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„Neutrino Masses in a Two-Higgs-Doublet Model with Classical Scale Invariance“

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Manuel Fink BSc BSc

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Abstract

This thesis deals with a conformal multi-Higgs-doublet model, in which a particular focus is put on neutrino masses. To this end we extensively discuss the theory of spontaneous symmetry breaking in models with classical scale-invariance. In particular, we present an explicit calculation of the one-loop effective potential in a general gauge theory. Furthermore, we explain the Coleman-Weinberg mechanism and the Gildener-Weinberg approach. The model under scrutiny comprises right-handed neutrino singlets, and it turns out that it is sufficient to add a single scalar singlet in order to generate a conformal symmetry-breaking Majorana mass term for them. We show that naturally light neutrino masses can be generated by combining the seesaw mechanism with radiative mass production. Finally, we analyze the simplest version of this model, which includes only two Higgs doublets and one right-handed neutrino singlet. In this model, one of the neutral scalars has almost Standard Model-like couplings, which suggests an identification with the H^0 particle discovered at 125 GeV at the LHC. Another attractive property of the model is that flavor-changing neutral interactions are "naturally" suppressed, because all other scalars have masses which are at least in the TeV region.

Zusammenfassung

In dieser Arbeit wird ein konformes Multi-Higgsdublett-Modell behandelt, wobei ein besonderer Fokus auf Neutrinomassen gelegt wird. Dafür wird die Theorie der spontanen Symmetriebrechung in Modellen mit klassischer Skaleninvarianz ausführlich besprochen. Insbesondere wird eine explizite Berechnung des effektiven Potentials auf Ein-Schleifen-Niveau präsentiert. Außerdem wird der Coleman-Weinberg-Mechanismus und die Gildener-Weinberg Methode erklärt. Das betrachtete Modell beinhaltet rechtshändige Neutrinosingletts, und es stellt sich heraus, dass es ausreicht ein einziges skalares Singlett einzführen, um einen, die konforme Symmetrie brechenden, Majorana-Massenterm für diese zu generieren. Es wird gezeigt, dass sich durch die Kombination des Seesaw-Mechanismus mit radiativer Massenerzeugung auf natürliche Weise kleine Neutrinomassen ergeben. Zum Schluss wird die einfachste Version des Modells analysiert, welche nur zwei Higgsdubletts und ein rechtshändiges Neutrinosinglett enthält. In diesem Modell hat eines der neutralen Skalarteilchen beinahe die gleichen Kopplungen wie das Higgs-Teilchen im Standard-Modell und es erscheint daher naheliegend es mit dem H^0-Teilchen zu identifizieren, welches mit einer Masse von 125 GeV am LHC entdeckt wurde. Eine weitere attraktive Eigenschaft des Modells besteht darin, dass Flavor-ändernde neutrale Wechselwirkungen auf "natürliche" Weise unterdrückt sind, weil alle anderen Skalarteilchen Massen besitzen die zumindest im TeV-Bereich liegen.

Contents

1. Introduction

Although the Standard Model (SM) of particle physics has been enormously successful in providing predictions that are in agreement with experiments, there are still some shortcomings and open questions. For example, it misses to provide a suitable dark matter particle, it fails to fully explain the matter-antimatter asymmetry problem, and it lacks an explanation for the tininess of neutrino masses. Furthermore, the scalar potential of the SM contains an explicit mass parameter, which leads to the so-called hierarchy problem. The hierarchy problem describes the issue that an immense amount of fine-tuning is necessary in order to cancel the quadratic radiative corrections to the bare mass parameter.

By imposing classical scale invariance, only terms with operator dimension four are allowed in the Lagrangian density, and thus the explicit mass term of the Higgs doublet is forbidden. Although this term is necessary for spontaneous symmetry breaking (SSB) at tree-level, a mechanism discovered by Coleman and Weinberg [1] allows to break gauge symmetry as well as scale invariance by radiative corrections. However, as the analysis by Gildener and Weinberg [2] reveals, additional scalar or gauge fields need to be introduced to obtain a stable classically scale-invariant version of the SM.

In order to explain small neutrino masses, Grimus and Neufeld [3] proposed a multi-Higgs-doublet model, which includes an arbitrary number of right-handed neutrino singlets, that combines the seesaw mechanism with radiative mass generation. In this thesis we build a classically scale-invariant version of this model, and, as it turns out, this can be achieved by introducing an additional scalar singlet, which allows to generate a Majorana mass term for the right-handed neutrinos via SSB. The simplest version of this model comprises two Higgs doublets and only one right-handed neutrino singlet.

This thesis is structured as follows. In chapter 2 we focus on the theoretical framework related to the process of SSB in theories with classical scale-invariance. For this purpose, we introduce the effective potential and explicitly show a calculation of the one-loop effective potential in a general gauge theory. Furthermore, we discuss the Coleman-Weinberg mechanism [1] and the Gildener-Weinberg approach [2]. In chapter 3 we build a scale-invariant version of the multi-Higgs doublet model proposed by Grimus and Neufeld [3], where we essentially follow the discussion of [4]. Thereafter, we focus on the simplest version of this model and close with a phenomenological analysis.

2. Theoretical framework

In this chapter we explain the theoretical framework which is needed to understand SSB in theories with classical scale-invariance. The tools we develop will be used in the next chapter to build and analyze classically scale-invariant models.

2.1. Functional methods

In this section we define some elementary quantities, like the generating functional, the generating functional of connected correlation functions, the effective action and the effective potential, which will be used in the next sections. To keep things simple, we consider only a theory with a single scalar field, but the concepts can easily be generalized to more complicated theories. The following discussion can also be found in most of the textbooks on quantum field theory (e.g. [5–7]).

If the theory we consider is characterized by a Lagrangian density $\mathcal{L}$, the n-point correlation function can be expressed in terms of functional integrals (see e.g. [8]):[1,2]

$$\langle 0| \, T\phi_{\mathrm{H}}(x_1)\ldots\phi_{\mathrm{H}}(x_n)\,|0\rangle = \lim_{T\to\infty(1-\mathrm{i}0^+)} \frac{\int[\mathrm{d}\phi]\phi(x_1)\ldots\phi(x_n)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{-T}^{T}\mathrm{d}^d x\,\mathcal{L}\right]}{\int[\mathrm{d}\phi]\exp\left[\frac{\mathrm{i}}{\hbar}\int_{-T}^{T}\mathrm{d}^d x\,\mathcal{L}\right]}\,. \tag{2.1}$$

In the presence of an external source $J(x)$, which couples to the field $\phi(x)$, the n-point correlation function is given by

$$\langle 0| \, T\phi_{\mathrm{H}}(x_1)\ldots\phi_{\mathrm{H}}(x_n)\,|0\rangle_J = \frac{\int[\mathrm{d}\phi]\phi(x_1)\ldots\phi(x_n)\exp\left[\frac{\mathrm{i}}{\hbar}\int\mathrm{d}^d x\,[\mathcal{L}+J(x)\phi(x)]\right]}{\int[\mathrm{d}\phi]\exp\left[\frac{\mathrm{i}}{\hbar}\int\mathrm{d}^d x\,[\mathcal{L}+J(x)\phi(x)]\right]}\,. \tag{2.2}$$

Let us now define the generating functional

$$Z[J] = \frac{1}{\mathcal{N}}\int[\mathrm{d}\phi]\exp\left[\frac{\mathrm{i}}{\hbar}\int\mathrm{d}^d x\,[\mathcal{L}+J(x)\phi(x)]\right], \tag{2.3}$$

with a normalization

$$\mathcal{N} = \int[\mathrm{d}\phi]\exp\left[\frac{\mathrm{i}}{\hbar}\int\mathrm{d}^d x\,\mathcal{L}\right], \tag{2.4}$$

[1] Hereafter, the $\mathrm{i}0^+$-terms are omitted, but they will be reconsidered once they become relevant.

[2] Note that $\phi(x)$ is an ordinary function, whereas $\phi_{\mathrm{H}}(x)$ is an operator.

which is chosen such that $Z[0] = 1$. With this definition the n-point correlation function in the presence of the external source can be generated by taking functional derivatives of the generating functional:

$$\langle 0| T\phi_{\mathrm{H}}(x_1)\dots\phi_{\mathrm{H}}(x_n)|0\rangle_J = \frac{(-\mathrm{i}\hbar)^n}{Z[J]}\frac{\delta^n}{\delta J(x_1)\dots\delta J(x_n)}Z[J]\,. \tag{2.5}$$

However, even in the absence of the external field, the n-point correlation function can be obtained from $Z[J]$:

$$\langle 0| T\phi_{\mathrm{H}}(x_1)\dots\phi_{\mathrm{H}}(x_n)|0\rangle = (-\mathrm{i}\hbar)^n\frac{\delta^n}{\delta J(x_1)\dots\delta J(x_n)}Z[J]\bigg|_{J=0}. \tag{2.6}$$

In terms of Feynman diagrams, the n-point correlation function corresponds to the sum of all connected and disconnected diagrams (without vacuum bubbles) with n external legs. If we are only interested in connected diagrams, it is useful to introduce the generating functional of connected correlation functions

$$W[J] = -\mathrm{i}\hbar\ln Z[J]\,. \tag{2.7}$$

The connected diagrams in the presence of the external field can then be obtained by taking derivatives of $W[J]$:

$$\langle 0| T\phi_{\mathrm{H}}(x_1)\dots\phi_{\mathrm{H}}(x_n)|0\rangle_J^{\mathrm{conn.}} = (-\mathrm{i}\hbar)^{n-1}\frac{\delta^n}{\delta J(x_1)\dots\delta J(x_n)}W[J]\,, \tag{2.8}$$

and in the absence of the external source we have

$$\langle 0| T\phi_{\mathrm{H}}(x_1)\dots\phi_{\mathrm{H}}(x_n)|0\rangle^{\mathrm{conn.}} = (-\mathrm{i}\hbar)^{n-1}\frac{\delta^n}{\delta J(x_1)\dots\delta J(x_n)}W[J]\bigg|_{J=0}. \tag{2.9}$$

While a full proof of connectivity can be found in [5], let us here only check the formula for $n = 1, 2$:

$$\langle 0|\phi_{\mathrm{H}}(x)|0\rangle_J^{\mathrm{conn.}} = \frac{\delta}{\delta J(x)}W[J] = \frac{-\mathrm{i}\hbar}{Z[J]}\frac{\delta}{\delta J(x)}Z[J] = \langle 0|\phi_{\mathrm{H}}(x)|0\rangle_J\,, \tag{2.10}$$

$$\begin{aligned}\langle 0| T\phi_{\mathrm{H}}(x_1)\phi_{\mathrm{H}}(x_2)|0\rangle_J^{\mathrm{conn.}} &= -\mathrm{i}\hbar\frac{\delta^2}{\delta J(x_1)\,\delta J(x_2)}W[J]\\ &= \frac{(-\mathrm{i}\hbar)^2}{Z[J]}\frac{\delta^2 Z[J]}{\delta J(x_1)\,\delta J(x_2)} - \frac{(-\mathrm{i}\hbar)^2}{Z[J]^2}\frac{\delta Z[J]}{\delta J(x_1)}\frac{\delta Z[J]}{\delta J(x_2)}\\ &= \langle 0| T\phi_{\mathrm{H}}(x_1)\phi_{\mathrm{H}}(x_2)|0\rangle_J - \langle 0|\phi_{\mathrm{H}}(x_1)|0\rangle_J\langle 0|\phi_{\mathrm{H}}(x_2)|0\rangle_J\,.\end{aligned} \tag{2.11}$$

The first equation is true because there are obviously no disconnected one-point correlation functions. In the second equation we see that all disconnected diagrams are subtracted from the full two-point correlation function, leaving only connected diagrams.

Besides connected diagrams, also one particle irreducible (1PI) diagrams are of particular interest. Note that the definition of one particle irreducibility that we are using here might differ from the common one. While usually by 1PI diagrams, one means diagrams that cannot be split into two parts by removing one internal line (see e.g. [8]), we refine the definition to diagrams that cannot be split into two parts, each of which is connected to at least one external point, by removing one internal line. This means that we also call diagrams with tadpole subdiagrams 1PI. If we are considering diagrams without tadpole subdiagrams, we will explicitly label them tadpole free (TF).

The effective action (or generating functional of 1PI diagrams) is defined by a functional Legendre transform of $W[J]$:

$$\Gamma[\varphi] = W[J] - \int \mathrm{d}^d x\, J(x)\varphi(x)\,, \tag{2.12}$$

where

$$\varphi(x) = \frac{\delta}{\delta J(x)} W[J]\,. \tag{2.13}$$

The inverse Legendre transform is given by

$$W[J] = \Gamma[\varphi] + \int \mathrm{d}^d x\, J(x)\varphi(x)\,, \tag{2.14}$$

where

$$J(x) = -\frac{\delta}{\delta\varphi(x)}\Gamma[\varphi]\,. \tag{2.15}$$

This allows us to obtain the connected 1PI n-point correlation function in the presence of an external field by taking derivatives of $\Gamma[\varphi]$:

$$\langle 0|\, T\phi_{\mathrm{H}}(x_1)\ldots\phi_{\mathrm{H}}(x_n)\,|0\rangle_J^{\mathrm{1PI}} = (-\mathrm{i}\hbar)^{-1}\frac{\delta^n}{\delta\varphi(x_1)\ldots\delta\varphi(x_n)}\Gamma[\varphi] \quad \text{for } n \geq 3\,. \tag{2.16}$$

Before checking this formula for $n = 3$, let us first see what happens if we take $n = 2$:

$$\begin{aligned}(-\mathrm{i}\hbar)^{-1}\frac{\delta^2}{\delta\varphi(x_1)\,\delta\varphi(x_2)}\Gamma[\varphi] &= (\mathrm{i}\hbar)^{-1}\frac{\delta J(x_2)}{\delta\varphi(x_1)} = \left(\mathrm{i}\hbar\frac{\delta\varphi(x_1)}{\delta J(x_2)}\right)^{-1} \\ &= -\left(-\mathrm{i}\hbar\frac{\delta^2}{\delta J(x_1)\,\delta J(x_2)}W[J]\right)^{-1} = -D_J^{-1}(x_1,x_2)\,,\end{aligned} \tag{2.17}$$

where $D_J(x_1,x_2) = \langle 0|\, T\phi_{\mathrm{H}}(x_1)\phi_{\mathrm{H}}(x_2)\,|0\rangle_J^{\mathrm{conn.}}$ is the propagator in the presence of the field $J(x)$. This shows that for $n = 2$ we do not get the 1PI correlation function, but the negative of the inverse propagator. For $n = 3$ we can use (2.17), the functional chain rule

$$\frac{\delta}{\delta\varphi(x)} = \int \mathrm{d}^d y\, \frac{\delta J(y)}{\delta\varphi(x)}\frac{\delta}{\delta J(y)} = -\mathrm{i}\hbar\int \mathrm{d}^d y\, D_J^{-1}(x,y)\frac{\delta}{\delta J(y)}\,, \tag{2.18}$$

and the functional version of the following rule for differentiating matrix inverses

$$\frac{\partial}{\partial \alpha} M^{-1} = -M^{-1} \frac{\partial M}{\partial \alpha} M^{-1} \,, \tag{2.19}$$

such that we have

$$\begin{aligned}
(-\mathrm{i}\hbar)^{-1} \frac{\delta^3}{\delta\varphi(x_1)\,\delta\varphi(x_2)\,\delta\varphi(x_3)} \Gamma[\varphi] &= -\int \mathrm{d}^d y\, D_J^{-1}(x_1, y) \frac{\delta}{\delta J(y)} \left(\frac{\delta^2 W[J]}{\delta J(x_2)\,\delta J(x_3)} \right)^{-1} \\
&= \int \mathrm{d}^d y\, \mathrm{d}^d z\, \mathrm{d}^d w\, D_J^{-1}(x_1, y) D_J^{-1}(x_2, z) D_J^{-1}(x_3, w) \left\langle 0 \right| T \phi_{\mathrm{H}}(y) \phi_{\mathrm{H}}(z) \phi_{\mathrm{H}}(w) \left| 0 \right\rangle_J^{\text{conn.}} \\
&= \left\langle 0 \right| T \phi_{\mathrm{H}}(x_1) \phi_{\mathrm{H}}(x_2) \phi_{\mathrm{H}}(x_3) \left| 0 \right\rangle_J^{(\text{conn., amp.})} .
\end{aligned} \tag{2.20}$$

This proves (2.16) for $n = 3$ since the connected amputated 3-point correlation function is the same as the 1PI 3-point correlation function.

Of course, we are also interested in the case where the external field vanishes. First note that in the limit $J(x) \to 0$ we have

$$\varphi(x)\bigg|_{J=0} = \frac{\delta}{\delta J(x)} W[J]\bigg|_{J=0} = \left\langle 0 \right| \phi_{\mathrm{H}}(x) \left| 0 \right\rangle . \tag{2.21}$$

Using (2.15), we can therefore find solutions for the vacuum expectation value (VEV) $\langle \phi_{\mathrm{H}}(x) \rangle$ by solving

$$\frac{\delta}{\delta\varphi(x)} \Gamma[\varphi]\bigg|_{\varphi(x) = \langle \phi_{\mathrm{H}}(x) \rangle} = 0 \,. \tag{2.22}$$

For a constant field $\varphi(x) = \varphi_0$ the effective action is an ordinary function and is proportional to the volume of spacetime $\mathcal{V}_d = \int \mathrm{d}^d x = (2\pi)^d \delta^{(d)}(k - k)$:

$$\Gamma[\varphi_0] = -\mathcal{V}_d V(\varphi_0) \,. \tag{2.23}$$

The function $V(\varphi_0)$ is called effective potential. If we assume that the vacuum state does not break Lorentz invariance, and therefore is translationally invariant, equation (2.22) reads

$$\frac{\mathrm{d}}{\mathrm{d}\varphi} V(\varphi_0)\bigg|_{\varphi_0 = \langle \phi_{\mathrm{H}} \rangle} = 0 \,. \tag{2.24}$$

It can be shown that the function $V(\varphi_0)$ corresponds to the expectation value of the energy density [9], and thus the physical VEV in a stable vacuum is really a minimum of the effective potential, rather than only a stationary point as suggested by (2.24). If we now take the limit $J(x) \to 0$ in (2.16) and (2.17), we get

$$\left\langle 0 \right| T \phi_{\mathrm{H}}(x_1) \ldots \phi_{\mathrm{H}}(x_n) \left| 0 \right\rangle^{\text{1PI}} = (-\mathrm{i}\hbar)^{-1} \frac{\delta^n}{\delta\varphi(x_1) \ldots \delta\varphi(x_n)} \Gamma[\varphi]\bigg|_{\varphi = \langle \phi_{\mathrm{H}} \rangle} \quad \text{for } n \geq 3 \,, \tag{2.25}$$

and

$$D^{-1}(x_1, x_2) = (\mathrm{i}\hbar)^{-1} \frac{\delta^2}{\delta\varphi(x_1)\,\delta\varphi(x_2)} \Gamma[\varphi] \bigg|_{\varphi = \langle\phi_\mathrm{H}\rangle}, \tag{2.26}$$

where $D(x_1, x_2) = \langle 0 |\, T\phi_\mathrm{H}(x_1)\phi_\mathrm{H}(x_2)\, | 0 \rangle^{\text{conn.}}$ is the propagator in the absence of the external field.

2.2. Calculation of the one-loop effective potential

Since it is in general not possible to calculate the full effective potential, one usually performs a loop expansion, which corresponds to an expansion in powers of $\hbar$. In this section we will calculate the effective potential at one-loop order. First, we will take a simple ϕ^4-theory as an example to show three different methods of calculating the effective potential, thereafter, we will use one of these methods to calculate the effective potential of a general gauge theory.

2.2.1. The one-loop effective potential of a simple ϕ^4-theory

The aim of this section is to calculate the one-loop effective potential of a theory with the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \Lambda^{4-d}\frac{\lambda}{4!}\phi^4, \tag{2.27}$$

where Λ is an arbitrary parameter of dimension length^{-1}, which was introduced to keep the dimension of the coupling constant (length$^{-1} \times$ energy^{-1}) fixed while using dimensional regularization.

2.2.1.1. Diagrammatic approach

Calculating the VEV directly gives $\langle\phi_\mathrm{H}\rangle = 0$, since it is not possible to write down a Feynman diagram with only one external leg at any loop order in our theory. Therefore, the effective potential has a stationary point at the origin, but we do not know if it is a minimum or a maximum. This should be, however, sufficient to justify performing a functional Taylor expansion of the effective action around $\varphi = \langle\phi_\mathrm{H}\rangle = 0$:

$$\Gamma[\varphi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}^d x_1 \ldots \mathrm{d}^d x_n \left[\frac{\delta}{\delta\varphi(x_1)} \cdots \frac{\delta}{\delta\varphi(x_n)} \Gamma[\varphi] \right] \bigg|_{\varphi=\langle\phi_\mathrm{H}\rangle} \varphi(x_1)\ldots\varphi(x_n)\,. \tag{2.28}$$

The $n = 1$ term vanishes due to the stationarity condition (2.22) and for the $n = 0$ term we have

$$\Gamma[\langle\phi_\mathrm{H}\rangle] = \Gamma[\varphi]\bigg|_{J=0} = W[0] = 0\,. \tag{2.29}$$

Hence, using (2.25) and (2.26) we get

$$\Gamma[\varphi] = \frac{\mathrm{i}\hbar}{2}\int \mathrm{d}^d x_1\, \mathrm{d}^d x_2\, D^{-1}(x_1,x_2)\varphi(x_1)\varphi(x_2) \\ - \sum_{n=3}^{\infty}\frac{\mathrm{i}\hbar}{n!}\int \mathrm{d}^d x_1 \ldots \mathrm{d}^d x_n \, \langle 0|\, T\phi_{\mathrm{H}}(x_1)\ldots\phi_{\mathrm{H}}(x_n)\,|0\rangle^{\mathrm{1PI}}\, \varphi(x_1)\ldots\varphi(x_n)\,. \quad (2.30)$$

The 1PI n-point correlation function in k-space $\tilde{\Gamma}^{(n)}(k_1,\ldots,k_n)$ is defined by

$$(2\pi)^d\delta^{(d)}(k_1+\cdots+k_n)\tilde{\Gamma}^{(n)}(k_1,\ldots,k_n) = \\ \int \mathrm{d}^d x_1 \ldots \mathrm{d}^d x_n \, \mathrm{e}^{\mathrm{i}k_1x_1}\ldots\mathrm{e}^{\mathrm{i}k_nx_n}\,\langle 0|\, T\phi_{\mathrm{H}}(x_1)\ldots\phi_{\mathrm{H}}(x_n)\,|0\rangle^{\mathrm{1PI}}\,, \quad (2.31)$$

and the inverse propagator in k-space is given by

$$(2\pi)^d\delta^{(d)}(k_1+k_2)\tilde{D}^{-1}(k_1,k_2) = \int \mathrm{d}^d x_1\, \mathrm{d}^d x_2\, \mathrm{e}^{\mathrm{i}k_1x_1}\mathrm{e}^{\mathrm{i}k_2x_2} D^{-1}(x_1,x_2)\,, \quad (2.32)$$

with

$$\tilde{D}^{-1}(k,-k) = (\mathrm{i}\hbar)^{-1}\left(k^2 - \Sigma(k^2) + \mathrm{i}0^+\right) = \tilde{\Gamma}^{(2)}(k,-k)\,, \quad (2.33)$$

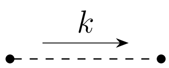
where $(\mathrm{i}\hbar)^{-1}\Sigma(k^2)$ is the self-energy.[3] Evaluating the effective action (2.30) at a constant field $\varphi(x) = \varphi_0$, we can use (2.31), (2.32) and (2.33) with $k_i = 0$ and get

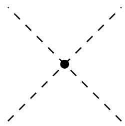
$$\Gamma[\varphi_0] = -(2\pi)^d\delta^{(d)}(0)\sum_{n=2}^{\infty}\frac{\mathrm{i}\hbar}{n!}\varphi_0^n\tilde{\Gamma}^{(n)}(0,\ldots,0)\,, \quad (2.34)$$

and thus, the effective potential reads

$$V(\varphi_0) = \sum_{n=2}^{\infty}\frac{\mathrm{i}\hbar}{n!}\varphi_0^n\tilde{\Gamma}^{(n)}(0,\ldots,0)\,. \quad (2.35)$$

With this formula we are in principle capable of calculating the effective potential at an arbitrary loop-order. We therefore have to draw all 1PI Feynman diagrams up to the desired loop-order with vanishing external momenta and calculate them using the Feynman rules

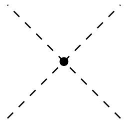


$= \dfrac{\mathrm{i}\hbar}{k^2+\mathrm{i}0^+}$ and  $= -\Lambda^{4-d}\dfrac{\mathrm{i}\lambda}{\hbar}\,.$ (2.36)

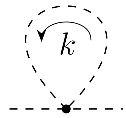
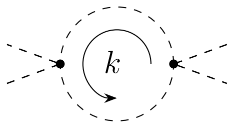
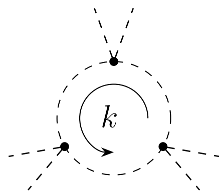
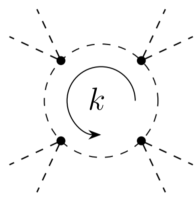
This means of course that in general we need to sum an infinite number of diagrams, but it turns out that, at least up to one-loop order, the sum can be carried out quite easily.

[3] Note that we do not have a mass term in our theory.

At tree-level there is just a single diagram and we thus have

$$V^{(\text{tree})}(\varphi_0) =$$

$$\times \frac{\mathrm{i}\hbar}{4!}\varphi_0^4 = \Lambda^{4-d}\frac{\lambda}{4!}\varphi_0^4 \,. \tag{2.37}$$

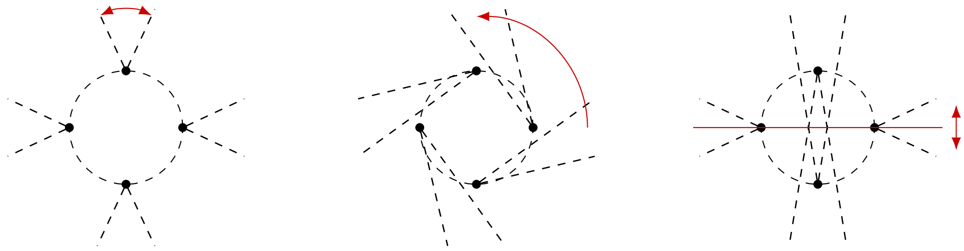
At one-loop order we get the contribution

$$V^{(1-\text{loop})}(\varphi_0) =$$

$$\times \frac{\mathrm{i}\hbar}{2}\varphi_0^2 +$$

$$\times \frac{\mathrm{i}\hbar}{8}\varphi_0^4$$
$$+$$

$$\times \frac{\mathrm{i}\hbar}{48}\varphi_0^6 +$$

$$\times \frac{\mathrm{i}\hbar}{128}\varphi_0^8 + \dots \tag{2.38}$$
$$= \mathrm{i}\hbar \int \frac{\mathrm{d}^d k}{(2\pi)^d} \sum_{m=1}^{\infty} \frac{1}{2m} \left(\frac{\Lambda^{4-d}\frac{\lambda}{2}\varphi_0^2}{k^2 + \mathrm{i}0^+} \right)^m .$$

The numerical factors can be explained in the following way (see [1]). Each 1PI one-loop diagram with n external legs has $m = n/2$ internal lines and m vertices. Because all external momenta vanish, all the diagrams with the same number of external legs give the same contribution. Therefore, the numerical factor is determined by the number of diagrams for a given number of external legs. There are in principle $n!$ ways to connect n external legs to n external points. However, exchanging the two legs which are connected to the same vertex reproduces the same diagram (see figure 2.1a), and thus we have to divide by a factor of 2^m to avoid double counting. Furthermore, simultaneously rotating all external lines by two external points, or reflecting them across a line connecting one vertex and the center of the diagram also reproduces the same diagram (see figure 2.1b and figure 2.1c). Therefore, we have to divide by another factor of $2m$. If we also consider the $1/n!$ in (2.35), we have in total a factor of $1/(2^{m+1}m)$ for each diagram. Note that this analysis does actually only hold for $m \geq 3$, since for $m = 1, 2$ the reflection is already included in the interchange of the legs connected to the same vertex. For these two diagrams we get the missing factor of $1/2$ from the symmetry factor according to the Feynman rules.

After performing a Wick-rotation, one finds that the infinite sum in (2.38) is the series representation of the logarithm:

$$\begin{aligned} V^{(1-\text{loop})}(\varphi_0) &= -\hbar \int \frac{\mathrm{d}^d k_\mathrm{E}}{(2\pi)^d} \sum_{m=1}^{\infty} \frac{1}{2m} \left(-\frac{\Lambda^{4-d}\lambda\varphi_0^2}{2k_\mathrm{E}^2} \right)^m \\ &= \frac{\hbar}{2} \int \frac{\mathrm{d}^d k_\mathrm{E}}{(2\pi)^d} \ln\!\left(1 + \frac{\Lambda^{4-d}\lambda\varphi_0^2}{2k_\mathrm{E}^2} \right), \end{aligned} \tag{2.39}$$



(a) (b) (c)

Figure 2.1.: Examples for rearrangements of external legs which reproduce the same diagram

and using (B.16) for the integration, we hence obtain

$$V^{(1-\text{loop})}(\varphi_0) = \frac{\hbar}{d}\left(\frac{\Lambda^{4-d}\lambda}{8\pi}\right)^{d/2}\Gamma\left(1-\frac{d}{2}\right)\varphi_0^d\,. \tag{2.40}$$

Adding up the tree-level and the one-loop contribution, the effective potential reads

$$V(\varphi_0) = \frac{\Lambda^{4-d}\lambda}{4!}\varphi_0^4 + \frac{\hbar}{d}\left(\frac{\Lambda^{4-d}\lambda}{8\pi}\right)^{d/2}\Gamma\left(1-\frac{d}{2}\right)\varphi_0^d + \mathcal{O}\left(\hbar^2\right). \tag{2.41}$$

2.2.1.2. Tadpole approach

As we already have discussed in the last section, calculating the VEV directly in a diagrammatic way, by summing up all tadpole diagrams, only yields one single solution and does not tell us if the solution is a minimum or a maximum of the effective potential. However, there is a procedure that allows us to find all stationary points and to calculate the effective potential, which has been first described by Lee and Sciaccaluga [10] and is further explained in [11]. We start by shifting the field by a constant value $\phi'(x) = \phi(x) - v$. Then, regarding $\langle 0|\phi'_{\mathrm{H}}(x)|0\rangle$ as a function of v, the stationary points are given by the zeroes of $\langle 0|\phi'_{\mathrm{H}}(x)|0\rangle$, since

$$\langle 0|\phi'_{\mathrm{H}}(x)|0\rangle = 0 \quad \Leftrightarrow \quad v = \langle 0|\phi_{\mathrm{H}}(x)|0\rangle\ . \tag{2.42}$$

Introducing a new generating functional

$$Z'[J'] = \frac{1}{\mathcal{N}}\int[\mathrm{d}\phi]\exp\left[\frac{\mathrm{i}}{\hbar}\int \mathrm{d}^d x\,[\mathcal{L}(\phi(x)) + J'(x)(\phi(x)-v)]\right] \tag{2.43}$$

$$= \frac{1}{\mathcal{N}}\int[\mathrm{d}\phi]\exp\left[\frac{\mathrm{i}}{\hbar}\int \mathrm{d}^d x\,[\mathcal{L}(\phi(x)+v) + J'(x)\phi(x)]\right], \tag{2.44}$$

with the normalization $\mathcal{N} = \int[\mathrm{d}\phi]\exp\left[\frac{\mathrm{i}}{\hbar}\int \mathrm{d}^d x\,\mathcal{L}(\phi(x))\right]$, enables us to calculate the VEV of the shifted field:

$$\langle 0|\phi'_{\mathrm{H}}(x)|0\rangle = \frac{\int[\mathrm{d}\phi]\phi'(x)\exp\left[\frac{\mathrm{i}}{\hbar}\int \mathrm{d}^d x\,\mathcal{L}\right]}{\int[\mathrm{d}\phi]\exp\left[\frac{\mathrm{i}}{\hbar}\int \mathrm{d}^d x\,\mathcal{L}\right]} = -\mathrm{i}\hbar\frac{\delta Z'[J']}{\delta J'(x)}\bigg|_{J'=0}. \tag{2.45}$$

This reveals that $\langle \phi'_{\mathrm{H}} \rangle$ is given by the sum of all tadpole diagrams, where the Feynman rules are determined by the Lagrangian density $\mathcal{L}'(\phi) = \mathcal{L}(\phi + v)$ instead of $\mathcal{L}(\phi)$. The new generating functional of connected correlation functions is defined as

$$W'[J'] = -\mathrm{i}\hbar \ln Z'[J'] \,, \tag{2.46}$$

and from equation (2.44) we see that we have

$$W'[J'] = W[J'] - \int \mathrm{d}^d x \, J'(x) v \,, \tag{2.47}$$

where $W[J]$ is the original generating functional of connected correlations functions. Furthermore, we can define the new effective action

$$\Gamma'[\varphi'] = W'[J'] - \int \mathrm{d}^d x \, J'(x)\varphi'(x) \,, \tag{2.48}$$

with

$$\varphi'(x) = \frac{\delta}{\delta J'(x)} W'[J'] \,, \tag{2.49}$$

and using (2.47) we get

$$\Gamma'[\varphi'] = W[J'] - \int \mathrm{d}^d x \, J'(x) \left(\varphi'(x) + v\right) = \Gamma[\varphi' + v] \,, \tag{2.50}$$

where $\Gamma[\varphi]$ is the original effective action. Using (2.8), the field $\varphi'(x)$ can be expressed in terms of connected diagrams as

$$\varphi'(x) = \, \langle 0|\phi_{\mathrm{H}}(x)|0\rangle'^{\,\mathrm{conn.}}_{J'} \,, \tag{2.51}$$

where the prime indicates that we are working with the Lagrangian density $\mathcal{L}'$.

Let us define the tadpole-free one-point function $\langle 0|\phi_{\mathrm{H}}(x)|0\rangle'^{\,\mathrm{TF}}$ as the amputated one-point function without any tadpole subdiagrams. We then see that at every place in one of the diagrams of $\langle 0|\phi_{\mathrm{H}}(x)|0\rangle'^{\,\mathrm{conn.}}_{J'}$ where we can put an external source $\frac{\mathrm{i}}{\hbar} J'(x)$, we can also put a tadpole-free one-point function. Therefore, if we choose

$$\mathrm{i}\hbar J'(x) = - \, \langle 0|\phi_{\mathrm{H}}(x)|0\rangle'^{\,\mathrm{TF}} \,, \tag{2.52}$$

we have $\varphi'(x) = 0$, and thus

$$J'(x)\Bigg|_{\varphi'=0} = \mathrm{i}\hbar \, \langle 0|\phi_{\mathrm{H}}(x)|0\rangle'^{\,\mathrm{TF}} \,. \tag{2.53}$$

In analogy to (2.31) we define the tadpole-free one-point function in k-space $\tilde{\mathcal{T}}(k)$ as

$$(2\pi)^d \delta^{(d)}(k) \tilde{\mathcal{T}}(k) = \int \mathrm{d}^d x \, \mathrm{e}^{\mathrm{i}kx} \, \langle 0|\phi_{\mathrm{H}}(x)|0\rangle'^{\,\mathrm{TF}} \,, \tag{2.54}$$

and hence we have

$$\langle 0|\phi_{\mathrm{H}}(x)|0\rangle'^{\mathrm{TF}} = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \mathrm{e}^{-\mathrm{i}kx} (2\pi)^d \delta^{(d)}(k) \tilde{\mathcal{T}}(k) = \tilde{\mathcal{T}}(0) \,. \tag{2.55}$$

Using what we have derived so far,

$$\left.\frac{\partial V}{\partial \varphi}\right|_{\varphi=v} \overset{(2.23)}{=} -\left.\frac{\delta \Gamma}{\delta \varphi}\right|_{\varphi=v} \overset{(2.50)}{=} -\left.\frac{\delta \Gamma'}{\delta \varphi'}\right|_{\varphi'=0} \overset{(2.15)}{=} \left. J' \right|_{\varphi'=0} \overset{(2.53)}{=} \mathrm{i}\hbar \, \langle 0|\phi_{\mathrm{H}}(x)|0\rangle'^{\mathrm{TF}} \overset{(2.55)}{=} -\mathrm{i}\hbar \tilde{\mathcal{T}}(0) \,, \tag{2.56}$$

shows that we can simply find the stationary points of the effective potential by looking for zeroes of $\tilde{\mathcal{T}}(0)$. We can even go one step further and actually calculate the effective potential:

$$V(\varphi_0) = -\mathrm{i}\hbar \int_0^{\varphi_0} \mathrm{d}v \, \tilde{\mathcal{T}}(0) + V(0) \,, \tag{2.57}$$

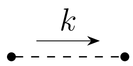
where the irrelevant constant $V(0)$ may be dropped.

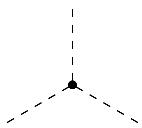
Let us now apply this approach to our simple ϕ^4-theory. From the shifted Lagrangian density

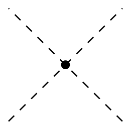
$$\mathcal{L}' = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \Lambda^{4-d} \frac{\lambda}{4!} \left(\phi^4 + 4v\phi^3 + 6v^2\phi^2 + 4v^3\phi + v^4 \right) \tag{2.58}$$

we get the k-space Feynman rules

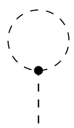
 $= -\Lambda^{4-d} \dfrac{\mathrm{i}\lambda v^3}{6\hbar}$,

 $= \dfrac{\mathrm{i}\hbar}{k^2 - \Lambda^{4-d}\lambda v^2/2 + \mathrm{i}0^+}$,

 $= -\Lambda^{4-d} \dfrac{\mathrm{i}\lambda v}{\hbar}$,

 $= -\Lambda^{4-d} \dfrac{\mathrm{i}\lambda}{\hbar}$, (2.59)

where the dot in the first diagram represents a one-point vertex but not an external point. We accordingly have

$\tilde{\mathcal{T}}(0) =$ $+$  $+ \ldots$

$$\begin{aligned} &= -\Lambda^{4-d} \frac{\mathrm{i}\lambda v^3}{6\hbar} - \Lambda^{4-d} \frac{\mathrm{i}\lambda v}{2\hbar} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{i}\hbar}{k^2 - \Lambda^{4-d}\frac{\lambda v^2}{2} + \mathrm{i}0^+} + \mathcal{O}(\hbar) \\ &= -\Lambda^{4-d} \frac{\mathrm{i}\lambda v^3}{6\hbar} - \Lambda^{4-d} \frac{\mathrm{i}\lambda v}{2} \int \frac{\mathrm{d}^d k_{\mathrm{E}}}{(2\pi)^d} \frac{1}{k_{\mathrm{E}}^2 + \Lambda^{4-d}\frac{\lambda v^2}{2}} + \mathcal{O}(\hbar) \\ &= -\Lambda^{4-d} \frac{\mathrm{i}\lambda v^3}{6\hbar} - \mathrm{i} \left(\frac{\Lambda^{4-d}\lambda}{8\pi} \right)^{d/2} \Gamma\left(1 - \frac{d}{2}\right) v^{d-1} + \mathcal{O}(\hbar) \,, \end{aligned} \tag{2.60}$$

where we performed a Wick rotation and used (B.15). If we integrate and drop the constant term, we get

$$V(\varphi_0) = -\mathrm{i}\hbar \int_0^{\varphi_0} \mathrm{d}v\, \tilde{\mathcal{T}}(0) = \frac{\Lambda^{4-d}\lambda}{4!}\varphi_0^4 + \frac{\hbar}{d}\left(\frac{\Lambda^{4-d}\lambda}{8\pi}\right)^{d/2}\Gamma\left(1-\frac{d}{2}\right)\varphi_0^d + \mathcal{O}\left(\hbar^2\right), \tag{2.61}$$

which agrees with the result that we have obtained using the diagrammatic approach (2.41).

2.2.1.3. Path integral approach

In this section we will calculate the path integral directly using the stationary phase approximation, where we basically follow [5]. In the limit $\hbar \to 0$ the generating functional is dominated by contributions of the path integral, for which the phase is stationary. Thus, we have at leading order

$$Z[J] = \exp\left[\frac{\mathrm{i}}{\hbar}\left(S[\phi_{\mathrm{cl}}] - S[\bar{\phi}_{\mathrm{cl}}] + \int \mathrm{d}^d x\, J(x)\phi_{\mathrm{cl}}(x) + \mathcal{O}(\hbar)\right)\right], \tag{2.62}$$

where $\phi_{\mathrm{cl}} = \phi_{\mathrm{cl}}[J]$ is the classical field, which is an implicit functional of $J(x)$ and is defined by

$$\left.\frac{\delta S}{\delta\phi(x)}\right|_{\phi=\phi_{\mathrm{cl}}} = -J(x)\,. \tag{2.63}$$

The $S[\bar{\phi}_{\mathrm{cl}}]$ term in the exponent stems from the normalization of the generating functional and we have defined $\bar{\phi}_{\mathrm{cl}} = \phi_{\mathrm{cl}}\big|_{J=0}$, which is constant due to Lorentz invariance. The generating functional of connected correlation functions accordingly reads

$$W[J] = W_0[J] + \mathcal{O}(\hbar)\,, \tag{2.64}$$

with

$$W_0[J] = S[\phi_{\mathrm{cl}}] - S[\bar{\phi}_{\mathrm{cl}}] + \int \mathrm{d}^d x\, J(x)\phi_{\mathrm{cl}}(x)\,. \tag{2.65}$$

To perform the Legendre transformation, we calculate

$$\varphi(x) = \frac{\delta}{\delta J(x)}W[J] = \int \mathrm{d}^d y\, \frac{\delta\phi_{\mathrm{cl}}(y)}{\delta J(x)}\left(\frac{\delta}{\delta\phi_{\mathrm{cl}}(y)}S[\phi_{\mathrm{cl}}] + J(y)\right) + \phi_{\mathrm{cl}}(x) + \mathcal{O}(\hbar)\,, \tag{2.66}$$

and using the saddle point equation (2.63) we thus obtain

$$\varphi(x) = \phi_{\mathrm{cl}}(x) + \mathcal{O}(\hbar). \tag{2.67}$$

Plugging this into the definition of the effective action (2.12), we find

$$\Gamma[\varphi] = \Gamma_0[\varphi] + \mathcal{O}(\hbar)\,, \tag{2.68}$$

with

$$\Gamma_0[\varphi] = S[\varphi] - S[\bar{\phi}_{\rm cl}]\,. \tag{2.69}$$

This immediately shows that, up to the irrelevant constant $S[\bar{\phi}_{\rm cl}]$, the effective potential at leading order is simply given by the negative of all non-derivative terms of the Lagrangian density.

