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„Non-smooth Spacetime Geometry:

Focusing, Energy conditions and Singualrity Theorems"

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Abstract

This thesis deals with the fields surrounding one of the main aspects of general relativity: the singularity theorems. In the 1960's Roger Penrose brought to live the field of singularity theorems by showing that under physically reasonable conditions, reflecting gravitational collapse, a generic spacetime must become singular. Shortly thereafter Hawking proved a similar result modelling an expanding universe and thereby predicting an initial singularity, the big bang.

The theory of general relativity over the last decades has shifted from a predominantly geometric point of view to one that is more PDE-centered. It focuses on the solvability and regularity of the Einstein equations. This, however, has also brought the issue of regularity of the solutions into the foreground. Furthermore, models like the Oppenheimer-Snyder model, matched spacetimes or impulsive gravitational waves all possess a metric of "low regularity", i.e. below C^2. These aspects have created more and more demand for results in Lorentzian geometry for non-smooth metrics, with particular focus on C^1 and $C^{1,1}$ differentiability classes.

In this thesis proofs of the Hawking-Penrose theorem as well as (a generalized version of) the Gannon-Lee theorem for C^1-metrics are provided. Moreover a result on causality of maximally null pseudoconvex spacetimes is given. Singularity theorems in low regularity sharpen the conclusion of the classical theorems, as the predicted singularity cannot just be the result of a drop in regularity of the metric, hence it excludes scenarios where "smoothly"-singular/incomplete spacetimes would have a C^1-extension.

In the course of this work we introduce non-branching conditions. They forbid branching of maximizing geodesics and turned out to be an essential tool for proving low regularity singularity theorems for non-globally hyperbolic spacetimes. They are similar to non-branching conditions in the metric geometry of length spaces. Along the way it is also shown that any causal maximizer in a C^1-spacetime must be a geodesic and hence a C^2-curve. Lastly the thesis contains an approach to distributional curvature with special attention to the C^1-case, where the curvatures are distributions of order 1. This is an essential ingredient in the proof of the Hawking-Penrose theorem for C^1-metrics.

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Contents

Chapter 1

Introduction

General relativity has been studied for more than a century and since its inception in 1915 has been both a physical and mathematical theory. One of its greatest achievements are the singularity theorems by Penrose and Hawking.

In his seminal 1965 paper [Pen65] Rodger Penrose established that the occurrence of spacetime singularities is a generic feature of general relativity. Previous to Penrose's result it was widely believed that the singularities present in many exact solutions of the theory, were only due to the high degree of symmetry inherent in these models. This position changed when Penrose proved that under physically meaningful conditions, reflecting the complete gravitational collapse of a massive object, spacetime necessarily becomes singular (in the sense of geodesic incompleteness). For a recent review of this landmark result and its impact see [SG15].
Shortly after Penrose's first result, Stephen Hawking (see e.g. [Haw67]) showed that an approximately homogeneous, isotropic cosmological solution to Einstein's equations must have an initial singularity. Thereafter followed several papers by Penrose, Hawking, Ellis, Geroch and others, which led to the development of the modern singularity theorems, see e.g. [HE73, Ch. 8]. We quote a pattern singularity theorem, due to Senovilla [Sen98], which shows the general structure inherent in the theorems.

Theorem 1 (Pattern Singularity Theorem). *If a smooth spacetime satisfies the following*

(i) an energy condition,

(ii) a causality condition and

(iii) a boundary or initial condition,

then it contains an incomplete causal geodesic, i.e. a geodesic which does not exist on all of $\mathbb{R}$.

Let us examine the structural points of the singularity theorems from theorem 1 and subsequently discuss in what way they have been generalized in the past. We will also highlight how the thesis at hand contributes to that progress.

The first point to examine are the energy conditions used in singularity theorems. Classically they are given pointwise

$$\mathrm{Ric}(X,X) \geq 0 \text{ for all causal/timelike/null } X \in \mathcal{X}(M) \tag{1.1}$$

and are called causal/timelike/null convergence condition. They correspond via Einstein's equations to conditions on the energy momentum tensor and are labeled differently, e.g. the timelike case above translates to the strong energy condition. Already in [EGJ65] incompatibility of the pointwise conditions on the energy density with quantum field theory was argued.

In [CE80] singularity theorems were proved under average energy conditions, i.e. where the Ricci tensor was not given a pointwise bound, but rather an average (integral) inequality along causal geodesics. Recent progress in this direction was made in [FG11, FK20], proving analogues of Penrose's and Hawkings theorem under energy conditions inspired by Quantum energy inequalities. These are more elaborate conditions taking into account Quantum field theory and allow for negative energy density of restricted magnitude and duration, see e.g. [FK19]. As both the average as well as the QEI inspired conditions allow for pointwise violations of the classical conditions equations (1.1), this opened an avenue for singularity theorems in a quantum realm, which we will discuss a bit more below.

In this thesis, in chapters 3 and 4 singularity theorems are proved for C^1 metrics under distributional energy conditions. As such these are not pointwise bounds and imply energy conditions for smooth approximating metrics which allow for small violations of the form

$$\mathrm{Ric}(X,X) \geq -\delta \text{ for small, positive } \delta \text{ on compact sets} \tag{1.2}$$

The method employed in proving low regularity singularity theorems (via approximation of the metrics by smooth ones) leads to focusing results under weak energy conditions of the above form and also allows for pointwise violations of the classical energy condition. In this sense the methods used to prove focusing under such conditions may be very useful in proving focusing results under QEIs.

The second point in theorem 1 is concerned with the causality of the spacetime. Causality theory itself has been an object of investigation in general relativity and Lorentzian geometry and many of its advances have translated to the singularity theorems [Min19]. Let us take a short look into causality theory and its connection to singularity theorems with particular attention to pseudoconvexity, a concept studied in chapter 2.

The theorems by Penrose and by Hawking assumed global hyperbolicity, the strongest causality condition, which stands on top of the so-called causal ladder [MS08]. Loosely it asserts predictability of events when given appropriate initial data on a so-called Cauchy surface. This condition itself in some ways excludes the existence of singularities, so called naked singularities [MS08, Rem. 3.72(2)].

Shortly after their individual contributions, Hawking and Penrose together proved a singularity theorem [HP70] under much weaker causality conditions, which merely forbid the existence of closed timelike curves. Further recently Minguzzi [Min20] proved a version of Penrose's singularity theorem without the assumption of global hyperbolicity, only using the much weaker condition of past reflectivity.

The notion of causal simplicity is slightly weaker than global hyperbolicity, but still excludes the case of a spacetime having holes (under an inextendibility assumption) [Min12]. Causal pseudoconvexity is a kind of internal completeness assumption weaker than global hyperbolicity with ties to the theory of PDE (similarly to global hyperbolicity) [BP83]. It was used in the context of Lorentzian geometry and causality theory in [BEE96, Thm. 7.35], [BP89, CeSF21] in proving results of causal geodesic connectedness of spacetimes and a Pseudo-Riemannian Hadamard-Cartan theorem. Further (maximal) null pseudoconvexity is closely linked to causal simplicity. Their relation has long been under investigation, with the only previous result being that maximal null pseudoconvexity follows from causal simplicity.

In chapter 2, pseudoconvexity for static spacetimes as well as Riemannian manifolds is analyzed. We provide a static spacetime which is maximal null pseudoconvex and causal but not causally simple, providing in particular a counterexample to a statement made in [VPE19].

Further we shed some light on whether a pseudoconvexity condition could occupy a step on the causal ladder and whether it could be used as an assumption in a singularity theorem. The last point is mainly due to the need for a replacement condition for causal simplicity in proving a Gannon-Lee type theorem akin to [CeS10], where it was incorrectly claimed that causal simplicity is lifted to covering spacetimes (for a counterexample see [CeSM20]), a crucial step in the proof of the main theorem. Causal and null pseudoconvexity are properties which lift to covering spacetimes and as such would be suited as a replacement condition. However as the exact relation between null pseudoconvexity and causal simplicity is still unclear, such an avenue is not yet open.

Coming back to the pattern theorem 1, the third point is an initial condition. In his 1965 paper Penrose used the notion of a closed trapped surface as an initial condition. It describes a spatial region in spacetime, where gravity is so strong that light cannot escape. Mathematically both the ingoing and outgoing congruences of null geodesics must initially converge. In the case of Hawkings theorem this place of the initial condition is taken by a Cauchy surface with a positive (future) contraction, i.e. signifying a contracting universe.

In some aspects already the theorem of Hawking and Penrose was an improvement on this condition and established some rigidity, since it dealt simultaneously with three different cases all leading to the formation of a so-called trapped set.

A different singularity theorem similar in methodology to Penrose's theorem was proven in [Gan75, Lee76] independently by Gannon and Lee. They replaced the trapped surface condition of Penrose by a topological condition, which roughly states that any nontrivial topology in a Cauchy surface must be contained within a sufficiently large sphere. The conclusion is existence of an incomplete null geodesic, given an appropriate energy condition. In chapter 3 this theorem is shown for non-globally hyperbolic C^1-spacetimes improving a previous theorem [CeSM20] to low regularity.

As portrayed in the pattern theorem the singularity theorems all conclude either timelike null or causal geodesic incompleteness. This notion was introduced first in Penroses theorem and is seen as a sufficient condition for a singularity in a spacetime to exist. Physically this corresponds to the worldline of a massive observer or a photon ceasing to exist after finite time. The narrative being that they encounter a point which cannot be described by the spacetime at hand. Mathematically this could be due to several reasons to be outlined below.

Traditionally singularities are associated with unbounded curvature. A curvature singularity, i.e. some curvature quantity becoming unbounded e.g. along a geodesic, is a sufficient condition for incompleteness, although not necessary. Such a singularity by the nature of general relativity, cannot lie in the spacetime, one rather thinks of a point on the boundary.

Further any extendible spacetime[1] is incomplete[2]. This, however, is a rather unwanted and unphysical kind of incompleteness, as it indicates, that given an extension the singularity might vanish. So one either needs to add the assumption of inextendibility to the

[1] A spacetime is extendible if it can be isometrically embedded into a larger spacetime.

[2] A fact which even holds for continuous spacetimes [GLS18]

theorems or better, find a way to rule it out. The second option leads to the strong cosmic censorship conjecture and would allow one to rigorously connect curvature singularities with incompleteness.

When approaching a singularity, it could be that quantum effects take over and classical energy conditions are violated and hence the theorems are no longer applicable. Hence Singularity theorems with energy conditions reflecting quantum field theory are in high demand as already outlined above. The quest to find common ground between general relativity and quantum theories is still ongoing, also in regard to finding appropriate energy conditions, justifiable from the point of view of quantum field theory.

Another point of consideration is that the classical theorems assert incompleteness of a smooth metric. Such a spacetime may be inextendible as a smooth spacetime, but could be extendible in a lower regularity. For example, if there were a $C^{1,1}$(locally Lipschitz continuous first derivatives)-extension of a smoothly inextendible spacetime, the curvature in the extension would still be locally bounded and so would hardly be considered singular as this corresponds to finite jumps in the matter variables. In fact there are physically relevant models like the Oppenheimer-Synder model of a collapsing star or general matched spacetimes [MS93] with metrics in that regularity and below. Even for C^1-spacetimes, curvature can be defined in a sensible way (and is done so in chapter 4) as a first order distribution. Establishing singularity theorems in this regularity class hence excludes the possibility of the singularity being merely due to C^1-extendability of the spacetime. A complementary argument is made in [Sbi18], where even the C^0-inextendability of the Schwarzschild spacetime was proved, showing the "strength" of the Schwarzschild singularity.

In addition, in the classical existence results for the field equations the metric tensor is of regularity H^s for $s > \frac{5}{2}$ and in case of formulations of cosmic censorship the metric merely possesses an L^2 connection. While proving singularity theorems under this differentiability remains open, the results in chapters 3 and 4, are a step in the right direction.

We give a brief history of singularity theorems in low regularity and the contributions of this thesis to it, below.

Already in [HE73] the idea of proving a singularity theorem for metrics of $C^{1,1}$ differentiability was put forward. In 2012 in [CG12] a method for regularizing continuous spacetime-metrics was introduced and after that used to show $C^{1,1}$-versions of the three classical singularity theorems of Hawking, Penrose and Hawking-Penrose in [KSSV15, KSV15, GGKS18]. In this differentiability class geodesics as solutions to the geodesic equation still exist and are unique for given initial data (as the connection is Lipschitz continuous) and the exponential map is locally a bi-Lipschitz homeomorphism [Min15, KSS14]. Further essentially by Rademachers theorem the curvature tensor is locally in L^∞, signifying the possibility of at most a finite jump in the energy momentum tensor of the spacetime.

In [Gra20] the singularity theorems by Penrose and Hawking were even proved for C^1 metrics. This had to be done without the use of the exponential map and with only an existence result on geodesics without uniqueness. Also the curvature quantities are only defined as (tensor) distributions. A careful analysis of the regularization procedure used previously for the $C^{1,1}$-theorems, still yielded energy conditions for the approximating metrics, which allow for a violation of the classical ones as in (1.2). Hence it was possible to utilize similar methods for proving the existence of conjugate points along geodesics as in the $C^{1,1}$ case. The result relied however on the spacetimes being globally hyperbolic to essentially only require to deal with geodesics which are limits of corresponding approximating geodesics.

In chapter 3 a Gannon-Lee type singularity theorem is shown for non-globally hyperbolic

C^1 metrics. This required on one hand the previously unknown fact, that any maximizing causal curve in a C^1-spacetime is a C^2-geodesic[3] as well as a non-branching condition on maximal null geodesics. The null non-branching condition physically ensures that a photon cannot suddenly split in two or be confronted with two equally valid paths to take in a spacetime, which from a physical viewpoint is quite a reasonable scenario to forbid. Put differently, branching in a spacetime would be similarly catastrophic as the occurrence of a singularity and hence is not too restrictive of a condition to add. A mathematically precise definition is given in chapter 3 for the null case and in chapter 4 for the causal case. They ensure that one can appropriately approximate a given null/causal geodesic in a C^1-spacetime by appropriate geodesics of the approximating metrics, achieving what was previously guaranteed by global hyperbolicity.

In chapter 4 similar methods as in chapter 3 are used to prove the Hawking-Penrose singularity theorem for C^1-metrics. In addition it features an abstract approach to tensor distributions in low regularity with particular attention to the C^1-case. There the curvature is a distribution of first order, an important aspect necessary in the proof of the theorem, in particular to formulate a version the genericity condition for C^1 metrics, which is consistent with the definitions in higher regularity. From there on the Hawking-Penrose theorem is proved along the lines of the previous proof for $C^{1,1}$-metrics. Special attention had to be given to the approximation procedure and its implication for the to approximation of causal geodesics in non-branching spacetimes. Further innovations are the definition of the distributional curvature quantities, like the genericity condition and their application to C^1 vector fields as well as a more elaborate focusing result.

[3]this was proven independently and in more generality in [LLS20] around the same time

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Chapter 2

Causal simplicity and (maximal) null pseudoconvexity

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This was a gradual joint collaboration. Contributions by B.S. were mainly made in chapters 2 and 3 regarding results on pseudoconvexity in Riemannian manifolds and in static spacetimes. Each Author made significant contributions.

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Note

Causal simplicity and (maximal) null pseudoconvexity

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Abstract
We consider pseudoconvexity properties in Lorentzian and Riemannian manifolds and their relationship in static spacetimes. We provide an example of a causally continuous and maximal null pseudoconvex spacetime that fails to be causally simple. Its Riemannian factor provides an analogous example of a manifold that is minimally pseudoconvex, but fails to be convex.

Keywords: causality theory, causal simplicity, null pseudoconvexity, static spacetime, convexity

1. Introduction

A pseudo-Riemannian manifold is said to have a pseudoconvex class of geodesics, if for each compact subset K there is a larger compact subset K', such that any geodesic segment of the class with endpoints in K lies entirely in K'. Causal pseudoconvexity is a kind of an 'internal completeness' assumption for spacetimes akin to, but strictly weaker than global hyperbolicity. Similar to the latter, it is a causality notion with strong ties to the theory of PDE. Pseudoconvexity of bicharacteristics characterizes the existence of a parametrix for pseudodifferential operators of real principal type, cf [12, section 6], [6].

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John Beem and his coauthors have used pseudoconvexity in several contexts in causality theory. In [4] causal pseudoconvexity together with causal geodesic non-imprisonment (inextendible causal geodesics leave every compact set) appear as sufficient conditions for the stability of causal geodesic completeness, cf [5, theorem 7.35]. Geodesic non-imprisonment and pseudoconvexity of all geodesics together with the absence of conjugate points implies geodesic connectedness and serve as conditions for a pseudo-Riemannian version of the Hadamard–Cartan theorem [7], [5, chapter 11]. For a recent generalization see [11].

Here we are especially interested in the relation between pseudoconvex and causally simple spacetimes, i.e. causal spacetimes with closed causal relation. Indeed, every causally simple spacetime is maximal null pseudoconvex by [8, theorem 1], for which we provide a simplified proof in theorem 2.7, below. Concerning the reverse implication it has been conjectured in [3, section 1] that strongly causal (i.e. there are no 'almost closed' causal curves) and null pseudoconvex spacetimes are causally simple. Finally, in [20, theorem 2] it was claimed that for strongly causal spacetimes maximal null pseudoconvexity and causal simplicity are equivalent. Here we provide a counterexample to this statement, i.e. we establish that

causal continuity and maximal null pseudoconvexity
do not imply causal simplicity.

Our counterexample is a static spacetime based on a corresponding Riemannian counter example, that enjoys a certain limiting property for minimizing geodesics, but fails to be convex. This counterexample adds to the list of recently found counterexamples involving the notion of causal simplicity [10, 14].

In section 2 we provide some general results on pseudoconvexity and in section 3 we specialize to static spacetimes. This will allow us to 'lift' the corresponding properties of the Riemannian counterexample to the spacetime level in section 4.

In the remainder of this introduction we fix some notations and conventions. All manifolds are assumed to be smooth, connected, Hausdorff, second countable, of arbitrary dimension $n \geqslant 2$, and without boundary. By a *minimizing geodesic* in a Riemannian manifold (Σ, h) we mean a geodesic whose length equals the distance $d^h(x, y)$ between its endpoints x and y. We call Σ *convex* if any pair of its points can be connected via a minimizing geodesic.

A spacetime (M, g) is a time oriented Lorentzian manifold, where we use the signature $(-, +, \ldots, +)$. A causal geodesic in M is called *maximizing* if its length equals the Lorentzian distance $d^g(p, q)$ between its endpoints p and q. We denote the chronological and the causal relation by I and J respectively. A spacetime is called *causal* if there are no closed causal curves. If in addition the causal relation J is closed, it is called *causally simple*. A spacetime is *non-total imprisoning* if no inextendible causal curve is contained in a compact set. We shall only consider causal spacetimes, in which case the non-total imprisonment property is equivalent with the causal geodesic non-imprisonment property mentioned above [17, proposition 4.41]. A spacetime is called *strongly causal* if for every point and for every neighborhood of the point there is a smaller neighborhood such that no causal curve intersects it more than once. A spacetime is called *causally continuous* if it is strongly causal and *reflective*: $I^+(q) \subset I^+(p) \Leftrightarrow I^-(p) \subset I^-(q)$. It is known that causal simplicity $\Rightarrow$ causal continuity $\Rightarrow$ strong causality $\Rightarrow$ non-total imprisonment $\Rightarrow$ causality. For other results and conventions on causality not explicitly mentioned in this work, we refer the reader to the review [17].

2. General results on pseudoconvexity

We start recalling some definitions, cf [5, chapters 7, 11].

Definition 2.1. A spacetime (M, g) is called (causal, null or maximally null) pseudoconvex, if for any compact set K, there exists another compact set K', such that each geodesic of the respective type with both endpoints in K must be entirely contained in K'.

Clearly pseudoconvexity implies causal pseudoconvexity which implies null pseudoconvexity which again is stronger than maximal null pseudoconvexity. There is also a Riemannian version of the notion:

Definition 2.2. A Riemannian manifold (Σ, h) is called (minimally) pseudoconvex, if for any compact set C, there exists another compact set C', such that each (minimal) geodesic with endpoints in C must be entirely contained in C'.

Clearly, pseudoconvexity implies minimal pseudoconvexity. Often it is useful to relate pseudoconvexity to a certain limiting property of geodesic segments which we define next.

Definition 2.3. We say a pseudo-Riemannian manifold M has the limit geodesic segment property (LGS) if the following holds true: given any pair of converging sequences of points $p_n \to p$ and $q_n \to q \neq p$ and any sequence σ_n of geodesic segments connecting p_n to q_n, there is a subsequence of σ_n (in a suitable affine reparametrization) converging (locally uniformly) to a geodesic σ from p to q.

In the Riemannian case we will speak of the minimal LGS if all σ_n (and hence σ) have the corresponding property. Similarly, in the Lorentzian case, we will speak of the causal, null, maximal null LGS, if all σ_n (and hence σ) have the corresponding property.

Recall from [7, definition 2] that a Riemannian manifold (Σ, h) is *disprisoning* if no forward inextensible geodesic $\gamma : [0, \omega) \to \Sigma$ has compact closure.

Lemma 2.4. *A Riemannian manifold (Σ, h) satisfies the LGS if and only if it is disprisoning and pseudoconvex.*

Proof. First assume that LGS holds. Let $C \subset \Sigma$ compact be given. Choose a compact exhaustion $\{C_n\}_{n\in\mathbb{N}}$ of Σ. If pseudoconvexity fails for C there exists a sequence $\gamma_n : [0, 1] \to \Sigma$ with endpoints in C such that γ_n leaves C_n. By the LGS $\{\gamma_n\}_{n\in\mathbb{N}}$ contains a convergent subsequence. The limit geodesic $\gamma : [0, 1] \to \Sigma$ is contained in some C_N. This clearly contradicts the initial assumption. Therefore (Σ, h) is pseudoconvex.

Next assume that LGS holds and there exists a inextensible geodesic ray $\gamma : [0, \omega) \to \Sigma$ with compact closure. Then we have $\omega = \infty$ and γ has infinite length. In that case the sequence $\gamma_n : [0, 1] \to \Sigma$ where $\gamma_n(t) := \gamma(nt)$ has no convergent subsequence.

Finally, assume that (Σ, h) is pseudoconvex and disprisoning. Let $p_n, q_n \in \Sigma$ be sequences converging to p and $q \neq p$ respectively and $\gamma_n : [0, 1] \to \Sigma$ be a sequence of geodesics from p_n to q_n. Let C be a compact set given by the union of a compact neighborhood of p with a compact neighborhood of q. Without loss of generality we can assume $p_n, q_n \in C$ for each n. There exists a compact set $C' \subset \Sigma$ such that $\gamma_n \subset C'$ for all $n \in \mathbb{N}$ by pseudoconvexity. Since the manifold is disprisoning the length $L^h(\gamma_n)$ is uniformly bounded from above. Then by a standard argument for geodesics it follows that there exists a convergent subsequence. □

By the previous result every compact (closed) Riemannian manifold is trivially pseudoconvex but fails to satisfy the LGS as, certainly, it is not disprisoning. For instance, T^2 is pseudoconvex but does not satisfy the LGS. Choosing a geodesic σ with irrational slope, which hence is dense in T^2, we may choose a point $x \notin \sigma$ and t_n such that $\sigma(t_n) \to x$. But the sequence of geodesic segments $\sigma|_{[0,t_n]}$ has no subsequence converging to a geodesic between $\sigma(0)$ and x.

In this connection we mention a theorem of Serre, see [19], implying that any pair of points in a noncontractible complete Riemannian manifold are connected by a sequence of geodesics whose lengths are diverging.

Completeness implies convexity by the Hopf–Rinow theorem [15], but completeness (hence convexity) does not imply pseudoconvexity (e.g. a complete surface with infinitely many holes), cf [18]. For another example of convex but not pseudoconvex space see theorem 2.7 in [14].

The next result is analogous to the previous one, but for the 'minimal' case.

Lemma 2.5. *A Riemannian manifold* (Σ, h) *satisfies the minimal LGS if and only if it is minimally pseudoconvex.*

Proof. We already know that if LGS holds then pseudoconvexity holds, which implies minimal pseudoconvexity.

For the converse, let Σ be minimally pseudoconvex. Further let $\sigma_n : [0, b_n] \to \Sigma$ be minimizing geodesics from x_n to y_n, parametrized by arc length, and let $x_n \to x$, $y_n \to y \neq x$. Let C be a compact set given by the union of a compact neighborhood of x with a compact neighborhood of y. Without loss of generality we can assume $x_n, y_n \in C$ for each n. By compactness $\sigma_n'(0)$ converges up to a subsequence to a unit vector $v \in T_x\Sigma$. Let $\sigma : [0, \beta) \to \Sigma$ be the maximally extended geodesic with initial vector v. Then σ_n converges to σ locally uniformly on $[0, \beta)$. By minimal pseudoconvexity all σ_n are contained in some compact C' and so $L^h(\sigma_n) \leqslant \mathrm{diam}^h(C')$. Hence, again up to a subsequence, b_n converge to some $b < \beta$. Hence $\sigma_n(b_n) \to \sigma(b) = q$ and we are done. $\square$

The next result is a Lorentzian analog to lemma 2.4. It slightly improves [5, lemma 11.20].

Lemma 2.6. (M, g) *satisfies the* (*maximal*) *null LGS iff it is non-total imprisoning and* (*maximal*) *null pseudoconvex.*

A similar result with 'causal' replacing the two instances of 'null' holds.

Proof. The proof of '(maximal) null LGS $\Rightarrow$ (maximal) null pseudoconvex', can be done in the same way as the proof of lemma 2.4 (first paragraph). If non-total imprisonment were violated we could find a future inextendible lightlike geodesic $\gamma : [0, a) \to M$, achronal hence maximizing, contained in a compact set C cf [17, theorem 2.77]. Let $a_n \to a$, and pass to a subsequence (denoted in the same way) so that $\gamma(a_n) \to q$. But then the maximizing lightlike geodesics $\gamma_n = \gamma|_{[0,a_n]}$ with $a_n \to a$, do not converge to a geodesic η connecting $p := \gamma(0)$ to q, for if that were the case, as each γ_n coincides with a segment of γ, η would coincide with a segment of γ, which is impossible since certainly γ_n converges on a domain larger than any subinterval $[0, b] \subset [0, a)$.

For the converse, assume non-total imprisonment and (maximal) null pseudoconvexity. Let σ_n be (maximal) null geodesics with endpoints p_n, q_n converging to p and q, respectively. Let C be a compact set given by the union of a compact neighborhood of p with a compact neighborhood of q. Without loss of generality we can assume $p_n, q_n \in C$ for each n. By (maximal) null pseudoconvexity there is a compact set C' containing all the curves σ_n. By the limit curve theorem (two endpoints case [17]), the sequence must admit a converging subsequence, otherwise we could find a future inextendible causal curve $\sigma^p \subset C'$ starting from p to which some, suitably parametrized, subsequence of σ_n converges uniformly on compact subset. But then σ^p would contradict non-total imprisonment. $\square$

In a globally hyperbolic spacetime for every compact subset K, $J^+(K) \cap J^-(K)$ is compact. As a consequence, global hyperbolicity trivially implies causal pseudoconvexity (and hence maximal causal pseudoconvexity). However, causal pseudoconvexity does not imply global hyperbolicity, e.g. a strip $|x| < 1$ in Minkowski 1 + 1 spacetime [5].

We now give a simplified proof of the next Lorentzian result originally proved in [8, theorem 1]. In the last section we shall show that the reverse implication does not hold.

Theorem 2.7. *If (M, g) is causally simple, then it is maximally null pseudoconvex (equivalently, it satisfies the maximal null LGS).*

The equivalence in the different formulations of the conclusion is given by lemma 2.6, since causal simplicity implies non-total imprisonment.

Proof. Suppose that the claim is false, then we can find a compact set K, and maximal null geodesic segments γ_n with endpoints $p_n, q_n \in K$, such that for some $r_n \in \gamma_n$, r_n escapes every compact set. Further we can assume $p_n \to p \in K$, and $q_n \to q \in K$. As $(p_n, q_n) \in J$ we have by causal simplicity $(p, q) \in J$. Moreover, it cannot be $(p, q) \in I$ otherwise for sufficiently large n, $(p_n, q_n) \in I$ which contradicts the maximality of γ_n. Thus $(p, q) \in E = J\backslash I$.

By the limit curve theorem [17, theorem 2.53(ii)] there are lightlike rays σ^p starting from p and σ^q ending at q such that for every $p' \in \sigma^p$ and $q' \in \sigma^q$, we have $(p', q') \in \bar{J} = J$ where we used causal simplicity. For every $p' \in \sigma^p\backslash\{p\}$ we have $(p', q) \in J$ but the causal curve connecting p' to q must be the prolongation of the lightlike ray σ^p otherwise $(p, q) \in I$, a contradiction. Thus $q \in \sigma^p$ and repeating the argument by taking p' along σ^p after q one gets that σ^p passes through q several times, which violates causality. □

Next we establish that for Riemannian manifolds convexity is stronger than the minimal LGS. Again, in the last section we shall show that the reverse implication does not hold.

Theorem 2.8. *If Σ is convex, then it satisfies the minimal LGS (equivalently, it is minimal pseudoconvex).*

The equivalence in the different formulations of the conclusion is given by lemma 2.5.

Proof. Let $\sigma_n : [0, a_n] \to \Sigma$ be minimizing unit speed geodesics such that $p_n := \sigma_n(0) \to p$ and $q_n := \sigma_n(a_n) \to q \neq p$, $p, q \in \Sigma$. Without loss of generality we can assume that $\dot{\sigma}_n(0) \to u \in T_p\Sigma$ and that $a_n \to a > 0$. Let $\sigma : I \to \Sigma$ be the unit speed geodesic that starts from p with tangent $\dot{\sigma}(0) = u$. For every $t \in I$ by the continuity of the exponential map

$$\sigma_n(t) = \exp_{p_n}(\dot{\sigma}_n(0)t) \to \exp_p(ut) = \sigma(t).$$

Moreover for every $c \in I$, $\sigma_n|_{[0,c]}$ is minimizing and so $\sigma|_{[0,c]}$ is minimizing. If there is $t \in I$ such that $\sigma(t) = q$, then it must be $t \geqslant a$ otherwise, as t is the length of a curve (the curve σ) connecting p to q, no σ_n could be minimizing for sufficiently large n. However, $t > a$ cannot happen since otherwise it would be shorter to go from p to q, passing from some σ_n, which would contradict that $\sigma|_{[0,t]}$ is minimizing. We conclude that if there is t such that $\sigma(t) = q$ then $t = a$.

We can now assume that $q \notin \sigma$ otherwise we have finished, hence we can assume $I = [0, b)$. Observe that $d(p_n, q_n) = a_n$ and by the continuity of distance $d(p, q) = a$. Now, for $0 < \epsilon < b$, we have $d(\sigma_n(\epsilon), q_n) = a_n - \epsilon$ as σ_n is minimizing, and hence $d(\sigma(\epsilon), q) = \lim_n d(\sigma_n(\epsilon), q_n) = a - \epsilon$ as d is continuous. Thus any chosen minimizing curve γ connecting $\sigma(\epsilon)$ with q has length $a - \epsilon$. But $d(p, q) < \ell(\sigma|_{[0,\epsilon]}) + \ell(\gamma) \leqslant \epsilon + a - \epsilon = a$ where the first inequality is strict because there must be a corner at $\sigma(\epsilon)$ between σ and γ, otherwise it would be $q \in \sigma$. The contradiction proves that the connecting case is the only option. □

We do not known if pseudoconvexity implies convexity. The next result represents an attempt in this direction. It is not used in what follows.

Proposition 2.9. *Let (Σ, h) be a Riemannian manifold which admits an equidimensional embedding into a complete manifold $(\tilde{\Sigma}, \tilde{h})$. Then (Σ, h) is minimally pseudoconvex if and only if it is convex.*

Remark 2.10. The proof shows that a positive lower bound on the convexity radius on bounded sets of (Σ, h) suffices at least for the conclusion that minimal pseudoconvexity implies convexity.

