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„Weighted blowups and stacks“

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Abstract

Weighted blowups have proven to be an important tool in finding resolutions of singularities of varieties. Since weighted blowups are usually singular, we present in this text two approaches to assign them a smooth structure. The first approach is to smoothen the singularities with a second blowup and uses methods from toric geometry. In the second part we introduce and discuss stacky blowups in the setting of stacks and will use this opportunity to give an introduction to stacks using the moduli stack of elliptic curves as motivation.

Zusammenfassung

Gewichtete Blowups sind wichtige Werkzeuge um Auflösungen von Singularitäten von Varietäten zu finden. Weil gewichtete Blowups in den meisten Fällen singulär sind, präsentieren wir in diesem Text zwei Möglichkeiten wie man ihnen eine glatte Struktur zuordnen kann. Der erste Ansatz ist die Singularitäten mit einem zweiten Blowup aufzulösen und er verwendet Methoden der torischen Geometrie. Im zweiten Teil führen wir stacky Blowups ein und diskutieren diese im Rahmen von Stacks. Wir werden diese Gelegenheit nützen um außerdem eine Einführung in die Theorie der Stacks zu geben, welche durch den Moduli Stack von elliptischen Kurven motiviert sein wird.

Contents

1 Introduction

Finding resolutions of singularities is a long standing problem in algebraic geometry. Recall that a resolution of a variety X is a smooth variety $\widetilde{X}$ together with a proper, birational morphism $\widetilde{X} \to X$. Given a variety X and a closed subvariety Y one can construct a scheme X' together with a birational, proper projection map $\pi\colon X' \to X$ called the blowup of X along Y. If one chooses Y wisely, one could hope that the singularities of X improve in some way. Hironaka proved in 1964 that if char $k = 0$ one can resolve a singular variety by a finite number of successive blowups, see [14]. In char $k = p > 0$ this question is still wide open and was only solved in special cases like toric varieties, curves, surfaces by Abhyankar, see [2], and threefolds by Cossart and Piltant, see [6]. Recently weighted blowups and stacky blowups were used by Abramovich et al, see [3], to approach this problem.

Throughout this text a variety will be a integral separated scheme of finite type over some field k. If not specified otherwise all rings will be assumed to be integral domains. We will quickly revisit one of the possible definitions of a blowup. A summary of the most important definitions and properties of blowups can be found in the beginning of Section 2. Let k be a field, R a finitely generated k-algebra and $I \subseteq R$ an ideal of R. The algebra $R[It] = \bigoplus_{k\geq 0} I^k t^k$ is graded with respect to t and called the Rees algebra. The variety $X' = \operatorname{Proj}(R[tI])$ comes with a projection $\pi\colon X' \to \operatorname{Spec}(R)$ and the pair (X', π) is called the blowup of X along I or the blowup of X along $\operatorname{Spec}(R/I)$.

A simple example of a blowup is the blowup of $\mathbb{A}^2_k$ along the origin, i.e $I = (x, y) \subseteq k[x, y]$. It consists of the subvariety $\mathbb{A}^{2\prime}_k$ of $\mathbb{A}^2_k \times \mathbb{P}^1_k$ given as the zeroset of the polynomial $xv - yw$, where x, y are the affine coordinates and $[v : w]$ the projective coordinates, and the restriction of the projection $\mathbb{A}^2_k \times \mathbb{P}^1_k \to \mathbb{A}^2_k$ to $\mathbb{A}^{2\prime}_k$, which gives the projection map $\pi\colon \mathbb{A}^{2\prime}_k \to \mathbb{A}^2_k$.

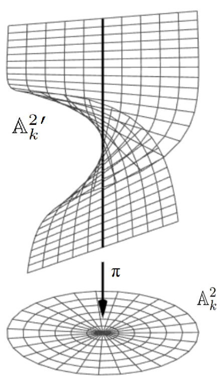

Figure 1: The blowup $\pi\colon \mathbb{A}^{2\prime}_k \to \mathbb{A}^2_k$

The underlying variety of the blowup of a nonsingular variety X along a closed nonsingular subvariety Y has again no singularities. However, if Y is not reduced, the underlying variety of the blowup of X along Y can be singular. A simple example is $X = \mathbb{A}_k^2$ and $Y = \operatorname{Spec}(k[x,y]/(x,y^2))$. If $\pi\colon X' \to X$ is the blowup of X along Y, the variety X' can be covered by two affine charts, one of which is the zeroset of $xy - z^2$ in $\mathbb{A}_k^3$, which has a singularity at the origin. We can generalize this example in the following way.

Definition. Let $m = (m_1, \ldots, m_n) \in \mathbb{N}^n$ and consider the blowup $\pi : \mathbb{A}_k^n(m) \to \mathbb{A}_k^n$ of $\mathbb{A}_k^n$ along the ideal $(x_1^{m_1}, \ldots, x_n^{m_n})$. This blowup is called the weighted blowup of $\mathbb{A}_k^n$ with weights $m_1, \ldots, m_n$.

For most m's $\mathbb{A}_k^n(m)$ is a singular variety. In the first part we try to find resolutions of $\mathbb{A}_k^n(m)$ using a single blowup. By a theorem of Bodnar, see Theorem 2.5, such a blowup implies the existence of an ideal $I \subseteq k[x_1, \ldots, x_n]$ such that the blowup of $\mathbb{A}_k^n$ along $(x_1^{m_1}, \ldots, x_n^{m_n}) \cdot I$ is smooth. In order to use tools from toric geometry, we will restrict ourselves to the case where I is a monomial ideal, i.e. generated by monomials. Following [8] we will first develop the necessary theory and then prove in Theorem 2.18 and Theorem 2.19 the existence of such an ideal for $n = 2$ and $n = 3$. In the second part we introduce stacks, using the moduli stack of elliptic curves as motivation following closely [11] and [4]. We then give the definition of a stacky blowup, a stack theoretic construction which generalizes the above definition and can be used as an alternative approach. This is also the starting point for the theory introduced in [3]. The second part is purely expository.

Throughout the text we will assume a basic understanding of category theory and algebraic geometry, for example as provided by [12]. The most important concepts appearing in the definitions and proofs will be shortly summarized in the appendices as well as some of the longer category theoretic proofs and constructions. Sometimes we will only sketch the definitions and refer the reader to the literature.

2 Toric approach

2.1 Blowups

In this section we quickly state the most important definitions and properties of blowups. We restrict ourselves to noetherian schemes. For a more detailed discussion see [12] or [13], where also all of the proofs can be found. We will start with the definition for affine varieties.

Definition. Let k be a field, R a finitely generated k-algebra and $I \subseteq R$ an ideal of R. The algebra

$$R[It] = \bigoplus_{k \geq 0} I^k t^k$$

is graded with respect to t and called the Rees algebra. The variety $X' = \operatorname{Proj}(R[tI])$ comes with a projection $\pi\colon X' \to \operatorname{Spec}(R)$, which is induced by the inclusion $R \hookrightarrow R[It]$. The pair (X', π) is called the blowup of X along I or the blowup of X along $\operatorname{Spec}(R/I)$. We will often write $\pi\colon X' \to X$ for the blowup.

Lemma 2.1. *In the above setting, if I is generated by $f_1, ..., f_r$, let $R_{f_i} = R[\frac{1}{f_i}]$ and $R_i = R[\frac{f_1}{f_i}, ..., \frac{f_r}{f_i}] \subseteq R_{f_i}, 1 \leq i \leq r$. X' can be covered by affine charts $U_i = \mathrm{Spec}(R_i), 1 \leq i \leq r$, which are glued along the inclusions $R_{f_i}, R_{f_j} \subseteq R_{f_i f_j}$.*

We can generalize the definition of blowups to schemes.

Definition. Let X be a scheme and $\mathscr{J}$ a quasi-coherent sheaf of ideals on X corresponding to a closed subscheme Y. Then the blowup of X along $\mathscr{J}$ is defined as the scheme

$$X' = \mathrm{Proj}(\mathscr{O}_X[\mathscr{J}t]),$$

together with the natural projection $\pi: X' \to X$. We will also sometime call it the blowup of X along Y. We say that the blowup of X along $\mathscr{J}$ is smooth if X' is smooth.

Alternatively, one can define blowups via a universal property. Note that in this case the blowup is only defined up to isomorphism and one can show that the above defined blowup satisfies the universal property.

Definition. Let X be a scheme and $\mathscr{J}$ a quasi-coherent sheaf of ideals on X. We call a scheme X' together with a morphism $\pi: X' \to X$ a blowup of X along $\mathscr{J}$ if the inverse image sheaf $\pi^{-1}\mathscr{J}\mathscr{O}_{X'}$ is invertible and for each scheme Z and morphism $f: Z \to X$ whose inverse image sheaf $f^{-1}\mathscr{J}\mathscr{O}_Z$ is invertible, there exists a unique map $g : Z \to X'$ such that $f = \pi \circ g$. If $\pi: X' \to X$ is a blowup of X, the divisor induced by $\pi^{-1}\mathscr{J}\mathscr{O}_{X'}$ is called the exceptional divisor.

Lemma 2.2. *Let $\pi : X' \to X$ be a blowup of a variety X along some closed subvariety Y.*

1. *X' is a variety*
2. *The projection π is a proper, surjective morphism and induces an isomorphism $\pi^{-1}(X - Y) \to X - Y$. It is therefore a birational morphism.*
3. *Let Z be a subvariety of X and Z' the Zariski closure of $\pi^{-1}(Z - Y)$ in X'. Then Z' together with the restriction of π to Z' is the blowup of Z along $Z \cap Y$.*
4. *Let $U \subseteq X$ be an open subvariety of X. Then $U' = \pi^{-1}(U)$ together with the restriction of π to U' is a blowup of U with center $Y \cap U$.*

Lemma 2.3. *Let X be a nonsingular variety, Y a closed, nonsingular subvariety and $\pi : X' \to X$ the blowup of X along Y. Then X' is nonsingular.*

As said in the introduction, in this section we want to find a resolution of the weighted blowup $\mathbb{A}_k^n(m)$ via a single blowup. The following theorem states that this amounts to finding an ideal $I \subseteq k[x_1, ..., x_n]$ such that the blowup of $\mathbb{A}_k^n$ along $I \cdot (x_1^{m_1}, ..., x_n^{m_n})$ is smooth.

If $S = \bigoplus_{k \geq 0} S_k$ is a graded ring generated by $x_0, ..., x_r \in S_1$ as an S_0-algebra,

$$I = \bigoplus_{k \geq 0} I_k$$

a homogeneous ideal and $d \in \mathbb{N}$, recall that the d-th inverse saturation of I is

$$I' = \bigoplus_{k \geq d} I_k$$

and the saturation of I is

$$\overline{I} = \{s \in S : \text{for all } 0 \leq i \leq r \text{ exists an } m \in \mathbb{N} : x_i^m s \in I\}.$$

To prove Theorem 2.5, we need the following lemma.

Lemma 2.4. *Let $S = \bigoplus_{k \geq 0} S_k$ be a graded ring generated by $x_0, \ldots, x_r \in S_1$ as an S_0-algebra, $I \subseteq S$ a homogeneous ideal generated by $g_1, \ldots, g_l$ and $d = \max_j \deg g_j$. Then I and the d-th inverse saturation I' of I induce the same subscheme of* $\mathrm{Proj}(S)$*. The blowups of* $\mathrm{Proj}(S)$ *along I and I' are isomorphic in the sense of the definition of the blowup via a universal property. Furthermore, I' is generated by elements of I_d.*

Proof. It is straightforward to see that I and the d-th inverse saturation of I have the same saturation, thus it is sufficient to prove that I and $\overline{I}$ induce the same subscheme of $\mathrm{Proj}(S)$. Let $I_{D(x_i)}$ resp. $\overline{I}_{D(x_i)}$ be the pullback of I resp. $\overline{I}$ to the affine subscheme

$$D(x_i) = \{\mathfrak{p} \text{ homogeneous prime ideal of } S : x_i \notin \mathfrak{p}\}$$

for $0 \leq i \leq r$. From the definition of $\overline{I}$ it follows immediately that $I_{D(x_i)} = \overline{I}_{D(x_i)}$ and therefore I and $\overline{I}$ induce the same closed subschemes on the affine subset $D(x_i)$. Therefore the induced subschemes on $\mathrm{Proj}(S)$ are the same, because the $D(x_i)$ cover $\mathrm{Proj}(S)$. Since the universal property can be checked locally, we also see that the two blowups of $\mathrm{Proj}(S)$ are isomorphic.

We are now going to show that the d-th inverse saturation of I is generated in degree d. Let $y \in I_{d+k}, k \geq 0$ be of the form $y = g_j z$ for some $1 \leq j \leq l$, where z is a monomial in the x_i's. Since $\deg g_j \leq d$, we can can choose a factor z' of z of degree $d - \deg g_j$. Then $g_j z' \in I_d$ and it is clear that the set of all $g_j z'$ obtained in such a way is a generating set of I', since $x_0, \ldots, x_r$ generates S. Moreover, S is noetherian and thus the d-th inverse saturation is finitely generated. ☐

Theorem 2.5 (Bodnar, [5])**.** *Let $X = \mathrm{Spec}(R)$ be an affine variety, $I \subseteq R$ an ideal and $\pi : X' \to X$ the blowup of X along I. Moreover, let*

$$J = \bigoplus_{k \geq 0} J_k$$

be a homogeneous ideal of $R[It]$ and consider the blowup $\pi' : X'' \to X'$ of X' along J. Let J be generated by homogeneous elements $h_1, \ldots, h_m$ and set $d =$ $\max_i \deg(h_i)$. Let

$$J' = \bigoplus_{k \geq d} J_k$$

be the d-th inverse saturation of J. Moreover, consider

$$\begin{aligned} \phi : R[It] &\to R \\ t &\mapsto 1 \end{aligned}$$

and set $\bar{J} = \phi(J')R$. If $\widetilde{\pi} : \widetilde{X} \to X$ is the blowup of X along the ideal $I \cdot \bar{J}$, then there exists an isomorphism $\psi : \widetilde{X} \to X''$ such that $\pi \circ \pi' \circ \psi = \widetilde{\pi}$.

Proof. Let I be generated by $f_1, ..., f_r$, $R_i = R[\frac{f_1}{f_i}, ..., \frac{f_r}{f_i}]$, $1 \le i \le r$ and $\alpha_i = \deg h_i$. Recall that X' is covered by the affine charts $U_i = \operatorname{Spec}(R_i)$, $1 \le i \le r$. Fix now $1 \le i \le r$. The pullback of $J = (h_1, ..., h_m)$ to U_i along this covering is

$$J\big|_{U_i} = (\frac{\bar{h}_1}{f_i^{\alpha_1}}, ..., \frac{\bar{h}_m}{f_i^{\alpha_m}}).$$

By Lemma 2.2 the pullback of the blowup of X' along J to U_i is isomorphic to U_i', the blowup of U_i along $J\big|_{U_i}$. The blowup U_i' can be covered by charts

$$V_{i,j} \cong \operatorname{Spec}(R_i[\frac{\bar{h}_1}{f_i^{\alpha_1}}\frac{f_i^{\alpha_j}}{\bar{h}_j}, ..., \frac{\bar{h}_m}{f_i^{\alpha_m}}\frac{f_i^{\alpha_j}}{\bar{h}_j}]), 1 \le j \le m,$$

where $\bar{h}_j = \phi(h_j)$.

Let $d = \max_i \alpha_i$. By Lemma 2.4, the d-th inverse saturation J' of J yields the same blowup and is generated by elements $g_1, ..., g_l$ with $\deg g_i = d$. Therefore we can replace J by J' and thus we can assume that U_i' can be covered by charts

$$V'_{i,k} \cong \operatorname{Spec}(R_i[\frac{\bar{g}_1}{\bar{g}_k}, ..., \frac{\bar{g}_l}{\bar{g}_k}]), 1 \le k \le l,$$

where again $\bar{g}_k = \phi(g_k)$.

Let $\widetilde{X}$ be the blowup of X along $I\bar{J} = (f_s\bar{g}_t, 1 \le s \le r, 1 \le t \le l)$. Then $\widetilde{X}$ can be covered by affine charts

$$\widetilde{U_{i,k}} \cong \operatorname{Spec}(R[\frac{f_s\bar{g}_t}{f_i\bar{g}_k}, 1 \le s \le r, 1 \le t \le l]) =$$

$$\operatorname{Spec}(R_i[\frac{\bar{g}_1}{\bar{g}_k}, ..., \frac{\bar{g}_l}{\bar{g}_k}]), 1 \le k \le l.$$

By computing the transition functions explicitly between the charts $\widetilde{U_{i,k}}$ and $\widetilde{U_{i,k'}}$ and $V'_{i,k}$ and $V'_{i,k'}$ one can easily check that they are identical, thus these local isomorphisms can be glued to a global isomorphism $\psi : \widetilde{X} \to X''$. Moreover, if we go back to the local picture, we see that the projections $\pi \circ \pi'$ and $\widetilde{\pi}$ both are induced by the inclusions $R \hookrightarrow R_i[\frac{\bar{g}_1}{\bar{g}_k}, ..., \frac{\bar{g}_l}{\bar{g}_k}], 1 \le i \le r, 1 \le k \le l$ and can deduce $\pi \circ \pi' \circ \psi = \widetilde{\pi}$ ☐

Example: Fix $1 \le i \le n$. The blowup $\pi \colon \mathbb{A}_k^{n\prime} \to \mathbb{A}_k^n$ of $\mathbb{A}_k^n$ along the ideal $(x_1, ..., x_n)$ followed by a blowup of the origin of the ith chart

$$U_i = \operatorname{Spec}(k[x_1, ..., x_n, \frac{x_1}{x_i}, ..., \frac{x_n}{x_i}]) \cong \mathbb{A}_k^n$$

of $\mathbb{A}_k^{n\prime}$ yields in above language $R = k[x_1, ..., x_n]$, $I = (x_1, ..., x_n)$,

$$J = (x_i, tx_1, ..., tx_{i-1}, tx_{i+1}, .., tx_n)$$

and

$$J' = (tx_1, .., tx_i^2, ..., tx_n).$$

Thus $\mathbb{A}_k^{n}{}''$ is isomorphic to the blowup of $\mathbb{A}_k^2$ along the ideal

$$I \cdot \bar{J} = (x_1, ..., x_n)(x_1, ..., x_i^2, ..., x_n).$$

Because of this theorem, a resolution of $\mathbb{A}_k^n(m)$ with one additional blowup implies the existence of an ideal I such that the blowup of $\mathbb{A}_k^n$ along $(x_1^{m_1}, ..., x_n^{m_n}) \cdot I$ is smooth. We will now restrict ourselves to the case where I is a monomial ideal. This allows us to work in the setting of toric geometry, which enables us to use combinatorial methods to prove our main results.

2.2 Blowups in monomial ideals

This section is based on [8], where also all the proofs are taken from. Let I be a monomial ideal of $k[x_1, .., x_n]$. The support of the ideal I is

$$\operatorname{supp}(I) = \{\alpha : x^\alpha \in I\}.$$

We will denote the convex hull of a subset $S \subseteq \mathbb{R}^n$ as $\operatorname{Conv}(S)$ and call the convex hull of $\operatorname{supp}(I)$ the Newton polyhedron of I, which will be denoted as $N(I) = \operatorname{Conv}(\operatorname{supp}(I))$. Recall that the Minkowski sum of two subsets $A, B \subseteq \mathbb{R}^n$ is defined as

$$A + B = \{a + b, a \in A, b \in B\}.$$

Lemma 2.6. *Let $A, B \subseteq \mathbb{R}^n$ with convex hulls $\operatorname{Conv}(A)$ resp. $\operatorname{Conv}(B)$. Then*

$$\operatorname{Conv}(A + B) = \operatorname{Conv}(A) + \operatorname{Conv}(B).$$

Proof. The set $\operatorname{Conv}(A) + \operatorname{Conv}(B)$ is convex: If

$$a', a'' \in \operatorname{Conv}(A), b', b'' \in \operatorname{Conv}(B),$$

then

$$\lambda(a'+b')+(1-\lambda)(a''+b'') = \lambda a'+(1-\lambda)a''+\lambda b'+(1-\lambda)b'' \in \operatorname{Conv}(A)+\operatorname{Conv}(B).$$

Thus $\operatorname{Conv}(A + B) \subseteq \operatorname{Conv}(A) + \operatorname{Conv}(B)$. On the other hand, any point $a' \in \operatorname{Conv}(A)$ can be written as a finite convex combination

$$a' = \sum_{i \in \Delta} \lambda_i a_i, \sum_{i \in \Delta} \lambda_i = 1$$

with $a_i \in A$ for $i \in \Delta$ and any point $b' \in \operatorname{Conv}(B)$ can be written similarly as a finite sum

$$b' = \sum_{j \in \Delta'} \mu_j b_j, \sum_{j \in \Delta'} \mu_j = 1$$

with $b_j \in B$ for $j \in \Delta'$. Since

$$a' + b' = \sum_{(i,j) \in \Delta \times \Delta'} \lambda_i \mu_j (a_i + b_j)$$

and

$$\sum_{(i,j) \in \Delta \times \Delta'} \lambda_i \mu_j = 1,$$

we obtain the other inclusion $\operatorname{Conv}(A) + \operatorname{Conv}(B) \subseteq \operatorname{Conv}(A + B)$. □

Lemma 2.7. *Let $I, J \subseteq k[x_1, .., x_n]$ be ideals with corresponding Newton polyhedra $N(I), N(J)$. Then the Newton polyhedron of IJ is $N(I) + N(J)$.*

Proof. This follows immediately from the above lemma. □

If $(\alpha_1, ..., \alpha_n) = \alpha \in \mathbb{N}^n$ we will write x^α for $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Let $\{e_1, ..., e_n\}$ be the standard basis of $\mathbb{R}^n$. If $S \subseteq \mathbb{R}^n$ is a finite set, we write $\langle S \rangle_{\mathbb{N}} = \{\sum_{s \in S} n_s s, n_s \in \mathbb{N}\}$ for the $\mathbb{N}$-span of S.

Definition. Let I be a monomial ideal of $k[x_1, ..., x_n]$. Let $\pi \colon \mathbb{A}_k^{n\prime} \to \mathbb{A}_k^n$ be the blowup of $\mathbb{A}_k^n$ along I and U_α be the affine chart of $\mathbb{A}_k^{n\prime}$ of the form

$$U_\alpha = \operatorname{Spec}(k[x_1, ..., x_n, \frac{x^{\alpha'}}{x^\alpha} : \alpha' \in A])$$

for some $\alpha \in \operatorname{supp}(I)$. Let

$$IT_\alpha = \langle e_1, ..., e_n, \alpha' - \alpha : \alpha' \in A \rangle_{\mathbb{N}}$$

be the ideal tangent cone at α of $N(I)$ and define $k[IT_\alpha] = k[x^v : v \in IT_\alpha]$. It is easy to see that $U_\alpha = \operatorname{Spec}(k[IT_\alpha])$. The ideal tangent cone IT_α is called pointed if $IT_\alpha \cap (-IT_\alpha) = \{0\}$. A generating system of IT_α are vectors $v_1, ..., v_r \in IT_\alpha$ such that $\langle v_1, ..., v_r \rangle_{\mathbb{N}} = IT_\alpha$. A generating system $v_1, ..., v_r$ of IT_α is called minimal if no v_i can be written as an $\mathbb{N}$-linear combination

$$v_i = \sum_{i \neq j} n_j v_j.$$

Lemma 2.8. *If IT_α is pointed it has a unique minimal generating system.*

Proof. Assume that there are two minimal generating systems $v_1, ..., v_r$ and $w_1, ..., w_s$. Then there exist $\alpha_{ij} \in \mathbb{N}$ such that

$$v_j = \sum_i \alpha_{ij} w_i, 1 \leq j \leq r$$

and $\beta_{ij} \in \mathbb{N}$ such that

$$w_i = \sum_j \beta_{ij} v_j, 1 \leq i \leq s.$$

Thus

$$v_k = \sum_{i,j} \alpha_{ik} \beta_{ij} v_j.$$

We have $\sum_i \alpha_{ik} \beta_{ik} \leq 1$, because otherwise

$$(1 - \sum_i \alpha_{ik} \beta_{ik}) v_k = \sum_{i, j \neq k} \alpha_{ik} \beta_{ij} v_j$$

would be a nonzero element of $IT_\alpha \cap (-IT_\alpha)$, which contradicts our assumption that IT_α is pointed. Similarly we see that $\sum_i \alpha_{ik} \beta_{ik} \geq 1$ because $v_1, ..., v_r$ is minimal. Hence

$$\sum_i \alpha_{ik} \beta_{ik} = 1, 1 \leq k \leq r$$

and

$$\sum_{i,j\neq k} \alpha_{ik}\beta_{ij}v_j = 0$$

and thus, again because IT_α is pointed, $\sum_i \alpha_{ik}\beta_{ij} = 0$ if $j \neq k$. Let A be the matrix with entries $A_{ij} = \alpha_{ij}$ and B the matrix with entries $B_{ij} = \beta_{ji}$. Both A and B have entries in the natural numbers and by the above reasoning A is the inverse of B and vice versa. Since all the nonzero entries of A and B have to be 1, we can write A and B as $A = P_1 + M_1$ and $B = P_2 + M_2$, where P_1 and P_2 are permutation matrices and M_1 and M_2 are matrices with entries in the natural numbers. Thus

$$P_1P_2 - I = P_1M_2 + M_1P_2 + M_1M_2.$$

All the entries of the matrix on the right are natural numbers and the entries of the matrix on the left are either all 0 or the exist a negative one. Hence $P_1P_2 = I$, $M_1 = M_2 = 0$ and $\{v_1, ..., v_r\} = \{w_1, ..., w_s\}$. □

Lemma 2.9. *The ideal tangent cone IT_α is pointed if and only if α is a vertex of the Newton polyhedron $N = N(I)$.*

Proof. We will first assume that IT_α is not pointed. By definition $IT_\alpha = \langle v_1, ..., v_r\rangle_{\mathbb{N}}$ with $v_i = e_i, 1 \leq i \leq n$ and $v_i = \beta_i - \alpha$, $\beta_i \in N, n+1 \leq i \leq r$. If IT_α is not pointed, there exist nontrivial $n_i \in \mathbb{N}$ such that $\sum_i n_iv_i = 0$. By inserting the definition of the v_i's and dividing by $\sum_{i=n+1}^r n_i$ we see there exist $\lambda_k, \mu_j \in \mathbb{Q}_{\geq 0}$ for $1 \leq j \leq n$ and $n+1 \leq k \leq r$ with $\sum_k \lambda_k = 1$ such that

$$\alpha = \sum_k \lambda_k\beta_k + \sum_j \mu_je_j.$$

Therefore α cannot be a vertex of N, because it is a convex linear combination of the β_i's plus an element in the cone generated by $e_1, ..., e_n$.

