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Abstract

The approximation of functions is a well established problem in numerical mathematics with a wide range of theory and applications. But instead of approximating functions of a function space, we are concerned with the more general problem of approximating vectors and inner products of a Hilbert Space. By assuming the existence of a closed linear subspace of a Hilbert Space for which the *Marcinkiewicz-Zygmund inequalities* hold, the corresponding approximation theorems can be derived in an elementary way, with special attention being paid to general Sobolev Spaces. Furthermore the behaviour of the underlying *frame*, with respect to the MZ-inequalities and different inner products, is analysed and afterwards extended to Riesz bases and orthonormal bases.

Abstrakt

Die Approximation von Funktionen ist ein allgegenwärtiges Problem in der numerischen Mathematik mit einem breiten Spektrum an Theorie und Anwendungen. Aber anstatt Funktionen eines Funktionenraumes zu approximieren, befassen wir uns mit dem allgemeineren Problem der Approximation von Vektoren eines Hilbertraumes. Durch die Annahme der Existenz eines abgeschlossenen linearen Teilraums eines Hilbertraumes für welchen die *Marcinkiewicz-Zygmund Ungleichungen* gelten, können die dazugehörigen Approximationssätze hergeleitet werden. Allgemeinen Sobolevräumen wird dabei besondere Aufmerksamkeit geschenkt. Darüber hinaus wird das Verhalten des zugrundeliegendem *Frames* in Bezug auf die MZ-Ungleichungen und verschiedene innere Produkte analysiert und anschließend auf Rieszbasen und orthonormale Basen ausgeweitet.

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Chapter 1

Introduction

A classical problem in applied and numerical mathematics is the approximation of functions of a certain space by simpler functions. This problem is a well known issue for a wide range of applications. Due to practical aspects, most literature and articles only make statements about approximation in special function spaces [1, 6, 8]. The recent paper [8] shows that approximation theorems follow automatically from the so-called *Marcinkiewicz-Zygmund inequalities*, i.e. for all functions f of a closed linear subspace Π of the underlying function space there exist constants $0 < A \leq B$ such that

$$A\|f\|_1^2 \leq \|f\|_2^2 \leq B\|f\|_1^2. \quad (1.1)$$

where $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on Π .

By (1.1) and taking a closer look at the results of Gröchenig [8, Theorem 3.5], it can be concluded that the necessary structure to derive approximation theorems is provided by a Hilbert Space. Hence the task of approximating functions in a function space can be generalized to approximating vectors in a Hilbert Space. A common way for further analysis would be to assume that there exists a basis for the Hilbert Space. Caused by practical reasons [3, Chapter 5] the use of basis is replaced by the more general concept of *frames*.

Since Hilbert Spaces are used for the generalization, it is natural to ask for an approximation of inner products. Therefore the goal of this thesis is to show that the MZ-inequalities provide the necessary assumption to formulate approximation theorems for inner products.

After this chapter, we give a short overview of Riesz bases and frames with basic statements in Chapter 2 and introduce the MZ-inequalities. Under the assumption that the MZ-inequalities hold, Chapter 3 provides results about the behaviours of Riesz bases and frames for different inner products. In Chapter 4 we prove the resulting approximation theorems for functions and inner products with respect to two different

settings. Chapter 5 is dedicated to analyse the so-called *worst case error* and compare the results from the present work with other popular theorems.

Chapter 2

Preliminaries

This chapter focuses on definitions and basic concepts that form the basis for deriving approximation theorems. The main goal of this chapter is to introduce the so-called *Marcinkiewicz-Zygmund inequalities* which are the key ingredients for further analysis and error estimation. Because such inequalities are closely related to frames, which also provide the necessary structure for the underlying setting, let us first give a short overview of frames and Riesz bases.

2.1 Frames and bases

Frames have established themselves as a very useful generalization for bases in applied mathematics. The popularity of frames has increased in recent decades due to the wide range of applications in applied and pure mathematics [2, 7, 9]. Therefore the goal of this section is to first highlight the connection between orthonormal bases and Riesz bases and then motivate the generalization of bases to frames. The following definitions and theorems are excerpts from [3] which also contains the proofs for interested readers.

Definition 2.1 (Basis [3, p. 68]). *Let B be a Banach Space and J be a countable index set. A sequence of vectors $\{\phi_j\}_{j \in J}$ in B is a (Schauder) basis for B if for every $f \in B$ there exist unique scalar coefficients $\{c_j(f)\}_{j \in J}$ such that*

$$f = \sum_{j \in J} c_j(f) \phi_j. \quad (2.1)$$

Definition 2.2 (Orthonormal system and orthonormal basis [3, p. 79]). *Let $\mathcal{H}$ be a Hilbert Space with inner product $\langle \cdot, \cdot \rangle$ and J be a countable index set. A sequence of vectors $\{\phi_j\}_{j \in J}$ in $\mathcal{H}$ is an orthonormal system if*

$$\langle \phi_j, \phi_k \rangle = \delta_{j,k}.$$

An orthonormal basis is an orthonormal system $\{\phi_j\}_{j \in J}$ which is a basis for $\mathcal{H}$.

Notice that (2.1) guarantees the convergence of the series with respect to the chosen order of elements [3]. Furthermore if an orthonormal basis has been found, all the other orthonormal bases can be characterized by the following theorem.

Theorem 2.3 (Characterization of orthonormal bases [3, p. 81]). *Let $\{\phi_j\}_{j \in J}$ be an orthonormal basis for $\mathcal{H}$. Then the orthonormal bases for $\mathcal{H}$ are precisely the sets $\{R\phi_j\}_{j \in J}$ where $R : \mathcal{H} \rightarrow \mathcal{H}$ is a unitary operator.*

Theorem 2.3 shows how all orthonormal bases are related to each other and by weakening the condition on the operator R one can define the so-called Riesz bases.

Definition 2.4 (Riesz Basis [3, p. 86]). *A Riesz basis for a Hilbert Space $\mathcal{H}$ is a family of the form $\{R\phi_j\}_{j \in J}$, where $\{\phi_j\}_{j \in J}$ is an orthonormal basis for $\mathcal{H}$ and $R : \mathcal{H} \rightarrow \mathcal{H}$ is a bounded bijective operator.*

Because in general it is hard to show that a given sequence is a Riesz basis just by using Definition 2.4, the following theorem provides a more practical equivalent condition.

Theorem 2.5 (Characterization of a Riesz basis [3, p. 89]). *For a sequence $\{\phi_j\}_{j \in J}$ in $\mathcal{H}$, the following conditions are equivalent:*

- (i) $\{\phi_j\}_{j \in J}$ is a Riesz basis in $\mathcal{H}$.
- (ii) $\{\phi_j\}_{j \in J}$ is complete in $\mathcal{H}$ (i.e. $\overline{\text{span}}\{\phi_j\}_{j \in J} = \mathcal{H}$), and there exist constants $0 < A \leq B$ such that for every $c \in \ell_2(J)$

$$A\|c\|_{\ell_2(J)} \leq \left\| \sum_{j \in J} c_j \phi_j \right\|^2 \leq B\|c\|_{\ell_2(J)}. \quad (2.2)$$

The numbers A, B are called Riesz basis bounds.

Note that condition (ii) from Theorem 2.5 is sometimes used from other authors as a definition for Riesz bases [11]. Nevertheless, with Definition 2.4 the step from orthonormal bases to Riesz base is more natural and provides information about which conditions are weakened for generalization. Notice if $\{\phi_j\}_{j \in J}$ is an orthonormal basis, then $A = B = 1$ and vice versa.

Since Definition 2.4 does not immediately indicate that a Riesz basis is actually a basis, the next theorem provides the natural expansion property.

Theorem 2.6 (Riesz basis expansion [3, p. 86]). *If $\{\phi_j\}_{j \in J}$ is a Riesz basis for $\mathcal{H}$, then there exists a unique sequence $\{\xi_j\}_{j \in J}$ in $\mathcal{H}$ such that*

$$f = \sum_{j \in J} \langle f, \xi_j \rangle \phi_j \quad \forall f \in \mathcal{H}.$$

In fact the conditions for being a basis are sometimes so strong that it is almost impossible to construct bases with certain properties. Another issue is that slight modifications of a basis might destroy the basis properties [3, p. 109]. Therefore the concept of bases is generalized to the so-called frames which are much easier to handle and open up a whole new field of theory [5, 10].

Definition 2.7 (Frame, tight frame and Parseval frame [3, Chapter 5]). *Let V be a vector space with sesquilinear product $\langle \cdot, \cdot \rangle$ which is linear in the first component. A countable set of elements $\{\phi_j\}_{j \in J}$ in V is a frame for V if there exist constants $0 < A \leq B$ such that*

$$A\|f\|^2 \leq \sum_{j \in J} |\langle f, \phi_j \rangle|^2 \leq B\|f\|^2 \quad \forall f \in V. \quad (2.3)$$

The numbers A, B are called frame bounds. A frame is tight if $A = B$. If $A = B = 1$ a frame is called a Parseval frame.

