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## Invertible extension of non-singular dynamical systems

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# Abstract

In this thesis we prove the existence of invertible extensions of dynamical systems. Basically we consider two cases. On the one hand, we use a measure preserving transformation and on the other hand, a non-singular one. Both proofs are related to each other in the sense that the procedure is the same. It is interesting to observe, which arguments hold for both cases and on which points the proofs differ from each other. Here we need some concepts of measure and integration theory, like the Standard Uniqueness Theorem for measures, the Radon-Nikodym derivative and the transformation formula. We will not only construct such a invertible extension, but even show that this extension can be chosen to be minimal. To do this, we use exhaustive partitions and factor maps between metric spaces which form a commutative diagram. Consequently, this factor map shall be invertible, such that those metric spaces become isomorphic.

In dieser Arbeit zeigen wir die Existenz einer invertierbaren Erweiterung eines dynamischen Systems. Im Prinzip behandeln wir zwei Fälle. Auf der einen Seite, soll die auf dem dynamischen System lebende Transformation maßtreu sein, auf der anderen Seite soll diese nicht-singulär sein. Beide Beweise sind verwandt miteinander, in dem Sinne, dass die Vorgangsweise dieselbe ist. Es ist interessant zu sehen, an welchen Stellen die Beweise übereinstimmen und an welchen Stellen andere Argumente verwendet müssen, um das Ziel zu erreichen. An dieser Stelle benötigen wir einige Resultate aus der Maß- und Integrationstheorie, wie zum Beispiel den Eindeutigkeitssatz für Maße, die Radon-Nikodym Ableitung und den Transformationssatz. Wir konstruieren nicht nur so eine invertierbare Erweiterung, sondern wir zeigen auch, dass diese Erweiterung minimal gewählt werden kann. Um das zu schaffen, verwenden wir ausschöpfende Partitionen und Faktorabbildungen zwischen metrischen Räumen, aus denen dann ein kommutatives Diagramm entsteht. Diese Faktorabbildung soll folglich invertierbar sein, damit diese metrischen Räume isomorph sind.



# 1 Introduction

In this thesis we are concerned with dynamical systems, which consist of a measure space  $(X, \mathcal{A}, \mu)$  and a measurable transformation  $S$ . A dynamical system is said to be invertible, if there exists  $S^{-1}$  s.t.  $S \circ S^{-1} = S^{-1} \circ S$ . Of course, there are dynamical systems which are noninvertible. The question then is, whether there exists another dynamical system  $(\tilde{X}, \tilde{\mathcal{A}}, \tilde{\mu}, T)$ , which is invertible (i.e.  $T$  is invertible) and inherits all properties of our initial system. If such an extension exists, we call  $T$  an invertible extension of  $S$ . There is a possibility that plenty of such invertible extensions exist, then one may ask, if there is a minimal invertible extension. Then this extension is called a natural extension.

Here  $S$  shall be an endomorphism, i.e.  $S$  is surjective. Two cases have been studied over the past 60 years, one where  $S$  is measure preserving and one where  $S$  is nonsingular. The first achievement was done by Rohlin (see [2] for more content) in the 1960s, as he managed to prove existence and even uniqueness.

In the non-singular case, Silva, Lambert and Beneteau proved existence of such a natural extension. Silva used in [4] the skew product construction to reduce to the measure preserving case, whereas Lambert managed to show in [9] existence for general measure spaces. However, his assumption on measurability of images of  $S$  is undesirable, since it fails even for a measure preserving  $S$ . Finally, Bénéteau used in [1] Lambert's proof as a template, without using the skew product, and succeeded in proving existence for standard Borel spaces.

The goal of this thesis is to discuss Bénéteau's result in a more comprehensive way. In order to do so, we will spend the first part of this work with a recap. Since we will be dealing with measure spaces and dynamical systems, one would suggest to recall some basic and less basic concepts from measure theory and ergodic theory.

After this is achieved, we will introduce factor spaces and define such a natural extension for standard Borel spaces.

Then it is time for the main topic of this thesis. Firstly, a proof for  $S$  being a measure preserving endomorphism, and secondly, a proof for  $S$  being a nonsingular endomorphism. We will prove the measure preserving case likewise with Bénéteau's nonsingular proof in [1]. Interesting here will be to see, how the proof changes when we go from measure preservation to nonsingularity.

## 2 Preliminaries

We consider the following setup. Let  $(X, \mathcal{A}, \mu, S)$  be a dynamical system, where  $(X, \mathcal{A}, \mu)$  is a measure space and  $S : X \rightarrow X$  is a measurable transformation, i.e.  $S^{-1}A \in \mathcal{A}$  for  $A \in \mathcal{A}$ . The goal of this chapter is to define some concepts and important tools, which will be used for the main purpose of this paper.

### 2.1 Background measure and integration theory

At the beginning, we want to analyze two measures  $\mu$  and  $\tilde{\mu}$  on  $(X, \mathcal{A})$ , which coincide on a smaller family  $\mathcal{E} \subseteq \mathcal{A}$ . The question is, which conditions must be asked for  $\mu, \tilde{\mu}$  and  $\mathcal{E}$  such that  $\mu = \tilde{\mu}$ . The answer to this question gives us the Standard Uniqueness Theorem for measures in [3].

**Theorem 2.1.1 :**

Let  $\mu$  and  $\tilde{\mu}$  be measures on  $(X, \mathcal{A})$  and  $\mathcal{E}$  be a generator of the  $\sigma$ -algebra  $\mathcal{A}$  which is stable under intersections, i.e.  $A, B \in \mathcal{E} \implies A \cap B \in \mathcal{E}$ . If

- (i)  $\mu(E) = \tilde{\mu}(E) \quad \forall E \in \mathcal{E}$
- (ii)  $\mu(X) = \tilde{\mu}(X) < \infty$  or  $\mu$  and  $\tilde{\mu}$  are  $\sigma$ -finite on  $\mathcal{E}$

then  $\mu = \tilde{\mu}$  on  $\sigma(\mathcal{E}) = \mathcal{A}$ .

One very important object in measure theory is the Radon-Nikodym derivative, which extends the well known concept of a derivative to measures. For more information about the Radon-Nikodym derivative and further content, see [3]. To state the general Radon-Nikodym theorem, firstly we need to introduce the concept of a signed measure.

**Definition 2.1.2 :**

A signed measure is a set function  $\nu : \mathcal{A} \rightarrow [-\infty, \infty)$  or  $\nu : \mathcal{A} \rightarrow (-\infty, \infty]$ , which satisfies

- (i)  $\nu(\emptyset) = 0$
- (ii) If  $(A_n)_{n \geq 1}$  is a disjoint sequence in  $\mathcal{A}$ , then
$$\nu(\bigcup_{n \geq 1} A_n) = \sum_{n \geq 1} \nu(A_n)$$

So basically, a signed measure is a measure, which allows negative values. Furthermore, basic properties of measures carry over to signed measures (e.g.

continuity from above/below).

One would like to sort of characterize one measure with respect to another measure, and one possible relation is absolute continuity of measures.

**Definition 2.1.3 :**

Let  $\mu$  and  $\nu$  be measures on  $(X, \mathcal{A})$ . Then  $\nu$  is absolutely continuous with respect to  $\mu$ , denoted by  $\nu \ll \mu$ , if

$$\mu(A) = 0 \Rightarrow \nu(A) = 0 \text{ for } A \in \mathcal{A}$$

In the case of signed measures, we have to add an extra condition, namely

$$\mu(A) = 0 \Rightarrow \nu(A) = 0 \text{ for } A \in \mathcal{A} \text{ and } \nu(B) = 0 \forall B \in \mathcal{A} \text{ s.t. } B \subseteq A$$

In other words, absolute continuity means that every  $\mu$ -null set is also a  $\nu$ -null set.

The importance of this property lies in the fact, that absolute continuity is sufficient for the existence of a density for  $\nu$ , namely if we restrict to  $\sigma$ -finite measures. This is guaranteed by the important Radon-Nikodym theorem, which is named after the mathematicians Johann Radon and Otto Nikodym.

**Theorem 2.1.4 :**

Let  $\mu$  be a  $\sigma$ -finite measure on  $(X, \mathcal{A})$  and  $\nu$  a signed measure s.t.  $\nu \ll \mu$ . Then  $\nu$  has a density with respect to  $\mu$ , i.e. there exists a quasi-integrable function  $f : X \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$  s.t. for any  $A \in \mathcal{A}$

$$\nu(A) = \int_A f d\mu$$

In addition, if  $\nu$  is  $\sigma$ -finite too,  $f$  is uniquely determined up to  $\mu$ -null sets. The density  $f$  of  $\nu$  with respect to  $\mu$  is called the Radon-Nikodym derivative and is denoted by  $f = \frac{d\nu}{d\mu}$ .

**Remark :**

This is the general Radon-Nikodym theorem for  $\mu$  being a measure and  $\nu$  being a signed measure. One can replace the signed measure by a measure, then, as a consequence, the density  $f$  becomes nonnegative. This is the case which will be used in the next chapters.

The assumption on  $\mu$  regarding  $\sigma$ -finiteness is necessary, without that the theorem would fail. We illustrate this behaviour with a simple example.

**Example 2.1.5 :**

Consider the trivial  $\sigma$ -algebra  $\mathcal{A} := \{\emptyset, X\}$  and define  $\nu(\emptyset) := 0$ ,  $\nu(X) := 1$  and  $\mu(\emptyset) := 0$ ,  $\mu(X) := \infty$ . Then clearly  $\nu \ll \mu$  holds, but  $\nu$  does not have a density with respect to  $\mu$ .

Next, we want to study the convergence of nonnegative measurable functions  $f : X \rightarrow \mathbb{R}_+ \cup \{\infty\}$  on  $(X, \mathcal{A}, \mu)$ . Here we study the pointwise convergence. Recall that the integral of such a measurable  $f \geq 0$  is defined via a sequence of nonnegative simple functions  $(f_n)_{n \geq 1}$  with  $f_n \nearrow f$  s.t.

$$\int_X f d\mu := \lim_{n \rightarrow \infty} \int_X f_n d\mu$$

One can show, that this definition of the integral is independent of the chosen sequence  $(f_n)_{n \geq 1}$ . For a proof, see again [3].

We want to extend this convergence property to an increasing sequence  $f_n \geq 0$  of measurable functions which converge to  $f \geq 0$ .

So, when does  $\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X f d\mu$  hold? This question is answered by one of the most important convergence theorems in integration theory, namely the monotone convergence theorem, see [6].

**Theorem 2.1.6 :**

Let  $(f_n)_{n \geq 1}$  be an increasing sequence of nonnegative measurable functions and  $\lim_{n \rightarrow \infty} f_n =: f$ , i.e.  $0 \leq f_1(x) \leq f_2(x) \leq \dots \leq f_n(x) \nearrow f(x)$ . Then

$$\int_X f d\mu = \int_X \lim_{n \rightarrow \infty} f_n d\mu = \lim_{n \rightarrow \infty} \int_X f_n d\mu \quad (1)$$

If we consider partial sums of a series consisting only of nonnegative terms (as seen in [6]), which is obviously again an increasing sequence, the monotone convergence theorem yields

**Corollary 2.1.7 :**

Let  $(h_n)_{n \geq 1}$  be a sequence of nonnegative measurable functions. Then

$$\int_X \left( \sum_{n \geq 1} h_n \right) d\mu = \sum_{n \geq 1} \int_X h_n d\mu$$

Without monotonicity, convergence fails even for simple functions:

**Example 2.1.8 :**

Consider the function  $f_n := \frac{1}{n} \cdot \mathbb{1}_{[0,n]}$  on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ , where  $\lambda$  is the Lebesgue measure. Then  $f_n$  converges pointwise to 0 but the integral

$$\int_X \frac{1}{n} \cdot \mathbb{1}_{[0,n]} d\lambda = \frac{1}{n} \cdot \lambda([0, n]) = 1 \neq 0$$

does not.

Of course there is a more general monotone convergence theorem, where one trades off pointwise convergence, monotonicity for almost everywhere convergence, monotonicity and gets in exchange arbitrary measurable functions  $f_n, f : X \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$  for which (1) holds. But for our purpose, Theorem 2.1.6 will be just fine.

Another important application in measure theory is the general transformation formula, see [7]. This formula is a generalization of the well known concept of integration by substitution. In order to denote this formula, we just introduce the image measure.

**Definition 2.1.9 :**

Let  $(X, \mathcal{A}, \mu)$  and  $(\tilde{X}, \tilde{\mathcal{A}}, \tilde{\mu})$  be two measure spaces and  $S : X \rightarrow \tilde{X}$  be a measurable mapping. Then  $\tilde{\mu} := \mu \circ S^{-1}$  is a measure on  $(\tilde{X}, \tilde{\mathcal{A}})$  and  $\mu \circ S^{-1}$  is called the image measure with respect to  $S$  or pushforward of  $\mu$ .

**Theorem 2.1.10 :**

Let  $(X, \mathcal{A})$  and  $(\tilde{X}, \tilde{\mathcal{A}})$  be two measure spaces and  $\mu$  a measure on  $(X, \mathcal{A})$ . Let  $S : X \rightarrow \tilde{X}$  be a measurable mapping and  $\mu \circ S^{-1}$  be the corresponding image measure. Then for a measurable function  $g : \tilde{X} \rightarrow [0, \infty]$

$$\int_X g \circ S d\mu = \int_{\tilde{X}} g d(\mu \circ S^{-1})$$

Of course, this transformation formula stays valid, if we don't integrate over the whole respective space, but over measurable subsets  $A \in \mathcal{A}$  or  $S^{-1}A \in \tilde{\mathcal{A}}$ , respectively.

$$\int_{S^{-1}A} g \circ S d\mu = \int_A g d(\mu \circ S^{-1})$$

We will close this subsection by introducing product measures, which will

play an important role from section 5 on. We restrict here to the two dimensional case, since higher dimensions can be obtained by induction from the two dimensional case. For more content, see [3].