We will now assume α is not a vertex. If α is not a vertex, we can write α as above

$$\alpha = \sum_k \lambda_k\beta_k + \sum_j \mu_je_j,$$

for $\lambda_k \in \mathbb{Q}_{\geq 0}$, $1 \leq k \leq l$ for some l with $\sum_i \lambda_k = 1$, $\mu_j \in \mathbb{N}$ for $1 \leq j \leq n$ and β_k some vertices of N. Thus after multiplying with a common denominator we obtain

$$d\alpha = \sum_k n_k\beta_k + \sum_j \mu_j'e_j,$$

with $\sum_k n_k = d$. This contradicts IT_α being pointed, since

$$0 = \sum_i n_i(\beta_i - \alpha) + \sum_j \mu_j'e_j.$$

□

Lemma 2.10. *Let α be a vertex of N and $I = (x^{\alpha_1}, ..., x^{\alpha_m})$. Then $\alpha \in \{\alpha_1, ..., \alpha_m\}$. If $\beta - \alpha$ is element of the minimal generating system of IT_α, we also have $\beta \in \{\alpha_1, ..., \alpha_m\}$.*

Proof. If α is a vertex, it is clear that $\alpha \in \operatorname{supp}(I)$. If $\alpha \notin \{\alpha_1, ..., \alpha_m\}$, we can write

$$\alpha = \sum_i n_i \alpha_i + \sum_j r_j e_j,$$

with $n_i, r_j \in \mathbb{N}$ for $1 \leq i \leq m, 1 \leq j \leq n$ as a nontrivial linear combination. Thus $(-\sum_i n_i)\alpha = \sum_i n_i(\alpha_i - \alpha) + \sum_j r_j e_j \in IT_\alpha$. This contradicts the fact that IT_α is pointed.

Assume $\beta \notin \{\alpha_1, ..., \alpha_m\}$. We can again write $\beta = \sum_i r_i \alpha_i, r_i \in \mathbb{N}$ for $1 \leq i \leq m$ as a nontrivial linear combination. Then $\beta - \alpha = \sum_i r_i(\alpha_i - \alpha) + ((\sum_i r_i) - 1)\alpha$. Since $\alpha \in \mathbb{N}^n$, we see $((\sum_i r_i) - 1)\alpha \in \langle e_1, ..., e_n \rangle_{\mathbb{N}}$ and therefore $\beta - \alpha$ is not element of the minimal generating system. □

If $v_1, ..., v_r$ is a generating system of IT_α, we see that $k[IT_\alpha] = k[x^{v_1}, ..., x^{v_r}]$, which is therefore a toric variety, see Appendix A.1 and A.2.

Lemma 2.11. *Let α be a vertex of N. If U_α is smooth it is isomorphic to $\mathbb{A}_k^n$.*

Proof. If U_α is a smooth toric variety $U_\alpha \cong \mathbb{A}_k^{n-d} \times k^{*d}$ by Lemma A.12. Because α is a vertex, IT_α is pointed, therefore $d = 0$. Indeed, if d would be nonzero, then w.l.o.g. both e_1 and $-e_1$ would be elements of IT_α, which contradicts IT_α being pointed. □

Let $I \subseteq k[x_1, ..., x_n]$ be a monomial ideal and $N = N(I)$ its Newton polyhedron. An ideal tangent cone is called simplicial if it is pointed and its minimal generating system consists of n vectors, or, equivalently, is a basis of $\mathbb{Z}^n$. We call a vector $v = (v_1, ..., v_n) \in \mathbb{Z}^n$ primitive if $\gcd(v_1, ..., v_n) = 1$.

The following theorem allows us to transform the question whether the blowup of $\mathbb{A}_k^n$ along an ideal generated by monomials is smooth or not into a combinatorial problem.

Theorem 2.12 (Faber, Westra,[8]). *Let $I \subseteq k[x_1, ..., x_m]$ be a monomial ideal, $N = N(I)$ its Newton polyhedron, $\pi\colon \mathbb{A}_k^{n\prime} \to \mathbb{A}_k^n$ the blowup of $\mathbb{A}_k^n$ along I and U_α the charts of $\mathbb{A}_k^{n\prime}$ for $\alpha \in \operatorname{supp}(I)$.*

1. *If α is not a vertex of N, then there exists a vertex α' of N such that U_α is contained in $U_{\alpha'}$.*
2. *If α is a vertex of N, U_α is smooth if and only if the ideal tangent IT_α is simplicial. Moreover, if U_α is smooth, $U_\alpha \cong \mathbb{A}_k^n$.*
3. *If α is a vertex of N and IT_α is simplicial, the generators of IT_α are primitive.*

Proof. (1) If α is not a vertex, we can write α as

$$\alpha = \sum_i \lambda_i \alpha_i + \sum_j \mu_j e_j,$$

with $\lambda_i \in \mathbb{Q}_{\geq 0}, \sum_i \lambda_i = 1, \mu_j \in \mathbb{N}, 1 \leq j \leq n, 1 \leq i \leq r$ for some r and α_i vertices of N. Thus, after multiplying with a common denominator and choosing any vertex α_j, we see that

$$\alpha - \alpha_j = \sum_{i \neq j} n_i(\alpha_i - \alpha) + (n_j - 1)(\alpha_j - 1) + \sum_j \mu'_j e_j \in IT_\alpha.$$

Hence for any $\alpha' \neq \alpha_j$ we get $(\alpha' - \alpha_j) = (\alpha' - \alpha) + (\alpha - \alpha_j) \in IT_\alpha$. This gives the desired inclusion $k[IT_{\alpha_j}] \hookrightarrow k[IT_\alpha]$.

(2) If U_α is smooth, $k[IT_\alpha] \cong \mathbb{A}^n_k$. Hence the unique generating system S of IT_α is a basis of $\mathbb{Z}^n$. If IT_α is simplicial with generating set $v_1, ..., v_n$, there is a morphism

$$\begin{aligned} \pi : k[x_1, ..., x_n] &\to k[IT_\alpha] \\ x_i &\mapsto x^{v_i} \end{aligned}$$

Surjectivity is clear. By Lemma A.6 the kernel of this map is generated by binomials. A binomial $x^{w'} - x^w$ is in the kernel if and only if $Vw' = Vw$, where V is the matrix with columns v_i. Since V is invertible, we conclude $w' = w$, hence π is an isomorphism.

(3) Suppose $v_1, ..., v_r$ is basis of $\mathbb{Z}^n$ and $v'_i = \frac{v_i}{d} \in \mathbb{Z}^n$. Let V' be the matrix where v_i is replaced by v'_i. If IT_α is simplicial, we have $\pm 1 = \det(V) = d \det(V') = \pm d$ and hence $d = \pm 1$. Thus IT_α has a primitive generating system. □

2.3 The saturation

Let $m = (m_1, ..., m_n)$ for fixed $m_i \in \mathbb{Z}^+$,

$$S = \{(a_1, ..., a_n) \in \mathbb{N}^n, 1 \leq a_1 \leq m_1, ..., 1 \leq a_n \leq m_n\}$$

and I the monomial ideal

$$I_{m_1,...,m_n} = \prod_{(a_1,...,a_n) \in S} (x_1^{a_1}, ..., x_n^{a_n}).$$

This will be our first candidate for giving us a resolution of $\mathbb{A}^n_k(m)$. The ideal $I_{m_1,...,m_n}$ is sometimes called the saturation of $(x^{m_1}, ..., x^{m_n})$. Note that this saturation is not the same as the one defined in the beginning of Section 2.

It is straightforward to see that that the set of all products $\prod_{s \in S} x^{\alpha_s}$, where $\alpha_{(a_1,...,a_n)} \in \{a_1, ..., a_n\}$, generate $I_{m_1,...,m_n}$. Thus, if α is a vertex of N, we see that x^α is of the form $\prod_{s \in S} x^{\alpha_s}$ with $\alpha_{(a_1,...,a_n)} \in \{a_1, ..., a_n\}$ by Lemma 2.10. If $x^\beta \in I$ and $\beta - \alpha$ should be part of a minimal generating system of IT_α, we can again assume by Lemma 2.10 that β is of the above form. If we choose β such that β_s differs from α_s only for $s = t$ for some fixed $t \in S$, then $\beta_t e_j - \alpha_t e_i \in IT_\alpha$, where j is the index of β_t in t and i is the index of α_t in t. In fact these elements together with $\{e_1, ..., e_n\}$ are a generating system of IT_α, because, if $x^\beta = \prod_{s \in S} x^{\beta_s}$, we see that

$$\beta - \alpha = \sum_{t : \alpha_t \neq \beta_t} \beta_t e_{j(t)} - \alpha_t e_{i(t)},$$

where $j(t)$ resp. $i(t)$ are the indices of β_t resp. α_t in t. We denote the generating system of IT_α consisting of vectors of the form $\beta_t e_j - \alpha_t e_i$ and $\{e_1, ..., e_n\}$ as $\{f_1, ..., f_r\}$.

Example: Let $n = 3$, $m_1 = 1, m_2 = 2, m_3 = 3$. Then

$$I_{1,2,3} = (x, y, z)(x, y^2, z)(x, y, z^2)(x, y^2, z^2)(x, y, z^3)(x, y^2, z^3)$$

and we choose $\alpha = (1, 2, 4)$. We can write $x^\alpha = \prod_{s\in S} x^{\alpha_s}$, where

$$\alpha_{1,1,1} = (0,0,1), \alpha_{1,2,1} = (0,0,1), \alpha_{1,1,2} = (0,1,0),$$

$$\alpha_{1,2,2} = (0,0,2), \alpha_{1,1,3} = (0,1,0), \alpha_{1,2,3} = (1,0,0).$$

Thus we can see that

$$k[IT_\alpha] = k[x, y, z, \frac{x}{z}, \frac{y}{z}, \frac{x}{z}, \frac{y^2}{z}, \frac{x}{y}, \frac{z^2}{y}, \frac{x}{z^2}, \frac{y^2}{z^2}, \frac{x}{y}, \frac{z^3}{y}, \frac{y^2}{x}, \frac{z^3}{x}].$$

Lemma 2.13. *Any minimal generating set $\{b_1, ..., b_n\}$ of IT_α is a subset of $\{f_1, ..., f_r\}$.*

Proof. Let $S \subseteq \{f_1, ..., f_r\}$ be a minimal generating set. By Lemma 2.8 $S = \{b_1, ..., b_n\}$. □

Lemma 2.14. *Let $\{h_1, ..., h_n\} \subseteq \{f_1, ..., f_r\}$ and assume that the matrix $C = (h_1 ... h_n)$ is invertible. Moreover, assume that there meet n edges $E_1, ... E_n$ in α in N and that $h_i = \alpha_i - \alpha$, where α_i is the nearest point to α in $E_i \cap supp(I)$. In this case U_α is smooth.*

Proof. If C is invertible, there cannot lie a point with integer coefficients in the interior of the parallelepiped corresponding to C. Any integer point in its interior would span a parallelepiped of smaller integer volume, but since C is invertible, the parallelepiped of C has already volume 1.

If $\alpha' \in IT_\alpha$, it can be written as a finite linear combination

$$\alpha' = \sum_i \lambda_i h_i, \lambda_i \in \mathbb{N}.$$

We can always find an α' such that all $\lambda_i < 1$, simply by subtracting h_i's. However, this would imply that α' lies in the parallelepiped C and hence $\lambda_i = 0$. Therefore, if $|\det(C)| = 1$, the h_i's generate IT_α and hence U_α is smooth. □

2.3.1 The Case $n = 2$

We will now show that for $n = 2$, the blowup of $\mathbb{A}^2_k$ along I_{m_1,m_2} will always be smooth. The main ingredient of the proof is Pick's theorem.

Theorem 2.15 (Pick's Theorem)**.** *If $P \subseteq \mathbb{R}^2$ is a convex polygon whose vertices have integer coefficients, the area A of P is*

$$A = i + \frac{b}{2} - 1,$$

where i is the number of lattice points in the interior of P and b the number of lattice points on the boundary of P.

A set of normal vectors of a polyhedron N with faces $F_1, ..., F_l$ is a set of vectors $\{v_1, ..., v_l\}$ such that v_i is a normal vector of F_i for $1 \leq i \leq l$.

Lemma 2.16. *Let $N = N(I_{m_1,m_2})$ be the Newton polyhedron of I_{m_1,m_2}. Then a set of normal vectors of N is*

$$\mathcal{N}(N) = \{(b,a) : 0 \leq b \leq m_2, 0 \leq a \leq m_1, \gcd(a,b) = 1\}.$$

Proof. By Lemma 2.7 we see that N is the Minkowski sum of all

$$N(x^a, y^b), 1 \leq a \leq m_1, 1 \leq b \leq m_2.$$

A set of normal vectors of $N(x^a, y^b)$ is $\{(1,0),(0,1),(b,a)\}$. It is well known that if A and B are convex 2 dimensional polyhedra with finitley many vertices and $\mathcal{N}(A)$ resp. $\mathcal{N}(B)$ are sets of normal vectors of A resp. B, then $\mathcal{N}(A) \cup \mathcal{N}(B)$ is a set of normal vectors of $A + B$. The lemma follows immediately from this observation. □

Lemma 2.17. *Let $k, l \in \mathbb{N}$, $(a,b),(c,d) \in \mathbb{N}^2$ with $0 < a, c \leq k$ and $b, d \leq l$. If $\frac{b}{a} \leq \frac{d}{c}$ and there exists no $(e,f) \in \mathbb{N}^2, 0 < e \leq k, f \leq l$ such that $\frac{b}{a} \leq \frac{f}{e} \leq \frac{d}{c}$, then $|ad - bc| = 1$.*

Proof. W.l.o.g. we can assume $d = l$. Moreover, we may assume $c \geq a$, since otherwise $\frac{d}{c} \geq \frac{l}{k} \geq \frac{b}{a}$. Thus we can also assume w.l.o.g. $c = k$. The triangle with vertices $(0,0),(a,b),(c,d)$ has by assumption no integer points in its interior, therefore its area is $\frac{1}{2}$ by Pick's theorem. Hence the triangle with vertices $(a,b),(c,d),(a+c,b+d)$ has area $\frac{1}{2}$ (see Figure 2). Since $|ad - bc|$ is the area of the parallelepiped spanned by (a,b) and (c,d), the lemma is proven.

□

Theorem 2.18. *Let $(m_1, m_2) \in \mathbb{N}^2$ and*

$$I_{m_1,m_2} = \prod_{1 \leq a \leq m_1, 1 \leq b \leq m_2} (x^a, y^b).$$

Then the blowup of $\mathbb{A}^2_k$ along I_{m_1,m_2} is smooth.

Proof. Let $N = N(I_{m_1,m_2})$ be the Newton polyhedron of I_{m_1,m_2}.

By Lemma 2.16

$$\mathcal{N}(N) = \{(b,a) : 0 \leq b \leq m_2, 0 \leq a \leq m_1, \gcd(a,b) = 1\}.$$

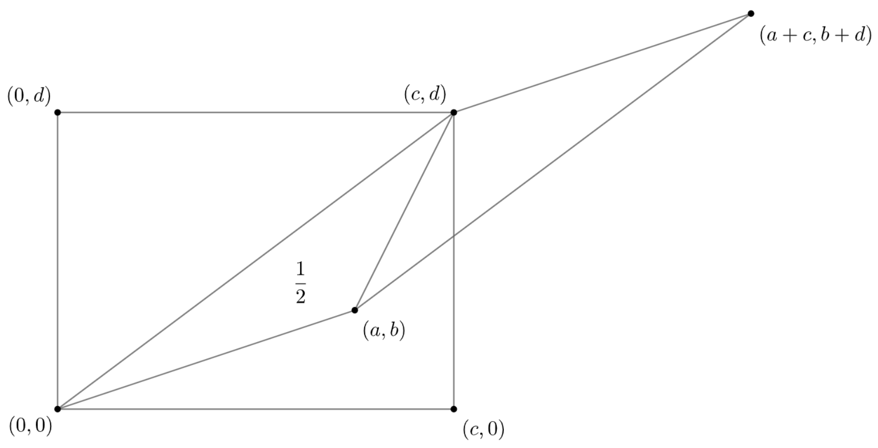

Figure 2: Picture of the parallelepiped spanned by (a, b) and (c, d)

Let α be a vertex of N and (b, a) and (d, c) the primitive normal vectors of the two edges meeting at α. Let α_1 and α_2 be the neighbours in $\operatorname{supp}(I)$ of α along these two edges. Hence $\alpha_1 = (-a, b)k_1 + \alpha$ and $\alpha_2 = (-c, d)k_2 + \alpha$ for $k_1, k_2 \in \mathbb{Z}$. By Lemma 2.17 $|ad - bc| = 1$, therefore, if $|k_1| = |k_2| = 1$, we are finished by Lemma 2.14.

Assume w.l.o.g that $k_1 > 1$. By assumption $(\frac{y^b}{x^a})^{k_1}$ is contained in $k[IT_\alpha]$. Since $a \leq k_1 a$ and $b \leq k_1 b$, the factor (x^a, y^b) appears in I, therefore either $(-a, b)$ or $(a, -b)$ are elements of IT_α. Because α is a vertex and thus IT_α is pointed, $(-b, a) \in IT_\alpha$. Hence $k_1 = 1$. The argument works completely analogously for k_2. □

2.3.2 The Case n>2

Unfortunately singular charts of weighted blowups can occur if $n = 3$. Let us consider the ideal

$$I_{1,2,3} = (x, y, z)(x, y^2, z)(x, y, z^2)(x, y^2, z^2)(x, y, z^3)(x, y^2, z^3)$$

and let $\alpha = (1, 2, 4)$. We consider the chart U_α of the blowup of $\mathbb{A}^3_k$ along $I_{1,2,3}$. As we have seen in the beginning of Section 2.3.

$$k[IT_\alpha] = k[x, y, z, \frac{x}{z}, \frac{y}{z}, \frac{x}{z}, \frac{y^2}{z}, \frac{x}{y}, \frac{z^2}{y}, \frac{x}{z^2}, \frac{y^2}{z^2}, \frac{x}{y}, \frac{z^3}{y}, \frac{y^2}{x}, \frac{z^3}{x}] = k[\frac{y}{z}, \frac{z^2}{y}, \frac{x}{z^2}, \frac{y^2}{x}, \frac{z^3}{x}]$$

Hence IT_α is generated by the five vectors

$$b_1 = (0, 1, -1), b_2 = (0, -1, 2), b_3 = (1, 0, -2), b_4 = (-1, 2, 0), b_5 = (-1, 0, 3).$$

It is straightforward to check that IT_α is pointed. Indeed, let $c_1, .., c_5 \in \mathbb{N}$ and assume $\sum_i c_i b_i = 0$. By looking at the entries we deduce $c_3 = c_4 + c_5$, $c_1 + 2c_4 = c_2$

and $2c_2 + 3c_5 = c_1 + 2c_3$. Inserting $c_4 + c_5$ for c_3 and $c_1 + 2c_4$ for c_2 into the third equation yields $2c_4 + c_5 = 0$ and therefore $c_4 = c_5 = 0$. It is then easy to see that also $c_1 = c_2 = c_3 = 0$. By *Theorem* 2.12 we conclude that α is a vertex of $N(I_{1,2,3})$. However, if U_α would be smooth, $\{b_1, b_2, b_3, b_4, b_5\}$ would contain a generating system consisting of three vectors and this is not possible since no b_i is contained in the $\mathbb{N}$-span of the other b_i's. Therefore U_α is singular.

Note that this example can easily be generalized to higher dimensions. Set $n_1 = 1, n_2 = 2, n_3 = 3$ and $n_i = 1$ for $i \geq 4$. Let $\alpha = (1, 2, 4, 0, ..., 0)$. Analogously to the case $n = 3$ we see that

$$k[IT_\alpha] = k[\frac{x_2}{x_3}, \frac{x_3^2}{x_2}, \frac{x_1}{x_3^2}, \frac{x_2^2}{x_1}, \frac{x_3^3}{x_1}, \frac{x_i}{x_1}, \frac{x_i}{x_2} : 4 \leq i \leq n].$$

It is again easy to see that α is a vertex of N and that there exists no system of n generators.

2.4 Difficulties in three variables

In this section we will showcase some new difficulties which arise in the 3-dimensional case and show that there exists a resolution of a weighted blowup of $\mathbb{A}_k^3$ with one blowup. As we have seen before, the saturation no longer provides a resolution of the singularities. Fix now a, b, c and consider the ideal $(x^a, y^b, z^c) \subseteq k[x, y, z]$. From now on we mean by normal vector of a face of some polyhedron a primitive normal vector with integer coefficients.

The first problem is that the Minkoswki sum is generally much more difficult to compute in higher dimensions. Since the goal is to find a monomial ideal I with Newton polyhedron $N = N(I)$ such that the Newton polyhedron $N + N(x^a, y^b, z^c)$ is simplicial, we are interested in polyhedra, whose primitive normal vectors of 3 intersecting faces meeting in a vertex form a basis of $\mathbb{Z}^3$. This is because 3 vectors v_1, v_2, v_3 have determinant ± 1 if and only if $v_1 \times v_2, v_1 \times v_3, v_2 \times v_3$ have determinant ± 1.

In dimension 2, if $\mathcal{N}(A), \mathcal{N}(B)$ are the set of normal vectors of convex polyhedra A and B with finitely many vertices, $\mathcal{N}(A + B) = \mathcal{N}(A) \cup \mathcal{N}(B)$. In dimension 3 or more, we have much less control on the set of normal vectors of the Minkowski sum of two polyhedra. This can already be seen if one computes the Minkowski sum

$$N(x^a, y^b, z^c) + N(x^d, y^e, z^f).$$

The normal vectors of $N(x^a, y^b, z^c)$ resp. $N(x^d, y^e, z^f)$ are $\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (bc, ac, ab)\}$ resp. $\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (ef, df, de)\}$. However $N(x^a, y^b, z^c) + N(x^d, y^e, z^f)$ has 7 normal vectors, but (bc, ac, ab) and (ef, df, de) do not necessarily need to appear.

The second problem is that the three dimensional analog of Pick's theorem is much more complicated. Let $\Delta \subseteq \mathbb{R}^n$ be a polytope with vertices in $\mathbb{Z}^n$ and consider the function

$$L(\Delta, k) = \#(\mathbb{Z}^n \cap k\Delta).$$

This function is called the Ehrhart polynomial of Δ and is, as the name suggests, a polynomial in k of degree n. Ehrhart showed that

$$L(\Delta, k) = a_n k^n + a_{n-1} k^{n-1} + ... + 1,$$

where $a_n = \mathrm{vol}(\Delta)$ and a_{n-1} is half of the sum of the volumes of the $n-1$-dimensional faces of Δ, where the volume of a k-dimensional face is measured with respect to the k-dimensional lattice plane spanned by the face, see [9]. In dimension 2 this yields Pick‘s theorem. One could hope that in dimension 3 the coefficient a_1 only depends on the volumes of the 1-dimensional faces of Δ, since this would allow us to proceed similarly as in dimension 2. However, this is not true, as one can see by considering the tetrahedra with vertices $(0,0,0),(1,0,0),(0,1,0),(1,1,k), k \in \mathbb{N}$. We obtain that $a_1 = 1 - \frac{1}{k}$, however the volumes of the one dimensional faces are always 1.

