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Refined Enumeration Formulae Related to Halved Monotone Triangles

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Contents

1 Introduction

An alternating sign matrix of order n is an $n \times n$ – square matrix with entries taken from the set $\{-1, 0, +1\}$ such that the entries in every row and in every column add up to 1 and that the nonzero elements alternate in sign along each row and each column. Every permutation matrix is obviously an alternating sign matrix. One example of a 6×6 – alternating sign matrix that is not a permutation matrix is

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The history of alternating sign matrices began in the early 1980s when Robbins and Rumsey studied a method for evaluating determinants by Charles Dodgson (better known under his pen name of Lewis Carroll). This method, called *condensation of determinants*, reduced the problem of evaluating determinants of large matrices to evaluating determinants of matrices of order 2 (see [Dod66]). Robbins and Rumsey generalized in [RR86] the notion of determinants by modifying Dodgson's condensation method with the help of alternating sign matrices. The problem of enumerating alternating sign matrices arose instantly.

In 1983, Mills, Robbins and Rumsey conjectured in [MRR83] that the number of alternating sign matrices of order n is exactly

$$\prod_{j=0}^{n-1} \frac{(3j+1)!}{(j+n)!}$$

without giving a proof. This was known as the *ASM conjecture*. However, they could show that alternating sign matrices of order n are in a bijective correspondence to certain triangular arrays of integers: monotone triangles with bottom row $\{1, 2, \ldots, n\}$ which Zeilberger named Gog-triangles. The triangle corresponding to the alternating sign matrix above is the following:

$$\begin{array}{ccccccccccc} & & & & & 4 & & & & & \\ & & & & 2 & & 5 & & & & \\ & & & 2 & & 4 & & 5 & & & \\ & & 2 & & 3 & & 5 & & 6 & & \\ & 1 & & 2 & & 4 & & 5 & & 6 & \\ 1 & & 2 & & 3 & & 4 & & 5 & & 6 \end{array}$$

Three years later, Mills, Robbins and Rumsey conjectured in [MRR86] that so-called self-complementary totally symmetric plane partitions are equinumerous to alternating sign matrices. A proof was missing again. But they pointed out that self-complementary totally symmetric plane partitions are in a one-to-one correspondence to another kind of triangular array of integers to which Zeilberger referred as Magog triangles. When Andrews proved in [And94] that self-complementary totally symmetric plane partitions fulfill the enumeration formula stated above, the last step towards a proof of the ASM conjecture was to show that the number of Gog-triangles and Magog-triangles coincide. It took some time until Zeilberger finally presented a proof in [Zei96a]. Zeilberger's proof, however, was very long, based on a massive use of computer algebra and over 80 mathematicians were involved in the process of proofreading. A bijective proof of the correspondence of Gog- and Magog-triangles is still unknown.
Shortly after that, Kuperberg was able to present another and much shorter proof of the ASM conjecture in [Kup96] using methods from statistical mechanics and the well-studied six-vertex-model of square ice.
The so-called six-vertex model with domain wall boundary condition provides possible shapes of water molecules arranged in a two-dimensional square lattice. These arrangements are in a one-to-one correspondence with alternating sign matrices. For example, the alternating sign matrix presented above corresponds to the following six-vertex-model of square ice:

```
H-O H-O H-O H-O-H O-H O-H
  |   |   |       |   |
  H   H   H   H   H   H
              |
H-O H-O-H O-H O H-O-H O-H
  |       |   |       |
  H   H   H   H   H   H
      |           |
H-O H-O H-O H-O-H O-H O-H
  |       |           |
  H   H   H   H   H   H
      |       |   |
H-O H-O H-O-H O H-O H-O-H
  |           |
  H   H   H   H   H   H
      |   |       |   |
H-O-H O-H O H-O-H O-H O-H
          |
  H   H   H   H   H   H
  |   |       |   |   |
H-O H-O H-O-H O-H O-H O-H
```

The techniques evovled from the six-vertex-model turned out to be very fruitful. They are still one of the important approaches in the investigation of alternating sign matrices. [Bre99] is highly recommended for further reading on the history of the ASM conjecture.
Besides the problem of the plain enumeration of alternating sign matrices which was posed in the ASM conjecture, one might be interested in refined enumeration problems. We want to introduce three different types of such refinements:
First, one can enumerate alternating sign matrices with given rows and columns. The problem of enumerating alternating sign matrices with respect to the first row was already

discussed in [MRR83] and solved in [Zei96b] with the help of the six-vertex-model. This is called the *refined ASM theorem*. A doubly refined enumeration formula of alternating sign matrices to control the first and the last row was derived in [Str02]. A formula to enumerate alternating sign matrices with respect to the first and the second row was presented in [KR10] and [Fis11].

Second, one can investigate alternating sign matrices with special symmetry properties. There are eight symmetry classes of the square. The corresponding symmetry classes of alternating sign matrices have been studied systematically. Kuperberg provided enumeration formulae for vertically symmetric alternating sign matrices, half-turn symmetric alternating sign matrices of even order and quarter-turn symmetric alternating sign matrices of even order in in [Kup02]. Four years later, Razumov and Stroganov enumerated half-turn symmetric alternating sign matrices of odd order in [RS06b] and quarter-turn symmetric alternating sign matrices in [RS06a]. In the same year, Okada proved an enumeration formula for vertically and horizontally symmetric alternating sign matrices in [Oka06]. Finally, Behrend, Fischer and Konvalinka enumerated diagonally and antidiagonally symmetric alternating sign matrices of odd order in [BFK15].

Third, one can enumerate alternating sign matrices with respect to certain entries, for example the entries -1. Mills, Robbins and Rumsey introduced in [MRR83] a generating function that enumerates alternating sign matrices depending on their total amount of entries equal to -1. This is the so-called Q-enumeration. Even though they were unable to prove the ASM conjectures, they managed to evaluate this generating function at $Q = 2$ (often referred to as the 2-*enumeration*). Alternating sign matrices with exactly one entry -1 were studied in [Lal02] and [Lal06] and those with more entries -1 were discussed in [Le 11].