Our setup are two measure spaces  $(X, \mathcal{A}, \mu)$  and  $(Y, \mathcal{B}, \nu)$ . From this, we want to construct the product space  $(X \times Y, \mathcal{A} \otimes \mathcal{B}, \mu \otimes \nu)$ , where

$$\mathcal{A} \otimes \mathcal{B} := \sigma(\pi_1^{-1}\mathcal{A} \cup \pi_2^{-1}\mathcal{B}) \quad \text{and} \\ \mu \otimes \nu(A \times B) := \mu(A) \cdot \nu(B)$$

$\pi_1 : X \times Y \rightarrow X$  and  $\pi_2 : X \times Y \rightarrow Y$  denote the canonical projections onto  $X$  and  $Y$ , respectively.

Obviously,  $\mathcal{A} \otimes \mathcal{B}$  is also generated by  $\mathcal{A} \times \mathcal{B} = \{A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$ , but for the sake of the main proof, we define the product  $\sigma$ -algebra as written before.

Such a measure  $\mu \otimes \nu$  always exists, see for example [3] (Theorem 1.2, p. 164):

**Theorem 2.1.11 :**

Let  $(X, \mathcal{A}, \mu)$  and  $(Y, \mathcal{B}, \nu)$  be two measure spaces. There exists a measure  $\mu \otimes \nu : \mathcal{A} \otimes \mathcal{B} \rightarrow [0, \infty]$  such that for any  $A \in \mathcal{A}$  and  $B \in \mathcal{B}$

$$\mu \otimes \nu(A \times B) := \mu(A) \cdot \nu(B)$$

At this point, we know how to measure sets of the form  $A \times B \in \mathcal{A} \otimes \mathcal{B}$ , but how do we measure a set  $Q \subseteq A \times B$ ? To answer that, we introduce the  $x$ -section and  $y$ -section of  $Q$ .

**Definition 2.1.12 :**

Let  $Q \subseteq A \times B \in \mathcal{A} \otimes \mathcal{B}$ . Then the  $x$ -section  $Q_x$  and  $y$ -section  $Q^y$  of  $Q$  are defined as

$$Q_x := \{y \mid (x, y) \in Q\} \subseteq Y \\ Q^y := \{x \mid (x, y) \in Q\} \subseteq X$$

One can show that sets of this form are measurable with respect to  $\mu$  or  $\nu$ , respectively. The last Theorem in this subsection will tell us, how to measure  $\mu \otimes \nu(Q)$  and under which circumstances the product measure is unique.

**Theorem 2.1.13 :**

Let  $(X, \mathcal{A}, \mu)$  and  $(Y, \mathcal{B}, \nu)$  be two measure spaces, where  $\mu$  and  $\nu$  are  $\sigma$ -finite. Then there exists a unique measure  $\mu \otimes \nu$  on  $\mathcal{A} \otimes \mathcal{B}$ , which is again  $\sigma$ -finite and for  $Q \in \mathcal{A} \otimes \mathcal{B}$  the following holds:

$$\mu \otimes \nu(Q) = \int_X \nu(Q_x) \, d\mu(x) = \int_Y \nu(Q^y) \, d\mu(y)$$

So, looking at Theorem 2.1.13 geometrically, on the one hand we could measure every infinitesimally thin vertical stripe in  $Q$  or, on the other hand, every infinitesimally thin horizontal stripe in  $Q$ . The sum of all stripes (i.e. the integral) in both cases is  $\mu \otimes \nu(Q)$ .

## 2.2 Borel spaces

At this point we learned quite a lot useful methods to work on measure spaces. But now it is time to specify the measure space  $(X, \mathcal{A}, \mu)$ . In the first definition of this subsection we introduce Polish spaces and the Borel  $\sigma$ -algebra. For more content on this topic, see [2] and [8].

**Definition 2.2.1 :**

- $X$  is called a Polish space if  $X$  is
  - (i) metrizable, i.e. there exists a metric  $d$  on  $X$
  - (ii) complete, i.e. every Cauchy-sequence converges in  $X$  with respect to  $d$
  - (iii) separable, i.e. there exists a at most countable set  $C \subseteq X$  s.t.  $C$  is dense in  $X$
- Let  $X$  be a Polish space. Then  $\mathcal{B}_X$  denotes the Borel  $\sigma$ -algebra of  $X$ . Sets in  $\mathcal{B}_X$  are called Borel sets.  $\mathcal{B}_X$  is the smallest  $\sigma$ -algebra containing all open subsets of  $X$ .

**Definition 2.2.2 :**

Let  $X$  be a Polish space and  $\mathcal{B}_X$  the corresponding Borel  $\sigma$ -algebra. Then  $(X, \mathcal{B}_X)$  is called a standard Borel space.

Every time we write  $\mathcal{B}_X$ , we indicate, that the Borel  $\sigma$ -algebra is used. Next we want to study the cartesian product of Borel spaces and its product  $\sigma$ -algebra. Luckily, the next definition tells us, that we obtain again a Borel space:

**Definition 2.2.3 :**

Let  $(X_i, \mathcal{B}_{X_i})$  be a Borel space for every  $i \geq 1$ . Then  $X = \prod_{i \geq 1} X_i$  is the cartesian product and  $\mathcal{B}_X = \bigotimes_{i \geq 1} \mathcal{B}_{X_i}$  is the product  $\sigma$ -algebra of  $X$ . Define  $\pi_i : X \rightarrow X_i$  to be the projection from  $X$  onto its  $i$ -th component. Then  $(X, \mathcal{B}_X)$  is again a Borel space where furthermore

$$\mathcal{B}_X = \bigotimes_{i \geq 1} \mathcal{B}_{X_i} := \sigma(\{\pi_i^{-1}(B_i) \mid B_i \in \mathcal{B}_{X_i}, i \geq 1\})$$

i.e.,  $\mathcal{B}_X$  is the smallest  $\sigma$ -algebra with respect to which all projections  $\pi_i$  are measurable.

**Remark :**

In the definition of the product  $\sigma$ -algebra, one can replace  $B_i \in \mathcal{B}_{X_i}$  by  $B_i \in \mathcal{E}_{X_i}$ , where  $\mathcal{E}_{X_i}$  is a generator of  $\mathcal{B}_{X_i}$ . This due to the fact, that  $\pi_i : X \rightarrow X_i$  is measurable as soon as  $\pi_i^{-1}\mathcal{E}_{X_i} \subseteq \mathcal{B}_X$  for some generator  $\mathcal{E}_{X_i}$  of  $\mathcal{B}_{X_i}$  (Proof:  $\pi_i^{-1}\mathcal{B}_{X_i} = \pi_i^{-1}\sigma(\mathcal{E}_{X_i}) = \sigma(\pi_i^{-1}\mathcal{E}_{X_i}) \subseteq \sigma(\mathcal{B}_X) = \mathcal{B}_X$ ).

## 2.3 Background ergodic theory

We spent the first two subsections on our measure space  $(X, \mathcal{A}, \mu)$ . Now the time has come to work on  $S$ . As already mentioned,  $S$  shall always be measurable, i.e.  $S^{-1}A \in \mathcal{A}$  for some  $A \in \mathcal{A}$ .

Furthermore,  $S$  shall have certain properties, which are respected by our measure  $\mu$ . See [8] for more information.

### Definition 2.3.1 :

Let  $S : X \rightarrow X$  be measurable. Then

- (i)  $S$  is called null-preserving, if  $\forall A \in \mathcal{A} : \mu(A) = 0 \implies \mu(S^{-1}A) = 0$ . Then  $(X, \mathcal{A}, \mu, S)$  is called a null-preserving dynamical system.
- (ii)  $S$  is non-singular, if  $\forall A \in \mathcal{A} : \mu(A) = 0 \iff \mu(S^{-1}A) = 0$ . Then  $(X, \mathcal{A}, \mu, S)$  is called a non-singular dynamical system.
- (iii)  $S$  is called measure preserving, if  $\forall A \in \mathcal{A} : \mu(A) = \mu(S^{-1}A)$ . Then  $(X, \mathcal{A}, \mu, S)$  is called a measure preserving dynamical system.

Note that we have the following implications:

measure preserving  $\implies$  non-singular  $\implies$  null-preserving.

In other words, if something is satisfied for a null-preserving dynamical system, all the more it is for non-singular and measure preserving ones.

### Definition 2.3.2 :

Let  $(X, \mathcal{A}, \mu, S)$  be a dynamical system. We say that  $(X, \mathcal{A}, \mu, S)$  is invertible, if  $S$  is invertible, i.e.  $\exists S^{-1}$  measurable s.t.  $S \circ S^{-1} = S^{-1} \circ S$ .

Next, we want to analyse a map between the state spaces  $X, \tilde{X}$  between two dynamical systems  $(X, \mathcal{A}, \mu, S)$  and  $(\tilde{X}, \tilde{\mathcal{A}}, \tilde{\mu}, \tilde{S})$ .

### Definition 2.3.3 :

Let  $(X, \mathcal{A}, \mu, S)$  and  $(\tilde{X}, \tilde{\mathcal{A}}, \tilde{\mu}, \tilde{S})$  be two null-preserving dynamical systems. We say that a map  $\phi : \tilde{X} \rightarrow X$  is a factor map, if it is surjective and the following diagram commutes:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{S}} & \tilde{X} \\ \phi \downarrow & & \downarrow \phi \\ X & \xrightarrow{S} & X \end{array}$$

Then  $(X, \mathcal{A}, \mu, S)$  is a factor space of  $(\tilde{X}, \tilde{\mathcal{A}}, \tilde{\mu}, \tilde{S})$ . Conversely,  $(\tilde{X}, \tilde{\mathcal{A}}, \tilde{\mu}, \tilde{S})$  is called an extension of  $(X, \mathcal{A}, \mu, S)$ . In addition, if  $\phi$  is invertible, then  $\phi$  is called an isomorphism of  $(X, \mathcal{A}, \mu, S)$  and  $(\tilde{X}, \tilde{\mathcal{A}}, \tilde{\mu}, \tilde{S})$ .

**Remark :**

- (i) If  $\phi$  is non-singular or measure preserving, we say that  $\phi$  is a non-singular or measure preserving factor map.
- (ii) One could write  $\phi : \tilde{S} \rightarrow S$ , indicating, that  $\tilde{S}$  is an extension of  $S$  and  $S$  is a factor of  $\tilde{S}$ . If  $\phi$  is bijective, we denote this isomorphism as  $\phi : \tilde{S} \longleftrightarrow S$ .

### 3 Towards the main theorem

In this section we do the last preparations to finally state the main theorem of this thesis. These preparations are taken from [1].

Let  $(Z, \Sigma, \nu, T)$  be a dynamical system, where  $(Z, \Sigma, \nu)$  is a measure space and  $T : Z \rightarrow Z$  is an automorphism of  $Z$ . Recall that an automorphism  $T$  is a bijective map and furthermore  $T$  and  $T^{-1}$  are endomorphisms. Let  $\xi$  be a partition of  $Z$ , i.e.  $\bigcup_{V \in \xi} V = Z$  and this union is disjoint. Since the preimage operator preserves unions and intersections,  $T^{-1}\xi$  is again a partition. We introduce the notation  $\mathfrak{B} \leq \mathfrak{A}$  meaning that  $\mathfrak{A}$  is a finer partition than  $\mathfrak{B}$ , i.e.  $\forall A \in \mathfrak{A} \exists B \in \mathfrak{B} \text{ s.t. } A \subseteq B$ .

**Remark :**

This finer-relation on the sets of partitions of a set  $Z$  forms a partial order, so the notation  $\leq$  is appropriate.

Now let  $T^{-1}\xi \leq \xi$ , i.e.  $\xi$  is invariant with respect to  $T$ . Applying  $T$  on those partitions, we get a sequence of ever-finer partitions

$$\xi \leq T\xi \leq T^2\xi \leq T^3\xi \dots$$

Let  $\prod_{k=0}^{\infty} T^k\xi$  be the coarsest partition, which is finer than  $T^k\xi$  for every  $k \geq 0$ .

**Definition 3.1 :**

Let  $\xi$  be a partition of  $Z$  and  $T : Z \rightarrow Z$  an automorphism. We say that  $\xi$  is exhaustive with respect to  $T$ , if  $\prod_{k=0}^{\infty} T^k\xi$  consists only of singletons.

Having such a partition  $\xi$  satisfying this property, we obtain another measure space  $(Z/\xi, \Sigma_\xi, \nu_\xi)$ , where  $Z/\xi \equiv \xi$  and  $G_\xi : Z \rightarrow Z/\xi$  is a factor map s.t.  $G_\xi(x) := [x]$ , where  $[x]$  denotes the unique (since any two sets in  $\xi$  are pairwise disjoint) set of the partition  $\xi$  containing  $x$ . Therefore, we call  $(Z/\xi, \Sigma_\xi, \nu_\xi)$  the factor space of  $Z$  with respect to  $\xi$ .

Next we define the  $\sigma$ -algebra  $\Sigma$  as follows:

$$A \in \Sigma_\xi \iff G_\xi^{-1}A \in \Sigma$$

This way defined,  $G_\xi$  is measurable and consequently a homomorphism. Last but not least, set  $\nu_\xi(A) = \nu(G_\xi^{-1}A)$ . By Definition 2.3.1,  $G_\xi$  is a measure preserving factor map.

We want to define a map  $T_\xi : Z/\xi \rightarrow Z/\xi$ , which is an endomorphism, s.t. we get the following commutative diagram:

$$\begin{array}{ccc} Z & \xrightarrow{T} & Z \\ G_\xi \downarrow & & \downarrow G_\xi \\ Z/\xi & \xrightarrow{T_\xi} & Z/\xi \end{array}$$

But how should  $T_\xi$  look like? Let's try

$$T_\xi(A) := B \iff T(A) \subseteq B \quad (2)$$

Our assumption  $T^{-1}\xi \leq \xi$  makes this map a well-defined map s.t.  $T_\xi \circ G_\xi = G_\xi \circ T$ . Since the right-hand side is measurable (since  $G_\xi$  and  $T$  are), the left-hand side is too and consequently  $T_\xi$  is.

We are now ready to define the natural extension of an endomorphism.