Proof.

(a) By theorem 2.8 convexity implies minimal pseudoconvexity without further assumptions.
(b) Assume that (Σ, h) is minimally pseudoconvex. We will show that the closure $\overline{\Sigma}$ of Σ in $\tilde{\Sigma}$ is locally convex, i.e. for any $x \in \overline{\Sigma}$ there exists $\epsilon > 0$ such that $\overline{\Sigma} \cap B^{\tilde{h}}_{\epsilon}(x)$ is convex in $\tilde{\Sigma}$ (i.e. any two points in this set are connected by a geodesic segment which is minimizing among all the connecting curves contained in the same set).

We define the convexity radius for $(\tilde{\Sigma}, \tilde{h})$ as in [16, corollary 1.9.11]. All we need to know is that this function $r : \tilde{\Sigma} \to (0, \infty]$ is continuous and that in $B^{\tilde{h}}_{r(\tilde{x})}(\tilde{x})$ any two points are connected by one and only one geodesic contained in the ball, and this geodesic is minimizing among all the connecting curves in $\tilde{\Sigma}$.

We claim that for all $\tilde{x} \in \tilde{\Sigma}$ and all y, z in the same connected component of $B^{\tilde{h}}_{r(\tilde{x})}(\tilde{x}) \cap \Sigma$ the unique minimal $\tilde{h}$-geodesic between y and z lies in Σ. This can be seen as follows. Let $\eta : [0, 1] \to B^{\tilde{h}}_{r(\tilde{x})}(\tilde{x}) \cap \Sigma$ be a curve from y to z. The set of parameters $t \in [0, 1]$ for which the unique minimal $\tilde{h}$-geodesic between y and $\eta(t)$ lies in Σ is nonempty. It is also open, since $\Sigma \subset \tilde{\Sigma}$ is open and $\exp^{\tilde{h}}_y$ is a diffeomorphism defined on ${\exp^{\tilde{h}}_y}^{-1}[B^{\tilde{h}}_{r(\tilde{x})}(\tilde{x})]$. Finally it is closed by minimal pseudoconvexity. Therefore the unique minimal $\tilde{h}$-geodesic between y and $\eta(1) = z$ lies in Σ. This geodesic is also a minimal geodesic for (Σ, h) as $\mathrm{dist}^{\tilde{h}} \leqslant \mathrm{dist}^{h}$, and hence the unique minimal geodesic for (Σ, h) connecting y and z. By considering limits of geodesics we infer that, for every $x, y \in B^{\tilde{h}}_{r(\tilde{x})}(\tilde{x})$ belonging to the closure of the same connected component C of $B^{\tilde{h}}_{r(\tilde{x})}(\tilde{x}) \cap \Sigma$ there is a $\tilde{h}$-minimizing geodesic entirely contained in $\overline{C} \subset \overline{\Sigma}$ connecting them.

Observe that for every point $\overline{x} \in \overline{\Sigma}$ we have that every connected component of $B^{\tilde{h}}_{r(\overline{x})}(\overline{x}) \cap \Sigma$ is convex, and that the established properties provide a 'local convexity' property for $\overline{\Sigma}$.

Now let $x, y \in \Sigma$ be given. We will show that there exists a minimal geodesic in Σ connecting the two points.

The following reasoning is similar to [1, Satz 2.8(i)]. Let $\gamma_n : [0, 1] \to \Sigma$ be a minimizing sequence of curves connecting x with y. According to the argument above we can assume that the curves are geodesic polygons and that the length of the individual arcs is bounded from below by some positive constant. By the completeness of $(\tilde{\Sigma}, \tilde{h})$ a subsequence converges to a geodesic polygon $\gamma : [0, 1] \to \overline{\Sigma}$. Using a standard argument involving the triangle inequality we see that γ is a $\tilde{h}$-geodesic.

We claim that γ does not intersect the boundary $\partial\Sigma := \overline{\Sigma}\backslash\Sigma$. Assume otherwise, i.e. there exists $t \in (0, 1)$ with $\gamma(t) \in \partial\Sigma$. W.l.o.g. we can assume that t is minimal in $(0, 1)$. Denote with Σ_t the connected component of $B^{\tilde{h}}_{r(\gamma(t))/4}(\gamma(t)) \cap \Sigma$ which contains a segment $\gamma|_{[s,t)}$ for some $s < t$. Note that Σ_t is convex with $\gamma(t) \in \partial\Sigma_t$. The following argument is an adaptation of [9, lemma 8.6] to the present situation. Choose an arbitrary small smooth hypersurface $W \subset \Sigma_t$

transversal to γ and containing $\gamma(s)$. Choose $u > t$ such that $\gamma|_{[t,u]} \subset B^{\tilde{h}}_{r(\gamma(t))/4}(\gamma(t))$ and consider the set

$$V := \{\exp_r(\lambda w)|\ \tilde{h}(w,w) < r(\gamma(t))^2,\ \exp_r(w) \in W,\ \lambda \in (0,1)\}.$$

Here $r \in \Sigma$ is a point sufficiently close to $\gamma(u) \in \overline{\Sigma}$ which stays in the connected component Σ_t (it is possible to show that it exists perturbing $\gamma_{[s,u]}$). The set V is open and contained in Σ_t by convexity. Further V is an open neighborhood of $\gamma(t)$, (remember that γ is a $\tilde{h}$-geodesic hence C^1) a contradiction to the assumption that $\gamma(t) \in \partial\Sigma_t$. Therefore γ is contained in Σ. The lower semicontinuity of the length functional implies that the curve is minimal in Σ. Note that γ is i.g. not minimal in $\tilde{\Sigma}$. □

3. Pseudoconvexity in static spacetimes

Let (Σ, h) be a Riemannian manifold. In this section we investigate the relation of convexity and pseudoconvexity properties of (Σ, h) to causality and pseudoconvexity properties of the static spacetime (M, g) with

$$M := \mathbb{R} \times \Sigma, \quad \text{and} \quad g = -\mathrm{d}t^2 + h. \tag{1}$$

The projection onto the first factor $t : M \to \mathbb{R}$ is a temporal function.

Observe that in semi-Riemannian product manifolds a path is, up to parametrisation, a geodesic if and only if the projections onto the factors are geodesics. Hence in our case, geodesics are always of the form $\gamma = (\alpha, \sigma)$ where α is some linear function in $\mathbb{R}$ and β is a geodesic in Σ. Moreover, clearly γ is causal if and only if $|\alpha'| \geqslant \|\sigma'\|_h$. We observe the following relation of minimizing and maximizing geodesics.

Lemma 3.1. *A causal geodesic $\gamma = (\alpha, \sigma)$ in M is maximizing if and only if σ is minimizing in Σ.*

Proof. It suffices to consider geodesics $\gamma = (\alpha, \sigma)$ of the form $\alpha : [0,1] \to \mathbb{R}$, $\alpha(s) = bt$, and $\sigma : [0,1] \to \Sigma$, i.e., $\|\sigma'(s)\|_h = L^h(\sigma)$, connecting $p = (0, x)$ and $q = (b, y)$ with $b > 0$. First observe that γ is causal iff $b \geqslant L^h(\sigma)$ and so

$$L^g(\gamma) = \sqrt{b^2 - L^h(\sigma)^2}. \tag{2}$$

Now suppose σ is not minimizing, then there is $\tilde{\sigma} : [0,1] \to \Sigma$ connecting x to y with $L^h(\tilde{\sigma}) < L^h(\sigma)$. But then for $\tilde{\gamma} = (bt, \tilde{\sigma})$ we find $L^g(\tilde{\gamma}) = \sqrt{b^2 - L^h(\tilde{\sigma})^2} > \sqrt{b^2 - L^h(\sigma)^2} = L^g(\gamma)$ and so γ is not maximizing.

Conversely, suppose that γ is not maximizing, then there is a future pointing timelike curve $\tilde{\gamma} : [0,1] \to M$, $\tilde{\gamma} = (bt, \tilde{\sigma})$ with $L^g(\tilde{\gamma}) > L^g(\gamma)$. Since $\tilde{\gamma}$ need not be a geodesic we can in general not parametrize it both linear in the first factor and constant speed in the second. However, we have

$$L^g(\tilde{\gamma}) = \int_0^1 \sqrt{b^2 - \|\tilde{\sigma}'\|_h^2}\mathrm{d}t > L^g(\gamma) = \sqrt{b^2 - (L^h(\sigma))^2}.$$

By applying the Cauchy–Schwarz inequality to $\int_0^1 \sqrt{b^2 - \|\tilde{\sigma}'\|_h^2}\mathrm{d}t$ and by using the previous inequality we get the second inequality in the next expression

$$L^h(\tilde{\sigma})^2 \leqslant \int_0^1 \|\tilde{\sigma}'\|_h^2 \mathrm{d}t < L^h(\sigma)^2.$$

The first inequality is obtained through another standard application of the Cauchy–Schwarz inequality. In conclusion, σ is not minimizing between x and y. $\square$

Lemma 3.2. *If M is maximally null pseudoconvex, then Σ is minimal pseudoconvex. Conversely, if Σ is minimal pseudoconvex, then M is maximally causal (hence null) pseudoconvex. M is pseudoconvex if and only if Σ is pseudoconvex.*

Proof. Let Σ be pseudoconvex and take any $K \subseteq M$. Then the projections of K onto $\mathbb{R}$ and Σ are compact, i.e. we have $K \subseteq [c, d] \times C$ for c, d $\in \mathbb{R}$ and some compact $C \subseteq \Sigma$. By pseudoconvexity of Σ there exists a compact set C' such that any geodesic starting and ending in C is contained in C'. Now let γ be a geodesic in M, starting and ending in K. We can write $\gamma(t) = (a + bt, \sigma(t))$, hence $c \leqslant a, a + b \leqslant d$ with σ a geodesic starting and ending in C. This however implies that $\gamma \subseteq K' := [c, d] \times C'$.

The proof that 'Σ is minimally pseudoconvex' implies 'M is maximally causal pseudoconvex' follows from a very similar argument which makes use of lemma 3.1.

For the converse direction, we first look at the maximal–minimal case: let M be maximally null pseudoconvex and let $C \subseteq \Sigma$ be compact. Any minimizing unit speed geodesic σ starting and ending in C fulfills $L^h(\sigma) \leqslant c := \mathrm{diam}^h(C)$. For the compact set $K := [0, c] \times C$ by maximal null pseudoconvexity there exists some compact set K' containing all maximal null geodesics starting and ending in K. Again by construction we must have $K' \subseteq [0, c] \times C'$ for some compact set C' in Σ. By lemma 3.1 the null geodesic $\gamma(t) := (t, \sigma(t))$ is maximizing and hence contained in K', but then also $\sigma \subseteq C'$.

If M is pseudoconvex then Σ is too by noticing that for any compact set $C \subseteq \Sigma$ the set $\{0\} \times C$ is compact in M. $\square$

Remark 3.3. While pseudoconvexity of Σ clearly implies causal and null pseudoconvexity of M the converse does not hold. This is due to the fact that any causal geodesic starting and ending in a compact set in M has a projection with a length bounded by the difference of the t components of the causal curve endpoints. So one cannot expect to gain control over sequences in Σ with unbounded lengths.

Lemma 3.4. *Σ is convex if and only if M is causally simple.*

Proof. Assume Σ is convex. Observe that (M, g) is causal as t is a time function. By translational invariance over the time fiber and by time reflection symmetry, we need only to show that $J^+(p)$ is closed for any point of the form $p = (0, x)$.

Let $q_n = (b_n, y_n) \in J^+(p)$ with $q_n = (b_n, y_n) \to (b, y) = q$. There exist causal paths σ_n from p to q_n of the form $(\frac{b_n}{d^h(x,y_n)}\, t, \sigma_n(t))$ with σ_n from x to y_n. The causality condition reads $\|\sigma_n'\|_h \leqslant b_n/d^h(x, y_n)$. Hence $d^h(x, y_n) \leqslant L^h(\sigma_n) = \int_0^{d^h(x,y_n)} \|\sigma_n'\|_h \leqslant b_n$, which by continuity gives $d^h(x, y) \leqslant b$.

By convexity there exists a unit speed geodesic $\sigma : [0, d^h(x, y)] \to \Sigma$ connecting x to y. Now $\gamma : [0, d^h(x, y)] \to M, \gamma(t) := (\frac{b}{d^h(x,y)}\, t, \sigma(t))$ is a path from p to q that is causal iff, $d^h(x, y) \leqslant b$, which is the case as we just proved. Thus $q \in J^+(p)$ and, by the arbitrariness of q, $J^+(p)$ is closed.

Conversely, let M be causally simple and let $x, y \in \Sigma$. There exist curves σ_n from x to y with $L^h(\sigma_n) =: l_n \searrow l := d^h(x, y)$ with $\|\sigma_n'\|_h = 1$. The curves $\gamma_n : [0, l_n] \to M, \gamma_n(t) := (t, \sigma_n(t))$ from $(0, x)$ to (l_n, y) are null and hence $(l, y) \in J^+((0, x))$ by causal simplicity. So there exists some causal curve $\alpha : [0, l] \to M, \alpha(t) = (t, \beta(t))$ with β a path in Σ from x to y. Moreover $0 \geqslant g(\alpha', \alpha') = -1 + \|\beta'\|_h^2$ and so $\|\beta'\|_h^2 \leqslant 1$, which leads to $L^h(\beta) \leqslant l$ and hence β must be a minimizing geodesic from x to y. $\square$

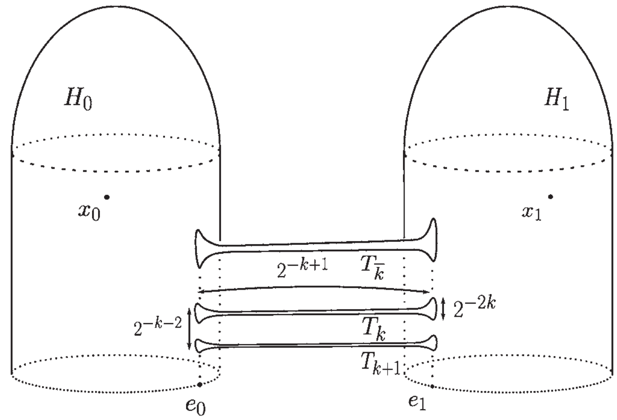

Figure 1. The counterexample consists of two cylinders H_0 and H_1, closed above with cups and connected by a sequence of immersed tubes T_k, $k \geqslant \bar{k}$. Adapted by permission from Springer Nature Customer Service Centre GmbH: [Kluwer Academic Publishers (Dordecht)] [Annals of Glocal Analysis and Geometry] [2, section 2.1], Copyright (2002).

4. The counterexamples

With theorem 2.8 we learned that on a Riemannian manifold convexity implies the minimal LGS. In this section we show that the converse does not hold.

Then we shall consider the Lorentzian theorem 2.7, analogous to theorem 2.8, according to which causal simplicity implies the maximal null LGS. By using the results of the previous section on static spacetimes, we shall show, once again, that the converse implication does not hold.

Let us give an example of Riemannian space (Σ, h) which satisfies the minimal LGS but is not convex. A very similar Riemannian space was constructed in [2, section 2.1] as a counterexample to another statement also related to the notions of convexity and connectedness in Riemannian spaces (figure 1).

Example 4.1 (A non-convex Riemannian manifold possessing the minimal LGS). The space Σ consists of two cylinders H_0 and H_1 closed above with cups and then connected in the flat region by a sequence of immersed tubes T_k, $k \geqslant \bar{k} > 0$. The Cauchy boundary of the space can be identified with the union of two circles, which are the lower boundaries of the cylinders used in the construction. The mouths of the tubes converge to points e_0 and e_1 on the Cauchy boundary of the space.

Let the geodesic distance between the center of the mouths of T_k and T_{k+1} in H_0 before the discs (tube mouths) are excised, be $\frac{1}{2^{k+2}}$, let the diameter of the mouths (discs) in the same geometry be $\frac{1}{4^k}$ (basically is goes so fast to zero that in our arguments this distance becomes negligible), and let the length of the tube T_k from mouth to mouth in (Σ, h) be $\frac{1}{2^{k-1}}$. Let analogous conditions hold in H_1. If γ_1 is a curve passing through T_k from mouth to mouth (here γ_1 might have endpoints belonging to some tubes that might coincide with T_k or not) then we can find a curve γ_2 connecting the same endpoints of γ_1 passing through T_{k+1} from mouth to mouth and such that

$$\ell(\gamma_2) \leqslant \ell(\gamma_1) + 2\frac{1}{2^{k+2}} - \frac{1}{2^{k-1}} + \frac{1}{2^k} + O\left(\frac{1}{4^k}\right) \leqslant \ell(\gamma_1) - \frac{1}{2^{k+1}} + O\left(\frac{1}{4^k}\right).$$

This implies that, provided the space is defined with $\bar{k}$ sufficiently large, $\ell(\gamma_2) < \ell(\gamma_1)$, hence a minimizing geodesic cannot pass through a tube from mouth to mouth and so that there are certainly pairs of points not connected by a minimizing geodesic, for instance any pair (x_0, x_1) where x_0 belongs to the first cylinder and x_1 belongs to the second cylinder. As a consequence, convexity does not hold (this fact was already pointed out in [2]).

We want to show that the minimal LGS holds. Let $\sigma_n : [0, a_n] \to \Sigma$ be minimizing unit speed geodesics such that $p_n := \sigma_n(0) \to p$ and $q_n := \sigma_n(a_n) \to q$, $p, q \in \Sigma$. Without loss of generality we can assume that the tangents $\dot{\sigma}_n$ converge to some unit vector $u \in T_p\Sigma$, and since the diameter of (Σ, h) is bounded we can also assume that $a_n \to a$. Let $\sigma : I \to \Sigma$ be the geodesic that starts from p with tangent u. If its domain interval includes $[0, a]$, then by the continuity of the exponential map we can conclude that

$$q_n = \exp_{p_n}(\dot{\sigma}_n(0)a_n) \to \exp_p(ua) = \sigma(a)$$

that is $q = \sigma(a)$. Observe that a_n is the length of σ_n and it converges to the length a of σ. Thus σ must be minimizing otherwise for sufficiently large n, σ_n would not be minimizing. This would give the desired result so we have only to show that the domain of σ includes $[0, a]$.

Suppose not. The maximal domain of σ is $[0, b)$ for some $b \leqslant a$, and by standard ODE theory [13] $\sigma(t)$ escapes every compact set as $t \to a$ (though possibly returning indefinitely to it).

Let $T^p = T_s$ be the tube whose mouth is closest to p (which could be the tube to which p belongs), and let $T^q = T_t$ be the tube whose mouth is closest to q (if there are more choices for the closest tube we choose that with larger s, resp. t). For sufficiently large k namely for $k > K > \max(s, t)$, with suitable $K > 0$ the minimizing geodesics σ_n cannot transverse T_k from mouth to mouth (as they are minimizing, cf the above argument).

Let us section the figure by cutting with a horizontal plane at a height $y > 0$ selected so that above it we have the points p, q, and the tubes T_k, $k \leqslant K$, thus including T^p and T^q. Then the geodesics σ_n stay entirely above the horizontal section (because if they cross the plane then they can be replaced by a shorter curve running on the plane for a geodesic segment, a fact which would contradict the minimizing property). As a consequence σ being a limit of σ_n is also entirely above the plane and so it cannot escape every compact set of Σ. The contradiction proves that the domain of σ is $[0, a]$ and so that the minimal LGS holds.

We are ready to give the Lorentzian example.

Example 4.2 (A maximally causal pseudoconvex and causally continuous but non-causally simple spacetime). We consider the static spacetime (M, g) with $M = \mathbb{R} \times \Sigma$ and $g = -\mathrm{d}t^2 + h$, where (Σ, h) is as in example 4.1. By construction, M is strongly causal (actually stably causal, because of the time function t). Due to the timelike Killing vector ∂_t it is reflective and hence causally continuous [17].

Now, since Σ possesses the minimal LGS, it is also minimal pseudoconvex by lemma 2.5 and hence M is maximal causal (hence null) pseudoconvex by lemma 3.2.

On the other hand Σ is not convex and so, by lemma 3.4, M is not causally simple.

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Data availability statement

No new data were created or analysed in this study.

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Chapter 3

A note on the Gannon-Lee theorem

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This was a gradual joint collaboration. The central Theorem 3.3 is due to B.S. with significant contributions to all other parts of the paper, in particular the topic of geodesic non-branching.

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A note on the Gannon–Lee theorem

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Abstract

We prove a Gannon–Lee theorem for non-globally hyperbolic Lorentzian metrics of regularity C^1, the most general regularity class currently available in the context of the classical singularity theorems. Along the way, we also prove that any maximizing causal curve in a C^1-spacetime is a geodesic and hence of C^2-regularity.

Keywords Lorentzian geometry · Singularity theorems · Low regularity · Geodesics · Causality · Branching

Mathematics Subject Classification 83C75 · 53C50 · 46F99

1 Introduction

In the mid-1970s, several years after the appearance of the singularity theorems of Penrose and Hawking (see, e.g., [24, Ch. 8]), Gannon [11,12] and Lee [31] independently derived a body of results that relate the singularities of a Lorentzian manifold to its topology. More precisely, in these results often dubbed Gannon–Lee theorem(s), they established that under an appropriate asymptotic flatness assumption, a non-trivial fundamental group of a (partial) Cauchy surface Σ necessarily leads to the existence of incomplete causal geodesics.

Most of these early results assumed global hyperbolicity, with the notable exception of [11, Thms. 2.1–2]. However, the proofs relied on the false deduction that maximizing geodesics in a covering spacetime project to maximizing geodesics of the base, as pointed out in [10]. In the same paper, Galloway proved a Gannon–Lee theorem without assuming global hyperbolicity by invoking the Hawking–Penrose theorem and heavily using a result from geometric measure theory. The latter fact was also

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reflected in the formulation of the main theorem, which assumed an extrinsic condition on the three surface Σ and the topological condition was that Σ is not a handlebody[1]. A related recent result for the globally hyperbolic case in a setting compatible with a positive cosmological constant was given in [17], providing a precise connection between the topology of a future expanding compact Cauchy surface and the existence of past singularities.

Another issue with the earlier results was that the nontrivial topology was confined to a compact region of a (partial) Cauchy surface bounded by a topological 2-sphere S. In the context of topological censorship [9], S is naturally interpreted as a section of an event horizon and in four dimensions the S^2 topology is only natural in the light of Hawking's black hole topology theorem [24, Sect. 9.2]. However, since the latter fails to hold in higher dimensions, with more complicated horizon topologies occurring (see [7], but [21] for corresponding restrictions), the demand for higher-dimensional Gannon–Lee-type results with more natural assumptions on the topology of S arises. Such a result was indeed given by Costa e Silva in [3] which also avoided the assumption of global hyperbolicity. More precisely the result was given for causally simple spacetimes, i.e., causal spacetimes where the causality relation is closed. However, we have recently discovered that this proof relies on an analogous false deduction: It uses that causal simplicity lifts to coverings, which is not true in general, as was explicitly shown in [4]. In the latter paper, Minguzzi and Costa e Silva also present a corrected result, which replaces the condition of causal simplicity by the assumption of past reflectivity, which also has to be supposed for certain covering spacetimes. The line of arguments using past reflectivity in the context of the Penrose singularity theorem was put forward in [37], which also argues that this condition holds true in a black hole evaporation scenario. In the causality part of our arguments, we will follow this path[2].

In this paper, we extend the validity of the Gannon–Lee theorems to Lorentzian metrics of low regularity, in particular to (certain) C^1-spacetimes. Having low regularity singularity theorems at hand is especially favorable in the context of extending spacetimes, as already noted in [24, Ch. 8]. In particular, they rule out the possibility to extend the spacetime to a complete one, even under mild regularity assumptions on the metric. Related results on C^0- and Lipschitz non-exendability have recently been given in [6,19,40,41].

Indeed, during the last couple of years, the classical singularity theorems of Penrose, Hawking, and Hawking–Penrose have been extended to $C^{1,1}$-regularity in [29], [30], and [13]. These results built upon extensions of Lorentzian causality theory to low regularity [5,28,34,45]. Most recently, Graf in [20] was able to further lower the regularity assumptions for the Penrose and the Hawking theorem to C^1. We will

[1] Under these conditions [38] guarantees the existence of a trapped surface within Σ.

[2] In an earlier version of this manuscript, we aimed at a different strategy, replacing causal simplicity by null pseudoconvexity which *does* lift to Lorentzian coverings. However, we have learned from Ettore Minguzzi that a crucial step in our argument, securing that maximal null pseudoconvexity plus strong causality implies causal simplicity, see also [46, Thm. 2], is false. An explicit counterexample was later given in [25]. However, we do not know whether null pseudoconvexity plus strong causality implies causal simplicity, a fact which would allow for an argument that allows to avoid assumptions on the causality of the cover.

extend her recent techniques, in particular, to the non-globally hyperbolic setting, to prove our result.

This note is organized in the following way: In the next section, we discuss preliminaries for our work, especially the intricacies arising in C^1-regularity and we state our results. In Sect. 3 we lay the analytical foundations of the proof of the main theorem. In particular, we will show that any causal maximizer in a C^1-spacetime is a geodesic and hence a C^2-curve. Finally, in Sect. 4, we provide new focusing results for null geodesics and collect all our results together to prove the main theorem.

2 Preliminaries and results

We begin by introducing our main notations and conventions, as well as some basic notions that are necessary to give a precise formulation of our results. Our main reference for all matters of Lorentzian geometry is [39].

We will assume that all manifolds M are smooth, Hausdorff, as well as second countable and of dimension n with $n \geq 3$. We will consider Lorentzian metrics g on M that are of regularity at least C^1 with signature $(-, +, \dots, +)$. Another important regularity class is $g \in C^{1,1}$ which means that g is C^1 and its first derivatives are locally Lipschitz continuous. A spacetime is a Lorentzian manifold with a time orientation, which we assume to be induced by a smooth timelike vector field.

A curve $\gamma : I \to M$ defined on some interval I is called timelike, causal, null, future or past directed, or spacelike, if γ is locally Lipschitz continuous and its velocity vector $\dot{\gamma}$ which exists Lebesgue almost everywhere has the respective property. We denote the timelike and causal relation by $p \ll q$ and $p \leq q$, respectively, and write $I^+(A)$ and $J^+(A)$ for the chronological and causal future of a set $A \subseteq M$. Finally we denote the future horismos of A by $E^+(A) := J^+(A) \setminus I^+(A)$. The respective past versions of these sets will be denoted by I^-, J^- and E^-, respectively. When we refer to these sets with regard to a particular metric g, it will appear in subscript, e.g. $E^+_g(A)$ denotes the future horismos of A w.r.t. g.

For Lorentzian metrics g_1, g_2 one says that g_1 has narrower lightcones than g_2 (respectively, g_2 has wider lightcones than g_1), denoted as $g_1 \prec g_2$, if $g_1(X, X) \leq 0$ implies $g_2(X, X) < 0$ for any $0 \neq X \in TM$.

Throughout we will fix a complete Riemannian background metric h and use its induced norm $\| \ \|_h$ and distance d_h. All local estimates will be independent of the choice of h.

We will denote the fundamental group of a manifold M by $\pi_1(M)$. Also if $i : N \to M$ is a continuous map, the induced homomorphism of the fundamental groups is denoted by $i_\# : \pi_1(N) \to \pi_1(M)$.

2.1 Low regularity

During the last couple of years, the bulk of Lorentzian causality theory has been transferred to $C^{1,1}$-spacetimes, where the exponential map and convex neighborhoods are still available. While convexity fails below that regularity [26,43], nevertheless, most aspects of causality theory can be maintained even under Lipschitz regularity of the metric. Further below some significant changes occur [5,16], while some robust features continue to hold even in more general settings [2,15,27,35]. In particular, for C^1-spacetimes, the push-up principle is still valid and $I^+(A)$ is open for any set $A \subseteq M$.

However, there are two essential features one loses when going from the $C^{1,1}$-setting to C^1-spacetimes: uniqueness of solutions of the geodesic equation and the local boundedness of the curvature tensor. Given the first fact, one has to make a choice concerning the definition of geodesic completeness. We will follow the natural approach of [20] and define a spacetime to be timelike (respectively null or causal) geodesically complete if *all*[3] inextendible timelike (respectively null or causal) geodesics are defined on $\mathbb{R}$.

Concerning the second issue, first note that C^1 is well within the maximal class of spacetimes allowing for a (stable definition of) distributional curvature, which is g locally in $H^1 \cap L^\infty$ [22,33,44]. The Riemann and the Ricci tensor are then tensor distributions in $\mathcal{D}'\mathcal{T}^1_3(M)$ and $\mathcal{D}'\mathcal{T}^0_2(M)$, respectively, where we recall that

$$\mathcal{D}'\mathcal{T}^r_s(M) := \Gamma_c\left(M, T^s_r M \otimes \mathrm{Vol}(M)\right)' = \mathcal{D}'(M) \otimes_{\mathcal{C}^\infty} \mathcal{T}^r_s(M)\,. \tag{1}$$

Here $\mathrm{Vol}(M)$ is the volume bundle over M, Γ_c denotes spaces of sections with compact support and $\mathcal{D}'(M)$ is the space of scalar distributions on M, i.e. the topological dual of the space of compactly supported volume densities $\Gamma_c(M, \mathrm{Vol}(M))$. We remark that, containing derivatives of the continuous connection, the distributional Riemann and Ricci tensors are of order one and that the usual coordinate formulae

$$\mathrm{Riem}^m{}_{ijk} := \partial_j \Gamma^m_{ik} - \partial_k \Gamma^m_{ij} + \Gamma^m_{js}\Gamma^s_{ik} - \Gamma^m_{ks}\Gamma^s_{ij}\,, \tag{2}$$

$$\mathrm{Ric}_{ij} := \partial_m \Gamma^m_{ij} - \partial_j \Gamma^m_{im} + \Gamma^m_{ij}\Gamma^k_{km} - \Gamma^m_{ik}\Gamma^k_{jm} \tag{3}$$

apply. For further details on tensor distributions see [14, Ch. 3.1].

Naturally, we define curvature bounds resp. energy conditions using the notion of positivity for distributions, i.e., $\mathcal{D}'(M) \ni u \geq 0$ ($u > 0$) if $\langle u, \mu\rangle \geq 0$ (> 0) for all non-negative (positive) volume densities $\mu \in \Gamma_c(M, \mathrm{Vol}(M))$. Then (M, g) is said to satisfy the strong energy condition (resp. to have nonnegative Ricci curvature) if the scalar distribution $\mathrm{Ric}(\mathcal{X}, \mathcal{X})$ is nonnegative for all smooth timelike vector fields $\mathcal{X}$. In the case of g being smooth, this condition coincides with the classical one, $\mathrm{Ric}(X, X) \geq 0$ for all timelike $X \in T_pM$ and all $p \in M$ by the fact that all such X can be extended to smooth timelike vector fields on M. For the same reason, the condition for $g \in C^{1,1}$ is equivalent to the condition $\mathrm{Ric}(\mathcal{X}, \mathcal{X}) \geq 0$ almost everywhere for all smooth timelike vector fields $\mathcal{X}$, used in the context of the $C^{1,1}$-singularity theorems. However, generalizing the null energy condition is more tricky due to the obstacles one encounters when extending null vectors. For a detailed discussion see [20, Sect. 5], and we define following her:

[3] The alternative would be to only demand the existence of *one* complete geodesic for every timelike (or null or causal) initial condition to the geodesic equation.

Definition 2.1 (*Distributional null energy condition*) A C^1-metric g satisfies the *distributional null energy condition*, if for any compact set $K \subseteq M$ and any $\delta > 0$ there exists $\epsilon(\delta, K)$ such that $\mathrm{Ric}(\mathcal{X}, \mathcal{X}) > -\delta$ (in the sense of distributions) for any local smooth vector field $\mathcal{X} \in \mathfrak{X}(U)$, $U \subseteq K$ with $\|\mathcal{X}\|_h = 1$ and which is $\epsilon(\delta, K)$ close to a C^1 g-null vector field $\mathcal{N}$ on U, i.e., $||\mathcal{X} - \mathcal{N}||_h < \epsilon(\delta, K)$ on U.