In this thesis, we have a closer look at so-called halved monotone triangles. They generalise vertically symmetric alternating sign matrices. Our main technique is the operator formula for monotone triangles going back to [Fis06]. In [Fis09], this formula was adapted to halved monotone triangles. We see how the weighted enumeration of alternating sign matrices with respect to the entries equal to -1 translates to monotone triangles. In [Fis10], Fischer provided an operator formula for the Q-enumeration of monotone triangles.

In the first part of this thesis, we will focus on monotone triangles: We will explain the correspondence between monotone triangles and alternating sign matrices and we will introduce the operator formula. In the second part, we generalise this concept to halved monotone triangles. Our main goal is to provide an operator formula for the Q-enumeration of halved monotone triangles. Furthermore, we give an explicit formula without operators for the 2-enumeration of halved monotone triangles with an odd number of rows. Finally, we present a Q-generating function for halved trees, i.e. for halved monotone triangles with truncated diagonals.

2 Monotone Triangles

In this chapter we present an overview over alternating sign matrices, some of their basic properties and the generalisation to monotone triangles. We explain how to prove the operator formula which can be used to enumerate alternating sign matrices.

2.1 Alternating Sign Matrices

We begin with the definition of alternating sign matrices:

Definition 1. An *alternating sign matrix (ASM) of order* n is an $n \times n$ – square matrix which fulfills the following conditions:

- The entries are -1, 0 or +1.
- The sum of the entries in each row and each column equals 1.
- The nonzero entries alternate in sign along each row and each column.

We denote the set of all alternating sign matrices of order n by $\mathcal{ASM}(n)$ and the cardinality of $\mathcal{ASM}(n)$ by $A(n)$. Example 2 lists all alternating sign matrices of order 1, 2 and 3.

Example 2.

n	$A(n)$	$\mathcal{ASM}(n)$
1	*1*	$\left\{\begin{pmatrix}1\end{pmatrix}\right\}$
2	*2*	$\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\right\}$
3	*7*	$\left\{\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1 & 0\\0 & 0 & 1\\1 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 1\\1 & 0 & 0\\0 & 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 1\\0 & 1 & 0\\1 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1 & 0\\1 & -1 & 1\\0 & 1 & 0\end{pmatrix}\right\}$

The following explicit formula for counting alternating sign matrices was conjectured in [MRR83] and proved for the first time in [Zei96a].

Theorem 3 (ASM Theorem)**.** *The number of alternating sign matrices of order* n *is given by*

$$A(n) = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(j+n)!}.$$

Every permutation matrix is obviously an alternating sign matrix. In fact, $\mathcal{ASM}(1)$ and $\mathcal{ASM}(2)$ only consist out of permutation matrices and all but one alternating sign matrix in $\mathcal{ASM}(3)$ are permutation matrices as well. There are exactly $n!$ permutation matrices of order n. The numbers $A(n)$, however, increase much faster:

$$1, 2, 7, 42, 429, 7436, 218348, \ldots.$$

Suppose there is an entry equal to -1 in the first row of an alternating sign matrix, for example in column i. Since the nonzero entries alternate in sign along each column, the sum of all entries in column i is either -1 or 0. This contradicts the definition of alternating sign matrices. Hence, the first row only contains one nonzero element, namely one unique entry equal to 1. This holds as well for the last row and for the first and the last column. We denote by $A_i(n)$ the number of alternating sign matrices of order n such that the unique 1 in the first row is located in the i^{th} column. Since the definition of alternating sign matrices is invariant under reflection along the vertical symmetry axis, it holds that

$$A_i(n) = A_{n-i}(n).$$

Instead of counting alternating sign matrices in general, we can enumerate them with respect to the position of the unique 1 in the first row and then sum up all these numbers. It is clear that the numbers $A(n)$ and $A_i(n)$ fulfill the following relation:

$$A(n) = \sum_{i=1}^{n} A_i(n).$$

Shortly after Kuperberg's proof of the ASM Theorem, Zeilberger presented in [Zei96b] the following formula for the refined enumeration using the six-vertex-model:

Theorem 4 ([Zei96b], Main Theorem)**.**

$$A_i(n) = A(n) \frac{\binom{n+i-2}{n-1}\binom{2n-1-i}{n-1}}{\binom{3n-2}{n-1}}$$

The study of alternating sign matrices has not only become popular beacuse it is a surprisingly difficult enumeration problem of rather simple objects, but because alternating sign matrices are in a one-to-one correspondence with many other combinatorial objects. An important example for these objects is the set of monotone triangles which were already defined in [MRR83]. Before defining the notion of a monotone triangle, we will illustrate how we can transform an alternating sign matrix into a triangular array of integers.

Example 5.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \longleftrightarrow \begin{array}{ccccccccccc} & & & & & 4 & & & & & \\ & & & & 2 & & 5 & & & & \\ & & & 2 & & 4 & & 5 & & & \\ & & 2 & & 3 & & 5 & & 6 & & \\ & 1 & & 2 & & 4 & & 5 & & 6 & \\ 1 & & 2 & & 3 & & 4 & & 5 & & 6 \end{array}$$

ASM *partial column sum matrix* *monotone triangle*

Given an alternating sign matrix, we first construct the *partial column sum matrix*. It has the same size as the alternating sign matrix and its k^{th} row consists of the sum of the first k rows of the alternating sign matrix.
By the definition of alternating sign matrices, we know that the partial column sum matrix has entries in the set $\{0, 1\}$ and the sum of the entries in the k^{th} row is exactly k. That is the reason why we can record the position of the non-zero entries in a triangular shape: The k^{th} row of the triangular array consists of a sequence of k distinct, strictly increasing integers, namely the numbers of the columns in which the positive entries are located.