**Definition 3.2 :**

Let  $(X, \mathcal{A}, \mu)$  be a measure space and  $S : X \rightarrow X$  be an endomorphism. We say that  $S$  has a natural extension  $T$  if there exist:

- (i) a measure space  $(Z, \Sigma, \nu)$
- (ii) an automorphism  $T : Z \rightarrow Z$
- (iii) a partition  $\xi$  of  $Z$  which is exhaustive with respect to  $T$

s.t.  $Z/\xi \cong X$  and  $T_\xi \cong S$ . This indicates the existence of a measure preserving isomorphism  $f$ , i.e.  $\mu(A) = \nu_\xi(f^{-1}(A)) \quad \forall A \in \mathcal{A}$ , s.t. the following diagram commutes:

$$\begin{array}{ccc} Z/\xi & \xrightarrow{T_\xi} & Z/\xi \\ f \downarrow & & \downarrow f \\ X & \xrightarrow{S} & X \end{array}$$

Now we know what a natural extension is. Natural in this sense means, that the measure space  $(Z, \Sigma, \nu)$  is the smallest possible space, where  $T$  is an invertible extension of  $S$  such that  $Z/\xi \cong X$  and  $T_\xi \cong S$ .

Finally, it is time to state the main theorem of this thesis.

**Theorem 3.3 :**

Let  $(X, \mathcal{B}_X, \mu)$  be a standard Borel space, where  $\mu$  is a continuous probability measure, i.e.  $\mu(X) = 1$ , and  $S : X \rightarrow X$  a nonsingular endomorphism. Then there exists a natural extension of  $S$ .

## 4 Motivating examples

Before starting with the first proof, we want to introduce two basic examples to illustrate, how such a natural extension could be constructed.

### Exapmle 4.1 :

Let  $X$  be the unit interval  $[0, 1]$ , equipped with  $\mathcal{B}_X = \mathcal{B}_{[0,1]}$  and  $\mu = \lambda^1$ , the Lebesgue measure. We define a transformation  $S : X \rightarrow X$  as

$$S(x) := 2x \bmod 1$$

Then  $S$  is clearly a non-singular endomorphism. Geometrically speaking,  $S$  does the following:

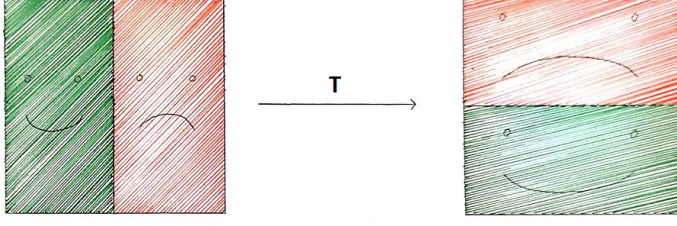


We cut the interval in the middle to get two intervals  $[0, \frac{1}{2}]$  and  $[\frac{1}{2}, 1]$ , where we put the equal amount of green and red mass, respectively. The interval  $[0, \frac{1}{2}]$  is expanded to  $[0, 1]$ , whereas the same happens for  $[\frac{1}{2}, 1]$ . Then we lay them on top of each other, where the expansion of  $[0, \frac{1}{2}]$  is at the bottom.

Obvioulsy,  $S$  is not invertible, since for any  $z \in SX$  there are always  $x_1 \in [0, \frac{1}{2}]$  and  $x_2 \in [\frac{1}{2}, 1]$  such that  $S(x_i) = z$  for  $i = 1, 2$ . We need information, in which mass a point  $z$  lies. Therefore, we have to introduce a new variable  $y$ , which represents the thickness of the red and green mass. For simplicity, assume that the thickness is 1, which gives us a new space  $Z = [0, 1]^2$  with  $\Sigma = \mathcal{B}_{[0,1]^2}$  and  $\nu = \lambda^2$ , the two-dimensional Lebesgue measure. Now define a map  $T : Z \rightarrow Z$  via

$$T(x, y) := \begin{cases} (S(x), \frac{y}{2} + \frac{1}{2}) & \text{if } x > \frac{1}{2} \\ (S(x), \frac{y}{2}) & \text{if } x \leq \frac{1}{2} \end{cases}$$

This map is called the baker's map, because the map does exactly what bakers do with dough, cut in half, stack together and compress:



In this setting,  $T$  is invertible and therefore an automorphism. Now consider the following family of sets:

$$E_x := \{x\} \times [0, 1]$$

$E_x$  represents a vertical line of length 1. Furthermore,  $(E_x)_{x \in X}$  forms a partition of  $Z$ , where  $E_{x_1} \cap E_{x_2} = \emptyset$  for  $x_1 \neq x_2$ . Applying  $T$  on these sets provides us

$$TE_x = \begin{cases} \{S(x)\} \times [0, \frac{1}{2}] & \text{if } x \leq \frac{1}{2} \\ \{S(x)\} \times [\frac{1}{2}, 1] & \text{if } x > \frac{1}{2} \end{cases}$$

In other words, we get again a vertical line, but this time with length  $\frac{1}{2}$ . Moreover, if we write  $\xi := \{E_x \mid x \in [0, 1]\}$  we have

$$\xi \leq T\xi$$

Iterating this step gives us new partitions, which get finer and finer, i.e.  $\xi$  is an exhaustive partition. Define a homomorphism  $G_\xi : Z \rightarrow Z/\xi$  via

$$G_\xi(x, y) := \{x\} \times [0, 1] = E_x$$

In this case we find an isomorphism  $f : Z/\xi \rightarrow X$ , i.e.  $Z/\xi \cong X$ , which basically is just a projection onto the first coordinate:

$$f(E_x) = x$$

Next define  $T_\xi : Z/\xi \rightarrow Z/\xi$  as  $T_\xi(E_x) := E_{S(x)}$ , meaning that  $T_\xi \cong S$ . As a consequence, we get the following commutative diagram:

$$\begin{array}{ccc} Z/\xi & \xrightarrow{T_\xi} & Z/\xi \\ f \downarrow & & \downarrow f \\ X & \xrightarrow{S} & X \end{array}$$

What is left to check is, that  $f$  preserves measure between  $(Z/\xi, \Sigma_\xi, \nu_\xi)$  and  $(X, \mathcal{B}_X, \lambda^1)$ , then  $T$  is a natural extension of  $S$ .

So assume that for any  $A \in \mathcal{A}$  we have  $\lambda^1(A) = a$ . Then, using  $\nu_\xi = \lambda^2 \circ G_\xi^{-1}$  we have

$$\begin{aligned}\nu_\xi(f^{-1}(A)) &= \lambda^2(G_\xi^{-1}(\{E_x \mid x \in A\})) \\ &= \lambda^2(A \times [0, 1]) \\ &= \lambda^1(A) \cdot \lambda^1([0, 1]) \\ &= a\end{aligned}$$

This makes  $f$  measure preserving and  $T$  a natural extension of  $S$ .

**Remark :**

One could look at this problem from a probability theoretic point of view. We can identify any point in  $X = [0, 1]$  with a infinite sequence of coin tosses.

$$X \cong \{\omega = (\omega_0, \omega_1, \dots) \mid \omega_i \in \{0, 1\}\}$$

At every step we divide  $[0, 1]$  in two intervals of the same length. Now 0 represents the left sided interval of every division and 1 every right sided interval. This way, every 0 – 1 sequence yields one point in  $X$ .

On this space,  $S$  acts like the right-shift, i.e.  $(S(\omega))_i := \omega_{i+1}$ . So  $S$  forgets about the past. On the other hand, we can identify  $Z$  via

$$Z \cong \{\tilde{\omega} = (\dots, \omega_{-2}, \omega_{-1}, \omega_0, \omega_1, \dots) \mid \omega_i \in \{0, 1\}\}$$

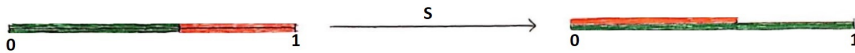
Here  $T$  is again the right-shift,  $(T(\omega))_i := \omega_{i+1}$ , but now, since the sequence is two-sided, we can always go to the past.

**Example 4.2 :**

This time we consider a slightly different setup. Let again be  $X = [0, 1]$  and  $\mathcal{B}_X = \mathcal{B}_{[0,1]}$ . Define

$$S(x) := \beta x \bmod 1$$

such that  $\beta > 1$  satisfies  $\beta - 1 = \frac{1}{\beta}$ . It is easy to calculate that  $\beta = \frac{1+\sqrt{5}}{2}$ , i.e. the golden ratio. Geometrically speaking, we have:



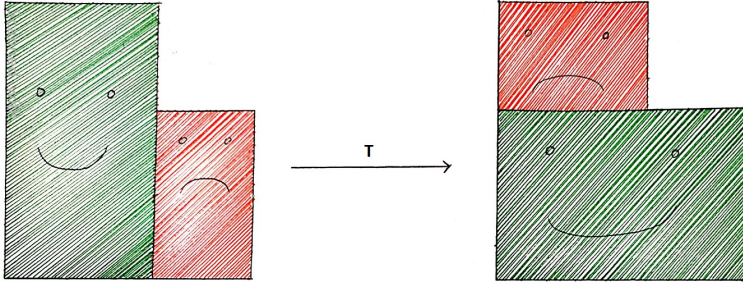
Basically, the same happens as in Example 4.1. The difference is, that now

we divide  $X$  into  $[0, \frac{1}{\beta}]$  (green) and  $[\frac{1}{\beta}, 1]$  (red).

$S$  expands both intervals by a factor  $\beta$ , such that the green mass covers  $X$  and the red mass covers  $[0, \frac{1}{\beta}]$ , since we lay again both intervals on top of each other and  $(1 - \frac{1}{\beta})\beta = \beta - 1 = \frac{1}{\beta}$ . Clearly,  $S$  is not invertible again. This time we use a different strategy. We introduce a space  $Z$  with a map  $T : Z \rightarrow Z$ , where we find a measure preserving measure  $\nu$ . From this, we will construct, assuming that  $T$  extends  $S$  naturally, a measure preserving measure  $\mu$  on  $(X, \mathcal{B}_X)$ .

Define  $Z := [0, \frac{1}{\beta}] \times [0, a] \cup [\frac{1}{\beta}, 1] \times [0, \frac{a}{\beta}]$ , where  $a > 1$  is a real number, with its Borel  $\sigma$ -algebra  $\mathcal{B}_Z$  and  $T : Z \rightarrow Z$  as

$$T(x, y) := \begin{cases} (S(x), \frac{y}{\beta} + \frac{a}{\beta}) & \text{if } x > \frac{1}{\beta} \\ (S(x), \frac{y}{\beta}) & \text{if } x \leq \frac{1}{\beta} \end{cases}$$



Again,  $T$  is invertible. On this space, define  $\nu := \lambda^2|_Z$ . Here we want  $\lambda^2|_Z(Z) = 1$ , which in order forces  $a$  to be

$$\begin{aligned} 1 = \lambda^2|_Z(Z) &= \lambda^2([0, \frac{1}{\beta}] \times [0, a] \cup [\frac{1}{\beta}, 1] \times [0, \frac{a}{\beta}]) \\ &= \lambda^2([0, \frac{1}{\beta}] \times [0, a]) + \lambda^2([\frac{1}{\beta}, 1] \times [0, \frac{a}{\beta}]) \\ &= \frac{a}{\beta} + (1 - \frac{1}{\beta}) \cdot \frac{a}{\beta} \\ &\iff a = \frac{\beta^2}{2\beta - 1} = \frac{1}{2} + \frac{3 \cdot \sqrt{5}}{10} \end{aligned}$$

So  $a > 1$  indeed. Moreover, we have that  $T$  is indeed measure preserving, i.e.  $\lambda^2|_Z \circ T^{-1} = \lambda^2|_Z$ . It is sufficient to show this equation for any rectangle  $M \times N$ . Since the family of all rectangles generates  $\mathcal{B}_Z$ , the result extends

for general sets in  $\mathcal{B}_Z$  by the Standard Uniqueness Theorem. Note that

$$\begin{aligned} T^{-1}(M \times N) = & \{x \in M \mid \frac{x}{\beta}\} \times \{y \in N \cap [0, \frac{a}{\beta}] \mid y \cdot \beta\} \\ & \cup \{x \in M \mid \frac{x}{\beta}\} \times \{y \in N \cap [\frac{a}{\beta}, a] \mid y \cdot \beta - \frac{a}{\beta}\} \end{aligned}$$

Now suppose that  $\lambda^1(M) := m$ ,  $\lambda^1(N) := n$ ,  $\lambda^1(N \cap [0, \frac{a}{\beta}]) := n_1$  and  $\lambda^1(N \cap [\frac{a}{\beta}, a]) := n_2$ . This provides us

$$\begin{aligned} \lambda^2|_Z(T^{-1}(M \times N)) &= \lambda^2(T^{-1}(M \times N)) \\ &= \lambda^2(\{x \in M \mid \frac{x}{\beta}\} \times \{y \in N \cap [0, \frac{a}{\beta}] \mid y \cdot \beta\}) \\ &\quad + \lambda^2(\{x \in M \mid \frac{x}{\beta}\} \times \{y \in N \cap [\frac{a}{\beta}, a] \mid y \cdot \beta - \frac{a}{\beta}\}) \\ &= \frac{m}{\beta} \cdot n_1\beta + \frac{m}{\beta} \cdot n_2\beta \\ &= m(n_1 + n_2) \\ &= m \cdot n \\ &= \lambda^1(M) \cdot \lambda^1(N) \\ &= \lambda^2(M \times N) \\ &= \lambda^2|_Z(M \times N) \end{aligned}$$

Indeed, we found a measure preserving measure under  $T$ . Before we construct  $\mu$ , we first observe that

$$\begin{array}{ccc} Z & \xrightarrow{T} & Z \\ G_\xi \downarrow & & \downarrow G_\xi \\ Z/\xi & \xrightarrow{T_\xi} & Z/\xi \\ f \downarrow & & \downarrow f \\ X & \xrightarrow{s} & X \end{array}$$

where  $Z/\xi := \{E_x \mid x \in [0, 1]\}$ ,  $G_\xi(x, y) := E_x$ ,  $f(E_x) := x$  and

$$E_x := \begin{cases} \{x\} \times [0, \frac{a}{\beta}] & \text{if } x > \frac{1}{\beta} \\ \{x\} \times [0, a] & \text{if } x \leq \frac{1}{\beta} \end{cases}$$

