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DISSERTATION / DOCTORAL THESIS

Titel der Dissertation /Title of the Doctoral Thesis

„Exploring the conceptual limits of quantum theory: the measurement problem, coherence and indeterminism“

verfasst von / submitted by

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Contents

Abstract

Despite its tremendous success in terms of empirical predictions, quantum theory still exhibits a number of interpretational problems. The aim of this thesis is to further contribute to the understanding of certain unintuitive features of quantum mechanics, exploring the conceptual limits of the theory.

The thesis is articulated in three parts, each contributing to clarifying and characterising signature features of quantum theory, namely the so-called quantum measurement problem, the property of preparing individual systems in a coherent superposition and indeterminism, respectively.

The first part addresses the intrinsic ambiguity of the quantum formalism on whether to apply the unitary evolution prescribed by the Schrödinger equation or the "collapse" dynamics at measurements. While in the daily routine in quantum laboratories it is unproblematic to discriminate between a measurement and the standard unitary evolution, the theory does not avoid leading to ambiguous situations: This is the case of the Wigner's friend gedankenexperiment (and extensions thereof), in which an observer who performs quantum measurements on a particle is in turn considered a quantum system, thus measured upon by a superobserver. We show that this problem can be recast in a timeless formulation of quantum theory and that in such a framework the ambiguity is pushed back into the choice of a reasonable rule to assign probabilities for outcomes at different times (a sort of extension of the Born rule). We moreover show, again considering a Wigner's friend scenario, that quantum theory violates a set of intuitive assumptions, which lead to conclude that the perceptions that Wigner's friend has of her own measurement outcomes relative to different times cannot "share the same reality".

In the second part of the thesis, we introduce a "coherence equality" that is fulfilled by any theory (such as classical mechanics) that does not exhibit the effect of coherence in individual systems, while it is violated in the presence of coherence (such as in quantum theory). We then characterise the possibility of detecting quantum coherence "at a distance", i.e., without the need of recombining a system in an interferometer.

Finally, in the third part, we propose an alternative interpretation of classical physics that renders that theory fundamentally indeterministic. This allows to scale down the fundamental distance between classical and quantum mechanics. We also extend these considerations about irreducible indeterminism to relativistic scenarios.

Zusammenfassung

Trotz ihres enormen Erfolgs in Bezug auf empirische Vorhersagen weist die Quantentheorie immer noch eine Reihe von Interpretationsproblemen auf. Ziel dieser Arbeit ist es, einen weiteren Beitrag zum Verständnis bestimmter unintuitiver Merkmale der Quantenmechanik zu leisten und die konzeptionellen Grenzen der Theorie auszuloten.

Die Arbeit gliedert sich in drei Teile, von denen jeder zur Erklärung und zur Charakterisierung signifikanter Merkmale der Quantentheorie beiträgt, nämlich des so genannten Messproblems, der Eigenschaft einzelne Systeme in eine kohärente Superposition zu versetzen sowie des Indeterminismus.

Der erste Teil befasst sich mit der dem Quantenformalismus innewohnenden Zweideutigkeit, ob die von der Schrödinger-Gleichung vorgeschriebene unitäre Zeitentwicklung oder die "Kollaps"-Dynamik bei Messungen anzuwenden ist. Während es in der täglichen Praxis in Quantenlaboren unproblematisch ist, zwischen einer Messung und der standardmäßigen unitären Zeitentwicklung zu unterscheiden, vermeidet die Theorie grundsätzlich nicht, dass mehrdeutigen Situationen auftreten: Dies ist der Fall beim Wigners-Freund-Gedankenexperiment (und dessen Erweiterungen), bei dem ein Beobachter, der Quantenmessungen an einem Teilchen durchführt, seinerseits als Quantensystem betrachtet wird, welches wiederum ein Superbeobachter misst. Wir zeigen, dass dieses Problem in einer zeitlosen Formulierung der Quantentheorie neu formuliert werden kann. In dieser Formulierung verschiebt sich die Mehrdeutigkeit in in der Dynamik zu mehreren Möglichkeiten für die Wahl einer vernünftigen Regel zur Zuweisung von Wahrscheinlichkeiten für Ergebnisse zu verschiedenen Zeiten (eine Art Erweiterung der Bornschen Regel). Darüber hinaus zeigen wir, wiederum unter Berücksichtigung des Szenarios von Wigners Freund, dass die Quantentheorie eine Reihe intuitiver Annahmen verletzt, die zu dem Schluss führen, dass die Wahrnehmungen, die Wigners Freund von den eigenen Messergebnissen zu verschiedenen Zeiten hat, nicht "die gleiche Realität teilen" können.

Im zweiten Teil der Arbeit führen wir eine "Kohärenzgleichung" ein, die von jeder Theorie erfüllt wird, die den Effekt der Kohärenz in einzelnen Systemen nicht aufweist (wie z.B. die klassische Mechanik), während sie bei Vorhandensein von Kohärenz (wie in der Quantentheorie) verletzt wird. Anschließend beschreiben wir die Möglichkeit, Quantenkohärenz "aus der Ferne" nachzuweisen, d. h. ohne dass ein System in einem Interferometer rekombiniert werden muss.

Schließlich schlagen wir im dritten Teil eine alternative Interpretation der klassischen Physik vor, die diese Theorie grundlegend indeterministisch macht. Dies ermöglicht es, den fundamentalen Unterschied zwischen klassischer und Quantenmechanik zu relativieren. Wir weiten diese Überlegungen zum irreduziblen Indeterminismus auch auf relativistische Szenarien aus.

Curriculum Vitae

Education

- PhD in Physics, University of Vienna
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- M.Sc. in Physics, University of Vienna
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Research interests

Foundations of Quantum Mechanics, Indeterminism, Quantum Information Theory, Quantum Communication, Foundations of Probability, Contemporary History of Physics, Philosophy of Physics, Epistemology.

Awards

- SISS 2020 National Prize (honorable mention), Italian Society for the History of Science (SISS).
- FQXi 2020 Essay Contest (third prize), Foundational Questions Institute (FQXi).
- History of Physics 2018 Essay Contest, FHP of the American Physical Society (APS).
- Best Paper Prize 2018 (honorable mention), Italian Society for the History of Physics (SISFA).
- "Alumni Hall of Fame" Dean's list 2017, University of Vienna.

Publications and preprints

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- F. Del Santo* and S. Horvat*. 2022. "Comment on 'Quantum principle of relativity'". Preprint arXiv:2203.03661.
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Selected invited talks

- "Fundamental indeterminism in classical mechanics and special relativity". ICQIF2022, on February 14th, 2022 (online).
- "Philosophysics: A case study of a successful interplay between physics and philosophy for the establishment of quantum foundations in Vienna". Atelier du LKB, Sorbonne University and Ecole Normale Supérieure, Paris on January 13th, 2022.

- “The foundations of quantum mechanics in post-war Italy’s cultural context”. Annual meeting of History of Science Society (online), November 2021.
- “The relativity of indeterminacy”. University of Innsbruck, Austria on March 11th, 2021 (online).
- “Striving for Realism, not for Determinism: Historical Misconceptions on Einstein and Bohm”. American Physical Society April Meeting 2021 (online) on April 20th, 2021.
- “Physics without determinism: alternative interpretations of classical physics”. University of Notre Dame, Notre Dame, IN (US) on September 30th, 2020 (online).
- “Physics without determinism: Alternative interpretations of classical physics”. Università della Svizzera Italiana, Lugano (Switzerland) on October 8th, 2019.
- “Indeterminism in quantum and classical physics: History and open problems”, Workshop “Experiencing Reality Directly - Philosophy and Science”, Jerusalem (Israel) on May 20th-22nd, 2019.
- “Comment on Healey’s Quantum theory and the limits of objectivity”, Workshop “Encapsulated agents in quantum theory: Re-examining Wigner’s friend”, University of Massachusetts, Boston, Massachusetts (US) on March 9th-10th, 2019.
- “Two-way communication with a single quantum particle”, Dartmouth College, Hanover, New Hampshire (US) on February 28th, 2019.
- “Two-way communication with a single quantum particle”, University of Geneva (Switzerland) on February 15th, 2019.
- “Karl Popper’s Contribution to the Foundation of Quantum Mechanics”, Max Planck Institute for the History of Science, Berlin (Germany), July 6th, 2018.

Preamble

Quantum mechanics is arguably the most successful physical theory ever in terms of experimental predictions. Yet, although we are close to celebrating the first centenary of its complete formalization, the conceptual problems it poses are far from being uncontroversially resolved; rather, the "disagreement about the meaning of the theory is stronger than ever. New interpretations appear every day. None ever disappear." [1]. It took indeed a short time before physicists realized that they were dealing not only with a completely new theory but with some of the most involved philosophical problems that physics ever had to face. It is often argued that quantum physics departs from some of the a priori ideas that have been at the foundations of natural sciences for centuries, requiring a new way of looking at the world.

To begin with, Heisenberg's uncertainty principle has led physicists away from the comfort zone of determinism, which had dominated physics since the times of Galileo and Newton: particles that were classically believed to always have well defined and (in principle) perfectly predictable trajectories, face now the fundamental epistemic limits imposed by quantum theory on the determinacy of their position and momentum. Instead, quantum mechanics merely allows to compute probabilities for the outcomes of measurements, even given the most complete possible knowledge of a physical situation (i.e., a pure state). And, to make things worse, Bell's theorem [2] has ruled out the possibility of explaining indeterminism away by introducing underlying local hidden variables.

On the other hand, even the concept of a particle itself became more blurred. Although quantum particles, too, always leave well-localised traces when observed –like a spot on a photographic film, or a single click in an array of detectors–, they also exhibit the phenomenon of interference, which is a signature characteristic of waves at the classical level but is fundamentally impossible for single classical particles. More precisely, quantum systems can be prepared in a coherent superposition of two or more possible values of any degree of freedom (position, spin, etc.), which is detectable by observing interference patterns. Yet, every time a measurement is carried out, the system is found in *one and only one* of its possible states that form the superposition,[1] seemingly "forcing" a system in a superposed state to "choose" a value among its potential alternatives.

However, despite measurements (or observations) playing such a pivotal role in quantum physics, the very concept of measurement is not derived from within the theory, but rather added "by hand" as an a priori theoretical notion. Therefore, understanding how, when and under what exact circumstances a measurement –as opposed to the standard unitary evolution governed by the Schrödinger equation–,

[1] Formally, a quantum (projective) measurement returns one of the eigenstates of the considered observable.

takes place remains one of the central issues for the understanding of quantum mechanics. This goes by the name of the quantum measurement problem, and it basically addresses the fundamental question: "*what makes a measurement a measurement*? [...] There is nothing in the theory to tell us which device in the laboratory corresponds to a unitary transformation and which to a projection! [...] What makes a photodiode a good detector for photons? And why is a beam splitter a bad detector?" [3]. This directly leads to the problem of the "roles" in quantum mechanics. In fact, the unitary (and deterministic) evolution of a system is drastically interrupted by the *action of a measurement* (which only gives probabilistic predictions) that "collapses" the state on one of the potential outcomes, and this seems to introduce a mysterious external entity that performs the measurement, a role traditionally called the "observer". The distinction between systems and observers has sparked considerable debate throughout the whole history of quantum theory, for it seems to induce an arbitrary hierarchical separation, despite the widespread belief that in principle there are no limits to the applicability of quantum theory (in terms of size, energy, mass, etc.), and hence *every* physical object should be fundamentally described by quantum theory. This even led in some cases to the belief (not shared by the present author) that is possibly the consciousness of the observer to cause the "collapse" of the quantum state [4], about which J. Bell ironically commented: "What exactly qualifies some subsystems to play this role? Was the world wave function waiting to jump for thousands of millions of years until a single-celled living creature appeared? Or did it have to wait a little longer for some more highly qualified measurer - with a Ph.D.?" [5].

The aim of this thesis is to contribute to the understanding of some of these unintuitive, perhaps problematic, and yet (allegedly) fundamental characteristics of quantum physics. In particular, my doctoral project was divided into three main different –yet intertwined– lines of research each dealing, respectively, with one of the aforementioned topics: the measurement problem, coherence exhibited by individual quantum systems, and the problem of indeterminism. Making a large use of the tools developed within quantum information theory, each of these lines of research aims, in a broad sense, to explore some "limits" of quantum theory –either internally, or in comparison to other theoretical frameworks.

A first meaning attributed to the word "limits" refers to intrinsic problematic issues of the theory itself, namely to the ambiguity that the formalism exhibits between the two prescribed dynamics –either unitary or "collapse"– when the applicability of quantum theory is pushed to the extreme. This is the case of the so-called *Wigner's friend thought experiment* [4], which emblematically exemplifies the measurement problem in quantum mechanics. This features a scenario in which an observer, traditionally called Wigner's friend, is situated in her laboratory wherein she performs a measurement on a standard quantum system (customarily a spin-1/2 particle) which is prepared in a superposed state. However, a "superobserver", traditionally called Wigner, is granted the possibility to perform quantum measurements on the whole laboratory, thus treating both the friend and the particle as a joint quantum system. This begs the question: *Which of the two dynamics should describe the friend's measurement?* In fact, since the friend observes a single outcome, she seems fully entitled to applying the "collapse" rule, while for Wigner, who regards the friend and the particle merely as two interacting components of a closed system, seems legit to describe the same physical process unitarily. New life was recently breathed into this topic, by the appearance of more sophisticated variants of the thought experiment that combined Wigner's friend scenarios with Bell-like arguments, which led to controversial claims such as that "quantum theory cannot consistently describe the use of itself" [6], or that "'facts', understood as 'immediate experiences of observers', [...] cannot be facts of the world per se, but only relative to an observer" [7] (see also [8, 9, 10, 11, 12]).

The measurement problem and extended Wigner's friend scenarios are the subjects of the first and main part of this thesis. We begin in Chapter 1 by studying the Wigner's friend gedankenexperiment in terms of the a "timeless" formulation of quantum theory called Page-Wootters Mechanism (PWM) [13]. The advantage of applying this approach to Wigner's friend scenarios is that it assigns an unambiguous timeless state about which all possible observers (notably Wigner and his friend) agree, and from which one can compute probabilities of an event conditioned on another, regardless of their temporal order. However, we find that the standard rules to assign two-time conditional probabilities within the PWM need in this case to be modified. We propose three main definitions to assign these probabilities, all of which give standard probabilities for non-Wigner's friend setups. We show, however, that in a Wigner's friend scenario these lead to different predictions, which can be justified by adopting different interpretations of quantum theory. In this way, the ambiguity between the two dynamical processes (i.e., unitary evolution versus "collapse") and the consequent state assignment is pushed back into the choice of the probability-assignment rule.

In Chapter 2, we put forward a new no-go theorem for Wigner's friend. We show that quantum theory can violate the conjunction of the following reasonable assumptions: (i) that quantum mechanics allows making single-time prediction for all the observers, (ii) that all interpretations of quantum theory should be empirically equivalent, and (iii) that the linearity of the Born rule (in the quantum state) can be extended to two-time events. From this, we conclude that the "perceptions" that the friend has of her own measurement outcomes cannot coexist at different points in time.

In Chapter 3, we rebut an argument recently put forward by R. Healey [11] to show that quantum mechanics cannot have "a unique, objective, physical outcome", and which builds on a thought experiment that combines Bell's inequalities with Wigner's friend scenarios, as proposed by Č. Brukner [3, 7]. In particular, we show that the claims based on Healey's modified protocol– which allegedly removes questionable implicit assumptions of the original argument– are unwarranted because they exclusively relate unobservable quantities.

The second part of the thesis is devoted to studying the feature of coherence in individual systems. In this case, the word "limits" refers to the possibility of using coherence as a test of non-classicality, in the same spirit of Bell's inequalities. In this sense, the limit is meant as a discrimination criterion between the classical theory and any theory that allows coherent superpositions of individual objects, such as quantum mechanics.

In Chapter 4, indeed, we propose a "coherence equality" that is satisfied by any possible theory that does not exhibit coherent superpositions of individual particles, such as classical theory. Phrasing the problem in terms of a "communication game", we show that quantum theory violates the maximal classical bound for the winning probability and hence the coherence equality. This is derived in the same spirit of Bell's inequalities, which are satisfied by any theory upholding local realism, while being violated by quantum mechanics. Interestingly, this chapter also contributes to a further understanding of quantum coherence per se. It is usually believed that to detect coherence one should carry out an indirect measurement, i.e. an interferometric experiment in which a particle is prepared in a superposition of separated spatial locations, picks up a phase difference between the two travelled paths, and finally it is recombined at the same location. The proposed coherence equality, on the other hand, allows detecting superposition "at a distance". Namely, exclusively by carrying out local operations at different locations and then computing a function of the correlations between the local measurement results –hence only by means of local operations and classical communication (LOCC).

In Chapter 5 we follow up on the same line of research, by analyzing more systematically the detection of coherence at a distance, thereby characterising which are the limits imposed by superselection rules for bosons and fermions, respectively, as well as for charged and uncharged particles. Moreover, we analyse the Aharonov-Bohm effect, showing that –contrarily to recent claims put forward by C. Marletto and V. Vedral [14]– the phase due to the interaction with the electromagnetic field is not easily explained as a local property (which would be a gauge-dependent quantity) but it is a property of a whole loop in space-time.

The third and last part of the thesis is devoted to exploring the possibility of providing new interpretations of classical mechanics and special relativity, such that these theoretical frameworks as well can be regarded as fundamentally indeterministic, therewith putting forward the idea that they can be more similar to quantum physics than usually believed. In this sense, the word "limits" refers again to a comparison between classical (also relativistic) theory and quantum mechanics, showing that one should not immediately jump to the conclusion that these two physical theories are necessarily so fundamentally different for what concerns (in)determinism.

In Chapter 6 we develop further the idea, proposed by N. Gisin [15], that classical physics is based on the implicit assumption of having infinite precision, leading to an infinite density of information. If one relaxes such an assumption, we show that also classical physics can be given an alternative interpretation as a fundamentally indeterministic theory on a par with quantum mechanics (that is, for what concerns this issue). Formally, this is achieved by replacing the real numbers, as the numerical field in which physical variables take value, with new mathematical entities, called finite information quantities (FIQs), which are defined through postulating objective probabilities (i.e., propensities) that evolve in time. Interestingly, we also show that this indeterministic interpretation leads to a "classical measurement problem", reminiscent of the quantum one (as expounded in the first part of this thesis).

Finally, in Chapter 7, we retain the assumption of finite information density and show that this hints at a formulation of physics that should be both indeterministic and relativistic. We then rebut a classical argument, independently put forth by C. Rietdijk [16] and H. Putnam [17], aimed at showing that special relativity and indeterminism are incompatible. To overcome this, we propose that (in)determinacy itself should be considered a relational quantity.

In addition to the seven publications which are included in this thesis, I have also coauthored other related works during my PhD [18, 19, 20, 21, 22], as well as carried out independent research on the history and philosophy of physics

[23, 24, 25, 26, 27, 28, 29].

Part I

The quantum measurement problem and extensions of the Wigner's friend gedankenexperiment

Chapter 1

Generalized probability rules from a timeless formulation of Wigner's friend scenarios

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The first three authors, listed in alphabetic order, share first authorship. I made essential contributions to all aspects of this work.

Abstract

The quantum measurement problem can be regarded as the tension between the two alternative dynamics prescribed by quantum mechanics: the unitary evolution of the wave function and the state-update rule (or "collapse") at the instant a measurement takes place. The notorious Wigner's friend gedankenexperiment constitutes the paradoxical scenario in which different observers (one of whom is observed by the other) describe one and the same interaction differently, one –the Friend– via state-update and the other –Wigner– unitarily. This can lead to Wigner and his friend assigning different probabilities to the outcome of the same subsequent measurement. In this paper, we apply the Page-Wootters mechanism (PWM) as a timeless description of Wigner's friend-like scenarios. We show that the standard rules to assign two-time conditional probabilities within the PWM need to be modified to deal with the Wigner's friend gedankenexperiment. We identify three main definitions of such modified rules to assign two-time conditional probabilities, all of which reduce to standard quantum theory for non-Wigner's friend scenarios. However, when applied to the Wigner's friend setup each rule assigns different conditional probabilities, potentially resolving the probability-assignment paradox in a different manner. Moreover, one rule imposes strict limits on when a joint probability distribution for the measurement outcomes of Wigner and his Friend is well-defined, which single out those cases where Wigner's measurement does not disturb the Friend's memory and such a probability has an operational meaning in terms of collectible statistics. Interestingly, the same limits guarantee that said measurement outcomes fulfill the consistency condition of the consistent histories framework.

1.1 Introduction

Standard quantum theory features two dynamical processes: the so-called "collapse of the wave function" that occurs during a measurement and the unitary evolution describing the propagation of the wave function in the absence of measurements. However, quantum theory itself does not specify when each process should apply, leading to the well-known quantum measurement problem[1] [31, 32].

This situation is illustrated by the gedankenexperiment of Wigner's friend [4]. An observer F (the Friend) performs a measurement on a quantum system S in a closed laboratory, the outcome of which is recorded by the click of a measuring apparatus and/or as a definite record in the Friend's memory inside the laboratory. After the measurement, F assigns a state to S conditioned on her measurement outcome (i.e., state update). Simultaneously, another observer W (Wigner) situated outside of the laboratory, would describe the entire measurement by F as unitary, resulting in him assigning a specific entangled state to F and S. This leads to the paradoxical situation that the two observers F and W assign different states after F's measurement and, hence, different probabilities to the outcomes of subsequent measurements. As shown in Refs. [3, 6, 7], the seemingly natural assumptions that quantum theory is universal, that different observers can each apply either of the two dynamical processes of the theory with respect to their description, and that the reasoning of different agents about one another can be freely combined to make statements about an objective reality lead to contradictions for Wigner's friend setups. These contradictions can be regarded as resulting from an observer dependent application of the state-update rule and involve inferences about the outcomes different observers obtain at different times.

We begin by formulating the Wigner's friend gedankenexperiment in terms of the timeless formulation of quantum theory put forward by Page and Wootters [13]. In such an approach time evolution emerges from quantum correlations between a clock and a system whose dynamics the clock will track. The advantage of the timeless approach in the context of the Wigner's friend gedankenexperiment is that it assigns a timeless state from which one can compute probabilities of an event conditioned on another, regardless of their temporal order as indicated by the clock. This allows us to ask the following two questions:

1. What is the probability that W measures outcome w at clock time t_2, given that F has measured outcome f at a previous clock time t_1?
2. What is the probability that F measures outcome f at clock time t_1, conditioned on the fact that W will measure outcome w at a later clock time t_2?

We will show that there exist more than one way to consistently assign conditional probabilities using the Page-Wootters mechanism (PWM), all of which reduce to standard quantum conditional probabilities for non-Wigner's friend scenarios. Depending on the choice of conditional-probability rule, however, the Wigner's friend gedankenexperiment leads to different probabilities for the results of Wigner and his Friend, thus resolving the paradox of the ambiguous probability

[1] In the vast literature devoted to this topic, the "quantum measurement problem" does not have a unique, clear definition. Intuitively, it can be thought of as the problem of *how*, *when* and *under what circumstances* definite values of physical variables are obtained [30]. However, in this paper we focus on the ambiguity of the application between unitary and "collapse" dynamics.

assignments in different manners. Using the PWM all observers will agree on the overall state and assuming they all use the same probability rule they will agree in their probability assignments. The first rule gives probabilities that correspond to applying the state-update rule after the Friend's measurement and, hence, suggests collapse dynamics for all observers. The other two rules are in accordance with unitary evolutionand give conditional probabilities different from the first rule. Moreover, they give well-defined probabilities only under certain conditions implying that the above two questions can only be meaningfully answered for specific settings. In case of the last rule, these conditions single out those settings for which Wigner's measurement does not disturb the Freind's memory and, hence, the respective probabilities correspond to collectible statistics for the Friend.

1.2 The Page and Wootters Mechanism

We now summarize the Page-Wootters mechanism [13, 33], which begins by considering a clock and system whose dynamics the clock tracks. The clock and system are described respectively by the Hilbert spaces $\mathcal{H}_C \simeq L^2(\mathbb{R})$ and $\mathcal{H}_S$. The joint state $|\Psi\rangle\rangle$ assigned to the clock and system is a solution to a Wheeler-DeWitt-like equation

$$\hat{H}\,|\Psi\rangle\rangle = \left(\hat{H}_C + \hat{H}_S\right)|\Psi\rangle\rangle = 0, \tag{1.1}$$

where $\hat{H}_S$ is the system Hamiltonian and $H_C = \hat{P}_C$ is the clock Hamiltonian, which is taken to be the momentum operator $\hat{P}_C$ on $\mathcal{H}_C$. In general, a Wheeler-DeWitt-like equation can in some cases be interpreted as arising from the canonical quantization of a gauge theory with a Hamiltonian constraint. General relativity is an example of such a gauge theory and it is for this reason that Page and Wootters first put forward their formulation of quantum theory to address the problem of time [34, 35]. Solutions to Eq. (1.1) can be obtained by acting on states in the *kinematical* Hilbert space $\mathcal{K} \simeq \mathcal{H}_C \otimes \mathcal{H}_S$ with the operator

$$P^{\text{ph}} := \int_{\mathbb{R}} ds\, e^{-is\hat{H}}, \tag{1.2}$$

that is, supposing $|\psi\rangle \in \mathcal{K}$, then $|\Psi\rangle\rangle = P^{\text{ph}}\,|\psi\rangle$ is a solution to Eq. (1.1). Physical states of the theory, $|\Psi\rangle\rangle \in \mathcal{H}$, are elements of the *physical* space $\mathcal{H}$, which is not a subspace of the kinematical Hilbert space $\mathcal{K}$. This is because the spectrum of $\hat{H}$ is continuous around zero, from which it follows that physical states are not normalizable in the kinematical inner product [36, 37]. Thus a new inner product must be used to normalize the physical states, which in turn defines the physical Hilbert space $\mathcal{H}$; such an inner product is defined in Eq. (1.6) and motivated in what follows.

First, one defines the time read by the clock as the measurement outcome of a time observable $\hat{T}_C$ that is covariant [38, 39] with respect to the clock Hamiltonian $\hat{H}_C$. By covariant it is meant that supposing the spectral decomposition of the time observable on $\mathcal{H}_C$ is

$$\hat{T}_C = \int_{\mathbb{R}} dt\, t\,|t\rangle\langle t|\,, \tag{1.3}$$

where $|t\rangle$ are eigenkets of $\hat{T}_C$ associated with the eigenvalue $t \in \mathbb{R}$, the eigenkets are connected to one another by the unitary generated by the clock Hamiltonian, $|t'\rangle = e^{-i\hat{H}_C(t'-t)}\,|t\rangle$. In this case, enforcing the covariance condition implies that $\hat{T}_C$ is canonically conjugate to the clock Hamiltonian, $[\hat{T}_C, \hat{H}_C] = i$, and thus is equivalent to the position operator on $\mathcal{H}_C$.

Now consider a system observable $\hat{M} = \sum_m m\Pi_m$, where $\{\Pi_m,\ \forall\, m \in \operatorname{Spec} \hat{M}\}$ defines a projective valued measure on $\mathcal{H}_S$. The probability that $\hat{M}$ takes the value m when the clock reads time t is given by the Born rule

$$\begin{aligned} P\left(\hat{M} = m \text{ when } \hat{T}_C = t\right) \\ = \frac{\langle\langle \Psi |\, \big(|t\rangle\langle t| \otimes \Pi_m\big)\, | \Psi \rangle\rangle}{\langle\langle \Psi |\, \big(|t\rangle\langle t| \otimes \mathbb{1}_S\big)\, | \Psi \rangle\rangle}. \end{aligned} \tag{1.4}$$

In what follows, we will use an abbreviated expression to refer to these probabilities in which we omit the operators, i.e., $P(\hat{M} = m \text{ when } \hat{T}_C = t) \equiv P(m \text{ when } t)$.

The form of Eq. (1.4) suggests defining the conditional state of the system given the clock reads the time t as

$$|\psi_S(t)\rangle := \big(\, \langle t| \otimes \mathbb{1}_S\big)\, | \Psi \rangle\rangle\,. \tag{1.5}$$

Further, demanding that the conditional state remains normalized for all t implies that the physical state $| \Psi \rangle\rangle$ must be normalized with respect to the physical inner product [40]

$$\langle\langle \Psi | \Psi \rangle\rangle_{\mathrm{PW}} := \langle\langle \Psi | \big(|t\rangle\langle t| \otimes \mathbb{1}_S\big) | \Psi \rangle\rangle = 1, \tag{1.6}$$

for all $t \in \mathbb{R}$, from which it follows that $\langle \psi_S(t) | \psi_S(t) \rangle = 1$. Given Eqs. (1.5) and (1.6), the probability in Eq. (1.4) may be expressed as

$$P\,(m \text{ when } t) = \langle \psi_S(t) | \Pi_m | \psi_S(t) \rangle\,. \tag{1.7}$$

Since Eq. (1.7) is identical to the probability assigned to the measurement outcome m given the state of the system is $|\psi_S(t)\rangle$ by the Born rule in the standard formulation of quantum theory, we are justified in identifying the conditional state in Eq. (1.5) as the standard time-dependent wave function. Further, the covariance condition satisfied by the eigenstates of $\hat{T}_C$ implies that the conditional state satisfies the Schrödinger equation [33].

We note that because the eigenstates of $\hat{T}_C$ can be used to form a resolution of the identity on the clock Hilbert space, $\mathbb{1}_C = \int dt\, |t\rangle\langle t|$, physical states may be expressed as

$$| \Psi \rangle\rangle = \int_{\mathbb{R}} dt\ |t\rangle\, |\psi_S(t)\rangle\,. \tag{1.8}$$

From the above equation one can immediately see that in general physical states describe an entangled state of the clock and the system. It is the correlations between the clock and system described by this entangled state that are responsible for the relative evolution of the system with respect to the clock.

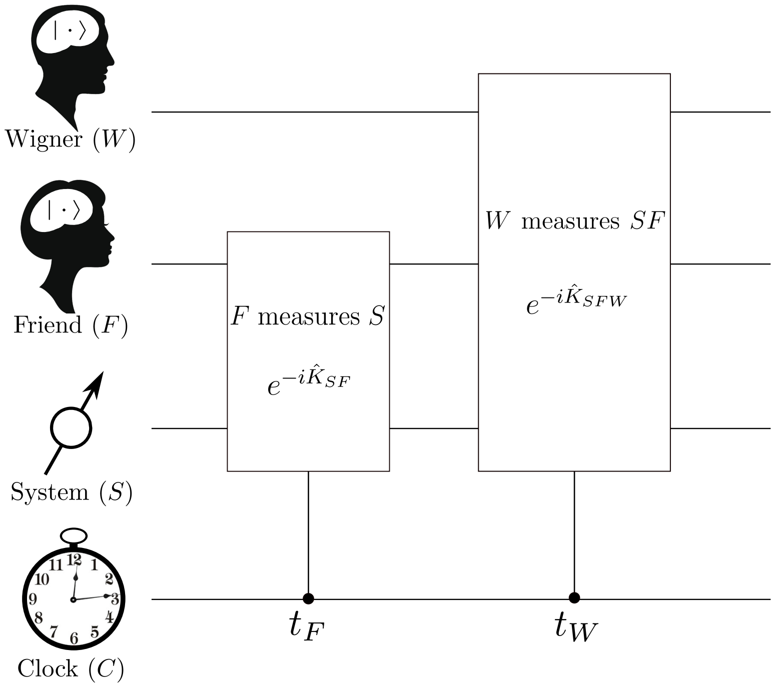

Figure 1.1: The circuit representation of the Wigner's friend experiment as encoded in the physical state $|\Psi\rangle\rangle$ given in Eq. (1.11). This can be thought of as the perspective of a hypothetical higher level observer whose experimental powers are even beyond that of Wigner. As described in the main text, at time t_F according to clock C the Friend performs a measurement and her memory gets entangled with the system S. At another time $t_W > t_F$ Wigner measures both S and F and his memory gets entangled with the joint system.

1.3 Timeless formulation of the Wigner's friend experiment

We now present a description of the Wigner's friend experiment depicted in Fig. 1.1 in terms of the Page and Wootters mechanism.

Consider a two-level system (S) associated with the Hilbert space $\mathcal{H}_S \simeq \mathbb{C}^2$ and a projective valued measurement of S that yields two possible outcomes, $\{\uparrow,\downarrow\}$. Such a measurement is described by the set of projectors $\{\Pi_\uparrow, \Pi_\downarrow\} \subset \mathcal{E}(\mathcal{H}_S)$, where $\Pi_\uparrow + \Pi_\downarrow = \mathbb{1}_S$ and $\mathcal{E}(\mathcal{H})$ denotes the space of effect operators on the Hilbert space $\mathcal{H}$. The Friend (F) makes such a measurement of S at the time t_F and the outcome is recorded by F's measuring device and eventually in her memory, which we associate with the Hilbert space $\mathcal{H}_F \simeq \mathbb{C}^3$. Situated outside of F's laboratory is Wigner (W), who makes a joint measurement of S and F with two possible outcomes, $\{\text{yes}, \text{no}\}$, at a time $t_W > t_F$. This measurement corresponds to the set of projectors $\{\Pi_{\text{yes}}, \Pi_{\text{no}}\} \subset \mathcal{E}(\mathcal{H}_S \otimes \mathcal{H}_F)$ with $\Pi_{\text{yes}} + \Pi_{\text{no}} = \mathbb{1}$, and its result is recorded in W's memory, the state of which is encoded in the Hilbert space $\mathcal{H}_W \simeq \mathbb{C}^3$. The times t_F and t_W are associated with a clock system C described by the Hilbert space $H_C \simeq L^2(\mathbb{R})$ as discussed in Sec. 1.2.

Following Hellmann *et al.* [41] and Giovannetti *et al.* [42], we use a von Neumann measurement model to describe F's and W's measurement within the PWM. The physical states describing the Wigner's friend experiment in Fig. 1.1

are solutions to

$$\Big(\hat{H}_C+\delta(\hat{T}-t_F)\hat{K}_{SF}+\delta(\hat{T}-t_W)\hat{K}_{SFW}\Big)|\,\Psi\rangle\rangle=0, \tag{1.9}$$

where $\hat{K}_{SF}$ and $\hat{K}_{SFW}$ are the interaction Hamiltonians coupling respectively S and F during F's measurement of S and S, F, and W during W's measurement of SF:

$$\begin{aligned}
e^{-i\hat{K}_{SF}}\,|\psi_S\rangle\,|R_F\rangle &= \sum_{f\in\{\uparrow,\downarrow\}} \Pi_f\,|\psi_S\rangle\,|f_F\rangle\,, \\
e^{-i\hat{K}_{SFW}}\,|\phi_{SF}\rangle\,|R_W\rangle &= \sum_{w\in\{\text{yes, no}\}} \Pi_w\,|\phi_{SF}\rangle\,|w_W\rangle\,,
\end{aligned}$$

where $|\psi_S\rangle$ is the state of the system for clock readings $t < t_F$ and $|R_F\rangle \in \mathcal{H}_F$, $|R_W\rangle \in \mathcal{H}_W$ are the pre-measurement "ready" states of the memories of F and W. The sets $\{|R_F\rangle\,, |\uparrow_F\rangle\,, |\downarrow_F\rangle\}$ and $\{|R_W\rangle\,, |\text{yes}_W\rangle\,, |\text{no}_W\rangle\}$ form orthonormal bases for $\mathcal{H}_F$ and $\mathcal{H}_W$ respectively. Using this measurement scheme we will calculate the probabilities of result m by projecting onto the memory entries of the respective observer. Hence, the formula in Eq. (1.4) for arbitrary time t becomes

$$P\,(m \text{ when } t) = \frac{\langle\langle\Psi\,|\,\big(|t\rangle\langle t|\otimes\Pi^m\big)\,|\,\Psi\rangle\rangle}{\langle\langle\Psi\,|\,\big(|t\rangle\langle t|\otimes\mathbb{1}\big)\,|\,\Psi\rangle\rangle}, \tag{1.10}$$

where Π^m now acts on $\mathcal{H}_M$, the Hilbert space of the apparatus or memory storing the result, and $|\,\Psi\rangle\rangle$ is a solution to Eq. (1.9). For simplicity we have assumed no free dynamics of S, F, and W in Eq. (1.9) in addition to their interaction with one another during the measurements. The solutions take the form

$$\begin{aligned}
|\Psi\rangle\rangle = &\int_{-\infty}^{t_F} dt\ |t\rangle\,|\psi_S\rangle\,|R_F\rangle\,|R_W\rangle + \\
&\int_{t_F}^{t_W} dt\ |t\rangle \sum_{f\in\{\uparrow,\downarrow\}} \Pi_f\,|\psi_S\rangle\,|f_F\rangle\,|R_W\rangle + \\
&\int_{t_W}^{\infty} dt\ |t\rangle \sum_{\substack{f\in\{\uparrow,\downarrow\}\\ w\in\{\text{yes, no}\}}} \Pi_w\Pi_f\,|\psi_S\rangle\,|f_F\rangle\,|w_W\rangle,
\end{aligned} \tag{1.11}$$

for arbitrary states $|\psi_S\rangle$ of the system. When expressed in the clock basis like in Eq. (1.11) the physical state encodes the complete history of system S and the measurements performed on it. In what follows we will explicitly consider states

$$|\psi_S\rangle = a\,|\uparrow_S\rangle + be^{i\phi_S}\,|\downarrow_S\rangle\,, \tag{1.12}$$

where $a, b, \phi_S \in \mathbb{R}$ and

$$|\text{yes}_{SF}\rangle = \alpha\,|\uparrow_S\rangle\,|\uparrow_F\rangle + \beta e^{i\phi_{SF}}\,|\downarrow_S\rangle\,|\downarrow_F\rangle\,, \tag{1.13}$$

where $\alpha, \beta, \phi_{SF} \in \mathbb{R}$, with $\Pi_{\text{yes}} = |\text{yes}\rangle\langle\text{yes}|$ and $\Pi_{\text{no}} = \mathbb{1} - |\text{yes}\rangle\langle\text{yes}|$, moreover, we will use $\phi := \phi_S - \phi_{SF}$. Note that a measurement by Wigner as implied by Eq. (1.13) will in general affect the Friend's memory state and potentially alter her perceived result.