Definition 6. A *monotone triangle* is an array of integers in a triangular shape of the following form

$$\begin{array}{ccccccccccccc} & & & & & & a_{1,1} & & & & & & \\ & & & & & a_{2,1} & & a_{2,2} & & & & & \\ & & & & \dots & & \dots & & \dots & & & & \\ & & & a_{n-3,1} & & \dots & & \dots & & a_{n-3,n-3} & & & \\ & & a_{n-2,1} & & a_{n-2,2} & & \dots & & \dots & & a_{n-2,n-2} & & \\ & a_{n-1,1} & & a_{n-1,2} & & a_{n-1,3} & & \dots & & \dots & & a_{n-1,n-1} & \\ a_{n,1} & & a_{n,2} & & a_{n,3} & & a_{n,4} & & \dots & & a_{n,n-1} & & a_{n,n} \end{array}$$

such that the entries

- strictly increase along rows, i.e. $a_{i,j} < a_{i,j+1}$ and
- weakly increase along northeast and southeast direction, i.e. $a_{i+1,j} \leq a_{i,j} \leq a_{i+1,j+1}$.

We denote the set of monotone triangles with bottom row $(k_1, \dots, k_n)$ by $\mathcal{MT}(n; k_1, \dots, k_n)$.

We have seen in Example 5 how to construct a monotone triangle with bottom row $(1, 2, 3, 4, 5, 6)$ from an alternating sign matrix of order 6. In fact, every alternating sign matrix of order n can be transformed into a monotone triangle with bottom row $(1, 2, \dots, n)$ and vice versa. We prove the following theorem:

Theorem 7. *There is a one-to-one correspondence between the sets $\mathcal{ASM}(n)$ and $\mathcal{MT}(n; 1, \dots, n)$.*

Proof. This proof is based on the proof of Proposition 1.2.5 in [Rie14]: Let $A \in \mathcal{ASM}(n)$ be an alternating sign matrix of order n. As we have seen above, A can be transformed into a triangular array of integers in the set $\{1, 2, \dots, n\}$ such that the entries in each row are strictly increasing. We still have to prove that entries are weakly increasing in northeast and southeast direction.
Due to our construction, there is an entry j in the i^{th} row of the triangle if the sum of the i topmost entries in the j^{th} column of A is equal to 1. Therefore, if $c_{i,j}$ denotes the number of columns within the j leftmost ones in which the i topmost rows sum up to 1, $c_{i,j}$ also counts the number of entries in the i^{th} row of the triangle which are less or equal than j. If we fix j, then $c_{1,j} \leq c_{2,j} \leq \cdots \leq c_{n,j}$. Since the rows in the triangle are strictly increasing, this means that entries along northeast diagonals are weakly increasing.
For the southeast diagonals, we consider the rightmost column of A and denote by $d_{i,j}$ the number of columns among the j rightmost ones in which the i topmost rows sum up to 1. This corresponds to the number of entries in the i^{th} row of the triangle which are greater or equal than j. Since $d_{1,j} \leq d_{2,j} \leq \cdots \leq d_{n,j}$ for fixed j, the entries along southeast diagonals are weakly increasing, too. We have thus shown that each alternating sign matrix of order n corresponds bijectively to a monotone triangle with bottom row $(1, 2, \dots, n)$.
Conversely, let us now suppose that we have a monotone triangle $\alpha \in \mathcal{MT}(n; 1, \dots, n)$ with bottom row $(1, 2, \dots, n)$. We construct a square matrix of order n by the following rule: If there is an entry j in the i^{th} row of α, then the sum of the i topmost entries in the j^{th} column of the matrix is equal to 1. Since α is a monotone triangle, the i^{th} row consists of i different integers less or equal than n. Therefore, the entries of the constructed matrix are in the set {-1,0,+1} and the nonzero elements in each column alternate in sign and sum up to 1. Let $c_{i,j}$ count how many entries in the i^{th} are less or equal than j. Then $c_{1,j} \leq c_{2,j} \leq \cdots \leq c_{n,j}$ because the entries in the northeast diagonals of the monotone triangle are weakly increasing. This implies that in each row of the matrix the nonzero elements alternate in sign and sum up to 1. So every monotone triangle with bottom row $(1, 2, \dots, n)$ corresponds to an alternating sign matrix of order n. □

If we allow any strictly increasing integer sequence $k_1 < k_2 < \cdots < k_n$ to be the bottom row of a monotone triangle, then any enumeration formula for monotone triangles with prescribed bottom row $(k_1, k_2, \dots, k_n)$ would surely enable us to enumerate alternating sign matrices.
Besides the fact that the notion of a monotone triangle suggests a simple generalisation of alternating sign matrices, monotone triangles have a recursive structure which is very useful for inductive proofs. This idea is presented in the next section.

2.2 The Summation Operator

Our goal is to enumerate monotone triangles with prescribed bottom row with the help of their recursive structure. The crucial observation is the following:
If we have a monotone triangle with n rows and delete the bottom row, we get another monotone triangle with $n-1$ rows. So instead of enumerating all the monotone triangles

with n rows and prescribed bottom row $(k_1, \dots, k_n)$, we can count all monotone triangles with $n-1$ rows and bottom rows $(l_1, \dots, l_{n-1})$ that can be an admissible penultimate row for the bottom row $(k_1, \dots, k_n)$; these are exactly all strictly increasing sequences $(l_1, \dots, l_{n-1}) \in \mathbb{N}^{n-1}$ such that $k_1 \le l_1 \le k_2 \le \dots \le k_{n-1} \le l_{n-1} \le k_n$.
We want to define a function $\alpha(n; k_1, \dots, k_n)$ inductively that counts the number of monotone triangles with n rows and prescribed bottom row $(k_1, \dots, k_n)$ if $(k_1, \dots, k_n)$ is a strictly increasing sequence of integers. That means it should fulfill the following property:

$$\alpha(n; k_1, \dots, k_n) = \sum_{\substack{(l_1,\dots,l_{n-1}) \in \mathbb{Z}^{n-1}, \\ l_1 < l_2 < \dots < l_{n-1}, \\ k_1 \le l_1 \le k_2 \le \dots \le k_{n-1} \le l_{n-1} \le k_n}} \alpha(n-1; l_1, \dots, l_{n-1}). \tag{2.1}$$

