



universität
wien

MASTERARBEIT / MASTER'S THESIS

Titel der Masterarbeit / Title of the Master's Thesis

„Analytic Foundations of low regularity
Lorentzian Geometry“

verfasst von / submitted by

Emina Hadzialic, BSc

angestrebter akademischer Grad / in partial fulfilment of the
requirements for the degree of
Master of Science (MSc)

Wien, 2019 / Vienna, 2019

Studienkennzahl lt. Studienblatt /
degree programme code as it appears on
the student record sheet

UA 066 821

Studienrichtung lt. Studienblatt /
degree programme code as it appears on
the student record sheet

Masterstudium Mathematik

Betreut von / Supervisor:

Univ.-Prof. Mag. Mag. Dr. Michael Kunzinger

STATUTORY DECLARATION

I declare that I have authored this thesis independently, that I have not used other than the declared sources / resources, and that I have explicitly marked all material which has been quoted either literally or by content from the used sources.

Vienna, December 12, 2019

.....
Signature

Danksagung

Jede Masterthesis trägt die Handschrift des Erstellers, und doch ist sie nie die Arbeit eines Einzelnen. Deswegen möchte ich mich an dieser Stelle bei all denjenigen bedanken, die mich in den letzten Monaten unterstützt haben.

Zuerst möchte ich mich bei meinen Eltern für die emotionale und finanzielle Unterstützung bedanken. Sie standen immer hinter mir und erleichterten es mir meine Ziele zu erreichen. Sie waren stets verständnisvoll, auch wenn ich manchmal etwas schwierig war.

Ein großes Dankeschön geht auch an meine große Schwester und an meinen Schwager. In schwierigen Phasen haben sie trotzdem immer an mich geglaubt und mich in meinem Vorhaben bestärkt. Sie hatten immer ein offenes Ohr für mich und standen mir mit viel Begeisterung und Motivation zur Seite.

Ich möchte mich auch bei meinen beiden Mädels, Aigerim und Vera, bedanken. Sie waren immer eine große Stütze für mich und haben nie an mir gezweifelt.

Ein besonderer Dank geht an meinen Betreuer für die kontinuierliche thematische Unterstützung. Er nahm sich immer sehr viel Zeit und stand mir mit viel Geduld und Verständnis zur Seite. Dies ermöglichte es einen thematischen Tiefgang zu erlangen.

Last, but not least, danke an Eva Breunig. Durch sie habe ich erst die Freude an der Mathematik entdeckt und mich dazu entschlossen diesen Weg zu gehen.

Danke!

Abriss

In dieser Arbeit werden semi-Riemannsche und Lorentzsche Mannigfaltigkeiten versehen mit Metriken niedriger Regularität betrachtet und es wird gezeigt, dass Standard Resultate aus der Differentialgeometrie weiterhin ihre Gültigkeit behalten, wenn die Regularität der Metrik auf $\mathcal{C}^{1,1}$ reduziert wird.

Das Streben nach geringer Regularität von Seiten der Physik und Analysis, andererseits die Notwendigkeit von höherer Regularität, um die Gültigkeit von Standard Resultaten aufrecht zu erhalten, ist weit verbreitet. $\mathcal{C}^{1,1}$ Regularität kann als die optimale Regularität angesehen werden, um das gewünschte Ziel zu erreichen.

Unter anderem wird auch gezeigt, dass die lokale Maximierung der Länge von kausalen Geodäten schon für $\mathcal{C}^{1,1}$ Metriken gültig ist.

Mithilfe der Picard Lindelöf Approximation wird die starke Differenzierbarkeit der Exponential Abbildung gezeigt. Außerdem wird bewiesen, dass auf einer glatten Mannigfaltigkeit die Exponential Abbildung eines Lipschitz Zusammenhanges einen lokalen bi-Lipschitz Homeomorphismus ist. Der Beweis beruht auf einer Version des Satzes über inverse Funktionen. Anschließend wird die Existenz von konvexen normalen Umgebungen bewiesen.

Schlussendlich wird eine Version des Gauss-Lemmas präsentiert, in welcher die Differenzierbarkeit der Exponential Abbildung nicht benötigt wird.

Abstract

In this thesis we consider semi-Riemannian and Lorentzian manifolds with metrics in low regularity and prove that many standard results of differential geometry remain valid if the regularity of the metric is reduced to $\mathcal{C}^{1,1}$. This search for low regularity from physics and analysis versus the need for higher regularity in order for the standard results from geometry to hold is widely noted. $\mathcal{C}^{1,1}$ differentiability conditions are the best choice to preserve most results in geometry.

In particular, we show that the local Lorentzian length maximization property of causal geodesics in the class of absolutely continuous causal curves already holds for $\mathcal{C}^{1,1}$ spacetime metrics.

Using a Picard-Lindelöf analysis, we can prove the strong differentiability of the exponential map at the origin. Furthermore, we establish that over a $\mathcal{C}^\infty$ -manifold the exponential map of any Lipschitz connection constitutes a local bi-Lipschitz homeomorphism. The proof rests on a version of the inverse function theorem. We refine the pointed version of the theorem by establishing and proving the existence of convex normal neighborhoods.

Finally, we also state a formulation of Gauss' Lemma, which does not require the differentiability of the exponential map.

Contents

1	Introduction to semi-Riemannian Geometry	4
1.1	Semi-Riemannian metrics	4
1.2	Levi-Civita connection	4
2	Preliminary statements for $\mathcal{C}^\infty$-metrics	8
2.1	Gauss Lemma for a metric of class $\mathcal{C}^\infty$	11
2.2	Main theorem 1 for a metric of class $\mathcal{C}^\infty$	12
3	Lorentzian Geometry	17
3.1	The causal character	17
3.2	Timecones	19
3.3	Local Lorentzian Geometry	20
3.4	Causality relations	21
3.5	Lorentzian length of a causal AC-curve	23
4	Introduction to the topic	29
4.1	Picard-Lindelöf analysis	29
4.2	Existence and Uniqueness of geodesics	30
4.3	The exponential map	38
4.3.1	Locally Lipschitzness	38
4.3.2	Lipschitzness of the inverse	46
5	Main Theorems	54
5.1	Strong differentiability of the exponential map	54
5.2	Position vector on convex neighborhoods	60
5.3	Gauss Lemma	64

Notation

The following notations are the most frequently used throughout this thesis:

M	$\mathbb{C}^\infty$ – manifold
p, q	points
x^μ	local chart
(x^μ, v^μ)	chart on tangent bundle π induced by the chart x^μ
π	tangent bundle
TU, TM	tangent fields
$\dot{x}$	$\frac{dx}{dt}$
H^μ	connection
$\gamma_v(t)$	geodesic, starting from $\pi(v)$ with velocity v
Ω	set of those v for which geodesics exists at least for $t \in [0, 1]$, star-shaped
Ω_p	$\Omega \cup \pi^{-1}(p)$, star-shaped
$\ \cdot\ $	Euclidean norm on $\mathbb{R}^n$
$B(p, r)$	open ball of radius r centred at p
$\overline{B}(p, r)$	closed ball with radius r centred at p
$\mathcal{C}^{n,k}$	n -times continuously differentiable and Hölder with $\alpha = k$
∇	Levi-Civita connection
V, W	vector spaces
X, Y	vector fields
$\nabla_X Y$	covariant derivative of Y in direction X
I	interval
$U, \tilde{U}$	neighborhoods
v	tangent vector
$P(r, q)$	position vector of q with respect to r

Preface

In this thesis we focus on both semi-Riemannian and Lorentzian manifolds endowed with a metric of low regularity. The goal is to show that many standard results of geometry remain valid if the regularity of the metric is reduced to $\mathcal{C}^{1,1}$. General relativity is governed by field equations and the regularity is essential in the context of solving the initial value problem. From the physical point of view non-smooth solutions are of interest, because one would like to study systems where different regions of space-time have different matter contents for example. On matching these regions the matter variables become discontinuous. This forces the differentiability of the metric to be lower than $\mathcal{C}^2$. For example choosing a metric of regularity $\mathcal{C}^{1,1}$ would correspond to finite jumps of the matter variables. In most of the literature in general relativity [3], [5], [7], [8], [10], [11], [19], [20], [21], [22] it is assumed that the differentiability of the metric is at least $\mathcal{C}^2$, because a $\mathcal{C}^2$ metric allows one to retain many of the standard properties of smooth metrics. The search for low regularity from physics and analysis versus the need for higher regularity in order for the standard results from geometry to hold is widely noted [16].

$\mathcal{C}^{1,1}$ regularity of the metric turned out to be the threshold of classical theory. For instance, an example of Hartman and Wintner shows that for connections of regularity $\mathcal{C}^{1,\lambda-1}$ with $1 < \lambda < 2$ the relation between minimizing curves and geodesics turns out to differ from the smooth situation if the regularity of the metric drops below $\mathcal{C}^{1,1}$. It is possible to develop some of the elements of geometry in low regularity.

With this background we start this thesis with an introduction to semi-Riemannian geometry. We then state some standard results that use a smooth metric, which we will generalize to results using a $\mathcal{C}^{1,1}$ metric. In preparation to proving that the exponential map of any Lipschitz connection is a local bi-Lipschitz homeomorphism, we first show that $\exp : TM \rightarrow M \times M$ is a diffeomorphism from a neighborhood of the zero section TM_0 of TM to a neighborhood of the diagonal in $M \times M$. We prove the Gauss Lemma considering a smooth metric. Later on we generalize this statement for Lipschitz connections and $\mathcal{C}^{1,1}$ metrics. We continue with an introduction to Lorentzian Geometry. In particular, we prove that the local Lorentzian length maximization property of causal geodesics in the class of absolutely continuous causal curves holds already for $\mathcal{C}^{1,1}$ space-time metrics. Indeed, we consider a future directed causal AC -curve σ and show that $\exp^{-1}(\sigma(s))$ is future directed causal for every $s > 0$ and that in case that $\exp_p^{-1}(\sigma(\hat{s}))$ is lightlike, σ restricted to the interval $[0, \hat{s}]$ is a future directed lightlike geodesic segment up to parametrizations. We also prove that the Lorentz length of σ is smaller than that of the unique future directed causal

geodesic connecting its endpoints, unless its image is the same as that of that geodesic.

The fourth chapter is an introduction to the topic, where we go through some necessary computations and definitions, which will be needed later on. For example we carry out a Picard-Lindelöf analysis in order to prove the strong differentiability of the exponential map at the origin. Moreover, we prove the existence and uniqueness of geodesics.

We finish this thesis with the three main theorems, which are a generalization of the standard results, mentioned in the beginning, for $\mathcal{C}^{1,1}$ metric. The first one we already mentioned. We establish that the exponential map of any Lipschitz connection is a local bi-Lipschitz homeomorphism. The proof rests on a version of the inverse function theorem, namely Leach's theorem. Then we refine the pointed case by proving the existence of convex normal neighborhoods, indeed the second main theorem states that the exponential map provides a bi-Lipschitz homeomorphism between an open star-shaped subset of the tangent bundle TC and $C \times C$, where C is a strictly convex normal neighborhood of p .

This thesis includes also a short subsection about position vectors on convex neighborhoods in preparation to the Gauss Lemma, the third main theorem. Under this differentiability condition the exponential map is just Lipschitz. Thus, the Gauss Lemma is not expected to hold and we show that it still holds true in a formulation which does not require the differentiability of the exponential map.

We assume all manifolds to be of class $\mathcal{C}^\infty$ and just lower the regularity of the metric in the chapters 3-5.

1 Introduction to semi-Riemannian Geometry

In this chapter we want to transfer the setting of elementary differential geometry to abstract manifolds.

In the differential geometry of hypersurfaces in $\mathbb{R}^n$ all properties of the surface depend on the scalar product. It is induced on the tangent spaces by the standard scalar product of the Euclidean space. The scalar product is usually chosen to be positive definite.

The semi-Riemannian Geometry uses a more general concept of nondegenerate bilinear forms, which are not necessarily positive definite.

A bilinear form $b : V \times V \longrightarrow \mathbb{R}$ on a finite dimensional vector space V is called nondegenerate if $b(v, w) = 0$ for all $w \in V$ implies $v = 0$.

The index r of a symmetric bilinear form on a finite dimensional vector space is given by

$$r := \max\{\dim W : W \subset V \text{ with } b|_W \text{ negative definite}\}.$$

Clearly, $0 \leq r \leq \dim V$ holds. If $r = 0$ then all g_p are inner products on the tangent space at p , meaning that g is a Riemannian metric.

For further informations, we refer to [15].

1.1 Semi-Riemannian metrics

In order to reach our goal of this chapter we have to endow every tangent space with a scalar product, which varies smoothly on the manifold.

Let M be a C^∞ -manifold. A semi-Riemannian metric g on M smoothly assigns a bilinear form $g(p) = g_p : T_p M \times T_p M \longrightarrow \mathbb{R}$ to each point p in M , which is symmetric nondegenerate and such that the index r_p of g_p is the same for all p . The pair (M, g) is called a semi-Riemannian manifold, where g is the metric on M .

For further informations, we refer to [15].

1.2 Levi-Civita connection

In order to construct a directional derivative of a vector field on semi-Riemannian manifolds we will use as a model the covariant derivative of hypersurfaces of $\mathbb{R}^n$. Note that this construction is using the structure of the ambient Euclidean space, which is not available anymore on general semi-Riemannian manifolds. Thus we have to generalise the covariant derivative and define it as an operation mapping a pair of vector fields to another vector field with the following properties.

Let us consider a smooth manifold M . Then a connection is a map

$$\begin{aligned}\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) &\longrightarrow \mathfrak{X}(M) \\ (X, Y) &\longmapsto \nabla_X Y\end{aligned}$$

such that the following hold:

($\nabla 1$) $\nabla_X Y$ is $\mathcal{C}^\infty(M)$ –linear in X

($\nabla 2$) $\nabla_X Y$ is $\mathbb{R}$ –linear in Y

($\nabla 3$) for all smooth functions f on M the Leibniz rule holds:

$$\nabla_X fY = X(f)Y + f\nabla_X Y,$$

where $\nabla_X Y$ is called the covariant derivative of Y in direction X with respect to the connection ∇ .

Let us take a closer look at the maps

$$X \longrightarrow \nabla_X Y \text{ and} \tag{1.1}$$

$$Y \longrightarrow \nabla_X Y. \tag{1.2}$$

Note that tensor fields are $\mathcal{C}^\infty(M)$ –multilinear maps taking one forms and vector fields to smooth functions, indeed $A : \Omega^1(M) \times \mathfrak{X}(M) \longrightarrow \mathcal{C}^\infty(M)$. This map is $\mathcal{C}^\infty(M)$ –multilinear.

Taking Y fixed, the map (1.1) can be identified with the one above and hence is a tensor field. Because of the $\mathbb{R}$ –linearity in Y of $\nabla_X Y$, this cannot be said for the map (1.2) for fixed X .

Now it remains to show that on any semi-Riemannian manifold there exists a unique connection which is compatible with the metric [15]. To this end, consider

Theorem 1.1. [15, Theorem 1.3.4]

Consider a semi-Riemannian manifold (M, g) . Then there is a unique connection ∇ on M such that for all $X, Y, Z \in \mathfrak{X}(M)$ the properties ($\nabla 1$), ($\nabla 2$), ($\nabla 3$), as well as

($\nabla 4$) *the torsion free condition: $[X, Y] = \nabla_X Y - \nabla_Y X$ and*

($\nabla 5$) *the metric property: $Z\langle X, Y \rangle = \langle \nabla_Z X, Y \rangle + \langle X, \nabla_Z Y \rangle$*

hold.

The Levi-Civita connection ∇ of (M, g) is uniquely determined by the Koszul-formula

$$2\langle \nabla_X Y, Z \rangle = X\langle Y, Z \rangle + Y\langle Z, X \rangle - Z\langle X, Y \rangle - \langle X, [Y, Z] \rangle + \langle Y, [Z, X] \rangle + \langle Z, [X, Y] \rangle.$$

Proof. Let $F(X, Y, Z)$ be the right-hand side of the Koszul-formula and let ∇ be a connection with additional properties $(\nabla 4), (\nabla 5)$. Then applying $(\nabla 5)$ to the first three terms and $(\nabla 4)$ to the last three gives

$$\begin{aligned} F(X, Y, Z) &= \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle + \langle \nabla_Y Z, X \rangle + \langle Z, \nabla_Y X \rangle - \langle \nabla_Z X, Y \rangle \\ &\quad - \langle X, \nabla_Z Y \rangle - \langle X, \nabla_Y Z \rangle + \langle X, \nabla_Z Y \rangle + \langle Y, \nabla_Z X \rangle - \langle Y, \nabla_X Z \rangle \\ &\quad + \langle Z, \nabla_X Y \rangle - \langle Z, \nabla_Y X \rangle = 2\langle \nabla_X Y, Z \rangle. \end{aligned}$$

Recall that the theorem about musical isomorphisms states that on a semi-Riemannian manifold M , any $X \in \mathfrak{X}(M)$ defines a one form $X^\flat \in \Omega^1(M)$ by $X^\flat(Y) := \langle X, Y \rangle$ for all $Y \in \mathfrak{X}(M)$ and that the mapping $X \mapsto X^\flat$ is a $\mathcal{C}^\infty(M)$ –linear isomorphism from $\mathfrak{X}(M)$ to $\Omega^1(M)$. So we conclude that by the injectivity of the map

$$\mathfrak{X}(M) \longrightarrow \Omega^1(M),$$

$\nabla_X Y$ is uniquely determined.

In order to prove existence we consider for fixed $X, Y \in \mathfrak{X}(M)$ the mapping $Z \rightarrow F(X, Y, Z)$. This mapping is $\mathcal{C}^\infty(M)$ –linear, because of the following property of the Lie bracket

$$\begin{aligned} [fX, gY] &= (fX)(gY) - (gY)(fX) \\ &= fX(gY) + fgXY - fgYX - gY(fX) \\ &= fX(gY) + fg[X, Y] - gY(f)X, \end{aligned}$$

where $f, g \in \mathcal{C}^\infty(M)$. Indeed,

$$\begin{aligned} F(X, Y, fZ) &= X(f\langle Y, Z \rangle) + Y(f\langle Z, X \rangle) - fZ(\langle X, Y \rangle) - \langle X, [Y, fZ] \rangle \\ &\quad + \langle Y, [fZ, X] \rangle + f\langle Z, [X, Y] \rangle = X(f)\langle Y, Z \rangle + fX(\langle Y, Z \rangle) + Y(f)\langle Z, X \rangle \\ &\quad + fY(\langle Z, X \rangle) - fZ(\langle X, Y \rangle) - \langle X, Y(f)Z + f[Y, Z] \rangle + \langle Y, f[Z, X] - X(f)Z \rangle \\ &\quad + f\langle Z, [X, Y] \rangle = fF(X, Y, Z) + X(f)\langle Y, Z \rangle + Y(f)\langle Z, X \rangle - Y(f)\langle X, Z \rangle \\ &\quad - X(f)\langle Y, Z \rangle = fF(X, Y, Z) \end{aligned}$$

Hence the mapping is a one-form on M and again by the theorem about musical isomorphism for all $Z \in \mathfrak{X}(M)$ there is a uniquely defined vector field $\nabla_X Y$ such that $2\langle \nabla_X Y, Z \rangle = F(X, Y, Z)$. By definition $\nabla_X Y$ fulfils the Koszul-formula and thus it remains to show that the properties $(\nabla 1) - (\nabla 5)$ hold.

Let us start with the first property:

We have that $F(X_1 + X_2, Y, Z) = F(X_1, Y, Z) + F(X_2, Y, Z)$, indeed

$$\begin{aligned}
F(X_1 + X_2, Y, Z) &= (X_1 + X_2)(\langle Y, Z \rangle) + Y(\langle Z, X_1 + X_2 \rangle) - Z(\langle X_1 + X_2, Y \rangle) \\
&\quad - \langle X_1 + X_2, [Y, Z] \rangle + \langle Y, [Z, X_1 + X_2] \rangle + \langle Z, [X_1 + X_2, Y] \rangle = X_1(\langle Y, Z \rangle) \\
&\quad + X_2(\langle Y, Z \rangle) + Y(\langle Z, X_1 \rangle) + Y(\langle Z, X_2 \rangle) - Z(\langle X_1, Y \rangle) - Z(\langle X_2, Y \rangle) \\
&\quad - \langle X_1, [Y, Z] \rangle - \langle X_2, [Y, Z] \rangle + \langle Y, [Z, X_1] \rangle + \langle Y, [Z, X_2] \rangle + \langle Z, [X_1, Y] \rangle \\
&\quad + \langle Z, [X_2, Y] \rangle = F(X_1, Y, Z) + F(X_2, Y, Z)
\end{aligned}$$

Thus we can conclude that $\nabla_{X_1+X_2}Y = \nabla_{X_1}Y + \nabla_{X_2}Y$. For $f \in \mathcal{C}^\infty(M)$ given, we get using a similar computation for $F(fX, Y, Z)$ as for $F(X, Y, fZ)$

$$\begin{aligned}
2\langle \nabla_{fX}Y - f\nabla_XY, Z \rangle &= F(fX, Y, Z) - fF(X, Y, Z) \\
&= fF(X, Y, Z) + Y(f)\langle Z, X \rangle - Z(f)\langle X, Y \rangle + \langle Y, Z(f)X \rangle - \langle Z, Y(f)X \rangle = 0.
\end{aligned}$$

Since $Y \mapsto F(X, Y, Z)$ is $\mathbb{R}$ -linear the property $(\nabla 2)$ follows. In order to prove $(\nabla 3)$ we proceed as follows:

$$\begin{aligned}
2\langle \nabla_X fY, Z \rangle &= F(X, fY, Z) = X(f)\langle Y, Z \rangle - Z(f)\langle X, Y \rangle + Z(f)\langle X, Y \rangle \\
&\quad + X(f)\langle Z, Y \rangle + fF(X, Y, Z) = 2\langle X(f)Y + f\nabla_XY, Z \rangle.
\end{aligned}$$

Apply again the theorem of musical isomorphisms and the claim follows. The next property is again a straightforward computation, indeed

$$\begin{aligned}
2\langle \nabla_XY - \nabla_YX, Z \rangle &= F(X, Y, Z) - F(Y, X, Z) \\
&= \langle \nabla_XY, Z \rangle + \langle Y, \nabla_XZ \rangle + \langle \nabla_YZ, X \rangle + \langle Z, \nabla_YX \rangle - \langle \nabla_ZX, Y \rangle - \langle X, \nabla_ZY \rangle \\
&\quad - \langle X, \nabla_YZ \rangle + \langle X, \nabla_ZY \rangle + \langle Y, \nabla_ZX \rangle - \langle Y, \nabla_XZ \rangle + \langle Z, \nabla_XY \rangle - \langle Z, \nabla_YX \rangle \\
&\quad - \langle \nabla_YX, Z \rangle + \langle X, \nabla_YZ \rangle + \langle \nabla_XZ, Y \rangle + \langle Z, \nabla_XY \rangle - \langle \nabla_ZY, X \rangle - \langle Y, \nabla_ZX \rangle \\
&\quad - \langle Y, \nabla_XZ \rangle + \langle Y, \nabla_ZX \rangle + \langle X, \nabla_ZY \rangle - \langle X, \nabla_YZ \rangle + \langle Z, \nabla_YX \rangle - \langle Z, \nabla_XY \rangle \\
&= 2\langle \nabla_XY, Z \rangle + 2\langle \nabla_YX, Z \rangle = 2\langle \nabla_XY + \nabla_YX, Z \rangle = 2\langle [X, Y], Z \rangle.
\end{aligned}$$

Apply the theorem about musical isomorphisms and the claim follows. In order to prove the last property we proceed similarly as above:

$$\begin{aligned}
2(\langle \nabla_ZX, Y \rangle + \langle X, \nabla_ZY \rangle) &= F(Z, X, Y) + F(Z, Y, X) \\
&= 2\langle \nabla_ZX, Y \rangle + 2\langle \nabla_ZY, X \rangle = 2Z\langle X, Y \rangle + 2Z\langle Y, X \rangle = 2Z(\langle X, Y \rangle).
\end{aligned}$$

□

For further information, we refer to [12] and [15].

2 Preliminary statements for $\mathcal{C}^\infty$ -metrics

Before we turn to the next chapter, where we concentrate on low regularity of the metric and prove that some standard results of geometry remain valid under $\mathcal{C}^{1,1}$ differentiability conditions, we present some preliminary statements, where we consider metrics of class $\mathcal{C}^\infty$.

