.22. *The DAHA $\ddot{H}_{q,t}(n)$ acts on $W_{(K,N,a,b)}$ by*

$$X_i \nu_\sigma = w_\sigma(i)\nu_\sigma, \qquad \pi\nu_\sigma = \nu_{\pi\sigma},$$

$$T_i\nu_\sigma = \begin{cases} q\nu_\sigma & \text{if } i \text{ is to the left of } i+1, \\ -\nu_\sigma & \text{if } i \text{ is on top of } i+1, \\ -\frac{1-q}{1-w_\sigma(i)(w_\sigma(i+1))^{-1}}\nu_\sigma + \frac{1-qw_\sigma(i)(w_\sigma(i+1))^{-1}}{1-w_\sigma(i)(w_\sigma(i+1))^{-1}}\nu_{s_i(\sigma)} & \text{otherwise.} \end{cases}$$

Proof. We show here the relations that can be immediately verified:

$$\pi X_i \nu_\sigma = w_\sigma(i)\nu_{\pi\sigma}, \qquad X_i \pi \nu_\sigma = X_i \nu_{\pi\sigma} = w_\sigma(i-1)\nu_{\pi\sigma} \;\Rightarrow\; \pi X_i = X_{i+1}\pi$$

$$\pi X_n = t^{-1}X_1\pi, \quad \text{where} \quad t = q^{-a-b}.$$

The remaining relations can be checked by direct computation (see also [Ram03b]).

□

Remark 1.3.23. Note that $T_i \nu_\sigma$ is simply a multiple of ν_σ when s_i is not allowed to act on σ and, when s_i is allowed to act on σ, it is a sum of a multiple of ν_σ and a non-zero multiple of $\nu_{s_i \sigma}$ by 1.3.20.

As we shall see, these representations have slightly different properties than the usual semisimple representations at generic q ant t. The weight spaces are easily seen to be no longer one-dimensional: for example, every diagonal shift of a given DPT is again a DPT and its content function will be the same. Therefore, every weight space is clearly infinite dimensional. Moreover the representations are no longer irreducible. However, we can change our perspective and look at $W_{(K,N,a,b)}$ as a graded X-semisimple representations, then it will still have many desirable properties. Moreover, as we shall see, representations of this form classify all graded X-semisimple representations.

Let us define a grading on DPT. We set

$$\deg(\sigma) = -\sum_{i=1}^{N} \lambda_i.$$

For a basis vector ν_σ, we define its degree to be

$$\deg(\nu_\sigma) = \deg(\sigma).$$

Let us take a moment to see why this is a sensible definition and it turns the representation $W_{(K,N,a,b)}$ into a graded X-semisimple representation.

Recall Remark 1.3.15, which tells us that the action of the (non-extended) affine Weyl group leaves the fundamental domain (hence the corresponding partition) of a DPT σ unaltered. This means that the actions of the generators $X_i^{\pm 1}$ and T_i do not change the degree of ν_σ. We now have to check that π shifts the degree by $+1$. Let us give a precise description of the action of π on a DPT. Let (x_n, y_n) be the cell in $\lambda[a,b] \setminus \lambda$ such that $\sigma(x_n, y_n) = n$. Pick $s \in \mathbb{Z}$ such that $(x_0, y_0) := (x_n, y_n) - (a, -b) + s(K, -N)$ has $0 \leq y_0 < N$. Then $x_0 = \lambda_{y_n + b - sN + 1} - 1$ and $\sigma(x_0, y_0) = 0$. Then we have

$$(\pi\lambda)_i = \begin{cases} \lambda_i - 1 & \text{if } i = y_n + b - sN + 1 \\ \lambda_i & \text{else.} \end{cases}$$

Example 1.3.24. For $(K, N, a, b) = (3, 2, 1, -1)$, we look at the tableau of Example 1.3.13 and we see that the actions of π changes the fundamental domain (in gray) and changes the partition associated to the tableau:

-7	-5	-3	-1	1	3
-4	-2	0	2	4	6
-1	1	3	5	7	9
2	4	6	8	10	12

$\xrightarrow{\pi}$

-6	-4	-2	0	2	4
-3	-1	1	3	5	7
0	2	4	6	8	10
3	5	7	9	11	13

We now want to show that $W_{(K,N,a,b)}$ it is irreducible. In order to do so, we will need the following easy lemma.

Lemma 1.3.25. *Let $\sigma, \tau \in \mathrm{DPT}(K, N, a, b)$. Then the following are equivalent:*

(1) $\sigma = \tau$*,*

(2) $\deg(\sigma) = \deg(\tau)$ *and* $C_\sigma = C_\tau$*.*

Proof. This is a consequence of Proposition 1.3.21. $\square$

We are now ready to prove irreducibility.

Theorem 1.3.26. *The representation $W_{(K,N,a,b)}$ is an irreducible graded X-semisimple representation of $\ddot{H}_{q,t}(n)$.*

Proof. The fact that $W_{(K,N,a,b)}$ is a graded X-semisimple representation is clear from the definition of the action in Proposition 1.3.22 and our choice of the grading. We have to prove only irreducibility. Take a non-zero graded X-semisimple submodule M. By Lemma 1.3.25, as M is graded, we can find a non-zero weight vector $\nu_\sigma \in M$, for some $\sigma \in \mathrm{DPT}(K, N, a, b)$. By Proposition 1.3.16, we know that, for any tableau $\tau \in \mathrm{DPT}(K, N, a, b)$, we can find an affine permutation f such that $f\sigma = \tau$. By Proposition 1.3.17 and Remark 1.3.23, the action of $T_f + \sum_{g \in \widehat{\mathfrak{S}}^e_n, g \prec f} h_g T_g$, for some coefficients $h_g \in \mathbb{C}[q^{\pm 1}, t^{\pm 1}][X_1^{\pm 1}, \dots, X_n^{\pm 1}]$, where $\prec$ denotes the Bruhat order, takes ν_σ to a non-zero multiple of ν_τ. We have then proven that every weight vector has to be contained in M, so $M = W_{(K,N,a,b)}$. $\square$

Remark 1.3.27. Note that a consequence of the previous theorem is that each graded piece $(W_{(K,N,a,b)})_i$ is an irreducible X-semisimple representation of the small DAHA $\ddot{H}_{q,t}(n)^s$ (recall Lemma 1.3.7).

1.3.5 Classification of graded X-semisimple representations

Analogously to the case of representations from periodic skew tableaux, the representations defined via DPT provide a classification of graded X-semisimple representations of the DAHA at roots of unity.

For $A \geq 3$ and $B \in \mathbb{Z}/A\mathbb{Z}$, we define $\Xi_n^{(A,B)}$ to be the set of isomorphism classes of irreducible graded X-semisimple representations of $\ddot{H}_{q,t}(n)$ where q acts as a primitive A-th root of unity and t as q^{-B}.

We can now state our classification theorem.

Theorem 1.3.28. *Let $A \geq 3$ and $B \in \mathbb{Z}/A\mathbb{Z}$. Let*

$$J_n^{(A,B)} := \{(K, N, a, b) : aN - bK = n, N + K = A, a + b = B\}/\sim,$$

where the relation $\sim$ is given by $(K, N, a, b) \sim (K', N', a', b')$ if $K = K', N = N', a = a' + rK, b = b' + rN$.

Then the map $(K, N, a, b) \to W_{(K,N,a,b)}$ induces a bijection

$$J_n^{(A,B)} \xrightarrow{\sim} \Xi_n^{(A,B)}. \tag{1.3.2}$$

We will need two intermediate results to prove this theorem. We have to show that, given a graded X-semisimple representation, it is isomorphic to a representation from DPT. Secondly, two representations $W_{(K,N,a,b)}$ and $W_{(K',N',a',b')}$ are isomorphic precisely when $(K,N,a,b) \sim (K',N',a',b')$, according to the relation $\sim$ in the statement.

For the first step we will use methods that are analogous to those used in [SV05] to prove the classification in the generic case.

For a $\ddot{H}_{q,t}(n)$-module M such that q acts as a primitive A-th root of unity, and t as q^{-B}, we extend the notion of weight periodically. For a weight $\underline{w}$ and $i \in \mathbb{Z}$, write $i = \underline{i} + kn$, with $\underline{i} \in [1,n]$, then

$$w_i = q^{kB} w_{\underline{i}}.$$

The map $\mathbb{Z} \to q^{\mathbb{Z}}, i \mapsto w_i$ will still be denoted by $\underline{w}$.

The following result is stated and proved by Suzuki-Vazirani in [SV05, Lemma 4.19] when q is not a root of unity, by induction on $j-i$, but can be adapted to our situation as long as q does not have order 2.

Lemma 1.3.29. *Let L be an irreducible X-semisimple $\ddot{H}_{q,t}(n)^s$-module, where q acts as a primitive A-th root of unity with $A \neq 2$. For any weight $\underline{w}$ and $i, j \in \mathbb{Z}$ such that $i < j$ and $w_i = w_j$, there exists $p_\pm \in [i+1, j-1]$ respectively satisfying:*

$$w_i = q^{\mp 1} w_{p_\pm}.$$

The following is the analogue in our situation of Proposition 3.20 of [SV05] and will be the key result to construct a tableau from a weight $w \in M$.

Proposition 1.3.30. *Let $A \in \mathbb{Z}_{>0}$, $B \in \mathbb{Z}/A\mathbb{Z}$, and $C : \mathbb{Z} \to \mathbb{Z}/A\mathbb{Z}$, satisfying*

(1) $C(i+n) = C(i) + B \mod A$;

(2) For $i, j \in C^{-1}(r), r \in \mathbb{Z}/A\mathbb{Z}$, such that there is no integer between i and j in $C^{-1}(r)$, there exist unique $i < p_\pm < j$ such that $p_\pm \in C^{-1}(r \pm 1)$.

Then there exist integers K, N, a, b such that $N+K = A, a+b = B, aN - Kb = n$ and $\sigma \in \mathrm{DPT}(K,N,a,b)$ such that $C = C_\sigma$.

Note that the content function C_σ of any $\sigma \in \mathrm{DPT}(K,N,a,b)$ clearly satisfies conditions (1) and (2), with $A = N+K$, $B = a+b$.

Proof. First of all, we see that conditions (1) and (2) imply that C is surjective, and that, for all $r \in \mathbb{Z}/A\mathbb{Z}$, $C^{-1}(r)$ is unbounded.

For all $r \in \mathbb{Z}/A\mathbb{Z}$, we write $C^{-1}(r) = \left\{ i_r^{(j)} \mid j \in \mathbb{Z} \right\}$, such that

$$\forall j \in \mathbb{Z}, i_r^{(j)} < i_r^{(j+1)}, \quad i_r^{(1)} = \min\{C^{-1}(r) \cap \mathbb{Z}_{>0}\}.$$

By condition (2) and induction on j, we have

(i) if $i_r^{(1)} < i_{r+1}^{(1)}$ then $\begin{cases} i_r^{(j)} < i_{r+1}^{(j)} \\ i_r^{(j)} > i_{r+1}^{(j-1)} \end{cases}$, for all $j \in \mathbb{Z}$,

(ii) if $i_r^{(1)} > i_{r+1}^{(1)}$ then $\begin{cases} i_r^{(j)} > i_{r+1}^{(j)} \\ i_r^{(j-1)} < i_{r+1}^{(j)} \end{cases}$, for all $j \in \mathbb{Z}$,

We will construct the DPT σ and its associated lattice path $\mathcal{L}$ recursively. First, let $\sigma(0,0) = i_0^{(1)}$, and $\mathcal{L}_0 = (0,0)$.

For all $0 \le r \le A-1$, we assume $\mathcal{L}_0, \ldots, \mathcal{L}_r$ are constructed and $\sigma(\mathcal{L}_0), \ldots, \sigma(\mathcal{L}_r)$ are defined. We distinguish two cases:

- if $i_{r+1}^{(1)} > i_r^{(1)}$, then $\mathcal{L}_{r+1} = \mathcal{L}_r + (1,0)$ and $\sigma(\mathcal{L}_{r+1}) = i_r^{(1)}$ $\begin{array}{|c|c|}\hline i_r^{(1)} & i_{r+1}^{(1)} \\ \hline \end{array}$,
- if $i_{r+1}^{(1)} < i_{r+1}^{(1)}$, then $\mathcal{L}_{r+1} = \mathcal{L}_r + (0,-1)$ and $\sigma(\mathcal{L}_{r+1}) = i_{r+1}^{(1)}$ $\begin{array}{|c|}\hline i_{r+1}^{(1)} \\ \hline i_r^{(1)} \\ \hline \end{array}$.

Once $\mathcal{L}_A$ is constructed, let $\mathcal{L}_A = (K, -N)$. Then, necessarily, $N + K = A$.

For $r > A$, we continue constructing $\mathcal{L}_r$ and $\sigma(\mathcal{L}_r)$ recursively (the values of $i_r^{(1)}$ being A-periodic). Similarly, we can build $\mathcal{L}_r$ and $\sigma(\mathcal{L}_r)$ for $r \ge 0$ by downward recursion (the new box is placed to the left or below the previous box). By construction, $\mathcal{L}$ is a (K,N)-periodic lattice path. We see that we have filled precisely one box in each diagonal $x - y = \text{cst}$.

Now, for all $(r,j) \in \mathbb{Z}^2$, let

$$\sigma(\mathcal{L}_r + (j-1, j-1)) = i_r^{(j)}.$$

This defines a (K,N)-periodic filling of the whole plane $\mathbb{Z}^2$. Moreover, for all $(x,y) \in \mathbb{Z}^2$, $\sigma(x,y) = i_r^{(j)}$ with $r = x - y$.

Moreover, this filling is standard. Indeed, for all $(x,y) \in \mathbb{Z}^2$, let $\sigma(x,y) = i_{x-y}^{(j)}$. Consider $\sigma(x+1,y) = i_{x-y+1}^{(j')}$. We have the following configuration

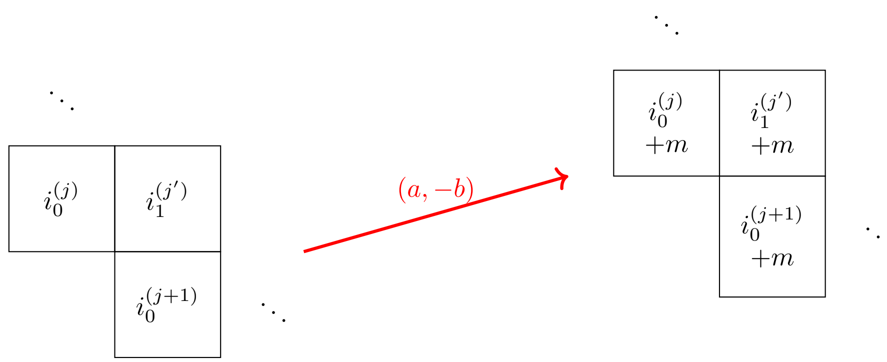
$$\begin{array}{|c|c|}\hline i_{x-y}^{(j)} & i_{x-y+1}^{(j')} \\ \hline \multicolumn{1}{c|}{} & i_{x-y}^{(j+1)} \\ \cline{2-2} \end{array}.$$

By definition of σ, by moving diagonally in the plane, we encounter these configurations

$$\begin{array}{|c|c|}\hline i_{x-y}^{(1)} & i_{x-y+1}^{(j'-j+1)} \\ \hline \multicolumn{1}{c|}{} & i_{x-y}^{(2)} \\ \cline{2-2} \end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline i_{x-y}^{(0)} & i_{x-y+1}^{(j'-j)} \\ \hline \multicolumn{1}{c|}{} & i_{x-y}^{(1)} \\ \cline{2-2} \end{array}.$$

Now, if $i^{(1)}_{x-y+1} > i^{(1)}_{x-y}$, then from the first configuration, $j' = j$ and by statement (i) from above, $\sigma(x+1,y) = i^{(j)}_{x-y+1} > \sigma(x,y)$. Otherwise, if $i^{(1)}_{x-y+1} < i^{(1)}_{x-y}$, then from the second configuration, we get $j' = j+1$ and by statement (ii), we obtain $\sigma(x+1,y) = i^{(j+1)}_{x-y+1} > \sigma(x,y)$. We can prove similarly that $\sigma(x,y+1) > \sigma(x,y)$.

Using condition (1), we see that $C(i^{(1)}_0 + n) = B$, so the value $i^{(1)}_0 + n$ appears in the B-th diagonal of σ. Let $(a,-b) \in \mathbb{Z}^2$ be the position of the box labelled $i^{(1)}_0 + n$ in that diagonal. We necessarily have $a+b = B$. Thus $\sigma(a,-b) = \sigma(0,0) + n$. From condition (1) again, all the values appearing the B-th diagonal are n plus the values appearing in the zeroth diagonal and both diagonals are ordered increasingly. Thus the translation sending i to $i+n$, between those two diagonals, is always $(a,-b)$ and we have $\sigma(x+a, x-b) = \sigma(x,x)+n$, for all $x \in \mathbb{Z}$. We now consider the diagonals 1 and $B+1$, whose values are n plus values of the former. From condition (2), the value $i^{(j')}_1$ is the unique value between $i^{(j)}_0$ and $i^{(j+1)}_0$ in diagonal 1 ($j' = j$ or $j+1$). Necessarily, the value $i^{(j')}_1 + n$ is the unique value between $i^{(j)}_0 + n$ and $i^{(j+1)}_0 + n$ in diagonal $B+1$. So the value $i^{(j')}_1 + n$ is also obtained from $i^{(j')}_1$ by translation by $(a,-b)$.



From one diagonal to the next, we show iteratively that, for all $(x,y) \in \mathbb{Z}^2$, $\sigma(x+a, y-b) = \sigma(x,y) + n$. Therefore, σ is indeed a doubly periodic tableau with respect to (K,N,a,b). From this reasoning, we also deduce that $\mathcal{L} < \mathcal{L}[a,b]$ and that the boxes between these lattice paths only have fillings between 1 and n, so $n = aN - bK$ is satisfied.

Finally, by construction, we clearly have $C = C_\sigma$.

$\square$

We are now ready to state the classification theorem for irreducible X-semisimple small DAHA representations.

Theorem 1.3.31. *Let M be an irreducible X-semisimple representation of $\ddot{H}_{q,t}(n)^s$ where q acts as a primitive A-th root of unity and t as q^{-B}. Then*

M is isomorphic to a graded piece $(W_{(K,N,a,b)})_i$ of $W_{(K,N,a,b)}$, for some choice of K, N, a, b such that $N + K = A$ and $a + b = B$.

As we mentioned above, the proof of this theorem mostly follows the proof of [SV05, Theorem 4.20]. The main difference lies in the combinatorial results concerning content functions which are necessary for the proof (compare our Proposition 1.3.30 to [SV05, Proposition 3.20]). We give a sketch of the proof and refer to [SV05] for more details.

Sketch of Proof: We start by picking a weight $\underline{w}$ of M. For $i \in \mathbb{Z}$, let $C(i)$ be such that $w_i = q^{C(i)}$. As q is an A-th root of unity, we can define a function $C : \mathbb{Z} \to \mathbb{Z}/A\mathbb{Z}$ sending i to $C(i)$. We note that C satisfies the hypothesis of Proposition 1.3.30. Indeed, the first condition is clearly fulfilled and the second follows by Lemma 1.3.29. Therefore, applying Proposition 1.3.30, we obtain a $\sigma \in \mathrm{DPT}(K, N, a, b)$ with $N + K = A, a + b = B$ and $C = C_\sigma$. We now want to show that M is isomorphic to $(W_{(K,N,a,b)})_i$, where i is the degree of v_σ in $W_{(K,N,a,b)}$.

We define $\psi : M \to (W_{(K,N,a,b)})_i$ as the map of representations sending $v_{\underline{w}}$ to v_σ, i.e. we set $\psi((\sum_f \alpha_f T_f)v_{\underline{w}}) = \sum_f \alpha_f T_f v_\sigma$. By the irreducibility of $(W_{(K,N,a,b)})_i$ (recall Remark 1.3.27) we can conclude that ψ is an isomorphism. □

Remark 1.3.32. The isomorphism ψ can be defined more explicitly using so-called DAHA intertwiners. Namely, using these elements of $\ddot{H}_{q,t}(n)$, one can give an explicit formula for the image of each weight vector in M (see the proof of [SV05, Theorem 4.20]).

We now have all the intermediate results we need for the proof of the classification result in Theorem 1.3.28.

Proof of Theorem 1.3.28. Let M be an irreducible graded X-semisimple representation of $\ddot{H}_{q,t}(n)$ where q acts as a primitive A-th root of unity and t as q^{-B}. We take the graded piece M_0 of M in degree 0. By Theorem 1.3.31, we have an isomorphism $\psi : M_0 \to W_{(K,N,a,b)}$ of X-semisimple $\ddot{H}_{q,t}(n)^s$-modules. We extend ψ to a $\ddot{H}_{q,t}(n)$-homomorphism. By the irreducibility of $W_{(K,N,a,b)}$ we obtain that ψ is an isomorphism.

Moreover, using Proposition 1.3.21, we can see that any two representations $W_{(K,N,a,b)}$ and $W_{(K',N',a',b')}$ are isomorphic if and only if $(K, N, a, b) \sim (K', N', a', b')$. □

1.3.6 A faithful DAHA representation

Constructing a faithful representation of the DAHA is a key step to prove that the DAHA is isomorphic to a certain tangle algebra of the torus, as conjectured [MS21]. We will show that by taking the direct sum of representations $W_{(K,N,a,b)}$, as N, K, a, b vary, we obtain a faithful representation. We decided to condense here the proof of this result, whose steps are taken along several different sections in [BCMN23].

Theorem 1.3.33. *The representation $\mathcal{W}$ is a faithful $\ddot{H}_{q,t}(n)$ representation.*

We want to reduce to proof of Theorem 1.3.33 to a much simpler statement concerning Zariski denseness of a certain set. We begin by studying a very important example of DPT and understanding for which values of K, N, a, b we can find such a tableau. It will be crucial for us to construct a family of representations $W_{(K,N,a,b)}$ that admit a map $\mathcal{P} \to W_{(K,N,a,b)}$. We can define such a map whenever $\mathrm{DPT}(K, N, a, b)$ contains a *line DPT* σ, by which we mean a tableau whose fundamental domain is a line, so, more precisely, that $\sigma(x, 0) = x + 1$, for $0 \le x < n$. Let us check for which values of K, N, a, b such a tableau appears in $\mathrm{DPT}(K, N, a, b)$.

As $\sigma(0, 1) = \alpha n + \beta + 1$, there exists $s \in \mathbb{Z}$ satisfying $(0, 1) = (\beta, 0) + \alpha(a, -b) + s(K, -N)$. Therefore, we have $\sigma(-\beta, 1) = \alpha n + 1$ and, more generally, $\sigma(-r\beta, r) = r\alpha n + 1$ for all $r \in \mathbb{Z}$. In particular, by setting $r = N$ we obtain $\sigma(-N\beta, N) = N\alpha n + 1$, which implies $\sigma(-N\beta - \alpha n, N) = 1$, so $K = N\beta + \alpha n$. Values of the form $kn + 1$ can only appear in positions $\sigma(-r\beta + tn, r) = r\alpha n + tNn + 1$. So, surjectivity implies that in the sequence $\{r\alpha\}_{r\in\mathbb{Z}}$ all residues modulo N must appear, so we need to have $\gcd(\alpha, N) = 1$.