There does exist a formula for a_1 in the 3-dimensional case if Δ is a simplex, see [17]. Recall thatnthe Dedekind sum for relative prime natural numbers p and q is

$$s(p,q) = \sum_{i=1}^{q} \Big(\Big(\frac{i}{q}\Big)\Big)\Big(\Big(\frac{ip}{q}\Big)\Big),$$

where

$$((x)) = begincasesx - [x] - \frac{1}{2} \text{ if } x \notin \mathbb{Z}0 \text{ if } x \in \mathbb{Z}$$

.

Let the vertices of Δ be $(0,0,0),(a,0,0),(0,b,0),(0,0,c), a,b,c \in \mathbb{N}, \gcd(a,b,c) = 1$ and $A = \gcd(b,c), B = \gcd(a,c), C = \gcd(b,c), d = ABC$. Then

$$a_1 = \frac{1}{12}\Big(\frac{ab}{c} + \frac{ac}{b} + \frac{bc}{a} + \frac{d^2}{abc}\Big) + \frac{a+b+c+A+B+C}{4} -$$

$$\Big(As\Big(\frac{bc}{d}, \frac{aA}{d}\Big) + Bs\Big(\frac{ac}{d}, \frac{bB}{d}\Big) + Cs\Big(\frac{ab}{d}, \frac{cC}{d}\Big)\Big).$$

For the formula for a general simplex see [17] .

This gives another reason why one should expect the saturation or any other "simple" set of normal vectors not to work in dimension 3. Such a set would give us constraints on the very complicated term a_1, which would be very surprising if it were possible with a simple description. However, we can at least show the existence of a resolution of weighted blowups with one blowup in dimension 3.

2.5 Existence of resolution

Theorem 2.19. *Let $a, b, c \in \mathbb{N}$ and consider $(x^a, y^b, z^c) \subseteq k[x,y,z]$. Then there exists a monomial ideal I such that the blowup of $\mathbb{A}^3_k$ along $(x^a, y^b, z^c) \cdot I$ is smooth. Moreover, the ideal I can be explicitly constructed.*

The idea is to show that we can construct a suitable polyhedron N such that in each vertex of N only 3 edges meet, the determinant of the normal vectors is ± 1 and there exists another polyhedron N' such that $N' + N(x^a, y^b, z^c) = N$. We will then show that there exists a monomial ideal $I \subseteq k[x,y,z]$ such that $N(I) = N'$ and at each vertex α of N' the ideal tangent cone IT_α is simplicial. Our proof does not generalize to arbitrary monomial ideals of $k[x,y,z]$.

Lemma 2.20. *For a finite set $V \subseteq \mathbb{N}^3$ such that*

$$(1,0,0),(0,1,0),(0,0,1) \in V,$$

there exists a monomial ideal $I \subseteq k[x,y,z]$ such that the set of normal vectors of the Newton polyhedron $N(I)$ is V. Moreover I can be chosen such that in each vertex of $N(I)$ only 3 faces meet.

Proof. If $V = \{(1,0,0),(0,1,0),(0,0,1)\}$ we can choose $I = k[x,y,z]$.

We will first show there exists a polyhedron N with normal vectors V, whose vertices have rational coordinates and only 3 edges are meeting in each vertex. Suppose we have already such a polyhedron N' for $V = \{v_1, ..., v_{n-1}\}$ and we want to have a new face with normal vector v_n. We will introduce a new face to N' by sliding the hyperplane with normal vector v_n along v_n a bit past its first intersection point with N' and cutting off N' along this plane.

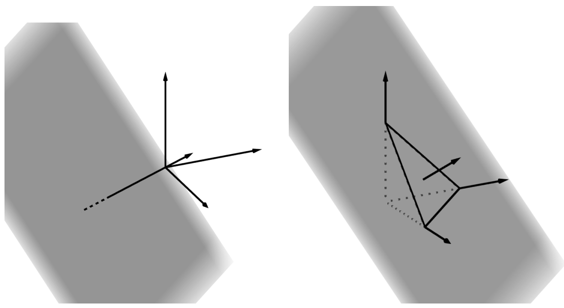

Figure 3: Introducing a new face

We formulate this process rigorously as follows. For $\lambda \in \mathbb{R}$ let

$$F_{v_n,\lambda} = \{x \in \mathbb{R}^3 : x \cdot v_n = x \cdot \lambda v_n\},$$

$$E_{v_n,\lambda} = \{x \in \mathbb{R}^3 : x \cdot v_n \geq x \cdot \lambda v_n\}$$

and $N_\lambda = E_{v_n,\lambda} \cap N'$. Note that for $\lambda \leq 0$ $N_\lambda = N'$ and for $\lambda >> 0$ $N_\lambda = E_{v_n,\lambda}$. Let μ be the largest λ such that $N_\lambda = N'$. Then $N' \cap F_{v_n,\mu}$ is either a vertex or an edge of N'. Let μ' be the smallest number bigger than μ such that $N' \cap F_{v_n,\mu'}$ contains a new vertex of N'. Choose any rational number q between μ and μ' and set $N = N_q$. It is easy to see that in both cases in all new vertices only 3 edges meet, since we assumed that in each vertex of N' only 3 edges meet.

Finally we multiply all elements of N with a sufficiently large natural number such that all vertices have integer coordinates and set $I = (x^\alpha : \alpha \in N \cap \mathbb{N}^3)$. This is indeed an ideal. If $\alpha \in N \cap \mathbb{N}^3$ we have to show

$$\alpha + (1,0,0), \alpha + (0,1,0), \alpha + (0,0,1) \in N \cap \mathbb{N}^3.$$

We can assume α to be a vertex of N and, after multiplying N with a large enough number, that $\alpha + (1,0,0)$ is in N if and only if

$$v_1 \cdot (\alpha + (1,0,0)) \geq v_1 \cdot \alpha, v_2 \cdot (\alpha + (1,0,0)) \geq v_2 \cdot \alpha, v_3 \cdot (\alpha + (1,0,0)) \geq v_3 \cdot \alpha,$$

where v_1, v_2, v_3 are the normal vectors of the faces meeting in α. This is always true since $v_1, v_2, v_3 \in \mathbb{N}^3$. □

We can now proof Theorem 2.19.

Proof. Let $N_1 = N(x^a, y^b, z^c)$ and note that the set of normal vectors of N_1 is

$$V = \{(1,0,0), (0,1,0), (0,0,1), (bc, ac, ab)\}.$$

We now choose a refinement V' of V as in the Appendix and construct a Newton polyhedron N with set of normal vectors V' as in above lemma. Note that in all vertices of N exactly three edges meet and the determinant of the normal vectors is ± 1. Moreover, $N_1 = N(x^a, y^b, z^c) + N(k[x,y,z])$. Let N_i be the Newton polyhedron in the i-th step in the construction of N, i.e. where the i-th facet is introduced. We will show now that if $N_i = N(x^a, y^b, z^c) + N_i'$ then there exists N_{i+1}' such that $N_{i+1} = N(x^a, y^b, z^c) + N_{i+1}'$, where we might have to rescale N_{i+1} with some natural number.

By the definition of the Minkowski sum, there exists a Newton polyhedron N_{i+1}' such that $N_{i+1} = N_{i+1}' + N(x^a, y^b, z^c)$ if and only if for each vertex p of N_{i+1} there exists a vertex q of N_{i+1}' such that $q = p - (a,0,0)$ or $q = p - (0,b,0)$ or $q = p - (0,0,c)$ and $q + (a,0,0), q + (0,b,0), q + (0,0,c) \in N_{i+1}$. Let

$$Q = \{q : q = p - (a,0,0) \text{ or } q = p - (0,b,0) \text{ or } q = p - (0,0,c) : p \text{ vertex of } N\}$$

and set

$$N_{i+1}' = \operatorname{Conv}(\{q + \alpha : \alpha \in \mathbb{N}^3, q \in Q\}).$$

Fix now a vertex p of N_{i+1} and let F_1, F_2, F_3 be the facets meeting in p with respective normal vectors v_1, v_2, v_3. We multiply N_{i+1} with a sufficiently large natural number such that the following holds. If q is the vertex of N_{i+1}' as mentioned above $q + (a,0,0), q + (0,b,0), q + (0,0,c) \in N_{i+1}$ if and only if

$$v_1 \cdot (q + (a,0,0)) \geq v_1 \cdot p, v_1 \cdot (q + (0,b,0)) \geq v_1 \cdot p, v_1 \cdot (q + (0,0,c)) \geq v_1 \cdot p,$$

$$v_2 \cdot (q + (a,0,0)) \geq v_2 \cdot p, v_2 \cdot (q + (0,b,0)) \geq v_2 \cdot p, v_2 \cdot (q + (0,0,c)) \geq v_2 \cdot p,$$

$$v_3 \cdot (q + (a,0,0)) \geq v_3 \cdot p, v_3 \cdot (q + (0,b,0)) \geq v_3 \cdot p, v_3 \cdot (q + (0,0,c)) \geq v_3 \cdot p.$$

We have to rescale N_{i+1} so the question whether these points lie in the interior of the polyhedron depends only on the three neighbouring faces.

Let F_1 be w.l.o.g. the new facet introduced in the $(i+1)$-th step and let p' be a vertex of N_i were F_1 first touches N_i. For all vertices of N_{i+1} which are

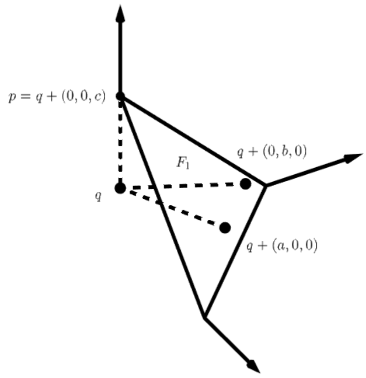

Figure 4: The three points that have to lie in the interior of N_{i+1}

not created by the slicing with F_1, we get that the above inequalities hold by assumption. If p is a vertex created by the slicing with F_1, we know that 2 of the normal vectors of the faces meeting in p' are v_2 and v_3. If the third normal vector in N_i is v we know by construction that $v_1 = \lambda v + \lambda_2 v_2 + \lambda_3 v_3, 0 \leq \lambda, \lambda_2, \lambda_3$. Let w.l.o.g. $q' = p' - (a,0,0)$ be the vertex of N_i' corresponding to p'. Setting then $q = p - (a,0,0)$ implies then that all inequalities are satisfied for v, v_2, v_3 and thus we have shown the existence of N_{i+1}'. Finally we note that all outward normal vectors of N_{i+1}' lie in $\mathbb{N}^3$. If $v = (v_1, v_2, v_3)$ would be normal vector such that w.l.o.g $v_1 < 0$ and p a vertex of the face of v, we would obtain that for all $k \in \mathbb{N}$, $p + kv \in N_{i+1}'$. However for sufficiently k the first coordinate of $p + kv$ is negative, hence a contradiction. Therefore there exists an ideal I such that $N(I) = N_{i+1}'$ as was shown in the proof of the previous lemma.

Finally, if α is a vertex of $N(I)$, IT_α is simplicial. If v_1, v_2, v_3 are the normal vectors of the three faces meeting at α with positive entries, the three edges meeting in α are spanned by the vectors $v_1 \times v_2$, $v_1 \times v_3$ and $v_2 \times v_3$. Therefore $\alpha + v_1 \times v_2, \alpha + v_1 \times v_3$ and $\alpha + v_2 \times v_3$ are elements of $\mathrm{supp}(I)$, since these are the nearest lattice points on the three edges meeting in α to α. Thus $v_1 \times v_2$, $v_1 \times v_3, v_2 \times v_3 \in IT_\alpha$. Since v_1, v_2, v_3 is a basis for $\mathbb{Z}^3$, the three vectors $v_1 \times v_2, v_1 \times v_3$ and $v_2 \times v_3$ span $\mathbb{Z}^3$ as well. $\square$

This proof is a generalization of the proof in 2 dimensions where we were in the lucky position that we had firstly an explicit description of a refinement of the set $\{(1,0),(0,1),(b,a)\}$ and secondly we could use that the Minkowski sum is very easy computable. Both these things are missing in higher dimensions. Moreover, this idea of showing the existence of such an ideal should carry without much work over to the general case.

3 Approach via stacks

Stacks have proven to be a very useful tool in modern mathematics. This text is meant as an introduction to them with the goal of using stacks to define weighted blowups. We start be motivating the necessity of stacks in moduli problems by looking at families of triangles and elliptic curves. From there on we will give first a purely category theoretic definition of stacks which we will then refine to Deligne-Mumford stacks. After revisiting the stack of elliptic curves we will go on and deduce some properties of stacks, which will hopefully show their possible applications. Finally we will focus on the definition of stacky projs and stacky blowups.

3.1 The moduli problem

The moduli problem is the problem of finding a space which parameterizes some objects up to some equivalence relations. For example one could be interested in finding such a space for triangles in $\mathbb{R}^2$ where two triangles are equivalent if they are similar. The moduli space of triangles up to similarity should somehow have for each equivalence class of triangles a point in it. Another example would the moduli space of elliptic curves up to isomorphism. We will first focus on the example of triangles to show the problems that occur if we want to find such a suitable moduli space. This introduction to stacks is based on [4].

3.1.1 The moduli space of triangles

Let $\Delta \subseteq \mathbb{R}^6$ be the set of all triangles in the plane. Since we want to paramterize triangles up to equivalence, we can determine any equivalence class of triangles by the lengths of its sides. Moreover, if we pick the representative with circumference 2, we see that for each class of similar triangles there is exactly one triple (a, b, c) in the set

$$\widetilde{T} = \{(a, b, c) \in \mathbb{R}^3 : b + c > a, a \geq b \geq c > 0, a + b + c = 2\}.$$

Thus this space is a possible candidate for the desire moduli space.

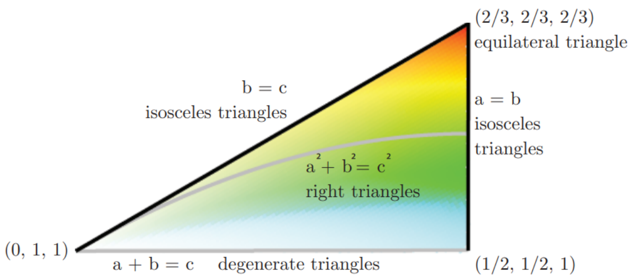

Figure 5: The moduli space $\widetilde{T}$. This image is taken from [4].

We can also construct a space $\mathcal{T} \subseteq \Delta$ together with a bijective continuous map $\mathcal{T} \to \widetilde{T}$, which sends each triangle in $\mathcal{T}$ to its corresponding triple. The space $\mathcal{T}$ will be called a universal family of $\widetilde{T}$.

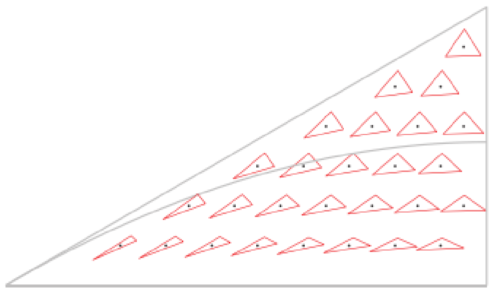

Figure 6: The universal family $\mathcal{T}$. This image is taken from [4].

A family of triangles over some topological space S is a degree 3 covering map $\mathcal{S} \to S$ together with a map $l: \mathcal{S} \to \mathbb{R}$ such that for all $(a, b, c) \in \mathcal{S}$ lying over some $s \in S$, $l(a) + l(b) > l(c), l(b) + l(c) > l(a), l(a) + l(c) > l(b), l(a), l(b), l(c) > 0$ and $l(a) + l(b) + l(c) = 2$. Thus a family of triangles over S is a space of triangles parameterized by S which is locally trivial. We say two families $\mathcal{S}$ and $\mathcal{S}'$ over S are isomorphic if there exists a homeomorphism $F: \mathcal{S} \to \mathcal{S}'$ such that

$$\begin{tikzcd}
\mathcal{S} \arrow[rr, "F"] \arrow[dr] & & \mathcal{S}' \arrow[dl] \\
& S &
\end{tikzcd}$$

commutes. If S can be covered by open sets U_i and over each U_i there is a family of triangles $\mathcal{S}_i$ for each U_i together with isomorphisms $\phi_{ij}: \mathcal{S}_i \to \mathcal{S}_j$ on $U_i \cap U_j$ which satisfy the cocycle condition, i.e. $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$ on $U_i \cap U_j \cap U_k$, the $\mathcal{S}_i$'s can be glued to a family of triangles $\mathcal{S}$ over S.

If $\mathcal{S} \to S$ is a family of triangles, we obtain a map $S \to \mathcal{T}$ by sending $s \in S$ to the triangle given by the fibre of s and therefore we also obtain a map $S \to T$. Similarly, if we start with a map $f: S \to \widetilde{T}$, we can pullback $\mathcal{T}$ along $f: S \to \widetilde{T}$ and obtain a family of triangles $f^*\mathcal{T}$ over S.

Definition. A coarse moduli space of triangles M is a space such that

1. The points of M are in bijection with equivalence classes of triangles.
2. If $\mathcal{S}$ is a family over an object S, the induced map $S \to M$ is continuous.
3. M parameterizes a family of triangles $\mathcal{M}$ whose induced map is id_M.

So far we have seen that $\widetilde{T}$ is indeed a coarse moduli problem. If M is a coarse moduli space and $S \to M$ is a continous map, we can pullback $\mathcal{M}$ like mentioned above.

Definition. A fine moduli space of triangles M is a coarse moduli space of triangles, such that each family $\mathcal{S}$ over S is isomorphic to the pullback $f^*\mathcal{M}$, where $f: S \to M$ is the map induced by the family $\mathcal{S}$.

The problem is that $\widetilde{T}$ is not a fine moduli space of triangles. To see this let $S = S^1$ be the circle and let $\mathcal{S}$ be the trivial family over S, where each fibre is the equilateral triangle, i.e. $\mathcal{S} = S \times \{a, b, c\}$ and $l(a) = l(b) = l(c) = \frac{2}{3}$. We can construct a second family over S by covering the circle with two copies U_1 and U_2 of the unit interval $[0, 1]$. We let $\mathcal{S}_1'$ and $\mathcal{S}_2'$ be the trivial families over U_1 and U_2 where again each fibre is the equilateral triangle. On one overlap we glue these families trivially, however on the other overlap we first twist the equilateral triangles over U_2 by $\frac{2\pi}{3}$ and than glue. The resulting family is a family over S^1 which has a twist in it. To see that they are non isomorphic, we fix a side of the equilateral triangle and follow it around the circle. In the first case it forms a cylinder and in the second one it forms a möbius strip. Thus those two families are not isomorphic.

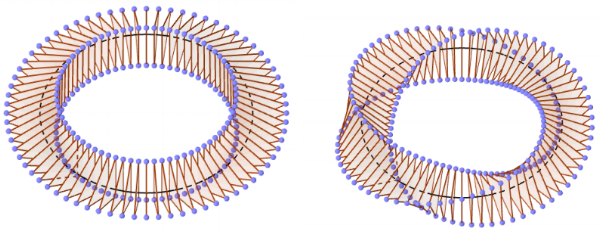

Figure 7: Two nonisomorphic families over S^1. This image is taken from [4].

The problem was that we had a nontrivial automorphism of an object, in this case the twist of the equilateral triangle. This is the main phenomenon which prevents us from finding a fine moduli space. Completely analogously we see that there is no fine moduli space of elliptic curves, since the automorphism group of every elliptic curve is non trivial, see [12], Corollary 4.7. Another problem of the space $\widetilde{T}$ is that it has a boundary and hence it is not entirely smooth, something which would be desirable, since this would allow us to do geometry more effectively on the moduli space. The goal of stacks is to remember those nontrivial automorphisms and provide some geometric structure on the moduli space.

3.1.2 The functor of points

We will now move on to the situation of schemes. The main idea is to pretend that we have a moduli space, formulate what properties it should have and then simulate the existence of such an object by describing how other objects see the moduli space.

The key observation, which even allows this point of view, is that we only have to care how other schemes "see" our moduli space and not what our moduli space actually is. Because of the Yoneda lemma, see Lemma D.3, we can try to understand the functor of points h_X of a scheme X instead of X itself. A morphism $f \in \mathrm{Hom}(T, X)$ is called a T-valued point of X. If we want some space M to exist, which does not exist, we can nonetheless pretend that M exists and define a functor, which should behave like the functor of points of M. If everything goes right we could hope that this functor allows us to do geometry on M by probing it with schemes, even though M does not exist in the first place.

3.2 The moduli space of elliptic curves

Fix some field k of char$k \neq 2, 3$. We will enrich the category Sch of schemes over k by an object $\widetilde{M_{ell}}$ which will act like the moduli space of elliptic curves. This has to be a new object, since we know there is no fine moduli space of elliptic curves.This section is based on [15]. We will treat the moduli stack of elliptic curves not very rigorously, since it will only be used as a motivation. For a more rigorous treatment we refer to appendix E.

3.2.1 Morphisms of families of elliptic curves

A morphism $S \to \widetilde{M_{ell}}$ should, according to our functor of points view, correspond to an S-point of $\widetilde{M_{ell}}$, which in turn is nothing more than a family of elliptic curves over S, i.e. a proper, smooth morphism $\pi: E \to S$ of relative dimension 1 such that each geometric fiber $E_s, s \in S$ is an elliptic curve together with a section $e: S \to E$, which we will omit from now on. In the appendix we will treat the moduli stack of elliptic curves more rigorously and there the section e will play a more important role.

Thus a morphism $S \to \widetilde{M_{ell}}$ is a family of elliptic curves $E \to S$. A morphism between two families of elliptic curves $E \to S$ and $E' \to S'$ with sections e and e' is a morphism $\alpha: E \to E'$ and a morphism $a: S \to S'$ such that

$$\begin{tikzcd} E \arrow[r, "\alpha"] \arrow[d] & E' \arrow[d] \\ S \arrow[r, "a"] & S' \end{tikzcd}$$

commutes and is cartesian, i.e. E is a pullback of E' along $a: S \to S'$. Moreover, we demand that $\alpha \circ e = e' \circ a$. We say that α lies over a.

We would also want that the following diagram commutes

$$\begin{tikzcd} S \arrow[rr, "a"] \arrow[dr, "E"'] & & S' \arrow[dl, "E'"] \\ & \widetilde{M_{ell}} & \end{tikzcd}$$

if and only if there exists a morphism between the families $E \to S$ and $E' \to S'$. We also have to describe how morphisms $S \to S'$ and $E' \colon S' \to \widetilde{M_{ell}}$ compose. Again by our functor of points point of view, we demand that

$$S \to S' \to \widetilde{M_{ell}}$$

should correspond to $E' \times_{S'} S$, which is a S-valued point of $\widetilde{M_{ell}}$.

Finally we also describe the morphism $\widetilde{M_{ell}} \to T$, defining them pointwise, i.e. on all families of elliptic curves. Again a point of $\widetilde{M_{ell}}$ is just a morphism $T \to \widetilde{M_{ell}}$. A morphism

$$F \colon \widetilde{M_{ell}} \to T$$

is hence nothing more than an assignment which associates to each family of elliptic curves $E \to S$ a morphism

$$F(E) \colon S \to T.$$

Moreover every commutative diagram

$$\begin{tikzcd} S \arrow[rr, "a"] \arrow[dr, "E"'] & & S' \arrow[dl, "E'"] \\ & \widetilde{M_{ell}} & \end{tikzcd}$$

should give a commutative diagram

$$\begin{tikzcd} S \arrow[rr, "a"] \arrow[dr, "F(E)"'] & & S' \arrow[dl, "f(E')"] \\ & T & \end{tikzcd}$$

We will later see that all these conditions can be easily captured in the language of fibred categories.

3.2.2 Gluing of families and morphisms

As in the example of triangles or elliptic curves, we can glue families of those objects, if they satisfy some cocycle conditions.

Assume that we have an étale or fpqc covering $\{e_i : S_i \to S\}$ of some scheme S and families of elliptic curves $E_i \to S_i$. We will write S_{ij} for $S_i \times_S S_j$ and S_{ijk} for $S_i \times_S S_j \times_S S_k$.

Assume that

$$\begin{tikzcd} S_{ij} \arrow[rr, "id"] \arrow[dr, "E_i \circ \pi_{S_j}"'] & & S' \arrow[dl, "E_J \circ \pi_{S_j}"] \\ & M_{ell} & \end{tikzcd}$$

Let α_{ij} lie over the identity maps on S_{ij} and $\pi_{ij} : S_{ijk} \to S_{ij}$ the projection. Similarly define α_{jk}, α_{ik}, π_{jk} and π_{ik}. We can glue the families over the S_i to a family over S if and only if they satisfy the cocycle condition

$$\pi_{ik}^* \alpha_{ik} = \pi_{jk}^* \alpha_{jk} \circ \pi_{ij}^* \alpha_{ij}.$$

If this is the case let $E \to S$ be the resulting family of elliptic curves. Let $E' \to S'$ be any family of elliptic curves and $f_i : E_i \to T$ be some morphisms over S_i. If the morphisms agree over S_{ij} for all i, j, we can glue this morphisms to a morphism $E \to E'$.