Let M be a $\mathcal{C}^\infty$ -manifold and g a $\mathcal{C}^\infty$ semi-Riemannian metric. Consider $v \in T_p M \subset TM$. There exists a neighborhood $\mathcal{N}$ of v in TM and an interval I containing 0 such that $(w, s) \mapsto \gamma_w(s)$ is well-defined and smooth map from $\mathcal{N} \times I$ to M , where $\gamma_w(s)$ is a geodesic. We know that γ_w is a solution of a second order ordinary differential equation, namely the geodesic equation. The solution depends smoothly on $t, \gamma_w(0) = p$ and $\gamma'_w(0) = w$. Rewriting this in a first order ordinary differential equation, one can prove that there exists a unique solution to our problem and thus prove that the map from $\mathcal{N} \times I$ to M is smooth [13], [15]. Hence, geodesics are flow lines of a vector field, indeed:

Theorem 2.1. [13, Theorem 2.3.11], [15, Theorem 2.1.12]

Consider a $\mathcal{C}^\infty$ -manifold M and a $\mathcal{C}^\infty$ semi-Riemannian metric g . There exists a uniquely defined vector field $X \in \mathfrak{X}(TM)$ with the following properties: The projection $\pi : TM \rightarrow M$ establishes a one-to-one correspondence between maximal integral curves of X and maximal geodesics of M . This vector field is called the geodesic field.

Proof. Consider $v \in TM$. Then $s \mapsto \gamma'_v(s)$ is a smooth curve in TM . Denote by $X_v = X(v)$ the initial velocity of this curve, that is

$$X_v = \left. \frac{d}{ds} \right|_{s=0} (\gamma'_v(s)) \in T_{\gamma'_v(0)=v}(TM).$$

So $X \in \mathfrak{X}(TM)$, because $(v, s) \mapsto \gamma_v(s)$ is well-defined and smooth.

We want to prove that if γ is a geodesic in M , then γ' is an integral curve of X . Let $\alpha(s) = \gamma'(s)$. Define $w := \gamma'(t)$ and $\beta(s) := \gamma'_w(s)$ for arbitrarily chosen s, t . We have that $\gamma_w(0) = \gamma(t)$ and $\gamma'_w(0) = w = \gamma'(t)$, meaning that two geodesics and their derivatives coincide at one point and thus they coincide everywhere. In other words $\gamma(s+t) = \gamma_w(s)$. Then differentiating by s gives $\alpha(t+s) = \gamma'_w(s) = \beta(s)$. Hence, in $T(TM)$ we have $\alpha'(t+s) = \beta'(s)$. In particular,

$$\alpha'(t) = \beta'(0) = X_w = X_{\alpha(t)},$$

meaning that α is an integral curve of X .

Now it remains to show that if α is an integral curve of X , then $\pi \circ \alpha$ is a geodesic in M . Indeed, consider $\alpha : I \rightarrow TM$ and $t \in I$. We have that $\alpha(t) \in TM$ and thus, as already shown in the first part of this proof, $s \mapsto \gamma'_{\alpha(t)}(s)$ is an integral

curve of X . At $s = 0$, $\gamma'_{\alpha(t)}$ and $s \mapsto \alpha(t + s)$ both have the same value $\alpha(t)$. Thus, they coincide on the whole domain where they are defined. For all s we have

$$\alpha(t + s) = \gamma'_{\alpha(t)}(s) \Rightarrow \pi(\alpha(t + s)) = \gamma_{\alpha(t)}(s).$$

Setting $t = 0$ we obtain that $\pi \circ \alpha$ is a geodesic in M . Altogether, from $\pi \circ \gamma' = \gamma$ and

$$(\pi \circ \alpha)' = (\pi \circ \alpha)'(t) = \frac{d}{ds} \Big|_{s=0} \pi(\alpha(t+s)) = \frac{d}{ds} \Big|_{s=0} \gamma_{\alpha(t)}(s) = \gamma'_{\alpha(t)}(0) = \alpha(t) = \alpha$$

we conclude that $\alpha \mapsto \pi \circ \alpha$ is inverse to $\gamma \mapsto \gamma'$. Therefore, X is uniquely defined [13], [15]. $\square$

For any $p \in M$ we define

$$\mathcal{D}_p := \{v \in T_p M : \gamma_v \text{ is defined at least on } [0, 1]\}.$$

The exponential map $\exp : \mathcal{D}_p \rightarrow M$ of M at p is given by $\exp_p(v) := \gamma_v(1)$. $\mathcal{D}_p$ is the maximal domain of the pointed exponential map. The geodesic $s \mapsto \gamma_v(ts)$ has initial velocity $t\gamma'_v(0) = tv$ and thus we get $\gamma_{vt}(s) = \gamma_v(ts)$ for all s, t for which at least one side of the equation is defined. Hence, for the exponential map we conclude $\exp_p(tv) = \gamma_{tv}(1) = \gamma_v(t)$ [13].

Theorem 2.2. [13, Theorem 2.3.14], [15, Theorem 2.1.14]

Consider a $\mathcal{C}^\infty$ -manifold M , $p \in M$ and a $\mathcal{C}^\infty$ semi-Riemannian metric g . For every p there exists a neighborhood $\tilde{U}$ of 0 in $T_p M$ and U of p in M such that $\exp_p : \tilde{U} \rightarrow U$ is a diffeomorphism.

Proof. By ODE theory [1], we have that $(w, s) \mapsto \gamma_w(s)$ is smooth in $\mathcal{N} \times I$, where $\mathcal{N}$ is an open neighborhood of TM and I is an interval around 0. Without loss of generality, we choose $I = (-a, a)$. Define

$$\mathcal{N}'_p := \mathcal{N} \cap T_p M$$

open in $T_p M$ and

$$\mathcal{N}_p := \left\{ \frac{v}{a'} : v \in \mathcal{N}'_p \right\},$$

where $a' > \frac{1}{a}$. When two geodesics and their derivatives coincide at one point, they coincide everywhere. So we can conclude that $(w, s) \mapsto \gamma_w(s)$ is defined on $\mathcal{N}_p \times J$, where $[0, 1] \subset J$. Indeed, for $w \in \mathcal{N}_p$, set $w = \frac{v}{a'}$ with $v \in \mathcal{N}'_p$. Then

$$\gamma_w(s) = \gamma_{\frac{v}{a'}}(s) = \gamma_v\left(\frac{s}{a'}\right),$$

meaning that $\frac{s}{a'} \in I = (-a, a)$ and this holds whenever $s \in a' \cdot I \supset [0, 1]$. Thus, $\exp_p : \mathcal{N}_p \rightarrow M, w \mapsto \gamma_w(1)$ is smooth.

Let us now show that

$$T_0(\exp_p) : T_0(T_p M) \rightarrow T_p M$$

is a linear isomorphism. In order to prove this let $v \in T_0(T_p M)$. $T_p M$ is a vector space and so we can identify $T_0(T_p M)$ with $T_p M$. Setting $\rho(t) := tv$, then

$$v = \rho'(0) = T_0 \rho \left(\frac{d}{dt} \Big|_{t=0} \right)$$

and we have $\exp_p(tv) = \gamma_v(t)$. Thus,

$$\begin{aligned} T_0(\exp_p)(v) &= T_0(\exp_p)(\rho'(0)) = T_0(\exp_p \circ \rho) \left(\frac{d}{dt} \Big|_{t=0} \right) = T_0(\gamma_v) \left(\frac{d}{dt} \Big|_{t=0} \right) \\ &= \gamma'_v(0) = v. \end{aligned}$$

The inverse function theorem states that there exist neighborhoods of p and $\exp_p(p)$ such that $\exp_p : \tilde{U} \rightarrow U$ is a diffeomorphism and the claim follows [13]. $\square$

Consider now again U and $\tilde{U}$ as in the previous theorem. If $\tilde{U}$ is star-shaped, then U is called a normal neighborhood of p [13]. Then U is also star-shaped in the following sense:

Proposition 2.3. [13, Proposition 2.3.15]

Consider a $\mathcal{C}^\infty$ -manifold M and let g be a $\mathcal{C}^\infty$ semi-Riemannian metric. Let U be a normal neighborhood of $p \in M$. Then there exists for every $q \in U$ a unique geodesic $\sigma : [0, 1] \rightarrow U$ from p to q in U . We have $\sigma'(0) = \exp_p^{-1}(q) \in \tilde{U}$ and σ is called a radial geodesic.

Proof. We proved that $\exp_p : \tilde{U} \rightarrow U$ is a diffeomorphism. Let $q \in U$ and $v := \exp_p^{-1}(q) \in \tilde{U}$. By assumption, $\tilde{U}$ is star-shaped at 0 and so we have that $\rho(t) := tv \in \tilde{U}$ for $0 \leq t \leq 1$. So the geodesic $\exp_p \circ \rho$ joining p and q lies completely in U . We have

$$\sigma'(0) = T_0 \exp_p(\rho'(0)) = T_0 \exp_p(v) = v = \exp_p^{-1}(q).$$

Let $\tau : [0, 1] \rightarrow U$ be an arbitrary geodesic in U from p to q and $w := \tau'(0)$. Then the geodesics $t \mapsto \exp_p(tw)$ and τ passing through p have the same initial velocity and thus they are equal. We want to prove that $w \in \tilde{U}$. By contradiction, we

assume that w is not in $\tilde{U}$. Define

$$\tilde{t} := \sup\{t \in [0, 1] : tw \in \tilde{U}\}.$$

Clearly, then $\tilde{t}w \in \partial\tilde{U}$. Furthermore, $\tau([0, 1])$ is compact in U and so also $(\exp_p|_{\tilde{U}})^{-1}(\tau([0, 1]))$ is compact in $\tilde{U}$. Thus it has a positive distance to $\partial\tilde{U}$ and hence also to $\tilde{t}w$. By definition of the supremum there exists $t_0 < \tilde{t}$ close enough to $\tilde{t}$ such that $t_0w \in \tilde{U}$ but $t_0w \notin (\exp_p|_{\tilde{U}})^{-1}(\tau([0, 1]))$. But then we get $\exp_p(t_0w) \notin \tau([0, 1])$ and this is a contraction to τ coinciding with $t \mapsto \exp_p(tw)$. Hence $w \in \tilde{U}$.

If we replace w by $\tilde{w} = sw$ for $s \in [0, 1]$ one can show that not only $w \in \tilde{U}$, but the whole segment $\{tw : t \in [0, 1]\} \in \tilde{U}$.

Moreover, from

$$\exp_p(w) = \tau(1) = q = \exp_p(v)$$

and the fact that $\exp_p|_{\tilde{U}}$ is injective, we conclude that $w = v$ and thus $\tau = \sigma$ [13]. $\square$

For further informations, we refer to [13] and [15].

2.1 Gauss Lemma for a metric of class $\mathcal{C}^\infty$

In the last chapter we will prove the Gauss Lemma. There we will consider a $\mathcal{C}^{1,1}$ metric giving rise to a Lipschitz connection and a normal neighborhood of p . The exponential map will be just Lipschitz. In preparation to this we will first prove this statement for a smooth metric.

Theorem 2.4. [13, Theorem 2.3.23], [15, Theorem 2.1.21]

Let M be a $\mathcal{C}^\infty$ -manifold, $p \in M$ and g a $\mathcal{C}^\infty$ semi-Riemannian metric. Consider $x \in T_p M$ with $x \neq 0$ such that $x \in \Omega_p$, the domain of the pointed exponential map. Let $v_x, w_x \in T_x(T_p M)$ and v_x be radial. Then

$$\langle (T_x \exp_p)(v_x), (T_x \exp_p)(w_x) \rangle = \langle v_x, w_x \rangle.$$

Proof. We have that v_x is radial, meaning that it is a scalar multiple of x and hence we can suppose $v_x = x$. Define

$$f(t, s) := \exp_p(t(v_x + sw_x)).$$

This map is continuous and maps $[0, 1] \times \{0\}$ into the domain of $\exp_p$ in $T_p M$. We proved that the domain of $\exp_p$ is open, hence there exists $\epsilon, \delta > 0$ such that

f is defined on

$$\mathcal{D} := \{(t, s) : -\delta < t < 1 + \delta, -\epsilon < s < \epsilon\}$$

and so f is a smooth map from $\mathcal{D}$ to M . Now we have

$$\begin{aligned} f_t(1, 0) &= T_{v_x} \exp_p(v_x) \text{ and} \\ f_s(1, 0) &= T_{v_x} \exp_p(w_x) \end{aligned}$$

and it suffices to show that $\langle f_t(1, 0), f_s(1, 0) \rangle = \langle v_x, w_x \rangle$. The curves $t \mapsto f(t, s)$ are geodesics and have initial speed $v_x + sw_x$. Clearly, $f_{tt} = 0$ and hence $\langle f_t, f_t \rangle$ is constant and equal to $\langle v_x + sw_x, v_x + sw_x \rangle$, since

$$f_t(0, s) = T_0 \exp_p(v_x + sw_x) = v_x + sw_x.$$

By smoothness, we have that $f_{st} = f_{ts}$ and so we conclude

$$\begin{aligned} \frac{\partial}{\partial t} \langle f_s, f_t \rangle &= \langle f_{st}, f_t \rangle + \langle f_s, f_{tt} \rangle = \langle f_{ts}, f_t \rangle = \frac{1}{2} \frac{\partial}{\partial s} \langle f_t, f_t \rangle \\ &= \frac{1}{2} \frac{\partial}{\partial s} \langle v_x + sw_x, v_x + sw_x \rangle = \langle v_x, w_x \rangle + s \langle w_x, w_x \rangle, \end{aligned}$$

and this implies

$$\left(\frac{\partial}{\partial t} \langle f_s, f_t \rangle \right) (t, 0) = \langle v_x, w_x \rangle$$

for all t . So $f(0, s) = \exp_p(0) = p$ for all s and $f_s(0, 0) = 0$ which gives $\langle f_s, f_t \rangle(0, 0) = 0$. Integrating the last equation gives $\langle f_s, f_t \rangle(t, 0) = t \langle v, w \rangle$ and setting $t = 1$ we get

$$\langle f_t(1, 0), f_s(1, 0) \rangle = \langle v_x, w_x \rangle.$$

[13], [15]

□

For further information, we refer to [13] and [15].

2.2 Main theorem 1 for a metric of class $\mathcal{C}^\infty$

Let $\pi : TM \rightarrow M$ be the base point map and

$$E : TM \rightarrow M \times M$$

given by $E(v) = (\pi(v), \exp_{\pi(v)} v)$. The maximal domain of E is given by

$$\mathcal{D} := \{v \in TM : \gamma_v \text{ exists at least on } [0, 1]\}.$$

Moreover, $\mathcal{D}_p = \mathcal{D} \cap T_p M$ and $\exp = \pi_2 \circ E$.

In the last chapter we will also prove that the exponential map is strongly differentiable over the zero section and provides a bi-Lipschitz homeomorphism between an open star-shaped neighborhood of the zero section and an open neighborhood of the diagonal [13]. In preparation to this we first prove

Theorem 2.5. [13, Theorem 2.4.6]

Let M be a $\mathcal{C}^\infty$ -manifold M and let g be a $\mathcal{C}^\infty$ semi-Riemannian metric. The map E is a diffeomorphism from a neighborhood of the zero section TM_0 of TM to a neighborhood of the diagonal Δ_M in $M \times M$.

Before we start with the proof of this theorem, let us prove the following proposition.

Proposition 2.6. [13, Proposition 2.4.3]

Let M be a $\mathcal{C}^\infty$ -manifold and let g be a $\mathcal{C}^\infty$ semi-Riemannian metric. The domain $\mathcal{D}$ of E is open in TM and the domain $\mathcal{D}_p$ of the pointed exponential map is open and star-shaped at 0 in $T_p M$.

Proof. Consider the geodesic field $X \in \mathfrak{X}(TM)$. We proved that the integral curves of X are the derivatives of the geodesics. The flow ψ of X is then given by

$$\psi_t(v) = \gamma'_v(t).$$

The maximal domain $\tilde{\mathcal{D}}$ of ψ is open by ODE theory [1] in $\mathbb{R} \times TM$. We have that $\tilde{\mathcal{D}}$ is also the maximal domain of the map

$$\pi \circ \psi : (t, v) \mapsto \gamma_v(t).$$

Consider

$$\begin{aligned} \Phi : TM &\rightarrow TM \times \mathbb{R}, \\ v &\mapsto (v, 1). \end{aligned}$$

This map is smooth and

$$\mathcal{D} = \{v \in TM : \exists \gamma_v(1)\} = \{v \in TM : (v, 1) \in \tilde{\mathcal{D}}\} = \Phi^{-1}(\tilde{\mathcal{D}})$$

is open. So $\mathcal{D}_p = \mathcal{D} \cap T_p M$ is open in $T_p M$. Finally, for $v \in \mathcal{D}_p$ we have that γ_v is defined on $[0, 1]$. We know that if two geodesics and their derivatives coincide in one point $a \in I$, I some interval, then they coincide everywhere. Thus, we have that $\gamma_{tv}(1) = \gamma_v(t)$ and so $tv \in \mathcal{D}_p$ for $t \in [0, 1]$. Therefore we conclude that $\mathcal{D}_p$ is star-shaped at 0 in $T_p M$ [13]. $\square$

Proof. First we are going to prove that every $x \in TM_0$ has a neighborhood where E is a diffeomorphism. We have $E(x) = (\pi(x), \exp_p(0)) = (p, p)$. By the inverse function theorem and the fact that $\dim TM = \dim(M \times M)$, it only remains to show that

$$T_x E : T_x(TM) \rightarrow T_{E(x)}(M \times M)$$

is injective and therefore bijective for all $x \in TM_0$. So let $x = 0_p \in TM_0, v \in T_x(TM)$ and $T_x E(v) = 0$. The goal is to prove that $v = 0$. By the base point map

$$\begin{aligned} \pi : TM &\rightarrow M \text{ and} \\ \pi_1 : M \times M &\rightarrow M, \end{aligned}$$

we have $\pi_1 \circ E = \pi$ and thus $T_x \pi(v) = T_{E(x)} \pi_1(T_x E(v)) = 0$, meaning that $v \in \ker(T_x \pi)$. Because $T_x(T_p M) = \ker(T_x \pi)$ we conclude that $v \in T_x(T_p M)$. Notice that by the inverse function theorem, $\pi_1|_{\{p\} \times M}$ is a diffeomorphism and thus we can identify $\{p\} \times M$ with M and we get $E|_{T_p M} = \exp_p$. Hence

$$0 = T_x E(v) = T_x(E|_{T_p M})(v) = T_{0_p}(\exp_p)(v) = v,$$

by theorem 2.2.

Let us now cover TM_0 by neighborhoods as above, which additionally have the form

$$W_i := (T\psi_i)^{-1}(\psi_i(U_i) \times B_{a_i}(0)),$$

where U_i is compact, $a_i > 0$ for charts (ψ_i, U_i) and $\{U_i : i \in \mathbb{N}\}$ is locally finite. Using Partition of unity, we choose a smooth function $f : M \rightarrow \mathbb{R}^+$ such that

$$\max_{\{(i,j): q \in U_i \cap U_j\}} \{1, \max_{p' \in U_i \cap U_j} \|T_{p'} \psi_i \circ (T_{p'} \psi_j)^{-1}\|\} f(q) < \min_{\{i: q \in U_i\}} a_i,$$

for all $q \in M$ and where $\|\cdot\|$ is the operator norm. Let

$$\begin{aligned} \tilde{W}_i &:= T\psi_i^{-1}(\psi(U_i) \times B_{\min_{U_i} f}(0)) \subset W_i \text{ and} \\ W &:= \bigcup_{i \in \mathbb{N}} \tilde{W}_i. \end{aligned}$$

Then W is a neighborhood of TM_0 in TM . It remains only to show that $E|_W$ is injective and our claim follows.

For this, let $w_1, w_2 \in W$ and $E(w_1) = E(w_2)$. Then $\pi(w_1) = \pi(w_2) = p$. Let i, j be such that $w_1 \in \tilde{W}_i, w_2 \in \tilde{W}_j$. Then $p \in U_i \cap U_j$.

Because of $w_1 \in W_i$ we have that

$$\|T_p \psi_i(w_1)\| < a_i.$$

Furthermore, using $\|T_p \psi_j(w_2)\| < f(p)$

$$\|T_p \psi_i(w_2)\| \leq \|T_p \psi_i \circ (T_p \psi_j)^{-1}\| \|T_p \psi_j(w_2)\| < a_i,$$

by construction. So we conclude that w_1 and w_2 are in W_i and because of the injectivity of $E|_{W_i}$ we have $w_1 = w_2$ [13]. $\square$

Now let us turn to convex sets. Convex sets are sets that are normal neighborhoods of each of its points.

Every point p in a semi-Riemannian manifold M possesses a basis of neighborhoods, which consists of convex sets. For δ small enough, we know that E is a diffeomorphism of an open neighborhood W of $0_p \in TM$ to $V(\delta) \times V(\delta)$, where V is a normal neighborhood and $V(\delta) := \{q \in V : N(q) < \delta\}$ diffeomorphic via normal coordinates to open balls in $\mathbb{R}^n$. $V(\delta)$ is a normal neighborhood of each of its points, indeed consider $q \in V(\delta)$ and set $W_q = W \cap T_p M$. Then $E(W_q) = \{q\} \times V(\delta)$. So

$$E|_{W_q} : W_q \rightarrow \{q\} \times V(\delta)$$

is a diffeomorphism and thus $\exp_q|_{W_q}$ onto $V(\delta)$. Let us now show that W_q is star-shaped.

For $q \neq \tilde{q} \in V(\delta)$, set $v := E^{-1}(q, \tilde{q}) = \exp_q^{-1}(\tilde{q}) \in W_q$. Then

$$\sigma : [0, 1] \rightarrow M$$

given by $\sigma(t) = \gamma_v(t)$ is a geodesic from q to $\tilde{q}$. If we prove that σ lies in $V(q)$, then we get from the proof of proposition 2.3 that $t\sigma \in W_q$ for all $t \in [0, 1]$. Since any $v \in W_q$ is of the form $E^{-1}(q, \tilde{q})$, we are done.

By contradiction, we assume that σ leaves $V(\delta)$. Clearly, then there is a $t \in (0, 1)$ with $N(\sigma(t)) \geq \delta$. But $N(q), N(\tilde{q}) < \delta$ and thus there is $t_0 \in (0, 1)$ such that $t \mapsto N \circ \sigma$ takes the maximum in t_0 . By the geodesic equation, we then find for $\sigma^i = x^i \circ \sigma$

$$\frac{d^2(N \circ \sigma)}{dt^2} = 2 \sum_i \left(\left(\frac{d\sigma^i}{dt} \right)^2 + \sigma^i \frac{d^2(\sigma^i)}{dt^2} \right) = 2 \sum_{jk} \left(\delta_{jk} - \sum_i \Gamma_{jk}^i \sigma^i \right) \frac{d\sigma^j}{dt} \frac{d\sigma^k}{dt}.$$

So

$$\frac{d^2(N \circ \sigma)}{dt^2}(t_0) = 2B(\sigma'(t_0), \sigma'(t_0)) > 0,$$

since $\sigma' \neq 0$. This is a contraction to $N \circ \sigma(t_0)$ maximal [13], [15].

For further information, we refer to [13] and [15].

Now we are ready to deal with the problems encountered when attempting to lower the regularity of the metric. We will see that some elements of geometry can be developed in low regularity.

Let us start with an introduction to Lorentzian geometry. Then we are going to state an important theorem, where we prove that the local Lorentzian length maximization property of causal geodesics in the class of absolutely continuous causal curves holds already for $\mathcal{C}^{1,1}$ spacetime metrics.

3 Lorentzian Geometry

A symmetric non-degenerate $(0, 2)$ tensor field, meaning a covariant tensor field of rank 2, of constant index on a smooth manifold M is a metric tensor g . It smoothly assigns a non-degenerate symmetric bilinear form g_p on the tangent space $T_p M$ to each point $p \in M$. The index of g_p is constant for all p .

A smooth manifold M endowed with such a metric tensor g such that the index of M is 1 is called a Lorentzian manifold. The tangent space to such a manifold is linearly isometric to Minkowski space $\mathbb{R}_1^n$. A Lorentzian vector space is defined as a scalar product space with index 1 and dimension bigger or equal 2.

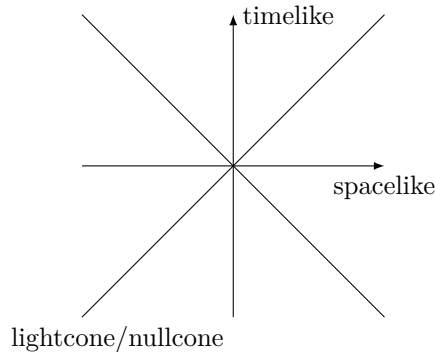
In the Relativity theory three space dimensions and one time dimension are merged into a four-dimensional Lorentzian manifold. We will take time to be the zero-th dimension. For instance, an orthonormal basis for a Lorentzian vector space will be denoted by $e_0, e_1, \dots, e_{n-1}$, where e_0 is the timelike vector [20], [24].