Again this condition is equivalent to the classical null energy condition if the metric is smooth. Moreover, in case $g \in C^{1,1}$, it is equivalent to the condition used in the $C^{1,1}$-setting, i.e., $\mathrm{Ric}(\mathcal{X}, \mathcal{X}) > 0$ for all Lipschitz-continuous local null vector fields $\mathcal{X}$.

One key technique in low regularity Lorentzian geometry is regularization. More specifically, Chrusciel and Grant in [5] have put forward a technique to regularize a continuous metric g by smooth metrics $\check{g}_\varepsilon$ with narrower lightcones resp. by a net $\hat{g}_\varepsilon$ with wider lightcones than g. The basic operation (denoted by $*$) is chartwise convolution with a standard mollifier ρ_ϵ, which is globalized using cutoffs and a partition of unity, cf. [14, Thm. 3.2.10]. To manipulate the lightcones in the desired way one has to add a "spacelike correction term." The most recent version of this construction, which also quantifies the rate of convergence in terms of ϵ is [20, Lem. 4.2], which we recall here.

Lemma 2.2 *Let (M, g) be a spacetime with a C^1-Lorentzian metric. Then for any $\epsilon > 0$, there exist smooth Lorentzian metrics $\check{g}_\epsilon$ and $\hat{g}_\epsilon$ with $\check{g}_\epsilon \prec g \prec \hat{g}_\epsilon$, both converging to g in C^1_{loc}. Additionally, on any compact set K, there is $c_k > 0$ such that for all small ϵ*

$$\|\check{g}_\epsilon - g * \rho_\epsilon\|_{\infty,K} \leq c_K \epsilon \quad \textit{and} \quad \|\hat{g}_\epsilon - g * \rho_\epsilon\|_{\infty,K} \leq c_K \epsilon \,. \tag{4}$$

A main step in the proof of singularity theorems in low regularity is to show that the energy condition (Definition 2.1, in our case) implies that the regularized metrics $\hat{g}_\epsilon$ and/or $\check{g}_\epsilon$ violate the classical energy conditions (the NEC, in our case) only by a small amount—small enough, such that null geodesics still tend to focus. Technically this is done by a Friedrichs-type lemma, which in the present case is [20, Lem. 4.5], and draws essentially from (4). The corresponding result is then [20, Lem. 5.5]:

Lemma 2.3 (Surrogate energy condition) *Let M be a C^1-spacetime where the distributional null energy condition holds. Given any compact set $K \subseteq M$ and $c_2 > c_1 > 0$, then for all $\delta > 0$, there is $\epsilon_0 > 0$ such that $\forall \epsilon < \epsilon_0$*

$$\begin{aligned} &\mathrm{Ric}[\check{g}_\epsilon](X, X) > -\delta \quad \forall X \in TM|_K \textit{ with } \check{g}_\epsilon(X, X) = 0 \\ &\quad \textit{and } 0 < c_1 \leq ||X||_h \leq c_2 \,. \end{aligned} \tag{5}$$

Here $\check{g}_\epsilon$ is as in Lemma 2.2 and $\mathrm{Ric}[\check{g}_\epsilon]$ is its Ricci tensor. We will use this result in an essential way, when showing compactness of the horismos of a certain set, which is needed for the causal/analytic part of the proof of our Gannon–Lee theorem. The main difference to the arguments in [20] is that we do not assume global hyperbolicity, but are able to compensate for it by assuming non-branching of null maximizers. Formally, we define:

Definition 2.4 Let M be a spacetime and let $\gamma : [0,1] \to M$ be a maximizing null curve. We say that γ branches if there exists another maximizing null curve $\sigma : [0,1] \to M$ such that $\gamma(t) = \sigma(t)$ for all $0 \leq t \leq a$ for some $0 < a < 1$ and $\gamma(t) \neq \sigma(t)$ for all $1 \geq t > a$. The point $\gamma(a)$ is called branching point. If no maximizing null curve branches, we say that there is *no null branching* in M.

In light of the fact that (even) for C^1-metrics causal maximizers are geodesics (to be proven in Theorem 3.3, below), we see that if null branching were to occur in $\gamma(a)$, there would be (at least) two different, maximizing solutions to the geodesic equations with initial values $\gamma(a)$ and $\gamma'(a)$.

Generally speaking, in the low regularity Riemannian setting, branching of maximizers is associated with unbounded sectional curvature from below. More precisely, in length spaces with a lower curvature bound, branching does not occur [42, Lem. 2.4]. In smooth Lorentzian manifolds sectional curvature bounds, although more delicate to handle, are still characterized by triangle comparison [1] and in the setting of Lorentzian length spaces [27] synthetic curvature bounds extending sectional curvature bounds for smooth spacetimes have been introduced. In [27, Sect. 4], the authors show that a synthetic curvature bound from below prevents the branching of timelike maximizers. Unfortunately, it is not clear at the moment how one could extend such a result to triangles with null sides.

However, in a merely C^1-spacetime the curvature is generically not locally bounded and so it seems natural that an additional condition as in Definition 2.4 is needed. Indeed, this condition enters in an essential way into our arguments. It will be a topic of future investigations to relate null branching to curvature bounds.

2.2 Results

Next we introduce the particular notions needed for the formulation of our results. In spirit, they reflect Gannon's [11] assumptions on the spacetime, inspired by asymptotic flatness, however, we stay close to the formulations of [3, Sect. 2].

Definition 2.5 An *asymptotically regular hypersurface* is a spacelike, smooth, connected partial Cauchy surface Σ which possesses an enclosing surface S, i.e. a compact, connected submanfiold of codim. 2 in M with the properties

(1) S separates Σ into two (open, sub-) manifolds Σ_+, Σ_- such that $\bar{\Sigma}_+$ is non-compact and Σ_- connected,
(2) The map $h_\# : \pi_1(S) \to \pi_1(\bar{\Sigma}_+)$, induced by the inclusion $h : S \to \bar{\Sigma}_+$ is surjective,
(3) $k_- > 0$ on S, i.e. S is inner trapped.

We further say that a surface Σ admits a *piercing*, if there exists a timelike vector field X on M such that every integral curve of X meets Σ exactly once.

To elaborate on this definition, we first fix some notation. Throughout let us denote the future directed timelike unit vector field perpendicular to Σ near S by U. Further as S is a hypersurface in Σ we denote by $N_\pm$ the unit normals to S in Σ such that N_- points into Σ_- and N_+ into Σ_+. We obtain future directed null normals to S via

$K_\pm := U|_S + N_\pm$. We will refer to $K_-(p)$ as the ingoing null vector. The convergence of a point in S is defined via $k_\pm := g(H_p, K_\pm(p))$, where H_p is the mean–curvature vector field of Σ at p. Observe that since $g \in C^1$, H_p is still continuous and all corresponding formulae hold "classically."

Also note that any piercing of Σ induces a continuous, open map $\rho_X : M \to \Sigma$, which maps any point p in M to the unique intersection of the integral curve of X through p with Σ. However, existence of a piercing is a strictly weaker condition than global hyperbolicity; see, e.g., [3, p. 4, 2nd paragraph], although it implies that M is homeomorphic to the product $\mathbb{R} \times \Sigma$ as pointed out in [4, below 3.2].

We are now ready to state our main result, a Gannon–Lee theorem for non-globally hyperbolic C^1-spacetimes without null branching. We will discuss several of its special cases below.

Theorem 2.6 (C^1-Gannon–Lee theorem) *Let (M, g) be a past reflecting, null geodesically complete C^1-spacetime without null branching and such that the distributional null energy condition holds. Let Σ be an asymptotically regular hypersurface (with enclosing surface S) admitting a piercing. Further let one of the following possibilities hold*

(i) Any covering spacetime of (M, g) is past reflecting, or
(ii) S is simply connected and the universal covering spacetime of (M, g) is past reflecting.

Then the map $i_\# : \pi_1(S) \to \pi_1(\Sigma)$, induced by the inclusion $i : S \to \Sigma$, is surjective.

Note that for $C^{1,1}$-metrics the geodesic equation is uniquely solvable and hence there can be no null branching. Moreover using that the distributional null energy condition reduces to the "almost everywhere condition" for $C^{1,1}$-metrics we immediately obtain the following $C^{1,1}$-Gannon–Lee theorem.

Corollary 2.7 ($C^{1,1}$-Gannon–Lee theorem) *Let (M, g) be a past reflecting, null geodesically complete $C^{1,1}$-spacetime such that the null energy condition $\mathrm{Ric}(\mathcal{X}, \mathcal{X}) \geq 0$ holds for all local Lipschitz-continuous null vector fields $\mathcal{X}$. Let Σ be an asymptotically regular hypersurface (with enclosing surface S) admitting a piercing. Further let one of the assumptions (i) or (ii) of Theorem 2.6 hold. Then the map $i_\# : \pi_1(S) \to \pi_1(\Sigma)$ is surjective.*

Going back to C^1 and assuming global hyperbolicity we clearly can skip past reflectivity and any assumption on the covering spacetime. Due to results in [20], the assumption of no null branching is also obsolete. Moreover, in this case also, some of the assumptions in [3] can be dropped, as they are implied by the existence of a Cauchy surface.

Corollary 2.8 (Globally hyperbolic C^1-Gannon–Lee theorem) *Let (M, g) be a globally hyperbolic, null geodesically complete C^1-spacetime such that the distributional null energy condition holds. Further let Σ be an asymptotically regular hypersurface (with enclosing surface S), then the map $i_\# : \pi_1(S) \to \pi_1(\Sigma)$ is surjective.*

A simpler formulation is obtained assuming that S is simply connected: the theorems then state that the entire spacetime is simply connected provided it is null complete.

Originally, the theorem of Gannon was given in contrapositive form, saying that if S is (topologically) a sphere and Σ is not simply connected, then M has to be null incomplete.

In proving our results we will follow the general layout of [3]. The proof consists of a causal and an analytic part as well as a purely topological part. At the heart of the causal part lies Proposition 4.1 of [3], which we will prove for past reflecting C^1-spacetimes without null branching in Proposition 4.3. It essentially states that the inside region Σ_- of an asymptotically regular hypersurface Σ is relatively compact. We will first establish the needed causality properties for C^1-spacetimes.

3 Maximizers and causality in C^1

In this section, we establish properties of geodesics and results on causality in C^1-spacetimes. Building on recent results of [18], [20] we will establish that causal maximizers in C^1-spacetimes are geodesics. Note that this result was also independently discovered very recently in [32].

First note that by [18, Thm. 1.1] maximizers have a causal character, even in Lipschitz spacetimes.

We start by showing that also in a C^1-spacetime broken causal geodesics are not maximizing. As a prerequisite, we use a variational argument similar to the one in [39], 10.45–46, which still holds true in our setting.

Lemma 3.1 *Let $c : [0, 1] \to M$ be a causal pieceweise C^2-curve in a C^1-spacetime M and let X be a piecewise C^1-vector field along c. Then there is a piecewise C^2-variation c_s of c with variation vector field X. Moreover, if $g(X', c') < 0$ along c then for any variation c_s of c with variation vector field X and small enough s, the longitudinal curve c_s is timelike and longer than c.*

Proof First we expand X to a vector field $\tilde{X}$ in a neighborhood of c and set $c_s(t) := \mathrm{Fl}^{\tilde{X}}_s(c(t))$, which is a variation of c with variation vector field X. For the second part note that $g(c_0'(t), c_0'(t)) \le 0$ for all t (except possible break points) as c is causal and further

$$\frac{\partial}{\partial s}|_0 g(c_s'(t), c_s'(t)) = 2g\left(\frac{\partial}{\partial s}|_0 \frac{\partial}{\partial t} c_s(t), c_0'(t)\right) = 2g\left(\frac{\partial}{\partial t} X(t), c'(t)\right) < 0$$

by assumption. So for small s, we have $g(c_s'(t), c_s'(t)) < g(c'(t), c'(t)) \le 0$ (for almost all t) and hence $L(c_s) = \int_0^1 (-g(c_s'(t), c_s'(t)))^{\frac{1}{2}}\, \mathrm{d}t > \int_0^1 (-g(c_0'(t), c_0'(t)))^{\frac{1}{2}}\, \mathrm{d}t = L(c)$. □

Lemma 3.2 *In a C^1-spacetime no broken causal geodesic is maximizing.*

Proof Let $\gamma : [0, 2] \to M$ be a broken causal geodesic with a break point at $\gamma(1)$. Hence $v := \lim_{t\uparrow 1} \gamma'(t)$ and $w := \lim_{t\downarrow 1} \gamma'(t)$ are linearly independent. Also since both vectors are either future or past pointing, we have $\langle v, w\rangle < 0$.

Let us show that the parameteriziation of γ can be chosen such that:

$$\langle v, v\rangle - \langle v, w\rangle > 0 \quad \text{as well as} \quad \langle v, w\rangle - \langle w, w\rangle < 0. \tag{6}$$

If both v and w are null, this is clear by $\langle v, w\rangle < 0$. If both vectors are timelike we can w.l.o.g. assume them to be unit vectors. By the reverse Cauchy–Schwarz inequality, we then have $|\langle v, w\rangle| > 1$ (recall that v and w are not colinear) and hence $\langle v, w\rangle < -1$. So $\langle v, v\rangle - \langle v, w\rangle > 0$, and in the same way it follows that $\langle v, w\rangle - \langle w, w\rangle < 0$.

We now show that there exists a timelike curve from $\gamma(0)$ to $\gamma(2)$ longer than γ. First note that by continuity of the the Christoffel symbols, we can solve the linear equations for parallel transport along γ (uniquely) and the solution is a C^1-vector field.

We set $y = v - w$ and let Y_1 and Y_2 be the vector fields along $\gamma|_{[0,1]}$ and $\gamma|_{[1,2]}$, obtained by parallel transport of y along $\gamma|_{[0,1]}$ and $\gamma|_{[1,2]}$, respectively. Next we define a piecewise C^1-vector field Y along γ by setting $Y|_{[0,1]} = Y_1$ and $Y|_{[1,2]} = Y_2$.

Since γ is a geodesic and Y is parallel, we have on $[0, 1]$ using (6)

$$\langle \gamma'(t), Y(t)\rangle = \langle v, v - w\rangle = \langle v, v\rangle - \langle v, w\rangle > 0 \tag{7}$$

and similarly on $[1, 2]$ we have $\langle \gamma'(t), Y\rangle < 0$.

Now let $f : [0, 2] \to [0, \infty)$ be a continuous, piecewise linear function such that $f(0) = 0 = f(2)$, $f'|_{[0,1)} = 1$, and $f'|_{(1,2]} = -1$ and set $X(t) := f(t)Y(t)$. Then X is a piecewise C^1-vector field along γ. Now consider the variation γ_s of γ with variation vector field X. By (7) we have $\langle \gamma', X'\rangle < 0$ and hence by Lemma 3.1 there exists some small s, such that γ_s is longer than γ. By the choice of f the endpoints agree and we have shown the statement. □

Theorem 3.3 *Let (M, g) be a C^1-spacetime, then any maximizing causal curve is a causal geodesic and hence a C^2-curve.*

Proof Observe that being a geodesic is a local property and that any part of a maximizing curve is maximizing. Moreover, since any point in a C^1-spacetime has a globally hyperbolic neighborhood,[4] we can assume M to be globally hyperbolic.

Let $\gamma : [0, 1] \to M$ be a maximizer and set $p = \gamma(0)$, $q = \gamma(1)$. By [20], Proposition 2.13, there exists a maximizing causal geodesic from p to q of the same causal character as γ. Also there exist maximizing, causal geodesics σ_1 from p to $\gamma(\frac{1}{2})$ and σ_2 from $\gamma(\frac{1}{2})$ to q. Note that since σ_i are maximizing, we have $L(\sigma_1) = L(\gamma|_{[0,\frac{1}{2}]})$ and $L(\sigma_2) = L(\gamma|_{[\frac{1}{2},1]})$. This means $L(\sigma_1 \circ \sigma_2) = L(\gamma)$. So the curve $\gamma_1 := \sigma_1 \circ \sigma_2$ is maximizing, and hence by Lemma 3.2, it is an unbroken geodesic.

This procedure can be iterated to obtain a sequence of maximizing causal geodesics γ_n from p to q, which meet γ at all parameter values $\frac{k}{2^n}$, for $\mathbb{N} \ni k \leq 2^n$. Observe that γ_n converge to γ uniformly: First, for any $\epsilon > 0$ we cover γ by finitely many open, causally convex, sets $V^{\epsilon}_{p_i}$ around $p_i \in \gamma$ of h-diameter at most ϵ. The union $V^{\epsilon} = \sup_i V^{\epsilon}_{p_i}$ is a neighborhood of γ, and there exist dyadic numbers $t_m, m = 0, \dots k$,

[4] Actually there exists a smooth metric with wider lightcones, which has a neighborhood base of globally hyperbolic sets, but these are also globally hyperbolic for g.

$t_0 = 0$ and $t_k = 1$, such that $\gamma(t_m)$ and $\gamma(t_{m+1})$ lie in a single $V^\epsilon_{p_i}$ for some i. Moreover, there exists some $N(\epsilon)$, such that for all $n \geq N$ all curves γ_n meet every $\gamma(t_m)$ and hence the segments of γ_n from t_m to t_{m+1} are contained in $V^\epsilon_{p_i}$. So we conclude that $d_h(\gamma(t), \gamma_n(t)) \leq \epsilon$, so $\gamma_n \to \gamma$ uniformly.

We can now parameterize γ_n, such that $\|\gamma_n'(0)\|_h = 1$ and hence pass to a subsequence (again denoted by γ_n) such that $\gamma_n'(0) \to v$. By [23, Chap. II Thm. 3.2,], there exists a subsequence of γ_n which converges uniformly on compact sets to a geodesic σ with initial values $\sigma(0) = p$ and $\sigma'(0) = v$. But as γ_n converges to γ, so must any subsequence and hence $\gamma = \sigma$ on the entire domain of γ. Finally, σ reaches q since otherwise, σ would agree with γ on its entire maximal domain of definition, would be future inextendible and contained in the compact set $\gamma([0, 1])$, a contradiction to non-imprisonment which holds in any globally hyperbolic C^1-spacetime. □

Assuming non-branching we are able to prove results on limits of maximizers needed in the following.

Proposition 3.4 *Let M be a C^1-spacetime without null branching. If two causal geodesic segments contained in an achronal set intersect, they are segments of the same geodesic or they intersect at the endpoints.*

Proof Suppose such a segment intersetcs a second one in the interior of its domain. Then either their tangents at the meeting point are not proportional and so by Lemma 3.2 their concatination, which is a broken causal geodesic, stops maximizing, contradicting the fact that both segments are contained in an achronal set. Or otherwise their tangents at the meeting point are proportional and hence null branching would occur, again a contradiction. □

Using Theorem 3.3 we may also give a slightly different formulation: If two different initially maximizing null curves starting at the same point meet again, they stop maximizing.

The final result of this section will be essential in the proof of the main theorem, and it is the main point at which we use the assumption that null branching does not occur. Again, $\check{g}_\epsilon$ is as in Lemma 2.2.

Corollary 3.5 *Let (M, g) be a C^1-spacetime without null branching. Let S be an enclosing surface in a partial Cauchy surface Σ and let $\gamma : [0, 1] \to E^+_g(S)$ be a g-null curve. Then for any $1 > \delta > 0$, $\gamma|_{[0,1-\delta]}$ is a limit of $\check{g}_{\epsilon_n}$-null curves contained in $E^+_{\check{g}_{\epsilon_n}}(S)$ for an appropriate subsequence of $\check{g}_{\epsilon_n}$.*

Proof Let γ be a future directed g-null S-maximizer starting at $p = \gamma(0) \in S$, so $\gamma \subseteq E^+(p)$. For any $1 > \delta > 0$, there exist points $q^\delta_n \in \partial I^+_n(S) := \partial I^+_{\check{g}_{\epsilon_n}}(S)$ converging to $\gamma(1-\delta) =: q^\delta$: To see this let U_k be a sequence of connected nested neighborhoods of q^δ with $\bigcap_k U_k = \{q^\delta\}$. Choose some $q^e_k \in U_k \backslash \overline{J^+(S)}$ and $q^i_k \in U_k \cap I^+(S)$. For large n, we can achieve $q^i_k \in I^+_n(S)$ and since also $q^e_k \in U_k \backslash \overline{J^+_n(S)}$ there exists a curve from q^i_k to q^e_k which starts in $I^+_n(S)$ and leaves it and hence must meet $\partial I^+_n(S)$ in a point which we call q^δ_n.

Hence there are future directed $\check{g}_{\epsilon_n}$-null maximizing geodesics γ^δ_n ending at q^δ_n and contained in $\partial I^+_n(S)$. Note that the γ^δ_n either meet S or are past inextendible.

Now by [23, Chap. II, Thm. 3.2] there exists a subsequence of γ_n^δ, denoted again by γ_n^δ, converging to a maximizing g-null geodesic σ^δ ending at q^δ which is entirely contained in $\partial I^+(S)$. By Theorem 3.3 γ is a geodesic and continues to be maximizing after q^δ. Also, by construction σ^δ is non-trivial, so γ and σ^δ coincide on some $\gamma|_{[a,1-\delta]}$ (with $0 \le a < 1-\delta$) by Proposition 3.4.

There are two possibilities: Either there exists a subsequence of $\check{g}_{\epsilon_n}$ such that any γ_n^δ meets S, in which case, by passing to this subsequence, also σ^δ meets S and hence $\sigma^\delta \supseteq \gamma|_{[0,1-\delta]}$ and we obtain the desired property.

The other possibility is that there is no subsequence, such that all γ_n^δ meet S. We show that this is impossible. If this were the case we could choose a subsequence γ_n^δ of past-inextendible curves. Then also σ^δ is past-inextendible and hence must leave $\gamma([0, 1-\delta])$. Thus again $\sigma^\delta \supseteq \gamma|_{[0,1-\delta]}$ and it remains to show that $\gamma_n^\delta \subseteq E_n^+(S)$ for large n.

To this end let $U := D(\Sigma)$, which is a globally hyperbolic, open, and causally convex neighborhood of $p = \gamma(0) \in S$ for the metric g [36, Cor. 3.36, Prop. 3.43][5] and hence also for g_{ε_n}[6]. Then there are points $p_n \in \gamma_n^\delta$ with $p_n \to p$ and so for large n, $p_n \in \partial I_n^+(S) \cap U = \partial I_n^+(S, U) = E_n^+(S, U)$. Hence there is a g_{ε_n}-null geodesic from S to p_n contained in $E_n^+(S, U)$ which by Proposition 3.4 coincides with (a part of) γ_n^δ. Hence γ_n^δ must meet S, a contradiction. □

Observe that Corollary 3.5 clearly holds true for $C^{1,1}$-spacetimes. For globally hyperbolic C^1-spacetimes (where, in principle, null branching can occur) a similar result was established in [20, Prop. 2.16], with slightly different assumptions on S and a weaker conclusion, which only guarantees that for any $p \in E_g^+(S)$, there is a geodesic segment which is an appropriate limit of approximating $\check{g}_\epsilon$-maximizers.

4 Proof of the main result

We will split the proof of Theorem 2.6 into two parts. The first, analytic part will be concerned with showing that the set Σ_- is relatively compact. Here we will generalize (the proof of) [3, Prop. 4.1] by proving new focusing statements for null geodesics using the results from Sect. 3.

The second, topological part uses the results from causality theory detailed in Sect. 3 to generalize (the proof of) [3, Thm. 2.1].

To be self-contained we will briefly sketch also those parts of the original (smooth) proofs which do not need major revision.

4.1 Analytic aspects

The main analytical ingredient of our proof is a generalization of [3, Prop. 4.1] to C^1-spacetimes, which we will give below in Proposition 4.3.

[5] The proofs carry over verbatim to C^1-metrics.

[6] This follows immediately since we approximate from the inside. However, global hyperbolicity is stable in the interval topology even for continuous metrics [45] and C^1-convergence is stronger.

In order to do so we need a focusing result for null geodesics in smooth spacetimes which violate the null energy condition by a small margin δ, as in Lemma 2.3. To this end, we apply a result of [8][7], which itself is a generalization of [39, Prop. 10.43].

Proposition 4.1 [8, Prop. 2.7] *Let S be a spacelike submanifold of co-dimension 2 in a smooth spacetime and let γ be a null geodesic joining $p \in S$ to $q \in J^+(S)$. If there exists a smooth function f on γ which is nonvanishing at p but vanishes at q and so that*

$$\int_\gamma \Big((n-2)(f')^2 - f^2\,\mathrm{Ric}(\gamma',\gamma')\Big) \;\leq\; (n-2)\,\langle f^2\,\gamma', H\rangle\rangle\,|_p\,, \tag{8}$$

then there is a focal point to S along γ.

Lemma 4.2 *Let S be a C^2-spacelike submanifold of codimension 2 in a smooth spacetime. Let γ be a geodesic starting at some $p \in S$ such that $\nu := \gamma'(0)$ is a future pointing null normal to S. Let the convergence*

$$c := k_S(\nu) := \langle H_{\gamma(0)}, \nu\rangle > 0 \tag{9}$$

and choose some $b > \frac{1}{c}$ and $0 < \delta(b,c) =: \delta \leq \frac{3}{b^2}(n-2)(bc-1)$. Now, if $Ric(\gamma',\gamma') \geq -\delta$ along γ, then $\gamma|_{[0,b]}$ cannot be maximizing to S, provided it exists that long.[8]

Proof We set $f(t) := 1 - \frac{t}{b}$ and check condition (8). For our choice of δ, we obtain

$$\begin{aligned}&\int_0^b (n-2)\,\frac{1}{b^2}\,\mathrm{d}t - \int_0^b \left(1 - 2\,\frac{t}{b} + \frac{t^2}{b^2}\right)\mathrm{Ric}(\gamma'(t),\gamma'(t))\,\mathrm{d}t\\ &\quad\leq \frac{n-2}{b} + \delta\,\frac{b}{3} \leq (n-2)\,c = (n-2)\,k_S(\nu) = (n-2)\,\langle f^2\gamma', H\rangle\rangle\,|_p\,,\end{aligned} \tag{10}$$

and hence $\gamma|_{[0,b]}$ cannot be maximizing. □

Recall that for C^1-metrics the mean curvature and the convergence of S are still continuous. The core of the following proof is largely from [4].

Proposition 4.3 *Let (M, g) be an n-dimensional (with $n \geq 3$), past reflecting, null geodesically complete C^1-spacetime without null branching which satisfies the distributional null energy condition and admits an asymptotically regular hypersurface Σ with a piercing. Then for any enclosing surface $S \subseteq \Sigma$ the closure of its inside, $\overline{\Sigma}_- = S \cup \Sigma_-$ is compact.*

We consider the closed set $T := \partial I^+(\Sigma_+)\backslash\Sigma_+$. Also, by Theorem 3.3 the set $E^+(S)$ really consists of null *geodesics* emanating from S and perpendicular to S, which follows as in [39, 10.45, 10.50] by using Lemma 3.1. So we define $\mathcal{H}^+$ as the

7 Observe the opposite signature convention used there.

8 One can specify the choice of b further to allow for bigger violations of δ, but we do not need this here.

subset of all points $p \subseteq E^+(S)$ on future directed null geodesics $\gamma : [0,1] \to M$ with $\gamma(0) \in S$, $\gamma'(0) = K_-(\gamma(0))$. Note that $S \subseteq \mathcal{H}^+$. Further, no point on $\mathcal{H}^+$ can lie on a null geodesic from S in direction of K_+, see e.g. [11, Lemma 1.1][9]

The proof consists in successively establishing the following three claims:

(1) $\mathcal{H}^+$ is relatively compact, (2) T is compact, (3) $\rho_X(T) = \overline{\Sigma}_-$.

Steps (1) and (2) combine the causality part of the proof with the analytical arguments which we have to provide in C^1-regularity. Some arguments in steps (2) and (3) do not require changes from the original proofs put forward in [4] resp. [3] but will be included as a sketch for the sake of completeness.

Proof (1) Any point p on $\mathcal{H}^+$ lies on a null geodesic emanating from S and its initial tangent vector is inward pointing, i.e. proportional to $K_-(q)$ for some $q \in S$. By continuity k_- possesses a minimum $c := \min_{p\in S} k_-(p) = \min_{p\in S}\langle H_p, K_-(p)\rangle$ on S. Also, the set $K := \{(p, \lambda K_-(p)) \in TS^\perp \,|\, 0 \le \lambda \le \frac{2}{c}\} \subseteq TM$ is compact and, by [20, Prop. 2.11] (or rather a simplified version without ϵ) the set

$$F := \bigcup_{\dot\gamma \text{ with } \dot\gamma(0)\in K} \mathrm{im}(\dot\gamma|_{[0,1]}) \tag{11}$$

is relatively compact (Here $\dot\gamma$ denotes the trajectory of a geodesic in TM with the specified initial conditions). We will show that $\mathcal{H}^+ \subseteq \pi(F)$, where $\pi : TM \to M$ is the projection. Assume the contrary and let $p \in \mathcal{H}^+ \backslash \pi(F)$. Let $\gamma : [0,1] \to M$ be a null geodesic from S to p maximizing the distance to S. As $\gamma \subseteq \mathcal{H}^+$ we know that $\gamma'(0) = \mu K_-(\gamma(0))$ for some $\mu > \frac{2}{c}$, since $\gamma'(0) \notin K$. This means that $k_S(\gamma'(0)) = \langle H(\gamma(0)), \mu K_-(\gamma(0))\rangle \ge \mu c > 2$. Let $\check g_{\epsilon_k}$ be as in Lemma 2.2. By Corollary 3.5 $\gamma|_{[0,1-\delta]}$ for some arbitrarily small δ is the C^1-limit of $\check g_{\epsilon_k}$-null geodesics $\gamma_{\epsilon_k} : [0, b_k] \to M$ with $b_k \to 1-\delta$ contained in $E^+_{\check g_{\epsilon_k}}(S)$. Further we can assume that all γ_{ϵ_k} are contained in a compact neighborhood $\tilde K$ of γ and that $c_1 < \|\gamma'_{\epsilon_k}\|_h < c_2$ for some $c_i > 0$. Additionally, for k large enough, we have $k_S^{\epsilon_k}(\gamma'_{\epsilon_k}(0)) := c_k > 2$ and by Lemma 2.3 we can also achieve $\mathrm{Ric}[g_{\epsilon_k}](\gamma'_{\epsilon_k}, \gamma'_{\epsilon_k}) \ge -3(n-2)$. In order to apply Lemma 4.2 for $b = 1$, we set $\delta_k = \frac{3}{b^2}(n-2)(b\,c_k - 1) = 3(n-2)(c_k - 1)$. Since $c_k > 2$ for all large k, we have $-\delta_k < -3(n-2)$ and hence $\mathrm{Ric}[g_{\epsilon_k}](\gamma'_{\epsilon_k}, \gamma'_{\epsilon_k}) \ge -3(n-2) > -\delta_k$. So by Lemma 4.2, γ_{ϵ_k} cannot be maximizing up to $\frac{1}{c_k} < \frac{1}{2} < 1-\delta < b$ but by construction it is maximizing up to $b_k \to 1-\delta$, a contradiction.

(2) We prove the inclusion $T \subseteq \mathcal{H}^+$ which gives that T is compact. Assume by contradiction that there were $q \in T\backslash\mathcal{H}^+$. By [4, Proof of 3.5] we can find $q_n \in I^+(S)$ such that $q_n \to q$ and future directed, future inextendible timelike curves $\sigma_n : [0,\infty) \to M$ parametrized by h-arclength starting at S with $\sigma_n(t_n) = q_n$. Further by the limit curve theorem, which is valid in C^1 spacetimes, see [35, Thm. 14] one obtains a future directed, future inextendible causal curve σ starting at S. If q were to lie on σ, it had to be a maximizing null geodesic perpendicular to

[9] The proof there is given for smooth spacetimes and one assumes S to be simply connected; however, the method of proof also works in our case.