Because [8] motivates an underlying Hilbert Space, the following statements and definitions are also made for Hilbert spaces. Further, since frames are only countable sets of elements, we restrict ourself to separable Hilbert spaces and assume that J is a countable index set.

One can suggest from the Inequalities (2.2) in Theorem 2.5 and Definition 2.7, that Riesz bases are related to frames. This relationship is established by the following theorem.

Theorem 2.8 (Riesz basis is frame [3, p. 130]). *A Riesz basis $\{\phi_j\}_{j \in J}$ for $\mathcal{H}$ is a frame for $\mathcal{H}$, and the Riesz basis bounds coincide with the frame bounds.*

Consequently statements with respect to frames can also be formalized for Riesz bases and even stronger results are possible. The fact that frames are actually a generalization of bases is provided by the next proposition.

Proposition 2.9 (Frames are generalized bases [3]). *Let $\mathcal{H}$ be a separable Hilbert Space. If $\{\phi_j\}_{j \in J}$ is a frame for $\mathcal{H}$ then $\overline{\text{span}}\{\phi_j\}_{j \in J} = \mathcal{H}$.*

Proof. Assume that $\overline{\text{span}}\{\phi_j\}_{j \in J} \neq \mathcal{H}$. It follows that $\overline{\text{span}}\{\phi_j\}_{j \in J} \subset \mathcal{H}$ and there exists a non-zero element $f \in (\overline{\text{span}}\{\phi_j\}_{j \in J})^\perp$. Hence for all $j \in J$ holds that $\langle f, \phi_j \rangle = 0$. Thus the lower frame bound A has to be zero and $\{\phi_j\}_{j \in J}$ cannot be a frame for $\mathcal{H}$. $\square$

The main difference to bases is that the elements of a frame need not be linear independent, which means that the representation of an element $f \in \mathcal{H}$ with the frame

elements ϕ_j is not unique. Therefore frames can be seen as an “overcomplete basis” and justifies the generalization.

Talking about frames, there are three main operators associated to frames which determine the analysis and reconstruction operation.

Definition 2.10 (Common operators in frame theory).

Let $\{\phi_j\}_{j \in J}$ be a frame for the separable Hilbert Space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$.

The synthesis operator $T : \ell_2(J) \rightarrow \mathcal{H}$ is defined as

$$Tc := \sum_{j \in J} c_j \phi_j. \quad (2.4)$$

The adjoint operator $T^* : \mathcal{H} \rightarrow \ell_2(J)$ is called analysis operator and reads

$$T^*f = \left\{ \langle f, \phi_j \rangle \right\}_{j \in J}. \quad (2.5)$$

The frame operator $S : \mathcal{H} \rightarrow \mathcal{H}$ is defined as

$$Sf := TT^*f = \sum_{j \in J} \langle f, \phi_j \rangle \phi_j. \quad (2.6)$$

In fact direct calculation for $c \in \ell_2(J)$ and $f \in \mathcal{H}$ leads to

$$\langle Tc, f \rangle = \sum_{j \in J} c_j \langle \phi_j, f \rangle = \sum_{j \in J} c_j \overline{\langle f, \phi_j \rangle} = \langle c, T^*f \rangle_{\ell_2(J)}$$

and hence the adjoint operator of (2.4) is the operator (2.5). Consequently the frame operator as a concatenation of the analysis and synthesis operator is selfadjoint and provides an important reconstruction property.

2.2 Marcinkiewicz-Zygmund inequalities

For quadrature rules and approximation theorems *Marcinkiewicz-Zygmund inequalities* (MZ-inequalities) and *Marcinkiewicz-Zygmund families* (MZ-families) are common tools for error analysis and estimation. The main difference between MZ-inequalities and MZ-families is that MZ-inequalities focus on a single subspace, whereas the concept of MZ-families is about families of increasing subspaces. It turns out that the existence of a MZ-family in L^2 provides quadrature rules and approximation theorems “for free” as pointed out in [8].

Definition 2.11 (Marcinkiewicz-Zygmund inequalities). *Let $\mathcal{H}_1$ be a Hilbert Space with inner product $\langle \cdot, \cdot \rangle_1$ and induced norm $\|\cdot\|_1$ and H_2 be a vector space with positive semi-definite sesquilinear product $\langle \cdot, \cdot \rangle_2$ that is linear in the first component and induced semi-norm $\|\cdot\|_2$. Further let $\Pi \subset \mathcal{H}_1 \cap H_2$ be a closed subspace. If there exist constants $0 < A \leq B$ such that the Inequalities*

$$A\|f\|_1^2 \leq \|f\|_2^2 \leq B\|f\|_1^2 \quad (2.7)$$

for all $f \in \Pi$ are fulfilled, these inequalities are called Marcinkiewicz-Zygmund inequalities for Π with respect to $\mathcal{H}_1$ and H_2 .

In terms of MZ-families the Inequalities (2.7) from Definition 2.11 holds for every subset of the corresponding family of increasing subspaces with the same constants A, B . Also notice that the semi-norm $\|\cdot\|_2$ is actually a norm on Π and the Inequalities (2.7) coincides with the fact that the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent on Π . If the Hilbert Space $\mathcal{H}_1$ is finite dimensional, also Π is finite dimensional and therefore any two norms are equivalent on Π . This conclusion cannot be applied to the infinitely dimensional case.

Since Definition 2.11 only makes demands on functions from the closed subspace Π , the focus for further analysis is on this subspace. A common way would be to look at a basis of Π . However, due to practical reasons [3, Chapter 5], the concept of bases is replaced by the more general framework of Riesz bases and frames from Section 2.1.

Now that we have defined the MZ-inequalities and built up a basic understanding for frames, the two main ingredients to derive the theory are given. So the next chapter is dedicated to the behaviour of frames and Riesz bases with respect to different inner products and by assuming that the MZ-inequalities hold.

Chapter 3

Marcinkiewicz-Zygmund assumption

Let us denote $\mathcal{H}_1$ as a Hilbert Space with inner product $\langle \cdot, \cdot \rangle_1$ and H_2 as a vector space with corresponding positive semi-definite sesquilinear product $\langle \cdot, \cdot \rangle_2$ which is linear in the first component. The *Marcinkiewicz-Zygmund assumption* says that for every f from a closed subspace $\Pi \subset \mathcal{H}_1 \cap H_2$ the MZ-inequalities hold, i.e. there exist constants $0 < A \leq B$ such that for all $f \in \Pi$ the inequalities

$$A\|f\|_1^2 \leq \|f\|_2^2 \leq B\|f\|_1^2 \quad (3.1)$$

are fulfilled. Additionally assume that $\{\phi_j\}_{j \in J}$ (J countable) is a frame for Π with respect to the inner product $\langle \cdot, \cdot \rangle_1$ and with frame bounds $0 < A' \leq B'$. Further let us define $\Pi_i := (\Pi, \langle \cdot, \cdot \rangle_i)$ for $i = 1, 2$ and the associated synthesis operators with respect to the spaces Π_1 and Π_2

$$\begin{aligned} U_1 : \ell_2(J) &\rightarrow \Pi_1 & U_1 c &:= \sum_{j \in J} c_j \phi_j \\ U_2 : \ell_2(J) &\rightarrow \Pi_2 & U_2 c &:= \sum_{j \in J} c_j \phi_j. \end{aligned}$$

For $c \in \ell_2(J)$, $f \in \Pi_1$ and $g \in \Pi_2$ the equalities

$$\begin{aligned} \langle U_1 c, f \rangle_1 &= \sum_{j \in J} c_j \langle \phi_j, f \rangle_1 = \langle c, U_1^* f \rangle_{\ell_2(J)} \\ \langle U_2 c, g \rangle_2 &= \sum_{j \in J} c_j \langle \phi_j, g \rangle_2 = \langle c, U_2^* g \rangle_{\ell_2(J)} \end{aligned}$$

hold and therefore the analysis operators for U_1 and U_2 , that are just the adjoint operators, read

$$\begin{aligned} U_1^* : \Pi_1 &\rightarrow \ell_2(J) & U_1^* f &= \left\{ \langle f, \phi_j \rangle_1 \right\}_{j \in J} \\ U_2^* : \Pi_2 &\rightarrow \ell_2(J) & U_2^* g &= \left\{ \langle g, \phi_j \rangle_2 \right\}_{j \in J}. \end{aligned}$$

Hence the associated frame operators S_1, S_2 , which are the concatenation of the corresponding analysis and synthesis operators, are defined as

$$\begin{aligned} S_1 : \Pi_1 &\rightarrow \Pi_1 & S_1 f &:= U_1 U_1^* = \sum_{j \in J} \langle f, \phi_j \rangle_1 \phi_j \\ S_2 : \Pi_2 &\rightarrow \Pi_2 & S_2 g &:= U_2 U_2^* = \sum_{j \in J} \langle g, \phi_j \rangle_2 \phi_j. \end{aligned}$$

Note that we called the operator S_2 the associated frame operator, which in turn would suggest that $\{\phi_j\}_{j \in J}$ also represents a frame for Π with respect to the sesquilinear product $\langle \cdot, \cdot \rangle_2$. This claim will be checked in the next section and statements about the associated frame bounds are going to be specified.