3.1 The causal character

Lorentzian geometry starts with the study of the causal character of vectors. We will call a tangent vector v to M

$$\begin{aligned} &\text{spacelike if } \langle v, v \rangle > 0 \text{ or } v = 0, \\ &\text{lightlike or null if } \langle v, v \rangle = 0 \text{ and } v \neq 0 \text{ and} \\ &\text{timelike if } \langle v, v \rangle < 0. \end{aligned}$$

The null cone at p is the set of all null vectors in $T_p M$. For example, the lightlike vectors in a Minkowski space are null vectors. The subspace $\mathbb{R}_v$ that is generated by v has the same causal character as v itself. For instance, the zero subspace is spacelike, like the zero vector.



We have that if v is a timelike vector in a Lorentzian vector space, then the subspace $v^\perp$ is spacelike. More generally, something similar holds also for a subspace W . We call W

- (1) spacelike, if $g|_W$ is positive definite, that is W is an inner product space.
- (2) timelike, if $g|_W$ is nondegenerate of index 1.
- (3) lightlike, if $g|_W$ is degenerate.

This will be called the causal character of W . Since $(W^\perp)^\perp = W$, timelike and spacelike can be reversed, indeed if W is timelike then $W^\perp$ is spacelike and W is lightlike if and only if $W^\perp$ is. The easiest case is if W is spacelike. For instance, then all of its subspaces are also spacelike and the Cauchy-Schwarz inequality holds. Another property of the causal character of tangent vectors is that it is preserved by local isometries [20], [24].

Let us now consider some criteria for a subspace to be timelike.

Lemma 3.1. [20, Lemma 27, Chapter 5]

Consider a subspace W of dimension ≥ 2 in a Lorentzian vector space. Then

- (1) W is timelike, thus it is itself a Lorentzian vector space.

$\iff$

- (2) There are two linearly independent null vectors contained in W .

$\iff$

- (3) There is a timelike vector contained in W .

Proof. In order to show $(1) \Rightarrow (2)$, consider a orthonormal basis $e_1, \dots, e_m$ for W and let e_1 be the timelike vector. Then $e_1 \pm e_2$ are two linearly independent null vectors, indeed

$$\langle e_1 \pm e_2, e_1 \pm e_2 \rangle = \langle e_1, e_1 \rangle + \langle \pm e_2, \pm e_2 \rangle = 0.$$

If u and v are independent null vectors, then $g(u, v) \neq 0$, meaning that $g(u, v) < 0$ respectively $g(u, v) > 0$ and thus one of the vectors $u \pm v$ has to be timelike. This proves the implication $(2) \Rightarrow (3)$. So for the last implication let v be timelike in W . Then $W^\perp$ is a subset of $v^\perp$, which is spacelike. So $W^\perp$ is also spacelike as a subset. Obviously, $W = (W^\perp)^\perp$ is then timelike [20]. $\square$

Lemma 3.2. [20, Lemma 28, Chapter 5]

Again consider a subspace W of a Lorentzian vector space. Then

(1) W is lightlike, so degenerate.

$\iff$

(2) There is a null vector, but not a timelike vector contained in W .

$\iff$

(3) For a one-dimensional subspace L and the null cone Λ of V we have

$$W \cap \Lambda = L - 0.$$

Proof. Let us start with the implication (1) $\Rightarrow$ (2). W contains a null vector, since it is lightlike and above we proved that there cannot be a timelike vector contained in W .

Assume now that W contains a null vector. Then the intersection $W \cap \Lambda$ is clearly non-empty. If there were two independent null vectors, again by the previous lemma, then a timelike vector must be contained in W . Obviously, for the last implication we have that W cannot be spacelike and also not timelike, by the previous lemma and so W must be lightlike [20]. $\square$

Let us now consider a submanifold P of a Lorentzian manifold M . Assume that for every $p \in P$ the subspace $T_p P$ has the same causal character in $T_p M$. Then this causal character is ascribed to P . Hence semi-Riemannian submanifolds are not lightlike. An example for a lightlike submanifold is the nullcone in $\mathbb{R}_1^n$. The causal character of submanifolds is preserved by local isometries [20]. For further information, we refer to [20] and [24].

3.2 Timecones

Denote the set of all timelike vectors in a Lorentzian vector space V by $\mathcal{T}$. Let $u \in \mathcal{T}$, then the timecone of V containing u is the set

$$C(u) = \{v \in \mathcal{T} : \langle u, v \rangle < 0\}.$$

$\mathcal{T}$ is the disjoint union of $C(u)$ and the opposite timecone $C(-u) = -C(u)$, since $u^\perp$ is spacelike [20].

Lemma 3.3. [20, Lemma 29, Chapter 5]

Two timelike vectors v, w in a Lorentzian vector space are in the same timecone if and only if $\langle v, w \rangle < 0$.

Proof. Assume $v \in C(u)$ and w is timelike. We want to show that $w \in C(u)$ if and only if $\langle v, w \rangle < 0$. We have $C(\frac{u}{|u|}) = C(u)$, thus we can assume that u is a

timelike unit vector. We can write v as $au + \bar{v}$ and w as $bu + \bar{w}$ with $\bar{v}, \bar{w} \in u^\perp$. These are timelike vectors, so we have $|a| > |\bar{v}|$ and $|b| > |\bar{w}|$. We have that $\langle u, u \rangle = -1$ and $\langle \bar{w}, u \rangle = \langle \bar{v}, u \rangle = 0$, thus

$$\langle v, w \rangle = \langle au + \bar{v}, bu + \bar{w} \rangle = \langle au, bu \rangle + \langle au, \bar{w} \rangle + \langle \bar{v}, bu \rangle + \langle \bar{v}, \bar{w} \rangle = -ab + \langle \bar{v}, \bar{w} \rangle.$$

Furthermore, $|\langle \bar{v}, \bar{w} \rangle| \leq |\bar{v}||\bar{w}| < |ab|$ holds, by the Cauchy-Schwarz inequality. So $\text{sgn}\langle v, w \rangle = \text{sgn}(-ab)$. Since v is contained in $C(u)$, we have that a is positive and thus $\langle v, w \rangle$ is equal to $-\text{sgn}(b)$. Thus,

$$\langle v, w \rangle < 0 \Leftrightarrow b > 0 \Leftrightarrow \langle u, w \rangle < 0 \Leftrightarrow w \in C(u).$$

[20]

□

Many properties of inner product spaces have analogues in the Lorentzian case. For instance, by the Schwarz inequality, that the definition of the angle between v and w is the unique number $0 \leq \theta \leq \pi$ such that $\cos \theta = \frac{\langle v, w \rangle}{|v||w|}$. Also the triangle inequality runs backwards, that is $|v| + |w| \leq |v + w|$.

There exist two timecones in each Lorentzian tangent space $T_p M$ and we cannot distinguish one from the other in an intrinsic way. Choosing one of them means to time-orient $T_p M$. But can every tangent space be time-oriented in a continuous way? Consider a function τ on M assigning to each point p a timecone τ_p in the tangent space of M at p . If for each p there exists a smooth vector field V on some neighborhood U of p such that $V_q \in \tau_q$ for each $q \in U$, then τ is smooth and called a time-orientation of M . A manifold, that admits a time-orientation is called time-orientable. Choosing a specific time-orientation on a manifold means to time-orient it. An example is the Minkowski space $\mathbb{R}_1^n$. Also all Lorentzian spheres S_1^n are time-orientable. A Lorentzian manifold is time-orientable if and only if there is a timelike vector field. A vector in a Lorentzian vector space that is either null or timelike is called causal. So for a timelike vector v the set $\overline{C}(v)$ is the causal cone that contains v . The velocity vectors of a causal curve in a Lorentzian manifold are all nonspacelike [20].

For further information, we refer to [20].

3.3 Local Lorentzian Geometry

Let us now compare two results, one from Riemannian geometry and the other from Lorentzian geometry. The local geometry of Riemannian geodesics is dominated by the fact that if we consider a normal neighborhood of a point q in a Riemannian manifold containing p then the radial geodesic segment mapping from $[0, 1]$ to the normal neighborhood is the unique shortest curve from q to p in

this neighborhood. We have an analogue in a Lorentzian manifold. Considering a timelike curve in a normal neighborhood from q to p , then the radial geodesic segment from q to p is the unique longest timelike curve from q to p in this neighborhood. We can refine the result by omitting the normal neighborhood and comparing the geodesic segment with timelike curves. This result implies that any timelike geodesic segment in the Minkowski space is the unique longest timelike curve that joins its endpoints. Observe here that in Minkowski space the entire manifold is a normal neighborhood of each point.

We want to point out one more analogue. For this, consider the Riemannian product manifold $S^1 \times \mathbb{R}$. It can be viewed as the cylinder $x^2 + y^2 = 1$ in $\mathbb{R}^3$. The map ϕ given by $(u, v) \rightarrow (\cos u, \sin u, v)$, that wraps the plane around the cylinder, is locally an isometry. Hence all the geodesics of the cylinder have the form

$$\gamma(t) = (\cos(at + b), \sin(at + b), ct + d).$$

In analogue to this, consider the Lorentzian surface $S^1_1 \times \mathbb{R}$. It can also be viewed as a cylinder in $\mathbb{R}^3_2$, that has the same connection, hence the same geodesics. Null cones on this cylinder are intersections of null geodesics in the Riemannian case, for instance $(\pm \cos s, \sin s, s + c)$ [20].

For further information, we refer to [20].

3.4 Causality relations

Which points in a Lorentzian manifold can be joined by causal curves, in other words which events can influence or can be influenced by a given event? In the study of causality a useful aid is a Lorentzian analogue of Riemannian distance. Thus, let us turn to causality relations. On a Lorentzian manifold M they are defined as follows. Let $p, q \in M$, then

- (1) The existence of a future-pointing timelike curve in M from p to q is denoted by $p \ll q$.
- (2) The existence of a future-pointing causal curve in M from p to q is denoted by $p < q$.

Obviously, (1) implies (2). These relations are transitive and for $p \ll q$ there exist infinitely many r such that $p \ll r \ll q$. This holds also for the second property.

Furthermore, let us define the chronological respectively causal future. For this,

consider a subset A of M . Then

$$I^+(A) = \{q \in M : \text{there exists } p \in A \text{ with } p \ll q\}$$

is the chronological future of A . The causal future of A is given by

$$J^+(A) = \{q \in M : \text{there exists } p \in A \text{ with } p \leq q\}.$$

We have $A \cup I^+(A) \subset J^+(A)$. For a single point the chronological future is given by $I^+(p) = \{q : p \ll q\}$ and analogously for the causal future. The past definitions follow from the future ones by reversing time-orientation. Observe that the chronological past of A is given by

$$I^-(A) = \{q \in M : \text{there exists } p \in A \text{ with } q \ll p\}.$$

Let us give an example. In the Minkowski space $\mathbb{R}_1^4$ the chronological future of a point p is the future timecone of p , indeed $\{q : \overline{pq}$ is timelike future-pointing $\}$. The causal future here is p together with $\{q : \overline{pq}$ is causal future-pointing $\}$. We can conclude that $I^+(p)$ is open and its closure is $J^+(p)$ and $J^+(p)$ is the union of the future nullcone $I^+(p)$ at p and p itself.

Another example is the Lorentzian cylinder $S_1^1 \times \mathbb{R}$. It has trivial causality. For a single point we have $I^+(p) = J^+(p)$, thus it is the entire manifold.

A subset $A \subset M$ is called causally convex in $B \subset M$ with $A \subset B$ if every $\mathcal{C}^1$ causal curve, that is contained in B and joining two points of A , must be contained in A . An open subset B is called strongly causal if every point p in B admits arbitrarily small open neighborhoods that are causally convex in B . Furthermore, an open subset S of M is causally simple if it is strongly causal and

$$J_S^+ \subset S \times S$$

is closed in the product topology. A causally simple subset S is called globally hyperbolic if

$$J_S^+(p) \cap J_S^-(p)$$

is compact for every $p, q \in S$.

Two points $x, y \in M$ are called chronologically related

$$y \in I_S^+(x), (x, y) \in I_S^+ \text{ or } x \ll_S y$$

in a set $S \subset M$, if there exists a future directed $\mathcal{C}^1$ timelike curve from x to y contained in S . Two points are causally related

$$y \in J_S^+(x), (x, y) \in J_S^+ \text{ or } \\ x \leq_S y,$$

if there exists a future directed $\mathcal{C}^1$ causal curve from x to y contained in S or $x = y \in S$. If $x \leq_S y$ for $x \neq y$ we write $x <_S y$ and if $S = M$ then we write simply $\ll, \leq$ and $<$.

3.5 Lorentzian length of a causal AC-curve

A curve $\sigma : [a, b] \rightarrow M$ is absolutely continuous if its components in one and hence every local chart are locally absolutely continuous. Such a curve is called an *AC-curve*. Let us introduce a complete Riemannian metric on M and denote by ρ the corresponding distance. Then, equivalently, σ is absolutely continuous if it satisfies locally the usual definition of absolute continuity between (topological) metric spaces. This definition does not depend on the metric chosen, since every pair of Riemannian metrics over a compact set are Lipschitz equivalent and M is locally compact. We can define analogously the concept of Lipschitz curve. An AC-curve

$$\sigma : [a, b] \rightarrow M, \\ t \mapsto \sigma(t)$$

is future directed causal if $\dot{\sigma}$ is future directed causal almost everywhere. We do not need to define a notion of absolutely continuous timelike curve, because they are assumed to be piecewise $\mathcal{C}^1$.

The Lorentzian length of a causal AC-curve is given by

$$l[\sigma] = \int_a^b \sqrt{-g_\sigma(\dot{\sigma}, \dot{\sigma})} dt.$$

The integrand belongs to $L^1([a, b])$ and thus the Lorentzian length of a causal AC-curve is finite. In coordinates we have

$$\dot{\sigma}^\mu \in L^1([a, b]) \text{ and } \sqrt{|\dot{\sigma}|^\mu} \in L^1([a, b]).$$

A Lorentzian distance between p and q with $p \leq q$ can also be defined as the supremum of the Lorentzian lengths of the causal AC-curves, that connects p to q [20], [18].

Theorem 3.4. [18, Theorem 6]

Consider a $\mathcal{C}^\infty$ manifold M endowed with a $\mathcal{C}^{1,1}$ metric g and a normal neighborhood N of $p \in M$. Let $\sigma : [0, 1] \rightarrow N$ be any future directed causal AC-curve starting from p . Then $\exp_p^{-1}(\sigma(s))$ is future directed causal for every $s > 0$ and in case that $\exp_p^{-1}(\sigma(\hat{s}))$ is lightlike then $\sigma|_{[0, \hat{s}]}$ is equal to the future directed lightlike geodesic segment up to parametrizations. The Lorentz length of σ is smaller than that of the unique future directed casual geodesic, that connects its endpoints, unless its image is the same as that of that geodesic. In the last case the affine parameter of the geodesic is absolutely continuous and increasing with s .

Proof. Assume that for some $\tilde{s}$, $D_p^2(\tilde{s}) < 0$, where D_p^2 is defined as

$$D_p^2(q) := g_p(\exp_p^{-1}(q), \exp_p^{-1}(q))$$

and $\mathcal{C}^{1,1}$ in q . The Lorentzian length of the geodesic that connects p and q is given by

$$D_p^L(q) := (-D_p^2(q))^{\frac{1}{2}}.$$

Indeed, consider a geodesic γ_v with $v = \exp_p^{-1}(q)$ and notice that $t \mapsto \langle \gamma'_v(t), \gamma'_v(t) \rangle$ is constant, then

$$\text{Length}(\gamma) = \int_0^1 \langle \gamma'_v(t), \gamma'_v(t) \rangle dt = \langle \gamma'_v(0), \gamma'_v(0) \rangle = (-D_p^2(q))^{\frac{1}{2}}.$$

In the region, where $D_p^2 < 0$, D_p^L is $\mathcal{C}^{1,1}$, since D_p^2 is. $D^L(\sigma(s))$ is a composition of a locally Lipschitz function and an absolutely continuous function and thus it is absolutely continuous. We have that $D_p^2(\tilde{s}) < 0$. By continuity, this holds also in an interval $[\tilde{s}, s]$, provided s is sufficiently close to $\tilde{s}$. Set $\hat{P} := \frac{P}{\sqrt{-g_P(P, P)}}$, note

$$\frac{1}{2} \frac{d}{ds} D_p^2(\sigma(s)) = g(P(p, \sigma(s)), \sigma'(s))$$

and $P(p, \sigma(s))$ is the position vector of $\sigma(s)$ with respect to p .

This will be proved in the proof of Gauss' Lemma in the last chapter. Then

observe

$$\begin{aligned}
D_p^L(\sigma(s)) - D_p^L(\sigma(\tilde{s})) &= \int_{\tilde{s}}^s \frac{dD_p^L(\sigma(t))}{dt} dt \\
&= - \int_{\tilde{s}}^s \frac{1}{D_p^L(\sigma(t))} g(P(p, \sigma(t)), \sigma'(t)) dt \\
&= - \int_{\tilde{s}}^s g(\hat{P}(p, \sigma(t)), \sigma'(t)) dt \geq \int_{\tilde{s}}^s \sqrt{-g_{\sigma(t)}(\sigma'(t), \sigma'(t))} dt \\
&\geq l[\sigma] \geq 0,
\end{aligned} \tag{3.1}$$

where in the last inequality we used the reverse Cauchy-Schwarz inequality: Consider v_1, v_2 to be causal and future directed, then

$$-g_p(v_1, v_2) \geq \sqrt{-g_p(v_1, v_1)} \sqrt{-g_p(v_2, v_2)}.$$

Equality holds if and only if v_1 and v_2 are proportional.

(3.1) shows that if σ once enters a region of the chronological future of p , denoted by $D_p^2 < 0$, then it remains there.

Consider now a small future directed timelike geodesic

$$\eta : [-\epsilon, 0] \rightarrow N$$

given by $\eta(0) = p$, contained in a convex normal neighborhood of p with $C \subset N$. We have that $\sigma(s) \in C$ for sufficiently small s . Concatenating η with σ , we get a curve, that connects $\eta(-\epsilon)$ to $\sigma(s)$ and starts with a timelike geodesic. Hence it enters the chronological future of $\eta(-\epsilon)$ and thus, by the argument above, there is a future directed timelike geodesic $\nu^{(\epsilon)}$, that connects $\eta(-\epsilon)$ with $\sigma(s)$. Let $\epsilon \rightarrow 0$ and use the continuity of the exponential map at $\sigma(s)$, then we can infer the existence of a geodesic that connects p and $\sigma(s)$. This must be future directed causal, by the continuity of $g(v, v)$ at $T_{\sigma(s)}M$. Indeed, $\nu^{(\epsilon)}(t) = \exp_{\sigma(s)}(-tw^{(\epsilon)})$, where $w^{(\epsilon)} := \exp_{\sigma(s)}^{-1}(\eta(-\epsilon))$, for $\epsilon \rightarrow 0$ we have $w^{(\epsilon)} \rightarrow w := \exp_{\sigma(s)}^{-1}(p)$ and $\nu^{(\epsilon)} \rightarrow \nu$ in $\mathcal{C}^1$, where $\nu(t) = \exp_{\sigma(s)}(-tw)$. All $w^{(\epsilon)}$ are timelike and so w is causal and also ν is causal. Since s is arbitrary we know that in a maximal closed interval $[0, b] \subset [0, 1]$ with $b > 0$, we have $D_p^2(\sigma(s)) \leq 0$.

Now we want to prove that if $D_p^2(\sigma(a)) = 0$ for $a \in (0, b]$, then $\sigma|_{[0, a]}$ is a lightlike geodesic up to parametrizations and so $D_p^2 = 0$ over $[0, a]$.

Note that D_p^2 is Lipschitz and thus $D_p^2(\sigma(s))$ is absolutely continuous

$$D_p^2(\sigma(a)) = \int_0^a \frac{dD_p^2(\sigma(s))}{ds} ds = 2 \int_0^a g_{P(p, \sigma(s))}(P(p, \sigma(s)), \sigma'(s)) ds.$$

By the reverse Cauchy-Schwarz inequality, since σ' is future directed causal

almost everywhere, we have

$$g_{P(p, \sigma(s))}(P(p, \sigma(s)), \sigma'(s)) \leq 0,$$

on the region $D_p^2 \leq 0$ for almost every s . Thus we have $D_p^2(\sigma(a)) = 0$ only if σ' is proportional to P for almost every s in $[0, a]$. Introduce now a Euclidean scalar product on $T_p M$ and associated spherical normal coordinates $(r, \Theta_1, \dots, \Theta_n)$ over N . Since this chart is $\mathcal{C}^1$ related to those of M we have that σ is still absolutely continuous in this chart. The angle coordinates cannot change over σ , since σ' is proportional to ∂_r almost everywhere. Otherwise one would get that σ' is not radial in a set of non-vanishing measure, since $\Theta_i(\sigma(s))$ is the integral of its own derivative. This is a contradiction. We obtain that $\sigma|_{[0, a]}$ is an integral curve of P and thus a lightlike geodesic issued from p .

Now let a be the maximal value of s , for which we have $D_p^2(\sigma(s)) = 0$. So it remains only to show that $b = 1$. Notice that b is the maximal value of s for which $D_p^2(\sigma(s)) \leq 0$. By contradiction, suppose $b \neq 1$, meaning $b < 1$. Then $a = b$, because otherwise $D_p^2(\sigma(b)) < 0$ and once σ enters the chronological future of p it remains there. But this means that the same inequality would also hold in $(b, 1]$. This is a contradiction to $b < 1$, because we have $\max_{s \in [0, 1]} D_p^2(\sigma(s)) \leq 0$ and clearly this implies $s = 1$.

Let $p' = \sigma(b)$ and consider a convex neighborhood C'' containing p with $C'' \subset N$. With the same argument as above we see that for any sufficiently small δ , p' is connected to $\sigma(b + \delta)$ by a future directed causal geodesic

$$\eta : [0, 1] \rightarrow C''.$$

This geodesic cannot be the prolongation of the lightlike geodesic $\sigma|_{[0, b]}$ for which we would get $D_p^2(\sigma(b + a\delta)) \leq 0$ with $a \in [0, 1]$, because this would be a contradiction to the maximality of b . Notice that $\eta(0) = \sigma(a) = \sigma(b)$. So $g(P(p, \eta(0)), \eta'(0)) < 0$, where $P(p, \eta(0)) = \exp_p^{-1}(\eta(0)) = \exp_p^{-1}(\sigma(b))$ and by continuity, the scalar product

$$g_p(P(p, \eta(t)), \eta'(t)) < 0$$

for $t > 0$ and t sufficiently small.

Observe that D_p^2 is Lipschitz and thus $D_p^2(\eta(t))$ is absolutely continuous. For sufficiently small t we then have

$$D_p^2(\eta(t)) = D_p^2(\sigma(b)) + \int_0^t \frac{dD_p^2(\eta(t))}{dt} dt = 2 \int_0^t g_p(P(p, \eta(t)), \eta'(t)) dt < 0.$$

Since the concatenation of $\sigma|_{[0, b]}$ with η is a causal AC -curve and since D_p^2 gets

negative on it at some point and it remains so, we have at the endpoint

$$D_p^2(\sigma(b + \delta)) = D_p^2(\eta(1)) < 0.$$

Since δ was chosen arbitrary we get a contradiction to the maximality of b and so $b = 1$.

In the case that σ is a lightlike geodesic up to parametrization, its Lorentzian length clearly vanishes and

$$D_p^L(\sigma(1)) \geq l(\sigma)$$

holds. Assume now that σ is not a lightlike geodesic up to parametrizations. Then $a < 1$ and its Lorentzian length is given by $\sigma_{[a,1]}$. Consider $\tilde{s} \in [a, 1]$ such that $D_p^2(\sigma(\tilde{s})) < 0$. By (3.1) we have

$$D_p^L(\sigma(1)) \geq l(\sigma_{[\tilde{s},1]}).$$

Letting $\tilde{s} \rightarrow a$ we get

$$D_p^L(\sigma(1)) \geq l(\sigma),$$

proving that σ has a Lorentzian length not larger than that of the geodesic that connects its endpoints.

By contradiction, we suppose that they have the same Lorentzian length and that σ is not a causal geodesic up to parametrizations. Then $a < 1$, because otherwise the geodesic would be lightlike. From (3.1) for $\tilde{s} > a$ we obtain

$$D_p^L(\sigma(1)) \geq D_p^L(\sigma(\tilde{s})) + l(\sigma_{[\tilde{s},1]}) \geq l(\sigma_{[0,\tilde{s}]}) + l(\sigma_{[\tilde{s},1]}) = l(\sigma).$$

Thus, we actually have the equality

$$D_p^L(\sigma(1)) = D_p^L(\sigma(\tilde{s})) + l(\sigma_{[\tilde{s},1]})$$

and this implies that

$$g_p(P(p, \sigma(s)), \sigma'(s)) = 0$$

for almost every $s \in [\tilde{s}, 1]$. Hence, we obtain, since $\tilde{s}$ is arbitrarily, that σ' is proportional to P for almost every $s \in [a, 1]$. Let us now again introduce spherical normal coordinates. Then arguing as above shows that the image of $\sigma|_{[a,1]}$ is an integral curve of P and hence the prolongation of $\sigma|_{[0,a]}$ if $a \neq 0$. Thus it is the image of a geodesic.