We introduce the following notation for this sum:

$$\alpha(n; k_1, \dots, k_n) = \sum_{(l_1,\dots,l_{n-1})}^{(k_1,\dots,k_n)} \alpha(n-1; l_1, \dots, l_{n-1}).$$

How can we construct an admissible penultimate row for a given bottom row? Figure 2.1 illustrates the last two rows of an monotone triangle.

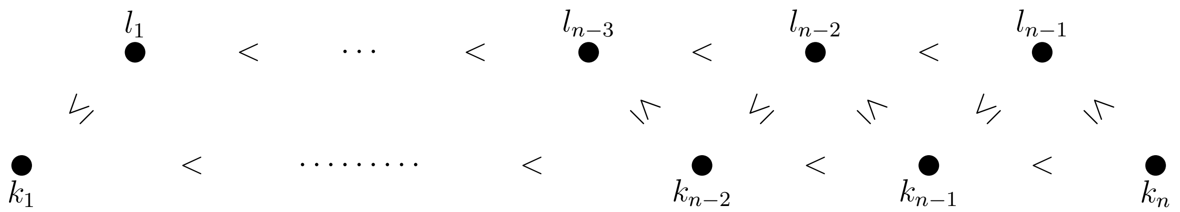

Figure 2.1: Bottom and penultimate row

Suppose we have already determined all appropriate entries $l_1, \dots, l_{n-2}$ for the penultimate row for a given bottom row $(k_1, \dots, k_n)$. The value for the last entry l_{n-1} can range from k_{n-1} up to k_n. In the case $l_{n-1} = k_{n-1}$ we have to make sure that $l_{n-2} < l_{n-1}$, i.e. $l_{n-2} \neq k_{n-1}$. This leads us to the following definition:

Definition 8. We define the summation operator $\sum\limits_{(l_1,\dots,l_{n-1})}^{(k_1,\dots,k_n)} F(l_1, \dots, l_{n-1})$ for a given $(k_1, \dots, k_n) \in \mathbb{Z}^n$ and a function F on $\mathbb{Z}^{n-1}$ by induction on n. For $n = 0$ we set $\sum\limits_{\emptyset}^{\emptyset} F := 0$, for $n = 1$ we set $\sum\limits_{\emptyset}^{k_1} F := F$ and for $n \ge 2$ we define:

$$\sum_{(l_1,\dots,l_{n-1})}^{(k_1,\dots,k_n)} F(l_1, \dots, l_{n-1}) := \left(\sum_{l_{n-1}=k_{n-1}}^{k_n} \sum_{(l_1,\dots,l_{n-2})}^{(k_1,\dots,k_{n-1})} F(l_1, \dots, l_{n-1}) \right) - \left(\sum_{(l_1,\dots,l_{n-3})}^{(k_1,\dots,k_{n-2})} F(l_1, \dots, l_{n-3}, k_{n-1}, k_{n-1}) \right).$$

Since we define this summation operator also for nonincreasing sequences $(k_1, \ldots, k_n)$, we have to define $\sum_{x=u}^{v} F(x)$ for $u > v$. In this case we set

$$\sum_{x=u}^{v} F(x) := - \sum_{x=v+1}^{u-1} F(x). \tag{2.2}$$

This definition is motivated by the simple observation that $\sum_{x=u}^{v} F(x)$ can be formally interpreted as $\sum_{x=u}^{v} F(x) = \sum_{x=u}^{\infty} F(x) - \sum_{x=v+1}^{\infty} F(x)$ for $u < v$.
We want to show that the summation operator preserves polynomiality: If the function $F(l_1, \ldots, l_{n-1})$ is polynomial in $l_1, \ldots, l_{n-1}$, then the mapping

$$(k_1, \ldots, k_n) \mapsto \sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} F(l_1, \ldots, l_{n-1})$$

is a polynomial funtion on $\mathbb{Z}^n$. Since the summation operator can be expressed only in terms of sums defined in (2.2), we show that these sums preserve polynomiality: Given any polynomial $p(x) = \sum_{i=0}^{n} c_i \binom{x}{i}$, we set $q(x) := \sum_{i=0}^{n} c_i \binom{x}{i+1}$. Then $q(x+1) - q(x) = p(x)$. It follows that $\sum_{x=u}^{v} p(x) = q(v+1) - q(u)$ for any integers $u \leq v$. But due to (2.2), this is also true for integers $u > v$. It follows that

$$\sum_{x=u}^{v} p(x) = q(v+1) - q(u)$$

for any $u, v \in \mathbb{Z}$. By induction, we see that the summation operator preserves polynomiality.
Moreover, it is clear that the summation operator extends (2.1), i.e. for any function F on $\mathbb{Z}^{n-1}$ and for any integers $k_1 < \cdots < k_n$ it holds that

$$\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} F(l_1, \ldots, l_{n-1}) = \sum_{\substack{(l_1,\ldots,l_{n-1}) \in \mathbb{Z}^{n-1}, \\ l_1 < l_2 < \cdots < l_{n-1}, \\ k_1 \leq l_1 \leq k_2 \leq \cdots \leq k_{n-1} \leq l_{n-1} \leq k_n}} F(l_1, \ldots, l_{n-1}).$$

We now define $\alpha(1; k_1) := 1$ for a given $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ and

$$\alpha(n; k_1, \ldots, k_n) := \sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} \alpha(n-1; l_1, \ldots, l_{n-1}).$$

It is clear that $\alpha(n; k_1, \ldots, k_n)$ is a polynomial in $(k_1, \ldots, k_n)$ which enumerates all monotone triangles with n rows and prescribed bottom row $(k_1, \ldots, k_n)$ if $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ and $k_1 < \cdots < k_n$.