3.1 Frame behaviour for different inner products

In the most general situation we assume that $\{\phi_j\}_{j \in J}$ is a frame for Π_1 . As already mentioned this frame is also a frame for Π_2 and the following theorem provides information about the corresponding frame bounds.

Theorem 3.1. *Let $\mathcal{H}_1$ be a Hilbert Space, H_2 be a vector space and $\Pi \subset \mathcal{H}_1 \cap H_2$ closed such that (3.1) is satisfied. If $\{\phi_j\}_{j \in J}$ (J countable) is a frame for Π_1 with frame bounds $0 < A' \leq B'$, then $\{\phi_j\}_{j \in J}$ is a frame for Π_2 with frame bounds $0 < AA' \leq BB'$. Further the associated frame operators S_1, S_2 are boundedly invertible.*

Proof. Since $\{\phi_j\}_{j \in J}$ is a frame for Π_1 with frame bounds A' and B' we have

$$A' \|f\|_1^2 \leq \langle S_1 f, f \rangle_1 \leq B' \|f\|_1$$

for all $f \in \Pi$. Therefore the selfadjoint operator S_1 is bounded and

$$\|I - \frac{1}{B'} S_1\| = \sup_{\|f\|_1=1} |\langle (I - \frac{1}{B'} S_1) f, f \rangle_1| \leq \frac{B' - A'}{B'} < 1$$

where I is the identity operator on Π_1 . Hence by Theorem A.1 the operator S_1 is

boundedly invertible and

$$\|U_1\| = \|U_1^*\| = \sqrt{\|S_1\|} \leq \sqrt{B'}. \quad (3.2)$$

Next let $f, g \in \Pi$ and since the inequalities

$$\begin{aligned} \langle f, g \rangle_2 &\leq \|f\|_2 \|g\|_2 \leq B \|f\|_1 \|g\|_1 \\ \langle f, f \rangle_2 &\geq A \|f\|_1^2 \end{aligned}$$

hold the sesquilinear product $\langle f, g \rangle_2$ is continuous and coercive in Π_1 . By the lemma of Lax-Milgram there exists a unique invertible operator $L : \Pi_1 \rightarrow \Pi_1$ such that $\langle f, g \rangle_2 = \langle f, Lg \rangle_1$. Direct calculations yield

$$\langle f, S_2 g \rangle_1 = \left\langle f, \sum_{j \in J} \langle g, \phi_j \rangle_2 \phi_j \right\rangle_1 = \left\langle \sum_{j \in J} \langle f, \phi_j \rangle_1 \phi_j, g \right\rangle_2 = \langle S_1 f, g \rangle_2 \quad (3.3)$$

and because S_1 is invertible, we can conclude that the operator $S_1^{-1} S_2$ is the unique invertible operator L from the lemma of Lax-Milgram. Since S_1 and $S_1^{-1} S_2$ are invertible, also $S_2 = S_1 (S_1^{-1} S_2)$ is invertible. Using (3.3) and by choosing $g' = S_2^{-1} g \in \Pi$ it follows that

$$\langle S_1^{-1} f, g \rangle_1 = \langle S_1^{-1} f, S_2 g' \rangle_1 = \langle S_1 S_1^{-1} f, g' \rangle_2 = \langle f, g' \rangle_2 = \langle f, S_2^{-1} g \rangle_2. \quad (3.4)$$

Due to the results of (3.1), (3.2) and (3.4) we get

$$\begin{aligned} \|S_2 f\|_2^2 &= \|U_2 U_2^* f\|_2^2 \leq B \|U_1 U_2^* f\|_1^2 \\ &\leq B B' \|U_2^* f\|_{\ell_2(J)}^2 = B B' \langle S_2 f, f \rangle_2 \\ &\leq B B' \|S_2 f\|_2 \|f\|_2 \end{aligned}$$

and

$$\|S_2^{-1}\| = \sup_{\|f\|_2=1} \langle S_2^{-1} f, f \rangle_2 = \sup_{\|f\|_2=1} \langle f, S_1^{-1} f \rangle_1 \leq \sup_{\|f\|_2=1} \frac{1}{A'} \|f\|_1^2 \leq \frac{1}{A A'}.$$

Therefore the resulting bounds for the operator norm $\|S_2\|$ are $A A' \leq \|S_2\| \leq B B'$. Hence the inequality

$$A A' \|f\|_2^2 \leq \langle f, S_2 f \rangle_2 \leq B B' \|f\|_2^2$$

holds and $\{\phi_j\}_{j \in J}$ is a frame for Π_2 with frame bounds $A A'$ and $B B'$. $\square$

By assuming that $\{\phi_j\}_{j \in J}$ is a Parseval frame for Π_1 , the following corollary is an immediate consequence of Theorem 3.1.

Corollary 3.2. *Let $\mathcal{H}_1$ be a Hilbert Space, H_2 be a vector space and $\Pi \subset \mathcal{H}_1 \cap H_2$ closed such that (3.1) is satisfied. If $\{\phi_j\}_{j \in J}$ (J countable) is a Parseval frame for Π_1 , then $\{\phi_j\}_{j \in J}$ is a frame for Π_2 with frame bounds $0 < A \leq B$. Further the associated frame operators S_1, S_2 are boundedly invertible and S_1 is equal to the identity operator on Π_1 .*

Proof. Because a Parseval frame is a frame with frame bounds $A' = B' = 1$, Theorem 3.1 can be applied. Hence $\{\phi_j\}_{j \in J}$ is a frame for Π_2 with frame bounds A, B and the frame operators S_1, S_2 are boundedly invertible. Since $\{\phi_j\}_{j \in J}$ is a Parseval frame for Π_1 we observe that for all $f \in \Pi$

$$\langle S_1 f, f \rangle_1 = \sum_{j \in J} |\langle f, \phi_j \rangle_1|^2 = \|f\|_1^2 = \langle f, f \rangle_1$$

which concludes the proof. $\square$

These statements can also be formulated for Riesz bases, which are in terms of generality closer to orthonormal bases than frames and will be covered in the next section.

3.2 Riesz basis behaviour for different inner products

Consider $\{\phi_j\}_{j \in J}$ as a Riesz basis for Π_1 . Hence $\{\phi_j\}_{j \in J}$ is also a frame for Π_1 and Theorem 3.1 can be applied. However, even better results can be achieved, as the following theorem clarifies.

Theorem 3.3. *Let $\mathcal{H}_1$ be a Hilbert Space, H_2 be a vector space and $\Pi \subset \mathcal{H}_1 \cap H_2$ closed such that (3.1) is satisfied. If $\{\phi_j\}_{j \in J}$ (J countable) is a Riesz basis for Π_1 with bounds $0 < A' \leq B'$, then $\{\phi_j\}_{j \in J}$ is a Riesz basis for Π_2 with bounds $0 < AA' \leq BB'$. Further the operators S_1, S_2 are boundedly invertible.*

Proof. Due to Theorem 2.8 and Theorem 3.1 the operators S_1, S_2 are boundedly invertible. Since $\{\phi_j\}_{j \in J}$ is a frame for Π_2 , by proposition 2.9 the sequence $\{\phi_j\}_{j \in J}$ is complete in Π_2 . Because $\{\phi_j\}_{j \in J}$ is a Riesz basis for Π_1 with bounds A' and B' , for

all $c \in \ell_2(J)$ the inequalities

$$\begin{aligned} \left\| \sum_{j \in J} c_j \phi_j \right\|_2^2 &\leq B \left\| \sum_{j \in J} c_j \phi_j \right\|_1^2 \leq BB' \|c\|_{\ell_2(J)}^2 \\ \left\| \sum_{j \in J} c_j \phi_j \right\|_2^2 &\geq A \left\| \sum_{j \in J} c_j \phi_j \right\|_1^2 \geq AA' \|c\|_{\ell_2(J)}^2 \end{aligned}$$

hold and by using Theorem 2.5, $\{\phi_j\}_{j \in J}$ is a Riesz basis for Π_2 with bounds AA' and BB' . $\square$

The following corollary is an immediate consequence from Theorem 3.3 for orthonormal bases.

