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The categorical notion of structure

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Abstract

This paper is about category theory and structuralism. According to structuralists, mathematical knowledge is knowledge of abstract structures which may be instantiated by abstract or physical objects rather than knowledge of a special class of ”mathematical” objects. Category theory is a branch of mathematics which reasons about abstract forms and relationships and categorical structuralists are those who think that category theory is an ideal framework in which to formalize the ”structures” of structuralism. Some structuralists reject categorical structuralism because of its apparent lack of assertions about existence and truth and its dependence on further foundations. In my view, categorical structuralism implies a new concept of structure that is different from its critics have in mind and this results in misunderstandings. In the following I talk about the concept of structure is implied by category theory and how it has been misunderstood by certain critics. Towards the end I reflect on differing positions among categorical structuralists.

Zussamenfassung

Dieser Aufsatz handelt sich um die Kategorienlehre und den Strukturalismus. Dem Strukturalismus zufolge, besteht mathematische Erkenntnis nicht in dem Wissen von einer bestimmten Klasse von „mathematischen“ Objekten, sondern in dem Wissen von abstrakten Strukturen, die sich in vielfachen Instanzen instanziieren lassen. Die Kategorienlehre ist besonders geeignet für die Formalisierung von abstrakten mathematischen Formen und auf diesem Grund glauben einige Strukturalisten, dass die Kategorienlehre ein gutes Framework für die Formalisierung von den „Strukturen“ des Strukturalismus ist. Einigie Strukturalisten lehnen den kategorischen Strukturalismus ab, weil (ihrer Meinung nach) kategorische Definitionen nichts über Existenz oder Wahrheit aussagen und die Kategorienlehre von weiteren Grundlagen abhängt. Meiner Meinung nach, handelt sich die Kategorienlehre und um einen neuen Begriff von Struktur, der sich von dem Begriff von ihren Kritikern unterscheidet. Ich glaube, die Unterschiede zwischen dem kategorischen Strukturbegriff und dem Strukturbegriff der Kritiker ist eine Quelle von Missverständnissen. Am Ende werde ich über den Konzept von Struktur sprechen, der von der Kategorienlehre und von dem kategoriuschen Struturalismus impliziert ist und wie und warum einige Kritiker diesen Konzept und den kategorischen Strukturalismus folglich missverstanden haben. Am Ende diskutiere ich über wichtige Unterschieden zwischen kategorischen Strukturalisten.

Contents

The merits of category theory as a framework for a structuralist interpretation of mathematics have been debated for some decades now and controversy continues to this day. My goals of this paper are, firstly, to clarify what categorical structuralism is all about and explain how it relates to and differs from other structuralist positions in the philosophy of mathematics. One will find that the literature on categorical structuralism contains more debate about whether category theory is or is not a suitable foundation for mathematics than explanation of what categorical structuralism means as a philosophical position, with the result that this debate takes precedence over the task of clarifying and elaborating on what categorical structuralism means in the first place. My first goal is to try to explain what categorical structuralism is about and what general general assertions are contained therein, and how categorical structuralism differs from other structuralist positions. I will attempt to answer some questions, namely:

- What are the key positions and features of categorical structuralism that we can piece together from various texts?
- How does category theory help us conceptualize and talk about structures and structural properties on a philosophical level?
- Does categorical structuralism represent an evolution and innovation of the notions of structure and structural property or does it merely offer a different framework for conceptualizing the same old notions?

I think that gaining clarity about the concept of structure implied by categorical structuralism is valuable because categorical structuralism seems to come with a new and different concept of structure than what we see among other forms of mathematical structuralism. I believe that some critics of categorical structuralism misunderstand categorical notions of structure or wrongly evaluate categorical structuralism in light of their own preconceptions of structure, which are fundamentally different. Such arguments, I believe, fall short because they misconstrue the position they are aimed at. Interestingly, we will see that categorists' responses to their critics reveal some important differences and disagreements among categorical structuralists themselves and reveal that categorical structuralism itself is not a singular position, but has variations that reflect prior philosphical convictions and conceptions of categorical structure. I will end with an investigation of these disagreements and some reflection on their nature.

Chapter 1

A sketch of structuralism

Before I begin my discussion of category theory I will give an outline of the structuralist premise in the philosophy of mathematics. The key thing I want to emphasize is that there is no universal consensus on that the term "structure" actually means. Rather, different versions of structuralism, including categorical structuralism, operate with different concepts of structure. Instead of presenting a widely agreed upon thesis and set of concepts, what we see in structuralist philosophy of mathematics are differing explorations of the very general idea that mathematics studies "structures" rather than "objects", and that "structures" are something qualitatively different from "objects". For instance, according to structuralist accounts, when we make assertions about numbers in arithmetic and number theory we are not making claims about some specific mathematical objects that we identify with particular numbers, but rather we are making claims about the pattern of the natural numbers which is instantiated in any collection of objects with the right mathematical properties. What is means that these and other mathematical assertions are to be taken as assertions about structures depends on how one views the ontology and epistemology of structures. Since we are interested in what categorical structuralism has to say and how it differs from other structuralisms, we need to first survey the landscape of mathematical structuralist positions and their classifications in order to see where it fits in. What I want to do in this section is examine some important interpretations of the general thesis of structuralism. Over the course of this paper I will discuss how category-theoretic structuralism differs from or relates to these interpretations.

1.1 Varieties of structuralism

To begin this discussion, I should emphasize that structuralism in the philosophy of mathematics consists in a variety of related, but often competing viewpoints. Reck and Price (2000) emphasize that structuralism is more of a general thesis or way of thinking rather than a unified position and describe structuralism being united mostly by cetain "intuitive theses" and "guiding ideas", which are namely

> (1) that mathematics is primarily concerned with "the investigation of structures" [quoting Hellman 1989, vii]; (2) that this involves an "abstraction from the nature of individual objects"; or even, (3) that mathematical objects "have no more to them than can be expressed in terms of the basic relations of the structure" [quoting Parsons 1990, p.303] (p.341).

Theses slogans are a decent starting point for understanding structuralism and its variants. In regard to (1), we might ask about the ontology of structures: if mathematics is about structures, then what are structures? The second thesis says something about the epistemology of structures, namely that knowledge of structures involves abstraction. There are different notions of abstraction used by structuralists and abstraction in structuralism is a heavily debated topic. The third thesis is that all we know about mathematical objects is their mathematical and thus structural features. One could think of a structure as a one of a kind entity with its own nature or a pattern that emerges and exists in and though different mediums. On the other hand, one could think of structures as creative abstractions without any existence of their own. Yet again, one could think of structures as something in between.

Structuralism has a purely mathematical form as well, having to do with developments in modern mathematics toward greater generality and abstractness, both in subject matter and method. For example, the Bourbaki movement is associated with the rise of a modern and structural approach to mathematics, and may be said to have the first formal account of what a structure is. Bourbaki aimed at characterizing the objects that mathematicians had already been working with on to a higher degree of rigor, abstractness, and generality. A "structure" is meant to be something implicitly defined by an axiomatic theory, namely a set with a structural organization:

> It can now be made clear what is to be understood, in general, by a mathematical structure. The common character of the different concepts designated by this generic name, is that they can be applied to sets of elements whose nature has not been specified; to define a structure, one takes as given one or several relations, into which these elements enter, then one postulates that the given relation, or relations, satisfy certain conditions which postulates that the given relation, or relations, satisfy certain conditions (which are explicitly stated and which are the axioms of the structure under consideration). To set up the axiomatic theory of a given structure amounts to the deduction of the logical consequences of the axioms of the structure, excluding every other hypothesis on the elements under consideration

> (in particular, every hypothesis as to their own nature). (pp. 225-226).

According to Bourbaki there are three classifications of so-called ”mother structures”: algebraic (e.g groups, fields, rings), order (e.g. partial or well order), and topological (p.227).

Whereas groups and rings once were primarily investigated in concrete contexts (e.g. groups of transformations of geometrical objects), in modern mathematics groups and rings and other kinds of structures are understood axiomatically. Thus our conceptualization of groups (and other structures) evolved from their initial appearance as concrete groups and developed into a theory of groups in general (i.e. without assumptions about what a group is a group of). The Bourbaki notion of structure could be thought of as a distillation of this tendency toward investigating general form in abstraction from concrete examples. A Bourbaki structure was a certain set satisfying certain conditions. While the Bourbaki movement was an important step in a structuralist direction, it was still a viewpoint that viewed mathematical entities as objects (i.e. set-theoretic models), even if these objects were viewed more abstractly than before. As Bell (1981) remarks:

> With the rise of abstract algebra... the attitude gradually emerged that the crucial characteristic of mathematical structure is not their internal constitution as set-theoretical entities but rather the relationship among them as embodied in the network of morphisms... However, although the account of mathematics they [Bourbaki] gave in their Elments was manifestly structuralist in intention, actually they still defined structures as sets of a certain kind, thereby failing to make them truly independent of their ’internal constitution’ (p.351).

A potential problem with the Bourbaki conception of structure, as we will see below, is that structured sets come with non-structural features. Many structured sets satify the axioms of a theory (e.g. of natural numbers), but they will not share all of the same properties. There will not be the set of natural numbers. Something else is needed to capture the concept of structure.

1.1.0.1 What are numbers?

Benacerraf's "What numbers could not be" (1965) raised questions that continue to frame debates about mathematical structuralism. Benacerraf took the premise that the natural numbers are a specific abstract object, a set, and shows how it leads to an impasse. If we want to identify the natural numbers with a set, we have to choose from one out of infinite possibilities. One model of the natural numbers is the set of Zermelo ordinals, where 0 is identified with the empty set and for any number n, $n+1 = \{n\}$:

$0 = \emptyset$

$1 = \{\emptyset\}$

$2 = \{\{\emptyset\}\}$

$3 = \{\{\{\emptyset\}\}\}$

...

Another set theoretic representation is the von Neumann ordinals, where we let $0 = \emptyset$ and recursively define the rest, so that $n+1 = n \cup \{n\}$:

$0 = \emptyset$

$1 = \{\emptyset\}$

$2 = \{\emptyset, \{\emptyset\}\}$

$3 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

...

If we want to identify numbers with sets then there is an obvious problem: is 3 $\{\{\{\emptyset\}\}\}$ or $\{\emptyset, \{\emptyset\}, \{\emptyset\}, \{\emptyset\}\}\}$? If multiple representations of numbers are possible, there is no justification for identifying numbers with these sets and not those sets. There are infinite models of Peano arithmetic and these can have properties that are not shared in common. The question arises, then: which one? This is the essence of Benacerraf's identification problem. The first step out of this impasse, according to Benacerraf, is to stop seeing mathematical entities as particular abstract objects, and instead see them as patterns which many objects can have:

> Arithmetic is therefore the science that elaborates the abstract structure that all progressions have in common merely in virtue of being progressions. It is not a science concerned with particular objects-the numbers. The search for which independently identifiable particular objects the numbers really are (sets? Julius Caesars? Quark stars?) is a misguided one. (p.70)

The natural numbers are no longer a set of elements ordered in a certain way, but something more abstract: the natural numbers are the system of the features that all discrete infinite systems share and none of the features that are not.

As is the case with the natural numbers, there are different ways that the real numbers can be constructed, for instance, through Dedekind cuts. Dedekind cuts allow us to define real numbers as partitions of $\mathbb{Q}$ into two sets (X,Y) where all elements of X are less than all elements of Y and X as no greatest element. For example, we can represent the square root of two as a Dedekind cut (A, B) where $A = \{a \in \mathbb{Q} : a^2 \vee a < 0\}$ and $B = \{b \in \mathbb{Q} : b^2 \geq 2 \wedge b\}$. Alternatively, we could construct the reals from the rationals as convergent Cauchy sequences. Choosing one approach over the other produces different kinds of objects. Thus we end up in the same kind of situation as Benacerraf pointed out above.

1.1.0.2 If numbers are a structure, what is this structure?

Is a structure itself a new kind of object apart from the objects that instantiate it, or something else entirely? Parsons (1990) distinguishes between eliminative and non-eliminative structuralism. On the one hand, non-eliminativists believe that there is a single, unique referent to the term "natural number structure". Whenever we study a structure our statements are about a unique abstract structure. Accordingly, the only mathematically relevant features of a mathematical object, for instance a natural number in a natural number sequence, is its structural properties which are shared by the corresponding position in other natural number sequences. Eliminative structuralists adopt a nominalist point of view and see structures as mere names for classes of systems with shared properties. From the eliminativist point of view, all this talk of structures is nothing more than a figure of speech, just a way of generalizing about systems with the same relevant features.

Eliminative structuralism, then, is the view that mathematical structures are shorthand ways of speaking about classes of systems sharing certain features. This position faces a problem.Since structures are ways of talking about classes of systems, it is assumed that those systems exist, and that there are enough of them. In other words, there is an ontological assumption here. But what if the systems and objects which form the stuff of such generalizations are not given? In this case mathematical statements, as understood according to eliminativism, turn out to vacuously true, since they aren't about anything at all.

Hellman (1989) sought to secure mathematical strucuturalism from the danger of vacuuity. Hellman wanted to show how we can avoid this danger by detaching structuralism from any commitment to any background ontology of objects and he thought that we can achieve this through modal logic to replace an actual ontology with a possible or permissible one. An eliminative structuralist who assumes a background ontology will view a theorem ϕ about natural numbers like this:

$$\forall X(X \models PA \implies \phi(X))$$

But understanding theorems in this way leaves the eliminative structuralist open to the objection that there might not be any X. According to Hellman we should instead think of a mathematical assertion like this:

$$\Box\forall X(X \models PA \implies \phi(X))$$

In order to avoid the problem of vacuity, Hellman defines universal quantification over domains of logically possible systems and augments the above assertion with another that asserts the possibility of the relevant systems:

$$\Diamond\exists X(X \text{ is an } \omega\text{-sequence and } X \models PA).$$

Thus even if there never have been nor ever will be natural number systems, their mere possibility allegedly guarantees that arithmetic propositions and theorems are not vacuous.

1.1.0.3 If structures "exist", then in what sense do they exist?

Shapiro (1997) distinguishes between ante rem and in re structuralism, which correspond roughly to non-eliminative and eliminative positions. Shapiro, an ante rem structuralist, sees structures as unique universals that are instantiated like Plato's forms, but exist independently of their instantiations. Shapiro has formulated an axiomatic theory of ante rem structures (p.90-97).