Example 9. *We want to derive the formula for $\alpha(3; k_1, k_2, k_3)$ by using our recursive definition:*

- $\alpha(1; k_1) = 1$.
- $\alpha(2; k_1, k_2) = \sum_{(l_1)}^{(k_1,k_2)} \alpha(1; l_1) = \sum_{l_1=k_1}^{k_2} \sum_{\emptyset}^{(k_1)} \alpha(1; l_1) - \sum_{\emptyset}^{\emptyset} \alpha(1; k_1) = \sum_{l_1=k_1}^{k_2} 1 - 0$
  $= k_2 - k_1 + 1.$

- $\alpha(3;k_1,k_2,k_3) = \sum\limits_{(l_1,l_2)}^{(k_1,k_2,k_3)} \alpha(2;l_1,l_2) = \sum_{l_2=k_2}^{k_3} \sum\limits_{(l_1)}^{(k_1,k_2)} \alpha(2;l_1,l_2) - \sum\limits_{\emptyset}^{(k_1)} \alpha(2;k_2,k_2)$
  $= \sum_{l_2=k_2}^{k_3} \left(\sum_{l_1=k_1}^{k_2} \sum\limits_{\emptyset}^{(k_1)} \alpha(2;l_1,l_2) - \sum\limits_{\emptyset}^{\emptyset} \alpha(2;k_1,k_1) \right) - \sum\limits_{\emptyset}^{(k_1)} \alpha(2;k_2,k_2)$
  $= \sum_{l_2=k_2}^{k_3} \sum_{l_1=k_1}^{k_2} (l_2 - l_1 + 1) - 1$
  $= \left(\binom{k_3+1}{2} - \binom{k_2}{2} \right) (k_2 - k_1 + 1) - (k_3 - k_2 + 1) \left(\binom{k_2+1}{2} - \binom{k_1}{2} \right)$
  $+ (k_3 - k_2 + 1)(k_2 - k_1 + 1) - 1.$

It follows that

$$\begin{aligned}\alpha(3;k_1,k_2,k_3) = -\frac{3}{2}k_1 + \frac{1}{2}k_1^2 + k_1k_2 - \frac{1}{2}k_1^2k_2 - k_2^2 + \frac{1}{2}k_1k_2^2 + \frac{3}{2}k_3 \\ - 2k_1k_3 + \frac{1}{2}k_1^2k_3 + k_2k_3 - \frac{1}{2}k_2^2k_3 + \frac{1}{2}k_3^2 - \frac{1}{2}k_1k_3^2 + \frac{1}{2}k_2k_3^2.\end{aligned}$$

We compute $\alpha(3;2,4,5) = 14$*. There are indeed 14 corresponding MTs:*

$$\begin{array}{ccccc} & & 2 & & \\ & 2 & & 4 & \\ 2 & & 4 & & 5 \end{array}, \quad \begin{array}{ccccc} & & 3 & & \\ & 2 & & 4 & \\ 2 & & 4 & & 5 \end{array}, \quad \begin{array}{ccccc} & & 4 & & \\ & 2 & & 4 & \\ 2 & & 4 & & 5 \end{array}, \quad \begin{array}{ccccc} & & 2 & & \\ & 2 & & 5 & \\ 2 & & 4 & & 5 \end{array}, \quad \begin{array}{ccccc} & & 3 & & \\ & 2 & & 5 & \\ 2 & & 4 & & 5 \end{array}, \quad \begin{array}{ccccc} & & 4 & & \\ & 2 & & 5 & \\ 2 & & 4 & & 5 \end{array}, \quad \begin{array}{ccccc} & & 5 & & \\ & 2 & & 5 & \\ 2 & & 4 & & 5 \end{array},$$

$$\begin{array}{ccccc} & & 3 & & \\ & 3 & & 4 & \\ 2 & & 4 & & 5 \end{array}, \quad \begin{array}{ccccc} & & 4 & & \\ & 3 & & 4 & \\ 2 & & 4 & & 5 \end{array}, \quad \begin{array}{ccccc} & & 3 & & \\ & 3 & & 5 & \\ 2 & & 4 & & 5 \end{array}, \quad \begin{array}{ccccc} & & 4 & & \\ & 3 & & 5 & \\ 2 & & 4 & & 5 \end{array}, \quad \begin{array}{ccccc} & & 5 & & \\ & 3 & & 5 & \\ 2 & & 4 & & 5 \end{array}, \quad \begin{array}{ccccc} & & 4 & & \\ & 4 & & 5 & \\ 2 & & 4 & & 5 \end{array}, \quad \begin{array}{ccccc} & & 5 & & \\ & 4 & & 5 & \\ 2 & & 4 & & 5 \end{array}.$$

By the combinatorial interpretation and the definition of the summation operator, it is clear that $\alpha(n;k_1,\ldots,k_n)$ does not depend on the actual values $(k_1,\ldots,k_n)$ but rather on the differences $(k_2-k_1,\ldots,k_n-k_{n-1})$. So it follows for any $m \in \mathbb{Z}$ that

$$\alpha(n;k_1,\ldots,k_n) = \alpha(n;k_1-m,\ldots,k_n-m).$$

Thus, it follows that $\alpha(3;1,3,4) = \alpha(3;2,4,5) = 14$ by Example 9.