Choose $u, v \in \mathbb{Z}$ such that $uN - v\alpha = 1$. Then we have

$$\sigma(-v\beta, v) = v\alpha n + 1 = (uN - 1)n + 1, \qquad \sigma(-v\beta - un, v) = -n + 1.$$

Therefore $(a, b) = (v\beta + un, v) \mod (K, N)$.

Conversely, suppose $n, N, \alpha, \beta \in \mathbb{Z}$ are given and satisfy $n, N > 0$, $\alpha \ge 0$, $\gcd(\alpha, N) = 1$, $0 \le \beta < n$, $(\alpha, \beta) \ne (0, 0)$. Pick $u, v \in \mathbb{Z}$ satisfying $uN - v\alpha = 1$. Set

$$K = N\beta + \alpha n, \qquad a = v\beta + un, \qquad b = v.$$

In the matrix form, we have

$$\begin{pmatrix} a & K \\ b & N \end{pmatrix} = \begin{pmatrix} n & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u & \alpha \\ v & N \end{pmatrix}.$$

Observe that a different choice of u, v gives an equivalent vector (a, b) modulo (K, N).

Set $\sigma(-r\beta + i, r) = r\alpha n + i + 1$ for all $r \in \mathbb{Z}$, $0 \le i < n$ and extend σ to the whole of $\mathbb{Z}^2$ using the fact that

$$\sigma(x + n, y) = \sigma(x + aN - bK, y - bN + bN) = \sigma(x, y) + Nn.$$

The resulting tableau satisfies

$$\sigma(x - \beta, y + 1) = \sigma(x, y) + \alpha n, \qquad \sigma(x + n, y) = \sigma(x, y) + nN. \tag{1.3.3}$$

Using $(K, N) = N(\beta, 1) + \alpha(n, 0)$, $(a, b) = v(\beta, 1) + u(n, 0)$, we get

$$\sigma(x + K, y - N) - \sigma(x, y) = -N\alpha n + \alpha n N = 0$$

and

$$\sigma(x + a, y - b) - \sigma(x, y) = -v\alpha n + unN = n.$$

Finally, to see that σ is standard we use

$$\sigma(i,1) = \alpha n + \beta + i + 1 > i + 1 = \sigma(i,0) \quad (0 \leq i < n - \beta),$$

$$\sigma(i,1) = \alpha n + \beta + i + 1 - n + Nn > i + 1 = \sigma(i,0) \quad (n - \beta \leq i < n),$$

and periodicity conditions (1.3.3).

Let us analyse which permutations are allowed to act on a line tableau. Clearly, no element of the finite permutation group $\mathfrak{S}_n$ is allowed to act on σ. Consider now a permutation f satisfying $f(1) < f(2) < \cdots < f(n)$. For N and α sufficiently large, more precisely for $N > \frac{f(n)-f(1)}{n}$ and $\alpha > \frac{f(n)}{n}$, the permutation f is allowed to act on σ. So, for a given f to be allowed to act on σ, we just need N and α to be sufficiently large. We state this rather oddly looking lemma, which will be a very important step in the proof of Theorem 1.3.33.

Lemma 1.3.34. *Fix $f \in \widehat{\mathfrak{S}}_n^e$ such that $f(1) < \cdots < f(n)$ and let K, N, a, b vary among the values for which $DPT(K,N,a,b)$ admits a line tableau σ and f is allowed to act σ. The set of pairs (q, q^{-a-b}) is Zariski dense in $\mathbb{C}^2$*

Proof. The above construction uses numbers N, α, u, v, β. We set $\beta = n-1$. The remaining numbers need to satisfy $uN - v\alpha = 1$. Computing K, a, b, we obtain

$$K + N = n(N + \alpha), \qquad a + b = n(u + v).$$

For fixed values of $u, v > 0$ satisfying $\gcd(u,v) = 1$, the values a and b are fixed and the set $\{(q, q^{-a-b})\}$, as N and α vary, is Zariski dense in the curve $\{(z, z^{-n(u+v)}), z \in \mathbb{C}\}$. Indeed, adding (u,v) to (α, N) makes the values of α, N and $K + N$ arbitrarily large, so q takes infinitely many values.

Next we show that a polynomial vanishing on all these curves has to be identically zero on $\mathbb{C}^2$. We take a polynomial $p(x,y)$ and write

$$p(x,y) = x^m p_m(y) + \cdots + p_0(y).$$

Fix a complex number $\eta \neq 0$ which is not a root of unity. We know that for all the solutions to the equations $z^{-n(u+v)} = \eta$, as u and v vary over pairs of positive relatively prime integers, the polynomial

$$p_\eta(x) = x^m p_m(\eta) + \cdots + p_0(\eta)$$

takes the value 0. Therefore p_η is a polynomial in one variable with infinitely many zeroes, so it is the zero polynomial, hence $p_i(\eta) = 0, i = 0, \ldots, m$. As this has to be true for any choice of η, we conclude that $p_i(y)$ is the zero polynomial for $i = 0, \ldots, m$, hence $p(x,y)$ is the zero polynomial. □

We are now ready to give a proof of Theorem 1.3.33

Proof of Theorem 1.3.33. We need to prove that any nonzero element ξ of $\ddot{H}_{q,t}(n)$ does not act as zero on the representation $\mathcal{W}$. We know (recall Theorem 1.2.14) that the polynomial representation $\mathcal{P}$ of $\ddot{H}_{q,t}(n)$ is faithful, so $\xi(p(Y)) \neq 0$, for some $p(Y) \in \mathcal{P}$. Moreover, the element $\xi(p(Y))$, possibly up to multiplication by a denominator, can be written as $\sum_{f \in S} \alpha_f T_f(1)$, where S is a finite subset of $\{f \in \widehat{\mathfrak{S}}_n^e \mid f(1) < \cdots < f(n)\}$ and each coefficient α_f is a nonzero polynomial in q and t. Now we want to restrict our attention to those values of N, K, a, b that admit the map $\mathcal{P} \to W_{(K,N,a,b)}$ sending 1 to the line tableau. We want to show that for some choice of these parameters K, N, a, b we have $\sum_{f \in S}(\alpha_f T_f)(\sigma_{(K,N,a,b)}) \neq 0$, which would then conclude our proof. Assume this is not the case, then $\sum_{f \in S}(\alpha_f T_f)(\sigma_{(K,N,a,b)}) = 0$ also for those values of K, N, a, b for which each f is allowed to act on σ. We deduce from Remark 1.3.23 that $\alpha_f(q, q^{-a-b}) = 0$, for q a primitive $N+K$-th root of unity, for all these choices of N, K, a, b. But we proved in Lemma 1.3.34 that the set of pairs (q, q^{-a-b}) forms a Zariski dense set in $\mathbb{C}$, so each α_f must be zero. $\square$

1.4 DAHA representations from intertwiners

As we have previously mentioned, in [JV18] we find a construction of a DAHA representation from quantum group intertwiners. In that context, a key ingredient is the combinatorial structure of the category of quantum group representations. Thanks to that, the authors are able to show that their *rectangular representation* is in fact one of the representations in [SV05] and is indeed rectangular in the sense that it corresponds to a periodic skew tableaux of rectangular shape. Here we want to do something similar in the root of unity context, which means that we have to change our setting slightly and have to deal with the fusion category, the category of certain representations of the quantum group $U_v(\mathfrak{gl}_N)$ at roots of 1, hoping to then recover our doubly periodic tableaux.

1.4.1 Crash course in fusion rings

We want to interpret our representations $W_{(N,K,a,b)}$ as spaces of intertwiners. In order to do so, we need to revise a few facts about fusion rings. We will be mostly interested in the combinatorial description of fusion rings, however we will quickly recall the definition. We refer the reader to [Kir96] and [AS14] for more details.

We want to consider the fusion ring at level K of the quantum group $U_v(\mathfrak{gl}_N)$. This is defined as the Grothendieck ring of the quotient of the category of tilting modules by negligible modules.

Let us spend a few more words on the construction. The quantum group $U_{\mathrm{q}}(\mathfrak{gl}_N)$ is the algebra over $\mathbb{Q}(\mathrm{q})$ generated by $E_i, F_i, K_i^{\pm 1}, i = 1, \ldots, N$ subject to certain relations (see for example [Jan]). We want to specialize the quantum parameter q to a root of unity v, therefore the construction uses the so-called *Lusztig divided power quantum group* $U_{\mathcal{A}}(\mathfrak{gl}_N)$. This is the $\mathcal{A} := \mathbb{Z}[\mathrm{q}, \mathrm{q}^{-1}]$-

subalgebra of $U_{\mathrm{q}}(\mathfrak{gl}_N)$ generated by $K_i^{\pm 1}$ and the divided powers

$$E_i^{(r)} = \frac{E_i^r}{[r]_{\mathrm{q}}}, F_i^{(r)} = \frac{F_i^r}{[r]_{\mathrm{q}}}, i = 1 \dots N, r \geq 0.$$

The quantum group we will be looking at is defined as $U_v(\mathfrak{gl}_N) := U_{\mathcal{A}}(\mathfrak{gl}_N) \otimes_{\mathcal{A}} \mathbb{C}$, where we are considering $\mathbb{C}$ as an $\mathcal{A}$-module via the specialization $\mathrm{q} \mapsto v$.

Let $\mathcal{C}_v$ be the category of type 1 modules over the quantum group $U_v(\mathfrak{gl}_N)$, where we choose v to be square root of q and therefore a primitive $K + N$-th root of 1, if $K + N$ is odd, or a $2(K + N)$-th root of 1, if $N + K$ is even. We consider the Grothendieck ring of $\mathcal{C}_v$, i.e. the abelian group generated by the isomorphism classes in $\mathcal{C}_v$ with relations $[A] + [B] = [C]$ whenever A, B, C fit into the exact sequence $0 \to A \to C \to B \to 0$. We can make $\mathcal{C}_v$ into a tensor category using the coproduct in $U_v(\mathfrak{gl}_N)$, defining effectively a natural ring structure on the Grothendieck group; this will be called the *Grothendieck ring* $\mathrm{Gr}(\mathcal{C}_v)$. We now want to restrict our attention to *tilting modules*. These are the modules V such that V and V^* admit a composition series with factors isomorphic Weyl modules. We refer to [Kir96] for the precise definitions. Let $X^+ = \{\lambda \in \mathbb{Z}^N : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N\}$. For each $\lambda \in X^+$, there is a unique tilting module of highest weight λ and these modules are all the indecomposable tilting modules in $\mathcal{C}_v$ up to isomorphism. To finally construct the fusion ring, we want to take the quotient of the subcategory of tilting modules in $\mathcal{C}_v$ by *negligible modules*. These are the tilting modules that have no summands with highest weight out of the fundamental alcove

$$\mathcal{A}_{K,N} = \left\{ \lambda \in X^+ \mid \lambda_1 - \lambda_N \leq K \right\}.$$

The fusion ring $\mathcal{F}_v$ is the Grothendieck ring of this quotient category. A $\mathbb{Z}$-basis for $\mathcal{F}_v$ is given by the image of the tilting modules with highest weight in $\mathcal{A}_{K,N}$. By abuse of notation, we will simply write $[\lambda]$ for the corresponding class in the fusion ring.

We will consider some special representations for which it is easy to compute tensor products: $V := (1, 0, \dots, 0), L := (K, 0, \dots, 0)$ and $D = (1, 1, \dots, 1)$.

Proposition 1.4.1 ([GW90], [MS12]). *For all* $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N) \in \mathcal{A}_{K,N}$,

$$\begin{aligned} V \otimes [\lambda] &= \sum_{\substack{1 \leq j \leq N \\ [\lambda + \varepsilon_j] \in \mathcal{A}_{K,N}}} [\lambda + \varepsilon_j], \\ L \otimes [\lambda] &= [(K + \lambda_N, \lambda_1, \lambda_2, \dots, \lambda_{N-1})], \\ D \otimes [\lambda] &= [\lambda + \omega_N]. \end{aligned}$$

As we can see, the proposition states that multiplying by L is equivalent to adding a line from below and multiplying by D shifts the diagram by one box to the right. The representations L and D have inverses $L^{-1} = (0, \dots, 0, -K)$ and $D = (-1, -1, \dots, -1)$.

Remark 1.4.2. If U, W are irreducible representations $W = [\lambda]$ and $U = [\mu]$, with $\lambda, \mu \in \mathcal{A}_{K,N}$, by Proposition 1.4.1, the intertwining space $M^{(n)}_{[\mu],[\lambda]}$ is non-trivial only if there exist $(i_1, i_2, \ldots, i_n) \in [\![1, N]\!]^n$ such that

$$\mu = \lambda + \sum_{\ell=1}^{m} \varepsilon_{i_\ell}.$$

1.4.2 Combinatorics of quantum group intertwiners

We want to give a combinatorial description of quantum group intertwiners. To this end, we introduce lattice paths. Once again, for more details on lattice paths we refer to [BCMN23]. We fix again $n = aN - bK > 0$.

Definition 1.4.3. A *lattice path* $\mathcal{L}$ in $\mathbb{Z}^2$ is a sequence $(\mathcal{L}_i)_{i\in\mathbb{Z}}$ where $\mathcal{L}_i \in \mathbb{Z}^2$, and for each $i \in \mathbb{Z}$, $\mathcal{L}_{i+1} = \mathcal{L}_i + (0, -1)$ or $\mathcal{L}_{i+1} = \mathcal{L}_i + (1, 0)$. The *image* of $\mathcal{L}$ is $\{\mathcal{L}_i\}_{i\in\mathbb{Z}}$. Given a lattice path $\mathcal{L}$ and integers a, b, form the (a, b)-shifted path $\mathcal{L}[a, b]$ where $\mathcal{L}[a, b]_i = \mathcal{L}_i + (a, -b)$. A lattice path $\mathcal{L}$ is (K, N)-*periodic* if $\mathcal{L}$ and $\mathcal{L}[K, N]$ have the same image.

For a (K, N)-periodic lattice path $\mathcal{L}$ and for each $y \in \mathbb{Z}$, we define $\mathcal{L}(y)$ to be the unique value of $x \in \mathbb{Z}$ such that both (x, y) and $(x, y + 1)$ are in the image of $\mathcal{L}$ (i.e. the x value of the vertical step from $y + 1$ to y on $\mathcal{L}$).

There is a partial ordering $\leq$ on the set of $\mathbb{Z}^2$ lattice paths given by $\mathcal{L} \leq \mathcal{M}$ when $\mathcal{L}(y) \leq \mathcal{M}(y)$ for all $y \in \mathbb{Z}$.

Proposition 1.4.4. *The following map is a bijection:*

$$\begin{array}{ccc} \left\{ \begin{array}{c} (K,N)\text{-}\textit{periodic} \\ \textit{lattice paths} \end{array} \right\} & \to & \mathcal{A}_{K,N}, \\ \mathcal{L} & \mapsto & \lambda(\mathcal{L}) := (\mathcal{L}(0), \mathcal{L}(1), \ldots, \mathcal{L}(N-1)). \end{array}$$

Proof. For all (K, N)-periodic lattice paths $\mathcal{L}$, $\mathcal{L}(0) \leq \mathcal{L}(-1) = \mathcal{L}(N - 1) + K$, so the tuple $(\mathcal{L}(0), \mathcal{L}(1), \ldots, \mathcal{L}(N - 1))$ is indeed in $\mathcal{A}_{K,N}$.

Moreover, for all $\lambda \in \mathcal{A}_{K,N}$, we can define a (K, N)-periodic lattice path $\mathcal{L}_\lambda$ via, for all $0 \leq i \leq N - 1$, $r \in \mathbb{Z}$,

$$\mathcal{L}_\lambda(i + rN) = \lambda_{i+1} - rK.$$

Then $\lambda \mapsto \mathcal{L}_\lambda$ is the inverse map of $\mathcal{L} \mapsto \lambda(\mathcal{L})$. $\square$

As a consequence of this result, we can consider the (K, N)-periodic lattice paths $(\mathcal{L}_\lambda)_{\lambda\in\mathcal{A}_{K,N}}$ as the basis for the fusion ring $\mathcal{F}_v$. This gives a ring structure to the set of (K, N)-periodic lattice paths, using the fusion ring structure. In particular, we can translate the Pieri rule in terms of lattice paths as follows.

Lemma 1.4.5. *For* $\lambda \in \mathcal{A}_{K,N}$*,*

$$\mathcal{L}_V \cdot \mathcal{L}_\lambda = \sum_{\substack{1\leq j\leq N \\ \mathcal{L}_\lambda(j-1)<\mathcal{L}_\lambda(j)}} \mathcal{L}_{\lambda+\varepsilon_j}. \tag{1.4.1}$$

Proof. We can obtain the lattice path $\mathcal{L}_{\lambda+\varepsilon_j}$ from the lattice path $\mathcal{L}_\lambda$ by replacing each $\mathcal{L}_\lambda(j-1+rN)$, for $r \in \mathbb{Z}$, by $\mathcal{L}_\lambda(j-1+rN)+1$ (adding one box in lines $j-1+rN$). Therefore, it is still a (K,N)-periodic lattice paths if and only if $\mathcal{L}_\lambda(j-1) < \mathcal{L}_\lambda(j)$. $\square$

Definition 1.4.6. For $\mathcal{L}$, $\mathcal{M}$ two (K,N)-periodic lattice paths such that $\mathcal{L} \leq \mathcal{M}$, a *standard* (K,N)*-periodic skew tableau of shape* $\mathcal{M}\setminus\mathcal{L}$ is a standard filling σ of

$$\mathcal{M}\setminus\mathcal{L} := \left\{(x,y)\in\mathbb{Z}^2 \mid \mathcal{L}(y) \leq x < \mathcal{M}(y)\right\},$$

such that:

(1) σ is (K,N)-periodic: for all $(x,y)\in\mathcal{M}\setminus\mathcal{L}$, we have $(x+K,y-N)\in\mathcal{M}\setminus\mathcal{L}$, and

$$\sigma(x+K,y-N) = \sigma(x,y).$$

(2) if n is the number of boxes of $\mathcal{M}\setminus\mathcal{L}$ in each N consecutive rows (or K consecutive columns), then all numbers $1,2\dots,n$ appear in σ.

Let $ST_{K,N}(\mathcal{M}\setminus\mathcal{L})$ denote the set of (K,N)-periodic skew tableaux of shape $\mathcal{M}\setminus\mathcal{L}$.

We want to give a description of the *intertwining space* $M_{U,W} := \mathrm{Hom}(U, V^{\otimes n}\otimes W)$, where U and W are classes in the fusion ring.

Let $\lambda,\mu\in\mathcal{A}_{K,N}$, if $\lambda_i \leq \mu_i$, for all $1\leq i\leq N$, then fix n

$$n = \sum_{i=1}^{N}(\mu_i-\lambda_i)\,, \tag{1.4.2}$$

the difference of the total number of boxes of μ and λ.

For any (K,N)-periodic skew tableau σ of shape $\mathcal{L}_\mu\setminus\mathcal{L}_\lambda$, one can associate a chain of $n+1$ dominant weights $\mathbf{u}=(u_0,u_1,\dots,u_n)$ as follows. Let $u_0=\lambda$, then for each $1\leq\ell\leq n$, let $u_\ell = u_{\ell-1}+\varepsilon_{i_\ell}$, where i_ℓ is the row between 1 and N where the value ℓ appears in σ. Then, from Remark 1.4.2,

$$u_n = \mu = \lambda + \sum_{\ell=1}^{n}\varepsilon_{i_\ell}.$$

We now want to show that

$$\dim M^{(n)}_{[\mu],[\lambda]} = \#ST_{K,N}(\mathcal{L}_\mu\setminus\mathcal{L}_\lambda).$$

This is done with a method very similar to that applied in [JV18]. The idea is to show that to each standard (K,N)-periodic skew tableau σ of shape $\mathcal{L}_\mu\setminus\mathcal{L}_\lambda$, one can associate a line L_σ in $M^{(n)}_{[\mu],[\lambda]}$ as follows.

Recall that we had the Pieri rule, according to which

$$V\otimes[\lambda] = \sum_{\lambda+\varepsilon_j\in\mathcal{A}_{K,N}}[\lambda+\varepsilon_j].$$

Therefore, for all j such that $\lambda+\varepsilon_j \in \mathcal{A}_{K,N}$, the $(\lambda+\varepsilon_j)$-isotypic component of $V \otimes [\lambda]$ has multiplicity 1 and $\dim \mathrm{Hom}([\lambda+\varepsilon_j], V \otimes [\lambda]) = 1$.

Similarly, for i, j such that $\lambda + \varepsilon_i \in \mathcal{A}_{K,N}$ and $\lambda + \varepsilon_i + \varepsilon_j \in \mathcal{A}_{K,N}$,

$$\mathrm{Hom}\left([\lambda+\varepsilon_i+\varepsilon_j], (V \otimes (V \otimes [\lambda])_{\lambda+\varepsilon_i}) \cap \left((V^{\otimes 2} \otimes [\lambda])_{\lambda+\varepsilon_i+\varepsilon_j}\right)\right)$$

is a one-dimensional subspace of $\mathrm{Hom}([\lambda+\varepsilon_i+\varepsilon_j], V^{\otimes 2} \otimes [\lambda])$. It keeps track of the fact that we have added first ε_i and then ε_j. Conversely, $\mathrm{Hom}([\lambda+\varepsilon_i+\varepsilon_j], V^{\otimes 2} \otimes [\lambda])$ is of dimension 1 or 2, depending on whether $i = j$ and on whether it is possible to also add first ε_j then ε_i.

Now, let $\sigma \in ST_{K,N}(\mathcal{L}_\mu \setminus \mathcal{L}_\lambda)$, and let $\mathbf{u}_\sigma = (u_0, \ldots, u_n)$ be the associated chain of dominant weights. Then

$$L_\sigma := \mathrm{Hom}\left([\mu], \bigcap_{i=0}^{n} V^{\otimes n-i} \otimes \left(V^{\otimes i} \otimes [\lambda]\right)_{u_i}\right) \tag{1.4.3}$$

is a one dimensional subspace of $M^{(n)}_{[\mu],[\lambda]}$.

We obtain the following theorem.

Theorem 1.4.7. *There is an isomorphism of vectors spaces*

$$M^{(n)}_{[\mu],[\lambda]} \simeq \bigoplus_{\sigma \in ST_{K,N}(\mathcal{L}_\mu \setminus \mathcal{L}_\lambda)} L_\sigma.$$

In particular, the dimension of $M^{(n)}_{[\mu],[\lambda]}$ is equal to the number of standard (K,N)-periodic fillings of the infinite skew tableau $\mathcal{L}_\mu \setminus \mathcal{L}_\lambda$.

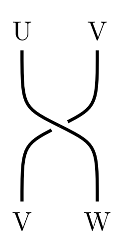
1.4.3 Affine Hecke algebra action on spaces of intertwiners

In this section, we will define an AHA action on the space $\mathrm{Hom}(U, V^{\otimes n} \otimes W)$. The fusion category is endowed with a braiding, which is a functorial isomorphism called the *R-matrix*

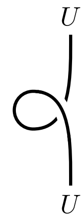
$$\check{R}_{U,V} : U \otimes W \to W \otimes U,$$

for any two classes U, V.