S[10]. It can however neither start inward going (in direction K_-) as then $q \in \mathcal{H}^+$ nor outward going (in direction K_+) as then $q \in I^+(\Sigma_+)$. Hence $q \notin \sigma$ and so $t_n \to \infty$. Since by (1) there is no inward pointing S-null ray, there is $b \in (0, \infty)$ with $\sigma(b) \in I^+(S)$. But then, again by the limit curve theorem, $q \in \overline{I^+(\sigma(b))}$. By past reflectivity, one obtains $\sigma(b) \in \overline{I^-(q)} \cap I^+(S)$, implying $q \in I^+(S)$ and hence $q \in I^+(\Sigma_+)$, contradicting $q \in T$.

(3) First $\rho_X(T) \subseteq \overline{\Sigma}_-$ (since otherwise $T \cap I^+(\Sigma_+) \neq \emptyset$) and $\overline{\Sigma}_- \setminus \rho_X(T) \subseteq \Sigma_-$ (since $S = \rho_X(S) \subseteq \rho_X(T)$). Now assuming indirectly that $\overline{\Sigma}_- \setminus \rho_X(T) \neq \emptyset$, there is $p \in \partial_\Sigma \rho_X(T) \cap \Sigma_-$ and we will reach a contradiction by showing that $p \in \operatorname{int}\rho_X(T)$. By compactness of T there is $q \in T$ with $\rho_X(q) = p$, and $q \notin S$ (otherwise $p = q \in S \cap \Sigma_- = \emptyset$). So $q \in T \setminus S = \partial I^+(\Sigma_+) \setminus \overline{\Sigma}_+$ which is a topological hypersurface (S being the edge of the achronal set T). So there is V_0, an M-neighborhood of q with $V_0 \cap T$ open in $T \setminus S$, $\rho_X(V_0) \subseteq \Sigma_-$, and $V_0 \cap \Sigma = \emptyset$. Next denote by Ψ the local flow of X and choose $\epsilon > 0$ and U_0, an open M-neighborhood of q, so small that $\Psi(U_0 \times (-\epsilon, \epsilon)) \subseteq V_0$. Further set $\Psi_0 := \Psi_{|(U_0 \cap T) \times (-\epsilon, \epsilon)}$, and $W := \operatorname{Im}\Psi_0$. By achronality of T and invariance of domain W is open and Ψ_0 is a homeomorphism. But then $p \in \rho_X(U_0 \times (-\epsilon, \epsilon)) = \rho_X(W)$ and the latter set is open by openness of ρ_X [39, 14.31] and so $p \in \operatorname{int}\rho_X(T)$.

□

4.2 Topological aspects

Finally, we invoke Proposition 4.3 to prove our main result. Here we will be brief on the topological aspects laid out already in the proof of [3, Thm. 2.1].

Proof (Proof of theorem 2.6)

Let $\Phi : \tilde{M} \to M$ be a connected (smooth) covering with $\Phi_\#(\pi_1(\tilde{M})) = j_\#(\pi_1(S))$, where j is the inclusion of S in M.

Note that w.l.o.g. one can assume the vector field of the piercing to be complete and by properties of its flow map easily show that $M \cong \mathbb{R} \times \Sigma$. Hence the inclusion m of Σ in M induces an isomorphism $m_\# : \pi_1(\Sigma) \to \pi_1(M)$. In particular, $\Phi_\Sigma := \Phi|_{\tilde{\Sigma}} : \Phi^{-1}(\Sigma) := \tilde{\Sigma} \to \Sigma$ is a Riemannian covering with $\tilde{\Sigma}$ connected.

In Lemma 4.4 below we will establish that Φ_Σ is trivial. Accepting this for the moment, we will show the theorem, i.e. for every $y \in \pi_1(\Sigma)$ there is $x \in \pi_1(S)$ with $i_\#(x) = y$. From the diagrams,

$$\begin{tikzcd}
 & M & \\
S \arrow[ur, "j"] \arrow[r, hook, "i"] & \Sigma \arrow[u, "m"] & \tilde{M} \arrow[ul, "\Phi"'] \\
 & \tilde{\Sigma} \arrow[u, leftrightarrow, "\Phi_\Sigma"] \arrow[ur, "\tilde{m}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & j_\#(\pi_1(S)) \arrow[d, hook] \\
\pi_1(S) \arrow[urr, bend left, "j_\#"] \arrow[r, "i_\#"] & \pi_1(\Sigma) \arrow[r, leftrightarrow, "m_\#"] & \pi_1(M) \\
 & \pi_1(\tilde{\Sigma}) \arrow[u] \arrow[r] & \pi_1(\tilde{M}) \arrow[u] \arrow[uu, bend right, "\Phi_\#"']
\end{tikzcd}$$

[10] by the same argument as above using [39, 10.45, 10.50] and Lemma 3.1

we see that

$$m_\#(y) = (\Phi \circ \tilde{m} \circ \Phi_\Sigma^{-1})_\#(y) = \Phi_\#(\tilde{m} \circ \Phi_\Sigma^{-1})_\#(y) \in \Phi_\#(\pi_1(\tilde{M})) = j_\#(\pi_1(S)). \tag{12}$$

Since $j = m \circ i$ we have $m_\#(y) = j_\#(x) = m_\#(i_\#(x))$, and since $m_\#$ is an isomorphism and we are done. □

Lemma 4.4 $\Phi_\Sigma := \Phi|_{\tilde{\Sigma}} : \tilde{\Sigma} := \Phi^{-1}(\Sigma) \to \Sigma$ *is a trivial covering.*

Proof First, by definition there is a local deformation $F : S \times (-1, 1) \to \Sigma$ of S. Further set $U_F^- := F(S \times (0, 1))$ and $V := U_F^- \cup S \cup \Sigma_+$. Then $\bar{\Sigma}_+$ is a deformation retract of V and hence $\pi_1(\bar{\Sigma}_+) \cong \pi_1(V)$.

Next we establish that on every connected component $\tilde{V}$ of $\Phi_\Sigma^{-1}(V)$ the map $\Phi_V := \Phi|_{\tilde{V}} : \tilde{V} \to V$ is a diffeomorphism.

We only have to show injectivity since Φ_V is a local diffeomorphism. Take $\tilde{p}, \tilde{q} \in \tilde{V}$ such that $\Phi_V(\tilde{p}) = \Phi_V(\tilde{q}) =: p \in V$. Let $\tilde{\alpha} : [0, 1] \to \tilde{V}$ be a path connecting these two points, then $\alpha := \Phi \circ \tilde{\alpha}$ is a loop in V, homotopic to a loop in S since $\pi_1(V) \cong \pi_1(\bar{\Sigma}_+) \cong \pi_1(S)$. Further since $\Phi_\#(\pi_1(\tilde{M})) = j_\#(\pi_1(S))$, there is a loop $\tilde{\beta}$ in $\tilde{M}$, fixed endpoint-homotopic to $\tilde{\alpha}$ and so we must have $\tilde{p} = \tilde{q}$.

In order to show that Φ_Σ is trivial we assume the converse, and, in particular that $\Phi_\Sigma^{-1}(S)$ has more than one component. Since $S \subseteq V$ each of these components is diffemorphic to S, and they separate $\tilde{\Sigma}$.

Let $\tilde{S}_1, \tilde{S}_2$ be two such different components and let $\tilde{V}_1, \tilde{V}_2$ be the respective copies of V containing them. Since $\bar{\Sigma}_+ \subseteq V$, each $\tilde{V}_i$ $(i = 1, 2)$ contains a diffeomorphic copy of $\bar{\Sigma}_+$ called $\tilde{C}_i$, which are closed and non-compact.

Further since $\tilde{\Sigma}$ is connected and separated by $\tilde{S}_1$, the set $\tilde{C}_2$ is contained in $\tilde{S}_1 \cup \tilde{\Sigma}_-^{(1)} := \tilde{S}_1 \cup (\tilde{\Sigma} \backslash \tilde{C}_1)$, since otherwise $\tilde{V}_1 \cap \tilde{V}_2 \neq \emptyset$.

So $\tilde{C}_2 \subseteq \tilde{S}_1 \cup \tilde{\Sigma}_-^{(1)}$. Being local properties, both null geodesic completeness and the distributional null convergence condition lift to $\tilde{M}$ and by assumption $\tilde{M}$ is past-reflecting. Further as null branching is a local property as well, it also cannot occur in $\tilde{M}$. Thus the assumptions of Proposition 4.3 are fulfilled for $\tilde{M}$, $\tilde{\Sigma}$, $\tilde{C}_1$ and $\tilde{\Sigma} \backslash \tilde{C}_1$ implying that $\tilde{S}_1 \cup (\tilde{\Sigma} \backslash \tilde{C}_1)$ is compact. However, this set contains the non-compact, closed set $\tilde{C}_2$, a contradiction and we are done. □

Finally we *sketch the proof of Corollary* 2.8, i.e. the globally hyperbolic C^1-Gannon–Lee theorem. We can proceed analogously to the one of Theorem 2.6, where most points will even be significantly easier. In fact, the topological part remains the same and the only aspect one has to pay attention to is proving that $\mathcal{H}^+$ is relatively compact, i.e., step (1) in Proposition 4.3: We have to prove that any maximizing g-null geodesic is a limit of maximizing $\check{g}_\epsilon$-null geodesics. Corollary 3.5 does not apply, but we can replace it by the "limiting-result" [20, Prop. 2.13]. Note that compared to Corollary 3.5 the result in [20, Prop. 2.13] only shows that there is one g-geodesic which can be approximated by $\check{g}_\epsilon$ maximizers, but this is sufficient for the proof to work out.

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Chapter 4

The Hawking-Penrose Singularity Theorem for C^1-Lorentzian Metrics

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This was a gradual joint collaboration. Contributions by B.S. were made throughout in close collaboration with the other authors, mainly Argam Ohanyan. In particular in chapter 2 regarding regularization results and the genericity condition (jointly with A.O.), chapter 3 on geodesic branching and its implication for regularization, chapter 4 in optimizing the focusing result and proving the no-lines theorems (jointly with A.O.) as well as chapter 6 and appendix B. All authors contributed significantly.

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The Hawking–Penrose Singularity Theorem for C^1-Lorentzian Metrics

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Abstract: We extend both the Hawking–Penrose theorem and its generalisation due to Galloway and Senovilla to Lorentzian metrics of regularity C^1. For metrics of such low regularity, two main obstacles have to be addressed. On the one hand, the Ricci tensor now is distributional, and on the other hand, unique solvability of the geodesic equation is lost. To deal with the first issue in a consistent way, we develop a theory of tensor distributions of finite order, which also provides a framework for the recent proofs of the theorems of Hawking and of Penrose for C^1-metrics (Graf in Commun Math Phys 378(2):1417–1450, 2020). For the second issue, we study geodesic branching and add a further alternative to causal geodesic incompleteness to the theorem, namely a condition of maximal causal non-branching. The genericity condition is re-cast in a distributional form that applies to the current reduced regularity while still being fully compatible with the smooth and $C^{1,1}$-settings. In addition, we develop refinements of the comparison techniques used in the proof of the $C^{1,1}$-version of the theorem (Graf in Commun Math Phys 360:1009–1042, 2018). The necessary results from low regularity causality theory are collected in an appendix.

1. Introduction

The classical singularity theorems[1] of General Relativity (GR) collect sufficient and physically reasonable conditions that lead to causal geodesic incompleteness of spacetime, hence to the occurrence of a singularity in the spirit of Roger Penrose's Nobel Prize-winning approach [40]. Being rigorous statements in pure Lorentzian geometry, they were first formulated within the smooth category. As they form a body of essential results in GR, the quest for low regularity versions of the theorems is eminent and was already explicitly discussed in [14, Sec. 8.4]. In fact, the (smooth) theorems can be read as predicting a mere drop of the differentiability of the metric below C^2, rather than incompleteness.

[1] See [14, Ch. 8], [21, Ch. 9], [46–48] for extensive treatments.

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However, only with the rather recent advent of systematic studies in low regularity GR [6] and, in particular, causality theory [5,24,31,41], a rigorous extension of the singularity theorems for metrics below the C^2-class became feasible, cf. [46, Sec. 6.2]. This applies first of all to metrics of regularity $C^{1,1}$, which model situations where the matter variables possess finite jumps. Indeed the Hawking, the Penrose, and, finally, the Hawking-Penrose theorem were proven in this regularity [8,25,26]. These results rule out that a smooth spacetime that is incomplete by the classical theorems, can be extended to a complete $C^{1,1}$-spacetime. Also, they imply that the spacetime either is incomplete or the differentiability of the metric is below $C^{1,1}$, and consequently the curvature becomes unbounded.

Technically, in $C^{1,1}$ the exponential map is still available [23,31] and the curvature is still locally bounded, which allows for a natural extension of the energy and the genericity conditions. However, the curvature tensor is only defined almost everywhere, which forbids the use of Jacobi fields and conjugate points, both essential tools in the classical proofs. Instead one uses regularisation techniques which allow one to derive weakened versions of the energy conditions for approximating smooth metrics with controlled causality. Using careful comparison techniques for the (matrix) Riccati equation, it is then possible to show that these (still) lead to the occurrence of conjugate or focal points along causal geodesics for the approximating metrics. This in turn forces the geodesics of the $C^{1,1}$-metric to stop maximising. Similarly, a natural extension of the initial and the energy conditions leads to the formation of a trapped set. This, together with an extension of the causal parts of the classical proofs, allows one to finally derive the results.

The next step in lowering the regularity assumptions was undertaken by Graf [7], who extended the Hawking and the Penrose theorems to C^1-metrics, with the Gannon-Lee theorem following in [43]. C^1-regularity is at the moment the lowest possible class where classical singularity theorems have been established, and in this paper we complete this effort by proving the most refined of these statements, namely the Hawking-Penrose theorem.

In this regularity class one faces the following added severe complications arising from the fact that the Levi-Civita connection is only continuous:

(a) The curvature is no longer locally bounded, but is merely a distribution *of order one*.
(b) The initial value problem for the geodesic equation is solvable, but *not uniquely* so.
(c) The exponential map is no longer defined.

The first item is especially relevant when formulating the energy conditions. While the strong energy condition can be extended in a straightforward manner, a formulation of the null energy condition is more subtle, cf. [7, Sec. 5], see also Sect. 2.3, below. The genericity condition needed for the Hawking-Penrose theorem is still more delicate. Actually it turns out to be necessary to apply the Ricci tensor to vector fields constructed via parallel transport, which means that they are merely of C^1-regularity, see Definition 2.15. This can only be done since the curvature is a distribution of order one, and we present the required distributional setting in Sect. 2.1, below. Whereas in the $C^{1,1}$-context the use of distributional methods could be avoided, it becomes essential in the present work, and at the same time has the benefit of providing a global framework also for the earlier results by Graf [7] on which we build.

Then, while we can use the weakened versions of the strong and null energy conditions for approximating smooth metrics derived in [7] (employing a refined version of the Friedrichs Lemma), we have to derive from the distributional genericity condition an appropriate weakened version for smooth approximations that allows us to employ an extension of the Riccati comparison techniques developed in [8], see Lemmas 2.17, and 2.18 . The key technique of regularisation by smooth metrics with adapted causality as put forward in [5] is recalled in Sect. 2.2.

The first issue connected to item (b) is that it necessitates a decision on how to extend the notion of geodesic completeness, and following [7] we say that a spacetime is complete if *every* inextendible solution to the geodesic equation is complete (rather than merely demanding only one complete solution for any choice of initial data).

Next, item (b) together with item (c) forbids the use of an essential argument in the context of approximating maximising causal geodesics by maximising causal geodesics of the approximating metrics. This issue has been solved in [7, Sec. 2] in the *globally hyperbolic* case. In the context of the present work, however, we have to establish more general results, see Sect. 3. In fact, we have to introduce an additional assumption, namely a non-branching condition for maximising causal geodesics (Definition 3.1), to derive the corresponding approximation results in Proposition 3.4. Here it is also essential to apply a classical ODE result (Proposition 2.5), which we reformulate for our purpose in Corollary 2.6. Moreover, since the interrelation between causal geodesics (i.e., solutions of the geodesic equation) and maximising curves becomes more subtle below $C^{1,1}$ [16,42], we rely in particular on the fact that maximising causal curves are indeed (unbroken) geodesics and hence C^2-curves [27,43].

The new non-branching assumption is well motivated by similar conditions used in metric geometry, where due to the lack of a differentiable structure the geodesic equation is not available. Also a null version of the condition was essential in the proof of the C^1-Gannon-Lee theorem, see [43, Sec. 3]. Moreover it adds a novel facet to the interpretation of the C^1-version of the theorem: It predicts either geodesic incompleteness or branching of maximising causal geodesics, with both alternatives physically signifying a catastrophic event for the observer corresponding to the geodesic. Of course, there is the alternative that the regularity of the metric drops below C^1, which renders the curvature a distribution of higher order. Again, the result also forbids the extension of the spacetime to a complete one of regularity C^1 without (maximal causal) geodesic branching. This once more complements the recent C^0-inextendibility results of [9,45].

Next we briefly discuss the further prospects of low-regularity singularity theorems. Concerning the causality part of the results it is expected that they extend to $C^{0,1}$-metrics[2], while some features of causality theory fail below this class [5,12]. Nevertheless, the causal core of some of the singularity theorems has been established in the very general setting of closed cone structures, see [35, Sec. 2.15].

On the analytic side, already in $C^{0,1}$ one faces the problem that the right hand side of the geodesic equation is merely locally bounded and one would have to resort to non-classical solution concepts. Also another analytical technique at the core of the arguments, i.e. the Friedrichs Lemma which is used to go from the distributional energy conditions of the singular metric to useful surrogate conditions for the smooth approximations, appears to be sharp, see [7, Lem. 4.8] for the currently most advanced version. This is actually a long way from the largest possible class where the curvature can be (stably) defined in an analytical (distributional) way, i.e. $H^1 \cap L^\infty$ locally [11,28,50]. However, the quest is there to at least go into the direction of regularity classes more closely linked to the classical existence results for the field equations, H^s locally for $s > 5/2$, or to current formulations of cosmic censorship, i.e. the connection lying locally in L^2.

[2] A class for which Hawking and Ellis [14, p. 268, 287] still speculate the singularity theorems to hold.

Very recently, also synthetic formulations of the singularity theorems have appeared: In [1], Lorentzian length spaces that are warped products are studied and a Hawking singularity theorem is derived from suitable sectional curvature bounds (implying Ricci curvature bounds in such geometries), based on triangle comparison. Moreover, Cavalletti and Mondino prove a version of Hawking's singularity theorem in Lorentzian metric measure spaces, where Ricci curvature bounds are implemented using methods from optimal transport [4]. However, it remains unclear to date how these results precisely relate to the analytical approach to low regularity pursued in this work.

The Hawking-Penrose theorem classically comes in a causal version [17, p. 538, Thm.], which asserts the incompatibility of the following three conditions for a C^2-spacetime (M, g): chronology, the fact that every inextendible causal geodesic stops maximising, and the existence of a (future or past) trapped set. The analytic main result is then a corollary [17, Sec. 3, Cor.] which collects sufficient conditions (energy and genericity conditions, and initial conditions) to derive the above incompatible items under the assumption of causal geodesic completeness. We will generally follow this path, but prove a more general version of the theorem due to Galloway and Senovilla [10], which adds the existence of a trapped submanifold of arbitrary codimension to the list of initial conditions.

Our version of the causal result is Theorem 6.2 and, for the sake of completeness and for the convenience of the reader we collect all results from C^1-causality theory[3] needed here (and elsewhere) in Appendix A. The main result is then Theorem 6.3. The results needed to infer from its analytic conditions (the distributional energy and genericity conditions) and the non-branching assumptions the inexistence of lines are proven in Sect. 4. The existence of a trapped set is derived from the various sets of initial conditions (and the energy conditions) in Sect. 5. Finally, some technical results on extending vector fields in C^1-spacetimes are collected in Appendix B. They allow us to put to use the distributional genericity condition in the context of regularisations.

To conclude this introduction, we fix the notations and conventions used throughout the paper. By a manifold M we will mean a smooth, connected, second countable Hausdorff manifold of dimension n (with $n \geq 3$). When such an M is endowed with a Lorentzian metric g of regularity C^1 of signature $(-, + \dots, +)$ and is time-oriented via a smooth timelike vector field we call it a C^1-spacetime (hence lowering the regularity will always apply to the spacetime metric only, not to the underlying manifold). The Levi-Civita connection of a smooth or C^1-spacetime will be denoted by ∇ (for its distributional definition in the C^1-case, see the next section). We will always assume that M is also endowed with a smooth (complete, background) Riemannian metric h with associated Riemannian distance function d_h and norm $\| \, \|_h$ which we use to estimate tensor fields. Since all such estimates will be on compact sets only, they are independent of the choice of h.

A curve $\gamma : I \to M$ is called timelike (causal, null, future or past directed) if it is locally Lipschitz and $\dot{\gamma}(t)$ is timelike (causal, null, future or past directed) almost everywhere. Concerning causality theory we use standard notation, cf. e.g., [34,39]. Thus $p \ll q$ (resp. $p \leq q$) means that there exists a future directed timelike (resp. causal) curve from p to q, $I^+(A) := \{q \in M : \ p \ll q \text{ for some } p \in A\}$ and $J^+(A) := \{q \in M : \ p \leq q \text{ for some } p \in A\}$. These definitions do not change if one defines the causality relations via piecewise smooth curves [24,31]. The Lorentzian distance (or time-separation function) associated to a Lorentzian metric g will be denoted by d_g. A C^1-spacetime (M, g) is globally hyperbolic if it is causal (i.e., contains no closed

[3] It should be mentioned that these results all follow from the more general approaches of [5,35,41].

causal curves) and $J(p,q) := J^+(p) \cap J^-(q)$ is compact for all $p, q \in M$. A Cauchy hypersurface in (M, g) is a closed acausal set that is met by each inextendible causal curve.

For the Riemann curvature tensor we use the convention $R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$ and the Ricci tensor is given by $\mathrm{Ric}(X,Y) = \sum_{i=1}^n \langle E_i, E_i\rangle \langle R(E_i, X)Y, E_i\rangle$ (again we refer to the next section for a distributional interpretation in the case of C^1-metrics). Here $(E_i)_{i=1}^n$ is a (local) orthonormal frame field, while by $(e_i)_{i=1}^n$ we will denote frames in individual tangent spaces T_pM. For an embedded submanifold N of M of codimension m we define the second fundamental form by $\mathrm{II}(V,W) = \mathrm{nor}(\nabla_V W)$ and the shape operator derived from a normal unit field ν by $S(X) = \nabla_X \nu$. Using (2.13) below it follows that for $g \in C^1$ and $V, W \in C^1$, $\mathrm{II}(V,W)$ is continuous.

For g smooth (resp. C^1), $1 < m < n$ and S a smooth (resp. C^2) spacelike $(n-m)$-dimensional submanifold let $e_1(q), \ldots, e_{n-m}(q)$ be an orthonormal basis for T_qS, varying smoothly (resp. C^1) with q in a neighborhood (in S) of $p \in S$. Then $H_S := \frac{1}{n-m}\sum_{i=1}^{n-m} \mathrm{II}(e_i, e_i)$ denotes the mean curvature vector field of S, and $\mathbf{k}_S(v) := g(H_S, v)$ the convergence of $v \in TM|_S$. For g in C^1, both H_S and $\mathbf{k}_S$ are continuous. A closed spacelike submanifold S is called *(future) trapped* if for any future-directed null vector $\nu \in TS^\perp$ the convergence $\mathbf{k}_S(\nu)$ is positive, or equivalently that the mean curvature vector field H_S is past pointing timelike on S.

One final convention we shall require is that of tensor classes (cf. [21, Sec. 4.6.3]): Given a causal geodesic γ in a C^1-spacetime (M, g), we set $[\dot\gamma(t)]^\perp := (\dot\gamma(t))^\perp/\mathbb{R}\dot\gamma(t)$. For γ null, $[\dot\gamma(t)]^\perp$ is an $(n-2)$-dimensional subspace of the hypersurface $(\dot\gamma(t))^\perp$. On the other hand, for γ timelike, $[\dot\gamma(t)]^\perp$ equals $(\dot\gamma(t))^\perp$. To obtain a unified notation, throughout the paper we will denote the dimension of $[\dot\gamma(t)]^\perp$ by d, which amounts to setting $d = n-2$ in the null case and $d = n-1$ in the timelike case. Furthermore, we set $[\dot\gamma]^\perp = \bigcup_t [\dot\gamma(t)]^\perp$. Then for every normal tensor field A along γ we obtain a well-defined tensor class $[A]$ along γ, as well as a well-defined tensor class representing the induced covariant derivative $\nabla_{\dot\gamma}$. We then set $[\dot A] = [\nabla_{\dot\gamma} A]$. The metric $g|_{[\dot\gamma]^\perp}$ is positive definite in both the null and the timelike case. For smooth metrics, the curvature (or tidal force) operator is given by $[R](t) : [\dot\gamma(t)]^\perp \to [\dot\gamma(t)]^\perp$, $[v] \mapsto [R(v, \dot\gamma(t))\dot\gamma(t)]$.

2. Distributional Curvature of C^1-Spacetimes

2.1. Curvature tensors of C^1-metrics. In what follows we discuss the general distributional framework in which to understand the curvature quantities associated to metric tensors (of arbitrary signature) of regularity below C^2, as is required for our further analysis. The general mathematical framework goes back to [29], while [11] provided the first study specific to GR. We mainly follow [28], cf. also [13,49], but put a special emphasis on metrics of regularity C^1 and distributions of finite order.

As in [13, Sec. 3.1], for $k \in \mathbb{N}_0 \cup \{\infty\}$ we denote by $\mathrm{Vol}(M)$ the volume bundle over M, and by $\Gamma_c^k(M, \mathrm{Vol}(M))$ the space of compactly supported one-densities on M (i.e., sections of $\mathrm{Vol}(M)$) that are k-times continuously differentiable. Then the space of distributions of order k on M is the topological dual of $\Gamma_c^k(M, \mathrm{Vol}(M))$,

$$\mathcal{D}'^{(k)}(M) := \Gamma_c^k(M, \mathrm{Vol}(M))'.$$

For $k = \infty$ we omit the superscript (k). Clearly there are topological embeddings $\mathcal{D}'^{(k)}(M) \hookrightarrow \mathcal{D}'^{(k+1)}(M) \hookrightarrow \mathcal{D}'(M)$ for all k. As we shall see, all curvature quantities of C^1-metrics are distributional tensor fields of order 1. The space of distributional (r, s)-tensor fields of order k is defined as

$$\mathcal{D}'^{(k)}\mathcal{T}^r_s(M) \equiv \mathcal{D}'^{(k)}(M, T^r_s M) := \Gamma^k_c(M, \mathcal{T}^s_r(M) \otimes \mathrm{Vol}(M))'. \tag{2.1}$$

By [13, 3.1.15],

$$\mathcal{D}'\mathcal{T}^r_s(M) \cong \mathcal{D}'(M) \otimes_{C^\infty(M)} \mathcal{T}^r_s(M) \cong L_{C^\infty(M)}(\Omega^1(M)^r \times \mathfrak{X}(M)^s; \mathcal{D}'(M)). \tag{2.2}$$

This algebraic isomorphism ultimately rests on the fact that the $C^\infty(M)$-module of C^∞-sections $\Gamma(M, F)$ in any smooth vector bundle $F \to M$ is finitely generated and projective, which allows one to apply [3, Ch. II §4.2, Prop. 2]. These properties persist if we consider $\Gamma_{C^k}(M, F)$ as a $C^k(M)$-module for any finite k, so we also have

$$\begin{aligned} \mathcal{D}'^{(k)}\mathcal{T}^r_s(M) &\cong \mathcal{D}'^{(k)}(M) \otimes_{C^k(M)} (\mathcal{T}^r_s)_{C^k}(M) \\ &\cong L_{C^k(M)}(\Omega^1_{C^k}(M)^r \times \mathfrak{X}_{C^k}(M)^s; \mathcal{D}'^{(k)}(M)). \end{aligned} \tag{2.3}$$

In other words, distributional tensor fields of order k act as C^k-balanced multilinear maps on one forms and vector fields of regularity C^k to give a scalar distribution of order k. For finite k, the multiplication $C^k(M) \times \mathcal{D}'^{(k)}(M) \to \mathcal{D}'^{(k)}(M)$ is jointly continuous (w.r.t. the strong topology), whereas for $k = \infty$ it is only hypocontinuous [20, p. 362], but still jointly sequentially continuous [37, p. 233] since $C^\infty(M)$ is Fréchet, hence barrelled. The corresponding continuity properties therefore also hold for the tensor operations introduced above. The isomorphisms (2.2), (2.3) are not topological, but still bornological by [36, Thm. 15] (which remains valid for spaces of C^k-sections with k finite).

Smooth, and in fact even L^1_{loc}-tensor fields are continuously and densely embedded via

$$\begin{aligned} &\mathcal{T}^r_s(M) \hookrightarrow \mathcal{D}'^{(k)}\mathcal{T}^r_s(M) \\ &\qquad t \mapsto [(\theta_1, \ldots, \theta_r, X_1, \ldots, X_s) \mapsto [\omega \mapsto \int_M t(\theta_1, \ldots, \theta_r, X_1, \ldots, X_s)\omega]]. \end{aligned}$$

Here, ω is a one-density. If M is orientable, densities can be canonically identified with n-forms. The fact that $\mathcal{T}^r_s(M)$ is dense in $\mathcal{D}'^{(k)}\mathcal{T}^r_s(M)$ uniquely fixes all the operations on distributional tensor fields to be introduced below in a way compatible with smooth pseudo-Riemannian geometry.

For any $t \in \mathcal{T}^r_s(M)$ there is a unique extension that accepts one distributional argument. E.g., if $\tilde{\theta}_1 \in \mathcal{D}'\mathcal{T}^0_1(M)$, then since $t(\,.\,, \theta_2, \ldots, X_s) \in \mathfrak{X}(M)$ we may set

$$t(\tilde{\theta}_1, \theta_2, \ldots, X_s) := \tilde{\theta}_1(t(\,.\,, \theta_2, \ldots, X_s)) \in \mathcal{D}'(M), \tag{2.4}$$

and analogously for the other slots.

Definition 2.1. A *distributional connection* is a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathcal{D}'\mathcal{T}^1_0(M)$ satisfying for $X, X', Y, Y' \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$ the usual computational rules: $\nabla_{fX+X'}Y = f\nabla_X Y + \nabla_{X'}Y$, $\nabla_X(Y + Y') = \nabla_X Y + \nabla_X Y'$, $\nabla_X(fY) = X(f)Y + f\nabla_X Y$.

Using (2.4) any distributional connection can be extended to the entire tensor algebra, i.e., to a map $\nabla : \mathfrak{X}(M) \times \mathcal{T}^r_s(M) \to \mathcal{D}'\mathcal{T}^r_s(M)$, e.g. for $\Theta \in \Omega^1(M)$ we set

$$(\nabla_X \Theta)(Y) := X(\Theta(Y)) - \Theta(\nabla_X Y) \quad (X, Y \in \mathfrak{X}(M)). \tag{2.5}$$

Let $\mathcal{G}$ be any of the spaces C^k $(0 \le k)$ or L^p_{loc} $(1 \le p)$, then we call a distributional connection a *$\mathcal{G}$-connection* if $\nabla_X Y$ is a $\mathcal{G}$-vector field for any $X, Y \in \mathfrak{X}(M)$. A particularly important case are L^2_{loc}-connections since they form the largest class that allows for a stable definition of the curvature tensor in distributions, cf. [11,28,49].