The gluing of morphisms and families will later be called the prestack resp. stack condition of $\widetilde{M_{ell}}$.

3.2.3 Fiber product of families of elliptic curves

We can also try to describe what the fibre product $S \times_{\widetilde{M_{ell}}} S'$ of two families $E \to S$, $E' \to S'$ over $\widetilde{M_{ell}}$ would be. This will be a scheme on which we can do geometry, hence this will be one of the key requirements to form a Deligne-Mumford stack.

We again describe $S \times_{\widetilde{M_{ell}}} S'$ from a functor of points point of view. A T-valued point of $S \times_{\widetilde{M_{ell}}} S'$ should consist of two maps

$$a : T \to S, \ a' : T \to S'$$

and an isomorphism

$$\alpha : E \times_S T \to E' \times_{S'} T.$$

The first two morphisms are motivated by the universal property of the fiber product. Since $S \times_{\widetilde{M_{ell}}} S'$ should correspond to the pairs (s, s') such that $E_s \cong E'_{s'}$, we require the existence of the isomorphism α by replacing the pair (s, s') by the T-valued points a and a' of S and S'. Since the T fiber of E is exactly $E \times_S T$ we arrive at the above situation.

Ideally we also would have a universal family $\mathcal{M}_{ell} \to \widetilde{M_{ell}}$. This would be a bijective map, which is problematic since $\widetilde{M_{ell}}$ does not actually exist. So we weaken the condition and demand instead the existence of a smooth, surjective morphism

$$S \to \widetilde{M_{ell}}$$

In this context smooth means that for every morphism $S' \to \widetilde{M_{ell}}$, the morphism

$$S' \times_{\widetilde{M_{ell}}} S \to S$$

is smooth in the category of schemes. An analogous approach is taken for the surjectivity.

3.3 Stacks

We are going to try to reformulate all above notions into a category theoretic language. For a more thorough treatment see [11], of which this section is an outline and where all the proofs are taken from. We will start with a short reminder on categories fibred in groupoids. Since most of the times the categories which we consider are small, i.e. the class of objects is a set, we will write $X \in \mathfrak{X}$ if X is an object of the category $\mathfrak{X}$. Recall that a groupoid is a category where all morphisms are isomorphisms.

3.3.1 Categories fibred in groupoids

Definition.

1. A category $\mathfrak{X}$ lying over $\mathfrak{S}$ is a functor $p\colon \mathfrak{X} \to \mathfrak{S}$ called the projection.
2. An object $X \in \mathfrak{X}$ lies over $S \in \mathfrak{S}$ if $p(X) = S$ and a morphism $\phi\colon X \to Y$ lies over $f\colon S \to S'$ if $p(\phi) = f$.
3. For $S \in \mathfrak{X}$ the fiber $\mathfrak{X}_S$ over S is the category consisting of objects lying over S and morphisms lying over $\mathrm{id_S}$.

Definition. Let $\mathfrak{X}$ be a category over $\mathfrak{S}$. Let $S, S' \in \mathfrak{S}$, $f\colon S' \to S$ a morphism and $X \in \mathfrak{X}$.

A object X' together with a morphism ϕ lying over f is called a pullback of X along $f\colon S' \to S$ if for any diagram

$$\begin{tikzcd}
Y \arrow[r, dotted, "\varphi"] \arrow[rr, bend left, "\psi"] \arrow[d, "p"] & X' \arrow[r, "\phi"] \arrow[d, "p"] & X \arrow[d, "p"] \\
T \arrow[r, "g"] & S' \arrow[r, "f"] & S
\end{tikzcd}$$

where ψ lies over $f \circ g$, there exists a unique morphism $\varphi\colon Y \to X'$ lying over g. X' is often denoted with f^*X and the morphism ϕ is called cartesian.

Definition. 1. $\mathfrak{X}$ is called a fibred category over $\mathfrak{S}$ if we have a notion of pullback along maps in $\mathfrak{S}$.

2. $\mathfrak{X}$ is called a category fibred in groupoids (CFG) if for all $S \in G$, $\mathfrak{X}_S$ is a groupoid.

3. A morphism between CFGs $\mathfrak{X}$ and $\mathfrak{Y}$ over $\mathfrak{S}$ is a functor $F: \mathfrak{X} \to \mathfrak{Y}$ such that $p_{\mathfrak{X}} = p_{\mathfrak{Y}} \circ F$ and cartesian morphisms are sent to cartesian morphisms, i.e. F respects pullbacks.

3.3.2 Examples

The previous example $\widetilde{M_{ell}}$ can be described as a CFG over Sch. We define the CFG M_{ell}.

$$\mathrm{Ob}(M_{ell}) = \{E \to S, \text{ family of elliptic curves}\}$$

$$\mathrm{Hom}(E' \to S', E \to S) =$$

$$= \{(a: S' \to S, \alpha: E' \to E \text{ as in the definition of morphism}$$

$$\text{between families of elliptic curves}\}$$

$$p(E \to S) = S, p((\alpha, a)) = a$$

If G is an affine group acting on some scheme X, we will later define the quotient stack $[X/G]$ over Sch, which acts a replacement of the quotient space X/G.

Definition. Let $\mathfrak{S}$ be a category and $S \in \mathfrak{S}$ an object. The category $\underline{S}$ has as objects morphisms $T \to S$ and morphisms between $T \to S$ and $T' \to S$ are morphisms $T \to T'$ such that

$$\begin{tikzcd} T \arrow[rr] \arrow[dr] & & T' \arrow[dl] \\ & S & \end{tikzcd}$$

commutes. The functor which sends an object $T \to S$ to T and a morphism of objects over S $T' \to T$ to $T' \to T$ is a projection of $\underline{S}$ to $\mathfrak{S}$.

Lemma 3.1. *For any category $\mathfrak{S}$ and every object $S \in \mathfrak{S}$, $\underline{S}$ is a CFG.*

Proof. The pullback of a morphism $T \to S$ along $T' \to T$ is the morphism $T' \to T \to S$. Moreover the only morphism lying over the identity is the identity. □

The Yoneda-lemma allows us to work with $\underline{S}$ instead of S for any scheme S. Going back to our example $\widetilde{M_{ell}}$, a morphism from $\widetilde{M_{ell}}$ to T corresponds to a functor $F: M_{ell} \to \underline{T}$.

3.3.3 Stack condition (Gluing of families and morphisms)

From now on we have to fix a Grothendieck topology of our fibred category, see appendix C. We are first going to deal with the gluing of morphisms.

3.3.4 Prestacks

Let $\mathfrak{X} \to \mathfrak{S}$ be a category fibred in groupoids over $\mathfrak{S}$. Let $S \in \mathfrak{S}$ and $X, Y \in \mathfrak{X}_S$ and make a choice of pullbacks.

Define

$$\underline{Hom}_{\mathfrak{X}}(X,Y) \colon \underline{S} \to Sets$$
$$\underline{Hom}_{\mathfrak{X}}(X,Y)(f \colon T \to S) \mapsto \mathrm{Hom}_{\mathfrak{X}_T}(f^*X, f^*Y)$$
$$\underline{Hom}_{\mathfrak{X}}(X,Y)(g \colon f \to f') \mapsto$$
$$(\mathrm{Hom}_{\mathfrak{X}_T}(f'^*X, f'^*Y) \to \mathrm{Hom}_{\mathfrak{X}_T}(f^*X, f^*Y) \cong \mathrm{Hom}_{\mathfrak{X}_T}(g^*f'^*X, g^*f'^*Y))$$

Lemma 3.2. *$\underline{Hom}_{\mathfrak{X}}(X,Y)$ is a presheaf on $\underline{S}$.*

Proof. By Lemma D.6 we can assume that $\mathfrak{X}$ is split, hence $\underline{Hom}_{\mathfrak{X}}(X,Y)$ is a presheaf. □

Definition. A fibred category $\mathfrak{X}$ over $\mathfrak{S}$ is called a prestack if all such $\underline{Hom}_{\mathfrak{X}}(X,Y)$ are in fact sheaves.

In the case of the fibred category of elliptic curves, this is just the statement that the local morphisms of families glue together to a global morphism of families if the agree on overlaps.

The sheaf $\underline{Isom}_{\mathfrak{X}}(X,Y)$ can be defined in exactly the same way, since isomorphisms glue to isomorphisms.

3.3.5 Descent data

All that is left to do is to describe how families glue. A stack should allow us to glue families of elliptic curves, if they agree on the overlap and satisfy the cocycle condition. To formulate this in a general context we have to introduce the notion of descent data.

For $T_i \to T$, $T_j \to T$ we will write T_{ij} for $T_i \times_T T_j$, π_n for the projection to the n-th component and π_{nm} for the projection to the product of the n-th and m-th component.

Let $(T_i \to T)_{i \in I}$ be a family of morphisms in $\mathfrak{S}$.

Definition. A descent datum of $\mathfrak{X}$ relative to $(T_i \to T)_{i\in I}$ are objects X_i over T_i and isomorphisms

$$\phi_{ij}: \pi_1^* X_i \to \pi_2^* X_j$$

which satisfy the cocycle condition

$$\begin{tikzcd}
\pi_{12}^*\pi_1^* X_i \arrow[r, "\pi_{12}^*\phi_{ij}"] \arrow[d, "\cong"] & \pi_{12}^*\pi_2^* X_j \arrow[r, "\cong"] & \pi_{23}^*\pi_2^* X_j \arrow[d, "\pi_{23}^*\phi_{jk}"] \\
\pi_{13}^*\pi_1^* X_i \arrow[r, "\pi_{13}^*\phi_{ik}"] & \pi_{13}^*\pi_3^* X_k \arrow[r, "\cong"] & \pi_{23}^*\pi_3^* X_k
\end{tikzcd}$$

where $\cong$ are the isomorphisms given by the universal property of the fibre product.

A morphism of descent data relative to $(T_i \to T)_{i\in I}$ from (X_i, ϕ_{ij}) to (X_i', ϕ_{ij}') are morphisms

$$\phi_i: X_i \to X_i'$$

such that

$$\begin{tikzcd}
\pi_1^* X_i \arrow[r, "\phi_{ij}"] \arrow[d, "\pi_1^*\phi_i"] & \pi_2^* X_j \arrow[d, "\pi_2^*\phi_j"] \\
\pi_1^* X_i' \arrow[r, "\phi_{ij}'"] & \pi_2^* X_j'
\end{tikzcd}$$

commutes. The category of descent data relative to $(T_i \to T)_{i\in I}$ is called $DD((T_i \to T)_{i\in I})$.

There is a notion of pulling back a descent data, which we need to speak of restricting families.

Definition. Let $(T_i \to T)_{i\in I}$ and $(T_j' \to T')_{j\in J}$ be two families of morphisms in $\mathfrak{S}$. Let $\alpha: J \to I$, $f: T' \to T$ and $f_j: T_j' \to T_{\alpha(J)}$.

We can define the pullback functor

$$f^*: DD((T_i \to T)_{i\in I}) \to DD((T_j' \to T')_J)$$

by sending

$$(X_i, \phi_{ij}) \mapsto (f_j^* X_{\alpha(j)}, (f_{jj'})^* \phi_{\alpha(j),\alpha(j')})$$

and sending morphisms

$$(\phi_i) \mapsto (f_j^* \phi_{\alpha(j)}).$$

This is indeed functorial and well defined. This is straightforward to check.

Definition. For $X \in \mathfrak{X}_T$ there is the trivial descent datum (X, id_X). For X lying over T and $(T_i \to T)_{i\in I}$ we can pullback X to the T_i to get a descent datum for $(T_i \to T)_{i\in I}$, the canonical descent datum. A descent datum is called effective if it is isomorphic to a canonical descent datum.

Definition. A prestack is called a stack if for every cover $(T_i \to T)_{i\in I}$ all descent data relative to $(T_i \to T)_{i\in I}$ are effective.

Translating this back to families of elliptic curves, this is the requirement that families that satisfy the cocycle condition can be glued together.

Finally, a morphism between two stacks is just a morphism of these stacks considered as CFGs.

3.3.6 Fibre product

Since we are working over categories, we will not work with the usual notion of fibre products, but with the notion of 2-fibre products, which is just the normal fiber product where we replaced the part of the unique morphism in the universal property by unique up to natural isomorphism. For a more precise definition see Appendix D.2

Definition. Let $F: \mathfrak{X} \to \mathfrak{Z}$ and $G: \mathfrak{Y} \to \mathfrak{Z}$ be functors between fibred categories over $\mathfrak{S}$. We define a fibred category $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Y}$ over $\mathfrak{S}$.

The objects of $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Y}$ are quadruples (S, X, Y, ϕ), where $S \in \mathfrak{S}$, $X \in \mathfrak{X}_S$, $Y \in \mathfrak{Y}_S$ and $\phi: F(X) \to G(Y)$ an isomorphism.

A morphism between

$$(S_1, X_1, Y_1, \phi_1) \to (S_2, X_2, Y_2, \phi_2)$$

are morphisms $a: X_1 \to X_2$, $b: Y_1 \to Y_2$ lying over the same morphism $S_1 \to S_2$ such that

$$\begin{tikzcd}
F(X_1) \arrow[r, "\phi_1"] \arrow[d, "F(a)"] & G(Y_1) \arrow[d, "G(b)"] \\
F(X_2) \arrow[r, "\phi_2"] & G(Y_2)
\end{tikzcd}$$

commutes. The projections $\pi_{\mathfrak{X}}$ and $\pi_{\mathfrak{Y}}$ are given by the forgetful functors.

Lemma 3.3. *The CFG $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Y}$ satisfies the universal property of the 2-fibre product in the categories of fibred categories. Moreover if $\mathfrak{X}, \mathfrak{Y}$ and $\mathfrak{Z}$ are CFGs, $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ is a CFG.*

Proof. See appendix Lemma D.10. □

Note that for functors $\underline{S} \to M_{ell}$, $\underline{S'} \to M_{ell}$ over Sch, this gives exactly the definition of the fiber product over elliptic curves.

By the 2-Yoneda lemma, see Lemma D.4, a functor $\underline{S} \to M_{ell}$ is nothing more than an element of the fibre $\mathfrak{X}_S$, i.e. a family of elliptic curves over S. Note that we have not dealt with the problem that $S \times_{M_{ell}} S'$ should be a scheme. This will be dealt with in the definition of Deligne-Mumford stacks.

Lemma 3.4. *The category of stacks over a category $\mathfrak{S}$ is closed under 2-fibre products.*

Proof. Let $F: \mathfrak{X} \to \mathfrak{Z}$ and $G: \mathfrak{Y} \to \mathfrak{Z}$ be functors between stacks. It is straightforward to see from the definition that

$$\underline{Hom}_{\mathfrak{X}}((S,X,Y,\phi),(S,X',Y',\phi)) = \underline{Hom}_{\mathfrak{X}}(X,X') \times_{\underline{Hom}_{\mathfrak{Z}}(F(X),F(X'))} \underline{Hom}_{\mathfrak{Y}}(Y,Y'),$$

where the second fibre product is in taken in the categories of presheaves. It is well known that the fibre product of sheaves in the category of presheaves is a sheaf, hence a 2-fibre product of prestacks is a prestack.

Let $(T_i \to T)_{i \in II}$ be a covering. Then a descent datum of $(T_i \to T)_{i \in I}$ in the 2-fibre category is of the form $(T_i, X_i, Y_i, \phi_i), (\psi_{ij}, \chi_{ij})$. The X_i's glue to an object X with gluing maps ψ_{ij} and the Y_i's glue to an object Y together with gluing maps χ_{ij}. Moreover, the ϕ_i's glue together. Since the fibre product is a prestack, the isomorphisms ϕ_i glue to an isomorphism $\phi: F(X) \to F(Y)$. □

3.4 Deligne-Mumford stacks

The final requirements we had on our stack is the existence of a smooth surjective cover and the representability of the fibre product. From now on we will work over the étale site ET/Δ of Sch, where $\Delta = \mathrm{Spec}(k)$ and by stack we mean a stack over Sch. The category of schemes over some fixed scheme S will be denoted by Sch/S.

Definition. A morphism $\mathfrak{X} \to \mathfrak{Y}$ of stacks is representable if for every scheme S and $\underline{S} \to \mathfrak{Y}$ the fibre product $\mathfrak{X} \times_{\mathfrak{Y}} \underline{S}$ is equivalent to $\underline{T}$ for some scheme T.

Let $\mathfrak{X} \to \mathfrak{Y}$ be representable. If P is a property of morphisms in Sch stable under base change and locally verifiable in the étale topology, a morphism between stacks $\mathfrak{X} \to \mathfrak{Y}$ has the property P if for every scheme S the morphism corresponding by the 2-Yoneda lemma to

$$\mathfrak{X} \times_{\mathfrak{Y}} \underline{S} \to \underline{S}$$

has P. Examples would include smooth, surjective, étale,...

Note that for M_{ell} this is precisely what we wanted.

Lemma 3.5 ([11])**.** *The composition, base change and product of representable morphisms is representable. If a morphism and its target are representable, its domain is representable. If $F \circ G$ and F are representable, G is representable.*

Proof. The first assertion follows immediately from the definitions and the identity $\mathfrak{X} \times_{\mathfrak{Y}} \underline{S} \cong \mathfrak{X} \times_{\mathfrak{Z}} (\mathfrak{Z} \times_{\mathfrak{Y}} \underline{S})$. The assertion on the domain follows from taking the identity $\underline{T} \to \underline{T}$.

For the last claim, consider the 2-cartesian diagram

$$\begin{tikzcd} \mathfrak{X} \times_{\mathfrak{Y}} \underline{S} \arrow[r] \arrow[d] & \underline{S} \arrow[d] & \\ \mathfrak{X} \times_{\mathfrak{Z}} \underline{S} \arrow[r] \arrow[d] & \mathfrak{Y} \times_{\mathfrak{Z}} \underline{S} \arrow[r] \arrow[d] & \underline{S} \arrow[d] \\ \mathfrak{X} \arrow[r] & \mathfrak{Y} \arrow[r] & \mathfrak{Z} \end{tikzcd}$$

All objects in the second row are representable, hence also $\mathfrak{X} \times_{\mathfrak{Y}} \underline{S}$. □

Lemma 3.6 ([11]). *The composition, base change and product of morphisms having P have P again. To check whether $\mathfrak{X} \to \mathfrak{Y}$ has P it suffices to check for P for base change along a étale and surjective morphism $\underline{T} \to \mathfrak{Y}$. If P is locally smooth at the target, étale can be relaxed to smooth.*

Proof. The first part is precisely above lemma. If $\underline{T'} \to \mathfrak{Y}$ is another test scheme, consider the 2 cartesian diagram

$$\begin{tikzcd} \underline{T'} \times_{\mathfrak{Y}} \mathfrak{X} \arrow[d] & \underline{T'} \times_{\mathfrak{Y}} \underline{T} \times_{\mathfrak{Y}} \mathfrak{X} \arrow[l] \arrow[r] \arrow[d] & \underline{T} \times_{\mathfrak{Y}} \mathfrak{X} \arrow[d] \\ \underline{T'} & \underline{T'} \times_{\mathfrak{Y}} \underline{T} \arrow[l] \arrow[r] & \underline{T} \end{tikzcd}$$

The right vertical arrow has P by assumption, the middle vertical arrow has P by base change and the left vertical arrow has P by locality on the target. □

If $\mathfrak{X}$ is a stack, let $\Delta_{\mathfrak{X}}$ be the diagonal $\mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{X}} \mathfrak{X}$.

Lemma 3.7 ([11]). *T.f.a.e.*

1. *$\Delta_{\mathfrak{X}}$ is representable*
2. *For all schemes S, T and $\underline{S} \to \mathfrak{X}$, $\underline{T} \to \mathfrak{X}$, $\underline{S} \times_{\mathfrak{X}} \underline{T}$ is representable*
3. *For all schemes S, $\underline{S} \to \mathfrak{X}$ is representable.*
4. *For all schemes S and $F: \underline{S} \Rightarrow \mathfrak{X}$, $G: \underline{S} \to \mathfrak{X}$, $\underline{S} \times_{\mathfrak{X}} \underline{S}$ is representable.*
5. *For any scheme S and X, Y objects over S, the sheaf $\underline{Isom}_{\mathfrak{X}}(X, Y)$ is representable by a scheme over S.*

Proof. The directions (2) ⇒ (3) ⇒ (4) are trivial. Since $\underline{S} \times_{\mathfrak{X}} \underline{T} \cong \underline{S \times T} \times_{\mathfrak{X} \times \mathfrak{X}} \mathfrak{X}$ by Lemma D.9, (1) ⇒ (2). From Lemma D.8 (5) ⇒ (1) follows. Finally (4) ⇒ (5)

follows also from Lemma D.8 by identifying X and Y as functors $\underline{S} \to \mathfrak{X}$ and using

$$\mathfrak{X} \times_{\mathfrak{X}\times\mathfrak{X}} \underline{S} \cong ((\underline{S \times S} \times_{\mathfrak{X}\times\mathfrak{X}} \mathfrak{X}) \times_{\underline{S\times S}} \underline{S} \cong (\underline{S} \times_{\mathfrak{X}} \underline{S}) \times_{\underline{S\times S}} \underline{S}.$$

□

Hence we reduce the problem of giving $S \times_{M_{ell}} S'$ the structure of a scheme to the representability of $\Delta_{M_{ell}}$.

Definition. A stack $\mathfrak{X}$ is called a Deligne-Mumford stack if

1. $\Delta_{\mathfrak{X}}$ is representable
2. There exists a scheme S and a surjective and étale morphism $\underline{S} \to \mathfrak{X}$ called an atlas or presentation.

Definition. A Deligne-Mumford stack $\mathfrak{X}$ is quasi-separated if $\Delta_{\mathfrak{X}}$ is separated and quasi-compact.

Sometimes it is better to require only that $\Delta_{\mathfrak{X}}$ is representable only over algebraic spaces and that the atlas is smooth instead of étale. However for quasi-separated stacks this is equivalent, which will always be the case in this treatment of stacks.

3.5 The stack of elliptic curves

We will now give the idea on how to show that the stack of elliptic curves is indeed a Deligne-Mumford stack. This is a brief summary of the introduction of [15]. We will only give a rough idea, for an exact proof we refer again to the appendix.

First we are going to address the fibre product. By Lemma 3.7 it suffices to show that for two families of elliptic curves paramatrized by the same scheme $\underline{Isom}_{M_{ell}}(E, E')$ is representable.

We are going to achieve this by first proving the local case and then using that we can glue representable functors. If $\pi: E \to S$ is a family of elliptic curves, Recall that for each $s \in S$ there exists an open affine neighbourhood U of s such that $\pi^{-1}(U) \to U$ is of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, a_i \in U$$

since π is smooth. An complete proof will be presented in appendix E.

Lemma 3.8 ([15])**.** *If E and E' are families of elliptic curves over S, let U be a trivializing set for E and E' as above. Then $Isom_{M_{ell}}(\pi^{-1}(U), \pi^{-1}(U))$ is representable.*

Proof. Recall that two elliptic curves over some field k in Weierstrass form

$$y^2 = x^3 + ax + b, y^2 = x^3 + a'x + b'$$

are isomorphic if and only if there exists a nonzero $u \in K$ with $a' = u^4 a$, $b' = u^6 b$. Hence two elliptic curves in general form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, y^2 = x^3 + ax + b$$

are isomorphic if and only if there exists u, r, s, t such that the coefficients coincide under the isomorphism

$$(x, y) \mapsto (u^2x + r, u^3y + u^2sx + t).$$

Thus u, s, r, t have to satisfy

$$a_1 = 2u^{-1}s, a_2 = u^{-2}(3r-s^2), a_3 = 2u^{-3}t, a_4 = u^{-4}(a-2st+3r^2), a_6 = u^{-6}(b+r^3+Ar-t^2).$$

Let $U = \mathrm{Spec}(A)$. By the above observation we can conclude that $\underline{Isom}_{M_{ell}}(\pi^{-1}(U), \pi^{-1}(U))$ can be represented by

$$\mathrm{Spec}(A[u, u^{-1}, r, t, s]/I),$$

where I is the ideal given by the requirements on u, r, s, t which are obtained just as above. □

By going back to our original definition of the fibre product as a functor which sends a scheme T to a triple of maps (a, a', α), we see that if we have an open cover U_i of T with functions (a_i, a_i', α_i), we can glue these and hence by [15], Lemma 26.15.4 the functor is representable.

This way of proving the representability condition can be done in more general situations.