One of the attractions of Shapiro's ante rem structuralism is that it preserves a "face value" interpretation of mathematical statements while allowing that that structures can be multiply instantiated: sometimes we speak of the individual positions in the natural number structure as though they are empty slots that can be inhabited by things like Zermelo or Von Neumann numerals or Roman or Chinese numerals (without being identified with these items) whereas other times we speak of positions as objects in their own right (Shapiro calls these the places as "offices" and places as "objects" perspectives, respectively (1997, p.10). Shapiro is currently one our main sources for the ante rem position.

According to Shapiro, in re structuralists invert the relationship between structures and their instantiations seen in ante rem structuralism. Much like Aristotle understood forms in relation to beings that have form, in re structuralists see instantiations as prior to structure, rather than the other way around. When we look at a certain model of the natural numbers, we can think of it as a sort of hylomorph, a compound of an abstract structure and a concrete system in which that structure exists. When we talk about structures we are really talking about all the systems that have those structures. Mathematical structures, in this view, are like the curved shape that bowls have in common. While we can refer the curvature and mentally abstract it from the ceramic or porcelain, we do not regard it as existing separately from the material that bears the curvature. Ante rem and in re structuralists agree that structures "exist", but they disagree about the relationship between strucure and its instantiations. For instance, both agree that something like the group structure exists, but ante rem structuralists believe that this structure is ontologically prior to instantiations, while in re structuralists believe that this structure is ontologically posterior: if all groups dissapear from reality, then so does the group structure itself. The in re structuralist thus thinks that there is a common structure that we can refer to, but this common structure is something that we abstract from instantiations.

Shapiro's distinction between ante rem and in re structuralism concerns two different approaches to non-eliminative structuralism. Both ante rem and in re structuralists are non-eliminativists. Both positions agree that when mathematicians formulate theorems about groups, they are making assertions about a universal group structure, and so each believe that there is a unique group structure that "exists" at the very minimum as a linguistic entity. In this unique universal structure the elements have no other properties than the relationships they have to other elements. Thus the natural number structure has no properties that would be specific to any particular model of natural numbers. The only properties that the 3 position has in a non-eliminative structure are the properties that the 3 position has for any and all natural number sequences. While both positions are committed to the existence of structures they disagree about the relation of ontological dependence between structures and their instantiations, with the in re structuralist taking an aristotelian approach to universals and the ante rem structuralist taking a platonic approach.

One can observe that in re and eliminative structuralism (with the affor-mentioned foundational background) are similar in some ways. For instance, in re and eliminative structuralists both reject a platonic view of structures, namely, the view of structures as separate (i.e. ante rem) entities and agree that talk of structures is meaningless without connection to systems that actually have those structures. In re and eliminative structuralists think of mathematical statements as generalizing over mathematical entities that have relevant features in common: when I assert something about prime numbers

I am asserting something about actual number systems. Moreover, both in re and eliminative structuralism face similar challenges. On the one hand, eliminative structuralism faces the problem that mathematical statements may turn out to be vacuously true if the background ontology is not given. On the other hand, in re structuralism faces the objection that if only those structures exist that are instantiated, there may be plausible (i.e. definable) structures that are not or could never be instantiated. Essentialy, both eliminative and in re structuralism rely on there existing particular examples of structures. However, there are fundamental differences between these positions. Eliminative structuralists do not think that structures literally exist, but rather, they think that classes of systems with the same relevant features exist. In re structuralists, on the other hand, think that structures do in fact exist, but only insofar as they are instantiated in some context or another.

Chapter 2

Categorical structuralism

2.1 Axioms of general category theory

Bennaceraf (1973) once distinguished between two different ways of talking about structures. In his example we may speak of the Empire State Building structure as this structure here, i.e. the building itself as a structure and the structure itself as a unique object. On the other hand, according to Benacerraf, we may refer the Empire State Building structure not as an object but as a structure that various objects may have, for example, if a wealthy client hires an architect to build an exact copy in Las Vegas or Dubai. Clearly we can see that Shapiro's ante rem structuralism prioritizes the former way of talking about structures, whereas in re and eliminative structuralism both prioritize the latter. Category theory is a framework that suits the latter way of thinking of structures well.

Rather than speaking of structure as a kind of entity like the Empire State Building, category theory provides a framework in which we can describe the structural features of objects of the same type in terms of structure preserving mappings. A category consists of:

- Objects, which are interpreted as models of a mathematical theory such as groups, data types in computer science, etc.

- Morphisms, which are interpreted as the allowed relationships between objects (i.e. the relations that preserve the type of object: functions take sets and give you other sets, homomorphisms take groups, monoids, rings, etc. and give you other groups, monoids, rings, etc.). These have a starting point (domain) and target (codomain), and are modeled by graphs in which morphisms are represented as edges and objects as their vertices. For any pair of objects A and B we call the collection[1] of morphisms from X to Y, sometimes called the homset (when it is a set) and signified $Hom(A,B)$.

- For any object A in a category C there is the identity morphism $id_A : A \to A$ with identical domain and codomain.

- Let $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ be morphisms in a category C. Then there is the composition $g \circ f$ which fuses these morphisms into one morphism: $A \xrightarrow{g \circ f} C$. The composition of f and g is represented as
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g\circ f"] & B \arrow[d, "g"] \\ & C \end{tikzcd}.$$

Morphisms, identity morphisms, and composition follow these two rules:

- (Identity) For any morphism $f : A \to B$, $id_B \circ f = f$ and $f \circ id_A = f$, i.e. $$\begin{tikzcd} A \arrow[loop left, "id_A"] \arrow[r, "f"] \arrow[r, bend left, dashed, "f\circ id_A"] \arrow[r, bend right, dashed, "id_b\circ f"'] & B \arrow[loop right, "id_B"] \end{tikzcd}$$

[1]Here "collection" does not mean anything determinate. A collection could be a set or as large as a proper class.

- Associativity: If we have morphisms $f : A \to B$, $g : B \to C$, $h : C \to D$, then $h \circ (g \circ f) = (h \circ g) \circ f$.

The abstract category defined by these axioms is a mathematical theory in broad outline: there are some kind of entities that the theory concerns, the important relationships between those entities, and rules governing composition of relationships. The terms ”category”, ”morphism”, and ”object” in the Eilenberg-Mac Lane axioms of category theory are placeholders, as the theory says what it means for anything to be a category. When we define a specific category these terms take on a definite meaning. For example, the category of groups consists in groups and group homomorphisms and the category of topological spaces consists in such spaces and continuous maps between them.

A morphism $f : A \to B$ is a relationship which preserves structure from A to B (e.g. a function with one parameter takes a set-structured object and gives you another set-structured object). In a category of topological spaces the morphisms are continuous functions and in a category of metric spaces those are metric maps and in a category of the times of day on a wall clock the morphisms are movements of the arms, etc. The composition of a morphism f and another morphism $g : B \to C$ carries over that structure: $g \circ f : A \to B \to C$ from A to C.

When we say that two objects are equivalent in a categorical setting we mean that they are isomorphic, a relationship with its own special definition involving morphisms and composition. Isomorphism captures a general concept of sameness of structure for objects seen as purely relational entities rather than concrete and particular entities. Different mathematical theories have their own concept of sameness. For instance, in ZFC set theory two sets are equivalent when there is a bijection between them. A function f is a bijection when f is injective and surjective. f is bijective if $\forall y \in Y \exists! x \in X(f(x) = y)$ so that every element of X is matched with just one element in Y and vice versa. Equivalently, if f is bijective if and only if there is an inverse function f^{-1} such that $f(X) = f^{-1}(Y)$. In projective geometry equivalence is isommetry and in topology it is homeomorphism.

What the notion of isomorphism accomplishes is to generalize the notion of A being the same as B in any theory. Isomorphism is defined for objects of categories as follows. Let $\mathbb{C}$ be a category and X and Y be objects of $\mathbb{C}$. $f : X \to Y$ is an isomorphism if there exists an inverse morphism $g : Y \to X$ such that $f \circ g = id_X$ and $g \circ f = id_Y$, as we see on the graph
$$\begin{tikzcd} X \arrow[loop left, "id_X"] \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \arrow[loop right, "id_Y"] \end{tikzcd}$$
We say that X and Y are isomorphic ($X \cong Y$) if there is an isomorphism between them. Reversibility is thus a central and definitive part of what it means for things to be the same: sameness is a two way relationship in which the entities have the same relationship to one another as to themselves.

2.2 What structure means for categorical structuralists

What both the eliminative and non-eliminative and in re and ante rem positions described above have in common, in spite of their opposing ideas about what a structure is, is a certain analytical perspective.[2] This perspective approaches structures from the perspective of the "structure" of the elements in a system, e.g. something like this:

[2]The positions I mean are those of Shapiro and Hellman, as well as other structuralist positions that share a similar basic concept of structure, although differing in regard to the application of that concept.

$$1 \xrightarrow{\leq} 2 \xrightarrow{\leq} 3 \xrightarrow{\leq} \dots$$

We can pretend that this graph is an ante rem structure, or one instance of a structured system that mathematical statements generalize about according to the in re interpretation. By contrast, category theory offers a completely different analytical perspective and a completely different concept of structure. Rather than looking at the "internal" structure of a system and its elements the "external" perspective of categorical structuralism takes systems that have a certain kind of structure and views them as elements in a more abstract system, a category:

$$\begin{tikzcd}
 & & & 1 & \\
0 \arrow[r] \arrow[loop above, "id"] \arrow[urrr, dotted] \arrow[rr, dashed, bend right] \arrow[rrr, dashed, bend right] \arrow[rrrr, dashed, bend right] & A \arrow[r, "f"] \arrow[loop above] \arrow[urr, dotted] \arrow[rr, bend right, "g\circ f"'] & B \arrow[r, "g"] \arrow[loop above] \arrow[ur, dotted] & C \arrow[r] \arrow[loop above] \arrow[u, dotted] & \dots \arrow[ul, dotted]
\end{tikzcd}$$

Thus, unlike in re or eliminative structuralism who characterize shared structure from an internal perspective, categorical structuralism characterizes a structure by the way that objects possessing that structure interact with other objects within a context (i.e. the category). This seems to me to be a significantly different way of think about structure.

2.2.1 Structure defined from the top down

According to Awodey (2004), category theory is a framework in which the abstract structure shared by kinds of concrete structures is defined from the top down and mathematical statements are understood as schematic expressions that have a conditional form.

A top down definition of a structure is one in which we specify only the minimum required information, which are, on a general level, the conditions that anything needs to satisfy to count as a group or natural number sequence or whatever. Mathematical axioms and theorems are those statements which apply primarily at the ”top” level where objects are characterized generally and apply downstream (i.e. ”down”) to particular objects that have that structure. For instance, A theorem about groups is a schematic statement about anything at all satisfying group theory axioms which be thought of a setting down a rule that governs particular examples and particular kinds of groups. What sets the top down approach of category theory apart from the set theoretic approach is that set theory sets out to build up a universe out of objects, i.e. sets, and interprets theorems as quantified assertions about them. Another way we can think of the top down conception of mathematics is that mathematical statements become more complex in the information they express when they are applied to a certain domain of objects and thus take on additional structure as compared to when they are stated at the top level.

Awodey regards mathematical theorems as having the general form $P(X) \implies Q(X)$. For example, if X is a group or ring or other object fitting certain specifications, then X has such and such properties. The property P in the antecedent serves to specify the context (satisfaction of the group-theoretic axioms) in which the statement of the antecedent ($Q(X)$) applies. At the top level no information about X is specified or assumed other than that it satisfies the antecedent condition. Thus, in a top down interpretation of a theorem we do not, as we do according to the ”bottom up” interpretation, quantify over any specific domain of objects. Whereas in the bottom up interpretation the antecedent and condition are truth-functional and range over an intended domain, in the top down interpretation there is no intended domain and no quantification, since the ”if” part of the expression applies to literally anything at all satisfying the condition. Thus the relationship between the "if" and "then" parts is perhaps better understood as one in which certain things can be deduced about something on the condition that that that something (whatever it is) meets certain requirements. The important thing here is that nothing needs to be known about the "something" other than it meeting those requirements!

Awodey's top down categorical structuralism has something in common with in re and eliminative structuralism, namely that categorical structures are instantiated by many mathematical systems or can be thought of as generalizations of mathematical systems. What makes categorical structure apart from other conceptualizations of structure is the top down part: whereas in re structuralism must make certain assumptions about an underlying ontology that constitutes the substance of mathematical assertions, the specification of a categorical structure makes no such assumptions, but rather outlines what it is to be an X in a certain context. The top down approach of categorical structuralism means that structures can be defined at an extremely level of generality where the emphasis is on the definition and the defining conditions and very few or no assumptions are made about the entities satisfying those conditions.

As an example of a structure defined from the top down, consider a product $A \times B$ for objects A and B of category $\mathbb{C}$. $A \times B$ is a type of object with projections π_A snd π_B obeying the following universal structural property. For any contender object S there is a unique morphism $h : S \to A \times B$ such that that $g = \pi_B \circ h$ and $f = \pi_B \circ h$, as visualized in this commutative diagram:

$$
\begin{tikzcd}
 & S \arrow[dl, "g"'] \arrow[d, dashed, "h"] \arrow[dr, "f"] & \\
A & A \times B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B
\end{tikzcd}
$$

Thus we have defined a product, the universal product for this context, such that any other entity that satisfies these rules is essentially identical to it (i.e. there is a unique isomorphism between both). A coproduct is the dual concept to the product. We form a coproduct by taking a product and reversing all of its arrows. For the same objects A and B, the coproduct is an object $A+B$ with projections π_A and π_B coming into it from A and B such that if there is a rival object S with projections f and g then there is a unique morphism h such that $g = h \circ \pi_A$ and $f = h \circ \pi_B$.

$$
\begin{tikzcd}
 & S & \\
A \arrow[ur, "g"] \arrow[r, "\pi_A"'] & A + B \arrow[u, dashed, "h"] & B \arrow[ul, "f"'] \arrow[l, "\pi_B"]
\end{tikzcd}
$$

Similarly, a group object for some category C with products and distinct terminal and initial objects is any object G with mappings for multiplication $m : G \times G \to G$, identity element $e : 1 \to G$, and inverses $\star^{-1} : G \to G$. Here we see e as unit for multiplication and elements being sent to their inverse, respectively:

$$
\begin{tikzcd}
G \arrow[r, "{(e,1_G)}"] \arrow[dr, "1_G"] & G \times G \arrow[d, "m"] \\
 & G
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "{(1_G,\star^{-1})}"] \arrow[dr, "e\circ!_G"'] & G \times G \arrow[d, "m"] & G \arrow[l, "{(\star^{-1},1_G)}"'] \arrow[dl, "e\circ!_G"] \\
 & G &
\end{tikzcd}
$$

And we see the associativity of multiplication:

$$
\begin{tikzcd}
G \times (G \times G) \arrow[r, "1_G \times m"] \arrow[d] & G \times G \arrow[dd, "m"] \\
(G \times G) \times G \arrow[d, "m \times 1_G"'] & \\
G \times G \arrow[r, "m"] & {}
\end{tikzcd}
$$

This definition of a group object defines the group structure at the highest level of generality possible by setting down a rule that things we call ”groups” have to obey in any mathematical framework. It is exactly this ability to define structures at the highest level of abstractness and generality that, according to Awodey, makes category theory so appealing as a framework for structuralism.