Using a local frame one easily sees that by virtue of the respective computational rules any $\mathcal{G}$-connection can be (uniquely) extended to a map $\nabla : \mathfrak{X}_{\mathcal{F}}(M) \times \mathfrak{X}_{\mathcal{H}}(M) \to \mathfrak{X}_{\mathcal{G}}(M)$, provided that $\mathcal{F}$ and $\mathcal{H}$ are function spaces such that $\mathcal{F} \cdot \mathcal{G} \subseteq \mathcal{G}$, $D\mathcal{H} \subseteq \mathcal{G}$, and $\mathcal{H} \cdot \mathcal{G} \subseteq \mathcal{G}$. Hence, in particular, each L^2_{loc}-connection extends in this way to a map

$$\nabla : \mathfrak{X}_{C^0}(M) \times \mathfrak{X}_{C^1}(M) \to \mathfrak{X}_{L^2_{\mathrm{loc}}}(M). \tag{2.6}$$

Moreover, by a similar reasoning, $\mathcal{G}$-connections allow one to insert even less regular vector fields in the first and second slot at the price of a less regular outcome. In particular, any L^2_{loc}-connection can be extended to a map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}_{L^2_{\mathrm{loc}}}(M) \to \mathcal{D}'\mathcal{T}^1_0(M). \tag{2.7}$$

Using this extension, we have [28, Def. 3.3]:

Definition 2.2. The distributional Riemann tensor of an L^2_{loc}-connection ∇ is the map $R : \mathfrak{X}(M)^3 \to \mathcal{D}'\mathcal{T}^1_0(M)$,

$$R(X, Y, Z)(\theta) \equiv (R(X, Y)Z)(\theta) := (\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z)(\theta)$$

for $X, Y, Z \in \mathfrak{X}(M)$ and $\theta \in \Omega^1(M)$,

If F_i is a local frame in $\mathfrak{X}(U)$ and $F^j \in \Omega^1(U)$ its dual frame, then the Ricci tensor corresponding to ∇ is given by

$$\mathrm{Ric}(X, Y) := (R(X, F_i)Y)(F^i) \in \mathcal{D}'(U) \qquad (X, Y \in \mathfrak{X}(U)). \tag{2.8}$$

Recall the Koszul formula for the Levi-Civita connection of a smooth metric g on M:

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ &\quad - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]) =: F(X, Y, Z) \end{aligned} \tag{2.9}$$

Focusing now on the case at hand in this work, suppose that g is merely C^1 (still allowing arbitrary signature). Then $F(X, Y, Z)$ is a well-defined element of $C^0(M)$, and

$$\nabla^\flat_X Y := Z \mapsto \frac{1}{2} F(X, Y, Z) \in \Omega^1_{C^0}(M) \subseteq \mathcal{D}'\mathcal{T}^0_1(M) \tag{2.10}$$

defines the *distributional Levi-Civita connection* of g [28, Def. 4.2]. Note that this is not (yet) a distributional connection in the sense of Definition 2.1 since it is of order $(0, 1)$ instead of $(1, 0)$. In addition to the standard product rules it also possesses the properties (corresponding to [39, Thm. 3.11])

$$\begin{aligned} &\nabla^\flat_X Y - \nabla^\flat_Y X = [X, Y]^\flat, \ \text{ i.e., } (\nabla^\flat_X Y - \nabla^\flat_Y X)(Z) = g([X, Y], Z), \\ &X(g(Y, Z)) = (\nabla^\flat_X Y)(Z) + (\nabla^\flat_X Z)(Y) \end{aligned} \tag{2.11}$$

for all $X, Y, Z \in \mathfrak{X}(M)$. These equalities hold distributionally, hence also in C^0.

In order to obtain an L^2_{loc}-connection from $\nabla^\flat$ (which in turn will allow us to define the curvature tensors we require) we need to raise the index via g, setting

$$g(\nabla_X Y, Z) := (\nabla^\flat_X Y)(Z) \qquad (X, Y, Z \in \mathfrak{X}(M)). \tag{2.12}$$

This defines a continuous vector field $\nabla_X Y$, hence even a C^0-connection. More generally, by [28, Sec. 4] this procedure yields an L^2_{loc}-connection even for the significantly larger Geroch-Traschen class of metrics ($g \in H^1_{\text{loc}} \cap L^\infty_{\text{loc}}$ and locally uniformly non-degenerate).

The Riemann tensor of $g \in C^1$ is now given by Definition 2.2. Furthermore, since in this case ∇ is a C^0-connection, it follows that $(R(X, Y)Z)(\theta) \in \mathcal{D}'^{(1)}(M)$, and consequently, $R \in \mathcal{D}'^{(1)}\mathcal{T}^1_3(M)$. Indeed by the same reasoning that led to (2.6), (2.7) we may extend the C^0-Levi-Civita connection of g to a map

$$\nabla : \mathfrak{X}_{C^0}(M) \times \mathfrak{X}_{C^1}(M) \to \mathfrak{X}_{C^0}(M), \text{ and } \nabla : \mathfrak{X}_{C^1}(M) \times \mathfrak{X}_{C^0}(M) \to \mathcal{D}'^{(1)}\mathcal{T}^1_0(M), \tag{2.13}$$

respectively. Here, the second equation shows that the term $([\nabla_X, \nabla_Y]Z)(\theta)$ yields an order one distribution, while $(\nabla_{[X,Y]}Z)(\theta) \in C^0$ since ∇ is a C^0-connection.

For $W, X, Y, Z \in \mathfrak{X}(M)$ we define $R(W, X, Y, Z) \in \mathcal{D}'^{(1)}(M)$ by

$$\begin{aligned} R(W, X, Y, Z) := {} & X(g(W, \nabla_Y Z)) - Y(g(W, \nabla_X Z)) \\ & - g(\nabla_X W, \nabla_Y Z) + g(\nabla_Y W, \nabla_X Z) - g(W, \nabla_{[X,Y]}Z). \end{aligned}$$

Using (2.4), (2.7) and (2.11) it is straightforward to check that cf. [28, Rem. 4.5]

$$R(W, Z, X, Y) = g(W, R(X, Y)Z).$$

Alternatively, one may verify this identity in local coordinates using the fact that there is a well-defined multiplication of distributions of first order with C^1-functions.

The Ricci tensor of a C^1 metric g is given by (2.8), hence, in particular, is again a distributional tensor field of order 1. Note that by (2.3) this allows us to apply it to C^1-vector fields, which will be essential in the context of the genericity condition below. Similarly, since the Riemann tensor, again by (2.3), acts on C^1-vector fields, we may in particular use g-orthonormal frames (which generically are only C^1) to calculate the Ricci tensor, which is also vital below. Explicitly, (2.8) holds for F_i a g-orthonormal frame and F^i its g-dual. All these facts can alternatively be derived directly from the extension of ∇ in (2.13).

Again, by the same observations, we have that the scalar curvature S is a distribution of order 1 (this answers a concern from [7, Sec. 3.2]) and that the standard local formulae hold in $\mathcal{D}'^{(1)}$:

$$\begin{aligned} R^m_{ijk} &= \partial_j \Gamma^m_{ik} - \partial_k \Gamma^m_{ij} + \Gamma^m_{js}\Gamma^s_{ik} - \Gamma^m_{ks}\Gamma^s_{ij}, \\ \text{Ric}_{ij} &= R^m_{imj}, \\ S &= \text{Ric}^i_i. \end{aligned}$$

This shows that the local definitions given in [7, (3.2),(3.3)] are compatible with the global approach developed here.

2.2. Regularisation. As in previous works on the generalisation of singularity theorems to metrics of differentiability below C^2 [7,24–26], an essential tool in our approach will be suitable regularisations, both of the metrics themselves and of the derived distributional curvature quantities. A general regularisation scheme based on chart-wise convolution with a mollifier $\rho \in \mathcal{D}(B_1(0))$, $\int \rho = 1$, $\rho \geq 0$ (cf. [13, 3.2.10], [24, Sec. 2], [7, Sec. 3.3]) is as follows: Cover M by a countable and locally finite family of relatively compact chart neighbourhoods (U_i, ψ_i) $(i \in \mathbb{N})$. Let $(\zeta_i)_i$ be a subordinate partition of unity with $\operatorname{supp}(\zeta_i) \subseteq U_i$ for all i and choose a family of cut-off functions $\chi_i \in \mathscr{D}(U_i)$ with $\chi_i \equiv 1$ on a neighbourhood of $\operatorname{supp}(\zeta_i)$. Finally, for $\varepsilon \in (0, 1]$ set $\rho_\varepsilon(x) := \varepsilon^{-n}\rho\left(\frac{x}{\varepsilon}\right)$. Then denoting by f_* (respectively f^*) push-forward (resp. pull-back) of distributions under a diffeomorphism f, for any $\mathcal{T} \in \mathcal{D}'\mathcal{T}^r_s(M)$ consider the expression

$$\mathcal{T} \star_M \rho_\varepsilon(x) := \sum_i \chi_i(x)\, \psi_i^*\Big(\big(\psi_{i\,*}(\zeta_i \cdot \mathcal{T})\big) * \rho_\varepsilon\Big)(x). \tag{2.14}$$

Here, $\psi_{i\,*}(\zeta_i \cdot \mathcal{T})$ is viewed as a compactly supported distributional tensor field on $\mathbb{R}^n$, so componentwise convolution with ρ_ε yields a smooth field on $\mathbb{R}^n$. The cut-off functions χ_i secure that $(\varepsilon, x) \mapsto \mathcal{T} \star_M \rho_\varepsilon(x)$ is a smooth map on $(0, 1] \times M$. For any compact set $K \Subset M$ there is an ε_K such that for all $\varepsilon < \varepsilon_K$ and all $x \in K$ (2.14) reduces to a finite sum with all $\chi_i \equiv 1$ (hence to be omitted from the formula), namely when ε_K is less than the distance between the support of $\zeta_i \circ \psi_i^{-1}$ and the boundary of $\psi_i(U_i)$ for all i with $U_i \cap K \neq \emptyset$.

Just as is the case for smoothing via convolution in the local setting, $\mathcal{T} \star_M \rho_\varepsilon$ converges to $\mathcal{T}$ as $\varepsilon \to 0$ in $\mathcal{D}'\mathcal{T}^r_s$, and indeed in C^k_{loc} or $W^{k,p}_{\text{loc}}$ $(p < \infty)$ if $\mathcal{T}$ is contained in these spaces [7, Prop. 3.5].

For a given C^1 Lorentzian metric g we will in addition require concrete regularisations that are adapted to the causal structure induced by g. This construction goes back to Chrusciel and Grant [5, Prop. 1.2], cf. also [24, Prop. 2.5]. The version we will use is [7, Lem. 4.2, Cor. 4.3]:

Lemma 2.3. *Let (M, g) be a C^1-spacetime. Then for any $\varepsilon > 0$ there exist smooth Lorentzian metrics $\check{g}_\varepsilon$, $\hat{g}_\varepsilon$ on M, time orientable by the same timelike vector field as g, and satisfying:*

(i) *$\check{g}_\varepsilon \prec g \prec \hat{g}_\varepsilon$ for all ε.*
(ii) *$\check{g}_\varepsilon$ and $\hat{g}_\varepsilon$ converge to g in C^1_{loc} as $\varepsilon \to 0$.*
(iii) *$\check{g}_\varepsilon - g \star_M \rho_\varepsilon \to 0$ in C^∞_{loc} and for any $K \Subset M$ there exist $c_K > 0$ and $\varepsilon_0(K)$ such that $\|\check{g}_\varepsilon - g \star_M \rho_\varepsilon\|_{\infty,K} \leq c_K\varepsilon$ and $\|g - g \star_M \rho_\varepsilon\|_{\infty,K} \leq c_K\varepsilon$ for all $\varepsilon < \varepsilon_0$. An analogous statement holds for $\hat{g}_\varepsilon$ as well as for the inverse metrics g^{-1}, $(\check{g}_\varepsilon)^{-1}$ and $(\hat{g}_\varepsilon)^{-1}$.*

In this work we will generally use $\check{g}_\varepsilon$, $\hat{g}_\varepsilon$ to denote the regularisations of Lemma 2.3, which are of utmost importance throughout.

Next we recall the following essential result on convergence of the curvature under regularisations, which is a special case of several so-called stability results, see [11, Thm. 2], [28, Thm. 4.6], [50, Thm. 5.1].

Lemma 2.4. *Let (M, g) be a C^1-spacetime, and let g_ε be either $\check{g}_\varepsilon$ or $\hat{g}_\varepsilon$. Then we have for the Riemann and the Ricci tensor*

$$R[g_\varepsilon] \to R[g] \quad \textit{and} \quad \operatorname{Ric}[g_\varepsilon] \to \operatorname{Ric}[g] \quad \textit{distributionally.} \tag{2.15}$$

In particular, the Riemann and the Ricci tensor possess the usual symmetries.

We will frequently need the following result from ODE theory (cf. [15, Ch.2, Thm. 3.2]), which we will mainly use in the context of regularisation given below in Corollary 2.6.

Proposition 2.5. *Let f and f_m, $m \in \mathbb{N}$ be continuous functions on an open set $E \subseteq \mathbb{R} \times \mathbb{R}^n$ with $f_m \to f$ in C^0_{loc}. Let $y_m = y_m(t)$ be a solution to the initial value problem*

$$y_m'(t) = f_m(t, y_m(t)), \quad y_m'(t_m) = y_{m0}$$

with data $(t_m, y_{m0}) \in E$ and maximal interval of existence (ω_m^-, ω_m^+). Suppose that the sequence of initial data converges,

$$(t_m, y_{m0}) \to (t_0, y_0) \in E.$$

Then there exist a solution $y = y(t)$ for the initial value problem

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0,$$

defined on a maximal interval (ω^-, ω^+), and a subsequence m_l with the property that for any $s_1 < s_2$ with $(s_1, s_2) \subseteq (\omega^-, \omega^+)$ we have $(s_1, s_2) \subseteq (\omega_{m_l}^-, \omega_{m_l}^+)$ for all large l, and $y_{m_l} \to y$ in $C^1_{\text{loc}}(\omega^-, \omega^+)$.[4] *In particular,*

$$\limsup_m \omega_m^- \leq \omega^- < \omega^+ \leq \liminf_m \omega_m^+.$$

We will mostly use the previous result to conclude that g_ε-geodesics, where $g_\varepsilon \in \{\check{g}_\varepsilon, \hat{g}_\varepsilon\}$, converge uniformly on compact intervals to a g-geodesic:

Corollary 2.6. *Let (M, g) be a C^1-spacetime and let $g_m = \check{g}_{\varepsilon_m}$ or $g_m = \hat{g}_{\varepsilon_m}$ with $\varepsilon_m \to 0$. Let $\gamma_m : (a_m, b_m) \to M$ be inextendible g_m-geodesics. Suppose that $\gamma_m(t_0) \to p \in M$ and $\dot{\gamma}_m(t_0) \to v \in T_pM$, where $t_0 \in \bigcap(a_m, b_m)$. Then there exist an inextendible g-geodesic $\gamma : (a, b) \to M$ with $\gamma(t_0) = p$, $\dot{\gamma}(t_0) = v$ and a subsequence m_l such that whenever $(s_1, s_2) \subseteq (a, b)$, we have $(s_1, s_2) \subseteq (a_{m_l}, b_{m_l})$ for all large l and γ_{m_l} converges in $C^2_{\text{loc}}(a, b)$ to γ. Moreover,*

$$\limsup_m a_m \leq a < b \leq \liminf_m b_m.$$

Proof. As this is a local statement, we may assume $M = \mathbb{R}^n$. We rewrite the geodesic equations for g_m as a first order initial value problem (with Γ_m the Christoffel symbols of g_m) and $1 \leq i \leq n$:

$$\begin{aligned}
\dot{\gamma}_m^i(t) &= \eta_m^i(t), \\
\dot{\eta}_m^i &= -(\Gamma_m)^i_{jk}(\gamma_m(t))\eta_m^j(t)\eta_m^k(t), \\
\gamma_m^i(t_0) &= p_m^i, \\
\eta_m^i(t_0) &= v_m^i.
\end{aligned}$$

We define the continuous functions $f_m : \mathbb{R}^{2n} \to \mathbb{R}$, $f_m(x, y) := (y, -(\Gamma_m)^i_{jk}(x)y^j y^k)$, and similarly $f : \mathbb{R}^{2n} \to \mathbb{R}$, $f(x, y) := (y, -\Gamma_{jk}(x)y^j y^k)$. Then $f_m \to f$ uniformly on compact subsets of $\mathbb{R}^{2n}$, and by assumption $(p_m, v_m) \to (p, v) \in \mathbb{R}^{2n}$. The existence of an inextendible g-geodesic sublimit γ of the sequence γ_m now follows from Proposition 2.5, and the C^2_{loc}-convergence follows suit. ◻

[4] In [15, Ch.2, Thm. 3.2], convergence is only stated in C^0_{loc}, but inserting this into the equivalent integral equations immediately yields the claim as given here.

In the case of only future or past inextendibility, an obvious analogue of Corollary 2.6 holds.

Remark 2.7. (Parallel transport) Under the assumptions of Corollary 2.6, let $w_m \in T_{\gamma_m(t_0)}M$, $w_m \to w \in T_pM$. Then since the ODEs governing parallel transport are linear, there are unique smooth vector fields W_m along γ_m that are global solutions of the corresponding initial value problem, and by the standard results on continuous dependence of solutions on the right hand side and the data, they converge in C^1_{loc} to the unique C^1-vector field W along γ resulting from g-parallel transport of w along γ.

2.3. Energy conditions. Let us briefly recall the timelike and null energy conditions for C^1-spacetimes. They are formulated in a way so that they imply corresponding conditions for the curvatures of approximating metrics. If the metric is $C^{1,1}$ or better, then the energy conditions presented here are equivalent to the usual pointwise (or pointwise a.e.) energy conditions. We follow [7, Sec. 3-5], to which we also refer for proofs.
We start with the timelike energy condition, whose distributional generalisation is straightforward, cf. [7, Def. 3.3].

Definition 2.8 (*Distributional timelike energy condition*). Let (M, g) be a C^1-spacetime. We say that (M, g) satisfies the *distributional timelike energy condition* if for any timelike $X \in \mathfrak{X}(M)$, $\text{Ric}(X, X) \in \mathcal{D}^{'(1)}(M)$ is a nonnegative scalar distribution on M.

Recall that a scalar distribution $u \in \mathcal{D}'(M)$ is nonnegative, $u \geq 0$, if $u(\omega) \equiv \langle u, \omega\rangle \geq 0$ for all nonnegative test densities $\omega \in \Gamma_c(M, \text{Vol}(M))$. A nonnegative distribution is always a measure [19, Thm. 2.1.7] and hence a distribution of order 0. Moreover, nonnegativity is stable with respect to regularisation [19, Thm. 2.1.9]. Finally, for $u, v \in \mathcal{D}'(M)$ we write $u \geq v$ if $u - v \geq 0$.

Remark 2.9. We note that an equivalent formulation of the distributional timelike energy condition consists in imposing that, for any $U \subseteq M$ open and any $X \in \mathfrak{X}(U)$ timelike, $\text{Ric}(X, X) \geq 0$ in $\mathcal{D}^{'(1)}(U)$. Indeed, given a timelike $X \in \mathfrak{X}(U)$, let $V \Subset U$ be open and choose a cut-off function $\chi \in \mathcal{D}(U)$ that equals 1 on a neighbourhood of $\overline{V}$. By time-orientability of M there exists some timelike $T \in \mathfrak{X}(M)$ of the same time-orientation as X, so $Y := \chi X + (1 - \chi)T$ is a global timelike vector field on M. Then for any non-negative $\omega \in \Gamma_c(M, \text{Vol}(M))$ with support in V we have $0 \leq \langle \text{Ric}(Y, Y), \omega\rangle = \langle \text{Ric}(X, X), \omega\rangle$, and by the sheaf property of $\mathcal{D}^{'(1)}$, we obtain $\text{Ric}(X, X) \geq 0$ in $\mathcal{D}^{'(1)}(U)$.

The usefulness of Definition 2.8 is highlighted by its implications for the curvatures of approximating metrics, cf. [7, Lem. 4.1, Lem. 4.6].

Lemma 2.10. *Let (M, g) be a C^1-spacetime satisfying the distributional timelike energy condition and let K be compact in M. Then*

$$\begin{aligned}&\forall C > 0\, \forall \delta > 0\, \forall \kappa < 0\, \exists \varepsilon_0 > 0\, \forall \varepsilon < \varepsilon_0\, \forall X \in TM|_K\\ &\textit{with } g(X, X) \leq \kappa \textit{ and } \|X\|_h \leq C:\ \text{Ric}[\check{g}_\varepsilon](X, X) > -\delta.\end{aligned}$$

The following is an immediate consequence of the previous result.

Corollary 2.11. *Let (M, g) be a C^1-spacetime satisfying the distributional timelike energy condition and let K be compact in M. Then*

$$\forall \delta > 0\, \exists \varepsilon_0 > 0\, \forall \varepsilon < \varepsilon_0\, \forall X \in TM|_K \textit{ with } \check{g}_\varepsilon(X, X) = -1:\ \text{Ric}[\check{g}_\varepsilon](X, X) > -\delta.$$

Let us now turn to the proper distributional formulation of the null energy condition. The naive approach of defining it in analogy to Definition 2.8 does not yield analogues of Lemma 2.10 and Corollary 2.11 because vector fields that are $\check{g}_\varepsilon$-null are only almost g-null. This necessitates a definition of the following form.

Definition 2.12 (*Distributional null energy condition*, [7, Def. 5.1]). Let (M, g) be a C^1-spacetime. We say (M, g) satisfies the *distributional null energy condition* if for any compact set K and any $\delta > 0$ there exists $\varepsilon = \varepsilon(\delta, K) > 0$ such that $\mathrm{Ric}(X, X) > -\delta$ (as distributions) for any local smooth vector field $X \in \mathfrak{X}(U)$ ($U \subseteq K$ open) with $\|X\|_h = 1$ and $|g(X, X)| < \varepsilon$ on U.

The above requirement $|g(X, X)| < \varepsilon$ may equivalently be replaced by $\|X - N\|_h < \varepsilon$, where N is a C^1 g-null vector field on U, cf. [7, Lem. 5.2]. Yet another useful equivalent reformulation is the following rescaled version, cf. [7, Def. 5.3].

Lemma 2.13. *A C^1-spacetime (M, g) satisfies the distributional null energy condition if and only if for any compact $K \subseteq M$, any $c_1, c_2 > 0$ and any $\delta > 0$ there is $\varepsilon = \varepsilon(\delta, K, c_1, c_2) > 0$ such that $\mathrm{Ric}(X, X) > -\delta$ (as distributions) for any local smooth vector field $X \in \mathfrak{X}(U)$, $U \subseteq K$, with $0 < c_1 \leq \|X\|_h \leq c_2$ and $|g(X, X)| < \varepsilon$ on U.*

The following analogue of Lemma 2.10 and Corollary 2.11 shows that the above definition of the null energy condition is the correct one in the C^1-case (see [7, Lem. 5.5]).

Lemma 2.14. *Let (M, g) be a C^1-spacetime satisfying the distributional null energy condition. Let $K \subseteq M$ be compact and let $c_1, c_2 > 0$. Then for all $\delta > 0$ there is $\varepsilon_0 = \varepsilon_0(\delta, K, c_1, c_2) > 0$ such that*

$$\begin{aligned}&\forall \varepsilon < \varepsilon_0\ \forall X \in TM|_K \text{ with } 0 < c_1 \leq \|X\|_h \leq c_2 \text{ and}\\ &\check{g}_\varepsilon(X, X) = 0 :\ \mathrm{Ric}[\check{g}_\varepsilon](X, X) > -\delta.\end{aligned}$$

The above distributional versions of the timelike and null energy conditions were successfully used in [7] to prove the singularity theorems of Hawking and of Penrose for C^1-spacetimes. For our C^1-proof of the Hawking-Penrose singularity theorem, we need a distributional version of the genericity condition along geodesics.

Definition 2.15 (*Distributional genericity condition*). Let (M, g) be a C^1-spacetime and let $\gamma : I \to M$ be a causal geodesic. We say that the *distributional genericity condition* holds at $\gamma(t_0)$, $t_0 \in I$, if there is a neighbourhood U of $\gamma(t_0)$, C^1-vector fields X, V on U with the property that $(X \circ \gamma)(t) = \dot{\gamma}(t)$, $(V \circ \gamma)(t) \perp \dot{\gamma}(t)$ for all $t \in I$ with $\gamma(t) \in U$, and there exist $c > 0$ and $\delta > 0$ such that for all C^1-vector fields $\tilde{X}, \tilde{V}$ on U with $\|X - \tilde{X}\|_h < \delta$ and $\|V - \tilde{V}\|_h < \delta$ we have

$$g(R(\tilde{X}, \tilde{V})\tilde{V}, \tilde{X}) > c \quad \text{in } \mathcal{D}'^{(1)}(U). \tag{2.16}$$

This condition extends the one used in the $C^{1,1}$-case in [8, Def. 2.2] as we shall discuss in Remark 2.19, below. Again it allows us to derive an approximate genericity condition for regularisations. Note, however, that it will turn out to be essential to use C^1-vector fields (and their approximations), which can be inserted into $R \in \mathcal{D}'^{(1)}\mathcal{T}^1_3(M)$, by (2.3).

Also the distributional genericity condition appropriately restricts to smaller sets, a technical point which we clarify first.

Remark 2.16. In Definition 2.15 we may shrink the neighbourhood U while retaining the constants c and δ. More precisely, let U, X, V, c, and δ be as in the definition and let $W \Subset U$, then (2.16) holds for all C^1-vector fields $\tilde{X}_W$, $\tilde{V}_W$ on W which satisfy $\|X|_W - \tilde{X}_W\|_h < \delta$ and $\|V|_W - \tilde{V}_W\|_h < \delta$. Indeed, by the sheaf property of distributions, it suffices to establish that (2.16) then holds locally around each point $x \in W$. To establish this let $\chi \in \mathcal{D}(W)$ be a plateau function that equals 1 in a neighbourhood W' of x and set $\tilde{X} = X + \chi(\tilde{X}_W - X)$ and likewise for $\tilde{V}$. Then $\tilde{X}$ and $\tilde{V}$ are C^1-vector fields on U with $\|X - \tilde{X}\|_h < \delta$ and $\|V - \tilde{V}\|_h < \delta$ (2.16). So we obtain on W'

$$c < g(R(\tilde{X}, \tilde{V})\tilde{V}, \tilde{X}) = g(R(\tilde{X}_W, \tilde{V}_W)\tilde{V}_W, \tilde{X}_W). \tag{2.17}$$

Lemma 2.17. *Let (M, g) be a C^1-spacetime and let g_ε be either $\check{g}_\varepsilon$ or $\hat{g}_\varepsilon$. Suppose γ is a g-causal g-geodesic and assume that the distributional genericity condition holds at $\gamma(t_0)$ (with neighbourhood U, vector fields X, V and constants c, δ). Suppose that X_ε, V_ε are C^1-vector fields on U with $X_\varepsilon, V_\varepsilon \to X, V$ in $C^1_{\mathrm{loc}}(U)$. Then for any compact subset $K \Subset U$ there is $\tilde{\varepsilon} > 0$ such that for all $\varepsilon < \tilde{\varepsilon}$:*

$$g_\varepsilon(R[g_\varepsilon](X_\varepsilon, V_\varepsilon)V_\varepsilon, X_\varepsilon) > \frac{c}{2}$$

on K.

Proof. We may take U to be relatively compact. By assumption, for all C^1-vector fields $\tilde{X}$, $\tilde{V}$ on U with $\|X - \tilde{X}\|_h < \delta$ and $\|V - \tilde{V}\|_h < \delta$, we have

$$g(R(\tilde{X}, \tilde{V})\tilde{V}, \tilde{X}) > c \quad \text{in } \mathcal{D}^{'(1)}(U).$$

Since $X_\varepsilon \to X$ and $V_\varepsilon \to V$ in C^1_{loc}, there is $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, $\|X_\varepsilon - X\|_h < \delta$ and $\|V_\varepsilon - V\|_h < \delta$. Let from now on $\varepsilon < \varepsilon_0$. Hence, the assumption gives

$$g(R(X_\varepsilon, V_\varepsilon)V_\varepsilon, X_\varepsilon) > c \quad \text{in } \mathcal{D}^{'(1)}(U).$$

Since $\star_M$-convolution with a non-negative mollifier ρ_ε respects positivity, we have

$$g(R(X_\varepsilon, V_\varepsilon)V_\varepsilon, X_\varepsilon) \star_M \rho_\varepsilon > c.$$

We claim that

$$g(R(X_\varepsilon, V_\varepsilon)V_\varepsilon, X_\varepsilon) \star_M \rho_\varepsilon - (g \star_M \rho_\varepsilon)(R[g \star_M \rho_\varepsilon](X_\varepsilon, V_\varepsilon)V_\varepsilon, X_\varepsilon) \to 0 \quad \text{in } C^0_{\mathrm{loc}}(U).$$

This is seen by first showing that

$$g(R(X_\varepsilon, V_\varepsilon)V_\varepsilon, X_\varepsilon) \star_M \rho_\varepsilon - g((R[g] \star_M \rho_\varepsilon)(X_\varepsilon, V_\varepsilon)V_\varepsilon, X_\varepsilon) \to 0 \quad \text{in } C^0_{\mathrm{loc}}(U).$$

The latter locally corresponds to the question whether given a net h_ε of C^1-functions that converges in C^1_{loc} to $h \in C^1$ and a first order distribution T we can show $(h_\varepsilon T) * \rho_\varepsilon - h_\varepsilon(T * \rho_\varepsilon) \to 0$ in C^0_{loc}. This is shown by arguing along the lines of [7, Lem. 4.8] and using that T can locally be written as a first order distributional derivative of a C^0-function. By a similar argument (noting the fast convergence speed of $g \star_M \rho_\varepsilon \to g$), the right hand side above is C^0_{loc}-equivalent (i.e. the difference goes to 0 in C^0_{loc}) to

$$(g \star_M \rho_\varepsilon)(R[g] \star_M \rho_\varepsilon(X_\varepsilon, V_\varepsilon)V_\varepsilon, X_\varepsilon).$$

This is C^0_{loc}-equivalent to

$$(g \star_M \rho_\varepsilon)(R[g \star_M \rho_\varepsilon](X_\varepsilon, V_\varepsilon)V_\varepsilon, X_\varepsilon)$$

by the same arguments as in [7, Lem. 4.6] and uniform boundedness of X_ε, V_ε.

Next, we show that

$$(g \star_M \rho_\varepsilon)(R[g \star_M \rho_\varepsilon](X_\varepsilon, V_\varepsilon)V_\varepsilon, X_\varepsilon) - g_\varepsilon(R[g_\varepsilon](X_\varepsilon, V_\varepsilon)V_\varepsilon, X_\varepsilon) \to 0$$

in $C^0_{\mathrm{loc}}(U)$. Upon writing everything out in coordinates, we see that only terms of the form

$$(g \star_M \rho_\varepsilon)\partial\Gamma[g \star_M \rho_\varepsilon]X_\varepsilon V_\varepsilon V_\varepsilon X_\varepsilon - g_\varepsilon \partial\Gamma[g_\varepsilon]X_\varepsilon V_\varepsilon V_\varepsilon X_\varepsilon$$

are of interest, because for any of the other terms in the difference, both summands converge in $C^0_{\mathrm{loc}}(U)$ to the respective expressions for g. Since the X_ε, V_ε are uniformly bounded, we may omit them in our estimates without loss of information. But for terms as above, we only need to consider those expressions where we have second derivatives of the metrics, since $\partial_j(g \star_M \rho_\varepsilon)^{mk} \to \partial_j g^{mk}$ in $C^0_{\mathrm{loc}}(U)$ and similarly for $\partial_j(g_\varepsilon)^{mk}$. Hence, we work in a chart, fix a compact set K and are left with estimating the following expression on K:

$$\begin{aligned}
&|(g \star_M \rho_\varepsilon)_{bc}(g \star_M \rho_\varepsilon)^{mk}\partial_s\partial_r(g \star_M \rho_\varepsilon)_{ij} - (g_\varepsilon)_{bc}(g_\varepsilon)^{mk}\partial_s\partial_r(g_\varepsilon)_{ij}| \\
&\leq |(g \star_M \rho_\varepsilon)_{bc}(g \star_M \rho_\varepsilon)^{mk}| \cdot |\partial_s\partial_r(g \star_M \rho_\varepsilon)_{ij} - \partial_s\partial_r(g_\varepsilon)_{ij}| + II.
\end{aligned}$$

The first term goes to 0 uniformly on K by Lemma 2.3(iii). We continue estimating the remaining term II:

$$\begin{aligned}
II &= |(g \star_M \rho_\varepsilon)_{bc}(g \star_M \rho_\varepsilon)^{mk}\partial_s\partial_r(g_\varepsilon)_{ij} - (g_\varepsilon)_{bc}(g_\varepsilon)^{mk}\partial_s\partial_r(g_\varepsilon)_{ij}| \\
&\leq |\partial_s\partial_r(g_\varepsilon)_{ij}| \cdot |(g \star_M \rho_\varepsilon)_{bc}(g \star_M \rho_\varepsilon)^{mk} - (g_\varepsilon)_{bc}(g_\varepsilon)^{mk}|.
\end{aligned}$$

To show that this goes to 0 uniformly on K, it suffices to show that the second factor can be estimated by a constant times ε by the same argument as in the proof of [7, Lem. 4.5]. Since $\|g_\varepsilon - (g \star_M \rho_\varepsilon)\|_{\infty,K}$ and $\|g_\varepsilon^{-1} - (g \star_M \rho_\varepsilon)^{-1}\|_{\infty,K}$ can both be estimated by a constant times ε again by Lemma 2.3(iii), this is easily seen to hold.