To obtain the effective action at order $\hbar$, we have to expand the field around the saddle point:

$$\phi(x) = \phi_{\rm cl}(x) + \sqrt{\hbar}\phi'(x)\,. \tag{2.70}$$

Expanding the exponent of the generating functional in powers of $\hbar$,

$$\begin{aligned} S[\phi] + \int \mathrm{d}^d x\, J(x)\phi(x) = S[\phi_{\rm cl}] &+ \int \mathrm{d}^d x\, J(x)\phi_{\rm cl}(x) \\ &+ \frac{\hbar}{2}\int \mathrm{d}^d x_1\, \mathrm{d}^d x_2\, \left.\frac{\delta^2 S}{\delta\phi(x_1)\delta\phi(x_2)}\right|_{\phi=\phi_{\rm cl}} \phi'(x_1)\phi'(x_2) + \mathcal{O}\left(\hbar^{3/2}\right) \end{aligned} \tag{2.71}$$

and doing the same for the normalization constant, we get

$$W[J] = W_0[J] + \hbar W_1[J] + \mathcal{O}\left(\hbar^2\right) \tag{2.72}$$

with

$$\begin{aligned} W_1[J] = -\mathrm{i}\ln &\left[\int [\mathrm{d}\phi']\exp\left(\frac{\mathrm{i}}{2}\int \mathrm{d}^d x_1\, \mathrm{d}^d x_2\, \left.\frac{\delta^2 S}{\delta\phi(x_1)\delta\phi(x_2)}\right|_{\phi=\phi_{\rm cl}} \phi'(x_1)\phi'(x_2)\right)\right] \\ + \mathrm{i}\ln &\left[\int [\mathrm{d}\phi']\exp\left(\frac{\mathrm{i}}{2}\int \mathrm{d}^d x_1\, \mathrm{d}^d x_2\, \left.\frac{\delta^2 S}{\delta\phi(x_1)\delta\phi(x_2)}\right|_{\phi=\bar{\phi}_{\rm cl}} \phi'(x_1)\phi'(x_2)\right)\right]. \end{aligned} \tag{2.73}$$

At the order we are working at, the effective action reads

$$\Gamma[\varphi] = W_0[J] + \hbar W_1[J] - \int \mathrm{d}^d x\, J(x)\varphi(x) + \mathcal{O}\left(\hbar^2\right). \tag{2.74}$$

Since the effective action is by definition stationary with respect to variations of $J(x)$ at fixed $\varphi(x)$, we have

$$0 = \frac{\delta W_0}{\delta J(x)} - \hbar\frac{\delta W_1}{\delta J(x)} - \varphi(x) + \mathcal{O}\left(\hbar^2\right) \tag{2.75}$$

and thus

$$\varphi(x) = \phi_{\rm cl}(x) + \hbar\frac{\delta W_1}{\delta J(x)} + \mathcal{O}\left(\hbar^2\right). \tag{2.76}$$

We can use this expansion in the definition of W_0,

$$
\begin{aligned}
W_0[J] &= S[\phi_{\mathrm{cl}}] - S[\bar{\phi}_{\mathrm{cl}}] + \int \mathrm{d}^d x\, J(x)\phi_{\mathrm{cl}}(x) \\
&= S[\varphi] - \hbar \int \mathrm{d}^d x\, \frac{\delta W_1}{\delta J(x)} \underbrace{\frac{\delta S}{\delta \phi}\bigg|_{\phi(x)=\varphi(x)}}_{=-J(x)+\mathcal{O}(\hbar)} - S[\bar{\phi}_{\mathrm{cl}}] \\
&\quad + \int \mathrm{d}^d x\, J(x) \left(\varphi - \hbar \frac{\delta W_1}{\delta J(x)}\right) + \mathcal{O}\left(\hbar^2\right) \\
&= S[\varphi] - S[\bar{\phi}_{\mathrm{cl}}] + \int \mathrm{d}^d x\, J(x)\varphi(x) + \mathcal{O}\left(\hbar^2\right) \\
&= \Gamma_0[\varphi] + \mathcal{O}\left(\hbar^2\right),
\end{aligned}
\tag{2.77}
$$

and realize that, at the order we are working at, we can simply exchange ϕ_{cl} with φ in W_0 to obtain $\Gamma_0[\varphi]$. This is also true for W_1 since it is already multiplied with $\hbar$ in (2.74), and thus corrections of order $\hbar$ do not give any contribution. Therefore, we have

$$
\Gamma[\varphi] = \Gamma_0[\varphi] + \hbar \Gamma_1[\varphi] + \mathcal{O}\left(\hbar^2\right), \tag{2.78}
$$

with

$$
\begin{aligned}
\Gamma_1[\varphi] = &- \mathrm{i} \ln \left[\int [\mathrm{d}\phi'] \exp\left(\frac{\mathrm{i}}{2} \int \mathrm{d}^d x_1\, \mathrm{d}^d x_2\, \frac{\delta^2 S}{\delta\phi(x_1)\delta\phi(x_2)}\bigg|_{\phi=\varphi} \phi'(x_1)\phi'(x_2) \right) \right] \\
&+ \mathrm{i} \ln \left[\int [\mathrm{d}\phi'] \exp\left(\frac{\mathrm{i}}{2} \int \mathrm{d}^d x_1\, \mathrm{d}^d x_2\, \frac{\delta^2 S}{\delta\phi(x_1)\delta\phi(x_2)}\bigg|_{\phi=\bar{\phi}_{\mathrm{cl}}} \phi'(x_1)\phi'(x_2) \right) \right].
\end{aligned}
\tag{2.79}
$$

Let us now apply this approach to the simple ϕ^4-theory. If we drop the irrelevant constant term $S[\bar{\phi}_{\mathrm{cl}}]$, the tree-level effective potential can simply be read off from the Lagrangian density:

$$
V^{(\mathrm{tree})}(\varphi_0) = \Lambda^{4-d} \frac{\lambda}{4!} \varphi_0^4. \tag{2.80}
$$

To calculate the one-loop term we need to plug

$$
\frac{\delta^2 S}{\delta\phi(x_1)\delta\phi(x_2)} = - \int \mathrm{d}^d x\, \delta^{(d)}(x-x_1) \Box \delta^{(d)}(x-x_2) - \Lambda^{4-d} \frac{\lambda}{2} \delta^{(d)}(x_1-x_2)\phi^2(x_1) \tag{2.81}
$$

into (2.79) and get

$$
\begin{aligned}
\Gamma_1[\varphi_0] = &- \mathrm{i} \ln \left[\int [\mathrm{d}\phi'] \exp\left(\frac{\mathrm{i}}{2} \int \mathrm{d}^d x\, \phi'(x) \left(-\Box - \Lambda^{4-d} \frac{\lambda}{2} \varphi_0^2 \right) \phi'(x) \right) \right] \\
&+ \mathrm{i} \ln \left[\int [\mathrm{d}\phi'] \exp\left(\frac{\mathrm{i}}{2} \int \mathrm{d}^d x\, \phi'(x) \left(-\Box - \Lambda^{4-d} \frac{\lambda}{2} \bar{\phi}_{\mathrm{cl}}^2 \right) \phi'(x) \right) \right].
\end{aligned}
\tag{2.82}
$$

Using (B.41) and (B.42) to solve the path integral we obtain

$$\begin{aligned}\Gamma_1[\varphi_0] &= \frac{\mathrm{i}}{2}\mathcal{V}_d \int \frac{\mathrm{d}^d k}{(2\pi)^d} \ln\left(\frac{k^2}{2} - \Lambda^{4-d}\frac{\lambda}{4}\varphi_0^2\right) - \frac{\mathrm{i}}{2}\mathcal{V}_d \int \frac{\mathrm{d}^d k}{(2\pi)^d} \ln\left(\frac{k^2}{2} - \Lambda^{4-d}\frac{\lambda}{4}\bar{\phi}_{\mathrm{cl}}^2\right) \\ &= \frac{\mathrm{i}}{2}\mathcal{V}_d \int \frac{\mathrm{d}^d k}{(2\pi)^d} \ln\left(1 - \frac{\Lambda^{4-d}\lambda\varphi_0^2}{2k^2}\right) - \frac{\mathrm{i}}{2}\mathcal{V}_d \int \frac{\mathrm{d}^d k}{(2\pi)^d} \ln\left(1 - \frac{\Lambda^{4-d}\lambda\bar{\phi}_{\mathrm{cl}}^2}{2k^2}\right) \\ &= -\frac{\mathcal{V}_d}{d}\left(\frac{\Lambda^{4-d}\lambda}{8\pi}\right)^{d/2}\Gamma\left(1-\frac{d}{2}\right)\varphi_0^d + \frac{\mathcal{V}_d}{d}\left(\frac{\Lambda^{4-d}\lambda}{8\pi}\right)^{d/2}\Gamma\left(1-\frac{d}{2}\right)\bar{\phi}_{\mathrm{cl}}^d\,,\end{aligned} \tag{2.83}$$

where we used (B.17) in the last step. Dropping the irrelevant constant term, which only depends on $\bar{\phi}_{\mathrm{cl}}$ and not on φ_0, we then get

$$V(\varphi_0) = \frac{\Lambda^{4-d}\lambda}{4!}\varphi_0^4 + \frac{\hbar}{d}\left(\frac{\Lambda^{4-d}\lambda}{8\pi}\right)^{d/2}\Gamma\left(1-\frac{d}{2}\right)\varphi_0^d + \mathcal{O}\left(\hbar^2\right). \tag{2.84}$$

which is in perfect agreement with the results of the diagrammatic and the tadpole approach.

2.2.1.4. Renormalization

In the last sections we have calculated the effective potential for a simple ϕ^4-theory using three different methods, but we still need to renormalize the result. Using the expansion $d = 4 - 2\epsilon$ we get

$$V(\varphi_0) = \frac{\Lambda^{2\epsilon}\lambda}{4!}\varphi_0^4 + \frac{\hbar\Lambda^{2\epsilon}\lambda^2\varphi_0^4}{256\pi^2}\left(-\frac{1}{\epsilon} + \gamma - \ln(4\pi)\right) + \frac{\hbar\lambda^2\varphi_0^4}{256\pi^2}\left(\ln\frac{\lambda\varphi_0^2}{\Lambda^2} - \frac{3}{2}\right). \tag{2.85}$$

At one-loop order we can use the expansion $\lambda = \lambda_{\mathrm{r}} + \hbar\lambda\delta Z_\lambda$, where λ_{r} is the renormalized coupling constant. According to the $\overline{\mathrm{MS}}$ scheme, terms proportional to $-1/\epsilon + \gamma - \ln(4\pi)$ are absorbed by counterterms and thus the renormalized effective potential at one-loop order reads

$$V(\varphi_0) = \frac{\lambda_{\mathrm{r}}}{4!}\varphi_0^4 + \frac{\hbar\lambda_{\mathrm{r}}^2\varphi_0^4}{256\pi^2}\left(\ln\frac{\lambda_{\mathrm{r}}\varphi_0^2}{2\Lambda^2} - \frac{3}{2}\right). \tag{2.86}$$

2.2.2. The one-loop effective potential of a general gauge theory

We will now use the path integral approach to calculate the effective potential of a general gauge theory. Let us assume that our gauge group is a compact reductive[4] Lie group $\mathcal{G}$ and that the field content is given by m real scalar fields $\phi_i(x)$, k_{F} left- and right-handed fermionic fields $\psi_{\mathrm{L}\,r}(x)$ and $\psi_{\mathrm{R}\,r}(x)$ and n gauge fields $A_{a\,\mu}$. The most general renormalizable, gauge- and Lorentz invariant Lagrangian density

[4] Reductive means that the Lie algebra of the Lie group is a direct sum of simple and trivial (one-dimensional) Lie algebras.

that we can write down is extensively discussed in appendix C and we will here recapitulate the important features.[5]

The Lagrangian density reads

$$\mathcal{L} = K_A + K_{\phi A} + K_{\psi A} - V_\phi - V_\psi - V_{\phi\psi} \,. \tag{2.87}$$

It includes a kinetic term for the gauge fields

$$K_A = -\frac{1}{4} F_{a\,\mu\nu} F_a^{\mu\nu} \,, \tag{2.88}$$

a gauge-kinetic term for the scalar fields

$$K_{\phi A} = \frac{1}{2}(D_\mu \phi_i)(D^\mu \phi_i) \,, \tag{2.89}$$

and a gauge-kinetic term for the fermionic fields

$$K_{\psi A} = \overline{\psi}(\mathbb{1}_{k_\mathrm{f}\times k_\mathrm{f}} \otimes \mathrm{i}\not{D})\psi \,, \tag{2.90}$$

where we have introduced the vector $\psi(x) = \psi_\mathrm{L}(x) + \psi_\mathrm{R}(x)$. Furthermore, we have a scalar interaction term

$$V_\phi = \mathcal{P}(\phi) = \Lambda^{d-4} f + \Lambda^{\frac{d-4}{2}} f_i \phi_i + f_{ij}\phi_i\phi_j + \Lambda^{\frac{4-d}{2}} f_{ijk}\phi_i\phi_j\phi_k + \Lambda^{4-d} f_{ijkl}\phi_i\phi_j\phi_k\phi_l \,, \tag{2.91}$$

where all coupling constants are real and totally symmetric; a fermionic mass term

$$V_\psi = \overline{\psi}(m'_\mathrm{D} \otimes P_\mathrm{L})\psi + \frac{1}{2}\overline{\psi^c}(m'_\mathrm{L} \otimes P_\mathrm{L})\psi + \frac{1}{2}\overline{\psi^c}(m'_\mathrm{R} \otimes P_\mathrm{R})\psi + \mathrm{h.c.} \,, \tag{2.92}$$

where m'_D is a general complex matrix and the complex matrices m'_L and m'_R are symmetric; and a Yukawa term

$$V_{\phi\psi} = \Lambda^{\frac{4-d}{2}} \phi_i \left[\overline{\psi}(\Gamma_{\mathrm{D}\,i} \otimes P_\mathrm{L})\psi + \frac{1}{2}\overline{\psi^c}(\Gamma_{\mathrm{L}\,i} \otimes P_\mathrm{L})\psi + \frac{1}{2}\overline{\psi^c}(\Gamma_{\mathrm{R}\,i} \otimes P_\mathrm{R})\psi + \mathrm{h.c.} \right] , \tag{2.93}$$

with a general complex matrix $\Gamma_{\mathrm{D}\,i}$ and the symmetric, complex matrices $\Gamma_{\mathrm{L}\,i}$ and $\Gamma_{\mathrm{R}\,i}$. In order to have gauge invariant interaction terms, the coupling constants need to fulfill certain conditions that are listed in (C.40) and (C.43). The respective covariant derivatives of the scalar and fermionic fields read

$$(D_\mu \phi)_i = \partial_\mu \phi_i + \Lambda^{\frac{4-d}{2}} \mathrm{i}(\theta_a)_{ij}\phi_j A_{a\,\mu} \,, \tag{2.94}$$

$$(D_\mu \psi_\mathrm{L})_r = \partial_\mu \psi_{\mathrm{L}\,r} + \Lambda^{\frac{4-d}{2}} \mathrm{i}(t_{\mathrm{L}\,a})_{rs}\psi_{\mathrm{L}\,s} A_{a\,\mu} \,, \tag{2.95}$$

$$(D_\mu \psi_\mathrm{R})_r = \partial_\mu \psi_{\mathrm{R}\,r} + \Lambda^{\frac{4-d}{2}} \mathrm{i}(t_{\mathrm{R}\,a})_{rs}\psi_{\mathrm{R}\,s} A_{a\,\mu} \,. \tag{2.96}$$

[5] Note that we are setting $\hbar = 1$ from now on to keep our expressions simpler. Once we perform the expansion in $\hbar$, we recall from the last sections, how the various terms depended on $\hbar$.

In appendix C.5 it is explained that, as long as we are dealing with quotients of path integrals, we may add the gauge-fixing term [6]

$$\mathcal{L}_{\rm GF} = -\frac{1}{2}(\partial_\mu A^\mu_a)\xi_{ab}(\partial_\nu A^\nu_b) + \frac{1}{2}\Lambda^{4-d}\xi^{-1}_{ab}(\theta_a\phi_{\rm cl})_i(\phi-\phi_{\rm cl})_i(\theta_b\phi_{\rm cl})_j(\phi-\phi_{\rm cl})_j \\ + \mathrm{i}\Lambda^{\frac{4-d}{2}}(\partial_\mu A^\mu_a)(\theta_a\phi_{\rm cl})_i(\phi-\phi_{\rm cl})_i\,. \quad (2.97)$$

and the Faddeev-Popov ghost term

$$\mathcal{L}_{\rm FP} = (\partial_\mu\omega^*_a)(\partial^\mu\omega_a) + (\partial_\mu\omega^*_a)\Lambda^{\frac{4-d}{2}}\xi^{\frac{1}{4}}_{ac}C_{ced}A^\mu_d\xi^{-\frac{1}{4}}_{eb}\omega_b - \omega^*_a\Lambda^{4-d}\xi^{-\frac{3}{4}}_{ac}(\theta_d\theta_c\phi_{\rm cl})_i\phi_i\xi^{-\frac{1}{4}}_{db}\omega_b\,, \quad (2.98)$$

to the Lagrangian density:

$$\mathcal{L}' = \mathcal{L} + \mathcal{L}_{\rm GF} + \mathcal{L}_{\rm FP}\,. \quad (2.99)$$

The action corresponding to the Lagrangian density $\mathcal{L}'$ is

$$S'[\Phi] = \int \mathrm{d}^d x\,\mathcal{L}'\,, \quad (2.100)$$

where we introduced a vector that contains all fields:

$$\Phi = \begin{pmatrix} \phi \\ \psi \\ \psi^{\rm c} \\ A \\ \omega \\ \omega^* \end{pmatrix}. \quad (2.101)$$

After fixing the gauge, the generating functional reads

$$Z[J] = \frac{1}{\mathcal{N}}\int[\mathrm{d}\Phi]\exp\left[\mathrm{i}S'[\Phi] + \mathrm{i}\int \mathrm{d}^d x\,J_i(x)\phi_i(x)\right], \quad (2.102)$$

with a normalization constant

$$\mathcal{N} = \int[\mathrm{d}\Phi]\exp\mathrm{i}S'[\Phi]\,, \quad (2.103)$$

which is chosen such that $Z[0] = 1$. Actually, we could also introduce external sources which couple to the fermionic and the gauge fields, but since we assume that these fields have a vanishing VEV, we do not need them here.

We start the calculation of the effective potential with an expansion around the classical field

$$\Phi(x) = \Phi_{\rm cl}(x) + \Phi'(x)\,, \quad (2.104)$$

[6] The field $\phi_{\rm cl}$ will be defined in a moment.

which is defined by [7]

$$\frac{\delta S'}{\delta\phi_i(x)}[\Phi_{\text{cl}}] = -J_i(x)\,, \tag{2.105}$$

with

$$\Phi_{\text{cl}} = \begin{pmatrix}\phi_{\text{cl}}\\0\\0\\0\\0\\0\end{pmatrix} \qquad \text{and} \qquad \Phi' = \begin{pmatrix}\phi'\\\psi'\\\psi'^{\text{c}}\\A'\\\omega'\\\omega'^{*}\end{pmatrix}. \tag{2.106}$$

In the exponent of the normalization constant we expand around $\bar{\phi}_{\text{cl}} = \phi_{\text{cl}}\big|_{J=0}$ instead of ϕ_{cl}. Therefore, we can use the stationary-phase approximation for the path integral in the generating functional and get

$$Z[J] = \exp\left(\mathrm{i}S[\Phi_{\text{cl}}] + \mathrm{i}S[\bar{\Phi}_{\text{cl}}] - \mathrm{i}\int \mathrm{d}^d x\, J_i\phi_{\text{cl}\,i}\right)\int [\mathrm{d}\Phi']\exp\left[\mathrm{i}S_{\text{qu}} - \mathrm{i}\bar{S}_{\text{qu}} + \mathcal{O}\left(\hbar^{3/2}\right)\right], \tag{2.107}$$

where S_{qu} contains all terms of $S'[\Phi]$ which are quadratic in ϕ', A', ψ' and ω' (terms of order $\hbar$, where $\phi', A', \psi', \omega' \sim \sqrt{\hbar}$). We use the notation that a bar over a quantity means that it is evaluated at $\phi_{\text{cl}} \to \bar{\phi}_{\text{cl}}$ (e.g. $\bar{S}_{\text{qu}} = S_{\text{qu}}\big|_{\phi_{\text{cl}}=\bar{\phi}_{\text{cl}}}$). In section 2.2.1.3 we have observed that if we want to calculate the effective potential up to order $\hbar$, we can exchange $\phi_{\text{cl}}(x)$ with φ_0 and to simplify the calculation we will hence already use $\partial_\mu\phi_{\text{cl}\,i}(x) = 0$. Therefore, we have

$$S[\phi_{\text{cl}}] = -\int \mathrm{d}^d x\, \mathcal{P}(\phi_{\text{cl}})\,, \tag{2.108}$$

and the Lagrangian density corresponding to S_{qu} is given by

$$\begin{aligned}
\mathcal{L}_{\text{qu}} = &-\frac{1}{4}(\partial_\mu A'_{a\,\nu} - \partial_\nu A'_{a\,\mu})(\partial^\mu A'^{\nu}_a - \partial^\nu A'^{\mu}_a) + \frac{1}{2}(\partial_\mu\phi'_i)(\partial^\mu\phi'_i)\\
&+ \mathrm{i}\Lambda^{\frac{4-d}{2}}(\partial_\mu\phi'_i)(\theta_a\phi_{\text{cl}})_i A'^{\mu}_a + \frac{1}{2}A'_{a\,\mu}(\mu^2)_{ab}A'^{\mu}_b + \mathrm{i}\overline{\psi'}\partial\!\!\!/\psi'\\
&- \left[\overline{\psi'}(m_{\text{D}}\otimes P_{\text{L}})\psi' + \frac{1}{2}\overline{\psi'^{\text{c}}}(m_{\text{L}}\otimes P_{\text{L}})\psi' + \frac{1}{2}\overline{\psi'^{\text{c}}}(m_{\text{R}}\otimes P_{\text{R}})\psi' + \text{h.c.}\right]\\
&- \frac{1}{2}\phi'_i(M^2)_{ij}\phi'_j - \frac{1}{2}(\partial_\mu A'^{\mu}_a)\xi_{ab}(\partial_\nu A'^{\nu}_b) + \mathrm{i}\Lambda^{\frac{4-d}{2}}(\partial_\mu A'^{\mu}_a)(\theta_a\phi_{\text{cl}})_i\phi'_i\\
&+ \frac{1}{2}\Lambda^{4-d}\phi'_i(\theta_a\phi_{\text{cl}})_i\xi^{-1}_{ab}(\theta_b\phi_{\text{cl}})_j\phi'_j + (\partial_\mu\omega'^{*}_a)(\partial^\mu\omega'_a) - \omega'^{*}_a\xi^{-\frac{3}{4}}_{ab}(\mu^2)_{bc}\xi^{-\frac{1}{4}}_{cd}\omega'_d\,,
\end{aligned} \tag{2.109}$$

[7] Since it is crucial for the derivation of the Faddeev-Popov term in appendix C.5, we should note that we assume that the external fields $J_i(x)$ do not transform under gauge transformations, and thus neither do $\phi_{\text{cl}\,i}$.

where we introduced the "mass" matrices

$$(\mu^2)_{ab} = -\Lambda^{4-d}(\theta_a\phi_{\text{cl}})_i(\theta_b\phi_{\text{cl}})_i = \Lambda^{4-d}\phi_{\text{cl}\,i}(\theta_a\theta_b\phi_{\text{cl}})_i\,, \tag{2.110}$$

$$(M^2)_{ij} = \left.\frac{\partial^2\mathcal{P}(\phi)}{\partial\phi_i\partial\phi_j}\right|_{\phi=\phi_{\text{cl}}}, \tag{2.111}$$

$$m_{\text{D}} = m'_{\text{D}} + \Lambda^{\frac{4-d}{2}}\Gamma_{\text{D}\,i}\phi_{\text{cl}\,i}\,, \tag{2.112}$$

$$m_{\text{L}} = m'_{\text{L}} + \Lambda^{\frac{4-d}{2}}\Gamma_{\text{L}\,i}\phi_{\text{cl}\,i}\,, \tag{2.113}$$

$$m_{\text{R}} = m'_{\text{R}} + \Lambda^{\frac{4-d}{2}}\Gamma_{\text{R}\,i}\phi_{\text{cl}\,i}\,. \tag{2.114}$$

If we do some partial integrations and discard various gradient terms, we get

$$\begin{aligned}\mathcal{L}_{\text{qu}} = {}& \frac{1}{2}A'_{a\,\mu}\left[\delta_{ab}g^{\mu\nu}\Box - (\delta_{ab}-\xi_{ab})g^{\mu\rho}\partial_\rho\partial_\sigma g^{\sigma\nu} + (\mu^2)_{ab}g^{\mu\nu}\right]A'_{b\,\nu}\\ &+ \frac{1}{2}\phi'_i\left[-\delta_{ij}\Box - (M^2)_{ij} + \Lambda^{4-d}(\theta_a\phi_{\text{cl}})_i\xi^{-1}_{ab}(\theta_b\phi_{\text{cl}})_j\right]\phi'_j\\ &+ \overline{\psi'}\mathrm{i}\partial\!\!\!/\psi' - \left[\overline{\psi'}(m_{\text{D}}\otimes P_{\text{L}})\psi' + \frac{1}{2}\overline{\psi'^{\text{c}}}(m_{\text{L}}\otimes P_{\text{L}})\psi' + \frac{1}{2}\overline{\psi'^{\text{c}}}(m_{\text{R}}\otimes P_{\text{R}})\psi' + \text{h.c.}\right]\\ &+ \omega'^*_a\left[-\delta_{ab}\Box - \xi_{ac}^{-\frac{3}{4}}(\mu^2)_{cd}\xi_{db}^{-\frac{1}{4}}\right]\omega'_b\end{aligned} \tag{2.115}$$

and we recognize that the terms which mix scalar and gauge fields conveniently vanish. This is not by accident, but rather we have chosen the gauge-fixing function in appendix C.5 such that this happens.

Introducing the vectors

$$\chi = \begin{pmatrix}\psi'\\ \psi'^{\text{c}}\end{pmatrix} \qquad \text{and} \qquad \chi^{\text{c}} = \begin{pmatrix}\psi'^{\text{c}}\\ \psi'\end{pmatrix}, \tag{2.116}$$

the term with the fermion fields can be written in a compact form:

$$\begin{aligned}&\overline{\psi'}\mathrm{i}\partial\!\!\!/\psi' - \left[\overline{\psi'}(m_{\text{D}}\otimes P_{\text{L}})\psi' + \frac{1}{2}\overline{\psi'^{\text{c}}}(m_{\text{L}}\otimes P_{\text{L}})\psi' + \frac{1}{2}\overline{\psi'^{\text{c}}}(m_{\text{R}}\otimes P_{\text{R}})\psi' + \text{h.c.}\right]\\ &= \frac{1}{2}\overline{\chi}\left(\mathbb{1}_{2k_{\text{f}}\times 2k_{\text{f}}}\otimes\mathrm{i}\partial\!\!\!/\right)\chi - \frac{1}{2}\overline{\chi^{\text{c}}}\left(m\otimes P_{\text{L}}\right)\chi - \frac{1}{2}\overline{\chi}\left(m^*\otimes P_{\text{R}}\right)\chi^{\text{c}}\\ &= \frac{1}{2}\chi^{\mathsf{T}}\left(\hat{C}\otimes C^{-1}\right)\left[-\mathbb{1}_{2k_{\text{f}}\times 2k_{\text{f}}}\otimes\mathrm{i}\partial\!\!\!/ + \left(\hat{C}\cdot m\right)\otimes P_{\text{L}} + \left(m^*\cdot\hat{C}\right)\otimes P_{\text{R}}\right]\chi\end{aligned} \tag{2.117}$$

with

$$m = \begin{pmatrix}m_{\text{L}} & m_{\text{D}}^{\mathsf{T}}\\ m_{\text{D}} & m_{\text{R}}\end{pmatrix} \qquad \text{and} \qquad \hat{C} = \begin{pmatrix}0_{k_{\text{f}}\times k_{\text{f}}} & \mathbb{1}_{k_{\text{f}}\times k_{\text{f}}}\\ \mathbb{1}_{k_{\text{f}}\times k_{\text{f}}} & 0_{k_{\text{f}}\times k_{\text{f}}}\end{pmatrix}. \tag{2.118}$$

The path integrals in (2.107) can now be solved using (B.41), (B.50) and (B.56):

$$\begin{aligned}Z[J] = \exp\!\left(\mathrm{i}S[\Phi_{\text{cl}}] - \mathrm{i}S[\bar{\Phi}_{\text{cl}}] - \mathrm{i}\int\mathrm{d}^dx\,J_i\phi_{\text{cl}\,i}\right)\left(\frac{\text{Det}\,D_A}{\text{Det}\,\bar{D}_A}\right)^{-\frac{1}{2}}\left(\frac{\text{Det}\,D_\phi}{\text{Det}\,\bar{D}_\phi}\right)^{-\frac{1}{2}}&\\ \times\left(\frac{\text{Det}\,D_\psi}{\text{Det}\,\bar{D}_\psi}\right)^{\frac{1}{2}}\left(\frac{\text{Det}\,D_\omega}{\text{Det}\,\bar{D}_\omega}\right),&\end{aligned} \tag{2.119}$$

where

$$D_A = \frac{1}{2}\left[\mathbb{1}_{n\times n}\otimes g\Box - (\mathbb{1}_{n\times n}-\xi)\otimes(g\partial\partial^{\mathsf{T}}g) + \mu^2\otimes g\right], \tag{2.120}$$

$$D_\phi = \frac{1}{2}\left[-\mathbb{1}_{m\times m}\Box - M^2 + \Lambda^{4-d}(\theta_a\phi_{\rm cl})\xi^{-1}_{ab}(\theta_b\phi_{\rm cl})^{\mathsf{T}}\right], \tag{2.121}$$

$$D_\psi = \frac{1}{2}\left(\hat{C}\otimes C^{-1}\right)\left[-\mathbb{1}_{2k_{\rm f}\times 2k_{\rm f}}\otimes \mathrm{i}\partial\!\!\!/ + \left(\hat{C}\cdot m\right)\otimes P_{\rm L} + \left(m^\dagger\cdot\hat{C}\right)\otimes P_{\rm R}\right], \tag{2.122}$$

$$D_\omega = -\mathbb{1}_{n\times n}\Box - \xi^{-\frac{3}{4}}\mu^2\xi^{-\frac{1}{4}}\,. \tag{2.123}$$

Accordingly, the generating functional of connected correlation functions $W[J]=-\mathrm{i}\ln Z[J]$ reads

$$W[J] = \underbrace{S[\Phi_{\rm cl}] - S[\bar{\Phi}_{\rm cl}] + \int \mathrm{d}^dx\, J_i\phi_{{\rm cl}\,i}}_{W_0[J]} + \underbrace{\frac{\mathrm{i}}{2}\ln\left(\frac{\mathrm{Det}\,D_A}{\mathrm{Det}\,\bar{D}_A}\right)}_{W_A[J]} + \underbrace{\frac{\mathrm{i}}{2}\ln\left(\frac{\mathrm{Det}\,D_\phi}{\mathrm{Det}\,\bar{D}_\phi}\right)}_{W_\phi[J]}$$
$$\underbrace{-\frac{\mathrm{i}}{2}\ln\left(\frac{\mathrm{Det}\,D_\psi}{\mathrm{Det}\,\bar{D}_\psi}\right)}_{W_\psi[J]}\underbrace{-\mathrm{i}\ln\left(\frac{\mathrm{Det}\,D_\omega}{\mathrm{Det}\,\bar{D}_\omega}\right)}_{W_\omega[J]} + \mathcal{O}\left(\hbar^2\right), \tag{2.124}$$

where $W_0[J]$ is of order $\hbar^0$ and the other terms are of order $\hbar$. The one-loop terms are calculated in appendix D.1. with the following result:

$$W_A[J] = -\frac{\mathcal{V}_d\Gamma\left(1-\frac{d}{2}\right)}{d(4\pi)^{\frac{d}{2}}}\left[(d-1)\,\mathrm{tr}\,\mu^d + \mathrm{tr}\big(\xi^{-d/2}\mu^d\big)\right] \tag{2.125}$$

$$+\frac{\mathcal{V}_d\Gamma\left(1-\frac{d}{2}\right)}{d(4\pi)^{\frac{d}{2}}}\left[(d-1)\,\mathrm{tr}\,\bar{\mu}^d + \mathrm{tr}\big(\xi^{-d/2}\bar{\mu}^d\big)\right], \tag{2.126}$$

$$W_\phi[J] = -\frac{\mathcal{V}_d\Gamma\left(1-\frac{d}{2}\right)}{d(4\pi)^{d/2}}\,\mathrm{tr}\left[M^2 + \Lambda^{4-d}(\theta_a\phi_{\rm cl})\xi^{-1}_{ab}(\theta_b\phi_{\rm cl})^{\mathsf{T}}\right]^{d/2} \tag{2.127}$$

$$+\frac{\mathcal{V}_d\Gamma\left(1-\frac{d}{2}\right)}{d(4\pi)^{d/2}}\,\mathrm{tr}\left[\bar{M}^2 + \Lambda^{4-d}(\theta_a\bar{\phi}_{\rm cl})\xi^{-1}_{ab}(\theta_b\bar{\phi}_{\rm cl})^{\mathsf{T}}\right]^{d/2}, \tag{2.128}$$

$$W_\psi[J] = \frac{2\mathcal{V}_d\Gamma\left(1-\frac{d}{2}\right)}{d(4\pi)^{d/2}}\,\mathrm{tr}\,(m^*m)^{d/2} - \frac{2\mathcal{V}_d\Gamma\left(1-\frac{d}{2}\right)}{d(4\pi)^{d/2}}\,\mathrm{tr}\,(\bar{m}^*\bar{m})^{d/2}, \tag{2.129}$$

$$W_\omega[J] = \frac{2\mathcal{V}_d\Gamma\left(1-\frac{d}{2}\right)}{d(4\pi)^{d/2}}\,\mathrm{tr}\left(\xi^{-d/2}\mu^d\right) - \frac{2\mathcal{V}_d\Gamma\left(1-\frac{d}{2}\right)}{d(4\pi)^{d/2}}\,\mathrm{tr}\left(\xi^{-d/2}\bar{\mu}^d\right). \tag{2.130}$$

To get the effective action, we have to perform a Legendre transformation,

$$\Gamma[\varphi] = W[J] - \int \mathrm{d}^dx\, J_i(x)\varphi_i(x)\,, \tag{2.131}$$

and therefore, we need a relation between φ and the external field $J(x)$. As already explained, we are allowed to use $\varphi(x)=\phi_{\rm cl}(x)$ at the order we are working at. If we use the expansion $d = 4 - 2\epsilon$, renormalize according to the $\overline{\mathrm{MS}}$-scheme[8] and drop irrelevant constants that depend only on $\bar{\phi}_{\mathrm{cl}}$, the effective potential reads

$$\begin{aligned} V(\varphi_0) = \mathcal{P}_{\mathrm{r}}(\varphi_0) + \frac{1}{64\pi^2} \Bigg\{ & 3\,\mathrm{tr}\left[\mu_\varphi^4 \left(\ln \frac{\mu_\varphi^2}{\Lambda^2} - \frac{5}{6}\mathbb{1}_{n\times n}\right)\right] \\ + \mathrm{tr}\left[\left(M_\varphi^2 - \xi_{ab}^{-1}(\theta_a\varphi_0)(\theta_b\varphi_0)^{\mathsf{T}}\right)^2 \left(\ln \frac{M_\varphi^2 - \xi_{ab}^{-1}(\theta_a\varphi_0)(\theta_b\varphi_0)^{\mathsf{T}}}{\Lambda^2} - \frac{3}{2}\mathbb{1}_{m\times m}\right)\right] & \\ - 2\,\mathrm{tr}\left[(m_\varphi^* m_\varphi)^2 \left(\ln \frac{m_\varphi^* m_\varphi}{\Lambda^2} - \frac{3}{2}\mathbb{1}_{2k_{\mathrm{f}}\times 2k_{\mathrm{f}}}\right)\right] & \\ - \mathrm{tr}\left[\xi^{-2}\mu_\varphi^4 \left(\ln \frac{\xi^{-1}\mu_\varphi^2}{\Lambda^2} - \frac{3}{2}\mathbb{1}_{n\times n}\right)\right] \Bigg\} & + \mathcal{O}\left(\hbar^2\right) , \end{aligned} \tag{2.132}$$

where we introduced the renormalized fourth-order polynomial

$$\mathcal{P}_{\mathrm{r}}(\varphi_0) = f^{\mathrm{r}} + f_i^{\mathrm{r}}\varphi_{0\,i} + f_{ij}^{\mathrm{r}}\varphi_{0\,i}\varphi_{0\,j} + f_{ijk}^{\mathrm{r}}\varphi_{0\,i}\varphi_{0\,j}\varphi_{0\,k} + f_{ijkl}^{\mathrm{r}}\varphi_{0\,i}\varphi_{0\,j}\varphi_{0\,k}\varphi_{0\,l}\,, \tag{2.133}$$

with renormalized coupling constants, and the "mass" matrices

$$(\mu_\varphi^2)_{ab} = -(\theta_a\varphi_0)_i(\theta_b\varphi_0)_i\,, \quad (M_\varphi^2)_{ij} = \frac{\partial^2\mathcal{P}_{\mathrm{r}}(\varphi_0)}{\partial\varphi_{0\,i}\partial\varphi_{0\,j}}\,, \quad m_\varphi = \begin{pmatrix} m_{\mathrm{L}}^\varphi & (m_{\mathrm{D}}^\varphi)^{\mathsf{T}} \\ m_{\mathrm{D}}^\varphi & m_{\mathrm{R}}^\varphi \end{pmatrix} \tag{2.134}$$

with

$$m_{\mathrm{D}}^\varphi = m_{\mathrm{D}}' + \Gamma_{\mathrm{D}\,i}\varphi_{0\,i}\,, \quad m_{\mathrm{L}}^\varphi = m_{\mathrm{L}}' + \Gamma_{\mathrm{L}\,i}\varphi_{0\,i}\,, \quad m_{\mathrm{R}}^\varphi = m_{\mathrm{R}}' + \Gamma_{\mathrm{R}\,i}\varphi_{0\,i}\,. \tag{2.135}$$

Let us now compare our final result for the effective potential (2.132) with other results from literature. Coleman and Weinberg [1] calculated the effective potential in Landau gauge ($\xi \to \infty$) using the diagrammatic approach, and except for different conventions and notations their result coincides with ours. Actually, to the best of the knowledge of the author, there exists no other calculation of the effective potential of a general gauge theory which uses a general R_ξ gauge, as we did.[9] Weinberg [13] used the tadpole approach and interestingly concluded that with a general R_ξ gauge it is not possible to find an effective potential at all. This is of course in contradiction with our result and it would be interesting to further investigate how the discrepancy arises. In Landau gauge, however, the result in [13] is in agreement with ours.

Andreassen [14] analyzed the gauge dependence of the SM effective potential, but he used a different gauge-fixing term. He also discussed our way of gauge-fixing and criticized it, because it depends on a function which is not known beforehand. Indeed, in the gauge-fixing function (C.55) we use the classical field ϕ_{cl}, which

[8] Terms proportional to $-\frac{1}{\epsilon} + \gamma - \ln(4\pi)$ are absorbed by counterterms.

[9] The only exception is the master's thesis of Lanschützer [12], with whom the calculation was done conjointly.

might be problematic. At the loop-order we are working at, the classical field is closely connected to the point φ_0 at which we evaluate the effective potential $V(\varphi_0)$: $\varphi_0 \simeq \phi_{\text{cl}}$. From this point of view, our gauge-fixing function depends on the point where the effective potential is evaluated at. This admittedly seems odd, but it is still not necessarily wrong.

2.3. Symmetry breaking in theories with classical scale invariance

The effective potential becomes particularly important in theories with classical scale invariance, since in such theories the symmetry cannot be broken at tree-level and thus the one-loop corrections to the effective potential become crucial.

2.3.1. Classical scale invariance

A theory is said to be classically scale-invariant if the field equations, or equivalently the action, is invariant under dilatations. Dilatations are scale transformations of the coordinates:

$$x^\mu \to x'^\mu = \rho x^\mu \qquad \rho > 0\,. \tag{2.136}$$

In $d = 4$ space-time dimensions, a field $\Phi_i(x)$ transforms according to

$$\Phi_i(x) \to \Phi'_i(x') = \rho^{-\Delta_i}\Phi(x) = \rho^{-\Delta_i}\Phi_i(\rho^{-1}x') \tag{2.137}$$

under a dilatation, where Δ_i is the mass dimension of the field (e.g. 1 for scalars and gauge bosons and 3/2 for spin-1/2 fermions). Therefore, only terms with operator dimension four are allowed in the Lagrangian density of classically scale-invariant theories and hence all terms which include a dimensionful constant are forbidden.

Conformal invariance is more restricting than classical scale invariance. In addition to Poincaré invariance and invariance under dilatations, conformal theories are also invariant under special conformal transformations. However, since almost all classically scale-invariant field theories are conformally invariant [15], we here use the two terms interchangeably.