Now we move onto the next term, namely schematic. What does Awodey mean when he says that mathematical statements are schematic? In general, a schematic statement in mathematics is not a single proposition, but the general form of a proposition with many instances. For instance, for certain logical reasons, in set theory, the axiom dealing with replacement is not a specific axiom making a singular assertion, but an axiom schema which has an instance for any definable function f with a possessing a set for a domain and this instance says that the codomain of f is also a set. Similarly, as we saw above in regard to the top down perspective, the ”if” part of a general mathematical statement is a ”schematic statement about a structure... which can have various instances” (p.4). We can think of the antecedent as a mathematical blank check, where the part that is left blank is the context or domain of application.

Awodey is careful to distinguish this view that theorems are conditional from another interpretation of mathematics called if-thenism, which is the view that mathematical theorems are hypothetically true and consist in logical deductions from a hypothetical premise. According to this view, saying that a certain number is a prime is essentially saying that in the event that there are systems satisfying Peano axioms, then a certain position in any of them is prime. Adopting this point of view means, of course, that mathematics is not, strictly speaking, true, but hypothetically true. Hoever, in contrast to if-thenism, Awodey does not hold the ”if” part to consist in a condition that might or might not be true.

Awodey sees this top down, schematic, and conditional interpretation of mathematical statements as capturing the essence of mathematical practice:

> This lack of specificity or determination is not an accidental feature of mathematics, to be described as universal quantification over all particular instances in a specific foundational system as the foundationalist would have it. . . rather it is characteristic of mathematical statements that the particular nature of the entities involved plays no role, but rather their relations, operations, etc.–the structures that they bear–are related, connected, and described in the statements and proofs of theorems.
>
> The ‘schematic’ element in mathematical theorems, definitions, and even proofs is not captured by treating the indeterminate objects involved as universally quantified variables, as quantification requires a fixed domain over which the range of the variable is restricted (p.7).

The opposite of the top down, namely the ”bottom up” and ”foundationalist” perspective on structures, as Awodey calls it, consists in isolating a certain kind of entity and interpreting mathematics (including structural interpretations of mathematics) in terms of that entity, much like we often think of statements about numbers as ultimately statements about sets. The key thing here is that the entity that is chosen is not a structural entity and the theory governing it is not a structural theory, but rather these make up the medium and substance in which structures emerge. As Hellman (2005) explains: Hellman (2005) has likewise said of set theory:

> Here we encounter a massive exception to the structuralist point of view, in that, on its face-value interpretation, set theory itself is not treated structurally: its axioms are not understood as defining conditions on structures of interest but are taken as assertions of truths in an absolute sense (p.540).

Similarly, Mayberry (2000) argues that since axiomatic mathematical structures depend on and are built up from set theory, it follows that the same axiomatic approach to mathematical structures elsewhere cannot be given to set theory itself:

> . . . the logical dependence of axiomatics on the set-theoretical concept of mathematical structure requires that set theory already be in place before an account of the axiomatic method, understood in the modern sense of axiomatic definition, can be given. It follows necessarily, therefore, that we cannot use the modern axiomatic method to establish the theory of sets (p.7).

If sets are building blocks for structure, they are matter and substrate, not form. One of the main tenets of categorical structuralism is that structures can be specified without specifying building blocks:

> As opposed to this one-universe, ”global foundational” view, the ”categorical structural” one we advocate is based on the idea of specifying, for a given theorem or theory only the required or relevant degree of information or structure, the essential features of a given situation, for the purpose at hand, without assuming some ultimate knowledge, specification, or determination of the ”objects” involved (2003, p.3).

What is a car? Someone might argue that a car is identical to its material components. I think it is more reasonable to define a car according to its use: a car is anything organized to function a specific set of ways. This definition does not say what a car is supposed to be made of, but there are implied restrictions. A car can be made of many materials, but not of paper or cookie dough. Top down definitions function in a similar manner: we can specify a general kind of structured thing without presupposing what it is made of other than that it is suitable. For example, Awodey (1996) pointed to the definition of a product object described above for any category as a perfect example of categorical generality: "The categorical definition of a product... is one of the first examples of category theory being used to give a purely structural characterization of an important basic mathematical notion" (p. 220). In this way the general structure of a product is defined independently of any reference the particular things that might make up a product:

> ...one and the same categorical definition describes also products of topological spaces, groups, vector bundles on a smooth manifold, or whatever. The definition... provides a uniform, structural characterization of a product of two objects in terms of their relations to other objects and morphisms [arrows] in a category, in contrast to 'material' set-theoretic definitions which depend on specific and often irrelevant features of the objects involved, introducing unwanted additional structure (p.220).

A definition like that of the categorical product defines a general structure that can emerge in any context in which it is possible.

As such definitions show, category theory is entirely agnostic in regard to the nature of structures themselves. Rather than attempting to give an account of what a structure is and what structures are made of, categorical structuralism gives an account of what it means for something to have a certain structure. This unique standpoint sets categorical structuralism apart from the interpretations of structuralism we have discussed above. At the very least, it is not exactly one to one with any of them, even though there are similarities. Like ante rem structuralism, category theory defines can structures independently of their instantiations. However, unlike ante rem structuralism, category theory based structuralism does not come with the assumption that structures are separate entities in and of themselves.

Concerning the truth of mathematical statements, however, category theory has little or nothing to say, since category theory consists in specifications that say when anything has a certain structure, rather than assertions about specific objects which must be true or false. In a similar vein, Mac Lane (2012) has spoken of mathematics as being not a matter of true or false assertions or even knowledge per se, but rather of correctness and incorrectness:

> This view means that the philosophy of Mathematics need not involve questions about epistemology or ontology. If Mathematical theorems do not assert truths about the world, we need not inquire as to how we know or would come to know such truths. (We of course do need to inquire how we recognize a correct proof, but getting the recognition is a major part of advanced education in Mathematics, and is usually not considered as part of epistemology.) This observation means that the philosophy of Mathematics cannot be much advanced by many of the books entitled ”Mathematical Knowledge”, in view of the observation that such a title usually covers a book which appears to involve little knowledge of Mathematics and much discussion of how Mathematicians can (or cannot) know the truth (p.443-444).

In a category of groups Gr if G is a group and P applies to groups then $P(G)$. However in set theory the corresponding assertion would range over a determinate domain of objects, i.e. for all G in domain D, if G is a group then $P(G)$. Category theory can leave the domain of application (the building blocks) blank unless you want it to be specified. In this sense, when we deal with structures like categorical groups or products we are not dealing with truth or falsity. True and false come in when structures are interpreted.

2.2.2 Structural properties in category theory

A central topic in structuralism is that of structural properties. From the point of view of structuralism, a the elements of a structured system have no mathematically relevant or interesting properties besides their structural ones. If category theory is a promising framework for structuralism, it should offer its own formulation of or insights into structural properties.

Structural properties are special kinds of predicates that are invariant under some concept of sameness or equality. Non-structural properties are predicates that can vary among equivalent objects. In short, structural properties are those properties which same structured objects always share. The structural properties of a theory are determined by the theory's form of isomorphism or equivalence. In the context of graph theory, for example, the relation of sameness is graph isomorphism. For graphs G and H ad graph isomorphism $f : G \to H$ is a bijection between the set of vertices for G and the set of vertices for H where vertices $x, y \in G$ are adjacent if and only if $f(x), f(y) \in H$ are adjacent. An example of a property that is invariant between isomorphic graphs is the number of vertices or edges.

A structural property should have nothing to do with the particular internal composition of structured systems and nothing to do with having or not having this or that element. Likewise, a structural property should hold for isomorphic objects. According to Mcdonald (2012, p.44-53) a structural property $F(X)$ for an object X of a category $\mathbb{C}$ is a mapping that satisfies two criteria:

1. F has no constants or free variables.

2. $F(X) \wedge X \cong X' \implies F(X')$

Constants and free variables are prohibited because they introduce non-invariant properties. Mcdonald gives an example of a property that fails (1), namely $p(X) :=$ "X has base element equal to 1", which is true of $N = (1,2,3,...)$ but false of $2N = (2,4,6,...)$. This property not structural according to McDonald because not all models of arithmetic have the same base element. A structural property of natural number systems will not have anything to do with containing some or any element. As an example of a property which fails to be structural on account of (2), the formula $v(X,m,n) :=$ "X has a base element m and its successor is n" with only n free lets us substitute values which will make v true of one system but not of another.

McDonald argues that category theory and categorical set theory have an advantage over regular set theory in regard to structuralist ideas. The advantage is that properties in category theory ignore internal composition of objects and are preserved under isomorphism unlike in set theory. Thus the broadest class of non-structural properties like those above simply cannot be formulated. Set theory does, according to McDonald, allow the formulation of properties which do not refer to specific elements such as $s(x) :=$ "all elements of x are singletons", but such properties are not structural. We see that this property is false of the Von Neumann ordinals where $0 = \emptyset$ and $n+1 = n \cup \{n\}$, but true of the Zermelo ordinals where $0 = \emptyset$ and $n+1 = \{n\}$. Category theory and categorical set theory abstract away the internal composition of objects and thus cannot formulate a class of non-structural properties that set theory can. For this reason category theory is thought by some to better capture the concept of a structural property than traditional set theory can.

Think about the features that define any set $\mathbb{N}$ with the natural number structure:

1. $\mathbb{N}$ has a base element $0 \in \mathbb{N}$.

2. $\mathbb{N}$ has a simple recursive structure given by the successor function $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\forall n \in \mathbb{N}[\sigma(n) = n + 1]$.

In a category of sets we can define the natural numbers as an object satisfying a structural property. The following theory is a function-based set theory known as the elementary theory of the category of sets (ETCS) based off of the accounts of Lawvere (2003), McLarty (1993), and also Linnebo and Pettigrew (2011, p.233-44) which describes a category Set whose objects are sets and morphisms are functions. Within this category we can define what it is to be the natural numbers (in accordance with the simple features above) in a set-theoretic context. One of the main arguments for categorical structuralism is that a categorical definition of the natural structure seems to avoid the problems that Benacerraf has brought up. We could think of categorical structuralism as presenting an answer to the challenge that Benaceraff's work presents us: since identifying numbers with sets results in impasse, how do we make sense of the natural numbers as a structure? We define a something that behaves in the ways we expect it to for the relevant contexts.

2.2.3 A category of sets

Let Set be category in which the objects are sets and morphisms are functions.

2.2.3.1 Initial and terminal objects (empty set and singleton)

Set has initial and terminal objects, which are defined as follows. Let $\mathbb{C}$ be a category. An initial object in $\mathbb{C}$ is any object 0 such that for any A there is a unique morphism $0 \to A$. A terminal object in $\mathbb{C}$ is any object 1 such that for any object A there is a unique morphism $A \to 1$. It can be proven that the terminal and initial objects of a category (where given) are uniquely defined up to isomorphism. Let both 1 and $1'$ be terminal objects of $\mathbb{C}$. By the definition of terminal objects, there must then be morphisms $f : 1 \to 1'$ and $g : 1' \to 1$, since each are terminal. But then $f \circ g = id_{1'}$ and $g \circ f = id_1$. Therefore 1 and $1'$ are isomorphic. Similarly, if 0 and 0' are initial objects in $\mathbb{C}$ then there are morphisms $p : 0 \to 0'$ and $q : 0' \to 0$ and thus $q \circ p = id_0$ and $p \circ q = 0'$ and thus 0 and 0' are isomorphic. Wherever they exist, terminal and initial objects are defined up to isomorphism. In the category of sets the initial and terminal objects are distinct from one another (i.e. not isomorphic). In the category of sets the initial and terminal objects are the empty set $\emptyset$ and singleton set $\{x\}$ for for any set x. The empty set and singleton set are essentially instances of initial and terminal objects, which are general structures that have many other instances in mathematics.

2.2.3.2 Elements

In ZFC and other membership-based set theories a set is defined by the composition of its elements and elementhood, or '∈' is global in scope, meaning that for any two sets a and b, $a \in b$ must be either true or false. The fact that elementhood is defined globally in these theories is a source of troublesome non-structural properties of sets. For example, if we define 2 and 3 as $\{\{\emptyset\}\}$ and $\{\{\{\emptyset\}\}\}$ then it is true that $2 \in 3$, which is a nonsensical statement from the point of view of the natural number structure. In ETCS we omit this troublesome '$\in$' relation entirely and redefine the elementhood relation functionally: an element is defined for any set x as the unique morphism (function) originating in the terminal object (singleton set) $e : 1 \to x$.

2.2.3.3 Epimorphisms and monomorphisms (surjections and injections)

A morphism $f : A \to B$ is said to be epic and is called an epimorphism if for every object C and any morphisms $g_1, g_2 : B \to C$ it holds that $g_1 \circ f = g_2 \circ f \implies g_1 = g_2$. A morphism $f : A \to B$ is said to be monic and is called a monomorphism if for every object Z and any morphisms $g_1, g_2 : Z \to X$ it holds that $f \circ g_1 = f \circ g_2 \implies g_1 = g_2$. In the context of set theory, epimorphisms and and monomorphisms are surjective and injective functions, respectively.

2.2.3.4 Cartesian products and coproducts

Set has products and coproducts (as defined in the previous section). These are exactly the same as we discussed above for an abstract category, only the objects are sets and morphisms functions.