1.4 Different definitions of conditional probabilities, different solutions of the Wigner's friend paradox

We want to express the dynamical content of a theory in a manifestly operational way, that is, in terms of probabilities of measurement outcomes. As emphasized by Kuchař [34], *the fundamental question in any dynamical theory* is: if one measures an observable at the time t_1 and obtains outcome a, what is the probability of a different measurement yielding the outcome b at the time $t_2 > t_1$? For non-Wigner's friend scenarios there have been two proposals for defining probability rules within the Page-Wootters mechanism [43, 42] in order to answer this fundamental question. Both approaches recover the standard quantum formalism for the situation of two observables measured at different times inquired above.

In order to deal with Wigner's friend scenarios (in particular with the setup depicted in Fig. 1.1), however, one needs to generalize the standard definition of the conditional-probability rule. The rationale behind this generalization is that the state of the Friend's memory is –except for particular cases– modified by Wigner's measurement. Hence, it is not any longer possible to simply read out the information in F's memory at the end of the protocol in order to learn what outcome she obtained. Thus, we will be interested in calculating two-time conditional probabilities at times t_1 (in between F's and W's measurement) and t_2 (after W's measurement), i.e., $t_F < t_1 < t_W < t_2$. We propose three main alternative conditional probability definitions, two of which are inspired by Refs. [43] and [42] respectively, as well as a third novel one. While all of these definitions give the standard quantum probabilities for non-Wigner's friend cases (as shown explicitly in Appendix 1.A), they give different conditional probabilities for the measurement results of Wigner and his Friend.

First we propose a conditional-probability rule similar to that put forward by Dolby [43], where measurement operators $|t_1\rangle\langle t_1| \otimes \Pi_m$ together with the operator P^{ph} (defined in Eq. (1.2)) are used to calculate two-time conditional probabilities. Specifically, after the application of one operator $|t_1\rangle\langle t_1| \otimes \Pi_m$ to the physical state $|\,\Psi\rangle\rangle$ the constraint in Eq. (1.1) is in general no longer fulfilled, i.e., $\hat{H}(|t_1\rangle\langle t_1| \otimes \Pi_m\,|\,\Psi\rangle\rangle) \neq 0$. As proposed in [43] one uses the operator P^{ph} to obtain again a physical state, $\hat{H}\big(P^{\mathrm{ph}}|t_1\rangle\langle t_1| \otimes \Pi_m\,|\,\Psi\rangle\rangle\,\big) = 0$, before applying a second measurement operator $|t_2\rangle\langle t_2| \otimes \Pi_n$. In order to describe Wigner's friend scenarios we account for the measurements as in Eqs. (1.9)-(1.11).

Definition 1 (two-time "collapse"). The conditional probability of result n at time t_2 given result m at time t_1 is

$$P_1\,(n \text{ when } t_2\,|\,m \text{ when } t_1) = \frac{\langle\langle\Psi\,|\,|t_1\rangle\langle t_1|\otimes\Pi^m P^{\mathrm{ph}}|t_2\rangle\langle t_2| \otimes \Pi^n P^{\mathrm{ph}}\,|t_1\rangle\langle t_1|\otimes\Pi^m|\Psi\rangle\rangle}{\langle\langle\Psi|\,|t_1\rangle\langle t_1|\otimes\Pi^m|\Psi\rangle\rangle}, \tag{1.14}$$

where Π^m and Π^n are now the projectors on the respective states of the apparatus or memories in $\mathcal{H}_M$ and $\mathcal{H}_N$, and P^{ph} was defined in Eq. (1.2).

Intuitively, the numerator in Eq. (1.14), in the fashion of Ref. [43], is the modulus square of the physical state $|\,\Psi\rangle\rangle$ which is first projected on Π^m at time t_1, then brought back into the physical space by applying P^{ph} and finally projected on Π^n at time t_2. Definition 1 always gives well-defined probabilities, and for the

$P_1\,(w \text{ when } t_2 \,|\, f \text{ when } t_1)$

$f \backslash w$	yes	no
$\uparrow$	α^2	β^2
$\downarrow$	β^2	α^2

$P_1\,(f \text{ when } t_1 \,|\, w \text{ when } t_2)$

$f \backslash w$	yes	no
$\uparrow$	α^2	β^2
$\downarrow$	β^2	α^2

Table 1.1: The conditional probabilities of Wigner seeing result w at time t_2 given that the Friend saw result f at time t_1 and of the Friend seeing f at t_1 given that Wigner will see w at t_2 according to definition 1. Note that the two conditional probabilities are equal and hence, applying Bayes' rule will in general *not* give rise to *one* joint probability expression.

results of Wigner and his Friend this leads to the conditional probabilities displayed in Table 1.1. We label this definition "two-time collapse" because any observer will agree on these probabilities assignments which correspond to applying the state-update rule ("collapse") after every measurement. Moreover, it is genuinely two-time, since the expression explicitly depends on both the times t_1 and t_2. Hence, this resolves the probabilistic paradox for Wigner's friend scenarios insofar as it suggests that both Wigner and the Friend should describe F's measurement with "collapse" dynamics. Note, however, that this rule does not give rise to a well-defined joint probability since $P_1\,(n \text{ when } t_2 \,|\, m \text{ when } t_1) \cdot P(m \text{ when } t_1) \neq P_1\,(m \text{ when } t_1 \,|\, n \text{ when } t_2) \cdot P(n \text{ when } t_2)$. This reflects the fact that the numerator in Eq. (1.14) is not invariant under the exchange of $|t_1\rangle\langle t_1| \otimes \Pi^m$ and $|t_2\rangle\langle t_2| \otimes \Pi^n$.

Alternatively, we consider the conditional probability rule proposed by Giovanetti *et al.* [42] in the context of Wigner's friend experiments. The authors of [42] include generalized measurements as described in Eq. (1.9)-(1.11) in the constraint Hamiltonian and apply projection operators on the measurement apparatus or memories at some time t after both measurements in question have been performed, i.e., $|t\rangle\langle t| \otimes \Pi^m \otimes \Pi^n$. This equates measurements at different times with one joint measurement of the records of the obtained results at a final single time, which is no longer a trivial thing to do when considering Wigner's friend scenarios. The original formulation in [42] states the following rule for calculating conditional probabilities.

Definition 2a (two-time unitary, uninterpreted conditions). The conditional probability (in the cases where it is well-defined) of result n at time t_2 given result m at time t_1 is

$$P_{2a}\,(n \text{ when } t_2 \,|\, m \text{ when } t_1) = \frac{\langle\langle \Psi \,|t_2\rangle \, \langle t_2| \otimes \Pi^n \otimes \Pi^m \,|\, \Psi\rangle\rangle}{\langle\langle \Psi \,|t_1\rangle \, \langle t_1| \otimes \Pi^m \,|\, \Psi\rangle\rangle}. \tag{1.15}$$

The intuition behind this definition is to read out both F's and W's memories at the final time t_2 and take the modulus square to define a joint probability, and use that to construct a conditional probability. Namely, to take the expectation value of the operators Π^m and Π^n at time t_2 with the physical state $|\Psi\rangle\rangle$ (numerator). This is then divided by the one-time probability of F finding outcome m at time t_1 (denominator), i.e., before W's measurement takes place. Equation (1.15) depends on both measurement outcomes and on both times t_1 and t_2, thus amounting to a genuine two-time probability. However, as shown in Appendix 1.B, in the Wigner's friend scenario of Fig. 1.1, these expressions are not always proper probabilities, for they are normalized only if

$$\begin{aligned}&2\alpha^2\beta^2\left(\frac{b}{a}\right)^2+2\cos(\phi)(\alpha^3\beta-\alpha\beta^3)\frac{b}{a}=\\&2\alpha^2\beta^2\left(\frac{a}{b}\right)^2-2\cos(\phi)(\alpha^3\beta-\alpha\beta^3)\frac{a}{b}=\\&1-\alpha^4-\beta^4.\end{aligned}\tag{1.16}$$

Under these conditions, definition 2a gives the probabilities stated in Table 1.2. While these expressions are formally probabilities when Eqs. (1.16) are satisfied, their operational meaning is admittedly not clear to us, except for special cases (see Sec. 1.5).

$\boldsymbol{P_{2a}\,(w \textbf{ when } t_2\,|\,f \textbf{ when } t_1)}$

f \ w	yes	no
$\uparrow$	$\frac{1+2\alpha^2+(\beta^4-\alpha^4)\cos(2\phi)\pm\chi}{4}$	$\frac{1+2\beta^2+(\alpha^4-\beta^4)\cos(2\phi)\mp\chi}{4}$
$\downarrow$	$\frac{1+2\beta^2+(\alpha^4-\beta^4)\cos(2\phi)\pm\chi}{4}$	$\frac{1+2\alpha^2+(\beta^4-\alpha^4)\cos(2\phi)\mp\chi}{4}$

with $\chi := 2\cos(\phi)\sqrt{1-(\alpha^2-\beta^2)^2\sin^2(\phi)}$

Table 1.2: The conditional probabilities of Wigner seeing result w at time t_2 given that the Friend saw result f at time t_1 according to definition 2a. The different signs correspond to the different solutions of the quadratic equations in Eqs.(1.16). Note that there is no sensible definition of $P_{2a}\,(f \text{ when } t_1\,|\,w \text{ when } t_2)$, since the numerator in Eq.(1.15) does not depend on t_1 and $\langle\langle\Psi\,|t_1\rangle\,\langle t_1|\,\Pi^w\otimes|\,\Psi\rangle\rangle=0$.

Considering the general arguments in [41] in favor of equating measurements at different times with one collective measurement at a final time, we propose a straightforward modification of definition 2a for conditional probabilities.

Definition 2b (one-time unitary). The conditional probability of result n given m is

$$P_{2b}\,(n \text{ when } t_2\,|\,m \text{ when } t_2) = \frac{\langle\langle\Psi\,|t_2\rangle\,\langle t_2|\otimes\Pi^n\otimes\Pi^m\,|\,\Psi\rangle\rangle}{\langle\langle\Psi\,|t_2\rangle\,\langle t_2|\otimes\Pi^m\,|\,\Psi\rangle\rangle},\tag{1.17}$$

where t_2 is some time after the second measurement has been performed.

These one-time probabilities are always well-defined and shown in Table 1.3. They correspond to jointly observable statistics after full unitary evolution (i.e., without state-update) and are equivalent to the proposal in [44]. This resolves the paradox of the ambiguous probability assignment in Wigner's friend scenarios insofar as it corresponds to all observers describing all measurements unitarily up to that point where they can all compare their records, that is, the entries in F's and W's memories after both measurements took place. It is important to note that F's memory will in general be altered by W's measurement and reading it out at the end will no longer correspond to what was encoded after F's measurement. Note that for non-Wigner's friend scenarios definitions 2a and 2b are equivalent, since in this case $\langle\langle \Psi \,|t_2\rangle \langle t_2| \otimes \Pi^m \,| \Psi\rangle\rangle = \langle\langle \Psi \,|t_1\rangle \langle t_1| \otimes \Pi^m \,| \Psi\rangle\rangle$ (see Appendix 1.A). This is due to the fact that the memory, in which the result m is stored at some time t_1, undergoes no further evolution until time t_2 and after.

$P_{2b}\,(w$ when $t_2\,|\,f$ when $t_2)$

f \ w	yes	no
$\uparrow$	$\frac{\frac{\alpha^2}{\beta^2}+2\frac{b\alpha}{a\beta}\cos(\phi)+\frac{b^2}{a^2}}{N_\uparrow}$	$\frac{\frac{\beta^2}{\alpha^2}-2\frac{b\beta}{a\alpha}\cos(\phi)+\frac{b^2}{a^2}}{N_\uparrow}$
$\downarrow$	$\frac{\frac{\beta^2}{\alpha^2}+2\frac{a\beta}{b\alpha}\cos(\phi)+\frac{a^2}{b^2}}{N_\downarrow}$	$\frac{\frac{\alpha^2}{\beta^2}+2\frac{a\alpha}{b\beta}\cos(\phi)+\frac{a^2}{b^2}}{N_\downarrow}$

with $N_\uparrow := \frac{\alpha^2}{\beta^2} + \frac{\beta^2}{\alpha^2} + 2\frac{b}{a}\cos(\phi)\left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right) + 2\frac{b^2}{a^2}$
and $N_\downarrow := \frac{\alpha^2}{\beta^2} + \frac{\beta^2}{\alpha^2} + 2\frac{a}{b}\cos(\phi)\left(\frac{\beta}{\alpha} - \frac{\alpha}{\beta}\right) + 2\frac{a^2}{b^2}$

$P_{2b}\,(f$ when $t_2\,|\,w$ when $t_2)$

f \ w	yes	no
$\uparrow$	α^2	β^2
$\downarrow$	β^2	α^2

Table 1.3: The conditional probabilities for results w of Wigner and f of the Friend according to definition 2b. The joint probability is well-defined and corresponds to the numerator in Eq. (1.17).

The fact that the Friend's memory is the *only* record of her observed result and is in general affected by Wigner's measurement raises the question whether it is operationally meaningful to assign a joint probability to the results f and w. By construction of the setup these two results are in general not jointly observable by any single observer. However, as discussed in Ref. [45], in the special case of the joint state of the quantum system and the Friend being in an eigenstate of Wigner's measurement, F's memory is not affected (a *non-disturbance measurement*) and she can, in principle, learn both results. In fact, in this case Wigner can send his observed outcomes to the Friend who can combine them with her own (left undisturbed by Wigner's measurement) and construct the joint probability distribution. We, thus, propose a third rule for calculating conditional two-time probabilities, which agrees with standard quantum theory for non-Wigner's friend scenarios and gives well-defined probabilities for Wigner's friend setups, in the cases where these probabilities are operationally meaningful.

Definition 3 (two-time unitary, consistency conditions). The conditional probability (in the cases where it is well-defined) of result n at time t_2 given result m at time t_1 is

$$P_3\left(n \text{ when } t_2 \,|\, m \text{ when } t_1\right) = \frac{\langle\langle\Psi|(|t_2\rangle\langle t_2| \otimes \Pi^n)P^{\text{ph}}(|t_1\rangle\langle t_1| \otimes \Pi^m)\,|\Psi\rangle\rangle}{\langle\langle\Psi\,|t_1\rangle\langle t_1| \otimes \Pi^m\,|\Psi\rangle\rangle}. \tag{1.18}$$

$P_3\,(w$ when $t_2 \,|\, f$ when $t_1)$

$f \backslash w$	yes	no
$\uparrow$	1 (or 0)	0 (or 1)
$\downarrow$	1 (or 0)	0 (or 1)

$P_3\,(f$ when $t_1 \,|\, w$ when $t_2)$

$f \backslash w$	yes	no
$\uparrow$	α^2 (or 0)	0 (or β^2)
$\downarrow$	β^2 (or 0)	0 (or α^2)

Table 1.4: The conditional probabilities of Wigner seeing result w at t_2 and the Friend seeing result f at time t_1 (above) and of the Friend seeing f at t_1 given that Wigner will see w at t_2 (below), according to definition 3. the probabilities are well-defined only in the case where Wigner's measurement is non-disturbing, i.e. $a = \alpha$ and $b = \beta$ (or $a = \beta$ and $b = -\alpha$).

This rule gives well-defined probabilities under the condition that the measurement operators commute on the physical state when compared at the same instant of time, i.e.

$$\left[\mathcal{U}(t_1, t_2)\Pi^m\mathcal{U}^\dagger(t_1, t_2), \Pi^n\right]|\Psi\rangle\rangle = 0, \tag{1.19}$$

where $\mathcal{U}(t_2, t_1) = \langle t_2|\,P^{\text{ph}}\,|t_1\rangle$.
We note, that condition (1.19) ensures that m and n are part of a family of consistent quantum histories [46] as shown in Appendix 1.5. According to the consistent histories framework only within such a family can one meaningfully assign a set of probabilities to the properties encoded in the respective histories; in the case considered here, the results of Wigner and his Friend. This agrees with the fact that definition 3 singles out the non-disturbing measurements by Wigner in the setup depicted in Fig. 1.1 (see Appendix 1.D for explicit calculations), namely

$$\phi = n\pi \qquad \text{and} \qquad \frac{a}{\alpha} = \frac{b}{\beta}. \tag{1.20}$$

The probabilities given by definition 3 are listed in Table 1.4 and coincide with those given by definitions 2a and 2b (i.e., without the application of the state-update). In contrast to definition 2b, however, definition 3 describes genuine two-time probabilities, which, contrary to those given by definition 2a, have a clear interpretation and operational meaning. However, note that collecting these statistics is not straight-forward, for also in the non-disturbance case the Friend cannot simply send out her observed results to Wigner without changing the probabilities for Wigner's result (see Ref. [45]). Yet, Wigner can in principle send inside the laboratory his measured data, from which the Friend can construct a joint (and conditional) probability distribution.

1.5 Discussion and outlook

We have presented three generalizations of the standard rule to assign probabilities to consecutive quantum measurements in a Wigner's friend scenario using the Page-Wootters mechanism. We showed how these probability rules potentially remove, in different manners, the ambiguity between the application of unitary dynamics and the state-update rule ("collapse"). While these represent a formal resolution (in fact a few distinct ones) of the probability-assignment paradox in Wigner's friend-like experiments, a number of key issues remain open: There are potentially more than just the three probability rules we presented, possibly leading to other resolutions of the paradox. Is there a way of classifying all of them? Furthermore, what is the operational meaning of each of the expressions that formally give probabilities?

Concerning the first question, we ought to stress that the only necessary condition that we have required from our definitions of two-time probabilities was, in fact, that they have to reduce to the standard Born rule in the non-Wigner's friend case, and that they give well-defined probabilities for at least certain cases of a Wigner's friend scenario. However, one can imagine further possible definitions that comply with these requirements, hence, a more systematic study would be desirable. In particular, one extension of the original proposal of Page and Wootters put forward by Gambine *et al.* is to compute conditional probabilities between averages over 'evolving constants of motion',[2] which would be interesting to examine in Wigner's friend setups.

To address the second question, we now discuss the interpretation and the limits of each definition of conditional probabilities presented in the last section (see Table 1.5 for a summary of the definitions and their properties, and Fig. 1.2 for a sketched graphical representation). As already noticed, definition 1 has a clear interpretation in terms of "objective collapse" probabilities, but does not give rise to a well-defined joint probability (see Sec. 1.4). However, one could maintain that at the operational level conditional probabilities come prior to joint probabilities, which are *a posteriori* constructions.

On the other hand, definition 2b gives measurement statistics consistent with standard quantum theory assuming S, F and W evolve unitarily and the Born rule is applied after W's measurement interaction has occurred. While this definition also has a clear interpretation, it fails in representing genuine two-time probabilities

[2]More precisely, one would construct a family of Dirac observables, which commute with the constraint in Eq. (1.1), comprised of a partial observables associated with a clock and observables on Wigner's and his friend's memories. Such a family of Dirac observables is parametrized by the times read by the clock. Conditional probabilities would be computed as in Eq. (1.4), with the important difference that the numerator and denominator are averaged over all readings of the clock. See Ref. [47] for more detail.

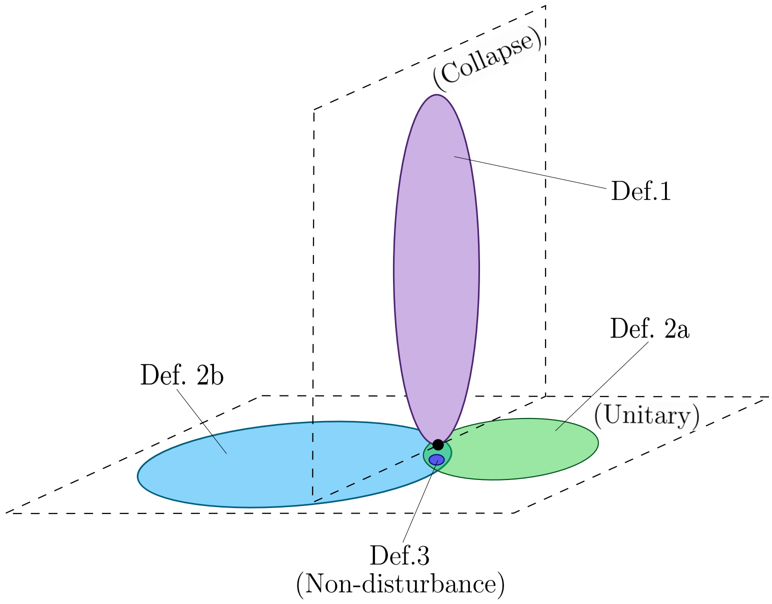

Figure 1.2: Suggestive graphical representation of the probabilities according to the different definitions in a Wigner's friend scenario (when they are well-defined). For non Wigner's friend scenarios all the definitions recover standard quantum probabilities, thus they fully overlap. The two planes distinguish between joint probabilities that are in accordance with applying (collapse) or not applying (unitary) the state-update rule after the Friend's measurement. In the non-disturbance case (i.e., $\phi = n\pi$ and $a/\alpha = \pm b/\beta$) definition 2a, definition 2b and definition 3 give the same unitary, conditional probabilities, that are different from the collapse ones given by definition 1. There is, however, at least one instance where definitions 1, 2a, and 2b coincide (i.e., $\phi = (n + 1/2)\pi$, $a = b = 1/\sqrt{2}$ and arbitrary α.

(only the final time t_2 appears in the rule to assign conditional probabilities for the measurements of F and W).

Definition 3 has the merit of being interpretable as a genuine two-time probability, however it is a well-defined probability (i.e. real and non-negative) only under the rather demanding "non-disturbance measurement" conditions, that is, when Wigner measures in a basis that contains the joint state of the system and the Friend (see Table 1.5). In this case, the probabilities are the same as definition 2b. The condition required for definition 3 to result in genuine probabilities can be interpreted as evidence that only in the non-disturbance case there exists a record of the past outcome f of F's measurement and the current outcome w of W's measurement. Thus, only in this case is a joint probability distribution associated with the measurement outcomes of W and F operationally meaningful, since it corresponds to collectible statistics for the friend. Remarkably, this limitation to the meaningful attribution of a joint probability of the outcomes of F and W emerges naturally from our formalism. This suggests a connection with a recent no-go theorem [3, 7] stating that it is in general not possible to construct a joint probability distribution associated with measurement outcomes of different observers in a (more complicated) Wigner's friend scenario. Moreover, we have shown that the conditions for definition 3 to give well-defined probabilities ensure that the measurement outcomes of Wigner and his Friend are elements of a family of consistent quantum histories [46, 48] (see Appendix 1.C).

Finally, definition 2a gives well-defined (i.e., normalized) probabilities only

	standard QT	positive	normalized	two-time	sym. joint	Conditions
Def. 1	✓	✓	✓	✓	✗	
Def. 2a	✓	✓	✗	✓	✓	normalized, if $\alpha^4+\beta^4+2\cos(\phi)(\alpha^3\beta-\alpha\beta^3)\frac{b}{a}+2\alpha^2\beta^2\left(\frac{b}{a}\right)^2=1$ and $\alpha^4+\beta^4-2\cos(\phi)(\alpha^3\beta-\alpha\beta^3)\frac{a}{b}+2\alpha^2\beta^2\left(\frac{a}{b}\right)^2=1$.
Def. 2b	✓	✓	✓	✗	✓	
Def. 3	✓	✗	✓	✓	✗	positive for non-disturbance measurement: $a=\alpha, b=\beta$ or $a=\beta, b=-\alpha$. And $\phi=n\pi$.

Table 1.5: Comparison of the different proposed conditional probability rules for describing Wigner's friend experiments. From the left to the right, the columns indicate whether each definition: (i) reduces to standard quantum theory for non-Wigner's friend scenarios; (ii) is positive; (iii) is normalized; (iv) is a genuine two-time expression; (v) gives rise to a well-defined joint probability distribution (i.e. $P(A)P(B|A)=P(B)P(A|B)$). Note, in fact, that definitions 2a and 3 become probabilities only under certain conditions, the blue ✗ indicates that the respective property is satisfied under certain conditions but not in general. The last column on the right shows these conditions (if any) under which the respective properties are satisfied.

under the conditions in Eq. (1.16). Note that the non-disturbance measurement satisfies these conditions (see Fig. 1.2), meaning that if Eq. (1.20) is satisfied, so is Eq. (1.16). In general, however, it is not clear to us how to interpret the probabilities given by definition 2a operationally. It seems that formally for any given initial state of the quantum system and fixed measurement of the Friend, there always exists a measurement choice of Wigner (besides the non-disturbance one) which ensures that definition 2a gives well-defined probabilities. However, we do not have an intuition of what is special about the measurement choices that satisfy Eqs. (1.16). Moreover, the probabilities provided by definition 2a (see Table 1.2) do not have a straightforward interpretation since they do not correspond to either "collapse" or full unitary evolution up to the final measurement of Wigner and unlike for definition 3 we do not find an operational principle distinguishing them. Again, a systematic study of all possible probability rules that reduce to standard quantum probabilities for non-Wigner's friend scenarios might give further insight. We leave it up for future work to do so.

Moreover, it will be fruitful to explicitly construct maps between the perspectives of Wigner and his friend [49] using the timeless description of the setup developed here.

In contrast to the standard quantum formalism –where there is a dichotomy between the assignment of probabilities at the moment of measurements and unitary evolution– the timeless formulation of quantum mechanics places the rule to assign probabilities to measurement outcomes (Born rule) logically prior to unitary dynamics. Thus, the timeless formulation is not directly concerned with the "evolution of quantum states" (in particular the transition from before to after a measurement), making it an appropriate tool to deal with the ambiguity of the standard formalism in describing quantum measurements.

As long as one considers typical quantum scenarios (i.e., a system measured one or more times), there have been at least two different, yet equivalent, proposal in the timeless formulation on how to assign probabilities to the outcomes of measurements [43, 42]. However, we have shown that to deal with Wigner's friend scenarios these two-time probabilities rules ought to be adapted. We presented two such adaptations and considered a third proposal, all of which resolve the ambiguity of Wigner's friend probability paradox in different ways. Thus, in this framework, the ambiguity between the two dynamical processes (i.e., unitary evolution versus "collapse") is pushed back to the choice of the probability-assignment rule: Finding compelling physical arguments to rule out all but one of the proposed two-time probability-assignments would significantly contribute to solving the Wigner's friend paradox.

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Appendix 1.A: Reproducing standard quantum theory in non-Wigner's friend setups

For non-Wigner's friend scenarios the two measurements are performed on the same quantum system instead of one of them measuring both the system and an observer. The constraint Hamiltonian then takes the form

$$\hat{H}' = \hat{P}_t + H_S + \delta(\hat{T} - t_M)\hat{K}_{SM} + \delta(\hat{T} - t_N)\hat{K}_{SN}, \tag{1.21}$$

where now M and N are apparatus or memories, in which the results of the respective measurements are encoded. This gives the physical state

$$
\begin{aligned}
|\Psi'\rangle\rangle = &\int_{-\infty}^{t_M} dt\ |t\rangle\, U_S(t,t_0)\, |\psi_S(t_0)\rangle\, |R_M\rangle\, |R_N\rangle + \\
&\int_{t_M}^{t_N} dt\ |t\rangle \sum_m U_S(t,t_M)\Pi_m\, |\psi_S(t_M)\rangle\, |m_M\rangle\, |R_N\rangle + \\
&\int_{t_N}^{\infty} dt\ |t\rangle \sum_{m,n} \big[U_S(t,t_N)\Pi_n U_S(t_N,t_M)\Pi_m\, |\psi_S(t_M)\rangle\, |m_M\rangle\, |n_N\rangle\big], \quad (1.22)
\end{aligned}
$$

with both Π_m and Π_n acting on $\mathcal{H}_S$. Moreover, we have

$$
\langle t|\, P^{\mathrm{ph}}\, |t_0\rangle\, |\phi(t_0)\rangle = \mathcal{U}(t,t_0)\, |\phi(t_0)\rangle \quad (1.23)
$$

for arbitrary $|\phi(t_0)\rangle \in \mathcal{H}_S \otimes \mathcal{H}_M \otimes \mathcal{H}_N$, where

$$
\mathcal{U}(t,t_0) = \begin{cases} U_S(t,t_0) & t_0 < t \\ U_S(t,t_M)U_M U_S(t_M,t_0) & t_0 < t_M < t \\ U_S(t,t_N)U_N U_S(t_N,t_0) & t_0 < t_N < t \\ U_S(t,t_N)U_N U_S(t_N,t_M)U_M U_S(t_M,t_0) & t_0 < t_M < t_N < t \end{cases} \quad (1.24)
$$

with $U_M = e^{-i\hat{K}_{SM}}$ and $U_N = e^{-i\hat{K}_{SN}}$ being the measurement unitaries that entagle the measured system with the respective memories:

$$
U_X\, |\psi_S\rangle\, |R_X\rangle = \sum_x \Pi_x\, |\psi_S\rangle\, |x_X\rangle\,,
$$

and $(x,X) \in \{(m,M),(n,N)\}$.

According to definition 1 the conditional probability of result n at time $t_2 \geq t_N$ given result m at time $t_1 \geq t_M$ is

$$
\frac{\langle\langle\Psi'\,|\,t_1\rangle\Pi^m\,\langle t_1|P^{\mathrm{ph}}|t_2\rangle\,\Pi^n\,\langle t_2|P^{\mathrm{ph}}|t_1\rangle\,\Pi^m\,\langle t_1|\Psi'\rangle\rangle}{\langle\langle\Psi'\,|\,t_1\rangle\Pi^m\,\langle t_1|\Psi'\rangle\rangle}, \quad (1.25)
$$

with Π^m and Π^n acting on $\mathcal{H}_M$ and $\mathcal{H}_N$ respectively. From Eq. (1.22) we see that $|\phi(t_1)\rangle := \langle t_1|\Psi'\rangle\rangle = \sum_{m'} U_S(t_1,t_M)\Pi_{m'}\,|\psi_S(t_M)\rangle\,|m'_M\rangle\,|R_N\rangle$, and the denominator in Eq.(1.25) is

$$
\begin{aligned}
\langle\phi(t_1)|\Pi^m|\phi(t_1)\rangle &= \sum_{m',m''} \langle m''_M|\Pi^m|m'_M\rangle\, \langle\psi_S(t_M)|\, \Pi_{m''}\Pi_{m'}\, |\psi_S(t_M)\rangle \\
&= |\,\langle m|\psi_S(t_M)\rangle\,|^2. \quad (1.26)
\end{aligned}
$$

Moreover, since

$$
\mathcal{U}(t_2,t_1)\Pi^m\,|\phi(t_1)\rangle = \sum_{n'} U_S(t_2,t_N)\Pi_{b'}U_S(t_N,t_M)\Pi_m\,|\psi_S(t_M)\rangle\,|m_M\rangle\,|n'_N\rangle\,, \quad (1.27)
$$

the numerator in Eq.(1.25) gives

$$
\begin{aligned}
&\langle\phi(t_1)|\Pi^m\mathcal{U}^\dagger(t_2,t_1)\Pi^n\mathcal{U}(t_2,t_1)\Pi^m|\phi(t_1)\rangle \\
&\quad=\sum_{n',n''}\langle n''_N|\Pi^n|n'_N\rangle\,\langle\psi_S(t_M)|\,\Pi_m U_S(t_M,t_N) \\
&\quad\cdot\Pi_{n''}U_S(t_N,t_N)\Pi_{n'}U_S(t_N,t_M)\Pi_m\,|\psi_S(t_M)\rangle \\
&\quad=\langle\psi_S(t_M)|\Pi_m U_S(t_M,t_N)\Pi_n U_S(t_N,t_M)\Pi_m|\psi_S(t_M)\rangle \\
&\quad=|\,\langle n|U_S(t_N,t_M)|m\rangle\,|^2|\,\langle m|\psi_S(t_M)\rangle\,|^2
\end{aligned}
\tag{1.28}
$$

and, hence,

$$
P_1\,(n\text{ when }t_2\,|\,m\text{ when }t_1)=|\,\langle n|U_S(t_N,t_M)|m\rangle\,|^2,
\tag{1.29}
$$

which is the standard quantum probabilities for two subsequent measurements on a quantum system S.

In case of non-Wigner's friend scenarios definition 2a and definition 2b are equal since

$$
\begin{aligned}
\langle\langle\Psi'\,|\,t_2\rangle\Pi^m\,\langle t_2|\Psi'\rangle\rangle &=\sum_{\substack{m',n\\ m'',n'}}\Big[\langle n'_N|n_N\rangle\,\langle m''_M|\Pi^m|m'_M\rangle\,\langle\psi_S(t_M)|\,\Pi_{m''}U_S(t_M,t_N) \\
&\qquad\cdot\Pi_{n'}U_S(t_N,t_2)U_S(t_2,t_N)\Pi_n U_S(t_N,t_M)\Pi_{m'}\,|\psi_S(t_M)\rangle\Big] \\
&=\langle\psi_S(t_M)|\,\Pi_m U_S(t_M,t_N)\left(\sum_n\Pi_n\right)U_S(t_N,t_M)\Pi_m\,|\psi_S(t_M)\rangle \\
&=|\,\langle m|\psi_S(t_M)\rangle\,|^2 \\
&=\langle\langle\Psi'\,|\,t_1\rangle\Pi^m\,\langle t_1|\Psi'\rangle\rangle.
\end{aligned}
$$

From Eq. (1.22) we get that

$$
|\phi(t_2)\rangle:=\langle t_2|\Psi'\rangle\rangle=\sum_{m',n'}U_S(t,t_N)\Pi_{n'}U_S(t_N,t_M)\Pi_{m'}\,|\psi_S(t_M)\rangle\,|m'_M\rangle\,|n'_N\rangle\,,
$$

and the numerator in Defs. 2 gives

$$
\begin{aligned}
&\langle\langle\Psi'\,|t_2\rangle\,\langle t_2|\otimes\Pi^n\otimes\Pi^m\,\big|\,\Psi'\rangle\rangle=\langle\phi(t_2)|\Pi^n\otimes\Pi^m|\phi(t_2)\rangle \\
&\quad=\sum_{\substack{m',n'\\ m'',n''}}\Big[\langle n''_N|\Pi^n|n'_N\rangle\,\langle m''_M|\Pi^m|m'_M\rangle\,\langle\psi_S(t_M)| \\
&\quad\cdot\Pi_{m''}U_S(t_M,t_N)\Pi_{n''}\Pi_{n'}U_S(t_N,t_M)\Pi_{m'}\,|\psi_S(t_M)\rangle\Big] \\
&\quad=|\,\langle b|U_S(t_N,t_M)|m\rangle\,|^2|\,\langle m|\psi_S(t_M)\rangle\,|^2
\end{aligned}
\tag{1.30}
$$

Therefore, both Defs. 2 give the standard quantum probabilities

$$
\begin{aligned}
P_{2a}\,(n\text{ when }t_2\,|\,m\text{ when }t_1)=|\,\langle n|U_S(t_N,t_M)|m\rangle\,|^2&= \\
P_{2b}\,(n\text{ when }t_2\,|\,m\text{ when }t_2)&.
\end{aligned}
$$

According to definition 3 the conditional probability of result n at time $t_2\geq t_N$ given result m at time $t_1\geq t_M$ is

$$
\frac{\langle\langle\Psi'\,|\,t_1\rangle\Pi^m\,\langle t_1|P^{\mathrm{ph}}|t_2\rangle\,\Pi^n\,\langle t_2|\Psi'\rangle\rangle}{\langle\langle\Psi'\,|\,t_1\rangle\Pi^m\,\langle t_1|\Psi'\rangle\rangle}.
\tag{1.31}
$$

The denominator is the same as in definition 1 and definition 2a and given by Eq. (1.26). The numerator of Eq. (1.31) gives

$$\begin{aligned}
\langle\phi(t_1)|\Pi^m\mathcal{U}(t_1,t_2)\Pi^n|\phi(t_2)\rangle = \langle R_N|\,\langle m_M|\,\langle\psi_S(t_M)|\,\Pi_m U_S(t_M,t_N)U_N^\dagger U_S(t_N,t_2)\cdot \\
\sum_{m'} U_S(t_2,t_N)\Pi_n U_S(t_N,t_M)\Pi_{m'}\,|\psi_S(t_M)\rangle\,|m'_M\rangle\,|n_N\rangle = \\
\sum_{m',n'} \langle n'_N|n_N\rangle\,\langle m_M|m'_M\rangle\,\langle\psi_S(t_M)|\,\Pi_m U_S(t_M,t_N)\Pi_{n'}\cdot \\
\Pi_n U_S(t_N,t_M)\Pi_{m'}\,|\psi_S(t_M)\rangle = \\
|\,\langle n|U_S(t_N,t_M)|m\rangle\,|^2|\,\langle m|\psi_S(t_M)\rangle\,|^2.
\end{aligned} \tag{1.32}$$

Hence, also definition 3 gives the standard quantum probabilities for non-Wigner's friend scenarios.