Note that  $\xi := \{E_x \mid x \in [0, 1]\}$  is again an exhaustive partition w.r.t.  $T$ . Now define  $\mu$  on  $([0, 1], \mathcal{B}_{[0,1]})$  via

$$\mu := \lambda^2 \mid_Z \circ (f \circ G_\xi)^{-1}$$

For simplicity, let  $\gamma := f \circ G_\xi$ . Note that

$$S \circ \gamma = \gamma \circ T$$

Altogether, this yields measure preservation of  $\mu$  under  $S$ , since for any  $A \in \mathcal{B}_{[0,1]}$

$$\begin{aligned} \mu \circ S^{-1}(A) &= \lambda^2 \mid_Z \circ \gamma^{-1} \circ S^{-1}(A) \\ &= \lambda^2 \mid_Z \circ T^{-1} \circ \gamma^{-1}(A) \\ &= \lambda^2 \mid_Z \circ \gamma^{-1}(A) \\ &= \mu(A) \end{aligned}$$

We want to identify, what measure  $\mu$  is as a matter of fact. Therefore, for  $A \in \mathcal{B}_{[0,1]}$  and  $A \subseteq [0, \frac{1}{\beta}]$ , calculate

$$\begin{aligned} \mu(A) &= \lambda^2 \mid_Z (G_\xi^{-1}(f^{-1}(A))) \\ &= \lambda^2 \mid_Z (G_\xi^{-1}(\{E_x \mid x \in A\})) \\ &= \lambda^2 \mid_Z (A \times [0, a]) \\ &= a \cdot \lambda^1(A) \end{aligned}$$

On the other hand, for  $A \in \mathcal{B}_{[0,1]}$  and  $A \subseteq [\frac{1}{\beta}, 1]$ , we find that

$$\begin{aligned} \mu(A) &= \lambda^2 \mid_Z (G_\xi^{-1}(f^{-1}(A))) \\ &= \lambda^2 \mid_Z (G_\xi^{-1}(\{E_x \mid x \in A\})) \\ &= \lambda^2 \mid_Z (A \times [0, \frac{a}{\beta}]) \\ &= \frac{a}{\beta} \cdot \lambda^1(A) \end{aligned}$$

As a consequence, we can define a density  $g := a \cdot \mathbb{1}_{[0, \frac{1}{\beta}]} + \frac{a}{\beta} \cdot \mathbb{1}_{[\frac{1}{\beta}, 1]}$ , such that, in the end we get

$$\mu = g \odot \lambda^1$$

So, using these measures  $\mu$  and  $\nu$ ,  $T$  is a natural extension of  $S$ , and we are done.

**Remark :**

Once again, we could use infinite 0 – 1 sequences to identify  $X$  and  $Z$ , with a small restriction this time, namely

$$X \cong \{\omega = (\omega_0, \omega_1, \omega_2, \dots) \mid \omega_i \in \{0, 1\} \text{ and if } \omega_i = 1, \text{ then } \omega_{i+1} = 0\}$$

and

$$Z \cong \{\omega = (\dots, \omega_{-1}, \omega_0, \omega_1, \dots) \mid \omega_i \in \{0, 1\} \text{ and if } \omega_i = 1, \text{ then } \omega_{i+1} = 0\}$$

## 5 The Proof

As mentioned in the introduction, we won't immediately prove the nonsingular case, but discuss the measure preserving case first. The measure preserving case is well known since V.A. Rohlin proved this case in the 1960s (see [5]). Furthermore, for  $S$  being measure preserving, the natural extension is even unique up to isomorphisms. However, uniqueness is not the goal of this thesis.

We will construct a proof for existence in the measure preserving case obtained from Beneteaus proof for nonsingularity ([1]) and then continue to prove the nonsingular case.

### 5.1 The measure preserving case

The proof will consist of two parts. Firstly, we construct a measure preserving automorphism  $T$ , which will represent the invertible extension, and secondly we look for the desired exhaustive partition  $\xi$ .

As a warm up, before starting with the first part of the proof, define for  $n \geq 1$  cartesian products

$$X^n := \prod_{k=0}^{n-1} X = \{(x_0, \dots, x_{n-1}) \mid x_k \in X \text{ for all } k \leq n-1\}$$

$$\bar{X} := \prod_{k \geq 0} X = \{(x_0, x_1, \dots) \mid x_k \in X \text{ for all } k\}$$

Define  $Z := \{(x_0, x_1, \dots) \mid x_i \in X, S(x_{i+1}) = x_i, i = 0, 1, \dots\}$ .  $Z$  is a subset of  $\bar{X}$ . Our goal is to define a measure  $\nu$  on  $\bar{X}$  which is concentrated on  $Z$ , i.e.  $\nu(Z) = 1$ . In order to do this, we have to be sure that  $Z$  is contained in the product  $\sigma$ -algebra  $\mathcal{B}_{\bar{X}}$ , which coincides with the Borel  $\sigma$ -algebra, see subsection 2.2.

**Lemma 5.1.1 :**

$Z \in \mathcal{B}_{\bar{X}}$ .

**Proof :**

Define  $Y^n := \{(x_0, x_1, \dots, x_{n-1}) \mid x_i \in X, S(x_{i+1}) = x_i, i = 0, 1, \dots, n-2\}$ . Now set  $Z^n := Y^n \times \prod_{k=n}^{\infty} X$ . Then obviously  $Z = \bigcap_{n \geq 0} Z^n$ . Consequently, if we prove that  $Y^n$  is measurable with respect to the finite product  $\sigma$ -algebra  $\mathcal{B}_{X^n}$ , then  $Z$  is automatically measurable. We do this by induction on  $n$ .

Let  $n = 0$ . Clearly  $Y^0 = X \in \mathcal{B}_X$ .

Let  $n = 1$ . Note that  $Y^1 = \{(x_0, x_1) \in X \times X \mid S(x_1) = x_0\} = \{(S(x), x) \mid x \in X\}$  is essentially the graph of a measurable function. Luckily, according to Theorem 3.3. in [2], such a graph is always measurable:

**Theorem 5.1.2 :**

Let  $f : X_1 \rightarrow X_2$  be a measurable map between two separable metric spaces  $X_1$  and  $X_2$ . Then  $E := \{(x_1, f(x_1)) \mid x_1 \in X_1\}$  is a Borel set of  $X_1 \times X_2$ , i.e.  $E \in \mathcal{B}_{X_1} \otimes \mathcal{B}_{X_2}$ .

Since  $X$  is Polish, we can apply Theorem 5.1.2 and therefore  $Y^1 \in \mathcal{B}_X \otimes \mathcal{B}_X$ . Now suppose that  $Y^{n-1} \in \mathcal{B}_{X^{n-1}}$ . We use our assumption  $S(x_{i+1}) = x_i$  to write  $Y^n$  as

$$\begin{aligned} Y^n &= \{(S^{n-1}(x), S^{n-2}(x), \dots, S(x), x) \mid x \in X\} \\ &\stackrel{!}{=} \{(S^{n-2}(z), S^{n-3}(z), \dots, S(z), y) \mid y, z \in X\} \\ &\cap \{y, S^{n-2}(x), \dots, S(x), x \mid x, y \in X\} \end{aligned}$$

Comparing the first component yields  $S^{n-2}(z) = y$ . Comparing the second component yields

$$\begin{aligned} S^{n-3}(z) = S^{n-2}(x) &\Leftrightarrow z = S(x) \Leftrightarrow S^{n-2}(z) = S^{n-1}(x) \\ &\Rightarrow S^{n-1}(x) = y \end{aligned}$$

by the first comparison. So, basically  $Y^n = (Y^{n-1} \times X) \cap (X \times Y^{n-1})$ . Since  $Y^{n-1} \in \mathcal{B}_{X^{n-1}}$  by our induction hypothesis,  $Y^n \in \mathcal{B}_{X^n}$ . This proves the Lemma.  $\square$

Now that we know that  $Z$  is measurable with respect to the product  $\sigma$ -algebra, we can define a measure  $\nu$  on  $\bar{X}$ . We do this by defining the following set function  $\nu^n$  on the finite product space  $X^n$ .

For  $B_0, B_1, \dots, B_{n-1} \in \mathcal{B}_X$  let

$$\nu^n(B_0 \times \dots \times B_{n-1}) := \int_X \mathbb{1}_{S^{-(n-1)}B_0 \cap S^{-(n-2)}B_1 \cap \dots \cap B_{n-1}} d\mu$$

We want to show that  $\nu^n$  is a measure on  $X^n$ , Firstly, clearly  $\nu^n(\emptyset) = 0$  since

$$\nu^n(\emptyset) = \nu^n(\emptyset \times \dots \times \emptyset) = \int_X \mathbb{1}_{S^{-(n-1)}\emptyset \cap \dots \cap \emptyset} d\mu = \int_X \mathbb{1}_{\emptyset} d\mu = \mu(\emptyset) = 0$$

Secondly, we have to show  $\sigma$ -additivity. In order to do this, it is enough to consider rectangles. Assume that

$$B_0 \times \dots \times B_{n-1} = \bigcup_{k \geq 1} (B_0^k \times \dots \times B_{n-1}^k) \quad (3)$$

i.e., the set  $B_0 \times \dots \times B_{n-1} \in \mathcal{B}_{X^n}$  is a union of measurable rectangles. To show  $\sigma$ -additivity, suppose that the union on the right-hand side is disjoint. For  $x_{n-1} \in X$ , we observe the following equivalence:

$$\begin{aligned} (S^{n-1}(x_{n-1}), S^{n-2}(x_{n-1}), \dots, x_{n-1}) \in B_0 \times B_1 \times \dots \times B_{n-1} &\iff \\ x_{n-1} \in S^{-(n-1)}B_0 \cap S^{-(n-2)}B_1 \cap \dots \cap B_{n-1} \end{aligned}$$

So (3) can be rewritten as

$$\begin{aligned} &S^{-(n-1)}B_0 \cap S^{-(n-2)}B_1 \cap \dots \cap B_{n-1} \\ &= \bigcup_{k \geq 1} (S^{-(n-1)}B_0^k \cap S^{-(n-2)}B_1^k \cap \dots \cap B_{n-1}^k) \end{aligned}$$

where again the union is disjoint. Recall that for a disjoint union  $A \cup B$  we have  $\mathbb{1}_{A \cup B} = \mathbb{1}_A + \mathbb{1}_B$ . Using this and Corollary 2.1.7 we finally get

$$\begin{aligned} \nu^n(B_0 \times \dots \times B_{n-1}) &= \int_X \mathbb{1}_{S^{-(n-1)}B_0 \cap S^{-(n-2)}B_1 \cap \dots \cap B_{n-1}} d\mu \\ &= \int_X \sum_{k \geq 1} \mathbb{1}_{S^{-(n-1)}B_0^k \cap S^{-(n-2)}B_1^k \cap \dots \cap B_{n-1}^k} d\mu \\ (\text{Corollary 2.1.7}) &= \sum_{k \geq 1} \int_X \mathbb{1}_{S^{-(n-1)}B_0^k \cap S^{-(n-2)}B_1^k \cap \dots \cap B_{n-1}^k} d\mu \\ &= \sum_{k \geq 1} \nu^n(B_0^k \times B_1^k \times \dots \times B_{n-1}^k) \end{aligned}$$

So  $\nu^n$  is  $\sigma$ -additive on rectangles. The family of all  $n$ -dimensional rectangles forms a semiring  $\mathcal{S}^n$ , which furthermore generates  $\mathcal{B}_{X^n}$ , i.e.  $\sigma(\mathcal{S}^n) = \mathcal{B}_{X^n}$ . Thus the pre measure  $\nu^n$  can be extended to a measure on  $(X^n, \mathcal{B}_{X^n})$ . By the Standard Uniqueness Theorem for measures, this extension is unique and we call (still) denote this measure as  $\nu^n$ .

But that is not enough, since our goal was to construct a measure on  $\bar{X}$ .

For the next step, we shall observe that the family  $(\nu^n)$  forms a consistent sequence of measures, namely

$$\nu^{n+1}(B_0 \times \dots \times B_{n-1} \times X) = \nu^n(B_0 \times \dots \times B_{n-1})$$

So let's calculate  $\nu^{n+1}(B_0 \times \dots \times B_{n-1} \times X)$ :

$$\begin{aligned} \nu^{n+1}(B_0 \times \dots \times B_{n-1} \times X) &= \int_X \mathbb{1}_{S^{-n}B_0 \cap S^{-(n-1)}B_1 \cap \dots \cap S^{-1}B_{n-1} \cap X} d\mu \\ (S^{-1}X = X, \text{ since } S \text{ surjective}) &= \int_X \mathbb{1}_{S^{-1}(S^{-(n-1)}B_0 \cap S^{-(n-2)}B_1 \cap \dots \cap B_{n-1})} d\mu \\ &= \int_X \mathbb{1}_{S^{-(n-1)}B_0 \cap S^{-(n-2)}B_1 \cap \dots \cap B_{n-1}} \circ S d\mu \\ (\text{Theorem 2.1.10}) &= \int_X \mathbb{1}_{S^{-(n-1)}B_0 \cap S^{-(n-2)}B_1 \cap \dots \cap B_{n-1}} d(\mu \circ S^{-1}) \\ (\mu = \mu \circ S^{-1}) &= \int_X \mathbb{1}_{S^{-(n-1)}B_0 \cap S^{-(n-2)}B_1 \cap \dots \cap B_{n-1}} d\mu \\ &= \nu^n(B_0 \times \dots \times B_{n-1}) \end{aligned}$$

So this proves consistency. Now we are done, because the Daniell Kolmogorov Consistency Theorem in [2] (Theorem 5.1) finishes our job:

**Theorem 5.1.3 :**

Let  $I$  be any index set and let  $(X_i, \mathcal{B}_{X_i})$  be a standard Borel space for  $i \in I$ . Let  $X := \prod_{i \in I} X_i$  and  $\mathcal{B}_X := \bigotimes_{i \in I} \mathcal{B}_{X_i}$ .