2.2.3.5 Well pointed

The category *Set* is a well-pointed category, which means that for any pair of morphisms and objects $f, g : A \to B$ where $f \neq g$ there is a morphism $p : 1 \to A$ such that $f \circ p \neq g \circ p$[3] The main point here is that if f and g differ then they must differ on some value. The core idea is that functions in set theory are defined by what elements they send. Functions differ by sending different elements.

[3] Conversely, if $f \circ p = g \circ \alpha$ for all $p : I \to A$ then $f = g$.

2.2.3.6 Equalizers

Set has an equalizer object, which is defined as follows. Say there are objects A and B and morphisms $f : A \to B$ and $g : A \to B$ The equalizer is an object E with morphism $eq : E \to A$ which classifies the values that f and g agree on,[4] i.e. $f \circ eq = g \circ eq$. Again, to prove the uniqueness of E we show that if there is a competing "equalizer" E' satisfying $f \circ eq' = g \circ eq'$, then we can define a unique morphism $h : E' \to E$ such that $eq \circ h = eq'$:

$$\begin{tikzcd}
E \arrow[r, "eq"] & A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & B \\
E' \arrow[u, dashed, "h"] \arrow[ur, "eq'"] & &
\end{tikzcd}$$

The equalizer object, being a generalization of equalizers in regular set theory, tells us when two functions are the same, namely, when they agree on all values.

Subobject classifiers The categorical correlate or generalization of the subset is called the subobject. A subobject classifier is a construction that says that one object is or is not a subobject of another. Set has subobject classifiers, which are defined as follows. Let S and A be sets with morphism $f : A \to S$. Let Ω denote the set of boolean values $\Omega = \{True, False\}$ and let $\chi : S \to \Omega$ be the characteristic function which sends elements of S to the boolean value *True* or *False* and $unit : A \to T$ be the arrow sending A to the terminal object. The subobject classifier is a monomorphism $true : T \to \Omega$. The pullback of *true* with χ defines both the subobject (subset) A and the

[4]In set theory, we call the equalizer of two functions f and g both having sets a and b in common as domain and codomain the set $E = \{x | x \in A \wedge f(x) = g(x)\}$. The equalizer set E is just the set of parameters for which the two functions yield the same values. E is the biggest set of $x \in A$ for which $f(x) = g(x)$.

injection (monomorphism) f embedding A in S:

$$\begin{tikzcd} A \arrow[r, "unit"] \arrow[d, "f"'] & T \arrow[d, "true"] \\ S \arrow[r, "\chi"] & \Omega \end{tikzcd}$$

Thus the subset relation between sets can be defined functionally and with no reference to elements.

With the correlate of subsets defined as above, the power object is defined as follows. For any set A there is an object $\mathcal{P}(A)$ that comes with the local membership relation for A, namely $\in_A: A \times \mathbb{P}(A) \to \Omega$, which we call the power object of A if for any monomorphism $i : D \to A \times C$ there is also a unique morphism $\chi'_i : C \to P(A)$ and the following commutes:

$$\begin{tikzcd} A \times P(A) \arrow[r, "\in_A"] & \Omega \\ A \times C \arrow[u, dashed, "id_A{,}\chi'_i"] \arrow[ur, "\chi_i"'] & \end{tikzcd}$$

2.2.3.7 Choice

The axiom of choice in set theory states that any non-empty family of disjoint sets has a choice function that ”chooses” a single element from each set. The set of these elements is called the choice set. In ETCS if there is an epimorphism $p : A \to B$ then there is also also $q : B \to A$ such that $p \circ q = id_B$.

2.2.3.8 The natural number object (NNO)

A natural number object is an object $\mathbb{N}$ which is defined by the following relations:

- $z : 1 \to \mathbb{N}$
- $\sigma : \mathbb{N} \to \mathbb{N}$

The successor morphism σ satisfies the following condition: For any rival object X with rival base element $z' : 1 \to \mathbb{N}$ and rival successor morphism $\sigma' : X \to X$ there exists a unique mapping $g : \mathbb{N} \to \mathbb{N}$ such that $z \circ \sigma = \sigma' \circ z$:

$$\begin{tikzcd}
1 \arrow[r, "z"] \arrow[dr, "z'"'] & \mathbb{N} \arrow[r, "\sigma"] \arrow[d, dashed] & \mathbb{N} \arrow[d, dashed] \\
& X \arrow[r, "\sigma'"] & X
\end{tikzcd}$$

If X is a natural number object, then any other object X' isomorphic to X is also a natural number object. The property defining the natural number object is thus a structural property and the structure defined is unique insofar as any other object with he same property is indistinguishable.Isomorphic objects in Set turn out to have all of the same structural properties and any structural properties of one object are shared by any objects isomorphic to it. The natural numbers are thus defined by a structural property which effectively captures what it means to be the natural numbers in a set-theoretic context while omitting any non-structural information. Any sets satisfying this property might be said to ”have” the natural number structure. McLarty writes:

> No model of the Peano axioms (or of any axioms) in ZF has only the properties that all have in common. That is Benacerraf’s point. But the point fails for categorical set theory. Sets there, like Benacerraf’s numbers, have only structural relations. (McLarty 1993, p. 495)

McLarty’s discussion of structural properties in the context of categorical set theory can thus be seen as just a specific example and application of a general account of structural properties of objects in any category.

2.2.4 Counterpoint (1): Tsementzis

Tsementzis comes up with an example of a property that can be formulated in ETCS which distinguishes between isomorphic objects. Let $e : 1 \to \mathbb{Z}$ be the "$e \in \mathbb{Z}$" morphism and let $\phi(x)$ be the property of $cod(e) = x$ where cod maps any morphism to its codomain. We see that $ETCS \models \phi(\mathbb{Z})$ (i.e. $cod(e) = \mathbb{Z}$), but then again $ETCS \not\models \phi(2\mathbb{Z})$. Thus in ETCS we have just formulated a property which is not invariant and we can see that ETCS, like ZFC, can make structurally irrelevant distinctions between isomorphic sets.

Tsementzis approaches the structuralist thesis as a "design constraint" on a foundation of mathematics stipulating that every expressible property of objects be invariant with respect to the relevant notion of equivalence. The defining principle for a "structural foundation of mathematics" S is that any branch of mathematics can be formulated in S and all "grammatical" properties of objects in S are "invariant under the relevant criterion of identity in that context . . ." (p.10). Tsementzis believes his example shows that ETCS is not a structural foundation of mathematics (SFOM) and refutes the view that the important difference, from a structuralist point of view, between ZFC set theory and ETCS is that the former has a global elementhood operator '$\in$'. The source of nonstructural properties in ZFC and ETCS is, according to Tsementzis, the untyped identity relation '$=$', rather than the global element relation '$\in$'.

Tsementzis is right that not all expressible properties in a category are structural, but that does not mean that category theory is no better than element-based set theory as an account of structural properties. In fact, we can see that there is something wrong with Tsementzis' example. As we discussed in the previous section, a property p is not structural if it contains constants or free variables. However, in Tsementzis' example ϕ is defined as

$\phi(x) \equiv cod(o) = x$, where $o : 1 \to \mathbb{Z}$. The predicate ϕ essentially essentially means "being equal to the codomain of o", i.e. being equal to $\mathbb{Z}$. Thus, because ϕ contains a constant term $\mathbb{Z}$, it is indeed ruled out by the criteria attending the definition of a structural property for objects of a category. Thus, while non-structural predicates like ϕ are expressible in ETCS, they are uniformly distinguishable from structural properties, and category theory maintains its advantage over set theory as an account of structural properties.

2.2.5 Counterpoint (2): Barton and Friedman

Does category theory have a monopoly on the concept of "structure" and "structural property"? That depends on how we define these terms. Barton and Friedman (2019) have challenged the idea of category theory as the sole arbiter of structural properties. In their view, if we consider cardinality to be a structural property then we should consider membership based set theory to be the best suited framework for investigating this structural property. Thus there is at least one structural property for whose investigation category theory is not demonstrably superior. I think that Barton and Friedman are right, inasmuch as there is no universally agreed upon concept of structure.

According to Barton and Friedman, since a set-theoretic account of cardinality can be given which is also isomorphism invariant, the categorical structuralist must either deny that cardinality is a structural property or admit category theory does not have a monopoly on the notion of structure, at least in regard to cardinality. I am inclined to agree that set theory in some form or other might be of use in exploring aspects of structure like cardinality for which category theory is not particularly suited.

I agree that it is best to take a pluralist approach to the concept of structure in which category theory, element-based set theory, and other frameworks investigate a variety of approaches to the concept of structure. I think that this perspective is also consistent with the view that category theory has a special role to play as a framework that is better suited for analyzing the concept of structure at the highest level of generality and in the widest possible range of contexts. This might necessarily mean that some fine-grained structural properties like cardinality are overlooked, but no framework can do everything.

2.3 Yonneda's lemma and the essence of structuralism

2.3.1 The social view of structures

A common intuition in structuralism, particularly the non-eliminative variant, is that that objects, like natural numbers, are in some sense incomplete and depend on one another for their identity. Let's call this the social view (after Manin (2008) below) of objects and structures:

Definition (Social view of structure). An object X is identified by its relationships to other objects in a system.

There are two importantant ways in which the social view occurs in structuralism. Coming from the "internal" perspective on structure, the social view applies to elements of a system, which all hang together as parts of a whole. For example, here the 1-position comes before the 2-position, which comes before the 3-position. You either have all these positions or none of them.:

$$1 \xrightarrow{\leq} 2 \xrightarrow{\leq} 3 \xrightarrow{\leq} \ldots$$

In non-eliminative structuralism the parts can be considered empty positions with only relational features, while in eliminative structuralism the parts are whatever plays these roles in some particular system.

When take the "external" perspective on structure like we do in category theory the social view applies to how any systems of a certain kind (e.g. sets, groups, topological spaces) relate to one another:

$$
\begin{tikzcd}
 & & & 1 & \\
0 \arrow[loop above, "id"] \arrow[r] \arrow[urrr, dotted] \arrow[rr, bend right, dashed] \arrow[rrr, bend right, dashed] \arrow[rrrr, bend right, dashed] & A \arrow[loop above] \arrow[r, "f"] \arrow[urr, dotted] \arrow[rr, bend right, "g\circ f"] & B \arrow[loop above] \arrow[r, "g"] \arrow[ur, dotted] & C \arrow[loop above] \arrow[r] \arrow[u, dotted] & \dots \arrow[ul, dotted]
\end{tikzcd}
$$

What have yet to do is explain how the notion of dependence is handled in categorical structuralism. The core principle is contained in this passage from Manin (2008):

> An object X of the category C can be identified with the functor it represents: $Y \mapsto Hom_C(Y,X)$. Thus, if C is small, initially structureless X becomes a structured set. This external, "sociological" characterization of a mathematical object defining it through its interaction with all objects of the same category rather than in terms of its intrinsic structure, proved to be extremely useful in all problems involving, e.g., moduli spaces in algebraic geometry. (p.50)

I will spend the rest of this section explaining what Manin refers to here, namely how the essence of an object is captured in category theory through its relationships to other objects. In order to do this we will have to move up in abstraction and take a quick journey through relationships between entire categories, namely functors and natural transformations. At the end of this journey we will hopefully see how concepts like representation an results like Yoneda's lemma shed light on or perhaps even serve to evolve an important intuition in structuralism.

2.3.2 The natural numbers in a functor

Mazur (2007) has us imagine that two of us have been using our own versions of the natural numbers in isolation and we are trying to make sense of the notion that my numbers $\mathbb{N}$ are the same as your numbers $\mathbb{N}*$ and discoveries about mine carry over as discoveries about yours. As Mazur explains, there are two opposite and extreme positions we could take when it comes to conceptualizing sameness in mathematics, each of which with their own problems. On the one hand, there is a logical approach, like that of Frege, where we understand a number as an equivalence class in a set of all sets. On the other hand, there is a conventionalist approach where we, for example, point to a particular sample and define five as the number of beers in my refrigerator (assuming this does not change). The first approach has the problem that it results in a pardadox and the second approach has problem that it is contingent. What we want is a middle approach that brings together some of the objectivity sought by the first and some of the flexibility sought by the second.

Mazur proposes that the middle approach consists in providing a way of translating the structure of my numbers and yours. For instance, we can agree that both of our "natural numbers" are models of Peano axioms and we can also agree that whatever these look like, they can be defined as the initial object of the Peano category $\mathbb{P}$.[5] The objects of $\mathbb{P}$ are triples $(N, 0, \sigma)$ where N is a set, 0 is a base element, and a morphism $\sigma : N \to N$ is succession. A

[5]In the context of ZFC set theory, we regard the natural numbers as the "smallest" inductive set ω whose existence follows from the axiom of infinity $\exists I(\emptyset \in I \wedge \forall x \in I((x \cup \{x\}) \in I))$. When I say the natural numbers are the "smallest" inductive set I mean that the intersection $\bigcap M_{i \in I}$ of the inductive set ω and any nonempty family of inductive sets $M_{i \in I}$ is equal to ω. Within the category $\mathbb{P}$ this "smallness" is expressed in property of initiality.

morphism such as $f:(N,o,\sigma)\to(N',o',\sigma')$ is function that maps N to N' so that $f(o)=o'$ and so that for all $n\in N, f(\sigma(n))=\sigma'(f(n))$. Let's say that I define my numbers as such an initial object in this category $(\mathfrak{N},0_{\mathfrak{N}},s_{\mathfrak{N}})$ of $\mathbb{P}$ where $0_{\mathfrak{N}}$ is my base element and $\mathfrak{N}=\{0_{\mathfrak{N}},1_{\mathfrak{N}},2_{\mathfrak{N}},3_{\mathfrak{N}},...n_{\mathfrak{N}}\}$ are my natural number sequence and the n-th natural number in my framework is given by composing $s_{\mathfrak{N}}$ on $0_{\mathfrak{N}}$ $n-1$ times. For example, $3_{\mathfrak{N}}=s_{\mathfrak{N}}\circ s_{\mathfrak{N}}\circ s_{\mathfrak{N}}(0_{\mathfrak{N}})$. For any other other object (X,m_X,tX) there is a unique morphism $i:\mathfrak{N},0_{\mathfrak{N}},s_{\mathfrak{N}})\to(X,m_X,t_X)$.

2.3.2.1 Functors and natural transformations

Mazur goes on to explain how the essence of such an object can be extracted from its relations to other objects. In order to do this we need to define functors and natural transformations, because these will form the sort of correspondence we need, namely the corespondence between the object of the Peano category and the set of its relationships to other objects, which itself is an object in a category of sets. A functor is a higher order morphism which maps one category to another.