Appendix 1.B: Conditions for Definition 2a

$$\frac{\langle\langle\boldsymbol{\Psi}|\boldsymbol{t_2}\rangle\langle\boldsymbol{t_2}|\otimes\boldsymbol{\Pi^w}\otimes\boldsymbol{\Pi^f}|\,\boldsymbol{\Psi}\rangle\rangle}{\langle\langle\boldsymbol{\Psi}|\boldsymbol{t_1}\rangle\langle\boldsymbol{t_1}|\otimes\boldsymbol{\Pi^f}|\,\boldsymbol{\Psi}\rangle\rangle}$$

	yes	no
$\uparrow$	$\frac{a^2\alpha^4+2ab\alpha^3\beta\cos(\phi)+b^2\alpha^2\beta^2}{a^2}$	$\frac{a^2\beta^4-2ab\alpha\beta^3\cos(\phi)+b^2\alpha^2\beta^2}{a^2}$
$\downarrow$	$\frac{b^2\beta^4+2ab\alpha\beta^3\cos(\phi)+a^2\alpha^2\beta^2}{b^2}$	$\frac{b^2\alpha^4-2ab\alpha^3\beta\cos(\phi)+a^2\alpha^2\beta^2}{b^2}$

Table 1.6: Definition 2a evaluated for arbitrary a, b, α, β and $\phi = \phi_S - \phi_{SF}$. These expressions constitute possible conditional probabilities of Wigner seeing result $w \in \{\text{yes}, \text{no}\}$ at time t_2 given that the Friend saw result $f \in \{\uparrow,\downarrow\}$ at time t_1 only if conditions (1.33) and (1.34) are both satisfied.

Evaluating definition 2a for the setup in Fig. 1.1 gives the terms listed in Table 1.6. Normalization requires that $\forall f : \sum_w \frac{\langle\langle\Psi|t_2\rangle\langle t_2|\otimes\Pi^w\otimes\Pi^f|\,\Psi\rangle\rangle}{\langle\langle\Psi|t_1\rangle\langle t_1|\otimes\Pi^f|\,\Psi\rangle\rangle} = 1$, which gives the following conditions

$$\alpha^4+\beta^4+2\cos(\phi)(\alpha^3\beta-\alpha\beta^3)\frac{b}{a}+2\alpha^2\beta^2\left(\frac{b}{a}\right)^2 = 1 \tag{1.33}$$

$$\alpha^4+\beta^4-2\cos(\phi)(\alpha^3\beta-\alpha\beta^3)\frac{a}{b}+2\alpha^2\beta^2\left(\frac{a}{b}\right)^2 = 1. \tag{1.34}$$

These quadratic equations in $\frac{b}{a}$ and $\frac{a}{b}$ have solutions

$$\left(\frac{b}{a}\right)_\pm = \frac{-\cos\phi(\alpha^2-\beta^2)\pm\sqrt{1-\sin^2\phi(\alpha^2-\beta^2)^2}}{2\alpha\beta}$$

and

$$\left(\frac{a}{b}\right)_\pm = \frac{\cos\phi(\alpha^2-\beta^2)\pm\sqrt{1-\sin^2\phi(\alpha^2-\beta^2)^2}}{2\alpha\beta},$$

which exist for any α, β and ϕ. However, requiring that $\left(\frac{b}{a}\right)_\pm = 1/\left(\frac{a}{b}\right)_\pm$ has solutions only when combining either the two "+" or the two "−" solutions. These combinations are the common solutions to both Eqs. (1.33) and (1.34) and, hence, give normalized probabilities for definition 2a.

Appendix 1.C: Consistent histories for Wigner's-friend setups

In the consistent histories framework sequences of physical properties are assigned to a closed quantum system. These sequences are represented by tensor products of orthogonal projectors (i.e. a *quantum history*)

$$Y^{\mathrm{i}} = \rho_0 \otimes P_1^{\mathrm{i}_1} \cdots \otimes P_f^{\mathrm{i}_\mathrm{f}}, \tag{1.35}$$

where ρ_0 is the initial state and each $P_k^{\mathrm{i}_\mathrm{k}}$ corresponds to some physical property at a certain time k. A consistent family of histories is a complete set of histories $\{Y^i\}$ which satisfy the consistency condition

$$\mathrm{tr}\left(K^\dagger(Y^{\mathrm{i}})\rho_0 K(Y^{\mathrm{i}'})\right) = 0 \quad \text{for } \mathrm{i} \neq \mathrm{i}' \tag{1.36}$$

where $\mathrm{i} = (\mathrm{i}_1 \dots \mathrm{i}_\mathrm{f})$ and K is the so called chain operator defined by

$$K(Y^{\mathrm{i}}) = P_0^{\mathrm{i}_1} \cdot P_0^{\mathrm{i}_2} \cdots \cdot P_0^{\mathrm{i}_\mathrm{f}}, \tag{1.37}$$

with $P_0^{\mathrm{i}_\mathrm{k}} = U(t_0, t_k) P_k^{\mathrm{i}_\mathrm{k}} U(t_k, t_0)$. Only within a consistent family the dynamics of quantum theory describe the respective properties over time.

For the simple Wigner's-friend setup in Fig. 1.1, these properties are the results observed by Wigner and his Friend, $\mathrm{i} = (\mathrm{f}, \mathrm{w})$ and

$$\rho_0 = |R_W\rangle\,|R_F\rangle\,|\psi_S\rangle\langle\psi_S|\,\langle R_F|\,\langle R_W|\,.$$

Condition (1.19) –under which definition 3 gives proper probabilities– implies that

$$\mathrm{tr}\left(K^\dagger(Y^{(f,w)})\rho_0 K(Y^{(f',w')})\right) = \delta_{ff'}\delta_{ww'}\,\mathrm{tr}\left(\rho(t_2)\Pi^w U(t_2,t_1)\Pi^f U(t_1,t_2)\right), \tag{1.38}$$

where $\rho(t_2) = \mathcal{U}(t_2,t_0)\rho_0\mathcal{U}(t_0,t_2)$. Hence, in this case, the consistency condition is satisfied.

The solutions to the conditions on definition 2a, however, in general do *not* satisfy Eq. (1.36). Consider the concrete counterexample of $|\psi_S\rangle = \sqrt{\frac{1}{2}}(|\uparrow\rangle + |\downarrow\rangle)$ and $|\mathrm{yes}\rangle = \alpha\,|\uparrow,\uparrow\rangle + i\beta\,|\downarrow,\downarrow\rangle$. In this case one obtains

$$\mathrm{tr}\left(K^\dagger(Y^i)\rho_0 K(Y^{i'})\right) = \pm\delta_{ww'}\frac{i}{2}\alpha\beta \quad \text{for } f \neq f', \tag{1.39}$$

which still satisfies the so called weak consistency condition

$$\mathrm{Re}\left[\mathrm{tr}\left(K^\dagger(Y^i)\rho_0 K(Y^{i'})\right)\right] = 0.$$

In contrast to the consistency condition of (1.36), however, this has been shown to be highly problematic concerning trivial combination of independent subsystems as well as dynamical stability, see Ref. [50].

$$\frac{\langle\langle \boldsymbol{\Psi}(|\boldsymbol{t_2}\rangle\langle \boldsymbol{t_2}|\otimes\boldsymbol{\Pi^w})\boldsymbol{P^{\mathrm{ph}}}(|\boldsymbol{t_1}\rangle\langle \boldsymbol{t_1}|\otimes\boldsymbol{\Pi^f})|\,\boldsymbol{\Psi}\rangle\rangle}{\langle\langle \boldsymbol{\Psi}|\boldsymbol{t_1}\rangle\langle \boldsymbol{t_1}|\otimes\boldsymbol{\Pi^f}|\,\boldsymbol{\Psi}\rangle\rangle}$$

	yes	no
$\uparrow$	$\alpha^2 + \frac{b}{a}\alpha\beta e^{-i(\phi)}$	$\beta^2 - \frac{b}{a}\alpha\beta e^{-i(\phi)}$
$\downarrow$	$\beta^2 + \frac{a}{b}\alpha\beta e^{-i(\phi)}$	$\alpha^2 - \frac{a}{b}\alpha\beta e^{-i(\phi)}$

$$\frac{\langle\langle \boldsymbol{\Psi}(|\boldsymbol{t_2}\rangle\langle \boldsymbol{t_2}|\otimes\boldsymbol{\Pi^w})\boldsymbol{P^{\mathrm{ph}}}(|\boldsymbol{t_1}\rangle\langle \boldsymbol{t_1}|\otimes\boldsymbol{\Pi^f})|\,\boldsymbol{\Psi}\rangle\rangle}{\langle\langle \boldsymbol{\Psi}|\boldsymbol{t_2}\rangle\langle \boldsymbol{t_2}|\otimes\boldsymbol{\Pi^w}|\,\boldsymbol{\Psi}\rangle\rangle}$$

	yes	no
$\uparrow$	$\frac{a\alpha}{a\alpha+b\beta e^{-i(\phi)}}$	$\frac{b\beta}{b\beta+a\alpha e^{i(\phi)}}$
$\downarrow$	$\frac{a^2\beta^2-ab\alpha\beta e^{i(\phi)}}{a^2\beta^2+b^2\alpha^2-2ab\alpha\beta\cos(\phi)}$	$\frac{b^2\alpha^2-ab\alpha\beta e^{-i(\phi)}}{a^2\beta^2+b^2\alpha^2-2ab\alpha\beta\cos(\phi)}$

Table 1.7: Definition 3 evaluated for arbitrary a, b, α, β and ϕ possibly constituting the conditional probabilities of Wigner seeing result $w \in \{\text{yes}, \text{no}\}$ at time t_2 given that the Friend saw result $f \in \{\uparrow, \downarrow\}$ at time t_1 and that of the Friend seeing f at t_1 given that Wigner will see w at t_2. These expressions are real and positive and, hence, probabilities only if conditions (1.42) and either (1.45) or (1.46) are satisfied.

Appendix 1.D: Conditions for Definition 3

Evaluating definition 3 for the setup in Fig. 1.1 gives the terms listed in Table 1.7. Those expressions are in general not probabilities, since they are neither necessarily real nor positive although they do add up to one. For them to be real numbers we require that

$$\phi = n\pi. \tag{1.40}$$

For all the terms in Table 1.7 to be positive, of the following conditions one each must hold. If either

$$\alpha^2 - \frac{a}{b}\alpha\beta \geq 0 \quad \text{and} \quad \beta^2 - \frac{b}{a}\alpha\beta \geq 0 \tag{1.41}$$

or

$$\alpha^2 - \frac{b}{a}\alpha\beta \geq 0 \quad \text{and} \quad \beta^2 - \frac{a}{b}\alpha\beta \geq 0, \tag{1.42}$$

the conditional probabilities for result w at t_2 given result f at t_1 are well-defined. And if, either

$$1 - \frac{b\alpha}{a\beta} \geq 0 \quad \text{and} \quad 1 - \frac{a\beta}{b\alpha} \geq 0 \tag{1.43}$$

or

$$1 - \frac{b\beta}{a\alpha} \geq 0 \quad \text{and} \quad 1 - \frac{a\alpha}{b\beta} \geq 0, \tag{1.44}$$

the conditional probabilities for result f at t_1 given result w at t_2 are well-defined. The solutions to both cases are

$$a = \alpha \quad , \quad b = \beta \tag{1.45}$$

or

$$a = \beta \quad , \quad b = -\alpha, \tag{1.46}$$

which means Wigner's measurement is aligned with the initial state. The joint system of F and S is in an eigenstate of W's observable, i.e., the measurement is non disturbing.

Chapter 2

A no-go theorem for the persistent reality of Wigner's friend's perception

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I gave conceptual contributions to this work and contributed to writing the paper.

Abstract

The notorious Wigner's friend thought experiment (and modifications thereof) has in recent years received renewed interest especially due to new arguments that force us to question some of the fundamental assumptions of quantum theory. In this paper, we formulate a no-go theorem for the persistent reality of Wigner's friend's perception, which allows us to conclude that the perceptions that the friend has of her own measurement outcomes at different times cannot "share the same reality", if seemingly natural quantum mechanical assumptions are met. More formally, this means that, in a Wigner's friend scenario, there is no joint probability distribution for the friend's perceived measurement outcomes at two different times, that depends linearly on the initial state of the measured system and whose marginals reproduce the predictions of unitary quantum theory. This theorem entails that one must either (1) propose a nonlinear modification of the Born rule for two-time predictions, (2) sometimes prohibit the use of present information to predict the future –thereby reducing the predictive power of quantum theory– or (3) deny that unitary quantum mechanics makes valid single-time predictions for all observers. We briefly discuss which of the theorem's assumptions are more likely to be dropped within various popular interpretations of quantum mechanics.

2.1 Introduction

One of the most puzzling scenarios one can encounter in quantum physics is the so-called Wigner's friend thought experiment [4, 51]. It allows to investigate the applicability of the quantum formalism beyond its usual limits, by considering a certain physical system –called “the friend”– simultaneously as a quantum system and as a user of quantum theory (observer). In this thought experiment, a superobserver (Wigner) describes, using the pure quantum formalism, his friend who is performing a quantum measurement on a spin system. After the friend's measurement has taken place, we are in a counter-intuitive situation where Wigner describes the friend in a quantum superposition of observing two different outcomes, while from the friend's perspective a definite outcome must be perceived.

There has been a number of recent works that cast new light onto this thought experiment [3, 7, 52, 44, 11, 53, 8, 54, 55, 56, 57, 10, 9, 58, 59], many of which originated as reactions to a paper by Frauchiger and Renner [6]. The latter work can be regarded as showing that, in quantum mechanics, it may be problematic to treat observational knowledge of other agents as if it were one's own, and to logically compare such indirect knowledge with that gained through direct observation. In the words of these authors, in a scenario where “multiple agents have access to different pieces of information, and draw conclusions by reasoning about the information held by others”, it can be shown that, “in the general context of quantum theory, the rules for such nested reasoning may be ambiguous” [6]; a conclusion that is reminescent of the QBist interpretation of quantum theory [60, 61]. Other works [3, 7, 8, 10], show that a no-go result can be obtained when the assumption that superobservers can treat other observer's outcomes as “facts of the worlds” is combined with a locality assumption. What all of the above-cited works have in common is that they reach their no-go results by combining the observations of multiple observers as if those all belonged to the same “classical reality”.

In the present paper, we put forward a *no-go theorem for the persistent reality of Wigner's friend's perception* that has perhaps more counter-intuitive and drastic consequences: Even a single observer, when making predictions about his or her observations at two different times, can conflict with the linear dependence of quantum mechanical probabilities on the density operator of the system being measured. This will occur if said observer is subjected to a measurement by a superobserver between these two times and uses unitary quantum mechanics (i.e. “no-collapse” quantum mechanics) to make his or her predictions. Our result is in line with a different understanding of the Frauchiger-Renner argument, where it is taken to primarily show that inferences made on the basis of a quantum state assigned at a certain time are not necessarily valid at later times, especially not if “someone Hadamards your brain” in between [62].

Indeed, one conclusion that can be drawn from this no-go theorem is that treating a piece of information from the past as if it was still presently existing (even when one takes into account a possible subjective uncertainty) cannot in general be upheld together with the conjunction of the above seemingly natural assumptions within the domain of quantum theory. We will show that in a particular instance of the Wigner's friend experiment, our assumptions imply that, even in cases where the theory states that no change would take place in the quantum state of the friend's laboratory, the perceptions of the friend have a non-zero probability to change from before to after Wigner's measurement. Finally, we will briefly discuss how different interpretations of quantum mechanics might comply with the no-go theorem, by identifying which of the assumptions are most likely to be abandoned within these interpretations.

2.2 Wigner's friend thought experiment

Let us begin by reviewing the Wigner's friend thought experiment within a unitary formalism to set up some basic notation, and to clarify what we mean by *unitary quantum mechanics* for the particular scenario that we consider. It is common to assume that the following description provided by unitary quantum mechanics is empirically adequate in all situations: any observer which uses this formalism will predict probabilities that match the relative frequencies that would be observed if the experiment was repeated many times.

The experiment features an observer, the friend (F), performing measurements on a qubit (e.g., a spin-1/2 particle), the system (S), in a sealed laboratory. The system is initialized in the state $|\psi\rangle_S = \alpha\,|\uparrow\rangle_S + \beta\,|\downarrow\rangle_S$, where α and β are complex numbers that obey $|\alpha|^2 + |\beta|^2 = 1$, and the possible outcomes of the measurement are recorded by the friend as U or D, respectively standing for "up" and "down". Another (super-)observer, Wigner (W), located outside the laboratory, performs a measurement on both the system and the friend. The initial state of the friend (which encompasses any other possible degree of freedom in the isolated lab) is known by Wigner, and is initially in a macroscopic "ready" state $|0\rangle_F$. The state of Wigner himself is also in a macroscopic "ready" state $|0\rangle_W$.

The initial state (at time t_0) of the whole setup is therefore

$$|\Psi(t_0)\rangle = |\psi\rangle_S|0\rangle_F|0\rangle_W = (\alpha\,|\uparrow\rangle_S + \beta\,|\downarrow\rangle_S)\,|0\rangle_F|0\rangle_W. \tag{2.1}$$

At time t_1, the friend measures the spin in the z-basis, and the state becomes

$$|\Psi(t_1)\rangle = (\alpha\,|\uparrow\rangle_S\,|U\rangle_F + \beta\,|\downarrow\rangle_S\,|D\rangle_F)\,|0\rangle_W, \tag{2.2}$$

where the states $|U\rangle_F\,, |D\rangle_F$ correspond to the friend having observed outcome "up" or "down" respectively. Later, at time t_W, Wigner measures the friend and system in some entangled basis, with binary outcomes [1] corresponding to the orthogonal states

$$\begin{aligned}|1\rangle_{SF} &= a\,|\uparrow, U\rangle + b\,|\downarrow, D\rangle\\ |2\rangle_{SF} &= b^*\,|\uparrow, U\rangle - a^*\,|\downarrow, D\rangle\,,\end{aligned}$$

with a, b being complex numbers obeying $|a|^2 + |b|^2 = 1$. At a slightly later time $t_2 > t_W$, the measurement is over and we have the final state

$$\begin{aligned}|\Psi(t_2)\rangle =&(\alpha a^* + \beta b^*)|1\rangle_{SF}|1\rangle_W + (\alpha b - \beta a)|2\rangle_{SF}|2\rangle_W\\ =&a(\alpha a^* + \beta b^*)\,|\uparrow\rangle_S\,|U\rangle_F|1\rangle_W\\ &+ b(\alpha a^* + \beta b^*)\,|\downarrow\rangle_S\,|D\rangle_F|1\rangle_W\\ &+ b^*(\alpha b - \beta a)\,|\uparrow\rangle_S\,|U\rangle_F|2\rangle_W\\ &- a^*(\alpha b - \beta a)\,|\downarrow\rangle_S\,|D\rangle_F|2\rangle_W,\end{aligned} \tag{2.3}$$

where $|1\rangle_W$ and $|2\rangle_W$ are pure quantum states corresponding to Wigner seeing the outcome "1" or "2" respectively. Note that the state $|\Psi(t_2)\rangle$ depends on the specific unitary realization of Wigner's measurement; different purifications can lead to different states $|\Psi(t_2)\rangle$.

[1] Strictly speaking, there should be two other outcomes corresponding to the rank-2 projector $|\uparrow, D\rangle\langle\uparrow, D| + |\downarrow, U\rangle\langle\downarrow, U|$, but these outcomes are never actualized in the experiment.

Using the states in Eqs. (2.1)-(2.3) and the Born rule, one can find the expected statistics for any of the friend's or Wigner's measurement outcomes using unitary quantum mechanics. This is achieved by applying a projector Π_x onto the state where the respective observer is seeing outcome x to the state at the time of interest, i.e.

$$p(x) = \mathrm{tr}\left(\Pi_x |\Psi(t)\rangle\langle\Psi(t)|\right), \tag{2.4}$$

where, in the case relevant for this work where there are two outcomes U and D, the probability of the friend seeing for example outcome $x = U$ is obtained with $\Pi_U = |U\rangle\langle U|_F$.

The states in Eqs. (2.1)-(2.3) represent the unitary evolution of the full quantum state at all times. While the latter is commonly associated with the many-worlds interpretation [63, 64], or with Bohmian mechanics [65, 66, 67], it is also compatible with a timeless formulation of quantum theory as introduced by Page and Wootters [13]. Even without necessarily accepting the picture of the world provided by the many-worlds interpretation, Eq. (2.4) can be used by any observer to make predictions about their observered outcome at some time. In particular, we assume that the friend has enough information about the experimental setup in order to use Eq. (2.4) for her probability assignments.

In the following, we will also be interested in cases where the initial state of the system is a mixed state ρ_S. Such a state can be decomposed as $\rho = \lambda|\psi\rangle\langle\psi| + (1-\lambda)|\phi\rangle\langle\phi|$, where $|\psi\rangle, |\phi\rangle$ are orthonormal states and $0 \leq \lambda \leq 1$. Then we have the analogue of expressions (2.1)-(2.3) for the mixed state $\Sigma(t)$ of the whole setup at different times

$$\Sigma(t) = \lambda|\Psi(t)\rangle\langle\Psi(t)| + (1-\lambda)|\Phi(t)\rangle\langle\Phi(t)|, \tag{2.5}$$

where $|\Psi(t)\rangle$ and $|\Phi(t)\rangle$ are states analogous to Eqs. (2.1)-(2.3) with initial system states $|\psi\rangle_S$ and $|\phi\rangle_S$ respectively. Furthermore, probabilities are now given by $p(x) = \mathrm{tr}\left(\Pi_x \Sigma(t)\right)$.

In the standard analysis of the Wigner's friend thought experiment, the friend is usually assumed to describe the dynamics of her lab by using the state-update rule instead of Eq. (2.2). She, therefore, would assign probabilities to Wigner's measurement that are different from those assigned by Wigner based on Eqs. (2.2) and (2.3), which leads to an inconsistency between the predictions of both observers (see for example Ref. [44]).

2.3 Probability assignments in a scientific theory

A necessary requirement for an empirically adequate scientific theory is that it should be able to give (quantitative) predictions, such as to have testable empirical content.[2] Namely, a theory should be able to associate a measure of likelihood to an event y to happen, given that certain conditions –that in turn are captured by another event x– have already occurred. In the words of Wigner,

> "One realises that *all* the information which the laws of physics provide consists of probability connections between subsequent impressions

[2]Throughout the paper, by the term *prediction* we merely mean the possibility of assigning a probability distribution, and we do not strictly commit to any particular (operational) interpretation of probability, such as in terms of betting quotients.

that a system makes on one if one interacts with it repeatedly, i.e., if one makes repeated measurements on it." [4]

The theory should thus be able to answer questions of the form: "given that I observed event x at time t_1, how likely is it that I will observe event y at a later time t_2?". Mathematically speaking, such a question is answered by specifying a conditional probability distributions $p(y|x)$. [3] On that note, a recent work about the emergence of physical laws is based on the idea that the primary purpose of such laws is to give the conditional probability distributions relating events perceived by an observer at two subsequent times [68].

In the context of the Wigner's friend thought experiment, we are thus interested in the friend's question: "given that I saw outcome f_1 at time t_1, what is the probability (attributed by using quantum theory) that I will see outcome f_2 at time t_2?". We assume that quantum mechanics is empirically adequate and is able to answer this question by providing a conditional probability distribution $p(f_2|f_1)$. Moreover, note that (unitary) quantum theory also prescribes how to assign a probability for observations at a single time (*one-time probabilities*) $p(f_1)$ by Eq. (2.4). [4] Given these elements, the standard axiomatization of probability theory allows the definition of a joint probability distribution through the identity $p(f_1, f_2) = p(f_1|f_2)p(f_2)$. Thus, any theory that, like quantum mechanics, prescribes rules to assign one-time probabilities and conditional probabilities, automatically allows to assign joint probability distributions. However, as we will see, such a joint probability distribution cannot simultaneously fulfill three seemingly natural assumptions in a Wigner's friend scenario.

2.4 No-go theorem for the persistent reality of Wigner's friend's perception

We now formulate a formal no-go theorem which shows that in Wigner's friend scenarios, the friend cannot treat her perceived measurement outcome as having *reality across multiple times* without contradicting what might appear to be core assumptions of quantum mechanics. Consider the following assumptions:

P1 The events f_1 and f_2, corresponding to the perceived measurement records of the friend at times t_1 and t_2, respectively, can be combined into a joint event to which is assigned a probability distribution $p(f_1, f_2)$. Moreover, the rules of the probability calculus imply that $p(f_1) = \sum_{f_2} p(f_1, f_2)$ and $p(f_2) = \sum_{f_1} p(f_1, f_2)$.

P2 One-time probabilities are assigned without resorting to the state-update rule (i.e., using unitary quantum theory, where no "collapse" is considered to

[3] In situations where a user of the theory does not have enough information to uniquely determine $p(y|x)$ (for example some other agent could intervene in an unknown way), the theory should be able to provide a list of all the information which, if it were known, would determine $p(y|x)$. For lack of a better alternative, the user of the theory can subjectively assign their best guess for a probability distribution over these unknown variables, which in turn allows to compute $p(y|x)$.

[4] It is worth stressing that one-time probabilities are fundamentally also themselves conditional probabilities, namely conditioned on all the possible past events that can influence the probability of the event that we are trying to predict.

occur). Thus, when the initial state of the qubit is $|\psi\rangle_S$,

$$p(f_i) = \mathrm{tr}(|f_i\rangle\langle f_i|_F|\Psi(t_i)\rangle\langle\Psi(t_i)|), \tag{2.6}$$

with $|\Psi(t_i)\rangle$ being the unitarily evolved global state according to Eqs. (2.2), (2.3).

P3 The joint probability of the friend's perceived outcomes $p(f_1, f_2)$ has a convex linear dependence on the initial state ρ_S of the system qubit.

We will motivate these assumptions in more detail in Section 2.4.1. We now show that these assumptions lead to a contradiction when applied to the friend in a Wigner's friend scenario.

Theorem 1. *The conjunction of the assumptions P1-P3 cannot be satisfied for the Wigner's friend thought experiment for a general choice of Wigner's measurement basis.*

Proof. Define the isometries $V_i : \mathcal{H}_S \to \mathcal{H}_S \otimes \mathcal{H}_F \otimes \mathcal{H}$, $i = 1, 2$ mapping the initial state of the spin $|\psi\rangle_S$ to the corresponding state at time t_i as $V_i|\psi\rangle_S = |\Psi(t_i)\rangle_{SFW}$. Using Eqs. (2.2) and (2.3), these are found to be

$$V_1 = |\uparrow, U, 0\rangle_{SFW} \langle\uparrow|_S + |\downarrow, D, 0\rangle_{SFW} \langle\downarrow|_S \tag{2.7}$$
$$V_2 = |1\rangle_{SF}|1\rangle_W\langle\phi_1|_S + |2\rangle_{SF}|2\rangle_W\langle\phi_2|_S \tag{2.8}$$

where $|\phi_1\rangle := a\,|\uparrow\rangle + b\,|\downarrow\rangle$ and $|\phi_2\rangle := b^*\,|\uparrow\rangle - a^*\,|\downarrow\rangle$.

By P2 (and using P3 to extend to mixed states) we have

$$p(f_1) = \mathrm{tr}\left((|f_1\rangle\langle f_1|_F \otimes \mathbb{I}_{SW})V_1\rho V_1^\dagger\right) \tag{2.9}$$
$$= \mathrm{tr}\left(V_1^\dagger(|f_1\rangle\langle f_1|_F \otimes \mathbb{I}_{SW})V_1\rho\right) = \mathrm{tr}(E^1_{f_1}\rho), \tag{2.10}$$

where we define $E^1_{f_1} := V_1^\dagger(|f_1\rangle\langle f_1|_F \otimes \mathbb{I}_{SW})V_1$, which can be understood as the "Heisenberg picture" operator [5] corresponding to measuring $|f_1\rangle\langle f_1|$ at time t_1. It is easily checked that $E^1_{f_1}$ is a positive operator on $\mathcal{H}_S$ and that $\sum_{f_1} E^1_{f_1} = \mathbb{I}_S$. Therefore $\{E^1_{f_1}\}$ is a positive operator-valued measure (POVM). Similarly, we have

$$p(f_2) = \mathrm{tr}(V_2^\dagger(|f_2\rangle\langle f_2|_F \otimes \mathbb{I}_{SW})V_2\rho) := \mathrm{tr}(E^2_{f_2}\rho). \tag{2.11}$$

The calculation of the POVM elements yields

$$E^1_U = |\uparrow\rangle\langle\uparrow| \tag{2.12}$$
$$E^1_D = |\downarrow\rangle\langle\downarrow| \tag{2.13}$$

and

$$E^2_U = |a|^2|\phi_1\rangle\langle\phi_1| + |b|^2|\phi_2\rangle\langle\phi_2| \tag{2.14}$$
$$E^2_D = |b|^2|\phi_1\rangle\langle\phi_1| + |a|^2|\phi_2\rangle\langle\phi_2|. \tag{2.15}$$

[5] This is not exactly the textbook Heisenberg picture, because V_1 is an isometry and not a unitary.

Assumptions P1 and P3 imply that there exists a joint POVM $\{G_{f_1 f_2}\}$ such that

$$p(f_1, f_2) = \mathrm{tr}(G_{f_1 f_2}\rho), \tag{2.16}$$

and requiring that the marginals obey P2 for all states means that $\sum_{f_1} G_{f_1 f_2} = E^1_{f_1}$ and $\sum_{f_2} G_{f_1 f_2} = E^2_{f_2}$. When there exists such a $\{G_{f_1 f_2}\}$, the POVMs $\{E^1_{f_1}\}$ and $\{E^2_{f_2}\}$ are called jointly measurable.

If (at least) one of the two POVM's considered is sharp, then joint measureability is equivalent to commutativity, and there is a unique joint observable $G_{f_1 f_2} = E^1_{f_1} E^2_{f_2}$ with the correct marginals (Proposition 8 of Ref. [69]). Since we are considering two-outcome POVMs, and since E^1 given by Eqs. (2.12) and (2.13), is sharp, joint measurability is equivalent to $[E^1_U, E^2_U] = 0$. Direct calculation yields

$$\begin{aligned}[E^1_U, E^2_U] = {} & (|a|^2 - |b|^2) a b^* \left|\uparrow\right\rangle \left\langle\downarrow\right| \\ & + (|b|^2 - |a|^2) a^* b \left|\downarrow\right\rangle \left\langle\uparrow\right| . \end{aligned} \tag{2.17}$$

So these two POVMs are not jointly measurable for general choices of a, b, which concludes the proof. □

In the following we make some conceptual remarks about the proof of the above theorem. Even though our proof uses the language of joint measurability, due to a formal equivalence with that problem, the physical interpretation of joint measurability is different in our scenario. Indeed, joint measurability usually refers to the possibility to "simultaneously" measure two POVMs via a third joint POVM. Two non-jointly measureable POVMs can nevertheless be measured sequentially, one after the other, but the first measurement will generally disturb the state which is the input to the second measurement. In our Wigner's friend scenario, we are likewise considering measurement operators that correspond to observed outcomes at two subsequent times. However, assumption P2 implies Eq. (2.11), where ρ is not affected by the measurement at t_1. Thus imposing P2 leads to a bypassing of the standard information-disturbance relations [70] and to a contradiction with assumptions P1 and P3.

Furthermore, note that it is not essential for Wigner to perform any measurement in order to derive a no-go result. Indeed, Wigner could instead perform a "Hadamard" unitary $|\uparrow, U\rangle \mapsto \frac{1}{\sqrt{2}}(|\uparrow, U\rangle + |\downarrow, D\rangle)$, $|\downarrow, D\rangle \mapsto \frac{1}{\sqrt{2}}(|\uparrow, U\rangle - |\downarrow, D\rangle)$, and the same theorem would follow after making the necessary modifications for the state $|\Psi(t_2)\rangle$.

2.4.1 Motivation of the assumptions

We attempt here to motivate each of the assumptions of our no-go theorem. We can only offer plausibility arguments, since, as we have already shown, these assumptions cannot in general all hold true in quantum mechanics. It should also be noted that the assumptions are not logically independent: for example one cannot hold P3 without at the same time assuming P1.

As discussed in Secs. 2.2 and 2.3, P1 is motivatived by the requirement that quantum theory –as any other predictive theory– should provide us with conditional probability distributions for the friend's perceptions before and after Wigner's measurement, i.e. $p(f_2|f_1)$, and P2 provides single time probabilities for the friend's perception: $p(f_1) = \mathrm{tr}\left(|f_1\rangle\langle f_1|_F |\Psi(t_1)\rangle\langle\Psi(t_1)|\right)$ and

$p(f_2) = \mathrm{tr}\left(|f_2\rangle\langle f_2|_F|\Psi(t_2)\rangle\langle\Psi(t_2)|\right)$. Thus we should in principle be able to construct a joint probability distribution for the friend's perceived outcomes at two different times, $p(f_1, f_2)$. Even if one initially only assigns probabilities to events directly perceived by an observer, such as f_1 and f_2, the requirement of predictability for a theory leads us to assign probabilities to the joint event (f_1, f_2), although this is not a directly perceivable event in its own (arguably, one cannot have direct perceptions about two different times).

Assumption P1 can also be understood as a special case of the general assumption that measurement records are *facts of the world* [3], or of the *absoluteness of observed events* (AOE) – the assumption that "an observed event is a real single event, and not relative to anything or anyone" [8, 10] – applied to events f_1 and f_2. It is important to emphasize that the negation of AOE is not necessarily the claim that measurements outcomes are observer-dependent. Indeed the observed events in assumption P1 are all associated with the same observer, and thus P1 is conceptually different from the version of AOE used in deriving the no-go theorem of Ref. [8], which is about joint probability assignments for the measurement outcomes of multiple observers.

P2 can be justified by appealing to the belief that interpretations of quantum mechanics should be empirically equivalent, i.e., that they all yield the same experimental predictions. Since P2 definitely holds in certain interpretations, notably in the Everett interpretation [63], one should then expect P2 to hold in general. Since P2 can in principle be tested empirically, it is appropriate to regard quantum mechanics with objective collapse [71] as a different physical theory from unitary quantum mechanics, and not merely a different interpretation [44].

Assumption P3 can be understood as a conservative extension of the Born rule –which assigns single-time probabilities linearly in the quantum state– to joint events at multiple times: P3 asks that the joint probabilities for events at multiple times must also depend linearly in the initial quantum state. P3 is true in typical laboratory situations where the usual quantum mechanical state-update rule can be used to calculate probabilities. Moreover, P3 can be motivated operationally, in a way that is customary in the context of generalized probabilistic theories [72, 73]. We can imagine that a third agent is preparing the initial state of the system qubit, independently from the friend and Wigner. One might assume that, after fully specifying all relevant details for the friend's and Wigner's measurement setups (this includes the measurement basis for both of them, the initial quantum state of the friend, etc.), the probabilities $p(f_1, f_2)$ only depend on the quantum state ρ_S but not on the way that the state was prepared. Suppose that $p_\sigma(f_1, f_2)$ and $p_\tau(f_1, f_2)$ are the probability distributions when the system state σ or τ is prepared. Since $\rho = \lambda\sigma + (1-\lambda)\tau$ can be prepared by tossing a biased coin which leads to prepare σ with probability λ, and τ otherwise, the linearity of probability implies that $p_\rho(f_1, f_2) = \lambda p_\sigma(f_1, f_2) + (1-\lambda)p_\tau(f_1, f_2)$. [6] Roughly speaking, upholding P1 while denying P3 amounts to the claim that quantum mechanics is "incomplete", in the sense that a full specification of the initial state ρ_S is not sufficient for computing $p(f_1, f_2)$. Furthermore, a convincing case against P3 should involve the prescription and justification of a specific non-linear two-time probability rule; Bohmian mechanics is an example of this strategy, as we discuss further in Section 2.5.1.