2.3.2. The Coleman-Weinberg mechanism

In classically scale-invariant theories, the only allowed terms in the scalar potential are quartic in the fields, and therefore the only stationary point of the tree-level potential is at the origin. However, Coleman and Weinberg [1] showed that, in such theories, SSB can still be induced by radiative corrections. They examined the conformal ϕ^4-theory which we have discussed in section 2.2.1. The effective potential up to one-loop order of this theory (2.86) is shown in figure 2.2. It seems as the one-loop contribution turns the tree-level minimum at the origin into a maximum and generates a new minimum away from the origin. However, it turns out that this

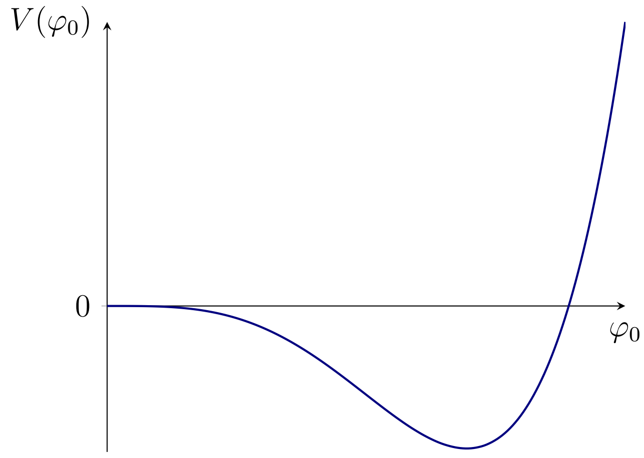


Figure 2.2.: One-loop effective potential of a conformal ϕ^4-theory

minimum is a spurious effect of perturbation theory, as Coleman and Weinberg [1] showed with a renormalization group analysis. To have SSB via radiative corrections, one needs apparently a more complex theory. Therefore, Coleman and Weinberg [1] considered massless scalar electrodynamics and showed that in such a theory it is indeed possible to find a perturbatively valid minimum, which breaks gauge symmetry as well as scale invariance.

2.3.3. The Gildener-Weinberg approach

As we have just discussed, it is possible to spontaneously break symmetries via radiative corrections in classically scale-invariant theories. While it is straightforward to find minima of the one-loop effective potential in theories with a single scalar, it is not obvious how to do the analysis for more involved theories with multiple scalars. Gildener and Weinberg [2], however, developed a method, with which multi-scalar theories can be analyzed systematically, and we will briefly recapitulate it here. While they used the Landau gauge for their calculations, we will use a general R_ξ gauge, as we already did in the previous sections. Furthermore, we will stick to our notation, which we introduced for a general gauge theory, but we set all dimensionful coupling constants to zero to ensure classical scale invariance.

For a conformal theory, the tree-level contribution to the effective potential (the first term in (2.132)) is of the form

$$V_0(\varphi_0) = \frac{1}{24}\lambda_{ijkl}\varphi_{0\,i}\varphi_{0\,j}\varphi_{0\,k}\varphi_{0\,l}. \tag{2.138}$$

We assume that the renormalized coupling constants λ_{ijkl} are all of order g^2 and that the Yukawa couplings are of order g, where $g \ll 1$ is a typical gauge coupling constant. Accordingly, the tree-level potential dominates over higher-order contributions (at least in the region where perturbation theory is valid), and the minimum of the potential lies at the origin. The situation changes, however, if the tree-level potential exhibits a flat direction. Along this direction the one-loop contribution becomes relevant and the symmetry might be broken spontaneously. The renormalized coupling constants λ_{ijkl} are functions of the renormalization scale Λ, and Gildener and Weinberg [2] pointed out that for generic initial values $\lambda_{ijkl}(\Lambda)$ it should be possible to find a value of Λ for which the tree-level potential develops a flat direction. This can be achieved by choosing $\Lambda = \Lambda_{\mathrm{GW}}$ such that the minimum value of $V_0(N)$ on the unit sphere vanishes:

$$\min_{N_m N_m = 1} (\lambda_{ijkl} N_i N_j N_k N_l) = 0\,. \tag{2.139}$$

If the minimum is attained for the unit vector $N_i = n_i$, the tree-level potential $V_0(\varphi_0)$ attains a minimum value of zero everywhere on the ray $\varphi_{0\,i} = n_i \phi$, because the tree-level potential is a homogeneous function of the field variables. So, it turns out that demanding that the tree-level potential exhibits a flat direction corresponds to fixing the renormalization scale $\Lambda = \Lambda_{\mathrm{GW}}$ and imposing a single constraint on the coupling constants. Therefore, we exchange a dimensionless parameter (one of the couplings constants) for a dimensionful one (the renormalization scale Λ_{GW}), what is commonly called dimensional transmutation.

The condition that the tree-level potential is stationary at $N = n$ reads

$$\lambda_{ijkl} n_j n_k n_l = 0\,, \tag{2.140}$$

and if we further require that this is a minimum we need to impose that the matrix

$$P_{ij} = \frac{1}{2} \lambda_{ijkl} n_k n_l \tag{2.141}$$

is positive semidefinite.

The one-loop contribution to the effective potential $V_1(\varphi_0)$ generates a small curvature in the radial direction, which singles out a specific value $\langle \phi \rangle$ as a minimum along the ray $\varphi_0 = n\phi$. Furthermore, the one-loop term slightly shifts the minimum away from the flat direction. The stationarity condition for a one-loop minimum at $\varphi_{0\,i} = n_i \langle \phi \rangle + \delta\varphi_i$ is

$$\left[\frac{\partial}{\partial \varphi_{0\,i}} \Big(V_0(\varphi_0) + V_1(\varphi_0) \Big) \right]_{n\langle \phi \rangle + \delta\varphi} = 0\,, \tag{2.142}$$

and if we expand it and keep only terms up to one-loop order, it reads

$$P_{ij} \delta\varphi_j \langle \phi \rangle^2 + \left[\frac{\partial V_1(\varphi_0)}{\partial \varphi_{0\,i}} \right]_{n\langle \phi \rangle} = 0\,. \tag{2.143}$$

Contracting the last equation with n_i and using the stationarity condition (2.140), we find

$$n_i \left[\frac{\partial V_1(\varphi_0)}{\partial \varphi_{0\,i}} \right]_{n\langle \phi \rangle} = \left[\frac{\partial V_1(n\phi)}{\partial \phi} \right]_{\langle \phi \rangle} = 0\,, \tag{2.144}$$

which determines $\langle\phi\rangle$.

From appendix C.4 we know that, in order to have a gauge invariant Lagrangian density, the condition [10]

$$(\theta_a\varphi_0)_i \frac{\partial V_0(\varphi_0)}{\partial \varphi_{0\,i}} = \mathcal{O}(\hbar) \tag{2.145}$$

has to be fulfilled. By differentiating the last equation with respect to $\varphi_{0\,j}$ and evaluating it at $\varphi_0 = n\phi$ we obtain

$$P_{ij}(\theta_a n)_j = \mathcal{O}(\hbar)\,. \tag{2.146}$$

If we use this condition in the one-loop contribution of the effective potential (2.132) along the flat direction, we get at the scale $\Lambda = \Lambda_{\mathrm{GW}}$:

$$\begin{aligned} V_1(n\phi) = \frac{1}{64\pi^2}\Bigg\{ &3\,\mathrm{tr}\left[\mu_\phi^4\left(\ln\frac{\mu_\phi^2}{\Lambda_{\mathrm{GW}}^2} - \frac{5}{6}\mathbb{1}_{n\times n}\right)\right] + \mathrm{tr}\left[M_\phi^4\left(\ln\frac{M_\phi^2}{\Lambda_{\mathrm{GW}}^2} - \frac{3}{2}\mathbb{1}_{m\times m}\right)\right] \\ &- 2\,\mathrm{tr}\left[(m_\phi^* m_\phi)^2\left(\ln\frac{m_\phi^* m_\phi}{\Lambda_{\mathrm{GW}}^2} - \frac{3}{2}\mathbb{1}_{2k_{\mathrm{f}}\times 2k_{\mathrm{f}}}\right)\right]\Bigg\}\,, \end{aligned} \tag{2.147}$$

with the "mass" matrices

$$\mu_\phi^2 = \mu_0^2\phi^2/\langle\phi\rangle^2\,, \tag{2.148}$$
$$M_\phi^2 = (M_0^2)_{ij}\phi^2/\langle\phi\rangle^2\,, \tag{2.149}$$
$$m_\phi^2 = m_0^2\phi^2/\langle\phi\rangle^2\,, \tag{2.150}$$

which are expressed in terms of the tree-level mass matrices

$$(\mu_0^2)_{ab} = -\langle\phi\rangle^2(\theta_a n)_i(\theta_b n)_i\,, \tag{2.151}$$
$$(M_0^2)_{ij} = \frac{1}{2}\langle\phi\rangle^2\lambda_{ijkl}n_k n_l\,, \tag{2.152}$$
$$m_0 = \langle\phi\rangle\begin{pmatrix}\Gamma_{\mathrm{L}\,i}n_i & (\Gamma_{\mathrm{D}\,i})^{\mathsf{T}}n_i \\ \Gamma_{\mathrm{D}\,i}n_i & \Gamma_{\mathrm{R}\,i}n_i\end{pmatrix}. \tag{2.153}$$

Interestingly, along the flat direction of the effective potential all dependencies on the gauge-fixing parameters drop out. We may rewrite the one-loop potential in the flat direction in the compact form

$$V_1(n\phi) = A\phi^4 + B\phi^4\ln\!\left(\phi^2/\Lambda_{\mathrm{GW}}^2\right), \tag{2.154}$$

[10] The reason why we have $\mathcal{O}(\hbar)$ instead of zero on the right-hand side is that V_0 is the renormalized tree-level potential, whereas in (C.40) we had the bare tree-level potential. However, this does not bother us since all terms in which we are going to use this condition are already of one-loop order.

with the dimensionless constants

$$A = \frac{1}{64\pi^2 \langle\phi\rangle^4} \left\{ 3\,\mathrm{tr}\left[\mu_0^4 \left(\ln \frac{\mu_0^2}{\langle\phi\rangle^2} - \frac{5}{6}\mathbb{1}_{n\times n}\right)\right] + \mathrm{tr}\left[M_0^4 \left(\ln \frac{M_0^2}{\langle\phi\rangle^2} - \frac{3}{2}\mathbb{1}_{m\times m}\right)\right] \right.$$
$$\left. - 2\,\mathrm{tr}\left[(m_0^* m_0^*)^2 \left(\ln \frac{m_0^* m_0}{\langle\phi\rangle^2} - \frac{3}{2}\mathbb{1}_{2k_\mathrm{f}\times 2k_\mathrm{f}}\right)\right] \right\} \quad (2.155)$$

and

$$B = \frac{1}{64\pi^2 \langle\phi\rangle^2} \left[3\,\mathrm{tr}\,\mu_0^4 + \mathrm{tr}\,M_0^4 - 2\,\mathrm{tr}(m_0^* m_0)^2\right]. \quad (2.156)$$

Therefore, the stationarity condition (2.144) reads

$$\ln \frac{\langle\phi\rangle^2}{\Lambda_{\mathrm{GW}}^2} = -\frac{1}{2} - \frac{A}{B}. \quad (2.157)$$

The stationary point, which is defined by this equation, is certainly only a minimum if

$$B > 0\,, \quad (2.158)$$

since for $B < 0$ the potential is not bounded from below and for $B = 0$ the one-loop potential stays purely quartic. If $B > 0$, we see that the value of the one-loop effective potential at its stationary point is less than its value at the origin $V(0) = 0$:

$$V(n\,\langle\phi\rangle) = -\frac{1}{2} B \langle\phi\rangle^4 < 0\,, \quad (2.159)$$

where we used (2.157) in (2.154).

The tree-level mass matrix of the physical scalar bosons is given by:

$$(M_0^2)_{ij} = \left[\frac{\partial^2 V_0(\varphi_0)}{\partial\varphi_{0\,i}\partial\varphi_{0\,j}}\right]_{n\langle\phi\rangle} = P_{ij}\langle\phi\rangle^2. \quad (2.160)$$

Due to (2.140) this tree-level mass matrix has an eigenvector n, which points into the flat direction, with eigenvalue zero. The particle corresponding to this eigenvector is called scalon [2]. When we go to one-loop order, the mass matrix gets shifted:

$$(M_0^2 + \delta M^2)_{ij} = \left[\frac{\partial^2}{\partial\varphi_{0\,i}\partial\varphi_{0\,j}}\left[V_0(\varphi_0) + V_1(\varphi_0)\right]\right]_{n\langle\phi\rangle+\delta\varphi}, \quad (2.161)$$

and doing an expansion and keeping only terms up to one-loop order, we get

$$(\delta M^2)_{ij} = \left[\frac{\partial^2 V_1(\varphi_0)}{\partial\varphi_{0\,i}\partial\varphi_{0\,j}}\right]_{n\langle\phi\rangle} + \lambda_{ijkl} n_k \delta\varphi_l \,\langle\phi\rangle\,. \quad (2.162)$$

From perturbation theory we know that the mass of the scalon at one-loop level is given by the expectation value of the one-loop mass matrix with respect to the tree-level eigenvector n (see appendix B.5). Accordingly, we obtain

$$\begin{aligned} M_S^2 = n_i n_j (\delta M^2)_{ij} = n_i n_j \left[\frac{\partial^2 V_1(\varphi_0)}{\partial \varphi_{0\,i} \partial \varphi_{0\,j}}\right]_{n\langle\phi\rangle} &= \left[\frac{\mathrm{d}^2}{\mathrm{d}\phi^2} V_1(n\phi)\right]_{\langle\phi\rangle} \\ &= 12\langle\phi\rangle^2 \left(B \ln \frac{\langle\phi\rangle^2}{\Lambda_{\mathrm{GW}}^2} + \frac{7}{6} B + A\right) = 8B\langle\phi\rangle^2 , \end{aligned} \tag{2.163}$$

where we used (2.140), (2.154) and (2.157). The one-loop mass of the scalon is positive because of (2.158) and it can be expressed in terms of tree-level masses of all other physical particles in the theory:

$$M_S^2 = \frac{1}{8\pi\langle\phi\rangle^2} \left(\sum_s M_s^4 + 3 \sum_g M_g^4 - 2 \sum_f m_f^4\right) , \tag{2.164}$$

where the summations run over scalars, gauge bosons and Weyl fermions (two-component spinors), respectively.

3. Building classically scale-invariant models

In this chapter we want to make use of the tools we have discussed in the last chapter. Our goal is to build a classically scale-invariant model which is close to the Standard Model (SM). The only term in the Lagrangian density of the SM that breaks conformal invariance is the explicit mass term of the Higgs doublet. If one naively tries to build a classically scale-invariant SM by simply dropping this term, the only relevant contributions to the mass of the scalon in (2.164) stem from the gauge boson masses and the mass of the top quark:

$$3M_Z^4 + 6M_W^4 - 12m_t^4 \simeq -(319\,\mathrm{GeV})^4\,. \tag{3.1}$$

They clearly lead to a negative result, and thus such a theory would be unstable. To fix this, we need to introduce new massive gauge bosons or scalar bosons, which deliver positive contributions to the scalon mass in (2.164). Since we want to stick to the SM gauge group, we here only extend the scalar sector. One of the simplest ways of doing so is to add additional Higgs doublets, and thus a lot of work on non-conformal multi-Higgs-doublet models, and especially on two-Higgs-doublet models, can already be found in literature (see e.g. [16] and the references therein).

Although we want to stay close to the SM, it would be nice if we could fix some of its deficiencies. One fundamental shortcoming of the SM is that it lacks masses for the neutrinos although experiments show that at least two of them have small but positive masses [17]. The most common explanation for the smallness of the neutrino masses is the seesaw mechanism. One usually assumes that the Dirac mass matrix of the neutrinos is of the order m_D, which is similar to other lepton masses $m_\mathrm{D} \sim m_\ell$, but much smaller than the order of the Majorana mass matrix of the right-handed neutrinos $m_\mathrm{D} \ll m_\mathrm{R}$. This leads to some light mass eigenstates with masses of the order $m_\mathrm{D}^2/m_\mathrm{R}$, and some heavy ones with masses of the order of m_R. Since heavy fermions spoil the stability of our theory (they contribute with a negative sign to the scalon mass (2.164)), it would be convenient to have only few heavy neutrinos. Grimus and Neufeld [3] proposed a multi-Higgs-doublet model that combines the seesaw mechanism with radiative neutrino mass production, and in the simplest version, it yields the necessary three light neutrinos, but only one heavy one.

To implement the seesaw mechanism one needs a Majorana mass term for the right-handed neutrinos, but explicit mass terms are forbidden in a classically scale-invariant model. However, it turns out that it is sufficient to add one single real scalar singlet, which allows to generate the Majorana mass term for the right-handed neutrinos via SSB.

We start with a discussion of a conformal multi-Higgs-doublet model with an arbitrary number of right-handed neutrinos in the next section, and thereafter we discuss the simplest version of this model with two Higgs doublets and a single right-handed neutrino.

3.1. Neutrino masses in a conformal multi-Higgs-doublet model

In this section we recapitulate the discussion of [4], in which a conformal multi-Higgs-doublet model is analyzed.

3.1.1. Description of the model

We consider a classically scale-invariant model that is invariant under the SM $\mathrm{SU}(3)_{\mathrm{c}} \times \mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_Y$ gauge group. The field content is given by:

- a real scalar singlet φ_0;
- an arbitrary number n_{H} of scalar doublets;

$$\Phi_k = \begin{pmatrix} \phi_k^+ \\ \phi_k^0 \end{pmatrix}, \quad 1 \le k \le n_{\mathrm{H}}\,; \tag{3.2}$$

- a not specified number of n_{R} right-handed neutrino singlets,

$$\nu_{\mathrm{R}\,s}\,, \quad 1 \le s \le n_{\mathrm{R}}\,; \tag{3.3}$$

- the usual SM lepton-, quark-, and gauge fields

$$L_r = \begin{pmatrix} \nu_{\mathrm{L}} \\ \ell'_{\mathrm{L}} \end{pmatrix}_r, \quad \ell'_{\mathrm{R}\,r}\,, \quad 1 \le r \le n_{\mathrm{L}} = 3\,, \tag{3.4}$$

$$Q_r = \begin{pmatrix} u'_{\mathrm{L}} \\ d'_{\mathrm{L}} \end{pmatrix}_r, \quad u'_{\mathrm{R}\,r}\,, \quad d'_{\mathrm{R}\,r}\,, \quad 1 \le r \le n_{\mathrm{L}}\,, \tag{3.5}$$

  and

$$B_\mu, \quad \vec{W}_\mu, \quad G^a_\mu, \quad 1 \le a \le 8\,. \tag{3.6}$$

The gauge fields transform in the adjoint representation, and the transformation properties of the scalar- and fermion fields are listed in table 3.1.

The covariant derivative is given by

$$D_\mu = \partial_\mu + \mathrm{i} g_{\mathrm{s}} \sum_{a=1}^{8} \mathcal{T}_a G^a_\mu + \mathrm{i} g \vec{T} \cdot \vec{W}_\mu + \mathrm{i} g' \frac{Y}{2} B_\mu\,, \tag{3.7}$$

	$\mathrm{SU}(3)_\mathrm{c}$	$\mathrm{SU}(2)_\mathrm{L}$	$\mathrm{U}(1)_Y$
φ_0	$\mathbf{1}$	$\mathbf{1}$	0
Φ_k	$\mathbf{1}$	$\mathbf{2}$	1
$\nu_{\mathrm{R}\,s}$	$\mathbf{1}$	$\mathbf{1}$	0
L_r	$\mathbf{1}$	$\mathbf{2}$	-1
$\ell'_{\mathrm{R}\,r}$	$\mathbf{1}$	$\mathbf{1}$	-2
Q_r	$\mathbf{3}$	$\mathbf{2}$	1/3
$u'_{\mathrm{R}\,r}$	$\mathbf{3}$	$\mathbf{1}$	4/3
$d'_{\mathrm{R}\,r}$	$\mathbf{3}$	$\mathbf{1}$	-2/3

Table 3.1.: Gauge-transformation properties of the various fields

with

$$\mathcal{T}_a = \begin{cases} \lambda_a/2 & \text{for SU(3) triplets} \\ 0 & \text{for SU(3) singlets} \end{cases}, \qquad \vec{T} = \begin{cases} \vec{\tau}/2 & \text{for SU(2) doublets} \\ 0 & \text{for SU(2) singlets} \end{cases}, \tag{3.8}$$

where λ_a are the Gell-Mann matrices and $\vec{\tau}$ denotes the Pauli matrices.

We can now write down the most general renormalizable Lagrangian density for this model that we can think of. The pure gauge part reads

$$\mathcal{L}_\mathrm{g} = -\frac{1}{4}\sum_{a=1}^{8} G^a_{\mu\nu} G^{\mu\nu}_a - \frac{1}{4}\vec{W}_{\mu\nu}\vec{W}^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}\,, \tag{3.9}$$

where we introduced field strength tensors for the various gauge fields. The gauge-kinetic terms of the scalars and fermions are respectively given by

$$\mathcal{L}_\mathrm{s} = \frac{1}{2}(\partial_\mu\varphi_0)(\partial^\mu\varphi_0) + \sum_{k=1}^{n_\mathrm{H}} \left(D_\mu\Phi_k\right)^\dagger \left(D^\mu\Phi_k\right) \tag{3.10}$$

and

$$\mathcal{L}_\mathrm{f} = \overline{L}\mathrm{i}\not{D}L + \overline{\ell'_\mathrm{R}}\mathrm{i}\not{D}\ell'_\mathrm{R} + \overline{\nu_\mathrm{R}}\mathrm{i}\not{\partial}\nu_\mathrm{R} + \overline{Q}\mathrm{i}\not{D}Q + +\overline{u'_\mathrm{R}}\mathrm{i}\not{D}u'_\mathrm{R} + \overline{d'_\mathrm{R}}\mathrm{i}\not{D}d'_\mathrm{R}\,, \tag{3.11}$$

where we used a vector notation for the fermion fields.

We can define the conjugate scalar doublets

$$\tilde{\Phi}_k = \mathrm{i}\tau_2\Phi_k^* = \begin{pmatrix} \left(\Phi_k^0\right)^* \\ -\left(\Phi_k^+\right)^* \end{pmatrix} \equiv \begin{pmatrix} \left(\Phi_k^0\right)^* \\ -\Phi_k^- \end{pmatrix}, \tag{3.12}$$

which transform in the same way under SU(2)-transformations as Φ_k, but have weak hypercharge $Y = -1$. Then, the Yukawa Lagrangian is given by

$$\begin{aligned}\mathcal{L}_\mathrm{Y} = -\frac{\varphi_0}{2}\overline{\nu_\mathrm{R}}Y_\mathrm{R}(\nu_\mathrm{R})^\mathrm{c} - \sum_{k=1}^{n_\mathrm{H}} \Big[\Big(\Phi_k^\dagger\overline{\ell'_\mathrm{R}}Y_k^{(\ell)} + \tilde{\Phi}_k^\dagger\overline{\nu_\mathrm{R}}Y_{\mathrm{D}\,k}\Big) L \\ + \Big(\Phi_k^\dagger\overline{d'_\mathrm{R}}Y_k^{(d)} + \tilde{\Phi}_k^\dagger\overline{u'_\mathrm{R}}Y_k^{(u)}\Big) Q\Big] + \mathrm{h.c.}\,,\end{aligned} \tag{3.13}$$

where $Y_k^{(\ell)}$, $Y_k^{(d)}$, $Y_k^{(u)}$ are complex $n_\mathrm{L} \times n_\mathrm{L}$ matrices, $Y_{\mathrm{D}\,k}$ are complex $n_\mathrm{R} \times n_\mathrm{L}$ matrices and $Y_\mathrm{R} = Y_\mathrm{R}^\mathsf{T}$ is a symmetric complex $n_\mathrm{R} \times n_\mathrm{R}$ matrix.

The scalar potential has the form

$$V_0(\varphi_0, \Phi_k) = \frac{\lambda_0}{4}\varphi_0^4 + \sum_{i,j,k,l} \lambda_{ijkl}(\Phi_i^\dagger \Phi_j)(\Phi_k^\dagger \Phi_l) + \varphi_0^2 \sum_{i,j} \kappa_{ij} \Phi_i^\dagger \Phi_j \,. \tag{3.14}$$

Since the Lagrangian density is supposed to be real, the following conditions hold

$$\lambda_0 = \lambda_0^* \,, \quad \lambda_{ijkl} = \lambda_{jilk}^* \,, \quad \kappa_{ij} = \kappa_{ji}^* \,, \tag{3.15}$$

and due to the symmetry of the second term we further assume

$$\lambda_{ijkl} = \lambda_{klij} \,. \tag{3.16}$$

In appendix D.2, it is shown that terms including $\widetilde{\Phi}_k$, which we could in principle add to the potential, do not yield any new contributions and they are thus omitted.

Finally, the full Lagrangian density reads

$$\mathcal{L} = \mathcal{L}_\mathrm{g} + \mathcal{L}_\mathrm{f} + \mathcal{L}_\mathrm{s} + \mathcal{L}_\mathrm{Y} - V_0 \,. \tag{3.17}$$

Note that, due to scale invariance, the explicit mass terms for the right-handed neutrino singlets and the scalars

$$-\frac{1}{2}\overline{\nu_\mathrm{R}} M_\mathrm{R} (\nu_\mathrm{R})^\mathrm{c} - \frac{1}{2}\overline{(\nu_\mathrm{R})^\mathrm{c}} M_\mathrm{R}^* \nu_\mathrm{R} - \frac{\mu_0^2}{2}\varphi_0^2 - \sum_{i,j} \mu_{ij}^2 \Phi_i^\dagger \Phi_j \tag{3.18}$$

are forbidden, although they would be gauge invariant. The Majorana mass term is, however, crucial if we want to implement the seesaw mechanism, and for this reason we have introduced the scalar singlet φ_0. After SSB, φ_0 acquires a VEV and a Majorana mass matrix $M_\mathrm{R} = \langle \varphi_0 \rangle Y_\mathrm{R}$ is produced due to the first term of (3.13).

3.1.2. Spontaneous symmetry breaking

The spontaneous breaking of gauge symmetry and scale invariance by radiative corrections is most efficiently analyzed with the Gildener-Weinberg approach, which we discussed in section 2.3.3.

First, we look for a minimum of the effective potential on the unit sphere, which is parametrized by

$$\varphi_0 = N_0 \in \mathbb{R} \,, \quad \Phi_k = \frac{1}{\sqrt{2}} N_k \in \mathbb{C}^2 \,, \tag{3.19}$$

with the constraint

$$N_0^2 + \sum_{k=1}^{n_\mathrm{H}} N_k^\dagger N_k = 1. \tag{3.20}$$

Since the electromagnetic charge ($Q_{\mathrm{em}} = T_3 + Y/2$) should remain unbroken, we further restrict the search for the minimum to the points

$$N_0 = n_0 > 0\,, \quad N_k = \begin{pmatrix} 0 \\ n_k \end{pmatrix}, \; n_k \in \mathbb{C}\,, \quad n_0^2 + \sum_{k=1}^{n_{\mathrm{H}}} n_k^* n_k = 1\,. \tag{3.21}$$

At a stationary point, the gradient of the potential and the gradient of the constraint (3.20) have to be parallel, what yields the conditions

$$\lambda_0 n_0^3 + n_0 \sum_{i,j} \kappa_{ij}^* n_i n_j^* = \lambda n_0 \tag{3.22}$$

$$\sum_{j,k,l} \frac{1}{\sqrt{2}} \lambda_{ijkl}^* n_j^* n_k n_l^* + n_0^2 \sum_j \frac{1}{\sqrt{2}} \kappa_{ij}^* n_j^* = \lambda \frac{1}{\sqrt{2}} n_i^*\,, \tag{3.23}$$

where we used (3.21). Eliminating the Lagrange multiplier λ, we then obtain the stationarity condition

$$\sum_{j,k,l} \lambda_{ijkl} n_j n_k^* n_l + n_0^2 \sum_j \kappa_{ij} n_j - n_i \left(\sum_{k,l} \kappa_{kl} n_k^* n_l + \lambda_0 n_0^2 \right) = 0\,. \tag{3.24}$$

Following the procedure described in [2], we can adjust the renormalization scale $\Lambda = \Lambda_{\mathrm{GW}}$, such that the minimum value of the tree-level potential on the unit sphere vanishes:

$$V_0 \left(n_0, \frac{n_i}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = 0\,. \tag{3.25}$$

Substituting the specific form of our potential, this condition reads

$$\lambda_0 n_0^4 + \sum_{i,j,k,l} \lambda_{ijkl} n_i^* n_j n_k^* n_l + 2 n_0^2 \sum_{i,j} \kappa_{ij} n_i^* n_j = 0\,. \tag{3.26}$$

From now on, we will assume that all coupling constants are taken at $\Lambda = \Lambda_{\mathrm{GW}}$.

If we contract (3.24) with n_i and subtract it from (3.26), we obtain

$$\sum_{i,j} \kappa_{ij} n_i^* n_j + \lambda_0 n_0^2 = 0\,, \tag{3.27}$$

and using this equation in (3.24) we get

$$\sum_j \left(\sum_{k,l} \lambda_{ijkl} n_k^* n_l + n_0^2 \kappa_{ij} \right) n_j = 0\,. \tag{3.28}$$

Therefore, the system of equations (3.24) and (3.26) is equivalent to the system of equations (3.27) and (3.28).

The flat direction of the tree-level potential can be parametrized by

$$\varphi_0 = \phi n_0 \,, \quad \Phi_k = \frac{\phi n_k}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \phi \in \mathbb{R} \,. \tag{3.29}$$

The position of the minimum along this ray $\langle \phi \rangle = V > 0$ is determined by higher-order corrections, and accordingly the VEVs of the neutral scalar fields read

$$\langle \varphi_0 \rangle = \underbrace{V n_0}_{v_0} + \delta v_0 \,, \quad \langle \Phi_k^0 \rangle = \frac{1}{\sqrt{2}} (\underbrace{V n_k}_{v_k} + \delta v_k) \,, \tag{3.30}$$

where δv_0 and δv_k account for the small shift away from the flat direction.

3.1.3. Tree-level masses

To get the tree-level masses of the model, the scalar fields are expanded around their VEVs. Considering only the leading contribution and omitting the small shifts in (3.30) we can write

$$\varphi_0 = v_0 + \rho_0 \,, \quad \Phi_k = \begin{pmatrix} \Phi_k^+ \\ \frac{1}{\sqrt{2}} (v_k + \rho_k + \mathrm{i}\sigma_k) \end{pmatrix}, \tag{3.31}$$

where we introduced the real fields ρ_0, ρ_k and σ_k.

3.1.3.1. Gauge bosons

The mass terms of the gauge bosons are obtained from the gauge-kinetic scalar term $\mathcal{L}_\mathrm{s}$ given in (3.10):

$$\frac{v^2}{8} \left[(-g W_{3\,\mu} + g' B_\mu)(-g W_3^\mu + g' B^\mu) + g^2 (W_{1\,\mu} + \mathrm{i} W_{2\,\mu})(W_1^\mu - \mathrm{i} W_2^\mu) \right] . \tag{3.32}$$

Therefore, the mass of the charged W boson eigenfield

$$W_\mu^\pm = \frac{(W_1 \mp \mathrm{i} W_2)_\mu}{\sqrt{2}} \tag{3.33}$$

is given by

$$M_W = \frac{vg}{2} \,, \quad \text{with} \quad v^2 = \sum_{k=1}^{n_\mathrm{H}} v_k^* v_k \,, \tag{3.34}$$

where $v = (\sqrt{2} G_\mathrm{F})^{-1/2} \simeq 246\,\mathrm{GeV}$ is determined by Fermi's interaction [18]. The eigenfields corresponding to the neutral Z-boson, with mass

$$M_Z = \frac{v\sqrt{g^2 + g'^2}}{2} \,, \tag{3.35}$$

and the massless photon are given by

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_\mathrm{W} & -\sin\theta_\mathrm{W} \\ \sin\theta_\mathrm{W} & \cos\theta_\mathrm{W} \end{pmatrix} \begin{pmatrix} W_{3\,\mu} \\ B_\mu \end{pmatrix} , \tag{3.36}$$

where we used the Weinberg angle θ_W, which is defined by $\sin\theta_\mathrm{W} = g'/\sqrt{g^2 + g'^2}$ and $\cos\theta_\mathrm{W} = g/\sqrt{g^2 + g'^2}$.

3.1.3.2. Charged scalars

The mass term of the charged scalars stems from the scalar potential (3.14) and reads

$$-\sum_{i,j} \Phi_i^- (M_+^2)_{ij} \Phi_j^+ \,, \tag{3.37}$$

with the hermitian matrix

$$(M_+^2)_{ij} = V^2 \left(\sum_{k,l} \lambda_{ijkl} n_k^* n_l + n_0^2 \kappa_{ij} \right) . \tag{3.38}$$

Due to (3.28) the vector $n = \begin{pmatrix} n_1 & \cdots & n_{n_\mathrm{H}} \end{pmatrix}^\mathsf{T}$ is an eigenvector of the mass matrix M_+^2 with eigenvalue zero:

$$M_+^2 n = 0 \,. \tag{3.39}$$

Since M_+^2 is hermitian, it can be diagonalized by a unitary matrix U:

$$U^\dagger M_+^2 U = \mathrm{diag}\left(\mu_1^2, \ldots, \mu_{n_\mathrm{H}-1}^2, 0\right) \,, \tag{3.40}$$

and the fields Φ_k can then be expressed in terms of the charged scalar eigenfields H_a^+:

$$\Phi_k^+ = \sum_{a=1}^{n_\mathrm{H}} U_{ka} H_a^+ \,. \tag{3.41}$$

Beside the $n_\mathrm{H} - 1$ physical charged scalars $H_1^\pm, \ldots, H_{n_\mathrm{H}-1}^\pm$, the model contains a charged would-be Goldstone boson $H_{n_\mathrm{H}}^\pm \equiv G^\pm$, which corresponds to the eigenvector n with eigenvalue zero and thus the last column of the unitary matrix U reads

$$U_{k\,n_\mathrm{H}} = \frac{n_k}{\sqrt{1-n_0^2}} \,. \tag{3.42}$$

3.1.3.3. Neutral scalars

The mass terms of the neutral scalars are again obtained from the scalar potential (3.14), and using (3.27), they read

$$-\frac{1}{2} \sum_{i,j=1}^{n_\mathrm{H}} (\rho_i A_{ij} \rho_j + \sigma_i B_{ij} \sigma_j + 2\rho_i C_{ij} \sigma_j) - \rho_0 \sum_{i=1}^{n_\mathrm{H}} (\rho_i \,\mathrm{Re}\, k_i + \sigma_i \,\mathrm{Im}\, k_i) - \lambda_0 v_0^2 \rho_0^2 \,, \tag{3.43}$$

with the n_H dimensional complex vector

$$k_i = 2V^2 n_0 \sum_{j=1}^{n_\mathrm{H}} \kappa_{ij} n_j \,, \tag{3.44}$$

and the real $n_\mathrm{H} \times n_\mathrm{H}$ matrices

$$A = \mathrm{Re}(M_+^2 + K' + K) \,, \quad B = \mathrm{Re}(M_+^2 + K' - K) \,, \quad C = \mathrm{Im}(-M_+^2 - K' + K) \,. \tag{3.45}$$

The symmetric matrix $K = K^\mathsf{T}$ and the hermitian matrix $K' = K'^\dagger$ have already been introduced in [19] and they are defined by

$$K_{ij} = V^2 \sum_{k,l} \lambda_{ikjl} n_k n_l \,, \quad K'_{ij} = V^2 \sum_{k,l} \lambda_{iklj} n_k n_l^* \,. \tag{3.46}$$

Adopting a vector notation, the mass term (3.43) can be rewritten as

$$-\frac{1}{2}\begin{pmatrix}\rho_0 & \rho^\mathsf{T} & \sigma^\mathsf{T}\end{pmatrix} \underbrace{\begin{pmatrix} 2\lambda_0 v_0^2 & \operatorname{Re} k^\mathsf{T} & \operatorname{Im} k^\mathsf{T} \\ \operatorname{Re} k & A & C \\ \operatorname{Im} k & C^\mathsf{T} & B \end{pmatrix}}_{\mathcal{M}_0^2} \begin{pmatrix}\rho_0 \\ \rho \\ \sigma\end{pmatrix} , \tag{3.47}$$

with $\rho^\mathsf{T} = \begin{pmatrix}\rho_1 & \cdots & \rho_{n_\mathrm{H}}\end{pmatrix}$ and $\sigma^\mathsf{T} = \begin{pmatrix}\sigma_1 & \cdots & \sigma_{n_\mathrm{H}}\end{pmatrix}$.

In appendix D.3, it is shown that the two orthogonal vectors

$$\begin{pmatrix} n_0 \\ \operatorname{Re} n \\ \operatorname{Im} n \end{pmatrix} , \quad \begin{pmatrix} 0 \\ -\operatorname{Im} n \\ \operatorname{Re} n \end{pmatrix} \in \mathbb{R}^{2n_\mathrm{H}+1} \tag{3.48}$$

are eigenvectors of $\mathcal{M}_0^2$ with eigenvalue zero. Since the first vector points into the flat direction of the tree-level potential, it is associated with the scalon S [2]. The second vector corresponds to the neutral would-be Goldstone boson G^0.

The mass matrix $\mathcal{M}_0^2$ is real and symmetric, and therefore it can be diagonalized by an orthogonal $(2n_\mathrm{H}+1) \times (2n_\mathrm{H}+1)$ matrix $\mathcal{R}$:

$$\mathcal{R}^\mathsf{T} \mathcal{M}_0^2 \mathcal{R} = \operatorname{diag}(0, M_1^2, \ldots, M_{2n_\mathrm{H}-1}^2, 0) \,. \tag{3.49}$$

Accordingly, the fields ρ_0, ρ and σ can be expressed in terms of the neutral scalar mass eigenfields S_b^0 $(b = 0, \ldots, 2n_\mathrm{H})$:

$$\begin{pmatrix}\rho_0 \\ \rho \\ \sigma\end{pmatrix} = \underbrace{\begin{pmatrix} r^\mathsf{T} \\ \operatorname{Re} R \\ \operatorname{Im} R \end{pmatrix}}_{\mathcal{R}} \begin{pmatrix} S_0^0 \\ \vdots \\ S_{2n_\mathrm{H}}^0 \end{pmatrix} . \tag{3.50}$$

We introduced a complex $n_\mathrm{H} \times (2n_\mathrm{H}+1)$ matrix $R = \operatorname{Re} R + \mathrm{i} \operatorname{Im} R$ and the vector $r \in \mathbb{R}^{2n_\mathrm{H}+1}$ to decompose $\mathcal{R}$. Using index notation, the decomposition reads

$$\mathcal{R}_{0b} = r_b \,, \quad \mathcal{R}_{kb} = \operatorname{Re} R_{kb} \,, \quad \mathcal{R}_{(k+n_\mathrm{H})b} = \operatorname{Im} R_{kb} \,, \quad 1 \le k \le n_\mathrm{H} \,, \quad 0 \le b \le 2n_\mathrm{H} \,. \tag{3.51}$$

We choose the numbering of the mass eigenfields such that $S_0^0 \equiv S$, and $S_{2n_\mathrm{H}}^0 \equiv G^0$, and therefore the first column of $\mathcal{R}$ is fixed to $r_0 = n_0$, $R_{k0} = n_k$, and the last one is determined by $r_{2n_\mathrm{H}} = 0$, $R_{k\,2n_\mathrm{H}} = \mathrm{i} n_k / \sqrt{1-n_0^2}$. Accordingly, equation (3.50) can be written as

$$\rho_0 = n_0 S + \sum_{b=1}^{2n_\mathrm{H}-1} r_b S_b^0 \,, \quad \rho_k + \mathrm{i}\sigma_k = n_k S + \sum_{b=1}^{2n_\mathrm{H}-1} R_{kb} S_b^0 + \frac{\mathrm{i} n_k}{\sqrt{1-n_0^2}} G^0 \,. \tag{3.52}$$

The matrix $\mathcal{R}$ is orthogonal and hence it has to fulfill the condition $\mathcal{R}^\mathsf{T}\mathcal{R} = \mathbb{1}_{(2n_\mathrm{H}+1)\times(2n_\mathrm{H}+1)}$, which explicitly written out reads

$$rr^\mathsf{T} + \mathrm{Re}(R^\dagger R) = \mathbb{1}_{(2n_\mathrm{H}+1)\times(2n_\mathrm{H}+1)}\,, \tag{3.53}$$

or the equivalent condition $\mathcal{R}\mathcal{R}^\mathsf{T} = \mathbb{1}_{(2n_\mathrm{H}+1)\times(2n_\mathrm{H}+1)}$, which is expressed explicitly by the relations

$$\begin{aligned} r^\mathsf{T} r = 1\,, \quad Rr = 0\,, \quad \mathrm{Re}\,R\,\mathrm{Re}\,R^\mathsf{T} = \mathbb{1}_{n_\mathrm{H}\times n_\mathrm{H}}\,, \\ \mathrm{Im}\,R\,\mathrm{Im}\,R^\mathsf{T} = \mathbb{1}_{n_\mathrm{H}\times n_\mathrm{H}}\,, \quad \mathrm{Re}\,R\,\mathrm{Im}\,R^\mathsf{T} = 0\,. \end{aligned} \tag{3.54}$$

As shown in appendix D.4, the last relations can be rewritten in the equivalent and more compact form

$$r^\mathsf{T} r = 1\,, \quad Rr = 0\,, \quad RR^\dagger = 2\cdot\mathbb{1}_{n_\mathrm{H}\times n_\mathrm{H}}\,, \quad RR^\mathrm{T} = 0\,. \tag{3.55}$$

3.1.3.4. Charged fermions

The mass terms of the charged fermions are obtained from the Yukawa coupling terms (3.13) and they read

$$-\,\overline{\ell'_\mathrm{R}} M_\ell \ell'_\mathrm{L} - \overline{d'_\mathrm{R}} M_d d'_\mathrm{L} - \overline{u'_\mathrm{R}} M_u u'_\mathrm{L} + \mathrm{h.c.} \tag{3.56}$$

with the mass matrices

$$M_\ell = \frac{1}{\sqrt{2}}\sum_{k=1}^{n_\mathrm{H}} v_k^* Y_k^{(\ell)}\,, \quad M_d = \frac{1}{\sqrt{2}}\sum_{k=1}^{n_\mathrm{H}} v_k^* Y_k^{(d)}\,, \quad M_u = \frac{1}{\sqrt{2}}\sum_{k=1}^{n_\mathrm{H}} v_k Y_k^{(u)}\,. \tag{3.57}$$

Using the fact that, by multiplication with unitary matrices from the left and from the right, every complex matrix can be brought to a diagonal form with non-negative entries [1], we can write

$$\hat{M}_\ell = U_\mathrm{R}^{(\ell)\,\dagger} M_\ell U_\mathrm{L}^{(\ell)} = \mathrm{diag}(m_e,\, m_\mu,\, m_\tau)\,, \tag{3.58}$$

$$\hat{M}_d = U_\mathrm{R}^{(d)\,\dagger} M_d U_\mathrm{L}^{(d)} = \mathrm{diag}(m_d,\, m_s,\, m_b)\,, \tag{3.59}$$

$$\hat{M}_u = U_\mathrm{R}^{(u)\,\dagger} M_u U_\mathrm{L}^{(u)} = \mathrm{diag}(m_u,\, m_c,\, m_t)\,, \tag{3.60}$$

where $U_\mathrm{R}^{(\ell)}$, $U_\mathrm{L}^{(\ell)}$, $U_\mathrm{R}^{(d)}$, $U_\mathrm{L}^{(d)}$, $U_\mathrm{R}^{(u)}$, $U_\mathrm{L}^{(u)}$ are unitary $n_\mathrm{L}\times n_\mathrm{L}$ matrices. Accordingly, we can express the fields ℓ'_L, ℓ'_R, d'_L, d'_R, u'_L, u'_R in terms of mass eigenfields:

$$\ell'_\mathrm{L} = U_\mathrm{L}^{(\ell)}\ell_\mathrm{L}\,, \quad d'_\mathrm{L} = U_\mathrm{L}^{(d)} d_\mathrm{L}\,, \quad u'_\mathrm{L} = U_\mathrm{L}^{(u)} u_\mathrm{L}\,, \tag{3.61}$$

$$\ell'_\mathrm{R} = U_\mathrm{R}^{(\ell)}\ell_\mathrm{R}\,, \quad d'_\mathrm{R} = U_\mathrm{R}^{(d)} d_\mathrm{R}\,, \quad u'_\mathrm{R} = U_\mathrm{R}^{(u)} u_\mathrm{R}\,. \tag{3.62}$$

The mass terms can then be written as

$$-\,\overline{\ell}\hat{M}_\ell \ell - \overline{d}\hat{M}_d d - \overline{u}\hat{M}_u u\,, \tag{3.63}$$

where we have introduced the Dirac fields $\ell = \ell_\mathrm{L} + \ell_\mathrm{R}$, $u = u_\mathrm{L} + u_\mathrm{R}$, $d = d_\mathrm{L} + d_\mathrm{R}$.