Let $\mathbb{C}$ and $\mathbb{D}$ be categories. A functor $F:\mathbb{C}\to\mathbb{D}$ maps any object X of $\mathbb{C}$ to an object $F(X)$ in $\mathbb{D}$. For any morphism $f:X\to Y$ of $\mathbb{C}$, F maps f to a morphism in $\mathbb{D}$: $F(f):F(X)\to F(Y)$. F embeds $\mathbb{C}$ in $\mathbb{D}$ and preserves essential algebraic structure of composition, identity, and associativity:

- $F(id_X)=id_F(X)$ for any $X\in\mathbb{C}$
- $F(g\circ f)=F(g)\circ F(f)$ for any $f:X\to Y$ and $g:Y\to Z$.
- $F((h\circ g)\circ f)=F(h\circ(g\circ f))$ for any $f:X\to Y$, any $g:Y\to Z$, and any $h:Z\to I$.

Let $\mathbb{C}$ and $\mathbb{D}$ be two categories and let F and G be two functors with domain and codomain $\mathbb{C} \to \mathbb{D}$ and let X and Y be any two objects of $\mathbb{C}$ and let f be a morphism $f : X \to Y$ in $\mathbb{C}$. For an object X the natural transformation $\mu : F \to G$ is a mapping of the functor F to the functor G:

$$\begin{tikzcd} \mathcal{C} \arrow[rr, bend left, "F"{name=U}] \arrow[rr, bend right, "G"'{name=D}] & & \mathcal{D} \arrow[Rightarrow, from=U, to=D, "\eta"] \end{tikzcd}$$

η maps the mappings of objects from one functor to the other. The natural tranformation must assign an object X to a mapping of mappings $\mu_X : F(X) \to G(X)$ which is called the component of η at X. For any morphism $f : X \to Y$ the components of X and Y must yield the composition $G(f) \circ \mu_X = \mu_Y \circ F(f)$:

$$\begin{tikzcd} F(X) \arrow[r, "\mu_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\mu_Y"'] & G(Y) \end{tikzcd}$$

Just as F and G embed the structure of category $\mathbb{C}$ in category $\mathbb{D}$, the natural transformation μ embeds the structure of functor F within G. Together, functors and natural transformations can form a category where functors are objects and natural transformations are morphisms. Isomorphism between functors, or natural isomorphism, is defined as follows. We say that μ is a natural isomorphism $\mu : F \to G$ and that F and G are isomorphic ($F \cong G$) if and only if there is an inverse $\nu : G \to F$ satisfying $\nu \circ \mu = 1_F$ and $\mu \circ \nu = 1_G$ where 1_F and 1_G are the identity functors of F and G.

We are coming closer to being able to describe an object in terms of its relationships to other objects. We now discuss how we associate an object to collections of relationships between it and other objects. Remember that for any objects X and Y in a category $\mathbb{C}$ the collection of morphisms with domain X and codomain Y is called the homset, i.e. $hom(X,Y)$. What sort of a collection is $hom(X,Y)$? For so called large categories it is a proper class, but for so called small categories $hom(X,Y)$ is a set. Thus for a small category $\mathbb{C}$ the homset $\mathbb{C}(X,Y)$ implies a mapping from $\mathbb{C}$ to a category of sets. For any object X there is a homfunctor $F_X : \mathbb{C} \to Set$ which maps any other object Y to their homset: $F_X(Y) = \mathbb{C}(X,Y)$. The homset $\mathbb{C}(X,Y)$ can be understand as an object in an underlying category of sets **Set** defined above:

$$\begin{tikzcd}
\mathbb{C} & X \arrow[dd, bend right=60] \arrow[dd, bend right=30] \arrow[dd] \arrow[dd, bend left=30] \arrow[dd, bend left=60] & & \textbf{Set} \\
& & {} \arrow[r, Rightarrow, "F_X(Y)"] & \mathbb{C}(X,Y) \\
& Y & &
\end{tikzcd}$$

Let's say that we have homsets $\mathbb{C}(X,Y)$ and $\mathbb{C}(X,Z)$. For any morphism $g : Y \to Z$ the homfunctor F_X assigns g to a function between homsets $F_X(g) : \mathbb{C}(X,Y) \to \mathbb{C}(X,Z)$, i.e. $F_X(f) : F_X(Y) \to F_X(Z)$ because for any element $f \in F_X(Y)$ $F_X(g)$ maps f to $g \circ g \in F_X(Y)$:

$$\begin{tikzcd}
\mathbb{C} & & Y \arrow[dddd, "g"] & & & \textbf{Set} \\
& & & {} \arrow[r, Rightarrow] & F_X(Y) = \mathbb{C}(X,Y) & f \in F_X(Y) \arrow[dd, dashed] \\
X \arrow[urr, bend left] \arrow[urr] \arrow[urr, dashed, bend right, "f"'] \arrow[drr, bend right] \arrow[drr] \arrow[drr, dashed, bend left, "g \circ f"] & & & & & \\
& & & {} \arrow[r, Rightarrow] & F_X(Z) = \mathbb{C}(X,Z) & g \circ f \in F_X(Y) \\
& & Z & & &
\end{tikzcd}$$

For any morphism $h : X \to X'$ there is the natural transformation $\eta : F_X \longrightarrow F_{X'}$ and for any object Y, h mapsgives rise to the mapping $\eta_Y : F_X(Y) \to F_{X'}(Y)$

which sends any morphism $f : X \to Y \in F_X(Y)$ to $f \circ h : X' \to Y \in F_{X'}(Y)$.

To summarize: the existence of any object X of category $\mathbb{C}$ implies the existence of a functor $F_X : \mathbb{C} \to \mathbf{Set}$ which distills the essence of X out of the entirety of its relationships to any and all other objects. The generalization of this idea is Yoneda's lemma which tells us that the elements of the homset $hom(X,Y)$ maps one to one to the set of natural transformations from F_Y to F_X.

2.3.2.2 Natural numbers defined

There is one more concept required for understanding Mazur's argument. A functor $G : \mathbb{C} \longrightarrow \mathbb{S}$ to the category of sets modeling C is said to be represented by an object X if there exists an isomorphism between it and the homfunctor for X, namely $G \cong F_X$. F is called representable if it can be represented by some object of C. As a consequence of this definition (from earlier results), if a functor F is representable by an object X then F determines the identity of X up to unique isomorphism.

Mazur defines a certain functor from the Peano category $\mathbb{P}$ to sets $I : \mathbb{P} \to Set$ which maps every object $X \in \mathbb{P}$ to the singleton set containing it as an element $I(X) = \{X\}$. For any morphism $f : X \to Y$ in $\mathbb{P}$, there is a corresponding function in Set between singletons $I(f) : I(X) = \{X\} \to I(Y) = \{Y\}$. Since I is representable if and only if $\mathbb{C}$ has an initial object, it turns out that the initial object of $\mathbb{P}$, which we defined earlier as the natural numbers in the context of that category, represents the functor I. Thus the natural numbers can be defined as as the representable functor I of the category $\mathbb{P}$. In the end, rather than thinking of the natural numbers as an object, we think of them as a functor which omits any information we could connect with the "internal" perspective in favor of a wholly "external" perspective. Thus our journey is complete: we have succeeded in giving a complete account of a kind of structure in category theory in accordance with the social view of objects described in the beginning of this section.

2.3.3 Connection to structuralist philosophy in general

The social view of mathematical objects implied by category theory has connections with structuralist tendencies elsewhere in philosophy.[6]. Conceived of generally, a structuralist approach to a topic arises from a common intuition that in order to achieve complete understanding of a phenomenon we should not look within that phenomonon, but we should instead look at is interactions with other phenomena. This intuition frequently appears in the form of a certain understanding of language and meaning. For example, the later Wittgenstein (1953) deconstructed a naiive view of language according to which words are essentially signs tied to things through acts of ostensive definition:

> But if the ostensive teaching has this effect,—am I to say that it effects an understanding of the word? Don't you understand the call "Slab!" if you act upon it in such-and-such a way?—Doubtless the ostensive teaching helped to bring this about; but only together with a particular training. With different training the same ostensive teaching of these words would have effected a quite different understanding.
>
> "I set the brake up by connecting up rod and lever."—Yes, given the whole of the rest of the mechanism. Only in conjunction with that is it a brake-lever, and separated from its support it is not even a lever; it may be anything, or nothing. (§6)

[6]See Harris (2003) on the relationship between Yoneda's lemma, mathematical structuralism, and structuralism elsewhere.

The mistake is in not realizing that ostensive definition is also a part of language: pointing to something and uttering a sound means nothing in isolation. Just as a brake lever is not a brake lever if there were no such things as automobiles, ostensive definition is a practice that can only be understood by looking at its role in a larger context of language and forms of life. Analogously, I think that category-theoretic structuralism can be understood as the view that a better approach to understanding mathematical entities is not by looking inside them to isolate their internal structure, but rather by looking outside towards their relationships to other entities in the contexts in which they appear.

2.4 Conclusions: What is categorical structuralism about?

I think we are now in a position to articulate what categorical structuralism is about. The category-theoretic structuralist position consists in these core claims:

1. As opposed to the internal structure of concrete structures, category sets concrete structures into a context where they relate to one another as elements in an abstract system (category). Within this greater context certain kinds of structures can be defined in terms of structural properties, for which a generic account may be given, which identify a certain sort of object by the role that it plays. Thus a general kind of structure may be defined by a very general property that applies to many categories or by more specific properties which apply to specific categories.

   a categorical structure is something we get by ignoring the internal nature of an object and focusing on its relationships to other objects.

the entirety of an object's relationships to other objects is the entirety of its structure and any unique structure has a unique structural property defining it. this concept of structure seems different to me from the notion of structure discussed by many structuralists, namely the internal pattern shared by different natural number sequences.

2. Category theory can define very general structures that cut accross different theories and leave the details of their instantiation open to interpretation. The categorical definition of a group, for instance, defines a group for virtually any context in which groups are possible. This is what Awodey referred to as the top down perspective of categorical structuralism.

3. Categorical structure consists not only in mappings between objects in a category, but also in mappings between categories (functors) and mappings between mappings between categories (natural transformations). As Mazur pointed out, this means that categorical structures can be almost entirely detached from concrete objects. Everything we need to know (structurally) about an object, according to Yoneda's lemma, can be gathered from that object's relationships to other objects, and the object itself is replaced by its own relationships.

Does categorical structuralism fit any of those classifications mentioned earlier, like eliminative, non-eliminative, in re, or ante rem structuralism? One might entertain the idea that structures defined by category theory are ante rem structures because they can be so abstract and general that they could be interpreted as platonic abstract entities. One could also see categorical structuralism as fitting an eliminative and nominalist take on structures, or the aristotelian in re position, since categorical structures seem to generalize features of many concrete systems. Category theory itself is a mathematical theory and so is neutral in regard to these classifications and philosophers interested in category theory are likely to see their own philosophical preconceptions reflected back at them. It is not the main topic of this writing so I will not discusss this in length. However, I find that categorical structuralism suits an aristotelian interpretation of structure. Something like the product object defines products in various different circumstances with different ”matter”. While the definition of the categorical product does not contain any specific predetermination of what ”matter” makes up a product, it does carry the implication that the ”matter” must be suitable. High level definitions of form correlate with matter on some level. Categorical structures seem to define forms with the utmost minimal demands on the matter.

2.5 Objections

The literature on category-theoretic structuralism has overwhelmingly revolved around questions about the viability of category theory as a structuralist foundation for mathematics. The general form of of these arguments are as follows:

1. A structuralist framework must provide a a foundation for mathematics.
2. Category theory, for certain reasons, is not a viable foundation for mathematics.
3. Therefore, category theory is not a viable structuralist framework.

Most of the debate around categorical structuralism has consisted in arguments of this sort and counterarguments which either deny the second premise and therefore the conclusion (for instance McLarty) or deny the first premise and therefore the therefore the entire argument (Awodey). These debates are complicated and can lead to impasses because there are many moving parts, and many many underlying topics that people can have differing assumptions about,chief among these being the notion of a foundation and the notion of structure itself.

2.5.1 Feferman

Feferman (1977) argued that category theory depends on notions of collection and operation and that proposing category theory as a foundation of mathematics amounts to wrongly assuming that these concepts are already understood or self evident, given that these are concepts that we want to account for in the first place and given that they must be assumed in order to formulate category theory at all:

> The point is simply that when explaining the general notion of structure and of particular kinds of structures such as groups, rings, categories, etc. we implicitly presume as understood the ideas of operation and collection; e.g. we say that a group consists of a collection of a objects together with a binary operation satisfying such and such conditions. Next, when explaining the notion of homomorphism for groups or functor for categories, etc., we must again understand the concept of operation... Thus at each step we must make use of the unstructured notions of operation and collection to explain the structural notions to be studied. The logical and psychological priority if not primacy of the notions of operation and collection is thus evident. It follows that a theory whose objects are supposed to be highly structured and which does not explicitly reveal assumptions about operations and collections cannot claim to constitute a foundation for mathematics, simply because those assumptions are unexamined. It is evidently

> begging the question to treat collections (and operations between them) as a category which is supposed to be one of the objects of the universe of the theory to be formulated (p.150).

What Feferman is getting at here is that that since any concrete structure is ultimately some sort of collection with some sorts of operations on it, in order to give an acount of structures we need to first have an account of what a collection and what an operation is. Since category theory is also a theory which talks about certain collections and operations on them (categories of objects, morphisms, and compositionality), the concepts of collection and operation are implied in the general definition of a category as well. Since collections and operations are what we want to account for in the first place, we certainly cannot rely on category theory, since it already presupposes them.

In response to Feferman I would say firstly that it is hard to see how any theory could avoid presuming such vastly general concepts, even set theory. What is a set, after all, but a collection of elements. What is an element? A member of a set. What is membership in a set? Elementhood. And so on and so forth we go in a circle. Sure, for mathematical purposes, set theory is an account of operation and collection. However, as far as these concepts are concerned, I do not see how set theory, or any theory for that matter, can independently address what a collection or operation is. I think that a categorical structuralist could respond by saying, first of all, that for most practical purposes we know how to interpret axioms and deduce propositions and conclusions from them, which indicates that we already have a solid grasp of what a collection or an operation is. Secondly, we are able to do this without having to justify our use of theses notions by giving an account of them.