[6] A further, independent justification for P3, i.e. allowing probabilistic mixtures, is that it implies that optimal compression is equivalent to linear compression [74, 75]

2.5 Implications of the no-go theorem

2.5.1 The no-go theorem in different interpretations of quantum mechanics

As mentioned above, strategies for coping with the no-go theorem Thm. 1, i.e. deciding which of the assumptions one is most likely to drop, will depend on one's interpretation of quantum theory. We believe that organizing interpretations according to which of the assumptions they reject can help to give a clearer understanding of the fundamental differences between them. In what follows we will go through each of the assumptions and for each give examples of a prominent interpretation that would reject it. We do not strive here for exhaustivity, but rather to give an impression of the variety of ways in which our theorem can be understood. In the interest of space, our representation of any interpretation will be rather superficial.

P1 According to our understanding, the Everett (or *many-worlds*) interepretation [63, 64] denies that it is meaningful to assign a joint probability $p(f_1, f_2)$ to the friend's observations at multiple times.[7] This is at least Bell's diagnostic:

> "Everett [...] tries to associate each particular branch at the present time with some particular branch at any past time in a tree-like structure, in such a way that each representative of an observer has actually lived through the particular past that he remembers. In my opinion this attempt does not succeed and is in any case against the spirit of Everett's emphasis on memory contents as the important thing. We have no access to the past, but only to present memories." [76]

Only in situations where a sufficient amount of decoherence is present is it possible to identify "worlds" branching in time, which would allow to meaningfully speak of $p(f_1, f_2)$. By construction, this is not the case in Wigner's friend scenario.

Further, note that operational approaches [3] might only allow for the assignment of probabilities that can in principle be measured by performing many trials of the experiment, or in situations where probability assignments can be related to rational bets. This is not the case for the joint event (f_1, f_2) here, because in a Wigner's friend experiment there is no "reliable record" of f_1 that remains available after time t_2.

P2 There are at least two ways that this assumption can be denied: objective collapse of the wave function, or subjective collapse of the wave function. In a theory with objective collapse [71], not only would P2 be false, but the predictions that Wigner makes using Eq. (2.3) would be verifiably wrong.

[7] In Chapter 7 of Ref. [64], Wallace concludes that the there are only two viable candidates for a correct theory of identity (i.e. for what it means to talk about the "same object" at two different times) within the many-worlds interpretation; he calls these canditates the *Lewisian view* and the *Stage view*. In the Lewisian view, the identity of an object holds over a period of time within a (decohered) history, while in the Stage view the identity of objects only refers to a single instant in time. When "worlds" are allowed to interefere with each other as in the Wigner's friends thought experiment, the Lewisian view appears less viable.

In a theory with subjective state assignments such as QBism [60], an agent is normatively constrained to use the Born rule for computing probabilities, but the quantum state used for doing so is up to the agent's good judgment; furthermore, QBism prohibits agents from assigning quantum states to themselves [61]. Therefore there can be subjective collapse: the friend would have the right to use the usual state update rule in order to calculate $p(f_2|f_1)$ – and thus not recover $p(f_2)$ according to Eq. (2.6) – while Wigner uses unitary evolution for his own predictions.

P3 In the de Broglie-Bohm interpretation [65, 66, 67], the memory of the friend has a definite and observer-independent value at all times and P1 holds. Furthermore, it can be proven that Bohmian mechanics recovers the same single-time predictions as unitary quantum mechanics so that P2 holds [65, 66]. Therefore it must be P3 that fails to hold in that interpretation. Indeed, in is known in the context of double-slit intereference that the Bohmian guidance equation is non-linear in the density operator [77, 78]. It would be interesting for future work to calculate $p(f_1, f_2)$ for this experiment within a Bohmian description.

2.5.2 The (non-)persistence of memory in special cases

Theorem 1 has been derived by assuming that in Wigner's choice of measurement basis, the parameters a, b are arbitrary. However, from Eq. (2.17) it is easy to see which particular choices of a, b allow the assumptions P1-P3 to be satisfied. In the following we compute $p(f_1, f_2)$ for these special cases. We shall see (point 2 below) that assumptions P1-P3 can lead to counterintuitive conclusions about the time-evolution of the friend's memory even in those cases. There are essentially two possibilities which make the commutator in Eq. (2.17) vanish, which is equivalent to satisfying all three assumptions.

1. $|a| = 1, b = 0$. (The case $a = 0, |b| = 1$ differs by a relabeling of the basis states). This corresponds to Wigner performing a measurement of the friend and system in the "computational basis" $|1\rangle = |\uparrow, U\rangle$, $|2\rangle = |\downarrow, D\rangle$ revealing to him which result the friend observed. In that case the unique probability distribution that satisfies the assumptions of the theorem is $p(f_1, f_2) = \mathrm{tr}(E^2_{f_1} E^2_{f_2} \rho)$, with
$$E^1_U = E^2_U = |\uparrow\rangle\langle\uparrow| \tag{2.18}$$
$$E^1_D = E^2_D = |\downarrow\rangle\langle\downarrow| \,. \tag{2.19}$$
It is easy to verify that $p(f_2|f_1) = \delta_{f_1 f_2}$, which means that the friends memory of the outcome is perfectly preserved.

2. $|a|^2 = |b|^2 = \frac{1}{2}$. This corresponds to Wigner performing a measurement in the "Bell basis" , for example $|1\rangle = \frac{1}{\sqrt{2}}(|\uparrow, U\rangle + |\downarrow, D\rangle)$, $|2\rangle = \frac{1}{\sqrt{2}}(|\uparrow, U\rangle - |\downarrow, D\rangle)$.[8] Eqs. (2.14) and (2.15) show that the relative phases do not matter, so it suffices to consider this example. We have in this case

[8] Here again we restrict our analysis to the only two out of the four "Bell's states" that are physically relevant in the described scenario.

$p(f_1, f_2) = \mathrm{tr}(E^1_{f_1} E^2_{f_2} \rho)$, with

$$E^1_U = |{\uparrow}\rangle\langle{\uparrow}| \tag{2.20}$$

$$E^1_D = |{\downarrow}\rangle\langle{\downarrow}| \tag{2.21}$$

$$E^2_U = E^2_D = \frac{\mathbb{I}}{2}, \tag{2.22}$$

and one can check that $p(f_2|f_1) = \frac{1}{2}$. This means that the friend's memory gets flipped with probability $\frac{1}{2}$, independently of the initial state ρ. This is particularly surprising in the case where the initial state is $|\psi\rangle = \frac{1}{\sqrt{2}}(|{\uparrow}\rangle + |{\downarrow}\rangle)$, because in that case Wigner performs a *non-disturbance measurement* [55, 56]. This means that the joint state of the friend and system $|\Psi(t_1)\rangle$ is actually an eigenstate of Wigner's measurement. One might expect that in this case, since the quantum state is not changed by Wigner's measurement, the friend's perceived result should remained unchanged as well; this is implicitly assumed in most discussions on the Wigner's friend thought experiment (and explicitly, for example, in Ref. [56]). However, this conflicts with the assumption of quantum mechanical linearity of probabilities: if $p(f_1, f_2)$ is linear in ρ, the friend's perceived outcome must get flipped with probability $\frac{1}{2}$, indepently of ρ.

2.6 Conclusion

From the point of view of Wigner, assuming that the friend's memory has a (unknown but definite) value is akin to assuming a hidden-variable model. Bell-type arguments involving two Wigners and two friends [3, 7, 8] have shown that if we further make a locality assumption on that hidden-variable model, it will not be possible to reproduce the quantum mechanical predictions. In this paper, we have shown that even from the friend's perspective, treating the memory of her measurement outcome as having a value throughout the experiment is in conflict with important features of quantum mechanics. More precisely, we have shown that it is not possible to assign a joint probability to her observed outcomes at two different times of the thought experiment, in a way that is compatible with unitary marginal probabilities and with the linear dependence of quantum mechanical probabilities on quantum states.

How to understand this theorem will depend on one's interpretation of quantum mechanics, but it seems that interesting lessons can be drawn from various interpretational points of view. Many popular interpretations (excluding hidden-variable interpretations like Bohmian mechanics) implicitly satisfy the principle that legitimate probability assignments should depend linearly on the initial quantum state. It appears in light of our theorem that the consequence of such a commitment is that one must in general either prohibit the use of present information to predict the future (drastically scaling down the predictive power of quantum theory), or deny that unitary quantum mechanics makes valid single-time predictions on all scales. That such a radical conclusion is necessary *in general* does not affect the fact that for all practical purposes, i.e. in normal conditions when sufficient amounts of decoherence is present, one can continue to successfully use present information for predictions.

Our results might also raise interesting questions about the persistence of identity for the friend. If it is not possible for the friend to use the Born rule

–or any other rule linear in the quantum state of the system– to assign a joint probability distribution to her observed outcomes before and after Wigner's measurement, then to what extent can the friend at these two different times be considered the same agent? It is conceivable, although counterintuitive, that the friend at t_1 and the friend at t_2 should be legitimately considered to be two distinct agents. [9] In that case one could reach similar conclusions to the ones of Ref. [10], and say that the friend's outcome at t_2 is *not an event* from the point of view of the friend at t_1, and vice-versa.

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[9] Conceptually speaking, this would be a costly conclusion to make in general, since these "two agents" share many common memories about their past.

Chapter 3

Comment on Healey's "Quantum Theory and the Limits of Objectivity"

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The first two authors, listed in alphabetic order, share first authorship. I made essential contributions to all aspects of this work.

Abstract

In this comment we critically review an argument against the existence of objective physical outcomes, recently proposed by Healey [1]. We show that his gedankenexperiment, based on a combination of "Wigner's friend" scenarios and Bell's inequalities, suffers from the main criticism, that the computed correlation functions entering the Bell's inequality are in principle experimentally inaccessible, and hence the author's claim is in principle not testable. We discuss perspectives for fixing that by adapting the proposed protocol and show that this, however, makes Healey's argument virtually equivalent to other previous, similar proposals that he explicitly criticises.

3.1 Introduction

In a recent paper [11], R. Healey proposes an argument to supposedly show that there exist situations in which quantum measurements cannot have "a unique, objective, physical outcome". At the same time the author critically analyses two similar arguments put forward in [7] and [6], concluding that their "dependence on questionable implicit assumptions severely limits their significance". Healey's argument, however, contains a series of problems and attempting to resolve them leaves it prone to the same type of criticism raised about [7].

The argument is based on the computation of a set of correlation functions that violate a Bell-like inequality. This violation should signify the non-existence of objective physical outcomes in quantum mechanics. The argument involves a sequence of successive measurements that have been performed by different observers. Two of these measurements are performed by observers Carol and Dan (who take the role of "Wigner's friends" [4]), the others by "superobservers" Alice and Bob, who describe the "friends' "measurements unitarily and are capable of undoing them.

In deriving his no-go theorem, Healey uses the standard quantum formalism to compute expressions for correlation functions, but gives no prescription on how to relate these expressions to observable quantities. Moreover, if one assumes the standard prescriptions of quantum theory to relate all correlations entering the Bell-like inequality with recorded counts, these counts would not give rise to violation of the inequality in the proposed protocol. Hence, in the protocol it is not possible, even in principle, to verify Healey's claim against the existence of objective physical outcomes in quantum mechanics. In this respect, we would like to stress that a physical theory is composed of theoretical symbols provided with a consistent calculus to combine them (the *formalism*), together with a series of *rules of correspondence*. In the words of Jammer [79], *"The formalism F, the logical skeleton of the theory, is a deductive, usually axiomatised calculus devoid of any empirical meaning; [...] To transform F into a hypothetic deductive system of empirical statements and to make it thus physically meaningful, some of the nonlogical terms, or some formulae in which they occur, have to be correlated with observable phenomena or empirical operations. [...] F without R is a meaningless game with symbols".*

The protocol proposed in Ref. [11] faces three main problems:

(i) The violation of the proposed Bell's inequality cannot, not even in principle, be tested, because in no region of space-time are the experimental data from which all the correlation functions can be extracted available. Hence, no observer can ever test the inequality. This applies in particular to the "superobservers" Alice and Bob who have no access to the counts registered in Carol's and Dan's measurements.

(ii) If one wants to make the argument testable, one needs to attribute an operational meaning to the computed expressions for the correlation functions. In standard quantum mechanics, the correlation functions correspond to the relative number of counts in respective measurements. However, in the protocol of Ref. [11] Alice and Bob could only evaluate all correlation functions, if they would register the counts from Carol's and Dan's measurements (i.e. if Alice and Bob actually perform measurements). In that case they need to use the standard "state-update rule" for the prediction of their subsequent measurements in contrast to the assumption of Ref. [11], wherein a measurement is assumed to proceed in accordance with a unitary interaction.[1] However then, there would be no violation of the Bell's inequality.

(iii) If Alice and Bob were to verify their predictions and actually violate a Bell's inequality with collected data, one would have to adapt the protocol in a way that renders the setup analogous to that proposed in Ref. [7]. In that case, however, one further has to require the standard assumptions for testing the violation of Bell's inequality, namely "freedom of choice" and "locality".

[1]In Ref. [11], Healey states "that a measurement causes no physical 'collapse' of the quantum state, so that each spin-measurement proceeds in accordance with a unitary interaction between the measured particle and the rest of the experimenter's lab, and that this is consistent with its having a definite, physical outcome recorded by the experimenter in that lab".

3.2 Description of the Protocol

Similarly to proposals [7] and [6], the *gedankenexperiment* introduced in [11] involves two "Wigners" (Alice and Bob) and their respective "friends" (Carol and Dan). The latter each receive one spin-1/2 particle (whose states are labeled by 1 and 2, respectively) of a maximally entangled pair. The protocol describes a series of operations applied by the four experimenters from the point of view of the two superobservers.

With respect to Alice's lab, the protocol is defined by the following steps:

0 State preparation:
$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2\right)|r\rangle_C|r\rangle_D$. Where $|r\rangle_C$ and $|r\rangle_D$ are the initial ("ready") states of Carol and Dan, respectively, that will record the outcome of their spin measurements (and therefore do not need to be initially further specified).

1 Dan measures: $|\psi(0)\rangle \to |\psi(1)\rangle = U_D|\psi(0)\rangle$; where $U_D|\downarrow_d\rangle_2|r\rangle_D =$ $|$"D sees 2 $\downarrow_d$ "$\rangle_{2D}$ (and equivalently for $|\uparrow_d\rangle_2$).

2 Carol measures: $|\psi(1)\rangle \to |\psi(2)\rangle = U_C|\psi(1)\rangle$; where $U_C|\downarrow_c\rangle_1|r\rangle_C =$ $|$"C sees 1 $\downarrow_c$ "$\rangle_{1C}$ (and equivalently for $|\uparrow_c\rangle_1$).

3 Alice undoes Carol's measurement by applying $U_C^\dagger$:
$|\psi(2)\rangle \to |\psi(3)\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_d\rangle_1|\text{"Dan sees 2} \downarrow_d \text{"}\rangle_{2D} -\right.$ $\left.|\downarrow_d\rangle_1|\text{"Dan sees 2} \uparrow_d \text{"}\rangle_{2D}\right)|r\rangle_C = |\psi(1)\rangle$.

4 Alice measures, corresponding to a unitary U_A. Analogous to U_C, the unitary $U_A|\downarrow_a\rangle_1|r\rangle_A = |$"A sees 1 $\downarrow_a$ "$\rangle_{1A}$ (and equivalently for $|\uparrow_a\rangle_1$)

5 Bob undoes Dan's measurement by applying $U_D^\dagger$.

6 Bob measures corresponding to a unitary U_B, which is defined as $U_B|\downarrow_d\rangle_2|r\rangle_B = |$"B sees 2 $\downarrow_b$ "$\rangle_{2B}$ (and equivalently for $|\uparrow_b\rangle_2$) resulting in the state $|\psi(6)\rangle = U_A U_B|\psi(0)\rangle$.

Healey claims that this protocol allows to predict, in every run, the correlation function between the outcomes of different observers.
After step **2**, Alice predicts

$$E(c,d) = -\cos(c-d)\ , \tag{3.1}$$

where $c-d$ is the relative angle between measurement directions $\vec{c}$ and $\vec{d}$.
After step **4**, she predicts

$$E(a,d) = -\cos(a-d)\ . \tag{3.2}$$

With respect to Bob's laboratory, in relative motion with respect to Alice's (see section "Analysis of the protocol" for a discussion), the protocol looks as follows.

0* State preparation:
$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2\right)|r\rangle_C|r\rangle_D$.

1* Carol measures: $|\psi(0)\rangle \to |\psi(1^*)\rangle = U_C|\psi(0)\rangle$.

2* Dan measures: $|\psi(1^*)\rangle \rightarrow |\psi(2^*)\rangle = U_D|\psi(1^*)\rangle$.

3* Bob undoes Dan's measurement by applying $U_D^\dagger$:
$|\psi(2^*)\rangle \rightarrow |\psi(3^*)\rangle = |\psi(1^*)\rangle$.

4* Bob measures, which is described by a unitary U_B.

5* Alice undoes Carol's measurement by applying $U_C^\dagger$.

6* Alice measures corresponding to a unitary U_A, resulting in the final state $|\psi(6^*)\rangle = U_A U_B|\psi(0)\rangle$.

After step **4***, Bob predicts

$$E(c, b) = -\cos(b - c) \; . \tag{3.3}$$

Finally, after step **6**, Alice (or alternatively Bob, after step **6***) can compute the correlation

$$E(a, b) = -cos(a - b) \tag{3.4}$$

for their observed outcomes a and b.

The main feature claimed by Healey —which supposedly improves the scheme in Ref. [7]— is that in the setup of [11] there exist, for every run of the protocol, four regions of space-time where each of the observables A_a, B_b, C_c and D_d takes a unique, definite value (although admittedly they never co-exist, in so far as the measurement outcomes of the observers Carol and Dan are erased by the superobservers Alice and Bob). These regions are illustrated in Fig. 3.1. For the sake of clarity and convenience for our proposed modification of the protocol —and in accordance with Healey's statement that the two laboratories are moving apart with constant velocities — we represent the space-time diagram in Fig. 3.1 with slight differences from the original in Ref. [11]. This leads to swapping points **1** (respectively **1***) with **2** (respectively **2***) interchanging the order of Carol's and Dan's measurements in both reference-frames, but does not affect tho argument in any way.

3.3 Analysis of the protocol

We present here what we consider the fundamental problems with the proposal in [11] and discuss perspectives for fixing them.

1. Unitary measurements and evaluating probabilities. In order to test whether quantum mechanics violates the objectivity of physical outcomes in the sense of Healey, one would need to find an operational way that would enable, at least in principle, to collect statistics from which the correlation functions could be computed and inserted into the Bell's expression.

In the protocol there is no space-time region where the measurement counts for all four measurement settings, which would enable the computation of the four correlations functions, are accessible in principle. Specifically, $E(a, d)$ is only accessible to Alice and Dan, $E(c, b)$ only to Carol and Bob, $E(c, d)$ only to Carol and Dan, and finally $E(a, b)$ only to Alice and Bob. In three of the four calculated correlation functions either Alice or Bob or both of them perform no measurement and hence obtain no data. Moreover, the three correlation functions remain to be inaccessible in principle to Alice and Bob after they undo their "friends' " measurements and, hence, there will be no records of Carols's and Dan's results and no means for Alice and Bob to ever evaluate any of (3.1), (3.2) and (3.3).

The lack of the link between computed correlation functions and observable quantities give rise to further inconsistencies in the argument. The correlation functions $E(a,d)$ and $E(c,b)$ are computed from two different reference frames, namely the one of Alice and of Bob respectively, see Fig 3.1. In fact, it is known that the notion of quantum state, and hence of entanglement as well as the correlation functions are reference-frame dependent. This is a consequence of the relativity of simultaneity in different reference frames [80]. It is thus not clear to us what combining expression (3.1)-(3.4), which are computed for different reference frames and thus for different quantum states, in a CHSH-like inequality is supposed to signify.

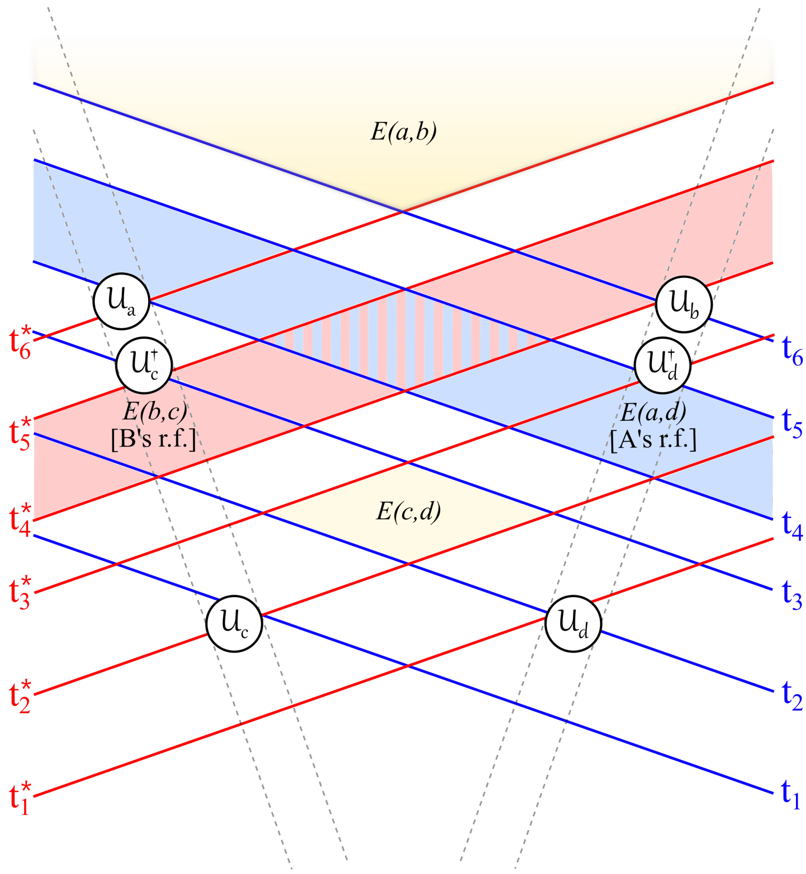

Figure 3.1: **Space-time diagram of Healey's protocol.** Two laboratories (containing Alice-Carol-system 1 and Bob-Dan-system 2, respectively) are moving apart with constant velocity. Four agents perform a series of measurements (the order of U_c and U_d has been changed from the original protocol to avoid misunderstanding and to conform to the modified protocol of Fig. 3.2). In every run of the protocol there supposedly exist four space-time regions wherein correlations between the results could be in principle established. The areas highlighted in yellow are those where correlations can be established in both reference frames, whereas the blue (red) areas are those where correlations are relative to Alice's (Bob's) reference frame only.

Moreover, if one would accept that combining expressions from different reference frames is non-problematic, one could equally assign different values to the correlation functions by an argument similar to that proposed in Ref. [81]. More concretely, with respect to Alice's reference frame, the calculated correlation function $E(c, b) = 0$ for all times. For times between t_1 and t_3 the register b is in a fixed pre-measurement state $|r\rangle_b$ and the results $c = +1$ and $c = -1$ occur with equal probability, whereas for times between t_3 and t_6 the result of c is erased and the register is reset to it initial state $|r\rangle_c$ and the results $b = +1$ and $b = -1$ occur with equal probability. The mean value of the product of c and b (i.e. the correlation function) is zero independently of choice of the value assigned to the fixed states of the registers. Similarly, Bob would predict $E(a, d) = 0$, since in his reference frame result d is erased before a is measured. Combining either of the two with (3.1), (3.4) and either (3.2) or (3.3) respectively still gives a violation of the CHSH-inequality, but with a different value of $3/\sqrt{2}$. However, combining the predictions of both reference frames, as done in [11], but using both $E(c, b) = 0$ and $E(a, d) = 0$ instead of (3.2) and (3.3) results in no violation of Bell's inequality:

$$\begin{aligned} |E(a, b) + E(b, c) + E(c, d) - E(a, d)| &= \\ |0 + E(b, c) + 0 - E(a, d)| &\leq 2. \end{aligned} \tag{3.5}$$

Without specifying how the computed expressions are related to observable quantities, there seems to be no physical motivation to favor the one combination of values which leads to the alleged violation, over the one that predicts no violation of the inequality. Clearly, all these inconsistencies appear because the computed expressions for correlation functions are not associated to observable quantities. If the four measurements would be identified with four space-time points in which counts are registered, then the correlations between these counts will be reference frame independent and there would be no inconsistencies. However, then no violation would be observed in the protocol.

2. A modified protocol that gives the expected probabilities suffers the same criticism that Healey raises. The protocol proposed in [11] allows to observe in principle only the correlations between a and b (since at the end of the protocol the measurement results of Carol and Dan have been erased). For the "superobservers" Alice and Bob to evaluate all the correlation functions and test the proposed CHSH-like inequality in [11], one is forced to adapt the protocol to one wherein Alice can choose to measure either a or c and Bob either b or d by deciding whether or not to erase the measurement outcome of their "friends". In the latter, each "superobserver" (Alice and Bob) performs the same measurement as their associated observer (Carol and Dan, respectively). Since in that case one allows for a "choice of setting" one has to ensure "locality" and "freedom of choice" as required in a standard Bell-inequality setup. In the original protocol of Healey, "locality" is not required, since all four measurements are fixed and performed in each round. Any assignment of four definite outcomes of course satisfies a Bell-like inequality by construction.

In the adapted protocol, Alice *not* erasing Carol's measurement, but rather measuring Carol's observable, allows for computing $E(b, c)$, whereas Bob *not* erasing Dan's measurement, but performing the same measurement instead will make $E(a, d)$ experimentally accessible; whereas, both the "superobservers" *not* undoing their friends' measurements allows for measuring $E(c, d)$. This adapted protocol is illustrated in Fig. 3.2. Note that in the rounds where Alice, Bob or both decide to measure the observables of Carol or Dan or both respectively, and register counts in the respective measurements, they cannot erase these counts in a unitary fashion to continue with the subsequent measurements. Upon registering the counts in the first measurements, Alice and Bob will observe a statistics in the subsequent measurement, which is compatible with the state-update rule and not with a unitary transformation. The continuation of the protocol in its original form [11] but with the application of the state-update rule would not lead to a violation of Bell's inequalities. It appears therefore that the only way to give an operational meaning to accessing the correlation functions and hence to the verification of Healey's argument is to assume that in each run Alice and Bob chooses one of the two measurement settings —a step that was criticised by Healey.

It, therefore, seems to us that this setup is subject to the same criticism as [7] when one requires the actual computation of the terms violating the CHSH-inequality to correspond to measured values. Otherwise, we think equations (3.1)-(3.4) need further justification.

In conclusion, the only possibility for the correlations to be in principle measured is to revise the scheme in a way that does not allow to be sure that the four values coexisted in each round, therefore suffering the very same criticism that was addressed towards proposal [7]. However, one can make a stronger assumption that these values are fixed even when they are not all co-measured, which then allows to derive the no-go theorem of Ref. [2].

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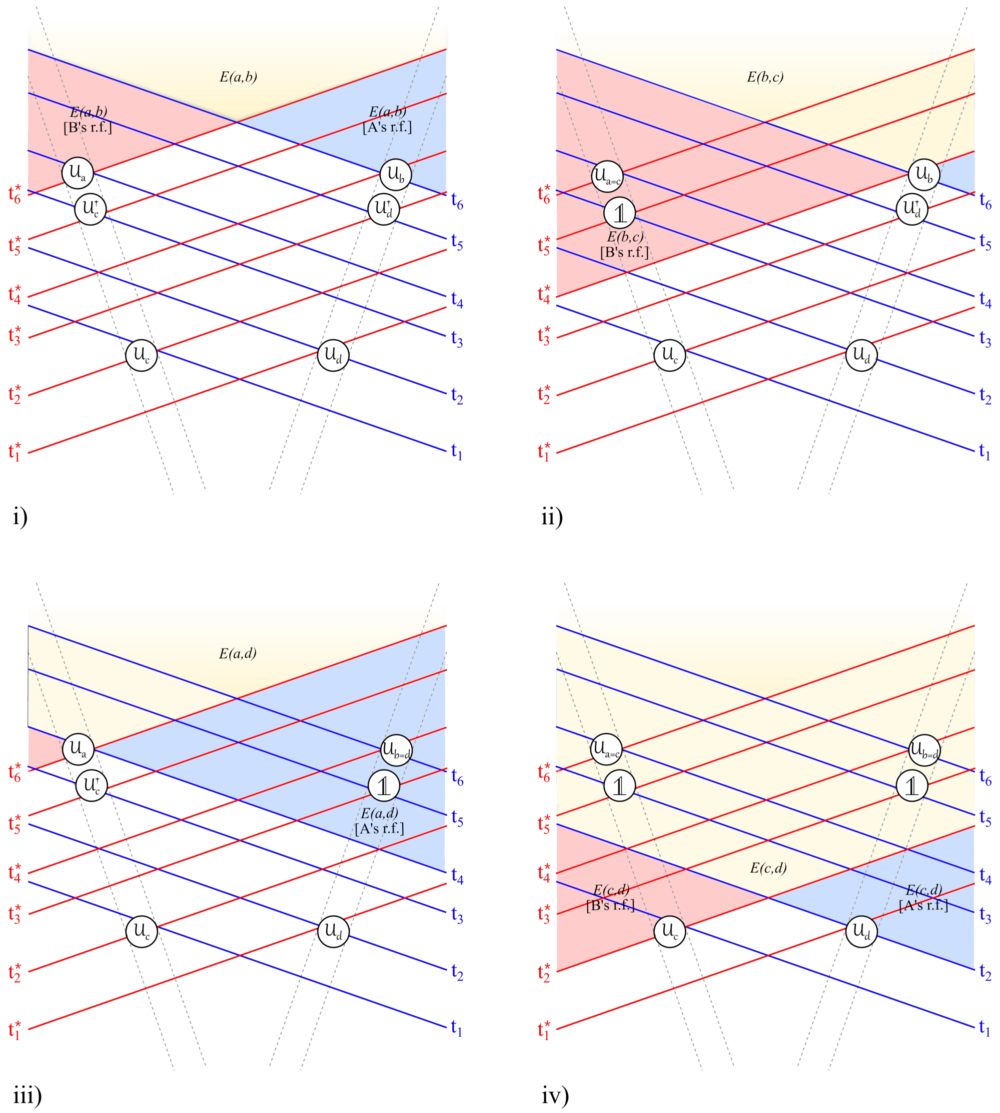

Figure 3.2: **Space-time diagrams of the modified protocol.** The modified protocol allows the observers to chose which operations to perform on their local system in each run. This allows to provide an operational meaning to the correlations between measured quantities. At the end of the protocol the highest-level observers, Alice and Bob, unambiguously agree on the correlation functions: i) $E(a,b)$, ii) $E(b,c)$, iii) $E(a,d)$, iv) $E(c,d)$. The colour legend is defined in Fig. 3.1.

Part II

Discriminating quantum physics from classical physics: the case of quantum coherence

Chapter 4

Coherence Equality and Communication in a Quantum Superposition

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I made essential contributions to all aspects of this work.

Abstract

In this paper, we introduce a "coherence equality" that is satisfied by any classical communication –i.e. conveyed by a localized carrier traveling along well-defined directions. In contrast, this equality is violated when the carrier is prepared in coherent quantum superposition of communication directions. This is phrased in terms of the success probability of a certain communication task, which results to be always constant and equal to 1/2 in the classical case. On the other hand, we develop two simple quantum schemes that deviate systematically from the classical value, thus violating the coherence equality. Such a violation can also be exploited as an operational way to witness spatial quantum superpositions without requiring to recombine the modes in a standard interferometer, but only by means of spatially separated local measurements.

4.1 Introduction

Quantum superposition principle states that an arbitrary linear combination of two physical states is still a valid quantum state. Such a principle lies at the core of genuine quantum behaviors. In fact, it even supervenes quantum entanglement, which can be regarded as a particular state of superposition that combines two or more joint degrees of freedom. Since the early days of quantum theory, physicists have been using the effect of quantum superposition (or coherence) as the foremost observable evidence to discriminate between the classical and the quantum domains. It has been also a prime conceptual tool for testing the limits of quantum mechanics, like in the notorious Schrödinger's cat gedankenexperiment [82].

In more recent years, effects based on coherence played a central role in the revolutions of quantum information and quantum technologies, allowing a plethora of novel achievements that are fundamentally unattainable in classical scenarios, such as secure communication [83], and algorithms with an exponential advantage over their classical counterpart [84, 85]. Moreover, it was shown that quantum superposition can be used as a resource for quantum communication [86, 87, 88, 89, 90, 18, 91] and can lead to novel effects such as the enhancement of a classical channel capacity [90], secure anonymous communication [18] or the doubling of the bandwidth of a classical channel [91].

However, it is well known that the direct observation of a quantum superposition of distinct states is not possible. For example, in the celebrated double-slit experiment, if one detects the presence of the particle at either of the two slits, no quantum effects are manifested, and even microscopic particles (e.g., electrons) resemble classical bullets. Accordingly, witnessing a quantum superposition requires a particular type of indirect observation, usually achieved by spatially separating the wave function before recombining them in an interference experiment. Consider, for instance, the aforementioned double-slit experiment, where each slit has the option to be open or closed, labeled by “0” and “1”, respectively. This leads to four possible configurations: “00”, “01”, “10”, or “11”, where the position of the digit indicates the first or the second slit, respectively. The “non-classicality”, i.e. the presence of a quantum superposition, can then be measured by the interference term [92]:

$$I := \sum_{i,j=0}^{1} (-1)^{i\oplus j} p_{ij}, \tag{4.1}$$

where p_{ij} is the conditional probability to find the particle on the observation screen, given the configuration ij. In a classical scenario $I = 0$ always, whereas a non-zero value would represent a witness of a coherent superposition. Exactly the same condition one finds in the case of a Mach-Zehnder interferometer (the simplest instantiation of the double-slit experiment), where a single particle is separated into two paths by a beam-splitter and interference fringes appear at the detector when $I \neq 0$. However, as in every other interferometric experiment, it is necessary to recombine the paths at a second beam-splitter (and to introduce a relative phase between the two paths) to observe the interference.

In this paper, we propose an operational way to witness a superposition only using local measurements conducted at separated locations, without the necessity of recombining the paths as in a standard interferometer. To achieve this, we derive a “coherence equality”, which is satisfied by any classical resource, but that can be violated by systems that exhibit coherence (i.e., in quantum superposition). We will phrase the problem using the modern language of information and communication tasks that present a systematic difference in the probabilities of success with their classical counterpart, when quantum resources are employed.

4.2 Coherence equality and interference

In this section, we derive a “coherence equality”, whose violation can be regarded as an operational procedure to witness quantum superposition without the use of a standard interferometer. That is, without the necessity of recombining the two arms to detect an interference pattern (thus effectively using a setup which is only half of a Mach-Zehnder interferometer).

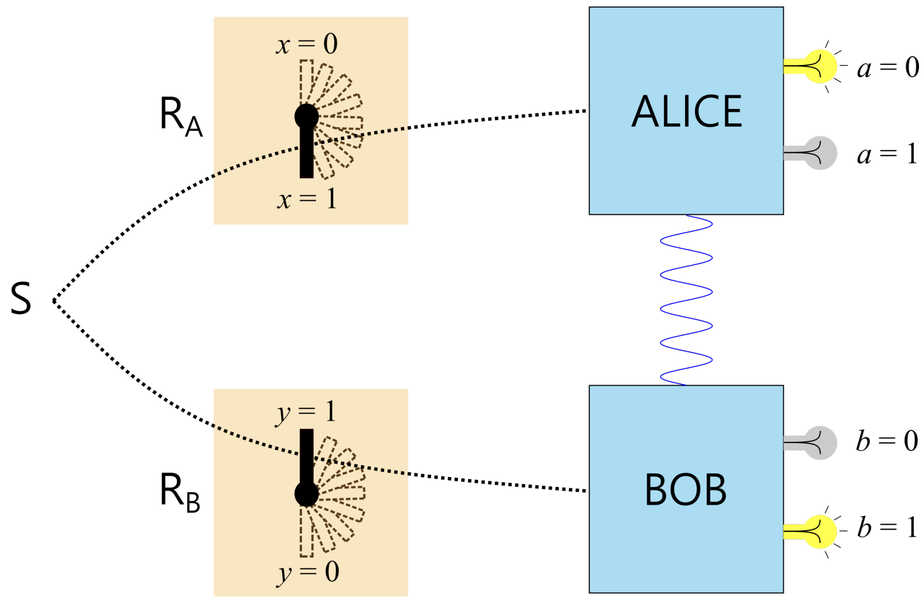

Figure 4.1: **Scheme of the "coherence without re-interference" communication game**. A source S produces a single "information carrier" which can be sent to two parties, Alice and Bob. The latter can share some resources (represented by the blue wavy line). A and B are asked by two referees to output one bit each (a and b, respectively) to fulfill a certain task. Movable "blockers" are situated along the channels $S-A$ and $S-B$ and their configurations, open or closed, function as encoded inputs, x on Alice's side, and y on Bob's. Quantum mechanics allows a violation of a fundamental classical bound to the probability of success (see main text).