Hence, there is $\varepsilon_1 > 0$ such that for all $\varepsilon < \min(\varepsilon_0, \varepsilon_1)$,

$$g_\varepsilon(R[g_\varepsilon](X_\varepsilon, V_\varepsilon)V_\varepsilon, X_\varepsilon) > \frac{c}{2} \quad \text{on } K.$$

□

Next we derive from the distributional genericity condition an estimate on the tidal force operator for the regularised metrics. It is this result which allows us to enter into the Riccati comparison techniques in Sect. 4.

Lemma 2.18. *Let the assumptions of Lemma 2.17 hold and let γ_ε be g_ε-causal g_ε-geodesics, whose g_ε-causal character is the same as the g-causal character of γ and which converge in C^1_{loc} to γ. Then there are $\tilde{c} > 0$, $r > 0$, $\varepsilon_0 > 0$ and a neighbourhood U of $\gamma(t_0)$ such that for any $0 < \varepsilon < \varepsilon_0$ there exist vector fields E_i^ε on U such that $E_i^\varepsilon \circ \gamma_\varepsilon$ constitutes a g_ε-orthonormal frame along γ_ε and such that $E_i^\varepsilon \to E_i$ in $C^1_{\mathrm{loc}}(U)$, where $E_i \circ \gamma$ is an orthonormal frame along γ. Further, there exists some $C = C(\varepsilon) > 0$ such* *that along γ_ε the ε-tidal force operator $[R_\varepsilon](t) := [R_\varepsilon(., \dot\gamma_\varepsilon(t))\dot\gamma_\varepsilon(t)] : [\dot\gamma_\varepsilon(t)]^\perp \to [\dot\gamma_\varepsilon(t)]^\perp$ fulfills*

$$[R_\varepsilon](t) > \operatorname{diag}(\tilde c, -C, \dots, -C) \text{ on } [t_0 - r, t_0 + r] \tag{2.18}$$

in terms of the frame E_i^ε and where we have used the shorthand R_ε for $R[g_\varepsilon]$.

Proof. Let w.l.o.g. $t_0 = 0$. We deal with the case of γ being null or timelike simultaneously and construct a g-orthonormal frame for $[\dot\gamma]^\perp$ along γ. First note that, if γ is timelike, $V \circ \gamma$ is nowhere proportional to $\dot\gamma$, but the same may not be true in the null case. First we will show that we can replace V by a vector field whose restriction to γ is parallel in $[\dot\gamma]^\perp$ and still retain (2.16).

There must exist some t such that $V \circ \gamma(t) \notin \operatorname{span}(\dot\gamma(t))$. Indeed, suppose to the contrary that $V(\gamma(t)) = f(t) \cdot X(\gamma(t))$ for some real-valued C^1 function f and all t. Then extending both $X \circ \gamma$ and $V \circ \gamma$ in a cylindrically constant fashion as given by Lemma B.1, we obtain vector fields $\tilde V$, $\tilde X$ on a neighbourhood W of $\gamma(0)$ that are proportional and can be made arbitrarily close to V resp. X in C^0 when shrinking W since they restrict to the same vector fields on γ. According to Definition 2.15 they therefore satisfy (2.16) close to $\gamma(0)$. However, by the symmetry properties of R (cf. Lemma 2.4) and the proportionality of $\tilde V$ and $\tilde X$ we must have $R(\tilde X, \tilde V)\tilde V = 0$ in $\mathcal{D}'(W)$, a contradiction.

Hence there exists some $v := V(t_1)$ with $v \in [\dot\gamma(t_1)]^\perp$. Defining V_1 as the parallel translate of v along γ we obtain $V_1 \in [\dot\gamma]^\perp$ along all of γ. By the same continuity argument as above, (2.16) then still holds for the vector fields V_1 and X (note that a t_1 as above must exist arbitrarily close to 0). In the timelike case we may assume that γ is parametrised to unit speed, and we set $e_n := \dot\gamma(0)$. If γ is null we can write $\dot\gamma(0) = e_{n-1} + e_n$ for orthonormal vectors e_{n-1}, e_n with e_n timelike.

In the null case, extend $e_1 := V_1(0) \in [\dot\gamma(0)]^\perp$ to a g-orthonormal basis $\{e_i\}_{i=1}^n$ (with e_{n-1}, e_n as above) and define E_i to be the g-parallel translate of e_i along γ (so that $E_1 = V_1$ and $X = E_{n-1} + E_n$ along γ). For γ timelike we also construct an orthonormal frame $E_1 = V_1, E_2, \dots, E_n = X$ along γ via parallel transport. As $\gamma_\varepsilon \to \gamma$ in C^1_{loc}, we can find a g_ε orthonormal basis $\{e_i^\varepsilon\}$ at $\gamma_\varepsilon(0)$ with $e_i^\varepsilon \to e_i$ in TM, and such that $\dot\gamma_\varepsilon(0) = e_n^\varepsilon$ if γ_ε is g_ε-timelike, while $\dot\gamma_\varepsilon(0) = e_{n-1}^\varepsilon + e_n^\varepsilon$ with e_n^ε timelike if γ_ε is g_ε-null. For every ε let E_i^ε be the g_ε-parallel translate of e_i^ε along γ_ε. By Remark 2.7, $E_i^\varepsilon \to E_i$ in C^1_{loc} (i.e. in $T(TM)$, uniformly on compact time intervals) and we can use Lemma B.2 to extend E_i^ε to vector fields on U, converging in $C^1_{\mathrm{loc}}(U)$ to the extensions of E_i to U.

As we have now found vector fields E_i^ε that restrict to a g_ε-frame along γ_ε and converge to the g-frame E_i along γ, we are in a position to apply Lemma 2.17 and hence estimate the g_ε-tidal force operator along γ_ε.

For the remainder of the proof we roughly follow the layout of the proof of [8, Prop. 3.6]. There, for a $C^{1,1}$-metric g the statement is first shown for the g-tidal force operator along γ and then carried over for small ε by an approximation argument. However, in the C^1-setting this is no longer possible, as R is now merely distributional and for a matrix with distributional entries eigenvalues are not defined. Also, a direct approximation procedure of that form is no longer feasible. Instead we use Lemma 2.17, and argue separately for any ε, making the procedure (and hence the constant C) dependent on ε[5]. More precisely:

Let c be the constant given by the genericity condition and let $\tilde c := \frac{c}{2}$ and $0 < c_1 < \tilde c$. Set $R_{ij}^\varepsilon := \langle R_\varepsilon(E_i^\varepsilon, X_\varepsilon)X_\varepsilon, E_j^\varepsilon\rangle$, $i, j = 1, \dots, d$, where $X_\varepsilon = E_n^\varepsilon$ in the timelike case

[5] This is, however, sufficient in our proof of the main result.

and $X_\varepsilon = E^\varepsilon_{n-1} + E^\varepsilon_n$ in the null case. As in the proof of (3.5) in [8, Lemma 3.7], we choose $C(\varepsilon) > 0$ such that $\lambda^\varepsilon_{\min}(x) \geq \frac{\|(R^\varepsilon_{1j})_j\|_e}{\tilde{c}-c_1}$ for all $x \in U$, where $\lambda^\varepsilon_{\min}(x)$ is the smallest eigenvalue of $(R^\varepsilon_{ij})^d_{i,j=2} + C(\varepsilon)\mathrm{Id}_{d-1}$ at $x \in U$. By Lemma 2.17 there exists some $\tilde{\varepsilon} > 0$ such that $R^\varepsilon_{11} > \tilde{c}$ for all $0 < \varepsilon < \tilde{\varepsilon}$ on U (shrinking U once more if necessary). For these ε it then follows that $R^\varepsilon_{ij} - \mathrm{diag}(c_1, -C(\varepsilon), \ldots -C(\varepsilon))$ is positive definite on U. Then picking $r > 0$ such that $\gamma_\varepsilon([-r, r]) \subseteq U$ for all ε small, evaluation along γ_ε gives the claim. $\square$

Remark 2.19. Note that while Definition 2.15 may at first sight appear to be stronger than a straightforward generalisation of the genericity condition given for $C^{1,1}$-metrics in [8], due to the local boundedness of R in that regularity it is actually equivalent:

For $C^{1,1}$-metrics Definition 2.15 clearly implies Definition 2.2 of [8]. For the reverse implication, let X, V be continuous vector fields in a neighbourhood of $\gamma(t_0)$ such that $X(\gamma(t)) = \dot{\gamma}(t)$ and $V(\gamma(t)) \in (\dot{\gamma}(t))^\perp$ for all t and such that $\langle R(V, X)X, V\rangle > c > 0$. Arguing exactly as in the proof of Lemma 2.18 we can then use parallel transport and cylindrically constant extension to find C^1-vector fields that are uniformly close to X and V. It therefore suffices to show that the genericity condition from [8] is stable under locally uniform perturbation. Thus let $\tilde{V}$, $\tilde{X}$ be C^1 vector fields that are uniformly close to V, X. Then one can locally in a coordinate system estimate $|\tilde{V}_i - V_i| \leq \delta$, $|\tilde{X}_i - X_i| \leq \delta$ for some small δ. Consequently,

$$\langle R(\tilde{V}, \tilde{X})\tilde{X}, \tilde{V}\rangle = g_{ij} R^i_{klm} \tilde{V}^l \tilde{X}^m \tilde{X}^k \tilde{V}^j \geq g_{ij} R^i_{klm} V^l X^m X^k V^j + \mathcal{O}(\delta)$$

where the last term collects all δ-contributions and takes into account the local boundedness of R. Consequently, choosing δ small enough (and appropriately shrinking the neighbourhood of $\gamma(t_0)$) we can secure that the sum still remains $> c/2$.

3. Branching

In metric geometry, where geodesics are defined as (local) minimisers of the length functional, local uniqueness of geodesics is expressed in the form of a *non-branching* condition. Here, a branch point is defined as an element of a minimiser at which the curve splits into two minimisers that on some positive parameter interval do not have another point in common, cf., e.g., [44,51]. Similarly, in the synthetic Lorentzian setting [22], the role of causal geodesics is taken on by maximising causal curves, and non-branching is formulated analogously. In both cases, lower synthetic sectional curvature bounds (formulated via triangle comparison in constant curvature model spaces) imply non-branching [22,44]. Synthetic Ricci curvature bounds in metric measure spaces (curvature-dimension conditions), on the other hand, do not imply non-branching of geodesics [38]. Similarly, in recent work on synthetic Ricci curvature bounds in Lorentzian pre-length spaces [4], a timelike non-branching condition is required in addition to a timelike-curvature dimension condition to obtain a version of the Hawking singularity theorem.

In the C^1-setting we are concerned with in this paper, the coincidence between causal local maximisers and geodesics that is familiar from smooth Lorentzian geometry ceases to hold (cf. Example 3.2 below), although it is still true that causal maximisers are geodesics (Lemma A.4). In addition, one generically cannot expect unique local solvability of the geodesic initial value problem. Nevertheless, on physical grounds, it seems reasonable to assign a privileged role to causal geodesics that *are* locally maximising.

We therefore introduce a (weak notion of a) non-branching condition that is intended to preclude locally maximising geodesics from branching (where we do not require the second branch to be maximising as well):

Definition 3.1 (*Non-branching conditions*). Let (M, g) be a C^1-spacetime. A geodesic $\gamma : [a, b] \to M$ branches at $t_0 \in (a, b)$ if there exist $\varepsilon > 0$ and some geodesic σ with $\gamma|_{[t_0-\varepsilon,t_0]} \subseteq \sigma$, but $\gamma|_{(t_0,t_0+\varepsilon)} \cap \sigma = \emptyset$.
(M, g) is called maximally causally (resp. timelike, resp. null) non-branching (MCNB, MTNB, MNNB), if no maximal causal (resp. timelike, resp. null) geodesic branches in the above sense.

Example 3.2. A C^1-spacetime in which maximal causal branching occurs can be constructed from the second Riemannian example given in [16]. There, a $C^{1,\alpha}$-Riemannian metric ($\alpha < 1$) in the (u, v)-plane is given with non-unique solutions to the geodesic initial value problem starting at $\{v = 0\}$. Further it is shown that the geodesic boundary value problem for geodesics γ starting at the surface $\{v = 0\}$ is uniquely solvable and hence such geodesics are at least initially minimizing: If they were not, by properties of any C^1-Riemannian manifold as a locally compact length space, there would exist a minimiser σ between two points on γ. However in C^1-Riemannian manifolds minimisers are geodesics and by uniqueness (of the boundary value problem) $\gamma = \sigma$ (cf. [42] for proofs and references).

In [18] it is shown that causal geodesics in static spacetimes are maximising if and only if their Riemannian parts are minimizing and so one can easily construct both maximising timelike and null geodesics that branch.

Lemma 3.3. *If the C^1-spacetime (M, g) is MCNB (resp. MTNB, MNNB), then any two maximal causal (resp. timelike, null) geodesics that meet tangentially at an interior point must coincide on their maximal domain of definition.*

Proof. We only show this for MCNB spacetimes, the other cases are proven in exactly the same way by replacing all occurrences of “causal" with “timelike” resp. “null”. Let γ, σ be two maximising, causal geodesics meeting tangentially at p, which is not an endpoint of either curve. W.l.o.g. we may assume that $\gamma(0) = \sigma(0) = p$. It suffices to show that there is $\varepsilon > 0$ such that $\gamma|_{[0,\varepsilon)} = \sigma|_{[0,\varepsilon)}$. Indeed since then γ and σ meet tangentially at $t = \varepsilon$, the maximal such interval coincides with the maximal domain of definition (to the right and analogously to the left).

Assume to the contrary that $\gamma|_{[0,\varepsilon)} \neq \sigma|_{[0,\varepsilon)}$ for any ε. As the curves both satisfy the geodesic equation at p and their tangents agree, so do their second derivatives. Thus γ and $\gamma|_{(-\varepsilon,0]} \cup \sigma|_{[0,\varepsilon)}$ are both (unbroken) geodesics and by MCNB γ cannot branch at $t = 0$. So for every $\varepsilon > 0$ we have $\gamma|_{(0,\varepsilon)} \cap \sigma|_{(0,\varepsilon)} \neq \emptyset$. Hence there exist $t_k \downarrow 0$ such that $\gamma(t_k) = \sigma(t_k)$, and by our indirect assumption there exists $s_k \downarrow 0$ such that $\gamma(s_k) \neq \sigma(s_k)$. For any k set

$$\eta_k := \sup\{a \in [0, s_k] : \gamma|_{(s_k-a,s_k]} \cap \sigma|_{(s_k-a,s_k]} = \emptyset\}.$$

Further choose some $\tilde{\eta}_k > 0$ such that for $I_k := (s_k-\eta_k, s_k+\tilde{\eta}_k)$ we have $\gamma|_{I_k} \cap \sigma|_{I_k} = \emptyset$. Since $t_k \downarrow 0$ and $\gamma(t_k) = \sigma(t_k)$, we know that $s_k > \eta_k$. However, $\gamma(s_k-\eta_k) = \sigma(s_k-\eta_k)$ by our choice of η_k. Set $s_k - \eta_k = r_k$ and for simplicity of notation assume $r_k < t_k$. If $\dot{\gamma}(r_k) \neq \dot{\sigma}(r_k)$, the curve $\alpha := \gamma|_{[0,r_k]} \cup \sigma|_{[r_k,t_k]}$ is a broken geodesic from p to $\gamma(t_k)$ and hence not maximising by [43, Lem. 3.2], i.e., $d(p, \gamma(t_k)) > L(\alpha)$. Since γ and σ are maximisers we have $d(p, \gamma(r_k)) = L(\gamma|_{[0,r_k]}) = L(\sigma|_{[0,r_k]})$ Consequently,

$$d(p, \sigma(t_k)) = L(\sigma|_{[0,t_k]}) = L(\sigma|_{[0,r_k]}) + L(\sigma|_{[r_k,t_k]}) = L(\alpha) < d(p, \sigma(t_k)),$$

a contradiction, and hence $\dot\gamma(r_k) = \dot\sigma(r_k)$.

Again employing the geodesic equation, also the second derivatives of γ and σ coincide at r_k, and so the curves $\gamma|_{[0,s_k]}$ and $\gamma|_{[0,r_k]} \cup \sigma|_{[r_k,s_k]}$ display maximal causal branching, a contradiction. □

The following result shows that in maximally causally non-branching spacetimes, maximal causal geodesics can always be approximated by geodesics for the regularised spacetimes $\hat g_\varepsilon$ and $\check g_\varepsilon$. Whether such a result is true in general is questionable and very likely regularisation dependent.

Proposition 3.4. *Let (M, g) be a C^1-spacetime that is MCNB.*

(i) *Suppose that M is globally hyperbolic and let $\varepsilon_k \searrow 0$ ($k \to \infty$). Set $g_k := \hat g_{\varepsilon_k}$ or $g_k := \check g_{\varepsilon_k}$ for all k. If $\gamma : [0, a] \to M$ is a maximising, timelike g-geodesic then for any small $\delta > 0$ there exists a subsequence g_{k_l} of g_k and g_{k_l}-maximising, timelike geodesics γ_l converging in C^1 to $\gamma|_{[0,a-\delta]}$.*

(ii) *Let $g_k = \check g_{\varepsilon_k}$ for all k. If M is causal and $\gamma : [0, a] \to M$ is a g-maximizing null geodesic, then for any small $\delta > 0$, there exists a subsequence g_{k_l} of g_k, $t_l \downarrow 0$ and g_{k_l}-null geodesics $\gamma_l : [t_l, a-\delta] \to M$ contained in $\partial I^+_l(\gamma(0))$ (hence in particular g_{k_l}-maximising), which converge in C^1_{loc} to $\gamma|_{[0,a-\delta]}$.*

If, furthermore, M is strongly causal and S is an acausal set such that $\gamma \subseteq E^+(S)$, then even $\gamma_l : [t_l, a - \delta] \to E^+_l(S)$ and $\gamma_l(t_l) \in S$ with $\gamma_l(t_l) \to \gamma(0)$.

Proof. Fix $\delta \in (0, a)$ and set $p := \gamma(0)$, $q := \gamma(a)$, and $q_\delta := \gamma(a - \delta)$.

(i) Since γ is timelike, $q_\delta \in I^+(p)$ and without loss of generality we may assume that also $q_\delta \in I^+_k(p)$ for all k. Since g is globally hyperbolic we can also assume w.l.o.g. that g_k are globally hyperbolic as well (see [26, Prop. 2.3 (iv)], which remains valid for $g \in C^1$). Thus there exist g_k-maximising, timelike geodesics γ_k from p to q_δ [41, Prop. 6.5]. By Corollary 2.6 an affinely reparametrised subsequence, without loss of generality γ_k itself, converges in C^1_{loc} to a g-causal geodesic σ. By Lemma A.13 σ is a g-maximising timelike geodesic from p to q_δ, in particular $d(p, q_\delta) = L(\gamma|_{[0,a-\delta]}) = L(\sigma)$. We cannot yet apply Lemma 3.3 since it is not clear whether σ is maximising beyond q_δ. However denote by $\eta := \sigma \cup \gamma|_{[a-\delta,a]}$ the concatenation of σ and γ. Since $d(p, q) = L(\gamma) = L(\gamma|_{[0,a-\delta]}) + L(\gamma|_{[a-\delta,a]}) = L(\sigma) + L(\gamma|_{[a-\delta,a]}) = L(\eta)$, it is a maximising, timelike geodesic from p to q. We can now apply Lemma 3.3 to the curves η and γ, to obtain $\eta = \gamma$ and hence $\sigma = \gamma|_{[0,a-\delta]}$.

(ii) A similar statement was shown in [43, Cor. 3.5], but we need to adapt some of the main points of the argument to our assumptions.
Choose a sequence of points $q^k_\delta \in \partial I^+_{g_k}(p)$ converging to q_δ. There exist future directed, g_k-null maximising geodesics γ_k ending at q^k_δ and contained in $\partial I^+_{g_k}(p)$. W.l.o.g we may assume their terminal velocities to be h-normalized, so we can apply Corollary 2.6 to obtain a subsequence of γ_k (denoted in the same way) converging to a g-null geodesic σ. Since $\partial I^+_{g_k}(p) \subseteq \overline{I^+(p)}$ (recall that $g_k = \check g_{\varepsilon_k}$) we know that $\sigma \subseteq \partial I^+(p)$, since otherwise $q_\delta \in I^+(p)$, hence σ is maximising (cf. Lemma A.15). Note that γ and σ meet tangentially at q_δ, because otherwise $\eta := \sigma \cup \gamma|_{[a-\delta,a]}$ would not be maximising by Lemma A.4 and so $q \in I^+(\sigma) \subseteq I^+(\overline{I^+(p)}) \subseteq I^+(p)$, a contradiction. Lemma 3.3 now shows that $\gamma = \eta$ and so $\sigma = \gamma$, wherever both curves are defined. There are two possibilities for σ, either it is past inextendible or it reaches p. However, if it is past inextendible, we must

have $\sigma \supseteq \gamma|_{[0,a-\delta]}$. Hence in both cases the γ_k converge to γ in C^1_{loc}. Further this means that there are $t_k \in [0, a-\delta]$ with $\gamma_k(t_k) \to p$, and we have $t_k \to t_0 = 0$ since otherwise $\gamma(t_0) = p = \gamma(0)$, contradicting the causality of M. Now assume M to be strongly causal and $\gamma \subseteq E^+(S)$ for S acausal, then there exists a g-causally convex, g-globally hyperbolic neighbourhood U of p by [32, Lem. 3.21] (this result holds also for C^1-spacetimes). In such neighbourhoods it holds that $I^+_U(S) = I^+(S) \cap U$, $J^+_U(S) = J^+(S) \cap U$ and hence also $E^+_U(S) = \partial I^+_U(S) = \partial I^+(S) \cap U$. Clearly U is also g_k-causally convex and g_k-globally hyperbolic and so these equalities also hold for g_k. Note that in the same fashion as for the case of a single point p one can show that $\gamma_k \to \gamma|_{[0,a-\delta]}$. If γ_k reaches S, we immediately obtain $\gamma_k \subseteq E^+_k(S)$. In the other case $\gamma_k \subseteq \partial I^+_k(S) \backslash J^+_k(S)$ is past inextendible. However, as they converge to $\gamma|_{[0,a-\delta]}$, there must be $p_k \in \gamma_k$, with $p_k \to p$. Hence, w.l.o.g. $p_k \in \partial I^+_k(S) \cap U = E^+_{U,k}(S)$ and so $p_k \in J^+_k(S)$, a contradiction. We have shown that γ_k reach S, and it only remains to show that they do so close to 0. Again let $\gamma_k(t_k) \in S$ with $\gamma_k(t_k) \to p$. If $t_k \to t_0$, then $\gamma(t_0) \in S$ and by acausality of S we must have $t_0 = 0$. $\square$

Remark 3.5. (i) As the proof of Proposition 3.4 shows, for (i) it suffices to assume (M, g) to be MTNB, while for (ii) MNNB would suffice.

(ii) We will make use of Proposition 3.4(ii) in two different scenarios: Thm. 4.3 and Proposition 5.5 below. In the first case (M causal) all we need to ensure is that γ_l are maximizing for long enough, i.e. $a - \delta - t_l$ is close to a. In the second (M strongly causal) it is important to also obtain that γ_l reaches S, in order to be able to use the initial conditions assumed on S.

4. No Lines

In this section we will prove that, under suitable causality and energy conditions, complete causal geodesics stop being maximising also in (MCNB) C^1-spacetimes. To this end we first prove the existence of conjugate points for causal geodesics in smooth spacetimes under weakened versions of the energy conditions. More precisely we will invoke the energy conditions derived for regularisations in Lemma 2.10 and in Lemma 2.14 from the distributional energy conditions, as well as in Lemma 2.18 from the distributional genericity condition. The following is a strengthened version of [8, Prop. 4.2]:

Lemma 4.1. *Let (M, g) be a smooth spacetime. Given some $c > 0$ and $0 < r < \frac{\pi}{4\sqrt{c}}$, there exist $\delta(c, r) > 0$ and $T(c, r) > 0$ such that any causal geodesic γ defined on $[-T, T]$ for which the following hold*

(i) *$Ric(\dot\gamma, \dot\gamma) \geq -\delta$ on $[-T, T]$ and*

(ii) *there exists a smooth parallel orthonormal frame for $[\dot\gamma]^\perp$ and some $C > 0$ such that w.r.t. this frame the tidal force operator satisfies $[R](t) > \operatorname{diag}(c, -C, \ldots, -C)$ on $[-r, r]$,*

possesses a pair of conjugate points on $[-T, T]$.

Proof. This actually follows by a more careful choice of constants in the proof of [8, Prop. 4.2]. To see this we briefly recall the main steps of that proof. The argument proceeds indirectly, assuming that for any $\delta > 0$ and $T > 0$ there is some γ satisfying (i) and (ii) without conjugate points in $[-T, T]$. Denote by $[A]$ the unique Jacobi tensor class along γ with $[A](-T) = 0$ and $[A](0) = \text{id}$. Taking $[E_1](t), \ldots, [E_d](t)$ as in (ii), linear endomorphisms of $[\dot\gamma]^\perp$ are written as matrices in this basis, and we set $[\tilde R](t) := \mathrm{diag}(c, -C, \dots, -C)$ with C as in (ii). Then by (ii), $[\tilde R](t) < [R](t)$ on $[-r, r]$.

The self-adjoint operator $[B] := [\dot A] \cdot [A]^{-1}$satisfies the matrix Riccati equation

$$[\dot B] + [B]^2 + [R] = 0, \tag{4.1}$$

and we denote by $[\tilde B]$ the solution to (4.1), with $[\tilde R]$ instead of $[R]$ and initial value prescribed at some $t_1 \in [-r, r]$. We may even choose t_1 in $[-r, 0]$ and $[\tilde B](t_1) := \tilde\beta(t_1) \cdot \mathrm{id}$, where $\tilde\beta(t_1)$ is greater or equal than the largest eigenvalue of $[B](t_1)$. More precisely, examining the Raychaudhuri equation

$$\dot\theta + \frac{1}{d}\theta^2 + \mathrm{tr}(\sigma^2) + \mathrm{tr}([R]) = 0, \tag{4.2}$$

for the expansion $\theta = \mathrm{tr}([B])$ (with $\sigma = [B] - \frac{1}{d}\theta \cdot \mathrm{id}$), it follows that one can pick $\tilde\beta(t_1) := f(\nu, \delta, r)$ and $[\tilde B](t_1) := f(\nu, \delta, r) \cdot \mathrm{id}$ to indeed achieve that $[B](t) \le [\tilde B](t)$ on $[t_1, r]$. Here, $f(\nu, \delta, r) = \sqrt{\frac{2\nu}{r} + \delta} + \frac{\nu}{d}$ with $\nu = 4d/T$.

Moreover, due to the fact that both $[\tilde R]$ and $[\tilde B](t_1)$ are diagonal, the Riccati equation for $[\tilde B]$ decouples. Indeed it can be explicitly solved by

$$[\tilde B](t) = \frac{1}{d}\mathrm{diag}(H_{c,f}(t), H_{-C,f}(t), \dots, H_{-C,f}(t)),$$

where

$$H_{c,f}(t) = d\sqrt{c}\cot(\sqrt{c}(t - t_1) + \mathrm{arccot}(f/\sqrt{c})),$$

and

$$H_{-C,f}(t) = d\sqrt{C}\tanh\left(\sqrt{C}(t - t_1) + \mathrm{artanh}(f/\sqrt{C})\right).$$

From this point on, the proof of [8, Prop. 4.2] proceeds by exclusively analysing the function $H_{c,f}$, and the only requirement on r turns out to be $4r\sqrt{c} < \pi$. Once this is granted, δ and T can be chosen depending only on c to arrive at the desired contradiction. □

Recall that a line is an inextendible causal geodesic maximising the Lorentzian distance between any of its points.

Theorem 4.2. (No timelike lines). *Let (M, g) be a globally hyperbolic, MTNB C^1-spacetime satisfying the distributional timelike energy condition and the distributional genericity condition along any inextendible timelike geodesic. Then there is no complete timelike line in M.*

Proof. Suppose $\gamma : \mathbb{R} \to M$ is a complete timelike line. We may assume that distributional genericity holds at $t_0 = 0$ along γ. We approximate g by $\check g_\varepsilon \prec g$, which are hence also globally hyperbolic. By Lemma 2.18 there exist $c > 0$, $0 < r < \frac{\pi}{4\sqrt{c}}$ with the following property: For any C^1-approximation γ_ε of γ by $\check g_\varepsilon$-timelike geodesics, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, there is $C = C(\varepsilon) > 0$ so that (ii) in Lemma 4.1 is satisfied for R_ε. Now for the above choice of c and r, pick $\delta > 0$ and $T > 0$ as in Lemma 4.1, and let $\tilde T > T$. By Proposition 3.4(i), for the curve $\gamma|_{[-T,\tilde T]}$

and $\delta < \tilde{T} - T$, we obtain a subsequence $\check{g}_{\varepsilon_k}$ and $\check{g}_{\varepsilon_k}$-maximal timelike geodesics γ_k from $\gamma(-T)$ to $\gamma(\tilde{T})$ converging to $\gamma|_{[-T,\tilde{T}]}$ in C^1.

Let K be a compact neighbourhood of $\gamma([-T, T])$. Since $\gamma_{\varepsilon_k} \to \gamma$ in $C^1([-T, T])$, there are $k_0 \in \mathbb{N}$, $\tilde{C} > 0$ and $\kappa < 0$ such that for all $k \geq k_0$, we have $\gamma_{\varepsilon_k}([-T, T]) \subseteq K$, $\|\dot{\gamma}_{\varepsilon_k}\|_h \leq \tilde{C}$ and $\check{g}_{\varepsilon_k}(\dot{\gamma}_{\varepsilon_k}, \dot{\gamma}_{\varepsilon_k}) < \kappa$ on $[-T, T]$. Hence by [7, Lem. 4.6] (and the remark preceding it) we have $\mathrm{Ric}_{\varepsilon_k}(\dot{\gamma}_{\varepsilon_k}, \dot{\gamma}_{\varepsilon_k}) \geq -\delta$ on $[-T, T]$ for large k. Therefore, also (i) in Lemma 4.1 is satisfied for γ_{ε_k}, yielding the existence of conjugate points for any γ_{ε_k} (k large) in $[-T, T]$, a contradiction since they are maximising even in the strictly larger interval $[-T, \tilde{T}]$. □

Theorem 4.3. *(No null lines) Let (M, g) be a causal, MNNB C^1-spacetime that satisfies the distributional null energy condition and the distributional genericity condition along any inextendible null geodesic. Then there is no complete null line in M.*

Proof. Suppose $\gamma : \mathbb{R} \to M$ is a complete null line. We will again make use of the approximation $\check{g}_\varepsilon$. Suppose without loss of generality that the distributional genericity condition holds along γ at $t_0 = 0$. By Lemma 2.18 there exist $c > 0$ and $0 < r < \frac{\pi}{4\sqrt{c}}$ satisfying: For any C^1-approximation γ_ε of γ by $\check{g}_\varepsilon$-null geodesics, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, there is $C = C(\varepsilon) > 0$ so that (ii) in Lemma 4.1 is satisfied for R_ε. Choose $\delta > 0$ and $T > 0$ as in Lemma 4.1 for this pair (c, r) and let $\tilde{T} > T$. Using Proposition 3.4(ii) for the curve $\gamma|_{[-T,\tilde{T}]}$ and $S = \gamma(-T)$ (cf. Remark 3.5(ii)), there exists a subsequence $\check{g}_{\varepsilon_k}$ and $\check{g}_{\varepsilon_k}$-maximal null geodesics $\gamma_k : [-T, \tilde{T}] \to M$ converging to $\gamma|_{[-T,\tilde{T}]}$ in C^1.