If  $\{\nu_J \mid J \subseteq I, J \text{ finite}\}$  forms a consistent family of probability measures, then there exists a unique probability measure  $\nu$  on  $(X, \mathcal{B}_X)$  s.t.  $\nu_J(A) = \nu(\pi_J^{-1}(A))$ , where  $\pi_J$  is the restriction from  $I$  to  $J$ , for all finite  $J \subseteq I$  and  $A \in \bigotimes_{j \in J} \mathcal{B}_{X_j}$ .

Since our  $(X, \mathcal{B}_X)$  is standard Borel by assumption, the family  $(\nu^n)$  extends to a unique measure on  $\bar{X}$ . We call this measure  $\nu$ . Note at this point, that  $\nu(\bar{X}) = \nu^1(X) = \mu(X) = 1$ .

Next, we want to show that  $\nu(Z) = 1$ , which is equivalent to the fact, that  $\nu$  is zero outside of  $Z$ . In [1], a calculation yields  $\nu(Z^n) = 1$ , and, due to  $Z_n \searrow Z$  it immediately follows  $\nu(Z) = 1$ . Since this calculation is vague, we try a different approach. Therefore, we introduce transition kernels.

**Definition 5.1.4 :**

Let  $(X, \mathcal{A})$  and  $(Y, \mathcal{B})$  be two measurable spaces. The map  $\kappa : X \times \mathcal{B} \rightarrow [0, \infty]$  is a finite transition kernel from  $X$  to  $Y$  if

- (i)  $x \mapsto \kappa(x, B)$  is  $\mathcal{A}$ -measurable for every  $B \in \mathcal{B}$ .
- (ii)  $B \mapsto \kappa(x, B)$  is a finite measure on  $(Y, \mathcal{B})$  for every  $x \in X$ .

If  $\kappa(x, Y) = 1$  for every  $x \in X$ , then  $\kappa$  is a Markov kernel. For more information about transition kernels, see [10].

We need this object for the proof of the following Lemma.

**Lemma 5.1.5 :**

Let  $(X, \mathcal{A}, \lambda)$  be a probability space and  $R : X \rightarrow X$  a measurable and null-preserving map. Then there exists a unique measure  $\eta$  on the product space  $(X^2, \mathcal{A}^2)$  such that for every  $A, B \in \mathcal{A}$

$$\eta(A \times B) = \lambda(A \cap R^{-1}B) \quad (4)$$

Moreover, let  $G := \{(x, R(x)) \mid x \in X\}$  be the graph of  $R$ . Then  $G$  is measurable and  $\eta$  satisfies  $\eta(G) = 1$

**Proof :**

We construct a measure  $\tilde{\eta}$ . For this, we use a specific kernel, namely  $\delta_{R(x)}$  to define for any  $Q \in \mathcal{A}^2$

$$\tilde{\eta}(Q) := \int_X \delta_{R(x)}(Q_x) d\lambda(x)$$

Since  $\delta_{R(x)}$  is a Dirac measure, it follows, by linearity of the integral, that  $\tilde{\eta}$  is a measure. Using  $X_x^2 = (X \times X)_x = X$  and  $\delta_{R(x)}(X) = 1$ , we observe that

$$\begin{aligned} \tilde{\eta}(X^2) &= \int_X \delta_{R(x)}(X_x^2) d\lambda(x) \\ &= \int_X \delta_{R(x)}(X) d\lambda(x) \\ &= \int_X 1 d\lambda(x) \\ &= \lambda(X) \\ &= 1 \end{aligned}$$

Therefore,  $\tilde{\eta}$  is a probability measure.

Next, we show (4). For this, write for  $A, B \in \mathcal{A}$

$$\tilde{\eta}(A \times B) = \int_X \delta_{R(x)}((A \times B)_x) d\lambda(x) \quad (5)$$

Note that, more generally,

$$(A \times B)_x = \begin{cases} B & x \in A \\ \emptyset & x \notin A \end{cases}$$

and

$$\delta_{R(x)}(B) = \mathbb{1}_B(R(x)) = \mathbb{1}_{R^{-1}B}(x)$$

Then the calculation of (5) continues as follows:

$$\begin{aligned} \tilde{\eta}(A \times B) &= \int_A \delta_{R(x)}(B) d\lambda(x) \\ &= \int_A \mathbb{1}_{R^{-1}B}(x) d\lambda(x) \\ &= \lambda(A \cap R^{-1}B) \end{aligned}$$

So  $\tilde{\eta}$  satisfies (4). Next, we are concerned with  $G$ . Measurability of  $G$  is given by Theorem 5.1.2. To show that also  $\tilde{\eta}(G) = 1$ , first observe that

$$G_x = \{R(x)\} \implies \delta_{R(x)}(\{R(x)\}) = 1$$

This observation provides us

$$\begin{aligned} \tilde{\eta}(G) &= \int_X \delta_{R(x)}(G_x) d\lambda(x) \\ &= \int_X 1 d\lambda(x) \\ &= 1 \end{aligned}$$

Using once again the fact, that  $\mathcal{S}^2 := \{A \times B \mid A, B \in \mathcal{A}\}$  is a  $\pi$ -system satisfying  $\sigma(\mathcal{S}^2) = \mathcal{A}^2$ , we must have  $\eta = \tilde{\eta}$  by the Standard Uniqueness Theorem, since  $\tilde{\eta}$  was a probability measure. □

By induction, the result follows for product spaces of any dimension. In our case  $S$  is measure preserving, in particular null-preserving. Therefore,

this argument applies here and for  $Z^n$  we get  $\nu^n(Z^n) = 1$ .

Using continuity from above while letting  $n \rightarrow \infty$ , we get

$$\nu(Z) = \lim_{n \rightarrow \infty} \nu(Z^n) = \lim_{n \rightarrow \infty} \nu^n(Z^n) = 1$$

since  $Z_n \searrow Z$ .

What is left to do to get a measure space  $(Z, \Sigma, \nu)$ , is to specify  $\Sigma$ . For this, define  $\Sigma := \{B \cap Z \mid B \in \mathcal{B}_{\bar{X}}\}$ , i.e.  $\Sigma$  is the trace  $\sigma$ -algebra on  $Z$ , which is a  $\sigma$ -algebra on  $Z$ .

Now that we finally have a measure space  $(Z, \Sigma, \nu)$ , we need to define a map  $T$  on  $Z$  s.t.  $T$  is an automorphism, which extends  $S$ . Define

$$\begin{aligned} T(x_0, x_1, x_2, \dots) &:= (S(x_0), S(x_1), S(x_2), \dots) \\ &= (S(x_0), x_0, x_1, \dots) \end{aligned}$$

Firstly, we show that  $T$  is bijective.

On the one hand, let  $(x_0, x_1, \dots) \neq (y_0, y_1, \dots) \in Z$ . Then

$$T((x_0, x_1, \dots)) = (S(x_0), x_0, x_1, \dots) \neq (S(y_0), y_0, y_1, \dots) = T((y_0, y_1, \dots))$$

which yields injectivity.

On the other hand, for surjectivity consider any  $y = (y_0, y_1, \dots) \in Z$ . We need  $x = (x_0, x_1, \dots) \in Z$  s.t.

$$T(x) = (S(x_0), x_0, x_1, \dots) \stackrel{!}{=} (y_0, y_1, \dots)$$

Take  $\tilde{x} := (y_1, y_2, \dots) \in Z$ , then  $T(\tilde{x}) = (S(y_1), y_1, y_2, \dots) = (y_0, y_1, y_2, \dots)$ . This shows that  $T$  is surjective and ultimately bijective.

Secondly, we claim that  $T$  is an automorphism. But before we start showing this, recall that a product  $\sigma$ -algebra is generated by preimages of measurable sets under  $\pi_i, i \geq 0$  (Definition 2.2.3). Here we have, for  $i \geq 0$  and  $B \in \mathcal{B}_X$ , that the

$$\begin{aligned} \pi_i^{-1}(B) &=: (B)_i = (X \times \dots \times B \times X \times \dots) \cap Z \\ &= \{(x_0, x_1, \dots) \in Z \mid x_i \in B\} \end{aligned} \tag{6}$$

generate  $\Sigma$ .

As a consequence, it is enough to show that  $T$  as well as  $T^{-1}$  sends such sets to measurable sets. For that, observe that

$$T^{-1}(x_0, x_1, x_2, \dots) = (x_1, x_2, \dots) \text{ for } (x_0, x_1, x_2, \dots) \in Z$$

For any  $i$ , we get

$$\begin{aligned} T(B)_i &= \{T(x_0, x_1, \dots) \mid (x_0, x_1, \dots) \in Z, x_i \in B\} \\ &= \{(S(x_0), x_0, x_1, \dots) \mid (x_0, x_1, \dots) \in Z, x_i \in B\} \\ &= \{(\tilde{x}_0, \tilde{x}_1, \dots) \in Z \mid x_{i+1} \in B\} = (B)_{i+1} \end{aligned}$$

The same way, for  $i \geq 1$  we get

$$\begin{aligned} T^{-1}(B)_i &= \{T^{-1}(x_0, x_1, x_2, \dots) \mid (x_0, x_1, x_2, \dots) \in Z, x_i \in B\} \\ &= \{(x_1, x_2, \dots) \in Z \mid x_i \in B\} \\ &= \{(\tilde{x}_0, \tilde{x}_1, \dots) \in Z \mid x_{i-1} \in B\} = (B)_{i-1} \end{aligned}$$

For the special case  $i = 0$ , we have

$$T^{-1}(B)_0 = \{(x_1, x_2, \dots) \mid (x_0, x_1, x_2, \dots) \in Z, S(x_1) = x_0 \in B\} = (S^{-1}B)_0$$

All this together gives us an automorphism  $T$ . Now we want to check, that  $T$  inherits the last important property of  $S$ , namely measure preservation. To check that, we first show that  $T$  is measure preserving on the family of measurable rectangles, i.e. for  $B_0, \dots, B_{n-1} \in \mathcal{B}_X$  let

$$\begin{aligned} B_0 \times \dots \times B_{n-1} &= \{(x_0, \dots, x_{n-1}) \mid x_i \in B_i \text{ for } i \leq n-1\} \\ B_0 \times \dots \times B_{n-1} \times \bar{X} &= \{(x_0, x_1, \dots) \mid x_i \in B_i \text{ for } i \leq n-1\} \end{aligned}$$

Recall that those families form semirings  $\mathcal{S}^n$  and  $(\Pi_n)^{-1}\mathcal{S}^n$ , respectively. Here  $\Pi_n$  is the projection from  $\bar{X}$  onto its first  $n$  components. Furthermore, define  $\mathcal{S} := \bigcup_{n \geq 1} (\Pi_n)^{-1}\mathcal{S}^n$ . Moreover,  $\mathcal{S}^n$  generates  $\mathcal{B}_{X^n}$  and  $\mathcal{S}$  generates  $\mathcal{B}_{\bar{X}}$ .

Our goal is to show  $\nu = \nu \circ T^{-1}$  on  $\mathcal{S}^n$ . Since  $\mathcal{S}^n$  is a semiring, it is in particular closed under intersections. In addition,  $\sigma(\mathcal{S}^n) = \mathcal{B}_{X^n}$  will allow us to use the Standard Uniqueness Theorem for measures (see Theorem 2.1.1), which will yield  $\nu = \nu \circ T^{-1}$  on  $\mathcal{B}_{X^n}$ . Letting  $n \rightarrow \infty$  will give us the desired result.

For  $B_0, \dots, B_{n-1} \in \mathcal{B}_X$ , calculate

$$\begin{aligned} \nu(B_0 \times \dots \times B_{n-1} \times \bar{X}) &= \nu^n(B_0 \times \dots \times B_{n-1}) \\ &= \int_X \mathbb{1}_{S^{-(n-1)}B_0 \cap S^{-(n-2)}B_1 \cap \dots \cap B_{n-1}} d\mu \end{aligned}$$

On the other hand, using

$$T^{-1}(B_0 \times \dots \times B_{n-1} \times \bar{X}) = ((S^{-1}B_0 \cap B_1) \times B_2 \times \dots \times B_{n-1} \times \bar{X})$$

we get

$$\begin{aligned}
\nu \circ T^{-1}(B_0 \times \dots \times B_{n-1} \times \bar{X}) &= \nu_{n-1}((S^{-1}B_0 \cap B_1) \times \dots \times B_{n-1}) \\
&= \int_X \mathbb{1}_{S^{-(n-2)}(S^{-1}B_0 \cap B_1) \cap S^{-(n-3)}B_2 \cap \dots \cap B_{n-1}} d\mu \\
&= \int_X \mathbb{1}_{S^{-(n-1)}B_0 \cap S^{-(n-2)}B_1 \cap \dots \cap B_{n-1}} d\mu
\end{aligned}$$

This shows measure preservation of  $T$ .

So we finally have found an invertible map  $T$ , which extends  $S$ . Recalling Definition 3.2, the last step that is remaining for  $T$  to become a natural extension, is to find an exhaustive decomposition  $\xi$  and a measure preserving isomorphism  $f$  such that the following diagram commutes

$$\begin{array}{ccc}
Z/\xi & \xrightarrow{T_\xi} & Z/\xi \\
f \downarrow & & \downarrow f \\
X & \xrightarrow{S} & X
\end{array}$$

In order to do, we first define for  $x = (x_0, x_1, \dots) \in Z$  an equivalence relation  $x \sim \tilde{x} : \iff x_0 = \tilde{x}_0$ . This way, we get a partition  $\xi$  such that

$$Z/\xi = \{[(x_0, \dots)] \mid x_0 \in X\}$$

Next we have to show that this partition is exhaustive with respect to  $T$ . The easiest way to do this, is to check that any point of  $Z$  can be obtained as an intersection  $\bigcap_{i=0}^{\infty} T^i J_i$  for  $J_i \in \xi$ .