Let us start by considering a scenario like the one depicted in Fig. 4.1. A source S produces a *single information carrier* (e.g., a single particle) in order to covey a piece of information to Alice (A) or Bob (B), who are separated at two different locations. (As we shall see, even if communication is allowed between the two agents –contrarily to non-local scenarios wherein space-like separation is enforced– the result would still hold unchanged).

The information to be communicated is a bit which is encoded along either of the paths traveled by the information carrier: $x \in \{0,1\}$ on the branch leading to Alice, and $y \in \{0,1\}$ in the branch leading to Bob. The operation of encoding consists in selecting one of the two configurations, open or closed, of a movable "blocker" (i.e., an ideally impenetrable barrier) on each of the two channels connecting S to A and S to B, respectively. We denote by $x=0$ ($x=1$) the configurations in which the blocker is open (closed) along the channel $S-A$; in the same fashion, the input $y=0$ ($y=1$) means that the blocker in channel $S-B$ is open (closed). No information can reach the receiver if a blocker is placed in the respective path (i.e., if $x=1$ and/or $y=1$).

The parties A and B then (potentially) receive the information carrier and they perform local measurements whose binary outputs are labeled by $a \in \{0,1\}$ and $b \in \{0,1\}$, respectively. Moreover, A and B are allowed to share some resources such as classical shared randomness, entanglement or coherence. Since in the classical case the information carrier is a well-localized object at any instant in time (which means that the particle will take only one definite path among

$S - A$ and $S - B$) the inputs x and y can causally influence the outputs a and b in two mutually exclusive, possible ways (see Fig. 4.2). Therefore, the joint

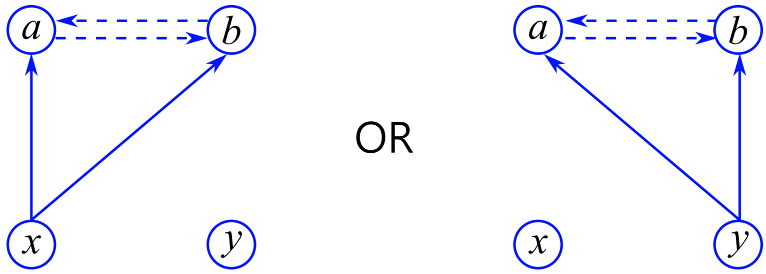

Figure 4.2: **Causal diagram**. Classically, a single information carrier can convey information only in one of two depicted scenarios. Either x or y influences a and b. The dashed arrows indicate that, in principle, there could be communication between a and b.

probability distribution of all the outputs, given the inputs, is a classical mixture of the distributions corresponding to one-way signaling distributions (either S communicates to A, or to B):

$$p(ab|xy) = \lambda_{S-A} p_{S-A}(ab|x) + \lambda_{S-B} p_{S-B}(ab|y), \tag{4.2}$$

where λ_{S-A} and λ_{S-B} are non-negative constants that add up to the unity. To measure interference effects, in analogy with Eq. (4.1), we define:

$$I_{ab} = \sum_{x,y=0}^{1} (-1)^{x \oplus y} p(ab|xy). \tag{4.3}$$

It follows immediately from (4.2) that for classical systems

$$I_{ab}^{Class} = 0. \tag{4.4}$$

We call the above expression "coherence equality", because any deviation from the value 0 would imply that the information carrier is a non-classical object that exhibit coherence. Note that communication between the parties is in principle allowed, since the derivation of Eq. (4.4) is independent of the separation between A and B (which could indeed be replaced by a single agent). However, for the argument here discussed, we maintain the parties separated at two different locations, because it is our main aim to show that spatially separated agents can certify the presence of quantum coherence only by means of local operations and classical communication, i.e. without closing a standard interferometer. Moreover, such a two-party scenario finds applications in cryptographic protocols [18, 93].

4.3 "Coherence without re-interference" communication game

Following a recent trend that aims at quantifying the discrepancy between classical and quantum scenarios by computing the success probability of quantum XOR non-local games [94, 95, 96], we define the "coherence without re-interference" game, that provides an operational procedure to demarcate the classical resources from the quantum ones.

Consider again the setup of Fig. 4.1, but where in this case two referees, R_A and R_B, encode inputs x and y (randomly assigned with uniform distribution) by opening or closing their blocker, as described above. The referees challenge the parties A and B to return outputs a and b, respectively, which ought to fulfill the following relation:

$$a \oplus b = x \oplus y. \tag{4.5}$$

The easiest classical strategy to play the game, would be for Alice and Bob to detect the presence of the particle at their respective positions, each outputting 1 if a particle is detected or 0 otherwise. From the distribution in (4.2), it is trivial to see that the (non-vanishing) probabilities read

$$\begin{aligned}
&p(10|00) = \lambda_{S-A}, \quad p(01|00) = \lambda_{S-B} \\
&p(10|01) = \lambda_{S-A}, \quad p(00|01) = \lambda_{S-B} \\
&p(00|10) = \lambda_{S-A}, \quad p(01|10) = \lambda_{S-B} \\
&p(00|11) = 1.
\end{aligned}$$

It follows that the probability of fulfilling relation (4.5) is given by $P_{win} = 1/4(\lambda_{S-A} + \lambda_{S-B} + 1) = 1/2$, which satisfies the coherence equality $I_{ab} = 0$ in Eq. (4.3), is satisfied.

In general, the probability of success, P_{win}, is given by the following expression:

$$\begin{aligned}
P_{win} &= \frac{1}{4}[p(00|00) + p(00|11) + p(01|01) + p(01|10) + \\
&\quad p(10|01) + p(10|10) + p(11|00) + p(11|11)] \\
&= \frac{1}{2} + \frac{1}{4}(I_{00} + I_{11}).
\end{aligned} \tag{4.6}$$

Since $I_{ab}^{Class} = 0$ for all a and b, we get $P_{win}^{Class} = \frac{1}{2}$. This means that, using only classical resources, A and B will always achieve the same probability of success of 1/2, regardless of the strategy they choose.

In practice, the probability of deviation of the relative frequency from 1/2 is exponentially suppressed with the number of experimental trials for any possible classical scenario (see Appendix 4.A).

4.4 Quantum task: Communication in quantum superposition

First example. Let us consider now the analogous quantum scenario. Let the state of the particle be an equal-weighted superposition of directions of communication, i.e. $|\psi\rangle_S = \frac{1}{\sqrt{2}}(|A\rangle_S + |B\rangle_S)$. The states $|A\rangle$ and $|B\rangle$ are taken to mean that the particle is in the path leading towards Alice or Bob, respectively, whereas the subscript S indicates that this particle was created at the source. Consider now that the internal mechanism of the measuring devices located at the two separate positions, A and B, have some pre-shared coherence. Namely, an ancillary identical (i.e. indistinguishable) particle, previously prepared in the superposition state $|\psi\rangle_M = \frac{1}{\sqrt{2}}(|A\rangle_M + |B\rangle_M)$, where the subscript M refers to the measurement device. The quantum particle produced in S travels from the source to A and B

and, after passing through the encoding ports, arrives to the measurement devices where it gets measured together with the particle M.

Consider a strategy where the players agree to output a random bit each time they detect two or no particles at their respective locations. This event occurs if both slits are open ($x = y = 0$) with probability 1/2; if one slit is open and the other is closed ($x \oplus y = 1$) it occurs with probability 3/4; finally, for both slits closed ($x = y = 1$) it occurs always. Overall, the player will output random bits in three quarterrs of the runs and corresponds to the classical probability of success of 1/2.

Let us now consider the cases when one particle is detected by Alice and the other by Bob, which would occur in the other quarter of the runs. Using the formalism of Fock space , we introduce four ladder operators $a^\dagger_{S/M}$ and $b^\dagger_{S/M}$, which can be either fermionic or bosonic, to designate the two different locations for each of the particles. For instance, the joint state of the particles (before measurement), for the case of both slits open ($x = y = 0$) is given by

$$\frac{1}{2}(a^\dagger_S + b^\dagger_S)(a^\dagger_M + b^\dagger_M) |0\rangle_{AS} |0\rangle_{AM} |0\rangle_{BS} |0\rangle_{BM} . \tag{4.7}$$

Here, for example, $|0\rangle_{AS}$ labels the vacuum state for the source particle located at Alice's side. Making explicit the action of the ladder operators on the vacuum states, allows to show the difference between fermionic and bosonic statistics:

$$\begin{aligned} &|1\rangle_{AS} |1\rangle_{AM} |0\rangle_{BS} |0\rangle_{BM} + |1\rangle_{AS} |0\rangle_{AM} |0\rangle_{BS} |1\rangle_{BM} \pm \\ &|0\rangle_{AS} |1\rangle_{AM} |1\rangle_{BS} |0\rangle_{BM} + |0\rangle_{AS} |0\rangle_{AM} |1\rangle_{BS} |1\rangle_{BM} , \end{aligned}$$

where the $\pm$ holds for bosons/fermions, respectively.

It is convenient to introduce the following "qubit" states for A and B,

$$|0\rangle_A = |1\rangle_{AS} |0\rangle_{AM} , \qquad |1\rangle_A = |0\rangle_{AS} |1\rangle_{AM} , \tag{4.8}$$

$$|0\rangle_B = |1\rangle_{BS} |0\rangle_{BM} , \qquad |1\rangle_B = |0\rangle_{BS} |1\rangle_{BM} . \tag{4.9}$$

These states constitute Alice's and Bob's qubits and can be fully manipulated locally, e.g. by means of linear-optical elements, such as beam-splitters and phase-shifters. Formally, this means to locally apply transformations of the group $SU(2)$ to these qubit states. To translate any matrix $u \in SU(2)$ in second quantization formalism, firstly one needs to identify its generator (Hamiltonian), i.e. $u = e^{i\hat{h}}$, with $\hat{h}_{ij}$ being a 2×2 Hermitian matrix. The corresponding Hamiltonian, in the Fock space representation, reads $\hat{H} = \sum_{ij} h_{ij} a^\dagger_i a_j$. This, in turn, defines the generic transformation of $SU(2)$ in Fock space as $U = e^{i\hat{H}}$. The transformation U preserves the qubit subspace spanned by $\{|0\rangle_A , |1\rangle_A\}$, and its action simply reduces to the action of $u \in SU(2)$ [97]. The same holds for the operators on Bob's side. Note also that these operators take the same form for both fermions and bosons.

The players will then perform the measurements within these qubit (sub)spaces, i.e. spanned by $\beta_A = \{|0\rangle_A , |1\rangle_A\}$ and $\beta_B = \{|0\rangle_B , |1\rangle_B\}$. Since we are interested only in cases for which the measurement reveals one particle per party, this situation occurs only when $x = y = 0$ or $x \oplus y = 1$, and we label the three post-selected states by $|\psi_{xy}\rangle$. A simple calculation shows

$$\begin{aligned} |\psi_{00}\rangle &= \frac{1}{\sqrt{2}}(|0\rangle_A |1\rangle_B \pm |1\rangle_A |0\rangle_B), && (4.10) \\ |\psi_{01}\rangle &= |0\rangle_A |1\rangle_B , && (4.11) \\ |\psi_{10}\rangle &= |1\rangle_A |0\rangle_B , && (4.12) \end{aligned}$$

where the $\pm$ refers to bosons/fermions, respectively. Now, we label local measurement projectors as $\Pi^{A/B}_{a/b} = \frac{1}{2}\left(\mathbb{1} + (-1)^{a/b}\sigma_{A/B}\right)$ for A and B, respectively. Here, $\sigma^2_{A/B} = \mathbb{1}$ are binary observables. These operators reside in the qubit subspaces spanned by β_A and β_B. To implement this measurements one can locally interfere the "M" and "S" particles at a beam splitter for each of the two locations. Notice however, that the particles on Bob's and Alice's sides will never be brought together again.

Let us analyze the probability of success case by case. For $x = y = 0$, in half of the cases A and B achieve $1/2$ (this accounts for the situations where local detectors register two or no particles). In the other two cases, the probability of success is given by $\langle\psi_{00}| \Pi^A_0 \otimes \Pi^B_0 + \Pi^A_1 \otimes \Pi^B_1 |\psi_{00}\rangle$. Hence,one has to average these two possibilities:

$$\begin{aligned} p(00|00) + p(11|00) &= \frac{1}{4} + \frac{1}{2} \langle\psi_{00}| \Pi^A_0 \otimes \Pi^B_0 + \Pi^A_1 \otimes \Pi^B_1 |\psi_{00}\rangle \\ &= \frac{1}{2} + \frac{1}{4} \langle\psi_{00}| \sigma_A \otimes \sigma_B |\psi_{00}\rangle . && (4.13) \end{aligned}$$

Similarly, for $x = 0$ and $y = 1$, the measurement reveals one particle per party in $1/4$ of the cases only (in other $3/4$ of the cases A and B achieve the success of $1/2$ by outputting random results), thus we have

$$\begin{aligned} p(01|01) + p(10|01) &= \frac{3}{8} + \frac{1}{4} \langle\psi_{01}| \Pi^A_0 \otimes \Pi^B_1 + \Pi^A_1 \otimes \Pi^B_0 |\psi_{01}\rangle \\ &= \frac{1}{2} - \frac{1}{8} \langle\psi_{01}| \sigma_A \otimes \sigma_B |\psi_{01}\rangle . && (4.14) \end{aligned}$$

In complete analogy, for $x = 1$ and $y = 0$ we get

$$\begin{aligned} p(01|10) + p(10|10) &= \frac{3}{8} + \frac{1}{4} \langle\psi_{10}| \Pi^A_0 \otimes \Pi^B_1 + \Pi^A_1 \otimes \Pi^B_0 |\psi_{10}\rangle \\ &= \frac{1}{2} - \frac{1}{8} \langle\psi_{10}| \sigma_A \otimes \sigma_B |\psi_{10}\rangle . && (4.15) \end{aligned}$$

Finally, for the case $x = y = 1$, A and B output random bits always, thus, the probability of success is $p(00|11) + p(11|11) = 1/2$. Putting everything together we have

$$P_{win} = \frac{1}{2} \pm \frac{1}{32} \left(\langle 0| \sigma_A |1\rangle \langle 1| \sigma_B |0\rangle + \langle 1| \sigma_A |0\rangle \langle 0| \sigma_B |1\rangle\right). \quad (4.16)$$

The maximal value is achieved for $\sigma_A = \pm\sigma_B = \sigma_x$, where $\pm$ refers again to bosons/fermions, respectively. Therefore, the optimal quantum value is $P^{Q1}_{win} = \frac{9}{16}$. We can also explicitly calculate all the coherence equalities, as defined in Eq. (4.3), that for this choice of measurements give a maximal violation of $I_{ab} = \frac{(-1)^{a\oplus b}}{8}$.

Second example. We introduce now an alternative example that, while not making use of pre-shared coherence, requires to violate the particle-number conservation, thereby making this example fundamentally unattainable for fermions [98]. Suppose that the initial state of the particles produced by the source is (in second quantization) $|\psi\rangle = s_0 |1\rangle_A |0\rangle_B + s_1 |0\rangle_A |1\rangle_B$, with $|s_0|^2 + |s_1|^2 = 1$. Here $|0\rangle_A$ and $|1\rangle_A$ are the states associated with, respectively, zero or one particle in mode A (i.e., in the path $S - A$). The analogous notation holds for mode B. The associated two-mode density matrix is given by $\rho^{AB} = |\psi\rangle\langle\psi|$. The operation associated to blocking the path is the "blocking" operator $\mathcal{B}$ [99]. When a system in mode A, prepared in some state ρ^A, arrives at the blocker, it undergoes the operation $\mathcal{B}_A(\rho^A) = |0\rangle\langle 0|_A$, for an arbitrary input state ρ^A. Adapted to the present scenario, we have the following blocking operators for modes A and B, respectively:

$$\begin{aligned} \mathcal{B}_A\left(\rho^{AB}\right) &= |0\rangle\langle 0|_A \otimes \rho^B, \\ \mathcal{B}_B\left(\rho^{AB}\right) &= \rho^A \otimes |0\rangle\langle 0|_B, \end{aligned} \tag{4.17}$$

where $\rho^A = Tr_B\left(\rho^{AB}\right)$ and $\rho^B = Tr_A\left(\rho^{AB}\right)$ are the reduced density matrices of the two subsystems B and A respectively. If no blocker is introduced, then the corresponding state does not undergo any influence, thus the identity transformation is applied.

The introduction of the blockers or otherwise transforms the input state ρ^{AB} into the state ρ_{xy}, which now encodes the inputs x and y (as defined above) by the transformation:

$$\rho_{xy} = (\mathcal{B}_A)^x (\mathcal{B}_B)^y \rho^{AB}. \tag{4.18}$$

Hence, one has $\rho_{00} = \rho^{AB}$, $\rho_{01} = \rho_A \otimes |0\rangle\langle 0|_B$, $\rho_{10} = |0\rangle\langle 0|_A \otimes \rho_B$ and $\rho_{11} = |0\rangle\langle 0|_A \otimes |0\rangle\langle 0|_B$. Let A and B perform binary measurements in their respective Fock spaces, defined by $\Pi_{a/b}^{A/B} = \frac{1}{2}\left(\mathbb{1} + (-1)^{a/b}\sigma_{A/B}\right)$. Here $\sigma_{A/B}$ are single-quibit operators that reside in local Fock spaces spanned by the vacuum $|0\rangle_{A/B}$ and single-particle state $|1\rangle_{A/B}$. The conditional probabilities are given by $p(ab|xy) = \mathrm{Tr}\left[\rho_{xy}\Pi_a^A \otimes \Pi_b^B\right]$ and

$$I_{ab} = \mathrm{Tr}\left[\Pi_a^A \otimes \Pi_b^B \sum_{xy=0}^{1} (-1)^{x\oplus y} \rho_{xy}\right] \tag{4.19}$$

$$= s_0 s_1^* \langle 0| \Pi_a^A |1\rangle \langle 1| \Pi_b^B |0\rangle + h.c. \tag{4.20}$$

The probability of success (4.6) evaluates to

$$P_{win} = \frac{1}{2} + \frac{1}{8}(s_0 s_1^* \langle 0| \sigma_A |1\rangle \langle 1| \sigma_B |0\rangle + h.c.). \tag{4.21}$$

The maximum is achieved for $\sigma_A = \sigma_B = \sigma_x$ and $s_0 = s_1 = \frac{1}{\sqrt{2}}$, for which we find:

$$P_{win}^{Q_2} = \frac{5}{8}. \tag{4.22}$$

One can then evaluate $I_{ab} = \frac{(-1)^{a\oplus b}}{4}$, which clearly violates all four coherence equalities (4.4).

Note that in this example the implementations of measurements $\Pi_{a/b}^{A/B}$ requires the read-out in superposition of the vacuum state and a single-particle excitation, thus requiring the violation of the particle number conservation. This makes the proposal demanding for bosonic particles, whereas the parity superselection rule completely forbids this for fermions [100, 98]. Nevertheless, such measurements are in principle physical for bosonic particles, and have, in fact, been implemented for single-photons by transferring the photonic excitations to atoms [101, 102, 103].

4.5 Conclusions

In this paper, we have investigated the possibility of witnessing a quantum superposition of communication, by means of probabilistic correlations between distant parties. Remarkably, this procedure does not require to recombine the beams to unambiguously detect a superposition state. Phrasing this problem in terms of a communication game, we have derived a coherence equality that is satisfied by any classical communication (i.e., when the information carrier is a well-localized particle). On the contrary, we have provided two concrete examples –experimentally implementable– where the use of quantum resources for communication allows a violation of the coherence equality, and therefore certifies the presence of quantum superposition. Remarkably, this is done by local measurements only.

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Appendix 4.A: Statistical test of the coherent equality

One could wonder how is the classical equality $P_{win}^{Class} = \frac{1}{2}$ testable in practice, as being only satisfied for one precise value instead of an interval (standardly one tests the Bell's inequality, i.e. $P_{win} \leq P_{Class}$). In order to make an operationally meaningful statement, suppose an experimenter has performed N repeated game trials, as defined above, and has recorded the sequence of outcomes f_i (for $i = 1 \cdots N$), where $f_i = 1$ means success and $f_i = 0$ failure of the i-th trial. The estimation of the probability of success is given by the relative frequency $F_N = \frac{1}{N}(f_1 + \cdots + f_N)$. It is clear that the classical equality $P_{win}^{Class} = 1/2$ guarantees F_N to be a martingale random variable, therefore by the straightforward application of the Azuma's theorem [104] we have

$$P\left[\left|F_N - \frac{1}{2}\right| \geq \varepsilon\right] \leq 2e^{-2N\varepsilon^2}, \tag{4.23}$$

for every $\varepsilon > 0$. This means that the probability of deviation of the relative frequency from 1/2 is exponentially suppressed with the number of experimental trials for any possible classical scenario. Finally, from a concrete data record one can use the inequality above to calculate the statistical significance of a (possible) violation.

Chapter 5

Probing quantum coherence at a distance and Aharonov-Bohm nonlocality

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I gave conceptual contributions to this work and contributed to writing a considerable part of the paper.

Abstract

In a standard interferometry experiment, one measures the phase difference between two paths by recombining the two wave packets on a beam-splitter. However, it has been recognized that the phase can also be estimated via local measurements, by using an ancillary particle in a known superposition state. In this work, we further analyse these protocols for different types of particles (bosons or fermions, charged or uncharged), with a particular emphasis on the subtleties that arise when the phase is due to the coupling to an abelian gauge field. In that case, we show that the measurable quantities are spacetime loop integrals of the 4-vector potential, enclosed by two identical particles or by a particle-antiparticle pair. Furthermore, we generalize our considerations to scenarios involving an arbitrary number of parties performing local measurements on a general charged fermionic state. Finally, as a concrete application, we analyze a recent proposal involving the time-dependent Aharonov-Bohm effect [14].

5.1 Introduction

One ubiquitous feature that distinguishes quantum and classical theory is the superposition principle. In order to detect the presence of a quantum superposition (coherence), one has to perform measurements of at least two incompatible observables and, in fact, it is sufficient to measure the relative phase between the two (or more) states supposedly in superposition, in addition to the absolute value of the amplitudes. This can for example be achieved by an interferometric experiment, whose simplest implementation is the Mach-Zehnder interferometer.

There, a single particle is prepared in superposition of two spatially separated paths (by sending it through a beam-splitter). The phase difference between the paths can be measured by recombining the paths at a second beam-splitter and by collecting the statistics of detectors placed at the output ports.

A celebrated variant of this interferometric experiment allows to detect a phase that is due to the interaction between a charged quantum particle (e.g. an electron) and the electromagnetic potential, known as the Aharonov-Bohm (A-B) effect [105, 106, 107]. In particular, this experiment features a Mach-Zehnder interferometer that encircles a solenoid such that, despite the electromagnetic field being zero everywhere on the paths visited by the electron, there is a measurable phase that is directly proportional to the magnetic flux through the surface crossed by the solenoid. It thus seems that a relative quantum phase can only be measured indirectly by recombining the two paths in a closed interferometer, and arguably this limitation looks even more dramatic in the case of Aharonov-Bohm-like phases, since they depend on the value of the total electromagnetic flux contained in a closed region. Despite these indications, a number of works have proposed protocols to detect quantum superpositions without needing to reinterfere the beams, i.e. by using only local operations and classical communication (LOCC) [108, 109, 14, 110, 111, 112].

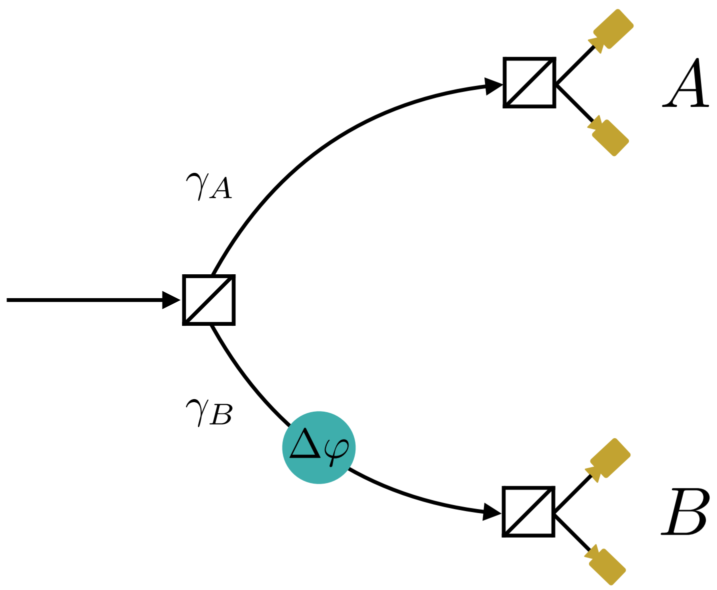

Figure 5.1: A particle is sent through a 50/50 beam-splitter, along two paths, γ_A and γ_B. For simplicity, we model the phase difference as arising due to a phase shifter placed along one of the paths. The goal of Alice and Bob is to measure the phase $\Delta\varphi$ using LOCC.

In this work, we further analyze these kinds of protocols, characterizing their domain of applicability to different types of particles (bosons or fermions, charged or uncharged) and emphasize the constraints imposed by superselection rules and gauge symmetries in determining which observables are measurable (as discussed before in, e.g. Refs. [113, 114, 115, 116]). After a brief analysis of uncharged bosons, we focus on uncharged fermions for which the parity superselection rule forbids certain measurements. Nevertheless, one can circumvent this limitation by using an ancillary system as a resource; indeed, we show that an arbitrary number of parties sharing a generic fermionic pure state, can perform full state tomography by means of a known delocalized ancillary state and LOCC.

We then proceed with the case of electromagnetically charged particles and explain issues arising from gauge coupling and gauge invariance. Contrarily to what the authors of Ref. [14] have recently proposed, we show that in protocols involving an ancillary system and local measurements, the measured phase corresponds always to the net phase picked up around a closed loop in spacetime. Thus, even if it is true that one can extract information about the surrounding electromagnetic field by means of only LOCC conducted at distant locations (i.e., without recombining the paths in an interferometer) and an ancillary system, its value is still equal to the electromagnetic flux through an hyper-surface enclosed by the paths travelled by the particle(s) in spacetime. We then generalize the latter result to an arbitrary number of sources and parties and show that all information about the gauge field that the parties can gather from LOCC can be reconstructed from bipartite loop integrals for all pairs of sources and parties.

In the final section, we explicitly apply our considerations to the scenario proposed in [14, 111], which deals with the time-dependent Aharonov-Bohm effect. We thus show that, even if it is true that the Aharonov-Bohm phase is detectable by means of only LOCC conducted at distant locations (i.e., without recombining the paths in an interferometer), its measurable value is still equal to the (gauge-independent) electromagnetic flux through the hyper-surface enclosed by the paths traveled by the particle(s) in spacetime and as such it is not acquired locally.[1]

5.2 Local measurements of the interferometric relative phase

In this section we review some protocols allowing to measure locally the relative phase acquired along two arms of an interferometer, without having to recombine the two paths. The setup we consider is illustrated in Figure 5.1 and is similar to the settings studied in Refs. [14, 110, 111] . Two experimenters, Alice and Bob, reside at two distant locations and a quantum particle is sent in an equal-weighted superposition of the two paths towards the parties. Most of the following applies to any particle type (boson or fermion, charged or uncharged); when specificities arise we will point them out explicitly.

We assume that the situation can be described as a superposition over classical paths, γ_A and γ_B, leading to Alice and Bob. In this case, the phase acquired along a path is simply related to the Lagrangian action S of the corresponding classical trajectory. More precisely, let the phase difference be defined as (we are using

[1]The term "non-locality" has several meanings in quantum theory (see e.g. [108] for a concise overview). Indeed, it most customarily refers to the impossibility of explaining the statistical correlations of quantum measurements conducted at distant locations by means of local hidden-variables. Yet, in the context of the Aharonov-Bohm effect –and thus throughout this paper– non-locality refers to a "topological" property of quantum theory. Namely, the fact that a measurable quantum phase in the presence of a gauge coupling is not acquired by summing up physically meaningful gauge-invariant quantities in a point-by-point (i.e. local) fashion along the path travelled by the particle. Rather, the phase due the gauge coupling can only be attributed to closed paths in spacetime, and it can be expressed as the net electromagnetic flux through a hypersurface enclosing that path.

units where $\hbar = 1$)

$$\Delta\varphi = i(S(\gamma_B) - S(\gamma_A)) = i \int_{t_i}^{t_f} dt \big(\mathcal{L}(\gamma_B(t)) - \mathcal{L}(\gamma_A(t))\big), \tag{5.1}$$

where t_i is the time when the particle is emitted from the beam-splitter and t_f is the time when the parties receive the particle. The shared state between the two parties at the final time t_f is then

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|A\rangle + e^{i\Delta\varphi} |B\rangle\right), \tag{5.2}$$

where $|A\rangle$ and $|B\rangle$ correspond to states where the particle is localised at Alice's or Bob's position, respectively. When dealing with interferometric experiments it is convenient to adopt a second-quantized notation:

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(a_A^\dagger + e^{i\Delta\varphi} a_B^\dagger\right) |0\rangle, \tag{5.3}$$

where $a_{A/B}^\dagger$ represent creation operators (either bosonic or fermionic), creating one particle at Alice's and Bob' location respectively, and $|0\rangle$ is the vacuum state.

The task consists in estimating the phase difference $\Delta\varphi$, using only LOCC. This would be straightforward if Alice and Bob were able to implement local projective measurements that involve the preparation of superposition of different number states, i.e., of the form $\Pi_\pm = |\pm\rangle \langle\pm|$, where

$$|\pm\rangle_{A/B} = \frac{1}{\sqrt{2}} \left(\mathbb{1} \pm a_{A/B}^\dagger\right) |0\rangle. \tag{5.4}$$

This, however, is only possible for uncharged bosons, because otherwise this measurement is forbidden by the parity and/or charge superselection rules [117, 118, 119, 100]. For photons, these measurements can be carried out by making each local mode interact with another system such as an atom, and then locally measuring the two atoms in an appropriate basis [101, 108, 120, 121], or by measuring local interference with a coherent state [108]. However, for general species of particles, the measurement of Eq. (5.4) is fundamentally forbidden (by specific superselection rules) and the determination of the relative phase is apparently impossible.

5.2.1 A general method for local phase measurements

Despite the above-mentioned issues, a possible solution to the apparent impossibility of locally measuring phase differences of fermions has been proposed in Refs. [108, 14]. The idea is to bypass the superselection rule by using an ancillary particle in a known superposition state as a resource which enables the phase measurement. The protocol proceeds as follows. Let us assume that Alice and Bob already possess an ancillary particle in spatial superposition of their locations and that the phase difference between the two components of the wave function is null.[2] After some time, the second particle is sent in spatial superpostition of

[2] As we will discuss in detail in Section 5.3, for charged particles this assumption is not as innocent as it may look, for it corresponds to fixing a gauge.

the two paths, of which we want to measure the relative phase difference. Upon reaching Alice and Bob, the total state of the two-particle system is

$$|\psi\rangle = \frac{1}{2}\left(a^\dagger_{A_1} + a^\dagger_{B_1}\right)\left(a^\dagger_{A_2} + e^{i\Delta\varphi}a^\dagger_{B_2}\right)|0\rangle\,, \tag{5.5}$$

where the mode corresponding to the ancillary particle is indicated by subscript 1 and the mode corresponding to the second one –whose relative phase needs to be estimated– by subscript 2. Both Alice and Bob now possess two "wave packets", one arising from each particle. Furthermore, suppose they possess local beam-splitters allowing them to locally interfere their wave packets, and detectors allowing to measure the outcome statistics at the output ports of the beam-splitters. In order for the parties to be able to perform the measurements, the particles must have the same parity and charge because of the associated superselection rules. We thus choose the particles to be identical. Now, suppose that Alice and Bob perform their measurements and discard the results if they detect either zero or two particles: the postselected quantum state of interest is then (prior to the interference through the beam-splitter)

$$|\psi\rangle_{PS} = \frac{1}{\sqrt{2}}\left(a^\dagger_{B_1}a^\dagger_{A_2} + e^{i\Delta\varphi}a^\dagger_{A_1}a^\dagger_{B_2}\right)|0\rangle\,, \tag{5.6}$$

The beam-splitters, followed by measurements at the output ports, implement local projective measurements of the form $\Pi_\pm = |\pm\rangle\langle\pm|$ acting on $|\psi\rangle_{PS}$, where

$$|\pm\rangle_{A/B} = \frac{1}{\sqrt{2}}\left(a^\dagger_{A_1/B_1} \pm a^\dagger_{A_2/B_2}\right)|0\rangle\,. \tag{5.7}$$

Note that the outcome probabilities do not depend on whether the particles are fermions or bosons, because we are only concerned with the subset of events in which a single particle is found at each of the two locations. The outcome statistics enables Alice and Bob to reconstruct the phase shifter's phase $\Delta\varphi$ using only LOCC despite issues caused by the parity superselection rule.

The latter considerations were concerned with the case of two parties sharing one particle. In Appendix 5.A, we provide a generalization to a scenario involving N parties sharing a general fermionic uncharged state, which can now involve an arbitrary number of excitations. Notice that the parity superselection rule allows the state to be in a superposition of states with different numbers of fermions (albeit with equal parity). We show that, analogously to the single-particle case, the parties can fully reconstruct an arbitrary fermionic pure state using LOCC and a common global state as a resource. The protocol essentially involves the usage of an auxiliary reference state prepared by an external party and local beam-splitter operations performed by the N parties. Even though the parties' measurements are local, the overall process involves the preparation of a global (entangled) state which cannot be generated by local means; this is in accord with the results of Refs. [122, 123].

5.3 Charged particles and the gauge-dependence of relative phases

Let us now turn to charged particles, for which we already pointed out that measurements of the type in expression (5.4) are prohibited by the charge superselection rule. Furthermore, in the general protocol of Section 5.2.1, we had to assume, in Eq. (5.5), that it is possible to prepare the ancillary particle in a known state. As we will emphasize in this section, since the phase acquired along a path is gauge-dependent, the condition of Eq. (5.5), namely that the ancillary state has zero relative phase, is not a gauge-independent property. In the following we show how the protocol can nevertheless be used to provide local measurements of gauge-invariant phases, i.e. phases that are acquired by integrating along a closed loop in spacetime. Hereinafter, we will focus on charged fermions, and, as a matter of simplicity, we will consider the particular case of electrons. Consider once again the setup of Fig. 5.1. The phase difference is given by Eq. (5.1), with Lagrangian

$$\mathcal{L} = \frac{1}{2m}\dot{\vec{x}}^2 - e\dot{\vec{x}} \cdot \vec{A} + eV(\vec{x}) \,. \tag{5.8}$$

We emphasize that setting the gauge potential to zero, even in the absence of external electric and magnetic fields, amounts to fixing a gauge. Hence, the accumulated phase on the two paths now necessarily depends also on the electromagnetic potential:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(e^{ie\int_{\gamma_A} A^\mu dx_\mu} |A\rangle + e^{ie\int_{\gamma_B} A^\mu dx_\mu} e^{i\beta} |B\rangle\right), \tag{5.9}$$

where β is the gauge-invariant "mechanical" phase difference, i.e. due to the kinetic term $\frac{1}{2m}\dot{\vec{x}}^2$ in the Lagrangian; whereas A^μ is the 4-vector potential, i.e. (in the units where $c = 1$) $A^\mu = (V, \vec{A})$, and we use a metric with signature $(+, -, -, -)$. Therefore, the phase difference between the two paths now reads

$$\Delta\varphi = \beta + e\int_{(\gamma_B - \gamma_A)} A^\mu dx_\mu, \tag{5.10}$$

which is an explicitly gauge-dependent quantity. We thus see why it is necessary to introduce the charge superselection rule which forbids us to implement the projectors of Eq. (5.4): if this rule did not exist we would be able to measure physically meaningless gauge-dependent quantities![3] This argument has two main consequences: (i) the principle of gauge invariance implies the charge superselection rule and prohibits this type of local tomographic protocols for a delocalized electron. Furthermore, (ii) the acquired phase difference between the two paths is not an observable since it is a gauge-dependent quantity: it might be equal to the mechanical phase shift β only in a specific gauge.

On the other hand, note that if the electron's wave packets are reinterfered on a beam-splitter as it happens in a standard Aharonov-Bohm experiment, the phase difference between the two paths connecting the two beam-splitters reduces to

$$\Delta\varphi = \beta + e\oint A^\mu dx_\mu, \tag{5.11}$$

where the loop integral is performed around the whole interferometer. The phase is now a gauge-independent quantity and it depends on the distribution of electromagnetic currents in spacetime. A specific case is a regular Mach-Zehnder interferometer, where, since there are no currents, the loop integral vanishes and, as expected, the total phase difference is equal to the mechanical phase shift β only.