Lemma 3.9. *Let $\mathfrak{X}$ be a stack, X, Y objects over $S \in Sch$. If there exists an étale cover $f: S' \to S$ such that $\underline{Isom}_{\mathfrak{X}}(f^*X, f^*Y)$ is represented by a scheme T', $Isom_{\mathfrak{X}}(X, Y)$ is represented by a scheme T, quasi-affine over S. Hence, if there exists such a cover for all S, the diagonal is quasi-affine.*

Proof. See [11], Proposition 5.16. □

Set

$$W = \mathrm{Spec}(k[a_1, ..., a_6, \frac{1}{\Delta}])$$

and let E_W be given by the equation

$$y^2z + a_1xyz + a_3yz^2 = x^3 + a_2x^2z + a_4xz^2 + a_6z^3$$

in $\mathbb{P}^2_W$. The polynomial Δ is the condition on the coefficients such that corresponding curve is smooth. Explicitly written out it looks like

$$\Delta = a_6a_1^6 + a_4a_3a_1^5 + ((-a_3^2 - 12a_6)a_2 + a_4^2)a_1^4 +$$
$$+(8a_4a_3a_2 + (a_3^3 + 36a_6a_3))a_1^3 +$$
$$+((-8a_3^2 - 48a_6)a_2^2 + 8a_4^2a_2 + (-30a_4a_3^2 + 72a_6a_4))a_1^2 +$$
$$+(16a_4a_3a_2^2 + (36a_3^3 + 144a_6a_3)a_2 - 96a_4^2a_3)a_1 +$$
$$+(-16a_3^2 - 64a_6)a_2^3 + 16a_4^2a_2^2 + (72a_4a_3^2 + 288a_6a_4)a_2 +$$
$$-27a_3^4 - 216a_6a_3^2 - 64a_4^3 - 432a_6^2$$

This would be a good candidate for an atlas, since every elliptic curve should be of form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

Since the statement that $W \times_{M_{ell}} S \to S$ is surjective and smooth is a local statement, we can use the explicit construction of the fibre product used in the proof of Lemma 3.8 and see that it is smooth and surjective.

3.6 Quotient stacks

Often stacks appear when a group G acts on some object X and we want to consider the quotient space X/G, which is not representable by a scheme or an algebraic space.

We will now motivate the definition of the quotient stack $[X/G]$. Let G be a quasi-affine group scheme and X a scheme where it acts on. Again we are using the functor of points approach. Lets first consider the fibre of a single point, i.e. what a point $\{*\} \to [X/G]$ should correspond to. This should be the points of X where 2 points are isomorphic if there are in the same G-orbit.

A point of X is nothing more than a G-equivariant map $G \to X$. In fact it is the same as the a map $T \to X$ where T is a G-torsor. Since two points should be isomorphic if they lie in the same G-orbit, a morphism between two maps $T \to X$, $T' \to X$ should be a G-equivariant map $T \to T'$, which is just multiplication by an element $g \in G$.

Since an S valued point should correspond to an S family of $\{*\}$ points of $[X/G]$, it should correspond to a family of G-torsors parameterized by S which is locally trivial. This is nothing more than a principal fibre bundle.

This leads us naturally to the definition of the quotient stack $[X/G]$.

Definition. The objects of $[X/G]$ are principal fibre bundles $E \to S$ together with an equivariant map $E \to X$.

A morphism $(E \to S) \to (E' \to S')$ is a diagram

$$\begin{array}{ccc} E & \longrightarrow & E' \\ \downarrow & & \downarrow \\ S & \longrightarrow & S' \end{array}$$

which commutes and is cartesian and such that

$$\begin{tikzcd} E \arrow[rr] \arrow[dr] & & E' \arrow[dl] \\ & X & \end{tikzcd}$$

commutes.

The functor $[X/G] \to Sch$ sends a principal fibre bundle $E \to S$ to S and a morphism $(E \to S) \to (E' \to S')$ to $S \to S'$.

Lemma 3.10. *The category $[X/G]$ is in fact a stack w.r.t the étale site.*

To prove that $[X/G]$ is in fact a Deligne-Mumford stack we need the following lemma.

Lemma 3.11 ([11])**.** *Let $\mathfrak{X}$ be a stack over a noetherian scheme $\underline{\Delta}$. If*

1. *The diagonal is representable and unramified*
2. *There exists a smooth, surjective cover $\underline{U} \to \mathfrak{X}$*
3. *U is of finite type over k*

there exists a scheme T of finite type and a étale, surjective morphism $\underline{T} \to \mathfrak{X}$ making $\underline{T}$ an atlas.

Proof. We will only sketch proof in the case where all residue fields of Δ are perfect.

Choose a closed point $u = \operatorname{Spec}(k) \in U$ and consider $\underline{V} = \underline{u} \times_{\mathfrak{X}} \underline{U}$. Let $z \in V$ be the point induced by $(\mathrm{id}_u, u \to U)$. Note that $V \to u \times_\Delta U$ is a base change of the diagonal along

$$\underline{u} \times_{\underline{\Delta}} \underline{U} \to \mathfrak{X} \times \underline{\Delta}\mathfrak{X}.$$

Moreover $u \to U \to \Delta$ is unramified, since U is of finite type over Δ and all residue fields of Δ are perfect. Thus $u \times_\Delta U \to U$ is perfect and hence by combining those two observations we obtain that $V \to U$ is unramified. From [1], 18.4.8 we obtain local neighbourhoods V' and U', such that $V \to U$ is locally given by inclusion $V' \hookrightarrow U'$. The point $z \in V'$ is cut out by a regular sequence of sections, which, after lifting them to U', define a closed subscheme Z_u of U.

The induced morphism $\underline{Z_u} \to \mathfrak{X}$ is étale. Indeed, we have to verify this at the base change along U. Since Z_u is a subscheme defined by a regular sequence, the immersion $Z_u \to U$ is regular and hence after a flat base change

$$\underline{Z_u} \times_{\mathfrak{X}} \underline{U} \to \underline{U'} \times_{\mathfrak{X}} \underline{U}$$

and using that $\underline{U'} \times_{\mathfrak{X}} \underline{U} \to \underline{U'}$ is smooth, we see that

$$\underline{Z_u} \times_{\mathfrak{X}} \underline{U} \to \underline{U}$$

is smooth in an neighbourhood of z. Thus all that remains is to show that the relative dimension of $\underline{Z_u} \times_{\mathfrak{X}} \underline{U} \to \underline{U}$ is 0. To see this, note that

$$\underline{z} \times_{\underline{U}} \underline{Z_u} \times_{\mathfrak{X}} \underline{U} \cong \underline{V'} \times_{\underline{U'}} \underline{Z_u} \cong \underline{z}.$$

Therefore the fibre over z is isomorphic to a point in an étale neighbourhood, hence the relative dimension is 0.

Thus we have constructed for each $u \in U$ a étale map of finite type over Δ $\underline{Z} \to \mathfrak{X}$. Taking the disjoint union of all Z_u we obtain a surjective and étale atlas, since the pullback along $\underline{U} \to \mathfrak{X}$ is open and contains every closed point. Since U is noetherian it suffices to choose finitely many Z_u's. $\square$

Lemma 3.12 ([11]). *Let X be a quasi-separated scheme of finite type over k. If G is étale or k is perfect and the stabilisers of G are finite and geometrically reduced, i.e. the local rings are reduced, $[X/G]$ is a Deligne-Mumford stack and the diagonal is quasi-affine.*

Proof. This proof is only a sketch. For the full proof see [11], Proposition 5.16, Example 5.17.

By similar arguments as in the case of elliptic curves, we can focus on the local case, hence we have to show that $\underline{Isom}_{[X/G]}(S \times G, S \times G)$, $S \times G$ quasi-affine over S, is represented by a scheme.

Therefore we are in the following situation. Let $a: T \to S$ be a morphism and

$$\alpha: T \times_S G \to T \times_S G$$

a G-equivariant isomorphism. Hence α is of the form

$$(t, g) \mapsto (t, \beta(t) g),$$

$\beta: T \to G$.

Moreover the equivariant morphisms $T \times G \to X$ are of the form

$$(t, g) \mapsto p_1(s) g, (t, g) \mapsto p_2(s) g$$

with $p_1, p_2: T \to X$. Hence the compatibility condition is

$$p_1(a(t)) g = p_2(f(t))(\beta(t) g)$$

for all $g \in G$.

We claim that $\underline{Isom}_{[X/G]}(S \times G, S \times G)$ is represented by $(S \times G) \times_{X \times X} X$, where $S \times G \to X \times X$ is given by $(s, g) \mapsto (p_1(s), p_2(s)g)$.

To prove this notice that

$$
\begin{gathered}
Hom(T, (S \times G) \times_{X \times X} X) = \\
= \{a{:}T \to S, b{:}T \to X, \beta{:}T \to G : f = a, p_1(a(t)) = b(t) = p_2(a(t))\beta(t)\} = \\
= \{\beta{:}T \to G{:} p_1(a(t)) = p_2(f(t))(\beta(t))\},
\end{gathered}
$$

which is exactly the above condition.

Finally, since $(S \times G) \times_{X \times X} X \to T \times G$ is a base change of $X \to X \times X$, it is quasi-separated, quasi-finite and quasi-compact. Hence $(S \times G) \times_{X \times X} X$ is quasi-affine.

To find an atlas, note first that the pullback of $\underline{X} \to [X/G]$ along $\underline{S} \to [X/G]$ is a principal homogeneous G-bundle $E \to S$. Hence $\underline{X} \to [X/G]$ is smooth and surjective.

If G is étale, we are finished. It remains to show that the diagonal is unramified by Lemma 3.11. This requires some work and will be skipped. □

We can also describe the 2-fibre product explicitly. Let $\underline{S} \to [X/G]$ with corresponding $E \in [X/G]_S$ be any morphism and let $\underline{X} \to [X/G]$ be given by sending a map $T \to X$ to the trivial G-bundle $T \times G$. Then

$$\underline{X} \times_{[X/G]} \underline{S} \cong \underline{E}.$$

Objects in $\underline{X} \times_{[X/G]} \underline{S}$ are morphisms $T \to S, T \to X$ and isomorphisms $E_T \cong T \times G$, where E_T is the pullback of E along $T \to S$. We construct a functor from $\underline{E}$ to $\underline{X} \times_{[X/G]} \underline{S}$ by sending a morphism $T \to E$ to the object

$$(T, T \to E \to S, T \to E \to X, E_T \cong T \times G),$$

where the isomorphism $E_T \cong T \times G$ is the one induced by the section $T \to E_T$ given by the universal property.

By construction this functor is dense. A morphism $(T' \to E) \to (T \to E)$ is sent to the morphism $T' \times G \to T \times G$ over S and X.

If we have a morphism $T' \to T$ over S and X in the fibre product, we get a 2-commutative diagram

$$
\begin{tikzcd}
T' \arrow[r] \arrow[d] & T' \times G \arrow[r, "\cong"] \arrow[d] & E'_T \arrow[r] \arrow[d] & E \arrow[d, "\mathrm{id}"] \\
T \arrow[r] & T \times G \arrow[r, "\cong"] & E_T \arrow[r] & E
\end{tikzcd}
$$

and hence a morphism $T' \to T$ over E. Thus the functor is also fully faithful.

3.6.1 M_{ell} as a quotient stack

We can now sketch an alternative proof that the stack of elliptic curves is indeed a Deligne-Mumford stack.

Proof. Let

$$W = \mathrm{Spec}(k[a_1, ..., a_6, \frac{1}{\Delta}])$$

and E_W given by the equation

$$y^2z + a_1xyz + a_3yz^2 = x^3 + a_2x^2z + a_4xz^2 + a_6z^3$$

in $\mathbb{P}^2_W$ as in the previous section. Let H be the following group.

$$H = \left\{ \begin{pmatrix} u^2 & s & 0 \\ 0 & u^3 & 0 \\ r & t & 1 \end{pmatrix}, u \text{ unit}, r, t, s \text{ arbitrary} \right\}$$

It can be shown that two families of elliptic curves lie locally in the same H orbit if and only if they are isomorphic. Hence $[W/H]$ is the moduli stack of elliptic curves, which is a Deligne-Mumford stack by the above lemmata.

For an exact proof see [15] or [11], Proposition 7.1. □

3.7 Basic concepts of Stacks

Definition. We say a stack $\mathfrak{X}$ has property P if its atlas $\underline{S} \to \mathfrak{X}$ has P and P is étale local. A stack is called noetherian if it is quasi-separated, locally-noetherian and quasi-compact.

Lemma 3.13. *If an atlas S has P, every other atlas S' has P.*

Proof. The projections of $\underline{S} \times_{\mathfrak{X}} \underline{S'}$ are ètale, surjective base changes and since $\underline{S} \times_{\mathfrak{X}} \underline{S'} \to \mathfrak{X}$ has P, $\underline{S'} \to \mathfrak{X}$ has P by locality. □

Definition. The disjoint union $\coprod_i \mathfrak{X}_i$ of stacks $\mathfrak{X}_i, i \in I$ is the stack with objects $(i, X), i \in I, X \in \mathfrak{X}_i$ and with morphisms $(i, X) \to (j, Y)$ $j = i$ and $X \to Y$ is a morphism in $\mathfrak{X}_i$. A stack is called connected if it cannot be written as a nontrivial union of stacks.

Lemma 3.14. *A noetherian stack can be written as a unique union of connected stacks.*

Definition. A morphism $\mathfrak{X} \to \mathfrak{Y}$ is called a open immersion/closed immersion/immersion if it is representable and an open immersion/closed immersion/immersion in the sense of Section 3.4. In this case $\mathfrak{X}$ is called a open/closed/locally closed substack of $\mathfrak{Y}$. Open/closed/locally closed substacks are closed under finite intersection. A stack is called irreducible if any two open substacks have non-empty intersection.

Lemma 3.15 ([11]). *The pullback of an open substack to an atlas S is an open subscheme $U \subseteq S$.*

Conversely, if U is an open subscheme of S, there exists an open substack $\mathfrak{U} \subseteq \mathfrak{X}$, whose pullback is U.

If $\mathfrak{X}_i, i \in I$ is a family of open substacks of $\mathfrak{X}$, there exists a union $\bigcup_i \mathfrak{X}_i$, which is an open substack of $\mathfrak{X}$ with open immersions $\mathfrak{X}_i \to \bigcup_I \mathfrak{X}_i$. Moreover the induced map

$$\coprod_i \mathfrak{X}_i \to \bigcup_i \mathfrak{X}_i$$

is surjective.

Proof. The pullback is by definition a subscheme U.

We define $\mathfrak{U}$ as the full subcategory of $\mathfrak{X}$ of objects $X \in \mathfrak{X}$ over some scheme T such that $\underline{T} \times_{\mathfrak{X}} \underline{U} \to \underline{T}$ is surjective. The morphism $\underline{T} \to \mathfrak{X}$ is the one induced by X. Let $f: T' \to T$ be a morphism and consider the diagram

$$\begin{tikzcd}
\underline{T'} \times_{\mathfrak{X}} \underline{U} \arrow[r] \arrow[d] & \underline{T} \times_{\mathfrak{X}} \underline{U} \arrow[r] \arrow[d] & \underline{U} \arrow[d] \\
\underline{T'} \arrow[r] & \underline{T} \arrow[r] & \mathfrak{X}
\end{tikzcd}$$

The morphism $\underline{T'} \to \mathfrak{X}$ is the one induced by f^*X. Thus $\mathfrak{U}$ is fibred in groupoids and the Isom-sheaves are representable. Moreover, $\mathfrak{U}$ is closed under descent of objects.

All that remains is a smooth ètale atlas. The objects of $\underline{T} \times_{\mathfrak{X}} \underline{U}$ are by definition tuples of the form $(f: T' \to T, Y \in \mathfrak{X}_{T'}, Y \cong f^*X)$. Thus by considering again above diagram, the middle vertical arrow is smooth, hence we let T_0 be its image, which is an open subscheme in T. Therefore, by the definition of $\mathfrak{U}$, $\underline{T} \times_{\mathfrak{X}} \underline{U} \cong \underline{T_0}$. Taking the inclusion $\underline{U} \hookrightarrow \underline{T}$, we obtain the following 2-commutative diagram

$$\begin{tikzcd}
\underline{U} \arrow[r] \arrow[d] & \underline{S} \arrow[d] \\
\mathfrak{U} \arrow[r] & \mathfrak{X}
\end{tikzcd}$$

Since $\underline{U} \times_{\mathfrak{X}} \mathfrak{U}$ is an open immersion together wit a section $\mathrm{id}_U, \underline{U} \to \mathfrak{U}$ it is an isomorphism. Hence $\underline{U} \to \mathfrak{U}$ is a base change of the inclusion $\underline{U} \hookrightarrow \underline{S}$ and therefore it is ètale. To see surjectivity, we let $\underline{T} \to \mathfrak{U}$ be any morphism and look at the diagram

$$\begin{tikzcd}
\underline{U}_{\times \mathfrak{X}} \underline{T} \arrow[r] \arrow[d] & \underline{U} \times \mathfrak{X} \mathfrak{U} \arrow[r] \arrow[d] & \underline{U} \arrow[d] \\
\underline{T} \arrow[r] & \mathfrak{U} \arrow[r] & \mathfrak{X}
\end{tikzcd}$$

The left vertical arrow is by construction surjective.

Let S_i be subscheme of S induced by the pullback of the subscheme $\mathfrak{X}_i$ of $\mathfrak{X}$, where S is the atlas of $\mathfrak{X}$. The union $\bigcup_i S_i$ induces a substack of $\mathfrak{X}$, which has the desired properties. □

3.7.1 Points of a stack

The above lemmata allow us to define a topological space associated to a stack.

Definition. Let $|\mathfrak{X}|$ denote the set $\coprod \mathfrak{X}_{\mathrm{Spec}(k')}/\sim$, where k' ranges over all field extensions of k and two $X \in \mathfrak{X}_{\mathrm{Spec}(k')}$ and $Y \in \mathfrak{X}_{\mathrm{Spec}(k'')}$ are equivalent if there exists a field extension K of k' and k'' such that the pullbacks of X and Y w.r.t these extensions are isomorphic. The elements of $|\mathfrak{X}|$ are called points of $\mathfrak{X}$.

A morphism $F\colon \mathfrak{X} \to \mathfrak{Y}$ induces a morphism $|F|\colon |\mathfrak{X}| \to |\mathfrak{Y}|$ by sending X to $F(X)$, which is independent of the chosen field.

This is a generalization of the concept of points of a scheme. The following lemma is a straightforward consequence of the definition.

Lemma 3.16. *1. $\mathfrak{X}$ is empty if $|\mathfrak{X}|$ is empty*

2. *A representable morphism F is surjective if and only if $|F|$ is surjective*
3. *$|\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}| \to |\mathfrak{X}| \times_{|\mathfrak{Z}|} |\mathfrak{Y}|$ is surjective*
4. *If $\mathfrak{X}$, $\mathfrak{Y}$ are locally closed substacks of $\mathfrak{Z}$ or $\mathfrak{Y}$ is open, $\mathfrak{X} \subseteq \mathfrak{Y}$ if and only $|\mathfrak{X}| \subseteq |\mathfrak{Y}|$*

We can equip $|\mathfrak{X}|$ with a topology by letting all $|\mathfrak{U}|$ be open subsets of $|\mathfrak{X}|$, where $\mathfrak{U}$ is an open substack of $\mathfrak{X}$.

Lemma 3.17. *If $F\colon \mathfrak{X} \to \mathfrak{Y}$ is a morphism, $|F|\colon |\mathfrak{X}| \to |\mathfrak{Y}|$ is continuous.*

Proof. Let S and T be the atlas of $\mathfrak{X}$ and $\mathfrak{Y}$. Hence there is a commutative diagram

$$\begin{tikzcd} |S| \arrow[r] \arrow[d] & |T| \arrow[d] \\ |\mathfrak{X}| \arrow[r] & |\mathfrak{Y}| \end{tikzcd}$$

By the properties of subschemes, we know that the vertical arrows and the the top arrow are continuous. Since the left horizontal arrow is surjective, we thus conclude that the bottom horizontal arrow is continous. □

This allows us to change between $|\mathfrak{X}|$ and $\mathfrak{X}$ if it comes to properties like being reducible.

Lemma 3.18. *Let G be an affine algebraic group acting on a scheme S and $[S/G]$ a Deligne-Mumford stack. Then $|[S/G]| = |S|/G$.*

Proof. See [15], Lemma 98.4.5. □

3.7.2 The coarse moduli space

If we start with a moduli problem and consider the resulting stack, we want to extract the coarse moduli space, which often exists, from our stack. It should be scheme, which differs the least from $\mathfrak{X}$ and whose closed points are the closed points of $\mathfrak{X}$. This leads to the following definition.

Definition. Let $\mathfrak{X}$ be a quasi-compact Deligne-Mumford stack with separated diagonal over $\mathbb{Z}$. A coarse moduli space of $\mathfrak{X}$ is a scheme X with a morphism $\pi\colon \mathfrak{X} \to X$ such that for every scheme Y and every morphism $f\colon \mathfrak{X} \to Y$ there exists a morphism $g\colon X \to Y$ with $g \circ \pi = f$ and for every algebraically closed field k, the set of k points of X, $X(k)$, is in bijection with the isomorphism classes in $\mathfrak{X}$ lying over k.

The following theorem gives us a criterion when such a coarse moduli space exists.

Definition. Let $\mathfrak{X}$ be a Deligne-Mumford stack lying over some scheme S. The inertia stack of $\mathfrak{X}$ is

$$I_S(\mathfrak{X}) = \mathfrak{X} \times_{\mathfrak{X} \times_S \mathfrak{X}} \mathfrak{X}.$$

Theorem 3.19. *[Keel-Mori,[19]] Let S be scheme and $\mathfrak{X}$ a Deligne-Mumford stack that is locally of finite presentation over S. Assume furthermore that the inertia stack $I_S(\mathfrak{X})$ is finite over $\mathfrak{X}$. Then there exists a coarse moduli space $\pi\colon \mathfrak{X} \to X$ of $\mathfrak{X}$ and it satisfies*

1. *The induced morphism $X \to S$ is separated of $\mathfrak{X} \to S$ is separated and it is locally of finite type if S is locally noetherian.*
2. *The morphism π is proper and quasi-finite.*
3. *If $X' \to X$ is a flat morphism, then*

   $$\pi'\colon \mathfrak{X}' = \mathfrak{X} \times_X X' \to X'$$

   is a coarse moduli space.

Proof. See [19]. □

3.7.3 Sheaves on stacks

In this section we will quickly sketch the notion of sheaves on stacks, however we will not provide any proofs and will skip most definitions. For a more thorough treatment see [15], 7.94.

Definition. Let $\mathfrak{X}$ be a stack over Sch/S. A presheaf of $\mathfrak{X}$ is a presheaf of the category $\mathfrak{X}$, i.e. a contravariant functor $\mathfrak{X} \to Sets$, $\mathfrak{X} \to Rings$,... . A morphism of presheaves on $\mathfrak{X}$ is a morphism of presheaves on the category $\mathfrak{X}$.

If $f: \mathfrak{X} \to \mathfrak{Y}$ we can define the pullback $f^{-1}\mathcal{G}$ of a presheaf on $\mathfrak{Y}$ and the pushforward $f_*\mathcal{F}$ of a presheaf on $\mathfrak{X}$. If we choose a suitable site on Sch/S, there is a way of endowing $\mathfrak{X}$ with a site compatible with the site on Sch/S. A presheaf on $\mathfrak{X}$ is called a sheaf on $\mathfrak{X}$ if it is a sheaf on this endowed site. The pullback and pushforward of a sheaf is again a sheaf.

Definition. The structure sheaf of a stack $p: \mathfrak{X} \to Sch$ is defined as

$$\mathscr{O}_{\mathfrak{X}} = p^{-1}\mathscr{O}.$$

The sheaf $\mathscr{O}$ on Sch/S is the sheaf defined by

$$U \to \Gamma(U, \mathscr{O}_U).$$

This allows us to define sheaves of modules on $\mathfrak{X}$.

Definition. Let $\mathfrak{X}$ be a stack with structure sheaf $\mathscr{O}_{\mathfrak{X}}$. A presheaf of modules on $\mathfrak{X}$ is a presheaf of $\mathscr{O}_{\mathfrak{X}}$-modules. A sheaf of modules on $\mathfrak{X}$ is a sheaf of $\mathscr{O}_{\mathfrak{X}}$-modules.

As in the case of schemes, we can define the tensor product of sheaves of modules and the pullback

$$f^*(-) = f^{-1}(-) \otimes_{f^{-1}\mathscr{O}_{\mathfrak{Y}}} \mathscr{O}_X.$$

If X is a scheme the notions of sheaf on X and sheaf on $\underline{X}$ are canonically equivalent. If $X \in \mathfrak{X}$ lies over scheme S, and $\mathcal{F}$ is a sheaf on $\mathfrak{X}$, we can restrict $\mathcal{F}$ to $\mathfrak{X}/X$, the subcategory of $\mathfrak{X}$ consisting of objects lying over X, i.e. morphisms in $\mathfrak{X}$ of the form $Y \to X$.