Corollary 3.4. *Let $\mathcal{H}_1$ be a Hilbert Space, H_2 be a vector space and $\Pi \subset \mathcal{H}_1 \cap H_2$ closed such that (3.1) is satisfied. If $\{\phi_j\}_{j \in J}$ (J countable) is an orthonormal basis for Π_1 , then $\{\phi_j\}_{j \in J}$ is a Riesz basis for Π_2 with bounds $0 < A \leq B$. Further the operators S_1 , S_2 and $T_2 := U_2^* U_2$ are boundedly invertible and S_1 is equal to the identity operator on Π_1 .*

Proof. Because $\{\phi_j\}_{j \in J}$ is an orthonormal basis for Π_1 , it is also a Riesz basis with bounds $A = B = 1$ and due to Theorem 3.3, $\{\phi_j\}_{j \in J}$ is a Riesz basis for Π_2 with bounds A, B . Further the operators S_1, S_2 are bounded invertible. Since $\{\phi_j\}_{j \in J}$ is an orthonormal basis for Π_1 , the operator S_1 is the identity operator on Π_1 . Further the operators U_2 and U_2^* are invertible and for $f \in \Pi$,

$$\|U_2^* f\|_{\ell_2(J)}^2 = \sum_{j \in J} |\langle f, \phi_j \rangle_2|^2 = \langle S_2 f, f \rangle_2.$$

Therefore T_2 is also invertible and because

$$\begin{aligned} A &\leq \|U_2\|^2 = \|U_2^*\|^2 = \|T_2\| \leq B \\ B^{-1} &\leq \|T_2^{-1}\| \leq A^{-1}. \end{aligned}$$

the operator $T_2 := U_2^* U_2$ is boundedly invertible. $\square$

Chapter 4

General approximation theorems

This chapter is devoted to a general derivation of approximation theorems for vectors and inner products, by keeping the underlying setting as general as possible and assuming that the MZ-inequalities hold.

Before we start to analyse the general theory for Hilbert spaces, let us consider the following function space setting to motivate the upcoming definitions and theorems. The example is also intended to provide an indication of how to proceed for the generalization and what assumptions have to be made such that meaningful results can be achieved.

Example 4.1. *Let σ be a Borel probability measure on an arbitrary space $\mathbb{X}$. Further let $\Pi \subset L^2(\mathbb{X}, \sigma)$ be a closed linear subset and assume that $\mathbf{1} \in \Pi \subset \mathcal{C}(\mathbb{X}) \subset L^2(\mathbb{X}, \sigma)$ where $\mathbf{1}$ is the constant function with value 1. Suppose that there is another finite Borel measure ν on $\mathbb{X}$ with two constants $0 < A \leq B$ such that*

$$A\|f\|_{\sigma}^2 \leq \|f\|_{\nu}^2 \leq B\|f\|_{\sigma}^2. \quad (4.1)$$

for all $f \in \Pi$ where

$$\langle f, g \rangle_{\sigma} := \langle f, g \rangle_{L^2(\mathbb{X}, \sigma)} = \int_{\mathbb{X}} f \bar{g} d\sigma \quad \langle f, g \rangle_{\nu} := \int_{\mathbb{X}} f \bar{g} d\nu. \quad (4.2)$$

Note that (4.1) implies that $\langle \cdot, \cdot \rangle_{\nu}$ is an inner product on Π . Due to this notation we are able to write integrals with respect to σ or ν as $\langle f, \mathbf{1} \rangle_{\sigma} = \int_{\mathbb{X}} f d\sigma$ and $\langle f, \mathbf{1} \rangle_{\nu} = \int_{\mathbb{X}} f d\nu$ with $\mathbf{1} \in \Pi$. Next let us denote the spaces Π equipped with one of the inner product as

$$\Pi_{\sigma} := (\Pi, \langle \cdot, \cdot \rangle_{\sigma}) \quad \Pi_{\nu} := (\Pi, \langle \cdot, \cdot \rangle_{\nu})$$

and assume that $\{\phi_j\}_{j \in J}$ is a frame for Π_{σ} with frame bounds $0 < A' \leq B'$ and $J \subset \mathbb{N}$.

Due to Theorem 3.1 $\{\phi_j\}_{j \in J}$ is a frame for Π_ν and the operator

$$S_\nu : \Pi_\nu \rightarrow \Pi_\nu \quad S_\nu f := \sum_{j \in J} \langle f, \phi_j \rangle_\nu \phi_j$$

is boundedly invertible. Hence for $f \in \mathcal{C}(\mathbb{X})$ direct calculation and (3.3) leads to

$$\begin{aligned} \left| \int_{\mathbb{X}} f d\sigma - \int_{\mathbb{X}} f S_\nu^{-1} \mathbf{1} d\nu \right| &= |\langle f, \mathbf{1} \rangle_\sigma - \langle f, S_\nu^{-1} \mathbf{1} \rangle_\nu| \\ &= |\langle f - P_\sigma f, \mathbf{1} \rangle_\sigma - \langle f - P_\sigma f, S_\nu^{-1} \mathbf{1} \rangle_\nu| \\ &\leq \|f - P_\sigma f\|_\sigma \|\mathbf{1}\|_\sigma + \|f - P_\sigma f\|_\nu \|S_\nu^{-1} \mathbf{1}\|_\nu \\ &\leq \|f - P_\sigma f\|_\sigma + \sqrt{A^{-1}} \|f - P_\sigma f\|_\nu \end{aligned} \quad (4.3)$$

where $P_\sigma : L^2(\mathbb{X}, \sigma) \rightarrow \Pi$ denotes the orthogonal projector onto Π with respect to the inner product $\langle \cdot, \cdot \rangle_\sigma$.

The most important fact that can be taken from the example is that all statements only refer to the Hilbert Space structure which means that we can use the same argumentation for generalized settings.

Therefore let $\mathcal{H}_1$ be a Hilbert Space with inner product $\langle \cdot, \cdot \rangle_1$ and $\Pi \subset \mathcal{H}_1$ be a closed subspace. Suppose H_2 is a vector space with positive semi-definite sesquilinear product $\langle \cdot, \cdot \rangle_2$ such that

$$\Pi \subset H_2 \subset \mathcal{H}_1.$$

Regarding to this framework, we assume that for Π the MZ-inequalities with respect to $\mathcal{H}_1$ and H_2 hold and denote the corresponding constants as before with A, B , i.e.

$$A\|f\|_1 \leq \|f\|_2 \leq B\|f\|_2 \quad \forall f \in \Pi.$$

To simplify the notation let us define $\Pi_i := (\Pi, \langle \cdot, \cdot \rangle_i)$ for $i = 1, 2$. As a consequence of these definitions and assumptions, we obtain the following proposition.

Proposition 4.2. *Suppose that the left inequality in (3.1) holds. Then for all $g \in \mathcal{H}_1$, $f \in H_2$ and $p, q \in \Pi$ the identity operator $I : \Pi_2 \rightarrow \Pi_1$, $p \mapsto p$ satisfies*

$$|\langle f, g \rangle_1 - \langle f, I^* q \rangle_2| \leq \|f\|_1 \|g - q\|_1 + \|q\|_1 (\|f - p\|_1 + \sqrt{A^{-1}} \|f - p\|_2). \quad (4.4)$$

Proof. Let $g \in \mathcal{H}_1$, $f \in H_2$ and $p, q \in \Pi$. By the Cauchy-Schwarz inequality, we get

$$\begin{aligned} |\langle f, g \rangle_1 - \langle f, I^*q \rangle_2| &= |\langle f, g - q \rangle_1 + \langle f, q \rangle_1 - \langle f, I^*q \rangle_2| \\ &\leq \|f\|_1 \|g - q\|_1 + |\langle f, q \rangle_1 - \langle f, I^*q \rangle_2| \end{aligned}$$

According to the left inequality in (3.1), I is continuous and the adjoint operator $I^* : \Pi_1 \rightarrow \Pi_2$ satisfies

$$\langle p, q \rangle_1 = \langle Ip, q \rangle_1 = \langle p, I^*q \rangle_2.$$

Further for the operator norms of I and I^* hold

$$\|I^*\| = \|I\| = \sup_{\|q\|_2=1} \|Iq\|_1 \leq \sup_{\|q\|_2=1} \|q\|_1 \leq \sup_{\|q\|_2=1} \sqrt{A^{-1}} \|q\|_2 = \sqrt{A^{-1}}.$$

Therefore we can conclude

$$\begin{aligned} |\langle f, q \rangle_1 - \langle f, I^*q \rangle_2| &= |\langle f - p, q \rangle_1 - \langle f - p, I^*q \rangle_2| \\ &\leq |\langle f - p, q \rangle_1| + |\langle f - p, I^*q \rangle_2| \\ &\leq \|f - p\|_1 \|q\|_1 + \|f - p\|_2 \|I^*q\|_2 \\ &\leq \|f - p\|_1 \|q\|_1 + \|f - p\|_2 \sqrt{A^{-1}} \|q\|_1. \quad \square \end{aligned}$$

With the help of Proposition 4.2 we are now able to formulate different approximation theorems. In this context, a distinction between two different settings is made.