2.5.2 Hellman

Hellman argues that since the axioms of category theory consist in schematic definitions which by nature make no assertions and have no definite subject matter, it cannot be a foundation for a structuralist interpretation of mathematics. He claims that ”...as usually presented, category theory is defective as a framework for structuralism in at least two major, interrelated ways: it lacks an external theory of relations, and it lacks substantive axioms of mathematical existence.” (p.12). Since category theory does not address the existence of structures and since categorical axioms are schematic definitions, we need an ”external theory of relations” like set theory or Hellman’s own modal theory of structures in order to “...make sense of talk of structures satisfying the axioms of category theory, i.e being categories or topoi” (p.8).

Hellman (2003), moreover, claims that category theory depends on concepts that are ultimately borrowed from set theory. For example, Hellman claims that the concept of a morphism is borrowed from the concept of a function in set theory:

> There is frank acknowledgement that the notion of function is presupposed, at least informally, in formulating category theory; indeed, category theory has been described as investigating the behavior of families of functions under the operation of composition. These are notions that the working mathematician understands and uses every day, and to build a theory on them is, in principle, no different from building a theory on the notion of objects, including sets, belonging to sets (p.133).

To summarize:

1. Morphisms are either functions in a different wording or depend ”at least informally” on the concept of a function.

2. Category theory cannot independently define functions without set theory (because of 1).

3. Category theory depends on set theory (because of 1 and 2).

With the qualifier ”at least informally” Hellman evidently includes the possibility that the concept of a morphism depends formally on the concept of a functions, but it is easy to see that this cannot be the case, as Pedroso (2009) explains. But if Hellman just means that the concept of a morphism depends informally on the concept of a function, then he evidently means that the concpet of a morphism simply motivates the concept of morphisms. But in this case, as McLarty points out, the fact that one concept motivates another concept does not imply that anything is actually presupposed (McLarty, 2005, p.50).

Because a foundation should provide us with both the objects of mathematics and a framework for making assertions about them, whereas category theory is unable to do either of these things, Hellman (2003) views the desiderata we have of a foundation as irreconcilable with the nature of category theory. While Hellman attributes to category theory an ability to characterize structures of some sort, the schematic nature of such structures leaves them disconnected from the rest of mathematics:

> Moving beyond the question of autonomy proper, we turn now to an equally important, intimately related, problem, that of mathematical existence. This problem as it confronts category theory can be put very simply: the question really just does not seem to be addressed! (We might dub this the problem of the ’home address’: where do categories come from and where do they live?) Of course, the first-order ’axioms for categories’ or those extended to include ’axioms for topoi’ do include existence claims, e.g., existence of enough morphisms,

> e.g., for pullbacks or pushouts, existence of terminals, existence of morphisms for well-pointedness, existence of a 'natural-number object', existence guaranteeing the Axiom of Choice, etc. But, as already said, these axioms are to be read 'structurally'... that is, as defining conditions, not as absolute assertions of (putative) truths based on established meanings of primitive terms (p.9)

Additionally, according to Hellman, the schematic or conditional nature of categories and categorical structures means that we cannot decide whether any of these entities exist on the basis of how they are defined:

> Surely category theorists do not intend their first-order axioms for categories or topoi to be read in any such fashion. It is not as if the category theorist thinks there is a specific 'real-world topos' that is being described by these axioms! But then, just as in the cases of more familiar algebraic theories, the question about mathematical existence can be put: what categories or topoi exist? Or, more formally, what axioms govern the existence of categories or topoi? On the assumption of autonomy, we're in the situation of the Walrus and the Carpenter, after the oysters were gone: '.. .but answer came there none...' (p.10).

Hence the need, according to Hellman, for a supplemental background theory of some sort, which could be ZFC or his own modal thheory of structures:

> As its proponents have maintained, category theory does offer an interesting structuralist perspective on most mathematics as we know it. But it needs to be supplemented and set within a yet wider framework. As explained above, category theory–with its non-assertory, algebraico-structural axioms–depends on a prior notions of structure (collection with relations) and satisfaction by structures to make sense of the very notions of 'category' and 'topos'. On the ontological side,

> we have argued that category theory is insufficiently articulate. To avoid a collapse to "if-then'ism", it requires substantive axioms of mathematical existence, at least of background universes as logico-mathematical possibilities. Although this usually manifests itself through an introductory appeal to an unspecified, ambient domain of sets, this is not necessary. Instead, one can develop a modalized theory of large domains relying on the more neutral and general or schematic notions of "part/whole" and plural quantification, as described above. Relative to such domains, both category theory and set theory can be developed side-by-side, with neither viewed as prior to the other. (p.29)

As we see here, Hellman proposes his own framework as an alternative to categorical and set-theoretic approaches to structuralism. The advantage of adopting this framework is that it offers an in re account of structure which avoids both the above problems with category theory and ontological problems associated with in re structuralism by allowing mathematical sentences to apply to domains of concrete structures whose existence need not be assumed actual, but merely possible.

2.5.3 Responding to Hellman and Feferman

2.5.3.1 Does category theory depend on set theory?

Hellman's criticism would seem to leave us with two choices: accept that either category theory makes use of a set thoery like ZFC or Hellman's own framework. In either case category theory does not offer an independent framework for structuralism. But I think that we need to accept neither option. Let us look at the claim that because category theory does not say anything exists or make any assertions, it must rely on a background theory, like ZFC set theory, that does these things for it. First of all, it is true that category theory cannot do without sets. The question is, which set theory or conception of sets should or can it use? It turns out that there is no hard-set answer to this question because there are a variety of set theories out there which we can use. Let's look at what a set theory needs to do for category theory. Mazur (2008, p.7-8) describes a "bare bones set theory", which defines the minimum that a set theory should be able to do for the purposes of category theory. A "bare bones" set theory should provide the following:

1. sets $\{X, Y, Z, \ldots\}$
2. elements $\{a, b, c, \ldots\}$
3. an "element of" relation
4. maps which send elements from a domain to a codomain set
5. composition of maps, which should be associative

Sound familiar? It should, because we are only one step of abstraction away from a category of sets $\mathbf{S}$, which consists in:

1. sets (objects of $\mathbf{Set}$)
2. functions (including $id_X : X \to X$ for any X) and collections of functions between sets (homsets)
3. functions compose and compositions associate

According to Mazur, we can simply regard the underlying set theory as a variable ranging over any acceptible set theory:

> Category theory doesn't legislate which set theory we are to use, nor does it even give ground-rules for what "a" set theory should be. As I have already hinted, one of the beautiful aspects of category theory is that it is up to you, the category-theory-user to supply "a" set theory, a bare category of sets S, for example. A category is a B.Y.O.S.T. party, i.e., you bring your own set theory to it. Or, you can adopt an even more curious stance: you can view $\mathcal{S}$ as something of a free variable, and consequently, end up by making no specific choice! (p.10)

Thus, even if category theory has to work with some version of set theory, that does not mean that it is rooted in a specific kind of set-theoretic foundation, especially not one like ZFC with a global concept of elementhood.

2.5.3.2 The nature of mathematical statements

One interesting response[7] to Hellman and Feferman is that their criticism contains the assumption that mathematical assertions are referential and fact stating even if at first glance they do not appear to be. I think there is something to this claim. Hellman's own framework is designed to ensure that all mathematical statements can be formulated as generalizations about systems, even if those systems do not exist! Benacerraf (1973) spoke of a tendency in the philosophy of mathematics to seek a "homogeneous semantical theory in which semantics for the propositions of mathematics parallel the semantics for the rest of the language" (p.661). An extreme version of this view of mathematical semantics is platonism, but a lesser version might be the kind of view that Hellman believes in, namely that mathematical assertions are generalizations about systems.

Hellman's arguments rest on the assumption that mathematical sentences

[7] See Voost (2015, p.92-94) and Kukita (2009, p.4-8).

are statements ranging over domains of systems and this must holds even in the case of category theory where there need not be any explicit reference to concrete systems. Hellman concludes that since category theory lacks an account of what its structures are structures of it must depend on some background theory that can supply this. Thus, according to Hellman, categorical structuralism does not provide an independent account of mathematical structures. This is Hellman's argument in a nutshell, and it is a strong one if we follow the initial premise about the meaning of mathematical propositions. But if we do not find this assumption plausible, we might not find his criticisms of categorical structuralism compelling. One could assume with equal plausibility that mathematical theorems are statements concerning structures that can be taken at face value or interpreted as generalizations if we so choose. This seems to be a categorical structuralist way of thinking. It is quite opposite to Hellman's way of thinking about math which his criticism of categorical structuralism rests on. We have here two completely different ways of thinking about mathematics that are mostly mutually incompatable. It appears that this disagreement is more fundamental than the topic of category theory.

2.5.3.3 Linnebo

Linnebo (2011) investigates Hellman's claims about the autonomy of categorical foundations vis a vis set theory in light of varying concepts of autonomy (loagical, conceptual, and justificatory) and concludes that ETCS passes as logically and conceptually autonomous from ZFC. Logically speaking, since all of the objects and constructions of ETCS are defined via morphisms and relational properties and do not depend on concepts or axioms from ZFC, ETCS can be assured to be autonomous from ZFC. However, Linnebo sheds doubt on the justificatory autonomy of ETCS. Linnebo defines justificatory autonomy as follows: "T_1 has justificatory autonomy with respect to T_2 if it is possible to motivate and justify the claims of T_1 without appealing to T_2, or to justifications that belong to T_2." (p.242).

The question in regards to categorical foundations is whether there are similar grounds which would independently justify choosing ETCS over ZFC. Linnebo recognizes the widespread acceptance and use of categorical methods and frameworks like ETCS in mathematical practice. Thus, he says, the opinions of mathematicians and existing mathematical practice could be a justification for choosing ETCS over ZFC.

Linnebo has two objections to this argument. Firstly, Linnebo claims that mathematical practice generally adheres to “explicitly orthodox set theory”, i.e. ZFC (2011, 249). Thus the appeal to practice is bound to fail regardless of whether such arguments from practice are sound or not. Secondly, Linnebo argues, appeal to practice as a justification is certainly independent; the question is whether it is sufficient enough as a justification. As Hellman says, appealing to practice and the opinions of mathematicians is a “strong” naturalism. In the philosophy of science "strong naturalism" is the view that philosophy should heed scientific practice and the opinions of scientists.

In response to the first point, I would ask what exactly Linnebo means by "orthodox" set theory. Linnebo might be right that, if asked, many mathematicians would claim that ZFC is the foundation of mathematics, however, it is less clear that the actual set theory in use is consistent only with ZFC.ZFC is generally only taught when the subject of instruction is set theory or the foundations of mathematics and it is true that if you asked a mathematics student what the foundations of math are, they would likely say ZFC set theory. However, beyond a working knowledge of practical set theory, an in depth knowledge of axiomatic set theory is not essential to much mathematical research. As Leinster (2014) puts it:

> Mathematicians manipulate sets with confidence almost every day of their working lives. We do so whenever we work with sets of real or complex numbers, or with vector spaces, topological spaces, groups, or any of the many other set-based structures. These underlying set-theoretic manipulations are so automatic that we seldom give them a thought, and it is rare that we make mistakes in what we do with sets. However, very few mathematicians could accurately quote what are often referred to as 'the' axioms of set theory. (p.1)

While (at least in principle) much of modern mathematics could be represented by axiomatic set theory, in actual practice, you do not see mathematicians trying to justify their work with overt reference to specific ZFC axioms. Regardless, Linnebo has said that ETCS has justificatory autonomy from ZFC only if we can take the opinions of mathematicians and mathematical practice as a justification.

In regard to Linnebo's second point, I have a few responses.

1. In regard to the soundness of justification through mathematical practice, I do not think that I have a better response than that of McLarty (2012), who balks at the idea that philosophers could know a "better mathematics than mathematicians" (p.113). We can look at practice and interpret it, but dictate how mathematicians justify their practice we cannot.

2. The assertion that mathematical work is really grounded in ZFC (rather than ETCS) is questionable. As McLarty points out, the use of sets in modern mathematical texts are consistent with both a ZFC and an ETCS version of set theory.

3. I think that given that the development of the iterative conception came after the development of ZFC (and not the other way around), trying to construe ZFC as developing in response to the motivating iterative conception of set is a hard sell.

2.6 Responses of categorical structuralists

2.6.1 Should category theory be a foundation?

Debates about categorical structuralism and whether or not it is a compelling position often revolve around its viability as a ”foundation” of mathematics. Foundation is kind of an open term: in general it means some theoretical framework that can unify mathematics as a subject matter provide (hopefully) a bedrock for justifying mathematics. Structuralism is not a foundational theory, but it is closely tied in many people's minds with the idea of foundations because it attempts to offer a philosophical unification of what mathematics is about.

Hellman rejects categorical structuralism because it does not attempt to offer any kind of foundation. There are two general responses so far from categorical structuralists to this point. Awodey (as we will discuss later) simply rejects the premise that structuralism provide foundations, while McLarty actually defends his own notion of categorical structuralism as a foundation. Category theory, according to McLarty, contains a world of real entities of its own. Rather than the traditional model of a foundation, like ZFC, which is a theory with a ”one universe” perspective on mathematics, McLarty sees category theory as providing a plurality of universes of objects for which it provides a powerful organizational framework.

McLarty focuses especially on ETCS and CCAF (category of categories and functors) as foundations. The category of sets defined by ETCS is offered as a direct alternative to ZFC as a foundation for the rest of mathematics and the category of categories offers a foundation for category theory itself. CCAF situates categories and functors as objects and morphisms in a metacategory. When we speak of ”categorical foundations” we are speaking of a specific category or topos which serves as a foundation for other mathematics. We do not mean the general theory of categories as defined by the Eilenberg-Mac Lane axioms.

Marquis (1995) has provided analysis of foundations according to particular goals that they have and distinguishes between cases where the predicate ”foundations of” is essentially a different predicate. Marquis concludes that debates about categorical vs. set theoretical foundations obscure the fact that advocates of either approach privilege different uses of the term:

> A philosophy of mathematics amounts to an ordering of the above relations [different ’foundations of’ predications]. Thus, within a philosophy of mathematics, some of these relations lose their foundational status, since they are presumably shown to follow from one or a few others, and some are ignored altogether. In particular, the debate surrounding the foundational status of category theory rests on the fact that the parties involved order these relations differently... (p.431)

Accordingly, arguing that set theory is better suited as a foundation than category theory amounts to liking some sense of foundations (e.g. semantic, ontological, and epistemological) more than others. In order to distinguish the sense in which category theory is thought of as a foundation we need to distinguish certain goals or purposes that distinguish different senses of foundation.