As the fusion category has a ribbon category structure, morphism can be drawn as directed tangles. The R-matrix is represented pictorially as

$\check{R}_{U,V} =$ 

Moreover, the fusion category is endowed with a so-called *ribbon element* θ_U

$\theta_U =$ 

For all $U \in \mathcal{F}_v$, $\theta_U : U \to U$ is a functorial isomorphism satisfying,

$$\theta_{U \otimes W} = \check{R}_{W,U} \check{R}_{U,W} \left(\theta_U \otimes \theta_W\right) \qquad (U, W \in \mathcal{F}_v). \tag{1.4.4}$$

Lemma 1.4.8 ([Kir96])**.** *For all $\lambda \in \mathcal{A}_{K,N}$,*

$$\theta_{[\lambda]} = v^{\langle \lambda, \lambda+2\rho \rangle} \mathrm{id}_{[\lambda]},$$

where

$$\rho = \frac{1}{2}\left((N-1)\varepsilon_1 + (N-3)\varepsilon_2 + \cdots + (-N+1)\varepsilon_N\right).$$

For any $U \in \mathcal{F}_v$, write the decomposition of U in the basis of the fusion ring:

$$U = \sum_{\lambda \in \mathcal{A}_{K,N}} c_{U,\lambda}[\lambda].$$

Let $U_\lambda := [\lambda^{\oplus c_{U,\lambda}}] = c_{U,\lambda}[\lambda]$ denote the λ-*isotypic component* of U.

Corollary 1.4.9. *For $\lambda, \mu \in \mathcal{A}_{K,N}$, if the decomposition of $[\lambda] \otimes [\mu]$ is given by*

$$[\lambda] \otimes [\mu] = \sum_{\nu \in \mathcal{A}_{K,N}} c^{\nu}_{\lambda,\mu}[\nu],$$

then $\check{R}_{[\mu],[\lambda]}\check{R}_{[\lambda],[\mu]}$ acts on the isotypic component $([\lambda] \otimes [\mu])_\nu = [\nu^{\oplus c^{\nu}_{\lambda,\mu}}]$ by multiplication by

$$v^{\langle \nu, \nu+2\rho \rangle - \langle \lambda, \lambda+2\rho \rangle - \langle \mu, \mu+2\rho \rangle}.$$

Proof. From relation (1.4.4), we get

$$\check{R}_{[\mu],[\lambda]}\check{R}_{[\lambda],[\mu]} = \theta_{[\lambda]\otimes[\mu]} \left(\theta_{[\lambda]} \otimes \theta_{[\mu]}\right)^{-1}.$$

We apply Lemma 1.4.3 to obtain the result. $\square$

We define the Hecke algebra action by postcomposition on the intertwining space $\mathrm{Hom}(U, V^{\otimes n} \otimes W)$ following [OR07] (see also [AS98]). Recall that, from Remark 1.2.12, the AHA $\dot{H}_q(n)$ is the subalgebra of $\ddot{H}_{q,t}(n)$ generated by $T_1, T_2, \ldots, T_{n-1}$ and $X_1^{\pm 1}$.

Let us define, for all $1 \leq i \leq n-1$,

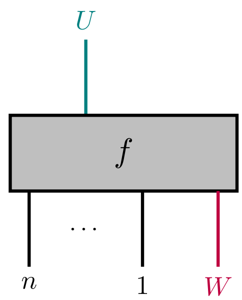
$$R_i := \mathrm{Id}_{V^{\otimes(m-i-1)}} \otimes \check{R}_{V,V} \otimes \mathrm{Id}_{V^{\otimes(i-1)}} \otimes \mathrm{Id}_W,$$
$$R_0^2 := \mathrm{Id}_{V^{\otimes(n-1)}} \otimes \left(\check{R}_{W,V}\check{R}_{V,W}\right).$$

Proposition 1.4.10. *The map*

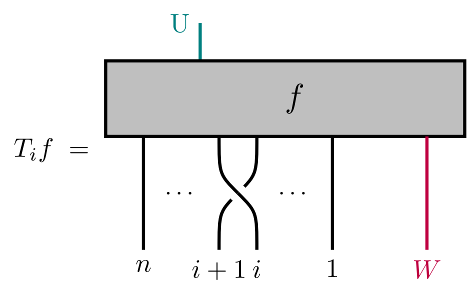
$$\begin{aligned} T_i &\mapsto v \cdot R_i, \quad \text{for} \quad 1 \le i \le n-1, \\ X_1 &\mapsto R_0^2, \end{aligned}$$

gives an action of $\dot{H}_q(n)$ *on* $\mathrm{Hom}(U, V^{\otimes n} \otimes W)$.

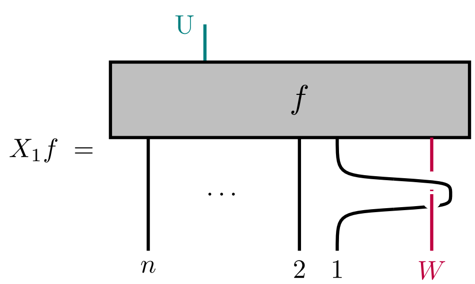
We want to give a pictorial interpretation to this action. A morphism in $\mathrm{Hom}(U, V^{\otimes n} \otimes W)$ can be represented as



The generator T_i acts on a morphism f by braiding the i-th and the $i+1$-st strand at the bottom

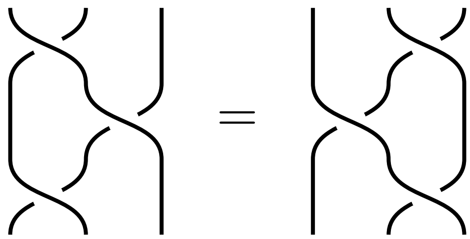


We draw the action of X_1 on f as



Proof of Proposition 1.4.10. The relations can be verified by using the tangle representation.

The R-matrix satisfied the Yang-Baxter equation, which we can draw as the braid relation:



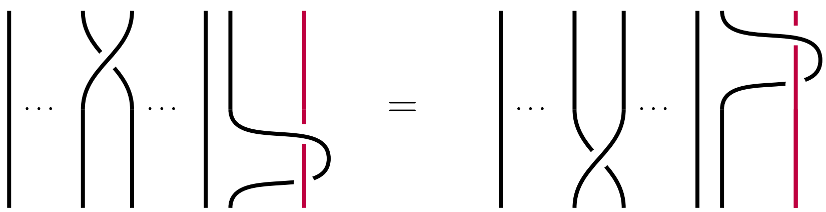
(1.4.5)

So, we obtain the braid relation ((2)) of the DAHA presentation.

Relation ((5)) corresponds to

$$R_i R_0^2 = R_0^2 R_i, \qquad (2 \leq i \leq n-1).$$

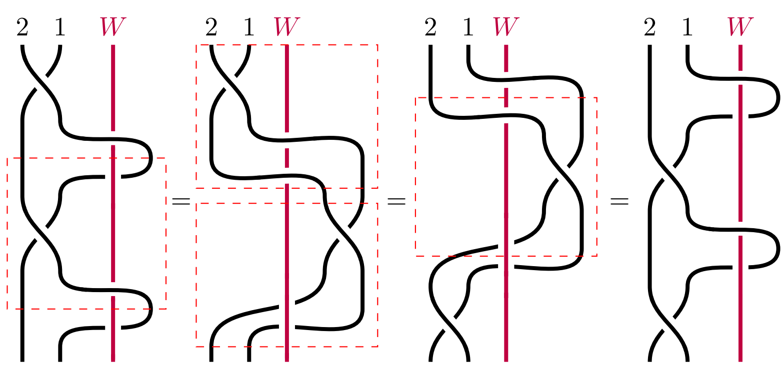
This is clear using the tangle representation:



As for relation ((8)), we have to check the following

$$R_1 R_0^2 R_1 R_0^2 = R_0^2 R_1 R_0^2 R_1.$$

This relation can be verified by applying several times the braid relation:



Finally, we need to take care of the Hecke relation ((1)) of Definition 1.2.11. As $v^2 = q$, these relations are, for all $1 \leq i \leq n-1$,

$$(T_i - v^2)(T_i + 1) = 0 \quad \Leftrightarrow \quad (R_i - v)(R_i + v^{-1}) = 0. \tag{1.4.6}$$

However, we know how the R-matrix of the quantum group $U_v(\mathfrak{gl}_N)$ acts on the tensor product $V \otimes V$ of two standard representations $V \simeq \mathbb{C}^N$. If $\{E_{ij}\}_{1\leq i,j\leq N}$ denotes the standard basis of $\mathrm{End}(V)$, then we can write:

$$\check{R}_{V,V} = v\sum_{i=1}^{N} E_{ii}\otimes E_{ii} + \sum_{i\neq j} E_{ij}\otimes E_{ji} + (v - v^{-1})\sum_{i>j} E_{ii}\otimes E_{jj}.$$

We can see that $\check{R}_{V,V}$ satisfies $(\check{R}_{V,V} - v\,\mathrm{Id}_{V\otimes V})(\check{R}_{V,V} + v^{-1}\,\mathrm{Id}_{V\otimes V}) = 0$. As a result, this relation is still valid in the fusion category and the R_i's satisfy the Hecke relation (1.4.6). □

We recall the content function of Definition 1.3.18, and extend it to standard (K,N)-periodic skew tableaux. The *content function* C_σ of a standard (K,N)-periodic skew tableaux σ is the function $\mathbb{Z} \to \mathbb{Z}/(N+K)\mathbb{Z}$ such that

$$C_\sigma(i) = x - y, \quad \text{for } (x,y)\in\mathbb{Z}^2 \text{ such that } \sigma(x,y) = i.$$

Proposition 1.4.11. *For all $\sigma \in ST_{K,N}(\mathcal{L}_\mu \setminus \mathcal{L}_\lambda)$, the generators X_i of $\dot{H}_q(n)$ act on L_σ as*

$$X_i \cdot L_\sigma = v^{2C_\sigma(i)} L_\sigma, \quad 1 \leq i \leq n-1. \tag{1.4.7}$$

Proof. For all $1\leq i\leq n-1$, we have

$$X_i = q^{-i+1} T_{i-1}\cdots T_2 T_1 X_1 T_1 T_2 \cdots T_{i-1}.$$

Therefore, each X_i acts on $M^{(n)}_{[\mu],[\lambda]}$ by $R_{i-1}\cdots R_2 R_1 R_0^2 R_1 R_2\cdots R_{i-1}$. By the properties of the $\mathcal{R}$-matrix and (1.4.4), we get

$$\begin{aligned} X_i &\mapsto \mathrm{Id}_{V^{\otimes n-i}} \otimes \left(\check{R}_{V^{\otimes i-1}\otimes[\lambda],V}\,\check{R}_{V,V^{\otimes i-1}\otimes[\lambda]}\right), \\ &= \mathrm{Id}_{V^{\otimes n-i}} \otimes \left(\theta_{V^{\otimes i}\otimes[\lambda]}\left(\theta_V\otimes\theta_{V^{\otimes i-1}\otimes[\lambda]}\right)^{-1}\right). \end{aligned}$$

Hence, using the definition (1.4.3) of L_σ and Lemma 1.4.3

$$X_i \cdot L_\sigma = v^{\langle u_i, u_i+2\rho\rangle - \langle u_{i-1}, u_{i-1}+2\rho\rangle - \langle\varepsilon_1,\varepsilon_1+2\rho\rangle} L_\sigma,$$

where u_i is the ith element of $\mathbf{u}_\sigma = (u_0, u_1, \ldots, u_n)$, the chain of dominant weights associated to σ. Writing $u_i = u_{i-1} + \varepsilon_{j_i}$ (the box numbered i is added to the j_ith row of u_{i-1}), we compute

$$\begin{aligned} &\langle u_i, u_i+2\rho\rangle - \langle u_{i-1}, u_{i-1}+2\rho\rangle - \langle\varepsilon_1,\varepsilon_1+2\rho\rangle \\ &\qquad = 2\left(\langle u_{i-1},\varepsilon_{j_i}\rangle + \langle\rho,\varepsilon_{j_i}\rangle - \langle\rho,\varepsilon_1\rangle\right) = 2\left((u_{i-1})_{j_i} + 1 - j_i\right). \end{aligned}$$

As $(u_{i-1})_{j_i}$ is the size of the j_ith row of u_{i-1}, $(u_{i-1})_{j_i}+1$ is the x coordinate of the added box i in u_i.

So the coordinates of the box numbered i in σ are $((u_{i-1})_{j_i}+1, j_i)$ and we have proved the result. □

1.4.4 Back to DPT

Our goal is to move from the combinatorics of lattice paths, which we used in the previous section, to that of DPT, which we used in Section 1.3.4. So, let us go back to the combinatorial setting which we used to define DPT by fixing $(a,b) \in \mathbb{Z}^2$ such that $n = aN - bK > 0$. Let us state the following useful lemma, whose proof can be found in [BCMN23].

Lemma 1.4.12. *Let $\mathcal{L}$ be a (K,N)-periodic lattice path such that $\mathcal{L} < \mathcal{L}[a,b]$. Then $\Delta(\mathcal{L})$ contains exactly n cells.*

Let us introduce the operators D and L acting on partitions $\lambda \in \mathcal{A}_{K,N}$:

$$D \cdot \lambda = (\lambda_1 + 1, \ldots, \lambda_N + 1) \tag{1.4.8}$$
$$L \cdot \lambda = (\lambda_N + K, \lambda_1, \lambda_2, \ldots, \lambda_{N-1}). \tag{1.4.9}$$

Note that this definition coincides with tensoring by D and L in the fusion ring as in Proposition 1.4.1, so it makes sense to use the same notation here. These actions are invertible and preserve the fundamental alcove $\mathcal{A}_{K,N}$, so, it makes sense to consider, for all $i \in \mathbb{Z}$, $D^i\lambda, L^i\lambda \in \mathcal{A}_{K,N}$. Also here, the actions of D and L correspond respectively to horizontal and vertical translations. Indeed, we have the following lemma.

Lemma 1.4.13. *For all periodic lattice paths $\mathcal{L}$ and all $(a,b) \in \mathbb{Z}^2$,*

$$\lambda(\mathcal{L}[a,b]) = D^a L^{-b} \lambda(\mathcal{L}).$$

Proof. For all (K,N)-periodic lattice path,

$$\lambda(\mathcal{L}[1,0]) = (\mathcal{L}(0)+1, \mathcal{L}(1)+1, \ldots, \ldots, \mathcal{L}(N-1)+1) = D\lambda(\mathcal{L}),$$

and

$$\lambda(\mathcal{L}[0,-1]) = (\mathcal{L}(N-1)+K, \mathcal{L}(0), \ldots, \mathcal{L}(N-2)) = L\lambda(\mathcal{L}). \qquad \square$$

We consider the shifted partition

$$\lambda[a,b] := D^a L^{-b} \lambda. \tag{1.4.10}$$

Given partitions λ, μ with N parts, if $\lambda_i \le \mu_i$ for all $1 \le i \le N$, recall the skew shape

$$\mu \setminus \lambda := \left\{ (x,y) \in \mathbb{Z}^2 \mid \begin{array}{c} 0 \le y \le N-1 \\ \lambda_{y+1} \le x < \mu_{y+1} \end{array} \right\}.$$

If $\lambda_i \le \lambda[a,b]_i$ for $1 \le i \le N$, it follows from Lemma 1.4.12 that the skew shape $\lambda[a,b] \setminus \lambda$ has exactly n cells. We define

$$\Delta(\lambda) = \lambda[a,b] \setminus \lambda, \tag{1.4.11}$$

$$\Delta'(\lambda) = \{(x + rK, y - rN) \in \mathbb{Z}^2 \mid (x,y) \in \Delta(\lambda), r \in \mathbb{Z}\}. \tag{1.4.12}$$

Notice that, for $\sigma \in \mathrm{DPT}(K, N, a, b)$, we have

$$\Delta(\sigma) = \Delta(\mathcal{L}(\sigma)) = \Delta(\lambda(\sigma)),$$
$$\text{and } \Delta'(\sigma) = \Delta'(\mathcal{L}(\sigma)) = \Delta'(\lambda(\sigma)).$$

We will now start making our way back to the doubly periodic tableaux of Section 1.3.4. We begin by stating the following two easy lemmas, whose proof can be found in Section 2.2 of [BCMN23]

Lemma 1.4.14. *Let $\sigma \in \mathrm{DPT}(K, N, a, b)$. Then*

$$y \mapsto \mathcal{L}(y) = \min\{x \in \mathbb{Z} \mid \sigma(x, y) \geq 1\}$$

defines a (K, N)-periodic lattice path $\mathcal{L} = \mathcal{L}(\sigma)$ such that $\mathcal{L} < \mathcal{L}[a, b]$ and $\Delta'(\sigma) = \Delta'(\mathcal{L})$.

Lemma 1.4.15. *Let $\mathcal{L}$ be a (K, N)-periodic lattice path such that $\mathcal{L} < \mathcal{L}[a, b]$. For any $(x, y) \in \mathbb{Z}^2$, there exists a unique $r \in \mathbb{Z}$ such that $(x - ra, y + rb) \in \Delta'(\mathcal{L})$.*

Lemma 1.4.16. *Let $\mathcal{L}$ be a (K, N)-periodic lattice path such that $\mathcal{L} < \mathcal{L}[a, b]$. For any $(x, y) \in \mathbb{Z}^2$ there exists a unique $(x', y', s, r) \in \mathbb{Z}^4$ such that $(x', y') \in \Delta(\mathcal{L})$ and $(x, y) = (x', y') + s(K, -N) + r(a, -b)$.*

We state and prove the following fact, which finally allows us to connect lattice paths with DPT.

Proposition 1.4.17. *The map*

$$\sigma \mapsto (\mathcal{L}(\sigma), \sigma|_{\Delta(\sigma)})$$

in Lemma 1.4.14, defines a bijection

$$\mathrm{DPT}(K, N, a, b) \simeq \Omega(K, N, a, b).$$

Proof. By definition, for all $\sigma \in \mathrm{DPT}(K, N, a, b)$, $\sigma|_{\Delta(\sigma)}$ is a standard filling of $\Delta(\mathcal{L}) = \Delta(\sigma)$ such that the periodic extension to $\Delta'(\mathcal{L}) = \Delta'(\sigma)$ is also standard. Hence $(\mathcal{L}(\sigma), \sigma|_{\Delta(\sigma)}) \in \Omega(K, N, a, b)$.

We have to define a map from $\Omega(K, N, a, b)$ to $\mathrm{DPT}(K, N, a, b)$ and show it is inverse to $\sigma \mapsto (\mathcal{L}(\sigma), \sigma|_{\Delta(\sigma)})$. Fix $(\mathcal{L}, \sigma) \in \Omega(K, N, a, b)$. The periodic extension σ' of σ to $\Delta'(\mathcal{L})$ is also standard. We then apply Lemma 1.4.15 to see that, for all $(x, y) \in \mathbb{Z}^2$, there exists unique $r \in \mathbb{Z}$ such that $(x - ra, y + rb) \in \Delta'(\mathcal{L})$. We set, for all $(x, y) \in \mathbb{Z}^2$,

$$\tilde{\sigma}(x, y) := \sigma'(x - ra, y + rb) + rn.$$

Observe that $rn < \tilde{\sigma}(x, y) \leq (r + 1)n$.

Let us check that $\tilde{\sigma} \in \mathrm{DPT}(K, N, a, b)$. By definition of σ', $\tilde{\sigma}$ is (K, N)-periodic. Moreover, we have $(x + a, y - b) \in \Delta'(\mathcal{L}[(r + 1)a, (r + 1)b])$, so

$$\tilde{\sigma}(x + a, y - b) = \tilde{\sigma}(x, y) + n.$$

Furthermore, we now show that the filling of $\tilde{\sigma}$ is standard.

If $(x+1-ra, y+rb) \in \Delta'(\mathcal{L})$, then

$$\tilde{\sigma}(x+1, y) > \tilde{\sigma}(x, y),$$

since the filling σ' of Δ' is supposed standard. Otherwise, $(x+1, y) \in \Delta'(\mathcal{L}[(r+1)a, (r+1)b])$, and $(r+1)n < \tilde{\sigma}(x+1, y) \leq (r+2)n$. In particular,

$$\tilde{\sigma}(x+1, y) > \tilde{\sigma}(x, y).$$

Similarly, if $(x-ra, y+1+rb) \in \Delta'(\mathcal{L})$, then $\tilde{\sigma}(x, y+1) > \tilde{\sigma}(x, y)$. Otherwise, as $z \mapsto \mathcal{L}(z)$ is strictly decreasing, then $(x, y+1) \in \Delta'(\mathcal{L}[(r+1)a, (r+1)b])$, and thus in any case

$$\tilde{\sigma}(x, y+1) > \tilde{\sigma}(x, y).$$

We now check that the map $(\mathcal{L}, \sigma) \mapsto \tilde{\sigma}$ is the inverse to $\sigma \mapsto (\mathcal{L}(\sigma), \sigma|_{\Delta(\sigma)})$.

Given $\sigma \in \mathrm{DPT}(K, N, a, b)$, we send $\sigma \mapsto (\mathcal{L}(\sigma), \sigma|_{\Delta(\sigma)}) \mapsto \tilde{\sigma}$. Given $(x, y) \in \mathbb{Z}^2$, using Lemma 1.4.16, we write $(x, y) = (x', y') + s(K, -N) + r(a, -b)$, where $(x', y') \in \Delta(\mathcal{L})$. Then $\tilde{\sigma}(x, y) = \sigma|_{\Delta(\sigma)}(x', y') + rn = \sigma(x, y)$. Going the other way, we send $(\mathcal{L}, \sigma) \mapsto \tilde{\sigma} \mapsto (\mathcal{L}(\tilde{\sigma}), \tilde{\sigma}|_{\Delta(\tilde{\sigma})})$. We immediately get the equality $\sigma = \tilde{\sigma}|_{\Delta(\tilde{\sigma})}$ since $\tilde{\sigma}$ is defined by extending σ. For $0 \leq y < N$, by definition, both $\mathcal{L}$ and $\mathcal{L}(\tilde{\sigma})$ coincide with the leftmost boundary of $\Delta(\tilde{\sigma})$. Since both lattice paths are (K, N) periodic, we see that $\mathcal{L} = \mathcal{L}(\tilde{\sigma})$. □

This result follows from the previous proposition.

Corollary 1.4.18. *The map*

$$\sigma \mapsto \sigma|_{\Delta(\sigma)}$$

as defined in Lemma 1.4.14, defines a bijection

$$\mathrm{DPT}(\mathcal{L}) \simeq \left\{ \begin{array}{c} \textit{standard fillings } \sigma \textit{ of } \Delta(\mathcal{L}) \\ \textit{such that } \sigma' \textit{ is standard} \end{array} \right\}.$$

Let us now go back to our fusion ring setting. For all dominant weights $\lambda \in \mathcal{A}_{K,N}$, we want to consider the representation

$$[\mu] = D^a \otimes L^{-b} \otimes [\lambda].$$

Using Lemma 1.4.13, we know that $\mathcal{L}_\mu = \mathcal{L}_\lambda[a, b]$. We can now put everything together, using Corollary 1.4.18, to get the bijection:

$$ST_{K,N}(\mathcal{L}_\lambda[a, b] \setminus \mathcal{L}_\lambda) \simeq \mathrm{DPT}(\mathcal{L}_\lambda).$$

Moreover, from Lemma 1.4.12, n is equal to the number of boxes in the skew diagram $\lambda[a, b] \setminus \lambda$ and it satisfies (1.4.2). Thus Theorem 1.4.7 gives in this case the following.