[1] This procedure is called singular value decomposition.

3.1.3.5. Neutrinos

The mass terms for the neutrinos are also obtained from the Yukawa coupling terms (3.13) and they read

$$-\frac{1}{2}\overline{\nu_{\mathrm{R}}}M_{\mathrm{R}}\left(\nu_{\mathrm{R}}\right)^{\mathrm{c}}-\overline{\nu_{\mathrm{R}}}M_{\mathrm{D}}\nu_{\mathrm{L}}+\text{h.c.} \tag{3.64}$$

with the mass matrices

$$M_{\mathrm{R}}=v_0 Y_{\mathrm{R}}\,,\quad M_{\mathrm{D}}=\frac{1}{\sqrt{2}}\sum_{k=1}^{n_{\mathrm{H}}}v_k Y_{\mathrm{D}\,k}\,. \tag{3.65}$$

If we define the left-handed field

$$\omega_{\mathrm{L}}=\begin{pmatrix}\nu_{\mathrm{L}}\\ (\nu_{\mathrm{R}})^{\mathrm{c}}\end{pmatrix}, \tag{3.66}$$

we can rewrite the neutrino mass terms (3.64) in the compact form

$$-\frac{1}{2}\overline{(\omega_{\mathrm{L}})^{\mathrm{c}}}\mathcal{M}_\nu\omega_{\mathrm{L}}+\text{h.c.} \tag{3.67}$$

with the symmetric $(n_{\mathrm{L}}+n_{\mathrm{R}})\times(n_{\mathrm{L}}+n_{\mathrm{R}})$ mass matrix

$$\mathcal{M}_\nu=\begin{pmatrix}0 & M_{\mathrm{D}}^{\mathsf{T}}\\ M_{\mathrm{D}} & M_{\mathrm{R}}\end{pmatrix}. \tag{3.68}$$

Since $\mathcal{M}_\nu$ is a complex symmetric matrix, it can be diagonalized by a unitary matrix $\mathcal{U}$ [20, 21]:

$$\mathcal{U}^{\mathsf{T}}\mathcal{M}_\nu\mathcal{U}=\hat{\mathcal{M}}_\nu\,, \tag{3.69}$$

where $\hat{\mathcal{M}}_\nu$ is diagonal and non-negative. The field ω_{L} can then be expressed in terms of the mass eigenfield $\hat{\omega}_{\mathrm{L}}$:

$$\omega_{\mathrm{L}}=\mathcal{U}\hat{\omega}_{\mathrm{L}}\,. \tag{3.70}$$

Following [3], we decompose the unitary $(n_{\mathrm{L}}+n_{\mathrm{R}})\times(n_{\mathrm{L}}+n_{\mathrm{R}})$ matrix $\mathcal{U}$ as

$$\mathcal{U}=\begin{pmatrix}U_{\mathrm{L}}\\ U_{\mathrm{R}}^*\end{pmatrix}. \tag{3.71}$$

The equivalent unitary conditions

$$\mathcal{U}^\dagger\mathcal{U}=\mathbb{1}_{(n_{\mathrm{L}}+n_{\mathrm{R}})\times(n_{\mathrm{L}}+n_{\mathrm{R}})}\quad\Leftrightarrow\quad\mathcal{U}\mathcal{U}^\dagger=\mathbb{1}_{(n_{\mathrm{L}}+n_{\mathrm{R}})\times(n_{\mathrm{L}}+n_{\mathrm{R}})} \tag{3.72}$$

may be written in terms of the $n_{\mathrm{L}}\times(n_{\mathrm{L}}+n_{\mathrm{R}})$ submatrix U_{L} and the $n_{\mathrm{R}}\times(n_{\mathrm{L}}+n_{\mathrm{R}})$ submatrix U_{R} [19]:

$$U_{\mathrm{L}}^\dagger U_{\mathrm{L}}+U_{\mathrm{R}}^{\mathsf{T}}U_{\mathrm{R}}^*=\mathbb{1}_{(n_{\mathrm{L}}+n_{\mathrm{R}})\times(n_{\mathrm{L}}+n_{\mathrm{R}})} \tag{3.73}$$

and

$$U_{\mathrm{L}}U_{\mathrm{L}}^\dagger=\mathbb{1}_{n_{\mathrm{L}}\times n_{\mathrm{L}}}\,,\quad U_{\mathrm{R}}U_{\mathrm{R}}^\dagger=\mathbb{1}_{n_{\mathrm{R}}\times n_{\mathrm{R}}}\,,\quad U_{\mathrm{L}}U_{\mathrm{R}}^{\mathsf{T}}=0_{n_{\mathrm{L}}\times n_{\mathrm{R}}}\,. \tag{3.74}$$

Defining the Majorana mass eigenfield

$$\chi = \hat{\omega}_{\mathrm{L}} + (\hat{\omega}_{\mathrm{L}})^{\mathrm{c}}\,, \tag{3.75}$$

the neutrino mass term (3.67) can be written as

$$-\frac{1}{2}\overline{\chi}\hat{\mathcal{M}}_\nu \chi\,. \tag{3.76}$$

Furthermore, the weak eigenfields ν_{L}, ν_{R} can be expressed in terms of Majorana mass eigenfields:

$$\nu_{\mathrm{L}} = U_{\mathrm{L}}\hat{\omega}_{\mathrm{L}} = U_{\mathrm{L}}P_{\mathrm{L}}\chi\,, \quad \nu_{\mathrm{R}} = U_{\mathrm{R}}(\hat{\omega}_{\mathrm{L}})^{\mathrm{c}} = U_{\mathrm{R}}P_{\mathrm{R}}\chi\,. \tag{3.77}$$

To get some useful relations, equation (3.69) can be written as $\mathcal{U}^{\mathsf{T}}\mathcal{M}_\nu = \hat{\mathcal{M}}_\nu\mathcal{U}^\dagger$, and using the definitions (3.71) and (3.68) we obtain

$$U_{\mathrm{R}}^\dagger M_{\mathrm{D}} = \hat{\mathcal{M}}_\nu U_{\mathrm{L}}^\dagger \quad \Leftrightarrow \quad M_{\mathrm{D}}^{\mathsf{T}} U_{\mathrm{R}}^* = U_{\mathrm{L}}^*\hat{\mathcal{M}}_\nu\,, \tag{3.78}$$

$$U_{\mathrm{L}}^{\mathsf{T}} M_{\mathrm{D}}^{\mathsf{T}} + U_{\mathrm{R}}^\dagger M_{\mathrm{R}} = \hat{\mathcal{M}}_\nu U_{\mathrm{R}}^{\mathsf{T}} \quad \Leftrightarrow \quad M_{\mathrm{D}}U_{\mathrm{L}} + M_{\mathrm{R}}U_{\mathrm{R}}^* = U_{\mathrm{R}}\hat{\mathcal{M}}_\nu\,. \tag{3.79}$$

Moreover, we can multiply the first equation of (3.78) by U_{L} from the right and the first equation of (3.79) by U_{R}^* from the right. Using (3.73), the sum of these two equations then reads

$$\hat{\mathcal{M}}_\nu = U_{\mathrm{R}}^\dagger M_{\mathrm{D}} U_{\mathrm{L}} + U_{\mathrm{L}}^{\mathsf{T}} M_{\mathrm{D}}^{\mathsf{T}} U_{\mathrm{R}}^* + U_{\mathrm{R}}^\dagger M_{\mathrm{R}} U_{\mathrm{R}}^*\,. \tag{3.80}$$

Let us now further analyze the diagonalization of the tree-level neutrino mass matrix (3.68). We closely follow the discussion in [3] and from now on assume $n_{\mathrm{L}} > n_{\mathrm{R}}$. The Dirac mass matrix M_{D} can be regarded as a linear mapping $M_{\mathrm{D}} : \mathbb{C}^{n_{\mathrm{L}}} \to \mathbb{C}^{n_{\mathrm{R}}}$. Using the rank-nullity theorem, we have

$$\dim\ker M_{\mathrm{D}} = \dim\ker \mathbb{C}^{n_{\mathrm{L}}} - \dim\mathrm{im}\, M_{\mathrm{D}} \geq n_{\mathrm{L}} - n_{\mathrm{R}}, \tag{3.81}$$

and therefore there exist at least $n_{\mathrm{L}} - n_{\mathrm{R}}$ orthonormal vectors $u_i' \in \mathbb{C}^{n_{\mathrm{L}}}$ $(1 \leq i \leq n_{\mathrm{L}} - n_{\mathrm{R}})$ which are eigenvectors of M_{D} with eigenvalue zero:

$$M_{\mathrm{D}} u_i' = 0\,. \tag{3.82}$$

Accordingly, the unitary matrix $\mathcal{U}$ can be written in the form

$$\mathcal{U} = \begin{pmatrix} u_1 & \cdots & u_{n_{\mathrm{L}}+n_{\mathrm{R}}} \end{pmatrix} \tag{3.83}$$

with $n_{\mathrm{L}}+n_{\mathrm{R}}$ orthonormal vectors $u_i \in \mathbb{C}^{n_{\mathrm{L}}+n_{\mathrm{R}}}$, where we can choose the first $n_{\mathrm{L}}-n_{\mathrm{R}}$ ones to be

$$u_i = \begin{pmatrix} u_i' \\ 0 \end{pmatrix}, \quad 1 \leq i \leq n_{\mathrm{L}} - n_{\mathrm{R}}\,. \tag{3.84}$$

The decomposition (3.71) then reads

$$\mathcal{U} = \begin{pmatrix} U'_{\mathrm{L}} & U''_{\mathrm{L}} \\ 0_{n_{\mathrm{R}}\times(n_{\mathrm{L}}-n_{\mathrm{R}})} & U''^{*}_{\mathrm{R}} \end{pmatrix} \tag{3.85}$$

with the $n_{\mathrm{L}} \times (n_{\mathrm{L}} - n_{\mathrm{R}})$ submatrix $U'_{\mathrm{L}} = \begin{pmatrix} u'_1 & \dots & u'_{n_{\mathrm{L}}-n_{\mathrm{R}}} \end{pmatrix}$, an $n_{\mathrm{L}} \times 2n_{\mathrm{R}}$ submatrix U''_{L} and an $n_{\mathrm{R}} \times 2n_{\mathrm{R}}$ submatrix U''^{*}_{R}. Plugging (3.85) into (3.69), we thus obtain [22]

$$\hat{\mathcal{M}}_\nu = \begin{pmatrix} 0 & U'^{\mathsf{T}}_{\mathrm{L}} M^{\mathsf{T}}_{\mathrm{D}} U''^{*}_{\mathrm{R}} \\ U''^{\dagger}_{\mathrm{R}} M_{\mathrm{D}} U'_{\mathrm{L}} & U''^{\mathsf{T}}_{\mathrm{L}} M^{\mathsf{T}}_{\mathrm{D}} U''^{*}_{\mathrm{R}} + U''^{\dagger}_{\mathrm{R}} M_{\mathrm{D}} U''_{\mathrm{L}} + U''^{\dagger}_{\mathrm{R}} M_{\mathrm{R}} U''^{*}_{\mathrm{R}} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \hat{M}' \end{pmatrix}, \tag{3.86}$$

where we used (3.82) and introduced the diagonal and positive $2n_{\mathrm{R}} \times 2n_{\mathrm{R}}$ matrix $\hat{M}'$. This shows that there are in general $n_{\mathrm{L}} - n_{\mathrm{R}}$ massless, and $2n_{\mathrm{R}}$ massive Majorana neutrinos at tree level. If we assume a much smaller mass scale for the Dirac mass matrix than for the Majorana mass matrix $m_{\mathrm{D}} \ll m_{\mathrm{R}}$, then n_{R} of the massive neutrinos are light with masses of order $m^2_{\mathrm{D}}/m_{\mathrm{R}}$, and n_{R} are heavy with masses of order m_{R} due to the seesaw mechanism [23–27].

3.1.4. Interaction terms

In this section we list the interaction terms of our theory expressed in terms of mass eigenfields. We omit the self-interaction terms of the gauge bosons and the strong interaction terms since they are not relevant for our considerations and they are the same as in the SM.

3.1.4.1. Fermionic gauge couplings

The fermionic gauge interaction terms are obtained from (3.11). The covariant derivative (3.7) can be expressed in terms of the vector boson mass eigenfields, and omitting the gluonic term it reads

$$D_\mu = \partial_\mu + \frac{\mathrm{i}g}{\sqrt{2}}\left(T_+ W^+_\mu + T_- W^-_\mu\right) + \frac{\mathrm{i}g}{\cos\theta_{\mathrm{W}}}\left(T_3 - \sin^2\theta_{\mathrm{W}} Q_{\mathrm{em}}\right) Z_\mu + \mathrm{i}e Q_{\mathrm{em}} A_\mu \tag{3.87}$$

with $T_\pm = T_1 \pm \mathrm{i}T_2$ and $e = g\sin\theta_{\mathrm{W}}$. In a straightforward manner, we then obtain the neutral current Lagrangian density

$$\begin{aligned}\mathcal{L}_{\mathrm{nc}} = -\frac{gZ_\mu}{\cos\theta_{\mathrm{W}}}\Bigg[&\frac{1}{4}\overline{\chi}\gamma^\mu\left(U^\dagger_{\mathrm{L}} U_{\mathrm{L}} P_{\mathrm{L}} - U^{\mathsf{T}}_{\mathrm{L}} U^{*}_{\mathrm{L}} P_{\mathrm{R}}\right)\chi + \overline{\ell}\gamma^\mu\left(-\frac{1}{2}P_{\mathrm{L}} + \sin^2\theta_{\mathrm{W}}\right)\ell \\ &+\overline{u}\gamma^\mu\left(\frac{1}{2}P_{\mathrm{L}} - \frac{2}{3}\sin^2\theta_{\mathrm{W}}\right)u + \overline{d}\gamma^\mu\left(-\frac{1}{2}P_{\mathrm{L}} + \frac{1}{3}\sin^2\theta_{\mathrm{W}}\right)d\Bigg];\end{aligned} \tag{3.88}$$

the charged current Lagrangian density

$$\mathcal{L}_{\mathrm{cc}} = -\frac{gW^+_\mu}{\sqrt{2}}\left(\overline{\chi}U^\dagger_{\mathrm{L}} U^{(\ell)}_{\mathrm{L}}\gamma^\mu P_{\mathrm{L}}\ell + \overline{u}V_{\mathrm{CKM}}\gamma^\mu P_{\mathrm{L}} d\right) + \mathrm{h.c.} \tag{3.89}$$

with the Cabibbo–Kobayashi–Maskawa matrix $V_{\mathrm{CKM}} = \left(U_{\mathrm{L}}^{(u)}\right)^{\dagger} U_{\mathrm{L}}^{(d)}$ [28, 29]; and the electromagnetic current Lagrangian density

$$\mathcal{L}_{\mathrm{em}} = -eA_{\mu}\left(\bar{\ell}\gamma^{\mu}\ell + \frac{1}{3}\bar{d}\gamma^{\mu}d - \frac{2}{3}\bar{u}\gamma^{\mu}u\right). \tag{3.90}$$

3.1.4.2. Scalar gauge couplings

The scalar gauge interactions for the non-conformal multi-Higgs-doublet model are calculated in [30] and we proceed in the same fashion for the conformal model. Using the covariant derivative (3.87) and some relations of section 3.1.3, the gauge-kinetic terms of the scalars (3.10) can be expressed in terms of mass eigenfields with the following result:

$$\begin{aligned}\mathcal{L}_{\mathrm{s}} = &\sum_{a=1}^{n_{\mathrm{H}}}\left(\partial_{\mu}H_a^-\right)\left(\partial^{\mu}H_a^+\right) + \frac{1}{2}\sum_{b=0}^{2n_{\mathrm{H}}}\left(\partial_{\mu}S_b^0\right)\left(\partial^{\mu}S_b^0\right) + M_W^2 W_{\mu}^+W^{-\,\mu} + \frac{1}{2}M_Z^2 Z_{\mu}Z^{\mu}\\ &+ \mathrm{i}M_W\left[W_{\mu}^+\left(\partial^{\mu}G^-\right) - W_{\mu}^-\left(\partial^{\mu}G^+\right)\right] - M_Z Z_{\mu}\left(\partial^{\mu}G^0\right) + \mathcal{L}_{VVS} + \mathcal{L}_{VSS} + \mathcal{L}_{VVSS}\,.\end{aligned} \tag{3.91}$$

The interaction terms which involve two gauge bosons and one scalar read

$$\begin{aligned}\mathcal{L}_{VVS} = &- g\left(M_W W_{\mu}^+W^{-\,\mu} + \frac{M_Z}{2\cos\theta_{\mathrm{W}}}Z_{\mu}Z^{\mu}\right)\sum_{b=0}^{2n_{\mathrm{H}}-1}\mathrm{Im}\left(R^{\dagger}R\right)_{2n_{\mathrm{H}}\,b}S_b^0\\ &+ \left(eM_W A^{\mu} - e\sin\theta_{\mathrm{W}}M_Z Z^{\mu}\right)\left(W_{\mu}^+G^- + W_{\mu}^-G^+\right),\end{aligned} \tag{3.92}$$

the terms with one gauge boson and two scalars are given by [2]

$$\begin{aligned}\mathcal{L}_{VSS} = &\frac{\mathrm{i}g}{2}\sum_{a=1}^{n_{\mathrm{H}}}\sum_{b=0}^{2n_{\mathrm{H}}}\left[\left(U^{\dagger}R\right)_{ab}W_{\mu}^+\left(S_b^0\overleftrightarrow{\partial}^{\mu}H_a^-\right) + \left(R^{\dagger}U\right)_{ba}W_{\mu}^-\left(H_a^+\overleftrightarrow{\partial}^{\mu}S_b^0\right)\right]\\ &+ \mathrm{i}\left(eA_{\mu} + \frac{g\cos(2\theta_{\mathrm{W}})}{2\cos\theta_{\mathrm{W}}}Z_{\mu}\right)\sum_{a=1}^{n_{\mathrm{H}}}H_a^+\overleftrightarrow{\partial}^{\mu}H_a^- + \frac{g}{4\cos\theta_{\mathrm{W}}}Z_{\mu}\sum_{b\neq b'=0}^{2n_{\mathrm{H}}}\mathrm{Im}(R^{\dagger}R)_{bb'}S_{b'}^0\overleftrightarrow{\partial}^{\mu}S_b^0\,,\end{aligned} \tag{3.93}$$

and finally we have interaction terms coupling two gauge bosons with two scalars:

$$\begin{aligned}\mathcal{L}_{VVSS} = &\left(\frac{g^2}{4}W_{\mu}^+W^{-\,\mu} + \frac{g^2}{8\cos^2\theta_{\mathrm{W}}}Z_{\mu}Z^{\mu}\right)\left(\sum_{b=0}^{2n_{\mathrm{H}}}(S_b^0)^2 - \sum_{b,b'=0}^{2n_{\mathrm{H}}-1}r_b r_{b'}S_b^0 S_{b'}^0\right)\\ &+ \left[\frac{g^2}{2}W_{\mu}^+W^{-\,\mu} + \frac{g^2\cos^2(2\theta_{\mathrm{W}})}{4\cos^2\theta_{\mathrm{W}}}Z_{\mu}Z^{\mu} + e^2A_{\mu}A^{\mu} + \frac{eg\cos(2\theta_{\mathrm{W}})}{\cos\theta_{\mathrm{W}}}Z_{\mu}A^{\mu}\right]\sum_{a=1}^{n_{\mathrm{H}}}H_a^+H_a^-\\ &+ \left(\frac{eg}{2}A_{\mu} - \frac{g^2\sin^2\theta_{\mathrm{W}}}{2\cos\theta_{\mathrm{W}}}Z_{\mu}\right)\sum_{a=1}^{n_{\mathrm{H}}}\sum_{b=0}^{2n_{\mathrm{H}}}S_b^0\left[\left(R^{\dagger}U\right)_{ba}W^{-\,\mu}H_a^+ + \left(U^{\dagger}R\right)_{ab}W^{+\,\mu}H_a^-\right].\end{aligned} \tag{3.94}$$

[2] We use the notation $S_{b'}^0\overleftrightarrow{\partial}^{\mu}S_b^0 = S_{b'}^0\partial^{\mu}S_b^0 - (\partial^{\mu}S_{b'}^0)S_b^0$.

Note that the terms mixing the gauge bosons with the Goldstone bosons,

$$\mathrm{i}M_W\left[W_\mu^+\left(\partial^\mu G^-\right)-W_\mu^-\left(\partial^\mu G^+\right)\right]-M_Z Z_\mu\left(\partial^\mu G^0\right), \tag{3.95}$$

get canceled once we add a gauge-fixing term (see appendix D.6).

3.1.4.3. Yukawa couplings

The Yukawa interactions are obtained from (3.13). The Yukawa couplings of the charged scalar $H_a^\pm$ $(1\le a\le n_\mathrm{H})$ are given by

$$\mathcal{L}_\mathrm{Y}(H_a^\pm)=H_a^+\left[\overline{\chi}\left(\widetilde{Y}_{\mathrm{D}\,a}P_\mathrm{L}-\widetilde{Y}_a^{(\ell)\dagger}P_\mathrm{R}\right)\ell+\overline{u}\left(\widetilde{Y}_a^{(u)}P_\mathrm{L}-\widetilde{Y}_a^{(d)\,\dagger}P_\mathrm{R}\right)d\right]+\mathrm{h.c.} \tag{3.96}$$

with

$$\widetilde{Y}_{\mathrm{D}\,a}=U_\mathrm{R}^\dagger\sum_{k=1}^{n_\mathrm{H}}Y_{\mathrm{D}\,k}U_{ka}U_\mathrm{L}^{(\ell)}\,,\quad \widetilde{Y}_a^{(\ell)}=U_\mathrm{R}^{(\ell)\,\dagger}\sum_{k=1}^{n_\mathrm{H}}U_{ak}^\dagger Y_k^{(\ell)}U_\mathrm{L}\,, \tag{3.97}$$

$$\widetilde{Y}_a^{(u)}=U_\mathrm{R}^{(u)\,\dagger}\sum_{k=1}^{n_\mathrm{H}}Y_k^{(u)}U_{ka}U_\mathrm{L}^{(d)}\,,\quad \widetilde{Y}_a^{(d)}=U_\mathrm{R}^{(d)\,\dagger}\sum_{k=1}^{n_\mathrm{H}}U_{ak}^\dagger Y_k^{(d)}U_\mathrm{L}^{(u)}\,. \tag{3.98}$$

The Yukawa couplings of the neutral scalar S_b^0 $(0\le b\le 2n_\mathrm{H})$ read

$$\begin{aligned}\mathcal{L}_\mathrm{Y}(S_b^0)=-\frac{S_b^0}{\sqrt{2}}\Bigg[&\overline{\ell}\left(\hat{Y}_b^{(\ell)}P_\mathrm{L}+\hat{Y}_b^{(\ell)\,\dagger}P_\mathrm{R}\right)\ell+\frac{1}{2}\overline{\chi}\left(F_bP_\mathrm{L}+F_b^\dagger P_\mathrm{R}\right)\chi\\&+\overline{d}\left(\hat{Y}_b^{(d)}P_\mathrm{L}+\hat{Y}_b^{(d)\,\dagger}P_\mathrm{R}\right)d+\overline{u}\left(\hat{Y}_b^{(u)}P_\mathrm{L}+\hat{Y}_b^{(u)\,\dagger}P_\mathrm{R}\right)u\Bigg],\end{aligned} \tag{3.99}$$

where

$$F_b=U_\mathrm{R}^\dagger\hat{Y}_{\mathrm{D}\,b}U_\mathrm{L}+U_\mathrm{L}^\intercal\hat{Y}_{\mathrm{D}\,b}^\intercal U_\mathrm{R}^*+\sqrt{2}U_\mathrm{R}^\dagger Y_\mathrm{R}U_\mathrm{R}^*r_\mathrm{b} \tag{3.100}$$

and

$$\hat{Y}_b^{(\ell)}=U_\mathrm{R}^{(\ell)\,\dagger}\sum_{k=1}^{n_\mathrm{H}}R_{bk}^\dagger Y_k^{(\ell)}U_\mathrm{L}^{(\ell)}\,,\quad \hat{Y}_{\mathrm{D}\,b}=\sum_{k=1}^{n_\mathrm{H}}Y_{\mathrm{D}\,k}R_{kb}\,, \tag{3.101}$$

$$\hat{Y}_b^{(d)}=U_\mathrm{R}^{(d)\,\dagger}\sum_{k=1}^{n_\mathrm{H}}Y_k^{(d)}R_{kb}^*U_\mathrm{L}^{(d)}\,,\quad \hat{Y}_b^{(u)}=U_\mathrm{R}^{(u)\,\dagger}\sum_{k=1}^{n_\mathrm{H}}Y_k^{(u)}R_{kb}U_\mathrm{L}^{(u)}\,. \tag{3.102}$$

3.1.4.4. Scalar self-interactions

Expressing the scalar potential (3.14) in terms of mass eigenfields (see [30] for the non-conformal multi-Higgs doublet model), we obtain, beside mass terms, the quartic interaction terms

$$\mathcal{L}_{SSSS} = - \sum_{a,a',c,c'=1}^{n_\mathrm{H}} \sum_{i,j,k,l=1}^{n_\mathrm{H}} \lambda_{ijkl} U^*_{ia} U_{ja'} U^*_{kc} U_{lc'} H^-_a H^+_{a'} H^-_c H^+_{c'}$$
$$- \sum_{b,b'=0}^{2n_\mathrm{H}} \sum_{a,a'=1}^{n_\mathrm{H}} \left(\sum_{i,j=1}^{n_\mathrm{H}} \kappa_{ij} U^*_{ia} U_{ja'} r_b r_{b'} + \sum_{i,j,k,l=1}^{n_\mathrm{H}} \lambda_{ijkl} U^*_{ia} U_{ja'} R^*_{kb} R_{lb'} \right) H^-_a H^+_{a'} S^0_b S^0_{b'}$$
$$- \sum_{b,b',d,d'=0}^{2n_\mathrm{H}} \left(\frac{\lambda_0}{4} r_b r_{b'} r_d r_{d'} + \sum_{i,j=1}^{n_\mathrm{H}} \frac{\kappa_{ij}}{2} r_b r_{b'} R^*_{id} R_{jd'} \right.$$
$$\left. + \sum_{i,j,k,l=1}^{n_\mathrm{H}} \frac{\lambda_{ijkl}}{4} R^*_{ib} R_{jb'} R^*_{kd} R_{ld'} \right) S^0_b S^0_{b'} S^0_d S^0_{d'} \,, \tag{3.103}$$

and the cubic interaction terms

$$\mathcal{L}_{SSS} = - \sum_{b=0}^{2n_\mathrm{H}} \sum_{a,a'=1}^{n_\mathrm{H}} \left[\sum_{i,j=1}^{n_\mathrm{H}} U^*_{ia} U_{ja'} \left(2 v_0 r_b \kappa_{ij} + \sum_{k,l=1}^{n_\mathrm{H}} \lambda_{ijkl} \left(v^*_k R_{lb} + R^*_{kb} v_l \right) \right) \right] H^-_a H^+_{a'} S^0_b$$
$$- \sum_{b,b',d=0}^{2n_\mathrm{H}} \left[\lambda_0 v_0 r_b r_{b'} r_d + \sum_{i,j=1}^{n_\mathrm{H}} \left(\kappa_{ij} v_0 r_b R^*_{ib'} R_{jd} + r_b r_{b'} \operatorname{Re}(\kappa_{ij} v^*_i R_{jd}) \right) \right.$$
$$\left. + \sum_{i,j,k,l=1}^{n_\mathrm{H}} \operatorname{Re}(\lambda_{ijkl} R^*_{ib} R_{jb'} v^*_k R_{ld}) \right] S^0_b S^0_{b'} S^0_d \,. \tag{3.104}$$

3.1.5. Scalon properties

The general structure of the scalon couplings has already been discussed in [2] and we recapitulate the calculation for our specific model here.

The couplings of the scalon to gauge bosons can be obtained from (3.91) by taking only interaction terms that include at least one scalon:

$$\left(M_W^2 W^+_\mu W^{-\,\mu} + \frac{1}{2} M_Z^2 Z_\mu Z^\mu \right) \left(\frac{2S}{V} + \frac{S^2}{V^2} - \frac{2n_0}{v^2} \sum_{b=1}^{2n_\mathrm{H}-1} r_b S^0_b S \right)$$
$$+ \mathrm{i} \frac{M_W}{V} W^+_\mu \left(S \overleftrightarrow{\partial}^\mu G^- \right) - \mathrm{i} \frac{M_W}{V} W^-_\mu \left(S \overleftrightarrow{\partial}^\mu G^+ \right) - \frac{M_Z}{V} Z_\mu \left(S \overleftrightarrow{\partial}^\mu G^0 \right) \tag{3.105}$$
$$+ \frac{eS}{V} \left(M_W A^\mu - \sin\theta_\mathrm{W} M_Z Z^\mu \right) \left(W^-_\mu G^+ + W^+_\mu G^- \right) .$$

The Yukawa couplings of the scalon are given by (3.99) with $b = 0$ and they explicitly read

$$\mathcal{L}_\mathrm{Y}(S) = -\frac{S}{V} \left(\overline{\ell} \hat{M}_\ell \ell + \frac{1}{2} \overline{\chi} \hat{\mathcal{M}}_\nu \chi + \overline{d} \hat{M}_d d + \overline{u} \hat{M}_u u \right) . \tag{3.106}$$

Finally, the scalar interactions that include at least one scalon are obtained from (3.103) and (3.104), and using (3.26-3.28), they read

$$-\left(\sum_{a=1}^{2n_{\mathrm{H}}-1}\mu_a^2 H_a^+ H_a^- + \frac{1}{2}\sum_{b=1}^{2n_{\mathrm{H}}-1} M_b^2\left(S_b^0\right)^2\right)\left(\frac{2S}{V}+\frac{S^2}{V^2}\right)+\frac{S}{V}\mathcal{L}'_{SSS}\,, \tag{3.107}$$

where $\mathcal{L}'_{SSS}$ is the cubic interaction term (3.104) without the terms which include the scalon $(b, b', d \neq 0)$. Note that the last term (the interaction of one scalon with three other scalars) is missing in [2].

The scalon interactions take a surprisingly simple form. This can be explained by the fact that the scalon S and the fundamental order parameter V enter the Lagrangian density only in the combination $\phi = V + S$, which has already been noticed in [2].

The scalon interactions (3.105-3.107) reveal that the preferred decay channel of the scalon is the decay into a pair of the heaviest possible particles. This is a model-independent feature of the scalon [2] and is also a property of the Higgs boson in the SM. This opens the question if the scalon could be identified with the H^0 particle with a mass of 125 GeV discovered at the LHC [31, 32]. The SM predictions for the decay properties of the Higgs particle are consistent with current experiments. Since the H^0 signal strength (for combined final states) of 1.10 ± 0.11 [17] is quite close to one, the couplings of the scalon should be similar to the couplings of the Higgs in the SM. However, they are related by a factor of v/V and since we want to implement the seesaw mechanism, we need to assume $|v_k| \ll v_0$, or equivalently $v \ll V$, to obtain the necessary mass hierarchy in the neutrino mass matrices (3.65). Therefore, the state S cannot correspond to the particle H^0 and hence one of the other neutral scalars S_b^0 $(1 \leq b \leq 2n_{\mathrm{H}} - 1)$ has to be identified with it.

The mass formula of the scalon (2.164) in the case of our model reads

$$M_S^2 = \frac{1}{8\pi V^2}\left(M_{H^0}^4 + 3M_Z^4 + 6M_W^4 - 12m_t^4 + \sum_{S_b^0 \neq H^0} M_b^4 + 2\sum_a \mu_a^4 - 2\,\mathrm{Tr}\,\hat{\mathcal{M}}_\nu^4\right), \tag{3.108}$$

where we omitted the tiny contributions from charged fermions lighter than the top quark. If we assume the mass hierarchy $m_{\mathrm{D}} \ll m_{\mathrm{R}}$ in the neutrino mass matrix (3.68), the contribution of the neutrinos to the scalon mass is mainly determined by the masses of the heavy neutrinos: $\mathrm{Tr}\,\hat{\mathcal{M}}_\nu^4 \simeq \mathrm{Tr}(M_{\mathrm{R}}^* M_{\mathrm{R}})^2$. As we do not know the masses of the scalars (beside the mass of H^0) and the masses of the neutrinos, equation (3.108) allows for a broad spectrum of scalon mass values; we only know that it is suppressed by the one-loop factor present in (3.108).

Demanding a positive squared scalon mass M_S^2, yields the condition

$$\sum_{S_b^0 \neq H^0} M_b^2 + 2\sum_a \mu_a^4 > \underbrace{-M_{H^0}^4 - 3M_Z^4 - 6M_W^4 + 12m_t^4}_{\simeq (317\,\mathrm{GeV})^4} + 2\,\mathrm{Tr}\,\hat{\mathcal{M}}_\nu^4\,. \tag{3.109}$$

Therefore, at least some of the scalar states need to be heavy, with masses of order m_{R} or larger, to compensate the contribution of the heavy neutrinos.

3.1.6. Neutrino masses at one-loop order

In this section we want to calculate the neutrino masses at one-loop order. However, in contrast to [33], we are only interested in the one-loop mass corrections of the neutrinos with vanishing tree-level masses.

At one-loop level the neutrino mass matrix gets shifted [33]

$$\mathcal{M}_\nu \to \mathcal{M}_\nu^{(1)} = \mathcal{M}_\nu + \delta\mathcal{M}_\nu\,, \quad \delta\mathcal{M}_\nu = \begin{pmatrix} \delta M_\mathrm{L} & \delta M_\mathrm{D}^\mathsf{T} \\ \delta M_\mathrm{D} & \delta M_\mathrm{D} \end{pmatrix}. \tag{3.110}$$

It can be diagonalized by a unitary transformation [3]

$$\hat{\mathcal{M}}_\nu^{(1)} = (\mathcal{U}\mathcal{V})^\mathsf{T}\,\mathcal{M}_\nu^{(1)}\mathcal{U}\mathcal{V} = \begin{pmatrix} \hat{M}_0 & 0 \\ 0 & \hat{M}'^{\,(1)} \end{pmatrix} = \hat{\mathcal{M}}_\nu + \delta\hat{\mathcal{M}}_\nu\,, \tag{3.111}$$

where $\mathcal{V} - \mathbb{1}$ is of one-loop order, such that the $(n_\mathrm{L} - n_\mathrm{R}) \times (n_\mathrm{L} - n_\mathrm{R})$ matrix

$$\hat{M}_0 = U_\mathrm{L}'^{\mathsf{T}}\delta M_\mathrm{L} U_\mathrm{L}' \tag{3.112}$$

is diagonal and non-negative and the $2n_\mathrm{R} \times 2n_\mathrm{R}$ matrix $\hat{M}'^{\,(1)}$ is diagonal and positive. The details of this transformation are given in appendix D.5.

The mass shifts $\delta\hat{\mathcal{M}}_\nu$ are determined by the neutrino self-energy matrix $\Sigma(p)$ in the basis of the tree-level mass eigenfields, which can be written as [33]

$$\Sigma(p) = A_\mathrm{L}(p^2)\not{p}P_\mathrm{L} + A_\mathrm{R}(p^2)\not{p}P_\mathrm{R} + B_\mathrm{L}(p^2)P_\mathrm{L} + B_\mathrm{R}(p^2)P_\mathrm{R}\,, \tag{3.113}$$

with the neutrino four-momentum p. The complex $(n_\mathrm{L} + n_\mathrm{R}) \times (n_\mathrm{L} + n_\mathrm{R})$ matrices A_L, A_R, B_L, B_R have to fulfill the conditions

$$A_\mathrm{L}^\dagger = A_\mathrm{L}\,, \quad A_\mathrm{R}^\dagger = A_\mathrm{R}\,, \quad B_\mathrm{L}^\dagger = B_\mathrm{R}\,, \quad A_\mathrm{L}^\mathsf{T} = A_\mathrm{R}\,, \quad B_\mathrm{L}^\mathsf{T} = B_\mathrm{L}\,, \quad B_\mathrm{R}^\mathsf{T} = B_\mathrm{R}\,. \tag{3.114}$$

According to [34, 35], the one-loop mass shifts $\delta\hat{\mathcal{M}}_\nu$ are given by

$$\delta\hat{\mathcal{M}}_{\nu\,ii} = \frac{1}{2}\left[\hat{\mathcal{M}}_{\nu\,ii}A_{\mathrm{L}\,ii}(\hat{\mathcal{M}}^2_{\nu\,ii}) + \hat{\mathcal{M}}_{\nu\,ii}A_{\mathrm{R}\,ii}(\hat{\mathcal{M}}^2_{\nu\,ii}) + B_{\mathrm{L}\,ii}(\hat{\mathcal{M}}^2_{\nu\,ii}) + B_{\mathrm{R}\,ii}(\hat{\mathcal{M}}^2_{\nu\,ii})\right]. \tag{3.115}$$

In particular, the masses of those neutrinos which were massless at tree-level are given by

$$\hat{M}_{0\,ii} = \mathrm{Re}\,B_{\mathrm{L}\,ii}(0)\,, \quad 1 \leq i \leq n_\mathrm{L} - n_\mathrm{R}\,. \tag{3.116}$$

Therefore, we can use (3.116) to directly calculate the one-loop neutrino masses $\hat{M}_0$ instead of following the calculations in [3, 4], where they are obtained by first computing the mass shift δM_L and subsequently using (3.112).

The Feynman diagrams contributing to the neutrino self-energy are shown in figure 3.1. The Feynman rules for the propagators are given in [36] and the Feynman rules for the vertices can be read off from the respective terms in section 3.1.4. Concerning the Feynman rules for Majorana neutrinos we rely on the discussion in [37].