4.1 General Hilbert-Vector-Space setting

Suppose that there exists a frame $\{\phi_j\}_{j \in J}$ for Π with respect to the inner product $\langle \cdot, \cdot \rangle_1$ and frame bounds $0 < A' \leq B'$. Due to Theorem 3.1 and because the MZ-inequalities with constants A, B hold, $\{\phi_j\}_{j \in J}$ is also a frame with frame bounds AA', BB' for Π with respect to the sesquilinear product $\langle \cdot, \cdot \rangle_2$. Further the associated frame operators

$$\begin{aligned} S_1 : \Pi_1 &\rightarrow \Pi_1 & S_1 f &:= \sum_{j \in J} \langle f, \phi_j \rangle_1 \phi_j \\ S_2 : \Pi_2 &\rightarrow \Pi_2 & S_2 f &:= \sum_{j \in J} \langle f, \phi_j \rangle_2 \phi_j. \end{aligned}$$

are boundedly invertible and the following theorem is an immediate consequence of Proposition 4.2.

Theorem 4.3. Assume that $\{\phi_j\}_{j \in J}$ is a frame for Π with frame bounds $0 < A' \leq B'$ with respect to the inner product $\langle \cdot, \cdot \rangle_1$ and the MZ-inequalities (3.1) for Π with respect to $\mathcal{H}_1$ and H_2 are satisfied. Then for $g \in \mathcal{H}_1$, $f \in H_2$ and $p, q \in \Pi$ we have

$$|\langle f, g \rangle_1 - \langle f, S_2^{-1} S_1 q \rangle_2| \leq \|f\|_1 \|g - q\|_1 + \|q\|_1 (\|f - p\|_1 + \sqrt{A^{-1}} \|f - p\|_2). \quad (4.5)$$

Proof. Due to Theorem 3.1 $\{\phi_j\}_{j \in J}$ is also a frame for Π with frame bounds AA', BB' regarding to the sesquilinear product $\langle \cdot, \cdot \rangle_2$. Consequently the operators S_1, S_2 are boundedly invertible. Moreover for all $p, q \in \Pi$ the equality

$$\langle p, S_2^{-1} S_1 q \rangle_2 = \langle p, q \rangle_1 = \langle p, I^* q \rangle_2 \quad (4.6)$$

holds and hence $S_2^{-1} S_1 q = I^* q$ for all $q \in \Pi$ so that Proposition (4.2) implies our claim. $\square$

Note that (4.6) and direct calculation lead to

$$\begin{aligned} \langle f, S_2^{-1} S_1 q \rangle_2 &= \sum_{k \in J} \overline{\langle q, \phi_k \rangle_1} \langle f, \psi_k \rangle_2 = \sum_{k \in J} \overline{\langle q, S_2^{-1} S_1 \phi_k \rangle_2} \langle f, \psi_k \rangle_2 \\ &= \sum_{k, j \in J} \langle \phi_k, \phi_j \rangle_1 \langle f, \psi_k \rangle_2 \langle \psi_j, q \rangle_2 \end{aligned}$$

where $\psi_j := S_2^{-1} \phi_j$. If $\{\phi_j\}_{j \in J}$ is a Parseval frame for Π with respect to the inner product $\langle \cdot, \cdot \rangle_1$, the operator S_1 is the identity on Π and

$$\langle f, S_2^{-1} q \rangle_2 = \sum_{j \in J} \langle f, \psi_j \rangle_2 \langle \psi_j, q \rangle_2.$$

Corollary 4.4. Let $\{\phi_j\}_{j \in J}$ (J countable) be a Parseval frame for Π with respect to the inner product $\langle \cdot, \cdot \rangle_1$. Then for $g \in \mathcal{H}_1$ and $f \in H_2$ we have

$$|\langle f, g \rangle_1 - \langle f, S_2^{-1} P_1 g \rangle_2| \leq \|f\|_1 \|g - P_1 g\|_1 + \|g\|_1 (\|f - P_1 f\|_1 + \sqrt{A^{-1}} \|f - P_1 f\|_2)$$

where the orthogonal projection $P_1 : \mathcal{H}_1 \rightarrow \Pi$ is defined as

$$P_1 g := \sum_{j \in J} \langle g, \phi_j \rangle_1 \phi_j.$$

Proof. Because $\{\phi_j\}_{j \in J}$ is a Parseval frame, P_1 is an orthogonal projection and an extension of S_1 to $\mathcal{H}_1$. Since $P_1 f, P_1 g \in \Pi$ our claim follows immediately from Theorem 4.3. $\square$

4.2 General Hilbert-Sobolev-Space setting

As in the previous section, let us consider the following function space setting analogous to [8] to motivate the upcoming definitions and theorems.

Example 4.5. Let σ be a Borel probability measure on a space $\mathbb{X}$. Suppose that $\{\phi_j\}_{j \in J}$ be an orthonormal basis for $L^2(\mathbb{X}, \sigma)$ and let $\{\lambda_j\}_{j \in J}$ be a nondecreasing sequence of positive numbers such that $\lim_{j \rightarrow \infty} \lambda_j = \infty$. The pairs (ϕ_j, λ_j) can be interpreted as eigenfunctions of a positive unbounded operator with corresponding eigenvalues [8]. Therefore the Sobolev Space $H^s(\mathbb{X})$ can be defined as

$$H^s(\mathbb{X}) := \left\{ f \in L^2(\mathbb{X}, \sigma) : \|f\|_{H^s}^2 := \sum_{j=0}^{\infty} (1 + \lambda_j)^s |\langle f, \phi_j \rangle_{\sigma}|^2 < \infty \right\}.$$

Further let us define the closed linear subspace

$$\mathfrak{V}_n := \text{span}\{\phi_j : \lambda_j \leq n\} \subset L^2(\mathbb{X}, \sigma) \quad (4.7)$$

with the orthogonal projector $P_{n,\sigma} : L^2(\mathbb{X}, \sigma) \rightarrow \mathfrak{V}_n$ onto $\mathfrak{V}_n$ with respect to the inner product $\langle \cdot, \cdot \rangle_{\sigma}$. By assuming

$$\sigma(\mathbb{X}) < \infty, \quad \mathbb{X} \text{ compact}, \quad \{\phi_j\}_{j \in J} \subset \mathcal{C}(\mathbb{X}),$$

$$\Phi_s(n) := \sup_{x \in \mathbb{X}} \left(\sum_{j: \lambda_j > n} (1 + \lambda_j)^{-s} |\phi_j(x)|^2 \right)^{1/2} < \infty \quad (4.8)$$

we are able to ensure that $H^s(\mathbb{X}) \hookrightarrow \mathcal{C}(\mathbb{X}) \subset L^2(\mathbb{X}, \sigma)$ [8]. Further let ν be a finite Borel measure on $\mathbb{X}$ and suppose that there are two constants $0 < A \leq B$ such that

$$A\|f\|_{\sigma}^2 \leq \|f\|_{\nu}^2 \leq B\|f\|_{\sigma}^2, \quad f \in \mathfrak{V}_n.$$

where $\|\cdot\|_{\sigma}^2$ and $\|\cdot\|_{\nu}^2$ are defined as (4.2). For $f \in H^s(\mathbb{X})$, same argumentation as in (4.3) leads to

$$|\langle f, \mathbf{1} \rangle_{\sigma} - \langle f, S_{\nu}^{-1} \mathbf{1} \rangle_{\nu}| \leq \|f - P_{\sigma} f\|_{\sigma} + \sqrt{A^{-1}} \|f - P_{\sigma} f\|_{\nu}.$$

Therefore the next step is to find bounds for $\|f - P_{n,\sigma} f\|_{\nu}$ and $\|f - P_{n,\sigma} f\|_{\sigma}$.