Corollary 1.4.19. *For all dominant weights $\lambda \in \mathcal{A}_{K,N}$ and $(a, b) \in \mathbb{Z}^2$ such that $m = aN - bK > 0$,*

$$\dim\left(\mathrm{Hom}_{\mathcal{C}_{\mathcal{F}}}\left(D^a \otimes L^{-b} \otimes [\lambda], V^{\otimes n} \otimes [\lambda]\right)\right) = \#\mathrm{DPT}(\mathcal{L}_\lambda). \tag{1.4.13}$$

For all dominant weights $\lambda \in \mathcal{A}_{K,N}$, we introduce the notation

$$M_\lambda := \mathrm{Hom}_{\mathcal{C}_\mathcal{F}}\left(D^a \otimes L^{-b} \otimes [\lambda], V^{\otimes n} \otimes [\lambda]\right).$$

For $\mathcal{L}$ a (K,N)-periodic lattice path such that $\mathcal{L} < \mathcal{L}[a,b]$, let $W_\mathcal{L}$ be the subspace of $W_{(K,N,a,b)}$ with basis ν_σ, for $\sigma \in \mathrm{DPT}(\mathcal{L})$.

Then, Corollary 1.4.19 gives an isomorphism of vector spaces

$$M_\lambda \simeq W_{\mathcal{L}_\lambda}. \tag{1.4.14}$$

The next result follows from the expressions given in Proposition 1.3.22, Remark 1.3.15 and [Ram03b, Theorem 4.1].

Proposition 1.4.20. *$W_\mathcal{L}$ is an irreducible finite-dimensional $\dot{H}_q(n)$-submodule of $W_{(K,N,a,b)}$.*

Proposition 1.4.21. *The isomorphism (1.4.14) is an isomorphism of $\dot{H}_q(n)$-modules.*

Proof. From Proposition 1.4.20, $W_{\mathcal{L}_\lambda}$ is an irreducible $\dot{H}_q(n)$-module. Using Proposition 1.4.11, we know that the $\dot{H}_q(n)$-modules M_λ and $W_{\mathcal{L}_\lambda}$ are both X-semisimple and have exactly the same weights and weight spaces. Therefore, they are isomorphic as $\dot{H}_q(n)$-modules. □

1.4.5 graded X-semisimple representations from spaces of intertwiners

We now want to define the action of π. In order to do so, we will need to resort to ribbon calculus. We will not present here the whole section concerning ribbon calculus in [BCMN23], however, we recall the fundamental fact that is proven there. Let $U = D^{-a} \otimes L^b \otimes V^{\otimes n}$. Then a homorphism from a representation W to $U \otimes W$ can be embedded into a solid torus and drawn as Figure 1.1.

More precisely, we have that $\otimes_{\lambda \in \mathcal{A}_{K,N}} \mathrm{Hom}([\lambda], [\lambda] \otimes U)$ is isomorphic to the *ribbon skein module*. The ribbon skein module is the vector space of linear combinations of ribbon diagrams in the punctured torus, where the puncture has been colored by the representation U (see Definition 5.4 in [BCMN23]). This Lemma 5.6 in [BCMN23], which shows that $\otimes_{\lambda \in \mathcal{A}_{K,N}} \mathrm{Hom}([\lambda], U \otimes [\lambda])$ is indeed isomorphic to the ribbon skein module associated to U.

Lemma 1.4.22. *If a ribbon category $\mathcal{C}$ is semi-simple, then for all objects U, the ribbon skein module S_U is isomorphic to*

$$\bigoplus_{\Gamma \in \mathrm{Irr}(\mathcal{C})} \mathrm{Hom}(\Gamma, U \otimes \Gamma).$$

We are now ready to define the action of π. For all $W \in \mathcal{F}_v$, and $f \in \mathrm{Hom}(W, U \otimes W)$, then πf and $\pi^{-1} f$ are given as follows.

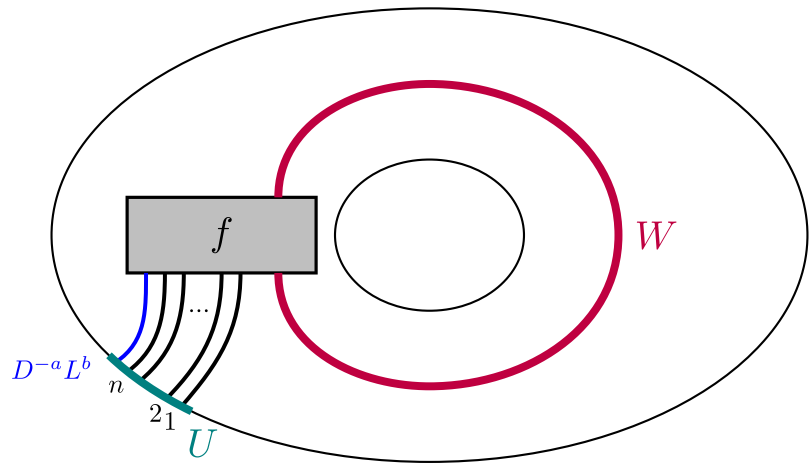
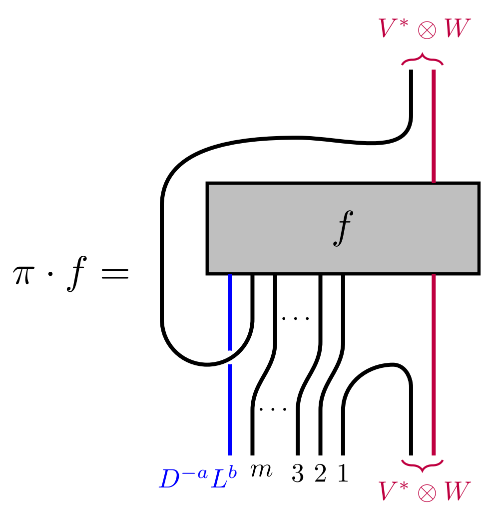
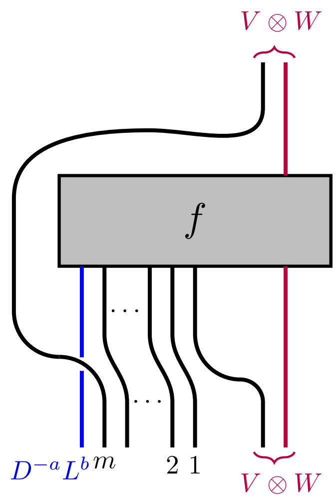


Figure 1.1: Embedding of $\mathrm{Hom}(W, U \otimes W)$ for $U = D^{-a} \otimes L^b \otimes V^{\otimes n}$ into the solid torus.



(1.4.15)

$$\pi^{-1} \cdot f =$$



(1.4.16)

Remark 1.4.23. Note here that the top and bottom double strands $V^* \otimes W$ and $V \otimes W$ actually go around the core of the torus and meet each other.

Proposition 1.4.24. *The action of $\pi^{\pm 1}$ given in (1.4.15) and (1.4.16) extends the $\dot{H}_q(m)$-module structure on S_U defined in Proposition 1.4.10 to a $\ddot{H}_{q,t}(n)$-module structure.*

Proof. Note that $U = D^{-a} \otimes L^b \otimes V^{\otimes n}$ here. The action of T_0 is given by the relation ((7)) $T_0 = \pi T_{n-1} \pi^{-1}$ of Definition 1.2.11. To prove the statement, we need to prove the following relations:

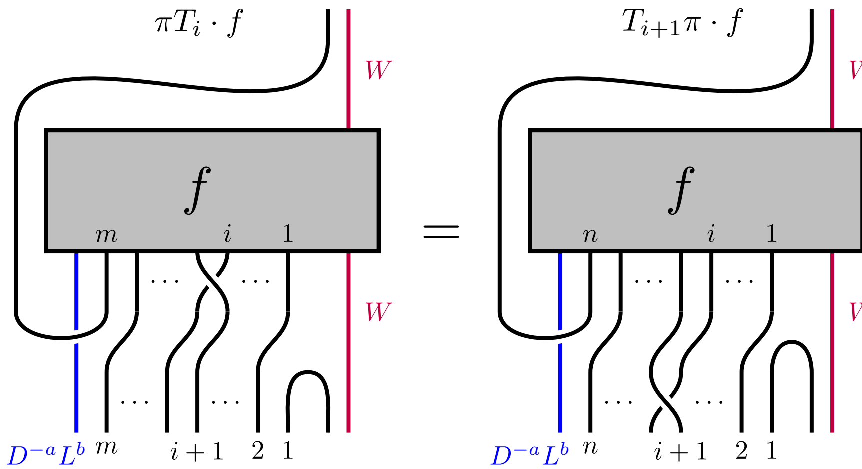
(1) $\pi T_i = T_{i+1}\pi$, for all $i \geq 1$,

(2) $\pi X_1 = X_2 \pi$,

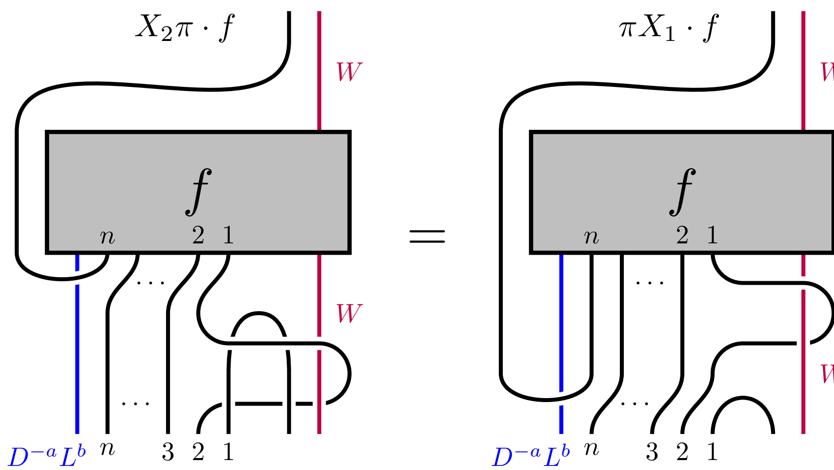
(3) $T_1 \pi^2 = \pi^2 T_{n-1}$,

(4) $\pi X_n = t^{-1} X_1 \pi$.

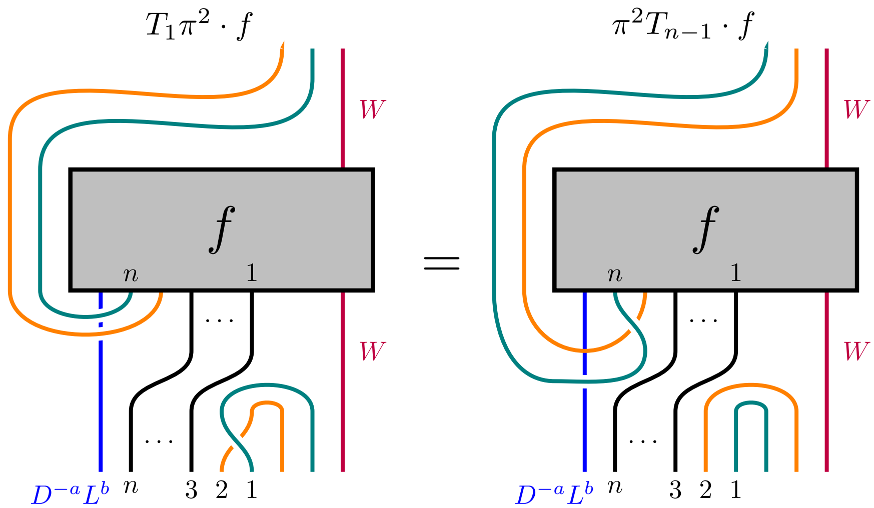
Relation ((1)) is easily verified from the diagrams. For all $W \in \mathcal{F}_v$ and $f \in \mathrm{Hom}(W, U \otimes W)$,



Relation ((2)) is also clearly satisfied.



The picture for Relation ((3)) is the following.

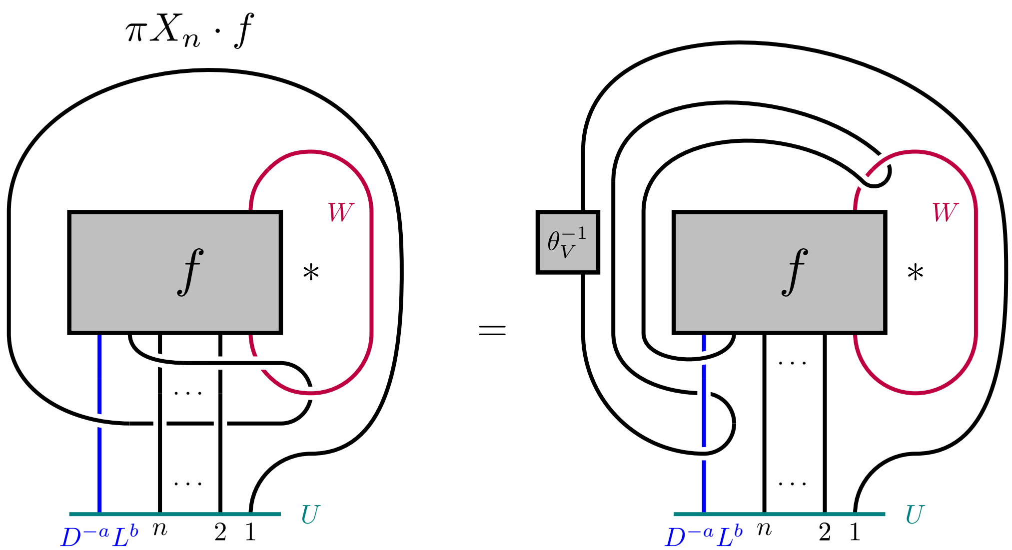


We can slide the crossing around the torus, or equivalently use the equivalence

$$(\mathrm{Id}_U \otimes \check{R}_{V,V} \otimes \mathrm{Id}_W) \circ (\mathrm{Id}_{V\otimes V} \otimes f) \sim (\mathrm{Id}_{V\otimes V} \otimes f) \circ (\check{R}_{V,V} \otimes \mathrm{Id}_W),$$

from ribbon calculus (see Lemma 5.5 [BCMN23]).

For relation ((4)), we draw the corresponding embeddings in the solid torus, as in Figure 1.1. In order to simplify the picture, the torus is only represented by its core, noted $*$, and the band U on its boundary. Here we have to keep in mind that the lines we draw are actually bands, so we have to take into account the twists which may appear when moving them around.



(1.4.17)

When performing this operation, a (negative) twist has appeared in the leftmost strand labelled by V, which corresponds to multiplying the strand by the inverse of the ribbon element θ_V^{-1}.

We can assume without loss of generality that $0 \le b < N$, then using (1.4.8) and (1.4.9), we get $D^{-a} \otimes L^b = L(\lambda)$ and $V \otimes D^{-a} \otimes L^b = L(\mu)$, where

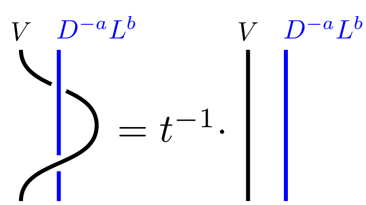
$$\lambda = (\underbrace{K-a, \ldots, K-a}_{b}, -a, -a, \ldots, -a),$$

$$\mu = (\underbrace{K-a, \ldots, K-a}_{b}, 1-a, -a, \ldots, -a).$$

Applying Corollary 1.4.9, we deduce that $\check{R}^{-1}_{D^{-a}\otimes L^b, V}\check{R}^{-1}_{V, D^{-a}\otimes L^b} = (\check{R}_{V, D^{-a}\otimes L^b}\check{R}_{D^{-a}\otimes L^b, V})^{-1}$ acts on $V \otimes D^{-a} \otimes L^b$ by multiplication by

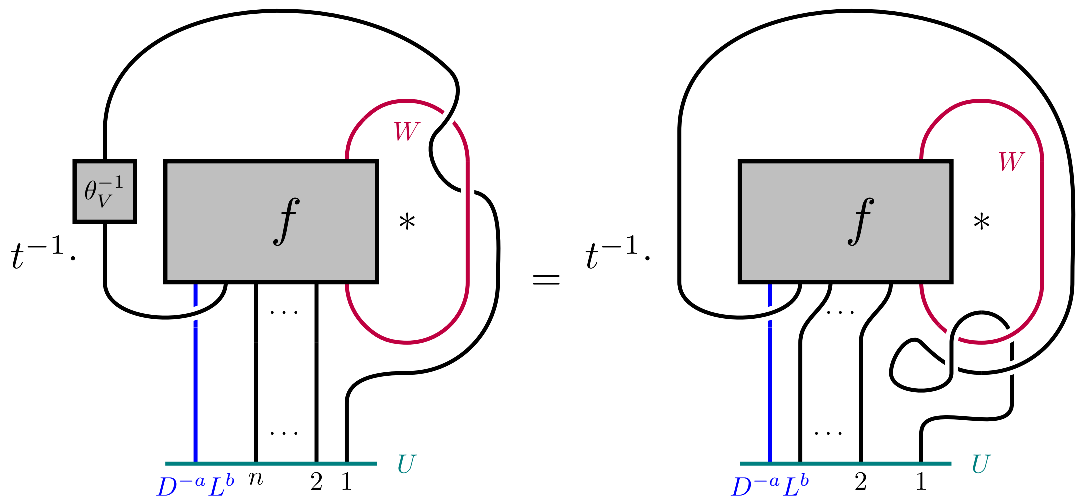
$$v^{\langle \varepsilon_1, \varepsilon_1 + 2\rho\rangle + \langle \lambda, \lambda + 2\rho\rangle - \langle \mu, \mu + 2\rho\rangle} = q^{a+b} = t^{-1}.$$

So



(1.4.18)

Hence, we can replace the picture on the right in (1.4.17) by



(1.4.19)

In the picture on the right of (1.4.19), the loop at the bottom introduces a new twist in the strand, which simplifies with the twist which previously appeared in (1.4.17). In the picture on the right, we recognize $X_1\pi \cdot f$, which concludes the proof.

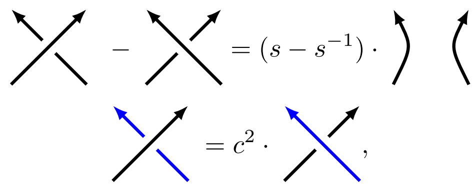
□

Let us make an observation that was perhaps not made explicit in [BCMN23]. We need to compare the action of π on tableaux with the action defined in terms of skeins. They are indeed the same, as we claim and use for the final application in the paper, and we have all the tools to show why. The key observation to make is simply that the skein representation is also graded. As a result, the skein action of π cannot but be the same as the DPT action of π by our classification theorem (Theorem 1.3.28).

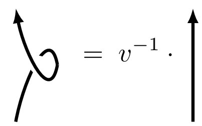
1.5 The DAHA as a skein algebra

The DAHA is proved to be isomorphic to a certain skein algebra of the torus in [MS21]. The authors give also a tangle algebra construction which they conjecture to be isomorphic to the DAHA. The main application of the DPT construction of DAHA representations is the proof of this conjecture.

Let us quickly recall the set up of the conjecture. The DAHA is isomorphic to the skein algebra of the torus given by the group algebra of braids on the punctured torus modulo the usual skein relation and a "crossing" relation for the special strand (see Theorem 3.5 in [MS21]). More explicitely, these relations are



where we draw the special strand in blue. In [MS21], we find a second candidate diagramatic presentation of the DAHA as a tangle algebra, where braids are endowed with a framing and subject to the change of framing relation



There is a natural map from the DAHA, thought of as the braid skein algebra, to this tangle algebra by framing the braids in the torus. This homomorphism

$$\ddot{H}_{q,t}(n) \otimes \mathbb{C}[v^{\pm 1}] \otimes_{\mathbb{C}[q^{\pm 1}, t^{\pm 1}]} \mathbb{C}(s,c) \to \mathrm{Sk}_n(T^2, *) \otimes_{\mathbb{C}[s^{\pm 1}, c^{\pm 1}]} \mathbb{C}(s,c)$$

is surjective, as shown in Theorem 4.1 in [MS21]. Here q, t are related to s, c as follows:

$$q = s^2, \qquad t = c^2.$$

Our goal is to prove the injectivity of this morphism. We can prove it with the following strategy. We want to show that the (injective) morphism from the DAHA to the space of endomorphisms of a certain faithful representation factors through this map. This would imply injectivity. We recall that we have cooked up a faithful representation by taking the direct sum $\mathcal{W} = \oplus_{aN-bK=n} W_{(K,N,a,b)}$. This representation has a also a skein theoretic interpretation obtained from the ribbon category structure of the fusion category.

Let φ be the homomorphism before taking the localization:

$$\varphi : \ddot{H}_{s,c,v}(n) \to \mathrm{Sk}_n(T^2, *).$$

Theorem 1.5.1. *The homomorphism φ is injective.*

Proof. Let us call a $\ddot{H}_{s,c,v}(n)$-representation W *ribbon* if the homomorphism $\ddot{H}_{s,c,v}(n) \to \mathrm{End}(W)$ factors through φ. Suppose we have a ribbon representation W which is faithful. Then the composition

$$\ddot{H}_{s,c,v}(n) \xrightarrow{\varphi} \mathrm{Sk}_n(T^2, *) \to \mathrm{End}(W)$$

is injective, which implies injectivity of φ, so we are done.

For each K, N, a, b, the representation $W_{(K,N,a,b)}$ after the base change to $\mathbb{C}[s^{\pm 1}, c^{\pm 1}]$ is isomorphic to the ribbon module S_U, for $U = D^{-a} \otimes L^b \otimes V^{\otimes n}$, which is naturally a module for the skein algebra $\mathrm{Sk}_n(T^2, *)$ as well: the action is obtained by gluing $T \times I$ along the boundary of the solid torus and connecting bands in the solid torus which represent elements of S_U with bands in $T \times I$ representing elements of $\mathrm{Sk}_n(T^2, *)$. The base string is labeled by the representation $D^{-a}L^b$ and therefore satisfies the local relation, see (1.4.18). Thus we see that $W_{(K,N,a,b)}$ is ribbon and therefore the direct sum $\mathcal{W} = \bigoplus_{aN-bK=n} W_{(K,N,a,b)}$ is ribbon.

By Theorem 1.3.33, $\mathcal{W}$ is a faithful representation of $\ddot{H}_{q,t}(n)$. We need to show that it is still a faithful representation when we pass to the larger algebra $\ddot{H}_{s,c,v}(n)$. Note that the action of v is given by s^{-N}. Similarly to the proof of Theorem 1.3.33, it suffices to show that the set of triples (s, c, v) is Zariski dense in $\mathbb{C}^3$ where $c = s^{n(u+v)}$, $v = s^{-N}$, $s^2 = q$, q is a primitive $n(N+\alpha)$-th root of unity and u, N, v, α range over the set of solutions of $uN - v\alpha = 1$, N and α sufficiently large. It is enough to prove that the image of this set under the map

$$(s, c, u) \to (s^{2n}, c^2, u^{-2n}) = (q^n, q^{n(u+v)}, q^{nN})$$

is dense, so, replacing q^n by q, we reduce the claim to the statement: The set of triples (q, q^{u+v}, q^N) is Zariski dense in $\mathbb{C}^3$, as q runs over the primitive $(N+\alpha)$-th roots of unity and u, v, N, α are as above.