A quasi-coherent module $\mathcal{F}$ on $\mathfrak{X}$ is a sheaf of modules on $\mathfrak{X}$, such that for each object U in $\mathfrak{X}$ there exists a covering $(U_i \to U)_{i \in I}$ in the chosen site such that $\mathcal{F}$ is trivial on the U_i's.

Definition. Let X be scheme and (G, m) an algebraic group acting on X by $a: G \times X \to X$. Moreover, let $\pi_1: G \times X \to X, \pi_2: G \times X \times X \to X$ be the projections. A G-equivariant quasi-coherent sheaf of $\mathscr{O}_X$-modules is a quasi-coherent module $\mathcal{F}$ on X together with an $\mathscr{O}_{G \times X}$-module map

$$\alpha: a^*\mathcal{F} \to \pi_1^*\mathcal{F}$$

such that

1. $$\begin{tikzcd} (\mathrm{id}_G \times a)^*\pi_1^*\mathcal{F} \arrow[r, "\pi_2^*\alpha"] & \pi_2^*\mathcal{F} \\ (\mathrm{id}_G \times a)^*a^*\mathcal{F} \arrow[u, "(\mathrm{id}_G \times a)^*\alpha"] \arrow[r, equal] & (m \times \mathrm{id}_X)^*a^*\mathcal{F} \arrow[u, "(m \times \mathrm{id}_X)^*\alpha"'] \end{tikzcd}$$

   commutes and
2. the pullback $(e \times \mathrm{id}_X)^*\alpha: \mathcal{F} \to \mathcal{F}$ is the identity.

Finally, if $\mathfrak{X} = [X/G]$ is a quotient stack, the category of quasi-coherent modules on $\mathfrak{X}$ is equivalent to the category of G equivariant quasi-coherent modules on X.

If $\mathfrak{X}$ is a stack and R is a graded, finitely generated $\mathscr{O}_{\mathfrak{X}}$ algebra, we can construct, like in case in the case of schemes, the Proj of R

$$\pi\colon \mathrm{Proj}(R) \to \mathfrak{X}.$$

3.8 Stacky Proj

This section is based on [3]. Recall that the classical projective space $\mathbb{P}^n_k$ is defined as

$$\mathbb{P}^n_k := k^{n+1} - \{0\}/k^*,$$

where k^* acts on $k^{n+1} - \{0\}$ by sending $(\lambda, x) \mapsto \lambda x$. Usually in algebraic geometry the scheme associated to this space is

$$\mathrm{Proj}(k[x_0, ..., x_n]).$$

By looking at $\mathbb{P}^n_k$ as a quotient space, one could also try to define Proj as a quotient or a quotient stack.

Let X be a scheme and $R = \bigoplus_{i=0}^{\infty} R_i$ a graded $\mathcal{O}_X$ algebra, $R_+ = \bigoplus_{i \geq 1} R_i$ and

$$\mathbb{G}_m := \mathrm{Spec}(\mathbb{Z}[x, x^{-1}])$$

the multiplicative group scheme. The algebraic group $\mathbb{G}_m$ acts on $\mathrm{Spec}(R)$ by

$$\begin{aligned} \mathbb{G}_m \times \mathrm{Spec}(R) &\to \mathrm{Spec}(R) \\ R[x, x^{-1}] &\leftarrow R \\ x^m a &\leftarrow\!\shortmid a \in R_m. \end{aligned}$$

Since the $\mathbb{G}_m$ represents the functor, which assigns to every scheme T, the units $\Gamma(T, \mathcal{O}_T^*)$ of $\Gamma(T, \mathcal{O}_T)$, this action is precisely the action of k^* on k^{n+1} discussed above, since it acts on the k-points of $\mathbb{A}^n_k$ by multiplication of k^*.

Definition. The stacky Proj of R is the quotient stack

$$\mathcal{P}roj(R) := [\mathrm{Spec}(R) - V(R_+)/\mathbb{G}_m]$$

together with the structure morphism $\pi\colon \mathcal{P}roj(R) \to X$, which is the morphism induced by the $\mathbb{G}_m$-equivariant morphism $\pi\colon \mathrm{Spec}(R) - V(R_+) \to X$

Since the morphism $\pi\colon \mathrm{Spec}(R) - V(R_+) \to X$ splits through $\mathrm{Spec}(R_0)$, we often can assume that $X = \mathrm{Spec}(R_0)$.

Assume that R is generated as an R_0-algebra in degree 1 by $f_1, \ldots, f_n$. Write $R_{f_i} = R[\frac{1}{f_i}]$ Then the $\mathbb{G}_m$ invariant open sets $\operatorname{Spec}(R_{f_i})$ cover $\operatorname{Spec}(R) - V(R_+)$. Since the map

$$R_{f_i} \otimes R_{f_i} \to R_{f_i}[x, x^{-1}]$$
$$a \otimes b \mapsto abx^{\deg a}$$

is surjective, the action is free on these open sets.

We will now give local charts of $\mathscr{P}\mathrm{roj}(R)$. Let R_+ be generated by homogeneous elements $f_1, \ldots, f_n$ of degree $d_1, \ldots, d_n$. Then $\operatorname{Spec}(R) - V(R_+)$ can be covered by the sets $\operatorname{Spec}(R_{f_i})$ and by Lemma 3.15 we now that $\mathcal{P}roj(R)$ can be covered by open sets

$$U_i = D_+(f_i) = [\operatorname{Spec}(R_{f_i})/\mathbb{G}_m] = [\operatorname{Spec}(\bigoplus_{d=0}^{d_i-1} (R_{f_i})_d[f_i, f_i^{-1}])/\mathbb{G}_m].$$

By [15], Lemma 76.24.2 we deduce that

$$U_i = [\operatorname{Spec}(\bigoplus_{d=0}^{d_i-1} (R_{f_i})_d)/\mu_{d_i}],$$

where μ_{d_i} is the group scheme $\operatorname{Spec}(\mathbb{Z}[x]/(x^{d_i} - 1))$. Again by *Lemma* 3.15 we see that $D_+(f_i) \cap D_+(f_j) = D_+(f_i f_j)$. If we moreover assume that the f_i are not zero divisors, we have the usual inclusions $U_i \cap U_j \hookrightarrow U_i$. Since μ_d is a finite group and the orbits of all points are contained in affine opens, we can use [10], Exposè V, Proposition 1.8 to show that the definition of the stacky Proj contains the usual definition of Proj. Similarly, if $\mathfrak{X}$ is a stack and R is a graded, finitely generated $\mathscr{O}_{\mathfrak{X}}$ algebra on $\mathfrak{X}$, we can define $\pi\colon \mathscr{P}\mathrm{roj}(R) \to \mathfrak{X}$, see [18].

As in the case of the usual $\operatorname{Proj}(R)$ with its line bundles $\mathscr{O}(d)$, we obtain line bundles $\mathscr{O}(d)$ from the shifts $R(d)$ of R as they are $\mathbb{G}_m$-equivariant. Moreover, we can pullback the components R_d of R via π and hence obtain morphisms $\pi^* R_d \to \mathscr{O}(d)$, which are induced by the multiplication morphisms $R \otimes R_d \to R(d)$. As usual, the sheaf $\mathscr{O}(1)$ is invertible on $\mathscr{P}\mathrm{roj}(R)$ and $\mathscr{O}(1)^{\otimes d} = \mathscr{O}(d)$.

Lemma 3.20. 1. *The diagonal of* $\mathscr{P}\mathrm{roj}(R)$ *is finite and hence there exists a moduli space, which is in fact* $\operatorname{Proj}(R)$*. The induced map* $\operatorname{Proj}(R) \to X$ *is the one induced by the* Proj *construction.*

2. *The map* $\mathscr{P}\mathrm{roj}(R) \to \operatorname{Spec}(R_0)$ *is proper.*

3. *If* $f\colon R \to S$ *is a graded morphism such that* $S_+ \subseteq \sqrt{f(R_+)}$*, there is an induced morphism*

$$\mathscr{P}\mathrm{roj}(S) \to \mathscr{P}\mathrm{roj}(R)$$

such that $f^* \mathscr{O}(1) = \mathscr{O}(1)$.

Proof. All these questions are of local nature, therefore we can assume X to be affine. For (1) we first note that $D_+(f_i f_j) \to D_+(f_i) \times D_+(f_j)$ is finite and hence the diagonal is finite. The fact that Proj is the coarse moduli space, follows from the construction in [19] as well as the assertion that the map is proper, see Theorem 3.19. Finally, a graded morphism f induces a a $\mathbb{G}_m$-equivariant morphism ϕ: $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ with $\phi^{-1}(V(R_+)) \subseteq V(S_+)$. Hence there is a $\mathbb{G}_m$ equivariant morphism $\mathrm{Spec}(S) - V(S_+) \to \mathrm{Spec}(R) - V(R_+)$ which induces the desired morphism. □

Lemma 3.21. *If X is quasi-compact and R is finitely generated as an $\mathscr{O}_X$-algebra, then there exists d such that $R^{(d)} = \bigoplus_n R^{nd}$ is generated in degree one. Thus $\mathscr{P}\mathrm{roj}(R^{(d)})$ coincides with $\mathrm{Proj}(R^{(d)})$ for the right d.*

Proof. Let R be generated by generators $f_1, ..., f_m$ with degree $d_1, ..., d_m$. Setting $d = \mathrm{lcm}(d_1, ..., d_n)$, we see that $R^{(d)}$ is generated is in degree 1. Let $\prod_i f_i^{\alpha_i}$ be a monomial with $\sum_i \alpha_i d_i = dk, k \in \mathbb{N}$. Since $d_j | \alpha_i, j \neq i$, $d | \alpha_i d_i$ and hence we can write $\prod_i f_i^{\alpha_i}$ as a product of elements of degree d. □

Definition. Let k be some field and $m_1, ..., m_n \in \mathbb{N}$. Then the weighted projective stack is defined as

$$\mathcal{P}(m_1, ..., m_n) = \mathscr{P}\mathrm{roj}(k[(x_1^{m_1}, ..., x_n^{m_n}]).$$

By the above lemma, the coarse moduli space of $\mathcal{P}(m_1, ..., m_n)$ is the, usually singular, conventional $\mathrm{Proj}(k[x(x_1^{m_1}, ..., x_n^{m_n}])$.

Definition. A Rees-algebra R on a stack $\mathfrak{X}$ is a finitely generated subalgebra of $\mathscr{O}_{\mathfrak{X}}[t]$ such that $R_0 = \mathscr{O}_{\mathfrak{X}}$ and $R_{i+1} \subseteq R_i$.

If X is a scheme and $\mathscr{I}$ a sheaf of ideals on X, then a Rees algebra on X is given by setting $R_k = t^k I^k$.

We can now define the weighted blowup of a stack.

Definition. Let $\mathfrak{X}$ be a Deligne-Mumford stack and R a Rees algebra on $\mathfrak{X}$. The weighted blowup of $\mathfrak{X}$ along R is given by

$$\pi: BL_{(R)}(X) = \mathscr{P}\mathrm{roj}(R) \to \mathfrak{X}.$$

The inclusion of the shift $R(1) \hookrightarrow R$ induces an inclusion of $\mathscr{O}(1) \hookrightarrow \mathscr{O}(0)$ and hence there exists a Cartier divisor E on $\mathfrak{X}$ with $\mathscr{O}(1) = \mathscr{O}(-E)$ called the exceptional divisor.

If R is generated in degree 1, i.e. $R_k = R_1^k$, and $X = \mathrm{Spec}(S)$ is an affine scheme, this definition coincides with the usual definition of the blowup along the ideal R_1.

From Lemma 3.21 we can deduce the following lemma

Lemma 3.22. *If X is quasi-compact, there exists d such that $R_{dk} = R_d^k$ for all k. Thus the coarse space of $BL_{(R^{(n)})}(X)$ is the usual $\mathrm{Proj}(R)$.*

The theory of stacky blowups and weighted blowups can now be used to gain additional information about the usual blowups, for example in [3].

Appendices

A Toric varieties

A.1 Affine toric varieties

The section on toric varieties is mostly based on [7], where most of the omitted proofs can be found. The proofs given in this section are not meant to be rigorous but to outline the main ideas. In this part we will give a definition of affine toric varieties using cones. Later on we will give the usual definition using the action of a torus.

Definition. Let $v_1, ..., v_r \in \mathbb{R}^n$. The set

$$\sigma = \text{Cone}(v_1, ..., v_r) = \{\sum_i \lambda_i v_i : \lambda_i \geq 0, \lambda_i \in \mathbb{R}\}$$

is called the (polyhedral) cone generated by $v_1, ..., v_r$. The dimension of σ is

$$\dim \sigma = \dim \text{Span}(\sigma).$$

From now on we fix a lattice $N \cong \mathbb{Z}^n$ in $\mathbb{R}^n$.

Definition. A cone σ is called rational if it is generated by elements of N. A cone σ is called strictly convex if $\sigma \cap (-\sigma) = \{0\}$, i.e. it contains no linear subspace of $\mathbb{R}^n$.

Let $\mathbb{R}^{n*}$ be the dual space of $\mathbb{R}^n$ together with the natural pairing

$$\langle .,. \rangle : \mathbb{R}^{n*} \times \mathbb{R}^n \to \mathbb{R}.$$

Definition. If σ is a cone, the dual cone of σ is

$$\sigma^\vee = \{u \in \mathbb{R}^{n*} : \langle u, v \rangle = 0\}.$$

Let M be the dual lattice of N in $\mathbb{Z}^{n*}$. It is then easy to see that the following lemma holds.

Lemma A.1. *If σ is a rational cone, $\sigma^\vee$ is also rational. If σ is convex, $\sigma^\vee$ is also convex.*

If $\sigma = \text{Cone}(v_1, ..., v_r)$, $\sigma^\vee = \bigcap_i \text{Cone}(v_i)^\vee$.

If $\sigma = \sigma_1 + \sigma_2$, $\sigma^\vee = \sigma_1^\vee \cap \sigma_2^\vee$.

Definition. Let σ be a cone and $\lambda \in \sigma^\vee \cap M$. Set $\lambda^\perp = \{v \in \sigma : \langle \lambda, v \rangle = 0\}$. The set $\tau = \sigma \cap \lambda^\perp$ is called a face of σ. We will write $\tau < \sigma$.

Lemma A.2. *Let σ be a rational, strictly convex cone. Then*

1. *Every face τ is again a rational strictly convex cone.*
2. *If τ, τ' are faces of σ, $\tau \cap \tau'$ is a face of σ.*
3. *A face of a face is a face.*
4. *If τ is face of σ, $\sigma^\vee \subseteq \tau^\perp$.*
5. *If $\tau = \sigma \cap \lambda^\perp$, $\tau = \sigma^\vee + \mathbb{R}_{\geq 0}(-\lambda)$.*

Lemma A.3. *If σ is a cone and $\tau < \sigma$, then $\sigma^\vee \cap \tau^\perp$ is a face of $\sigma^\vee$ and*

$$\dim \tau + \dim \sigma^\vee \cap \tau^\perp = n.$$

This induces a bijection of faces of σ and $\sigma^\vee$.

Recall that a monoid is a commutative semi-group S with a neutral element and a cancellation law

$$s + t = s' + t \Rightarrow s = s'.$$

We will call S finitely generated if there exists $a_1, ..., a_r \in S$ such that

$$\langle a_1, ..., a_r \rangle_{\mathbb{N}} = S.$$

It is easy to see that if σ is a cone, $\sigma \cap N$ is a monoid.

Lemma A.4 (Gordon's lemma)**.** *If σ is a rational cone, $\sigma \cap N$ is a finitely generated monoid.*

Proof. Let $\sigma = \mathrm{Cone}(v_1, ..., v_r)$ and set $K = \{\sum_i \lambda_i v_i : 0 \leq \lambda_i \leq 1, \lambda_i \in \mathbb{R}\}$. Since N is discrete and K is compact, $N \cap K$ is finite. It is straightforward to see that $N \cap K$ generates $\sigma \cap N$. $\square$

From now one let $S_\sigma = \sigma^\vee \cap M$.

Lemma A.5. *If $\tau = \sigma \cap \lambda^\perp$ is a face of a cone, $S_\tau = S_\sigma + \mathbb{N}(-\lambda)$.*

Definition. Let

$$k[z, z^{-1}] = k[z_1, ..., z_n, z_1^{-1}, ..., z_n^{-1}]$$

be the ring of Laurent polynomials. If $v = (v_1, ..., v_n) \in \mathbb{Z}^n$ define $z^v = z_1^{v_1} ... _n^{v_n} \in k[z, z^{-1}]$. If $f = \sum_v \lambda_v z^v \in k[z, z^{-1}]$, let

$$\mathrm{supp}(f) = \{v : \lambda_v \neq 0\}.$$

Definition. Let σ be a rational, strictly convex cone. The affine toric variety corresponding to σ is $X_\sigma = \mathrm{Spec}(R_\sigma)$, where

$$R_\sigma = \{f \in k[z, z^{-1}] : \mathrm{supp}(f) \in S_\sigma\}.$$

By Gordon's lemma X_σ is a affine variety of finite type over $\mathrm{Spec}(k)$.

Lemma A.6. *If S_σ is generated by $v_1, ..., v_r$,*

$$R_\sigma \cong V(I_\sigma) = k[x_1, ..., x_r]/I_\sigma,$$

where I_σ is generated by monomials $x^\alpha - x^\beta$.

Proof. The ideal I_σ is the kernel of the map

$$\pi: k[x_1, ..., x_r] \to k[z^{v_1}, ..., z^{v_r}] = R_\sigma$$
$$x_i \mapsto z^{v_i}$$

Let $f = \sum_v \lambda_v z^v \in \ker \pi$ and $A = (v_1 ... v_r)$ the matrix with columns v_i. Since $\pi(f) = \sum_v \lambda_v z^{Av}$, we see that $\sum_{v:Aw=v} \lambda_v = 0$. Choose $v \in \operatorname{supp}(f)$. There exists $v' \in \operatorname{supp}(f)$ with $Av = Av'$. The Laurent polynomial $f - \lambda_v(z^v - z^{v'})$ has less monomial terms than f. By induction we conclude that we can write f as linear combination of binomials. $\square$

Because of the embedding $k[z^{v_1}, ..., z^{v_r}] \hookrightarrow k[z, z^{-1}]$ the affine toric variety X_σ contains a torus $T \cong k^{*n}$ as an open, dense subset.

Theorem A.7. *An affine toric variety X_σ is irreducible and normal.*

Proof. Every affine toric variety is irreducible, since R_σ is integral. If $\sigma = \operatorname{Cone}(v_1, ..., v_r)$, $\sigma^\vee = \bigcap_i \operatorname{Cone}(v_i)^\vee$. Hence $R_\sigma = \bigcap_i R_{\operatorname{Cone}(v_i)}$, therefore it suffices to prove that $R_{\operatorname{Cone}(v_i)}$ is normal. From now on we can fix $v_i \in N \cap \mathbb{R}_{\geq 0}$ and assume it to be primitive. Extend v_i to a basis of N and hence, after a base change, we will assume $V_i = e_1$. Thus

$$R_{\operatorname{Cone}(v_i)} = k[x_1, ..., x_n, x_2^{-1}, ..., x_n^{-1}],$$

which is a localization of $K[x_1, ..., x_n]$ which is a UFD and hence normal. $\square$

A.2 Toric varieties

Definition. A fan Δ in $\mathbb{R}^n$ is a finite union of rational, strictly convex cones such that

1. If σ is a cone of Δ and $\tau < \sigma$, τ is also a cone of Δ.
2. If σ and σ' are cones of Δ, $\sigma \cap \sigma'$ is a face of σ and σ'.

Recall that if $\tau = \sigma \cap \lambda^\perp$ is a face of σ, $S_\tau = S_\sigma + \mathbb{N}(-\lambda)$. Thus there is a natural identification of X_τ with $(X_\sigma)_{z^\lambda} = X_\sigma - V(z^\lambda)$.

Definition. Let Δ be a fan in $\mathbb{R}^n$. The toric variety X_Δ associated to Δ is the variety obtained by gluing the X_σ, σ a cone of Δ along above identifications.

Theorem A.8. *A toric variety X_Σ contains a copy of the torus k^{*n} as a open, dense subset. Moreover X_σ is irreducible, separated and normal.*

Proof. The 0-face $\{0\}$ of every cone of Δ corresponds to a torus, which is embedded in every affine chart of X_σ. Hence it is an open, dense subset of X_σ. It is irreducible and normal since all U_σ are irreducible and normal. To see that it is separated, consider for all cones σ_1, σ_2, $\tau = \sigma_1 \cap \sigma_2$ of Δ the diagonal map

$$\Delta\colon U_\tau \to U_{\sigma_1} \times U_{\sigma_2}.$$

Thus, if $S_\tau = S_{\sigma_1} + S_{\sigma_2}$, the corresponding ring map is surjective and hence the image would be closed in $U_{\sigma_1} \times U_{\sigma_2}$. The inclusion $S_{\sigma_1} + S_{\sigma_2} \subseteq S_\tau$ follows from $\sigma_1^\vee + \sigma_2^\vee = \tau^\vee$. On the other hand, if $p \in S_\tau$, we can assume $p = q + l(-\lambda)$ for $q \in S_{\sigma_1}$ and $\lambda \in \sigma_1^\vee \cap (-\sigma_2^\vee) \cap M$. But then $(-\lambda) \in S_{\sigma_2}$, therefore we have equality. Since separateness is a local property, we are finished. ☐

We will now describe the action of the embedded torus $T = k^{*n}$ on a toric variety X_Δ. Let σ be a cone of Δ and let S_σ be generated by $v_1, ..., v_r$. Then we have an action of T on X_σ which extends the action of T on itself given by

$$\begin{aligned} T \times X_\sigma &\to X_\sigma \\ (t, x) &\mapsto (t^{v_1} x_1, ..., t^{v_r} x_r) \end{aligned}$$

Since this action is compatible with the restriction to faces, we obtain an action $T \times X_\Delta \to X_\Delta$.

From now on we will assume $k = \mathbb{C}$. We can identify closed points of X_σ with semi-group morphisms $\mathrm{Hom}(S_\sigma, \mathbb{C})$, where $\mathbb{C}$ is a semi-group w.r.t its multiplication. If S_σ is generated by $v_1, ..., v_r$, the closed point corresponding to a morphism ϕ is $(\phi(v_1), ..., \phi(v_n))$.

Definition. Let X_σ be an affine toric variety and $\tau < \sigma$. The distinguished point x_τ of τ is the point corresponding to the morphism

$$\begin{aligned} \phi\colon S_\sigma &\to \mathbb{C} \\ v &\mapsto \begin{cases} 1, & \text{if } v \in \tau^\perp \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

The orbit O_τ of a face $\tau < \sigma$ is the orbit of the point x_τ under the action of T.

Lemma A.9. *If X_Δ is a toric variety and σ is a cone of Δ, then*

1. $X_\sigma = \coprod_{\tau < \sigma} O_\tau$.
2. $V_\tau = \coprod_{\tau < \sigma} O_\sigma$*, where* V_τ *is the closure of* O_τ.
3. $O_\tau = V_\tau - \bigcap_{\tau < \sigma, \tau \neq \sigma} O_\sigma$.

Hence V_τ is the subscheme cut out by the equations $z_i = 0$ if $v_i \notin \tau^\perp$.

We will now see how to recover the fan from a toric variety. Let N be lattice with dual lattice M. Then we have a torus given by

$$T = \mathrm{Hom}(M, \mathbb{C}^*)$$

and with a choice of basis an isomorphism

$$\mathrm{Hom}(\mathbb{C}^*, T) \cong N$$

and

$$\mathrm{Hom}(T, \mathbb{C}) \cong M.$$

With this we can recover the lattices M and N from T. To recover a cone σ, we have to work a bit harder.

Lemma A.10. *Let $\lambda_v : \mathbb{C}^* \to T$ be a one-parameter subgroup of T given by $v \in N$. If $v \in \Delta$ and τ is the cone containing v, then $\lim_{z \to 0} \lambda_v(z) = x_\tau$.*

Proof. We can assume that τ is a face of some cone σ and thus we can work in the affine toric variety X_σ. We know that λ_v is identified with the morphism $u \mapsto z^{\langle u,v \rangle}$ and hence for $u \in S_\sigma$, $\langle u, v\rangle \geq 0$ with equality if and only if $u \in \tau^\perp$. Hence, by the definition of x_τ through the morphism ϕ, the lemma is proven. □

Lemma A.11. *If v does not belong to any cone of Δ, $\lim_{z \to 0} \lambda_v(z)$ does not exist.*

Proof. Again we can work in the case $\Delta = \sigma$. Since $v \notin \Delta$, there is $u \in \sigma^\vee$ such that $\langle v, u \rangle < 0$, therefore the limit goes to infinity. □

Thus the set $\sigma \cap N$ is given by vectors v such that the $\lambda_v(z)$ has a limit in X_σ. This gives the second definition of a toric variety.