For $f \in H^s(\mathbb{X})$ and $s > 0$, we have

$$\begin{aligned} \int_{\mathbb{X}} |f - P_{n,\sigma} f|^2 d\nu &= \|f - P_{n,\sigma} f\|_{\nu}^2 \\ &= \sum_{i,j:\lambda_i,\lambda_j > n} \langle f, \phi_i \rangle_{\sigma} \overline{\langle f, \phi_j \rangle_{\sigma}} \langle \phi_i, \phi_j \rangle_{\nu} \\ &\leq \nu(\mathbb{X}) \Phi_s^2(n) \|f\|_{H^s}^2 \end{aligned}$$

and if $\mathbf{1} \in \mathfrak{V}_n$ we deduce

$$\int_{\mathbb{X}} |f - P_{n,\sigma} f|^2 d\nu \leq \nu(\mathbb{X}) \Phi_s^2(n) \|f\|_{H^s}^2 \leq B \Phi_s^2(n) \|f\|_{H^s}^2. \quad (4.9)$$

The same estimation can also be done with respect to the norm $\|\cdot\|_{\sigma}$ to get

$$\begin{aligned} \|f - P_{n,\sigma} f\|_{\sigma}^2 &= \sum_{i,j:\lambda_i,\lambda_j > n} \langle f, \phi_i \rangle_{\sigma} \overline{\langle f, \phi_j \rangle_{\sigma}} \langle \phi_i, \phi_j \rangle_{\sigma} \\ &= \sum_{j:\lambda_j > n} |\langle f, \phi_j \rangle_{\sigma}|^2 \|\phi_j\|_{\sigma}^2 \\ &\leq \sum_{j:\lambda_j > n} (1 + \lambda_j)^{-s} \|f\|_{H_s}^2 \\ &\leq (1 + n)^{-s} \|f\|_{H_s}^2. \end{aligned}$$

Finally by summarizing the above results, we have

$$|\langle f, \mathbf{1} \rangle_{\sigma} - \langle f, S_{n,\nu}^{-1} \mathbf{1} \rangle_{\nu}| \leq \left((1 + n)^{-s/2} + \sqrt{\frac{B}{A}} \Phi_s(n) \right) \|f\|_{H^s}. \quad (4.10)$$

Since all statements only refer to the Hilbert Space structure we can use the same arguments again for generalized settings.

Therefore assume that $\{\phi_j\}_{j \in J}$ is an orthonormal basis for $\mathcal{H}_1$ and let $\{\lambda_j\}_{j \in J}$ be a nondecreasing sequence of positive numbers such that $\lim_{j \rightarrow \infty} \lambda_j = \infty$. The associated Sobolev Space H^s is defined as

$$H^s := \left\{ f \in \mathcal{H}_1 : \sum_{j=0}^{\infty} (1 + \lambda_j)^s |\langle f, \phi_j \rangle_1|^2 < \infty \right\}$$

with corresponding norm

$$\|f\|_{H^s}^2 := \sum_{j=0}^{\infty} (1 + \lambda_j)^s |\langle f, \phi_j \rangle_1|^2.$$

Suppose H_2 is a vector space with positive semi-definite sesquilinear product $\langle \cdot, \cdot \rangle_2$ which is linear in the first component such that $H^s \hookrightarrow H_2 \subset \mathcal{H}_1$, i.e. there exists a constant $C > 0$ such that $\| \cdot \|_2 \leq C \| \cdot \|_{H^s}$. Further for $n \in \mathbb{N}$ define the remainder function $\Psi_s(n)$ and the closed linear subspace $\mathfrak{H}_n$ with corresponding orthogonal projection $P_{n,1}$ as

$$\Psi_s(n) := \left(\sum_{j:\lambda_j > n} (1 + \lambda_j)^{-s} \|\phi_j\|_2^2 \right)^{1/2}, \quad (4.11)$$

$$\mathfrak{H}_n := \text{span}\{\phi_j : \lambda_j \leq n\} \subset \mathcal{H}_1, \quad (4.12)$$

$$P_{n,1} : \mathcal{H}_1 \rightarrow \mathfrak{H}_n \quad P_{n,1}f := \sum_{j:\lambda_j \leq n} \langle f, \phi_j \rangle_1 \phi_j. \quad (4.13)$$

Lastly, we assume that the MZ-inequalities for $\mathfrak{H}_n$ with respect to $\mathcal{H}_1$ and H_2 are fulfilled. Due to these assumptions and definitions, approximation theorems for vectors and inner products can be formulated.

4.2.1 Approximation of vectors

This section is devoted to the estimation of the approximation error due to projection for vectors from H^s . Similar to Example 4.5 we are initially interested in bounds for $\|f - P_{n,1}f\|_1$ and $\|f - P_{n,1}f\|_2$ which will also be used to approximate inner products. Due to the construction and direct calculation, the following lemma provides information about the bound for $\|f - P_{n,1}f\|_1$.

Lemma 4.6. *Let $f \in H^s$. Then the inequality*

$$\|f - P_{n,1}f\|_1^2 \leq (1 + n)^{-s} \|f\|_{H^s}^2$$

holds, where $P_{n,1}$ is defined as (4.13).

Proof. Let $f \in H^s$. Then

$$\begin{aligned} \|f - P_{n,1}f\|_1^2 &= \left\| \sum_{j:\lambda_j > n} \langle f, \phi_j \rangle_1 \phi_j \right\|_1^2 = \sum_{j:\lambda_j > n} |\langle f, \phi_j \rangle_1|^2 \|\phi_j\|_1^2 \\ &= \sum_{j:\lambda_j > n} (1 + \lambda_j)^{-s} (1 + \lambda_j)^s |\langle f, \phi_j \rangle_1|^2 \\ &\leq (1 + n)^{-s} \sum_{j:\lambda_j > n} (1 + \lambda_j)^s |\langle f, \phi_j \rangle_1|^2 \\ &\leq (1 + n)^{-s} \|f\|_{H^s}^2 \end{aligned}$$

□

To estimate the projection error with respect to $\|\cdot\|_2$, an additional assumption is necessary.

Theorem 4.7. *Let $f \in H^s$ and $\Psi_s(n)$ defined as (4.11). If $\Psi_s(n) < \infty$ then the inequality*

$$\|f - P_{n,1}f\|_2^2 \leq \Psi_s^2(n) \|f\|_{H^s}^2$$

holds, where $P_{n,1}$ is defined as (4.13).

Proof. Let $f \in H^s$ and define $\psi_j := (1 + \lambda_j)^{-s/2} \phi_j$. Then

$$\begin{aligned} \|f - P_{n,1}f\|_2^2 &= \left\| \sum_{j:\lambda_j > n} \langle f, \phi_j \rangle_1 \phi_j \right\|_2^2 \\ &= \sum_{i,j:\lambda_i, \lambda_j > n} \langle f, \phi_i \rangle_1 \overline{\langle f, \phi_j \rangle_1} \langle \phi_i, \phi_j \rangle_2 \\ &= \sum_{i,j:\lambda_i, \lambda_j > n} (1 + \lambda_i)^{s/2} \langle f, \phi_j \rangle_1 (1 + \lambda_j)^{s/2} \overline{\langle f, \phi_i \rangle_1} \langle \psi_i, \psi_j \rangle_2 \\ &\leq \|f\|_{H^s}^2 \|(\langle \psi_i, \psi_j \rangle_2)_{i,j:\lambda_i, \lambda_j > n}\|_{\ell_2 \rightarrow \ell_2} \end{aligned}$$

where the last inequality holds by considering

$$\sum_{i,j:\lambda_i, \lambda_j > n} (1 + \lambda_i)^{s/2} \langle f, \phi_j \rangle_1 (1 + \lambda_j)^{s/2} \overline{\langle f, \phi_i \rangle_1} \langle \psi_i, \psi_j \rangle_2$$

as a vector-matrix-vector-multiplication. By using the Cauchy-Schwarz inequality for $(c_l)_{l=n}^\infty$ we get

$$\begin{aligned} \sum_{j:\lambda_j > n} \left| \sum_{l:\lambda_l > n} \langle \psi_j, \psi_l \rangle_2 c_l \right|^2 &= \sum_{j:\lambda_j > n} \left| \sum_{l:\lambda_l > n} (1 + \lambda_j)^{-s/2} (1 + \lambda_l)^{-s/2} \langle \phi_j, \phi_l \rangle_2 c_l \right|^2 \\ &\leq \sum_{j:\lambda_j > n} (1 + \lambda_j)^{-s} \|c\|_{\ell_2}^2 \sum_{l:\lambda_l > n} (1 + \lambda_l)^{-s} \|\phi_j\|_2^2 \|\phi_l\|_2^2 \\ &= \|c\|_{\ell_2}^2 \left(\sum_{k:\lambda_k > n} (1 + \lambda_k)^{-s} \|\phi_k\|_2^2 \right)^2. \end{aligned}$$

Consequently we get

$$\begin{aligned} \|f - P_{n,1}f\|_2^2 &\leq \|f\|_{H^s}^2 \sum_{k:\lambda_k > n} (1 + \lambda_k)^{-s} \|\phi_k\|_2^2 \\ &= \|f\|_{H^s}^2 \Psi_s^2(n) \end{aligned}$$

□

4.2.2 Approximation of inner products

This section provides the main theorem for approximating inner products, which results from the previous statements. Because $\{\phi_j\}_{j \in J}$ is an orthonormal basis for $\mathcal{H}_1$, $\{\phi_j\}_{j: \lambda_j \leq n}$ is also an orthonormal basis for $\mathfrak{H}_n$ with respect to the inner product $\langle \cdot, \cdot \rangle_1$. As before to simplify the notation, define the spaces $\mathfrak{H}_n$ endowed with one of the inner products as $\mathfrak{H}_{n,i} := (\mathfrak{H}_n, \langle \cdot, \cdot \rangle_i)$ for $i = 1, 2$. Therefore Corollary 3.4 guarantees that the operator

$$S_{n,2} : \Pi_{n,2} \rightarrow \Pi_{n,2} \quad S_{n,2}f := \sum_{j: \lambda_j \leq n} \langle f, \phi_j \rangle_2 \phi_j$$

is boundedly invertible and the following theorem holds.