We have $(u+v)N - v(\alpha + N) = 1$. In particular, $u+v$ and $\alpha + N$ are relatively prime and we can replace q by q^{u+v}. We then have

$$(q^{u+v}, q^{(u+v)^2}, q^{N(u+v)}) = (q^{u+v}, q^{(u+v)^2}, q).$$

We can make a similar observation to that in the proof of Theorem 1.3.33: such triples for fixed u, v are Zariski dense on the curve $\{(z^{u+v}, z^{(u+v)^2}, z) \mid z \in \mathbb{C}\}$. So, we are left to show that the union of curves

$$C_r = \{(z, z^r, z^{r^2}) \mid z \in \mathbb{C}\},$$

where r ranges over positive integers, are Zariski dense in $\mathbb{C}^3$. Suppose this is not the case and there exists a nonzero polynomial

$$P(x, y, z) = \sum_{i,j,k<d} c_{ijk} x^i y^j z^k \qquad (c_{ijk} \in \mathbb{C})$$

vanishing on all these curves. Take r larger than d. Then $i + rj + r^2k$ are all distinct for distinct triples (i, j, k) satisfying $i, j, k < d$ and therefore $P(z, z^r, z^{r^2})$ cannot vanish. A contradiction. □

Corollary 1.5.2. *The homomorphism φ in Theorem 1.5.1 is an isomorphism.*

Proof. Since we have shown that φ is injective and localization is flat, the homomorphism in Theorem 1.5.1 is injective. □

Remark 1.5.3. Note that Theorem 5.7 in [MS21] on (a subalgebra of) the elliptic Hall algebra is stated and proved assuming the tangle algebra presentation of the DAHA is correct. So, our result implies that the theorem is true.

1.6 Open questions

While we feel that the role of DPT in the representation theory of DAHA is somehow fully understood, there are still some purely combinatorial questions about these objects that are still open. We decided not to venture into an extensive analysis of the combinatorial properties of DPT, however, we do think that they are interesting in their own right. For example, they are interestingly related to Dyck paths (see Theorem 2.51 in [BCMN23]).
Let us observe that the number of DPT is clearly infinite, however, we can consider the sets $\mathrm{DPT}(K, N, a, b)/\langle \pi \rangle$ or, equivalently (Theorem 2.48 [BCMN23]), $\mathrm{DPT}(K, N, a, b)/\langle L, D \rangle$. These sets are now finite, so it does make sense to speak of counting DPT if we refer to these quotients. It is still unclear how to count or just give a bound on the number of DPT, as only few special cases have been worked out (Proposition 2.53 and Proposition 2.54 in [BCMN23]). Therefore, we believe it would be interesting to explore this topic further and extend these results to other choices of the parameters K, N, a, b.

Chapter 2

The Dyck path algebra associated to a surface

2.1 Introduction

The first part of this chapter is based on Anton Mellit's unpublished notes on torus skeins. These notes contain the definition of the $\mathbb{A}_{q,t}$ algebra associated to a surface. This algebra should coincide with the usual $\mathbb{A}_q$ algebra (with inverses of the variables) when the input surface is the annulus and with the double Dyck path algebra $\mathbb{A}_{q,t}$ (with inverses) when the input surface is the torus. Inspired by this definition, in the second part of the chapter, we consider the algebra $\mathbb{A}_q(D)$, the $\mathbb{A}_q$ algebra associated to a disc. We give a basis for this algebra, proving a Schur-Weyl duality-like result about some quantum group representations. We then move on to deal with the $\mathbb{A}_q$ algebra associated to the annulus and construct a representation using intertwiners. Finally, we consider the $\mathbb{A}_{q,t}$ of the torus and show it acts on similar spaces of intertwiners.

2.2 Preliminaries on the double Dyck path algebra

In this section, we recall the definition of the $\mathbb{A}_{q,t}$ algebra following [CGM20] (note that this has slightly different conventions than the ones in the very first definition in [CM18]).

As we have mentioned in the previous chapter, the DAHA is obtained by gluing two copies of an affine Hecke algebra; similarly, the $\mathbb{A}_{q,t}$ algebra is obtained by gluing two copies of the so-called $\mathbb{A}_q$ algebra. So, let us start by defining the $\mathbb{A}_q$ algebra.

Definition 2.2.1. The algebra $\mathbb{A}_q$ is the linear $\mathbb{Q}(q)$-algebra generated by orthogonal idempotents $\mathbb{1}_n$ indexed by $\mathbb{Z}_{\geq 0}$ and elements $\mathbb{1}_{n+1}d_+\mathbb{1}_n, \mathbb{1}_{n-2}d_-\mathbb{1}_n$,

$\mathbb{1}_n T_i \mathbb{1}_n, \mathbb{1}_n y_1 \mathbb{1}_n$, for $1 \leq i \leq n$, which we will simply denote by d_+, d_-, T_i, y_i leaving the indeces of the idempotents implied, subject to relations

(1) Affine Hecke algebra relations

$$(T_i - 1)(T_i + q),\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1},\ T_i T_j = T_j T_i,\ \text{if } |i-j| > 1$$

$$T_i y_{i+1} T_i = q y_i,\ y_i T_j = T_j y_i,\ y_i y_j = y_j y_i;$$

(2) $d_-^2 T_{n-1} = d_-^2,\ d_- T_i = T_i d_-,$ for $1 \leq i \leq n-2,\ d_- y_i = y_i d_-,$ for $1 \leq i \leq n$;

(3) $T_1 d_+^2 = d_+^2,\ d_+ T_i = T_{i+1} d_+,$ for $1 \leq i \leq n-1$;

(4) $d_+ y_i = T_1 T_2 \ldots, T_i y_i T_i^{-1} \ldots T_2^{-1} T_1^{-1} d - +$, for $1 \leq i \leq n$;

(5) $[d_+, d_-] = (q-1) T_1 T_2 \ldots T_{n-1} y_n$.

Definition 2.2.2. The $\mathbb{A}_{q,t}$ algebra is the $\mathbb{Q}(q,t)$-algebra generated by $d_+ d_+^*, d_-, T_i, 1 \leq i \leq n-1, y_i, z_i, 1 \leq i \leq n$ subject to

(1) relations for $\mathbb{A}_q$ for d_-, d_+, T_i, y_i;

(2) relations for $\mathbb{A}_{q^{-1}}$ for $d_-, d_+^*, T_i^{-1}, z_i$;

(3) $d_+ z_i = z_{i+1} d_+,\ d_+^* y_i = y_{i+1} d_+^*,\ 1 \leq i \leq n,\ z_1 d_+ = -t q^{n+1} y_1 d_+^*$.

Remark 2.2.3. While it is clear that, for each n, the (positive part of the) affine Hecke algebra $\dot{H}_n$ sits inside the $\mathbb{A}_q$ algebra, the DAHA does not embed into $\mathbb{A}_{q,t}$. We refer to Remark 7.2 in [CM18] for a thorough discussion on the issue.

The $\mathbb{A}_{q,t}$ famously has a polynomial representation, defined in [CM18], on the space $\otimes_{n \geq 0} V_n$, where $V_n = \Lambda \otimes \mathbb{C}(q,t)[y_1, \ldots, y_n]$ and Λ denotes the ring of symmetric functions in $x_1, x_2, \ldots$. Until recently, this was essentially the only example of $\mathbb{A}_{q,t}$ representation. In [GGS23], we find the construction of so-called calibrated representations for an extension of the algebra $\mathbb{B}_{q,t}$ of [CGM20], which is equivalent to $\mathbb{A}_{q,t}$ in the sense of Theorem 3.2.7 of [CGM20]. Moreover, in [Wei24], we can find modules for subalgebras $\mathbb{B}_{q,t}^{(n)}$ of $\mathbb{B}_{q,t}$ constructed from graded DAHA representations.

Warning. This is not the definition of $\mathbb{A}_{q,t}$ that we will be using. We will define the algebra we will be studying later in this chapter.

2.3 The Dyck path algebra as a skein algebra

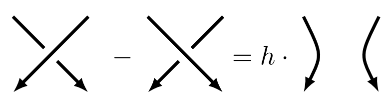
In this section, we want to define the $\mathbb{A}_{q,t}$ algebra of a given surface. This construction is fully based on unpublished notes [Mel19] written by Anton Mellit in 2019, following discussions with David Jordan and Peter Samuelson. Therefore, the ideas in this section are not original, however, we feel it is useful to explain them in some degree of detail, as the reader has (most probably) no access to these notes.

The construction is somewhat similar to that of the DAHA in [MS21], however, we want to replace the special strand by what will be called "the beam". This is also a strand which behaves differently than the others: as we want to be able to vary the number of incoming and outcoming strands, we will be allowed to pull a strand out of the beam (to obtain the d_+ operator) and put a strand into the beam (to get the d_- operator). We remark that it was already suggested by the work in [Mel21] that the $\mathbb{A}_{q,t}$ algebra should have an interpretation in skein theoretical terms.

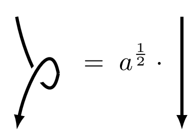
2.3.1 Preliminaries on skein algebras

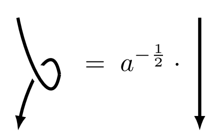
Let us fix the setting for our discussion. Let X be a compact connected oriented 3-manifold with boundary, or, more generally, with corners. A tangle in X is an embedding of several copies of circles S^1, which are called *knots*, and segments $[0, 1]$, which we call *arcs*. Our tangles are framed in the sense that at each point we choose a nonzero normal vector. The ends of the arcs lie on the boundary of X, while the knots and the interiors of the arcs lie in the interior of X. When drawing tangles, we use the following convention: we assume the framing is always parallel to the plane and pointing towards the right of the direction vector. We fix the base ring to be $R := \mathbb{Q}(q^{\frac{1}{2}})[a^{\frac{1}{2}}, a^{\frac{1}{2}}]$ and we will use the notation $h := q^{\frac{1}{2}} - q^{-\frac{1}{2}}$. We consider the free module on the set of tangles over R modulo isotopies that fix the boundary of X and (local) relations:

(1) Skein relations

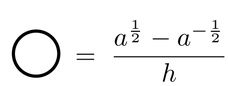


(2) Unwinding relations





(3) Unknot relations



We will fix the inputs and outputs of the arcs; we will denote the input points by $x^+ = (x_1^+, \ldots, x_n^+)$ and the output points by $x^- = (x_1^-, \ldots, x_n^-)$. The skein module with these fixed inputs and outputs will be denoted by $\mathrm{Sk}(X, x^+, x^-)$.

Remark 2.3.1. Suppose a manifold X can be obtained by gluing X_1 and X_2 along the common part of their boundaries $Z = \partial X_1 \cap \partial X_2$ so that on Z the inputs in X_1 correspond to the outputs of X_2 and vice versa. Then tangles can be glued to obtain a bilinear map

$$\mathrm{Sk}(X_1, -, -) \otimes \mathrm{Sk}(X_2, -, -) \to \mathrm{Sk}(X, -, -).$$

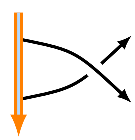
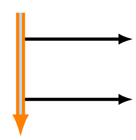
Moreover, every tangle in X can be deformed to a tangle transverse to Z, so that it can always be constructed from tangles in X_1 and X_2.

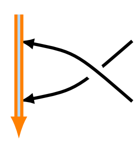
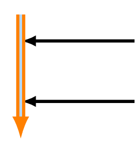
Let us consider a surface S with a choice of distinct points $x = (x_1, \ldots, x_n)$ with nonzero tangent vectors. Let $X = S \times [0,1]$, then $x_i^+ = (x_i, 0)$ and $x_i^- = (x_i, 1)$ are a choice of inputs and outputs in ∂X. By Remark 2.3.1, $\mathrm{Sk}(X, x^+, x^-)$ admits an algebra structure. We will denote this skein algebra by $\mathrm{Sk}(S, x)$. We will simply write $\mathrm{Sk}(X, n)$ when the choice of x is clear.

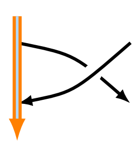
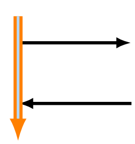
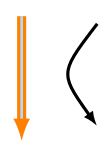
Example 2.3.2. For $S = D$, the disc, and n points on a line with tangent vectors parallel to the line, the skein algebra $\mathrm{Sk}(D, n)$ is isomorphic to the Hecke algebra over R.

2.3.2 The beam

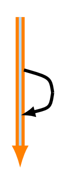
We now want to introduce the so-called *beam*. Tangles can have endpoints along this special strand. The skein algebra we want to build will be made of tangles with inputs and outputs in fixed sets of points in $\partial X \times \{0\}$ and $\partial X \times \{1\}$ and any number of inputs and outputs on the beam. Furthermore, we want to impose the following relations (here the beam is drawn as a thick orange line)

 $= q^{\frac{1}{2}} \cdot$  (2.3.1)

 $= q^{\frac{1}{2}} \cdot$  (2.3.2)

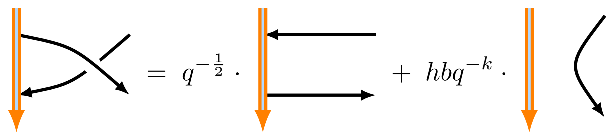
 $= q^{-\frac{1}{2}} \cdot$  $+ \; h \cdot$  (2.3.3)

We now introduce a new variable b. We want to introduce the following relation for adding an arc to the beam

 $= (1 - bq^{-k}) \cdot$  (2.3.4)

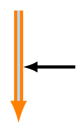
where k is the difference between the number of outgoing endpoints and the number of incoming endpoints. In other words multiplication by $1-b$ corresponds to adding an extra arc at the beginning of I and $(1-q^{-k}b)$ at an arbitrary location on I.

Note that putting together (2.3.3) and (2.3.4) we obtain

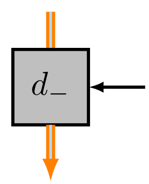


(2.3.5)

Let us make sense of relations (2.3.2), (2.3.1) and (2.3.3). The operation of "absorbing" a strand into the beam will be called d_-. The drawing

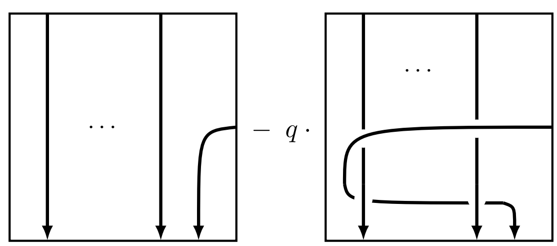


actually means

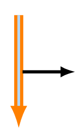


where

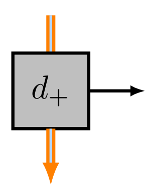
$d_- =$



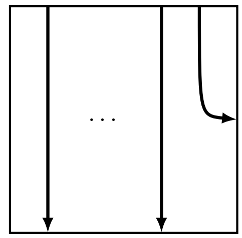
Similarly, the drawing



is interpreted as

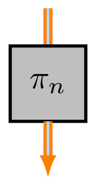


where

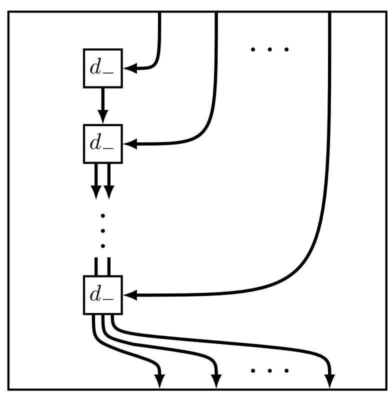
$d_+ =$ 

or zero when there is no strand to add.

For the sake of the discussion that follows, let us assume for a moment that the beam is an interval I embedded in the boundary of X. The beam is *symmetrized*: this means that we can imagine that, at its beginning, its strands are symmetrized so that swapping any of them applying a T_i has the result of simply multiplying by $q^{\frac{1}{2}}$. Graphically, we can picture this as having at the top of the beam



where the box π_n is given by

$\pi_n =$ 

Let us verify that the strands are indeed symmetrized. To prove this, we consider d_-. Let us label the strands in the square by $1, \dots, n$ and the incoming strand is labelled $n+1$. We denote by $T_1, \dots, T_n$ the usual generators of the braid group on $n+1$ strand. Then d_- inside the square looks like the element $\gamma_n = 1 - qT_n \dots T_2 T_1 T_2 \dots T_n$. It is easy to see that γ_n commutes with the T_i's for $i = 1, \dots, n-1$. For $i = n$, we rewrite γ_n as

$$\gamma_n = -qh(q^{-\frac{1}{2}} + T_n + T_n T_{n-1} T_n + \cdots + T_1 \dots T_{n-1} T_n T_{n-1} \dots T_1).$$

If we assume the n strands before γ_n were symmetrized, we can replace γ_n by

$$-qh(q^{-\frac{1}{2}} + T_n + q^{\frac{1}{2}} T_{n-1} T_n + \cdots + q^{\frac{n-1}{2}} T_1 \dots T_{n-1} T_n).$$

Multiplying by $T_n - q^{\frac{1}{2}}$ on the left eliminates the summand $q^{-\frac{1}{2}} + T_n$. Finally, we can see that $T_n - q^{\frac{1}{2}}$ kills the other terms by using our assumption and

$$T_n(T_i T_{i+1} \dots T_{n-1} T_n) = (T_i T_{i+1} \dots T_{n-1} T_n) T_{n-1}.$$

This proves that relation (2.3.1) is satisfied. Let us show that relation (2.3.2) is also satisfied. Recall the Jones-Wenzl idempotent of the Hecke algebra, which is given by

$$\pi_n = \frac{\gamma_{n-1}\gamma_{n-2}\cdots\gamma_0}{(1-q)(1-q^2)\dots(1-q^n)}$$

and satisfies $T_i \pi_n = q^{\frac{1}{2}} \pi_n$. This together with the fact that n strands are symmetrized if and only if applying π_n produces an equivalent element in the skein module proves the relation holds. The last relation to verify is (2.3.3). This corresponds to checking

$$\gamma_{n-1} T_n \pi_n = q^{-\frac{1}{2}} \gamma_n \pi_n + h\pi_n,$$

which is true as

$$q^{-\frac{1}{2}} \gamma_n \pi_n = T_n^{-1} \gamma_n \pi_n = (T_n^{-1} - qT_{n-1} \dots T_1 T_1 \dots T_n)\pi_n$$

and

$$\gamma_{n-1} T_n \pi_n = (T_n - qT_{n-1} \dots T_1 T_1 \dots T_n)\pi_n.$$

Remark 2.3.3. When we are trying to apply d_+ but there are no strands to add, all relations are trivially verified except the third one, which becomes

$$0 = q^{-\frac{1}{2}}(1-q) + h,$$

which is also true.

Let us fix some notation before stating what we have proved so far. We denote here by $\mathrm{Sk}_I(X, N, N')$ the $R[b, b^{-1}]$-module on tangles with inputs x^+ of size N and x^- of size N' and any number of inputs and outputs on the beam I module skein relations and (2.3.2), (2.3.1) and (2.3.3).

Proposition 2.3.4. *For every $m \geq 0$ the construction above gives a well-defined map $\mathrm{Sk}_I(X, N, N')$ to $\mathrm{Sk}(X, N+m)$ compatible with the morphism from $R[b, b^{-1}]$ to R sending b to q^m.*

It is possible to change the orientation of the beam introducing some extra factors. We denote by I' the same interval as I with flipped orientation and, for a tangle s, we set $o(s)$ to be the number of outgoing endpoints on I and $k_i(s)$ the number of outgoing endpoints minus the number of incoming endpoints between the beginning of the interval I and the i-th outgoing endpoint. We denote by $k(s)$ the total of outgoing endpoints on I minus the total number of incoming endpoints on I. We will see that the correction factor is

$$c(s) = \prod_{i=1}^{o(s)} (-a^{-\frac{1}{2}} b^{-1} q^{k_i(s) + \frac{1}{2}}).$$

Proposition 2.3.5. *There is a well-defined morphism over R from $\mathrm{Sk}_{I'}(X,N,N')$ to $\mathrm{Sk}_I(X,N,N')$ sending a tangle s to $c(s)s$ compatible with the morphism $R[b,b^{-1}] \to R[b,b^{-1}]$ taking b to $a^{-1}b^{-1}q^{N'-N}$.*

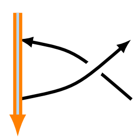
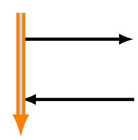
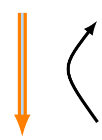
Proof. Compatibility with (2.3.2),(2.3.1) is clear. Let us apply our map to relation (2.3.4) at the beginning of the beam for I' to obtain

$-a^{-\frac{1}{2}}b^{-1}q^{k(s)-\frac{1}{2}} \cdot$  $= (1-a^{-1}b^{-1}q^{k(s)}) \cdot$ 

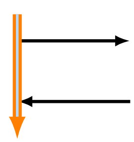
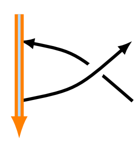
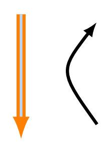
This is equivalent to

$q^{-\frac{1}{2}} \cdot$  $= (a^{-\frac{1}{2}} - a^{\frac{1}{2}}bq^{-k(s)}) \cdot$ 

which one can show in Sk_I by connecting the ends of (2.3.3) and using (2.3.5). Similarly, when looking at the map applied to (2.3.3), we get

$-a^{\frac{1}{2}}b^{-1}q^{k-\frac{1}{2}} \cdot$  $= (-a^{-\frac{1}{2}}b^{-1}q^{k+\frac{1}{2}})q^{-\frac{1}{2}} \cdot$  $+\ h \cdot$ 

where k is the difference between the number of outgoing endpoints and that of incoming endpoints between the beginning of I and the region we are looking at. This is the same as

 $= q^{-\frac{1}{2}} \cdot$  $+\ ha^{\frac{1}{2}}bq^{-k} \cdot$ 

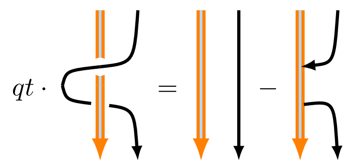
which can be verified by twisting the ends of (2.3.5). $\square$

Corollary 2.3.6. *The modules $\mathrm{Sk}_I(X,N,N')$ and $\mathrm{Sk}_{I'}(X,N',N)$ are isomorphic over R.*

The following is a combination of Proposition 2.3.4 and Proposition 2.3.5.