2.3 The Operator Formula for Monotone Triangles

We are able to develop an explicit formula for the enumeration of monotone triangles by dint of their recursive structure. To this end, we still need a special operator to formulate the theorem: the *shift operator* E_x. The shift operator increases the argument of a function by one, i.e. $\mathrm{E}_x\, f(x) := f(x+1)$ for any function $f(x)$.
The shift operator E_x is invertible and its inverse is $\mathrm{E}_x^{-1}\, f(x) = f(x-1)$. Moreover, shift operators commute: $\mathrm{E}_x\, \mathrm{E}_y\, f(x,y) = \mathrm{E}_y\, \mathrm{E}_x\, f(x,y)$.
In later sections, we use more operators. For the sake of completeness, we introduce them now. Table 2.1 provides an overview of the operators we need.

identity operator id	$\mathrm{id}\, f(x) := f(x)$
swap operator $\mathrm{S}_{x,y}$	$\mathrm{S}_{x,y}\, f(x,y) := f(y,x)$
shift operator E_x	$\mathrm{E}_x\, f(x) := f(x+1)$
forward difference operator Δ_x	$\Delta_x := \mathrm{E}_x - \mathrm{id}$
backward difference operator δ_x	$\delta_x := \mathrm{id} - \mathrm{E}_x^{-1}$
W-operator $\mathrm{W}_{x,y}$	$\mathrm{W}_{x,y} := \mathrm{E}_x + \Delta_x\, \Delta_y$
I-operator $\mathrm{I}_{x,y}$	$\mathrm{I}_{x,y} := \mathrm{id} - \delta_x - \delta_y$
Strict-operator $\mathrm{Strict}_{x,y}$	$\mathrm{Strict}_{x,y} := \mathrm{E}_x^{-1} + \mathrm{E}_y - \mathrm{E}_x^{-1}\, \mathrm{E}_y$

Table 2.1: Some operators

The product of two operators denotes the composition of operators. Since the shift operator commutes, all other operators from table 2.1 – except the swap operator – consequently commute pairwise.

Theorem 10 ([Fis06], Theorem 1)**.** *It holds for $n \in \mathbb{N}$ and $(k_1, \dots, k_n) \in \mathbb{Z}^n$ that*

$$\alpha\,(n; k_1, \dots, k_n) = \prod_{1 \le s < t \le n} (\mathrm{id} - \mathrm{E}_{k_s} + \mathrm{E}_{k_s}\, \mathrm{E}_{k_t}) \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i}.$$

This formula was first proved in [Fis06]. The number of alternating sign matrices of order n (Theorem 3) can be derived from this operator formula by computing $\alpha\,(n; 1, \dots, n)$ as shown in [Fis07]. Since then, Fischer has developed several simplifications and refinements of this proof such as in [Fis16]. [Fis10] provides a generalised approach dealing amongst others with the so-called Q-enumeration. We follow this approach and proof an operator formula for the Q-enumeration of monotone triangles (see Theorem 15). Theorem 10 is then the special case for $Q = 1$.

2.4 The Q-Enumeration of Monotone Triangles

Let us consider three adjacent entries of a monotone triangle arranged in a triangular shape, i.e. an entry $a_{i,j}$ and its southwest neighbour $a_{i+1,j}$ and its southeast neighbour $a_{i+1,j+1}$. We can distinguish three possible cases like in Figure 2.2.

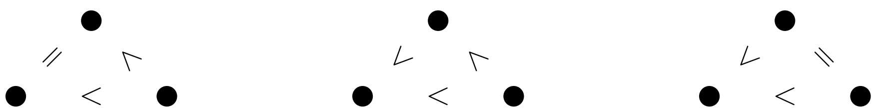

Figure 2.2: General relations between three adjacent entries

We are interested in the second of these three cases, namely when an entry is strictly lying between its neighbours in the row beneath: $a_{i+1,j} < a_{i,j} < a_{i+1,j+1}$. Our goal is to enumerate monotone triangles with respect to the number of entries strictly lying between its southwest and southeast neighbour in terms of a generating function. Therefore we need to define the Q-weight of a monotone triangle.

Definition 11. If $\alpha \in \mathcal{MT}(n; k_1, \dots, k_n)$ is a monotone triangle, then the Q-*weight* of α is Q raised to the cardinality of the set $\{a_{i,j} \mid a_{i+1,j} < a_{i,j} < a_{i+1,j+1}, 1 \le i \le n-1, 1 \le j \le i\}$. It is denoted by $Q(\alpha)$.

The power of Q counts the entries that are lying strictly between their southwest and southeast neighbours in the row beneath. The generating function for monotone triangles with respect to the Q-weight – or simply Q-generating function – is then

$$\sum_{\alpha \in \mathcal{MT}(n;k_1,\dots,k_n)} Q(\alpha).$$

Example 12. *Let us calculate the Q-generating function for the monotone triangles with 3 rows and bottom row $(2,4,5)$ that we discussed in example 9. We mark the special entries in every monotone triangle which contribute to the Q-weight by a circle:*