Definition. A toric variety X is an normal variety X with an open and dense embedded torus T together with an extension of the natural action of T on itself $T \times X \to X$.

Definition. A cone generated by vectors $v_1, ..., v_r$ is called a simplex if the v_i are linearly independent. A fan Δ is called simplicial if all its cones are simplices.

A cone is called regular, if its generating vectors $v_1, ..., v_r$ can be extended to a basis of N. A fan Δ is called regular if all its cones are regular.

A fan Δ is called complete if it covers $\mathbb{R}^n$. A fan Δ is called polytopal if it spanned by a polytope P in $\mathbb{R}^n$, i.e. $\Delta = \mathrm{Cone}(P \times \{1\}) \subseteq \mathbb{R}^{n+1}$.

Lemma A.12. *A toric variety X_Δ is smooth if and only if Δ is regular.*

Proof. Since this is a local question, we can assume $X_\Delta = X_\sigma$ to be an affine toric variety. If σ is regular, we can assume it is generated by $(e_1, ..., e_r)$ and hence $X_\sigma \cong k^{*\,n-r} \times \mathbb{A}^r_k$.

If X_σ is smooth, we will first assume $\dim \sigma = \dim U_\sigma = \dim N$. Let x_σ be the distinguished point of σ. Let $\mathfrak{m}$ be the maximal ideal in the point x_σ. Then $\dim \mathfrak{m}/\mathfrak{m}^2 = n$ since X_σ is smooth. The ideal $\mathfrak{m}$ is generated by $z^v, v \in S_\sigma - \{0\}$ and $\mathfrak{m}^2$ is generated by z^v, where v can be written as a nontrivial sum of elements of $S_\sigma - \{0\}$. Thus the first primitive vectors on the rays of $\sigma^\vee$ in M correspond to elements of a basis of $\mathfrak{m}/\mathfrak{m}$. Hence there cannot be more than n of those and those n have to generate S_σ. Moreover, since σ is strictly convex, those vectors generate a basis of M and hence induce a basis of N. Thus σ is regular and $X_\sigma \cong \mathbb{A}^n_k$.

If $r = \dim \sigma < n$, we consider $N_\sigma = (\sigma \cap N) + (-\sigma \cap N)$. By the structure theorem of finitely generated modules over a P.I.D. we can decompose $N = N_\sigma \oplus N'$ such that $\sigma = \sigma' \oplus \{0\}$ and N' is a sublattice of rank $n-r$. Thus $X_\sigma \cong k^{*n-r} \times \mathbb{A}^r_k$. □

A.3 Resolutions of toric varieties

In this section we will give a rough sketch on how to prove resolution of singularities for toric varieties.

We first are going to sketch the affine case. If σ is the cone of an affine toric variety and $\sigma = \mathrm{Cone}(v_1, ..., v_r)$ we will first embed σ in an simplicial fan Δ. This is achieved by adding new vectors v_i' which lie in the convex span of σ. The fan induced by this subdivision can be assumed to be simplicial. In Figure 8 the cone $\mathrm{Cone}(v_1, v_2, v_3, v_4)$ is subdivided into the fan induced by $\{\mathrm{Cone}(v_1, v_2, v), \mathrm{Cone}(v_2, v_3, v), \mathrm{Cone}(v_3, v_4, v), \mathrm{Cone}(v_4, v_1, v)\}$.

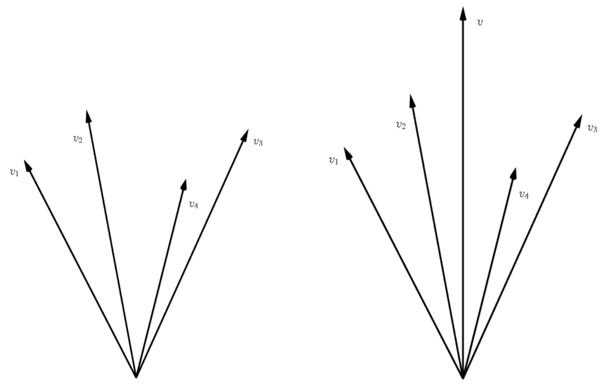

Figure 8: Subdivision of a cone

Lemma A.13. *A subdivision $\Delta \hookrightarrow \Delta$ induces a proper map $X_{\Delta'} \to X_{\Delta'}$.*

Proof. The map is locally given by the identity on the affine charts. Properness follows from the valuative criterion of properness, see [12]. □

Thus we can assume the fan to be simplicial, since the new variety coincides with the old one on an open, dense subset.

Next we will find a subdivision such that all cones of Δ are regular. For this we define the multiplicity of a cone.

Definition. Let σ be a cone of Δ and $v_1, ..., v_r$ the primitive vectors of the rays spanning σ. Recall that $N_\sigma = (\sigma \cap N) + (-\sigma \cap N)$. The multiplicity of σ is

$$\mathrm{mult}(\sigma) = [N_\sigma : \mathbb{Z}v_1 \oplus ... \oplus \mathbb{Z}v_r].$$

It is straightforward to see that $\mathrm{mult}(\sigma) = 1$ if and only if σ is regular. By the following lemma we can choose a subdivision of σ such that every cone of the resulting fan has strictly smaller multiplicity than σ. Thus, after a finite number of subdivisions, we obtain a regular fan and hence a resolution of singularities.

Lemma A.14. *Let $M \in \mathrm{Mat}_{n\times n}(\mathbb{N})$ and $\det(M) > 1$. The there exists an integral point in the parallelepiped spanned by M different from the vertices.*

To prove this we need Minkwoski's theorem.

Lemma A.15 (Minkwoski's theorem)**.** *Let $S \subseteq \mathbb{R}^n$ be symmetric w.r.t to the origin and assume $\mathrm{vol}(S) > 2^n$. Then S contains an integral point, i.e. $\mathbb{Z}^n \cap S \neq \varnothing$.*

Proof of Lemma A.14. Let P be the parallelepiped spanned by $M = (v_1, ..., v_n)$ and consider the parallelepiped P' spanned by $(2v_1, ..., 2v_n)$ and shifted such that it is symmetric w.r.t to the origin. Then $\mathrm{Vol}(P') > 2^n$ by assumption and by Minkwoski's theorem we obtain an integral point in the interior of P' since the boundary has volume 0. Thus there exists a integral point in P. ☐

Proof of existence of a resolution. To see that there exists a subdivision which lowers the multiplicity, we choose a ray induced by a lattice vector $v = \sum_i \lambda_i v_i, 0 \leq \lambda_i < 1$. We can then subdivide σ according to the subdivision induced by v as before. Then the cone with rays induced by $v_1, ...v_{i-1}, v, v_{i+1}, ..., v_r$ has multiplicity $\lambda_i \mathrm{mult}(\sigma) < \mathrm{mult}(\sigma)$. ☐

B Ètale and fpqc coverings

Definition. Let R be a ring and let $R \to S$ be a ring map, where

$$S \cong R[x_1, ..., x_n]/(f_1, ..., f_n)$$

and the map is the one induced by the inclusion $R \hookrightarrow R[x_1, ..., x_n]/(f_1, ..., f_n)$. Then $R \to S$ is called étale. A morphism between schemes $X \to Y$ is called étale if there exists for each $x \in X$ an affine open neighbourhood $\mathrm{Spec}(S) = U$ of x and an affine open neighbourhood $\mathrm{Spec}(R) = V$ of $f(x)$ such that $f(U) \subseteq V$ and restriction of f to U is given by an étale ring map.

Lemma B.1. *Let $f: X \to Y$ be a morphism of schemes. T.f.a.e*

1. *f is étale*
2. *f is smooth of relative dimension* 0

Proof. See [15], 29.35. □

Lemma B.2. *1. The composition of étale morphisms is étale.*

2. Base changes of étale morphisms are étale.

3. Fibre products of étale morphisms are étale.

4. Ètale morphisms are open.

Proof. See [15], 29.35. □

Definition. Let S be a scheme. An fpqc, *fidèlement plat (faithfully flat) quasi-compact*, covering of of S is a family of morphisms $\{\phi_i : S_i \to S)_{i \in I}$ such that

1. Each ϕ_i is flat and $\bigcup_i \phi_i(S_i) = S$.
2. For each open affine set $U \subseteq S$ there exists a finite subset $J \subseteq I$ such that $\bigcup_{j \in J} \phi_j(S_j) = U$.

C Grothendieck topologies and sheaves

Motivated by the covering of a topological space by open maps or of a scheme by étale maps, we introduce the notion of a Grothendieck topology of a category $\mathfrak{C}$

Definition. Let $\mathfrak{C}$ be a category which admits fibre products. A class C of morphism or *coverings* $(U_i \to U)_{i \in I}$ is called a Grothendieck topology of $\mathfrak{C}$

1. If for all coverings $(U_i \to U)_{i \in I}$ in C and morphisms $V \to U$, $(U_i \times_U V \to V)_{i \in I}$ is in C.
2. If for all $(U_i \to U)_{i \in I}$ and $(V_{ij} \to U_i)_{j \in J}$ in C, $(V_{ij} \to U_i \to U)_{(i,j) \in I \times J}$ is in C.
3. If $U \to U$ is an isomorphism, $U \to U$ is in C.

The category $\mathfrak{C}$ together with a chosen Grothendieck topology is called a site.

Definition. Let $\mathfrak{C}$ be a category with a Grothendieck topology. A $\mathfrak{D}$-valued presheaf on $\mathfrak{C}$ is a contravariant functor $F : \mathfrak{C} \to \mathfrak{D}$.

A $\mathfrak{D}$-valued sheaf is a $\mathfrak{D}$-valued presheaf $F : \mathfrak{C} \to \mathfrak{D}$ such that for all coverings $(U_i \to U)_{i \in I}$

$$F(U) \to \prod_I F(U_i) \rightrightarrows \prod_{I \times I} F(U_i \times_U U_j)$$

is an equalizer.

If $\mathfrak{Top}$ is the category of topological spaces with the Grothendieck topology given by open coverings and $\mathfrak{Rings}$ is the category of commutative rings, a $\mathfrak{Ring}$-sheaf on $\mathfrak{Top}$ is a usual sheaf.

Definition. Let P be a property of morphisms between schemes over some scheme X. The big site on the category Sch/X has as covers all morphisms $\{U_i \to U\}_{i \in I}$ satisfying P such that their open images cover U. The small site will be the restriction to all objects which have a morphism to X satisfying P.

If P is being an open immersion, being étale or being fpqc, we obtain a Grothendieck topology on the category of schemes over X. We call a site sub canonical if all representable presheaves on this site are in fact sheaves. Both the ètale and the fpqc site are subcanonical, see [15], 58.15 and 58.27.

D 2-Categories

A 2-category $\mathfrak{C}$ consists of the following data. All proofs are taken from [11] and [15].

1. A class of objects $\mathrm{Ob}(\mathfrak{C})$.
2. For all objects X, Y of $\mathfrak{C}$ a category $\mathrm{Hom}(X, Y)$, whose objects are called 1-morphisms and whose morphisms are called 2-morphisms. The composition of 2-morphism is called vertical composition.
3. A composition functor.
$$\mathrm{Hom}(Y, Z) \times \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z).$$
The composition of 2-morphism given by this functor is called horizontal composition.
4. $\mathrm{Ob}(\mathfrak{C})$ together with the 1-morphisms form a category. A 1-morphism F between objects X and Y is denoted as $F\colon X \to Y$. A 2-morphism α between 1-morphisms F and G is denoted $\alpha\colon F \Rightarrow G$.
5. Horizontal composition of associative and $\mathrm{id}_{\mathrm{id_X}}$ is an identity for horizontal composition.

The category $\mathfrak{Cat}$ of categories is a 2-category, with 1-morphisms as functors and 2-morphisms as natural transformations. A $(2,1)$-category is a 2-category where all 2-morphisms are isomorphisms. Note that any 2-category can be restricted to a $(2,1)$-category by forgetting all non-isomorphisms.

Two objects X, Y in a 2-category are called equivalent, if there exists 1-morphisms $F\colon X \to Y$, $G\colon Y \to X$ and 2-morphisms such that $F \circ G \Rightarrow \mathrm{id}_{\mathfrak{X}}$ and $G \circ F \Rightarrow \mathrm{id}_{\mathfrak{Y}}$.

Lemma D.1. *If $F\colon \mathfrak{X} \to \mathfrak{Y}$ is a functor and the class of objects $\mathrm{Ob}(\mathfrak{Y})$ is a set, then F is an equivalence in the 2-category $\mathfrak{Cat}$ if and only if F is fully faithful and dense, i.e. the induced morphism*
$$\mathrm{Hom}(X, X') \to \mathrm{Hom}(F(X), F(X'))$$

is a bijection and for each object $Y \in \mathfrak{Y}$ there exists an object $X \in \mathfrak{X}$ such that $F(X) \cong Y$.

Proof. We construct G in the following way. Choose for each $Y \in \mathfrak{Y}$ an object $X \in \mathfrak{X}$ such that $F(X) \cong Y$ and define $G(Y) = X$. A morphism $g: Y \to Y'$ is sent to the morphism $f: G(Y) \to G(Y')$ given by the bijection

$$\mathrm{Hom}(G(Y), G(Y')) \to \mathrm{Hom}(F(G(Y)), F(G(Y'))) \cong \mathrm{Hom}(Y, Y').$$

It is now straightforward to check that $F \circ G \Rightarrow \mathrm{id}_{\mathfrak{Y}}$ and $G \circ F \Rightarrow\Rightarrow \mathrm{id}_{\mathfrak{X}}$. □

From now on we will always assume that the class of objects is a set and we can use the axiom of choice.

Lemma D.2. *Let $F: \mathfrak{X} \to \mathfrak{Y}$ be a functor of fibred groupoids over $\mathfrak{S}$. T.f.a.e*

1. *F is an equivalence of categories over $\mathfrak{S}$.*
2. *F is an equivalence of categories.*
3. *$F_S: \mathfrak{X}_S \to \mathfrak{Y}_S$ is an equivalence of categories for all $S \in \mathfrak{S}$.*
4. *F is fully faithful and an equivalence on fibres.*

Proof. (1) ⇒ (2) and (1) ⇒ (3) are trivial. From Lemma D.1 we see (2) ⇒ (4).

Next we are going to show (3) ⇒ (4). Let $\phi: F(X) \to F(Y)$ over $f: S \to T$. Let $\psi: f^*Y \to Y$ be the morphism obtain by pulling Y back along f. Then $F(f^*Y) \cong F(Y)$ and hence we have a morphism $\mu: F(X) \to F(f^*Y)$ such that $\phi = (\psi) \circ \mu$. Because F_S is an equivalence, we obtain a morphism $\mu': X \to f^*Y$ and hence $\psi = F(\psi \circ \mu')$. This construction is forced, hence F is fully faithful.

We will now prove (4) ⇒ (1). By *Lemma D.1* it suffices to show that F is dense, i.e. for each object Y in $\mathfrak{Y}$ there exists an object X in $\mathfrak{X}$ such that $F(X) \cong Y$. Since F_S is an equivalence for all $S \in \mathfrak{S}$, F is dense.

□

D.1 The Yoneda lemma

The Yoneda lemma is one of the most important lemmas in category theory. If we want to understand some object X in a category, it allows us to instead try to understand how other objects "see" X.

Let C be a category, whose morphisms $\mathrm{Hom}(X,Y)$ are sets and F a functor $F: C \to Sets$. Let $X \in C$ be an object and define

$$\mathrm{Hom}(X, -) = h^X: C \to Set$$

$$Y \mapsto \mathrm{Hom}(X, Y)$$

$$(f:Y \to Y') \mapsto (\mathrm{Hom}(X,Y) \to \mathrm{Hom}(X,Y'), g \mapsto f \circ g).$$

Similarly we define the functor $h_X = \mathrm{Hom}(-,X)$.

Lemma D.3 (Yoneda lemma)**.** *The class of natural transformations between h^X and F, $\mathrm{Nat}(h^X,F)$, is a set and there exists a bijection*

$$\mathrm{Nat}(h^X,F) \cong F(X).$$

Proof. Let α be natural transformation $\alpha: h^X \to F$, Y an object in C and $f:X \to Y$. Then the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}(X,X) \arrow[r, "h^X(f)"] \arrow[d, "\alpha_X"'] & h^X(Y) \arrow[d, "\alpha_Y"] \\
F(X) \arrow[r, "F(f)"] & F(Y)
\end{tikzcd}$$

Let $u = \alpha_X(id_X)$. This is equivalent to $F(f)(u) = \alpha_Y(f)$. This implies that any natural transformation is defined by its value on id_X and moreover any $u \in F(X)$ defines a unique natural transformation. $\square$

If Y is another object in C, the Yoneda lemma allows us to construct an embedding of the category C^{op} into the category Set^C of functors from C to Set. The Yoneda embedding

$$h: C^{op} \to Set^C$$

sends objects X to the functor h^X and morphisms to the natural transformations given by the Yoneda lemma.

Moreover, if F is a presheaf, i.e. $F \in Set^C$ and representable, i.e. $F \cong h^X$, then the Yoneda lemma shows the uniqueness of X up to isomorphism.

Lemma D.4 (2-Yoneda lemma,[11])**.** *Let $\mathfrak{X}$ be a CFG over $\mathfrak{S}$ and let $S \in \mathfrak{S}$. Then there exists a functor*

$$y_S: \mathrm{Hom}(\underline{S}, \mathfrak{X}) \to \mathfrak{X}_S$$

which is an equivalence.

Proof. We define y_S by sending a functor F to $F(\mathrm{id}_S)$ and a natural transformation $\alpha: F \Rightarrow G$ to $\alpha_{\mathrm{id}_S}: F(\mathrm{id}_S) \to G(\mathrm{id}_S)$. This is indeed a functor.

First we are going to prove that y_S is dense. We make a choice of pullbacks such that $\mathrm{id}_S^* X = X$ for $X \in \mathfrak{X}_S$. Fix now $X \in \mathfrak{X}_S$ and define a functor $F: \underline{S} \to \mathfrak{X}$ by sending a morphism $f: T \to S$ to the object $f^* X \in \mathfrak{X}_T$. A morphism

$$\begin{tikzcd}
T \arrow[rr, "\phi"] \arrow[dr, "f"'] & & T' \arrow[dl, "g'"] \\
& S &
\end{tikzcd}$$

is sent to $f^*X \cong \phi^* g^* X \to g^*X$.

It is straightforward to check that this is indeed a functor.

To prove fully faithfulness assume that $\alpha: F \Rightarrow G$ is a 2-morphism and $\alpha(\mathrm{id}_S) = \beta(\mathrm{id}_S)$. So if $f: T \to S$ is an object in $\underline{S}$, we also can consider it as a morphism $f \to \mathrm{id}_S$. To see now that $\alpha(\mathrm{id}_S)$ uniquely determines α, we consider the diagram

$$\begin{tikzcd}
 & F(\mathrm{id}_S) \arrow[dr, bend left, "\alpha(\mathrm{id}_S)=\beta(\mathrm{id}_S)"] & \\
F(f) \arrow[ur, bend left, "F(f\to\mathrm{id}_S)"] \arrow[r, dotted] \arrow[d, "p"] & G(f) \arrow[r, "G(f\to\mathrm{id}_S)"] \arrow[d, "p"] & G(\mathrm{id}_S) \arrow[d, "p"] \\
T \arrow[r, "\mathrm{id}"] & T \arrow[r, "f"] & S'
\end{tikzcd}$$

□

The dotted morphism is now either $\alpha(f)$ or $\beta(f)$. Since it is unique, α is determined by $\alpha(\mathrm{id}_S)$. Moreover, we can construct easily a 2-morphism with this unique property for given $\alpha(\mathrm{id}_S)$.

Definition. Let be a category and $F: \mathfrak{S}^{op} \to \mathfrak{Cat}$ a functor. We will define a category $\mathfrak{S}_F$ over S.

$$\mathrm{Ob}(\mathfrak{S}) = \{(U,X), U \in \mathfrak{S}, X \in F(U)\}$$
$$\mathrm{Hom}((U',X'),(U,X)) = \{(f,\phi), F: U' \to U, \phi: X' \to F(f)(X)\}$$

Lemma D.5. *For any functor F and category $\mathfrak{S}$ $\mathfrak{S}_F$ is a fibred category. If F goes in the full subcategory of groupoids, $\mathfrak{S}_F$ is a CFG. Moreover $\mathfrak{S}_F$ is a split fibred category, i.e. $(g \circ f)^* = f^* \circ g^*$ for all morphisms.*

Proof. The fibre of over an object U is $F(U)$, which is a groupoid if F goes to groupoids. Moreover the pullback of (U,X) along $f: V \to U$ is given by $(f, \mathrm{id}_{F(f)(X)}(V, F(f)(X)) \to (U,X)$. Since F is a functor, $\mathfrak{S}_F$ is a split fibred category. □

Lemma D.6. *Any fibred category is equivalent to some category of above form, i.e. $\mathfrak{S}_F$.*

Proof. We define $F: \mathfrak{S} \to \mathfrak{Cat}$ as follows. For $S \in \mathfrak{S}$ we set

$$F(S) = \mathrm{Hom}(\underline{S}, \mathfrak{X})$$

and for $f: T \to S$ we let

$$F(f): \mathrm{Hom}(\underline{T}, \mathfrak{X}) \to \mathrm{Hom}(\underline{S}, \mathfrak{X})$$

be the morphism defined by precomposition with f.

Now define $G\colon \mathfrak{S}_F \to \mathfrak{X}$ by setting $G(U,X) = X(\mathrm{id}_U)$ and

$$G(f,\phi\colon (U,X) \to (V,Y))) = (Y(f) \to Y(\mathrm{id}_V)).$$

Since

$$G(U) = \mathrm{Hom}(\underline{U}, \mathfrak{X}) \to \underline{U}$$

G is an equivalence by the 2-Yoneda lemma and Lemma D.2. □

Lemma D.7. *A functor $F\colon \mathfrak{S}^{op} \to Sets$ is a sheaf, if and only if $\mathfrak{S}_F$ is a stack.*

Proof. Let $(T_i \to T)_{i \in II}$ be a covering. We will write T_{ij} for $T_i \times_T T_j$ and if X lies over T $X\big|_{T_i}$ for the pullback. Then $\mathfrak{S}_F$ is a prestack if and only if

$$\mathrm{Hom}_{F(T)}(X,Y) \to \prod_I \mathrm{Hom}_{F(T_i)}(X\big|_{T_i}, Y\big|_{T_i}) \rightrightarrows \prod_{I\times I} Hom_{F(T_{ij})}(X\big|_{T_{ij}}, Y\big|_{T_{ij}})$$

is an equalizer for all $X, Y \in F(T)$. Since F goes into $Sets$, this can only be the case if for any two $X, Y \in F(T)$ which agree on all pullbacks $X = Y$.

And $\mathfrak{S}_F$ satisfies the condition on the descent data if and only if the elements in the $F(T)$'s can be glued together if the agree on the T_{ij}'s.

Thus $\mathfrak{S}_F$ is a stack if and only if F is a sheaf. □

Lemma D.8. *Let $\mathfrak{X}$ be a stack over $\mathfrak{S}$. Recall that $\underline{Isom}_{\mathfrak{X}}(X,Y)$ is a sheaf for all objects X,Y lying over $S \in \mathfrak{S}$. Then*

$$\mathfrak{S}_{\underline{Isom}_{\mathfrak{X}}(X,Y)} \cong \mathfrak{X} \times_{\mathfrak{X}\times\mathfrak{X}} \underline{S},$$

where are X and Y are interpreted as morphisms into $\mathfrak{X}$ by the 2-Yoneda lemma.

Proof. The objects of $\mathfrak{X} \times_{\mathfrak{X}\times\mathfrak{X}} \underline{S}$ are of the form

$$(T, Z, f\colon T \to S, \alpha\colon Z \to f^*X, \beta\colon Z \to f^*Y).$$

Since the diagram

$$\begin{array}{ccc} (Z,Z) & \xrightarrow{(\alpha,\beta)} & (f^*X, f^*Y) \\ \big\downarrow{\scriptstyle(\alpha,\alpha)} & & \big\downarrow{\scriptstyle(\mathrm{id}_{f^*X},\mathrm{id}_{f^*Y})} \\ (f^*X, f^*X) & \xrightarrow{(\mathrm{id}_{f^*X},\beta\alpha^{-1})} & (f^*X, f^*Y) \end{array}$$

commutes, we obtain an isomorphism $(T, Z, f, (\alpha,\beta)) \cong (T, f^*X, (\mathrm{id}_{f^*X}, \beta\alpha^{-1}))$. Let $\mathfrak{Y}$ denote the full subcategory of objects of form $(T, f^*X, (\mathrm{id}_{f^*X}, \phi))$. By construction $\mathfrak{Y} \cong \mathfrak{X} \times_{\mathfrak{X}\times\mathfrak{X}} \underline{S}$.