Corollary 2.3.7. *For every $m \geq 0$, there is a well-defined map from $\mathrm{Sk}_I(X,N,N')$ to $\mathrm{Sk}(X,N+m)$ compatible with the morphism $R[b,b^{-1}]$ sending b to $a^{-1}q^{-m}$. Here the new inputs and outputs are created in a neighborhood of $1 \in I$ and $0 \in I$, respectively.*

Eventually, we want the beam to have its endpoints located on the boundary ∂X and its interior in the interior of X, so, from now on, we will assume this to be the case. The interval I is equipped with a framing. We want to extend the coefficient ring to $R[b, b^{-1}, t, t^{-1}]$. Now the tangles are allowed to wrap around the beam. The relation for going around the beam are



(2.3.6)

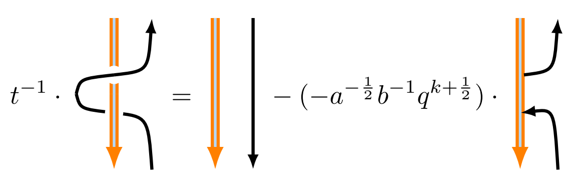
We have the following result which we can see as a translation of Proposition 2.3.4 in this new setting.

Proposition 2.3.8. *Denote by* $\mathrm{Sk}_I(X, N, N')$ *the free* $R[t, t^{-1}, b, b^{-1}]$*-module on tangles whose inputs and outputs are fixed sets* x^+, x^- *of sizes* N, N'*, plus any number of inputs and outputs on* I *modulo skein relations and relations (2.3.2),(2.3.1),(2.3.3),(2.3.4), (2.3.6). For every* $m \geq 0$ *we have a well-defined map from* $\mathrm{Sk}_I(X, N, N')$ *to* $\mathrm{Sk}(X, N + m)$ *compatible with the homomorphism* $R[t, t^{-1}, b, b^{-1}] \to R$ *sending* b *to* q^m *and* t *to* 1*.*

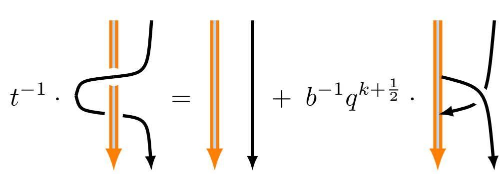
We have an analogue to Proposition 2.3.5 in this context.

Proposition 2.3.9. *There is a well-defined morphism over* R *from* $\mathrm{Sk}_{I'}(X, N, N')$ *to* $\mathrm{Sk}_I(X, N, N')$ *sending a tangle* s *to* $c(s)s$*, compatible with the homomorphism* $R[t, t^{-1}, b, b^{-1}] \to R[t, t^{-1}, b, b^{-1}]$ *taking* b *to* $a^{-1}b^{-1}q^{N'-N}$ *and* t *to* $q^{-1}t^{-1}$*.*

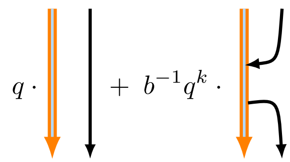
Proof. We begin by applying the map to relation (2.3.6):



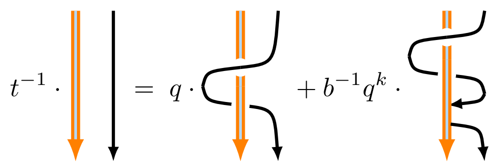
Here k is the difference between the number of outgoing and incoming endpoints in the interval above the diagram. We need to check this relation holds. We start by twisting the endpoints of the strand to obtain a negative crossing, to then apply the unwinding relation to get



We will apply (2.3.5) to see that this is equal to



Finally, we twist the top end of the thin strand around the thick strand to turn the equation into



This is equal to the left hand side by applying (2.3.6) and (2.3.4). $\square$

2.3.3 The surface Hecke algebra

Let Σ be an oriented compact surface with at least one boundary component. We think of Σ as several polygons glued along the sides with a small disc removed around the vertices. We denote by $2d$ the number of sides of the polygon. We label the sides with numbers in

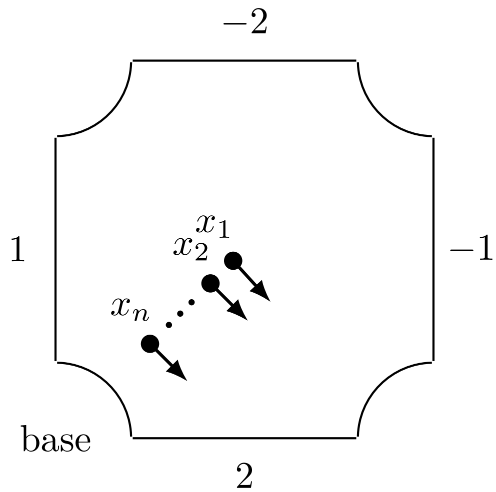
$$\Delta = \{1, \dots, d, -1, \dots, -d\},$$

so that the sides i and $-i$ are glued together for all i. We pick a vertex of the polygon and call it the base. Let the labels of the sides be $w_1, \dots w_{2d}$, where w_1 is immediately to the left of the base, w_2 is next to the left of w_1 and so on. We then have a linear order on Δ by setting

$$w_1 < \dots < w_{2d}.$$

We place points $x_1, \dots, x_n$ along a straight line connected to the base, so that the closest point to the base is x_n. The framing vectors are assumed to be perpendicular to the line along which the points lie.

Example 2.3.10. For $d = 2$, order $1 < -2 < -1 < 2$, we get the punctured torus.



The fundamental group of Σ is given by

$$\pi_1(\Sigma) = \langle g_i : i \in \Delta \rangle / \langle g_i g_{-i} = g_{-i} g_i \rangle,$$

where the generator g_i denotes the path that crosses the side labelled i to reappear on the side $-i$ and goes back to the original position. Let us describe the braid group of Σ, which we denote by $\mathrm{Br}_n(\Sigma)$, for $n \geq 0$. It is the trivial group for $n = 0$ and simply the fundamental group when $n = 1$. For arbitrary n, it is generated by $T_1, \ldots, T_{n-1}$ and g_i, $i \in \Delta$, subject to the following relations

(1) Braid group relations for the T_i's;

(2) $g_i T_1^{\sigma(i,-i)} g_i T_1 = T_1 g_i T_1^{\sigma(i,-i)} g_i$, for $i \in \Delta$;

(3) $g_i T_1^{\sigma(i,-j)} g_j T_1^{\sigma(j,i)} = T_1^{\sigma(-i,-j)} g_j T_1^{\sigma(j,-i)} g_i$, for $i, j \in \Delta, |i| \neq |j|$;

(4) $g_i g_{-i} = g_{-i} g_i = 1, i \in \Delta$,

where

$$\sigma(i,j) = \begin{cases} 1 & (i < j) \\ -1 & (i > j). \end{cases}$$

Here g_i is as above and T_i twists the points x_i and x_{i+1} clockwise. To take into account the framing when performing g_i, we always assume the point x_1 moves towards the center of the polygon, moves towards the designated edge, crosses it, goes back to the center and returns to its original place from the center. When returning to its position x_1 has to perform a U-turn; we chose opposite turns for g_i and g_{-i}, so that relation (4) is true also when taking framing into account.

Definition 2.3.11. The *surface Hecke algebra* $H_n(\Sigma)$ is the quotient of the group algebra of $\mathrm{Br}_n(\Sigma)$ by the Hecke relations $(T_i - q^{\frac{1}{2}})(T_i + q^{-\frac{1}{2}}) = 0$.

The algebra H_n comes with a natural morphism

$$H_n(\Sigma) \to \mathrm{Sk}(\Sigma, n).$$

2.3.4 The $\mathbb{A}_{q,t}$ algebra of a surface

We want to finally define the $\mathbb{A}_{q,t}$ algebra of a surface. We begin by fixing some notation. We denote by $\bar{\Sigma}$ the surface obtained from Σ by gluing a disc to the base. We place the beam in $\bar{\Sigma} \times [0,1]$ in the middle of the base. We would like to interpret algebraically our skein theoretic construction of $\mathrm{Sk}(\bar{\Sigma} \times [0,1], n, n')$. In this section, we fix the base ring to be $R = \mathbb{Q}(q,t)[a, a^{-1}, b, b^{-1}]$.

Definition 2.3.12. We set $\mathbb{A}_q(\Sigma)$ to be the R-linear category whose objects are indexed by $\mathbb{Z}_{\geq 0}$ and denoted by $[0], [1], \ldots$. The morphisms have generators

(1) The generators of the surface Hecke algebra $H_n(\Sigma)$ can be seen as endomorphisms of $[n]$;

(2) $d_+ : [n] \to [n+1]$ and $d_- : [n+1] \to [n]$, for each $n \in \mathbb{Z}_{\geq 0}$.

The relations are

(1) The relations for the surface Hecke algebra $H_n(\Sigma)$ for each $n \in \mathbb{Z}_{>0}$.

(2) The relations $d_+ f = \iota(f) d_+$ and $f d_- = \iota(f) d_-$, for each $n \in \mathbb{Z}_{>0}$ and each $f \in H_n(\Sigma)$, where $\iota : H_n(\Sigma) \to H_{n+1}(\Sigma)$ is the natural homomorphism of adding a strand.

(3) The relations corresponding to (2.3.2), (2.3.1), (2.3.3), (2.3.4).

$$T_{n+1} d_+^2 = q^{\frac{1}{2}} d_+,\ d_-^2 T_{n-1} = q^{\frac{1}{2}} d_-$$

$$d_- T_n d_+ = q^{-\frac{1}{2}} d_+ d_- + h\ d_- d_+ = 1 - b q^{-n}.$$

We define $\mathbb{A}_{q,t}(\bar{\Sigma})$ to be the category $\mathbb{A}_q(\Sigma)$ quotiented by

(4) The relation corresponding to (2.3.6):

$$1 - d_+ d_- = qt T_{n-1}^{-1} \ldots T_1^{-1} \iota(L) T_1^{-1} \ldots T_{n-1}^{-1},$$

where $\iota : H_1(\Sigma) \to H_n(\Sigma)$ is the natural map and $L \in H_1(\Sigma)$ is the element corresponding to a strand going clockwise around the base with constant framing.

Remark 2.3.13. We see that the right hand side of the last relation is invertible. The inverse to the left hand side is in fact given by

$$1 + b^{-1} q^{n-1} d_+ d_-.$$

By construction we have natural maps compatible with composition

$$\mathrm{Hom}_{\mathbb{A}_{q,t}(\bar{\Sigma})}([n], [n']) \to \mathrm{Sk}_I(\bar{\Sigma} \times [0,1], n, n'),\ n, n' \in \mathbb{Z}_{>0}.$$

2.3.5 The $\mathbb{A}_{q,t}$ algebra of a punctured surface

We consider her the case in which Σ has at least two boundary components.

Proposition 2.3.14. *The natural functor $\mathbb{A}_q(\bar{\Sigma}) \to \mathbb{A}_{q,t}(\bar{\Sigma})$ is an isomorphism.*

Proof. We begin by fixing some notation. Let us assume that the last two sides of the polygon are $-d, d$, i.e. $w_{2d-1} = -d, w_{2d} = d$. The surface Σ can be represented by gluing the sides of the $2d-2$-gon with labels $w_1, ..., w_{2d-2}$. In this setting L has the form

$$L = L' g_d.$$

So we can apply relation (4) to eliminate g_d. We write (in $\mathbb{A}_{q,t}(\bar{\Sigma})$):

$$g_d = (qt)^{-1} \iota(L'^{-1}) T_1 \dots T_{n-1}(1 - d_+ d_-) T_{n-1} \dots T_1.$$

We need to show that the map sending g_d to $\mathbb{A}_{q,t}(\bar{\Sigma})$, thought of as a morphism of $\mathbb{A}_q(\bar{\Sigma})$, extends to a functor. Here, by abuse of notation we will denote the right hand side of this equation by g_d in $\mathbb{A}_q(\bar{\Sigma})$. So we need to show the following relations (these are all the relations featuring g_d):

$$g_d d_- = d_- g_d,\ g_d d_+ = d_+ g_d,\ g_d T_1^{-1} g_d T_1 = T_1 g_d T_1^{-1} g_d,$$

$$g_i T_1 g_d T_1^{-1} = T_1 g_d T_1^{-1} g_i\ (i = 1, \dots, d-1).$$

It is easy to observe that the relations involving g_i^{-1} and g_d^{-1} or g_d and g_i in a different order are redundant. So we go on to check the relations listed above. The first relation is equivalent to

$$(1 - d_+ d_-) d_- = d_- T_{n-1}(1 - d_+ d_-) T_{n-1}. \tag{2.3.7}$$

We can write the left hand side as

$$h d_- T_{n-1} + d_- - (q^{-\frac{1}{2}} d_+ d_- + h) d_- T_{n-1} = d_- - d_+ d_- d_-,$$

which is indeed equal to the right hand side. The relation for g_d and d_+ translates to

$$d_+(1 - d_+ d_-) = T_n(1 - d_+ d_-) T_n d_+, \tag{2.3.8}$$

which can be verified in a similar fashion. For the third relation, we can write

$$g_d T_1^{-1} g_d = (qt)^{-2} \iota(L'^{-1}) T_1 \iota(L'^{-1}) T_2 \dots T_{n-1}(1 - d_+ d_-)$$
$$\times T_{n-1} \dots T_1 \dots T_{n-1}(1 - d_+ d_-) T_{n-1} \dots T_1.$$

We now observe that

$$T_{n-1} \dots T_1 \dots T_{n-1} = T_1 \dots T_{n-1} \dots T_1.$$

Therefore

$$g_d T_1^{-1} g_d = (qt)^{-2}\iota(L'^{-1})T_1\iota(L'^{-1})T_2\dots T_{n-1}T_1\dots T_{n-2}$$
$$\times(1-d_+d_-)T_{n-1}(1-d_+d_-)T_{n-2}\dots T_1T_{n-1}\dots T_1.$$

Therefore, the third identity follows from the following relations:

$$T_1\iota(L'^{-1})T_1\iota(L'^{-1}) = \iota(L'^{-1})T_1\iota(L')T_1, \tag{2.3.9}$$

$$T_1(T_2\dots T_{n-1}T_1\dots T_{n-2}) = (T_2\dots T_{n-1}T_1\dots T_{n-2})T_{n-1}, \tag{2.3.10}$$

$$T_{n-1}(T_{n-2}\dots T_1T_{n-1}\dots T_1) = (T_{n-2}\dots T_1T_{n-1}\dots T_1)T_1, \tag{2.3.11}$$

$$T_{n-1}(1-d_+d_-)T_{n-1}(1-d_+d_-) = (1-d_+d_-)T_{n-1}(1-d_+d_-)T_{n-1}. \tag{2.3.12}$$

Among these (2.3.9) holds because L' is a multiple of the element of going clockwise around the puncture of $\bar{\Sigma}$. The relations (2.3.10) and (2.3.11) are obtained from the classical braid group relations. Finally, we have that (2.3.12) follows (2.3.7) and (2.3.8) as follows:

$$d_+d_-T_{n-1}(1-d_+d_-)T_{n-1} = d_+(1-d_+d_-)d_- = T_{n-1}(1-d_+d_-)T_{n-1}d_+d_-.$$

The last relation amounts to checking the following:

$$T_1\iota(L'^{-1})T_1g_i = giT_1\iota(L'^{-1})T_1,$$

which again is true because L' is a multiple of the element of going clockwise around the puncture. □

2.3.6 The $\mathbb{A}_{q,t}$ algebra of the torus

Let $x_1, \dots, x_n$ be the fixed endpoints on the surface of the torus. The element T_i swaps x_i and x_{i+1} clockwise. We denote by y_1 and z_1 the generators of the fundamental group of the torus. Then, if we set

$$y_i = T_{i-1}\dots T_1y_1T_1\dots T_{i-1}\ ,\ z_i = T_{i-1}^{-1}\dots T_1^{-1}z_1T_1^{-1}\dots T_{i-1}^{-1},$$

we have the affine Hecke algebra relations for T_i, y_i and T_i, z_i. The element L is given by $y_1^{-1}z_1^{-1}y_1z_1$. So relation (4) translates to

$$1-d_+d_- = qtT_{n-1}^{-1}\dots T_1^{-1}y_1^{-1}z_1^{-1}y_1z_1T_1^{-1}\dots T_{n-1}^{-1}. \tag{2.3.13}$$

We will use the element φ, which is given by

$$\varphi = T_1\dots T_{n-1}z_n.$$

We can easily see from the pictorial representation that φ satisfies the following relations

(1) $\varphi T_i = T_{i+1}\varphi$, for $i = 1, \dots n-2$;

(2) $\varphi^2 T_{n-1} = T_1\varphi^2$;

(3) $\varphi d_- = d_-\varphi T_{n-1}$;

(4) $\varphi T_n d_+ = d_+\varphi$;

(5) $\varphi^{-1} d_- \varphi^{-1} = d_-\varphi^{-2} T_1$;

(6) $\varphi y_i = y_{i+1}\varphi$, for $i = 1, \ldots, n-1$;

(7) $\varphi y_n(1 - d_+d_-) = qty_1\varphi$.

We remark that relation (2) follows from (4) and (5) (see Lemma 3.1.11 in [CGM20]).

Based on this construction, we now want to give the following definition.

Definition 2.3.15. The $\mathbb{A}_{q,t}$ algebra of the torus is the algebra generated by $\varphi^{\pm 1}, d_+, d_-, T_i, y_i^{\pm 1}$ satisfying relations

(1) $T_{n+1}d_+^2 = q^{1/2}d_+^2$, $d_+T_i = T_id_+$;

(2) $d_-^2 T_{n-1} = q^{1/2}d_-^2$, $d_-T_i = T_id_-$;

(3) $d_-T_nd_+ = q^{-1/2}d_+d_- + h$;

(4) $d_-d_+ = 1 - bq^{-n}$;

(5) Affine Hecke algebra relations

$$T_i^2 - hT_i - 1 = 0,\ T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1},\ T_iT_j = T_jT_i, \text{ if } |i-j| > 1$$

$$T_iy_{i+1}T_i = y_i,\ y_iT_j = T_jy_i\ y_iy_j = y_jy_i;$$

(6) $d_-y_i = y_id_-$, $d_+y_i = y_id_+$;

(7) $\varphi T_i = T_{i+1}\varphi$, for $i = 1, \ldots n-2$;

(8) $\varphi d_- = d_-\varphi T_{n-1}$;

(9) $\varphi T_n d_+ = d_+\varphi$;

(10) $\varphi y_i = y_{i+1}\varphi$, for $i = 1, \ldots, n-1$;

(11) $\varphi^{-1}d_-\varphi^{-1} = d_-\varphi^{-2}T_1^{-1}$;

(12) $\varphi y_n(1 - d_+d_-) = qty_1\varphi$.

Remark 2.3.16. Note that our variables y_i and the element φ are invertible, unlike in the usual $\mathbb{A}_{q,t}$.

2.4 The $\mathbb{A}_q$ algebra of the disc: A basis and Schur-Weyl duality

Here we want to consider the $\mathbb{A}_q$ algebra of the disc, which we can look at as the subalgebra of the $\mathbb{A}_{q,t}$ of the torus generated by $d_\pm, T_i$.

Definition 2.4.1. The algebra $\mathbb{A}_q(D)$ is the algebra generated by d_+, d_-, T_i subject to relations

(1) $T_{n+1}d_+^2 = q^{1/2}d_+^2$, $d_+T_i = T_id_+$;

(2) $d_-^2T_{n-1} = q^{1/2}d_-^2$, $d_-T_i = T_id_-$;

(3) $d_-T_nd_+ = q^{-1/2}d_+d_- + h$;

(4) $d_-d_+ = 1 - bq^{-n}$;

(5) $T_i^2 - hT_i - 1 = 0$;

(6) $T_iT_j = T_jT_i$, if $|i-j| > 1$, $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$.

Our first goal is to give a basis for this algebra. To do so, we proceed as follows. We begin by considering certain subalgebras $\mathbb{A}_q^{(n)}$, for a non-negative integer n, and we give a basis for these algebras. To conclude this step, we will make use of a Schur-Weyl duality result. In fact, we will prove that $\mathbb{A}_q^{(n)}$ is isomorphic to the space of endomorphisms of a certain quantum group representation. The next step will be to consider a bigger algebra $\mathbb{A}_q^{\leq N}$, which has the property that any element in $\mathbb{A}_q(D)$ is contained in it for sufficiently large N. We will use our previous results to give a basis for $\mathbb{A}_q^{\leq N}$, which will in turn give us a basis for the whole $\mathbb{A}_q(D)$.

The algebra $\mathbb{A}_q^{(n)}$ is made of operators that take in n strands and give as output n strands.

Remark 2.4.2. It will be useful to observe that $\mathbb{A}_q^{(n-1)}$ is naturally contained in $\mathbb{A}_q^{(n)}$. More explicitly, given an element of $\mathbb{A}_q^{(n-1)}$, we can interpret it as an element of $\mathbb{A}_q^{(n)}$ by simply adding a strand to the right and shift the labels of the preexisting strands by 1.

Theorem 2.4.3. *The algebra $\mathbb{A}_q^{(n)}$ has a basis given by $T_{w_1}T_{w_2}d_+^kd_-^kT_{w_3}$, where $w_1 \in S_n/(S_k \times S_{n-k}), w_2 \in S_{n-k}, w_3 \in (S_k \times S_{n-k})\backslash S_n$, for $k = 0, \dots, n$.*

We observe that it is not difficult to see that any element can be written as a linear combination of elements of this form. The key observation is that we can always slide d_+ and d_- into this position. The only problem we really have to deal with is what to do when a d_+ appears to the right of a d_- and among the T_i's there is a T_n, which is the only T generator that a d_+ cannot slide past towards the left. In this case, we can pull to the right the closest d_- on the left of the d_+ until it meets the T_n and apply relation (3). The crucial remark here is that this is always possible because any T_w can be written in such a way that it presents only one T_n.

We note that we could have to chosen to place the T_{w_2} on the right, as this gathers the T's that can slide past the d_+'s and d_-'s in the middle. So, what we actually need to show is that these elements are linearly independent. Here is where Schur-Weyl duality will come in handy.

Remark 2.4.4. The number of the basis elements is $\sum_{k=0}^{n} \frac{(n!)^2}{(k!)^2(n-k)!}$.

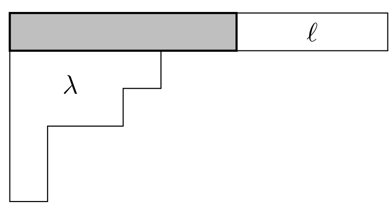
We now want to state and prove our first Schur-Weyl duality-like result. We want to consider representations of a quantum group $U_q(\mathfrak{gl}_M)$, where $M > n$ and q is generic.

Let us consider the following endomorphism space

$$\mathrm{End}(\mathrm{Sym}(V)^{L-n} \otimes V^{\otimes n}),$$

for $L >> n$.

To simplify the notation we will set $\mathrm{Sym}(V)^{L-n} \otimes V^{\otimes n} =: \mu_n$ We want to give a combinatorial interpretation of this space. According to the Pieri rule, we can think of an element of $\mathrm{End}(\mu_n)$ as a pair of standard tableaux, both of shape



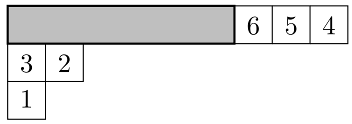
with $|\lambda| + |\ell| = n$.

Remark 2.4.5. By the combinatorial description, we can see that $\dim \mathrm{End}(\mu_n) = \sum_{k=0}^{n} \frac{(n!)^2}{(k!)^2(n-k)!}$. In other words, it conveniently matches our candidate for the dimension of $\mathbb{A}_q^{(n)}$.