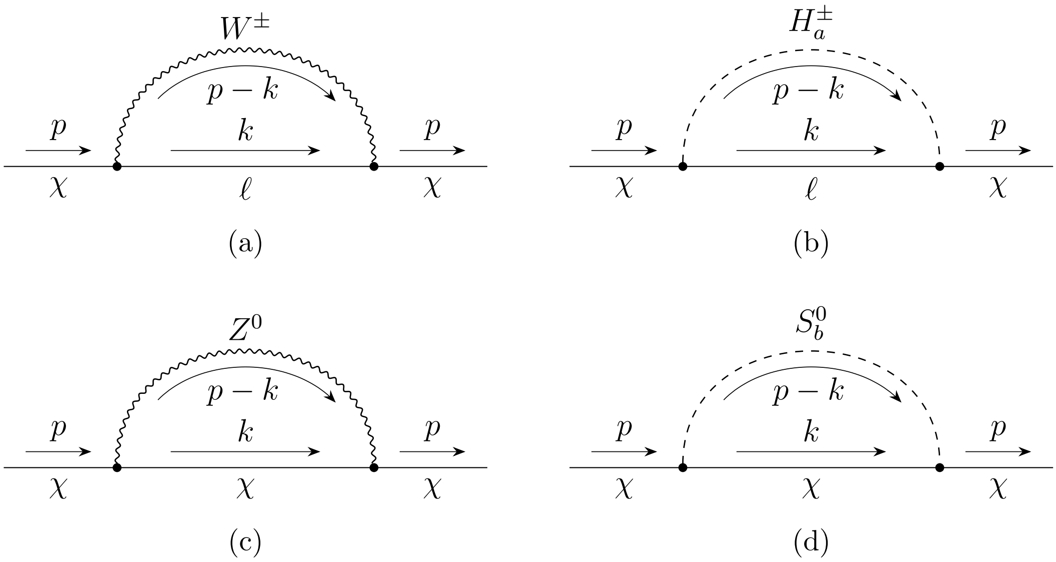


Figure 3.1.: Contributions to the neutrino self-energy at one-loop level

Note that we have not discussed gauge-fixing yet; or phrasing it differently, we have worked in Landau gauge so far. To be consistent with [4, 33], we want to perform our calculations in a general R_ξ gauge. Following [36], we add an appropriate gauge-fixing term to our Lagrangian density (see appendix D.6). Thereafter, the Goldstone bosons G^0 and $G^\pm$ get the unphysical masses $\sqrt{\xi_Z}M_Z$ and $\sqrt{\xi_W}M_W$, and the propagators of the gauge bosons in this gauge are given by (3.118) and (3.123).

The contribution to the neutrino self-energy from the $W^\pm$ boson (see figure 3.1a) reads

$$\Sigma_{ij}^{(W^\pm)}(p) = \frac{\mathrm{i}g^2}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\sum_{l=1}^{n_\mathrm{L}} \frac{S^W_{\mu\nu}(p-k)}{k^2-\hat{M}^2_{\ell\, ll}}\gamma^\mu \not{k}\gamma^\nu \left[\left(U_\mathrm{L}^\dagger U_\mathrm{L}^{(\ell)}\right)_{il}\left(U_\mathrm{L}^{(\ell)\,\dagger}U_\mathrm{L}\right)_{lj}P_\mathrm{L}\right.$$
$$\left. + \left(U_\mathrm{L}^\mathsf{T} U_\mathrm{L}^{(\ell)\,\mathsf{T}}\right)_{il}\left(U_\mathrm{L}^{(\ell)\,*}U_\mathrm{L}^*\right)_{lj}P_\mathrm{R}\right] \quad (3.117)$$

with the $W^\pm$ propagator [36]

$$\mathrm{i}S^W_{\mu\nu}(k) = \frac{-\mathrm{i}\eta_{\mu\nu}}{k^2-M_W^2} + \frac{\mathrm{i}k_\mu k_\nu}{M_W^2}\left(\frac{1}{k^2-M_W^2} - \frac{1}{k^2-\xi_W M_W^2}\right). \quad (3.118)$$

For our considerations, we are interested in the case where $p=0$. Then, the integrand is an odd function of k and hence $\Sigma_{ij}^{(W^\pm)}(0)=0$.

The neutrino self-energy contribution involving the charged scalar $H_a^\pm$ (see figure

3.1b) is given by

$$\Sigma_{ij}^{(H_a^\pm)}(p) = \mathrm{i} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k-p)^2 - \mu_a^2} \sum_{l=1}^{n_\mathrm{L}} \frac{1}{k^2 - \hat{M}_{\ell\, ll}^2}$$
$$\times \Big[\Big(\big(\tilde{Y}_{\mathrm{D}\,a}\big)_{il} P_\mathrm{L} - \big(\tilde{Y}_a^{(\ell)\,\dagger}\big)_{il} P_\mathrm{R} \Big) (\not{k} + \hat{M}_{\ell\, ll}) \Big(\big(\tilde{Y}_{\mathrm{D}\,a}^\dagger\big)_{lj} P_\mathrm{R} - \big(\tilde{Y}_a^{(\ell)}\big)_{lj} P_\mathrm{L} \Big)$$
$$+ \Big(\big(\tilde{Y}_{\mathrm{D}\,a}^*\big)_{il} P_\mathrm{R} - \big(\tilde{Y}_a^{(\ell)\,\mathsf{T}}\big)_{il} P_\mathrm{L} \Big) (\not{k} + \hat{M}_{\ell\, ll}) \Big(\big(\tilde{Y}_{\mathrm{D}\,a}^\mathsf{T}\big)_{lj} P_\mathrm{L} - \big(\tilde{Y}_a^{(\ell)\,*}\big)_{lj} P_\mathrm{R} \Big) \Big] . \tag{3.119}$$

For $p = 0$ the terms proportional to $\not{k}$ vanish and we are left with

$$\Sigma_{ii}^{(H_a^\pm)}(0) = -2\mathrm{i} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2 - \mu_a^2} \sum_{l=1}^{n_\mathrm{L}} \frac{1}{k^2 - \hat{M}_{\ell\, ll}^2} \Big[\big(\tilde{Y}_{\mathrm{D}\,a}\big)_{il} \hat{M}_{\ell\, ll} \big(\tilde{Y}_a^{(\ell)}\big)_{li} P_\mathrm{L}$$
$$+ \big(\tilde{Y}_a^{(\ell)\,\dagger}\big)_{il} \hat{M}_{\ell\, ll} \big(\tilde{Y}_{\mathrm{D}\,a}^\dagger\big)_{li} P_\mathrm{R} \Big] \tag{3.120}$$

for the diagonal elements. We are only interested in the diagonal elements with $1 \le i \le n_\mathrm{L} - n_\mathrm{R}$ and thus, using the definition (3.97) of the matrix $\tilde{Y}_{\mathrm{D\,a}}$ and the decomposition (3.85) of the matrix $\mathcal{U}$, the contribution from the charged scalars vanish:

$$\Sigma_{ii}^{(H_a^\pm)}(0) = 0 \,, \quad 1 \le i \le n_\mathrm{L} - n_\mathrm{R} \,. \tag{3.121}$$

For the neutrino self-energy induced by the Z^0 boson (see figure 3.1c), we get

$$\Sigma_{ij}^{(Z^0)}(p) = \frac{\mathrm{i} g^2}{4 \cos^2 \theta_\mathrm{W}} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \sum_{n=1}^{n_\mathrm{L}+n_\mathrm{R}} \frac{S_{\mu\nu}^Z(k-p)}{k^2 - \hat{\mathcal{M}}_{\nu\, nn}^2} \gamma^\mu \Big[P_\mathrm{L} (U_\mathrm{L}^\dagger U_\mathrm{L})_{in} - P_\mathrm{R} \big(U_\mathrm{L}^\mathsf{T} U_\mathrm{L}^*\big)_{in} \Big]$$
$$\times (\not{k} + \hat{\mathcal{M}}_{\nu\, nn}) \gamma^\nu \Big[P_\mathrm{L} (U_\mathrm{L}^\dagger U_\mathrm{L})_{nj} - P_\mathrm{R} \big(U_\mathrm{L}^\mathsf{T} U_\mathrm{L}^*\big)_{nj} \Big] \tag{3.122}$$

with the Z^0 propagator [36]

$$\mathrm{i} S_{\mu\nu}^Z(k) = \frac{-\mathrm{i} \eta_{\mu\nu}}{k^2 - M_Z^2} + \frac{\mathrm{i} k_\mu k_\nu}{M_Z^2} \left(\frac{1}{k^2 - M_Z^2} - \frac{1}{k^2 - \xi_Z M_Z^2} \right) . \tag{3.123}$$

Setting $p = 0$ and considering only the diagonal elements with $1 \le i \le n_\mathrm{L} - n_\mathrm{R}$, we then obtain [3]

$$\Sigma_{ii}^{(Z^0)}(0) = \frac{\mathrm{i} g^2}{4 \cos^2 \theta_\mathrm{W}} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \left[\frac{d}{k^2 - M_Z^2} + \frac{k^2}{M_Z^2} \left(\frac{1}{k^2 - \xi_Z M_Z^2} - \frac{1}{k^2 - M_Z^2} \right) \right]$$
$$\times \left(U_\mathrm{L}'^{\mathsf{T}} U_\mathrm{L}^* \frac{\hat{\mathcal{M}}_\nu}{k^2 - \hat{\mathcal{M}}_\nu^2} U_\mathrm{L}^\dagger U_\mathrm{L}' P_\mathrm{L} + U_\mathrm{L}'^{\dagger} U_\mathrm{L} \frac{\hat{\mathcal{M}}_\nu}{k^2 - \hat{\mathcal{M}}_\nu^2} U_\mathrm{L}^\mathsf{T} U_\mathrm{L}'^{*} P_\mathrm{R} \right)_{ii} = 0 \,, \tag{3.124}$$

[3] We use the notation $f(\hat{\mathcal{M}}_\nu) = \mathrm{diag}\Big(f(\hat{\mathcal{M}}_{\nu\, 11}), \ldots, f(\hat{\mathcal{M}}_{\nu\, (n_\mathrm{L}+n_\mathrm{R})(n_\mathrm{L}+n_\mathrm{R})}) \Big)$

where we used

$$\hat{\mathcal{M}}_\nu U_\mathrm{L}^\dagger U_\mathrm{L}' \overset{(3.78)}{=} U_\mathrm{R}^\dagger M_\mathrm{D} U_\mathrm{L}' \overset{(3.82)}{=} 0\,. \tag{3.125}$$

The contribution to the neutrino self-energy from the neutral scalar S_b^0 (see figure 3.1d) reads

$$\Sigma_{ij}^{(S_b^0)}(p) = \frac{\mathrm{i}}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{(k-p)^2 - M_b^2}\sum_{n=1}^{n_\mathrm{L}+n_\mathrm{R}} \frac{1}{k^2 - \hat{\mathcal{M}}_{\nu\,nn}^2}\Big[\big(F_b\big)_{in}P_\mathrm{L} + \big(F_b\big)_{in}^\dagger P_\mathrm{R}\Big]$$
$$\times\,(\not{k} + \hat{\mathcal{M}}_{\nu\,nn})\Big[\big(F_b\big)_{nj}P_\mathrm{L} + \big(F_b^\dagger\big)_{nj}P_\mathrm{R}\Big]. \tag{3.126}$$

Setting $p=0$ and considering only diagonal elements with $1 \le i \le n_\mathrm{L} - n_\mathrm{R}$, we then get for the terms proportional to P_L,

$$\begin{aligned}
B_{\mathrm{L}\,ii}^{(S_b^0)}(0) &= \frac{i}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{k^2 - M_b^2}\left(U_\mathrm{L}'^\mathsf{T}\hat{Y}_{\mathrm{D}\,b}^\mathsf{T}U_\mathrm{R}^*\frac{\hat{\mathcal{M}}_\nu}{k^2-\hat{\mathcal{M}}_\nu^2}U_\mathrm{R}^\dagger\hat{Y}_{\mathrm{D}\,b}U_\mathrm{L}'\right)_{ii}\\
&= -\frac{1}{2}\int\frac{\mathrm{d}^d k_\mathrm{E}}{(2\pi)^d}\int_0^1 \mathrm{d}\alpha\left(U_\mathrm{L}'^\mathsf{T}\hat{Y}_{\mathrm{D}\,b}^\mathsf{T}U_\mathrm{R}^*\frac{\hat{\mathcal{M}}_\nu}{\Big[k_\mathrm{E}^2 + \alpha\hat{\mathcal{M}}_\nu^2 + (1-\alpha)M_b^2\Big]^2}U_\mathrm{R}^\dagger\hat{Y}_{\mathrm{D}\,b}U_\mathrm{L}'\right)_{ii}\\
&= -\frac{\Gamma\left(2-\frac{d}{2}\right)}{2(4\pi)^{d/2}}\int_0^1\mathrm{d}\alpha\left(U_\mathrm{L}'^\mathsf{T}\hat{Y}_{\mathrm{D}\,b}^\mathsf{T}U_\mathrm{R}^*\hat{\mathcal{M}}_\nu\Big[\alpha\hat{\mathcal{M}}_\nu^2 + (1-\alpha)M_b^2\Big]^{\frac{d-4}{2}}U_\mathrm{R}^\dagger\hat{Y}_{\mathrm{D}\,b}U_\mathrm{L}'\right)_{ii}\\
&= \frac{1}{2(4\pi)^2}\left(U_\mathrm{L}'^\mathsf{T}\hat{Y}_{\mathrm{D}\,b}^\mathsf{T}U_\mathrm{R}^*\hat{\mathcal{M}}_\nu\int_0^1\mathrm{d}\alpha\left[\ln\Big(\alpha\hat{\mathcal{M}}_\nu^2 + (1-\alpha)M_b^2\Big)\right]U_\mathrm{R}^\dagger\hat{Y}_{\mathrm{D}\,b}U_\mathrm{L}'\right)_{ii}\\
&\quad + \frac{-\frac{1}{\epsilon}+\gamma-\ln(4\pi)}{2(4\pi)^2}\Big(U_\mathrm{L}'^\mathsf{T}\hat{Y}_{\mathrm{D}\,b}^\mathsf{T}U_\mathrm{R}^*\hat{\mathcal{M}}_\nu U_\mathrm{R}^\dagger\hat{Y}_{\mathrm{D}\,b}U_\mathrm{L}'\Big)_{ii}\,,
\end{aligned} \tag{3.127}$$

where in the first step we used the Feynman parametrization and performed a Wick rotation[4], in the second step we used (B.15) and in the last step we expanded around $d = 4 - 2\epsilon$. The remaining integral can readily be solved

$$\int_0^1 \mathrm{d}\alpha\ln\Big[\alpha\hat{\mathcal{M}}_\nu^2 + (1-\alpha)M_b^2\Big] = \frac{\ln\Big(\hat{\mathcal{M}}_\nu^2/M_b^2\Big)}{\hat{\mathcal{M}}_\nu^2/M_b^2 - 1} + \ln\hat{\mathcal{M}}_\nu^2 - 1\,. \tag{3.128}$$

If we add up all neutral scalar contributions, the terms independent of M_b^2 drop out, since

$$\sum_{b=0}^{2n_\mathrm{H}}\hat{Y}_{\mathrm{D}\,b}^\mathsf{T}\hat{Y}_{\mathrm{D}\,b} \overset{(3.101)}{=} \sum_{b=0}^{2n_\mathrm{H}}\sum_{k,k'=1}^{n_\mathrm{H}} Y_{\mathrm{D}\,k}^\mathsf{T}R_{kb}Y_{\mathrm{D}\,k'}R_{k'b} \overset{(3.55)}{=} 0\,. \tag{3.129}$$

[4]So far we have omitted the $\mathrm{i}0^+$ terms in the propagators. For the Wick rotation we need a prescription of how to close the contour and therefore we recall that we actually have $k^2 \to k^2 + \mathrm{i}0^+$ in the denominators.

We therefore get

$$\sum_{b=0}^{2n_{\rm H}} B_{ii}^{(S_b^0)}(0) = \sum_{b=1}^{2n_{\rm H}-1} \frac{1}{2(4\pi)^2} \left(U_{\rm L}'^{\mathsf{T}} \hat{Y}_{{\rm D}\,b}^{\mathsf{T}} U_{\rm R}^* \hat{\mathcal{M}}_\nu \frac{\ln\left(\hat{\mathcal{M}}_\nu^2/M_b^2\right)}{\hat{\mathcal{M}}_\nu^2/M_b^2 - 1} U_{\rm R}^\dagger \hat{Y}_{{\rm D}\,b} U_{\rm L}' \right)_{ii} , \tag{3.130}$$

where we used that the contributions from the scalon and the neutral Goldstone boson vanish due to $\hat{Y}_{{\rm D}\,0}, \hat{Y}_{{\rm D}\,2n_{\rm H}} \propto M_{\rm D}$ and $M_{\rm D} U_{\rm L}' \overset{(3.82)}{=} 0$. Using that the $n_{\rm L} - n_{\rm R}$ orthonormal vectors in $U_{\rm L}' = \begin{pmatrix} u_1' & \cdots & u_{n_{\rm L}-n_{\rm R}}' \end{pmatrix}$ can be chosen such that the matrix

$$\frac{1}{2(4\pi)^2} U_{\rm L}'^{\mathsf{T}} \left(\sum_{b=1}^{2n_{\rm H}-1} \hat{Y}_{{\rm D}\,b}^{\mathsf{T}} U_{\rm R}^* \hat{\mathcal{M}}_\nu \frac{\ln\left(\hat{\mathcal{M}}_\nu^2/M_b^2\right)}{\hat{\mathcal{M}}_\nu^2/M_b^2 - 1} U_{\rm R}^\dagger \hat{Y}_{{\rm D}\,b} \right) U_{\rm L}' \tag{3.131}$$

is diagonal and non-negative without spoiling the tree-level condition $M_{\rm D} u_i' = 0$,[5] we finally get

$$\hat{M}_0 = \frac{1}{2(4\pi)^2} U_{\rm L}'^{\mathsf{T}} \left(\sum_{b=1}^{2n_{\rm H}-1} \hat{Y}_{{\rm D}\,b}^{\mathsf{T}} U_{\rm R}^* \hat{\mathcal{M}}_\nu \frac{\ln\left(\hat{\mathcal{M}}_\nu^2/M_b^2\right)}{\hat{\mathcal{M}}_\nu^2/M_b^2 - 1} U_{\rm R}^\dagger \hat{Y}_{{\rm D}\,b} \right) U_{\rm L}' \,. \tag{3.132}$$

Note that our result is gauge-independent and finite, as one would expect. If we assume the mass hierarchy $m_{\rm D} \ll m_{\rm R}$, (3.132) can be approximated as (see appendix D.7)

$$\hat{M}_0 \simeq \frac{1}{2(4\pi)^2} U_{\rm L}'^{\mathsf{T}} \left(\sum_{b=1}^{2n_{\rm H}-1} \hat{Y}_{{\rm D}\,b}^{\mathsf{T}} M_{\rm R}^* \frac{\ln(M_{\rm R} M_{\rm R}^*/M_b^2)}{M_{\rm R} M_{\rm R}^*/M_b^2 - 1} \hat{Y}_{{\rm D}\,b} \right) U_{\rm L}' \,. \tag{3.133}$$

The equations (3.132) and (3.133) are in agreement with the results in [3, 4]; in particular there are no additional contributions from the scalon compared to the non-conformal multi-Higgs-doublet model discussed in [3].

We have achieved our goal to find a conformal model which combines the seesaw mechanism with radiative neutrino mass generation. The analysis concerning the neutrino mass spectrum discussed in [3] still holds for our model. Therefore, in general (i.e. without any relations among the Yukawa matrices $\hat{Y}_{{\rm D}\,b}$), $n_{\rm L} - n_{\rm R} - \max(0, n_{\rm L} - n_{\rm H} n_{\rm R})$ of the $n_{\rm L} - n_{\rm R}$ neutrinos with vanishing tree-level mass acquire a mass at one-loop level, whereas $\max(0, n_{\rm L} - n_{\rm H} n_{\rm R})$ stay massless.

Although our final result (3.133) has the same form as the result for the non-conformal model in [3], the one-loop masses $\hat{M}_0$ are actually slightly different since the transformation matrix R is altered due to the additional scalar singlet. For example, the third equation in (3.55) $RR^{\mathsf{T}} = 0$ can be rewritten as

$$\sum_{b=1}^{2n_{\rm H}-1} R_{kb} R_{k'b} = -R_{k\,2n_{\rm H}} R_{k'\,2n_{\rm H}} - R_{k0} R_{k'0} = \frac{v_k v_{k'}}{v^2} \left(1 - \frac{v^2}{V^2}\right) , \tag{3.134}$$

which differs by a factor of $1 - v^2/V^2 \simeq 1$ from the corresponding equation (3.12) in [3].

[5] See appendix D.5 for details.

3.2. A conformal two-Higgs-doublet model

In this section we analyze the simplest version of the model we discussed in the last section in more detail. This means that we consider two Higgs-doublets ($n_{\mathrm{H}} = 2$)[6] and only one right-handed neutrino singlet ($n_{\mathrm{R}} = 1$).

3.2.1. Scalar sector

The scalar potential of this model reads

$$\begin{aligned} V_0 = \frac{\lambda_0}{4}\varphi_0^4 &+ \lambda_1\left(\Phi_1^\dagger\Phi_1\right)^2 + \lambda_2\left(\Phi_2^\dagger\Phi_2\right)^2 + 2\lambda_3\Phi_1^\dagger\Phi_1\Phi_2^\dagger\Phi_2 \\ &+ 2\lambda_4\Phi_1^\dagger\Phi_2\Phi_2^\dagger\Phi_1 + \left[\lambda_5\left(\Phi_1^\dagger\Phi_2\right)^2 + \lambda_5^*\left(\Phi_2^\dagger\Phi_1\right)^2\right] \\ &+ 2\left(\lambda_6\Phi_1^\dagger\Phi_2 + \lambda_6^*\Phi_2^\dagger\Phi_1\right)\Phi_1^\dagger\Phi_1 + 2\left(\lambda_7\Phi_1^\dagger\Phi_2 + \lambda_7^*\Phi_2^\dagger\Phi_1\right)\Phi_2^\dagger\Phi_2 \\ &+ \varphi_0^2\left(\kappa_1\Phi_1^\dagger\Phi_1 + \kappa_2\Phi_2^\dagger\Phi_2 + \kappa_3\Phi_1^\dagger\Phi_2 + \kappa_3^*\Phi_2^\dagger\Phi_1\right), \end{aligned} \tag{3.135}$$

where we introduced the coupling constants $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \kappa_1, \kappa_2 \in \mathbb{R}$ and $\lambda_5, \lambda_6, \lambda_7$, $\kappa_3 \in \mathbb{C}$. The relations between these new coupling constants and the couplings λ_{ijkl} and κ_{ij} used in the last section are given by

$$\begin{aligned} &\lambda_1 = \lambda_{1111}\,, \quad \lambda_2 = \lambda_{2222}\,, \quad \lambda_3 = \lambda_{1122}\,, \quad \lambda_4 = \lambda_{1221}\,, \quad \lambda_5 = \lambda_{1212}\,, \\ &\lambda_6 = \lambda_{1112}\,, \quad \lambda_7 = \lambda_{1222}\,, \quad \kappa_1 = \kappa_{11}\,, \quad \kappa_2 = \kappa_{22}\,, \quad \kappa_3 = \kappa_{12}\,. \end{aligned} \tag{3.136}$$

In general, both neutral components of the two scalar doublets acquire a VEV:

$$\langle\Phi_1\rangle = \begin{pmatrix} 0 \\ v_1 \end{pmatrix}, \quad \langle\Phi_2\rangle = \begin{pmatrix} 0 \\ v_2 \end{pmatrix} \quad \text{with} \quad v_1, v_2 \in \mathbb{C}\,, \quad |v_1|^2 + |v_2|^2 = v^2\,. \tag{3.137}$$

It is, however, convenient to switch to the Higgs basis [36, 38, 39], in which the VEVs of the doublets are given by

$$\langle\Phi_1'\rangle = \begin{pmatrix} 0 \\ v \end{pmatrix}, \quad \langle\Phi_2'\rangle = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.138}$$

so that we can make the expansion

$$\Phi_1' = \begin{pmatrix} \Phi_1'^+ \\ \frac{1}{\sqrt{2}}(v + \rho_1' + \mathrm{i}\sigma_1') \end{pmatrix}, \qquad \Phi_2' = \begin{pmatrix} \Phi_2'^+ \\ \frac{1}{\sqrt{2}}(\rho_2' + \mathrm{i}\sigma_2') \end{pmatrix}. \tag{3.139}$$

The transformation to the Higgs basis is uniquely determined up to the phase of Φ_2' [40], and is given by

$$\Phi_1' = \frac{v_1^*\Phi_1 + v_2^*\Phi_2}{v}\,, \qquad \Phi_2' = \frac{-v_2\Phi_1 + v_1\Phi_2}{v}\,. \tag{3.140}$$

[6] Note that if we took $n_{\mathrm{H}} = 1$, we would not get any new neutrino masses by one-loop corrections.

The inverse transformation reads

$$\Phi_1 = \frac{v_1\Phi_1' - v_2^*\Phi_2'}{v}\,, \qquad \Phi_2 = \frac{v_2\Phi_1' + v_1^*\Phi_2'}{v}\,. \tag{3.141}$$

Inserting (3.141) into (3.135), we get the scalar potential in the Higgs basis,

$$\begin{aligned} V_0 = \frac{\lambda_0}{4}\varphi_0^4 &+ \lambda_1'\left(\Phi_1'^\dagger\Phi_1'\right)^2 + \lambda_2'\left(\Phi_2'^\dagger\Phi_2'\right)^2 + 2\lambda_3'\Phi_1'^\dagger\Phi_1'\Phi_2'^\dagger\Phi_2' \\ &+ 2\lambda_4'\Phi_1'^\dagger\Phi_2'\Phi_2'^\dagger\Phi_1' + \left[\lambda_5'\left(\Phi_1'^\dagger\Phi_2'\right)^2 + \lambda_5'^*\left(\Phi_2'^\dagger\Phi_1'\right)^2\right] \\ &+ \left(\lambda_6'\Phi_1'^\dagger\Phi_2' + \lambda_6'^*\Phi_2'^\dagger\Phi_1'\right)\Phi_1'^\dagger\Phi_1' + \left(\lambda_7'\Phi_1'^\dagger\Phi_2' + \lambda_7'^*\Phi_2'^\dagger\Phi_1'\right)\Phi_2'^\dagger\Phi_2' \\ &+ \varphi_0^2\left(\kappa_1'\Phi_1'^\dagger\Phi_1' + \kappa_2'\Phi_2'^\dagger\Phi_2' + \kappa_3'\Phi_1'^\dagger\Phi_2' + \kappa_3'^*\Phi_2'^\dagger\Phi_1'\right)\,, \end{aligned} \tag{3.142}$$

where (see [40, 41] for the non-conformal 2HDM)

$$\begin{aligned} \lambda_1' = \frac{1}{v^4}\Big[&\lambda_1|v_1|^4 + \lambda_2|v_2|^4 + 2(\lambda_3+\lambda_4)|v_1|^2|v_2|^2 + 2\,\mathrm{Re}\left(\lambda_5 v_1^{*\,2}v_2^2\right) \\ &+ 4\,\mathrm{Re}\left(\lambda_6|v_1|^2v_1^*v_2 + \lambda_7|v_2|^2v_1^*v_2\right)\Big]\,, \end{aligned} \tag{3.143}$$

$$\begin{aligned} \lambda_2' = \frac{1}{v^4}\Big[&\lambda_1|v_2|^4 + \lambda_2|v_1|^4 + 2(\lambda_3+\lambda_4)|v_1|^2|v_2|^2 + 2\,\mathrm{Re}\left(\lambda_5 v_1^{*\,2}v_2^2\right) \\ &- 4\,\mathrm{Re}\left(\lambda_6|v_2|^2v_1^*v_2 + \lambda_7|v_1|^2v_1^*v_2\right)\Big]\,, \end{aligned} \tag{3.144}$$

$$\begin{aligned} \lambda_3' = \lambda_3 + \frac{1}{v^4}\Big\{&(\lambda_1+\lambda_2-2\lambda_3-2\lambda_4)|v_1|^2|v_2|^2 - 2\,\mathrm{Re}(\lambda_5 v_1^{*\,2}v_2^2) \\ &- 2\left(|v_1|^2-|v_2|^2\right)\mathrm{Re}\left[(\lambda_6-\lambda_7)v_1^*v_2\right]\Big\}\,, \end{aligned} \tag{3.145}$$

$$\begin{aligned} \lambda_4' = \lambda_4 + \frac{1}{v^4}\Big\{&(\lambda_1+\lambda_2-2\lambda_3-2\lambda_4)|v_1|^2|v_2|^2 - 2\,\mathrm{Re}\left(\lambda_5 v_1^{*\,2}v_2^2\right) \\ &- 2\left(|v_1|^2-|v_2|^2\right)\mathrm{Re}\left[(\lambda_6-\lambda_7)v_1^*v_2\right]\Big\}\,, \end{aligned} \tag{3.146}$$

$$\begin{aligned} \lambda_5' = \frac{1}{v^4}\Big\{&(\lambda_1+\lambda_2-2\lambda_3-2\lambda_4)v_1^{*\,2}v_2^{*\,2} + \lambda_5 v_1^{*\,4} + \lambda_5^* v_2^{*\,4} \\ &- 2v_1^*v_2^*\left[(\lambda_6-\lambda_7)v_1^{*\,2} - (\lambda_6^*-\lambda_7^*)v_2^{*\,2}\right]\Big\}\,, \end{aligned} \tag{3.147}$$

$$\begin{aligned} \lambda_6' = \frac{1}{v^4}\Big[&\left(-\lambda_1|v_1|^2+\lambda_2|v_2|^2\right)v_1^*v_2^* + (\lambda_3+\lambda_4)\left(|v_1|^2-|v_2|^2\right)v_1^*v_2^* + \lambda_5 v_1^{*\,3}v_2 \\ &- \lambda_5^*v_1v_2^{*\,3} + \left(|v_1|^2-|v_2|^2\right)\left(\lambda_6 v_1^{*\,2} + \lambda_7^* v_2^{*\,2}\right) - 2\lambda_6^*|v_1|^2v_2^{*\,2} + 2\lambda_7 v_1^{*\,2}|v_2|^2\Big]\,, \end{aligned} \tag{3.148}$$

$$\begin{aligned} \lambda_7' = \frac{1}{v^4}\Big[&\left(-\lambda_1|v_2|^2+\lambda_2|v_1|^2\right)v_1^*v_2^* - (\lambda_3+\lambda_4)\left(|v_1|^2-|v_2|^2\right)v_1^*v_2^* - \lambda_5 v_1^{*\,3}v_2 \\ &+ \lambda_5^*v_1v_2^{*\,3} + \left(|v_1|^2-|v_2|^2\right)\left(\lambda_6^* v_2^{*\,2} + \lambda_7 v_1^{*\,2}\right) + 2\lambda_6 v_1^{*\,2}|v_2|^2 - 2\lambda_7^*|v_1|^2v_2^{*\,2}\Big]\,, \end{aligned} \tag{3.149}$$

$$\kappa_1' = \frac{1}{v^2}\left[\kappa_1|v_1|^2 + \kappa_2|v_2|^2 + 2\,\mathrm{Re}(\kappa_3 v_1^*v_2)\right]\,, \tag{3.150}$$

$$\kappa_2' = \frac{1}{v^2}\left[\kappa_1|v_2|^2 + \kappa_2|v_1|^2 - 2\,\mathrm{Re}(\kappa_3 v_1^*v_2)\right]\,, \tag{3.151}$$

$$\kappa_3' = \frac{1}{v^2}\left[(\kappa_2-\kappa_1)v_1^*v_2^* + \kappa_3 v_1^{*\,2} - \kappa_3^* v_2^{*\,2}\right]\,. \tag{3.152}$$

The relations between the Yukawa couplings in the Higgs basis and the Yukawa couplings in the original basis are given by

$$Y_1^{(\ell)\prime} = \frac{v_1^* Y_1^{(\ell)} + v_2^* Y_2^{(\ell)}}{v}\,, \qquad Y_2^{(\ell)\prime} = \frac{v_1 Y_2^{(\ell)} - v_2 Y_1^{(\ell)}}{v}\,, \tag{3.153}$$

$$Y_1^{(d)\prime} = \frac{v_1^* Y_1^{(d)} + v_2^* Y_2^{(d)}}{v}\,, \qquad Y_2^{(d)\prime} = \frac{v_1 Y_2^{(d)} - v_2 Y_1^{(d)}}{v}\,, \tag{3.154}$$

$$Y_1^{(u)\prime} = \frac{v_1 Y_1^{(u)} + v_2 Y_2^{(u)}}{v}\,, \qquad Y_2^{(u)\prime} = \frac{v_1^* Y_2^{(\ell)} - v_2^* Y_1^{(u)}}{v}\,, \tag{3.155}$$

$$Y_{\mathrm{D}\,1}' = \frac{v_1 Y_{\mathrm{D}\,1} + v_2 Y_{\mathrm{D}\,2}}{v}\,, \qquad Y_{\mathrm{D}\,2}' = \frac{v_1^* Y_{\mathrm{D}\,2} - v_2^* Y_{\mathrm{D}\,1}}{v}\,. \tag{3.156}$$

The stationarity conditions of the general model (3.24) now read

$$\lambda_1' v^2 + \kappa_1' v_0^2 - \kappa_1' v^2 - \lambda_0' v_0^2 = 0\,, \tag{3.157}$$

$$\lambda_6' v^2 + \kappa_3' v_0^2 = 0\,, \tag{3.158}$$

and the Gildener-Weinberg condition (3.26) is given by

$$\lambda_1' v^4 + 2\kappa_1' v_0^2 v^2 + \lambda_0' v_0^4 = 0\,. \tag{3.159}$$

The conditions (3.157-3.159) are equivalent to the relations [7]

$$\kappa_1' v^2 + \lambda_0' v_0^2 = 0\,, \tag{3.160}$$

$$\lambda_1' v^2 + \kappa_1' v_0^2 = 0\,, \tag{3.161}$$

$$\lambda_6' v^2 + \kappa_3' v_0^2 = 0\,, \tag{3.162}$$

which correspond to (3.27) and (3.28) in the multi-Higgs-doublet model.

The squared mass matrix of the charged scalars is given by

$$M_+^2 = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_3' v^2 + \kappa_2' v_0^2 \end{pmatrix}\,, \tag{3.163}$$

where we used (3.161) and (3.162). Therefore, we have a massless would-be Goldstone boson $G^\pm \equiv \Phi_1'^\pm$, and a physical, charged scalar $H^\pm \equiv \Phi_2'^\pm$ with a squared mass of

$$\mu^2 = \kappa_2' v_0^2 + \lambda_3' v^2\,. \tag{3.164}$$

The squared mass matrix of the neutral scalars is given by

$$\mathcal{M}_0^2 = \begin{pmatrix} 2\lambda_0' v_0^2 & 2\kappa_1' v v_0 & 2\,\mathrm{Re}(\kappa_3') v v_0 & 0 & -2\,\mathrm{Im}(\kappa_3') v v_0 \\ 2\kappa_1' v v_0 & 2\lambda_1' v^2 & 2\,\mathrm{Re}(\lambda_6') v^2 & 0 & -2\,\mathrm{Im}(\lambda_6') v^2 \\ 2\,\mathrm{Re}(\kappa_3') v v_0 & 2\,\mathrm{Re}(\lambda_6') v^2 & \kappa_2' v_0^2 + \lambda_{34+5}' v^2 & 0 & -\mathrm{Im}(\lambda_5') v^2 \\ 0 & 0 & 0 & 0 & 0 \\ -2\,\mathrm{Im}(\kappa_3') v v_0 & -2\,\mathrm{Im}(\lambda_6') v^2 & -\mathrm{Im}(\lambda_5') v^2 & 0 & \kappa_2' v_0^2 + \lambda_{34-5}' v^2 \end{pmatrix} \tag{3.165}$$

[7] It might appear quite unnatural that the coupling constants $\kappa_{1,3}'$ are suppressed, while κ_2' is unconstrained. However, it should be stressed that this is merely the case because we have chosen to work in the Higgs basis.

with

$$\lambda'_{34\pm5} = \lambda'_3 + \lambda'_4 \pm \operatorname{Re}\lambda'_5\,. \tag{3.166}$$

The massless tree-level eigenfield σ'_1 corresponds to the neutral would-be Goldstone boson, $G^0 \equiv \sigma'_1$. Furthermore, we already know that there is another massless scalar particle at tree-level, the scalon. This becomes more apparent once we make the transformation

$$\tilde{\mathcal{M}}_0^2 = \mathcal{R}_1^{\mathsf{T}} \mathcal{M}_0^2 \mathcal{R}_1 \tag{3.167}$$

with the orthogonal matrix

$$\mathcal{R}_1 = \frac{1}{V}\begin{pmatrix} v_0 & -v & 0 & 0 & 0 \\ v & v_0 & 0 & 0 & 0 \\ 0 & 0 & V & 0 & 0 \\ 0 & 0 & 0 & 0 & V \\ 0 & 0 & 0 & V & 0 \end{pmatrix}. \tag{3.168}$$

Using (3.160-3.162), the transformed squared mass matrix reads

$$\tilde{\mathcal{M}}_0^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2(\lambda'_1 - \kappa'_1)v^2 & 2\operatorname{Re}(\lambda'_6)v^2V/v_0 & -2\operatorname{Im}(\lambda'_6)v^2V/v_0 & 0 \\ 0 & 2\operatorname{Re}(\lambda'_6)v^2V/v_0 & \kappa'_2 v_0^2 + \lambda'_{34+5}v^2 & -\operatorname{Im}(\lambda'_5)v^2 & 0 \\ 0 & -2\operatorname{Im}(\lambda'_6)v^2V/v_0 & -\operatorname{Im}(\lambda'_5)v^2 & \kappa'_2 v_0^2 + \lambda'_{34-5}v^2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.169}$$

and, therefore, the scalon is given by

$$S = \frac{v_0\rho_0 + v\rho'_1}{V}\,. \tag{3.170}$$

In principle, the matrix $\tilde{\mathcal{M}}_0^2$ can be diagonalized by an orthogonal matrix:

$$\hat{\mathcal{M}}_0^2 = \mathcal{R}_2^{\mathsf{T}} \tilde{\mathcal{M}}_0^2 \mathcal{R}_2\,. \tag{3.171}$$

However, since this effectively corresponds to diagonalizing a general symmetric 3×3 matrix, it is an algebraically dreadful task and we refrain from determining the orthogonal matrix $\mathcal{R}_2$ directly. Instead, we seek for further simplifications. As a first step, we choose the phase of Φ'_2 such that $\operatorname{Im}\lambda'_5 = 0$. Furthermore, we use the hierarchy $v \ll v_0$ to make the expansion [8]

$$\tilde{\mathcal{M}}_0^2/v_0^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \kappa_2' & 0 & 0 \\ 0 & 0 & 0 & \kappa_2' & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \frac{v^2}{v_0^2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2\lambda_1' & 2\,\mathrm{Re}\,\lambda_6' & -2\,\mathrm{Im}\,\lambda_6' & 0 \\ 0 & 2\,\mathrm{Re}\,\lambda_6' & \lambda_{34+5}' & 0 & 0 \\ 0 & -2\,\mathrm{Im}\,\lambda_6' & 0 & \lambda_{34-5}' & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$+ \frac{v^4}{v_0^4} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2\lambda_1' & \mathrm{Re}\,\lambda_6' & -\,\mathrm{Im}\,\lambda_6' & 0 \\ 0 & \mathrm{Re}\,\lambda_6' & 0 & 0 & 0 \\ 0 & -\,\mathrm{Im}\,\lambda_6' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \mathcal{O}\left(\frac{v^6}{v_0^6}\right). \tag{3.172}$$

Using the methods described in appendix B.5, we thus find the approximations

$$\mathcal{R}_2 = \mathbb{1}_{5\times5} + \frac{v^2}{v_0^2}\frac{1}{\kappa_2'} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\,\mathrm{Re}\,\lambda_6' & -2\,\mathrm{Im}\,\lambda_6' & 0 \\ 0 & -2\,\mathrm{Re}\,\lambda_6' & 0 & \frac{2\,\mathrm{Re}(\lambda_6')\,\mathrm{Im}(\lambda_6')}{\mathrm{Re}\,\lambda_5'} & 0 \\ 0 & 2\,\mathrm{Im}\,\lambda_6' & -\frac{2\,\mathrm{Re}(\lambda_6')\,\mathrm{Im}(\lambda_6')}{\mathrm{Re}\,\lambda_5'} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \mathcal{O}\left(\frac{v^4}{v_0^4}\right) \tag{3.173}$$

and

$$\hat{\mathcal{M}}_0^2/v_0^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \kappa_2' & 0 & 0 \\ 0 & 0 & 0 & \kappa_2' & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \frac{v^2}{v_0^2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2\lambda_1' & 0 & 0 & 0 \\ 0 & 0 & \lambda_{34+5}' & 0 & 0 \\ 0 & 0 & 0 & \lambda_{34-5}' & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \mathcal{O}\left(\frac{v^4}{v_0^4}\right). \tag{3.174}$$

Therefore, our model includes a light neutral scalar S_1 with a squared mass of

$$M_1^2 \simeq 2\lambda_1' v^2\,, \tag{3.175}$$

and two heavy neutral scalars S_2, S_3 with squared masses

$$M_2^2 \simeq \kappa_2' v_0^2 + \lambda_{34+5}' v^2\,, \quad M_3^2 \simeq \kappa_2' v_0^2 + \lambda_{34-5}' v^2\,. \tag{3.176}$$

[8] Note that we implicitly assume that there are no large hierarchies among the coupling constants beside those governed by (3.160-3.162).