Now suppose that the image of σ coincides with that of a causal geodesic η . The composition $r(s)$ is absolutely continuous, since the coordinates of the spherical

normal chart are Lipschitz functions. So, by definition r is an affine parameter over η . The map $r(s)$ is increasing for if $r(s_2) \leq r(s_1)$ for $s_1 < s_2$. So we would have $r' \leq 0$ and thus also in a subset of measure different from zero on $[s_1, s_2]$. This again would imply that

$$\frac{d}{ds}\sigma = \left(\frac{d}{dr}\eta\right)r'$$

is not future directed causal in a set of measure different from zero. This is a contradiction to the definition of future directed causal AC -curve and thus our claim follows and the theorem is proved [18]. $\square$

For further information, we refer to [18] and [20].

We now continue with an introduction, where we go through some Picard Lindelöf analysis in order to prove the strong differentiability of the exponential map later on. By the Hartman and Wintner example, we will see that the relation between minimizing curves and geodesics turns out to differ from the smooth situation. Furthermore, we state a formulation of the inverse function theorem, namely Leach's theorem. This will be helpful in the proof of the main theorems 1 and 2.

4 Introduction to the topic

First consider a second order ordinary differential equation:

$$\frac{dx^\mu}{dt} = v^\mu, \quad (4.1)$$

$$\frac{dv^\mu}{dt} = H^\mu(x, v), \quad (4.2)$$

where x^μ is the coordinate chart in a neighborhood U of p in M , H^μ is locally Lipschitz and positive homogeneous of second degree in v , that is for every $s > 0$ we have

$$H^\mu(x, sv) = s^2 H^\mu(x, v).$$

Let us go through a Picard-Lindelöf analysis, which will be needed later on in order to prove the strong differentiability of the exponential map.

4.1 Picard-Lindelöf analysis

To this end assume without loss of generality that $x^\mu(p) = 0$. Furthermore, consider the closed ball $\overline{B}(p, r) = \{q : \|q - p\| \leq r\}$ and choose r such that the ball is a subset of U . The local coordinate system on the tangent bundle will be denoted by $\{x^\mu, \dot{x}^\mu\}$ and the image $x^\mu(U)$ of the coordinate chart is an open subset of $\mathbb{R}^n$. Let the Lipschitz condition hold in $\overline{B}(p, r) \times \{v : \|v\| \leq 1\}$.

Now applying the fundamental theorem of calculus on the system above we can rewrite it in integral form with the initial condition $(x_0, \dot{x}_0)$ at time $t = 0$ plugged in

$$x^\mu(t, x_0, \dot{x}_0) = x_0^\mu + \int_0^t \dot{x}^\mu(t, x_0, \dot{x}_0) dt, \quad (4.3)$$

$$\dot{x}^\mu(t, x_0, \dot{x}_0) = \dot{x}_0^\mu + \int_0^t H^\mu(x(t, x_0, \dot{x}_0), \dot{x}(t, x_0, \dot{x}_0)) dt. \quad (4.4)$$

We suppose that the initial condition satisfies $\max\{\|x_0\|, \|\dot{x}_0\|\} < \delta$, where δ is a constant which is positive with $\delta < \frac{1}{M}(1 - e^{-\frac{Mr}{2}}) \leq \frac{r}{2}$.

We will use Picard iteration starting from $k = -1$ to construct approximating solutions, meaning that we recursively define x_{k+1}^μ in terms of x_k^μ . So for $k = -1$ we define $x_{-1}(t) = x_0$ and $\dot{x}_{-1}(t) = 0$. They are well defined and continuously differentiable. Inductively so are the functions

$$x_{k+1}^\mu(t) = x_0^\mu + \int_0^t \dot{x}_k^\mu(t) dt, \quad (4.5)$$

$$\dot{x}_{k+1}^\mu(t) = \dot{x}_0^\mu + \int_0^t H^\mu(x_k(t), \dot{x}_k(t)) dt. \quad (4.6)$$

We are going to show that on $[0, 1]$, $(x_k, \dot{x}_k)(t) \in \pi^{-1}(\overline{B}(p, r))$ for every $t \in [0, 1]$ and each $k \geq -1$. Indeed, using the fact that the length of a continuously differentiable curve on $\mathbb{R}^n$ is greater than the distance between its endpoints and letting $M = \sup_{x \in \overline{B}(p, r)} \sup_{\|v\|=1} \|H(x, v)\|$, we derive using the homogeneity of H^μ that

$$\|x_{k+1}(t)\| \leq \|x_0\| + \int_0^t \|\dot{x}_k(t)\| dt, \quad (4.7)$$

$$\|\dot{x}_{k+1}(t)\| \leq \|\dot{x}_0\| + \int_0^t M \|\dot{x}_k(t)\|^2 dt. \quad (4.8)$$

From the second inequality we get the following estimate by induction:

$$\|\dot{x}_{k+1}(t)\| \leq \frac{\|\dot{x}_0\|}{1 - \|\dot{x}_0\|Mt}.$$

By plugging this into the first equation it gives us in turn

$$\|x_{k+1}(t)\| \leq \|x_0\| - \frac{1}{M} \ln(1 - \|\dot{x}_0\|Mt) \leq \delta - \frac{1}{M} \ln(1 - \delta M),$$

where in the last inequality we used that $\max\{\|x_0\|, \|\dot{x}_0\|\} < \delta$. Additionally, applying the estimate of δ , namely $\delta \leq \frac{r}{2}$, we get for every $k \geq 0$ that

$$\begin{aligned} \|x_k(t)\| &< r \text{ and} \\ \|\dot{x}_k(t)\| &< \frac{\delta}{1 - \delta M}, \end{aligned}$$

and thus they belong to $\pi^{-1}(\overline{B}(p, r))$. For any $t_1, t_2 \in [0, 1]$ and any $v_1 := \dot{x}_i(t_1)$, $v_2 := \dot{x}_j(t_2)$ we have that $\frac{\delta}{1 - \delta M} > \max\{\|v_1\|, \|v_2\|\}$. For further information, we refer to [18].

4.2 Existence and Uniqueness of geodesics

The non-stationary solutions to the equations (4.1) and (4.2) will be called geodesics. We will use the computations in subsection 4.1 to prove the existence of geodesics.

First, choose δ sufficiently small such that

$$\begin{aligned} \frac{\delta}{1 - \delta M} &\leq 1 \text{ and} \\ D := \frac{\beta\delta}{2(1 - \delta M)} \left(1 + \sqrt{1 + \frac{4\alpha}{\beta^2}} \right) &\leq 1. \end{aligned}$$

This positive constant fulfils the equation

$$\frac{[\frac{\delta}{1-\delta M}]^2 \alpha}{D} + \frac{\beta \delta}{1-\delta M} = D.$$

By the telescopic sum we have: $x_n - x_0 = \sum_{k=1}^n x_k - x_{k-1}$. Moreover, we have shown that $\frac{\delta}{1-\delta M} > \max\{\|v_1\|, \|v_2\|\}$ and we know that the inequalities (4.7) and (4.8) hold. Thus we can subtract two elements and rewrite the Lipschitz condition in a better form

$$\begin{aligned} \|\dot{x}_k(t) - \dot{x}_{k-1}(t)\| &\leq \int_0^t \|H(x_{k-1}(t), \dot{x}_{k-1}(t)) - H(x_{k-2}(t), \dot{x}_{k-2}(t))\| dt \quad (4.9) \\ &\leq \int_0^t \left\{ \left[\frac{\delta}{1-\delta M} \right]^2 \alpha \|x_{k-1}(t) - x_{k-2}(t)\| + \frac{\beta \delta}{1-\delta M} \|\dot{x}_{k-1}(t) - \dot{x}_{k-2}(t)\| \right\} dt, \end{aligned}$$

where we used that H is homogeneous of second degree in v and Lipschitz, that is $\|H(x_2, v_2) - H(x_1, v_1)\| \leq \alpha \|x_2 - x_1\| + \beta \|v_2 - v_1\|$. Indeed, for $V = \frac{\delta}{1-\delta M}$ we have the easy computation

$$\begin{aligned} \|H(x_2, v_2) - H(x_1, v_1)\| &= V^2 \left\| H\left(x_2, \frac{v_2}{V}\right) - H\left(x_1, \frac{v_1}{V}\right) \right\| \\ &\leq V^2 \left\{ \alpha \|x_2 - x_1\| + \beta \left\| \frac{v_2}{V} - \frac{v_1}{V} \right\| \right\} \\ &\leq \alpha V^2 \|x_2 - x_1\| + \beta V \|v_2 - v_1\|. \end{aligned}$$

Furthermore, observe that

$$\|x_{k+1}(t) - x_k(t)\| \leq \int_0^t \|\dot{x}_k(t) - \dot{x}_{k-1}(t)\| dt. \quad (4.10)$$

By induction we obtain an estimate for this expression. We will proceed in parallel and construct the following bounds

$$\|x_k(t) - x_{k-1}(t)\| \leq \frac{1}{D} \frac{(Dt)^k}{k!} \text{ for } k \geq 0 \quad (4.11)$$

$$\|\dot{x}_k(t) - \dot{x}_{k-1}(t)\| \leq \frac{(Dt)^k}{k!} \text{ for } k \geq 1. \quad (4.12)$$

Let us start with the first estimate. Note that we can rewrite $x_0(t)$ as

$$x_0 + \int_0^t \dot{x}_{-1}(t) dt$$

and recall that $x_{-1}(t) = x_0$ and $\dot{x}_{-1}(t) = 0$ hold. Then for $k = 0$ we get $\|x_0(t) - x_{-1}(t)\| = \|x_0 - x_0\| = 0$ and this is in any case smaller or equal to

$\frac{1}{D} \frac{(Dt)^0}{0!}$. Using this and the estimate in (4.9) we get (4.12) for $k = 1$:

$$\begin{aligned} \|\dot{x}_1(t) - \dot{x}_0(t)\| &\leq \int_0^t \left[\frac{\delta}{1-\delta M} \right]^2 \alpha \|x_0(t) - x_{-1}(t)\| + \frac{\beta\delta}{1-\delta M} \|\dot{x}_0(t) - \dot{x}_{-1}(t)\| dt \\ &\leq \int_0^t \frac{A}{D} + B dt = \int_0^t D dt = tD = \frac{(Dt)^1}{1!}, \end{aligned}$$

where we set $A = (\frac{\delta}{1-\delta M})^2 \alpha$ and $B = \frac{\beta\delta}{1-\delta M}$. Now we again switch to (4.11) and show by using (4.10) the induction hypothesis for $k + 1$.

$$\|x_{k+1}(t) - x_k(t)\| \leq \int_0^t \|\dot{x}_k - \dot{x}_{k-1}\| dt \leq \int_0^t \frac{(Dt)^k}{k!} dt = \frac{1}{D} \frac{(Dt)^{k+1}}{(k+1)!}.$$

For (4.12) we obtain:

$$\begin{aligned} \|\dot{x}_{k+1}(t) - \dot{x}_k(t)\| &\leq \int_0^t A \|x_k(t) - x_{k-1}(t)\| + B \|\dot{x}_k(t) - \dot{x}_{k-1}(t)\| dt \\ &\leq \int_0^t \frac{1}{D} \frac{(Dt)^k}{k!} + \frac{(Dt)^k}{k!} dt \leq \int_0^t \left(\frac{A}{D} + B \right) \frac{(Dt)^k}{k!} dt = \frac{(Dt)^{k+1}}{(k+1)!}. \end{aligned}$$

Using this we define a sequence of continuous functions

$$\begin{aligned} x_k^\mu(t) &= x_0^\mu + \sum_{k=0}^{n-1} [x_{k+1}^\mu(t) - x_k^\mu(t)], \\ \dot{x}_k^\mu(t) &= \dot{x}_0^\mu + \sum_{k=0}^{n-1} [\dot{x}_{k+1}^\mu(t) - \dot{x}_k^\mu(t)] \end{aligned}$$

converging uniformly to $x^\mu(t)$ and $\dot{x}^\mu(t)$ over $[0, 1]$. Notice that $H^\mu(x, \dot{x})$ is continuous over $\overline{B}(p, r) \times \{v : \|v\| \leq \frac{\delta}{1-\delta M}\}$, which is compact. For this reason it is also uniformly continuous. Using uniform convergence we can take the limit in the equations (4.5) and (4.6), which solve (4.3) and (4.4) [18].

In order to prove uniqueness let Y be another solution of the equations (4.1) and (4.2) on $[0, 1]$. To simplify the following, we introduce a new notation.

Let for now

$$\begin{aligned} X(t) &= \begin{pmatrix} x^\mu(t) \\ \dot{x}^\mu(t) \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \text{ and} \\ \dot{X}(t) &= \begin{pmatrix} X_2(t) \\ H^\mu(X_1(t), X_2(t)) \end{pmatrix} = F(t, X(t)). \end{aligned}$$

Let $\mathcal{M} := \max_{X_1 \in \overline{B}(p, r), X_2 \in \overline{B}_1(0)} \|F(t, X(t))\|$. Furthermore, we always can choose $\|X\| := \|X_1\| + \|X_2\|$.

Then, by (4.5), we have

$$Y(t) = X^{(0)} + \int_0^t F(s, Y(s)) ds.$$

By induction we get

$$\|X^{(k)}(t) - Y(t)\| \leq \mathcal{M} \frac{1}{D} \frac{(Dt)^k}{k!}.$$

Indeed, for $k = 0$ we have

$$\|X^{(0)}(t) - Y(t)\| \leq \int_0^t \|F(s, Y(s))\| ds \leq \mathcal{M} \cdot t,$$

where we used that $\|F(s, Y(s))\| \leq \mathcal{M}$. We have

$$\begin{aligned} & \|F(t, X(t)) - F(t, Y(t))\| \\ &= \|X_2(t) - Y_2(t)\| + \|H^\mu(X_1(t), X_2(t)) - H^\mu(Y_1(t), Y_2(t))\| \\ &\leq \|X_2(t) - Y_2(t)\| + \alpha \|X_1(t) - Y_1(t)\| + \beta \|X_2(t) - Y_2(t)\| \\ &\leq \alpha \|X_1(t) - Y_1(t)\| + (1 + \beta) \|X_2(t) - Y_2(t)\| \\ &\leq (\alpha + (1 + \beta)) (\|X_1(t) - Y_1(t)\| + \|X_2(t) - Y_2(t)\|) = D \|X(t) - Y(t)\|, \end{aligned}$$

where we used that H^μ is Lipschitz. Assuming that the induction hypothesis holds for some k and using the computation above, we can prove it for $k \rightarrow k+1$:

$$\begin{aligned} \|X^{(k+1)}(t) - Y(t)\| &\leq \int_0^t \|F(s, X^{(k)}(s)) - F(s, Y(s))\| ds \\ &\leq \int_0^t D \|X^{(k)}(s) - Y(s)\| ds \leq \int_0^t D \frac{1}{D} \mathcal{M} \frac{(Ds)^k}{k!} ds = \mathcal{M} \frac{1}{D} \frac{(Dt)^{k+1}}{(k+1)!}. \end{aligned}$$

By the induction hypothesis, we have for all t

$$\|X(t) - Y(t)\| = \lim_{k \rightarrow \infty} \|X^{(k)} - Y\| \leq \lim_{k \rightarrow \infty} \mathcal{M} \frac{1}{D} \frac{(Dt)^k}{k!} = 0.$$

This means that $X = Y$ and thus we have uniqueness [14].

Notice that uniqueness need not to hold if the metric is of lower regularity than $\mathcal{C}^{1,1}$. We will show this by the following example:

Consider $M := (-1, 1) \times \mathbb{R}$ and the metric $g = dx^2 + (1 - |x|^\lambda) dy^2$, where $1 < \lambda < 2$. The regularity of the metric is $\mathcal{C}^{1, \lambda-1}$. In order to state our geodesic equations let us look at the non-vanishing Christoffel symbols, given by $\Gamma_{i,j}^k = \frac{1}{2} g^{kl} (\partial_j g_{li} + \partial_i g_{lj} + \partial_l g_{ij})$. So let $i, j, k, l \in \{1, 2\}$ and notice that for all $r \neq s$, $g_{rs} = 0 = g^{rs}$, where g^{rs} is the inverse of g_{rs} . Then we obtain

Case 1: $k = 1, i = j = 2$

$$\Gamma_{22}^1 = \frac{1}{2}g^{11}(\partial_2 g_{12} + \partial_2 g_{12} - \partial_1 g_{22}) = \frac{1}{2} \cdot 1 \cdot (\lambda|x|^{\lambda-1} \operatorname{sgn} x),$$

Case 2: $k = 2, i = 2, j = 1$

$$\Gamma_{21}^2 = \frac{1}{2}g^{22}(\partial_1 g_{22} + \partial_2 g_{21} + \partial_2 g_{21}) = \frac{1}{2}(1 - |x|^\lambda)^{-1}(-\lambda|x|^{\lambda-1} \operatorname{sgn} x).$$

By symmetry of the Christoffel symbols we have $\Gamma_{21}^2 = \Gamma_{12}^2$. Thus, we get our geodesic equations

$$\begin{aligned} (x\text{-equation}) \quad & \ddot{x} + \frac{\lambda}{2}|x|^{\lambda-1} \operatorname{sgn} x (\dot{y})^2 = 0, \\ (y\text{-equation}) \quad & \ddot{y} - \frac{\lambda}{2} \frac{|x|^{\lambda-1} \operatorname{sgn}(x)}{1 - |x|^\lambda} \dot{x} \dot{y} = 0. \end{aligned}$$

The y -equation is equivalent to $((1 - |x|^\lambda)\dot{y})' = 0$, meaning

$$(1 - |x|^\lambda)\dot{y} = C \tag{4.13}$$

where C is a constant. So let us distinguish between the case where $C = 0$ and $C \neq 0$. In the first case we clearly get $\dot{y} = 0$ and thus y is constant. Then the x -equation gives $\ddot{x} = 0$, meaning x is linear in t , indeed $x(t) = at + b$. In the second case we conclude $\dot{y} \neq 0$ and thus that y is strictly monotonous along any geodesic. Since the metric is $\mathcal{C}^1$, $g_\gamma(\dot{\gamma}, \dot{\gamma})$ is constant, where γ is a geodesic. We always can choose the constant to be 1 and parametrize by arclength

$$\begin{aligned} g_{ij}\dot{x}^i\dot{x}^j &= 1 \\ \dot{x}^2 + (1 - |x|^\lambda)(\dot{y})^2 &= 1 \\ \dot{x}^2 + \frac{C^2}{1 - |x|^\lambda} &= 1, \end{aligned}$$

and finally get

$$\dot{x} = \pm \left(1 - \frac{C^2}{1 - |x|^\lambda}\right)^{\frac{1}{2}}, \tag{4.14}$$

where $C \in [-1, 1]$. The metric g is smooth for $x \neq 0$ and doesn't depend on y , so we only have to consider the initial value $(x_0, y_0) = (0, 0)$. Let us now again distinguish between two cases:

Case 1: $C^2 = 1$

Case 2: $C^2 < 1$.

The Case $C^2 > 1$ would imply a negative value under the square root and thus it will not be considered. In the first case we get $x \equiv 0$. This is the only solution for (4.14), because otherwise the radicand would be negative. In the second case the right-hand side of equation (4.14) is C^1 and hence there exists a unique solution with initial condition $x(0) = 0$. Assume x is this unique solution of equation (4.14), then by equation (4.13) we get

$$\dot{y} = \frac{C}{1 - |x|^\lambda}$$

and by integrating we conclude

$$y(t) = \int_0^t \frac{C}{1 - |x(s)|^\lambda} ds.$$

Summarising, at $(0, 0)$ we have for every direction a unique geodesic, meaning that the initial value problem is uniquely solvable, by symmetry.

Now we want to determine the shape of the geodesic. In order to do this, we introduce a new variable via $C = \pm\sqrt{1-\epsilon}$ for $\epsilon \in [0, 1]$. Since $x(0) = 0$ and $\dot{y}(0) = C$, where $C > 0$, we conclude by symmetry and noticing that the metric norm is 1 at $x(0) = 0$ that the initial velocity is $(\dot{x}(0), \dot{y}(0)) = (\pm\sqrt{\epsilon}, \sqrt{1-\epsilon})$. Then we get

$$\dot{x} = \pm \left(1 - \frac{1-\epsilon}{1-|x|^\lambda}\right)^{\frac{1}{2}} \quad \text{for } |x| < \sqrt[\lambda]{\epsilon} \quad (4.15)$$

$$\dot{y} = \frac{\sqrt{1-\epsilon}}{1-|x|^\lambda}. \quad (4.16)$$

Let now $\gamma_{\pm\epsilon}$ be the unique geodesic with initial velocity $(\pm\sqrt{\epsilon}, \sqrt{1-\epsilon})$. We know that equation (4.15) only makes sense for $|x| < \sqrt[\lambda]{\epsilon}$. So we are interested what happens at $|x| = \sqrt[\lambda]{\epsilon}$. Hence, consider

$$\frac{dy}{dx} = \frac{\sqrt{1-\epsilon}}{1-|x|^\lambda} \left(1 - \frac{1-\epsilon}{1-|x|^\lambda}\right)^{-\frac{1}{2}} = \sqrt{\frac{1-\epsilon}{(1-x^\lambda)(\epsilon-x^\lambda)}}.$$

Integrating gives

$$\int_0^{\sqrt[\lambda]{\epsilon}} \frac{dy}{dx} \leq \int_0^{\sqrt[\lambda]{\epsilon}} \frac{dx}{\sqrt{\epsilon-x^\lambda}}. \quad (4.17)$$

Substituting $z = \epsilon - x^\lambda$ gives

$$(4.17) = \frac{1}{\lambda} \int_0^\epsilon \frac{dz}{\sqrt{z(\epsilon-z)^{1-\frac{1}{\lambda}}}}, \quad (4.18)$$

and again substituting $w = z - \frac{\epsilon}{2}$ gives

$$(4.18) = \frac{1}{\lambda} \int_{-\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} \frac{dw}{(\sqrt{w + \frac{\epsilon}{2}})(\frac{\epsilon}{2} - w)^{1-\frac{1}{\lambda}}}. \quad (4.19)$$

Set $\alpha := 1 - \frac{1}{\lambda} \in (0, \frac{1}{2})$ and continue

$$(4.19) = \frac{1}{\lambda} \left[\int_{-\frac{\epsilon}{2}}^0 \frac{dw}{(\sqrt{w + \frac{\epsilon}{2}})(\frac{\epsilon}{2} - w)^\alpha} + \int_0^{\frac{\epsilon}{2}} \left(\frac{\epsilon}{2} - w\right)^\alpha \right]. \quad (4.20)$$

Notice here that $(\frac{\epsilon}{2} - w)^\alpha \geq (\frac{\epsilon}{2})^\alpha$ and $\sqrt{w + \frac{\epsilon}{2}} \geq \sqrt{\frac{\epsilon}{2}}$, thus we get

$$(4.20) \leq \frac{2\alpha}{\epsilon\alpha} \int_{-\frac{\epsilon}{2}}^0 \frac{dw}{\sqrt{w + \frac{\epsilon}{2}}} + \sqrt{\frac{2}{\epsilon}} \int_0^{\frac{\epsilon}{2}} \frac{dw}{(\frac{\epsilon}{2} - w)^\alpha},$$

and for $\alpha < 1$ this integral is finite, meaning that γ_ϵ reaches $x = \sqrt[3]{\epsilon}$ in finite time.

So now let s_0, y_1 be such that $\gamma_\epsilon(s_0) = (\sqrt[3]{\epsilon}, y_1)$. Then differentiating gives $\gamma'_\epsilon(s_0) = (0, \frac{1}{\sqrt{1-\epsilon}})$. It is obvious that the tangent is vertical at the point s_0 . So since the metric is independent of y , we can reflect γ_ϵ at $y = y_1$ and get the geodesic $\gamma_{-\epsilon}$ that starts at $(0, 2y_1)$. In the end the point $(0, 0)$ is connected to $(0, 2y_1)$ by three geodesics, namely $\gamma_\epsilon, \gamma_{-\epsilon}$ and $x \equiv 0$. Then

$$y_1 = y_1(\epsilon) = \int_0^{\sqrt[3]{\epsilon}} \frac{dy}{dx} dx = \sqrt{1-\epsilon} \int_0^{\sqrt[3]{\epsilon}} \frac{dx}{\sqrt{(1-|x|^\lambda)(\epsilon-|x|^\lambda)}}.$$

Now we claim that $y_1(\epsilon) \rightarrow 0$ for $\epsilon \rightarrow 0$ and prove this as follows. Denote by s the arclength along the geodesic γ_ϵ . Then for $\epsilon < \frac{1}{2}$ we get

$$\dot{y} \leq \frac{\sqrt{1-\epsilon}}{1-\epsilon} = \frac{1}{\sqrt{1-\epsilon}} < \sqrt{2} \leq 2.$$

Then we conclude that $y(s) \leq 2s$, since we starting at $(0, 0)$ and hence also $y_1 = y(s_0) \leq 2s_0$. So it suffices to show that $s_0(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. We have by equation (4.15)

$$s_0(\epsilon) = \int_0^{s_0(\epsilon)} 1 dt = \int_0^{s_0(\epsilon)} \frac{dx}{\dot{x}} = \int_0^{\sqrt[3]{\epsilon}} (1 - (1-\epsilon)(1-x^\lambda)^{-1})^{-\frac{1}{2}} dx \quad (4.21)$$

and this converges to 0 for $\lambda < 2$.