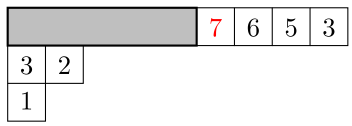
We now want to define an action of $\mathbb{A}_q^{(n)}$ on μ_n, which will give us a map $\varphi : \mathbb{A}_q \to \mathrm{End}(\mu_n)$. In fact, to prove Theorem 2.4.3, we will show that this map is surjective and this will suffice by Remark 2.4.4 and 2.4.5.

For our convenience, we will consider tableaux whose entries that are decreasing from left to right and top to bottom. This will make it easier to define the actions of the $d_\pm$ operators. For a tableau $\sigma = (\ell, \lambda)$ define $d_-\sigma$ for a tableau to be 0 whenever $\boxed{n} \notin \ell$ and $(\ell \setminus \boxed{n}, \lambda)$ when the label n is in ℓ. Here the notation $\ell \setminus \boxed{n}$ means that we have unlabelled the box with entry n in ℓ. As for d_+, it will simply label the rightmost unlabelled box of the first row and label it by $n + 1$.

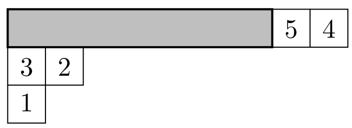
Example 2.4.6. Consider for example the tableaux σ given by



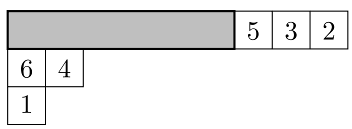
The operator d_+ will send σ to $d_+\sigma$ given by



On the other hand, $d_-\sigma$ will be, up to a constant, given by



However, the operator d_- will send the tableau



to 0.

Finally, we want to define the action of the T_i's. In order to do so, we need to define the weight of a box of a tableaux σ. The weight of i is given by $w(i) = q^{c_\sigma(i)}$, where the content c_σ is the usual content for the boxes of λ and on the first row we require that the first labelled box has weight bq^{-n-1}. We are now ready to define the action of each T_i:

$$T_i\nu_\sigma = \begin{cases} q^{\frac{1}{2}}\nu_\sigma & \text{if } i \text{ is to the left of } i+1, \\ -q^{-\frac{1}{2}}\nu_\sigma & \text{if } i \text{ is on top of } i+1, \\ \frac{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}{1-w(i)w^{-1}(i+1)}\nu_\sigma + \frac{q^{-\frac{1}{2}}-q^{\frac{1}{2}}w(i)^{-1}w(i+1)}{1-w(i)^{-1}w(i+1)}\nu_{s_i(\sigma)} & \text{otherwise.} \end{cases}$$

The fact that this is indeed an action can be checked by direct computation.

Remark 2.4.7. Note that the action of $\mathbb{A}_q(D)$ leaves the overall shape of the tableau unchanged.

We are now ready to prove that φ is an isomorphism. As we already mentioned above, Theorem 2.4.3 will follow.

Theorem 2.4.8. *The algebra $\mathbb{A}_q^{(n)}$ is isomorphic to* $\mathrm{End}(\mu_n)$ *via the map φ.*

Proof. As we observed, we just need to prove that the map is surjective since the dimension of $\mathbb{A}_q^{(n)}$ is less or equal to that of $\mathrm{End}(\mu_n)$.

We show that, given two standard pairs of tableaux $\sigma_1 = (\lambda_1, \ell_1)$ and $\sigma_2 = (\lambda_2, \ell_2)$, we can construct an element $h \in \mathbb{A}_q^{(n)}$ such that $h\nu_{\sigma_1} = \nu_{\sigma_2}$. Suppose that for each standard pair σ we can construct an element h_σ such that $h_\sigma \nu_\sigma = \nu_\sigma$ and $h_\sigma \nu_\tau = 0$ for all $\tau \neq \sigma$. Then, if $w \in S_n$ is the permutation taking σ_1 to σ_2, we can set $h = h_{\sigma_2} T_w h_{\sigma_1}$. So our claim amounts to showing the existence of h_σ for a given σ. We prove this by induction on n. We can easily show that this is true for $\mathbb{A}_q^{(1)}$. In fact, we only have to consider the diagrams $(\square, \varnothing)$ and $(\varnothing, \square)$. The operator d_+d_- multiplies the first by $1 - bq^{-1}$ and annihilates the second, so the sought operators are $\frac{d_+d_-}{1-bq^{-1}}$ and $1 - \frac{d_+d_-}{1-bq^{-1}}$. Now, suppose this is true for $n-1$. Given σ, we know there exists $h_\sigma^{(n-1)} \in \mathbb{A}_q^{(n-1)}$ such that $h_\sigma^{(n-1)}$ annihilates ν_τ for all τ except those coinciding with σ on all boxes except the one with label 1. By abuse of notation we will also denote by $h_\sigma^{(n-1)}$ the corresponding element of $\mathbb{A}_q^{(n)}$, according to Remark 2.4.2. We note that T_1 acts on these tableaux as $T_1 \nu_\tau = c_\tau \nu_\tau + c'_\tau \nu_{s_1\tau}$, where c_τ is nonzero and c'_τ is zero when $s_1\tau$ is not a standard pair. Then the element we are looking for is

$$h_\sigma = \prod_{\tau \neq \sigma} \frac{h_\sigma^{(n-1)} T_1 h_\sigma^{(n-1)} - c_\tau}{c_\sigma - c_\tau}.$$

$\square$

We now want to proceed to consider a bigger algebra, which we call $\mathbb{A}_q^{\leq N}$. This will be the algebra containing all the elements whose number of input strands and number of output strands are bounded by N. We want to give a basis for this algebra. We will again use Schur-Weyl duality to find this basis. Our expectation is now that

$$\mathbb{A}_q^{\leq N} \simeq \mathrm{End}(\oplus_{n=0}^{N} \mu_n),$$

where $L >> N$. As the direct sum is finite, we can decompose the space on the right hand side into a direct sum of endomorphism spaces $\mathrm{End}(\mu_n)$ and into homomorphism spaces $\mathrm{Hom}(\mu_n, \mu_{n+j})$ and $\mathrm{Hom}(\mu_{n+j}, \mu_n)$. As we have already dealt with the endomorphism spaces, we want to understand these homomorphism spaces to deduce a basis for $\mathbb{A}_q^{\leq N}$. We start with $\mathrm{Hom}(\mu_n, \mu_{n+1})$; we shall see later that, once we get a grasp of what happen in this case, we can easily deduce the remaining cases.

Let us begin by giving a combinatorial interpretation of these spaces. An element of $\mathrm{Hom}(\mu_n, \mu_{n+1})$ can be interpreted as a pair of tableaux of the same overall shape with n and $n+1$ labelled boxes, respectively. It is easy to count how many such pairs we have. So, the dimension of $\mathrm{Hom}(\mu_n, \mu_{n+1})$ is given by the formula

$$\sum_{\lambda, k=0}^{k=n} \frac{((n-k)!)^2}{\prod h_\lambda(i,j)^2} \binom{n}{k} \binom{n+1}{k+1},$$

which can be simplified to

$$\sum_{k=0}^{n}(n-k)!\binom{n}{k}\binom{n+1}{k+1}.$$

Let us look at how we can relate these spaces to $\mathbb{A}_q^{\leq N}$. On the one hand, we know that d_+ sends an element of μ_n to an element of μ_{n+1}, on the other hand, the elements of $\mathbb{A}_q^{(n+1)}$ give us endomorphisms of μ_{n+1} (actually all of them!). This observation suggests that $\mathrm{Hom}(\mu_n, \mu_{n+1}) \simeq \mathbb{A}_q^{(n+1)} d_+$.

We now want to understand the elements of $\mathbb{A}_q^{(n+1)} d_+$. We see that we can act as we did for $\mathbb{A}_q^{(n)}$ to see that any element of $\mathbb{A}_q^{(n+1)} d_+$ can be put into the form

$$T_{w_1} T_{w_2} d_+^{k+1} d_-^k T_{w_3},$$

where $w_1 \in S_{n+1}/(S_{n-k} \times S_{k+1}), w_2 \in S_{n-k}, w_3 \in (S_{n-k} \times S_k)\backslash S_n$, for $k = 0 \dots n$. Clearly, we have a map $\varphi_{n,1}^+ : \mathbb{A}_q^{(n+1)} \to \mathrm{Hom}(\mu_n, \mu_{n+1})$, which is surjective by Theorem 2.4.8. As the number of elements of our candidate basis and the dimension of the hom space coincide, we deduce that it is an isomorphism and therefore the elements listed above form indeed a basis.

It is not difficult to see that we can generalize our argument for $\mathbb{A}_q^{(n+j)} d_+^j$. This will have a basis of elements

$$T_{w_1} T_{w_2} d_+^{k+j} d_-^k T_{w_3},$$

where $w_1 \in S_{n+j}/(S_{n-k} \times S_{k+j}), w_2 \in S_{n-k}, w_3 \in (S_{n-k} \times S_k)\backslash S_n$, for $k = 0 \dots n$, and we will have an isomorphism $\varphi_{n,j}^+ : \mathbb{A}_q^{(n+j)} \to \mathrm{Hom}(\mu_n, \mu_{n+j})$.

Let us now treat the case of $\mathrm{Hom}(\mu_{n+j}, \mu_n)$. Here we expect this space to be isomorphic to $d_-^j \mathbb{A}_q^{(n+j)}$. First of all, we observe that the count of the dimension of $\mathrm{Hom}(\mu_{n+j}, \mu_n)$ is the same as for $\mathrm{Hom}(\mu_n, \mu_{n+j})$. Secondly, we see that an element of $d_-^j \mathbb{A}_q^{(n+j)}$ can be written in the form

$$T_{w_1} T_{w_2} d_+^k d_-^{k+j} T_{w_3}.$$

Therefore, we have again an isomorphism $\varphi_{n,j}^- : d_-^j \mathbb{A}_q^{(n+j)} \to \mathrm{Hom}(\mu_{n+j}, \mu_n)$, which allows us to prove that these elements form a basis.

We have obtained the following.

Theorem 2.4.9. *The algebra $\mathbb{A}_q^{\leq N}$ has a basis given by*

$$T_{w_1} T_{w_2} d_+^{k_1} d_-^{k_2} T_{w_3},$$

where $w_1 \in S_{n+(k_1-k_2)}/(S_{n-\min(k_1,k_2)} \times S_{\min(k_1,k_2)+(k_1-k_2)}), w_2 \in S_{n-\min(k_1,k_2)}, w_3 \in S_{n-\min(k_1,k_2)} \times S_{\min(k_1,k_2)} \backslash S_n$ and $n, n+(k_1-k_2) \leq N$.

As a direct consequence of Theorem 2.4.9, we now have a canonical way to write down any element of $\mathbb{A}_q(D)$, since all elements of $\mathbb{A}_q(D)$ belong to $\mathbb{A}_q(D)^{\leq N}$ for some N.

2.5 The $\mathbb{A}_q$ algebra of the annulus

We now look at the $\mathbb{A}_q$ algebra of the annulus, which we see as the subalgebra of the $\mathbb{A}_q$ of the torus generated by $d_\pm$ and the T and y generators.

Definition 2.5.1. The algebra $\mathbb{A}_q(A)$ is generated by $d_\pm, T_i, y_i^{\pm 1}$ subject to relations

(1) $T_{n+1}d_+^2 = q^{1/2}d_+^2$, $d_+T_i = T_id_+$;

(2) $d_-^2T_{n-1} = q^{1/2}d_-^2$, $d_-T_i = T_id_-$;

(3) $d_-T_nd_+ = q^{-1/2}d_+d_- + h$;

(4) $d_-d_+ = 1 - bq^{-n}$;

(5) $T_i^2 - hT_i - 1 = 0$;

(6) $T_iT_j = T_jT_i$, if $|i-j| > 1$, $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$;

(7) $T_iy_iT_i = y_i$;

(8) $y_iy_j = y_jy_i$;

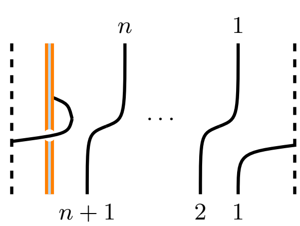
(9) $d_-y_i = y_id_-$;

(10) $d_+y_i = y_id_+$.

Remark 2.5.2. We can observe that the y variables and the T_i's form an affine Hecke algebra.

Let us take a moment to compare $\mathbb{A}_q(A)$ with the usual Dyck path algebra $\mathbb{A}_q$.

The d_+ of $\mathbb{A}_q$ in Definition 2.2.1 can be described as a skein as a multiple of the diagram



So, according to this definition, the d_+ is also looping around the annulus, unlike our skein version of d_+, which simply adds a strand. We will give more details on this comparison in Section 2.6.

We now want to define the action of $\mathbb{A}_q(A)$ on a space of quantum group intertwiners for the quantum group $U_q(\mathfrak{gl}_M)$. We consider the space $\mathrm{Hom}([\lambda] \otimes \det^k, \oplus_{n=0}^N([\lambda] \otimes \mathrm{Sym}(V)^{N-n} \otimes V^{\otimes n}))$, where λ is a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_M)$ with *big gaps*, meaning $\lambda_i - \lambda_{i+1} \geq k$, and $k \cdot M = N$ (the only case in which this space is non-zero). Using the Pieri rule, this space is generated by skew

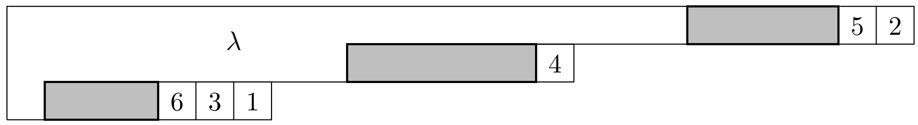


Figure 2.1: Example of horizontal strip representing an intertwiner in the space $\mathrm{Hom}(\lambda \otimes \det^6, \lambda \otimes \mathrm{Sym}(V)^{12} \otimes V^{\otimes 6})$

tableaux which are horizontal strips. In fact, every tableaux will have the same overall shape, by which we mean again the shape of the tableaux considering the labelled and the unlabelled boxes. We can see in Figure 2.5 how such a tableau will look like.

Again, to make the action of $d_\pm$ more natural, we use tableaux which are decreasing left to right and top to bottom. The content of a box is again the difference between its row and its column.

Let us define the action of $d_\pm$. We set $d_+\nu_\sigma = \sum_{\sigma'} \nu_{\sigma'}$, where σ' varies along all tableaux obtained by labelling by $n+1$ exactly one box to the left of a labelled box of σ. When there are no boxes to add, d_+ acts as zero.

As for d_-, its action is given by $d_-\nu_\sigma = K(\sigma)\nu_{\sigma''}$, where σ'' is obtained by deleting the label of the box filled by n in σ and $K(\sigma)$ is defined as follows. It will be convenient to define $K(\sigma)$ as the residue of a function. This will later allow us to easily verify the relations. Let us denote by $\mathrm{Unlab}(\sigma)$ the unlabelled boxes in σ. We say an unlabelled box in σ is addable if it is located immediately to the left of a labelled box. We denote addable boxes by w_i, then

$$K(\sigma + w_i) = \prod_{w \in \mathrm{Unlab}(\sigma)} \frac{1 - q^{1 + c_\sigma(w) - c_\sigma(w_i)}}{1 - q^{c_\sigma(w) - c_\sigma(w_i)}} (1-q).$$

This can be written as a multiple of the residue of the function

$$\psi_\sigma(z) = \prod_{w \in \mathrm{Unlab}(\sigma)} \frac{1 - q^{1 + c_\sigma(w)} z}{1 - q^{c_\sigma(w)} z}$$

at $z = q^{-c_\sigma(w_i)}$. More precisely, we have

$$K(\sigma + w_i) = \lim_{z \to q^{-c_\sigma(w_i)}} (1 - q^{c_\sigma(w_i)} z)\psi_\sigma(z)$$

The variables y_i act semisimply, multiplying ν_σ by $q^{c_\sigma(i)}$.

Note that, for $n = N$, the relation $d_-d_+ = 1 - bq^{-n}$ gives us $0 = 1 - bq^{-N}$, so $b = q^N$.

We set

$$T_i\nu_\sigma = \begin{cases} q^{\frac{1}{2}}\nu_\sigma & \text{if } i \text{ is to the right of } i+1, \\ \frac{h}{1 - w(i)w(i+1)^{-1}}\nu_\sigma + \frac{q^{-\frac{1}{2}} - q^{\frac{1}{2}} w(i)^{-1} w(i+1)}{1 - w(i)^{-1} w(i+1)} \nu_{s_i(\sigma)} & \text{otherwise.} \end{cases}$$

Proposition 2.5.3. *The action of* $\mathrm{d}_\pm, T_i, y_i$ *described above gives an action of* $\mathbb{A}_q(A)$ *on* $\mathrm{Hom}([\lambda]\otimes\det^k, \oplus_{n=0}^N([\lambda]\otimes\mathrm{Sym}(V)^{N-n}\otimes V^{\otimes n}))$.

Proof. The relations (1),(2),(5)-(10) are straightforward to verify (or classical for AHA calibrated representations, see [Ram03a]). Let us take a look at relation (3). This gives us two formulas which $K(.)$ has to satisfy, namely

$$q^{-\frac{1}{2}}K(\sigma)+h=\sum_{w_i\mathrm{addable}}A^{(1)}_{\sigma+w_i}K(\sigma+w_i) \tag{2.5.1}$$

and

$$A^{(2)}_{\sigma+w_i}K(s_n(\sigma+w_i))=q^{-\frac{1}{2}}K(\sigma), \tag{2.5.2}$$

where we denote by $A^{(1)}_{\sigma+w_i}$ and $A^{(2)}_{\sigma+w_i}$ the coefficients of $\nu_{\sigma+w_i}$ and $\nu_{s_n(\sigma+w_i)}$, respectively, in $T_n\nu_{\sigma+w_i}$. The second formula follows immediately from the definition of $K(.)$. The first formula is not as straightforward. In fact, we will prove it by applying the residue theorem. We make use of the following auxiliary function

$$f(z):=\frac{1}{z}\cdot\frac{h}{1-q^{c_\sigma(n)}z}\cdot\prod_{w\in\mathrm{Unlab}(\sigma)}\frac{1-q^{1+c_\sigma(w)}z}{1-q^{c_\sigma(w)}z}.$$

We observe that $f(z)$ has simple poles at $0, q^{-c_\sigma(n)}, q^{-c_\sigma(w)}$, for every addable box w. It is easy to see that its residue at ∞ is 0, therefore we have that $-\sum_{w_i\text{ addable}}\mathrm{Res}(f(z),q^{-c_\sigma(w_i)})=\mathrm{Res}(f(z),q^{-c_\sigma(n)})+\mathrm{Res}(f(z),0)$. We see that $\mathrm{Res}(f(z),q^{-c_\sigma(w_i)})$ is

$$\lim_{z\to q^{-c_\sigma(w_i)}}\frac{(z-q^{-c_\sigma(w_i)})(1-q^{1+c_\sigma(w_i)}z)}{z(-q^{c_\sigma(w_i)})(z-q^{-c_\sigma(w_i)})}\cdot\frac{h}{1-q^{c_\sigma(n)}z}\prod_{w\in\mathrm{Unlab}(\sigma)\setminus\{w_i\}}\frac{1-q^{1+c_\sigma(w)}z}{1-q^{c_\sigma(w)}z},$$

which is in fact equal to

$$-A^{(1)}_{\sigma+w_i}K(\sigma+w_i).$$

The residue at 0 is $\lim_{z\to 0}z\cdot f(z)=h$. Finally, the residue at $q^{-c_\sigma(n)}$ is

$$\lim_{z\to q^{-c_\sigma(n)}}\frac{(z-q^{c_\sigma(n)})}{z\cdot(-q^{c_\sigma(n)})}\cdot\frac{h}{z-q^{-c_\sigma(n)}}\prod_{w\in\mathrm{Unlab}(\sigma)}\frac{1-q^{1+c_\sigma(w)}z}{1-q^{c_\sigma(w)}z},$$

which is equal to $\frac{-h}{1-q}K(\sigma)=q^{-\frac{1}{2}}K(\sigma)$. ◻

Proposition 2.5.4. *The representation* $\mathrm{Hom}([\lambda]\otimes\det^k, \oplus_{n=0}^N([\lambda]\otimes\mathrm{Sym}(V)^{N-n}\otimes V^{\otimes n}))$ *is irreducible.*

Proof. We want to prove that, given a tableaux σ, we can reach any other tableaux τ by acting with elements of $\mathbb{A}_q(A)$. This is true because of two crucial observations. First of all, the AHA action is transitive on tableaux with the same labelled shape (the skew diagram drawn by the labelled boxes), so we just need to make sure that we can act on σ and obtain the labelled shape of τ.

The transitivity of the AHA action enables us to unlabel a box where needed by applying d_-, so we simply have to verify that we can add a box where we need to. Now we observe that this is indeed possible because all boxes in the overall shape of tableaux in $\mathrm{Hom}([\lambda] \otimes \det^k, [\lambda] \otimes \mathrm{Sym}(V)^{N-n} \otimes V^{\otimes n})$ have distinct content. As a consequence, once we have applied d_+ to σ, we can choose our preferred labelled shape $\sigma + w_j$ by multiplying by $\prod_{i \neq j}(y_{n+1} - w_i)$. $\square$

Let us take a moment to discuss a possible candidate basis. Following the $\mathbb{A}_q(D)$ case, we would like the basis of $\mathbb{A}_q(A)$ to be made of elements of the form

$$T_{w_1} y^{a_1} T_{w_2} d_+^k d_-^k T_{w_3} y^{a_3},$$

with the w_i's ranging in the appropriate sets, like in the discussion in the previous section. To write any element as a combination of elements of this kind, we can repeat the procedure we applied for $\mathbb{A}_q(D)$, however, we can observe that we meet an obstacle when we encounter the elements $L_a := d_- y_{n+1}^a d_+$. These elements L_a are somewhat special as they commute with each other and, as we shall see later, can be slightly modified to elements $\tilde{L}_a$ which are supposed to be central and still commute with each other.

Let us examine then the elements $L_a = d_- y_{n+1}^a d_+$. We see that we have the following relation

$$d_+ L_a = (L_a + (1-q)(y_{n+1} L_{a-1} + \cdots + y_{n+1}^{a-1} L_1 + y_{n+1}^a)) d_+$$

and a similar relation for d_-.

Remark 2.5.5. Note that we can see L_a as an element of $\mathbb{A}_q(A)^{(n)}$ for each n.