If $h: g \to f$ is a morphism, any morphism

$$(T, g^*X, (\mathrm{id}_{g^*X}, \psi)) \to (T, f^*X, (\mathrm{id}_{f^*X}, \phi))$$

has to fit in the commutative diagram

$$\begin{array}{ccc}
(g^*X, g^*X) & \xrightarrow{(\mathrm{id}_{g^*X}, \psi)} & (g^*X, g^*Y) \\
\downarrow & & \downarrow \\
(f^*X, f^*X) & \xrightarrow{(\mathrm{id}_{f^*X}, \phi)} & (f^*X, f^*Y)
\end{array}$$

Thus the only possible morphism lying over h is the morphism induced by $g^*X \to f^*X$. Hence $\mathfrak{Y}$ is fibred in sets and we obtain a natural isomorphism

$$Isom_{\mathfrak{X}}(X, Y) \cong \mathrm{Hom}(-, \mathfrak{Y}), (\phi: f^*X \to f^*Y) \mapsto (T, f^*X, f, (\mathrm{id}_{f^*X}, \phi))$$

of sheaves. Since $\mathfrak{S}_{\mathrm{Hom}(-,\mathfrak{Y})} \cong \mathfrak{Y}$ we are finished. □

D.2 2-fibre product

Definition. Let $\mathfrak{X}$ be a $(2,1)$-category. Let D be a diagram in $\mathfrak{X}$. The diagram D is said to be 2-commutative if

1. Its 1-morphisms commute up to 2-isomorphism.
2. The diagram D' having as vertices the 1-morphisms of D and as arrows the respective 2-isomorphisms between them is commutative.

Definition. Let $\mathfrak{X}$ be a $(2,1)$-category. Let $X, Y, Z \in \mathfrak{X}$ be objects and $f: X \to Z, g: Y \to Z$ 1-morphisms. An object T together with morphism $\pi_X: T \to X, \pi_Y: T \to Y$ is called a 2-fibre product of X, Y, Z if

$$\begin{array}{ccc}
T & \xrightarrow{\pi_Y} & Y \\
\downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle g} \\
X & \xrightarrow{f} & Z
\end{array}$$

is 2-commutative and if for every T' and maps $\pi'_X: T' \to X, \pi'_Y: T' \to Y$ such that the diagram

$$\begin{array}{ccc}
T' & \xrightarrow{\pi'_Y} & Y \\
\downarrow{\scriptstyle \pi'_X} & & \downarrow{\scriptstyle g} \\
X & \xrightarrow{f} & Z
\end{array}$$

is 2-commutative, there exists a morphism $\phi: T' \to T$ such that the diagram

$$\begin{tikzcd}
T' \arrow[dr, dotted, "\phi"] \arrow[drr, bend left, "\pi'_X"] \arrow[ddr, bend right, "\pi'_Y"'] & & \\
& T \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"] & Z
\end{tikzcd}$$

is 2-commutative and ϕ is unique up to 2-morphism.

Lemma D.9 ([11]). *Let $X \to Y$, $X' \to Y'$, $Z \to Y$ and $Z \to Y'$ be 1-morphisms in a 2-category $\mathfrak{X}$ over an object S. Then*

$$(X \times_S X') \times_{Y \times_S Y'} Z \cong (X \times Y Z) \times_Z (X' \times_{Y'} Z)$$

if the 2-fibre products appearing in these expressions exists. As a special case we obtain

$$(X \times_S X') \times_{Y \times_S Y} Y \cong (X' \times_Y X).$$

Proof. The assertion follows from the following diagram. The dashed arrows are the ones obtained by the universal property.

$$\begin{tikzcd}
(X \times_Y Z) \times_Z (X' \times_{Y'} Z) \arrow[d, dotted, leftrightarrow] \arrow[dr, dotted] \arrow[drr, dotted] \arrow[drr, bend left] \arrow[ddr] & & & & \\
(X \times_S X') \times_{Y \times_S Y'} Z \arrow[r] \arrow[rr, dotted, bend left] \arrow[drr, bend left] \arrow[dr, dotted] & X \times_S X' \arrow[rrr, bend left] \arrow[ddd, bend right] \arrow[ddrr, dotted] & X' \times_{Y'} Z \arrow[rr] \arrow[d] & & X' \arrow[dd] \\
& X \times_Y Z \arrow[r] \arrow[ddd] & Z \arrow[dr, dotted] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& & & Y \times_S Y' \arrow[r] \arrow[d] & Y' \arrow[d] \\
& X \arrow[rr] & & Y \arrow[r] & S
\end{tikzcd}$$

□

Lemma D.10. *The CFG $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Y}$ satisfies the universal property of the 2-fibre product in the categories of fibred categories. Moreover, $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}_S \cong \mathfrak{X}_S \times_{\mathfrak{Z}_S} \mathfrak{Y}_S$ and if $\mathfrak{X}, \mathfrak{Y}$ and $\mathfrak{Z}$ are CFGs, $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ is a CFG.*

Proof. We define the 2-morphism $\alpha: F \circ \pi_{\mathfrak{X}} \Rightarrow G \circ \pi_{\mathfrak{Y}}$ by $\alpha(S, X, Y, \phi) = \phi: F(X) \to F(Y)$. Hence

$$\begin{tikzcd}
\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y} \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "g"] \\
X \arrow[r, "f"] & Z
\end{tikzcd}$$

is 2-commutative. If $p: \mathfrak{T} \to \mathfrak{S}$, $\pi'_{\mathfrak{X}}: \mathfrak{T} \to \mathfrak{X}$, $\pi'_{\mathfrak{Y}}: \mathfrak{T} \to \mathfrak{Y}$, $\alpha': F \circ \pi'_{\mathfrak{X}} \Rightarrow \circ \pi'_{\mathfrak{Y}}$ is a test object define $\phi: \mathfrak{T} \to \mathfrak{X} \times \mathfrak{Z}\mathfrak{Y}$ as $\phi(T) = (p(T), \pi'_{\mathfrak{X}}(T), \pi'_{\mathfrak{Y}}(T), \alpha'(T))$.

If $\phi'(T) = (p(T), \pi''_{\mathfrak{X}}(T), \pi''_{\mathfrak{Y}}(T), \alpha''(T))$ is a second such morphism the isomorphisms

$$(a,b): (\pi''_{\mathfrak{X}}(T), \pi''_{\mathfrak{Y}}(T)) \to (\pi'_{\mathfrak{X}}(T), \pi'_{\mathfrak{Y}}(T))$$

induce a 2-morphism $(a,b): \phi' \Rightarrow \phi$. Thus $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ is indeed a 2-fibre product. By construction $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}_S \cong \mathfrak{X}_S \times_{\mathfrak{Z}_S} \mathfrak{Y}_S$.

It remains to check that $\mathfrak{X} \times \mathfrak{Z}\mathfrak{Y}$ has pullbacks.

Let (S, X, Y, ϕ) be an object over $S \in \mathfrak{S}$ and $f: T \to S$ a morphism in $\mathfrak{S}$. Define $f^*(S, X, Y, \phi) := (T, f^*X, f^*Y, f^*\phi)$, where $f^*\phi$ is yet to be defined. We have cartesian maps $a: f^*X \to X$ and $b: f^*Y \to Y$ lying in the commutative diagram

$$\begin{tikzcd}
F(f^*X) \arrow[r, dotted, "f^*\phi"] \arrow[d, "F(a)"] & G(f^*Y) \arrow[d, "G(b)"] \\
F(X) \arrow[r, "\phi"] & G(Y)
\end{tikzcd}$$

where $f^*\phi$ is the induced isomorphism.

Let $(a', b'): (T, X', Y', \phi') \to (S, X, Y, \phi)$ be a morphism lying over f. This gives us unique morphisms $c: X' \to f^*X$ and $d: Y' \to f^*Y$, which fit in the diagram

$$\begin{tikzcd}
F(X') \arrow[r, dotted, "\phi'"] \arrow[d, "F(c)"] \arrow[dd, bend right=60, "F(a')"'] & G(Y') \arrow[d, "G(c)"'] \arrow[dd, bend left=60, "F(b')"] \\
F(f^*X) \arrow[r, "f^*\phi"] \arrow[d, "F(a)"] & G(f^*Y) \arrow[d, "G(b)"'] \\
F(X) \arrow[r, "\phi"] & G(Y)
\end{tikzcd}$$

This diagram is commutative by the uniqueness of all the morphisms, hence we are finished.

By construction if $\mathfrak{X}, \mathfrak{Y}$ and $\mathfrak{Z}$ are CFGs the 2-fibre product is also a fibred groupoid and all the morphisms over a fibre are isomorphisms by construction. $\square$

E The stack of elliptic curves

In this section we will give an outline of the proof of the fact that the stack of elliptic curves previously introduced is in fact a stack, which is based on [16]. However, we will still skip some parts and refer to other literature, since the main goal of this section is to show where the difficulties in this proof lie. Recall the definition of a family of elliptic curves and morphisms between them.

Definition. Let S be any scheme and $p: E \to S$ a smooth, proper morphism. The family $p: E \to S$ is called an elliptic curve if there exists a section $e: S \to E$ and every geometric fibre is an elliptic curve.

The next two lemmata will be useful later on, when we proof that every family of elliptic curves has locally a Weierstrass form. For a proof see [16], Lemma 2.2, Lemma 2.5.

Lemma E.1. *If $k \subseteq L$ are both algebraically closed fields, then $E \times \mathrm{Spec}(k)$ is an elliptic curve if and only if $E \times \mathrm{Spec}(L)$ is an elliptic curve.*

Lemma E.2. *Let $E \to S$ be a family of elliptic curves over an affine scheme S. Than there exists a family of elliptic curves $E \to S'$ over a noetherian affine scheme S' and a map $S \to S'$ such that $E \to S$ is the pullback of $E' \to S'$.*

Definition. A morphism between two families of elliptic curves $E \to S$ and $E' \to S'$ with sections e and e' is a pair of morphisms $\alpha: E \to E'$ and $a: S \to S'$ such that

$$\begin{tikzcd} E \arrow[r, "\alpha"] \arrow[d] & E' \arrow[d] \\ S \arrow[r, "a"] & S' \end{tikzcd}$$

is cartesian and $\alpha \circ e = e' \circ a$.

Definition. The moduli stack of elliptic curves M_{ell} is the category over Sch where

1. The objects are $p: E \to S$ families of elliptic curves over some scheme S with section $e: S \to E$.
2. The morphisms between two families of elliptic curves are morphisms as described above.

Theorem E.3 ([16]). *The category M_{ell} is a stack in the $fpqc$ site on Sch.*

Proof. We will first prove that M_{ell} is a fpqc prestack. Let $\{f_i: U_i \to U\}_{i \in I}$ be a fpqc covering, $M_{ell}(U)$ the fibre of M_{ell} over U and $M_{ell}(\{f_i: U_i \to U\}_{i \in I})$ the corresponding category of descent data. To show that M_{ell} is a prestack we have to show that the functor induced by the pullbacks,

$$M_{ell}(U) \to M_{ell}(\{f_i: U_i \to U\}_{i \in I})$$

is fully faithful.

We first show faithfulness. Let (id, α) and (id, α') be two morphisms of families of elliptic curves $E \to U$, $E' \to U$ over U. Recall that the pullback functors $F_i^*: M_{ell}(U) \to M_{ell}(U_i)$ satisfy $f_i^*(\alpha) = f_i^*(\alpha')$. Hence $\alpha = \alpha'$ since the fpqc site is subcanonical.

To see that it is full, let $E \to U$ and $E' \to U$ be two families together with local morphisms

$$\alpha_i : f_i^* E \to f_i^* E'.$$

Let $\tau_i : f_i^* E \to E, \tau_i' : f_i^* E' \to E'$ be the morphisms induced by the pullback. Since fpqc is again a subcanonical site, we obtain a map $\alpha : E \to E'$ such that $\alpha \circ \tau_i = \tau_i' \circ \alpha_i$ and hence it is full.

To show that M_{ell} is a stack we need the following lemma.

Lemma E.4. *The prestack M_{ell} is a stack in the fpqc topology, if*

1. *M_{ell} is a stack in the Zariski topology.*
2. *For any flat surjective morphisms of affine schemes $V \to U$, the induced funtor $M_{ell}(U) \to M_{ell}(V \to U)$ is an equivalence.*

Proof. See [21], Lemma 4.25. □

We will start by showing that M_{ell} is a stack in the Zariski topology. Let Mor be the stack of morphisms of schemes in the Zariski topology. To see that this is indeed a stack see [21], Section 4.3 . So lets assume we are given local families $E_i \to S_i$ with isomorphisms of families of elliptic curves $\phi_{ij} : \pi_2^* E_j \to \pi_1^* E_i$. Since Mor is a stack, we obtain a map $E \to S$ compatible with the cover, which is smooth and proper, since these properties can be checked Zarisiki local. Similarly we obtain a section $e : S \to S$. Finally, we need to check that the geometric fibres are elliptic curves. Because every map $\mathrm{Spec}(k) \to S$ factors through some $S_i \to S$, we are done. This can therefore again be checked locally, which is given by assumption.

The second part of above lemma is the harder one to proof. For this step the existence of a section $e : E \to S$ is crucial, since without it, the gluing could lead to algebraic spaces and not schemes.

In order to proceed we need the following two lemmata.

Lemma E.5. *Let $E \to S$ be a family of elliptic curves with section $e : S \to E$. Then e is a closed immersion and induces a sheaf of ideals $\mathcal{J}$. If $E' \to S'$ is another family of elliptic curves with section $e' : S' \to E'$ with induced sheaf of ideals $\mathcal{J}'$ and $(\alpha, a) : (E \to S) \to (E' \to S')$ a morphism of families of elliptic curves, then $g^* \mathcal{J}' \cong \mathcal{J}$ and $\mathcal{J}^{-1}$ is an ample line bundle over S. Moreover, the isomorphisms are compatible with composition.*

Proof. See [16], Proposition 2.8. □

Lemma E.6. *The category Pol has objects $f : X \to Y, \mathcal{L}$, where f is a flat morphism of finite representation and $\mathcal{L}$ is a relatively ample line bundle on X.*

A morphism between two objects are morphisms (a, b, ϵ) *such that the diagram*

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"] & X' \arrow[d, "f'"] \\ Y \arrow[r, "b"] & Y' \end{tikzcd}$$

is cartesian and $\epsilon\colon a^*\mathcal{L} \to \mathcal{L}'$ *is an isomorphism of sheaves on* X'. *The category Pol is a stack w.r.t the fqpc topology.*

Proof. See [21], Theorem 4.35. $\square$

Let $f\colon V \to U$ be a morphism of affine schemes. To show that functor

$$M_{ell}(U) \to M_{ell}(V \to U)$$

is an equivalence it remains to show that it is dense. So let $E \to V$ be a family of elliptic curves with section $e\colon V \to E$, together with descent data $\alpha\colon \pi_2^* E \to_{1^*E}$ over $V \times_U V$. By the above two lemmata we have a 2-commutative diagram

$$\begin{tikzcd} M_{ell}(U) \arrow[r] \arrow[d] & M_{ell}(V \to U) \arrow[d] \\ Pol(U) \arrow[r] & Pol(V \to U) \end{tikzcd}$$

This gives us the existence of a scheme and a flat, proper morphism of finite representation $q\colon X \to U$ and an isomorphism $\sigma\colon f^*X \to U$. Moreover, X is compatible with the descent data, so all that misses is a section for q. We already have a map

$$V \to E \to f^*X \to X.$$

It is straightforward that this map is compatible with the descent data, thus there is a map $U \to X$, which is the desired section. Note that $q\colon X \to U$ is indeed smooth and proper, since every pullback of a faithfully and quasi-compact map is smooth.

Finally, we only need to show that $X \to U$ is indeed a family of elliptic curves. So let $\mathrm{Spec}(k)$ be a point. Let $V = \mathrm{Spec}(B), U = \mathrm{Spec}(A)$, and thus $\mathrm{Spec}(k) \times_U V \cong \mathrm{Spec}(B \otimes_A k)$. Let m be a maximal ideal of B and L be the closure of the quotient field of $B \otimes_A k/m$. Let E_L be the pullback of E along $\mathrm{Spec}(L) \to V$. Note that $E_L \cong X_k \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L)$ and therefore by Lemma E.1 X_k is a elliptic curve. $\square$

We will now show that M_{ell} is indeed a Deligne-Mumford stack, i.e. the diagonal is representable and there exists a smooth atlas of M_{ell}. This will be done by showing that locally, elliptic curves can be described by a Weierstrass equation. This already implies the above assertion, since the other part of the proof was already done in Section 3.5.

Lemma E.7. *Let R be a ring and consider the closed subscheme of $\mathbb{P}^2_R$ cut out by the equation*

$$y^2 + a_1xyz + a_3yz^2 = x^3 + a_2x^2z + a_4xz^2 + a_6z^3.$$

This scheme is a family of elliptic curves over $\mathrm{Spec}(R)$ *if*

$$\begin{gathered}\Delta = a_6a_1^6 + a_4a_3a_1^5 + ((-a_3^2 - 12a_6)a_2 + a_4^2)a_1^4 + \\ +(8a_4a_3a_2 + (a_3^3 + 36a_6a_3))a_1^3 + \\ +((-8a_3^2 - 48a_6)a_2^2 + 8a_4^2a_2 + (-30a_4a_3^2 + 72a_6a_4))a_1^2 + \\ +(16a_4a_3a_2^2 + (36a_3^3 + 144a_6a_3)a_2 - 96a_4^2a_3)a_1 + \\ +(-16a_3^2 - 64a_6)a_2^3 + 16a_4^2a_2^2 + (72a_4a_3^2 + 288a_6a_4)a_2 + \\ -27a_3^4 - 216a_6a_3^2 - 64a_4^3 - 432a_6^2\end{gathered}$$

is invertible in R.

Proof. See[20], Proposition III.1.4. □

Theorem E.8 ([16])**.** *Every family of elliptic curves has Zariski locally a Weierstrass form as above.*

Proof. Not all arguments in this proof are entirely layed out, for a more complete treatment we refer to [16]. By Lemma E.2 we can restrict ourselves to the case of a family of elliptic curves $p: E \to \mathrm{Spec}(R) = S, e: \mathrm{Spec}(R) \to E$, with R noetherian. Let $\mathcal{I}$ be the by e induced sheaf of ideals, $\mathcal{L} = \mathcal{I}^{-1}$ and consider the exact sequence

$$0 \to \mathcal{I} \to \mathscr{O}_E \to e_*\mathscr{O}_S \to 0.$$

Tensoring with $\mathcal{L}^{\otimes n+1}$ yields

$$0 \to \mathcal{L}^{\otimes n} \to \mathcal{L}^{\otimes n+1} p_*\mathcal{L}^{\otimes n} \to e_*\mathscr{O}_S \otimes \mathcal{L}^{\otimes n+1}.$$

We now want to show that $p_*\mathcal{L}^{\otimes n}$ is quasi coherent for $n \geq 1$. By [21], Lemma 4.37 it is enough to show that $H^1(E_k, \mathcal{L}_k^{\otimes n}) = 0$ for every algebraically closed field k, where E_k is the fibre of E over a point $\mathrm{Spec}(k)$ of S and $\mathcal{L}_k^{\otimes n}$ is the restriction of $\mathcal{L}^{\otimes n}$ to E_k. Since the canonical divisor of an elliptic curve is trivial, Serre duality gives us an isomorphism

$$H^1(E_k, \mathcal{L}_k^{\otimes n}) \cong H^0(E_k, \mathcal{L}_k^{\otimes -n}).$$

We can further identify $\mathcal{L}_k$ with $\mathscr{O}_{E_k}(e_{\mathrm{Spec}(k)})$ and therefore its negative powers have no global sections. Finally, $p_*\mathcal{L}^{\otimes n}$ has rank n by the Riemann-Roch theorem.

The next step is to show that the quotient of the map

$$p_*\mathcal{L}^{\otimes n} \to p_*\mathcal{L}^{\otimes n+1}$$

is an invertible sheaf. We can focus on the local case and hence assume that both sheaves are trivial. If $\mathrm{Spec}(A)$ is such an open covering, we obtain an exact sequence

$$0 \to A^n \to A^{n+1} \to M \to 0,$$

where M is the A-module corresponding to the quotient. Pulling back to another scheme $\mathrm{Spec}(B)$, we obtain a different family of elliptic curves and a analogous short exact sequence. By Lemma E.5 the formation of $p_*\mathcal{L}^{\otimes n}$ commutes with base change and therefore $\mathrm{Tor}_A^1(M, N) = 0$ for all A-modules N. Thus M is flat and, using that M is finitely generated over A and A is noetherian, we conclude that $M = A$, see [15], Lemma 10.77.2. From now on we restrict S such that

$$p_*\mathcal{L}, p_*\mathcal{L}^{\otimes 2}, p_*\mathcal{L}^{\otimes 3}, p_*\mathcal{L}^{\otimes 6}$$

and

$$p_*\mathcal{L}^{\otimes 2}/p_*\mathcal{L}, p_*\mathcal{L}^{\otimes 3}/p_*\mathcal{L}^{\otimes 2}$$

are trivial over $\mathrm{Spec}(A)$. We can thus choose a basis $1, x$ for $p_*\mathcal{L}^{\otimes 2}$ and a basis $1, x, y$ for $p_*\mathcal{L}^{\otimes 3}$ by above observations. Therefore $1, x, y, x^2, xy, y^2, x^3$ are elements of $p_*\mathcal{L}^{\otimes 6}$ and induce a map $f\colon A^7 \to A^6$. We can now show that x and y have to satisfy the relation

$$\alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^3 = 0$$

with $\alpha_6\alpha_7 \neq 0$ after a base change to an algebraically closed field k. The rough argument goes as follows. Using the Riemann-Roch theorem, one shows that all divisors of degree 3 are ample and give an embedding $E_k \hookrightarrow \mathbb{P}_k^2$. So let P be the point on E_k given by e. By the Riemann-Roch theorem we know that $\dim_k H^0(\mathcal{O}_{E_k}(6P)) = 6$ and since $1, x, y, x^2, xy, y^2, x^3$ all are elements of $6P$, we have a linear relation between them. Moreover, $\alpha_6\alpha_7 \neq 0$, since only y^2 and x^3 have a pole of order 6 at P.

Any other relation would imply $\alpha_6\alpha_7 = 0$, thus the map is surjective over every such k. Using Nakayama's lemma we conclude that f is surjective. Since α_7 is always a unit after a base change to a field k, we will assume from now on that $\alpha_7 = 1$.

We can now define a closed immersion $j\colon E_A = E \times_S \mathrm{Spec}(A) \to \mathbb{P}_A^2$ given by above equation, i.e.

$$F = \alpha_1 z^3 + \alpha_2 xz^2 + \alpha_3 yz^2 + \alpha_4 x^2 z + \alpha_5 xyz + \alpha_6 y^2 z + \alpha_7 x^3.$$

For the exact argument see [16], Lemma 4.5. Let $V(F) = \mathrm{Proj}(A[x, y, z]/F)$. We will now show that $E_A \cong V(F)$ by showing that the sheaf of ideals $\mathcal{J}$ in the exact sequence

$$0 \to \mathcal{J} \to \mathcal{O}_{V(F)} \to j_*\mathcal{O}_{E_A} \to 0$$

vanishes. On suitable affine opens we write the sequence as a short exact sequence of A-modules

$$0 \to J \to S \to N \to 0,$$

where S is a finitely generated A-algebra. Will first show that the pullback $\mathcal{J}_k$ of $\mathcal{J}$ along a base change with an algebraically closed field k vanishes. The resulting exact sequence is

$$0 \to \mathcal{J}_k \to \mathcal{O}_{V(f_k)} \to j_{k*}\mathcal{O}_{E_k} \to 0.$$

It can be shown that the inclusion j_k given by the sections $1, x, y \in \mathcal{L}_k^{\otimes 3}$ is an isomorphism over k, see [16], Lemma 4.5.

Thus we know that $J \otimes_A k = 0$ for all algebraically closed fields k, and therefore all fields k, and morphisms $A \to k$. So let K be a field and $S \to K$ a morphisms. This induces a morphism $A \to K$, thus $J \otimes_A K = 0$ and since $J \otimes_S K$ is a quotient of $J \otimes_A K$, $J \otimes_S K$ also vanishes. Letting $p \subseteq S$ be any prime ideal and setting $K = S_p / pS_p$, we conclude by Nakayama's lemma that $J = 0$.

To finish the proof, we consider the section of $V(F)$ given by $[0:1:0]$, i.e. the section induced by the ring map

$$\begin{aligned} A[x,y,z]/(F, y-1) &\to A \\ x &\mapsto 0 \\ z &\mapsto 0 \end{aligned}$$

By carefully checking the construction of the isomorphism $E_A \to V(F)$, one can show that it maps the section of E_A to the above section. Hence the proof is complete. $\square$

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