Since $\gamma_k \to \gamma$ in $C^1([-T, T])$, once we choose a compact neighbourhood K of $\gamma([-T, T])$, there are $k_0 \in \mathbb{N}$ and $\tilde{C}_2 > \tilde{C}_1 > 0$ such that for all $k \geq k_0$, we have $\gamma_{\varepsilon_k}([-T, T]) \subseteq K$, $\tilde{C}_1 < \|\dot{\gamma}_k\|_h < \tilde{C}_2$ in $[-T, T]$. Hence, Lemma 2.14 implies that $\mathrm{Ric}_{\varepsilon_k}(\dot{\gamma}_k, \dot{\gamma}_k) > -\delta$ in $[-T, T]$. But then Lemma 4.1 may be invoked to give the existence of conjugate points on $\gamma_k|_{[-T,T]}$ for large k, a contradiction since they were supposed to be maximising even on $[-T, \tilde{T}]$. □

5. Initial Conditions

The classical Hawking-Penrose theorem for spacetimes of dimension 4 assumes three alternative initial conditions: the existence of a compact, spacelike hypersurface, a trapped (2-)surface, or a "trapped point", i.e., a point from where all future (or past) null geodesics converge. The second condition was later generalised in [10] to trapped submanifolds of arbitrary codimension m with $1 < m < n = \dim(M)$.

While the first condition does not need any special attention here we will start by generalising the trapped submanifold-case to C^1-spacetimes. As for $C^{1,1}$-spacetimes (see [8, Sec. 6.2]), trapped submanifolds are defined in the support sense. However, to show that normal null geodesics emanating from them stop being maximising is more delicate now due to the lack of an exponential map, cf. (the proof of) Proposition 5.5. At the end of this section we will deal with the case of a "trapped point".

Definition 5.1 (*Support submanifolds*). Let (M, g) be a C^1-spacetime and let $S, \tilde{S} \subseteq M$ be submanifolds. We say that $\tilde{S}$ is a *future support submanifold for S at $q \in S$* if $\dim S = \dim \tilde{S}$, $q \in \tilde{S}$, and there is a neighbourhood U of q in M such that $\tilde{S} \cap U \subseteq J^+_U(S)$.

Definition 5.2 (Trapped C^0-submanifolds). Let (M, g) be a C^1-spacetime of dimension n. We say that a C^0-submanifold $S \subseteq M$ of codimension $1 \leq m < n$ is a *future trapped submanifold* if it is compact without boundary and for any $p \in S$ there exists a neighbourhood U of p such that $U \cap S$ is achronal in U and moreover S has past pointing timelike mean curvature in the sense of support submanifolds, i.e. for any $q \in S$ there exists a future C^2-support submanifold $\tilde{S}$ for S at q whose mean curvature vector at q is past-pointing timelike.

Now, to show that lightrays from trapped submanifolds stop maximising, we reduce the problem to a question about smooth approximating metrics, which are understood much better. The essential results that deal with the corresponding situations for smooth metrics are [8, Lem. 6.4] and [7, Lem. 5.6]. To give a precise formulation we first introduce some notation.

Suppose $S \subseteq M$ is a spacelike C^2-submanifold of codimension $1 < m < n$, $p \in S$ and $\nu \in T_pM$ is a future null normal to S. Let γ be a geodesic with $\gamma(0) = p$, $\dot{\gamma}(0) = \nu$. Let $e_1, \ldots, e_{n-m}$ be an ON-basis on S around p. Further let $E_1, \ldots, E_{n-m}$ be the parallel translates of $e_1(p), \ldots, e_{n-m}(p)$ along γ, which are C^1 since they satisfy the parallel transport equation. We will use this notation for the following results. It will be clear from the context which surface the E_i refer to.

Lemma 5.3. *Let (M, g) be a smooth spacetime and let S be a codimension-m $(1 < m < n)$ spacelike C^2 submanifold of M. Let γ be a geodesic starting in S such that $\nu := \dot{\gamma}(0) \in TM|_S$ is a future null normal to S. Suppose that $c := \mathbf{k}_S(\nu) > 0$ and let $b > 1/c$. Then there is $\delta = \delta(b, c) > 0$ such that, if for E_i as described above*

$$\sum_{i=1}^{n-m} g(R(E_i, \dot{\gamma})\dot{\gamma}, E_i) \geq -\delta$$

along γ, then $\gamma|_{[0,b]}$ is not maximising from S, provided that γ exists up to $t = b$.

Lemma 5.4. *Let (M, g) be a smooth spacetime and let S be a codimension-2 spacelike C^2 submanifold of M. Let γ be a geodesic starting in S such that $\nu := \dot{\gamma}(0)$ is a future null normal to S. Suppose that $c := \mathbf{k}_S(\nu) > 0$ and let $b > 1/c$. Then there is $\delta = \delta(b, c) > 0$ such that, if*

$$\mathrm{Ric}(\dot{\gamma}, \dot{\gamma}) \geq -\delta$$

along γ, then $\gamma|_{[0,b]}$ is not maximising from S, provided that γ exists up to $t = b$.

The following result is the C^1-analogue of [8, Prop. 6.5]. As was the case for the distributional genericity condition, cf. Definition 2.15, also here we have to assume that the distributional curvature condition at hand is stable under C^0-perturbations of the vector fields involved. This ensures that we can derive the necessary curvature conditions for the smooth approximating metrics in order to be able to use Lemmas 5.3 and 5.4 .

Proposition 5.5 *(Light rays from a submanifold). Let (M, g) be a strongly causal, MNNB C^1-spacetime and let $\tilde{S} \subseteq M$ be a C^2-spacelike submanifold of codimension $1 < m < n$. Suppose $\mathbf{k}_{\tilde{S}}(\nu) = g(\nu, H_p) > c > 0$ and let $b > 1/c$. Suppose there is a null geodesic γ with $\dot{\gamma}(0) = \nu$, a neighbourhood U of $\gamma|_{[0,b]}$ and C^1-extensions $\overline{E}_i$ of E_i and $\overline{N}$ of $\dot{\gamma}$ to U such that for each $\delta > 0$ there exists $\eta > 0$ such that for all collections of C^1-vector* *fields* $\{\tilde{E}_1, \dots, \tilde{E}_{n-m}, \tilde{N}\}$ *on* U *with* $\|\tilde{E}_i - \overline{E}_i\|_h < \eta$ *for all* i *and* $\|\tilde{N} - \overline{N}\|_h < \eta$*, we have*

$$\sum_{i=1}^{n-m} g(R(\tilde{E}_i, \tilde{N})\tilde{N}, \tilde{E}_i) \geq -\delta \quad \text{in } \mathcal{D}'^{(1)}(U). \tag{5.1}$$

Then $\gamma|_{[0,b]}$ *is not maximising from* $\tilde{S}$*.*

Note that condition (5.1) localises in a similar manner as the genericity condition. For the latter see Remark 2.16.

Proof. Let $g_\varepsilon = \check{g}_\varepsilon$, $R_\varepsilon = R[\check{g}_\varepsilon]$. Clearly, $\mathbf{k}_{\tilde{S}}$ is continuous and $\mathbf{k}_{\tilde{S},\varepsilon} \to \mathbf{k}_{\tilde{S}}$ uniformly on compact sets. Hence, there is a neighbourhood V of ν in $TM|_{\tilde{S}}$ and ε_0 such that $\forall \varepsilon \leq \varepsilon_0$ and $\forall v \in V$: $\mathbf{k}_{\tilde{S},\varepsilon}(v) > c$. Note that g-spacelikeness implies g_ε-spacelikeness. Hence $U \cap \tilde{S}$ is g_ε-spacelike and we may assume that $W := \pi(V)$ is contained in $U \cap \tilde{S}$.

Suppose now to the contrary that $\gamma|_{[0,b]}$ maximises the distance from $\overline{U} \cap \tilde{S}$. Let $1/c < b' < b'' < b$ and let $q = \gamma(b'')$. Find $q_k \in \partial J_k^+(\overline{U} \cap \tilde{S})$ (where $\partial J_k := \partial J_{g_{\varepsilon_k}}$ and $\varepsilon_k \to 0$) with $q_k \to q$. By Corollary A.12 there are g_{ε_k}-null geodesics $\gamma_k : I_k \to M$ (in future directed parametrisation) with $\gamma_k(b'') = q_k$ either intersecting $\overline{U} \cap \tilde{S}$ or past inextendible. We may assume that $\|\dot{\gamma}_k(b'')\|_h$ are all equal to $\|\dot{\gamma}(b'')\|_h$, hence we may assume that the $\dot{\gamma}_k(b'')$ converge to a g-null vector v and the geodesics converge to a g-null geodesic γ_v in C^2_{loc} by Corollary 2.6. Strong causality allows us to invoke Lemma A.30, so we can find a neighbourhood W of $\gamma(0)$ with $W \subseteq U$ such that $W \cap \tilde{S}$ is acausal in M. By (the proof of) Proposition 3.4(ii) (see also Remark 3.5(i)) we obtain that $\gamma = \gamma_v$, and that (up to picking a subsequence) γ_k reaches $\tilde{S}$ for k large at $t_k \downarrow 0$.

Since any of the g_{ε_k}-geodesics γ_k reaches $\tilde{S}$ for all $\varepsilon_k < \varepsilon_0$ (for some suitable $\varepsilon_0 > 0$) at points $\gamma_k(t_k) \in \tilde{S}$, by C^1_{loc} convergence of γ_k to γ we also have $\dot{\gamma}_k(t_k) \in V$. Note that, due to $\gamma_k(t_k) \to \gamma(0)$, we can pick g_{ε_k}-orthonormal bases for $T_{\gamma_k(t_k)}\tilde{S}$ converging to a g-orthonormal basis for $T_{\gamma(0)}\tilde{S}$. Denote by $E_i^{\varepsilon_k}$ and E_i the g_{ε_k}- and g-parallel transports of these bases along γ_k and γ, respectively. By Lemma B.2 there exist C^1-extensions $\tilde{E}_i^{\varepsilon_k}$ and $\tilde{E}_i$ to the (possibly shrunk) neighbourhood U. By a similar argument one can extend the velocity vector fields along γ_k and γ to all of U, they will be denoted by $\tilde{N}^\varepsilon$ and $\tilde{N}$, respectively.

Assume also without loss of generality that U is relatively compact and that $\gamma_k([t_k, b]) \subseteq U$ for large k. Pick $\delta = \delta(b', c)$ according to Lemma 5.3. Since they restrict to the same vector fields along γ, by shrinking U and by continuity of $\overline{E_i}$ and $\tilde{E}_i$ resp. $\overline{N}$ and $\tilde{N}$, we can assume these vector fields to be arbitrarily close in $\|\,.\,\|_h$ on U. Moreover, by the convergence of $\tilde{E}_i^{\varepsilon_k}$ and $\tilde{N}^{\varepsilon_k}$ to $\tilde{E}_i$ and $\tilde{N}$, respectively, also these vector fields can be made arbitrarily $\|\,.\,\|_h$-close to $\overline{E_i}$ and $\overline{N}$, respectively, on U for large k. The assumption of the Proposition therefore implies that

$$\sum_{i=1}^{n-m} g(R(\tilde{E}_i^{\varepsilon_k}, \tilde{N}^{\varepsilon_k})\tilde{N}^{\varepsilon_k}, \tilde{E}_i^{\varepsilon_k}) \geq -\delta/2$$

in $\mathcal{D}'(U)$ for k large. Since $\star_M \rho_{\varepsilon_k}$ respects inequalities, we conclude that also

$$\sum_{i=1}^{n-m} g(R(\tilde{E}_i^{\varepsilon_k}, \tilde{N}^{\varepsilon_k})\tilde{N}^{\varepsilon_k}, \tilde{E}_i^{\varepsilon_k}) \star_M \rho_{\varepsilon_k} \geq -\delta/2$$

on U for large k.

By the same reasoning as in Lemma 2.17 we obtain (once again writing R_{ε_k} for $R[g_{\varepsilon_k}]$)

$$g(R(\tilde{E}_i^{\varepsilon_k}, \tilde{N}^{\varepsilon_k})\tilde{N}^{\varepsilon_k}, \tilde{E}_i^{\varepsilon_k}) \star_M \rho_{\varepsilon_k} - g_{\varepsilon_k}(R_{\varepsilon_k}(\tilde{E}_i^{\varepsilon_k}, \tilde{N}^{\varepsilon_k})\tilde{N}^{\varepsilon_k}, \tilde{E}_i^{\varepsilon_k}) \to 0$$

uniformly on U as $\varepsilon_k \to 0$, $i = 1, \ldots, n-m$. But then, for large k,

$$\sum_{i=1}^{n-m} g_{\varepsilon_k}(R_{\varepsilon_k}(\tilde{E}_i^{\varepsilon_k}, \tilde{N}^{\varepsilon_k})\tilde{N}^{\varepsilon_k}, \tilde{E}_i^{\varepsilon_k}) \geq -\delta.$$

In particular, the above holds along γ_k, so

$$\sum_{i=1}^{n-m} g_{\varepsilon_k}(R_{\varepsilon_k}(E_i^{\varepsilon_k}(t), \dot{\gamma}_k(t))\dot{\gamma}_k(t), E_i^{\varepsilon_k}(t)) \geq -\delta$$

for all $t \in [t_k, b']$, where we may assume that k is so large that all γ_k are defined on $[t_k, b']$.

Due to our choice of δ, Lemma 5.3 implies that, for k sufficiently large, each γ_k stops maximising at parameter $b' + t_k$ at the latest (if γ_k is not g_ε-normal to $\tilde{S}$, then γ_k stops maximising the distance immediately, see Remark 5.9 below). By construction the γ_k maximize from t_k to b'', hence if k is so large that $b' + t_k < b''$ we obtain a contradiction. □

Remark 5.6. By arguments analogous to those given for the genericity condition, cf. Remark 2.19, namely essentially by boundedness of R on compact sets, one can see that if $g \in C^{1,1}$, then the curvature condition (5.1) in Proposition 5.5 is equivalent to the one given in [8, Prop. 6.5].

Proposition 5.7 *(The case of trapped surfaces). Let (M, g) be a strongly causal, MNNB C^1-spacetime satisfying the distributional null energy condition. Let $\tilde{S} \subseteq M$ be a codimension-2 C^2-spacelike submanifold such that $\mathbf{k}_{\tilde{S}}(\nu) > c > 0$, where ν is a null normal to $\tilde{S}$, and let $b > 1/c$. If γ is a null geodesic with $\dot{\gamma}(0) = \nu$, then $\gamma|_{[0,b]}$ is not maximising from $\tilde{S}$.*

Proof. The proof is completely analogous to that of Proposition 5.5, with the sole difference that one needs to use Lemma 5.4 instead of Lemma 5.3. Let us give a brief outline: Approximate g again by $g_{\varepsilon_k} \equiv \check{g}_{\varepsilon_k}$ and the compact segment $\gamma|_{[0,b]}$ as before by g_{ε_k}-null geodesics γ_k. Then by Lemma 2.14 (see also the proof of [7, Thm. 5.7]), for all $\delta > 0$ and large k we have $\mathrm{Ric}_{\varepsilon_k}(\dot{\gamma}_k, \dot{\gamma}_k) > -\delta$. Using this, combined with Lemma 5.4 and proceeding as in Proposition 5.5, one arrives at a contradiction as before. □

Remark 5.8. The corresponding $C^{1,1}$-version of Proposition 5.7 was dealt with in [8, Remark 6.6(i)], however, a similar line of reasoning is not available in our case of C^1-metrics: While it is still possible to extend an ON-frame along a curve to a neighbourhood, this does not suffice to derive condition (5.1) of Proposition 5.5, as the needed rigidity of the condition cannot be guaranteed. More precisely: For any ON-frame $\{\bar{E}_i, \bar{N}\}$ on U the equality (as before $\bar{N}$ denotes the extension of γ' in the frame)

$$\sum_{i=1}^{n-m} g(R(\bar{E}_i, \bar{N})\bar{N}, \bar{E}_i) = \mathrm{Ric}[g](\bar{N}, \bar{N}) \geq -\delta$$

still holds and the final inequality is due to the distributional null energy condition. However, to carry out a proof similar to the one of Proposition 5.5, this estimate would need to hold for all collections of vector fields close to the frame in the sense given in the Proposition. This cannot be guaranteed for C^1-metrics anymore as R is, in general, unbounded. So instead, the slightly different focusing result used in Proposition 5.7 is required.

Remark 5.9. If, in the situation of Proposition 5.5, ν is null but not a null normal, then geodesics in that direction immediately enter I^+: This is shown in the same way as in the smooth case, e.g. by using ideas from [39, Lem. 10.45, Lem. 10.50]: Let c be a geodesic in direction ν, hence in particular C^2. By [43, Lem. 3.1] for a (piecewise) C^1-vector field X along c with $g(X', \dot{c}) < 0$, any variation c_s with variation field X is timelike and longer than c (for small s).

Now for any vector v at p not a null normal to a spacelike surface S, w.l.o.g. there exists $y \in T_pS$ with $g(y, v) > 0$. Define $V(t) = (1 - \frac{t}{b})Y(t)$, where Y is the parallel translate of y along c. A straightforward calculation shows $g(V', \dot{c}) < 0$. Now choose a variation c_s of c with variational vector field V such that $c_s(0) \in S$. We obtain that c_s is timelike from S to $c(b)$ and as b was arbitrary the statement follows.

The main result on trapped submanifolds in C^1-spacetimes now is the following:

Proposition 5.10. *Let (M, g) be a strongly causal, MNNB C^1-spacetime and let $S \subseteq M$ be a trapped C^0-submanifold of codimension m, $1 < m < \dim(M) = n$. If $m = 2$, suppose that (M, g) satisfies the distributional null energy condition, and if $m \neq 2$, suppose that any support submanifold $\tilde{S}$ of S satisfies the condition in Proposition 5.5 in any null normal direction ν and for any null geodesic γ with $\dot{\gamma}(0) = \nu$. Then $E^+(S)$ is compact or (M, g) is null geodesically incomplete.*

Proof. Suppose (M, g) is null geodesically complete.

$E^+(S)$ is relatively compact: Assume, to the contrary, that there are $q_k \in E^+(S)$ with $q_k \to \infty$. Lemma A.5 guarantees the existence of future directed maximising null geodesics $\gamma_k : [0, t_k] \to M$ connecting $p_k \in S$ to q_k, which we may assume to be parametrised by h-unit speed. By compactness we may further assume that $\gamma_k(0) = p_k \to p \in S$, and we write $\gamma_k(t_k) = q_k$. Due to $q_k \to \infty$, only finitely many γ_k are contained in a fixed neighbourhood U of p. So we may apply Theorem A.10, implying that (a subsequence, not relabeled, of) γ_k converges to a g-null curve $\gamma : [0, \infty) \to M$ (in h-unit speed parametrisation) from p. Since the γ_k are S-maximising, so is γ. In particular, γ is a g-null pre-geodesic. Reparametrising γ as a geodesic, it is still defined on $[0, \infty)$ due to null geodesic completeness and so $\gamma : [0, \infty) \to M$ is a future complete g-null S-ray. Let $\tilde{S}$ be a future support submanifold of S at p, then γ maximises the distance from $\tilde{S}$ everywhere, because otherwise there would be some T such that $\gamma(T) \in I^+(\tilde{S}) \subseteq I^+(S)$, contradicting the fact that γ is an S-ray. As this cannot happen due to Proposition 5.5 resp. Proposition 5.7, $E^+(S)$ is relatively compact.

$E^+(S)$ is closed: Let $q_k \in E^+(S)$, $q_k \to q$. Let $\gamma_k : [0, t_k] \to M$ be null pre-geodesics (in h-unit speed parametrisation) from some $p_k \in S$ to q_k. By compactness of S, we may assume that $p_k \to p \in S$. If $q = p$, then we are done. Otherwise, there are two possibilities: If the t_k are bounded, then by going over to subsequences, $t_k \to t \in (0, \infty)$. Since $q \neq p$, we are in a position to invoke Theorem A.11 to get a maximising causal limit geodesic $\gamma : [0, t] \to M$ connecting p to q. γ cannot be timelike, as this would imply $q \in I^+(S)$ and hence $q_k \in I^+(S)$ for large k. Hence $q \in J^+(S) \setminus I^+(S) = E^+(S)$. The other possibility is that the t_k are unbounded, i.e. w.l.o.g. $t_k \to \infty$. Since (M, g) is

strongly causal, this means that the γ_k leave every compact set, i.e., $\gamma_k(t_k) \to \infty$ (after possibly relabelling t_k). However, as $E^+(S)$ is contained in some compact set, this is not possible. □

Corollary 5.11. *Let the assumptions be as in Proposition 5.10. Then $E^+(S) \cap S$ is achronal, and $E^+(E^+(S) \cap S)$ is compact or M is null geodesically incomplete.*

Proof. This is shown as in the smooth case, see [10, Prop. 4] or [46, Prop. 4.3] and using that for any $p \in S$ there is some neighbourhood U such that $S \cap U$ is achronal in U. □

We now start with our discussion of "trapped points". In [8, Sec. 6.3], a faithful way to generalise the classical condition, which uses Jacobi tensor classes—a tool no longer available already for $C^{1,1}$-metrics—is presented: It uses the mean curvature of spacelike 2-surfaces given as the level sets of the exponential map that generate the light cone. While the latter tool presently is not at our disposal, the very formulation of the corresponding condition again uses support manifolds and can be adopted verbatim. However, it is now more delicate to derive that the horismos of a trapped point is a trapped set.

Definition 5.12 (*Trapped points*).
A point $p \in M$ is *future trapped* if for any future pointing null vector $v \in T_pM$ and for any null geodesic γ with $\gamma(0) = p$, $\dot\gamma(0) = v$, there exists a parameter t and a spacelike C^2-submanifold of codimension $m = 2$ with $\tilde S \subseteq J^+(p)$, $\gamma(t) \in \tilde S$ and $\mathbf{k}_{\tilde S}(\dot\gamma(t)) > 0$.

Proposition 5.13. *Let (M, g) be a strongly causal, MNNB C^1-spacetime satisfying the distributional null energy condition. If $p \in M$ is a future trapped point and (M, g) is null geodesically complete then $E^+(p)$ is compact.*

Proof. Since trapped points are defined by means of codimension-2 submanifolds $\tilde S$, we will use Proposition 5.7 for which (only) the distributional null energy condition is needed. The proof is analogous to that of Proposition 5.10.

$E^+(p)$ is relatively compact: Suppose $q_j \in E^+(p)$, $q_j \to \infty$. Let $\gamma_j : [0, t_j] \to M$ be maximising null geodesics connecting p to q_j, cf. Lemma A.5. Then, up to a subsequence, the γ_j converge to a g-null ray $\gamma : [0, \infty) \to M$ from p (the domain is $[0, \infty)$ even in g-affine parametrisation since (M, g) is null geodesically complete). By assumption, there is a C^2-spacelike submanifold $\tilde S$ of codimension 2 and a parameter t such that $\mathbf{k}_{\tilde S}(\dot\gamma(t)) > c > 0$ and we let $b > 1/c$. γ is indeed an $\tilde S$-ray: If not, it would enter $I^+(\tilde S) \subseteq I^+(p)$, a contradiction. But by the previous results in this section, γ cannot maximise from $\tilde S$ past b, a contradiction since it is a ray.

$E^+(p)$ is closed: This is shown in exactly the same way as closedness was shown in the proof of Proposition 5.10. □

6. The Main Result

Given the results established so far, the general mechanics of the proof of the Hawking–Penrose theorem remains the same as in the smooth case. There do, however, remain notable differences in the details in this lower regularity (e.g. the use of [7, Prop. 2.13] in the proof of Theorem 6.2), so we include the full argument.

Lemma 6.1. *Let (M, g) be a strongly causal C^1-spacetime such that no inextendible null geodesic in M is maximising. Let $A \subseteq M$ be achronal such that $E^+(A)$ is compact, and let γ be a future inextendible timelike curve contained in $D^+(E^+(A))^\circ$. Then $F := E^+(A) \cap \overline{J^-(\gamma)}$ is achronal and $E^-(F)$ is compact.*

Proof. Due to Lemma A.16 we may assume that $A = \overline{A}$. Since $E^+(A)$ is always achronal and $F \subseteq E^+(A)$, it is also achronal. By compactness of $E^+(A)$, F is compact. Note that $E^-(F) \subseteq F \cup \partial J^-(\gamma)$ by similar arguments as in [21, Lem. 9.3.4], so to show compactness of $E^-(F)$, it suffices to show compactness of $E^-(F) \cap \partial J^-(\gamma)$.

Suppose now that $v \in TM|_F$ is past pointing causal and let $c_v : [0, a) \to M$ be any past inextendible, past directed geodesic with $\dot{c}_v(0) = v$ (recall that in C^1-spacetimes we no longer have unique solvability of the geodesic initial value problem). Then $c_v \subseteq \overline{J^-(\gamma)}$, and $\overline{J^-(\gamma)} = I^-(\gamma) \cup \partial J^-(\gamma)$. We show that c_v meets $I^-(\gamma)$: Suppose, to the contrary, that c_v is a null geodesic entirely contained in $\partial J^-(\gamma)$. Since γ is a future inextendible timelike curve, we have $J^-(\gamma) = I^-(\gamma)$, hence c_v lies entirely in $\partial J^-(\gamma) \setminus J^-(\gamma)$. By Corollary A.12, there is a future directed, future inextendible null geodesic λ starting at $c_v(0)$ entirely in $\partial J^-(\gamma)$ (γ is closed in M by Corollary A.33 and Lemma A.32). This means that either $c_v\lambda$ is an inextendible broken null geodesic, hence not maximising by Lemma A.4, or an inextendible unbroken maximising null geodesic. By assumption such a geodesic cannot exist, hence in either case $c_v\lambda$ cannot lie entirely in $\partial J^-(\gamma)$ and so c_v must enter $I^-(\gamma)$.

Thus we have shown that for any v and any geodesic c_v as above there is $t_v = t_v(v, c_v)$ (for which c_v is still defined) such that $c_v(t_v) \in I^-(\gamma)$. It remains to show that $E^-(F) \cap \partial J^-(\gamma)$ is compact.

$E^-(F) \cap \partial J^-(\gamma)$ is relatively compact: Suppose there are $q_k \in E^-(F) \cap \partial J^-(\gamma)$ with $q_k \to \infty$. Then there are past-directed maximising null pre-geodesics $c_k : [0, t_k] \to M$ in h-unit speed parametrisation with $c_k(0) = p_k \in F$ and $c_k(t_k) = q_k$. Since $q_k \to \infty$, also $t_k \to \infty$ By compactness, we may assume $p_k \to p \in F$. Since $q_k \to \infty$, there is a neighbourhood U which almost all c_k leave and we are in a position to apply the Theorem A.10. We get a past-directed inextendible null limit curve $c : [0, \infty) \to M$ from p and a subsequence of the c_k (w.l.o.g. c_k itself) which converges to c locally uniformly. It is easy to see that all c_k have to lie entirely in $\partial J^-(\gamma)$, hence so does the limit c. In particular, c is everywhere maximising and may be reparametrised as an inextendible geodesic $c : [0, a) \to M$ entirely contained in $\partial J^-(\gamma)$. This is a contradiction since we showed before that each past directed null geodesic with initial velocity in $TM|_F$ has to enter $I^-(\gamma)$.

$E^-(F) \cap \partial J^-(\gamma)$ is closed: Let $q_j \in E^-(F) \cap \partial J^-(\gamma)$, $q_j \to q$, where we may assume $q \notin F$, otherwise there is nothing to prove. Connect $F \ni p_j \to q_j$ via maximising null geodesics. Up to a subsequence, their limit has to be a maximising null geodesic from p to q, hence $q \in E^-(F) \cap \partial J^-(\gamma)$, because otherwise one would again obtain an inextendible maximising null geodesic entirely in $\partial J^-(\gamma)$, which would lead to the same contradiction as before. □

Now we are ready to give the C^1-version of the causal Hawking-Penrose theorem. It is the direct analog of [17, p. 538, Thm.] and [8, Thm. 7.4], respectively.

Theorem 6.2. *Let (M, g) be a C^1-spacetime. Then the following conditions cannot all hold:*

(i) *(M, g) is causal.*
(ii) *No inextendible timelike geodesic in an open globally hyperbolic subset is everywhere maximising.*
(iii) *No inextendible null geodesic is everywhere maximising.*
(iv) *There is an achronal set $A \subseteq M$ such that $E^+(A)$ or $E^-(A)$ is compact.*

Proof. Suppose all of the above conditions hold. Then by Lemma A.31, (i) and (iii) imply that (M, g) is strongly causal. W.l.o.g. we assume that $E^+(A)$ is compact. By

Lemma A.35 there exists a future inextendible (in M) timelike curve $\gamma \subseteq D^+(E^+(A))^\circ$. Set $F := E^+(A) \cap \overline{J^-(\gamma)}$, which by Lemma 6.1 is achronal and $E^-(F)$ is compact. Once more by Lemma A.35 there is some past-inextendible timelike curve $\lambda \subseteq D^-(E^-(F))^\circ$. Exactly as in the smooth case, see e.g. the second paragraph of [8, Proof of Thm. 7.4], one can show $\gamma \subseteq D^+(E^-(F))^\circ$, by invoking Proposition A.20 and Lemma A.24. So $\gamma, \lambda \subseteq D(E^-(F))^\circ$ which by Proposition A.28 is globally hyperbolic. Now we pick sequences $p_k = \lambda(s_k)$ and $q_k = \gamma(t_k)$ with $t_k, s_k \nearrow \infty$, leaving every compact set, which is possible because γ and λ are inextendible curves in M contained in a strongly causal set (Lemma A.32). Also because $\lambda \subseteq J^-(E^-(F)) \subseteq J^-(F) \subseteq J^-(\overline{J^-(\gamma)})$ and because γ is timelike, we can choose $p_k \in I^-(q_k)$.

By [7, Prop. 2.13] there exist maximising timelike geodesics $\tilde{\gamma}_k$ in $D(E^-(F))^\circ$ from p_k to q_k, which must intersect $E^-(F)$, say at r_k. By compactness of $E^-(F)$ we can assume $r_k \to r$ and by Corollary 2.6 there exists an inextendible, causal limit geodesic $\tilde{\gamma}$, which is also maximising (as a limit of maximisers).

By (iii), $\tilde{\gamma}$ cannot be null, so it must be timelike. As a limit of curves in $D(E^-(F))^\circ$ it is contained in $\overline{D(E^-(F))}$. Again using Proposition A.20 and Lemma A.24 one can now proceed as in the last step in [8, Proof of Thm. 7.4] to arrive at the desired contradiction (to (ii)). □

Finally, we formulate and prove the main result of this work, the analytical Hawking-Penrose theorem for maximally causally non-branching C^1-spacetimes.

Theorem 6.3. *(The Hawking-Penrose singularity theorem for C^1-metrics)*
Let (M, g) be a C^1-spacetime of dimension n with the following properties.