The matrix $\mathcal{R}$, which is defined in (3.49), is given by

$$\mathcal{R} = \mathcal{R}_1\mathcal{R}_2$$
$$= \begin{pmatrix} 1-\frac{v^2}{2v_0^2} & -\frac{v}{v_0}\left(1-\frac{v^2}{2v_0^2}\right) & -\frac{2\,\mathrm{Re}\,\lambda_6'}{\kappa_2'}\frac{v^3}{v_0^3} & \frac{2\,\mathrm{Im}\,\lambda_6'}{\kappa_2'}\frac{v^3}{v_0^3} & 0 \\ \frac{v}{v_0}\left(1-\frac{v^2}{2v_0^2}\right) & 1-\frac{v^2}{2v_0^2} & \frac{2\,\mathrm{Re}\,\lambda_6'}{\kappa_2'}\frac{v^2}{v_0^2} & -\frac{2\,\mathrm{Im}\,\lambda_6'}{\kappa_2'}\frac{v^2}{v_0^2} & 0 \\ 0 & -\frac{2\,\mathrm{Re}\,\lambda_6'}{\kappa_2'}\frac{v^2}{v_0^2} & 1 & \frac{2\,\mathrm{Re}(\lambda_6')\,\mathrm{Im}(\lambda_6')}{\kappa_2'\,\mathrm{Re}\,\lambda_5'} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{2\,\mathrm{Im}\,\lambda_6'}{\kappa_2'}\frac{v^2}{v_0^2} & -\frac{2\,\mathrm{Re}(\lambda_6')\,\mathrm{Im}(\lambda_6')}{\kappa_2'\,\mathrm{Re}\,\lambda_5'} & 1 & 0 \end{pmatrix} + \mathcal{O}\left(\frac{v^4}{v_0^4}\right), \tag{3.177}$$

and allows us to express the fields ρ_0, ρ_i' and σ_i' in terms of the mass-eigenfields:

$$\rho_0' \simeq \left(1-\frac{v^2}{2v_0^2}\right)\left(S-\frac{v}{v_0}S_1\right) + \frac{2v^3}{\kappa_2' v_0^3}\left[-\mathrm{Re}(\lambda_6')S_2 + \mathrm{Im}(\lambda_6')S_3\right], \tag{3.178}$$
$$\rho_1' \simeq \left(1-\frac{v^2}{2v_0^2}\right)\left(S_1+\frac{v}{v_0}S\right) + \frac{2v^2}{\kappa_2' v_0^2}\left[\mathrm{Re}(\lambda_6')S_2 - \mathrm{Im}(\lambda_6')S_3\right], \tag{3.179}$$
$$\rho_2' \simeq S_2 + \frac{2\,\mathrm{Re}\,\lambda_6'}{\kappa_2'}\frac{v^2}{v_0^2}\left(-S_1+\frac{\mathrm{Im}\,\lambda_6'}{\mathrm{Re}\,\lambda_5'}S_3\right), \tag{3.180}$$
$$\sigma_1' = G^0\,, \tag{3.181}$$
$$\sigma_2' \simeq S_3 + \frac{2\,\mathrm{Im}\,\lambda_6'}{\kappa_2'}\frac{v^2}{v_0^2}\left(S_1-\frac{\mathrm{Re}\,\lambda_6'}{\mathrm{Re}\,\lambda_5'}S_2\right). \tag{3.182}$$

Now that we have determined the matrix $\mathcal{R}$, it is a straightforward task to compute the interaction terms of section 3.1.4 for the case of two Higgs doublets.

3.2.2. Neutrino masses

The tree-level neutrino mass matrix $\mathcal{M}_\nu$ is diagonalized by the unitary matrix [42]

$$\mathcal{U} = \begin{pmatrix} u_1' & u_2' & \mathrm{i}e^{\mathrm{i}\arg(M_\mathrm{R})/2}\cos\theta\, u_3' & e^{\mathrm{i}\arg(M_\mathrm{R})/2}\sin\theta\, u_3' \\ 0 & 0 & -\mathrm{i}e^{-\mathrm{i}\arg(M_\mathrm{R})/2}\sin\theta & e^{-\mathrm{i}\arg(M_\mathrm{R})/2}\cos\theta \end{pmatrix}, \tag{3.183}$$

where

$$u_{1,2}' \perp M_\mathrm{D}^\dagger\,, \quad u_3' = \frac{M_\mathrm{D}^\dagger}{m_\mathrm{D}}\,, \quad \text{with} \quad m_\mathrm{D} = \sqrt{M_\mathrm{D}M_\mathrm{D}^\dagger}\,, \tag{3.184}$$

and with

$$\tan 2\theta = \frac{2m_\mathrm{D}}{|M_\mathrm{R}|}\,. \tag{3.185}$$

The diagonalized mass matrix of the two massive neutrinos is then given by

$$\hat{M}' = \frac{1}{2}\begin{pmatrix} \sqrt{|M_\mathrm{R}|^2+4m_\mathrm{D}^2}-|M_\mathrm{R}| & 0 \\ 0 & \sqrt{|M_\mathrm{R}|^2+4m_\mathrm{D}^2}+|M_\mathrm{R}| \end{pmatrix}. \tag{3.186}$$

Therefore, we obtain at tree-level, beside the two massless neutrinos, one light neutrino with a mass of $m_3 \simeq m_\mathrm{D}^2/|M_\mathrm{R}|$ and one almost sterile heavy neutrino with a mass of $m_4 \simeq |M_\mathrm{R}|$.

The one-loop mass corrections to the massless tree-level neutrinos are given by (3.133). In order to calculate them, we first need to compute the matrices $\hat{Y}_{\mathrm{D}\,b}$ defined in (3.101). Using the matrix

$$R = \begin{pmatrix} \frac{v}{v_0}\left(1-\frac{v^2}{2v_0^2}\right) & 1-\frac{v^2}{2v_0^2} & \frac{2\,\mathrm{Re}\,\lambda_6'}{\kappa_2'}\frac{v^2}{v_0^2} & -\frac{2\,\mathrm{Im}\,\lambda_6'}{\kappa_2'}\frac{v^2}{v_0^2} & \mathrm{i} \\ 0 & -\frac{2\lambda_6'^{\,*}}{\kappa_2'}\frac{v^2}{v_0^2} & 1-\mathrm{i}\frac{2\,\mathrm{Re}(\lambda_6')\,\mathrm{Im}(\lambda_6')}{\kappa_2'\,\mathrm{Re}\,\lambda_5'}\frac{v^2}{v_0^2} & \mathrm{i}+\frac{2\,\mathrm{Re}(\lambda_6')\,\mathrm{Im}(\lambda_6')}{\kappa_2'\,\mathrm{Re}\,\lambda_5'}\frac{v^2}{v_0^2} & 0 \end{pmatrix} + \mathcal{O}\left(\frac{v^4}{v_0^4}\right), \quad (3.187)$$

which is defined by (3.50) and can be read off from (3.177), we obtain

$$\hat{Y}_{\mathrm{D}\,1} \simeq \left(1-\frac{v^2}{2v_0^2}\right)Y'_{\mathrm{D}\,1} - \frac{2\lambda_6'^{\,*}}{\kappa_2'}\frac{v^2}{v_0^2}Y'_{\mathrm{D}\,2}\,, \quad (3.188)$$

$$\hat{Y}_{\mathrm{D}\,2} \simeq \frac{2\,\mathrm{Re}\,\lambda_6'}{\kappa_2'}\frac{v^2}{v_0^2}Y'_{\mathrm{D}\,1} + \left(1-\mathrm{i}\frac{2\,\mathrm{Re}(\lambda_6')\,\mathrm{Im}(\lambda_6')}{\kappa_2'\,\mathrm{Re}\,\lambda_5'}\frac{v^2}{v_0^2}\right)Y'_{\mathrm{D}\,2}\,, \quad (3.189)$$

$$\hat{Y}_{\mathrm{D}\,3} \simeq -\frac{2\,\mathrm{Im}\,\lambda_6'}{\kappa_2'}\frac{v^2}{v_0^2}Y'_{\mathrm{D}\,1} + \left(\mathrm{i}+\frac{2\,\mathrm{Re}(\lambda_6')\,\mathrm{Im}(\lambda_6')}{\kappa_2'\,\mathrm{Re}\,\lambda_5'}\frac{v^2}{v_0^2}\right)Y'_{\mathrm{D}\,2}\,. \quad (3.190)$$

Since $u'_{1,2} \perp M_\mathrm{D}^\dagger \propto Y'^{\dagger}_{\mathrm{D}\,1}$, only terms proportional to $Y_{\mathrm{D}\,2}$ contribute to the one-loop neutrino masses. Furthermore, by choosing $u'_1 \perp Y'^{\dagger}_{\mathrm{D}\,2}$ we find that one of the neutrinos is still massless at one-loop level: $m_1 = 0$. However, since there is no symmetry that protects this property, we expect that it acquires a mass at two-loop level [42]. The vector u'_2 has to be perpendicular to u'_1 and must thus lie in the plane spanned by the vectors $Y'^{\dagger}_{\mathrm{D}\,1,2}$. Using the tree-level condition $u'_2 \perp Y'^{\dagger}_{\mathrm{D}\,1}$, we then get

$$u'_2 \propto Y'^{\dagger}_{\mathrm{D}\,2} - Y'^{\dagger}_{\mathrm{D}\,1}\frac{Y'_{\mathrm{D}\,1}Y'^{\dagger}_{\mathrm{D}\,2}}{Y'_{\mathrm{D}\,1}Y'^{\dagger}_{\mathrm{D}\,1}}\,. \quad (3.191)$$

The global phase of u'_2 has to be chosen such that the one-loop mass correction is positive, and it turns out that it is given by $\mathrm{e}^{\mathrm{i}\arg(M_\mathrm{R})/2}$. Then, the mass of the neutrino which acquires a mass at one-loop level reads

$$m_2 \simeq \frac{\mathrm{Re}\,\lambda_5'}{(4\pi)^2}\left(\frac{Y'_{\mathrm{D}\,2}Y'^{\dagger}_{\mathrm{D}\,2}}{Y'_{\mathrm{D}\,1}Y'^{\dagger}_{\mathrm{D}\,1}} - \left|\frac{Y'_{\mathrm{D}\,1}Y'^{\dagger}_{\mathrm{D}\,2}}{Y'_{\mathrm{D}\,1}Y'^{\dagger}_{\mathrm{D}\,1}}\right|^2\right)\frac{\kappa_2'/|Y_\mathrm{R}|^2 - 1 - \ln\left(\kappa_2'/|Y_\mathrm{R}|^2\right)}{\left(\kappa_2'/|Y_\mathrm{R}|^2-1\right)^2}m_3\,, \quad (3.192)$$

where we expanded in v^2/v_0^2 and only took the leading contribution.

3.2.3. Phenomenological analysis

If we neglect contributions suppressed by a factor of v^2/v_0^2, the condition that ensures a positive scalon mass is given by

$$4M^4 \gtrsim 2|M_\mathrm{R}|^4\,, \quad (3.193)$$

where $M^2 = \kappa_2' v_0^2 \simeq \mu^2 \simeq M_{2,3}^2$. In terms of coupling constants this condition reads

$$\frac{\kappa_2'}{|Y_\mathrm{R}|^2} \gtrsim \frac{1}{\sqrt{2}} \,. \tag{3.194}$$

Therefore, the last factor in (3.192) is constrained by

$$\frac{\kappa_2'/|Y_\mathrm{R}|^2 - 1 - \ln\left(\kappa_2'/|Y_\mathrm{R}|^2\right)}{\left(\kappa_2'/|Y_\mathrm{R}|^2 - 1\right)^2} \lesssim 0.6 \,. \tag{3.195}$$

Moreover, if we assume

$$\frac{Y'_{\mathrm{D}\,2} Y'^{\dagger}_{\mathrm{D}\,2}}{Y'_{\mathrm{D}\,1} Y'^{\dagger}_{\mathrm{D}\,1}} \sim 1 \,, \tag{3.196}$$

we get the hierarchy $m_2 \ll m_3$, as long as the coupling constant $\mathrm{Re}\,\lambda_5'$ is in the perturbative region.

The neutrino masses are experimentally undetermined, but the differences of the squared masses for the active neutrinos are known from neutrino oscillations. As we have just discussed, it seems natural to assume the normal hierarchical spectrum

$$m_1 < m_2 \ll m_3 \tag{3.197}$$

in the case of our model. The differences of the squared masses are then given by [17]

$$\Delta m_{32}^2 = m_3^2 - m_2^2 = (2.51 \pm 0.05) \times 10^{-3}\,\mathrm{eV}^2 \,, \tag{3.198}$$
$$\Delta m_{21}^2 = m_2^2 - m_1^2 = (7.53 \pm 0.18) \times 10^{-5}\,\mathrm{eV}^2 \,, \tag{3.199}$$

and we hence get

$$m_3 > \sqrt{\Delta m_{32}^2} \simeq 0.05\,\mathrm{eV} \,, \quad m_2 > \sqrt{\Delta m_{21}^2} \simeq 0.0087\,\mathrm{eV} \tag{3.200}$$

as lower limits for the neutrino masses. Due to the normal hierarchy assumption (3.197) we actually expect

$$m_3 \simeq 0.05\,\mathrm{eV} \,, \quad m_2 \simeq 0.009\,\mathrm{eV} \,. \tag{3.201}$$

Plugging these values into (3.192), we get the following constraint for the coupling constants:

$$\mathrm{Re}\,\lambda_5' \frac{Y'_{\mathrm{D}\,2} Y'^{\dagger}_{\mathrm{D}\,2}}{Y'_{\mathrm{D}\,1} Y'^{\dagger}_{\mathrm{D}\,1}} \gtrsim 40 \,. \tag{3.202}$$

The Majorana mass scale $|M_\mathrm{R}|$ and the Dirac mass scale m_R are both undetermined. However, the reason why we implemented the seesaw mechanism was to allow for Dirac mass scales that are in the region of the other charged fermion masses. It is thus reasonable to assume $m_{\mathrm{D}} \gtrsim m_e$, and we accordingly get a lower limit for the Majorana mass scale,

$$|M_{\mathrm{R}}| \simeq \frac{m_{\mathrm{D}}^2}{m_3} \gtrsim 5.2\,\mathrm{TeV}\,, \tag{3.203}$$

as well as limits for the masses of the heavy scalars due to (3.193):

$$\mu \simeq M_{2,3} \gtrsim 4.4\,\mathrm{TeV}\,. \tag{3.204}$$

Note that these limits heavily depend on the limit we chose for m_{D}. We could for example also use the assumption $m_{\mathrm{D}} \gtrsim m_\tau$, which might be seen as more natural, what would lead to much larger masses:

$$|M_{\mathrm{R}}| \gtrsim 6.3 \times 10^7\,\mathrm{TeV}\,, \quad \mu, M_{2,3} \gtrsim 5.3 \times 10^7\,\mathrm{TeV}\,. \tag{3.205}$$

The only neutral scalar that can be identified with the H^0 particle measured at the LHC is the field S_1, since the scalars $S_{2,3}$ are too heavy, and we have already excluded the scalon in section 3.1.5. We thus conclude [17]

$$M_1 = (125.18 \pm 0.16)\,\mathrm{GeV}\,. \tag{3.206}$$

It is then easy to check that the field S_1 has almost SM-like couplings to all SM particles, where the corrections are of order v^2/v_0^2. Looking at the scalar interactions that include at least one scalon (see equation (3.107)) reveals that the field S_1 cannot decay into a scalon. Moreover, the decay into a heavy neutrino is kinematically forbidden, and the decay into light neutrinos is highly suppressed due to small coupling constants. Therefore, the decay rate of the scalar S_1 is quite close to the decay rate of the SM Higgs, and hence it is in good agreement with experimental data.

In contrast to the SM, our model allows for tree-level flavor-changing neutral interactions (FCNI), which are highly constrained by experiment. The simplest and most common approach to get rid of tree-level FCNI in the non-conformal two-Higgs-doublet model is to introduce discrete symmetries (see [16, 41] and the references therein). However, as it turns out, this may not be necessary for our model. First of all, note that the neutral scalar field S_1 interacts almost flavor diagonal, with corrections of order v^2/v_0^2, and that the scalon S has exactly flavor diagonal Yukawa couplings. The tree-level Yukawa interactions of the scalars $S_{2,3}$ are unconstrained, and thus in principle allow for FCNI. However, these particles are heavy, with masses at least in the TeV region, and therefore the experimental constraints may not be spoiled. The corresponding case in the non-conformal two-Higgs-doublet model, where one light scalar interacts almost flavor diagonal, whereas two heavy scalars exhibit FCNI at tree-level, is known as the decoupling limit and is discussed in [43]. Further phenomenological analysis is necessary to obtain precise limits on the heavy masses of the neutral scalars. Also note that the Yukawa matrices of the scalars $S_{2,3}$ are not independent. They are related by

$$\hat{Y}_2^{(\ell)} \simeq \mathrm{i}\hat{Y}_3^{(\ell)}\,, \quad \hat{Y}_2^{(d)} \simeq \mathrm{i}\hat{Y}_3^{(d)}\,, \quad \hat{Y}_2^{(u)} \simeq -\mathrm{i}\hat{Y}_3^{(u)}\,, \tag{3.207}$$

with corrections of order v^2/v_0^2, and this could lead to cancellations in some processes.

3.3. Conclusion and outlook

We started this chapter by considering a conformal multi-Higgs-doublet model with an arbitrary number of right-handed neutrino singlets, where we introduced a scalar singlet in order to generate a Majorana mass term for the right-handed neutrinos. We found that this model predicts a scalon, which couples similarly as the SM-Higgs to all SM particles, but the couplings are suppressed by a factor of $v/V \ll 1$ [2]. We expressed the scalon mass in terms of the masses of all other particles, but since the relevant contributions are unknown, the scalon mass stays undetermined. Subsequently, we analyzed the neutrino mass spectrum. We calculated the one-loop masses of the neutrinos which where massless at tree-level and found that the result is essentially the same as in the non-conformal model [3]. In particular, there are no relevant contributions from the additional scalar singlet.

In the last section we focused on the simplest version of this model with two Higgs doublets and only one right-handed neutrino singlet. We found that the scalar sector includes three heavy states (one charged and two neutral ones) with almost degenerate masses. The mass scale of these particles is of the order of the mass of the right-handed neutrino singlet, and therefore at least in the TeV region. Furthermore, we found one light state, which we identified with the H^0 particle discovered at the LHC. It turned out that this particle indeed has couplings that are almost SM-like. Another attractive feature of our model is that it automatically leads to the so-called decoupling limit [40] and therefore "naturally" provides a mechanism that suppresses flavor-changing neutral interactions. In the neutrino sector, our model includes, beside the two states that are massless at tree-level, one light ($m_3 \simeq m_{\mathrm{D}}^2/|M_{\mathrm{R}}|$) and one heavy ($m_4 \simeq |M_{\mathrm{R}}|$) state. At one-loop level, one of the neutrinos is still massless, whereas the other one acquires a mass, which we were able to express in terms of the tree-level parameters of our theory.

Of course, there is still a lot of work that could be done. In particular the phenomenological analysis of the simplest version of the model could be extended in various ways. For example, the mass of the neutrino which is massless at one-loop level could be calculated at the two-loop level. If the result would be incompatible with the experimental data, one could consider a model with three Higgs doublets. In such a model, two of the neutrinos acquire masses at one-loop level, and this might provide a natural explanation for a normal hierarchical neutrino mass spectrum. Moreover, it could be interesting to analyze the renormalization group equations of this model. The way we imposed the Gildener-Weinberg condition (3.26) was actually quite ad hoc, and it would be nice to check how it behaves under varying the renormalization scale.

A. Notation and conventions

As it is standard in the field of particle physics, we set $c = 1$, and, with exception of some parts of chapter 2, $\hbar = 1$.

When we refer to 4-vectors, we use Greek indices. The sign convention that we use for the Minkowski metric is

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{A.1}$$

The Einstein summation convention is always used for Greek indices, whereas for Latin indices it is only used in chapter 2 and the corresponding sections in the appendix.

In some sections we use the Kronecker product, denoted by $\otimes$, to explicitly keep track of the structure of matrices. The Kronecker product of an $m \times n$ matrix A and a $p \times q$ matrix B is given by a $mp \times nq$ block matrix:

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}. \tag{A.2}$$

B. Mathematical tools

B.1. The functional derivative

The functional (or variational) derivative is defined such that it obeys the basic axiom (see e.g. [5])

$$\frac{\delta}{\delta f(y)} f(x) = \delta(x-y)\,, \tag{B.1}$$

in addition to the usual rules for differential operators:

$$\frac{\delta}{\delta f(x)}\left(F_1[f] + F_2[f]\right) = \frac{\delta}{\delta f(x)} F_1[f] + \frac{\delta}{\delta f(x)} F_2[f]\,, \tag{B.2}$$

$$\frac{\delta}{\delta f(x)}\left(F_1[f] F_2[f]\right) = F_1[f] \frac{\delta}{\delta f(x)} F_2[f] + F_2[f] \frac{\delta}{\delta f(x)} F_1[f]\,. \tag{B.3}$$

The functional Taylor series of the functional $F[f]$ around the function $g(x)$ can subsequently be defined as

$$F[f] = \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}x_1 \ldots \mathrm{d}x_n\, F_g^{(n)}(x_1,\ldots,x_n)(f(x_1) - g(x_1)) \ldots (f(x_n) - g(x_n))\,, \tag{B.4}$$

where $F_g^{(n)}(x_1,\ldots,x_n) = \left[\frac{\delta}{\delta f(x_1)} \cdots \frac{\delta}{\delta f(x_n)} F[f]\right]\Big|_{f(x)=g(x)}$.

B.2. Grassmann numbers

Grassmann numbers are numbers that anti-commute with each other:

$$\theta_1\theta_2 = -\theta_2\theta_1\,, \tag{B.5}$$

but commute with ordinary numbers:

$$x\theta = \theta x\,. \tag{B.6}$$

Furthermore, they obey

$$(\theta_1\theta_2)^{\mathsf{T}} = \theta_1\theta_2 = -\theta_2\theta_1 \qquad \text{and} \qquad (\theta_1\theta_2)^* = \theta_2^*\theta_1^* = -\theta_1^*\theta_2^*\,. \tag{B.7}$$

Every function of a Grassmann variable can be expanded in a Taylor series which includes only a constant and a linear term, since terms of higher order vanish due to (B.5):

$$f(\theta) = A + B\theta\,. \tag{B.8}$$

We can define an integral over Grassmann numbers (known as Berezin integral) as a linear functional which satisfies the properties

$$\int \mathrm{d}\theta\, \theta = 1 \qquad \text{and} \qquad \int \mathrm{d}\theta = 0\,. \tag{B.9}$$

To avoid sign ambiguities, we adopt the convention that the innermost integral is performed first:

$$\int \mathrm{d}\theta_1 \int \mathrm{d}\theta_2\, \theta_2\theta_1 = 1\,. \tag{B.10}$$

We can also use (B.9) for complex Grassmann numbers and if we assume that θ and θ^* are independent we further have

$$\int \mathrm{d}\theta\, \theta^* = 0 \qquad \text{and} \qquad \int \mathrm{d}\theta^*\, \theta = 0\,. \tag{B.11}$$

B.3. Integrals

B.3.1. One-loop integrals

To solve the Euclidean d-dimensional k-space integral

$$\int \frac{\mathrm{d}^d k_\mathrm{E}}{(2\pi)^d} \frac{1}{(k_\mathrm{E}^2 + x)^n}\,,$$

it is convenient to introduce spherical coordinates. The integration over the $d-1$ angles simply yields a factor corresponding to the surface area of a d-dimensional unit sphere

$$S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}\,, \tag{B.12}$$

where $\Gamma(x)$ is the gamma function, and hence we are only left with the radial integration over the variable $r = \sqrt{k_\mathrm{E}^2}$:

$$\int \frac{\mathrm{d}^d k_\mathrm{E}}{(2\pi)^d} \frac{1}{(k_\mathrm{E}^2 + x)^n} = \frac{2}{(4\pi)^{d/2}\Gamma(d/2)} \int_0^\infty \mathrm{d}r\, r^{d-1} \frac{1}{(r^2 + x)^n}\,. \tag{B.13}$$

Making the substitution $r^2 = xz$, the integral can be solved using the integral representation of the beta function

$$B(x,y) = \int_0^\infty \mathrm{d}t\, \frac{t^{x-1}}{(1+t)^{x+y}}\,, \tag{B.14}$$

and the identity $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$:

$$\int \frac{\mathrm{d}^d k_\mathrm{E}}{(2\pi)^d} \frac{1}{(k_\mathrm{E}^2 + x)^n} = \frac{x^{d/2-n}}{(4\pi)^{d/2}\Gamma(d/2)} \int_0^\infty \mathrm{d}z\, z^{d/2-1} \frac{1}{(1+z)^n}$$
$$= \frac{x^{d/2-n}}{(4\pi)^{d/2}\Gamma(n)} \Gamma\left(n - \frac{d}{2}\right) . \tag{B.15}$$

Subsequently, it is easy to solve the integral

$$\begin{aligned} \int \frac{\mathrm{d}^d k_\mathrm{E}}{(2\pi)^d} \ln\left(1 + \frac{a}{k_\mathrm{E}^2}\right) &= \int_0^a \mathrm{d}x \int \frac{\mathrm{d}^d k_\mathrm{E}}{(2\pi)^d} \frac{1}{k_\mathrm{E}^2 + x} \\ &= \int_0^a \mathrm{d}x\, \frac{x^{d/2-1}}{(4\pi)^{d/2}} \Gamma\left(1 - \frac{d}{2}\right) \\ &= \frac{2a^{d/2}}{d(4\pi)^{d/2}} \Gamma\left(1 - \frac{d}{2}\right) , \end{aligned} \tag{B.16}$$

where we used (B.15) in the second step.

Furthermore, the following integral, with a hermitian $n \times n$ matrix A, which can be diagonalized by a unitary matrix U to $\hat{A} = U^\dagger A U = \mathrm{diag}(a_1, \ldots, a_n)$, can readily be solved:

$$\begin{aligned} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \ln\det\left(\mathbb{1} - \frac{A}{k^2}\right) &= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \ln\det\left(\mathbb{1} - \frac{\hat{A}}{k^2}\right) \\ &= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \operatorname{tr}\ln\left(\mathbb{1} - \frac{\hat{A}}{k^2}\right) \\ &= \sum_i \mathrm{i} \int \frac{\mathrm{d}^d k_\mathrm{E}}{(2\pi)^d} \ln\left(1 + \frac{a_i}{k_\mathrm{E}^2}\right) \\ &= \sum_i \mathrm{i} \frac{2}{d} \frac{1}{(4\pi)^{d/2}} \Gamma\left(1 - \frac{d}{2}\right) (a_i)^{d/2} \\ &= \mathrm{i} \frac{2}{d} \frac{\Gamma\left(1 - \frac{d}{2}\right)}{(4\pi)^{d/2}} \operatorname{tr} A^{d/2} . \end{aligned} \tag{B.17}$$

In the third step we performed a Wick rotation and in the fourth step we used (B.16). For the Wick rotation we need a prescription of how to close the contour. Going back to equation (2.1) we see that in the path integrals, which we are considering in this work, the time integration is actually always slightly rotated: $x_0 \to x_0(1 - \mathrm{i}0^+)$. Thus, after the Fourier transformation this corresponds to $k_0 \to k_0(1 + \mathrm{i}0^+)$ or $k_0^2 \to k_0^2 + \mathrm{i}0^+$, which slightly shifts the poles and tells us how to close the contour.

B.3.2. Gaussian integrals

The Gaussian integral

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-ax^2} = \sqrt{\frac{\pi}{a}}\,, \tag{B.18}$$

where $a \in \mathbb{C}$ and $\operatorname{Re} a > 0$, can be generalized to the n-dimensional integral

$$\int_{\mathbb{R}^n} \mathrm{d}^n x\, \mathrm{e}^{-x_i A_{ij} x_j}\,,$$

where A is a complex symmetric matrix with eigenvalues a_i, which satisfy $\operatorname{Re} a_i > 0$. Since A is symmetric, it can be diagonalized by a unitary matrix U [20, 21]:

$$U^{\mathsf{T}} A U = \operatorname{diag}(a_1, \ldots, a_n)\,. \tag{B.19}$$

Making the substitution $x_i \to U_{ij} x_j$ and using that the modulus of the Jacobian determinant is one, we hence find

$$\int_{\mathbb{R}^n} \mathrm{d}^n x\, \mathrm{e}^{-x_i A_{ij} x_j} = \int_{\mathbb{R}^n} \mathrm{d}^n x\, \mathrm{e}^{-\sum_i a_i x_i^2} = \prod_i \int \mathrm{d}x_i\, \mathrm{e}^{-a_i x_i^2} = \prod_i \sqrt{\frac{\pi}{a_i}} = \left(\det \frac{A}{\pi}\right)^{-\frac{1}{2}}. \tag{B.20}$$

B.3.3. Gaussian integrals over Grassmann numbers

Let us consider the integral

$$\int \mathrm{d}\theta_1\, \mathrm{d}\theta_2 \ldots \mathrm{d}\theta_{2n} \exp\left(-\frac{1}{2}\theta_i A_{ij} \theta_j\right), \tag{B.21}$$

where A is an antisymmetric $2n \times 2n$ matrix. If we expand the exponential in a Taylor series, we see that only the term of order n survives the integration:

$$\begin{aligned} &\int \mathrm{d}\theta_1\, \mathrm{d}\theta_2 \ldots \mathrm{d}\theta_{2n} \exp\left(-\frac{1}{2}\theta_i A_{ij} \theta_j\right) = \\ &\qquad \frac{(-1)^n}{2^n n!} \int \mathrm{d}\theta_1\, \mathrm{d}\theta_2 \ldots \mathrm{d}\theta_{2n}\, (\theta_{i_1} A_{i_1 j_1} \theta_{j_1})(\theta_{i_2} A_{i_2 j_2} \theta_{j_2}) \ldots (\theta_{i_n} A_{i_n j_n} \theta_{j_n})\,. \end{aligned} \tag{B.22}$$

Due to the anti-commutative property of Grassmann variables we have

$$\theta_{i_1}\theta_{j_1}\theta_{i_2}\theta_{j_2} \ldots \theta_{i_n}\theta_{j_n} = \epsilon_{i_1 j_1 i_2 j_2 \ldots i_n j_n} \theta_1 \theta_2 \ldots \theta_{2n} = (-1)^{-n} \epsilon_{i_1 j_1 i_2 j_2 \ldots i_n j_n} \theta_{2n} \theta_{2n-1} \ldots \theta_1\,, \tag{B.23}$$

where in the last step we performed $(2n-1) + (2n-2) + \cdots + 2 + 1 = n(2n-1)$ interchanges of adjoining Grassmann variables. Therefore, we find

$$\int \mathrm{d}\theta_1\, \mathrm{d}\theta_2 \ldots \mathrm{d}\theta_{2n} \exp\left(-\frac{1}{2}\theta_i A_{ij} \theta_j\right) = \frac{1}{2^n n!} \epsilon_{i_1 j_1 i_2 j_2 \ldots i_n j_n} A_{i_1 j_1} A_{i_2 j_2} \ldots A_{i_n j_n} = \operatorname{pf} A\,, \tag{B.24}$$

where the definition of the Pfaffian (see e.g. [44]) was used in the last step.

Furthermore, we may consider the integral

$$\int \mathrm{d}\eta_1 \ldots \mathrm{d}\eta_n \, \mathrm{d}\chi_1 \ldots \mathrm{d}\chi_n \exp\left(\chi_i B_{ij} \eta_j\right), \tag{B.25}$$

where B is a $n \times n$ matrix and η_j and χ_i are independent Grassmann variables. Due to the anti-commutative property of Grassmann variables we can rewrite the exponent as

$$\chi_i B_{ij} \eta_j = \frac{1}{2} \chi_i B_{ij} \eta_j - \frac{1}{2} \eta_i B^{\mathsf{T}}_{ij} \chi_j \,. \tag{B.26}$$

Defining the matrix

$$A = \begin{pmatrix} 0_{n\times n} & B^{\mathsf{T}} \\ -B & 0_{n\times n} \end{pmatrix}, \tag{B.27}$$

and the vector

$$\theta = \begin{pmatrix} \eta \\ \chi \end{pmatrix}, \tag{B.28}$$

the integral can be solved using (B.24):

$$\begin{aligned} &\int \mathrm{d}\eta_1 \ldots \mathrm{d}\eta_n \, \mathrm{d}\chi_1 \ldots \mathrm{d}\chi_n \exp\left(\chi_i B_{ij} \eta_j\right) \\ &\qquad = \int \mathrm{d}\theta_1 \, \mathrm{d}\theta_2 \ldots \mathrm{d}\theta_{2n} \exp\left(-\frac{1}{2} \theta_i A_{ij} \theta_j\right) = \operatorname{pf} A \,. \end{aligned} \tag{B.29}$$

Accordingly, using the identity (see e.g. [44])

$$\operatorname{pf} \begin{pmatrix} 0_{n\times n} & B^{\mathsf{T}} \\ -B & 0_{n\times n} \end{pmatrix} = (-1)^{n(n-1)/2} \det B \,, \tag{B.30}$$

we finally have

$$\int \mathrm{d}\eta_1 \ldots \mathrm{d}\eta_n \, \mathrm{d}\chi_1 \ldots \mathrm{d}\chi_n \exp\left(\chi_i B_{ij} \eta_j\right) = (-1)^{n(n-1)/2} \det B \,, \tag{B.31}$$

and especially for $\chi = \eta^*$

$$\int \mathrm{d}\eta_1 \ldots \mathrm{d}\eta_n \, \mathrm{d}\eta_1^* \ldots \mathrm{d}\eta_n^* \exp\left(\eta_i^* B_{ij} \eta_j\right) = (-1)^{n(n-1)/2} \det B \,. \tag{B.32}$$

Moreover, rearranging the integration variables in (B.31) we obtain

$$\int \mathrm{d}\eta_1 \, \mathrm{d}\chi_1 \, \mathrm{d}\eta_2 \, \mathrm{d}\chi_2 \ldots \mathrm{d}\eta_n \, \mathrm{d}\chi_n \exp\left(\chi_i B_{ij} \eta_j\right) = \det B \,. \tag{B.33}$$

B.4. Path integrals

B.4.1. Bosonic Gaussian integral

Let us consider the path integral

$$\int \left(\prod_{\ell} [\mathrm{d}\phi_\ell] \right) \exp\left(\frac{\mathrm{i}}{2} \int \mathrm{d}^d x \, \phi_i(x) \left(A_{ij}(\partial) \right) \phi_j(x) \right) , \tag{B.34}$$

where the integral is over n real bosonic fields $\phi_i(x)$ and the $n \times n$ matrix $A(\partial)$, which may depend on the derivatives ∂_μ, is real and satisfies the properties

$$A_{ij}(\partial) = A_{ji}(\partial) = A_{ij}(-\partial) \, . \tag{B.35}$$

Using the Fourier transform

$$\phi_i(x) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \mathrm{e}^{-\mathrm{i}kx} \tilde{\phi}_i(k) \, , \tag{B.36}$$

we can rewrite the action in the exponent as

$$\begin{aligned}
S &= \frac{1}{2} \int \mathrm{d}^d x \, \phi_i(x) A_{ij}(\partial) \phi_j(x) \\
&= \frac{1}{2} \int \mathrm{d}^d x \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d k'}{(2\pi)^d} \mathrm{e}^{-\mathrm{i}x(k+k')} \tilde{\phi}_i(k') A_{ij}(-\mathrm{i}k) \tilde{\phi}_j(k) \\
&= \frac{1}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d k'}{(2\pi)^d} (2\pi)^d \delta^{(d)}(k+k') \tilde{\phi}_i(k') A_{ij}(\mathrm{i}k) \tilde{\phi}_j(k) \\
&= \frac{1}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \tilde{\phi}_i(-k) A_{ij}(\mathrm{i}k) \tilde{\phi}_j(k) \\
&= \frac{1}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \Big(\mathrm{Re}\, \tilde{\phi}_i(k) \, \mathrm{Re}\, \tilde{\phi}_j(k) + \mathrm{Im}\, \tilde{\phi}_i(k) \, \mathrm{Im}\, \tilde{\phi}_j(k) \Big) A_{ij}(\mathrm{i}k) \, ,
\end{aligned} \tag{B.37}$$

where in the last step we used (B.35) and $\tilde{\phi}_i(k) = \tilde{\phi}_i^*(-k)$, which follows from (B.36) and the fact that $\phi_i(x)$ is real.

To solve the path integral we should actually discretize, but here we simply treat the continuous variables x and k formally as discrete without going into detail, as also done in [45]. This allows for example, to make the correspondence

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \to \frac{1}{\mathcal{V}_d} \sum_k \, , \tag{B.38}$$

where $\mathcal{V}_d = \int \mathrm{d}^d x$ is the infinite space-time volume in d dimensions.

Therefore, we get for the action [1]

$$S = \frac{1}{2\mathcal{V}_d}\operatorname{Re}\tilde{\phi}_i(0)A_{ij}(0)\operatorname{Re}\tilde{\phi}_j(0) + \frac{1}{\mathcal{V}_d}\sum_{k>0}\Big(\operatorname{Re}\tilde{\phi}_i(k)\operatorname{Re}\tilde{\phi}_j(k) + \operatorname{Im}\tilde{\phi}_i(k)\operatorname{Im}\tilde{\phi}_j(k)\Big)A_{ij}(\mathrm{i}k)\,, \tag{B.39}$$

where we used $\operatorname{Re}\tilde{\phi}_i(k) = \operatorname{Re}\tilde{\phi}_i(-k)$ and $\operatorname{Im}\tilde{\phi}_i(k) = -\operatorname{Im}\tilde{\phi}_i(-k)$.

Furthermore, if we make the variable transformation $\phi_i(x) \to \tilde{\phi}_i(k)$, the measure of the path integral can be rewritten as [2]

$$\prod_i [\mathrm{d}\phi_i] = \prod_i\prod_x \mathrm{d}\phi_i(x) = C\prod_i \mathrm{d}\operatorname{Re}\tilde{\phi}_i(0)\prod_{k>0}\mathrm{d}\operatorname{Re}\tilde{\phi}_i(k)\,\mathrm{d}\operatorname{Im}\tilde{\phi}_i(k)\,, \tag{B.40}$$

where the constant C accounts for any Jacobian determinant we might get.

Using (B.39) and (B.40), we are now able to calculate the path integral:

$$\begin{aligned}
&\int\left(\prod_\ell[\mathrm{d}\phi_\ell]\right)\exp\left(\frac{\mathrm{i}}{2}\int \mathrm{d}^d x\,\phi_i(x)A_{ij}(\partial)\phi_j(x)\right)\\
&= C\int\left(\prod_\ell \mathrm{d}\operatorname{Re}\tilde{\phi}_\ell(0)\right)\exp\left(\frac{\mathrm{i}}{2\mathcal{V}_d}\operatorname{Re}\tilde{\phi}_i(0)A_{ij}(0)\operatorname{Re}\tilde{\phi}_j(0)\right)\\
&\quad\times\int\left(\prod_\ell\prod_{k>0}\mathrm{d}\operatorname{Im}\tilde{\phi}_\ell(k')\right)\exp\left(\frac{\mathrm{i}}{\mathcal{V}_d}\sum_{k>0}\operatorname{Im}\tilde{\phi}_i(k)A_{ij}(\mathrm{i}k)\right)\operatorname{Im}\tilde{\phi}_j(k))\\
&\quad\times\int\left(\prod_\ell\prod_{k>0}\mathrm{d}\operatorname{Re}\tilde{\phi}_\ell(k')\right)\exp\left(\frac{\mathrm{i}}{\mathcal{V}_d}\sum_{k>0}\operatorname{Re}\tilde{\phi}_i(k)A_{ij}(\mathrm{i}k)\operatorname{Re}\tilde{\phi}_j(k)\right)\\
&= \quad C\int\left(\prod_\ell \mathrm{d}\operatorname{Re}\tilde{\phi}_\ell(0)\right)\exp\left(\frac{\mathrm{i}}{2\mathcal{V}_d}\operatorname{Re}\tilde{\phi}_i(0)A_{ij}(0)\operatorname{Re}\tilde{\phi}_j(0)\right)\\
&\quad\times\prod_{k>0}\left[\int\left(\prod_\ell \mathrm{d}\operatorname{Re}\tilde{\phi}_\ell(k)\right)\exp\left(\frac{\mathrm{i}}{\mathcal{V}_d}\operatorname{Re}\tilde{\phi}_i(k)A_{ij}(\mathrm{i}k)\operatorname{Re}\tilde{\phi}_j(k)\right)\right.\\
&\quad\left.\times\int\left(\prod_\ell \mathrm{d}\operatorname{Im}\tilde{\phi}_\ell(k)\right)\exp\left(\frac{\mathrm{i}}{\mathcal{V}_d}\operatorname{Im}\tilde{\phi}_i(k)A_{ij}(\mathrm{i}k)\operatorname{Im}\tilde{\phi}_j(k)\right)\right]\\
&= C\left(\det\frac{A(0)}{\mathrm{i}2\pi\mathcal{V}_d}\right)^{-\frac{1}{2}}\prod_{k>0}\left(\det\frac{A(\mathrm{i}k)}{\mathrm{i}\pi\mathcal{V}_d}\right)^{-\frac{1}{2}}\left(\det\frac{A(\mathrm{i}k}{\mathrm{i}\pi\mathcal{V}_d}\right)^{-\frac{1}{2}}\\
&= C2^{n/2}\prod_k\left(\det\frac{A(\mathrm{i}k)}{\mathrm{i}\pi\mathcal{V}_d}\right)^{-\frac{1}{2}} \propto (\operatorname{Det}A(\partial))^{-\frac{1}{2}}\,,
\end{aligned} \tag{B.41}$$

[1] What we actually mean by $k > 0$ is $k \in \mathcal{A}$, where $\mathcal{A}$ is an arbitrary subset of the d-dimensional Minkowski space $\mathbb{M}^d$ which satisfies the conditions $\mathcal{A}\cap\mathcal{A}' = \{\}$ and $\mathcal{A}\cup\mathcal{A}' = \mathbb{M}^d\setminus 0$, where $\mathcal{A}' = \{k\in\mathcal{M}^d \mid -k\in\mathcal{A}\}$ is the mirror image of a reflection of the region $\mathcal{A}$ through the origin.

[2] There is no integration over $\mathrm{d}\operatorname{Im}\tilde{\phi}_i(0)$ since $\operatorname{Im}\tilde{\phi}_i(0) = 0$.

where we used (B.20) and defined the functional determinant $\mathrm{Det}\, A(\partial) = \prod_k \det A(-\mathrm{i}k)$ in the last step. Actually, to use (B.20), the matrix $A(\mathrm{i}k)$ needs to be negative definite. In (2.1) the time integration is slightly rotated: $x_0 \to x_0(1 - \mathrm{i}0^+)$. After the Fourier transformation this corresponds to $k_0 \to k_0(1 + \mathrm{i}0^+)$ or $k_0^2 \to k_0^2 + \mathrm{i}0^+$, and for the path integrals in this work, this should be sufficient to guarantee that $A(\mathrm{i}k)$ is negative definite.