Theorem 4.8. *If $\Psi_s(n) < \infty$, then for $f, g \in H^s$,*

$$|\langle f, g \rangle_1 - \langle f, S_{n,2}^{-1} P_{n,1} g \rangle_2| \leq \left(2(1+n)^{-s/2} + \sqrt{A^{-1}} \Psi_s(n) \right) \|f\|_{H^s} \|g\|_{H^s}. \quad (4.14)$$

Proof. By Lemma 4.6, Corollary 4.4, Theorem 4.7 and since $\|\cdot\|_1 \leq \|\cdot\|_{H^s}$

$$\begin{aligned} |\langle f, g \rangle_1 - \langle f, S_{n,2}^{-1} P_{n,1} g \rangle_2| &\leq \|f\|_1 \|g - P_{n,1} g\|_1 + \|g\|_1 \left(\|f - P_{n,1} f\|_1 + \sqrt{A^{-1}} \|f - P_{n,1} f\|_2 \right) \\ &\leq \left(2(1+n)^{-s/2} + \sqrt{A^{-1}} \Psi_s(n) \right) \|f\|_{H^s} \|g\|_{H^s} \quad \square \end{aligned}$$

Theorem 4.8 shows how the approximation error for inner products behaves with respect to the chosen orthonormal basis $\{\phi_j\}_{j \in J}$, the sequence $\{\lambda_j\}_{j \in J}$ and n . Further, if $g \in \mathfrak{H}_n$ then (4.14) reads

$$|\langle f, g \rangle_1 - \langle f, S_{n,2}^{-1} g \rangle_2| \leq \left((1+n)^{-s/2} + \sqrt{A^{-1}} \Psi_s(n) \right) \|f\|_{H^s} \|g\|_{H^s}.$$

Since in practice the bounds for the so-called *worst case error* are also of interest, the next section is dedicated to this estimation. Afterwards we compare the results of the present work with other popular theorems.

Chapter 5

Error estimation and comparison with more specific results

As soon as the approximation error is analyzed in more detail, it is also natural to look closer at the so-called *worst case error*. Therefore the goal of this chapter is to find bounds for this error with respect to the space H^s defined as (4.7). Fortunately, this estimation is an immediate consequence of Theorem 4.8. Hence this chapter serves only a summary of the previous statements and translate the already shown theorems into results with respect to the worst case error. In addition, statements from the present work are compared with popular results from [1] and [8] and the main differences are highlighted.

5.1 Worst case error estimations

Due to the underlying setting, there are several ways to define a *worst case error* [4]. Therefore let us first define the *worst case error* for our purposes.

Definition 5.1 (Worst case error). *Let $\mathcal{H}_1$ be a Hilbert Space with inner product $\langle \cdot, \cdot \rangle_1$ and H_2 be a vector space with corresponding positive semi-definite sesquilinear product $\langle \cdot, \cdot \rangle_2$ which is linear in the first component. Further let H^s be the Sobolev Space defined as (4.7) and $g \in H^s$. The worst case error for g with respect to $H^s, \mathcal{H}_1$ and H_2 is defined as*

$$wce(g, H^s, \mathcal{H}_1, H_2) := \sup_{\substack{f \in H^s \\ \|f\|_{H^s} \leq 1}} |\langle f, g \rangle_1 - \langle f, S_{n,2}^{-1} P_{n,1} g \rangle_2|.$$

Note that for special frameworks like in Example 4.5, this *worst case error* is sometimes also called *discrepancy* [6]. Furthermore the following corollary is a direct consequence of Theorem 4.8 and provides general bounds for the *worst case error*.

Corollary 5.2. *Let $g \in H^s$, $n \in \mathbb{N}$ and $\Psi_s(n)$ be defined as (4.11). If $\Psi_s(n) < \infty$ then*

$$wce(g, H^s, \mathcal{H}_1, H_2) \leq \left(2(1+n)^{-s/2} + \sqrt{A^{-1}} \Psi_s(n) \right) \|g\|_{H^s}$$

Proof. Use Definition 5.1 and Theorem 4.8. □

Because in practice it is more natural to look at function space settings [1, 8], the previous corollary can be improved for the result of the motivational Example 4.5. By assuming that $\mathbb{1} \in \mathfrak{V}_n$ and setting $g = \mathbb{1}$, the *worst case error* can be interpreted as the worst possible error that can occur by integrating $f \in H^s(\mathbb{X})$ with respect to the different measures σ and ν .

Corollary 5.3. *Let $\mathbb{1} \in \mathfrak{V}_n$, $n \in \mathbb{N}$ and $\Phi_s(n)$ be defined as (4.8). If $\Phi_s(n) < \infty$ then*

$$wce(\mathbb{1}, H^s(\mathbb{X}), L_2(\mathbb{X}, \sigma), \mathcal{C}(\mathbb{X})) \leq \left((1+n)^{-s/2} + \sqrt{\frac{B}{A}} \Phi_s(n) \right)$$

Proof. Use Definition 5.1 and (4.10) from Example 4.5. □

If we look at Corollary 5.3, the results from [1] and [8] are slightly different due to alternative estimations and stronger assumptions. Therefore the next step is to analyse the differences between the present work and the results from [1] and [8] in more detail. But first let us summarize all the important assumptions and definitions of Example 4.5.

Let σ be a Borel probability measure on a space $\mathbb{X}$. Suppose that $\{\phi_j\}_{j \in J}$ be an orthonormal basis for $L^2(\mathbb{X}, \sigma)$ and let $\{\lambda_j\}_{j \in J}$ be a nondecreasing sequence of positive numbers such that $\lim_{j \rightarrow \infty} \lambda_j = \infty$. The Sobolev Space $H^s(\mathbb{X})$, the closed subspace $\mathfrak{V}_n \subset L^2(\mathbb{X}, \sigma)$ and the orthogonal projection $P_{n, \sigma}$ are defined as in Example 4.5. By assuming

$$\sigma(\mathbb{X}) < \infty, \quad \mathbb{X} \text{ compact}, \quad \{\phi_j\}_{j \in J} \subset \mathcal{C}(\mathbb{X}), \quad (5.1)$$

$$\mathbb{1} \in \mathfrak{V}_n, \quad \Phi_s(n) < \infty \quad (5.2)$$

we are able to ensure that $H^s(\mathbb{X}) \hookrightarrow \mathcal{C}(\mathbb{X}) \subset L^2(\mathbb{X}, \sigma)$ [8]. Further let ν be a finite Borel measure on $\mathbb{X}$ and suppose that there are two constants $0 < A \leq B$ such that the *MZ-inequalities*

$$A\|f\|_{\sigma}^2 \leq \|f\|_{\nu}^2 \leq B\|f\|_{\sigma}^2, \quad f \in \mathfrak{V}_n.$$

hold, where $\|\cdot\|_\sigma^2$ and $\|\cdot\|_\nu^2$ are defined as (4.2).

5.2 Comparison with L_2 function space setting

Because the generalization of the present work is motivated by [8], the assumptions from the specialized setting of Example 4.5 are identical to [8]. By translating one of the main results from [8] into the language of our framework, we get the following theorem.

Theorem 5.4 (Gröchenig [8, Theorem 3.5]). *Let $f \in H^s(\mathbb{X})$. If the assumptions (5.1), (5.2) are satisfied, then*

$$\left| \int_{\mathbb{X}} f d\sigma - \int_{\mathbb{X}} f S_{n,\nu}^{-1} \mathbf{1} d\nu \right| \leq \left(1 + \sqrt{B/A}\right) \Phi_s(n) \|f\|_{H_s} \quad (5.3)$$

where $\nu = \sum_{j=1}^m \delta_{x_j}$ is an atomic probability measure with $x_j \in \mathbb{X}$ and δ_{x_j} a Dirac measure for all $j < m$.