[3] A similar argument has already been invoked for example in [124]; for a more formal treatment of the relationship between the charge superselection rules and gauge invariance, see [125].

5.3.1 Measuring phase differences at a distance

We now show how, as in the protocol of Section 5.2.1, we can exploit an auxiliary identical particle (another electron) in order to circumvent the limitation imposed by the charge superselection rule and gauge invariance. The crucial difference with respect to the previous case is that, as already emphasized, there is an inevitable coupling to the gauge potential regardless of the surrounding electromagnetic sources, which implies that even the ancillary particle necessarily acquires a gauge-dependent phase difference which may be null only in a specific gauge (i.e. we cannot assume that Alice and Bob can prepare the ancillary particle in a known state without fixing a gauge). After the particles are sent to the parties, the two-particle state is

$$\begin{aligned} |\psi\rangle = \frac{1}{2}&\left(e^{ie\int_{\gamma_{A_1}} A^\mu dx_\mu} a^\dagger_{A_1} + e^{ie\int_{\gamma_{B_1}} A^\mu dx_\mu} e^{i\beta_1} a^\dagger_{B_1}\right) \cdot \\ &\left(e^{ie\int_{\gamma_{A_2}} A^\mu dx_\mu} a^\dagger_{A_2} + e^{ie\int_{\gamma_{B_2}} A^\mu dx_\mu} e^{i\beta_2} a^\dagger_{B_2}\right) |0\rangle\,, \end{aligned} \tag{5.12}$$

where the mode corresponding to the first particle is indicated by subscript 1 and the second one by subscript 2, the paths $\gamma_{A_1}, \gamma_{A_2}, \gamma_{B_1}$ and γ_{B_2} are defined in Fig. 5.2, and β_i is the mechanical phase difference between paths γ_{B_i} and γ_{A_i}. Alice and Bob again perform measurements with local beam-splitters and postselect on the one-particle subsector:

$$|\psi\rangle_{PS} = \frac{1}{\sqrt{2}}\left(a^\dagger_{A_2} a^\dagger_{B_1} + e^{i\Delta\varphi} a^\dagger_{A_1} a^\dagger_{B_2}\right)|0\rangle\,, \tag{5.13}$$

where the cumulative phase difference is now given by

$$\Delta\varphi = \beta_2 - \beta_1 + \varphi_{A_1} + \varphi_{B_2} - \varphi_{A_2} - \varphi_{B_1}, \tag{5.14}$$

with

$$\varphi_{A_i} \equiv e\int_{\gamma_{A_i}} A^\mu dx_\mu, \tag{5.15}$$

where $i \in \{1, 2\}$ labels the paths on Alice's side, and with an analogous expression for φ_{B_i}. As in Section 5.2.1, the quantity $\Delta\varphi$ can be inferred from the statistics of the detection clicks at the two beam-splitters, respectively situated at Alice's and Bob's locations.

In Figure 5.2 we portray the paths of the two particles in a spacetime diagram and show that they enclose a closed path. Therefore, the additional phase accumulated because of the interaction with the electromagnetic potential is gauge invariant. If there are no surrounding currents or fields, the loop integral vanishes and only the mechanical phase shift remains. On the contrary, if the particles are surrounded by an arbitrary distribution of currents, the loop integral does not generally vanish and can depend explicitly on the trajectory travelled by the ancillary particle (even if it is not acted on by classical forces).

Had the two particles different charges e_1 and e_2, the total phase $\Delta\varphi$ would involve the sum of two paths which do not compose into a spacetime loop integral as in (5.14) and would thus not be a gauge invariant quantity. However, in that case, the required measurement would not be possible since non-identical particles do not interfere with each other. Therefore, we see a consistency between gauge

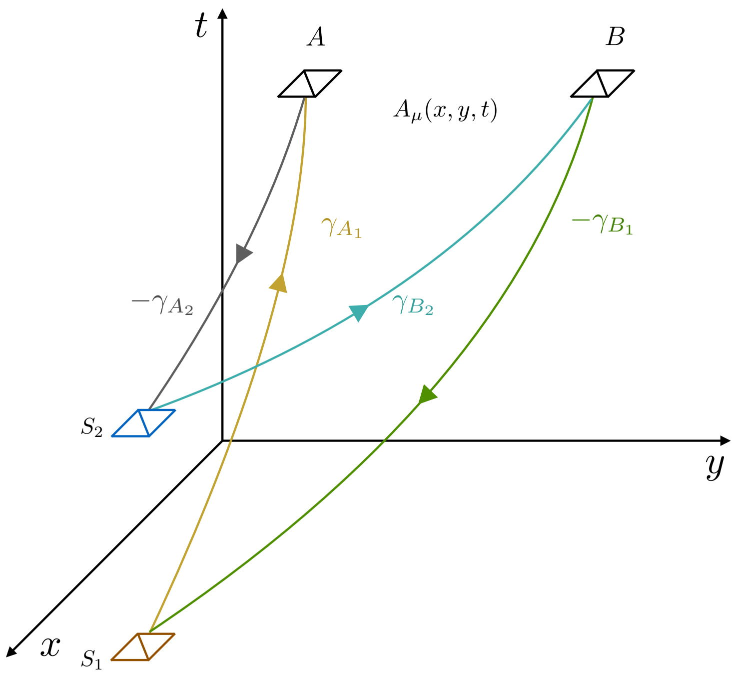

Figure 5.2: Spacetime diagram of the protocol described in Section 5.3.1. Two identical particles are prepared at locations S_1 (resp. S_2) in an equal superposition of paths γ_{A_1} and γ_{B_1} (resp. γ_{A_2}, γ_{B_2}). The parties perform local measurements which allow them to reconstruct the phase $\Delta\varphi$ of Eq. (5.14), which is a gauge-invariant spacetime loop integral around the shown path (the arrows are drawn in the direction of integration of the loop integral).

invariance, superselection rules and the operational attainability of the required local measurements.

In Appendix 5.B, we analyse a variation of the above experiment introduced by Aharonov and Vaidman in [108], which instead of two identical particles (e.g. two electrons) involves a particle-antiparticle pair (e.g. an electron and a positron). We show that the same quantity $\Delta\varphi$ from Eq. 5.14 can be estimated by measuring photons resulting from the annihilation of the electron with the positron.
From these considerations we draw the following conclusions:

(i) loops (in spacetime) can be closed by states involving two identical particles or by a particle-antiparticle pair, i.e. by two different excitations of the same quantum field (contrary to the standard Aharonov-Bohm effect where the loop is closed by a single electron);

(ii) the paths traced by the two particles can be "glued" together via local (interference) measurements, or via particle-antiparticle annihilation;

(iii) in the presence of gauge coupling, the distinction between "primary" particle (whose phase we are trying to estimate) and "ancillary" particle (which serves to circumvent the superselection rule) is not well defined and can be manifested only in a specific gauge; the measured quantity is a collective property of the two excitations, namely the spacetime loop integral.

5.3.2 General case

In Section 5.2.1 and in Appendix 5.A we saw the possibility of performing state tomography of an arbitrary uncharged fermionic state using LOCC and an ancillary system. Now, we want to analyse a similar scenario in the case of charged particles. However, since the concepts of "primary" and "ancillary" systems are not well defined in the presence of gauge coupling, here we ask a different question. Suppose that an arbitrary number of parties perform local measurements on an arbitrary state of multiple charged particles: what are the spacetime loop integrals that the outcome probabilities depend on? Can all probabilities be reconstructed from loops similar to the one depicted in Fig. 5.2? In Appendix 5.C we show that this is indeed the case. More precisely, suppose that we have d sources emitting single identical charged fermions at d spacetime points. Each source prepares a single particle in an arbitrary superposition of spatial trajectories. Moreover, suppose that there are N parties who perform local number-preserving (linear) operations and measurements at N space-like separated points located on a hypersurface which lies in the future of the d sources. We show that the joint probability of the local measurement outcomes can be fully reconstructed from loop integrals as in Fig. 5.2 for all pairs of sources and all pairs of parties. This result shows that all information about the gauge field that is acquirable via local measurements on single particle excitations can be reduced to simple experiments involving two parties and two sources as in Fig. 5.2.

5.4 Non-local generation of the Aharonov-Bohm phase

The traditional interpretation of the Aharonov-Bohm effect [105, 106, 107] is that the electric and magnetic fields are in general not sufficient for describing the physics of certain (quantum) scenarios and that the gauge-theoretic electromagnetic potential is in fact indispensable. Going against the received view, Vaidman has argued that the Aharonov-Bohm phase can be explained without the introduction of potentials if one takes into account the quantum nature of the solenoid [109]. A weakness in his treatment is that it relies on an instantaneous interaction between the solenoid and the electron. Marletto and Vedral [14] –followed by further developments by Saldanha [111]– have recently addressed this problem, concluding that the Aharonov-Bohm phase is acquired locally. In this section we follow up on these recent developments by applying the analysis from the previous section to the specific case where the only source of electromagnetic fields is a solenoid with a time-dependent current. We show that that the Aharonov-Bohm phase is not acquired locally, in the sense that the only measureable quantities involved in this type of experiments correspond to the integral of the 4-vector potential around whole *spacetime loops*. The phase acquired along smaller portions of the particle's path is not measureable.

The scenario studied in Ref. [14] is a special case of the general situation in Fig.5.2. During the journey of the ancillary particle towards the parties, the flux through the coil is set to zero. After the two wave packets of the ancillary particle arrive at the location of the parties, they are trapped using some external field. The current (and thus the flux) in the coil is then increased slowly until it reaches a stationary state, following which the second particle is sent to the parties and the local measurements of (5.7) are performed. The detection probabilities depend

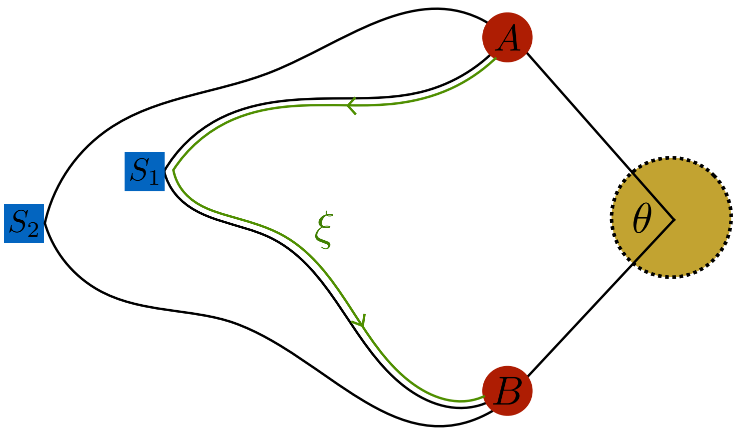

Figure 5.3: Spatial representation of the scenario of Fig. 5.2, for the special case studied in Ref. [14], where the source of A^μ is a solenoid with a time-dependent current. At time $t = 0$ the flux through the solenoid is zero, and a particle is prepared at S_1 in an equal superposition of paths γ_{A_1} and γ_{B_1}. Later, this particle is trapped at locations A and B, following which the flux is increased up to Φ_f. Finally, the second particle is sent at S_2 and each party performs a local measurement on their "wave packets".

on $\Delta\varphi$ as defined in (5.14), where (in the Coulomb gauge) the vector potential in the region outside the coil is given by

$$\vec{A}(\vec{r}, t) = \frac{\Phi(t)}{2\pi(x^2 + y^2)} \vec{r} \times \hat{z}, \tag{5.16}$$

where $\Phi(t)$ is the magnetic flux through the solenoid at time t, and $\vec{r}$ is the position vector of the particle (with the origin of the coordinates centered in the solenoid). Note that the electric field is not zero outside the solenoid at all times, because of the time-dependence of the current in the solenoid. Since the charge density is zero everywhere and at all time, we have that the scalar potential is zero in the Coulomb gauge, and thus the phase that the ancillary particle accumulates due to the solenoid during the time when it is trapped in Fig. 5.2 is zero (in this gauge). Furthermore, when the ancillary electron is moving, there is no current in the coil and consequently no 4-vector potential is interacting with the charge, so the contribution of the solenoid to the phase accumulated along γ_{A_1/B_1} is zero. Hence, we are left with the phase acquired by the second particle

$$\Delta\varphi = e \int_\xi \vec{A} \cdot d\vec{l}, \tag{5.17}$$

where ξ is the path shown in Figure 5.3. Using the expression (5.16) for the vector potential and switching to polar coordinates yields

$$\Delta\varphi = \int_0^\theta \frac{\Phi_f}{2\pi} d\theta' = \frac{\Phi_f}{2\pi}\theta, \tag{5.18}$$

where Φ_f is the final flux through the solenoid and θ is the angle defined in Figure 5.3. The probabilities for the measurement described in Section 5.3.1 depend on $\Delta\varphi$ and thus the outcome statistics allows to estimate $\Delta\varphi$ using LOCC.

It is tempting to interpret the calculation that we have just done, in particular Eq. (5.17), as showing that the phase is locally accumulated along the path ξ. However, this apparent localization of the phase accumulation is merely a consequence of our (arbitrary) choice to work in the Coulomb gauge. In other gauges, the phase accumulated by the ancillary particle is not necessarily zero: only the full loop integral is gauge-independent. Notice that, even though the particles seemingly enclose a spatial region with no electromagnetic fields, the measured quantity $\Delta\varphi$ corresponds to the spacetime loop integral which is non-zero due to the induced electric field (which arises due to the time dependence of the current) piercing through the hypersurface enclosed by the spacetime loop.

5.5 Conclusions and Outlook

We have studied a general protocol allowing to locally measure the relative phases of the state of a particle that is prepared, by using a beam-splitter, in a superposition of different spatial locations. In order to perform this protocol, the localised parties must share a known superposition state of an ancillary particle. We have shown how, using this known state as a resource, one can estimate the relative phase by performing only local measurements, despite the restrictions imposed by the parity superselection rule. Moreover, we extended the protocol to general fermionic systems shared by an arbitrary number of parties and we showed how the parties can perform full state tomography using LOCC and a global state as a resource. We proceeded by addressing the case of electromagnetically charged particles, where the unavoidable coupling to the gauge field makes it impossible to control the relative phase of the ancillary state without fixing the gauge. We have shown that the protocol nevertheless measures a gauge-independent quantity, namely a spacetime loop integral of the 4-vector potential. The protocol requires the two particles to possess equal charges in order to obtain gauge invariant quantities; however, the required measurements would be physically impossible on two different charges due to the charge superselection rule. Alternatively, one can employ a slightly modified protocol that involves particles with opposite charges (e.g. electron and positron) which can annihilate into a pair of uncharged particles (photons): the obtained measurement results yield the same gauge invariant phase as in the former protocol. The latter discussion shows the tight relation between charge conservation, gauge invariance and superselection rules. We then proceeded with the general scenario involving many source and many parties performing LOCC and showed that all probabilities arising from such experiments can be reduced to simple combinations of bipartite loop integrals involving two sources and two parties. Finally, we have applied the protocol to the case of local measurements of phases in the time-dependent Aharonov-Bohm effect, and, in particular, we have demonstrated its application to the setups of Refs. [14, 111]. Since all probabilities obtained in these experiments depend on loop integrals which are explicitly non-local quantities, the interpretation that the phase is acquired locally is not viable, and is apparently manifested only in a specific gauge.

In this manuscript we focused on scenarios which involve quantum particles coupled to Abelian gauge fields; it would be interesting to see to what extent our analysis can be extended to the non-Abelian case, where the probabilities would depend on gauge-invariant functionals of Wilson loop operators. In particular, one should inspect whether the result of section 5.3.2 holds in any gauge theory or only in the Abelian ones.

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Appendix 5.A: Local measurements on generic fermionic uncharged systems

Let us start by considering a simple generalisation of Fig. 5.1 to N parties. The particle goes through a beam-splitter with N output ports, and along each arm it acquires an unknown phase φ_i, so that the state received by the observers is $\frac{1}{\sqrt{N}}\sum_{i=1}^{N} e^{i\varphi_i} c_i^\dagger |0\rangle$, where $c_i^\dagger$ is an operator that creates a particle at the location of observer i. We would like to perform a procedure for estimating the phases φ_i that uses only local operations and classical communication. This can be achieved if the parties share a second particle in a known state $\frac{1}{\sqrt{N}}\sum_{i=1}^{N} c_i^\dagger |0\rangle$. Let each party apply a beam-splitter locally and measure at the click at the output ports, and post-select on cases where the two particles are found at different locations; this happens with probability $1-\frac{1}{N}$. Supposing that the particles are found at positions i and j, the probabilities for each of the 4 possible outcomes are the same as in the case of Fig. 5.1, so the above protocol allows to give an estimate for $\varphi_i - \varphi_j$. After performing many rounds it will be possible to reconstruct all the phases φ_k with a good accuracy. An interesting feature of this protocol is that the post-selection probability goes to one as the number of parties becomes large, which means that in this limit almost all rounds of the experiment yield useful information (in contrast with the two arm case, where half of the rounds have to be discarded).

Turning now to the general case, the most general fermionic state shared by N parties is

$$\begin{aligned} |\psi\rangle = \{ \sum_{\vec{x}_1,\dots,\vec{x}_N} \lambda\left(\vec{x}_1,\dots,\vec{x}_N\right) e^{i\varphi(\vec{x}_1,\dots,\vec{x}_N)} \prod_{j_1} \left(c_{j_1}^{(1)\dagger}\right)^{x_{1 j_1}} \prod_{j_2} \cdot \\ \left(c_{j_2}^{(2)\dagger}\right)^{x_{2 j_2}} \dots \prod_{j_N} \left(c_{j_N}^{(N)\dagger}\right)^{x_{N j_N}} \} |0\rangle \, . \end{aligned} \tag{5.19}$$

In the latter expression the sum ranges over all bit strings, the length of which depends on the maximal number of local modes available to each of the parties (the bit string notation automatically implements the fact that there can be no more than one excitation per mode). $\lambda(\vec{x}_1, ..., \vec{x}_N)$ are real amplitudes, $\varphi(\vec{x}_1, ..., \vec{x}_N)$ are the mechanical phases that we want to estimate, and $c_{j_k}^{(k)\dagger}$ denote fermionic creation operators that create one fermion in the j_k-th mode of the k-th party.
In order to measure the weights $\lambda(\vec{x}_1, ..., \vec{x}_N)$, each party performs a local projective measurement on their local modes, yielding the desired information via

$$\lambda(\vec{x}_1, ..., \vec{x}_N) = |\langle\psi| \prod_{j_1} \left(c_{j_1}^{(1)\dagger}\right)^{x_{1j_1}} ... \prod_{j_N} \left(c_{j_N}^{(N)\dagger}\right)^{x_{Nj_N}} |0\rangle|. \tag{5.20}$$

The parties communicate their local results to an external party who estimates the amplitudes $\lambda(\vec{x}_1, ..., \vec{x}_N)$, prepares a uniform superposition over states with non-zero amplitude and sends it towards the N parties, who store it in separate modes from the ones occupied by the original state. The "copied" state is thus

$$|\tilde{\psi}\rangle = \frac{1}{\sqrt{M}} \left\{ \sum_{\vec{x}_1,...,\vec{x}_N} \prod_{j_1} \left(\tilde{c}_{j_1}^{(1)\dagger}\right)^{x_{1j_1}} ... \prod_{j_N} \left(\tilde{c}_{j_N}^{(N)\dagger}\right)^{x_{Nj_N}} \right\} |0\rangle, \tag{5.21}$$

where $\tilde{c}_{j_k}^{(k)\dagger} \equiv c_{n_k+j_k}^{(k)\dagger}$, with n_k being the maximum number of modes present at the k-th party's location and M is the total number of non-zero components in the original state (5.19).
Next, each party interferes each of the original modes with the corresponding copied modes on local beam-splitters, i.e. $\forall k = 1, ..., N$ and $\forall j_k = 1, ..., n_k$

$$\begin{aligned} c_{j_k}^{(k)\dagger} &\to \frac{1}{\sqrt{2}} \left(c_{j_k}^{(k)\dagger} + \tilde{c}_{j_k}^{(k)\dagger}\right), \\ \tilde{c}_{j_k}^{(k)\dagger} &\to \frac{1}{\sqrt{2}} \left(c_{j_k}^{(k)\dagger} - \tilde{c}_{j_k}^{(k)\dagger}\right). \end{aligned} \tag{5.22}$$

The final joint state, after undergoing the beam-splitter operations, is thus

$$|\psi'\rangle = \{\frac{1}{\sqrt{M}} \sum_{\substack{\vec{x}_1,...,\vec{x}_N \\ \vec{x}'_1,...,\vec{x}'_N}} 2^{-\frac{1}{2}[\sum_{nj}(x_{nj}+x'_{nj})]} \lambda_x e^{i\varphi_x} \cdot$$

$$\prod_{k=1}^{N} \left[\prod_{j_k=1}^{n_k} \left(c_{j_k}^{(k)\dagger} + \tilde{c}_{j_k}^{(k)\dagger}\right)^{x_{kj_k}}\right] \prod_{k'=1}^{N} \left[\prod_{j'_k=1}^{n'_k} \left(c_{j'_k}^{(k')\dagger} - \tilde{c}_{j'_k}^{(k')\dagger}\right)^{x'_{k'j'_k}}\right] \} |0\rangle, \tag{5.23}$$

where λ_x and φ_x stand for $\lambda(\vec{x}_1, ..., \vec{x}_N)$ and $\varphi(\vec{x}_1, ..., \vec{x}_N)$.
Finally, the parties perform local projective measurements of their modes in the occupation number basis; the outcome probabilities depend explicitly on the required phases:

$$P_{xy} = \frac{1}{M 2^{[\sum_{n,m}(x_{nm}+y_{nm})]}} |\lambda_x + (-1)^{s_{xy}} e^{i(\varphi_y - \varphi_x)} \lambda_y|^2, \tag{5.24}$$

where $x = (\vec{x}_1, ..., \vec{x}_N)$ and $y = (\vec{y}_1, ..., \vec{y}_N)$ are two different bit strings, P_{xy} is the probability of projecting on state

$$\left\{ \prod_{k=1}^{N} \left[\prod_{j_k=1}^{n_k} \left(c_{j_k}^{(k)\dagger}\right)^{x_{kj_k}}\right] \prod_{k'=1}^{N} \left[\prod_{j'_k=1}^{n'_k} \left(\tilde{c}_{j'_k}^{(k')\dagger}\right)^{y_{k'j'_k}}\right] \right\} |0\rangle,$$

and s_{xy} is an integer that arises due to fermionic anticommutation relations. The parties can thus estimate all the unknown phases $\varphi(\vec{x}_1, ..., \vec{x}_N)$ using LOCC and a shared global ancillary state.

Appendix 5.B: The protocol involving a particle-antiparticle pair

Instead of two identical particles, one can choose the two particles used in the bipartite protocol of Section 5.3 to be a particle-antiparticle pair, for instance, an electron and a positron (this possibility was discussed already in Ref. [108]). In this case, Alice and Bob can annihilate their wave packets into photon pairs and measure the phase difference between the two components by performing local tomography on the resulting photons in a similar fashion as in section 5.3.1. More precisely, the state of the two particles before the annihilation process is the same as in protocol involving two identical particles, up to a minus sign in the phases acquired by the antiparticle (the "ancillary" system is now a positron with charge $-e$):

$$\begin{aligned}|\psi\rangle = \frac{1}{2}&\left(e^{-ie\int_{\gamma_{A_1}} A^\mu dx_\mu} b_A^\dagger + e^{-ie\int_{\gamma_{B_1}} A^\mu dx_\mu} b_B^\dagger\right) \otimes \\ &\left(e^{ie\int_{\gamma_{A_2}} A^\mu dx_\mu} c_A^\dagger + e^{ie\int_{\gamma_{B_2}} A^\mu dx_\mu} e^{i\beta} c_B^\dagger\right)|0\rangle\,, \end{aligned} \tag{5.25}$$

where $b^\dagger$ and $c^\dagger$ are respectively positron and electron creation operators.
Next, Alice and Bob let the particles interact and they postselect exclusively on processes which give rise to photons, thereby discarding the one-particle sector and those processes in which the pairs scatter without annihilating (e.g. the Bhabba scattering). The annihilation processes give rise to photon pairs of different momenta $\vec{k}$ and $\vec{k'}$:

$$b_i^\dagger c_i^\dagger \to a_{i,\vec{k}}^\dagger a_{i,\vec{k'}}^\dagger, \tag{5.26}$$

where $a_{i,\vec{k}}^\dagger$ denotes a (suitably smeared) creation operator for a photon of momentum $\vec{k}$ produced at location i. The quantum state after the interaction and post-selection is thus

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(a_{A,\vec{k}_A}^\dagger a_{A,\vec{k}'_A}^\dagger + e^{i\Delta\varphi'} a_{B,\vec{k}_B}^\dagger a_{B,\vec{k}'_B}^\dagger\right)|0\rangle\,, \tag{5.27}$$

where the phase difference is now

$$\Delta\varphi' = \beta + \varphi'_{B_1} + \varphi'_{B_2} - \varphi'_{A_2} - \varphi'_{A_1}, \quad \varphi'_{i_j} \equiv e_j \int_{\gamma_{i_j}} A^\mu dx_\mu. \tag{5.28}$$

Since the electric charges of the two particles are $e_1 = -e$ and $e_2 = e$, the phase difference acquired due to the interaction with the gauge potential is given by a spacetime loop integral of the gauge potential and is equal to Eq. (5.14). Once Alice and Bob possess their photons, they can estimate the phase from measurement results of local projections on states

$$|\pm\rangle_i = \frac{1}{\sqrt{2}}\left(\mathbb{1} \pm a_{\vec{k}_i}^\dagger a_{\vec{k}'_i}^\dagger\right)|0\rangle\,. \tag{5.29}$$

Therefore, we found a different procedure which yields the same gauge invariant phase as the one obtained in the protocol involving two identical particles.

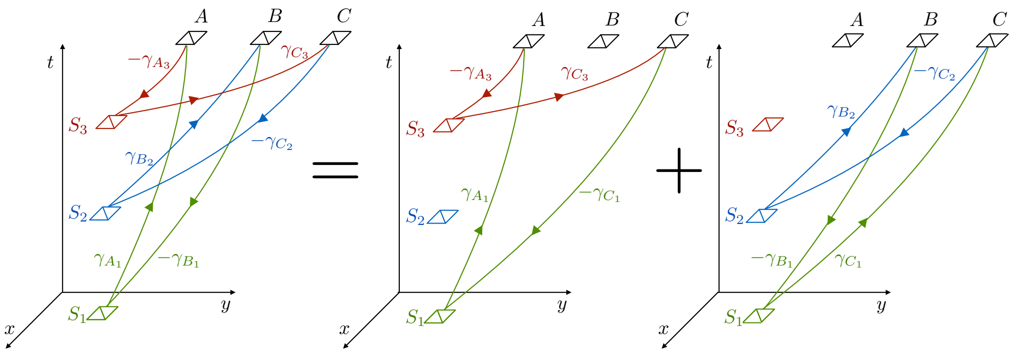

Figure 5.4: The loop integral on the left hand side corresponds to the quantity $\phi_{123} - \phi_{312}$ which can be estimated from one of the outcome probabilities arising in the experiment involving three sources and three parties. The right hand side shows that the latter integral can be decomposed as a sum of two bipartite integrals similar to the ones from Fig. 5.2.

Appendix 5.C: Local measurements on a general charged fermionic state

Before tackling the fully general case, let us first analyse the following simpler case. Suppose that the scenario involves three sources which emit three electrons in spatial superposition towards three parties labelled as A, B and C. Upon receiving the particles, the parties perform local unitary transformations on their pertaining wave packets, detect the particles and postselect on the cases in which each party detects one excitation. The postselected state of interest (prior to the local transformations) is thus

$$|\psi\rangle_{PS} = \frac{1}{\sqrt{6}} \sum_{\substack{i,j,k=1 \\ i\neq j\neq k\neq i}}^{3} e^{i\phi_{ijk}} c^\dagger_{A_i} c^\dagger_{B_j} c^\dagger_{C_k} |0\rangle \,, \tag{5.30}$$

where $c^\dagger$ are fermionic creation operators (for example $c^\dagger_{A_i}$ creates the i-th particle at A's location). The phase ϕ_{ijk} arises due to the coupling to the gauge potential A^μ (for simplicity we omit the mechanical phases):

$$\phi_{ijk} = e\left(\int_{\gamma_{A_i}} A^\mu dx_\mu + \int_{\gamma_{B_j}} A^\mu dx_\mu + \int_{\gamma_{C_k}} A^\mu dx_\mu \right), \tag{5.31}$$

where e.g. γ_{A_i} is the trajectory traced by the first particle towards A's location.

The parties apply local unitary transformations

$$c^\dagger_{P_i} \to \sum_{j=1}^{3} U^{(P)}_{ij} c^\dagger_{P_j}, \quad P = A, B, C, \tag{5.32}$$

where $U^{(P)}_{ij}$ are matrix elements of the transformations.
The final state before the measurement is thus

$$|\psi\rangle_{PS} = \frac{1}{\sqrt{6}} \sum_{lmn} \sum_{\substack{i,j,k=1 \\ i\neq j\neq k\neq i}}^{3} e^{i\phi_{ijk}} U^{(A)}_{il} U^{(B)}_{jm} U^{(C)}_{kn} c^\dagger_{A_l} c^\dagger_{B_m} c^\dagger_{C_n} |0\rangle \,. \tag{5.33}$$

Finally, the parties perform local projective measurements; the probability of measuring the state $c^{\dagger}_{A_l} c^{\dagger}_{B_m} c^{\dagger}_{C_n} |0\rangle$ is

$$P(A_l B_m C_n) = \frac{1}{6} | \sum_{\substack{i,j,k=1 \\ i \neq j \neq k \neq i}}^{3} e^{i\phi_{ijk}} U^{(A)}_{il} U^{(B)}_{jm} U^{(C)}_{kn} |^2. \tag{5.34}$$

One of the phase differences that appear in Eq. (5.34) is for instance $\phi_{123} - \phi_{312}$, which can be expressed as a sum of two bipartite loop integrals, as shown in Fig. 5.4. The analogous holds for all other outcome probabilities, i.e. all information that one can gather from local measurements can be reconstructed from bipartite loop integrals.

Let us now turn to the general case and prove that an analogous decomposition to the one in Fig. 5.4 can be made for experiments involving any number of sources and any number of parties. Suppose that d sources emit single identical charged fermions (with charge e) in arbitrary superpositions of spatial trajectories towards N parties. Adopting the same notation as in Appendix 5.A, the state shared by the parties upon receiving the particles is thus

$$|\psi_0\rangle = \sum_{\substack{\vec{x}_1,\ldots,\vec{x}_N \\ \sum_n x_{nj}=1, \forall j}} \lambda_x e^{i \sum_{n,j} x_{nj} \varphi(n,j)} \prod_{k=1}^{N} \left[\prod_{j=1}^{d} \left(c^{(k)\dagger}_j \right)^{x_{kj}} \right] |0\rangle ,$$

$$\varphi(n,j) \equiv e \int_{\gamma_{n_j}} A^{\mu} dx_{\mu}.$$

The bit strings $\{\vec{x}_1, \ldots, \vec{x}_N\}$ automatically implement the fact that each state can be occupied by at most one fermion; $c^{\dagger}$ are fermionic creation operators (i.e. $c^{(k)\dagger}_j$ creates one fermion in the j-th mode of k-th party), and γ_{n_j} indicates the path connecting the j-th source to the n-th party. Each phase $\varphi(n,j)$ corresponds to the phase picked up by the particle travelling from the j-th source towards the n-th party. As before, λ_x denote normalized real amplitudes.
Next, the parties perform local linear operations:

$$c^{(k)\dagger}_i \rightarrow \sum_j T^{(k)}_{ij} c^{(k)\dagger}_j, \tag{5.35}$$

where $T^{(k)}_{ij}$ are arbitrary coefficients (e.g. elements of a unitary matrix). The transformed state is thus

$$|\psi_f\rangle = \sum_{\substack{\vec{x}_1,\ldots,\vec{x}_N \\ \sum_n x_{nj}=1, \forall j}} \lambda_x e^{i \sum_{n,j} x_{nj} \varphi(n,j)} \prod_{k=1}^{N} \left[\prod_{i=1}^{d} \left(\sum_j T^{(k)}_{ij} c^{(k)\dagger}_j \right)^{x_{kj}} \right] |0\rangle . \tag{5.36}$$

Finally, the parties perform local projective measurements giving rise to the following probability distribution

$$P(y) = | \langle \psi_f | \prod_{k=1}^{N} \left[\prod_{j=1}^{d} \left(c^{(k)\dagger}_j \right)^{y_{kj}} \right] |0\rangle |^2. \tag{5.37}$$

After some inspection, one sees that the probabilities involve only interference terms of components with same local particle numbers, i.e. we can write them as

$$P(y) = \sum_{x,x'} \tilde{\lambda}_x \tilde{\lambda}^*_{x'} e^{i\sum_{n,j}(x_{nj}-x'_{nj})\varphi(n,j)}, \qquad \sum_j x_{nj} = \sum_j x'_{nj} = \sum_j y_{nj}, \quad \forall n, \tag{5.38}$$

where $\tilde{\lambda}_x$ are coefficients, the exact form of which we leave unspecified. The probabilities thus depend on the following phases

$$\Delta\varphi(x,x') = \sum_{n,j}(x_{nj}-x'_{nj})\varphi(n,j), \quad \sum_j (x_{nj}-x'_{nj}) = 0. \tag{5.39}$$

Now we want to show that any such phase can be written as a combination of bipartite loop integrals like the one shown in Fig. 5.2. More precisely, we want to prove that the phases can be cast in the following form

$$\sum_{n,n',j,j'} \left[\varphi(n,j) - \varphi(n,j') + \varphi(n',j') - \varphi(n',j)\right], \tag{5.40}$$

where each square bracket in the sum represents one such bipartite loop integral. Let us start by introducing the functions n_j, such that $x_{kj} = \delta_{k,n_j}$ (this can be done because each source emits one particle which can then be found at most at one party's location). Then, there exists a permutation π, such that the phase can be written in the following concise form

$$\Delta\varphi(x,x') = \sum_j \left[\varphi(n_j,j) - \varphi(n_j,\pi(j))\right]. \tag{5.41}$$

The possibility of writing the phase in this form is essentially enabled by the property that states corresponding to x and x' have same local particle numbers: therefore, for every term (n_j, j) arising from bit-string x there must exist a corresponding term $(n_j, f(j))$ arising from bit-string x', where f is a function that maps sources into sources; furthermore, since the particle emitted from each source can be found at only one location, the function f must be one-to-one, and hence a permutation. Now we will prove by construction that the phase can be cast in the form given by (5.40).

Let $G = \langle\pi\rangle$ be the cyclic subgroup of the full permutation group that is generated by π, where we assume π is not the identity permutation. G has a natural group action on the set $\mathbb{Z}_d$. It is well-known that the orbits of a set under the action of a group form a partition of the set; which we denote by $Y := \mathbb{Z}_d/G$. For any equivalence class $[y]$ in Y choose a representative y in $\mathbb{Z}_j$, with the further requirement that the representative of $[1]$ is 1. Furthermore let k_y be the cardinality of the equivalence class $[y]$; equivalently k_y is the smallest positive integer such that $\pi^{k_y}(y) = y$. With all this notation established, it is straightforward to show that

$$\begin{aligned}\Delta\varphi(x,x') = \sum_{[y]\in Y} \sum_{l=0}^{k_y-2} [&\varphi(n_{\pi^l(y)}, y) - \varphi(n_{\pi^l(y)}, \pi^{l+1}(y)) + \\ &\varphi(n_{\pi^{l+1}(y)}, \pi^{l+1}(y)) - \varphi(n_{\pi^{l+1}(y)}, y)].\end{aligned} \tag{5.42}$$

The proof proceeds by writing out each term in the sum over Y explicitly and comparing with Eq. (5.41). For example the portion of the sum with $y = 1$ is

$$\begin{aligned}&\big[\varphi(n_1, 1) - \varphi(n_1, \pi(1)) + \varphi(n_{\pi(1)}, \pi(1)) - \varphi(n_{\pi(1)}, 1)\big] + \\ &\big[\varphi(n_{\pi(1)}, 1) - \varphi(n_{\pi(1)}, \pi^2(1)) + \varphi(n_{\pi^2(1)}, \pi^2(1)) - \varphi(n_{\pi^2(1)}, 1)\big] + ... \\ &+ [\varphi(n_{\pi^{k_1-2}(1)}, 1) - \varphi(n_{\pi^{k_1-2}(1)}, \pi^{k_1-1}(1)) + \varphi(n_{\pi^{k_1-1}(1)}, \pi^{k_1-1}(1)) - \\ &\varphi(n_{\pi^{k_1-1}(1)}, 1)].\end{aligned} \tag{5.43}$$

We see that in the sum above, the last term of every bracket is cancelled by the first term of the succeeding bracket. Looking at the third term of every bracket, and the first term of the first bracket, we see that every term $\varphi(n_s, s)$ appears. Finally, the second term of every bracket plus the last term of the last bracket account for all terms $-\varphi(n_s, \pi(s))$.