Summarising, we have that the boundary value problem for geodesics is not uniquely solvable in any neighborhood of any point on $x = 0$. Any two points on $x = 0$ are connected by infinitely many geodesics.

Let us now check if these geodesics are minimizing. By Hilbert [4], for sufficiently small y_1 there exists a minimizing curve γ from $(0,0)$ to $(0,2y_1)$ and γ is a geodesic. So now we claim that γ is not a segment of $x \equiv 0$. In order to prove this we proceed as follows.

Since the metric restricted to $x = 0$ is the identity, we get for the length $L_0^{2y_1}(x \equiv 0) = 2y_1$. By definition we also have $L_0^{2s_0}(\gamma_\epsilon) = 2s_0$. Thus it suffices to show that $s_0(\epsilon) < y_1(\epsilon)$ for ϵ small, because $x \equiv 0$ is not minimizing. Consider,

$$y_1 = \int_0^{s_0} \dot{y}(t) dt = (1 - \epsilon)^{\frac{1}{2}} \int_0^{s_0} (1 - x^\lambda)^{-1} dt. \quad (4.22)$$

Substituting $x = x(t)$ and equation (4.15) gives

$$(4.22) = (1 - \epsilon)^{\frac{1}{2}} \int_0^{\sqrt[\lambda]{\epsilon}} (1 - x^\lambda)^{-\frac{1}{2}} (\epsilon - x^\lambda)^{-\frac{1}{2}} dx.$$

Setting $s := (\epsilon - x^\lambda)^{\frac{1}{2}} \epsilon^{-\frac{1}{2}}$ gives

$$\frac{1}{2} \lambda y_1 = \epsilon^{-\frac{1}{2} + \frac{1}{\lambda}} (1 - \epsilon)^{\frac{1}{2}} \int_0^1 (1 - \epsilon(1 - s^2))^{-\frac{1}{2}} (1 - s^2)^{-1 + \frac{1}{\lambda}} ds.$$

Set $\frac{1}{2} \lambda \epsilon^{\frac{1}{2} - \frac{1}{\lambda}} y_1(\epsilon) = a(\epsilon)$ and $\frac{1}{2} \lambda \epsilon^{\frac{1}{2} - \frac{1}{\lambda}} s_0(\epsilon) = b(\epsilon)$ and conclude by equation (4.21)

$$\lim_{\epsilon \rightarrow 0} a(\epsilon) = \lim_{\epsilon \rightarrow 0} b(\epsilon) =: \alpha.$$

Now extend a and b to $\epsilon = 0$. Then we get $a(0) = b(0)$ and for $\lambda > 2$

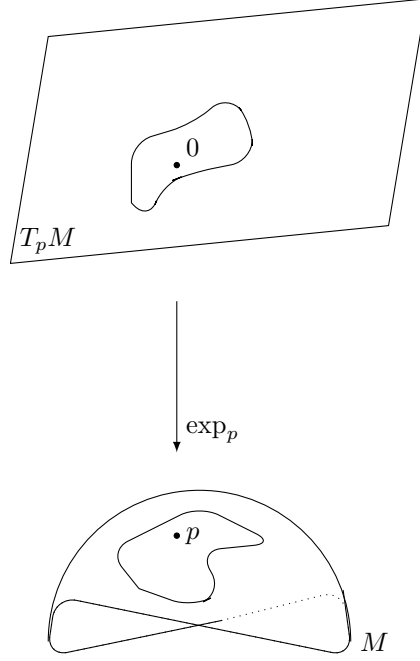
$$a'(0) = -\frac{1}{4} \lambda \int_0^1 (1 - t^2)^{\frac{1}{\lambda}} dt > -\frac{1}{2} \int_0^1 (1 - t^2)^{\frac{1}{\lambda}} dt = b'(0),$$

thus $a(\epsilon) > b(\epsilon)$ and hence also $y_1(\epsilon) > s_0(\epsilon)$ for small ϵ .

So we proved that the geodesic $x \equiv 0$ is not minimizing between any two points. The geodesics $\gamma_{\pm\epsilon}$ are minimizing between their endpoints and thus we conclude that even locally minimizing curves are not unique. Furthermore, there is no minimizing curve with initial direction $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ because otherwise by Hilbert [4] it would have to be a geodesic [9], [23].

For further information, we refer to [4], [9], [13], [14], [18] and [23].

4.3 The exponential map



Let us define $\gamma_v(t)$ to be the unique geodesic for the connection H^μ . This geodesic has velocity v and starts from $\pi(v)$. The exponential map is then given by

$$\begin{aligned} \exp : \Omega &\longrightarrow M \times M \\ v &\longmapsto (\pi(v), \gamma_v(1)), \end{aligned}$$

and the pointed exponential map at $p \in M$ is given by

$$\begin{aligned} \exp_p : \Omega_p &\longrightarrow M \\ v &\longmapsto \gamma_v(1), \end{aligned}$$

where $\gamma_v(1) = \pi_2(\exp v)$. The homogeneity of H^μ on velocities implies

$$\gamma_{sv}(t) = \gamma_v(st),$$

and so we can conclude that Ω and Ω_p are star-shaped [18].

4.3.1 Locally Lipschitzness

We now would like to show that this exponential map and the pointed version are locally Lipschitz, thus let us go through some definitions about Lipschitz continuity.

Let O be an open subset of $\mathbb{R}^n$. A function $f : O \longrightarrow \mathbb{R}^n$ is called Lipschitz if

$$\|f(p) - f(q)\| < K\|p - q\|,$$

for some $K > 0$ and every $p, q \in O$. If this inequality is valid for every compact subset of O and K is dependent of the compact subset, then we call f locally Lipschitz. In the case that f depends on another variable z and the Lipschitz constant does not depend on z then f is uniformly Lipschitz in z . Locally uniformly Lipschitz means that if we choose any compact set in the domain of z then the Lipschitz constant can be chosen such that it only depends on the compact set in place of z . So this means, for example, if we take a Lipschitz function depending on two variables then it is Lipschitz in one variable, uniformly in the other. A bi-Lipschitz homeomorphism f is defined as a homeomorphism, for which f itself and the inverse function f^{-1} are locally Lipschitz. In this work we will take f to be defined on the open subset $O \subset \mathbb{R}^n$ and $f(O)$ as an open subset of $\mathbb{R}^n$ [18].

By Picard-Lindelöf the ordinary differential equation $\dot{x} = f(x)$ with f Lipschitz has unique solutions. By the Lipschitz dependence on the initial conditions we will conclude that the exponential map is locally Lipschitz. Indeed, we have

Theorem 4.1. [18, Theorem 1]

Let H^μ be a Lipschitz connection on a C^∞ -manifold M , $W \subset [0, \infty) \times TM$ the maximal domain of

$$\begin{aligned} \psi : W &\longrightarrow TM \\ (t, v) &\longmapsto \gamma'_v(t), \end{aligned}$$

the geodesic flow map. In order for the right hand side to make sense, this map is defined for those $(t, v) \in \mathbb{R} \times TM$ for which the geodesic γ_v exists up to time t . Then W is open in $[0, \infty) \times TM$ such that $[0, 1] \times \Omega$ is a subset of W and W star-shaped. Equivalently, Ω is an open subset in TM and star-shaped.

Furthermore, the map ψ is locally Lipschitz and $\psi(\cdot, v) \in C^{1,1}$ and there exists a star-shaped subset $\tilde{\Omega}$ of Ω such that the Lebesgue measure of $\Omega \setminus \tilde{\Omega}$ is zero and such that $\psi(t, \cdot)$ is differentiable at v for every $v \in \tilde{\Omega}$ and every $t \in [0, 1]$ and the Jacobian $\partial_2 \psi(\cdot, v)$ is locally Lipschitz in t , locally uniformly in $v \in \tilde{\Omega}$. For any v in $\tilde{\Omega}$ the following holds:

$$\partial_1 \partial_2 \psi = \partial_2 \partial_1 \psi$$

for almost every $t \in [0, 1]$. These mixed differentials exist, coincide and are locally bounded. We can restrict ψ to some m -dimensional submanifold N of TM , which is Lipschitz and the statement above continues to hold. The

differential ∂_2 then refers to the variables of the Lipschitz chart on N and $\partial_2\psi$ exists almost everywhere in the sense of the m -dimensional Lebesgue measure of N .

Before we prove this theorem, we will first state the definition of Gâteaux differentiability and then go through two important statements. Indeed, consider two Banach spaces X and Y . A function $f : X \rightarrow Y$ is called Gâteaux differentiable at some point x if there exists a bounded linear operator $G_x \in \mathcal{B}(X, Y)$ such that for all $v \in X$

$$\|f(x + tv) - f(x) - tG_x v\| = o(|t|),$$

where G_x is the Gâteaux derivative of f at x [2]. Now we can prove the following statement.

Proposition 4.2. [18, Proposition 2]

Any Lipschitz function $f : O \rightarrow \mathbb{R}$ with $O \subset \mathbb{R}^n$ that is Gâteaux differentiable at p is differentiable at p .

Proof. We will prove this by contradiction. So let G be the Gâteaux differential at p and assume G is not the differential of f at p , meaning there exists $\epsilon > 0$ and a sequence $v_n \in \mathbb{R}^n$ with $v_n \neq 0$ and $v_n \rightarrow 0$ such that

$$\|f(p + v_n) - f(p) - G(v_n)\| > \epsilon \|v_n\|. \quad (4.23)$$

We set $e_n = \frac{v_n}{\|v_n\|}$. The unit-sphere S^{n-1} of $\mathbb{R}^n$ is compact and so there exists a converging subsequence of e_n and thus we can assume that $e_n \rightarrow e$, where $\|e\| = 1$. Pick now a scalar a_n and a vector b_n so as to decompose v_n in a component $a_n e$ which is parallel and a component b_n that is perpendicular to the unit vector e :

$$v_n = a_n e + b_n.$$

Plugging this into the equation (4.23) we get

$$\begin{aligned} \|f(p + v_n) - f(p) - G(v_n)\| &\leq \|f(p + v_n) - f(p + a_n e)\| \\ &\quad + \|f(p + a_n e) - f(p) - G(a_n e)\| + \|G(a_n e) - G(v_n)\| \\ &= K \|b_n\| + o(a_n) + \|G\| \|b_n\|, \end{aligned}$$

where the last equality follows from the locally Lipschitz condition of f and K is the Lipschitz constant and $\|G\|$ is the operator norm of G . We have $e_n = \frac{v_n}{\|v_n\|} = \frac{a_n}{\|v_n\|} e + \frac{b_n}{\|v_n\|}$. In order for $e_n \rightarrow e$ to hold, we conclude that

$\frac{a_n}{\|v_n\|} \rightarrow 1$ and $\frac{b_n}{\|v_n\|} \rightarrow 0$. So we get

$$\frac{\|f(p + v_n) - f(p) - G(v_n)\|}{\|v_n\|} \leq \frac{o(a_n)}{\|v_n\|} = \frac{o(a_n)}{a_n} \frac{a_n}{\|v_n\|},$$

which converges clearly to 0 and is a contraction to equation (4.23) for n large enough [18]. $\square$

Theorem 4.3. [18, Theorem 15]

Consider a continuous function

$$\begin{aligned} f : [a, b] \times \mathbb{R}^k &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto f(x, y). \end{aligned}$$

Let f be locally Lipschitz in y , uniformly in x . Then for almost every y the differential d_2f exists at (u, y) for almost every $u \in [a, b]$, it is integrable in $[a, b]$ and for every $x \in [a, b]$ we have:

$$\left(d_2 \int_a^x f(u, y) du \right) = \int_a^x d_2f(u, y) du.$$

Proof. Before we start, let us recall the statement of Rademacher's theorem, that is if we consider an open subset of a Euclidean space and a Lipschitz continuous function, then this function is differentiable almost everywhere.

So let $\bar{y} \in \mathbb{R}^n$, then there exists a relatively compact neighborhood $O = \prod_i (a_i, b_i)$ of $\bar{y}$, where (a_i, b_i) are open intervals. The function $f(x, \cdot)$ restricted to O is locally Lipschitz in y with constant K , uniformly in x as assumed. So now we can apply Rademacher's theorem and obtain that the function $f(x, \cdot)$ is differentiable almost everywhere for any $x \in [a, b]$, meaning that f is differentiable everywhere except on a set of Lebesgue measure zero. Now let $O \setminus E(x)$ be this set of Lebesgue measure zero and so the subset $E(x)$ is such that $f(x, \cdot)|_O$ is differentiable. In other words, $y \mapsto f(x, y)$ is locally Lipschitz and thus differentiable almost everywhere and the differential $d_2f(x, y)$ exists for all $y \in E(x)$.

Consider the characteristic function

$$\chi_A(x, y) = \begin{cases} 1, & (x, y) \in A \\ 0, & \text{else} \end{cases}$$

of $A = \bigcup_{x \in [a, b]} \{x\} \times E(x) \subset [a, b] \times O$, where $O = \prod_{i=2}^k (a_i, b_i)$. Integrating this

we get

$$\begin{aligned}\int_a^b \int_O \chi_A(x, y) dy dx &= \int_a^b \int_O \chi_{E(x)}(y) dy dx \\ &= \int_a^b \lambda(E(x)) dx = \int_a^b \lambda(O) dx \\ &= (b-a)\lambda(O) < \infty,\end{aligned}$$

where λ denotes the Lebesgue measure. By Fubini, we conclude that

$$\int_O \int_a^b \chi_A(x, y) dx dy = (b-a)\lambda(O).$$

Indeed, we know that $\int_a^b \chi_A(x, y) dx \leq b-a$ because $\chi \leq 1$. Then

$$(b-a) - \int_a^b \chi_A(x, y) dx \geq 0$$

and almost everywhere equal to 0. Hence

$$\int_O \left[(b-a) - \int_a^b \chi_A(x, y) dx \right] dy = 0$$

holds almost everywhere. Thus, $\int_a^b \chi_A(x, y) dx = b-a$ for almost every y . Summarising, for almost every $y \in O$ we have $\chi_A(x, y) = 1$ for almost every $x \in [a, b]$, meaning $y \in E(x)$, so for almost every $y \in O$ the differential $d_2 f$ exists at (x, y) for almost every $x \in [a, b]$.

Every differentiable function is also Gâteaux differentiable, so we obtain that for almost every $y \in O$ and for every $v \in \mathbb{R}^k$ the partial derivative $\partial_v f(x, y)$ exists for almost every $x \in [a, b]$. Knowing that f is locally Lipschitz and letting $\epsilon_n \rightarrow 0$ we get $\partial_v f(x, y) = \lim_{n \rightarrow \infty} \frac{1}{\epsilon_n} (f(x, y + v\epsilon_n) - f(x, y))$ and $|\lim_{n \rightarrow \infty} \frac{1}{\epsilon_n} (f(x, y + v\epsilon_n) - f(x, y))| \leq K\|v\|$. Hence $\|\partial_v f(x, y)\| \leq K\|v\|$. Then applying the dominated convergence theorem we conclude that $\partial_v f(x, y)$ is integrable and

$$\partial_v \int_a^x f(u, y) du = \int_a^x \partial_v f(u, y) du.$$

The right-hand side is the Gâteaux differential at y of $\int_a^x f(u, y) du$ applied to v . This function is also Lipschitz in y . Thus, we conclude by the previous proposition that $\int_a^x f(u, y) du$ is differentiable and that the Gâteaux differential coincides with its differential [18]. $\square$

Now we are ready to prove theorem 4.1:

Proof. The first step will be to prove that the geodesic flow map over TM is

locally Lipschitz and that W and Ω are open sets which fulfil the conditions of the theorem.

Recall that every geodesic segment can be covered by finitely many coordinate patches. If we prove that we have locally Lipschitz dependence on the initial condition we can then conclude that the geodesic flow over TM is locally Lipschitz. Furthermore, we will show that W and Ω are open and that they fulfil the claim of the theorem.

For the locally Lipschitz dependence on the initial condition to be proven we will use Picard-Lindelöf approximation.

Consider two solutions $x^\mu(t)$ and $\tilde{x}^\mu(t)$ of the geodesic equation with initial conditions $(x_0^\mu, \dot{x}_0^\mu)$ and $(\tilde{x}_0^\mu, \dot{\tilde{x}}_0^\mu)$ are given. Assume that they both belong to $\max\{\|x_0\|, \|\dot{x}_0\|\} < \delta$. As above by induction we can define the functions $x_k^\mu(t)$ and $\tilde{x}_k^\mu(t)$ and write them in the same form as the equations (4.5) and (4.6). Moreover, we will look at the chart-trivialized tangent bundle as the direct sum of $\mathbb{R}^n \oplus \mathbb{R}^n$ with the norm $\|(x, \dot{x})\| := \max\{\|x\|, \|\dot{x}\|\}$. Proceeding similarly as above we now show the following estimates by induction

$$\|x_k(t) - y_k(t)\| \leq \max\{D\|x_0 - y_0\|, \|\dot{x}_0 - \dot{y}_0\|\} \frac{1}{D} \sum_{j=0}^k \frac{(Dt)^j}{j!} \quad (4.24)$$

$$\|\dot{x}_k(t) - \dot{y}_k(t)\| \leq \max\{D\|x_0 - y_0\|, \|\dot{x}_0 - \dot{y}_0\|\} \sum_{j=0}^k \frac{(Dt)^j}{j!}. \quad (4.25)$$

So again we proceed in parallel and first consider (4.24) for $k = 0$, that is

$$\|x_0(t) - y_0(t)\| = \|x_0 - y_0\| \leq D\|x_0 - y_0\| \frac{1}{D} \left(\sum_{j=0}^0 \frac{(Dt)^j}{j!} \right).$$

For (4.25) and $k = 0$ we get

$$\|\dot{x}_0(t) - \dot{y}_0(t)\| = \|\dot{x}_0 - \dot{y}_0\| \leq \|\dot{x}_0 - \dot{y}_0\| \left(\sum_{j=0}^0 \frac{(Dt)^j}{j!} \right).$$

Assuming this holds true for some k , we show (4.24) for $k \rightarrow k+1$

$$\begin{aligned}
\|x_{k+1}(t) - y_{k+1}(t)\| &\leq \|x_0 - y_0\| + \int_0^t \|\dot{x}_k(s) - \dot{y}_k(s)\| ds \\
&\leq \|x_0 - y_0\| + \int_0^t \|\dot{x}_0 - \dot{y}_0\| \sum_{j=0}^k \frac{(Ds)^j}{j!} ds \\
&= \|x_0 - y_0\| + \|\dot{x}_0 - \dot{y}_0\| \frac{1}{D} \left(\sum_{j=0}^k \frac{(Ds)^{j+1}}{(j+1)!} \right) \Big|_{s=0}^t \\
&\leq \max\{D\|x_0 - y_0\|, \|\dot{x}_0 - \dot{y}_0\|\} \left(1 + Dt + \frac{(Dt)^2}{2!} + \cdots + \frac{(Dt)^{k+1}}{(k+1)!} \right) \\
&= \max\{D\|x_0 - y_0\|, \|\dot{x}_0 - \dot{y}_0\|\} \frac{1}{D} \left(\sum_{j=0}^{k+1} \frac{(Dt)^j}{j!} \right)
\end{aligned}$$

and for (4.25)

$$\begin{aligned}
\|\dot{x}_{k+1}(t) - \dot{y}_{k+1}(t)\| &\leq \|\dot{x}_0 - \dot{y}_0\| + \int_0^t A\|x_k(s) - y_k(s)\| + B\|\dot{x}_k(s) - \dot{y}_k(s)\| ds \\
&\leq \|\dot{x}_0 - \dot{y}_0\| + \max\{D\|x_0 - y_0\|, \|\dot{x}_0 - \dot{y}_0\|\} \left(\frac{A}{D} + B \right) \int_0^t \left(\sum_{j=0}^k \frac{(Ds)^j}{j!} \right) ds \\
&= \|\dot{x}_0 - \dot{y}_0\| + \max\{D\|x_0 - y_0\|, \|\dot{x}_0 - \dot{y}_0\|\} \left(\sum_{j=0}^k \frac{(Dt)^{j+1}}{(j+1)!} \right) \\
&\leq \max\{D\|x_0 - y_0\|, \|\dot{x}_0 - \dot{y}_0\|\} \left(1 + \sum_{j=0}^k \frac{(Dt)^{j+1}}{(j+1)!} \right) \\
&= \max\{D\|x_0 - y_0\|, \|\dot{x}_0 - \dot{y}_0\|\} \left(\sum_{j=0}^{k+1} \frac{(Dt)^j}{j!} \right).
\end{aligned}$$

So our claim follows. Notice that $\sum_{j=0}^k \frac{(Dt)^j}{j!} \leq e^{Dt}$ and so even more holds, namely

$$\begin{aligned}
\|x_k(t) - y_k(t)\| &\leq \max\{D\|x_0 - y_0\|, \|\dot{x}_0 - \dot{y}_0\|\} \frac{1}{D} e^{Dt} \\
\|\dot{x}_k(t) - \dot{y}_k(t)\| &\leq \max\{D\|x_0 - y_0\|, \|\dot{x}_0 - \dot{y}_0\|\} e^{Dt}.
\end{aligned}$$

Furthermore, we have $D \leq 1$ and can infer

$$\|(x, \dot{x}(t)) - (\tilde{x}, \dot{\tilde{x}})(t)\| \leq \|(x_0, \dot{x}_0) - (\tilde{x}_0, \dot{\tilde{x}}_0)\| \frac{e^{Dt}}{D},$$

which shows the Lipschitz dependence on the initial conditions.

The inequality

$$\|(x, \dot{x})(t') - (\tilde{x}, \dot{\tilde{x}})(t)\| \leq \|(x, \dot{x})(t') - (x, \dot{x})(t)\| + \|(x, \dot{x})(t) - (\tilde{x}, \dot{\tilde{x}})(t)\|$$

holds for t, t' in some Interval $[a, b]$ since the dependence of $(x, \dot{x})(t)$ on t is $\mathcal{C}^{1,1}$ since by (4.1) and (4.2) the dependence of $(x, \dot{x})(t)$ on t is $\mathcal{C}^{1,1}$. Altogether, we obtain local Lipschitz dependence on $(t, x_0, \dot{x}_0)$.

Now let us prove that W from theorem 4.1 is open in $[0, \infty) \times TM$ such that $[0, 1] \times \Omega$ is a subset of W and star-shaped.

W is the maximal domain of the geodesic flow map ψ , thus we can define $W := \{(t, v) : t_-^v < t < t_+^v\}$, where t_-^v and t_+^v denote the minimal respectively the maximal value of t . So for $v \in TM$, let $I_v := (t_-^v, t_+^v)$ and

$I'_v = \{t \in \mathbb{R} : \psi \text{ is defined and smooth in a neighborhood of } [0, t] \times \{v\} \text{ for } t \geq 0$
and respectively of $[t, 0] \times \{v\} \text{ for } t < 0\}$.

Then obviously $I'_v \subset I_v$, $0 \in I'_v$ and I'_v is an open interval. We now want to show that $I_v = I'_v$. By contradiction, assume I'_v is a proper subset of I_v . Without loss of generality, we suppose $t_0 = \inf\{t > 0 : t \in I_v \setminus I'_v\} > 0$. Thus t_0 is not contained in I'_v , since I'_v is open. We know that the geodesic flow map is defined and smooth on a neighborhood U of $(0, \psi_{t_0}(v)) \in \mathbb{R} \times TM$. Choose some δ with $0 < \delta < t_0$ and a neighborhood V of v in TM such that $(-\delta, 2\delta) \times \psi_{t_0-\delta}(V) \subset U$ and such that $q \mapsto \psi_{t_0-\delta}(q)$ is smooth on V . Then the map

$$(s, w) \mapsto \psi_s(\psi_{t_0-\delta}(w)) = \psi_{s+t_0-\delta}(w),$$

where the equality follows from the property of the flow map, is smooth on the neighborhood $(-\delta, 2\delta) \times V$ of $[0, \delta] \times \{v\}$, so ψ is smooth on the neighborhood $(t_0 - 2\delta, t_0 + \delta) \times V$ of $[t_0 - \delta, t_0] \times \{v\}$. Moreover $t_0 - \delta \in I'_v$, so ψ is smooth on a neighborhood of $[0, t_0 - \delta] \times \{v\}$. Summarizing, ψ is smooth on a neighborhood of $([0, t_0 - \delta] \cup [t_0 - \delta, t_0]) \times \{v\} = [0, t_0] \times \{v\}$. But according to the definition of I'_v this means that $t_0 \in I'_v$, which is a contradiction and thus we have $I_v = I'_v$. So our claim follows. W is also the maximal domain of $\pi \circ \psi$. This map is given by $(t, v) \mapsto \gamma_v(t)$. Let $\phi : TM \rightarrow TM \times \mathbb{R}$ be the smooth map given by $v \mapsto (v, 1)$ and $\Omega = \{v \in TM : \exists \gamma_v(1)\} = \{v \in TM : (v, 1) \in W\} = \phi^{-1}(W)$ is also open. So Ω_v is open and if $v \in \Omega_v$, then γ_v is defined on $[0, 1]$. We know $\gamma_{tv}(1) = \gamma_v(t)$ and thus $tv \in \Omega_v$ for $t \in [0, 1]$. Hence Ω_v is star-shaped. Furthermore, for $(t, v) \in W$ we can conclude from $\gamma_{tv}(1) = \gamma_v(t)$ that $[0, t] \times v \subset W$ holds whenever $[0, 1] \times \{tv\} \subset W$. Let us show that for $\lambda \in [0, 1]$ and $\omega \in \Omega$, $\lambda \cdot \omega \in \Omega$. For $\omega \in \Omega$ we have $(1, \omega) \in W$ and so we conclude that $[0, 1] \times \{\omega\} \subset W$ and thus $(\lambda, \omega) \in W$. This is equivalent to $(1, \lambda\omega) \in W$ and this holds whenever

$\lambda \cdot \omega \in \Omega$. Hence the first part of theorem 4.1 follows.