Taking the logarithm of the last equation, we get

$$\begin{aligned} \ln\left[(\mathrm{Det}\, A(\partial))^{-\frac{1}{2}}\right] &= \ln\left(\prod_k (\det A(-\mathrm{i}k))^{-\frac{1}{2}}\right) = \sum_k \ln\left(\det A(-\mathrm{i}k)\right)^{-\frac{1}{2}} \\ &= -\frac{1}{2}\mathcal{V}_d \int \frac{\mathrm{d}^d k}{(2\pi)^d} \ln\det A(-\mathrm{i}k)\,. \end{aligned} \tag{B.42}$$

B.4.2. Fermionic Gaussian integrals

Let us consider the path integral

$$\int \left(\prod_\ell [\mathrm{d}\psi_\ell]\right) \exp\left(\frac{1}{2}\int \mathrm{d}^d x\, \psi_i(x) A_{ij}(\partial)\psi_j\right), \tag{B.43}$$

where the integral is over $2n$ fermionic fields $\psi_i(x)$ and $A(\partial)$ is a $2n \times 2n$ matrix, which may depend on the derivatives ∂_μ and has the property $A(\partial) = -A^\mathsf{T}(-\partial)$. Using the Fourier transform

$$\psi_i(x) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \mathrm{e}^{-\mathrm{i}kx}\tilde{\psi}_i(k)\,, \tag{B.44}$$

we can rewrite the exponent as

$$\begin{aligned} \frac{1}{2}\int \mathrm{d}^d x\, \psi_i(x) A_{ij}(\partial)\psi_j(x) &= \frac{1}{2}\int \mathrm{d}^d x \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{d}^d k'}{(2\pi)^d} \mathrm{e}^{-\mathrm{i}x(k+k')}\tilde{\psi}_i(k') A_{ij}(-\mathrm{i}k)\tilde{\psi}_j(k) \\ &= \frac{1}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{d}^d k'}{(2\pi)^d} (2\pi)^d \delta^{(d)}(k+k')\tilde{\psi}_i(k') A_{ij}(-\mathrm{i}k)\tilde{\psi}_j(k) \\ &= \frac{1}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d} \tilde{\psi}_i(-k) A_{ij}(-\mathrm{i}k)\tilde{\psi}_j(k)\,. \end{aligned} \tag{B.45}$$

If we again assume formal discretization and consider the variable transformation $\psi_i(x) \to \tilde{\psi}_i(k)$, the measure of the path integral can be rewritten as

$$\prod_i [\mathrm{d}\psi_i] = \prod_i \prod_x \mathrm{d}\psi_i(x) = C \prod_i \mathrm{d}\tilde{\psi}_i(0) \prod_{k>0} \mathrm{d}\tilde{\psi}_i(k)\, \mathrm{d}\tilde{\psi}_i(-k)\,, \tag{B.46}$$

where the constant C accounts for any Jacobian determinant we might get. Furthermore, after discretization we get for the exponent:

$$\frac{1}{2}\int \mathrm{d}^d x\, \psi_i(x) A_{ij}(\partial)\psi_j(x) = \frac{1}{2}\tilde{\psi}_i(0) A_{ij}(0)\tilde{\psi}_j(0) + \sum_{k>0} \tilde{\psi}_i(-k) A_{ij}(-\mathrm{i}k)\tilde{\psi}_j(k)\,. \tag{B.47}$$

Therefore, the path integral gives

$$\begin{aligned}
&\int \left(\prod_\ell [\mathrm{d}\psi_\ell]\right) \exp\left(\frac{1}{2}\int \mathrm{d}^d x\, \psi_i(x) A_{ij}(\partial)\psi_j\right) = \\
&= C\int \left(\prod_\ell \mathrm{d}\tilde{\psi}_\ell\,(0)\right) \exp\left(\frac{1}{2\mathcal{V}_d}\tilde{\psi}_i(0)A_{ij}(0)\tilde{\psi}_j(0)\right) \\
&\quad\times \int \left(\prod_\ell \prod_{k>0} \mathrm{d}\tilde{\psi}_\ell(k')\,\mathrm{d}\tilde{\psi}_\ell(-k')\right) \exp\left(\frac{1}{\mathcal{V}_d}\sum_{k>0}\tilde{\psi}_i(-k)A_{ij}(-\mathrm{i}k)\tilde{\psi}_j(k)\right) \\
&= C\int \left(\prod_\ell \mathrm{d}\tilde{\psi}_\ell\,(0)\right) \exp\left(\frac{1}{2\mathcal{V}_d}\tilde{\psi}_i(0)A_{ij}(0)\tilde{\psi}_j(0)\right) \\
&\quad\times \prod_{k>0}\int \left(\prod_\ell \mathrm{d}\tilde{\psi}_\ell(k)\,\mathrm{d}\tilde{\psi}_\ell(-k)\right) \exp\left(\frac{1}{\mathcal{V}_d}\tilde{\psi}_i(-k)A_{ij}(-\mathrm{i}k)\tilde{\psi}_j(k)\right) \\
&= C\,\mathrm{pf}\,\frac{A^{\mathsf{T}}(0)}{\mathcal{V}_d}\prod_{k>0}\det\frac{A(-ik)}{\mathcal{V}_d}
\end{aligned} \tag{B.48}$$

where we used (B.24) and (B.33). With the following relation (see e.g. [44]):

$$\mathrm{pf}\,A = \pm\sqrt{\det A}\,, \tag{B.49}$$

we finally obtain

$$\int \left(\prod_\ell [\mathrm{d}\psi_\ell]\right) \exp\left(\frac{1}{2}\int \mathrm{d}^d x\, \psi_i(x)A_{ij}(\partial)\psi_j\right) \propto \prod_k \sqrt{\det A(-\mathrm{i}k)} = \sqrt{\mathrm{Det}\,A(\partial)}\,, \tag{B.50}$$

and taking the logarithm, we get

$$\begin{aligned}
\ln\sqrt{\mathrm{Det}\,A(\partial)} = \ln\left(\prod_k \sqrt{\det A(-\mathrm{i}k)}\right) &= \sum_k \ln\sqrt{\det A(-\mathrm{i}k)} \\
&= \frac{1}{2}\mathcal{V}_d\int\frac{\mathrm{d}^d k}{(2\pi)^d}\ln\det A(-\mathrm{i}k)\,.
\end{aligned} \tag{B.51}$$

Furthermore, let us consider the path integral

$$\int \left(\prod_\ell [\mathrm{d}\psi_\ell]\right)\left(\prod_m [\mathrm{d}\psi_m^*]\right) \exp\left(\int \mathrm{d}^d x\, \psi_i^*(x)A_{ij}(\partial)\psi_j\right), \tag{B.52}$$

where the integral is over the n fermionic fields $\psi_i(x)$ and their complex conjugated fields $\psi_i^*(x)$, and $A(\partial)$ is a $n \times n$ matrix, which may depend on the derivatives ∂_μ. Using the Fourier transform

$$\psi_i(x) = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\mathrm{e}^{-\mathrm{i}kx}\tilde{\psi}_i(k)\,, \tag{B.53}$$

we can rewrite the exponent as

$$\begin{aligned}\int \mathrm{d}^d x\, \psi_i^*(x) A_{ij}(\partial)\psi_j(x) &= \int \mathrm{d}^d x \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{d}^d k'}{(2\pi)^d} \mathrm{e}^{-\mathrm{i}x(k-k')}\tilde{\psi}_i^*(k')A_{ij}(-\mathrm{i}k)\tilde{\psi}_j(k) \\ &= \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{d}^d k'}{(2\pi)^d}(2\pi)^d\delta^{(d)}(k-k')\tilde{\psi}_i^*(k')A_{ij}(-\mathrm{i}k)\tilde{\psi}_j(k) \\ &= \int \frac{\mathrm{d}^d k}{(2\pi)^d}\tilde{\psi}_i^*(k)A_{ij}(-\mathrm{i}k)\tilde{\psi}_j(k)\,.\end{aligned} \tag{B.54}$$

If we again assume formal discretization and consider the variable transformation $\psi_i(x) \to \tilde{\psi}_i(k)$, the measure of the path integral may be rewritten as

$$\begin{aligned}\left(\prod_\ell [\mathrm{d}\psi_\ell]\right)\left(\prod_m [\mathrm{d}\psi_m^*]\right) &= \prod_x \left(\prod_\ell \mathrm{d}\psi_\ell(x)\right)\left(\prod_m \mathrm{d}\psi_m^*(x)\right) \\ &= C\prod_k \left(\prod_\ell \mathrm{d}\tilde{\psi}_\ell(k)\right)\left(\prod_m \mathrm{d}\tilde{\psi}_m^*(k)\right).\end{aligned} \tag{B.55}$$

Therefore, we can calculate the path integral using (B.54) and (B.55):

$$\begin{aligned}&\int \left(\prod_\ell [\mathrm{d}\psi_\ell]\right)\left(\prod_m [\mathrm{d}\psi_m^*]\right)\exp\left(\int \mathrm{d}^d x\, \psi_i^*(x)A_{ij}(\partial)\psi_j\right) \\ &= C\int \left[\prod_{k'}\left(\prod_\ell \mathrm{d}\tilde{\psi}_\ell(k')\right)\left(\prod_m \mathrm{d}\tilde{\psi}_m^*(k')\right)\right]\exp\left(\frac{1}{\mathcal{V}_d}\sum_k \tilde{\psi}_i^*(k)A_{ij}(-\mathrm{i}k)\tilde{\psi}_j(k)\right) \\ &= C\prod_k \int \left(\prod_\ell \mathrm{d}\tilde{\psi}_\ell(k)\right)\left(\prod_m \mathrm{d}\tilde{\psi}_m^*(k)\right)\exp\left(\frac{1}{\mathcal{V}_d}\tilde{\psi}_i^*(k)A_{ij}(-\mathrm{i}k)\tilde{\psi}_j(k)\right) \\ &= C\prod_k (-1)^{n(n-1)/2}\det\frac{A(-\mathrm{i}k)}{\mathcal{V}_d} \propto \prod_k \det A(-\mathrm{i}k) = \operatorname{Det} A(\partial)\,,\end{aligned} \tag{B.56}$$

where we used (B.32). Taking the logarithm, we subsequently get

$$\ln \operatorname{Det} A(\partial) = \ln\left(\prod_k \det A(-\mathrm{i}k)\right) = \sum_k \ln\det A(-\mathrm{i}k) = \mathcal{V}_d\int \frac{\mathrm{d}^d k}{(2\pi)^d}\ln\det A(-\mathrm{i}k). \tag{B.57}$$

B.5. Eigenvalue perturbation

Consider the eigenvalue problem [3]

$$Au_i = \lambda_i u_i\,, \qquad 1 \le i \le n\,, \tag{B.58}$$

[3]Note that we do not sum over repeated indices in this section.

where the symmetric $n \times n$ matrix A can be written as

$$A = A_0 + \epsilon A_1 + \epsilon^2 A_2 \,, \tag{B.59}$$

with a small parameter $\epsilon \ll 1$ and symmetric matrices A_0, A_1 and A_2. Let us assume that the eigenvalues $\lambda_i^{(0)}$ and eigenvectors $u_i^{(0)}$ of the symmetric matrix A_0 are known,

$$A_0 u_i^{(0)} = \lambda_i^{(0)} u_i^{(0)} \,, \quad 1 \leq i \leq n \,, \tag{B.60}$$

and that the eigenvalues are non-degenerate and the eigenvectors are normalized according to

$$u_i^{(0)\,\mathsf{T}} u_j^{(0)} = \delta_{ij} \,. \tag{B.61}$$

Furthermore, let us assume that we can make the expansions

$$\lambda = \lambda_i^{(0)} + \epsilon \lambda_i^{(1)} + \epsilon^2 \lambda_i^{(2)} + \cdots \,, \tag{B.62}$$

$$u = u_i^{(0)} + \epsilon u_i^{(1)} + \epsilon^2 u_i^{(2)} + \cdots \,. \tag{B.63}$$

Then, taking only terms of order ϵ, the eigenvalue equation (B.58) reads

$$A_0 u_i^{(1)} + A_1 u_i^{(0)} = \lambda_i^{(0)} u_i^{(1)} + \lambda_i^{(1)} u_i^{(0)} \tag{B.64}$$

and multiplying this by $u_i^{(0)\,\mathsf{T}}$ from the left we find

$$\lambda_1^{(1)} = u_i^{(0)\,\mathsf{T}} A_1 u_i^{(0)} \,. \tag{B.65}$$

If we multiply (B.64) by $u_j^{(0)\,\mathsf{T}}$ (where $j \neq i$) from the left, we obtain

$$u_j^{(0)\,\mathsf{T}} u_i^{(1)} = \frac{u_j^{(0)\,\mathsf{T}} A_1 u_i^{(0)}}{\lambda_i^{(0)} - \lambda_j^{(0)}} \,, \quad j \neq i \,. \tag{B.66}$$

Furthermore, demanding that the eigenvectors u_i are unit vectors,

$$u_i^{\mathsf{T}} u_i = u_i^{(0)\,\mathsf{T}} u_i^{(0)} + 2\epsilon u_i^{(0)\,\mathsf{T}} u_i^{(1)} + \cdots \stackrel{!}{=} 1 \,, \tag{B.67}$$

we get the condition

$$u_i^{(0)\,\mathsf{T}} u_i^{(1)} = 0 \,. \tag{B.68}$$

Therefore, using the completeness relation

$$\mathbb{1}_{n \times n} = \sum_i u_i^{(0)} u_i^{(0)\,\mathsf{T}} \,, \tag{B.69}$$

we finally find

$$u_i^{(1)} = \sum_j u_j^{(0)} u_j^{(0)\,\mathsf{T}} u_i^{(1)} = \sum_{j \neq i} u_j^{(0)} \frac{u_j^{(0)\,\mathsf{T}} A_1 u_i^{(0)}}{\lambda_i^{(0)} - \lambda_j^{(0)}} \,. \tag{B.70}$$

Now let us consider the case where one of the eigenvalues of A_0 is degenerate, where the degree of degeneracy is r:

$$\lambda \equiv \lambda_1^{(0)} = \cdots = \lambda_r^{(0)} . \tag{B.71}$$

Then, the eigenvectors $u_i^{(0)}$ $(1 \leq i \leq r)$ are not uniquely determined by (B.60) anymore. We can make the ansatz

$$u_i^{(0)} = \sum_{j=1}^{r} R_{ij} u'_j \tag{B.72}$$

with an orthogonal $r \times r$ matrix R, where the vectors u'_j build an orthonormal basis of the eigenspace corresponding to the eigenvalue λ. Inserting this into (B.64) and multiplying by $u_j^{(0)\,\mathsf{T}}$ from the left, we obtain

$$\sum_{k,l=1}^{r} R_{jk} u_k'^{\mathsf{T}} A_1 u'_l R_{li}^{\mathsf{T}} = \lambda_i^{(1)} \delta_{ij} , \quad \text{for} \quad i, j = 1, \ldots, r . \tag{B.73}$$

Writing the last equation in matrix form we have

$$R \Lambda R^{\mathsf{T}} = \operatorname{diag}(\lambda_1^{(1)}, \ldots, \lambda_r^{(1)}) \tag{B.74}$$

with the matrix

$$\Lambda_{kl} = u_k'^{\mathsf{T}} A_1 u'_l . \tag{B.75}$$

Therefore, the first order corrections $\lambda_i^{(1)}$ $(1 \leq i \leq r)$ are determined by the eigenvalues of Λ, and the rows of the matrix R are given by the eigenvectors of Λ.

While equation (B.70) is still valid for $r < i \leq n$, there is obviously a problem for $1 \leq i \leq r$. To determine the coefficients $u_i^{(0)\,\mathsf{T}} u_j^{(1)}$ for $1 \leq i, j \leq r$, we have to use the condition that we obtain by considering only terms of order ϵ^2 in the eigenvalue equation (B.58):

$$A_0 u_i^{(2)} + A_1 u_i^{(1)} + A_2 u_i^{(0)} = \lambda_i^{(0)} u_i^{(2)} + \lambda_i^{(1)} u_i^{(1)} + \lambda_i^{(2)} u_i^{(0)} . \tag{B.76}$$

Multiplying by $u_j^{(0)}$ (where $j \neq i$) from the left and using (B.66) and (B.73) we obtain

$$u_j^{(0)\,\mathsf{T}} u_i^{(1)} = \frac{u_j^{(0)\,\mathsf{T}} A_2 u_i^{(0)}}{\lambda_i^{(1)} - \lambda_j^{(1)}} + \sum_{k=r+1}^{n} \frac{u_j^{(0)\,\mathsf{T}} A_1 u_k^{(0)} u_k^{(0)\,\mathsf{T}} A_1 u_i^{(0)}}{\left(\lambda_i^{(1)} - \lambda_j^{(1)}\right)\left(\lambda_i^{(0)} - \lambda_k^{(0)}\right)} \quad \text{for} \quad \begin{cases} i, j = 1, \ldots, r \\ i \neq j \end{cases} . \tag{B.77}$$

Therefore, for $1 \leq i \leq r$ we get

$$\begin{aligned} u_i^{(1)} = \sum_j u_j^{(0)} u_j^{(0)\,\mathsf{T}} u_i^{(1)} &= \sum_{j=r+1}^{n} u_j^{(0)} \frac{u_j^{(0)\,\mathsf{T}} A_1 u_i^{(0)}}{\lambda_i^{(0)} - \lambda_j^{(0)}} \\ &+ \sum_{\substack{j=1 \\ j \neq i}}^{r} u_j^{(0)} \left[\frac{u_j^{(0)\,\mathsf{T}} A_2 u_i^{(0)}}{\lambda_i^{(1)} - \lambda_j^{(1)}} + \sum_{k=r+1}^{n} \frac{u_j^{(0)\,\mathsf{T}} A_1 u_k^{(0)} u_k^{(0)\,\mathsf{T}} A_1 u_i^{(0)}}{\left(\lambda_i^{(1)} - \lambda_j^{(1)}\right)\left(\lambda_i^{(0)} - \lambda_k^{(0)}\right)} \right] . \end{aligned} \tag{B.78}$$

C. The most general renormalizable Lagrangian density

In this section we want to find the most general renormalizable, Lorentz invariant Lagrangian density for a theory with a field content that includes m scalar fields $\phi_i(x)$, and k_L left-handed and k_R right-handed fermionic fields $\psi_{\mathrm{L}\,r}(x)$ and $\psi_{\mathrm{R}\,s}(x)$. In section C.4 we will introduce a gauge symmetry and therefore we will add n real gauge fields $A_a(x)$. We adopt a formalism that was introduced by Weinberg [13, 46, 47], which conveniently allows to describe a general gauge theory.

C.1. Scalar fields

In principle the considered scalar fields could be complex or real, but since it is always possible to rewrite a complex scalar field as two independent real scalar fields, we will assume that the m independent scalar fields ϕ_i are real in the first place.

C.1.1. Kinetic term

The only renormalizable and Lorentz invariant term, which includes derivatives of scalar fields, is of the form

$$K_\phi = (\partial_\mu \phi_i) A_{ij} (\partial^\mu \phi_j)\,. \tag{C.1}$$

The Lagrangian density is supposed to be real and thus the matrix A has to be real. Furthermore, we can assume A to be symmetric since the antisymmetric part drops out in equation (C.1) anyway. A real symmetric matrix can be diagonalized by an orthogonal Matrix O:

$$O^\mathsf{T} A O = \mathrm{diag}(a_1, \ldots, a_m) = \hat{A}\,. \tag{C.2}$$

If we redefine the fields according to $\sum_j \sqrt{2a_i} O^\mathsf{T}_{ij} \phi_j \to \phi_i$ (without summation over the index i) equation (C.1) reads

$$K_\phi = \frac{1}{2} (\partial_\mu \phi_i)(\partial^\mu \phi_i)\,. \tag{C.3}$$

C.1.2. Interaction term

The self-interaction terms of the scalar fields are usually called scalar potential. The most general renormalizable term is

$$V_\phi = \mathcal{P}(\phi)\,, \tag{C.4}$$

where $\mathcal{P}(\phi)$ is a fourth order polynomial in ϕ_i:

$$\mathcal{P}(\phi) = \Lambda^{d-4} f + \Lambda^{\frac{d-4}{2}} f_i \phi_i + f_{ij}\phi_i\phi_j + \Lambda^{\frac{4-d}{2}} f_{ijk}\phi_i\phi_j\phi_k + \Lambda^{4-d} f_{ijkl}\phi_i\phi_j\phi_k\phi_l\,, \tag{C.5}$$

where Λ is an arbitrary parameter of mass dimension one, which was introduced to keep the mass dimensions of the coupling constants ($[f] = 4$, $[f_i] = 3$, $[f_{ij}] = 2$, $[f_{ijk}] = 1$, $[f_{ijkl}] = 0$) fixed while using dimensional regularization. All coupling constants are real and can be assumed to be totally symmetric since any antisymmetric part would drop out anyway.

C.2. Fermionic fields

The k_L left-handed and k_R right-handed fermionic fields respectively fulfill the conditions

$$\psi_{\mathrm{L}\,r} = P_\mathrm{L}\psi_{\mathrm{L}\,r} \quad \text{and} \quad \psi_{\mathrm{R}\,r} = P_\mathrm{R}\psi_{\mathrm{R}\,r}\,, \tag{C.6}$$

where the projection operators $P_\mathrm{L} = \frac{1}{2}(\mathbb{1}_{4\times4} - \gamma_5)$ and $P_\mathrm{R} = \frac{1}{2}(\mathbb{1}_{4\times4} + \gamma_5)$ were introduced. For every field ψ we consider, there is a charge conjugated field

$$\psi^\mathrm{c} = C\left(\overline{\psi}\right)^\mathsf{T}\,, \tag{C.7}$$

where C is the unitary and antisymmetric charge conjugation matrix, defined by the relation

$$(\gamma^\mu)^\mathsf{T} = -C^{-1}\gamma^\mu C\,. \tag{C.8}$$

Note that charge conjugation changes the chirality of a spinor:

$$(\psi_\mathrm{L})^\mathrm{c} = P_\mathrm{R}(\psi_\mathrm{L})^\mathrm{c} \quad \text{and} \quad (\psi_\mathrm{R})^\mathrm{c} = P_\mathrm{L}(\psi_\mathrm{R})^\mathrm{c}\,. \tag{C.9}$$

C.2.1. Kinetic term

The most general Lorentz invariant kinetic term that we can write down is of the form

$$\begin{aligned} K_\psi = {}& \overline{\psi_\mathrm{L}}(A_1 \otimes \mathrm{i}\partial\!\!\!/)\psi_\mathrm{L} + \overline{\psi_\mathrm{L}}(A_2 \otimes \mathrm{i}\partial\!\!\!/)(\psi_\mathrm{R})^\mathrm{c} + \overline{(\psi_\mathrm{R})^\mathrm{c}}(A_3 \otimes \mathrm{i}\partial\!\!\!/)\psi_\mathrm{L} \\ &+ \overline{(\psi_\mathrm{R})^\mathrm{c}}(A_4 \otimes \mathrm{i}\partial\!\!\!/)(\psi_\mathrm{R})^\mathrm{c}\,, \end{aligned} \tag{C.10}$$

where we have introduced two vectors containing respectively all left-handed and all right-handed fields:

$$\psi_\mathrm{L} = \begin{pmatrix} \psi_{\mathrm{L}\,1} \\ \vdots \\ \psi_{\mathrm{L}\,k_\mathrm{L}} \end{pmatrix}, \quad \psi_\mathrm{R} = \begin{pmatrix} \psi_{\mathrm{R}\,1} \\ \vdots \\ \psi_{\mathrm{R}\,k_\mathrm{R}} \end{pmatrix}. \tag{C.11}$$

Of course, we could also add the terms $\overline{(\psi_{\mathrm{L}})^{\mathrm{c}}}(B_1 \otimes \mathrm{i}\partial\!\!\!/)(\psi_{\mathrm{L}})^{\mathrm{c}}$, $\overline{\psi_{\mathrm{R}}}(B_2 \otimes \mathrm{i}\partial\!\!\!/)(\psi_{\mathrm{L}})^{\mathrm{c}}$, $\overline{(\psi_{\mathrm{L}})^{\mathrm{c}}}(B_3 \otimes \mathrm{i}\partial\!\!\!/)\psi_{\mathrm{R}}$ and $\overline{\psi_{\mathrm{R}}}(B_2 \otimes \mathrm{i}\partial\!\!\!/)\psi_{\mathrm{R}}$, but using the identity

$$\overline{\psi_1}(A \otimes \mathrm{i}\partial\!\!\!/)\psi_2 = \left[\overline{\psi_1}(A \otimes \mathrm{i}\partial\!\!\!/)\psi_2\right]^{\mathsf{T}} = \overline{\psi_2^{\mathrm{c}}}(A^{\mathsf{T}} \otimes \mathrm{i}\partial\!\!\!/)\psi_1^{\mathrm{c}}\,, \tag{C.12}$$

and with the redefinitions $A_i + B_i^{\mathsf{T}} \to A_i$, we recover the original form.

To ensure that K_ψ is real, we can add the complex conjugate of all terms in (C.10). Using the identity

$$\left[\overline{\psi_1}(A \otimes \mathrm{i}\partial\!\!\!/)\psi_2\right]^* = \left[\overline{\psi_1}(A \otimes \mathrm{i}\partial\!\!\!/)\psi_2\right]^\dagger = \overline{\psi_2}(A^\dagger \otimes \mathrm{i}\partial\!\!\!/)\psi_1\,, \tag{C.13}$$

and redefining the matrices according to

$$A_1 + A_1^\dagger \to A_1\,, \quad A_2 + A_3^\dagger \to A_2\,, \quad A_2^\dagger + A_3 \to A_3\,, \quad A_4 + A_4^\dagger \to A_4\,, \tag{C.14}$$

we then again obtain the original form of equation (C.10), but now with the conditions $A_1 = A_1^\dagger$, $A_2 = A_3^\dagger$ and $A_4 = A_4^\dagger$.

If we construct the left-handed vector

$$\omega_{\mathrm{L}} = \begin{pmatrix} \psi_{\mathrm{L}} \\ (\psi_{\mathrm{R}})^{\mathrm{c}} \end{pmatrix}, \tag{C.15}$$

we can rewrite equation (C.10) as

$$K_\psi = \overline{\omega_{\mathrm{L}}}(A \otimes \mathrm{i}\partial\!\!\!/)\omega_{\mathrm{L}}\,, \tag{C.16}$$

with

$$A = \begin{pmatrix} A_1 & A_2 \\ A_2^\dagger & A_4 \end{pmatrix}. \tag{C.17}$$

Since A is hermitian, there exists a unitary matrix U that diagonalizes A:

$$U^\dagger A U = \mathrm{diag}(a_1, \ldots, a_{k_{\mathrm{L}}+k_{\mathrm{R}}}) = \hat{A}\,. \tag{C.18}$$

If we redefine the fields according to $\sqrt{a_r}\left[(U^\dagger \otimes \mathbb{1}_{4\times 4})\omega_{\mathrm{L}}\right]_r \to \omega_{\mathrm{L}\,r}$ (without summation over r) we get

$$\begin{aligned} K_\psi &= \overline{\omega_{\mathrm{L}}}(\mathbb{1}_{(k_{\mathrm{L}}+k_{\mathrm{R}})\times(k_{\mathrm{L}}+k_{\mathrm{R}})} \otimes \mathrm{i}\partial\!\!\!/)\omega_{\mathrm{L}} = \overline{\psi_{\mathrm{L}}}(\mathbb{1}_{k_{\mathrm{L}}\times k_{\mathrm{L}}} \otimes \mathrm{i}\partial\!\!\!/)\psi_{\mathrm{L}} + \overline{(\psi_{\mathrm{R}})^{\mathrm{c}}}(\mathbb{1}_{k_{\mathrm{R}}\times k_{\mathrm{R}}} \otimes \mathrm{i}\partial\!\!\!/)(\psi_{\mathrm{R}})^{\mathrm{c}} \\ &= \overline{\psi_{\mathrm{L}}}(\mathbb{1}_{k_{\mathrm{L}}\times k_{\mathrm{L}}} \otimes \mathrm{i}\partial\!\!\!/)\psi_{\mathrm{L}} + \overline{\psi_{\mathrm{R}}}(\mathbb{1}_{k_{\mathrm{R}}\times k_{\mathrm{R}}} \otimes \mathrm{i}\partial\!\!\!/)\psi_{\mathrm{R}}\,, \end{aligned} \tag{C.19}$$

where we used (C.12). It seems at this point as it does not matter if we consider a left-handed field or a right-handed one. Indeed, a charge conjugated right-handed field is left-handed and thus, we actually could have started with $k_{\mathrm{L}}+k_{\mathrm{R}}$ left-handed and no right-handed fields with the same result. The only difference is which name we assign to the fields: $(\psi_{\mathrm{R}\,r})^{\mathrm{c}}$ or $\psi_{\mathrm{L}\,r}$. Once we include Dirac mass terms and switch to the mass basis, we can construct Dirac fields out of respectively one left-handed and one right-handed field. Hence, at that point it really makes sense to speak of a left-handed and a right-handed component of a field.

C.2.2. Interaction term

The only pure fermionic, renormalizable interaction terms are mass terms of the form

$$
\begin{aligned}
V_\psi = \overline{\psi_{\mathrm{R}}}(m_1 \otimes \mathbb{1}_{4\times4})\psi_{\mathrm{L}} + \overline{\psi_{\mathrm{L}}}(m_2 \otimes \mathbb{1}_{4\times4})\psi_{\mathrm{R}} + \frac{1}{2}\overline{(\psi_{\mathrm{L}})^{\mathrm{c}}}(m_3 \otimes \mathbb{1}_{4\times4})\psi_{\mathrm{L}} \\
+ \frac{1}{2}\overline{\psi_{\mathrm{L}}}(m_4 \otimes \mathbb{1}_{4\times4})(\psi_{\mathrm{L}})^{\mathrm{c}} + \frac{1}{2}\overline{(\psi_{\mathrm{R}})^{\mathrm{c}}}(m_5 \otimes \mathbb{1}_{4\times4})\psi_{\mathrm{R}} + \frac{1}{2}\overline{\psi_{\mathrm{R}}}(m_6 \otimes \mathbb{1}_{4\times4})(\psi_{\mathrm{R}})^{\mathrm{c}} .
\end{aligned}
\tag{C.20}
$$

Of course, we could also add the terms $\overline{(\psi_{\mathrm{L}})^{\mathrm{c}}}(m_7 \otimes \mathbb{1}_{4\times4})(\psi_{\mathrm{R}})^{\mathrm{c}}$ and $\overline{(\psi_{\mathrm{R}})^{\mathrm{c}}}(m_8 \otimes \mathbb{1}_{4\times4})(\psi_{\mathrm{L}})^{\mathrm{c}}$, but using the identity

$$
\overline{\psi_1}(M \otimes \mathbb{1}_{4\times4})\psi_2 = \left[\overline{\psi_1}(M \otimes \mathbb{1}_{4\times4})\psi_2\right]^{\mathsf{T}} = \overline{(\psi_2)^{\mathrm{c}}}(M^{\mathsf{T}} \otimes \mathbb{1}_{4\times4})(\psi_1)^{\mathrm{c}} , \tag{C.21}
$$

and with the redefinitions $m_1 + m_7^{\mathsf{T}} \to m_1$ and $m_2 + m_8^{\mathsf{T}} \to m_2$, we recover the original form. Further, due to equation (C.21) the antisymmetric parts of m_3, m_4, m_5 and m_6 drop out in equation (C.20) and thus we can assume them to be symmetric.

To ensure that K_ψ is real, we add the complex conjugate of all terms in equation C.20. Using the identity

$$
\left[\overline{\psi_1}(M \otimes \mathbb{1}_{4\times4})\psi_2\right]^* = \left[\overline{\psi_1}(M \otimes \mathbb{1}_{4\times4})\psi_2\right]^\dagger = \overline{\psi_2}(M^\dagger \otimes \mathbb{1}_{4\times4})\psi_1 , \tag{C.22}
$$

and with the redefinitions $m_1 + m_2^\dagger \to m'_{\mathrm{D}}$, $m_3 + m_4^\dagger \to m'_{\mathrm{L}}$ and $m_5 + m_6^\dagger \to m'_{\mathrm{R}}$ we subsequently get

$$
V_\psi = \overline{\psi_{\mathrm{R}}}(m'_{\mathrm{D}} \otimes \mathbb{1}_{4\times4})\psi_{\mathrm{L}} + \frac{1}{2}\overline{(\psi_{\mathrm{L}})^{\mathrm{c}}}(m'_{\mathrm{L}} \otimes \mathbb{1}_{4\times4})\psi_{\mathrm{L}} + \frac{1}{2}\overline{(\psi_{\mathrm{R}})^{\mathrm{c}}}(m'_{\mathrm{R}} \otimes \mathbb{1}_{4\times4})\psi_{\mathrm{R}} + \text{h.c.} , \tag{C.23}
$$

where m'_{L} and m'_{R} are symmetric matrices.

C.3. Yukawa interactions

The interactions between scalar and fermionic fields are called Yukawa interactions. The only allowed terms are of the same form as the fermionic mass terms V_ψ, where we just have to replace the mass matrices by Yukawa matrices, coupling the fermions to the scalar fields:

$$
\begin{aligned}
V_{\phi\psi} = \Lambda^{\frac{4-d}{2}} \phi_i \Bigg[\overline{\psi_{\mathrm{R}}}(\Gamma_{\mathrm{D}\,i} \otimes \mathbb{1}_{4\times4})\psi_{\mathrm{L}} + \frac{1}{2}\overline{(\psi_{\mathrm{L}})^{\mathrm{c}}}(\Gamma_{\mathrm{L}\,i} \otimes \mathbb{1}_{4\times4})\psi_{\mathrm{L}} \\
+ \frac{1}{2}\overline{(\psi_{\mathrm{R}})^{\mathrm{c}}}(\Gamma_{\mathrm{R}\,i} \otimes \mathbb{1}_{4\times4})\psi_{\mathrm{R}} + \text{h.c.}\Bigg] ,
\end{aligned}
\tag{C.24}
$$

where the matrices $\Gamma_{\mathrm{L}\,i}$ and $\Gamma_{\mathrm{R}\,i}$ are symmetric.

C.4. Gauge fields

Let us now introduce a gauge symmetry, where the symmetry group is a compact reductive[1] Lie group $\mathcal{G}$. The group generators T_a satisfy the commutation relations

$$[T_a, T_b] = \mathrm{i} C_{abc} T_c \,, \tag{C.25}$$

where C_{abc} are the structure constants, which can be chosen to be totally antisymmetric. Moreover, we can choose the generators to be hermitian and the structure constants to be real without loss of generality. The scalar fields and the left- and right-handed fermionic fields transform under an infinitesimal gauge transformation, characterized by the real infinitesimal parameters $\alpha_a(x)$, as

$$\phi_i \to [\delta_{ij} - \mathrm{i}\alpha_a(x)(\theta_a)_{ij}]\, \phi_j \,, \tag{C.26}$$

$$\psi_{\mathrm{L}\,r} \to [\delta_{rs} - \mathrm{i}\alpha_a(x)(t_{\mathrm{L}\,a})_{rs}]\, \psi_{\mathrm{L}\,s} \,, \tag{C.27}$$

$$\psi_{\mathrm{R}\,r} \to [\delta_{rs} - \mathrm{i}\alpha_a(x)(t_{\mathrm{R}\,a})_{rs}]\, \psi_{\mathrm{R}\,s} \,, \tag{C.28}$$

where the generators θ_a, $t_{\mathrm{L}\,a}$ and $t_{\mathrm{R}\,a}$ are the matrices representing the abstract generators T_a in the representations furnished by the fields ϕ_i, $\psi_{\mathrm{L}\,r}$ and $\psi_{\mathrm{R}\,r}$ respectively. Note that we do not write the gauge couplings explicitly, they are included in the matrices θ_a, $t_{\mathrm{L}\,a}$, $t_{\mathrm{R}\,a}$ and in the structure constant C_{abc}. Since we want the scalar field to stay real under a gauge transformation we have to impose

$$\theta_a^* = -\theta_a \,. \tag{C.29}$$

Furthermore, since θ_a, $t_{\mathrm{L}\,a}$ and $t_{\mathrm{R}\,a}$ are matrices representing the generators T_a, they have to be hermitian and satisfy the same commutation relations as T_a:

$$\theta_a^\dagger = \theta_a \,, \qquad [\theta_a\,, \theta_b] = \mathrm{i} C_{abc} \theta_c \,, \tag{C.30}$$

$$t_{\mathrm{L}\,a}^\dagger = t_{\mathrm{L}\,a} \,, \qquad [t_{\mathrm{L}\,a}, t_{\mathrm{L}\,b}] = \mathrm{i} C_{abc} t_{\mathrm{L}\,c} \,, \tag{C.31}$$

$$t_{\mathrm{R}\,a}^\dagger = t_{\mathrm{R}\,a} \,, \qquad [t_{\mathrm{R}\,a}, t_{\mathrm{R}\,b}] = \mathrm{i} C_{abc} t_{\mathrm{R}\,c} \,. \tag{C.32}$$

To define a covariant derivative, we need to introduce the real gauge fields A_a^μ, which transform according to

$$A_{a\,\mu} \to A_{a\,\mu} + \Lambda^{\frac{d-4}{2}} \partial_\mu \alpha_a(x) - C_{abc} A_{b\,\mu} \alpha_c(x) \,. \tag{C.33}$$

With this definition, the respective covariant derivatives of the fields:

$$(D_\mu \phi)_i = \partial_\mu \phi_i + \Lambda^{\frac{4-d}{2}} \mathrm{i} (\theta_a)_{ij} \phi_j A_{a\,\mu} \,, \tag{C.34}$$

$$(D_\mu \psi_{\mathrm{L}})_r = \partial_\mu \psi_{\mathrm{L}\,r} + \Lambda^{\frac{4-d}{2}} \mathrm{i} (t_{\mathrm{L}\,a})_{rs} \psi_{\mathrm{L}\,s} A_{a\,\mu} \,, \tag{C.35}$$

$$(D_\mu \psi_{\mathrm{R}})_r = \partial_\mu \psi_{\mathrm{R}\,r} + \Lambda^{\frac{4-d}{2}} \mathrm{i} (t_{\mathrm{R}\,a})_{rs} \psi_{\mathrm{R}\,s} A_{a\,\mu} \,, \tag{C.36}$$

[1]Reductive means that the Lie algebra of the Lie group is a direct sum of simple and trivial (one-dimensional) Lie algebras.

transform in the same way as the original fields. If we exchange the partial derivative ∂_μ with the covariant derivative D_μ in the scalar and fermionic kinetic terms, one can readily check that these terms are gauge invariant. Therefore, the gauge-kinetic terms for scalars and fermions read

$$K_{\phi A} = \frac{1}{2}(D_\mu\phi)_i(D^\mu\phi)_i\,, \tag{C.37}$$

and

$$K_{\psi A} = \overline{\psi_{\mathrm{L}}}(\mathbb{1}_{k_{\mathrm{L}}\times k_{\mathrm{L}}}\otimes \mathrm{i}\not{D})\psi_{\mathrm{L}} + \overline{\psi_{\mathrm{R}}}(\mathbb{1}_{k_{\mathrm{R}}\times k_{\mathrm{R}}}\otimes \mathrm{i}\not{D})\psi_{\mathrm{R}}\,. \tag{C.38}$$

Applying an infinitesimal gauge transformation to the scalar interaction term and demanding that it is gauge invariant:

$$\mathcal{P}(\phi) \to \mathcal{P}(\phi) - \mathrm{i}\alpha_a(x)(\theta_a\phi)_i\frac{\partial\mathcal{P}(\phi)}{\partial\phi_i} \stackrel{!}{=} \mathcal{P}(\phi)\,, \tag{C.39}$$

leads to the condition

$$(\theta_a\phi)_i\frac{\partial\mathcal{P}(\phi)}{\partial\phi_i} = 0\,. \tag{C.40}$$

Proceeding in the same way with the fermionic interaction terms yields the following conditions for the mass and Yukawa matrices:

$$m'_{\mathrm{D}}t_{\mathrm{L}\,a} - t_{\mathrm{R}\,a}m'_{\mathrm{D}} = 0\,, \qquad \Gamma_{\mathrm{D}\,i}t_{\mathrm{L}\,a} - t_{\mathrm{R}\,a}\Gamma_{\mathrm{D}\,i} = (\theta_a)_{ij}\Gamma_{\mathrm{D}\,j}\,, \tag{C.41}$$

$$t^*_{\mathrm{L}\,a}m'_{\mathrm{L}} + m'_{\mathrm{L}}t_{\mathrm{L}\,a} = 0\,, \qquad \Gamma_{\mathrm{L}\,i}t_{\mathrm{L}\,a} + t^*_{\mathrm{L}\,a}\Gamma_{\mathrm{L}\,i} = (\theta_a)_{ij}\Gamma_{\mathrm{L}\,j}\,, \tag{C.42}$$

$$t^*_{\mathrm{R}\,a}m'_{\mathrm{R}} + m'_{\mathrm{R}}t_{\mathrm{R}\,a} = 0\,, \qquad \Gamma_{\mathrm{R}\,i}t_{\mathrm{R}\,a} + t^*_{\mathrm{R}\,a}\Gamma_{\mathrm{R}\,i} = (\theta_a)_{ij}\Gamma_{\mathrm{R}\,j}\,. \tag{C.43}$$

We have already discussed the terms describing interactions of gauge fields with other fields, but we still need kinetic and self-interaction terms for the gauge fields. The only terms that are renormalizable, gauge- and Lorentz invariant and include only gauge fields are of the form $aF_{\mathrm{a}\,\mu\nu}F_a^{\mu\nu}$ and $bF_{\mathrm{a}\,\mu\nu}\epsilon^{\mu\nu\rho\sigma}F_{\mathrm{a}\,\rho\sigma}$, where we introduced the field strength tensor

$$F_{\mathrm{a}\,\mu\nu} = \partial_\mu A_{a\,\nu} - \partial_\nu A_{a\,\mu} - \Lambda^{\frac{4-d}{2}}C_{abc}A_{b\,\mu}A_{c\,\nu}\,. \tag{C.44}$$

It can be shown that $bF_{\mathrm{a}\,\mu\nu}\epsilon^{\mu\nu\rho\sigma}F_{\mathrm{a}\,\rho\sigma}$ is a total derivative, and therefore, it does not contribute to the action and we hence drop it. This leaves us with

$$K_A = -\frac{1}{4}F_{\mathrm{a}\,\mu\nu}F_a^{\mu\nu}\,, \tag{C.45}$$

where we have chosen a specific normalization for the fields $A_{a\,\mu}$.