α	$\begin{smallmatrix} & & 2 & & \\ & 2 & & 4 & \\ 2 & & 4 & & 5 \end{smallmatrix}$,	$\begin{smallmatrix} & & 3 & & \\ & 2 & & 4 & \\ 2 & & 4 & & 5 \end{smallmatrix}$,	$\begin{smallmatrix} & & 4 & & \\ & 2 & & 4 & \\ 2 & & 4 & & 5 \end{smallmatrix}$,	$\begin{smallmatrix} & & 2 & & \\ & 2 & & 5 & \\ 2 & & 4 & & 5 \end{smallmatrix}$,	$\begin{smallmatrix} & & 3 & & \\ & 2 & & 5 & \\ 2 & & 4 & & 5 \end{smallmatrix}$,	$\begin{smallmatrix} & & 4 & & \\ & 2 & & 5 & \\ 2 & & 4 & & 5 \end{smallmatrix}$,	$\begin{smallmatrix} & & 5 & & \\ & 2 & & 5 & \\ 2 & & 4 & & 5 \end{smallmatrix}$,
$Q(\alpha)$	1	Q	1	1	Q	Q	1
α	$\begin{smallmatrix} & & 3 & & \\ & 3 & & 4 & \\ 2 & & 4 & & 5 \end{smallmatrix}$,	$\begin{smallmatrix} & & 4 & & \\ & 3 & & 4 & \\ 2 & & 4 & & 5 \end{smallmatrix}$,	$\begin{smallmatrix} & & 3 & & \\ & 3 & & 5 & \\ 2 & & 4 & & 5 \end{smallmatrix}$,	$\begin{smallmatrix} & & 4 & & \\ & 3 & & 5 & \\ 2 & & 4 & & 5 \end{smallmatrix}$,	$\begin{smallmatrix} & & 5 & & \\ & 3 & & 5 & \\ 2 & & 4 & & 5 \end{smallmatrix}$,	$\begin{smallmatrix} & & 4 & & \\ & 4 & & 5 & \\ 2 & & 4 & & 5 \end{smallmatrix}$,	$\begin{smallmatrix} & & 5 & & \\ & 4 & & 5 & \\ 2 & & 4 & & 5 \end{smallmatrix}$.
$Q(\alpha)$	Q	Q	Q	Q^2	Q	1	1

If we sum up the corresponding Q-weights, we get $6+7Q+Q^2$ as Q-generating function.

The Q-generating function of monotone triangles corresponds to the enumeration of alternating sign matrices with respect to the number of the entries equal to -1. This is proved in the following theorem:

Theorem 13. *There is a bijection between the entries -1 of an alternating sign matrix and the entries of the corresponding monotone triangle that are strictly lying between their southwest and southeast neighbours in the row below.*

Proof. Take any alternating sign matrix $A \in \mathcal{ASM}(n)$ and suppose we have an entry -1 at the position (i,j). This entry maps to a 0 at the position (i,j) in the corresponding partial column sum matrix. It is clear by the definition of the alternating sign matrix that there is at least an entry 1 to the left, to the right and above the entry -1 at (i,j) in A (and possibly some entries 0 in between). Because of that, there is at least one entry 1 to the left, to the right and directly above the entry 0 at (i,j) in the partial column sum matrix. Suppose those 1s that are closest to 0 at (i,j) have coordinates $(i, j-m_1)$, $(i, j+m_2)$ and $(i-1, j)$ for some positive integers m_1 and m_2. This corresponds to the following configuration in the monotone triangle:

$$\begin{array}{ccccc} & & j & & \\ & & \bullet & & \\ & \swarrow\!\!< & & >\!\!\nwarrow & \\ j-m_1 \; \bullet & & < & & \bullet \; j+m_2 \end{array}$$

For the converse direction of the bijection we argue that the construction above is invertible. If we have – just as given in the figure above – a monotone triangle with an entry that is strictly lying between its neigbours in the row below, then this corresponds to an entry 0 at position (i, j) (for some i) in the partial column sum matrix with one entry 1 in the position $(i-1, j)$ directly above. This means we have an entry -1 at (i, j) in the corresponding alternating sign matrix. □

Fischer presented in [Fis10] an explicit formula for the Q-generating function for monotone triangles. It is embedded in a much more general setting which we don't need here. We only present it restricted to our case.
To this end, we introduce in Table 2.2 several operators on functions that are used in the following. They are defined with respect to the indeterminant Q. Therefore, we call these operators generalised operators. By setting $Q = 1$, we obtain the corresponding ordinary operators from Table 2.1. Note that all of these operators commute pairwise.

Q-identity operator ${}^{Q}\mathrm{id}_x$	${}^{Q}\mathrm{id}_x := Q\,\mathrm{E}_x - \Delta_x$
Q-shift operator ${}^{Q}\mathrm{E}_x$	${}^{Q}\mathrm{E}_x := Q\,\mathrm{id} + \Delta_x$
Q-inverse shift operator ${}^{Q}\mathrm{E}_x^{-1}$	${}^{Q}\mathrm{E}_x^{-1} := Q\,\mathrm{id} - \delta_x$
Q-(forward) difference operator ${}^{Q}\Delta_x$	${}^{Q}\Delta_x := Q^{-1}((1-Q^{-1})\,\mathrm{E}_x + Q^{-1})^{-1}\,\Delta_x$
Q-W-operator ${}^{Q}\mathrm{W}_{x,y}$	${}^{Q}\mathrm{W}_{x,y} := {}^{Q}\mathrm{E}_x\, {}^{Q}\mathrm{id}_y + \Delta_x\,\Delta_y$
Q-I-operator ${}^{Q}\mathrm{I}_{x,y}$	${}^{Q}\mathrm{I}_{x,y} := Q\,\mathrm{id} - \delta_x - \delta_y$
Q-Strict-operator ${}^{Q}\mathrm{Strict}_{x,y}$	${}^{Q}\mathrm{Strict}_{x,y} := Q\,\mathrm{E}_x^{-1}\,\mathrm{E}_y + \delta_x\,\mathrm{E}_y - \mathrm{E}_x^{-1}\,\Delta_y$

Table 2.2: Generalised Q-operators[1]

We define a generalised version of the summation operator as our next step. It is obtained similarly as in Definition 8 but taking into account whether an entry of the penultimate row is strictly lying between its neighbours in the bottom row or not, i.e. it should hold for any function F on $\mathbb{Z}^{n-1}$ and for any integers $k_1 < \cdots < k_n$ that

$$\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} F(l_1,\ldots,l_{n-1}) = \sum_{\substack{(l_1,\ldots,l_{n-1})\in\mathbb{Z}^{n-1},\\ l_1<l_2<\cdots<l_{n-1},\\ k_1\le l_1\le k_2\le\cdots\le k_{n-1}\le l_{n-1}\le k_n}} Q^{[k_1<l_1<k_2]+\cdots+[k_{n-1}<l_{n-1}<k_n]} F(l_1,\ldots,l_{n-1})$$

where we use the *Iverson bracket*: For any logical proposition P, $[P] = 1$ if P is satisfied and $[P] = 0$ otherwise.