Now we want to show that the geodesic flow map is differentiable at v and the differential is locally Lipschitz in t , uniformly in $v \in \tilde{\Omega}$.

We again consider the equations (4.3) and (4.4). We proved above that the dependence $(x, \dot{x})(t, x_0, \dot{x}_0)$ is Lipschitz and so we can replace x by t , y by $(x_0, \dot{x}_0)$ and apply theorem 4.3. Thus, for almost every $(x_0, \dot{x}_0)$ in the domain $\max\{\|x_0\|, \|\dot{x}_0\|\} < \delta$ satisfying $\delta < \frac{1}{M}(1 - e^{-\frac{Mr}{2}}) \leq \frac{r}{2}$ the differential

$$\begin{aligned} d_{(x_0, \dot{x}_0)}(x, \dot{x}) &= d_{(x_0, \dot{x}_0)}(x_0, \dot{x}_0) \\ &+ \int_0^t d_{(x_0, \dot{x}_0)}(\dot{x}^\mu(u, x_0, \dot{x}_0), H^\mu(x(u, x_0, \dot{x}_0), \dot{x}(u, x_0, \dot{x}_0))) du \end{aligned} \quad (4.26)$$

exists, where for almost every $(x_0, \dot{x}_0)$, almost every u in some interval and every $t \in [0, 1]$ the differential $d_{(x_0, \dot{x}_0)}(\dot{x}^\mu(t, x_0, \dot{x}_0), H^\mu(x(t, x_0, \dot{x}_0), \dot{x}(t, x_0, \dot{x}_0)))(u)$ is integrable.

Clearly, since the functions $\dot{x}^\mu(t, x_0, \dot{x}_0)$ and $H^\mu(x(t, x_0, \dot{x}_0), \dot{x}(t, x_0, \dot{x}_0))$ are locally Lipschitz the differential is bounded. Hence, $d_{(x_0, \dot{x}_0)}(x, \dot{x})$ depends on $(t, x_0, \dot{x}_0)$ and is Lipschitz in t , uniformly in $(x_0, \dot{x}_0)$.

The next step is to show that for any $v \in \tilde{\Omega}$ and for almost every $t \in [0, 1]$ the following mixed differentials exist and are equal to each other. We can again apply Rademacher's theorem and differentiate (4.26) with $(x_0, \dot{x}_0) \in \tilde{\Omega}$, where $\Omega \setminus \tilde{\Omega}$ has Lebesgue measure zero and $\tilde{\Omega}$ is star-shaped by the homogeneity of H^μ on velocities. So we get for almost every t

$$\begin{aligned} \frac{d}{dt} d_{(x_0, \dot{x}_0)}(x, \dot{x}) &= d_{(x_0, \dot{x}_0)}(\dot{x}^\mu(t, x_0, \dot{x}_0), H^\mu(x(t, x_0, \dot{x}_0), \dot{x}(t, x_0, \dot{x}_0))) \\ &= d_{(x_0, \dot{x}_0)} \frac{d}{dt}(x, \dot{x}). \end{aligned}$$

For the last argument of the theorem to hold, consider z to be a local coordinate on the submanifold N . Then proceeding again as above, introducing the Lipschitz dependence $(x_0, \dot{x}_0)(z)$, securing that $(x, \dot{x})(t, z)$ is locally Lipschitz and noticing $\partial_z \psi$ to exist almost everywhere in the Lebesgue measure of N then the whole argument holds and the theorem follows [12], [18]. $\square$

For further information, we refer to [2], [12] and [18].

4.3.2 Lipschitzness of the inverse

Our next goal is to show that the exponential map admits a Lipschitz inverse. To reach this we will use an improved version of the inverse function theorem. So let start with a brief introduction to strong differentials.

Assume a Banach space E and let $f : O \longrightarrow F$ be a function that maps an open

set $O \subset E$ to a Banach space F . Then the strong differential of f at $p \in O$ is a bounded linear transformation L mapping E to a Banach space F such that for every $\epsilon > 0$ there exists $\delta > 0$ such that if $|q_1 - p| \leq \delta$ and $|q_2 - p| \leq \delta$, then

$$|f(q_1) - f(q_2) - L(q_1 - q_2)| \leq \epsilon |q_1 - q_2|.$$

It is easy to see that for $q_2 = p$ we have Fréchet differentiability. For the reason that we are only considering the finite dimensional case, strong differentiation and the strong differential are independent of the norm.

The strong differentiation has the following properties:

1. Assuming f to be strongly differentiable at p , then f fulfils a Lipschitz condition in a neighborhood of p . This does not mean that every Lipschitz function is strongly differentiable almost everywhere, even in the case of functions on the real line.
2. The converse is only valid if we add the condition that the differential is continuous at p , indeed if f is differentiable in a neighborhood of p and the differential is continuous at p then it is strongly differentiable at p .
3. Let f be strongly differentiable on a subset A of the Banach space E . Then the strong differential is continuous on A with respect to the induced topology. These last two conditions show that a function is once continuously differentiable on an open set only in case it is strongly differentiable on it and vice versa.
4. Let $f : U_1 \times U_2 \longrightarrow V$ be a continuous function on a product of Banach spaces and assume that f admits strong partial differentials at p when keeping the other variable constant. Then it is strongly differentiable at p . The total differential is expressed as usual in terms of partial differentials.
5. Strong differentiability is invariant under composition.
6. If mixed partial strong derivatives exist, then they coincide.
7. Assuming $f : \mathbb{R} \longrightarrow \mathbb{R}$ to have positive strong differential at p , then f is continuous and increasing in a neighborhood of p [18].

Many key theorems in analysis, which include a $\mathcal{C}^1$ -condition on a neighborhood of p can be proved using this weaker condition. In addition to this, we are now going to state Leach's inversion theorem [18].

Theorem 4.4 (Leach's theorem). [18, Theorem 2]

Let f be a function mapping an open subset O of $\mathbb{R}^n$ to $\mathbb{R}^n$ such that f has strong

differential $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$ at $p \in O$. In the case that L is invertible there is an open neighborhood N_1 of p and another open neighborhood N_2 of $f(p)$ such that there exists a function $g : N_2 \rightarrow \mathbb{R}^n$ with $f(N_1) = N_2$, $g(N_2) = N_1$ and $f|_{N_1}$ is the inverse of g and vice versa. The latter two functions are both Lipschitz and g has strong differential L^{-1} at $f(p)$. Furthermore, the following holds:

f is (strongly) differentiable at $q \in N_1 \Leftrightarrow g$ is (strongly) differentiable at $f(q)$.

In this case the differentials are invertible.

The Lipschitz condition in this theorem follows from the first property above.

Proof. Without loss of generality, let $p = 0$ and $f(0) = 0$. Then note that for any y in N_2 , define sequences $\{x_n\}$ and $\{y_n\}$ by $x_0 = 0$ and

$$y_n = f(x_n), \quad x_{n+1} = x_n + L^{-1}(y - y_n), \quad n = 0, 1, 2, \dots \quad (4.27)$$

Let us define $g(y)$ to be the limit of $\{x_n\}$ for all y such that the limit exists and set $g(y) = 0$ otherwise.

Now we have to prove convergence of $\{x_n\}$ for y near 0. For this, consider $\epsilon > 0$ such that $\epsilon \|L^{-1}\| < \frac{1}{2}$, where L^{-1} is the inverse of L , and choose $\delta > 0$ such that f has strong differential at the origin of N_1 . By induction, we show that

$$\begin{aligned} \|x_n\| &< \delta \text{ and} \\ \|x_{n+1} - x_n\| &< (\epsilon \|L^{-1}\|)^n \frac{\delta}{2}, \end{aligned}$$

if $\|y\| < \frac{\delta}{2} \|L^{-1}\|$ for $n = 0, 1, 2, \dots$. Indeed, let us start with $n = 0$. For $\|y\| < \frac{\delta}{2} \|L^{-1}\|$, we conclude that

$$\|x_1 - x_0\| \leq \|L^{-1}\| \|y - y_0\|$$

by equation (4.27). By induction hypothesis, we have for some n :

$$\|x_n - x_{n-1}\| \leq (\epsilon \|L^{-1}\|)^{n-1} \frac{\delta}{2}.$$

So, by the telescopic sum

$$\|x_n\| \leq \sum_{m=0}^{n-1} (\epsilon \|L^{-1}\|)^m \frac{\delta}{2} < \delta,$$

since $(\epsilon \|L^{-1}\|) < \frac{1}{2}$. Using (4.27) and the relation $L^{-1}LL^{-1} = L^{-1}$, for $n \geq 1$

we get:

$$\begin{aligned}
x_{n+1} - x_n &= L^{-1}(y - y_n) = L^{-1}(y - y_{n-1}) - L^{-1}(y_{n-1} - y_n) \\
&= L^{-1}LL^{-1}(y - y_{n-1}) - L^{-1}(y_{n-1} - y_n) \\
&= L^{-1}[L(x_n - x_{n-1}) - f(x_{n-1}) + f(x_n)].
\end{aligned}$$

On the left side we see that the part in brackets coincides with the one in the definition of the strong differential of f by replacing q_1 by x_n and q_2 by x_{n-1} . Therefore, we have that

$$\|x_{n+1} - x_n\| \leq \|L^{-1}\|\epsilon\|x_n - x_{n-1}\| \leq \|L^{-1}\|\epsilon(\epsilon\|L^{-1}\|)^{n-1}\frac{\delta}{2} = \epsilon^n\|L^{-1}\|^n\frac{\delta}{2},$$

since x_n and x_{n-1} fall within δ of 0. Thus, we get the asserted bound on $\|x_{n+1} - x_n\|$ and if $\|y\| < \frac{\delta}{2}\|L^{-1}\|$, the sequence $\{x_n\}$ then converges uniformly to $g(y)$.

Now we want to prove the fact that

$$(1) \quad L^{-1}(f(g(y))) = L^{-1}(y).$$

We can derive them easily for all y such that $\|y\| < \frac{\delta}{2}\|L^{-1}\|$. Indeed, take $n \rightarrow \infty$ in (4.27) and notice that by the continuity of f we have $f(x_n) \rightarrow f(g(y))$. We have $g(y) = g(y) + L^{-1}(y - f(g(y)))$ and this implies our first claim $L^{-1}fg = L^{-1}$ for $\|y\| < \frac{\delta}{2}\|L^{-1}\|$.

The next few steps involves estimating $q_1 - q_2$ for some choice of q_1, q_2 nearer than δ to 0 in N_1 .

First let us prove a Lipschitz condition on g . If $\tilde{y}$ and $\tilde{\tilde{y}}$ are nearer than $\frac{\delta}{2}\|L^{-1}\|$ to 0 in N_2 , $g(\tilde{y})$ and $g(\tilde{\tilde{y}})$ are nearer than δ to 0 in N_1 . By the definition of the strong differential we conclude:

$$\|f(g(\tilde{y})) - f(g(\tilde{\tilde{y}})) - L(g(\tilde{y}) - g(\tilde{\tilde{y}}))\| \leq \epsilon\|g(\tilde{y}) - g(\tilde{\tilde{y}})\|.$$

Inserting L^{-1} on both sides, leads to

$$\|L^{-1}(f(g(\tilde{y}))) - L^{-1}(f(g(\tilde{\tilde{y}}))) - L^{-1}L(g(\tilde{y}) - g(\tilde{\tilde{y}}))\| \leq \epsilon\|L^{-1}\|\|g(\tilde{y}) - g(\tilde{\tilde{y}})\|.$$

We already proved that $L^{-1}fg = L^{-1}$ and $L^{-1}Lg = g$ hold. Using this the left side simplifies to:

$$\|L^{-1}(\tilde{y} - \tilde{\tilde{y}}) - (g(\tilde{y}) - g(\tilde{\tilde{y}}))\| \leq \epsilon\|L^{-1}\|\|g(\tilde{y}) - g(\tilde{\tilde{y}})\|.$$

Applying the triangle inequality on this relation, we obtain

$$\|L^{-1}(\tilde{y} - \tilde{\tilde{y}})\| \geq \|g(\tilde{y}) - g(\tilde{\tilde{y}})\| - \epsilon\|L^{-1}\|\|g(\tilde{y}) - g(\tilde{\tilde{y}})\|.$$

Since $\|L^{-1}(\tilde{y} - \tilde{\tilde{y}})\| \leq \|L^{-1}\|\|\tilde{y} - \tilde{\tilde{y}}\|$,

$$\begin{aligned} \|g(\tilde{y}) - g(\tilde{\tilde{y}})\| - \epsilon\|L^{-1}\|\|g(\tilde{y}) - g(\tilde{\tilde{y}})\| &= (1 - \epsilon\|L^{-1}\|)\|g(\tilde{y}) - g(\tilde{\tilde{y}})\| \\ &\leq \|L^{-1}(\tilde{y} - \tilde{\tilde{y}})\| \leq \|L^{-1}\|\|\tilde{y} - \tilde{\tilde{y}}\| \end{aligned}$$

and $1 - \epsilon\|L^{-1}\| > 0$, we obtain the Lipschitz condition

$$\|g(\tilde{y}) - g(\tilde{\tilde{y}})\| \leq \|L^{-1}\| \frac{\|\tilde{y} - \tilde{\tilde{y}}\|}{(1 - \epsilon\|L^{-1}\|)}, \quad (4.28)$$

which is valid if $\tilde{y}$ and $\tilde{\tilde{y}}$ are nearer than $\frac{\delta}{2}\|L^{-1}\|$ to 0.

So we proved the Lipschitz condition for g . Now we want to prove the differentiability of g at any y in N_2 such that $\|y\| < \frac{\delta}{2}\|L^{-1}\|$, provided f is differentiable at $g(y)$ with differential L_1 .

For $\tilde{y}$ and $\tilde{\tilde{y}}$ near to y and within $\frac{\delta}{2}\|L^{-1}\|$ of 0, $g(\tilde{y})$ and $g(\tilde{\tilde{y}})$ are near to $g(y)$ and within δ to 0 and near to $g(y)$. For any $\epsilon_1 > 0$, differentiability of f at $g(y)$ implies the relation on f valid for $\tilde{y}, \tilde{\tilde{y}}$ near y :

$$\|fg(\tilde{y}) - fg(\tilde{\tilde{y}}) - L_1(g(\tilde{y}) - g(\tilde{\tilde{y}}))\| \leq \epsilon_1\|g(\tilde{y}) - g(\tilde{\tilde{y}})\|.$$

Applying L^{-1} on the left side gives:

$$\|L^{-1}(\tilde{y} - \tilde{\tilde{y}}) - (1 + L^{-1}(L_1 - L))(g(\tilde{y}) - g(\tilde{\tilde{y}}))\| \leq \epsilon_1\|L^{-1}\|\|g(\tilde{y}) - g(\tilde{\tilde{y}})\|.$$

In order to simplify the left side of the inequality, we apply $L^{-1}fg = L^{-1}$. We write $L^{-1}L_1 = L^{-1}L + L^{-1}(L_1 - L)$. Since $L^{-1}Lg = \text{id}$ we conclude:

$$\|L^{-1}(\tilde{y} - \tilde{\tilde{y}}) - (1 + L^{-1}(L_1 - L))(g(\tilde{y}) - g(\tilde{\tilde{y}}))\| \leq \epsilon_1\|L^{-1}\|\|g(\tilde{y}) - g(\tilde{\tilde{y}})\|.$$

We know that the relation

$$\|f(q_1) - f(q_2) - L(q_1 - q_2)\| \leq \|q_1 - q_2\|$$

implies for any ϵ and δ a Lipschitz condition and thus continuity of f in the sphere $\{\|q - p\| < \delta\}$. If f has a Fréchet differential L_1 at a point p_1 of that sphere and $\eta > 0$, then for q_2 nearer than δ to 0 and

$$|f(p_1) - f(q_2) - L_1(q - q_2)| \leq \eta|p_1 - q_2|$$

we get

$$\begin{aligned} |(L_1 - L)(p_1 - q_2)| &= |f(p_1) - f(q_2) - L(p_1 - q_2) - (f(p_1) - f(q_2))| \\ &\leq (\epsilon + \eta)|p_1 - q_2|. \end{aligned}$$

We conclude $\|L_1 - L\| \leq \epsilon + \eta$ for all $\eta > 0$ and thus obviously we have

$$\|L_1 - L\| \leq \epsilon.$$

Then $\|L^{-1}(L_1 - L)\| < \epsilon\|L^{-1}\| < \frac{1}{2}$. Hence, by an easy computation,

$$1 + L^{-1}(L_1 - L)$$

has a bounded inverse and its norm is smaller or equal to $\frac{1}{(1-\epsilon\|L^{-1}\|)}$. Apply now $-(1 + L^{-1}(L_1 - L))^{-1}$ to the left side above and change the order of the terms, then

$$\begin{aligned} &\|g(y') - g(y'') - [1 + L^{-1}(L_1 - L)]^{-1}(L^{-1}(y' - y''))\| \\ &\leq \epsilon_1\|L^{-1}\| \frac{\|g(y') - g(y'')\|}{1 - \epsilon\|L^{-1}\|}. \end{aligned} \tag{4.29}$$

If $\tilde{y}$ and $\tilde{\tilde{y}}$ are near to y , we replace the right side in (4.28) by $\epsilon_1\|L^{-1}\|^2 \frac{\|\tilde{y} - \tilde{\tilde{y}}\|}{(1 - \epsilon\|L^{-1}\|)^2}$. Then g is differentiable at y with differential $[1 + L^{-1}(L_1 - L)]^{-1}L^{-1}$, since the coefficient of $\|\tilde{y} - \tilde{\tilde{y}}\|$ approaches 0 as $\epsilon_1 \rightarrow 0$. This reduces to L^{-1} at $y = 0$, because $L = L_1$ at $y = 0$.

Finally, let us prove

$$(2) \quad g(f(L^{-1}(y))) = L^{-1}(y).$$

For if y is near to 0 we have that both y and $f(L^{-1}(y))$ are within $\frac{\delta}{2}\|L^{-1}\|$ of 0. In the proofs of the first two equations we replace q_1 and q_2 by $\tilde{y}$ and $\tilde{\tilde{y}}$ in the definition of the strong differential. Here we will write $g(f(L^{-1}(y)))$ and $L^{-1}(y)$ instead of q_1 and q_2 . Indeed, look at the relation with $L^{-1}f$ applied on the left side:

$$\begin{aligned} &\|L^{-1}(f(g(f(L^{-1}(y))))) - L^{-1}(f(L^{-1}(y))) - L^{-1}L(g(f(L^{-1}(y)))) - L^{-1}(y)\| \\ &\leq \epsilon\|L^{-1}\|\|g(f(L^{-1}(y))) - L^{-1}(y)\|. \end{aligned}$$

We have $L^{-1}fg = L^{-1}$ and this cancels the first two terms on the left side of

the inequality. Using $L^{-1}Lg = g$ and $L^{-1}LL^{-1} = L^{-1}$, we can simplify to

$$\|g(f(L^{-1}(y))) - L^{-1}(y)\| \leq \epsilon \|L^{-1}\| \|g(f(L^{-1}(y))) - L^{-1}(y)\|,$$

which is a contradiction unless $g(f(L^{-1}(y))) = L^{-1}(y)$, since $\epsilon \|L^{-1}\| < \frac{1}{2}$.

So now it remains to show uniqueness of g . Assuming $\tilde{g}$ has similar properties, then it coincides with g at y in a neighborhood of 0. Consider y so near to 0 that g and $\tilde{g}$ satisfy the first two proven equalities and such that $\|g(y)\|$ and $\|\tilde{g}(y)\|$ are nearer than δ to 0 in N_1 . This time we apply the definition of the strong differential on $g(y)$ and $\tilde{g}(y)$ and obtain the relation:

$$\|L^{-1}(f(g(y))) - L^{-1}(f(\tilde{g}(y))) - L^{-1}L(g(y) - \tilde{g}(y))\| \leq \epsilon \|L^{-1}\| \|g(y) - \tilde{g}(y)\|$$

Again we have, using $L^{-1}(f(g(y))) = L^{-1}(y) = L^{-1}(f(\tilde{g}(y)))$, that the first two terms on the left side cancel and that the expression simplifies to

$$\|g(y) - \tilde{g}(y)\| \leq \epsilon \|L^{-1}\| \|g(y) - \tilde{g}(y)\|,$$

by using $L^{-1}Lg = g$ and $L^{-1}L\tilde{g} = \tilde{g}$. So $g(y) = \tilde{g}(y)$ [17]. $\square$

We recall the notion of Clarke's [6] generalized differential for locally Lipschitz functions in order to make clear the connection between the inversion theorem above and Clarke's theorem [6, Theorem 1], [18].

Consider a locally Lipschitz function $f : O \rightarrow \mathbb{R}^n$, where $O \subset \mathbb{R}^k$. Then the generalized Jacobian $\partial f(p)$ at p is the convex hull of all matrices M of the form $M = \lim_{p_i \rightarrow p} df(p_i)$, where p_i converges to p , f is differentiable at p_i for each i and df is the usual Jacobian [18]. Furthermore, consider

Proposition 4.5. [18, Proposition 1]

Let $f : O \rightarrow \mathbb{R}^n$ with $O \subset \mathbb{R}^k$ be strongly differentiable at p , then $\partial f(p) = \{df(p)\}$.

Proof. Take any $\epsilon > 0$ and $\delta > 0$ such that if $|q_1 - p| \leq \delta$ and $|q_2 - p| \leq \delta$, then

$$\|f(q_1) - f(q_2) - L(q_1 - q_2)\| \leq \epsilon \|q_1 - q_2\|.$$

Consider the sequence $J_i = df(p_i)$ and let J be the limit for $p_i \rightarrow p$. We assume for each i that $\|p_i - p\| \leq \frac{\delta}{2}$ and let e be any normalized unit vector. We can find some $0 < \delta_i \leq \frac{\delta}{2}$ for each i such that for every q_i with $\|q_i - p_i\| \leq \delta_i$ we have

$$\|f(q_i) - f(p_i) - df(p_i)(q_i - p_i)\| \leq \epsilon \|q_i - p_i\|.$$

Set $q_i = p_i + \delta_i e$ and consider

$$\begin{aligned} \delta_i \|(L - J_i)e\| &= \|(L - J_i)(q_i - p_i)\| \\ &\leq \|f(q_i) - f(p_i) - J_i(q_i - p_i)\| + \|f(q_i) - f(p_i) - L(q_i - p_i)\| \\ &\leq 2\epsilon \|q_i - p_i\| = 2\epsilon \delta_i. \end{aligned}$$

Here we can cancel δ_i and if we take the limit $i \rightarrow \infty$ and use that ϵ and e were arbitrary then we conclude that $J = df(p)$ and thus $\partial f(p) = \{df(p)\}$ [18]. $\square$

By Clarke's theorem [6, Theorem 1], we have that if $k = n$ and ∂f admits only invertible elements then f is a local bi-Lipschitz homeomorphism. But Leach [17], [18, Theorem 2] states something more, namely if f is strongly differentiable at p then the inverse is strongly differentiable at $f(p)$. By means of this we will deduce the Lipschitzness of the inverse of the exponential map in the next chapter. We also refine the pointed version of the first main theorem. After this we state the definition of a position vector on convex neighborhoods which leads us to the Gauss Lemma. In particular, we present a version of the Gauss Lemma which does not require the differentiability of the exponential map. For further informations, we refer to [6], [17] and [18].

5 Main Theorems

In this chapter we are going to state the three main theorems of this thesis.