Organizational foundations for mathematics distinguish mathematical work in general or work in a specific area through the characteristic methods and tools which distinguish mathematics in general as an activity and subject matter from non-mathematics or distinguish a particular area of mathematics from other areas. To understand the purpose of organizational foundations we have only to ask how we distinguish mathematics from non-mathematics: in subjects, for example, theoretical computer science, the edges can become blurry, since algorithms, programs, and data types are considered in abstraction from engineering as mathematical constructs and, of course, computer science makes free use of mathematical logic and even category theory. Obviously the worst way to distinguish mathematics from non-mathematics would be to start a big running list of things that are considered mathematical, e.g. those things which mathematicians say they study and which are contained in mathematical textbooks, courses, conferences, or blogs. But this approach is circular, arbitrary, and also inefficient, much like Euthyphro's attempts at defining piety. Probably the best place to begin if we wanted to differentiate the study of mathematics from the study of other subjects is with the axiomatic method.

The foremost organizational innovation in the last centuries in mathematics has been the axiomatic method. From this standpoint we have the criterion that such and such investigations and claims are mathematical if they are investigating what we can logically infer from definitions in an axiomatic theory. Hilbert commented on the "organizational" sense of mathematical foundations as seen in a formal, axiomatic approach:

> Every science takes its starting point from a sufficiently coherent body of facts as given. It takes form, however, only by organizing this body of facts. This organization takes place through the axiomatic method, i.e., one constructs a logical structure of concepts so that

> the relationship between the concepts corresponds to the relationship between the facts to be organized. (Hilbert 1902, in Hallett and Majer, 2004 [as quoted in Landry (2011, p.446)])

The axiomatic method is organizational in a few important ways. For one, relates to what many mathematicians are doing most of the time, namely studying the implications of axiomatic theories interpreted in some context or another. Certainly this distinguishes mathematics from non-mathematical fields such as computer science, which are ultimately empirical in orientation to their subject matter. Another important sense in which the idea of axiomatic method is organizational is that axioms draw a line around specific areas of mathematics by implicitly defining and classifying objects according their shared features. Such was Hilbert's position regarding the importance regarding the axiomatic method, which Bernay (1967) characterizes as follows:

> [the] main feature of Hilbert's axiomatization of geometry is that the axiomatic method . . . consists in. . . understanding the assertions (theorems) of the axiomatized theory in a hypothetical sense, that is, as holding true for any interpretation. . . for which the axioms are satisfied. Thus, an axiom system is regarded not as a system of statements about a subject matter but as a system of conditions for what might be called a relational structure. (p.497 [as quoted in Landry (2011, p.438)])

When we say that axioms are schematic we mean something like what is said here:

> . . . it is certainly obvious that every theory is only a scaffolding or schema of concepts together with their necessary relations to one another, and that the basic elements can be thought of in any way

> one likes . . . . (Hilbert, 1899, pp. 40–41 [as quoted in Landry (2011, p.437)]).

In line with this understanding of axiomatic methods as providing organizational ”foundations”, Mac Lane (1996) spoke of foundations as ”proposals for the organization of mathematics” (2012, p.407) and characterized axiomatic structure in the following manner:

> ...a structure is essentially a list of operations and relations and their required properties, commonly given as axioms, and often so formulated as to be properties shared by a number of possibly quite different specific mathematical objects... a mathematical object ‘has’ a particular structure when specified aspects of the objects satisfy the (standard) list of axioms for the structure. This notion of ‘structure’ is clearly an outgrowth of the widespread use of the axiomatic method in mathematics (p.174 and p.176 [as quoted in Landry (2011, p.444)]).

Thus category theory can be thought of as an extension and generalization of attempts at organizing mathematical practice according to axiomatic methods. Inasmuch as the organization of mathematics in this way can be spoken of as ”foundational”, category theory could be spoken of as providing ”organizational foundations”. In any case it is safe to say that category theory is not foundational according to the classical senses of the term foundation: it is not a source of answers to ontological or epistemological questions about mathematics. Category theory does not speak to what objects in mathematics exist and how we know that mathematical assertions are true. However, category theory potentially offers tools for the organization of mathematics and does offer crucial specific methods and tools for new areas of research that directly depend on categories and it offers compelling heuristics that organize mathematical research (e.g. look for the adjoint functors in different mathematical contexts).

2.6.1.1 McLarty and categorical foundations

Mclarty sees foundations as a project of organizing mathematics and takes inspiration from Mac-Lane and Lawvere who advocated for category theory as a unifying framework. As a unifying framework, category theory is not meant to play a role like set theory, that is, as framework which provides the epistemological justification for mathematics. Rather, Mac Lane and Lawvere often thought of of category theory as offering the advantage of a common framework that forms conceptual bridges between separate areas of mathematics. By allowing a common language for things, they thought that category theory offers powerful heuristic tools which allow us to see how apparently separate phenomena are related on a more general level, (e.g. look for adjoint functors). Having a common framework should also allow for more transparency and help in the "debugging" of mathematical proofs.

McLarty responds to Hellman's claim that owing to its inherently algebraic character, category theory contains no real assertions. McLarty responds to Hellman's criticisms of categorical foundations by pointing out that

1. the Eilenberg-Mac Lane axioms were never really meant to be foundations at all, since they do not define any specific category. However, ETCS, CCAF, and synthetic-differential geometry (SDG) actually play a foundational role with regard to their subject matter;

2. the axioms of ETCS, CCAF, and SDG can be understood as consisting in assertion about their own objects, and ETCS provides a general theory of sets for mathematics;

3. axiomatic foundations like ETCS actually have specific answers to the

> question of existence (e.g. functions and sets ”exist” from the point of view of ETCS and functors and categories ”exist” from the point of view of CCAF).

Thus Hellman’s objection that categorical foundations have nothing to say about existence, according to McLarty, is wrong. According to McLarty, the difference between frameworks like ZFC and ETCS is not that ZFC is consists in assertions while ETCS consists in schematic definitions, but rather that ZFC can only be interpreted as a system of assertions, while ETCS and other categorical foundations can be interpreted schematically or assertorically. Mcarty rejects, therefore, Hellman’s premise that we can only view category theory, including categorical foundations for both mathematics and category theory, schematically. McLarty (2005) says the axioms of specific categories like ETCS can be taken as assertory and self contained foundations both for the objects of category theory and sets, much like ZFC:

> More importantly, each proposed categorical foundation is a proposed answer. Categorical set theory says: the sets and functions posited by the axioms exist (and thus the categories constructed from those sets exist). Synthetic differential geometry (SDG), taken as a foundation, says: the smooth spaces and maps posited by the axioms exist (and thus also the categories constructed from them).
>
> . . . Indeed category theory per se has no such axioms [i.e. existential assertions], but that is no lack, since category theory per se is a general theory applicable to many structures. Each specific categorical foundation offers various quite strong existence axioms. (p.42)

McLarty, employing this distinction in his response to Feferman, agrees that a foundation of mathematics must give an account of collection and operation and agrees that the Eilenberg-Mac Lane axioms do not provide this. However, McLarty points out that the ETCS and CCAF axioms provide autonomous accounts of operations and collections. While the ETCS axioms do not have the expressive power of ZFC, McLarty points out that a reflection principle R which can be used to extend ETCS and yield a theory that can be proven to have exactly the same expressive power as ZFC. Refering to examples from various mathematical textbooks, McLarty makes a reasonable claim that the set theory used in them both lacks explicit connection to ZFC set theory and is compatible with categorical set theory.

Regarding the ultimate question about the home address of categories, i.e. what are the foundations for category theory, McLarty argues that the concept of a category is implicitly defined by the axioms of CCAF which give rise to a category of categories and functors between them:

> The key point to grasp here is precisely that categorical foundations for category theory are not set-theoretical foundations for category theory. When we axiomatize a meta-category of categories, by the axioms of CCAF, the categories are not 'anything satisfying the algebraic axioms of category theory'-i.e., the Eilenberg-Mac Lane axioms. They are anything whose existence follows from the CCAF axioms. They are precisely not sets satisfying the Eilenberg axioms. They are categories as described by Lawvere's CCAF axioms (McLarty, 2005, p.52).

I have my own doubts about what this actually achieves, which I will get into toward the end of this chapter where I address foundational and anti-foundational structuralism.

2.6.1.2 Awodey: no foundations, no problem

Awodey argues that Hellman's "home address" objection only makes sense coming from the "bottom-up" perspective, which focuses on "construction" and “content” rather than “form” and “description” (p.55). This is because Awodey does not view category theory as a foundation for mathematics. Recall how we summarized the general approach of the “bottom-up” and “foundational” criticism of categorical structuralism above:

1. Category theory is not a foundation of mathematics (i.e. does not tell us anything about the nature or existence of structures as entities).

2. Thus, category theory is not an andequate framework for the structuralist hypothesis.

Awodey’s top down, anti-foundational perspective does not allow that (2) follow from (1) for three reasons:

1. Category theory is not a foundational framework.

2. Structuralism is autonomous, at least in principle, from the foundations of mathematics.

3. The viability of category theory as a framework for structuralism is not contingent on the viability of category theory as a foundation.

Whereas McLarty and Landry offer a direct refutation of Hellman’s argument, Awodey simply disregards Hellman’s objections such as the mismatch or home address objection as irrelevant: such criticisms begin with what he considers to be the incorrect assumption that mathematics, together with its philosophical interpretation in structuralism, must be understood in a bottom-up manner.

For Awodey (2004) the features of category theory which Hellman and Feferman view as a weakness are actually a strength. Category theory’s apparent lack of existence claims allows it to characterize structures without being held back by commitment to particular foundations. For Awodey, it is precisely because category theory implies no fixed foundations or fixed universe of mathematical objects and because of its inherent generality and indeterminacy that it can freely characterize structures from such an abstract, indefinite, and universal vantage point. If I understand Awodey right, his understanding of categorical structuralism is that

1. Mathematics is the study of structures (the general structuralist premise).

2. Structures are presented in in many contexts

3. Mathematical theorems are schematic statements about any objects with a certain structure rather than assertions about a certain predetermined domain of objects.

4. Category theory is the current best framework for structuralism because it can define a structure common to a certain classification of object and formulate statements about that structure independently of objects that have that it.

When, for instance, we define a product or a group object for any category satisfying the basic conditions, we effectively define these structures on the highest level of generality possible in mathematics, i.e. independently of what groups are groups of or products are products of. While categories can be concrete if we want, the flexibility of category theory that Awodey praises is owed to the fact that we can understand them as an abstract scaffolding and we can understand the structures characterized in them schematically.For instance,defining a product for any category fitting the minimal requirements is like writing a blank check where the blank part left to be filled out is the interpretation. Thus, as I understand Awodey's categorical structuralism, category theory is the best framework, because of the characteristics alluded to, for a general account of general structures that more broad than any attempt at capturing them in a specific foundation.

The indeterminacy and schematic character of category theory, which leads Hellman to the home address objection, is, according to Awodey, is also a trait of modern mathematics itself, which he described as the schematic and conditional character of mathematical statements. Theorems in abstract mathematics are often expressed on a high level of generality. If we are being consistent, we should regard the ”home address” objection as not only a problem for category theory, but for much of modern mathematics as a whole.

Essentially, what Hellman calls the “home address” for a category or a construction in category theory, is a a bit of “ambient structure” that is added on. When we “go down” and incorporate ambient structure, what we are doing is, on the one hand, increasing the complexity of the tree by adding on more nodes and leaves, and, on the other hand, increasing specificity and concreteness. Awodey considers an example where we talk about groups in a category C and then a category of those groups and then a category of functors from the category of groups in C back to C:

$$\begin{array}{c} \text{a group } G \text{ in an arbitrary category C} \\ \downarrow \\ \text{the category Group(C) of all groups in C} \\ \downarrow \\ \text{the category } C^{Group(C)} \text{ of all functors from Group(C)} \to \text{C} \end{array}$$

From the bottom up perspective, it appears that we are considering increasingly complex structures: G is contained in C and G is contained in $Group(C)$ where G is represented as a map in $C^{Group(C)}$. It's like we are talking about electrons with atoms within compounds within gas clouds within solar systems within galaxies within galactic clusters. We have to account for how we form these collections and in what greater context/collection we put them in. This is a bottom-up growth in complexity where we designate certain things as the leaves and move upwards forming ever more encompassing nodes.

From a top-down perspective we could simply imagine a category C within an ambient, cartesian closed category with the property of having an object G, which, again, is a structure considered as an element (object) in an abstract system (category) with only structural properties and move "down", i.e. add on ambient structure. From the former perspective we are making stronger and stronger existential assumption as we frame what we are talking about as a fixed object within more and more encompassing collections of objects. On the top-down view we make no existential assumptions and frame what we are talking about as a configuration which can be set in various contexts if we want to. What are, according to the bottom up perspective, a concrete structure embeded in a hierarchy of increasing strong existential assumptions, are, according according to the top down and categorical account, a schematic structure together with the incorporation of more and more ambient structure.

Adjoint functors allow us to translate between simpler structures and more complex structures built from them. For instance, the pair of free and forgetful functors between set and group let us freely construct groups from sets (or drop the extra structure from groups). According to the top down perspective, we "build downwards" from simple to complex. For instance, groups are sets that are organized in a certain way, that is to say, a group is a set with additional structure. We can represent the structural relationship between the set and the group structure with the forgetful functor $FO : Gr \to Set$ and its inverse, the free functor $FR : Set \to Gr$. FO takes groups and maps them to their underlying set and takes group homomorphisms and maps them to set functions. The forgetful functor thus takes the more complex group structure back "up" to the simpler set structure. The free functor FR takes the set structure and "adds on" the additional structure of the group. While there is only one way to "forget" extra structure (a group only has its underlying set), there are unlimited ways to "freely" create groups from sets, when we take a set as input and, limiting ourselves only by the axioms of group theory, generate a group as output.