Thus we have established Eq. (5.42), which shows that $\Delta\varphi(x, x')$, and therefore all probabilities arising from local linear operations/measurements on single charged particles in superposition of spatial trajectories can be deduced from bipartite loop integrals as the one depicted in Fig. 5.2.

Part III

Fundamental indeterminism in classical, quantum and relativistic physics

Chapter 6

Physics without determinism: Alternative interpretations of classical physics

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I made essential contributions to all aspects of this work.

Abstract

Classical physics is generally regarded as deterministic, as opposed to quantum mechanics that is considered the first theory to have introduced genuine indeterminism into physics. We challenge this view by arguing that the alleged determinism of classical physics relies on the tacit, metaphysical assumption that there exists an actual value of every physical quantity, with its infinite predetermined digits (which we name *principle of infinite precision*). Building on recent information-theoretic arguments showing that the principle of infinite precision (which translates into the attribution of a physical meaning to mathematical real numbers) leads to unphysical consequences, we consider possible alternative indeterministic interpretations of classical physics. We also link those to well-known interpretations of quantum mechanics. In particular, we propose a model of classical indeterminism based on *finite information quantities* (FIQs). Moreover, we discuss the perspectives that an indeterministic physics could open (such as strong emergence), as well as some potential problematic issues. Finally, we make evident that any indeterministic interpretation of physics would have to deal with the problem of explaining how the indeterminate values become determinate, a problem known in the context of quantum mechanics as (part of) the "quantum measurement problem". We discuss some similarities between the classical and the quantum measurement problems, and propose ideas for possible solutions (e.g., "collapse models" and "top-down causation").

6.1 Why is classical physics (in)deterministic?

It is generally accepted that classical physics (i.e., Newton's mechanics and Maxwell's electrodynamics) is *deterministic*. Restating a famous argument due to Laplace (known as Laplace's demon), determinism is usually assumed to be the "view that a sufficient knowledge of the laws of nature and appropriate boundary conditions will enable a superior intelligence to predict the future states of the physical world and to retrodict its past states with infinite precision" [126].

Yet, stimulated by the development of statistical physics (which is taken to introduce indeterminacy as merely epistemic), one could find notable exceptions to this deterministic view, remarkably in the works of preeminent physicists the likes of L. Boltzmann, F. Exner, E. Schrödinger and M. Born, who –admittedly with different standpoints– all argued for genuine indeterminism in classical physics (see [26] and references thereof). These doubts about classical determinism were fostered in the second half of the twentieth century, after the theory of chaotic systems was systematically developed and its implications fully understood [127, 128].

However, it is only in recent years that new life has been breathed into the critique of determinism in classical physics, showing that the advocacy of determinism leads to severe conceptual difficulties based on information-theoretic arguments [129, 15, 130], and that determinism might even be incompatible with the derivation of the second law of thermodynamics [131]. In fact, the hypothetical "superior intelligence" (demon), supposedly able to perfectly predict the future, is required to have complete information about the state of the universe and then use it to compute the subsequent evolution. Recent developments of information theory and its application to physics (mainly blossomed within the frameworks of quantum information and quantum thermodynamics) led to the conclusion that the abstract, mathematically well-formalized concept of information acquires a meaningful value in the natural sciences only if the information is embodied into a physical system (encoding), allowing it to be manipulated (computation) and transmitted (communication). As such, these processes are subject to the same limitations imposed by the laws of physics (*Landauer's principle* [132]). In the face of this, Laplace's demon, and hence determinism, leads to two categories of problems: the problem of *infinities* and the problem of *infinitesimals*.[1]

The former of these problems is directly related to the memory capability of the physical systems that are supposed to encode the information of the whole Universe, then to be manipulated to compute the subsequent evolution. About this, Blundell concludes: "If such a demon were (even hypothetically) to be constructed in our physical world, it would be subject to physical constraints which would include a limit on the number of atoms it could contain, bounded from above by the number of particles in the observable Universe. [...] Hence there is insufficient physical resource in the entire Universe to allow for the operation of a Laplacian demon able to analyze even a relatively limited macroscopic physical system" [130].[2]

[1] Similar problems have been recently discussed in a more general context and without resort to information-theoretic arguments in Ref. [133].

[2] The problem with (physical) infinities is connected to the so-called *Hilbert's Hotel paradox*, proposed by D. Hilbert in 1924 [134]. This paradox illustrates the possibility for a hypothetical hotel with (countably) infinite rooms, all of which already occupied, to allocate (countably) infinitely many more guests.

The problem of infinitesimals, instead, is related to the question of whether it would be possible to know, even in principle, the necessary boundary conditions with infinite precision. Moreover, do these infinite-precision conditions (i.e., with infinite predetermined digits) exist at all? As Drossel pointed out, this is related to the problem of determinism in so far as the "idea of a deterministic time evolution represented by a trajectory in phase space can only be upheld within the framework of classical mechanics if a point in phase space has infinite precision" [131]. To address the problem of infinitesimals, and in doing so challenging determinism both at the epistemic and ontic levels, one can again use an argument that relates information and physics, namely the fact that finite volumes can contain only a finite amount of information (*Bekenstein bound*, see [15]).

In this respect, one should realize that classical physics is not inherently deterministic just because its formalism is a set of deterministic functions (differential equations), but rather its alleged deterministic character is based on the metaphysical, unwarranted assumption of "infinite precision". Such a hidden assumption can be formulated as a principle –tacitly assumed in classical physics– which consists of two different aspects:

Principle of infinite precision:

1. (Ontological) – there exists an actual value of every physical quantity, with its infinite determined digits (in any arbitrary numerical base).

2. (Epistemological) – despite it might not be possible to know all the digits of a physical quantity (through measurements), it is possible to know an arbitrarily large number of digits.

In this paper, we further develop the argument put forward in Ref. [15], wherein it has been outlined a concrete possibility to replace the *principle of infinite precision.* According to this view, the limits of this principle rely on the faulty assumption of granting a physical significance to mathematical real numbers. We would like to stress that such an assumption cannot be whatsoever justified at the operational level, as already stressed by Born, as early as 1955: "Statements like 'a quantity x has a completely definite value' (expressed by a real number and represented by a point in the mathematical continuum) seem to me to have no physical meaning" [135]. Relaxing the assumption of the physical significance of mathematical real numbers, allows one to regard classical physics as a fundamentally indeterministic theory, contrarily to its standard formulation. The latter can be considered, in this view, a deterministic completion in terms of (tacitly) posited *hidden variables.* This situation resembles (without however being completely analogous) the contraposition between the standard formulation of quantum mechanics –which considers indeterminism an irrefutable part of quantum theory– and Bohm's [65] or Gudder's [136] hidden variable models, which provide a deterministic description of quantum phenomena –adding in principle inaccessible supplementary variables.

Before further discussing, in what follows, the arguments for alternative interpretations of classical physics without real numbers –and the issues that this can cause– some general remarks on indeterminism seem due. It appears, in fact, that a common misconception concerning physical indeterminism, is that –in the eyes of some physicists and philosophers– this is taken to imply that any kind of regularity or predictive power looks unwarranted –the supporter of determinism would ask: how can you explain the incredible predictive success of our laws of physics without causal determinism? Yet, an indeterministic description of the world does not (at least necessarily) entail an "anarchistic indeterminism", namely a complete chaos devoid of *any* laws or regularities (quantum mechanics with its probabilistic predictions provides a prime example of these indeterministic regularities). We would like to define indeterminism trough the sufficient condition of the existence of *some* events that are not fully determined by their past states; in the words of K. Popper, "indeterminism merely asserts that there exists at least one event (or perhaps, one kind of events [...]) which is not predetermined" [137]. Such a remark is important because, historically, classical mechanics allowed to predict with a tremendous precision the motion, for instance, of the planets in the Solar System and this led Laplace to formulate his ideas on determinism, which became the standard view among physicists and remained for a long time unchallenged. In fact, thinking that physics is deterministic seems totally legit, in so far as certain physical systems exhibit extremely stable dynamics –e.g. a frictionless pendulum, or the (Newtonian) gravitational two-body problem, and in general any integrable systems. However, this justification of determinism can be challenged on the basis of two considerations. On the one hand, as already remarked, the existence of *some* very stable systems (for all practical purposes treated as deterministic) does not undermine the possibility of indeterminism in the natural world. On the other hand, in the last one century a systematic study of chaotic systems –which are not integrable– has been carried out, giving us good reasons to doubt determinism. Indeed, chaotic systems are not stable under perturbations, meaning that an arbitrarily small change in the initial conditions would lead to a significantly different future behavior, thus making the principle of infinite precision even more operationally unjustified, and therefore representing a concrete challenge to determinism.

Incidentally, it is interesting to stress that, in the context of quantum physics, Bell's inequalities [2] have given us good reasons to believe, if not directly in indeterminism –which indeed cannot be confirmed or disproved within the domain of science (see further)– that having at least one not predetermined event in the history of the Universe could have tremendous consequences on the following evolution. In fact, an experimental violation of Bell's inequalities guarantees that if the inputs (measurement settings) were independent of the physical state shared by two distant parties, then the outcomes would be genuinely random (i.e., they cannot have predetermined values). Yet, the amount of random numbers generated in a Bell's test can be greater than the number of the corresponding inputs (in terms of bits of information): Bell's tests performed on quantum entangled states can be thought of as "machines" to increase the amount of randomness in the Universe (see also [138]). Surprisingly enough, recent results [139, 140] showed that it is not even necessary to have a single genuinely random bit from the outset, but it is sufficient to introduce an arbitrarily small amount of initial randomness (i.e., of measurement independence) to generate virtually unbounded randomness. Hence, if one single event in the past history of the Universe was not fully causally determined beforehand, there is an operationally well defined procedure that allows to arbitrarily multiply the amount of indeterministic events in the future. However, the initial arbitrarily small amount of randomness (or of indeterministic events) cannot be demonstrated by physics and its justification can only come from metaphysical arguments (see [141]).

6.2 Forms of indeterminism in classical and quantum physics

Before proposing some possible models of indeterministic classical physics, in this section we shortly discuss some general features of deterministic and indeterministic theories. In doing so we aim at clarifying possible similarities between classical and quantum physics. Both classical and quantum mechanics are, in fact, formalized by a set of differential equations (laws of motion) that govern the dynamics of systems, together with appropriate initial conditions (IC) that fix the free parameters of these equations. Thus, if one aims at eliminating determinism as an unfounded interpretational element, there seems to be different possibilities, either involving the laws of physics or the characterization of the IC:

1. *The laws of motions are fundamentally stochastic.* In this case, however, we cannot speak of an interpretation of the theory, but an actual modification of the formalism is required. In fact, in this case not only chaotic systems but also integrable ones would exhibit noisy outcomes, leading to experimentally inequivalent predictions. This case is the analogue of spontaneous collapse theories in quantum mechanics [142, 143, 144, 145], which modify the Schrödinger's equation with additional non-unitary terms.

2. *The IC cannot, in principle, be fully known.* Without any ontological commitments, one can take seriously the epistemological statements of the *principle of infinite precision* and push it to the extreme, namely asserting that there are in principle limits to the possibility of knowing (or measuring) certain quantities. This is what is entailed by the standard interpretation of the Heisenberg uncertainty principle that is usually stated as: there is in quantum physics a fundamental limit to the precision with which canonically conjugated variables *can be known*. Such an uncertainty can be introduced in classical physics as well, and would be characterized by a new natural constant ϵ (e.g., the standard deviation of a Gaussian function centered in the considered point). This would set an epistemic (yet fundamental) limit of precision, with which physical variables could be determined, as a classical analogue of Heisenberg's uncertainty relations in quantum theory. This viewpoint appears similar to the one proposed in Ref. [131]. Notice, however, that since this approach is agnostic with respect to the underlying ontology, it is fully compatible with a realist position that takes this uncertainty as being an ontological indeterminacy. [3]

3. *The IC* are *not fully determined.* One can think that the fundamental limit of precision in determining physical quantities is not merely epistemic, but actually is an objective, ontologic indeterminacy that depends on the system and its interactions at a certain time. This view is the one that will be pursued in what follows, when we will propose ways to remove real numbers $\mathbb{R}$ from the domain of physics. Although this case does not seem to be the analogue of any

[3] As a supporting argument for this fundamental epistemic limit, one can even think in terms of theoretical reduction (i.e. when a theory is supervened by a more fundamental one, of which it represent an approximation). In fact, determining positions in classical physics with higher and higher precision, means to access digits that are relevant at the microscopic scale, and thus the Heisenberg's uncertainty principle needs to be applied (leading to the identification $\epsilon = \hbar$). However, it ought to be remarked that in certain classical chaotic systems the digits that are to become relevant are not necessarily the ones in the microscopic domain, but could be those far away in time (such as in weather forecasts). So more general arguments than theoretical reduction would be desirable.

specific interpretation of quantum theory, it clearly goes in the direction of realistic interpretations. It ought to be stressed, however, that if the initial conditions are not fully determined, this even makes (quantum) Bohmian mechanics becoming indeterministic.

6.3 Indeterministic classical physics without real numbers

In Ref. [15], a model of indeterministic classical mechanics has been sketched, which, while leaving the dynamical equations unchanged, proposes a critical revision of the assumptions on the initial condition. In this view, the standard interpretation of classical mechanics has always tacitly assumed as hidden variables the predetermined values taken by physical quantities in the domain of mathematical real numbers, $\mathbb{R}$. The physical existence of an infinite amount of predetermined digits could lead to unphysical situations, such as the aforementioned infinite information density, as explained in detail in Refs. [129] and [15].

In this section, we discuss some possible solutions to eliminate these unwanted features, namely possible ways of carrying out the relaxation of the postulate according to which physical quantities take values in the real numbers. These solutions are however intended to be merely different interpretations of classical mechanics, i.e. they ought to be empirically indistinguishable from the standard predictions (in the same way as interpretations of quantum mechanics are).

6.3.1 "Truncated real numbers".

A first possibility is to consider physical variables as taking values in a set of "truncated real numbers". This, as already noted by Born, would ensure the empirical indistinguishability from the standard classical physics: "a statement like $x = \pi$ cm would have a physical meaning only if one could distinguish between it and $x = \pi_n$ cm for every n, where π_n is the approximation of π by the first n decimals. This, however, is impossible; and even if we suppose that the accuracy of measurement will be increased in the future, n can always be chosen so large that no experimental distinction is possible" [135]. If one, however, wishes to attribute an ontological value to such an interpretation has to identify n with a new universal constant, that, independent of how big it could be, sets a limitation to the length of physically significant numbers, ultimately including the life of the Universe, if time too is to be considered a physical quantity. Another problematic issue is that n would be dependent on the units in which one expresses the considered physical variables. This leads to consider a second possible solution.

6.3.2 Rational numbers.

Another possibility is to consider that physical quantities take value in the rational numbers, $\mathbb{Q}$. Even if this sounds somewhat strange, one can argue that, in practice, physical measurements are in fact only described by rational numbers. For instance, a measurement of length is obtained by comparing a rod that has been carefully divided into equal parts (i.e. a ruler) with the object to be measured and determining the best fit within its (rational) divisions. And even probabilities are obtained as limits of frequencies of events' occurrences (i.e. ratios of counts).

However, while rational numbers do eliminate the unwanted infinite information density, they do not seem to remove determinism in so far as *all* the digits are fully predetermined. Moreover, the use of rational numbers leads to those that can be named "Pitagora's no-go theorems". Indeed, positing a physics based on rational numbers, would rule out the possibility of constructing a physical object with the shape of a perfect square with unit edge or a perfect circle with unit diameter. In fact, by means of elementary mathematical theorems, their diagonal and circumference, respectively, would measure $\sqrt{2}$ and π, hence resulting to be physically unacceptable. Additionally, if one plugs in the equations of motion initial conditions and time both taking values in the rational numbers, the solutions are not in general rational numbers. These problematic issues lead to consider yet another possible solution.

6.3.3 "Computable real numbers".

A further alternative is to substitute the domain of physically meaningful numbers from (mathematically) real numbers to the proper subset thereof of "computable real numbers"; that is, to keep all real numbers deprived of the irrational, uncomputable ones. In fact, even irrational, computable real numbers can encode at most the same amount of non-trivial information (in bits) as the length of the shortest algorithm used to output their bits (i.e., the *Kolmogorov complexity*). Uncomputable real numbers are in this model instead substituted by genuinely random numbers, thus introducing fundamental randomness also in classical physics. These numbers, together with chaotic systems, lay the foundations of an alternative classical indeterministic physics, which removes the paradox of infinite information density. However, this proposal could be considered an *ad hoc* solution, since it maintains a field of mathematical numbers as physically significant, but removes "by hand" those that are problematic (which admittedly are almost all).

6.3.4 "Finite information quantities" (FIQs).

Developing further the proposal in [15], we put forward an alternative class of random numbers which are for all practical purposes (in terms of empirical predictions) equivalent to real numbers, but that have actually zero overlap with them (they are not a mathematical number field, nor a proper subset thereof). We refer to them as "finite-information quantities" (FIQs). In order to illustrate this possible alternative solution to overcome the problems with the principle of infinite precision, let us consider again the standard interpretation in greater formal detail. A physical quantity γ (which may be the scalar parameter time, a universal constant, as well as a one-dimensional component of the position or of the momentum, etc.) is assumed to take values in the domain of real numbers, i.e., $\gamma \in \mathbb{R}$. Without loss of generality, but as a matter of simplicity, let us consider γ to be between 0 and 1, and that its digits (bits) are expressed in binary base:

$$\gamma = 0.\gamma_1\gamma_2 \cdots \gamma_j \cdots ,$$

where each $\gamma_j \in \{0, 1\}$, $\forall j \in \mathbb{N}^+$. This means that, being $\gamma \in \mathbb{R}$, its infinite bits are *all* given at once and each one of them takes as a value either 0 or 1.

In an indeterministic world, however, not all the digits should be determined at all times, yet we require this model to give the same empirical predictions of the standard one. We therefore require a physical quantity to have the first (more significant) N digits fully determined –and to be the same as those that give the standard deterministic predictions– at time t, and we write $\gamma(N(t))$, whereas the following infinite digits are not yet determined. This reads:

$$\gamma\left(N(t)\right) = 0.\gamma_1\gamma_2\cdots\gamma_{N(t)}?_{N(t)+1}\cdots ?_k\cdots ,$$

where each $\gamma_j \in \{0,1\}$, $\forall j \leq N(t)$, and the symbol $?_k$ here means that the kth digit is a not yet actualized binary digit (see further).

Despite the element of randomness introduced, the transition between the actualized values and the random values still to be realized does not need to be a sharp one. In fact, one can conceive an objective property that quantifies the (possibly unbalanced) disposition of every digit to take one of the two possible values, 0 or 1. This property is reminiscent of Popper's propensities [146],[4] and it can be seen as the element of objective reality of this alternative interpretation:

Definition - ***propensities***
There exist (in the sense of being ontologically real) physical properties that we call *propensities* $q_j \in [0,1]\cap\mathbb{Q}$, for each digit j of a physical quantity $\gamma(N(t))$. A propensity quantifies the tendency or disposition of the jth binary digit to take the value 1.

For example, $q_j = 1$ means that the jth digit will take value 1 with certainty. On the other hand, a $q_k = 0.3$ means that there is an objective tendency of the kth digit to take the value 1, quantified by 0.3, and thus the complementary propensity of taking the value 0 would be 0.7. It also follows from the definition, that the propensities q_j for the first $N(t)$ digits of a quantity $\gamma(N(t))$ at time t are all either 0 or 1, i.e. $q_j \in \{0,1\}$, $\forall j \in [1, N(t)]$.

Making use of propensities, we can now refine our definition of physical quantities:

Definition - ***FIQs***
A *finite-information quantity* (FIQ) is an ordered list of propensities $\{q_1, q_2, \cdots, q_j, \cdots\}$, that satisfies the conditions:
(i) $\exists M(t) \in \mathbb{N}$ such that $q_j = \frac{1}{2}$, $\forall j > M(t)$ (sufficient condition);
(ii) $\sum_j I_j < \infty$ (necessary condition), where $I_j = 1 - H(q_j)$ is the information content of the propensity, and H is the binary entropy function of its argument. This ensures that the information content of FIQs is bounded from above.

It ought to be stressed that this view grants a prior fundamentality to the potential property of becoming actual (a list of propensities, FIQ), more that to the already actualized number (a list of determined bits). However, the merit of this view is that the bits are realized univocally and irreversibly as time passes, but the information content of a FIQ is always bounded, contrarily to that of a real number. A physical quantity γ reads in this interpretation as follows:

$$\gamma\left(N(t), M(t)\right) = 0.\underbrace{\gamma_1\gamma_2\cdots\gamma_{N(t)}}_{\text{determined } \gamma_j\in\{0,1\}}\overbrace{?_{N(t)+1}\cdots ?_{M(t)}}^{?_k,\text{ with } q_k\in(0,1)}\underbrace{?_{M(t)+1}\cdots}_{?_l,\text{ with } q_l=\frac{1}{2}}.$$

Notice that none of the FIQs is a mathematical number, but they capture the tendency (propensity) of each bit of a physical quantity to take the value 0 or 1 at

[4]While propensities were for Popper an interpretation of mathematical probabilities proper, we are not here necessarily requiring them to satisfy Kolmogorov's axioms, as discussed in Ref. [147].

the following instant in time. This admittedly leads to problematic issues, such as the problem of how and when the actualization of the digits from their propensity take place: it thus introduces the analogue of the quantum measurement problem also in classical physics (see further).

Moreover, FIQs partly even out the fundamental differences between classical and quantum physics, making both of them indeterministic (and making so even Bohmian interpretation).[5] Table 6.1 compares some possible combinations of deterministic and indeterministic interpretations of quantum and classical physics.

	Classical	Quantum
Indeterministic	IC $\in$ FIQs Newton's equation	$\lvert\psi\rangle \in \mathcal{L}^2(\mathbb{R}^N)$ Measurement postulate
		IC (position) $\in$ FIQs and $\lvert\psi\rangle \in \mathcal{L}^2(\mathbb{R}^N)$ Bohm's guidance equation admitted
Deterministic	IC $\in \mathbb{R}$ Newton's equation	IC (position) $\in \mathbb{R}$ and $\lvert\psi\rangle \in \mathcal{L}^2(\mathbb{R}^N)$ Bohm's guidance equation Schrödinger's equation
		Many worlds interpretation (?)

Table 6.1: A table comparing deterministic and indeterministic interpretations of classical and quantum physics. Note that the substitution of FIQs in the place of real numbers makes not only classical physics indeterministic, but also Bohm's interpretation of quantum physics (which is usually taken to restore determinism).

6.3.5 5. "Choice sequences".

Finite Information Quantities are not numbers is the usual sense, because their digits are not all given at once. On the contrary, the "bits" of FIQs evolve as time passes, they start from the value $\frac{1}{2}$ and evolve until they acquire a bit value of either 0 or 1. In a nutshell, FIQs are processes that develop in time. Interestingly, in intuitionistic mathematics, the continuum is filled by "choice sequences", as Brouwer, the father of intuitionism, and followers named them [148]. This is not the place to present intuitionistic mathematics (see, e.g., [149]), but let us emphasize that this alternative to classical (Platonistic) mathematics allows one to formalize "dynamical numbers" that resemble much our FIQs [150]. Interestingly, using the language of intuitionistic mathematics makes it much easier to talk of indeterminism [151].

[5]There is yet another deterministic interpretation of quantum mechanics, the so-called *many-worlds interpretation* that grants physical reality to the wave function of the Universe, which always evolve unitarily. If FIQs are introduced in that interpretation, then the realization of *all* the values of the bits would actually take place, each of which being real in a different "world".

6.4 Strong emergence

The argument for determinism seems to rely, to a certain extent, on the tacit assumption of reductionism in its stronger form of microphysicalism, i.e. the view that every entity and phenomenon are ultimately reducible to fundamental interactions between elementary building blocs of physics (e.g., particles). In fact, in a completely deterministic picture, every particular phenomenon can be traced back to the interactions between its primitive components, along a (finite) chain of causally predetermined events. In this way any form of strong emergence seems to be ruled out, and it becomes only apparent (i.e. a weak or epistemic emergence). On the other hand, admitting genuine randomness in the universe, allows in our opinion the possibility of strong emergence.[6]
As a concrete example, consider the kinetic theory of gases. If one starts from a molecular description of the ideal gas, from the perspective of standard, deterministic classical mechanics, the stochasticity is only epistemic (i.e. only an apparent effect due to the lack of complete information regarding positions and momenta of every single molecule). Thus, the deterministic behavior of the law of the ideal gas is not expected to be a strong emergent feature, but solely a retrieving at the macroscopic scale of the fundamental determinism of the microscopic components. In the perspective of the alternative indeterministic interpretation (based on FIQs), instead, the deterministic law of the ideal gas, ruling the behavior at the macroscopic level, emerges as a novel and not reducible feature, from fundamental randomness.[7]

Notice that the historical debate on the apparent incompatibility between Poincaré's recurrence theorem and Boltzmann's kinetic theory of gases does not arise in the framework of FIQs. Poincaré's recurrence theorem, in fact, states that continuous-state systems (i.e., in which the state variables change continuously in time) return to an arbitrarily small neighborhood of the initial state in phase space. However, Poincaré's theorem relies on the fact that the initial state is perfectly determined (i.e., it is a mathematical point identified by a set of coordinates which take values in the real numbers) in phase space. Thus in a FIQ-based alternative physics the theorem simply cannot be derived. In fact, FIQs interpretation features genuinely irreversible physical processes.

Similarly, Drossel has recently pointed out that, in a physics where it is impossible to determine points of phase space with infinite precision, "the time evolution of thermodynamics is undetermined by classical mechanics [...]. Thus, the second law of thermodynamics is an emergent law in the strong sense; it is not contained in the microscopic laws of classical mechanics" [131].

At this point one should ask oneself whether there are examples of emergence that possibly go beyond some form of the law of large numbers. Admittedly, we are unsure about this. Clearly, in all indeterministic physical theories, the law of large numbers will play an important role and lead to some stability and hence to

[6]For a definition of strong emergence, see, for instance, Ref. [152]: "We can say that a high-level phenomenon is strongly emergent with respect to a low-level domain when the high-level phenomenon arises from the low-level domain, but truths concerning that phenomenon are not deducible even in principle from truths in the low-level domain".

[7]It is true that also the law of the ideal gas would not be *perfectly* deterministic in a FIQ-based physics, however its stability makes it almost deterministic for all practical purposes, whereas at the microscopic level, chaotic behaviors multiply the fundamental uncertainty of the single molecules.

some form of determinism at the larger scale (or higher-level description). It seems that this question is closely related to possible top-down causation, the topic of the next section.

6.5 Top-down causation

The idea of strong emergence, including emergent determinism, is related to the concept of "top-down causation" [153, 154]. In this view, microphysicalism is not necessarily rejected ontologically (i.e., it admits that complex structures are hierarchical modular compositions of simpler ones), but the fact that the behavior of macroscopic events is fully determined by the interactions of the macroscopic entities is revised. Top-down causation maintains that the interactions between microscopic entities do not causally supervene the macroscopic phenomena, but rather it posits a mutual interaction where also the macroscopic (strongly emergent) laws impose constraints on the behavior of their constituents. Note that top-down causation requires indeterminism (at least at the lower level of the constituents) to be in principle conceivable [155]; this was already remarked by Popper, when stating: "For it seems that, were the universe *per impossible* a perfect determinist clockwork, there would be no heat production and no layers and therefore no such dominating influence would occur. This suggests that the emergence of hierarchical levels or layers, and of an interaction between them, depends upon a fundamental indeterminism of the physical universe. Each level is open to causal influences coming from lower and from higher levels" [156].

Concerning indeterministic interpretations of physical theories, top-down causation could help to understand how the determination of dynamical variables (i.e., the actualization of their values) occurs in the context of indeterministic theories. Namely, the reason why –and under what circumstances– a single definite value is realized among all the possible ones. In the FIQ-based indeterministic interpretation of classical physics here introduced, this translates into the understanding of how the bits of physical variables becomes fully determined, namely how their propensities become either 0 or 1. We envision two possible mechanisms that could explain the actualization of the variables:

1. The actualizations happens spontaneously as time passes. This view is compatible with reductionism and it does not necessarily require any effects of top-down causation. Note that this mechanism resembles, in the context of quantum mechanics, objective collapse models such as the "continuous spontaneous localization" (CSL) [143, 145].

2. The actualization happens when a higher level requires it. This means that when a higher level of description (e.g., the macroscopic measurement apparatus) requires some physical quantity pertaining to the lower-level description to acquire a determined value, then the lower level must get determined. In quantum mechanics a similar explanation is provided by the Copenhagen interpretation and, more explicitly, by the model in Ref. [154].

In fact, the latter mechanisms are strongly related to what has been discussed at length in the context of quantum theory, namely the long-standing "quantum measurement problem". This comprises the problem of "explaining why a certain outcome –as opposed to its alternatives– occurs in a particular run of an experiment" [3]. In fact, some of the most commonly accepted interpretations of quantum mechanics (e.g. the Copenhagen interpretation) uphold the view that it is the act of measurement to impose to microscopic (quantum) objects to actualize one determined value, out of the possible ones.

Note that, despite it has been already remarked in the literature that every indeterministic theory has to deal with a "measurement problem" (see e.g. [3]), it seems that there has hardly been any consideration of this issue in the context of other indeterministic theories than quantum mechanics. In the next subsection we will discuss what we call, by analogy, the "classical measurement problem'. We will then draw a connection with top-down causation, and show how this could help to shed light on this problem.

6.5.1 The "classical measurement problem"

It is a very corroborated experimental fact that if a quantity is measured twice with the same instrument, we expect a certain amount of digits to remain unchanged, and it is essential to scientific investigation that such a knowledge is intersubjectively available (up to the digit corresponding to the measurement accuracy). Moreover, if a more accurate measurement instrument is utilized, we expect not only the previous digits to remain unchanged, but also to determine some new digits that then become intersubjectively available. How to reconcile this stability of the measured digits with a fundamental uncertainty in the determination of a physical quantity? How does potentiality become actuality?

In the proposed FIQ-based indeterministic interpretation of classical physics, too, one has to carefully define how the digits of physical quantities realize themselves from the propensity of taking that (or another) possible value. Consider for example the chaotic systems analyzed in [15] (a simplified version of the *baker's map*). One can then think of a "faster" dynamics that, at every time step, shifts the bits by not only one digit toward the more significant position, but, say, it shifts them by 1000 digits (or any other arbitrarily large finite number). This clearly entails that the rate of change of propensities depends on the dynamical system under consideration, and cannot be thought of as a universal constant of "spontaneous" actualization. A possible solution is to introduce a model of measurement that makes the digits becoming actual (and therefore stable) up to the corresponding precision. This clearly resembles the solution to the quantum measurement problem provided by the *objective collapse models*, such as the CSL [143, 145] or the GRW [142, 144]. The latter model, indeed, posits a modification of the standard Schrödinger's equation which accounts for a spontaneous random "collapse" of the wave function, occurring with a certain natural rate. Under certain assumptions, this model leads to a mechanism that changes the rate of spontaneous collapse, which increases linearly with the number of components of a system (thus during a measurement the wave function of a microscopic system in contact with a macroscopic apparatus collapses extremely fast). An analogous solution can be in principle proposed for the rate of actualization of the propensities that define the FIQs. However, this seems to mean that the dynamical equations need to be modified (in the same fashion as the GWR model modifies the Schrödinger's equation), thus leading to a different formalism and not only an interpretation.

Coming back to top-down causation, this could explain why every time one performs a measurement the determined digits remain stable. In fact, the act of a measurement can be regarded as the direct action performed at the higher level which imposes to the lower level to get determinate. This is very similar to what is taken to be the solution to the quantum measurement problem within the Copenhagen interpretation, wherein the higher level is the macroscopic measurement apparatus, whereas the lower level is the measured microscopic system. However, this kind of solutions lacks a clear definition of what is to be considered a measurement and how to identify higher and lower levels of description.

As a matter of fact, the "classical measurement problem" here introduced remains so far unresolved, as well as the quantum measurement problem and, more in general, the problem of the actualization of physical variables in any indeterministic theory. Yet, it is desirable that the topic of the measurement problem should find room in the debate on foundations of physics, in more general discussions than those centered on quantum mechanics only.

6.6 Conclusions

We have discussed arguments –primarily based on the modern application of information theory to the foundations of physics– against the standard view that classical physics is necessarily deterministic. We have also discussed concrete perspectives to reinterpret classical physics in an indeterministic fashion. We have then compared our indeterministic proposals with some interpretations of quantum physics. However, it seems clear that the empirical results of both classical and quantum mechanics can fit in either a deterministic or indeterministic framework. Furthermore, there are compelling arguments (see e.g., [15, 141, 157]) to support the view that the same conclusion can be reached for any given physical theory –a trivial way to make an indeterministic theory fully determined is to "complete" the theory with all the results of every possible experiments that can be performed.

In conclusion, although the problem of determinism versus indeterminism is in our opinion central to science, the hope to resolve this problem within science itself has faded, and this is ultimately to be decided on the basis of metaphysical arguments.

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Chapter 7

The relativity of indeterminacy

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I made essential contributions to all aspects of this work.

Abstract

A long-standing tradition, largely present in both the physical and the philosophical literature, regards the advent of (special) relativity –with its block-universe picture– as the failure of any indeterministic program in physics. On the contrary, in this paper, we note that upholding reasonable principles of finiteness of information hints at a picture of the physical world that should be both relativistic and indeterministic. We thus rebut the block-universe picture by assuming that fundamental indeterminacy itself should as well be regarded as a relative property when considered in a relativistic scenario. We discuss the consequence that this view may have when correlated randomness is introduced, both in the classical case and in the quantum one.

7.1 introduction

In recent years, new attention has been brought to the problem of having infinite quantities in physical theories [133, 158], with a special emphasis on the connection between physics and information [159, 15, 160, 129, 131, 151, 161, 26, 162].[1] In particular, in Refs. [15, 160, 151, 147, 26, 162], we have pointed out that the usual understanding of classical (non-relativistic) mechanics rests on the unwarranted implicit assumption of "infinite precision", namely, that every physical quantity has an actual value with its *infinite* determined digits (which formally translates into the assumption that physical variables take values in the real numbers). This led to the general, supposedly uncontroversial, understanding of classical physics as a fundamentally deterministic theory.

To overcome this problem of "infinities", in Refs. [15, 160], we have assumed the "finiteness of information density" (i.e., finite regions of space cannot contain but finite information) as a foundational principle, and provided a model in which physical variables do not take values in the real numbers but rather in what

[1] Many of these approaches stem from, *inter alia*, Landauer's pivotal work that supports the view according to which "information is physical" [132].

we have called “finite information quantities” (FIQs). These are mathematical entities whose digits are not infinitely determined at every instant of time, but only the objective probability (propensity) of each digit to take a certain value is determined (and evolves in time). We then showed how the assumption of FIQs together with the existence of chaotic systems leads to an alternative fundamentally indeterministic interpretation of classical physics.

This led us to assume a primitive notion of “creative” time that openly contrast a widely accepted view according to which time is an evolution parameter in a set of deterministic laws of physics. The latter view about time has particularly crystallized among physicists with the advent of special relativity, which proposes a geometrization of space-time in a single “block-universe picture”. This fostered the conviction that time is a co-ordinate to describe events which however are bound to be fully pre-determined in the block-universe, and leaves no space to genuine indeterminism in physics.

In this paper we show that upholding a principle of finiteness of information hints at a picture of the physical world that should be both relativistic and indeterministic. We thus propose a new way out of the conundrum of reconciling indeterminism and the flow of time with (special) relativity –both in the classical block-universe picture and in its quantum adaptation (Page-Wootters [13] and Many-World views [64]). It is the aim of this paper to show that today’s physics, and in particular the theory of relativity, does not lead to accepting the block-universe as a necessity, but there exist at least an alternative consistent way to fit fundamental indeterminism within relativity. In particular, we rephrase by means of a physically intuitive scenario a by now classical argument independently put forward by Rietdijk [16] and Putnam [17], that “illustrate[s] the power of the block-universe picture” [163] by allegedly showing the incompatibility between (special) relativity and indeterminism. We then discuss its implicit assumptions, pointing out that one of them –which seems innocuous in Newtonian physics– is problematic in special relativity, thus making their argument untenable. As we shall see, the consequences of putting together the concepts of indeterminism and relativity would lead us to conclude that the state of determinacy of a physical (random) variable is relative as well. The relativity of indeterminacy will be analyzed in the different scenarios of classical independent randomness, classically correlated randomness and quantum correlated randomness.