So let  $x = (x_0, x_1, \dots)$  be an arbitrary point in  $Z$ . Define  $J_0 = [(x_0, \dots)]$ , then of course  $x \in J_0$ . Next define  $J_1 = [(x_1, \dots)]$ . Applying  $T$  gives us

$$TJ_1 = \{\tilde{x} = (\tilde{x}_0, \tilde{x}_1, \dots) \mid \tilde{x}_0 = x_0, \tilde{x}_1 = x_1\} \quad (7)$$

Since any arbitrary point  $y$  belonging to  $TJ_1$  obviously also belongs to  $J_0$ , we have  $TJ_1 \subseteq J_0$ . Furthermore,  $x \in TJ_1$  (since  $x$  satisfies the conditions of (7)). Iterating this step, we get for general  $i$  a set  $T^i J_i = \{\tilde{x} = (\tilde{x}_0, \tilde{x}_1, \dots) \mid \tilde{x}_0 = x_0, \dots, \tilde{x}_i = x_i\}$ , where again  $T^i J_i \subseteq T^{i-1} J_{i-1}$  and  $x \in T^i J_i$ . Intersecting over every  $i \geq 0$  gives us

$$\bigcap_{i=0}^{\infty} T^i J_i = \{x\}$$

i.e. the claim, which we wanted to prove.

Secondly, we shall find a measure preserving isomorphism  $f$ . For this, define  $f : Z/\xi \rightarrow X$  as  $f([x_0, \dots]) := x_0$ . Then  $f$  is easily seen to be bijective. We shall observe that this  $f$  gives us the following diagram:

$$\begin{array}{ccc} (Z, \nu) & \xrightarrow{G_\xi} & (Z/\xi, \nu_\xi) \\ & \searrow \pi_0 \quad \swarrow f & \\ & (X, \mu) & \end{array}$$

$\pi_0$  denotes again the projection onto the first coordinate. Moreover,  $G_\xi(x) := [x]$  and  $\nu_\xi(B) = \nu(G_\xi^{-1}B)$  as introduced in section 3.

Since we have  $\pi_0 = f \circ G_\xi$ ,  $f$  is obviously a homomorphism and, due to bijectivity, an isomorphism.

Next we want to check that  $f$  is measure preserving. On the one hand, for some  $B \in \mathcal{B}_X$ ,  $f^{-1}B$  is measurable if and only if  $G_\xi^{-1}f^{-1}B$  is measurable in  $Z$ . Looking at the diagram above, we have that

$$G_\xi^{-1}f^{-1}B = \pi_0^{-1}B \stackrel{(3)}{=} (B)_0 \in \Sigma$$

Now that  $f^{-1}B$  is measurable, we can calculate

$$\begin{aligned} \nu_\xi(f^{-1}B) &= \nu(G_\xi^{-1}f^{-1}B) \\ &= \nu((B)_0) \\ &= \nu(B \times X \times X \times \dots) \\ &= \nu_0(B) \\ &= \int_X \mathbb{1}_B d\mu = \mu(B) \end{aligned}$$

On the other hand, suppose that  $B$  is measurable in  $Z/\xi$ , which means that  $G_\xi^{-1}B$  is measurable in  $Z$ . According to the diagram above, we shall observe that  $\pi_0(G_\xi^{-1}B) = f(B)$ . Using the fact, that the measure  $\nu$  is concentrated on  $Z$  the sets

$$f(B) \times X \times X \cdots \quad \text{and} \quad (f(B) \times X \times X \cdots) \cap Z$$

differ only by a set of measure zero. This makes  $f(B) \times X \times X \cdots$  measurable in the product space  $\bar{X}$ . Note here that, since  $\pi_0(G_\xi^{-1}B) = f(B)$ , we have

$(f(B) \times X \times X \cdots) \cap Z = G_\xi^{-1}B$ . Altogether we get measurability of  $f(B)$  and furthermore

$$\begin{aligned}\mu(f(B)) &= \mu(\pi_0(G_\xi^{-1}B)) \\ &= \nu((f(B))_0) \\ &= \nu(G_\xi^{-1}B) \\ &= \nu_\xi(B)\end{aligned}$$

All this adds up to

$$\begin{aligned}\nu_\xi(f^{-1}B) &= \mu(B) \quad \text{and} \\ \mu(f(B)) &= \nu_\xi(B)\end{aligned}$$

implying, that  $f$  is indeed measure preserving.

Finally, to finish the proof, observe that

$$T([(x_0, \dots)]) = [(S(x_0), x_0, \dots)] \subseteq [(S(x_0), \dots)]$$

By (2), this gives us  $T_\xi([(x_0, \dots)]) = [(S(x_0), \dots)]$ , which in order makes the following diagram commutative:

$$\begin{array}{ccc} Z/\xi & \xrightarrow{T_\xi} & Z/\xi \\ f \downarrow & & \downarrow f \\ X & \xrightarrow{S} & X \end{array}$$

So we have  $T_\xi \cong S$ , because  $Z/\xi \cong X$ . By Definition 3.2,  $T$  is a natural extension of  $S$  and the proof is done.

□

## 5.2 The non-singular case

So now we proceed the same way as before, except that  $S$  is now non-singular. So let again  $Z := \{(x_0, x_1, \dots) \mid x_i \in X, S(x_{i+1}) = x_i, i = 0, 1, \dots\}$ , then  $Z \in \mathcal{B}_{\bar{X}}$  by Lemma 5.1.1, since this result was independent of  $S$  being measure preserving.

The next step is to define  $\nu$  on  $\bar{X}$ . Unfortunately, this is not as easy as before. By assumption,  $\mu$  is a normalized measure, therefore  $\sigma$ -finite.  $S$  being non-singular gives us  $\mu \circ S^{-1} \ll \mu$ . The Radon-Nikodym theorem (Theorem 2.1.5) guarantees the existence of an integrable and measurable function  $h : X \rightarrow \mathbb{R} \cup \{\infty\}$  s.t.

$$\mu \circ S^{-1}(A) = \int_A h \, d\mu$$

(recall that  $h$  is in this case nonnegative, since  $\mu \circ S^{-1}$  is a measure).

On the other hand, non-singularity also gives us  $\mu \ll \mu \circ S^{-1}$ . Altogether

$$\mu(A) = 0 \iff \mu \circ S^{-1}(A) = 0$$

In other words,  $\mu \circ S^{-1}$  only vanishes if and only if  $\mu$  vanishes. This implies  $0 < h < \infty$  almost everywhere.

Using this Radon-Nikodym derivative, we define a sequence of functions, which was introduced by Lambert (see [9]):

$$H_i(x) := \frac{1}{(h \circ S)(x) \cdot (h \circ S^2)(x) \cdots (h \circ S^i)(x)} \quad i \geq 1 \quad (8)$$

$$H_0(x) = 1 \quad \forall x \in X$$

Since the  $H_i$  consist of compositions of measurable functions, the  $H_i$  are again measurable. Moreover, the  $H_i$  are defined almost everywhere since  $0 < h < \infty$  a.e. This also gives us  $H_i > 0$  for all  $i \geq 0$ . We shall observe and prove some useful properties of this sequence.

**Theorem 5.2.1 :**

$$H_{i+1} = \frac{H_i \circ S}{h \circ S} \quad \forall i > 0$$

**Proof :**

We just calculate both sides, after we considered the special case  $i = 0$ .

$$H_1(x) = \frac{H_0 \circ S(x)}{h \circ S(x)} = \frac{1 \circ S(x)}{h \circ S(x)} = \frac{1}{h \circ S(x)}$$

Now let  $i \in \mathbb{N}$ . On the left-hand side we have

$$H_{i+1}(x) = \frac{1}{(h \circ S)(x) \cdots (h \circ S^{i+1})(x)}$$

On the right-hand side,

$$\begin{aligned} H_i(S(x)) \cdot \frac{1}{(h \circ S)(x)} &= \frac{1}{(h \circ S)(S(x)) \cdots (h \circ S^i)(S(x))} \cdot \frac{1}{(h \circ S)(x)} \\ &= \frac{1}{(h \circ S^2)(x) \cdots (h \circ S^{i+1})(x)} \cdot \frac{1}{(h \circ S)(x)} \end{aligned}$$

This does the job. □

**Lemma 5.2.2 :**

Let  $B \in \mathcal{B}_X$ . For every  $k$  and  $i \geq 0$  we have

$$\int_B H_i \, d\mu = \int_{S^{-k}B} H_{i+k} \, d\mu$$

**Proof :**

We only prove the lemma for  $k = 1$ , the rest follows by induction on  $k$ .

Recall that  $h = \frac{d(\mu \circ S^{-1})}{d\mu}$ . Using this and Theorem 5.2.1 we have

$$\begin{aligned} \int_{S^{-1}B} H_{i+1} \, d\mu &= \int_{S^{-1}B} \frac{H_i \circ S}{h \circ S} \, d\mu = \int_{S^{-1}B} \frac{H_i}{h} \circ S \, d\mu \\ &\stackrel{(\text{Theorem 2.1.10})}{=} \int_B \frac{H_i}{h} \, d(\mu \circ S^{-1}) \\ &\stackrel{(h = \frac{d(\mu \circ S^{-1})}{d\mu})}{=} \int_B \frac{H_i}{h} \cdot h \, d\mu \\ &= \int_B H_i \, d\mu \end{aligned}$$

□

**Theorem 5.2.3 :**

For all  $i \geq 0$  we have

$$\int_X H_i \, d\mu = 1$$

i.e.  $(H_i)_{i \geq 0}$  forms a sequence of probability densities.

**Proof :**

We prove this theorem by induction on  $i$ . For  $i = 0$  we have obviously

$$\int_X H_0 \, d\mu = \int_X 1 \, d\mu = \mu(X) = 1$$

Now let  $i \in \mathbb{N}$ , then

$$\begin{aligned} \int_X H_i \, d\mu &= \int_{S^{-1}X} H_i \, d\mu \\ (\text{Lemma 5.2.2}) &= \int_X H_{i-1} \, d\mu \\ (\text{Induction hypothesis}) &= 1 \end{aligned}$$

□

Since we know that  $H_i$  is a probability density for any  $i \in \mathbb{N}_0$ , we could try to define  $\nu^n$  the following way. For  $B_0, B_1, \dots, B_{n-1} \in \mathcal{B}_X$  let

$$\nu^n(B_0 \times \dots \times B_{n-1}) := \int_X \mathbb{1}_{S^{-(n-1)}B_0 \cap S^{-(n-2)}B_1 \cap \dots \cap B_{n-1}} H_{n-1} \, d\mu$$

We want to check that  $\nu^n$  is again a measure. We have  $\nu^n(\emptyset) = 0$  since

$$\begin{aligned} \nu_n(\emptyset \times \dots \times \emptyset) &= \int_X \mathbb{1}_{S^{-(n-1)}\emptyset \cap \dots \cap \emptyset} H_{n-1} \, d\mu = \int_X \mathbb{1}_{\emptyset} H_{n-1} \, d\mu \\ &= \int_{\emptyset} H_{n-1} \, d\mu \\ &= 0 \end{aligned}$$

For  $\sigma$ -additivity, consider a set  $B_0 \times \dots \times B_{n-1}$  in  $X^n$ , which is again a disjoint union of rectangles:

$$B_0 \times \dots \times B_{n-1} = \bigcup_{k \geq 1} (B_0^k \times \dots \times B_{n-1}^k)$$

With the same calculation as in the measure preserving case, we get that

$$\nu^n(B_0 \times \dots \times B_{n-1}) = \sum_{k \geq 1} \nu^n(B_0^k \times B_1^k \times \dots \times B_{n-1}^k)$$

The last piece, which is missing to use Theorem 5.1.3, is consistency. Define for simplicity  $K := S^{-(n-1)}B_0 \cap S^{-(n-2)}B_1 \cap \dots \cap B_{n-1}$ . Then

$$\begin{aligned} \nu^{n+1}(B_0 \times \dots \times B_{n-1} \times X) &= \int_X \mathbb{1}_{S^{-n}B_0 \cap S^{-(n-1)}B_1 \cap \dots \cap S^{-1}B_{n-1} \cap X} H_n \, d\mu \\ (S^{-1}X = X) &= \int_X \mathbb{1}_{S^{-1}(S^{-(n-1)}B_0 \cap S^{-(n-2)}B_1 \cap \dots \cap B_{n-1})} H_n \, d\mu \\ &= \int_X \mathbb{1}_{S^{-1}K} H_n \, d\mu \\ &= \int_{S^{-1}K} H_n \, d\mu \\ (\text{Lemma 5.2.2}) &= \int_K H_{n-1} \, d\mu \\ &= \int_X \mathbb{1}_{S^{-(n-1)}B_0 \cap S^{-(n-2)}B_1 \cap \dots \cap B_{n-1}} H_{n-1} \, d\mu \\ &= \nu^n(B_0 \times \dots \times B_{n-1}) \end{aligned}$$

Now we can use Theorem 5.1.3 and Lemma 5.1.5 (non-singularity implies null-preservation) to get a measure  $\nu$  on  $\bar{X}$ . With the same argument as in the previous subsection, luckily also in this case we have

$$\nu(Z) = 1$$

We consider the same measure space as in the measure preserving case, namely  $(Z, \Sigma, \nu)$ , where again  $\Sigma := \{B \cap Z \mid B \in \mathcal{B}_{\bar{X}}\}$ .

Also,

$$\begin{aligned} T(x_0, x_1, x_2, \dots) &:= (S(x_0), S(x_1), S(x_2), \dots) \\ &= (S(x_0), x_0, x_1, \dots) \end{aligned}$$

is again a automorphism, where  $T$  and  $T^{-1}$  are measurable (since all calculations before were independent of measure preservation).