FR and FO are adjoint to one another and such pairs of free and forgetful functors are called adjoint functors and can be thought of as "conceptual inverses" (Marquis). Applying $FO \circ FR$ to a set X does not give you the original set back, but there is a function $\eta : X \to FO \circ FR(X)$ which maps elements of X to elements in $FO \circ FR(X)$ and this function satisfies the property that for for any set function $g : X \to FO(G)$ there is a group homomorphism $f : FR(X) \to G$ that is unique up to isomorphism and satisfies $FO(h) \circ \eta = g$. $FO \circ FR(X)$ is "the best possible solution to the problem of inserting elements of X into a group" (Marquis). Another example of adjunction is the abelianization functor, which takes a group and freely constructs a abelian group. Adjunction relates, one might say "translates" (Category Theory for the Sciences) between different levels of structure. In category theory, the simpler structures, the more abstract or general structures like Set in this example are the "simple" elements in the top down way of thinking. The top-down view of mathematical assertions takes such assertions at "face value", e.g., takes a statement about the group structure as a statement about the group structure, rather than as an implicit quantification about systems that have the group structure.

2.7 Foundations and antifoundations

We can see that there is an important difference between the sort of category-theoretic structuralism envisaged by Awodey and McLarty. McLarty considers categorical structuralism to be a project of using category theory to give foundations for a structural interpretation of mathematics, while Awodey insists that it is the view that category theory offers a schematic and top down interpretation of mathematical structure which suits the schematic and top down character of mathematical practice itself. We will see how Awodey and McLarty's positions diverge in their approach to responding to Hellman.

McLarty disagrees with the premise that category theory has nothing to say about mathematical existence by pointing to various individual categories which seem to have their own answer to what exists (e.g. sets exist for CCAF and the category of sets) and responds to the home address objection by arguing that a metacategory, a category of categories as axiomatized by CCAF, can form a home address in which categories can live.

McLarty advocates for a version of categorical structuralism in which certain categories and topoi form various mathematical foundations. While category theory does not form a unified perspective on the objects of mathematics like that which set theory achieves, overall, the language of category theory has a unifying role with respect to a plurality of different mathematical domains. Different categorical theories provide specific foundations for their own objects, for instance ETCS+R, which is proposed an alternative to ZFC foundations and CCAF, which is proposed as an organizational foundation for category theory. Mathematical statements are thought of as ranging over not one universe, as is the case in set-theoretic foundations, but many, each corresponding to an object in the category of categories defined by CCAF.

Awodey, on the other hand, does not disagree with Hellman that category theory is schematic and says nothing about mathematical existence. Category theory, in Awodey's view, is not meant to provide foundations for areas of mathematics, as McLarty claims, but should instead be considered a language of mathematical structure. Awodey admits that certain categories like that of sets defined by ETCS can effectively translate traditional foundations, but that is not their purpose: their purpose is to show people who are used to thinking set-theoretically that these familiar frameworks can be captured categorically. The usefulness of category theory, however, does not lie in its being just a replacement that does the same thing as ZFC set theory does. Rather, category theory should be thought of as a new paradigm that should enable us to move beyond the set-theoretic and foundational way of thinking.

2.7.1 McLarty vs Awodey

Perhaps Awodey and Hellman's differences come down to semantics. After all, as we explained earlier, "foundations" is a term that does not have one singular meaning, but rather a family of loosely related meanings. McLarty thinks of category theory as an organizational framework of mathematics. Obviously, Awodey is not opposed to the idea of the language of category theory as a tool for organizing mathematics. Perhaps the main difference between Awodey and McLarty is that they use the word "foundations" differently. This, however, is not the case. There is more separating their positions than semantics. The difference between them is subtle, but important. It comes down to how mathematical statements are interpreted and in particular, McLarty's use of CCAF as a bedrock for categories themselves.

According to McLarty (2005), a basic foundational problem is articulating the ambient structure of mathematical objects and statements. While Awodey thinks that questions about the overall coherence of a theory boil down to assessing the consequences of the axioms and seeing if they are consistent. But there is a problem with this, in McLarty's view: even the consistency of the axioms depends on features of ambient structure, for instance which choices of logic you make. Moreover, a large portion of reference to categories in mathematics are to specific categories with intended interpretations, that is, with some explicit ambient structure:

> When checked recently, forty-one of the latest fifty references to 'category' in Mathematical Reviews were to specific categories, i.e., they did have intended interpretations—insofar as anything in mathematics ever does. These uses of 'category' and 'functor' are as specific, as un-'algebraic', as upper-level mathematics ever gets. They are meant as already situated within a foundation and not themselves open to foundational re-interpretation. Even if we go with Awodey's idea of mathematics as schematic, these papers take the ambient structure as a datum and not an explicit variable of interest to their work. (p.50)

McLarty disagrees with Awodey that we can do away with foundational problems altogether. Speaking directly to what he sees as Awodey's relaxed attitude toward the background structures as well as to his claim that the coherence of a mathematical statement should not be a foundational concern because it can be shown merely by testing the consistency of the axioms, McLarty says:

> That is a fine way to work for some purposes but Hellman is right that we also have foundational concerns. When we pursue those we cannot be satisfied with Awodey's equation, where he says 'the question of whether the conditions [for a given theorem] are ever satisfied' is just the question of 'whether they are consistent' ([2004], p. 60). Different logical presuppositions make different theories consistent. On

> Awodey's view (p. 62) when stronger existence assumptions are used this merely means 'specifying more of the ambient structure to be taken into account'. I am very sympathetic to that. But it does posit an ambient structure and another indispensable part of mathematics (and not only the philosophy of mathematics) is the effort to articulate the ambient structure for any body of work.

Thus McLarty's point is that it is important that such ambient structures should be accounted for. Take for example a general statement about a general kind of mathematical structure, like the statement that the identity element of any group is unique. Let us consider several ways of interpreting this statement. The traditional way of interpreting it (by which I mean the set-theoretic interpretation) is to specify a domain of sets $(G, \star, e)$ governed by group multiplacation $\star$ and possessing identity elements e and inverses for any element of G. The statement is a generalization about the objects of the specified domain.

Now let us think about the statement using McLarty's categorical foundations. Since the proposition is a general statement without any specific intended domain, we think of it as applying in any context (i.e. category) in which we can define a group. Obviously it applies to objects of the category of groups but it also applies in the category of sets and many other categories as well. Thus the statement about groups is really a schematic statement ranging over all these categories and then ranging over the objects of each category in turn. But there is a crucial caveat in regard to McLarty's categorical structuralism. CCAF and its metacategory Cat of categories and functors are supposed to be the ambient setting for abstract structures and their properties. Here is why that is a problem. McLarty, as we mentioned, sees CCAF as providing the answer to the "home address" objection of Hellman (2005). In turn, Hellman raised another concern: how do we know that the category of categories have itself as an object? This would lead to something like Russell's paradox in naiive set theory. In response to this concern, McLarty (2005) drew an important limitation:

> The key point to grasp here is precisely that categorical foundations for category theory are not set-theoretic foundations for category theory. When we axiomatize a metacategory of categories by the axioms CCAF, the categories are not 'anything satisfying the algebraic axioms of category theory'—i.e., the Eilenberg-Mac Lane axioms. They are anything whose existence follows from the CCAF axioms. They are precisely not sets satisfying the Eilenberg-Mac Lane axioms. They are categories as described by Lawvere's CCAF axioms. The third question raises two separate issues. Self-applicability is the question of whether these axioms can be extended to posit a 'category of all categories' as for example New Foundations set theory posits a set of all sets. Not much is yet known about that and I raised the issue only briefly in the article ([1991], p. 1243). But even if there is such a category it will not be the category of absolutely all categories any more than some extension of New Foundations posits the set of absolutely all sets. (p.52)

But what are we to make of this point that the objects of the meta category are "not 'anything satisfying the algebraic axioms of category theory'—i.e., the Eilenberg-Mac Lane axioms"? Firstly, let us first remark that the objects of Cat are only those whose existence is implied by the axioms of CCAF. Such categories qua objects of the metacategory are not, for instance, the category of certain topological spaces or the category of all groups within another category or any category with the group of transformations of the equilateral triangle, but are instead generic and featureless. The objects of a category of categories are not just any category (i.e. anything at all that can be characterized as satisfying the Eilenberg Mac Lane axioms), but are rather anything whose existence follows from the CCAF axioms.

Secondly, let us think about what this means for interpreting mathematical statements. If mathematical statements range over objects of Cat and for each of these range over their objects. But there is a certain problem here. The problem is that broad as this scope may be (it is very broad, since it offers a mathematical multiverse rather than a single universe, like ZFC does), it is still very narrow. As Mcdonald (2012, p.120-29) points out, certain things that we can define as categories are not objects of the metacategory. Very often we are interested in the the specific structural properties that certain general structures have when interpreted in this or that particular context. Such categories are not objects of Cat as specified by McLarty because their existence is not a consequence of the CCAF axioms.

There is a similarity between Shapiro's ante rem structuralism and Awodey's schematic, top down structuralism, namely the idea that a structure can be captured independently of its instantiations. Of course, the difference is that Shapiro's structures exist independently of instantiations, while in categorical structuralism they do not exist independently. When we specify a structure in this schematic, top down manner I think that what this does is capture structure in a way that matches a naiive philosophical conception of structure. According to this naiive conception of structure, when we speak of the natural numbers or groups on a general level, as when we formulate a theorem about all groups, we are talking about the natural number structure or the group structure as patterns or structures that can occur in any allowable context whatsoever. When we think about structures in this way, we are viewing them from a standpoint of the totality of mathematics. The axioms defining a general structure define it as it can appear anywhere in mathematics. From this perspective, there are no limitations on scope.

Now, of course, someone with Hellman's perspective will point out that not every context is a suitable home for the natural number pattern. Unless we make sure the conditions are actually met, the idea that category theory captures structures independently of instantiations is moot and uninteresting, if such structures are not guaranteed, with the aid of an assertory background theory, to be free of vacuity. However, from Awodey's perspective, this is not a practical issue: we usually do not doubt whether groups or rings or other structures are "given" or not, but should there be, we can test the consequences of the axioms and see if they are consistent. Thu Hellman sees our ability to meaningfully talk about structures as dependent on our ability to secure reference to systems that have them, which is not furnished by category theory, and this forms a major objection to categorical structuralism. Awodey acknowledges the fact that we might not know if conditions are fulfilled, but neither sees this as a major nor a general problem. Rather, it is something to be handled on a practical and case by case basis:

> There is usually no question about whether such conditions are ever satisfied; rather, like axiomatic definitions, they serve to specify the range of application of the subsequent statement... In cases where we are not sure whether the conditions are ever satisfied, i.e., whether they are consistent, we have no recourse but to investigate their consequences in order to gain more information. (2004, p. 60)

Awodey simply avoids the issue altogether: there is almost never a question about whether the structures are instantiated. In fact, Awodey takes it as an axiom that, as far as the structures that occur in mathematical practice are concerned, they already given. For Awodey the real question is what is the best language to use for reasoning about universal and general structures. That language turns out to be category theory.

Foundations organize mathematics by distinguishing its subject matter and by distinguishing mathematics from non-mathematics. Foundations are also thought to give an ontology and semantics (what terms refer to, quantify over, are predicated of, etc.) for mathematics so that mathematical statements are statements about something rather than being vacuous. A foundation of mathematics is also thought to give you a framework in which mathematical assertions can be justified.

It is easy to see why set theory has long been considered the foundation of mathematics. Much of mathematics can be represented, systematized, and grounded in terms of a framework of sets. Category theory clearly has little or nothing to say about the epistemology or semantics, but much can be said about how category theory organizes mathematical practice. Certainly, both Awodey and McLarty view category theory as providing organizational unification of mathematical practice, although Awodey does not like speaking of ”organizational foundations”.

Unlike Awodey, McLarty believes that categorical structuralism is meant to provide foundations and provide content or substance for mathematical assertions. According to McLarty (2013) the foundations of mathematics are an ongoing project. There is, firstly, the task of giving practical foundations that mathematicians actually work with. Then there is the task of taking these practical foundations and then turning them into general mathematical foundations of the kind that logicians and philosophers look for. Finally, the philosophy of mathematics has the task of providing philosophical analysis of these general foundations. McLarty seems to be trying to assimilate category theory and categorical structuralism into a more traditional notion of structuralism based on providing an alternative foundation for mathematics (in the above sense of providing the ontological and semantic substance of mathematical expressions and generalizations).

McLarty and Awodey thus present divergent views of structuralism even though category theory plays a central role for both. We may conclude that categorical structuralism is not an entirely unified position, but admits of differing conceptualizations of structure and differing ways of applying category theory for philosophical analysis. Awodey has an uncompromisingly algebraic conception of mathematical structuralism and believes that the value of category theory is that it allows that structures can be specified from the top down without being tied to how they are instantiated in particular contexts. For Awodey, attaching structuralism to foundations, even pluralist foundations like those of McLarty's, is seemingly antithetical to the notion of structure. Structuralism is about the abstract forms that can be defined and do appear in various contexts, it is not about the building blocks and bearers of these forms. Awodey's position is motivated by the most general view of structures possible, namely the view that a structure should be concieved as free to appear in virtually any context, as long as that context satisfy rules that implicitly define said structure.

Hellman's criticisms come from a way of thinking about the concept of structure that clashes with that of categorical structuralism. This concept of structure is that of a class of structured collections of elements with shared characteristics. There is no concieving of a structure without working out what that structure is a structure of. The concept of structure in category theory is not that of an internal structure of an object that can be mapped onto other objects, but that of the collection of abstract relationships that an object has to other objects. Hellman's criticism of categorical structuralism essentially accuses categorical structuralists of not accounting for structures in the sense that he understands them. However, category theory comes with a conception of structure that is fundamentally different from that of Hellman and others. Hellman's criticism thus fails to properly address the position that it is aimed at.

As far as critics of categorical structuralism go, Tsementzis (2016) and Barton and Friedman (2019) are rare examples of critics that address categorical structuralism on its own terms. In regard to these critics, I have said that they make strong points that are difficult not to concede to. However, I have also said that we can concede to some of their points and still maintain that category theory is the most viable framework for articulating mathematical structure of an algebraic and highly general nature.Tsementzis and Barton and Friedman's criticisms are valuable because by answering them the categorical structuralist makes their position more precise. In forming a response to Tsementzis, we have to make clear how a categorical account of structre works and how it is different from that of a set-theoretic account. Barton and Friedman effectively challenge the assumption that category theory is the sole arbiter of structural properties. I think the correct response to this is to allow that there may be different concepts of structure, but clarify the concept of structure which category theory is best at articulating.

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