The comparison of the inequality (5.3) and (4.10) reveals some small differences. By the estimation

$$\begin{aligned} \|f - P_{n,\sigma} f\|_\sigma^2 &= \sum_{j:\lambda_j > n} |\langle f, \phi_j \rangle_\sigma|^2 \|\phi_j\|_\sigma^2 \\ &\leq \sup_{x \in \mathbb{X}} \sum_{j:\lambda_j > n} |\langle f, \phi_j \rangle_\sigma|^2 (1 + \lambda_j)^s (1 + \lambda_j)^{-s} |\phi_j(x)|^2 \\ &\leq \Phi_s(n)^2 \|f\|_{H_s}^2 \end{aligned} \quad (5.4)$$

we are able to fill this gap. Hence Theorem 5.4 follows instantaneously from Theorem 4.8 what can be expected due to the close connection. By interpreting the pairs (ϕ_j, λ_j) as eigenfunctions of a positive unbounded operator with corresponding eigenvalues, the following definition and proposition provides additional information about $\Phi_s(n)$.

Definition 5.5 (Weyl's Law [8, Def. 3.6]). *We say the orthonormal basis $\{\phi_j : j \in \mathbb{N}\}$ and the eigenvalues $\{\lambda_j : k \in \mathbb{N}\}$ satisfy Weyl's law, if there exist constants $d = d_{\phi,\lambda} > 0$ and $C = C_{\phi,\lambda} > 0$ such that*

$$\sum_{j:\lambda_j \leq n} |\phi_j|^2 \leq C n^d.$$

Proposition 5.6 ([8, Prop. 3.7]). *Let $\{\phi_j : j \in \mathbb{N}\}$ be an orthonormal basis for $L^2(\mathbb{X}, \sigma)$ and $\lambda_j \geq 0$ be a non-decreasing sequence with $\lim_{j \rightarrow \infty} \lambda_j = \infty$. Assume that*

(ϕ_j, λ_j) satisfies Weyl's law with exponent d . Then for $s > d/2$

$$C_s^2 = \sup_{x \in \mathbb{X}} \sum_{j=1}^{\infty} (1 + \lambda_j)^{-s} |\phi_j(x)|^2 < \infty.$$

Moreover, the error function is

$$\Phi_s(n) = C_s n^{-s+d/2}.$$

By using Proposition 5.6, i.e. assume that (ϕ_j, λ_j) satisfies Weyl's law with exponent d then

$$\sum_{j: \lambda_j > n} (1 + \lambda_j)^{-s} |\phi_j(x)|^2 \leq C_s n^{-2s+d}$$

and for (4.10) we get

$$|\langle f, \mathbb{1} \rangle_\sigma - \langle f, S_{n,\nu}^{-1} \mathbb{1} \rangle_\nu| \leq \left((1+n)^{-s/2} + \sqrt{\frac{B}{A}} C_s n^{-s+d/2} \right) \|f\|_{H^s}.$$

We have seen that the approximation theorems from [8] are an immediate consequence from the present work and how the results are connected to each other. One remarkable fact about [8] is that only a few assumptions are needed to derive adequate estimations. The most important assumptions are that we are able to embed the Sobolev Space $H^s(\mathbb{X})$ into $\mathcal{C}(\mathbb{X})$ and that there exists a closed subspace $\mathfrak{N}_n$ such that the Marcinkiewicz-Zygmund inequalities hold.

5.3 Comparison with compact, connected Riemannian manifold setting

Another popular paper is [1] and can also be related to (4.10). Notice that in contrast to our setting where $\mathbb{X}$ is an arbitrary space, in [1] the space $\mathbb{X}$ is a compact connected Riemannian manifold. Nevertheless, this fact is initially negligible for the comparison. In addition to the previous assumptions (5.1), (5.2) we assume that ν is also a Borel probability measure and for all j with $\lambda_j \leq n$ that

$$\int_{\mathbb{X}} \phi_j d\sigma = \int_{\mathbb{X}} \phi_j d\nu. \quad (5.5)$$

By translating one of the main results from [1] into the language of our framework, we get the following theorem.

Theorem 5.7 (Brandolini et al [1, Theorem 2.12]). *Let $f \in H^s(\mathbb{X})$ and ν be a Borel probability measure. If the assumptions (5.1), (5.2) and (5.5) are satisfied, then*

$$\left| \int_{\mathbb{X}} f d\sigma - \int_{\mathbb{X}} f d\nu \right| \leq C n^{-s} \|f\|_{H^s}. \quad (5.6)$$

By concerning the measure ν as an atomic probability measure, Equation (5.5) is equivalent to say that for all j with $\lambda_j \leq n$ the quadrature rule given by ν is exact for the functions ϕ_j . Hence for $\mathbb{1} \in \mathfrak{V}_n$

$$\mathbb{1} = S_{n,\sigma} \mathbb{1} = \sum_{j=1}^n \langle \mathbb{1}, \phi_j \rangle_{\sigma} \phi_j = \sum_{j=1}^n \langle \mathbb{1}, \phi_j \rangle_{\nu} \phi_j = S_{n,\nu} \mathbb{1}$$

and similar to (5.4) we get

$$\begin{aligned} |\langle f, \mathbb{1} \rangle_{\sigma} - \langle f, S_{n,\nu}^{-1} \mathbb{1} \rangle_{\nu}| &= |\langle f, \mathbb{1} \rangle_{\sigma} - \langle f, \mathbb{1} \rangle_{\nu}| \\ &\leq (1 + \sqrt{B}) \Phi_s(n) \|f\|_{H^s}. \end{aligned}$$

Further if ν is a probability measure, by (4.9) we have

$$|\langle f, \mathbb{1} \rangle_{\sigma} - \langle f, S_{n,\nu}^{-1} \mathbb{1} \rangle_{\nu}| = |\langle f, \mathbb{1} \rangle_{\sigma} - \langle f, \mathbb{1} \rangle_{\nu}| \leq 2\Phi_s(n) \|f\|_{H^s}.$$

By using Proposition 5.6, i.e. assume that (ϕ_j, λ_j) satisfies Weyl's law with exponent d then

$$\sum_{j: \lambda_j > n} (1 + \lambda_j)^{-s} |\phi_j(x)|^2 \leq C_s n^{-2s+d},$$

we get

$$|\langle f, \mathbb{1} \rangle_{\sigma} - \langle f, \mathbb{1} \rangle_{\nu}| \leq 2C_s n^{-s+d/2} \|f\|_{H^s} \quad (5.7)$$

which is similar to Theorem 5.7. If we drop (5.5) we get

$$|\langle f, \mathbb{1} \rangle_{\sigma} - \langle f, S_{n,\nu}^{-1} \mathbb{1} \rangle_{\nu}| \leq (1 + \sqrt{A^{-1}}) C_s n^{-s+d/2} \|f\|_{H^s}.$$

Notice that the exponent of n in (5.7) is slightly worse than (5.6). This is because of proofs which are tailored to Riemannian manifolds and the fact that the pairs (ϕ_j, λ_j) are eigenfunctions with eigenvalues of a special operator called the Laplace-Beltrami operator [1]. For such pairs (ϕ_j, λ_j) even better statements and estimations are possible [1, Lemma 2.10].

As a conclusion of Section 5.2 and 5.3, the generalization provides similar results to [8] and [1], which approves the chosen general setting. Due to the more specific setting in [1], the exponent of n in (5.4) reduces to $-s$. Hence slightly better results can be achieved by using statements of the underlying structure and tailored proofs for the Riemannian manifold. Furthermore, the comparison highlights the aspects of the different results and reveals the relationships between them.

Conclusion

This thesis has established general approximation theorems for inner products and vectors of a Hilbert Space. Every estimation is based on the existence of a closed linear subset Π such that the *Marcinkiewicz-Zygmund inequalities* hold. Further the present work is characterized by the simplicity of the proofs and lead to insights on the necessary properties which are provided by the MZ-inequalities. In addition, the MZ-inequalities allows us to consider a given frame for the subset Π to different inner products and make statements about the corresponding bounds.

It is possible to successfully apply the derived approximation theorems to the special framework of [8] with almost identical results. By making the same assumptions as [1], similar results can also be achieved. These facts approve the validity of the general approximation theorems and highlights various aspects of the recent results.

Appendix A

Appendix

Theorem A.1 (Invertibility of bounded operators [3]). *Let X be a Banach Space. If $U : X \rightarrow X$ is bounded and $\|I - U\| < 1$ where I is the identity operator, then U is invertible, and*

$$U^{-1} = \sum_{k=0}^{\infty} (I - U)^k.$$

Furthermore,

$$\|U^{-1}\| \leq \frac{1}{1 - \|I - U\|}.$$

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