(i) *(M, g) is causal.*
(ii) *(M, g) satisfies the distributional timelike and null energy conditions.*
(iii) *(M, g) satisfies the distributional genericity condition along any inextendible causal geodesic.*
(iv) *(M, g) is maximally causally non-branching.*
(v) *(M, g) contains one of the following.*
  (a) *a compact, achronal, edgeless set.*
  (b) *a future trapped point.*
  (c) *a future trapped C^0-submanifold of codimension 2.*
  (d) *a future trapped C^0-submanifold of codimension $2 < m < n$ such that its support submanifolds satisfy the condition in Proposition 5.5.*

Then (M, g) is causally geodesically incomplete.

Proof. Assuming causal geodesic completeness, we would like to deduce a contradiction by using Theorem 6.2.

Note that due to Theorems 4.2 and 4.3 the assumptions (ii) and (iii) of Theorem 6.2 are satisfied and it remains to establish the existence of a future or past trapped set, i.e. an achronal set with compact future or past horismos. This will be implied by (v), but we have to distinguish the different cases:

By Proposition A.36, (a) implies the existence of a trapped set. Note that by Lemma A.31 (which can be applied by Theorem 4.3) (M, g) is strongly causal and hence Proposition 5.13 covers the case (b), whereas Proposition 5.10 covers cases (c) and (d). □

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A. Appendix: Causality Results in C^1-Spacetimes

In this section, we collect various results on the causality of C^1-Lorentzian metrics that will be needed in the proof of the main result. Unless stated otherwise, throughout this section (M, g) consists of a connected smooth (Hausdorff and second countable) manifold M, and a C^1-Lorentzian metric g that is time-orientable (i.e. there exists a smooth timelike vector field on M). Many of the results in this section hold in even greater generality, namely either for Lorentzian metrics that are merely Lipschitz or even in the general setting of closed cone structures [35]. Although we will occasionally note this, in proving the results in this section we will stay close to smooth causality theory (mainly based on the comprehensive reference work [34]) and only make those changes from classical proofs that are required due to the absence of certain tools (e.g., convex normal neighbourhoods) in the C^1-setting.

We begin with two elementary results, namely the openness of timelike futures and pasts and the push-up property. Both of these hold for causally plain spacetimes, which include spacetimes with Lipschitz continuous metrics [5, Cor. 1.17].

Lemma A.1. *For any $A \subseteq M$, the sets $I^\pm(A)$ are open in M.*

Proof. See [5, Prop. 1.21]. □

Lemma A.2. *Let $p, q, r \in M$ such that $p \le q \ll r$ or $p \ll q \le r$. Then $p \ll r$.*

Proof. See [5, Lem. 1.22]. □

Remark A.3. Note that from this one can show as in the smooth case, e.g. [39, Lem. 14.03][6], that the timelike relation is open, i.e., if $p \ll q$ then there are neighbourhoods U_p, U_q of p and q respectively, such that for all $x \in U_p$ and all $y \in U_q$, we have $x \ll y$.

Lemma A.4. *Let (M, g) be a C^1-spacetime and let $\gamma : [a, b] \to M$ be a maximising causal curve. Then γ is a (reparametrisation of a) causal geodesic. In particular, it is a C^2-curve and has a causal character.*

Proof. See [27, Thm. 1.1] or [43, Thm. 3.3]. □

For any point $q \in M$, we denote by $E^+(q) := J^+(q) \setminus I^+(q)$ its *future horismos*. Its *past horismos* $E^-(q)$ is defined analogously.

Lemma A.5. *If $p \in E^+(q)$ then there exists a maximising null geodesic segment from q to p.*

[6] the use of convex sets is not necessary once openness of I^+ in is established

Proof. Let c be a future causal curve from q to p. Since $p \notin I^+(q)$, c is null and maximising from q to p (because of push-up), hence it is (a reparametrisation of) a maximising null geodesic by Lemma A.4. $\square$

Lemma A.6. *For any subset $A \subseteq M$, we have $I^\pm(A) = J^\pm(A)^\circ$ and $\partial I^\pm(A) = \partial J^\pm(A)$. Moreover, the sets $E^\pm(A)$ and $\partial J^\pm(A)$ are achronal.*

Proof. This follows by the same proofs as in the smooth case (cf. [34, Thm. 2.27], [39, Cor. 14.27]). $\square$

Definition A.7. (Edge)
Let $A \subseteq M$ be achronal. The *edge* of A, denoted by $\mathrm{edge}(A)$, is the set of all $x \in \overline{A}$ such that for any neighbourhood U of x, there is a timelike curve from $I^-_U(x)$ to $I^+_U(x)$ that does not meet A.

Lemma A.8. *Let (M, g) be a C^1-spacetime. An achronal set $A \subseteq M$ is a topological hypersurface if and only if $A \cap \mathrm{edge}(A) = \emptyset$. Moreover, A is a closed topological hypersurface if and only if $\mathrm{edge}(A) = \emptyset$.*

Proof. The proofs for smooth metrics (cf. [39, Prop. 14.25, Cor. 14.26]) still hold in C^1. $\square$

Lemma A.9. *For any $A \subseteq M$, $\partial J^+(A)$ is a (topologically) closed, achronal topological hypersurface.*

Proof. This follows from the fact that $\partial J^+(A)$ is achronal and $\mathrm{edge}(\partial J^+(A)) = \emptyset$, which is proven in precisely the same way as in the smooth case. $\square$

Recall the notion of a limit maximising sequence of causal curves: A sequence $\gamma_k : [a_k, b_k] \to M$ of future directed causal curves is called *limit maximising* if there is $\varepsilon_k \to 0$ such that

$$L(\gamma_k) \geq d_g(\gamma(a_k), \gamma(b_k)) - \varepsilon_k,$$

where d_g is the time separation function induced by g. In particular a sequence of maximising curves is limit maximising. We shall employ the following version of the limit curve theorem:

Theorem A.10. *(Limit Curve Theorem)*
Let h be a complete Riemannian metric on M and let $\gamma_k : [a_k, b_k] \to M$ be future directed, h-arc length parametrised causal curves with $0 \in [a_k, b_k]$ such that the sequence $\{\gamma_k(0)\}$ has an accumulation point y. Suppose there are $a \leq 0$ and $b \geq 0$ such that $a_k \to a$, $b_k \to b$. If there is a neighbourhood U of y such that almost all γ_k leave U, then a subsequence of γ_k converges h-uniformly on compact sets to a future directed causal Lipschitz curve $\gamma : [a, b] \to M$. γ is future resp. past inextendible iff $b = \infty$ resp. $a = -\infty$. If $\{\gamma_k\}$ is limit maximising, then γ is maximising.

Proof. This follows from [30, Thm. 3.1] and [5, Thm. 1.6] by the same proof as in [8, Thm. A.6]. $\square$

For a much more general version of the limit curve theorem, valid in closed cone structures, we refer to [35, Thm. 2.14]. We shall also require the following variant of the limit curve theorem for a sequence of curves with converging past and future endpoints, cf. [30, Thm. 3.1(2)].

Theorem A.11 *(Limit Curve Theorem: two converging endpoints).*
Let $\gamma_k : [0, b_k] \to M$ be a sequence of future-directed, causal curves in h-arc length parametrisation connecting x_k to z_k, with $x_k \to x$ and $z_k \to z$. Suppose that $b_k \to b \in (0, \infty)$. If there is a neighbourhood U of x such that only a finite number of γ_k are entirely contained in U, then there is a Lipschitz future-directed, causal curve $\gamma : [0, b] \to M$ connecting x to z such that a subsequence of the γ_k converges to γ h-uniformly on compact sets. Moreover, if $\{\gamma_k\}$ is limit maximising, then γ is maximising.

Proof. This is contained in the statement of [30, Thm. 3.1(2)], whose proof is easily seen to hold for C^1-metrics once Theorem A.10 is established. □

Corollary A.12. *Let $A \subseteq M$. For any point $x \in \partial J^+(A) \setminus \overline{A}$, there is a causal curve $\gamma \subseteq \partial J^+(A)$ with future endpoint x that is either past inextendible and does not meet $\overline{A}$ or has a past endpoint in $\overline{A}$. It is a (reparametrisation of a) maximising null geodesic. If A is closed and $x \notin J^+(A)$, then $\gamma \subseteq \partial J^+(A) \setminus J^+(A)$ and it is past inextendible.*

Proof. Using Theorem A.10 and Lemma A.5, this follows as in [8, Prop. A.7]. □

Lemma A.13. *Let (M, g) be a C^1-spacetime and let $\{g_k\}$ be either the sequence $\{\hat{g}_{\varepsilon_k}\}$ or $\{\check{g}_{\varepsilon_k}\}$ with $\varepsilon_k \downarrow 0$. Let $p, q \in M$ and let γ_k be a g_k-maximising curve from p to q, so γ_k is in particular a g_k-geodesic. Suppose that γ_k converge to a curve γ in C^1_{loc}. If $\{g_k\} = \{\check{g}_{\varepsilon_k}\}$, suppose in addition that (M, g) is globally hyperbolic and $p \ll_g q$. Then γ is a g-maximising geodesic from p to q.*

Proof. The case $\{g_k\} = \{\hat{g}_{\varepsilon_k}\}$ follows immediately from [41, Prop. 6.5], so we will only consider the case $\{g_k\} = \{\check{g}_{\varepsilon_k}\}$ together with the additional assumptions of (M, g) being globally hyperbolic and $p \ll_g q$ in this case. Due to global hyperbolicity, the space $C(p, q)$ of future directed causal curves from p to q (considered up to orientation preserving reparametrisations) is compact with respect to its natural topology (see [41, Thm. 3.2]). Consequently, the h-speeds of curves in $C(p, q)$ are uniformly bounded by some constant $C_1 > 0$, where h is some complete Riemannian metric on M.

Since (M, g) is globally hyperbolic, there is an L_g-maximising g-geodesic $c : [0, 1] \to M$ with $c(0) = p$, $c(1) = q$ [41, Prop. 6.4]. Also, $p \ll_g q$ implies that c is g-timelike. By compactness, there is a constant $C_2 > 0$ such that $g(\dot{c}, \dot{c}) < -C_2$ on $[0, 1]$, and hence $g_k(\dot{c}, \dot{c}) < 0$ for k large, implying that c is g_k-timelike for such k. If we set $\delta_k := d_h(g, g_k)$ (the h-distance of g, g_k as in [5, (1.6)]), then

$$\begin{aligned} L_g(c) &= \int_0^1 \sqrt{-g(\dot{c}, \dot{c})}dt \leq \int_0^1 \sqrt{-g_k(\dot{c}, \dot{c}) + C_1^2 \delta_k}dt \leq L_{g_k}(c) + C_1\sqrt{\delta_k} \\ &\leq d_{g_k}(p, q) + C_1\sqrt{\delta_k} = L_{g_k}(\gamma_k) + C_1\sqrt{\delta_k} \to L_g(\gamma) \qquad (k \to \infty), \end{aligned}$$

so $L_g(\gamma) \geq L_g(c)$, which proves that also γ is maximising from p to q. □

Lemma A.14. *Let $\gamma_k \subseteq \overline{J^+(p)}$ be causal curves converging locally uniformly to a causal curve γ with future endpoint q. If $d_g(p, q) = 0$, then γ is a maximising null geodesic.*

Proof. Clearly $\gamma \subseteq \overline{J^+(p)}$. If γ would meet $I^+(p)$, then there would exist some $r \in I^+(p) \cap J^-(q)$ and using push-up we could conclude that $q \in I^+(p)$, contradicting $d_g(p, q) = 0$. Hence $\gamma \subseteq \partial J^+(p)$, which is achronal and γ is a maximising null geodesic by Lemma A.4. □

Lemma A.15. *Let $g_k = \check{g}_{\varepsilon_k}$ and let $\gamma_k \subseteq \overline{J^+(p)}$ be g_k-null geodesics converging in C^1_{loc} to a g-geodesic γ that ends in q. Assume that $d_g(p, q) = 0$. Then γ is a maximising g-null geodesic.*

Proof. This follows immediately from Lemma A.14 upon noting that any g_k-null geodesic is g causal. $\square$

Lemma A.16. *Let A be achronal. Then so is $\overline{A}$. Moreover, if $E^+(A)$ is closed, then $E^{\pm}(A) = E^{\pm}(\overline{A})$.*

Proof. This follows by the same proof as in [8, Lem. A.8]. $\square$

Lemma A.17. *Let (M, g) be a C^1-spacetime. Then every point in M has a neighbourhood base of globally hyperbolic neighbourhoods.*

Proof. Fix any smooth, time-orientable Lorentzian metric $\hat{g}$ with $g \prec \hat{g}$. For $x \in M$, [34, Thm. 2.7] gives a neighbourhood base V_m of neighbourhoods of x that are globally hyperbolic with respect to $\hat{g}|_{V_m}$. Any Cauchy hypersurface for $\hat{g}$ in V_m is also a Cauchy hypersurface for g, so the claim follows from [41, Thm. 5.7]. $\square$

Lemma A.18. *Let S be a spacelike (C^2-) hypersurface and let $p \in S$. Then there exists a neighbourhood V of p in M such that $V \cap S$ is a Cauchy hypersurface in V.*

Proof. The proof of the analogous result [25, Lemma A.25] also works in the C^1-setting. $\square$

Definition A.19. (Cauchy development and Cauchy horizon)
Let A be an achronal set. The *future Cauchy development* of A is the set

$$D^+(A) := \{x \in M : \text{ every past inextendible causal curve through } x \text{ meets } A\}.$$

The *future Cauchy horizon* of A is defined as

$$H^+(A) := \overline{D^+(A)} \setminus I^-(D^+(A)).$$

The *past Cauchy development and past Cauchy horizon* are defined analogously.

The first part of the following result is partly a C^1-analogue of [39, Lem. 14.51], where it is proved using convex normal neighbourhoods, a tool that we do not have at our disposal in the C^1-setting. The use of such neighbourhoods can, however, be avoided, as our arguments below illustrate.

Proposition A.20. *Let A be a closed, achronal set. Then*

$$\overline{D^+(A)} = \{x \in M : \textit{every p.i. t.l. curve through } x \textit{ meets } A\}$$

and

$$\partial D^+(A) = A \cup H^+(A).$$

Proof. Let T be the set on the right hand side in the first equation.
$\overline{D^+(A)} \subseteq T$: Suppose there is a point $p \in \overline{D^+(A)} \setminus T$. Then there is some past inextendible timelike (future directed) curve $\alpha : (a, 0] \to M$ with $\alpha(0) = p$ not meeting A. Let $p_k \in D^+(A)$, $p_k \to p$ and let U_k be open neighbourhoods of p with $U_k \to \{p\}$. By choosing subsequences, we may assume that $p_k \in I^+_{U_k}(\alpha(-1/k))$ and $\alpha|_{[-1/k,0]} \subseteq U_k$ for all k.
Construct now past inextendible causal curves α_k by connecting $\alpha(-1/k)$ to p_k in a timelike way such that these curves stay in U_k, and let the rest of α_k (i.e. the part to the past of $\alpha(-1/k)$) be the original curve α. Then the α_k are all past inextendible timelike, and since $p_k \in D^+(A)$ they must meet A at some a_k. Since α does not meet A, the

a_k have to lie on the replaced piece. But these pieces are entirely in U_k, which go to $\{p\}$, so we see that $a_k \to p$. Since A is closed, we have $p \in A$, which is absurd since $p \notin T \supseteq A$ by assumption.
$\overline{D^+(A)} \supseteq T$: Suppose $q \notin \overline{D^+(A)}$, and let $r \in I^-_{M\setminus\overline{D^+(A)}}(q)$. Since in particular $r \notin D^+(A)$, there is a past inextendible causal curve α from r missing A. Precisely as in the smooth case [39, Lem. 14.30(1)], noting that pushup still holds in C^1), there is a past inextendible timelike curve from q not meeting A. Thus $q \notin T$.

Finally, $\partial D^+(A) = A \cup H^+(A)$ follows exactly as in the smooth case (cf. [39, Lem. 14.52]). □

We note that the previous result in fact remains true even for locally Lipschitz proper cone structures [35, Thm. 2.35, 2.36].

Lemma A.21. *Let A be achronal. Then $H^+(A)$ is a closed, achronal set.*

Proof. The proof of [34, Prop. 3.15] carries over unchanged to C^1 metrics. □

Lemma A.22. *Let A be closed and achronal. If $x \in D^+(A) \setminus H^+(A)$, then every past inextendible causal curve through x meets $I^-(A)$. Moreover, if $x \in D(A)^\circ$, then every future (resp. past) inextendible future (resp. past) directed causal curve emanating from x intersects $I^+(A)$ (resp. $I^-(A)$).*

Proof. This follows from [34, Prop. 3.27] and [34, Prop. 3.42], which still hold for C^1-metrics. □

Lemma A.23. *Let A be closed, achronal. Let $x \in J^+(A) \setminus D^+(A)$ or $x \in I^+(A) \setminus D^+(A)^\circ$. Then every causal curve from x to A meets $H^+(A)$.*

Proof. The proof for $C^{1,1}$-metrics [8, Lem. A.12] still holds in C^1 because of Lemma A.1, Lemma A.2, and Proposition A.20. □

Using these results, one establishes the following formula for the timelike future of the Cauchy horizon of a closed, achronal set precisely as in the $C^{1,1}$-case, cf. [8, Lem. A.13].

Lemma A.24. *Let A be closed and achronal. Then $I^+(H^+(A)) = I^+(A) \setminus \overline{D^+(A)}$.*

Lemma A.25. *Let A be closed and achronal. Then* $\mathrm{edge}(H^+(A)) \subseteq \mathrm{edge}(A)$.

Proof. Based on Lemmas A.23 and A.24 , the proof of [8, Lem. A.14] carries over to the C^1-setting. □

Lemma A.26. *Let A be an achronal set, then $H^+(\partial J^+(A))$ is a closed, achronal, topological hypersurface.*

Proof. $H^+(\partial J^+(A))$ is closed and achronal by Lemma A.21, and by the previous result, combined with Lemmas A.8 and A.9 , $\mathrm{edge}(H^+(\partial J^+(A))) \subseteq \mathrm{edge}(\partial J^+(A)) = \emptyset$, so the claim follows from Lemma A.8. □

Lemma A.27. *Let A be closed, achronal. Then $H^+(\overline{E^+(A)}) \subseteq H^+(\partial J^+(A))$.*

Proof. The proof of [8, Lem. A.16] carries over, using Corollary A.12, Proposition A.20 and Lemmas A.23, A.24. □

Proposition A.28. *Let (M, g) be a C^1-spacetime and let $A \subseteq M$ be a closed, achronal set. Then the interior of its Cauchy development, i.e. the set $D(A)^\circ = (D^+(A) \cup D^-(A))^\circ$, if nonempty, is a globally hyperbolic C^1-spacetime when considered with the induced metric.*

Proof. Based on Theorem A.10 and Lemma A.22, this follows from the same proof as in [34, Thm. 3.45]. □

Definition A.29. (Strong causality)
A C^1-spacetime (M, g) is called *strongly causal at* $p \in M$ if for any neighbourhood U of p there is a neighbourhood V of p with $V \subseteq U$ such that any causal curve starting and ending in V is contained in U. (M, g) is called *strongly causal* if it is strongly causal at every point.

Lemma A.30. *(M, g) is strongly causal at p if and only if for any neighbourhood U of p there is a neighbourhood V of p with $V \subseteq U$ such that no causal curve leaving V ever returns.*

Proof. This follows from [33, Lem. 3.21], which continues to hold in the C^1-case. □

Lemma A.31. *Let (M, g) be chronological and suppose there are no inextendible maximising null geodesics in M. Then (M, g) is strongly causal.*

Proof. Using Theorem A.10, this follows by the same argument as [8, Lem. A.19]. □

Lemma A.32. *A strongly causal C^1-spacetime (M, g) is non-totally and non-partially imprisoning: No future or past inextendible causal curve is contained in a compact set and no future or past inextendible causal curve returns to a compact set infinitely often.*

Proof. The classical proof for the smooth case (see [39, Lem. 14.13]) holds even for continuous metrics. □

Corollary A.33. *Let (M, g) be a non-totally and non-partially imprisoning C^1-spacetime. Then the image of every inextendible causal curve is closed in M.*

Proof. Let γ be any inextendible causal curve. Since M is a manifold, the topology on M is compactly generated. Let $K \subseteq M$ be any compact subset. By non-total and non-partial imprisonment, $\gamma \cap K$ is a finite union of closed segments of γ, hence $\gamma \cap K$ is closed in K. Since K was arbitrary, γ is closed in M. □

Lemma A.34. *Let (M, g) be a strongly causal C^1-spacetime and let $A \subseteq M$ be a closed, achronal subset. If $H^+(\overline{E^+(A)})$ is nonempty, then it is noncompact.*

Proof. The proof of the smooth case goes through, because all the necessary preliminary results continue to hold in C^1 (see the outline in [8, Lem. 7.1]). □

Lemma A.35. *Let (M, g) be a strongly causal C^1-spacetime. Let $A \subseteq M$ be an achronal subset such that $E^+(A)$ is compact. Then there is a future inextendible timelike curve γ contained in $D^+(E^+(A))^\circ$.*

Proof. The proof of the smooth case goes through in C^1, see e.g. [17, Lem. 2.12], or also [21, Lem. 9.3.3], combined with Lemma A.16. □

Proposition A.36. *Let (M, g) be a C^1-spacetime and let $A \subseteq M$ be achronal and edgeless. Then $E^+(A) = A$, in particular if A is compact so is $E^+(A)$.*

Proof. This was shown e.g. in [34, Cor. 2.145] and the proof carries over to C^1 spacetimes. □

B. Appendix: Extending Vector Fields in C^1-Spacetimes Uniformly

When applying the genericity condition (Definition 2.15) we need to extend vectors or vector fields along curves to neighbourhoods, due to the fact that distributional (or also L^∞) curvature quantities are not well defined on points or along curves (as these are sets of measure zero). It is rather straightforward to extend a vector field along a curve to a neighbourhood. For C^1-metrics one can still use parallel transport along certain curves spanning an open set. Alternatively, one can also use "cylindrically constant" (see below) extensions in coordinates around a (part of a) curve.

In our case, we need to simultaneously extend vector fields along a sequence of converging curves. To be more precise, let γ_k be a sequence converging in C^1 to some γ and let V_k be vector fields along γ_k and V along γ such that $V_k \to V$ in C^1 as $k \to \infty$. Then given a point on γ, does it possess a neighbourhood U such that all V_k can be extended to vector fields on U and such that their extensions converge in C^1 to the extension of V?

To begin with, we consider the situation of extending vector fields around a point of a curve to a neighbourhood by cylindrically constant extension in coordinates.

Lemma B.1 *(Cylindrically constant extension of vector fields on curves). Let $\gamma : I \to M$ be a C^1-curve which is regular at 0, i.e. $\dot\gamma(0) \neq 0$ and set $p := \gamma(0)$. Then there exists a chart $(U, (x^1, \ldots, x^n))$ around p with the following property: For any C^1 vector field V along γ there exists a C^1 extension $\tilde V$ of V to U such that, in these coordinates, $\tilde V$ is independent of $x^2, \ldots, x^n$.*

Proof. Since the claim is local we may assume that $M = \mathbb{R}^n$, that $p = 0$, and that $\gamma = (\gamma^1, \ldots, \gamma^n)$ with $\gamma_1'(t) > c > 0$ for some c and all $t \in (a, b)$, where $a < 0 < b$. Thus γ^1 is a C^1-diffeomorphism from (a, b) onto some interval J. Then we set $U := J \times B_R^{n-1}(0)$, where $R > 0$ is such that $\gamma((a, b)) \subseteq U$. Given a C^1-vector field V along γ, i.e., a C^1-map $V : (a, b) \ni t \mapsto (\gamma(t), v(t))$ with $v \in C^1((a, b), \mathbb{R}^n)$, set $\tilde V : U \to U \times \mathbb{R}^n$,

$$\tilde V(x) := (x, v((\gamma^1)^{-1}(x_1)))$$

to obtain the desired extension. □

Now that we have set up a way of extending vector fields in a cylindrically constant fashion, we can implement an analogous procedure uniformly on a sequence of curves.

Lemma B.2. *Let $\gamma_k : [-1, 1] \to M$ be a sequence of C^1-curves converging in $C^1([-1, 1])$ to the regular C^1-curve $\gamma : [-1, 1] \to M$ and let V_k and V be C^1 vector fields along γ_k and γ, respectively. Further, let $V_k \to V$ in C^1, i.e., both $V_k \to V$ and $V_k' \to V'$ in TM, uniformly on $[-1, 1]$. Then there exists some open neighbourhood U of $\gamma(0)$ and C^1 extensions $\tilde V_k$ of V_k and $\tilde V$ of V to U such that $\tilde V_k \to \tilde V$ in $C^1_{\mathrm{loc}}(U)$.*

Proof. Again we may assume that $M = \mathbb{R}^n$. As in the previous Lemma we write $\gamma = (\gamma^1, \ldots, \gamma^n)$ and can assume without loss of generality that $\gamma(0) = 0$, and that $(\gamma^1)'(t) > c > 0$ for some c and all $t \in (a, b)$, where $a < 0 < b$. Due to the assumption on the C^1-convergence of the γ_k, we may additionally suppose that the same inequality holds on (a, b) for each $(\gamma_k^1)'$. Thus each γ_k^1 and γ^1 itself are C^1-diffeomorphisms from (a, b) onto their respective image. In addition (restricting to n large if necessary), there exists a nontrivial interval J around 0 that is contained in $\bigcap_k \gamma_k^1((a, b)) \cap \gamma((a, b))$. Then we set $U := J \times B_R^{n-1}(0)$, where $R > 0$ is chosen such that $\bigcup_k \gamma_k((a,b)) \cup \gamma((a,b)) \subseteq U$. We can write V in the form $t \mapsto (\gamma(t), v(t))$ with $v \in C^1((a,b), \mathbb{R}^n)$, and analogously $V_k(t) = (\gamma_k(t), v_k(t))$, with $v_k \to v$ in C^1. For $x \in U$ we set

$$\tilde{V}(x) := (x, v((\gamma^1)^{-1}(x^1))) \qquad \tilde{V}_k(x) := (x, v_k((\gamma_k^1)^{-1}(x^1))),$$

giving C^1-extensions of V resp. V_k to U. By [2, Cor. 1] we have that $(\gamma_k^1)^{-1}$ converges to $(\gamma^1)^{-1}$ locally uniformly, and that the same is true for the first derivatives. Consequently, $\tilde{V}_k \to \tilde{V}$ in $C^1_{\mathrm{loc}}(U)$. $\square$

As the proof shows, if the V_k converge to V merely in C^0_{loc} then one can still find C^1-extensions to U that also converge in C^0_{loc}.

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Zusammenfassung

Diese Dissertation befasst sich mit den Themen rund um eines der Hauptgebieter der allgemeinen Relativitätstheorie: den Singularitätentheoremen. In den 1960ern zeigte Roger Penrose, dass unter physikalisch realistischen Bedingungen, welche einen gravitationellen Kollaps wiederspiegeln, eine generische Raumzeit singulär werden muss und erschuf damit das Gebiet der Singularitätentheoreme. Kurz darauf bewies Stephen Hawking ein ähnliches Resultat, welches ein sich ausdehnendes Universum modellierte und sagte damit eine initiale Singularität voraus, den Urknall.

Die Allgemeine Relativitätstheorie wandelte sich in den letzten Jahrzehnten von einem hautsächlich durch einen geometrischen Zugang geprägtem Gebiet zu einem PDE orientierten. Dies führte dazu, dass die Regularität von Raumzeiten mehr Aufmerksamkeit bekam, da in diesem Zugang Lösbarkeit und Regularität der Einsteingleichungen im Mittelpunkt stehen. Außerdem haben Raumzeiten wie das Oppenheimer-Snyder Modell, "matched spacetimes" und impulsive Gravitationswellen allesamt Metriken niedriger Regularität, d.h. unter C^2. Durch diese Einflüsse sind Resultate der Lorentzgeometrie für nicht glatte Metriken immer mehr von Nöten, insbesondere für $C^{1,1}$ und C^1 Differenzierbarkeit.

In dieser Dissertation werden Beweise des Hawking-Penrose als auch (einer Variante) des Gannon-Lee Theorems für C^1-Metriken vorgestellt. Weiters ist ein Resultat über die Kausalität von maximal Nullpseudokonvexen Raumzeiten enthalten. Singuaritätentheoreme in niedriger Regularität präzisieren die Implikationen ihrer klassichern Vorgänger, da die vorhergesagte Singularität nicht schlicht durch ein Abfallen der Regularität der Metrik entstehen kann. Es wird somit ausgeschlossen, dass singuläre/unvollständige, glatte Raumzeiten C^1-Fortsetzungen haben.

Im Rahmen dieser Arbeit werden Nicht-Verzweigungsbedingungen ("non-branching conditions") definiert. Diese verbieten Verzweigungen von maximierenden Geodäten und haben sich als essentielles Werkzeug in den Beweisen von Singularitätentheoremen in niedriger Regularität für nicht global hyperbolische Raumzeiten herausgestellt. Sie ähneln in ihrer Art Nicht-Verzweigungsbedingungen aus der metrischen Geometrie der Längenräume. Nebenbei wird auch gezeigt, dass jede kausale Maximierende in einer C^1-Raumzeit eine C^2-Kurve ist. Letztlich enthält diese Arbeit einen abstrakten Zugang zu distributioneller Krümmung, besondere Aufmerkasmkeit wird dabei dem C^1-Fall geschenkt, in welchem die Krümmungsgrößen Distributionen erster Ordnung sind. Dabei handelt es sich um eine wichtigen Bestandteil im Beweis des Hawking-Penrose Theorems für C^1-Metriken.

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Education and Employment

since October 2019	**employed by University of Vienna**, Uni:Docs Stipend Program
since 2018	**Ph.D. Studies Mathematics**, University of Vienna
2016-2018	**Master of Science in Mathematics**, University of Vienna Passed with distinction
2013-2016	**B.Sc. in Mathematics**, University of Vienna Undergraduate theses: Convergence in Topological Spaces; Skew-Polynomial Rings
2011	**Matura** (High School Graduation), HBLVA f. chemische Industrie, Rosensteingasse, Wien, Passed with distinction.

Dissertation

Title	Non-smooth Spacetime Geometry Focusing, Energy Conditions and Singularity Theorems
Supervisor	Prof. Roland Steinbauer
Description	I study Lorentzian geometry for spacetime metrics of low Regularity and aim to extend recent results of singularity theorems for this class to weaker energy assumptions. Such assumptions are interesting to general relativity from a quantum point of view. Further I also work on proving a Gannon-Lee type singularity theorem in a low regularity setting.

Master's thesis

Title	Length Structures and Geodesics on Riemannian Manifolds of Low Regularity
Supervisor	Prof. Roland Steinbauer
Description	In this thesis I dealt with Riemannian geometry for metrics with low regularity. The approach taken is to rely on concepts from metric geometry such as length structures and shortest paths as well as regularization and comparison geometry

Experience

27. August 2018 - 31. August 2018	talk at International conference on generalized functions in Novi Sad, Serbia
14. November 2018 - 18. November 2018	visit at the department of theoretical physics, Prague
15. July 2019 - 19. July 2019	attended Domoschool - internatinal Summer school in Mathmatics and Physics, Domodossola, Italy
20. February 2020- 22-February 2020	visit at Central European Relativity Seminar, Potsdam-Golm, Germany
14. June 2021 - 18. June 2021	talk at SCRI21 (online) on on pseudoconvex spacetimes

Publications

1. *Causal simplicity and (maximal) null pseudoconvexity* - pusblished in Classical Quantum Gravity, joint work with Jakob Hedicke, Ettore Minguzzi, Stefan Suhr and Roland Steinbauer
2. *A note on the Gannon-Lee Theorem* - published in Letters in Mathematical Physics, joint work with Roland Steinbauer
3. *The Hawking-Penrose theorem for C^1-Lorentzian metrics*, published in Communications in Mathematical Physics, joint work with Argam Ohanyan, Michael Kunzinger and Roland Steinbauer