5.1 Strong differentiability of the exponential map

First we are going to prove the strong differentiability of the exponential map. This will be helpful to show the Lipschitzness of the inverse of a function which is strongly differentiable at some point p . Indeed, consider

Theorem 5.1 (Main Theorem 1). *[18, Theorem 3]*

Let M be a C^∞ -manifold endowed with a locally Lipschitz connection. Then we can conclude the following for the (pointed) exponential map.

The set Ω is open in the topology of TM , the exponential map $\exp : \Omega \rightarrow M \times M$ is locally Lipschitz. The exponential map is strongly differentiable over the zero section $TM_0 := \{0_p : p \in M\}$ and provides a bi-Lipschitz homeomorphism between an open star-shaped neighborhood of the zero section and an open neighborhood of the diagonal $\Delta_M := \{(p, p) : p \in M\}$ of $M \times M$.

The set Ω_p is open in the topology of T_pM for every $p \in M$. The pointed exponential map $\exp_p$ maps from Ω_p to M , where Ω_p is a subset of T_pM , is locally Lipschitz and strongly differentiable at the origin and provides a local bi-Lipschitz homeomorphism from a star-shaped open subset of Ω_p onto an open neighborhood of p .

Proof. As already mentioned in the previous chapter, we will use Picard-Lindelöf in order to prove the strong differentiability of the exponential map. So again starting as in the proof of theorem 5.1, let us consider geodesics x^μ and $\tilde{x}^\mu$ and the corresponding initial conditions $(x_0^\mu, \dot{x}_0^\mu), (\tilde{x}_0^\mu, \dot{\tilde{x}}_0^\mu)$. Again use Picard-Lindelöf in order to define the functions $x_k^\mu(t)$ and $\tilde{x}_k^\mu(t)$. We get the following two inequalities, where in the first one we add $(x_0 - y_0)t$ on both sides

$$\begin{aligned} \|x_{k+1}(t) - y_{k+1}(t) - (x_0 - y_0) - (\dot{x}_0 - \dot{y}_0)t\| &\leq \int_0^t \|\dot{x}_k(t) - \dot{y}_k(t) - (x_0 - y_0)\| dt, \\ \|\dot{x}_{k+1}(t) - \dot{y}_{k+1}(t) - (\dot{x}_0 - \dot{y}_0)\| &\leq \int_0^t A\|x_k - y_k\| + B\|\dot{x}_k - \dot{y}_k\| dt. \end{aligned}$$

We are going to prove that

$$\begin{aligned} & \|x_k(t) - y_k(t) - (x_0 - y_0) - (\dot{x}_0 - \dot{y}_0)t\| \\ & \leq \max\{D\|x_0 - y_0\|, \|\dot{x}_0 - \dot{y}_0\|\} \left(\frac{1}{D} \sum_{j=0}^k \frac{(Dt)^j}{j!} - \frac{1}{D} - t \right), \end{aligned} \quad (5.1)$$

$$\begin{aligned} & \|\dot{x}_k(t) - \dot{y}_k(t) - (\dot{x}_0 - \dot{y}_0)\| \\ & \leq \max\{D\|x_0 - y_0\|, \|\dot{x}_0 - \dot{y}_0\|\} \left(\sum_{j=0}^k \frac{(Dt)^j}{j!} - 1 \right). \end{aligned} \quad (5.2)$$

hold by induction. So, let us start with equation (5.1) and $k = 1$:

$$\begin{aligned} & \|x_1(t) - y_1(t) - (x_0 - y_0) - (\dot{x}_0 - \dot{y}_0)t\| = 0 \\ & \leq \max\{D\|x_0 - y_0\|, \|\dot{x}_0 - \dot{y}_0\|\} \left(\frac{1}{D} \sum_{j=0}^1 \frac{(Dt)^j}{j!} - \frac{1}{D} - t \right), \end{aligned}$$

where $x_1(t) = x_0 + t\dot{x}_0$ and $y_1(t) = y_0 + t\dot{y}_0$. For (5.2) and $k = 1$ we have:

$$\begin{aligned} & \|\dot{x}_1(t) - \dot{y}_1(t) - (\dot{x}_0 - \dot{y}_0)\| \leq \int_0^t A\|x_0 - y_0\| + B\|\dot{x}_0 - \dot{y}_0\| \\ & \leq \max\{D\|x_0 - y_0\|, \|\dot{x}_0 - \dot{y}_0\|\} \int_0^t \frac{A}{D} \sum_{j=0}^0 \frac{(Ds)^j}{j!} + B \sum_{j=0}^0 \frac{(Ds)^j}{j!} ds \\ & = \max\{D\|x_0 - y_0\|, \|\dot{x}_0 - \dot{y}_0\|\} \left(\frac{A}{D} + B \right) \int_0^t 1 ds \\ & = \max\{D\|x_0 - y_0\|, \|\dot{x}_0 - \dot{y}_0\|\} Dt. \end{aligned}$$

Assuming this holds true for some k , we now show (5.2) for $k \rightarrow k + 1$

$$\begin{aligned} & \|\dot{x}_{k+1}(t) - \dot{y}_{k+1}(t) - (\dot{x}_0 - \dot{y}_0)\| \leq \int_0^t A\|x_k(s) - y_k(s)\| + B\|\dot{x}_k(s) - \dot{y}_k(s)\| ds \\ & \leq \max\{D\|x_0 - y_0\|, \|\dot{x}_0 - \dot{y}_0\|\} \left(\frac{A}{D} + B \right) \int_0^t \left(\sum_{j=0}^k \frac{(Ds)^j}{j!} \right) ds \\ & = \max\{D\|x_0 - y_0\|, \|\dot{x}_0 - \dot{y}_0\|\} D \frac{1}{D} \sum_{j=0}^k \frac{(Dt)^{j+1}}{(j+1)!} \\ & = \max\{D\|x_0 - y_0\|, \|\dot{x}_0 - \dot{y}_0\|\} \left(\sum_{j=0}^{k+1} \frac{(Dt)^j}{j!} - 1 \right). \end{aligned}$$

Then for equation (5.1) we get for $k \rightarrow k+1$

$$\begin{aligned}
& \|x_{k+1}(t) - y_{k+1}(t) - (x_0 - y_0) - (\dot{x}_0 - \dot{y}_0)t\| \leq \int_0^t \|\dot{x}_k(s) - \dot{y}_k(s) - (\dot{x}_0 - \dot{y}_0)\| ds \\
& \leq \int_0^t \max\{D\|x_0 - y_0\|, \|\dot{x}_0 - \dot{y}_0\|\} \left(\sum_{j=0}^k \frac{(Dt)^j}{j!} - 1 \right) \\
& = \max\{D\|x_0 - y_0\|, \|\dot{x}_0 - \dot{y}_0\|\} \frac{1}{D} \left(\sum_{j=0}^k \frac{(Dt)^{j+1}}{(j+1)!} - t \right) \\
& = \max\{D\|x_0 - y_0\|, \|\dot{x}_0 - \dot{y}_0\|\} \frac{1}{D} \left(\sum_{j=0}^{k+1} \frac{(Dt)^j}{j!} - 1 \right) \\
& - \max\{D\|x_0 - y_0\|, \|\dot{x}_0 - \dot{y}_0\|\} t \\
& = \max\{D\|x_0 - y_0\|, \|\dot{x}_0 - \dot{y}_0\|\} \left(\frac{1}{D} \sum_{j=0}^{k+1} \frac{(Dt)^j}{j!} - \frac{1}{D} - t \right).
\end{aligned}$$

So our claim follows and because of $\sum_{j=0}^k \frac{(Dt)^j}{j!} \leq e^{Dt}$ even more holds, namely

$$\begin{aligned}
& \|x_k(t) - y_k(t) - (x_0 - y_0) - (\dot{x}_0 - \dot{y}_0)t\| \\
& \leq \max\{D\|x_0 - y_0\|, \|\dot{x}_0 - \dot{y}_0\|\} \left(\frac{e^{Dt}}{D} - \frac{1}{D} - t \right), \\
& \|\dot{x}_k(t) - \dot{y}_k(t) - (\dot{x}_0 - \dot{y}_0)\| \\
& \leq \max\{D\|x_0 - y_0\|, \|\dot{x}_0 - \dot{y}_0\|\} (e^{Dt} - 1).
\end{aligned}$$

Letting $k \rightarrow \infty$ we obtain

$$\begin{aligned}
& \|x(t) - y(t) - (x_0 - y_0) - (\dot{x}_0 - \dot{y}_0)t\| \\
& \leq \max\{D\|x_0 - y_0\|, \|\dot{x}_0 - \dot{y}_0\|\} \left(\frac{e^{Dt}}{D} - \frac{1}{D} - t \right), \\
& \|\dot{x}(t) - \dot{y}(t) - (\dot{x}_0 - \dot{y}_0)\| \\
& \leq \max\{D\|x_0 - y_0\|, \|\dot{x}_0 - \dot{y}_0\|\} (e^{Dt} - 1).
\end{aligned}$$

Let us now express the exponential map in coordinates, indeed, consider the function $f : X \rightarrow X$ on $X = \mathbb{R}^{2n}$ given by $f(x_0, \dot{x}_0) = (x_0, x(1))$. Furthermore, let the linear map $L : X \rightarrow X$ be given by

$$L = \begin{pmatrix} I & 0 \\ I & I \end{pmatrix},$$

where I is the $n \times n$ identity matrix and look at the trivial inequality

$$\|x_0 - y_0 - (x_0 - y_0)\| \leq \max\{D\|x_0 - y_0\|, \|\dot{x}_0 - \dot{y}_0\|\}(e^{Dt} - 1 - Dt). \quad (5.3)$$

Notice that for $t = 1$ we have

$$L \begin{pmatrix} x_0 - y_0 \\ \dot{x}_0 - \dot{y}_0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ I & I \end{pmatrix} \begin{pmatrix} x_0 - y_0 \\ \dot{x}_0 - \dot{y}_0 \end{pmatrix} = \begin{pmatrix} x_0 - y_0 \\ x_0 - y_0 + (\dot{x}_0 - \dot{y}_0) \end{pmatrix}.$$

Thus, we can rewrite the left side of the equations (5.1) and (5.3) as follows

$$\begin{pmatrix} x_0 - y_0 \\ x(1) - y(1) \end{pmatrix} - \begin{pmatrix} x_0 - y_0 \\ x_0 - y_0 + (\dot{x}_0 - \dot{y}_0) \end{pmatrix} = \begin{pmatrix} x_0 - y_0 \\ x(1) - y(1) \end{pmatrix} - L \begin{pmatrix} x_0 - y_0 \\ \dot{x}_0 - \dot{y}_0 \end{pmatrix}$$

and recalling here that $D \leq 1$ and by definition $\|(x, \dot{x})\| = \max\{\|x\|, \|\dot{x}\|\}$ we conclude $\max\{\|x_0 - y_0\|, \|\dot{x}_0 - \dot{y}_0\|\} \leq \|(x_0, \dot{x}_0) - (y_0, \dot{y}_0)\|$. Then we get

$$\|f(x_0, \dot{x}_0) - f(y_0, \dot{y}_0) - L((x_0, \dot{x}_0) - (y_0, \dot{y}_0))\| \leq \|(x_0, \dot{x}_0) - (y_0, \dot{y}_0)\| \left(\frac{e^D - 1}{D} - 1 \right)$$

for every $(x_0, \dot{x}_0)$ and $(y_0, \dot{y}_0)$ such that $\|(x_0, \dot{x}_0)\| < \delta$ and $\|(y_0, \dot{y}_0)\| < \delta$. We know that D is a function of δ and satisfies $\lim_{\delta \rightarrow 0} D(\delta) = 0$, thus we see from the equation above that f is strongly differentiable at $(0, 0)$ with strong differential L . The exponential map is differentiable at any point in the image of $p \rightarrow 0_p$, since $M \cong \{0_p : p \in M\} \subset TM$, meaning that any point $p \in M$ corresponds to zero tangent coordinates for some chart compatible with the atlas. Observe that if a function $f(x, y)$ is strongly differentiable, then keeping x fixed, we obtain that $f(x, \cdot)$ is strongly differentiable. Moreover, the composition of strongly differentiable functions is strong differentiable. So we can conclude that the pointed exponential map, which is given by $\exp_p := \pi_2(f(0, \cdot))$ is strongly differentiable at the origin, where π_2 is the projection $(a, b) \mapsto b$. The strong differential L is invertible and so the pointed exponential map establishes a local bi-Lipschitz homeomorphism and the theorem is proved. [18] $\square$

In order to refine the pointed case we will go through some definitions and then state our next theorem.

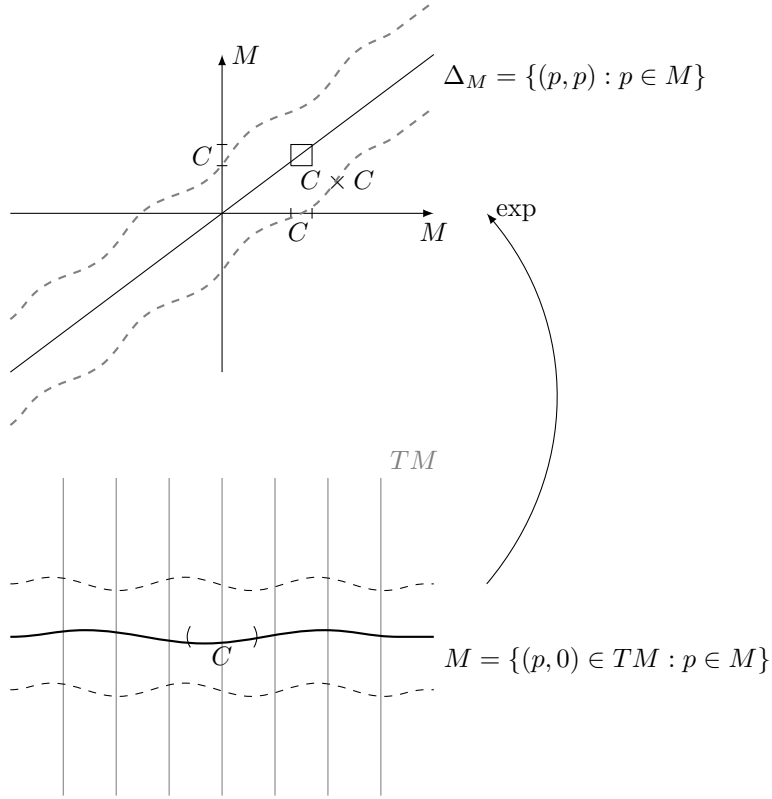
Let N be an open neighborhood of $p \in M$. N is called normal if there exists an open star-shaped subset $N_p \subset \Omega_p$ such that $\exp_p : N_p \rightarrow N$ is a bi-Lipschitz homeomorphism. We will call C convex normal if it fulfils the conditions above for each of its points, where $C \subset M$ is an open set. $\overline{C}$ is called strictly convex normal if C is convex normal and any two points of $\overline{C}$ are connected by a unique geodesic, which is contained in C [18].

Now we are ready to prove the following theorem:

Theorem 5.2 (Main Theorem 2). [18, Theorem 4]

Let again M be a C^∞ –manifold endowed with a locally Lipschitz connection and O an open neighborhood of p in M . Then in O there exists a strictly convex normal neighborhood C of p such that the exponential map provides a bi-Lipschitz homeomorphism between an open star-shaped subset of the tangent bundle TC and $C \times C$.

Furthermore, let $B(p, \delta)$ be an open ball defined through the Euclidean norm induced by the coordinates for any sufficiently small δ . Then C can be chosen such that it is equal to $B(p, \delta)$ for every chart x^μ which is defined in a neighborhood of p .



Proof. In the proof of the previous theorem we proved the strong differentiability of the exponential map, so we already know that there exists an open neighborhood O of (p, p) with $O \subset M \times M$ that is bi-Lipschitz homeomorphic to an open neighborhood U of the zero section $0_p \in TM$.

Let now δ be sufficiently small such that $C = B(p, \delta)$ for some chart x^μ which is defined in a neighborhood of p . We start this proof by showing that the ball $\overline{B}(p, \delta)$ is strictly convex normal for sufficiently small δ .

Choosing δ as above and sufficiently small such that $\overline{B}(p, \delta)^2 \subset O$ we have

$$N_q := (\exp^{-1} B(p, \delta)^2) \cap \pi^{-1}(q) = [\exp^{-1}(\{q\} \times B(p, \delta))] \cap \pi^{-1}(q) = \exp_q^{-1} B(p, \delta)$$

for $q \in B(p, \delta)$.

The exponential map is continuous and $\pi^{-1}(q) = T_q M$, so the first equality shows that N_p is open in the topology of $T_q M$. The existence of such a neighborhood is a consequence of Leach's theorem. Hence for each $q \in B(p, \delta)$ the open set N_q is chosen such that $\exp_q : N_q \rightarrow B(p, \delta)$ is a bi-Lipschitz homeomorphism. It suffices to show that $\overline{B}(p, \delta)$ is strictly convex and thus N_q is star-shaped.

Consider now $\tilde{q} \in B(p, \delta)$, $\tilde{q} \neq q$ and $v := \exp_q^{-1}(\tilde{q}) \in N_p$. Let $x : [0, 1] \rightarrow M$, $t \mapsto \gamma_v(t)$ be the geodesic from q to $\tilde{q}$. If x lies completely in $B(p, \delta)$, then $\gamma_v(t) \in N_q$. But $\gamma_v(t) = \exp_q(tv) \in N_q$ for all $t \in [0, 1]$, and thus $tv \in N_q$, meaning that N_q is star-shaped. So it suffices to show that x stays in $B(p, \delta)$. By contradiction, assume that x does not lie completely in $B(p, \delta)$. Then there must be $t \in (0, 1)$ such that $x(t) \geq \delta$. But $\|q\| < \delta$ and $\|\tilde{q}\| < \delta$, thus the function $t \mapsto x(t)$ must take a maximum at $t_0 \in (0, 1)$.

Any geodesic $x(t)$ is $\mathcal{C}^{2,1}$ and so is $\|x\|^2(t)$. Furthermore, remember that by definition a geodesic is regular, meaning $\dot{x} \neq 0$. Thus we can differentiate twice and get

$$\frac{d^2\|x\|^2}{dt^2} = 2 \left(\|\dot{x}\|^2 + x \frac{d^2x}{dt^2} \right) = 2(\|\dot{x}\|^2 + xH(x, \dot{x})). \quad (5.4)$$

Defining a continuous function z defined on $\pi^{-1}(\overline{B}(p, \delta))$ by

$$z(x, v) = \|\dot{x}\|^2 + xH(x, v),$$

restricting it to the unit tangent bundle $\overline{B}(p, \delta) \times S^{n-1}$ and recalling that $M = \sup_{x \in \overline{B}(p, r)} \sup_{\|v\|=1} \|H(x, v)\|$ we get

$$z(x, e) = 1 + xH(x, e) \geq 1 - \delta M > 0. \quad (5.5)$$

Using this we then have that (5.4) is equal to

$$2\|\dot{x}\|^2 \left(1 + xH \left(x, \frac{\dot{x}}{\|\dot{x}\|} \right) \right) = 2z \left(x, \frac{\dot{x}}{\|\dot{x}\|} \right) \|\dot{x}\|^2$$

and this is positive because of (5.5).

This is a contradiction to our assumption and thus we have that x stays in $B(p, \delta)$.

So we have that $\|x\|^2(t)$ obtains its maximum at the boundary of its interval of definition, which are the endpoints of the geodesic segment. We deal with a strict inequality, so if its endpoints are at the boundary of the ball then its interior points are in $B(p, \delta)$, meaning that the ball $\overline{B}(p, \delta)$ is strictly convex normal. From $B(p, \delta)^2 \subset O$, we can infer that the exponential map establishes a

bi-Lipschitz homeomorphism between an open subset of TC and $B(p, \delta)^2$. Thus the theorem is proved [18]. $\square$

For further information, we refer to [18].

5.2 Position vector on convex neighborhoods

Before we state Gauss' Lemma, we first define a position vector and take a closer look at its behaviour on a convex neighborhood. For this purpose, consider C to be a convex normal set and let $p, q \in C$. The unique geodesic connecting them is given by $x : [0, 1] \rightarrow C$, $x(0) = p, x(1) = q$. Then the vector $\dot{x}(1)$, denoted by $P(p, q)$, is called position vector [18].

Theorem 5.3. [18, Theorem 17]

As already proved, for ever $\epsilon < 0$ and sufficiently small δ we have that $C = B(p, \delta)$ satisfies the conditions of theorem 6.2. The coordinate system $\{x^\mu\}$ in the neighborhood U of p induces an affine structure that allows us to compare tangent vectors at different points of U . Furthermore we have that for every $q_1, q_2, q'_1, q'_2, q \in C$

$$\begin{aligned} \|[P(q'_1, q'_2) - (q'_2 - q'_1)] - [P(q_1, q_2) - (q_2 - q_1)]\| &\leq \epsilon \max\{\|q'_1 - q_1\|, \|q'_2 - q_2\|\}, \\ \|P(q, q_2) - P(q, q_1) - (q_2 - q_1)\| &\leq \epsilon \|q_2 - q_1\|, \end{aligned} \quad (5.6)$$

$$\|P(q_1, q_2) - (q_2 - q_1)\| \leq \epsilon \|q_2 - q_1\|, \quad (5.7)$$

$$\|P(q_1, q_2)\| \leq \epsilon. \quad (5.8)$$

If the coordinate chart is chosen such that the Christoffel symbols $\Gamma^\mu_{\alpha\beta}(p)$ vanish at p , then δ can be chosen such that the following holds

$$\|P(q_1, q_2) + P(q_2, q_1)\| \leq \epsilon \|q_2 - q_1\|^2.$$

This theorem will be proved by the next two statements.

Now consider the geodesic $x(t)$ with initial conditions satisfying the condition $\max\{\|x_0\|, \|\dot{x}_0\|\} = \|(x_0, \dot{x}_0)\| < \delta$ as before. Keep x_0 fixed. First we show that the map $\dot{x}_0 \mapsto \dot{x}(1)$ is injective. Recall that $\dot{x}(1) = P(x_0, x(1)) = P(x_0, \exp_{x_0} \dot{x}_0)$ [18]. Then we have:

Lemma 5.4. [18, Lemma 3]

For fixed x_0 , the map $\dot{x}_0 \mapsto \dot{x}(1)$ is strongly differentiable at the origin, where the strong differential is the $n \times n$ identity matrix I . Furthermore, the map $(x_0, \dot{x}_0) \mapsto (x_0, \dot{x}(1))$ is strongly differentiable whenever $\dot{x}_0 = 0$ and the strong

differential is the matrix

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

So we have for sufficiently small δ that both maps are injective and bi-Lipschitz with inverse strongly differentiable whenever $\dot{x}(1) = 0$.

Proof. Consider the exponential map of the vectors $\dot{x}_0, \dot{y}_0$ with base point $x_0 = y_0$. By equation (5.2) for $t = 1$ we get

$$\|\dot{x}(1) - \dot{y}(1) - (\dot{x}_0 - \dot{y}_0)\| \leq \|\dot{x}_0 - \dot{y}_0\|(e^D - 1).$$

We know $\lim_{\delta \rightarrow 0} D(\delta) = 0$ and thus we conclude that the map $\dot{x}_0 \mapsto \dot{x}(1)$ is strongly differentiable at the origin with strong differential the identity matrix. For the second part of the theorem we look at the following two inequalities:
The trivial inequality

$$\|x_0 - y_0 - (x_0 - y_0)\| \leq \|x_0 - y_0\|(e^D - 1),$$

and the inequality from above

$$\|\dot{x}(1) - \dot{y}(1) - (\dot{x}_0 - \dot{y}_0)\| \leq \|\dot{x}_0 - \dot{y}_0\|(e^D - 1).$$

Thus, we can rewrite them as follows

$$\left\| \begin{pmatrix} x_0 - y_0 \\ \dot{x}(1) - \dot{y}(1) \end{pmatrix} - \begin{pmatrix} x_0 - y_0 \\ \dot{x}_0 - \dot{y}_0 \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} x_0 - y_0 \\ \dot{x}_0 - \dot{y}_0 \end{pmatrix} \right\| (e^D - 1).$$

and thus we get

$$\|(x_0, \dot{x}(1)) - (y_0, \dot{y}(1)) - (x_0, \dot{x}_0) - (y_0, \dot{y}_0)\| \leq \|(x_0, \dot{x}_0) - (y_0, \dot{y}_0)\|(e^D - 1)$$

for every $(x_0, \dot{x}_0)$ and $(y_0, \dot{y}_0)$ such that $\|(x_0, \dot{x}_0)\| < \delta$ and $\|(y_0, \dot{y}_0)\| < \delta$. Since $\lim_{\delta \rightarrow 0} D(\delta) = 0$ we can conclude that the map $(x_0, \dot{x}_0) \mapsto (x_0, \dot{x}(1))$ is strongly differentiable at $(0, 0)$ with strong differential the identity matrix.