7.2 Relativity from information principles

The theory of special relativity can be regarded as stemming from an attempt to remove a problem with infinities that is intrinsic in classical physics. In fact, in Newtonian mechanics the interaction of particles is described by the potential energy of interaction which is a function of only the positions of the interacting particles, hence, as stressed also by Landau and Lifshitz, such a theory “contains the assumption of instantaneous propagation of interactions” [164]. They thus motivate the development of the theory of special relativity (SR) as to overcome a long-standing problem of “infinities” (the infinite speed of propagation of the interactions in this case), which affected classical physics.[2]

[2]Note that SR only deals with mechanics and electromagnetism, whereas to address similar problems about gravity one has to consider general relativity.

Remarkably, also the problem of infinities raised by Landau and Lifshitz can be addressed by invoking the same principle of "finiteness of information density" introduced in [160] as recalled in the introduction, thus assuming it as an alternative axiom from which deriving SR. Consider the following two axioms:

P1 – Principle of relativity: The laws of physics have the same form in every inertial frame of reference.

P2 – Principle of finiteness of information density: A finite volume of space can only contain a finite amount of information.[3] From P2 one infers that only a finite amount of information can be transmitted in a finite time, otherwise this would require to "move" an infinite volume of space. Hence, the *signal velocity* (i.e. the speed of propagation of information) necessarily needs to be finite too. Now, if there is a maximal velocity,[4] from P1, this must be the same in every inertial reference frame [164]. The former considerations are enough to derive the whole theory of special relativity (possibly with the further but quite innocuous assumptions of homogeneity of space and time and isotropy of space). Incidentally, note that the fact that this maximal signal velocity is identified with the speed of light in vacuum does not follow from physical principles, nor is it required at all to derive SR.[5]

The Principle of finiteness of information density (P2) has thus two physical consequences: On the one hand, it imposes that, in general, the values taken by physical quantities at future times (say the position of a particle) are not fully determined by their present state. Consequently, physical variables do not have determined values at all possible times (or, alternatively, that the truth value of certain empirical statements, such as "a particle is located at $\bar{x}$ at a certain instant $\bar{t}$", is indeterminate). On the other hand, P2 also imposes a dynamical limit to the propagation of information (signal velocity) which, together with P1, allows to fully derive the theory of SR. Hence, **upholding the Principle of finiteness of information density gives us a hint that physics should be at the same time indeterministic and relativistic**. But are these two worldviews compatible?

In what follows, we will show that special relativity and fundamental indeterminism can be reconciled by postulating the existence of true random events –i.e. genuine acts of creations whose outcome is indeterminate before it actually happens– whose indeterminacy is to be considered a relative property as well.

[3]Note that assumption P2 is supported, in the context of general relativity, by the *Bekenstein's bound* [165], which intuitively states that since information is associated with a certain amount of energy, unbound densities of energy would degenerate into black holes.

[4]Note that from P2 it is only possible to infer that the signal velocity cannot be infinite, but this does not mean that there is a single maximal value thereof. In fact, it is in principle possible that there is a set of different signal velocities, each of them finite if P2 holds, but without a maximum (i.e. the set is unbounded). Despite this theoretical possibility, we restrict our analysis to the case where there exists a maximal velocity.

[5]However, a body moving at the maximal signal velocity would be required by SR to be massless (see e.g. [166], p. 589). This, together with the empirical demonstration that Maxwell's equations are invariant under Lorentz transformations, leads to the identification of the maximal velocity with the speed of light in vacuum, c. Note that c is not only identified with the speed of light but it also appears as a natural constant in several other theoretical frameworks (see [167]), most notably as the speed of propagation of gravitational waves in vacuum. The numerical equivalence of these speeds has been recently empirically verified with high accuracy [168].

7.3 Indeterminacy is relative

7.3.1 Locally and independently generated randomness

To discuss indeterminism in the framework of SR, and rephrase Rietdijk's and Putnam's argument, we introduce the concept of a True Random Number Generator (TRNG), by which we mean an abstract device that outputs genuinely random bits. Namely, before each bit is output, its value is not only unknown (epistemic uncertainty), but it actually has no determined value (ontic indeterminacy), even though there might be a probability distribution associated to the outcome to be realized (in principle, both at the epistemic and at the ontic level). More precisely, even having complete knowledge of (i) the *state*, i.e. the values of all the variables that may influence the outcome of the TRNG (this can be in principle everything that lies in the past light cone of the event associated to the generation of the bit), and (ii) of the *dynamical laws* that rule the evolution of each and every of said variables, there is no way to predict with certainty which will be value of the bit output by the TRNG, not even in principle (see Table 7.1).

Epistemic / Ontic	Known a	Unknown a
Determinate a	✓	✓
Indeterminate a	✗	✓

Table 7.1: Differences between epistemic (un)certainty and ontic (in)determinacy. The value of a variable a can be known only if it is determinate.

This is what we mean by true randomness, which clearly entails indeterminism. In what follows, we will be concerned solely with the (ontic) indeterminacy that each bit has before it is output and acquires a definite value (on the contrary, we will not consider what dynamical laws –deterministic or otherwise– govern the evolution of the considered systems). This very concept can be expressed in terms of truth values of empirical propositions, such that the statement, say, "the value of the bit j output by the TRNG at time $\bar{t}$ (in a certain reference frame) is $j = 0$" has a definite truth value, either true or false, only after $\bar{t}$, whereas it was (ontologically) indeterminate before, i.e. its truth value is neither true nor false (see also [169]).[6] Note that this makes the law of the excluded middle of classical logic fail, and it so relates the concept of indeterminacy to mathematical intuitionism (see [162, 170]). Turning now to SR, with reference to Fig. 7.1, let us consider four inertial observers, Alice, Bob, Charlie and Debbie. Assume that Alice has a TRNG that outputs a fresh random bit every minute (locally in her inertial reference frame). Assume also that Bob, at rest in Alice's frame, is located at a certain fix distance from Alice, say, one minute away at light speed (one light-minute). Charlie and Debbie, at relative rest, move at a constant speed v with respect to Alice's and Bob's frame. All parties know that Alice's TRNG outputs a fresh random bit every minute in her frame, i.e. every $1/\gamma = 1/\sqrt{1 - v^2/c^2}$ minutes in

[6]We refer to propositions (or statements) as *empirical* because they are about the values taken by physical variables, i.e. outcomes of hypothetical experiments. However, in principle, we will not immediately disregard also those statements that refer to not directly empirically accessible quantities (e.g., a joint proposition about two outcomes of experiments conducted in space-like separation), which instead would be meaningless to genuine empiricists because they are unverifiable.

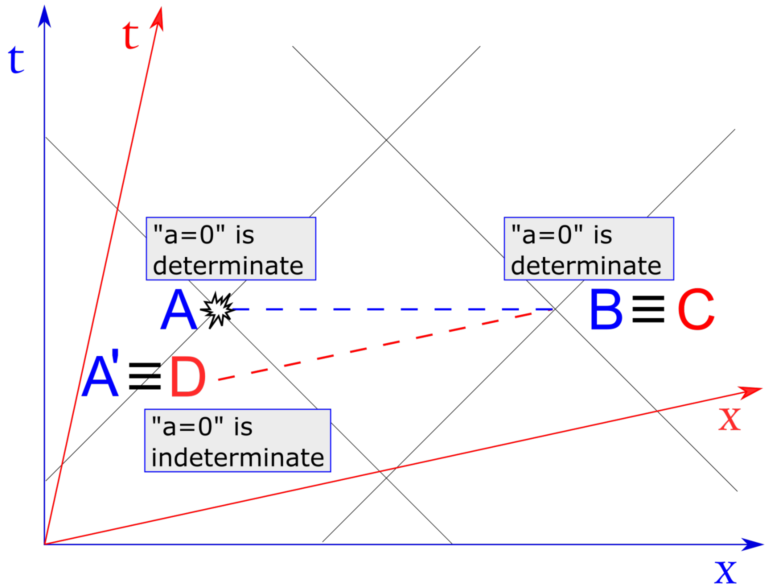

Figure 7.1: Space-time diagram (in 1+1 dimensions) illustrating Rietdijk's and Putnam's argument for the alleged incompatibility between special relativity and indeterminism. The observers Alice (A) and Bob (B) are at rest in the blue reference frame, whereas Charlie (C) and Debbie (D), also at relative rest, move at a constant speed in the positive x direction (their transformed reference frame is depicted in red). The blue dotted line represents the plane of simultaneity in Alice's and Bob's rest frame at the instant in which Alice's True Random Number Generator outputs the bit a (i.e. when it becomes determinate; event A). In the moving reference frame (red), however, when Charlie overlaps with Bob, he is simultaneous with Debbie, which in turn overlaps with Alice's past (event A') when a was not yet determinate.

Charlie's and Debbie's reference frame. Moreover, the positions and velocity v are tailored such that at 1:00 pm in both reference frames Charlie's position lies on Bob's world-line (point $B \equiv C$ in Fig. 7.1), while Debbie's position overlaps with Alice's world-line (point $A' \equiv D$ in Fig. 7.1).

Let us denote by $a \in \{0, 1\}$ the bit output by Alice's TRNG at 1:00 pm in her reference frame. As explained above, by the very nature of a TRNG, at any time instant before 1:00 pm, the proposition "$a = 0$" has no truth value for Alice (nor for any possible observer for that matter) but rather is fundamentally indeterminate. Yet, at 1:00 pm, Alice's TRNG outputs a fresh bit that now gets determined for Alice. Is the value of this bit determined for Bob, too? One may be tempted to answer in the positive (as in fact Rietdijk and Putnam do), since the events happening at Alice's and Bob's locations, respectively, are simultaneous in their rest frames. Of course, Bob may not know the value of a because Alice may not have communicated it, or, more fundamentally, because Bob is space-like separated from Alice. But does the proposition "$a = 0$" have a truth value or is it indeterminate for Bob?

Since at 1:00 pm Bob and Charlie are next to each other –ideally they exactly overlap at the same location– it is plausible to assume that the proposition "$a = 0$" has the same truth value for both of them (since this is empirically testable there is actually no room for any alternatives at the ontological level). Hence, let's assume it has a determinate truth value for Charlie as well. But then, by applying the same reasoning as before, this implies that the proposition also has a truth value at all locations simultaneous to Charlie in his inertial frame, thus it must be determinate for Debbie too. Yet, given the relative motion between the two reference frames, Debbie overlaps with Alice's world line at the space-time point A' where her TRNG has not yet output the bit. Hence, following this chain of inferences, "$a = 0$" had already a definite truth value before Alice's TRNG outputs the bit a, which is a contradiction with the assumption of a TRNG. From this contradiction, Rietdijk and Putnam concluded that, in general, the structure of SR does not allow for indeterminate events.

The previous argument, however, relies on two assumptions that seem *prima facie* innocuous, which originate from an intuitive extension of classical concepts to a relativistic scenario. In fact, these assumptions are introduced to characterize what it means for two (or more) observers to share a determinate reality (in terms of truth values of propositions). Their assumptions can be made explicit as follows:

1. Local reality: Any two observers that locally overlap attribute the same truth values to empirical propositions (including the value "indeterminate").

2. Present reality: Any two distant observers at relative rest attribute the same truth values (including the value "indeterminate") to empirical propositions about present events (i.e., lying on the same plane of simultaneity in their rest frame).

However, while the former of these assumptions may still be upheld in SR because any two observers can operationally verify the consistency of their attributed truth values locally, the latter assumption becomes questionable. Indeed, we maintain that the finiteness of the maximum speed of propagation of any piece of information renders also determinacy (i.e. the definiteness of the truth values of empirical statements) a property relative to specific regions of space-time. This can be seen by taking any binary function (e.g. the sum modulo 2) of the statements describing the outcomes of local physical processes taking place at distant locations (e.g. the output bits of two distant TRNG's). It thus becomes a straightforward assumption that determinacy itself propagates in space at the maximal signal velocity, i.e. instantaneously in Newtonian physics, but at a finite speed (that of light) according to relativity. This is where the Rietdijk's and Putnam's argument fails, in the assumption that truth values are always shared by observers lying on the same plane of simultaneity, even though this is not verifiable except for the region enclosed within the intersection of their future light cones. Hence, in short, (in)determinacy is relative.

In our simple example, this means that at 1:00 pm the proposition "$a = 0$" has no definite truth value for Bob, nor for Charlie, but it is for them still indeterminate, as it was for Alice before 1:00 pm. It is only one minute later, i.e. at 1:01 pm, that "$a = 0$" acquires a definite truth value for Bob, though Bob may still not know this value. For instance, if Bob also holds a TRNG –here assumed to generate randomness locally and independently of Alice's TRNG– which also outputs a fresh random bit, denoted by b, every minute, then the value of their sum modulo 2, $a \oplus b$, is indeterminate everywhere until 30 seconds after 1:00 pm (in their common inertial frame), and the proposition "$a = b$" has no truth value, i.e. is indeterminate, until then (see Fig. 7.2).

The present rebuttal of Rietdijk's and Putnam's argument is in agreement with the one put forward by Stein [171] who also argued for the notion of indeterminacy relative to a space-time point and pointed out the inadequacy of what we have

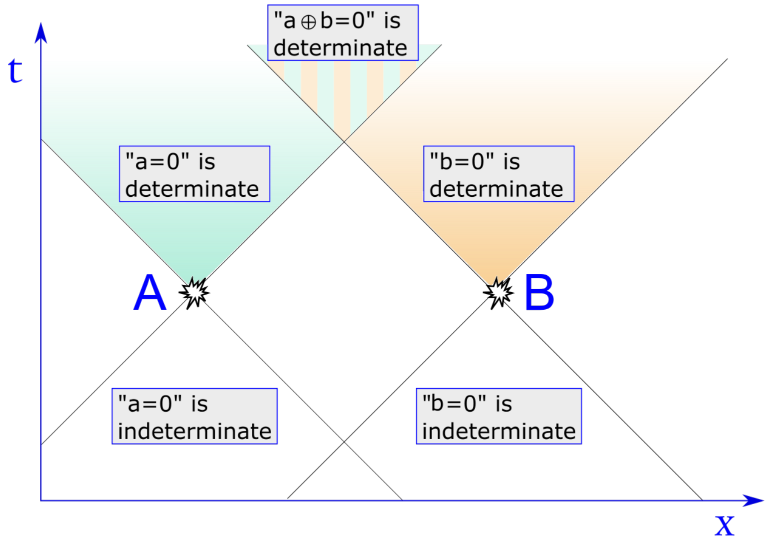

Figure 7.2: Space-time diagram (in 1+1 dimensions) showing that for distant observers (in)determinacy is relative. Even if each of their local TRNG outputs a bit, becoming determinate from indeterminate, it is only in the overlap of their future light cones that both bits become determinate. Note that both $a = 0$ and $b = 0$ are indeterminate in the entire white region.

named *present reality*. However, his argument differed from ours inasmuch as it was based on some anthropic considerations on the distance that light can travel within an interval of time that we can perceive as present. Moreover, our view of relative determinacy is also similar to that of Savitt, who –without however directly addressing Rietdijk's and Putnam's argument– put forward the idea that "special relativistic transience is the successive occurrence of local *nows* along a timelike curve." [172]. Also Myrwold [173] has discussed the compatibility between relativity and the openness of future (indeterminacy) in the context of objective collapse models of quantum mechanics. Finally, Healey [174] reached a similar conclusion to our relativity of (in)determinacy. However, he used a pragmatist view (which, in contrast to our position, takes probabilities to be relative to hypothetical agents in a specific physical situation, rather than objective propensities) to claim that the assignment of quantum states (and thus the probabilistic predictions they entail) should be specified relative to different hypothetical observers located at distant space-time regions, in a relativistic scenario.

7.3.2 "Classical" correlated randomness

In the previous section, we have argued that contrarily to a well-known philosophical argument, indeterminism and relativity remain compatible, insofar as the (in)determinacy of the values taken by physical variables, each measured by a distinct relativistic observer, is relative. So far, however, we limited our analysis to the assumption that each TRNG generates random bits at its local position and independently from any other random generators. Yet, one can in principle envision a form of "classical" correlated randomness between two or more TRNG's, which will help bridge the gap between the classical and the quantum case.[7]

[7]Here by "classical" we mean without introducing the structure of quantum probabilities, but merely consider systems whose outcomes are correlated through randomness.

Let us thus consider once more two distant observers, Alice and Bob, each provided with a TRNG such that their respective bits a and b take one of their two possible values with equal probability, i.e. $p(a = 0) = p(a = 1) = \frac{1}{2} = p(b = 0) = p(b = 1)$. However, when considered jointly, some correlations between the bits output by the two TRNG may have been established.[8] For example, although at the local level the values of the bits are random with uniform distribution, there could be a bias towards certain joint results. For instance, the cases when their values are the same could occur with a higher probability, say $p(a = b) = \frac{3}{4}$. One can think that at the ontological level there is a physical property such as a (non-local) propensity –i.e. an objective tendency that quantifies the bias towards the possible realization of an indeterministic outcome– that accounts for the correlations between the two TRNG.[9] Due to the space-like separation between Alice and Bob, there exist inertial reference frames in which Alice's TRNG generates a random bit, say $a = 0$, before Bob's one (which thus remains indeterminate in such reference frames). But because of the correlations between TRNG's, within the future light cone of Alice, the propensity for Bob to find the outcome 0 gets updated from $p(b = 0) = \frac{1}{2}$ to $p(b = 0|a = 0) = \frac{3}{4}$. On the other hand, in a reference frame in which the realization of a fresh bit by Bob's TRNG comes first, it is the propensity for the outcome of Alice's TRNG that gets updated according to the established correlations. Note that the realization of a fresh bit by a local TRNG, say, on Alice's side, does not directly affect the value of the bits generated on Bob's side (otherwise this could be used to signal). It is Alice's propensity for the outcome of Bob's TRNG that gets updated, such that if she obtains $a = 0$, within her future light cone the propensity associated with the statement $b = 0$ will change (to $3/4$ in this case). In this way, when Alice's bit gets determinate, also the propensity for the value of b relative to Alice's future light cone gets determinate, but not the value of b. It is only within the intersection of the future light cones of Alice and Bob that their outcomes unambiguously assume a definite value which ought to comply with this non-local propensity.

The reader acquainted with quantum physics would have already noticed the similarity of this "classical" non-local randomness with quantum correlations. However, we wanted here to express a hypothetical property of indeterminacy which can be conceptually introduced independently of quantum theory (although we do observe it experimentally in quantum systems and not in classical ones). Before moving to the quantum case, we deem it useful to consider the so-called Popescu-Rohrlich (PR) boxes [175], because, contrarily to the indeterministic scenarios so far discussed (in terms of TRNG), they admit the choice of inputs that is crucial in the quantum case (i.e., the choice of the measurement basis). PR boxes are a theoretical model of an operational setup that displays the maximal amount of correlated (non-local) randomness which however does not lead to instantaneous signaling (i.e. it respects the no-signaling conditions). In terms of correlations between two distant parties –which carry out local operations (i.e. they chose an input bit x and y, respectively) and measure an outcome of some not fully specified random process (indicated by the bits a and b, respectively)– the no-signaling conditions are given by the fact that the input choice (which in turn could be picked

[8] We do not discuss here possible physical mechanisms that may establish this kind of correlations, but we assume that in principle these can exist.

[9] This is inspired by the propensity interpretation of probability which was introduced by Popper [146]; see also [160] for a recent application of this concept to indeterministic physics.

according to local and independent TRNGs) of one party cannot directly influence the outcome of the other one. PR boxes are a set of correlations $p(a, b|x, y)$ defined as follows: if the pair of inputs, $(x, y) \in \{(0, 0), (0, 1), (1, 0)\}$, then the outputs are perfectly correlated, i.e., $p(0, 0|x, y) = p(1, 1|x, y) = 1/2$. Otherwise, if the input pair $(x, y) = (1, 1)$, the outcomes are perfectly anti-correlated, i.e., $p(0, 1|1, 1) = p(1, 0|1, 1) = 1/2$. It is straightforward to show that these correlations respect the no-signaling conditions. Accordingly, if, for example, $x = 0$ and $a = 0$, then the propensity $p^A(b|x, y, a) = 1$ (i.e., in this case $b = 0$), where the superscript index indicates Alice's future light cone, but $p(b|y) = 1/2$ everywhere else.

7.3.3 Quantum correlated randomness

In the previous sections, we have discussed different "classical" scenarios that bring together true randomness (i.e., indeterminism) and special relativity, reaching the conclusion that (in)determinacy is relative. Moreover, in the case of correlated randomness between two TRNGs, there are regions of space-time in which different inertial observers attribute, in general, different probabilities for measurement outcomes (corresponding to different objective propensities), and it is only in the overlap between their future light cones that their predictions match (where they also become testable). However, it would be impossible to discuss indeterminism without addressing quantum physics, which not only is the most successful theory ever in terms of predictions, but also a theory that hints at the indeterministic nature of our world. Indeed, the violation of Bell's inequalities [2] has proven that if there is at least a random event in the universe, then there can be arbitrarily many of them [176] (see also [160] for a discussion), and even that classical trajectories of particles cannot exist predetermined (if one upholds locality) [177]. Moreover, in a recent work, Dragan and Ekert showed that elementary special relativistic considerations can lead to quantum randomness [178].

Hence, we now introduce quantum (non-local) correlations, that is, what happens if Alice's and Bob's TRNGs are entangled? Assume an initial maximally entangled state, say the singlet $|\Psi^-\rangle = 1/\sqrt{2}(|0\rangle |1\rangle - |1\rangle |0\rangle)$, where $|0\rangle$ and $|1\rangle$ are eigenstates of σ_z, and the first qbit is with Alice and the second with Bob. When Alice measures her local qbit in an arbitrary basis, say the x-basis (recall that $|\Psi^-\rangle$ has the same form in every basis), she obtains, for example, the outcome $a = 0$, corresponding to the state $|0\rangle_x$. While Bob's local quantum state remains unchanged, Alice describes, after her measurement, the global state using the projection postulate, i.e. it becomes $|0\rangle_x |1\rangle_x$. Symmetrically, however, Bob locally measures his qbit, which is still in the initial entangled state $|\Psi^-\rangle$, in the y-basis, and obtains, say, outcome $b = 0$, corresponding to $|0\rangle_y$. By using the projection postulate, he updates the global state to $|1\rangle_y |0\rangle_y$, while Alice's state remains unchanged. Since Alice's and Bob's measurements are carried out in space-like separation, there are inertial reference frames in which one measurement occurs before the other and vice versa, from which we conclude that there are certain regions of space-time in which the global quantum state as assigned by Alice and Bob differ. At the intersection of the respective future light cones, however, the two bits acquire their value and the quantum state is reduced to $|0\rangle_x |0\rangle_y$, which reconciles with the usual projection postulate. In the way here described, the state is well defined at every point in space-time, but in every inertial frame there are regions of space-time where two different states are attributed to the two qbits. Notice that the probabilities of the two outcomes a and b are correlated, as usual in quantum mechanics (if one performs local measurement in two bases that are not mutually unbiased): this is non-local randomness [179], but the states change only locally.

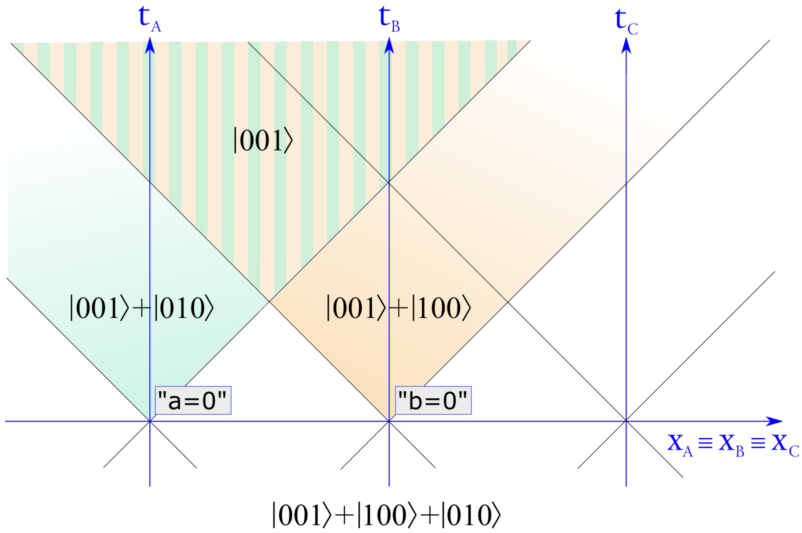

Figure 7.3: Space-time diagram (in 1+1 dimensions) showing a relativistic scenario (readapted from Ref. [180]) in which different global quantum states are assigned in different regions of space-time.

Our view is similar to that of Aharonov and Albert, who, in a series of papers [181, 180, 182], pointed out the inadequacy of the standard quantum state-vector description for multipartite systems in a relativistic scenario. We thus deem it worth to rephrase, in the language of the present paper, one of the examples they gave in Ref. [180]. With reference to Fig. 7.3, let Alice, Bob and Charlie be located at three distant locations and share a tripartite entangled W-state $|\alpha\rangle \propto |1\rangle|0\rangle|0\rangle+|0\rangle|1\rangle|0\rangle+|0\rangle|0\rangle|1\rangle$, (normalization is omitted) which can be seen as the Fock representation of a single particle in a quantum superposition between the three different locations (entanglement with vacuum). In a certain reference frame, a measurement is performed on the first qbit at t_1 revealing that the particle is not located at Alice's position (formally this means to apply to Alice's qbit the projector $|0\rangle\langle 0|$). The global state thus gets updated to $|\beta\rangle \propto |0\rangle(|1\rangle|0\rangle + |0\rangle|1\rangle)$. A second measurement is then performed, at time t_2, on the second qbit, revealing that the particles is also not located at Bob's location. This leaves the global state in $|\gamma\rangle = |0\rangle|0\rangle|1\rangle$. Given the space-like separation of these events, however, there exists another reference frame, in which the measurements happen in reversed order. In that frame, the initial state is the Lorentz transformed of $|\alpha\rangle$, which we indicate by $|\alpha'\rangle$. The first measurement happens at t_2' and leads to the updated state $|\eta\rangle \propto |1'\rangle|0'\rangle|0'\rangle + |0'\rangle|0'\rangle|1'\rangle)$ and then, after the second measurement at time t_1', to the final global state $|\gamma'\rangle = |0'\rangle|0'\rangle|1'\rangle$. This shows that the intermediate states $|\beta\rangle$ and $|\eta\rangle$, which describe the tripartite system in the period in between the two measurements in their respective reference frames, differ in a way that they are not the Lorenz transformed of one another.

In conclusion, the examples here discussed show that both at the classical level (yet assuming ontic indeterminacy and associated propensities) and in quantum mechanics, (in)determinacy is in general well defined only with respect to specific space-time regions. In particular, in this section we showed that different (global) quantum states pertain to two (or more) space-like separated regions. it is only at the intersection of the future light cones of the events that locally determine the truth value of a previously indeterminate empirical statement (i.e. the outcome of a measurement in quantum physics, or the generation of a bit in a TRNG in our previous scenarios) that the state becomes unique and reconciles with the standard quantum state-update rule.

7.4 The block-universe picture(s)

The proposal expounded in the previous sections shows a possible way to reconcile fundamental indeterminacy and SR. As already recalled, the theory of SR, by putting on the same footing space and time, led to a widely accepted view of a universe that is at the fundamental level a "block" in which nothing can really flow or evolve. In this way, the fundamental structure of the universe is regarded as a geometrical (Minkowski) space whose points are events. These events are fixed in this geometrical block and SR only allows to relate one another and the intuitive notion of time is retrieved by fixing a foliation of that space (see Fig. 7.4). We have shown, however, that if one considers the state of (in)determinacy itself as a relative property, then it is not necessary to adopt the block-universe picture.

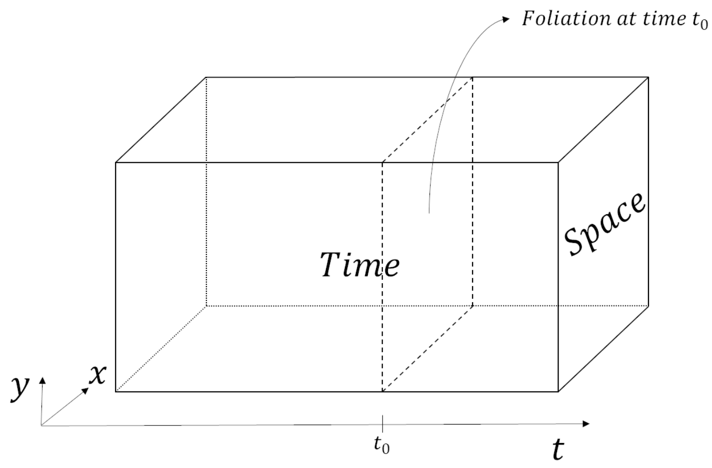

Figure 7.4: Graphical representation of the "classical" block-universe (in 2+1 dimensions).

In more recent years, in the attempt to bring quantum physics into the relativistic world-view, modifications of the block-universe picture have been proposed. In particular, D. Page, W. Wootters and others [13, 42], put forward the view that time in quantum theory –which is usually regarded as a classical scalar parameter entering the Schrödinger's equation– is described by the correlations between a clock (i.e. a physical quantum system) and the (quantum) system whose evolution the clock tracks. In this way, the fundamental object of physics is taken to be a timeless "history state" $|\Psi\rangle\rangle$, which annihilates the total Hamiltonian, i.e.

$$H|\Psi\rangle\rangle = (H_C + H_S)|\Psi\rangle\rangle = 0, \tag{7.1}$$

where H_C and H_S are the Hamiltonians of the clock and the system, respectively. The standard quantum state is obtained by conditioning the timeless state on the time read by the clock: $|\psi_S(t)\rangle = (|t\rangle \otimes \mathbb{1}_S \Psi\rangle\rangle$. Here $|t\rangle$ is an eigenstate of the operator T, such that $[T, H_C] = i\hbar$. The unitary evolution of quantum mechanics is retrieved by conditioning the timeless state on two different clock eigenstates (which is however not unambiguous, see [55]).

Assuming the universality of quantum theory (i.e. that there is in principle no limit to its domain of validity), this leads to regarding the timeless "history state of the universe" as *the* fundamental entity of nature. This creates a novel "quantum" block-universe picture, in which what really exists is a single huge quantum "history" state, which encapsulates every event in the universe. The standard notion of time emerges from correlations between clocks and systems (ultimately all the components of the universe), in a similar fashion to the foliation of the classical block-universe (see Fig. 7.5).

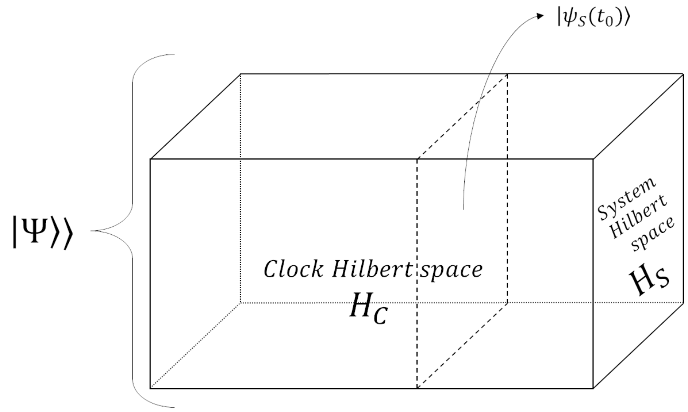

Figure 7.5: Graphical representation of the "quantum" block-universe as envisioned by the Page-Wootters formalism and the many-worlds interpretation of quantum physics.

Note that at the interpretational level, the Page-Wootters formalism fits Everett's relative state interpretation, popularized as the Many-World interpretation of quantum theory [64]. It ought to be remarked that our proposed solution for quantum scenarios (as discussed in subsection 7.3.3) is formally similar to the Many-World interpretation, for both assume the coexistence of different quantum states for the same considered system. However, there is an irreconcilable fundamental difference in our views: we take as a tenet that the outcome of a measurement always yields a single outcome (out of its possible ones), whereas in the Many-World interpretation there is a timeless state of the universe that is branched, bringing into existence at the same time all the possible outcomes at once in parallel "worlds" or realities.[10] The metaphysical consequence of these two views are thus not only different but actually antithetical. While in quantum block-universe picture the main object is the single universal wave function and time plays no fundamental role, in our view measurement outcomes (e.g. the actualized bits of a TRNG) are genuine acts of creation that transform potentiality into actuality. To this extent, our view departs from any received block-universe picture(s), taking ("creative") time as a primitive notion.

As such, our relative indeterminacy and the block-universe picture(s), should be considered two alternative world-views that stem from interpreting the theory

[10]Note that in parlance the realities are said to be parallel to indicate that the alternative realities co-exist, at the formal level, the different branches of the wave function associated with these realities are *orthogonal* in the Hilbert space.

of special relativity. This resembles the alternative interpretations of the quantum formalism, or the alternative interpretations –deterministic or otherwise– of classical physic as we proposed in Ref. [160].

7.5 Outlook

We have argued that, under the reasonable assumption of finiteness of information density, physics should comply with both indeterminism and relativity. Notwithstanding some historical criticisms, we have shown that these two views are compatible if one regards (in)determinacy itself as relative. Our analysis was based on the formulation of indeterminacy in terms of a third logical truth value (besides "true" and "false") for empirical propositions. This is thus reminiscent of mathematical intuitionism, where the law of the excluded middle fails [162, 170].

Despite these promising preliminary conclusions, a number of fundamental questions concerning a hypothetical physical framework that would bring together special relativity and indeterminism (possibly independently of quantum mechanics) remain open. For example, if we enforce the principle of finiteness of information density, the concept of a relativistic "event", usually defined as a (mathematical) point in the Minkowski space-time, would need to be substituted by a finite hypervolume. This implies that also light cones would not be perfectly determined, but their edges would be somehow blurred; does this mean that the future determinacy of physical variables is also not perfectly determined? To this end, following the principle of finiteness of information density, we have introduced in previous works [15, 160] a concrete model of indeterministic classical (non-relativistic) physics. This was achieved by replacing the usual real numbers –which are customarily assumed to be the values taken by physical quantities– with what we named "finite information quantities" (FIQs). It would be interesting to analyze in details what would be the consequences of introducing FIQs in the context of special relativity.

Furthermore, in an indeterministic worldview, one would have to reconcile the apparent need for a discretized time –due to a series of genuine acts of creation– with the continuous geometry of space-time entailed by relativity (see [183] for a work in this direction). Note, however, that we do not argue for a fundamental discretization of space-time, we agree that continuity is an irreplaceable feature in relativity theory, but we maintain that the continuum is retrieved similarly to Brouwer's "viscous continuum" in intuitionistic mathematics, wherein numbers that constitute the continuum are processes that get more and more determinate as time passes [162]. Can this proposed approach in terms of finite information help go beyond the standard relativistic block-universe picture, in which time is a mere illusion, and support instead Reichenbach's view according to which "the parallelism [between space and time] does not exist objectively and that in natural science time is more fundamental than space"? [184, 163].

Acknowledgements

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Conclusion

Despite an almost century-long history, the foundations of quantum theory remain territory for fierce debate. While one of its most problematic features, the measurement problem, is far from being uncontroversially resolved, this thesis has clarified certain issues with considering Wigner's friend scenarios –a topic that came back into fashion in recent years, yet together with a number of misconceptions. A major drawback of these lines of research is that the technology to uncontroversially implement a Wigner's friend scenario in an experiment is unforeseeably beyond reach –this would require to be able to measure superpositions, if not of human observers, of measurement devices such as photodiodes. However, being able to understand the conceptual limits of quantum theory becomes more and more important, as technology leads us slowly but steadily where this debate may find some answers from nature.

On the other hand, testing quantum coherence at a distance –namely only by means of local operations and classical communication, without recombining a particle in an interferometer– is totally feasible within current technology, and this thesis has helped characterise the possibilities, as well as the intrinsic limitations, of such experiments. Our proposals also allowed clarifying how the Aharonov-Bohm effect is far from being easily resolved as a "normal" quantum phase and requires further investigation.

While the first part of this thesis hopefully contributed to clarifying an intrinsic ambiguity on the application of quantum theory to extreme scenarios, and the second part characterised the distinction between classical and quantum mechanics, the last part had a more reuniting role between classical and quantum physics. In particular, we showed that both can be interpreted either indeterministically or otherwise and that this difference cannot be decided on empirical grounds. This leads to ask a more general question: *what is really quantum in quantum physics*? On this note, it would be advisable to systematically analyse which of the "mysterious" features strictly belong to quantum mechanics, and which ones can be retrieved in other (perhaps classical) theories, if one overcomes historical prejudices.

Looking at the whole history of quantum foundations, it is fascinating to contemplate how enormous the development of this field has been but at the same time how little it was achieved in terms of novel understanding. It would be desirable that the field of foundations of quantum mechanics keeps the momentum that has gained in the last few decades such that, perhaps, a larger and larger community could eventually succeed where even the founding fathers of the theory have failed.

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