Showing that  $T$  is non-singular gets a little bit tricky. Beneteau does in [1] the following:

Consider the generator of  $\Sigma$  consisting of the preimages under all projections,

namely all sets  $(B)_i$  for  $i \geq 0$  (recall Definition 2.2.3 and (6)). Then for any  $i \geq 0$  we have

$$\begin{aligned}\nu((B)_i) &= \nu^{i+1}((B)_i) = \int_X \mathbb{1}_{S^{-i}X \cap \dots \cap B} H_i \, d\mu = \int_B H_i \, d\mu \\ \nu(T^{-1}(B)_i) &= \nu^i((B)_{i-1}) = \int_X \mathbb{1}_{S^{-(i-1)}X \cap \dots \cap B} H_{i-1} \, d\mu = \int_B H_{i-1} \, d\mu\end{aligned}$$

Then it follows that  $\mu(B) = 0 \iff \nu((B)_i) = 0$  and  $\nu(T^{-1}(B)_i) = 0$ . This gives non-singularity for preimages of all projections. Now Beneteau uses an approximation argument, since all  $(B)_i$  generate  $\Sigma$ . The problem here lies in the fact, that this family of sets is not stable under intersections (see Theorem 2.1.1). To realize this, consider for  $A, B \in \mathcal{B}_X$  and  $i \neq j$

$$\begin{aligned}(A)_i \cap (B)_j &= ((X \times \dots \times X \times A \times X \dots) \cap Z) \cap ((X \times \dots \times X \times B \times X \dots) \cap Z) \\ &= (X \times \dots \times X \times A \times X \times \dots \times X \times B \times X \times \dots) \cap Z\end{aligned}$$

which is obviously not an element of this generator. Therefore, the argument fails and we don't have non-singularity. So we have to show this in a different way.

A special thanks at this point to my supervisor Roland Zweimüller, who managed to find a different way to solve this problem. We just need a different family of sets. In particular, we need a family of sets, which is slightly bigger. The easiest sets, which are suitable for this problem to work with, are the measurable rectangles

$$\begin{aligned}B_0 \times \dots \times B_{n-1} &= \{(x_0, \dots, x_{n-1}) \mid x_i \in B_i \text{ for } i \leq n-1\} \\ B_0 \times \dots \times B_{n-1} \times \bar{X} &= \{(x_0, x_1, \dots) \mid x_i \in B_i \text{ for } i \leq n-1\}\end{aligned}$$

Before we are able to show non-singularity of  $T$ , we need some preparations on  $(X, \mathcal{B}_X, \mu, S)$ . Using the  $H_i(x)$  defined in (8), we want to define measures on every time component. For any  $n \geq 0$  set

$$\begin{aligned}\mu_n &:= H_n \odot \mu = \int H_n \, d\mu \\ \mu_0 &:= \mu\end{aligned}\tag{9}$$

$\mu_n$  has an important property, which we want to keep hold of in the next lemma.

**Lemma 5.2.4 :**

The measures  $(\mu_n)$  as defined in (9) are preimages of  $\mu$  under the push-forward by  $T$ , meaning

$$\mu_n \circ T^{-i} = \mu_{n-i} \text{ for } n \geq 0$$

Note that for  $n = i$  we have  $\mu_n \circ T^{-n} = \mu$ .

**Proof :**

Use Lemma 5.2.2 for any  $B \in \mathcal{B}_X$  and  $i \geq 0$  to get

$$\begin{aligned} \mu_n \circ T^{-i}(B) &= (H_n \odot \mu)(T^{-i}B) \\ &= \int_{T^{-i}B} H_n \, d\mu \\ &= \int_B H_{n-i} \, d\mu \\ &= (H_{n-i} \odot \mu)(B) \\ &= \mu_{n-i}(B) \end{aligned}$$

□

Next we want to study two measures on the product space  $(X^n, \mathcal{B}_{X^n})$ . On the one hand, we have the already well studied measure  $\nu^n$ , which we can now rewrite using (9)

$$\nu^n(B_0 \times \dots \times B_{n-1}) = \int_X \mathbb{1}_{S^{-(n-1)}B_0 \cap \dots \cap B_{n-1}} \, d\mu_{n-1}$$

On the other hand, we want to use another related measure as an auxiliary function. This measure was introduced in section 5.1 and had also the name  $\nu^n$ . Therefore, to avoid confusion, we rename it as

$$\rho^n(B_0 \times \dots \times B_{n-1}) = \int_X \mathbb{1}_{S^{-(n-1)}B_0 \cap \dots \cap B_{n-1}} \, d\mu$$

Our goal is to find measures, which have a density with respect to  $\rho^n$ . For this, consider any  $n$ -tuple of measurable functions  $(V_0, \dots, V_{n-1})$  where  $V_i : X \rightarrow \mathbb{R}$  for every  $0 \leq i \leq n-1$ . Define  $V_0 \otimes \dots \otimes V_{n-1} : X^n \rightarrow \mathbb{R}$  as

$$V_0 \otimes \dots \otimes V_{n-1}(x_0, \dots, x_{n-1}) := V_0(x_0) \cdots V_{n-1}(x_{n-1})$$

Note that the map  $V_i \mapsto V_0 \otimes \dots \otimes V_i \otimes \dots \otimes V_{n-1}$  is linear. To see that, set  $V_i = \tilde{V}_i + a\bar{V}_i$  and we get

$$\begin{aligned} & V_0 \otimes \dots \otimes (\tilde{V}_i + a\bar{V}_i) \otimes \dots \otimes V_{n-1}(x_0, \dots, x_{n-1}) \\ &= V_0(x_0) \cdots (\tilde{V}_i + a\bar{V}_i)(x_i) \cdots V_{n-1}(x_{n-1}) \\ &= V_0(x_0) \cdots \tilde{V}_i(x_i) \cdots V_{n-1}(x_{n-1}) + V_0(x_0) \cdots a\bar{V}_i(x_i) \cdots V_{n-1}(x_{n-1}) \\ &= (V_0 \otimes \dots \otimes \tilde{V}_i \otimes \dots \otimes V_{n-1}) + a(V_0 \otimes \dots \otimes \bar{V}_i \otimes \dots \otimes V_{n-1}) \end{aligned}$$

Let  $\mathcal{V}^n := \{V_0 \otimes \dots \otimes V_{n-1} \mid V_i : X \rightarrow \mathbb{R} \text{ measurable for every } 0 \leq i \leq n-1\}$  be the set containing all such functions. This way we can consider a map  $\Phi^n : \mathcal{V}^n \rightarrow \mathcal{V}^0$  defined by

$$\begin{aligned} \Phi^n(V_0 \otimes \dots \otimes V_{n-1})(x) &:= V_0 \otimes \dots \otimes V_{n-1}(S^{n-1}(x), S^{n-2}(x), \dots, x) \\ &= (V_0 \circ S^{n-1})(x)(V_1 \circ S^{n-2})(x) \cdots (V_{n-2} \circ S)(x)V_{n-1}(x) \end{aligned}$$

This  $\Phi^n$  is multiplicative on  $\mathcal{V}^n$  in the sense that for some  $V^n := V_0 \otimes \dots \otimes V_{n-1}$  and  $W^n := W_0 \otimes \dots \otimes W_{n-1} \in \mathcal{V}^n$  we have

$$\Phi^n(V^n \cdot W^n) = \Phi^n(V^n) \cdot \Phi^n(W^n) \quad (10)$$

Just calculate both sides to see that

$$\begin{aligned} \Phi^n(V^n \cdot W^n) &= \Phi^n((V_0 \otimes \dots \otimes V_{n-1}) \cdot (W_0 \otimes \dots \otimes W_{n-1})) \\ &= \Phi^n((V_0 \cdot W_0) \otimes \dots \otimes (V_{n-1} \cdot W_{n-1})) \\ &= ((V_0 \cdot W_0) \circ S^n) \cdots ((V_{n-2} \cdot W_{n-2}) \circ S)(V_{n-1} \cdot W_{n-1}) \end{aligned}$$

and

$$\Phi^n(V^n) \cdot \Phi^n(W^n) = (V_0 \circ S^{n-1}) \cdots (V_{n-2} \circ S)V_{n-1} \cdot (W_0 \circ S^{n-1}) \cdots (W_{n-2} \circ S)W_{n-1}$$

coincide. Furthermore, the map

$$(V_0, \dots, V_{n-1}) \mapsto \Phi^n(V_0 \otimes \dots \otimes V_{n-1})$$

is linear in each  $V_i$ . For this, just write in the  $i$ -th component  $V_i = \tilde{V}_i + a\bar{V}_i$  and we get

$$\begin{aligned} & \Phi^n(V_0 \otimes \dots \otimes (\tilde{V}_i + a\bar{V}_i) \otimes \dots \otimes V_{n-1}) \\ &= (V_0 \circ S^{n-1}) \cdots ((\tilde{V}_i + a\bar{V}_i) \circ S^{n-(i+1)}) \cdots V_{n-1} \\ &= (V_0 \circ S^{n-1}) \cdots (\tilde{V}_i \circ S^{n-(i+1)} + a\bar{V}_i \circ S^{n-(i+1)}) \cdots V_{n-1} \\ &= (V_0 \circ S^{n-1}) \cdots (\tilde{V}_i \circ S^{n-(i+1)}) \cdots V_{n-1} + (V_0 \circ S^{n-1}) \cdots (a\bar{V}_i \circ S^{n-(i+1)}) \cdots V_{n-1} \\ &= \Phi^n(V_0 \otimes \dots \otimes \tilde{V}_i \otimes \dots \otimes V_{n-1}) + a \cdot \Phi^n(V_0 \otimes \dots \otimes \bar{V}_i \otimes \dots \otimes V_{n-1}) \end{aligned}$$

Next we want to examine which specific functions belong to  $\mathcal{V}^n$ . On the one hand we have for  $B_0, \dots, B_{n-1} \in \mathcal{B}_X$  the indicator function  $\mathbb{1}_{B_0 \times \dots \times B_{n-1}}$  which belongs to  $\mathcal{V}^n$  since

$$\mathbb{1}_{B_0 \times \dots \times B_{n-1}}(x_0, \dots, x_{n-1}) = \mathbb{1}_{B_0}(x_0) \cdots \mathbb{1}_{B_{n-1}}(x_{n-1})$$

Applying  $\Phi^n$  and  $\mathbb{1}_{A \cap B} = \mathbb{1}_A \cdot \mathbb{1}_B$  in this case gives

$$\begin{aligned} \Phi^n(\mathbb{1}_{B_0 \times \dots \times B_{n-1}}) &= (\mathbb{1}_{B_0} \circ S^{n-1}) \cdot (\mathbb{1}_{B_1} \circ S^{n-2}) \cdots (\mathbb{1}_{B_{n-2}} \circ S) \mathbb{1}_{B_{n-1}} \\ &= \mathbb{1}_{S^{-(n-1)}B_0} \cdot \mathbb{1}_{S^{-(n-2)}B_1} \cdots \mathbb{1}_{B_{n-1}} \\ &= \mathbb{1}_{S^{-(n-1)}B_0 \cap S^{-(n-2)}B_1 \cap \dots \cap B_{n-1}} \end{aligned}$$

Another important example is the function given by

$$\begin{aligned} H^n &:= \frac{1}{h} \otimes \frac{1}{h} \otimes \dots \otimes 1 \in \mathcal{V}^n \quad \text{for } n \geq 2 \text{ and} \\ H^1 &:= 1 \in \mathcal{V}^0 \end{aligned}$$

A further application of  $\Phi^n$  yields

$$\begin{aligned} \Phi^n(H^n) &= \left(\frac{1}{h} \circ S^{n-1}\right) \left(\frac{1}{h} \circ S^{n-2}\right) \cdots \left(\frac{1}{h} \circ S\right) \cdot 1 \\ &= \frac{1}{(h \circ S) \cdots (h \circ S^{n-1})} \\ &= H_{n-1} \end{aligned}$$

Combining these results and (10) provides the following:

$$\Phi^n(\mathbb{1}_{B_0 \times \dots \times B_{n-1}} H^n) = \mathbb{1}_{S^{-(n-1)}B_0 \cap S^{-(n-2)}B_1 \cap \dots \cap B_{n-1}} H_{n-1}$$

With the aid of these identities, we can rewrite the defining properties of  $\nu^n$  and  $\rho^n$  respectively as

$$\nu^n(B_0 \times \dots \times B_{n-1}) = \int_X \Phi^n(\mathbb{1}_{B_0 \times \dots \times B_{n-1}}) \cdot \Phi^n(H^n) \, d\mu \quad (11)$$

and

$$\rho^n(B_0 \times \dots \times B_{n-1}) = \int_X \Phi^n(\mathbb{1}_{B_0 \times \dots \times B_{n-1}}) \, d\mu \quad (12)$$

The need of writing  $\nu^n$  and  $\rho^n$  that way will get clear, once we establish the following two results:

**Lemma 5.2.5 :**

For any non-negative  $V^n = V_0 \otimes \dots \otimes V_{n-1} \in \mathcal{V}^n$  we have

$$\int_{X^n} V^n d\rho^n = \int_X \Phi^n(V^n) d\mu \quad (13)$$

So this characterizes how integration with respect to  $\rho^n$  would work.

**Proof :**

The proof will be done like most proofs in measure theory. We start with indicator functions  $V^n = \mathbb{1}_{B_0 \times \dots \times B_{n-1}}$ , where  $B_0, \dots, B_{n-1} \in \mathcal{B}_X$ . By (12) we have

$$\begin{aligned} \int_{X^n} \mathbb{1}_{B_0 \times \dots \times B_{n-1}} d\rho^n &= \int_{B_0 \times \dots \times B_{n-1}} d\rho^n \\ &= \rho^n(B_0 \times \dots \times B_{n-1}) = \int_X \Phi^n(\mathbb{1}_{B_0 \times \dots \times B_{n-1}}) d\mu \end{aligned}$$

In the next step we show that (13) stays valid for  $V^n = V_0 \otimes \dots \otimes V_{n-1}$ , where each  $V_i \geq 0$  is a simple function on  $X$ .

We already know that for any fixed  $i \in \{0, \dots, n-1\}$  and given  $V_0, \dots, V_{i-1}, V_{i+1}, \dots, V_{n-1}$  the map  $V_i \mapsto V_0 \otimes \dots \otimes V_i \otimes \dots \otimes V_{n-1}$  is linear. Therefore it suffices to show that for functions  $V_0 \otimes \dots \otimes \tilde{V}_i \otimes \dots \otimes V_{n-1}$  and  $V_0 \otimes \dots \otimes \bar{V}_i \otimes \dots \otimes V_{n-1}$ , which satisfy (13), the same is true for  $V_0 \otimes \dots \otimes (a\tilde{V}_i + b\bar{V}_i) \otimes \dots \otimes V_{n-1}$  for any  $a, b \geq 0$ .