Finally, the most general Lagrangian density reads

$$\mathcal{L} = K_A + K_{\phi A} + K_{\psi A} - V_\phi - V_\psi - V_{\phi\psi}\,. \tag{C.46}$$

For the calculations in section 2.2.2 it is convenient to have the same number of left- and right-handed fermionic fields. Since fields that only have a kinetic term and do not have any interaction with other fields are physically irrelevant, it is always possible to introduce new fields such that the number of left- and right-handed fermionic fields coincides. Therefore, we will now assume that we have the same number $k_\mathrm{f} = k_\mathrm{L} = k_\mathrm{R}$ of left- and right-handed fermionic fields. We can then define the field vector

$$\psi(x) = \psi_\mathrm{L}(x) + \psi_\mathrm{R}(x)\,. \tag{C.47}$$

and rewrite the gauge-kinetic fermion term:

$$K_{\psi A} = \overline{\psi}(\mathbb{1}_{k_\mathrm{f}\times k_\mathrm{f}} \otimes \mathrm{i}\not{D})\psi\,, \tag{C.48}$$

the mass term:

$$V_\psi = \overline{\psi}(m'_\mathrm{D} \otimes P_\mathrm{L})\psi + \frac{1}{2}\overline{\psi^\mathrm{c}}(m'_\mathrm{L} \otimes P_\mathrm{L})\psi + \frac{1}{2}\overline{\psi^\mathrm{c}}(m'_\mathrm{R} \otimes P_\mathrm{R})\psi + \text{h.c.}\,, \tag{C.49}$$

and the Yukawa term:

$$V_{\phi\psi} = \Lambda^{\frac{4-d}{2}}\phi_i\left[\overline{\psi}(\Gamma_{\mathrm{D}\,i} \otimes P_\mathrm{L})\psi + \frac{1}{2}\overline{\psi^\mathrm{c}}(\Gamma_{\mathrm{L}\,i} \otimes P_\mathrm{L})\psi + \frac{1}{2}\overline{\psi^\mathrm{c}}(\Gamma_{\mathrm{R}\,i} \otimes P_\mathrm{R})\psi + \text{h.c.}\right]. \tag{C.50}$$

C.5. Gauge-fixing and Faddeev-Popov ghosts

It was shown by Faddeev and Popov [48] that if we have a quotient of path integrals with an action $S[\Phi]$ in the exponent, which depends on a set of bosonic and fermionic fields Φ_i and is invariant under a gauge transformation, we may replace the action by

$$S'[\Phi] = S[\Phi] + S_\mathrm{GF} + S_\mathrm{FP}\,, \tag{C.51}$$

with the gauge-fixing part

$$S_\mathrm{GF} = -\frac{1}{2}\int \mathrm{d}^d x\, f_a(x) f_a(x)\,, \tag{C.52}$$

and the Faddeev-Popov ghost term

$$S_\mathrm{FP} = -\int \mathrm{d}^d x\, \mathrm{d}^d y\, \omega^*(y)\frac{\delta f'_a(x)}{\delta\alpha_b(y)}\omega(x)\,, \tag{C.53}$$

The function $f_a(x)$ is an arbitrary function of the fields $\Phi_i(x)$ and their derivatives, which determines how the gauge is fixed. The function $f'_a(x)$ is given by an infinitesimal gauge transformation of $f_a(x) \to f'_a(x)$. The fields $\omega(x)$ are anti-commuting quantities and are named ghost fields, because they are not physical.

Let us now consider the action corresponding to the Lagrangian density of the last section (C.46)

$$S[\phi, \psi, A] = \int \mathrm{d}^d x\, \mathcal{L}\,. \tag{C.54}$$

For a reason that becomes clear in section 2.2.2 we choose the gauge-fixing function

$$f_a(x) = \xi_{ab}^{\frac{1}{2}} \partial_\mu A_b^\mu - \mathrm{i}\Lambda^{\frac{4-d}{2}} \xi_{ab}^{-\frac{1}{2}} (\theta_b \phi_{\mathrm{cl}})_i (\phi - \phi_{\mathrm{cl}})_i \,, \tag{C.55}$$

where ξ_a are free, non-negative gauge-fixing parameters, which form the diagonal matrix $\xi = \mathrm{diag}(\xi_1, \ldots, \xi_n)$. The field $\phi_{\mathrm{cl}\,i}$ is defined in section 2.2.2, but the only thing that is relevant here is that it does not transform under gauge transformations. Using the transformation behavior of scalar and gauge-fields under infinitesimal gauge transformations, (C.26) and (C.33), the function f_a transforms as

$$f_a \to f_a' = f_a + \Lambda^{\frac{d-4}{2}} \xi_{ab}^{\frac{1}{2}} \Box \alpha_b + \xi_{ab}^{\frac{1}{2}} C_{bcd} \partial_\mu \left(A_d^\mu \alpha_c\right) - \Lambda^{\frac{4-d}{2}} \xi_{ab}^{-\frac{1}{2}} (\theta_b \phi_{\mathrm{cl}})_i (\theta_c)_{ij} \phi_j \alpha_c \,. \tag{C.56}$$

Plugging the functional derivative

$$\begin{aligned}\frac{\delta f_a'(x)}{\delta \alpha_b(y)} = \Lambda^{\frac{d-4}{2}} \xi_{ab}^{\frac{1}{2}} \Box \delta^{(d)}(x-y) + \xi_{ac}^{\frac{1}{2}} C_{cbd} \partial_\mu \left[A_d^\mu(x) \delta^{(d)}(x-y)\right] \\ + \Lambda^{\frac{4-d}{2}} \xi_{ac}^{-\frac{1}{2}} \left[\theta_b \theta_c \phi_{\mathrm{cl}}(x)\right]_i \phi_i(x) \delta^{(d)}(x-y)\end{aligned} \tag{C.57}$$

into (C.53), we get the Faddeev-Popov Lagrangian density

$$\mathcal{L}_{\mathrm{FP}} = (\partial_\mu \omega_a^*)(\partial^\mu \omega_a) + (\partial_\mu \omega_a^*) \Lambda^{\frac{4-d}{2}} \xi_{ac}^{\frac{1}{4}} C_{ced} A_d^\mu \xi_{eb}^{-\frac{1}{4}} \omega_b - \omega_a^* \Lambda^{4-d} \xi_{ac}^{-\frac{3}{4}} (\theta_d \theta_c \phi_{\mathrm{cl}})_i \phi_i \xi_{db}^{-\frac{1}{4}} \omega_b \,, \tag{C.58}$$

where we have made the transformation $\omega_a \to \Lambda^{\frac{4-d}{4}} \xi_{ab}^{-\frac{1}{4}} \omega_b$, such that the kinetic term is normalized in the usual way. This is allowed since we are dealing with quotients of path integrals and therefore the transformation constant of the measure $[\mathrm{d}\omega][\mathrm{d}\omega^*] \to \mathrm{const.}[\mathrm{d}\omega][\mathrm{d}\omega^*]$ cancels.

For the specific gauge-fixing function we have chosen the gauge-fixing term reads

$$\begin{aligned}\mathcal{L}_{\mathrm{GF}} = -\frac{1}{2}(\partial_\mu A_a^\mu) \xi_{ab} (\partial_\nu A_b^\nu) + \frac{1}{2} \Lambda^{4-d} \xi_{ab}^{-1} (\theta_a \phi_{\mathrm{cl}})_i (\phi - \phi_{\mathrm{cl}})_i (\theta_b \phi_{\mathrm{cl}})_j (\phi - \phi_{\mathrm{cl}})_j \\ + \mathrm{i}\Lambda^{\frac{4-d}{2}} (\partial_\mu A_a^\mu)(\theta_a \phi_{\mathrm{cl}})_i (\phi - \phi_{\mathrm{cl}})_i \,.\end{aligned} \tag{C.59}$$

D. Calculations

In this chapter we gather calculations and derivations which do not fit in the main part because they are too long or too cumbersome.

D.1. Generating functional of connected correlation functions

In chapter 2.2.2 we found that, for a general gauge theory, the one-loop contribution to the generating functional of connected correlation functions decomposes into terms which depend on only one type of fields, and hence we can calculate them separately.

D.1.1. Scalar term

The scalar contribution is given by

$$W_\phi[J] = \frac{\mathrm{i}}{2} \ln\left(\frac{\mathrm{Det}\, D_\phi}{\mathrm{Det}\, \bar{D}_\phi}\right), \tag{D.1}$$

with

$$D_\phi = \frac{1}{2}\left[-\mathbb{1}_{m\times m}\Box - M^2 + \Lambda^{4-d}(\theta_a\phi_{\mathrm{cl}})\xi^{-1}_{ab}(\theta_b\phi_{\mathrm{cl}})^\mathsf{T}\right]. \tag{D.2}$$

Using (B.42) and canceling some factors, we get

$$\begin{aligned} W_\phi[J] = \frac{\mathrm{i}\mathcal{V}_d}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d} \ln\det\left(\mathbb{1}_{m\times m} - \frac{M^2 + \Lambda^{4-d}(\theta_a\phi_{\mathrm{cl}})\xi^{-1}_{ab}(\theta_b\phi_{\mathrm{cl}})^\mathsf{T}}{k^2}\right) \\ - \frac{\mathrm{i}\mathcal{V}_d}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d} \ln\det\left(\mathbb{1}_{m\times m} - \frac{\bar{M}^2 + \Lambda^{4-d}(\theta_a\bar{\phi}_{\mathrm{cl}})\xi^{-1}_{ab}(\theta_b\bar{\phi}_{\mathrm{cl}})^\mathsf{T}}{k^2}\right), \end{aligned} \tag{D.3}$$

where the second term is the same as the first one, but with ϕ_{cl} exchanged with $\bar{\phi}_{\mathrm{cl}}$. Performing the integration with the help of (B.17), we obtain

$$\begin{aligned} W_\phi[J] = -\frac{\mathcal{V}_d\Gamma\left(1-\frac{d}{2}\right)}{d(4\pi)^{d/2}} \mathrm{tr}\left[M^2 + \Lambda^{4-d}(\theta_a\phi_{\mathrm{cl}})\xi^{-1}_{ab}(\theta_b\phi_{\mathrm{cl}})^\mathsf{T}\right]^{d/2} \\ + \frac{\mathcal{V}_d\Gamma\left(1-\frac{d}{2}\right)}{d(4\pi)^{d/2}} \mathrm{tr}\left[\bar{M}^2 + \Lambda^{4-d}(\theta_a\bar{\phi}_{\mathrm{cl}})\xi^{-1}_{ab}(\theta_b\bar{\phi}_{\mathrm{cl}})^\mathsf{T}\right]^{d/2}. \end{aligned} \tag{D.4}$$

D.1.2. Gauge term

The gauge term reads

$$W_A[J] = \frac{\mathrm{i}}{2}\ln\left(\frac{\operatorname{Det} D_A}{\operatorname{Det} \bar{D}_A}\right), \tag{D.5}$$

with

$$D_A = \frac{1}{2}\left[\mathbb{1}_{n\times n}\otimes g\Box - (\mathbb{1}_{n\times n} - \xi)\otimes (g\partial\partial^\mathsf{T} g) + \mu^2 \otimes g\right]. \tag{D.6}$$

Using (B.42) and canceling some factors we get

$$\begin{aligned} W_A[J] = \frac{\mathrm{i}\mathcal{V}_d}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\ln\det\left(\mathbb{1}_{n\times n}\otimes \mathbb{1}_{d\times d} - (\mathbb{1}_{n\times n} - \xi)\otimes \frac{kk^\mathsf{T} g}{k^2} - \frac{\mu^2}{k^2}\otimes \mathbb{1}_{n\times n}\right) \\ - \frac{\mathrm{i}\mathcal{V}_d}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\ln\det\left(\mathbb{1}_{n\times n}\otimes \mathbb{1}_{d\times d} - (\mathbb{1}_{n\times n} - \xi)\otimes \frac{kk^\mathsf{T} g}{k^2} - \frac{\bar{\mu}^2}{k^2}\otimes \mathbb{1}_{n\times n}\right). \end{aligned} \tag{D.7}$$

The $d\times d$ matrix $kk^\mathsf{T} g/k^2$ has the eigenvector k with eigenvalue 1. Furthermore, there exist $d-1$ linearly independent vectors, which are orthogonal to the vector gk and are thus eigenvectors of the matrix $kk^\mathsf{T} g/k^2$ with eigenvalue 0. Therefore, there exists a matrix V such that

$$V^{-1}\frac{kk^\mathsf{T} g}{k^2}V = \begin{pmatrix} 1 & 0 \\ 0 & 0_{(d-1)\times(d-1)}\end{pmatrix}. \tag{D.8}$$

Multiplying the determinant in (D.7) by $1 = \det(\mathbb{1}_{n\times n}\otimes V^{-1})\det(\mathbb{1}_{n\times n}\otimes V)$, we get

$$\begin{aligned} &\det\left(\mathbb{1}_{n\times n}\otimes \mathbb{1}_{d\times d} - (\mathbb{1}_{n\times n} - \xi)\otimes \frac{kk^\mathsf{T} g}{k^2} - \frac{\mu^2}{k^2}\otimes \mathbb{1}_{n\times n}\right) \\ &\quad = \det\left[\mathbb{1}_{n\times n}\otimes \mathbb{1}_{d\times d} - (\mathbb{1}_{n\times n} - \xi)\otimes \begin{pmatrix} 1 & 0 \\ 0 & 0_{(d-1)\times(d-1)}\end{pmatrix} - \frac{\mu^2}{k^2}\otimes \mathbb{1}_{n\times n}\right] \\ &\quad = \det\left[\mathbb{1}_{n\times n}\otimes \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_{(d-1)\times(d-1)}\end{pmatrix} + \xi\otimes \begin{pmatrix} 1 & 0 \\ 0 & 0_{(d-1)\times(d-1)}\end{pmatrix}\right] \\ &\quad\quad \times \det\left[\mathbb{1}_{n\times n}\otimes \mathbb{1}_{d\times d} - \frac{\mu^2}{k^2}\otimes \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_{(d-1)\times(d-1)}\end{pmatrix} - \frac{\xi^{-1}\mu^2}{k^2}\otimes \begin{pmatrix} 1 & 0 \\ 0 & 0_{(d-1)\times(d-1)}\end{pmatrix}\right]. \end{aligned} \tag{D.9}$$

Inserting this back in (D.7), the first determinant cancels and using (B.17) to perform the integration, we get

$$\begin{aligned} W_A[J] = &-\frac{\mathcal{V}_d\Gamma\left(1-\frac{d}{2}\right)}{d(4\pi)^{\frac{d}{2}}}\left[(d-1)\operatorname{tr}\mu^d + \operatorname{tr}\left(\xi^{-d/2}\mu^d\right)\right] \\ &+ \frac{\mathcal{V}_d\Gamma\left(1-\frac{d}{2}\right)}{d(4\pi)^{\frac{d}{2}}}\left[(d-1)\operatorname{tr}\bar{\mu}^d + \operatorname{tr}\left(\xi^{-d/2}\bar{\mu}^d\right)\right]. \end{aligned} \tag{D.10}$$

D.1.3. Fermion term

The fermion contribution is given by

$$W_\psi[J] = -\frac{\mathrm{i}}{2}\ln\left(\frac{\mathrm{Det}\, D_\psi}{\mathrm{Det}\, \bar{D}_\psi}\right), \tag{D.11}$$

with

$$D_\psi = \frac{1}{2}\left(\hat{C}\otimes C^{-1}\right)\left[-\mathbb{1}_{2k_\mathrm{f}\times 2k_\mathrm{f}}\otimes \mathrm{i}\partial\!\!\!/ + \left(\hat{C}\cdot m\right)\otimes P_\mathrm{L} + \left(m^\dagger\cdot\hat{C}\right)\otimes P_\mathrm{R}\right]. \tag{D.12}$$

Using (B.51) and canceling some factors we get

$$\begin{aligned} W_\psi[J] = -\frac{\mathrm{i}\mathcal{V}_d}{2}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\ln\det\Bigg[&\mathbb{1}_{2k_\mathrm{f}\times 2k_\mathrm{f}}\otimes\mathbb{1}_{4\times 4} - \left(\hat{C}\cdot m\right)\otimes\left(P_\mathrm{L}\cdot\frac{k\!\!\!/}{k^2}\right) \\ &- \left(m^*\cdot\hat{C}\right)\otimes\left(P_\mathrm{R}\cdot\frac{k\!\!\!/}{k^2}\right)\Bigg] \\ +\frac{\mathrm{i}\mathcal{V}_d}{2}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\ln\det\Bigg[&\mathbb{1}_{2k_\mathrm{f}\times 2k_\mathrm{f}}\otimes\mathbb{1}_{4\times 4} - \left(\hat{C}\cdot\bar{m}\right)\otimes\left(P_\mathrm{L}\cdot\frac{k\!\!\!/}{k^2}\right) \\ &- \left(\bar{m}^*\cdot\hat{C}\right)\otimes\left(P_\mathrm{R}\cdot\frac{k\!\!\!/}{k^2}\right)\Bigg]. \end{aligned} \tag{D.13}$$

After some calculation, the integral of the last equation looks much more compact:

$$\begin{aligned} &\int\frac{\mathrm{d}^d k}{(2\pi)^d}\ln\det\Bigg[\mathbb{1}_{2k_\mathrm{f}\times 2k_\mathrm{f}}\otimes\mathbb{1}_{4\times 4} - \left(\hat{C}\cdot m\right)\otimes\left(P_\mathrm{L}\cdot\frac{k\!\!\!/}{k^2}\right) \\ &\qquad\qquad - \left(m^*\cdot\hat{C}\right)\otimes\left(P_\mathrm{R}\cdot\frac{k\!\!\!/}{k^2}\right)\Bigg] \\ &= \int\frac{\mathrm{d}^d k}{(2\pi)^d}\,\mathrm{tr}\sum_{j=1}^{\infty}(-1)\frac{1}{j}\left[\left(\hat{C}\cdot m\right)\otimes\left(P_\mathrm{L}\cdot\frac{k\!\!\!/}{k^2}\right) + \left(m^*\cdot\hat{C}\right)\otimes\left(P_\mathrm{R}\cdot\frac{k\!\!\!/}{k^2}\right)\right]^j \\ &= \int\frac{\mathrm{d}^d k}{(2\pi)^d}\,\mathrm{tr}\sum_{j=1}^{\infty}(-1)\frac{1}{2j}\left[\left(\hat{C}\cdot m\right)\otimes\left(P_\mathrm{L}\cdot\frac{k\!\!\!/}{k^2}\right) + \left(m^*\cdot\hat{C}\right)\otimes\left(P_\mathrm{R}\cdot\frac{k\!\!\!/}{k^2}\right)\right]^{2j} \\ &= \int\frac{\mathrm{d}^d k}{(2\pi)^d}\,\mathrm{tr}\sum_{j=1}^{\infty}(-1)\frac{1}{2j}\left[\left(\frac{\hat{C}\cdot m\cdot m^*\cdot\hat{C}}{k^2}\right)^j\otimes P_\mathrm{L} + \left(\frac{m^*\cdot m}{k^2}\right)^j\otimes P_\mathrm{R}\right] \\ &= 2\int\frac{\mathrm{d}^d k}{(2\pi)^d}\,\mathrm{tr}\sum_{j=1}^{\infty}(-1)\frac{1}{j}\left(\frac{m^* m}{k^2}\right)^j \\ &= 2\int\frac{\mathrm{d}^d k}{(2\pi)^d}\,\mathrm{tr}\ln\left(\mathbb{1}_{2k_\mathrm{f}\times 2k_\mathrm{f}} - \frac{m^* m}{k^2}\right) \\ &= 2\int\frac{\mathrm{d}^d k}{(2\pi)^d}\ln\det\left(\mathbb{1}_{2k_\mathrm{f}\times 2k_\mathrm{f}} - \frac{m^* m}{k^2}\right), \end{aligned} \tag{D.14}$$

where we used the identity

$$\ln\det A = \operatorname{tr}\ln A\,, \tag{D.15}$$

and the series expansion of the logarithm. Moreover, in the second step we used that the odd powers in the expansion vanish after integration, because they are odd functions of k^μ. In the third step we used some properties of the Dirac matrices and in the fourth step we made use of the cyclicity of the trace. Using (B.17) to perform the integration, we accordingly get

$$W_\psi[J] = \frac{2\mathcal{V}_d\Gamma\left(1-\frac{d}{2}\right)}{d(4\pi)^{d/2}}\operatorname{tr}\left(m^* m\right)^{d/2} - \frac{2\mathcal{V}_d\Gamma\left(1-\frac{d}{2}\right)}{d(4\pi)^{d/2}}\operatorname{tr}\left(\bar{m}^*\bar{m}\right)^{d/2}\,. \tag{D.16}$$

D.1.4. Ghost term

The ghost term reads

$$W_\omega[J] = -\mathrm{i}\ln\left(\frac{\operatorname{Det} D_\omega}{\operatorname{Det}\bar{D}_\omega}\right)\,, \tag{D.17}$$

with

$$D_\omega = -\mathbb{1}_{n\times n}\Box - \xi^{-\frac{3}{4}}\mu^2\xi^{-\frac{1}{4}}\,. \tag{D.18}$$

Using (B.57) and canceling some factors, we get

$$\begin{aligned} W_\omega[J] = &-\mathrm{i}\mathcal{V}_d\int\frac{\mathrm{d}^d k}{(2\pi)^d}\ln\det\left(\mathbb{1}_{n\times n} - \frac{\xi^{-1}\mu^2}{k^2}\right) \\ &+\mathrm{i}\mathcal{V}_d\int\frac{\mathrm{d}^d k}{(2\pi)^d}\ln\det\left(\mathbb{1}_{n\times n} - \frac{\xi^{-1}\bar{\mu}^2}{k^2}\right)\,, \end{aligned} \tag{D.19}$$

and subsequently, using (B.17), we obtain

$$W_\omega[J] = \frac{2\mathcal{V}_d\Gamma\left(1-\frac{d}{2}\right)}{d(4\pi)^{d/2}}\operatorname{tr}\left(\xi^{-d/2}\mu^d\right) - \frac{2\mathcal{V}_d\Gamma\left(1-\frac{d}{2}\right)}{d(4\pi)^{d/2}}\operatorname{tr}\left(\xi^{-d/2}\bar{\mu}^d\right)\,. \tag{D.20}$$

D.2. Terms including $\tilde{\Phi}_k$ in the scalar potential

In section 3.1.1 we claimed that the most general form of the scalar potential is given by equation (3.14). We could, however, add the following gauge- and scale-invariant terms

$$\sum_{i,j,k,l}\left[\bar{\lambda}_{ijkl}(\Phi_i^\dagger\tilde{\Phi}_j)(\tilde{\Phi}_k^\dagger\Phi_l) + \tilde{\lambda}_{ijkl}(\tilde{\Phi}_i^\dagger\tilde{\Phi}_j)(\tilde{\Phi}_k^\dagger\tilde{\Phi}_l)\right] + \varphi_0^2\sum_{i,j}\tilde{\kappa}_{ij}\tilde{\Phi}_i^\dagger\tilde{\Phi}_j\,, \tag{D.21}$$

but the identity

$$\tilde{\Phi}_i^\dagger\tilde{\Phi}_j = \Phi_i^\mathsf{T}(\mathrm{i}\tau_2)^\dagger\mathrm{i}\tau_2\Phi_j^* = \Phi_i^\mathsf{T}\Phi_j^* = \Phi_j^\dagger\Phi_i \tag{D.22}$$

immediately reveals that the last two terms in (D.21) can be absorbed into the original potential by a proper redefinition of the coupling constants. Furthermore, it is easy to check that the components of the second Pauli matrix fulfill the condition

$$\tau_{2\,ab}\tau_{2\,cd} = \delta_{ad}\delta_{bc} - \delta_{ac}\delta_{bd}, \tag{D.23}$$

and therefore we may write

$$\begin{aligned}(\Phi_i^\dagger\tilde{\Phi}_j)(\tilde{\Phi}_k^\dagger\Phi_l) = \Phi^*_{i\,a}\mathrm{i}\tau_{2\,ab}\Phi^*_{j\,b}\Phi_{k\,c}(-\mathrm{i}\tau_{2\,cd})\Phi_{l\,d} = \Phi^*_{i\,a}\Phi^*_{j\,b}\Phi_{k\,b}\Phi_{l\,a} - \Phi^*_{i\,a}\Phi^*_{j\,b}\Phi_{k\,a}\Phi_{l\,b} \\ = (\Phi_i^\dagger\Phi_l)(\Phi_j^\dagger\Phi_k) - (\Phi_i^\dagger\Phi_k)(\Phi_j^\dagger\Phi_l)\,,\end{aligned} \tag{D.24}$$

what shows that also the first term of (D.21) adds nothing new to the potential.

D.3. Eigenvectors of the mass matrix $\mathcal{M}_0^2$

In section 3.1.3.3 we claimed that the two vectors

$$\begin{pmatrix} n_0 \\ \mathrm{Re}\, n \\ \mathrm{Im}\, n \end{pmatrix},\quad \begin{pmatrix} 0 \\ -\mathrm{Im}\, n \\ \mathrm{Re}\, n \end{pmatrix} \in \mathbb{R}^{2n_\mathrm{H}+1} \tag{D.25}$$

are eigenvectors of the mass matrix of the neutral scalars,

$$\mathcal{M}_0^2 = \begin{pmatrix} 2\lambda_0 v_0^2 & \mathrm{Re}\, k^\mathsf{T} & \mathrm{Im}\, k^\mathsf{T} \\ \mathrm{Re}\, k & A & C \\ \mathrm{Im}\, k & C^\mathsf{T} & B \end{pmatrix}, \tag{D.26}$$

with eigenvalue zero and we want to check this here.

For the first vector we get the three equations

$$2\lambda_0 v_0^2 n_0 + \mathrm{Re}(k^\mathsf{T} n^*) = 0\,, \tag{D.27}$$
$$\mathrm{Re}\left[n_0 k + K' n + K n^*\right] = 0\,, \tag{D.28}$$
$$\mathrm{Im}\left[n_0 k + K' n + K n^*\right] = 0\,, \tag{D.29}$$

where we used the definitions (3.45) of the matrices A, B, and C and the fact that n is an eigenvector of M_+^2 with eigenvalue zero (3.39). Equation (D.27) holds due to (3.27):

$$2\lambda_0 v_0^2 n_0 + \mathrm{Re}(k^\mathsf{T} n^*) = 2\lambda_0 v_0^2 n_0 + 2V^2 n_0 \sum_{i,j} \kappa_{ij} n_i^* n_j = 0\,, \tag{D.30}$$

where we also used the definition (3.44) of the vector k . Equations (D.28) and (D.29) are satisfied because

$$(n_0 k + K' n + K n^*)_i = 2V^2 n_0^2 \sum_j \kappa_{ij} n_j + V^2 \sum_{j,k,l} \left(\lambda_{iklj} n_k n_l^* n_j + \lambda_{ikjl} n_k n_l n_j^*\right) = 0\,, \tag{D.31}$$

where we used (3.28) and again the definition (3.44) of the vector k, and the definitions (3.46) of the matrices K and K'.

For the second vector the three equations read

$$\mathrm{Im}(k^{\mathsf{T}} n^*) = 0\,, \tag{D.32}$$
$$\mathrm{Im}\,(K n^* - K' n) = 0\,, \tag{D.33}$$
$$\mathrm{Re}\,(-K n^* + K' n) = 0\,, \tag{D.34}$$

where we again used the definitions (3.45) of the matrices A, B and C and the fact that n is an eigenvector of M_+^2 with eigenvalue zero (3.39). Equation (D.32) is satisfied because $k^{\mathsf{T}} n^* = 2V^2 n_0 \sum_{i,j} \kappa_{ij} n_i^* n_j$ is real. Equations (D.33) and (D.34) hold since

$$(K n^* - K' n) = V^2 \sum_{j,k,l} \left(\lambda_{ikjl} n_k n_l n_j^* - \lambda_{iklj} n_k n_l^* n_j\right) = 0\,, \tag{D.35}$$

where we used the definitions (3.46) of the matrices K and K'.

D.4. Putting the orthogonality relations of $\mathcal{R}$ in a compact form

In section 3.1.3.3 we claimed that the relations

$$\mathrm{Re}\,R\,\mathrm{Re}\,R^{\mathsf{T}} = \mathbb{1}_{n_{\mathrm{H}}\times n_{\mathrm{H}}}\,, \quad \mathrm{Im}\,R\,\mathrm{Im}\,R^{\mathsf{T}} = \mathbb{1}_{n_{\mathrm{H}}\times n_{\mathrm{H}}}\,, \quad \mathrm{Re}\,R\,\mathrm{Im}\,R^{\mathsf{T}} = 0 \tag{D.36}$$

are equivalent to

$$R R^{\dagger} = 2\cdot \mathbb{1}_{n_{\mathrm{H}}\times n_{\mathrm{H}}}\,, \quad R R^{\mathsf{T}} = 0\,, \tag{D.37}$$

and we are going to proof this here. The first two equations of (D.36) are equivalent to

$$\begin{cases} \mathrm{Re}\,R\,\mathrm{Re}\,R^{\mathsf{T}} + \mathrm{Im}\,R\,\mathrm{Im}\,R^{\mathsf{T}} = 2\cdot \mathbb{1}_{n_{\mathrm{H}}\times n_{\mathrm{H}}} \\ \mathrm{Re}\,R\,\mathrm{Re}\,R^{\mathsf{T}} - \mathrm{Im}\,R\,\mathrm{Im}\,R^{\mathsf{T}} = 0 \end{cases}, \tag{D.38}$$

and they can in turn be rewritten as

$$\begin{cases} \mathrm{Re}(R R^{\dagger}) = 2\cdot \mathbb{1}_{n_{\mathrm{H}}\times n_{\mathrm{H}}} \\ \mathrm{Re}(R R^{\mathsf{T}}) = 0 \end{cases}. \tag{D.39}$$

The last equation of (D.36) is equivalent to

$$\begin{cases} \mathrm{Re}\,R\,\mathrm{Im}\,R^{\mathsf{T}} + \mathrm{Im}\,R\,\mathrm{Re}\,R^{\mathsf{T}} = 0 \\ -\mathrm{Re}\,R\,\mathrm{Im}\,R^{\mathsf{T}} + \mathrm{Im}\,R\,\mathrm{Re}\,R^{\mathsf{T}} = 0 \end{cases}, \tag{D.40}$$

what can be rewritten as

$$\begin{cases} \mathrm{Im}(R R^{\dagger}) = 0 \\ \mathrm{Im}(R R^{\mathsf{T}}) = 0 \end{cases}. \tag{D.41}$$

Finally, the combination of (D.39) and (D.41) is equivalent to (D.37).

D.5. Diagonalization of the one-loop neutrino mass matrix

The one-loop neutrino mass matrix can be brought to a diagonal form with non-negative entries by successively applying three unitary matrices:

$$\hat{\mathcal{M}}_\nu^{(1)} = \left(\mathcal{U}\mathcal{U}'\mathcal{V}\right)^\mathsf{T} \mathcal{M}_\nu^{(1)}\mathcal{U}\mathcal{U}'\mathcal{V} = \begin{pmatrix} \hat{M}_0 & 0 \\ 0 & \hat{M}'^{(1)} \end{pmatrix} . \tag{D.42}$$

The first transformation matrix is given by (3.85) and yields

$$\mathcal{U}^\mathsf{T}\mathcal{M}_\nu^{(1)}\mathcal{U} = \begin{pmatrix} 0 & 0 \\ 0 & \hat{M}' \end{pmatrix} + \delta\mathcal{M}'_\nu \tag{D.43}$$

with

$$\delta\mathcal{M}'_\nu = \begin{pmatrix} M_0 & M''^\mathsf{T} \\ M'' & \delta M' \end{pmatrix} , \tag{D.44}$$

where

$$M_0 = U_\mathrm{L}'^\mathsf{T}\delta M_\mathrm{L} U'_\mathrm{L} \,, \quad M'' = U_\mathrm{L}''^\mathsf{T}\delta M_\mathrm{L} U'_\mathrm{L} + U_\mathrm{R}''^\dagger \delta M_\mathrm{D} U'_\mathrm{L} \,, \tag{D.45}$$

$$\delta M' = U_\mathrm{L}''^\mathsf{T}\delta M_\mathrm{L} U''_\mathrm{L} + U_\mathrm{L}''^\mathsf{T}\delta M_\mathrm{D}^\mathsf{T} U_\mathrm{R}''^* + U_\mathrm{R}''^\dagger \delta M_\mathrm{D} U''_\mathrm{L} + U_\mathrm{R}''^\dagger \delta M_\mathrm{R} U_\mathrm{R}''^* \,. \tag{D.46}$$

The second unitary transformation matrix is given by

$$\mathcal{U}' = \begin{pmatrix} U & 0 \\ 0 & \mathbb{1}_{2n_\mathrm{R}\times 2n_\mathrm{R}} \end{pmatrix} , \tag{D.47}$$

with a $(n_\mathrm{L}-n_\mathrm{R})\times(n_\mathrm{L}-n_\mathrm{R})$ unitary matrix U. It effectively leads to the substitution $U'_\mathrm{L} \to U'_\mathrm{L} U$ in (D.45) and hence the unitary matrix U can be chosen such that $\hat{M}_0 = U^\mathsf{T} U_\mathrm{L}'^\mathsf{T}\delta M_\mathrm{L} U'_\mathrm{L} U$ is diagonal and non-negative [20, 21]. However, the matrix $U'_\mathrm{L} = \begin{pmatrix} u'_1 & \dots & u'_{n_\mathrm{L}-n_\mathrm{R}} \end{pmatrix}$ is until now only constrained by the condition $M_\mathrm{D} u'_i = 0$, which obviously still holds after the redefinition $U'_\mathrm{L} U \to U'_\mathrm{L}$. Therefore, we are free to choose U'_L such that M_0 is diagonal and non-negative in the first place and then the transformation with the matrix $\mathcal{U}' = \mathbb{1}_{(n_\mathrm{L}+n_\mathrm{R})\times(n_\mathrm{L}+n_\mathrm{R})}$ is trivial.

The last transformation matrix can be approximated by

$$\mathcal{V} \simeq \mathbb{1}_{(n_\mathrm{L}+n_\mathrm{R})\times(n_\mathrm{L}+n_\mathrm{R})} + \mathrm{i}\Omega = \mathbb{1}_{(n_\mathrm{L}+n_\mathrm{R})\times(n_\mathrm{L}+n_\mathrm{R})} + \mathrm{i}\begin{pmatrix} 0 & \Omega_1 \\ \Omega_1^\dagger & \Omega_2 \end{pmatrix} , \tag{D.48}$$

where the hermitian matrix Ω is of one-loop order [3]. Keeping only terms up to one-loop order we then get

$$\mathcal{V}^\mathsf{T}\mathcal{U}^\mathsf{T}\mathcal{M}_\nu^{(1)}\mathcal{U}\mathcal{V} \simeq \begin{pmatrix} \hat{M}_0 & 0 \\ 0 & \hat{M}' \end{pmatrix} + \begin{pmatrix} 0 & M''^\mathsf{T} + \mathrm{i}\Omega_1^*\hat{M}' \\ M'' + \mathrm{i}\hat{M}'\Omega_1^\dagger & \delta M' + \mathrm{i}\hat{M}'\Omega_2 + \mathrm{i}\Omega_2^\mathsf{T}\hat{M}' \end{pmatrix} . \tag{D.49}$$

If we assume that the diagonal entries of the tree-level mass matrix $\hat{M}'$ are positive and non-degenerate, the matrix Ω is determined by

$$(\Omega_1)_{ij} = -\mathrm{i}\frac{M''^{\dagger}_{ij}}{\hat{M}'_{jj}}\,, \quad \mathrm{Re}(\Omega_2)_{ij} = -\frac{\mathrm{Im}\,\delta M'_{ij}}{\hat{M}'_{ii} + \hat{M}'_{jj}} \quad \forall i,j\,, \tag{D.50}$$

$$\mathrm{Im}(\Omega_2)_{ij} = \frac{\mathrm{Re}\,\delta M'_{ij}}{\hat{M}'_{ii} - \hat{M}'_{jj}} \quad \forall i \neq j\,, \tag{D.51}$$

such that all off-diagonal terms vanish in (D.49) and the diagonal is real and positive.

D.6. Gauge-fixing in the conformal multi-Higgs-doublet model

To fix the gauge in our model we follow [36] and add three gauge-fixing terms to our Lagrangian density.[1]

The first gauge-fixing term is given by

$$-\frac{1}{2\xi_Z}\left(\partial_\mu Z^\mu + \xi_Z M_Z G^0\right)^2 = -\frac{1}{2\xi_Z}(\partial_\mu Z^\mu)(\partial_\nu Z^\nu) - M_Z G^0 \partial_\mu Z^\mu - \frac{1}{2}\xi_Z M_Z^2 (G^0)^2\,. \tag{D.52}$$

The first term of (D.52) contributes to the propagator of the Z^0 boson. After a partial integration the second term cancels the G^0-Z^0 mixing term in (3.91) and due to the last term the neutral Goldstone boson G^0 acquires an unphysical mass of $\sqrt{\xi_Z}M_Z$.

The second gauge-fixing term reads

$$\begin{aligned}&-\frac{1}{2\xi_W}\left(\partial^\mu W^+_\mu + \mathrm{i}\xi_W M_W G^+\right)\left(\partial^\nu W^-_\nu - \mathrm{i}\xi_W M_W G^-\right)\\ &\quad = -\frac{1}{\xi_W}(\partial^\mu W^+_\mu)(\partial^\nu W^-_\nu) - \mathrm{i}M_W\left(G^+\partial^\mu W^-_\mu - G^-\partial^\mu W^+_\mu\right) - \xi_\mu M_W^2 G^+ G^-\end{aligned} \tag{D.53}$$

The first term contributes to the propagator of the $W^\pm$ boson. The second term cancels, after a partial integration, the $G^\pm$-$W^\pm$ mixing term in (3.91) and due to the last term the charged Goldstone boson $G^\pm$ acquires an unphysical mass of $\sqrt{\xi_W}M_W$.

Finally, we have a gauge-fixing term for the photon field:

$$-\frac{1}{2\xi_A}(\partial^\mu A_\mu)(\partial^\nu A_\nu)\,. \tag{D.54}$$

We refrain from computing the Faddeev-Popov Lagrangian density corresponding to these gauge-fixing terms since it is not relevant for our considerations.

[1] We are only discussing $SU(2)_\mathrm{L} \times U(1)_Y$ gauge fixing here.

D.7. Making use of the mass hierarchy $m_{\mathrm{D}} \ll m_{\mathrm{R}}$ in (3.133)

We want to show that

$$U_{\mathrm{R}}^{*} \hat{\mathcal{M}}_{\nu} f(\hat{\mathcal{M}}_{\nu}^{2}) U_{\mathrm{R}}^{\dagger} \simeq M_{\mathrm{R}}^{*} f(M_{\mathrm{R}} M_{\mathrm{R}}^{*}) \tag{D.55}$$

is a good approximation if we assume the mass hierarchy $m_{\mathrm{D}} \ll m_{\mathrm{R}}$.

The complex conjugate of the first equation of (3.79) reads

$$\hat{\mathcal{M}}_{\nu} U_{\mathrm{R}}^{\dagger} = U_{\mathrm{L}}^{\dagger} M_{\mathrm{D}}^{\dagger} + U_{\mathrm{R}}^{\mathsf{T}} M_{\mathrm{R}}^{*} \,. \tag{D.56}$$

If we multiply this by $\hat{\mathcal{M}}_{\nu}$ in the form of (3.80) from the left and use (3.74) we obtain

$$\hat{\mathcal{M}}_{\nu}^{2} U_{\mathrm{R}}^{\dagger} = U_{\mathrm{R}}^{\dagger} M_{\mathrm{D}} M_{\mathrm{D}}^{\dagger} + U_{\mathrm{L}}^{\mathsf{T}} M_{\mathrm{D}}^{\mathsf{T}} M_{\mathrm{R}}^{*} + U_{\mathrm{R}}^{\dagger} M_{\mathrm{R}} M_{\mathrm{R}}^{*} \simeq \hat{\mathcal{M}}_{\nu} U_{\mathrm{R}}^{\mathsf{T}} M_{\mathrm{R}}^{*} \,, \tag{D.57}$$

where we neglected the first against the third term and used (D.56) in the last step. If we again multiply this by $\hat{\mathcal{M}}_{\nu}$ from the left and use the complex conjugate of (D.57) we get

$$\hat{\mathcal{M}}_{\nu}^{3} U_{\mathrm{R}}^{\dagger} \simeq \hat{\mathcal{M}}_{\nu}^{2} U_{\mathrm{R}}^{\mathsf{T}} M_{\mathrm{R}}^{*} \simeq \hat{\mathcal{M}}_{\nu} U_{\mathrm{R}}^{\dagger} M_{\mathrm{R}} M_{\mathrm{R}}^{*}, \tag{D.58}$$

and this can readily be generalized to

$$\hat{\mathcal{M}}_{\nu} \hat{\mathcal{M}}_{\nu}^{2n} U_{\mathrm{R}}^{\dagger} \simeq \hat{\mathcal{M}}_{\nu} U_{\mathrm{R}}^{\dagger} \left(M_{\mathrm{R}} M_{\mathrm{R}}^{*}\right)^{n} \,. \tag{D.59}$$

Multiplying by U_{R}^{*} from the left and using (D.56) and (D.57) we finally have

$$U_{\mathrm{R}}^{*} \hat{\mathcal{M}}_{\nu} \hat{\mathcal{M}}_{\nu}^{2n} U_{\mathrm{R}}^{\dagger} \simeq M_{\mathrm{R}}^{*} \left(M_{\mathrm{R}} M_{\mathrm{R}}^{*}\right)^{n} \,, \tag{D.60}$$

and it is then obvious that (D.55) is a good approximation if $f(x)$ can be expanded in a power series of the form

$$f(x) = \sum_{n=0}^{\infty} a_{n} x^{n} \,. \tag{D.61}$$

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