Again by Leach's inverse function theorem we conclude that for sufficiently small δ both maps are injective and bi-Lipschitz with inverse strongly differentiable whenever $\dot{x}(1) = 0$ [18]. $\square$

Furthermore, we obtain:

Proposition 5.5. [18, Proposition 3]

Consider $p \in M$, $\{x^\mu\}$ a chart in a neighborhood U of p . Denote the position vector of q with respect to r by $P(r, q)$. There exists an open convex neighborhood

C containing p such that for every $q_2, q_1 \in C$ the map

$$(q_1, q_2) \mapsto P(q_1, q_2) - (q_2 - q_1),$$

interpreted with the affine structure induced by $\{x^\mu\}$, is strongly differentiable on the diagonal $q_1 = q_2$. The strong differential is zero.

Proof. In order to prove this, let C be a convex neighborhood of p such that the exponential map establishes a bi-Lipschitz homeomorphism between an open subset of TC and $C \times C$. Furthermore, let $x^\mu : C \rightarrow \mathbb{R}^n$ be $\mathcal{C}^{2,1}$ coordinates on C such that $x^\mu(p) = 0$.

As already proved, the exponential map is strongly differentiable on the zero section of TC with strong differential $L = \begin{pmatrix} I & 0 \\ I & I \end{pmatrix}$. By Leach's inverse function theorem, we conclude that $\exp^{-1}$ is strongly differentiable on the diagonal of $C \times C$ with strong differential $L^{-1} = \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix}$. In other words, the map written in coordinates $(q_1, q_2) \mapsto (q_1, \exp_{q_1}^{-1} q_2) = (q_1, P(q_1, q_2))$ is strongly differentiable on the diagonal with strong differential L^{-1} .

The strong differential of the map $(q_1, q_2) \mapsto (q_1, q_2 - q_1)$ is also L^{-1} , where we use the affine structure induced by the coordinate chart in order for the difference to make sense. So clearly then the strong differential of the map $(q_1, q_2) \mapsto (q_1, P(q_1, q_2) - (q_2 - q_1))$ vanishes on the diagonal of $C \times C$ and this implies that the strong differential of the map $(q_1, q_2) \mapsto P(q_1, q_2) - (q_2 - q_1)$ vanishes on the diagonal of $C \times C$. In particular, this map is strongly differentiable at (p, p) and so for every $\epsilon > 0$ the constant $\delta > 0$ and hence the convex neighborhood $C = B(p, \delta)$ can be chosen such that for every $q_1, q_2, q'_1, q'_2 \in C$

$$\| [P(q'_1, q'_2) - (q'_2 - q'_1)] - [P(q_1, q_2) - (q_2 - q_1)] \| \leq \epsilon \max\{\|q'_1 - q_1\|, \|q'_2 - q_2\|\}.$$

And hence for every $q_1, q_2, q \in C$, by replacing q'_1 by q , q'_2 by q_2 , q_2 by q_1 and q_1 by q we get

$$\|P(q, q_2) - P(q, q_1) - (q_2 - q_1)\| \leq \epsilon \|q_2 - q_1\|.$$

By setting $q = q_1$ we conclude

$$\|P(q_1, q_2) - (q_2 - q_1)\| \leq \epsilon \|q_2 - q_1\|.$$

By the triangle inequality we conclude that $\|P(q_1, q_2)\| \leq (1 + \epsilon)\|q_2 - q_1\|$. The diameter of C can be chosen sufficiently small such that $\|P(q_1, q_2)\| \leq \epsilon$.

Consider the geodesic $x : [0, 1] \rightarrow C$ given by $x(0) = q'_1, x(1) = q_1$ and let

$q_2 = q'_2 = x(t)$ with $t \in [0, 1]$. Then equation (5.6) gives:

$$\|P(x(0), x(t)) - P(x(1), x(t)) - (x(1) - x(0))\| \leq \epsilon \|x(1) - x(0)\|.$$

Observe that $P(p, q) = \gamma'_{\exp_p^{-1}(q)}(1)$. This implies $P(x(0), x(t)) = \gamma'_{\exp_{x(0)}^{-1}(x(t))}(1)$.

We have that $\exp_{x(0)}(t\dot{x}(0)) = x(t)$ and therefore $\exp_{x(0)}^{-1}(x(t)) = t\dot{x}(0)$. This again implies that $\gamma_{\exp_{x(0)}^{-1}(x(t))}(s) = \exp_{x(0)}(st\dot{x}(0)) = x(st)$. So we can conclude that

$$\gamma'_{\exp_{x(0)}^{-1}(x(t))}(1) = P(x(0), x(t)) = t\dot{x}(t).$$

Also, $P(x(1), x(t)) = \gamma_{\exp_{x(1)}^{-1}(x(t))}(1)$. We have $\exp_{x(1)}(-t\dot{x}(1)) = x(1-t)$ and therefore we conclude by writing $1-t$ instead of t that $\exp_{x(1)}(-(1-t)\dot{x}(1)) = x(t)$. Thus, $\exp_{x(1)}^{-1}(x(t)) = -(1-t)\dot{x}(1)$.

Then, $\gamma_{\exp_{x(1)}^{-1}(x(t))}(s) = \exp_{x(1)}(-s(1-t)\dot{x}(1)) = x(1-s(1-t))$. Differentiating by s gives

$$\gamma'_{\exp_{x(1)}^{-1}(x(t))}(1) = P(x(1), x(t)) = \dot{x}(t)(-(1-t)).$$

Hence for $t \in [0, 1]$

$$\|\dot{x}(t) - (x(1) - x(0))\| \leq \epsilon \|x(1) - x(0)\|.$$

For $t = 1$ we have $P(x(0), x(1)) = \dot{x}(1)$ and $P(x(1), x(0)) = -\dot{x}(0)$ for $t = 0$. Set $r(t) = \dot{x}(t) - (x(1) - x(0))$. Let us assume that the coordinate system is chosen such that for every normalized vector e and C sufficiently small such that on C the following hold:

$$\|H\| := \sup_{x \in C} \sup_{\|e\|=1} \|H(x, e)\| \leq \epsilon.$$

This is always possible by a coordinate transformation and using that H^μ is homogeneous of second order. Then we conclude

$$\begin{aligned} & \|P(x(0), x(1)) + P(x(1), x(0))\| = \|\dot{x}(1) - \dot{x}(0)\| \\ &= \left\| \int_0^1 H(x(s), \dot{x}(s)) ds \right\| \leq \int_0^1 \|H(x(s), \dot{x}(s))\| \|\dot{x}(s)\| ds \leq \epsilon \int_0^1 \|\dot{x}(s)\|^2 ds \\ &\leq \epsilon \left[\int_0^1 \|r(s)\|^2 ds + \int_0^1 \|x(1) - x(0)\|^2 ds + 2[x(1) - x(0)] \int_0^1 r(s) ds \right] \\ &\leq \epsilon(1 + \epsilon^2) \|x(1) - x(0)\|^2. \end{aligned}$$

A redefinition of ϵ gives the last inequality of theorem 5.3 [18]. □

For further information, we refer to [18].

5.3 Gauss Lemma

So now let us go back to Gauss' Lemma. In order to prove it we will need the following theorem.

Theorem 5.6. [18, Theorem 16]

Consider a Lipschitz function f defined on an open subset O of $\mathbb{R}^n$ and let $J : O \rightarrow \mathbb{R}^n$ be a continuous function. If the differential of f exists and is equal to J almost everywhere then f is $\mathcal{C}^1$ and the differential is J .

Proof. This theorem is easy to prove for $n = 1$. Indeed, let us look at

$$f(x) - f(a) - \int_a^x J(u) du,$$

where $a \in O$. Obviously, this function is Lipschitz, hence absolutely continuous and so it has a derivative, which vanishes almost everywhere. Thus it is constant. If we set $a = x$, we see that the constant is zero, hence we have

$$f(x) = f(a) + \int_a^x J(u) du$$

and this implies that f is $\mathcal{C}^1$ and the derivative is J .

For $n > 1$ we will proceed as follows. Without loss of generality, we assume $O = \mathbb{R}^n$. The goal is to show that at every $x \in O$, $\partial_v f = J(v)$, where $v \in \mathbb{R}^n$. This would imply that f is Gâteaux differentiable at each point with Gâteaux differential J and by proposition 4.2 our claim would follow.

For almost every hyperplane that is perpendicular to v , the function f is almost everywhere differentiable at the points that belongs to the hyperplane, by the theorems of Fubini and Rademacher.

We introduce coordinates $\{y^0, \dots, y^n\}$ in which such a hyperplane has equation $y^0 = 0$ and $v = \partial_{y^0}$ and thus is perpendicular to v . Consider

$$g(y_1, \dots, y_n) := f(0, y_1, \dots, y_n),$$

the Lipschitz restriction of f to the hyperplane. So by the choice of the hyperplane y^0 and $v = \partial_{y^0}$, this holds in our case and the differential of the function f is J . Let $(t, y) \in \mathbb{R}^n$ and U be an open neighborhood of (t, y) . Now consider a line parallel to the y^0 -axis, determined by its intersection with $y^0 = 0$. Then by Fubini's theorem, for almost every such line the function f is almost everywhere differentiable with differential J , where almost everywhere differentiable means that it is differentiable in the Lebesgue $(n - 1)$ -dimensional measure of this hyperplane. Hence there exists some $(t', y') \in U$ passing through one such parallel line.

By assumption, we have that the function $J(t, y)$ is continuous and thus uniformly continuous over compact subsets. Thus,

$$F(t, y) = g(y) + \int_0^t J(t, y; e_0) dt$$

is continuous in (t, y) and $\mathcal{C}^1$ in t .

Consider now the function $f - F$ above such a line is Lipschitz and differentiable almost everywhere and the derivative is zero. Hence it is a constant. But we have $f - F = 0$ at $y^0 = 0$ and so the constant vanishes and $f(t', y') = F(t', y')$. U was chosen arbitrary and both functions are continuous, so we conclude $f(t, y) = F(t, y)$, meaning $f = F$. Furthermore, $\partial_v f = \partial_v F = \partial_{y^0} F = J(e_0)$ and thus our claim follows [18]. $\square$

Lemma 5.7. [18, Gauss' Lemma under sufficient differentiability conditions, Section 1.3]

Consider a $\mathcal{C}^{1,1}$ metric g giving rise to a Lipschitz connection. Let $p \in M$, N a normal neighborhood of p and $v_x \in \exp_p^{-1} N$ with $x \neq 0$. Furthermore, let $w_x \in T_p M$. Then

$$g_{\exp_p(x)}(T_x \exp_p(v_x), T_x \exp_p(w_x)) = g_p(v_x, w_x).$$

Under this differentiability hypothesis the exponential map is just Lipschitz, meaning the Gauss Lemma is not expected to hold. So we show that the Lemma holds in a formulation that does not require the differentiability of the exponential map. Thus, let us prove

Theorem 5.8 (Main Theorem 3). [18, Theorem 5]

Consider a $\mathcal{C}^\infty$ -manifold M and a $\mathcal{C}^{1,1}$ -metric $g : TM \setminus \{0\} \rightarrow T^*M \otimes_M T^*M$. Let N be a normal neighborhood of p in M and remember that $L : TM \rightarrow \mathbb{R}$ with $L(p, v) = \frac{1}{2}g_p(v, v)$ for $v \neq 0$. Then

$$D_p^2(q) := 2L(p, \exp_p^{-1}(q)) = g_p(\exp_p^{-1}(q), \exp_p^{-1}(q)) \quad (5.9)$$

is $\mathcal{C}^{1,1}$ in q . Moreover,

$$T_q D_p^2 = 2g_q(P(p, q), \cdot), \quad (5.10)$$

where the position vector of q with respect to p is given by $P(p, q) = \gamma'_p(1)$. This last equation represents a geodesic connecting p and q , which is orthogonal to any level set $D_p^2 = \text{constant}$. Hence the level sets of D_p^2 are orthogonal to the geodesics issued from p . Furthermore, the $(-t)$ -time flow map of the position vector is a bi-Lipschitz homeomorphism from $(D_p^2)^{-1}(s)$ to its image on

$(D_p^2)^{-1}(se^{-2t})$ for $t, s > 0$. So Gauss' Lemma holds almost everywhere, indeed whenever $\exp_p : \exp_p^{-1} N \rightarrow N$ is differentiable.

Before we prove this theorem, let us add some remarks. D_p^2 can be negative in the case that the metric is not positive definite. In equation (5.9) we can see that D_p^2 stems from the inverse of the exponential map. Thus, one would expect that D_p^2 is Lipschitz, but we can check that the differential is almost everywhere as in (5.10) and that it can be extended everywhere, by the Lipschitzness of D_p^2 and continuity of the position vector [18].

Finally, let us prove theorem 5.8.

Proof. First, we are going to prove equation (5.10). Obviously, D_p^2 stems from the inverse of the exponential map and since $\exp_p^{-1}$ is Lipschitz, so is D_p^2 . By theorem 4.1, the function $2g_{P(p,q)}(P(p,q), \cdot)$ is Lipschitz in q , where the position vector is given by $\gamma'_{\exp_p^{-1} q}(1)$. By theorem 5.6, it suffices to show that the differential of D_p^2 exists and that equation (5.10) holds almost everywhere on N . We know that the exponential map is differentiable almost everywhere over the star-shaped open set $\exp^{-1} N \subset T_p M$. Hence, by Fubini, it is almost everywhere differentiable on almost every radial line passing through the origin, where almost everywhere is meant in the sense of the $(n-1)$ -dimensional Lebesgue measure of a Euclidean sphere contained in $\exp^{-1} N$. So we now need to take q on the exponential image of one of these geodesics.

Indeed, let $w \in T_q M$. Consider a $\mathcal{C}^{2,1}$ geodesic segment

$$\sigma : [-a, a] \rightarrow N$$

with $\sigma(0) = q, \sigma'(0) = w$ and let

$$\begin{aligned} x^{(s)} : [0, 1] &\rightarrow N, \\ t &\mapsto x^{(s)}(t) \end{aligned}$$

be the unique geodesic with $x^{(s)}(0) = p, x^{(s)}(1) = \sigma(s)$. Let $v(s) = \exp_p^{-1} \sigma(s)$, where $v : [-a, a] \rightarrow T_p M$ is Lipschitz, since $\exp_p^{-1}$ is Lipschitz. Especially,

$$S = \{u \in T_p M : u = tv(s), t \in [0, 1], s \in [-a, a]\}$$

is a Lipschitz submanifold of $T_p M$ and thus again by theorem 4.1

$$x(t, s) := x^{(s)}(t) = \exp_p(tv(s))$$

is Lipschitz in (t, s) .

The map $t \mapsto x^{(s)}(t)$ is a geodesic and for an arbitrarily geodesic α we have

that $\langle \alpha', \alpha' \rangle$ is constant because $\frac{d}{dt} \langle \alpha', \alpha' \rangle = 2 \langle \alpha'', \alpha' \rangle = 0$, due to have $\alpha'' = 0$. Therefore, the quantity

$$L(x^{(s)}(t), x_t^{(s)}(t)) = \langle x_t^{(s)}(t), x_t^{(s)}(t) \rangle$$

is independent of t and we can choose $t = 0$ and so it is equal to $L(p, v(s))$. The Lipschitz function $v(s)$ is differentiable almost everywhere. So now let s be such that $v'(s)$ exists and recall that by theorem 4.1 we have for almost every s , for every $t \in [0, 1]$, $x(t, \cdot)$ is differentiable at s and $\partial_2 x(t, s)$ is locally Lipschitz in t , locally uniformly with respect to those s where it is defined. Furthermore, for any such s we have that for almost every $t \in [0, 1]$, $\partial_1 \partial_2 x(t, s), \partial_2 \partial_1 x(t, s)$ are locally bounded and coincide. Then for t arbitrarily

$$\begin{aligned} \frac{1}{2} \partial_{\sigma'(s)} D_p^2 &= \frac{1}{2} \frac{d}{ds} D_p^2(\sigma(s)) = \frac{d}{ds} L(p, v(s)) \\ &= \frac{\partial}{\partial s} L(x(t, s), x_t(t, s)) = x_s^\mu \frac{\partial L}{\partial x^\mu} + \frac{\partial x_t^\mu}{\partial s} \frac{\partial L}{\partial x_t^\mu} \\ &= \frac{\partial}{\partial t} \left(x_s^\mu \frac{\partial L}{\partial x_t^\mu} \right) - x_s^\mu \left(\frac{\partial}{\partial t} \frac{\partial L}{\partial x_t^\mu} - \frac{\partial L}{\partial x^\mu} \right) + \left(\frac{\partial x_t^\mu}{\partial s} - \frac{\partial x_s^\mu}{\partial t} \right) \frac{\partial L}{\partial x_t^\mu}. \end{aligned}$$

By theorem 4.1, we have that mixed derivatives coincide and thus for almost every t the last term vanishes. The second term of the right-hand side vanishes because $x^{(s)}(t)$ is a geodesic. Thus, it solves the Euler-Lagrange equation for the Lagrangian L . By the definition of $D_p^2(q)$ in (5.9) we have $L(p, v) = \frac{1}{2} g(v, v)$ and this implies $\frac{\partial}{\partial v} L = g(v, \cdot)$. Using this and integrating in t over the $[0, 1]$ we obtain, taking into account that the left-hand side does not depend on t :

$$\begin{aligned} \frac{1}{2} \partial_{\sigma'(s)} D_p^2 &= \int_0^1 \frac{1}{2} \partial_{\sigma'(s)} D_p^2 = \int_0^1 \frac{d}{dt} g(x_t, x_s) dt \\ &= g(x_t(1, s), \sigma'(s)) - g(x_t(0, s), x_s(0, s)) = g(\gamma'_{v(s)}(1), \sigma'(s)), \end{aligned} \quad (5.11)$$

where we used that $x_s(1, s) = \partial_s \exp_p v(s) = \partial_s \sigma(s) = \sigma'(s)$ and

$$x_t(1, s) = \partial_t|_{t=1} \exp_p(tv(s)) = \partial_t|_{t=1} \gamma_{v(s)}(t) = \gamma'_{v(s)}(1)$$

and that $x(0, s) = p$, hence $x_s(0, s) = 0$. Evaluating the previous expression at $s = 0$ gives

$$\partial_s|_{s=0} D_p^2(\sigma(s)) = T_{\sigma(0)} D_p^2(\sigma'(0)) = g(\gamma'_{v(0)}(1), \sigma'(0)) = g(P(p, q), w),$$

where in the last equality we used that $v(0) = \exp_p^{-1}(q) = \exp_p^{-1}(\sigma(0))$ and thus this proves equation (5.10).

Consider an integral curve

$$\begin{aligned}\alpha &: [0, a) \rightarrow N, \\ t &\mapsto \alpha(t)\end{aligned}$$

of $P(p, \cdot)$ and differentiate D_p^2 along it

$$\begin{aligned}\frac{dD_p^2(\alpha(t))}{dt} &= 2 \frac{d}{dt} g(\exp_p^{-1}(\alpha(t)), \exp_p^{-1}(\alpha(t))) = 2g(P(p, \alpha(t)), \dot{\alpha}(t)) \\ &= 2g(P(p, \alpha(t)), P(p, \alpha(t))) = 2g(\gamma'_{\exp_p^{-1}(\alpha(t))}(1), \gamma'_{\exp_p^{-1}(\alpha(t))}(1)) \\ &= 2g(\gamma'_{\exp_p^{-1}(\alpha(t))}(0), \gamma'_{\exp_p^{-1}(\alpha(t))}(0)) = 2g(\exp_p^{-1}(\alpha(t)), \exp_p^{-1}(\alpha(t))) = 2D_p^2(\alpha(t)),\end{aligned}$$

where we used $P(p, \alpha(t)) = \gamma'_{\exp_p^{-1}(\alpha(t))}(1)$ and (5.10). Obviously, we obtain

$$D_p^2(\alpha(t)) = D_p^2(\alpha(0)) \exp(2t).$$

Since $\exp^{-1} N$ is star-shaped the $-t$ -time flow maps $(D_p^2)^{-1}(s)$ to $(D_p^2)^{-1}(se^{-2t})$ for $t > 0$. Since P is Lipschitz, by the mentioned result, the flow map is Lipschitz. Furthermore, since it is injective and invertible it is actually a bi-Lipschitz homeomorphism between $(D_p^2)^{-1}(s)$ and its image on $(D_p^2)^{-1}(se^{-2t})$, by Leach's theorem. In order to prove the last part of the theorem, we suppose that $\exp_p$ is differentiable at $v \in T_p M \setminus \{0\}$. Observe that $\gamma'_v(1) = T_x \exp_p(v_x)$ and let $w_x \in T_p M$. Define $v(s) := v_x + sw_x$ and $\sigma(s) := \exp_p(v_x + sw_x)$, such that $\sigma'(s) = T_x \exp_p(w_x)$. We have

$$D_p^2(\sigma(s)) = 2L(p, \exp_p^{-1}(\sigma(s))) = 2L(p, v + sw),$$

by the choice of $\sigma(s)$. Thus again by the fact that $\frac{\partial}{\partial v} L = g(v, \cdot)$ we obtain

$$\frac{1}{2} \frac{dD_p^2(\sigma(s))}{ds} \Big|_{s=0} = g_p(v_x, w_x),$$

while by the computations from the first part of the proof we have

$$\frac{1}{2} \frac{dD_p^2(\sigma(s))}{ds} \Big|_{s=0} = g_{\exp_p(x)}(T_x \exp_p(v_x), T_x \exp_p(w_x)).$$

So these two expressions coincide and the theorem is proved [18]. $\square$

For further informations, we refer to [18].

References

- [1] Amann, H. *Ordinary Differential Equations, An introduction to nonlinear analysis*. Berlin, New York, Walter de Gruyter, 1990.
- [2] Andrews, B., Hopper, C. *The Ricci Flow in Riemannian Geometry*. Berlin Heidelberg, Springer-Verlag, 2011.
- [3] Beem, J.K., Ehrlich, P.E., Easley, K.L. *Global Lorentzian Geometry*. New York, Dekker, 1996.
- [4] Bolza, O. *Lectures on the calculus of variations*. New York, N.Y., Chelsea Publishing Company, 1904.
- [5] Chrusciel, P.T. *Elements of causality theory*. arXiv:1110.6706., 2011.
- [6] Clarke, F.H. *On the inverse function theorem*. Pacific J. Math. 64 (1976), no.1, 97-102.
- [7] García-Parrado, A., Senovilla, J. M. M. *Causal structures and causal boundaries*. Classical Quantum Gravity 22 (2005), no. 9, R1-R84.
- [8] Griffiths, D.J., Podolský, J. *Exact Space-Times in Einstein's General Relativity*. Cambridge University Press, 2009.
- [9] Hartman, P., Wintner, A. *On the problems of geodesics in the small*. Amer. J. Math. 73 (1951), 132-148.
- [10] Hawking, S.W., Ellis, G.F.R. *The large scale structure of space-time*. Cambridge University Press, 1973.
- [11] Kriele, M. *Spacetime. Foundations of general relativity and differential geometry. Lecture Notes in Physics 59*. Springer, 1999.
- [12] Kunzinger, M. *Differential Geometry 1*. University of Vienna, 2008.
- [13] Kunzinger, M. *Differentialgeometrie 2*. Universität Wien, 2008.
- [14] Kunzinger, M. *Differentialgleichungen für LehramtskandidatInnen*. Universität Wien, 2012.
- [15] Kunzinger, M., Steinbauer, R. *Riemannian Geometry*. University of Vienna, 2017.
- [16] Kunzinger, M., Steinbauer, R., Stojković, M., Vickers, J.A. *A regularisation approach to causality theory for $C^{1,1}$ -Lorentzian metrics*. Gen. Relativ. Gravit. (2014) 46:1738.

- [17] Leach, E.B. *A note on inverse function theorems*. Proc. Amer. Math. Soc. 12 (1961), 694-697.
- [18] Minguzzi, E. *Convex neighborhoods for Lipschitz connections and sprays*. Monatsh. Math. 177 (2015), no. 4, 569-625.
- [19] Minguzzi, E., Sánchez, M. *The causal hierarchy of spacetimes*. Recent developments in pseudo-Riemannian geometry, 299-358, ESI Lect. Math. Phys., Eur. Math. Soc. Publ. House, Zürich, 2008.
- [20] O'Neill, B. *Semi-Riemannian Geometry with applications to relativity*. San Diego, New York, Boston, London, Sydney, Tokyo, Toronto, Academic Press, 1983.
- [21] Penrose, R. *Techniques of Differential Topology in Relativity*. Philadelphia, PA: SIAM, 1972.
- [22] Senovilla, J. M. M. *Singularity theorems and their consequences*. Gen. Relativity Gravitation 30 (1998), no 5, 701-848.
- [23] Sämann, C., Steinbauer, R. *On geodesics in low regularity*. J.Phys. Conf. Ser. 968 (2018), 012010, 14 pp.
- [24] Yat-Sen, W. *Lorentz Geometry and Its Applications*. The Chinese University of Hong Kong, 2009.