Proposition 2.5.6 ([Mel19]). *The elements L_a and y_i commute among each other in $\mathbb{A}_q(A)^{(n)}$*

Proof. It is clear that the y_i's commute with each other and with each L_a. What remains to prove is that $L_a L_b = L_b L_a$. Explicitly, we want to say that the expression

$$d_- y_{n+1}^a d_+ d_- y_{n+1}^b d_-$$

is symmetric in a and b. We begin by applying relation (3) to get

$$d_- y_{n+1}^a d_+ d_- y_{n+1}^b d_- = (1-q) d_- y_{n+1}^{a+b} d_+ + q^{\frac{1}{2}} d_-^2 y_{n+1}^a T_{n+1} y_{n+1}^b d_+^2.$$

It is immediate to see that the first summand of the right hand side is symmetric in k and j, so we can restrict our attention the second summand. Combining (5) and (7), we obtain the equality $T_{n+1} y_{n+1} = y_{n+2}(T_{n+1} - h)$, which we can repeatedly apply. together with (1), to get

$$d_-^2 y_{n+1}^a f_b(y_{n+1}, y_{n+2}) d_+^2,$$

for some Laurent polynomial f_b. Now, the observation we need to make is that to the same expression we can apply $y_{n+1} T_{n+1} = (T_{n+1} - h) y_{n+2}$ and $d_-^2 T_{n+1} = q^{\frac{1}{2}} d_-^2$ to write it as

$$d_-^2 f_a(y_{n+1}, y_{n+2}) y_{n+1}^b d_+^2,$$

where the polynomial f_a is constructed in the same fashion as f_b. It is now easy to see that swapping a and b interchanges between these two ways of writing this summand. $\square$

Ultimately, we would like to modify the elements L_a so that they will commute with the whole of $\mathbb{A}_q(A)$.

We want to see L_a as some generalized complete homogeneous functions in the following way. We consider the function

$$\phi(z) = \frac{\prod_{i=1}^M (1 - zv_i)}{\prod_{i=1}^M (1 - zu_i)}.$$

Then we write

$$h_a[X] = (\mathrm{Res}_{z=0} + \mathrm{Res}_{z=\infty})\phi(z)z^{-a}\frac{dz}{z},$$

where $X = \sum_i = 1^M u_i - \sum_{i=1}^M v_i$. We want to identify L_a with the function h_a. We want to change L_a by slightly modifying ϕ to

$$\tilde{\phi} = \frac{\prod_{i=1}^M (1 - zv_i)}{\prod_{i=1}^M (1 - zu_i)} \cdot \frac{\prod_{i=1}^M (1 - zqy_i)}{\prod_{i=1}^M (1 - zy_i)}.$$

Now, the idea is that, for the particular case of our representation, this corresponds to including in the product the factors corresponding to the labelled boxes. The modified elements $\tilde{L}_k = \tilde{h}_a$, defined analogously to h_a, should now be central. Note that, as operators, they clearly commute with the other elements of $\mathbb{A}_q(A)$.

It is clear that $\mathbb{A}_q(A)^{(n)}$ can be seen as a module over $\Lambda^{(n)} := \mathbb{C}[L_a, y_1 \ldots, y_n]$. Now, the key observation is that one can go from $\Lambda^{(n)}$ to $\tilde{\Lambda}^{(n)} := \mathbb{C}[\tilde{L}_a, y_1, \ldots, y_n]$ and back. On the other hand, $\tilde{\Lambda}$ acts of $\mathrm{Hom}(\lambda \otimes \det^k, \lambda \otimes \mathrm{Sym}(V)^{N-n} \otimes V^{\otimes n})$ and its action commutes with that of $\mathbb{A}_q(A)^{(n)}$. Using the symmetric function interpretation, one can prove that the $\tilde{\Lambda}^{(n)}$ acts faithfully on the sum over λ of our hom spaces (to appear in [BMN]), however, it is not yet clear how to prove that the $\mathbb{A}_q$ action is also faithful. We think proving this fact could result in proof of the following conjecture.

Conjecture 2.5.7. The algebra $\mathbb{A}_q(A)^{(n)}$ admits a basis

$$T_{w_1}\underline{y}^{a_1}T_{w_2}\underline{y}^{a_2}d_+^k d_-^k T_{w_3}\underline{y}^{a_3}$$

over $\mathbb{C}(q)[\tilde{L_a}]$, where $k = 0, \ldots, n$ and $T_{w_1}\underline{y}^{a_1}, T_{w_3}\underline{y}_{a_3}$ range in the appropriate AHA quotients, analogously to Theorem 2.4.3, and $T_{w_2}\underline{y}^{a_2} \in \dot{H}_q(n-k)$.

2.6 The $\mathbb{A}_{q,t}$ algebra of the torus: Schur-Weyl duality

We have already introduced this algebra in Section 2.3.6. We want to give a conclusion to this work by putting together all the Schur-Weyl duality-like results that have appeared so far to give yet another statement of this kind for the $\mathbb{A}_{q,t}$ of the torus.

We want to understand the action of the element $\varphi \in \mathbb{A}_{q,t}$ in Section 2.3.6, which corresponds in a sense (or at least pictorially) to the generator π of the DAHA. We would like this element to take an intertwiner in $\mu_\lambda := \mathrm{Hom}([\lambda] \otimes \det^k, [\lambda] \otimes V^{\otimes n} \otimes \mathrm{Sym}(V)^{N-n})$ to an intertwiner in $\mu_{\lambda \otimes V^*}$. Recall this element is given by $\varphi := T_1 \dots T_n z_n$.

Remark 2.6.1. Usually (see [CGM20]), φ is defined as (a multiple of) the commutator of d_+ and d_-. As we have pointed out, our d_+ is different, as it does not contain a loop. This is not the only difference in conventions though: we are using a different normalization of the T_i's, so when comparing the "old" d_+, which we call $\tilde{d}_+$, to our d_+, we have to take this into account. Indeed, the diagram for $\tilde{d}_+$ will be given by $T_1, \dots, T_n z_n d_+$, but this will have a coefficient given by $q^{\frac{n}{2}}$, as each T_1 will contribute to a factor $q^{\frac{1}{2}}$. One can see that for us φ is a multiple of $\tilde{d}_+ d_- - q^{-\frac{1}{2}} d_- \tilde{d}_+$.

It will be convenient to determine the action of φ^{-1} instead of φ. Let us take a look at the relations that are satisfied by the element φ. We have already stated them in Section 2.3.6, but we quickly recall them here in terms of φ^{-1}. We have the following relations

(1) $T_i \varphi^{-1} = \varphi^{-1} T_{i+1}$ for $i = 2, \dots, n-2$;

(2) $d_- T_{n-1}^{-1} \varphi^{-1} = \varphi^{-1} d_-$;

(3) $T_n d_+ \varphi^{-1} = \varphi^{-1} d_+$;

(4) $y_i \varphi^{-1} = \varphi^{-1} y_{i+1}$ for $i = 1, \dots, n-1$;

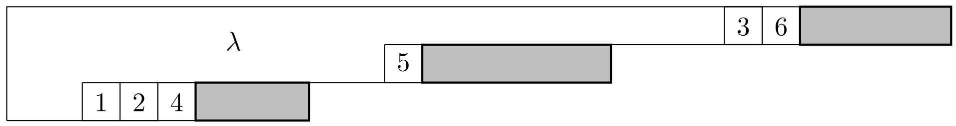
(5) $\varphi^{-1} d_- \varphi^{-1} = d_- \varphi^{-2} T_1^{-1}$;

(6) $qt\varphi^{-1} y_1 = y_n (1 - d_+ d_-) \varphi^{-1}$.

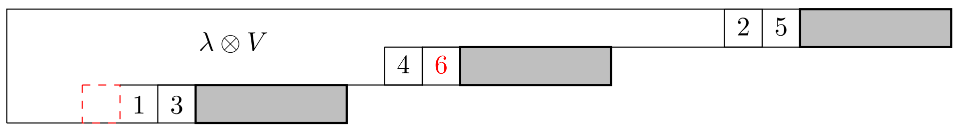
In this section, we consider the spaces $\oplus_\lambda \oplus_{n=0}^N \mathrm{Hom}([\lambda] \otimes \det^k, [\lambda] \otimes V^{\otimes n} \otimes \mathrm{Sym}(V)^{N-n})$, where λ ranges among the partitions with big gaps. The reason for our choice is that we feel it is easier to understand the action of φ^{-1}. This will require a change of conventions in our tableaux. Namely, the gray boxes corresponding to $\mathrm{Sym}(V)$ will now be on the right of the labelled boxes. Moreover, the labels will now be increasing from left to right. As a result, we will take the opposite of our previous definition of content (so it is now increasing left to right on the same row).

The idea is that φ^{-1} will decrease the entries $2, \dots, n$ by 1 and the box labelled 1 will be the box added to the shape of λ. Moreover, we now need to place the box labelled n: the image of σ will be a sum $\sum x_i \tau_i$, where the sums goes over the lines of σ, and in τ_i we have placed the box labelled n in the i-th row.

Example 2.6.2. Let us take for example the tableau



One of the summands in the image of this tableau looks like



where by abuse of notation we write $\lambda \otimes V$ for the summand of the corresponding representation in which we have put an extra box in λ in the row where 1 was located in σ.

Note that if we keep applying φ^{-1}, we might end up with a partition that does not have big gaps. In such case, we set $\varphi^{-1}\lambda = 0$.

We now have to determine the coefficients x_i. We begin by looking at relation (6):

$$qt\varphi^{-1}y_1 = y_n(1 - d_+d_-)\varphi^{-1}.$$

Suppose we apply this to a diagram σ:

$$qtq^{c_\sigma(1)} \sum_i x_i\tau_i = q^{c_{\tau_i}(n)}(\sum_i x_i\tau_i) - (\sum_i x_iK(\tau_i))\tau_i),$$

where we use the notation $\varphi^{-1}\sigma = \sum_i x_i\tau_i$. Let us set $\sum_i x_iK(\tau_i) =: C$. Then, comparing the coefficient of τ_i on each side, we can write

$$x_i = \frac{q^{c_{\tau_i}(n)}C}{q^{c_{\tau_i}(n)} - qtq^{c_\sigma(1)}}$$

Remark 2.6.3. We see that if we try to compute this equation on a tableau σ for which $n = N$, we obtain

$$qtq^{c_\sigma(1)} = q^{k+1}q^{c_\sigma(1)},$$

so $t = q^k$.

We now need to determine the constant C. To this end, we will use relation (3):

$$T_nd_+\varphi^{-1} = \varphi^{-1}d_+.$$

This gives us a way to relate the constant C for σ and the constant C_j for $\sigma + w_j$, where w_j is an addable box in σ. After computing both sides of the relation on σ, we see that for each C_j we get j equations. As for each j the equations look the same, we will write them for $j = 1$ to simplify the notation. We have one equation for the case in which the box added by applying φ^{-1} and d_+ are situated in the same row, which looks like

$$\frac{Cq^{c_\sigma(w_1)}}{q^{c_\sigma(w_1)}-qtq^{c_\sigma(1)}}\cdot q^{\frac{1}{2}}=\frac{C_1q^{c_\sigma(v_1)}}{q^{c_\sigma(v_1)}-qtq^{c_\sigma(1)}},$$

where v_1 is the box directly to the right of w_1. Then we have $j-1$ equation for C_1 involving the weight of w_j for $j\neq i$. Once again, for simplicity, we write this for the first and second row:

$$\frac{C_1q^{c_\sigma(w_2)}}{q^{c_\sigma(w_2)}-qtq^{c_\sigma(1)}}=\frac{Cq^{c_\sigma(w_1)}}{q^{c_\sigma(w_1)}-qtq^{c_\sigma}}\cdot A^{(1)}_{w_1,w_2}+\frac{Cq^{c_\sigma(w_2)}}{q^{c_\sigma(w_2)}-qtq^{c_\sigma(1)}}\cdot A^{(2)}_{w_2,w_1},$$

where $A^{(1)}_{w_1,w_2}$ is the first coefficient of T_n applied to a tableau in which w_1 has label n and w_2 has label $n+1$ and $A^{(2)}_{w_2,w_1}$ is the second coefficient of T_n applied to a tableau with label n in w_2 and $n+1$ in w_1.

From the first equation, we easily see that

$$C_1=q^{\frac{1}{2}}C\cdot\frac{q^{c_\sigma(w_1)}-tq^{c_\sigma(1)}}{q^{c_\sigma(w_1)}-qtq^{c_\sigma(1)}}. \tag{2.6.1}$$

We need to check that this is compatible with the second equation. We can use $A^{(2)}_{w_2,w_1}=q^{\frac{1}{2}}-A^{(1)}_{w_1,w_2}$ to simplify the equation to

$$A^{(1)}_{w_1,w_2}\left(\frac{Cq^{c_\sigma(w_1)}}{q^{c_\sigma(w_1)}-qtq^{c_\sigma(1)}}-\frac{Cq^{c_\sigma(w_2)}}{q^{c_\sigma(w_2)}-qtq^{c_\sigma(1)}}\right)+q^{\frac{1}{2}}\frac{Cq^{c_\sigma(w_2)}}{q^{c_\sigma(w_2)}-qtq^{c_\sigma(1)}}$$
$$=\frac{C_1q^{c_\sigma(w_2)}}{q^{c_\sigma(w_2)}-qtq^{c_\sigma(1)}}.$$

The coefficient $A^{(1)}_{w_1,w_2}$ is equal to $\frac{hq^{c_\sigma(w_2)}}{q^{c_\sigma(w_2)}-q^{c_\sigma(w_1)}}$, so by inserting this expression and computing common denominator, we get that the left hand side is

$$\frac{hq^{c_\sigma(w_2)}qtq^{c_\sigma(1)}C+q^{\frac{1}{2}}q^{c_\sigma(w_2)}C(q^{c_\sigma(w_1)}-qtq^{c_\sigma(1)})}{(q^{c_\sigma(w_1)}-qtq^{c_\sigma(1)})(q^{c_\sigma(w_2)}-qtq^{c_\sigma(1)})}.$$

As the numerator is equal to

$$Cq^{\frac{1}{2}}q^{c_\sigma(w_2)}(q^{c_\sigma(w_1)}-tq^{c_\sigma(1)}),$$

we see that by replacing C_1 on the left hand side using (2.6.1) we obtain an identity.

We now need to take care of the remaining equations. We observe that (1) and (4) are immediate to verify. We now consider (2). Let us fix some notation. Here the box labelled n in σ will be denoted by v; the coefficients of φ^{-1} when applied to $\frac{1}{K(\sigma)}d_-\sigma$ will be denoted by x'_i and x_i will be the coefficients for $\varphi^{-1}\sigma$. Once we unravel relation (2), we get the following equations:

$$A^{(2)}_{v,w_i}x_iK(s_{n-1}\tau_i)=K(\sigma)x'_i,$$

when i is not the row of v, and

$$K(\sigma)x'_j = \sum x_i K(\tau_i)(A^{(1)}_{v,w_i} - h),$$

when j is the row of v.

We now want to show that these equations follow from 2.5.2 and 2.5.1, respectively. The crucial observation we have to make is that we can easily compare $K(\tau_i)$ with $K(\sigma + w_i)$. Note that the coefficient K only depends of the unlabelled boxes of the diagram and, in this case, the set of unlabelled boxes are almost the same for τ_i and $\sigma + w_i$. They differ precisely by the fact that τ_i has one more unlabelled box located in the row where we deleted the box labelled 1 and put it into the shape of λ. This box has content $tq^{c_\sigma(1)}$. So we have

$$K(\tau_i) = K(\sigma + w_i) \cdot \frac{q^{c_\sigma(w_i)} - qtq^{c_\sigma(1)}}{q^{c_\sigma(w_i)} - tq^{c_\sigma(1)}}. \tag{2.6.2}$$

The first equation can now be easily seen to be equivalent to 2.5.2. The second equation needs some work. We begin by using 2.6.1, 2.6.2 and the fact that the content of the bracket is $\frac{q^{c_\sigma(v)}h}{q^{c_\sigma(w_i)} - q^{c_\sigma(v)}}$ to simplify the equation to

$$q^{-\frac{1}{2}}K(\sigma) = \sum A^{(1)}_{v,w_i} \cdot \frac{q^{c_\sigma(v)} - tq^{c_\sigma(1)}}{q^{c_\sigma(w_i)} - tq^{c_\sigma(1)}} K(\sigma + w_i).$$

We now see that

$$\frac{q^{c_\sigma(v)} - tq^{c_\sigma(1)}}{q^{c_\sigma(w_i)} - tq^{c_\sigma(1)}} = 1 + \frac{q^{c_\sigma(v)} - q^{c_\sigma(w_i)}}{q^{c_\sigma(w_i)} - tq^{c_\sigma(1)}}.$$

Therefore, it's now enough to show that

$$\sum A^{(1)}_{v,w_i} \cdot \frac{q^{c_\sigma(v)} - q^{c_\sigma(w_i)}}{q^{c_\sigma(w_i)} - tq^{c_\sigma(1)}} K(\sigma + w_i) = -h.$$

We can show this is true by replacing again $K(\sigma + w_i)$ according to equation 2.6.2 and using

$$\sum \frac{q^{c_\sigma(w_i)}}{q^{c_\sigma(w_i)} - qtq^{c_\sigma(1)}} K(\tau_i) = 1,$$

which can be verified by writing

$$\sum \frac{q^{c_\sigma(w_i)}C}{q^{c_\sigma(w_i)} - qtq^{c_\sigma(1)}} K(\tau_i) = \sum x_i K(\tau_i) = C$$

and simplifying C.

We now have to take care of relation (5). We have to stop a moment and make a crucial remark here. We have understood what happens to the constant of a given tableau when we apply $d_\pm$ to it. However, we have not figured out yet what changes as we vary the overall shape of the tableau, i.e. if we apply φ^{-1} to it. This will come from the relation (5). Let

$$\tilde{d}_- := d_-\varphi^{-1}.$$

We see that relation (5) is equivalent to

$$\varphi^{-1}\tilde{d}_- = \tilde{d}_-\varphi^{-1}T_1^{-1}.$$

The above relation will indeed give us a way to determine how constants for different shapes relate to each other.

The discussion that will now follow is very similar to that in the proof of Theorem 3.25 in [GGS23].

We want to introduce some notation to make the expressions that will follow more readable. We set $a = q^{c_\sigma(1)}, b = q^{c_\sigma(2)}$, $\tilde{d}_-\sigma = \tilde{\sigma}$ and, by abuse of notation, w_i will be used in place of $q^{c_\sigma(w_i)}$. Note that $\tilde{d}_-$ has the effect of deleting the box labelled 1 in σ and putting it in the shape of λ, while all the labels of σ are shifted by -1. Assume for now that 1 and 2 are not adjacent in σ. By applying the relation above to a tableau σ, we obtain the following relation for each i:

$$\begin{aligned} C(\sigma)C(\tilde{\sigma}) \cdot \frac{b-qta}{b-ta}\frac{w_i}{w_i-qtb} = \\ C(\sigma)C(\tilde{\sigma}) \cdot \frac{(q-1)a}{b-a} \cdot \frac{(b-qta)(w_i-tb)w_i}{(b-ta)(w-qtb)(w-qta)} + \\ +C(s_1\sigma)C(\tilde{s_1\sigma}) \cdot \frac{qb-a}{b-a} \cdot \frac{(a-qtb)(w-ta)(w)}{(a-tb)(w-qta)(w-qtb)}. \end{aligned}$$

This gives us

$$C(\sigma)C(\tilde{\sigma}) \cdot \frac{(b-qa)(b-qta)}{(b-ta)} = C(s_1\sigma)C(\tilde{s_1\sigma}) \cdot \frac{(qb-a)(a-qtb)}{(a-tb)}. \qquad (2.6.3)$$

So, to completely determine the action of φ^{-1} we are looking for a function that satisfies this condition. This condition is completely analogous to that in [GGS23] for the so-called *edge function*, where the authors also show that such a function exists for horizontal strips like ours.

Lemma 2.6.4 (Theorem 3.26 [GGS23])**.** *There exists an edge function $C(\sigma) \in \mathbb{C}(q,t)^*$ satisfying condition 2.6.3. Moreover, any choice of such function leads to the same representation up to isomorphism.*

The case in which 1 and 2 are adjacent simply gives us an identity, which we can easily verify by observing that $b = qa$ in this case. Indeed the second term on the right disappears and the first summand has coefficient

$$(q^{\frac{1}{2}} - h) \cdot q^{\frac{1}{2}} \cdot \frac{(q-1)a}{b-a} \cdot \frac{(b-qta)(w_i-tb)w_i}{(b-ta)(w-qtb)(w-qta)},$$

which equals the one on the left hand side once we replace $w_i - tb$ by $w_i - qta$ and simplify.

We proved the following result.

Theorem 2.6.5. *The algebra $\mathbb{A}_{q,q^k}$ of the torus acts on the space $\oplus_\lambda \oplus_{n=0}^N$ $\mathrm{Hom}([\lambda] \otimes \det^k, [\lambda] \otimes \mathrm{Sym}(V)^{N-n} \otimes V^n)$.*

2.7 Conclusion and future projects

We want to begin this section by remarking that we see that our conventions have been somewhat incoherent throughout this chapter when we inverted our diagrams in beginning of the previous section. The reason for this is that we tried to maintain the notation we used for the disc, which was very useful for that case. Indeed, our ordering for the tensor product allowed us to easily count the dimension of the endomorphism space and to define the action of $d_\pm$ without using shifts on the labels. However, we found it difficult to define the action of φ^{-1} on our "inverted" diagrams. We decided to switch our conventions at this point, hoping that this would not be confusing, and to maintain the conventions for the annulus to keep the passage from disc to annulus clearer.

We want to conclude this chapter with a few ideas on how our work on $\mathbb{A}_{q,t}$ could be extended further. It would be interesting to continue the parallel with the DAHA and see if it is possible to construct a faithful representation for $\mathbb{A}_q$ and $\mathbb{A}_{q,t}$ using representations from intertwiners. Let us recall that it is not known whether their polynomial representation is faithful. This was a key ingredient in proving that the representation $\mathcal{W}$ in Section 1.3.6 is faithful. However, we do believe that by taking the direct sum of our representations from intertwiners, just like we did for the DAHA, one can get a faithful representation. Moreover, let us mention that the PBW theorem for the DAHA is also based on the faithfulness of the polynomial representation. For $\mathbb{A}_q$ and $\mathbb{A}_{q,t}$ there is no PBW theorem available and we hope that our work can give a hint of what a basis should look like (see Conjecture 2.5.7). For $\mathbb{A}_q(D)$ the proof turned out to be a lot more accessible as we could count dimensions, but this is clearly no longer the case when we consider the annulus and the torus case.
Furthermore, we would like to have an equivalent to Corollary 1.5.2 (i.e. the proof of Conjeture 1.6 in [MS21]). Even if our definition of $\mathbb{A}_{q,t}$ of the torus is based on the skein theoretic construction, we cannot state an equivalent to that result. We remark that, also here, our proof was based on a faithfulness result, which at present we are still lacking on the $\mathbb{A}_{q,t}$ side. In a nutshell, there are a few representation theoretic results for the DAHA that are still missing a counterpart for $\mathbb{A}_q$ and $\mathbb{A}_{q,t}$, however, we hope that our work can be somewhat useful to close this gap.
We have one last point that we would like to bring up in this final section. As we have briefly mentioned when introducing $\mathbb{A}_{q,t}$, authors were usually only considering the polynomial representation of $\mathbb{A}_{q,t}$ until the appearance of [GGS23], which studies calibrated representations of an extension of $\mathbb{B}_{q,t}$. These representations are calibrated in the sense that a set of variables (and some extra operators) act semisimply. In our representations of $\mathbb{A}_{q,t}$ a set of variables also acts semisimply, therefore it would be interesting to compare the constructions and see if there is a way to pass from one to the other. As we pointed out at the end of the last section, our computations are indeed very similar, even if we are looking at slightly different algebras.

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