A straightforward calculation gives us

$$\begin{aligned} &\int_{X^n} V_0 \otimes \dots \otimes (a\tilde{V}_i + b\bar{V}_i) \otimes \dots \otimes V_{n-1} d\rho^n \\ &= a \int_{X^n} V_0 \otimes \dots \otimes \tilde{V}_i \otimes \dots \otimes V_{n-1} d\rho^n + b \int_{X^n} V_0 \otimes \dots \otimes \bar{V}_i \otimes \dots \otimes V_{n-1} d\rho^n \\ &\stackrel{(13)}{=} a \int_X \Phi^n(V_0 \otimes \dots \otimes \tilde{V}_i \otimes \dots \otimes V_{n-1}) d\mu + b \int_X \Phi^n(V_0 \otimes \dots \otimes \bar{V}_i \otimes \dots \otimes V_{n-1}) d\mu \\ &= \int_X \Phi^n(V_0 \otimes \dots \otimes (a\tilde{V}_i) \otimes \dots \otimes V_{n-1}) d\mu + \int_X \Phi^n(V_0 \otimes \dots \otimes (b\bar{V}_i) \otimes \dots \otimes V_{n-1}) d\mu \\ &= \int_X \Phi^n(V_0 \otimes \dots \otimes (a\tilde{V}_i + b\bar{V}_i) \otimes \dots \otimes V_{n-1}) d\mu \end{aligned}$$

where the last two equalities are resulting from linearity of  $\Phi^n$ .

In the last step we consider a general  $V^n = V_0 \otimes \dots \otimes V_{n-1} \in \mathcal{V}^n$  with non-negative  $V_i$ ,  $0 \leq i \leq n-1$ . Since all the  $V_i$  are non-negative, there exists a sequence of simple functions  $V_i^{(k)} \geq 0$  on  $X$  such that  $V_i^{(k)} \nearrow V_i$  as  $k \rightarrow \infty$  for every  $i \in \{0, \dots, n-1\}$ . As a consequence, we have

$$V_0^{(k)} \otimes \dots \otimes V_{n-1}^{(k)} \nearrow V_0 \otimes \dots \otimes V_{n-1} \text{ on } X^n \text{ as } k \rightarrow \infty$$

Moreover, this implies

$$\Phi^n(V_0^{(k)} \otimes \dots \otimes V_{n-1}^{(k)}) \nearrow \Phi^n(V_0 \otimes \dots \otimes V_{n-1}) \text{ on } X \text{ as } k \rightarrow \infty$$

Altogether, Theorem 2.1.6 and (13) for simple functions yield

$$\begin{aligned} \int_{X^n} V_0 \otimes \dots \otimes V_{n-1} d\rho^n &= \int_{X^n} \lim_{k \rightarrow \infty} V_0^{(k)} \otimes \dots \otimes V_{n-1}^{(k)} d\rho^n \\ &= \lim_{k \rightarrow \infty} \int_{X^n} V_0^{(k)} \otimes \dots \otimes V_{n-1}^{(k)} d\rho^n \\ &= \lim_{k \rightarrow \infty} \int_X \Phi^n(V_0^{(k)} \otimes \dots \otimes V_{n-1}^{(k)}) d\mu \\ &= \int_X \lim_{k \rightarrow \infty} \Phi^n(V_0^{(k)} \otimes \dots \otimes V_{n-1}^{(k)}) d\mu \\ &= \int_X \Phi^n(V_0 \otimes \dots \otimes V_{n-1}) d\mu \end{aligned}$$

This completes the proof of the lemma.  $\square$

**Lemma 5.2.6 :**

Let  $V^n \in \mathcal{V}^n$  be a non-negative function and  $\lambda^n$  be a finite measure on  $(X^n, \mathcal{B}_{X^n})$ . Suppose that for all  $B_0, \dots, B_{n-1} \in \mathcal{B}_X$ ,

$$\lambda^n(B_0 \times \dots \times B_{n-1}) = \int_X \mathbb{1}_{S^{-(n-1)}B_0 \cap S^{-(n-2)}B_1 \cap \dots \cap B_{n-1}} \Phi^n(V^n) d\mu \quad (14)$$

This implies that  $\lambda^n$  has density  $V^n$  with respect to  $\rho^n$ , namely

$$\lambda^n = V^n \odot \rho^n$$

**Proof :**

Rewrite (14) via  $\Phi^n(\mathbb{1}_{B_0 \times \dots \times B_{n-1}}) = \mathbb{1}_{S^{-(n-1)}B_0 \cap S^{-(n-2)}B_1 \cap \dots \cap B_{n-1}}$ , then (10)

and Lemma 5.2.5 give us

$$\begin{aligned}
\lambda^n(B_0 \times \dots \times B_{n-1}) &= \int_X \mathbb{1}_{S^{-(n-1)}B_0 \cap S^{-(n-2)}B_1 \cap \dots \cap B_{n-1}} \Phi^n(V^n) d\mu \\
&= \int_X \Phi^n(\mathbb{1}_{B_0 \times \dots \times B_{n-1}}) \cdot \Phi^n(V^n) d\mu \\
&= \int_X \Phi^n(\mathbb{1}_{B_0 \times \dots \times B_{n-1}} V^n) d\mu \\
&= \int_{X^n} \mathbb{1}_{B_0 \times \dots \times B_{n-1}} V^n d\rho^n \\
&= \int_{B_0 \times \dots \times B_{n-1}} V^n d\rho^n
\end{aligned}$$

As a result, the measures  $\lambda^n$  and  $V^n \odot \rho^n$  coincide on the semiring  $\mathcal{S}^n$ , which generates  $\mathcal{B}_{X^n}$ . Thus,  $\lambda^n = V^n \odot \rho^n$  on  $\mathcal{B}_{X^n}$  by the standard uniqueness theorem for measures.  $\square$

Now apply Lemma 5.2.6 on the rewritten  $\nu^n$  in (11) to realize that  $V^n = H^n$  in this case. This gives

$$\nu^n = H^n \odot \rho^n \quad (15)$$

Now we know, when a measure  $\nu^n$  on  $(X^n, \mathcal{B}_{X^n})$  possesses a density with respect to  $\rho^n$ . Since we want to show that  $T$  is non-singular with respect to  $\nu$ , we shall find a condition for measures on  $(\bar{X}, \mathcal{B}_{\bar{X}})$  to have a density with respect to  $\nu$ .

**Lemma 5.2.7 :**

Let  $\lambda$  be a finite measure on  $(\bar{X}, \mathcal{B}_{\bar{X}})$  and let  $g : X \rightarrow [0, \infty)$  be a measurable function. If for every  $n \geq 1$

$$\lambda \circ (\Pi_n)^{-1} = (g \circ \pi_0) \odot (\nu \circ (\Pi_n)^{-1})$$

then

$$\lambda = (g \circ \pi_0) \odot \nu$$

**Proof :**

Like in the proof of Lemma 5.2.5, it is enough to show that the measures  $\lambda$  and  $(g \circ \pi_0) \odot \nu$  coincide on  $\mathcal{S}$ , where  $\sigma(\mathcal{S}) = \mathcal{B}_{\bar{X}}$ . Then finiteness of  $\lambda$  entails finiteness of  $(g \circ \pi_0) \odot \nu$ , so that the uniqueness theorem for measures

applies. Now we know that  $\mathcal{S} = \bigcup_{n \geq 1} (\Pi_n)^{-1} \mathcal{S}^n$ .

But our assumption means precisely that  $\lambda = (g \circ \pi_0) \odot \nu$  on  $(\Pi_n)^{-1} \mathcal{S}^n$  for every  $n \geq 1$ . This proves the lemma.  $\square$

We want to check which function  $g$  is going to be, if we use  $\nu \circ T^{-1}$ , which is by assumption finite. It turns out that the density of  $\nu \circ T^{-1}$  with respect to  $\nu$  is the following.

**Lemma 5.2.8 :**

$$\nu \circ T^{-1} = (h \circ \pi_0) \odot \nu \quad (16)$$

**Proof :**

Fix any  $n \geq 1$ . We start with the finite-dimensional projection onto the first  $n$  components, namely  $\nu \circ T^{-1} \circ (\Pi_n)^{-1}$ . Therefore, choose  $B_0, \dots, B_{n-1} \in \mathcal{B}_X$  to calculate

$$\begin{aligned} \nu \circ T^{-1} \circ (\Pi_n)^{-1}(B_0 \times \dots \times B_{n-1}) &= \nu \circ T^{-1}(B_0 \times \dots \times B_{n-1} \times \bar{X}) \\ &= \nu((S^{-1}B_0 \cap B_1) \times \dots \times B_{n-1} \times \bar{X}) \\ &= \nu^{n-1}((S^{-1}B_0 \cap B_1) \times \dots \times B_{n-1}) \\ &= \int_X \mathbb{1}_{S^{-(n-2)}(S^{-1}B_0 \cap B_1) \cap \dots \cap B_{n-1}} d\mu_{n-2} \\ &= \int_X \mathbb{1}_{S^{-(n-1)}B_0 \cap S^{-(n-2)}B_1 \cap \dots \cap B_{n-1}} d\mu_{n-2} \\ &\stackrel{(9)}{=} \int_X \mathbb{1}_{S^{-(n-1)}B_0 \cap \dots \cap B_{n-1}} H_{n-2} d\mu \\ &= \int_X \mathbb{1}_{S^{-(n-1)}B_0 \cap \dots \cap B_n} K_{n-1} d\mu \end{aligned}$$

where  $K_{n-1} := H_{n-2} = H_{n-1}(h \circ S^{n-1})$ . Furthermore, set

$$K^n := 1 \otimes \frac{1}{h} \otimes \dots \otimes \frac{1}{h} \otimes 1 \in \mathcal{V}^n$$

Then we have  $K_{n-1} = \Phi^n(K^n)$ . This in turn gives us

$$\nu \circ T^{-1} \circ (\Pi_n)^{-1}(B_0 \times \dots \times B_{n-1}) = \int_X \mathbb{1}_{S^{-(n-1)}B_0 \cap \dots \cap B_{n-1}} \Phi^n(K^n) d\mu$$

Applying Lemma 5.2.6 gives us

$$\nu \circ T^{-1} \circ (\Pi_n)^{-1} = K^n \odot \rho^n$$

On the other hand, according to (15) we have

$$\nu \circ (\Pi_n)^{-1} = \nu^n = H^n \odot \rho^n$$

Also, note that

$$\frac{K^n}{H^n} = \frac{1 \otimes \frac{1}{h} \otimes \dots \otimes \frac{1}{h} \otimes 1}{\frac{1}{h} \otimes \frac{1}{h} \otimes \dots \otimes \frac{1}{h} \otimes 1} = h \circ \pi_0 \quad (17)$$

A combination of all this results yields

$$\begin{aligned} \nu \circ T^{-1} \circ (\Pi_n)^{-1} &= K^n \odot \rho^n \\ &= \frac{K^n}{H^n} \cdot H^n \odot \rho^n \\ &\stackrel{(15)}{=} \frac{K^n}{H^n} \odot \nu^n \\ &\stackrel{(17)}{=} (h \circ \pi_0) \odot (\nu \circ (\Pi_n)^{-1}) \end{aligned}$$

Using Lemma 5.2.7 for  $g = h$  finishes the proof, we obtain

$$\nu \circ T^{-1} = (h \circ \pi_0) \odot \nu$$

□

In other terms, Lemma 5.2.8 implies that

$$\nu \circ T^{-1} << \nu$$

The only thing that is missing for non-singularity of  $T$  is

$$\nu << \nu \circ T^{-1} \quad (18)$$

Instead, we will show  $\nu \circ T << \nu$ , which is equivalent to (18). For the beginning, we divide by  $h \circ \pi_0$  (since  $0 < h < \infty$ ) in (16) to obtain

$$\nu = \frac{1}{h \circ \pi_0} \odot (\nu \circ T^{-1})$$

Applying  $T$  on both sides gives us

$$\nu \circ T = \left( \frac{1}{h \circ \pi_0} \odot (\nu \circ T^{-1}) \right) \circ T \quad (19)$$

We claim, that (19) is equal to

$$\nu \circ T = \frac{1}{h \circ \pi_0 \circ T} \odot \nu \quad (20)$$

For this, consider an invertible function  $g : X \rightarrow X$ , a function  $f : X \rightarrow \mathbb{R}$  and a density  $u : X \rightarrow \mathbb{R}$ . Moreover, consider for measures  $\lambda$  and  $u \odot \lambda$  the corresponding image measures with respect to  $g$ ,  $\lambda \circ g^{-1}$  and  $(u \odot \lambda) \circ g^{-1}$  respectively. Then, by using Theorem 2.1.10 twice, we have

$$\begin{aligned}
\int_X f \, d((u \odot \lambda) \circ g^{-1}) &= \int_X f \circ g \, d(u \odot \lambda) \\
&= \int_X (f \circ g) \cdot u \, d\lambda \\
&= \int_X [f \cdot (u \circ g^{-1})] \circ g \, d\lambda \\
&= \int_X f \cdot (u \circ g^{-1}) \, d(\lambda \circ g^{-1})
\end{aligned}$$

In other words, this calculation gives us

$$(u \odot \lambda) \circ g^{-1} = (u \circ g^{-1}) \odot (\lambda \circ g^{-1}) \quad (21)$$

Now set  $g^{-1} := T$ ,  $\lambda := \nu \circ T^{-1}$ ,  $u := \frac{1}{h \circ \pi_0}$  to rewrite (21)

$$\left(\frac{1}{h \circ \pi_0} \odot (\nu \circ T^{-1})\right) \circ T = \frac{1}{h \circ \pi_0 \circ T} \odot \nu$$

which is equal to (20), so we get  $\nu \circ T \ll \nu$ , or equivalently

$$\nu \ll \nu \circ T^{-1}$$

This proves that  $T$  is again non-singular.

To make this invertible extension a natural one, finish the proof like in the previous subsection.

□

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