[1]Some comments on the notation: Fischer introduced in [Fis10] operators ${}^{Q}_{P}\mathrm{E}_x$ and ${}^{Q}_{P}\mathrm{id}_x$ with respect to the indeterminants P and Q. If we set $P = 1$, the operators simplify to our case, i.e. ${}^{Q}_{1}\mathrm{E}_x = {}^{Q}\mathrm{E}_x$ and ${}^{Q}_{1}\mathrm{id}_x = {}^{Q}\mathrm{id}_x$. In addition, she defined an operator $\mathrm{V}_{x,y}$ with respect to two indeterminants P and Q, a fixed set $S \subseteq \mathbb{Z}^2$ and a function $f : S \mapsto \mathbb{C}$. We obtain the operator ${}^{Q}\mathrm{W}_{x,y}$ by setting $P = 1$, $S = (0, 0)$ and $f \equiv -1$.

Furthermore, note that the Q-inverse shift operator ${}^{Q}\mathrm{E}_x^{-1}$ should rather be read as ${}^{Q}(\mathrm{E}_x^{-1})$ instead of $({}^{Q}\mathrm{E}_x)^{-1}$ because it is not the inverse operator to ${}^{Q}\mathrm{E}_x$ but a Q-generalisation of E_x^{-1} as mentioned above.

Definition 14. We define the generalised summation operator $\sum\limits_{(l_1,\dots,l_{n-1})}^{(k_1,\dots,k_n)} F(l_1,\dots,l_{n-1})$ for a given $(k_1,\dots,k_n)\in\mathbb{Z}^n$ and a function $F(l_1,\dots,l_{n-1})$ on $\mathbb{Z}^{n-1}$ by induction on n. For $n=0$ we set $\sum\limits_{\emptyset}^{\emptyset} F := 0$ and for $n=1$ we set $\sum\limits_{\emptyset}^{k_1} F := F$. Then we can define for $n\geq 2$:

$$\begin{aligned}&\sum_{(l_1,\dots,l_{n-1})}^{(k_1,\dots,k_n)} F(l_1,\dots,l_{n-1})\\ &\qquad := \left(Q^{-1}\,{}^{Q}\mathrm{E}_{k_n}\,{}^{Q}\mathrm{id}_{k^*_{n-1}} \sum_{l_{n-1}=k^*_{n-1}}^{k_n-1}\ \sum_{(l_1,\dots,l_{n-2})}^{(k_1,\dots,k_{n-1})} F(l_1,\dots,l_{n-1})\right)\Bigg|_{k^*_{n-1}=k_{n-1}}\\ &\qquad\qquad - \sum_{(l_1,\dots,l_{n-3})}^{(k_1,\dots,k_{n-2})} F(l_1,\dots,l_{n-3},k_{n-1},k_{n-1}).\end{aligned}$$

We use the same symbol for the generalised summation operator and the ordinary one from Definition 8. The context always clearly indicates which version of the summation operator we are using.
We rewrite this definition of the summation operator to make it clear that it really meets our purpose.

$$\begin{aligned}&\sum_{(l_1,\dots,l_{n-1})}^{(k_1,\dots,k_n)} F(l_1,\dots,l_{n-1}) = \Bigg(\Big(Q\,\mathrm{E}_{k^*_{n-1}} - \Delta_{k^*_{n-1}} + \Delta_{k_n}\,\mathrm{E}_{k^*_{n-1}} - \Delta_{k_n}\,\Delta_{k^*_{n-1}}\Big)\\ &\times \sum_{l_{n-1}=k^*_{n-1}}^{k_n-1}\ \sum_{(l_1,\dots,l_{n-2})}^{(k_1,\dots,k_{n-1})} F(l_1,\dots,l_{n-1})\Bigg)\Bigg|_{k^*_{n-1}=k_{n-1}} - \sum_{(l_1,\dots,l_{n-3})}^{(k_1,\dots,k_{n-2})} F(l_1,\dots,l_{n-3},k_{n-1},k_{n-1}).\end{aligned}$$

Note that the operator $\Delta_{k_n}\,\Delta_{k^*_{n-1}}$ applied to $\sum_{l_{n-1}=k^*_{n-1}}^{k_n-1} \sum\limits_{(l_1,\dots,l_{n-2})}^{(k_1,\dots,k_{n-1})} F(l_1,\dots,l_{n-1})$ gives zero. It follows that

$$\begin{aligned}&\sum_{(l_1,\dots,l_{n-1})}^{(k_1,\dots,k_n)} F(l_1,\dots,l_{n-1})\\ &\qquad = Q\sum_{l_{n-1}=k_{n-1}+1}^{k_n-1}\ \sum_{(l_1,\dots,l_{n-2})}^{(k_1,\dots,k_{n-1})} F(k_1,\dots,k_{n-1}) + \sum_{(l_1,\dots,l_{n-2})}^{(k_1,\dots,k_{n-1})} F(k_1,\dots,k_{n-1})\\ &\qquad\qquad + \sum_{(l_1,\dots,l_{n-2})}^{(k_1,\dots,k_{n-1})} F(k_1,\dots,k_n) - \sum_{(l_1,\dots,l_{n-3})}^{(k_1,\dots,k_{n-2})} F(l_1,\dots,l_{n-3},k_{n-1},k_{n-1}).\end{aligned}$$

It is now easy to see that the generalised summation operator preserves polynomiality and that both summation operators indeed coincide for $Q=1$.
We define inductively ${}^{Q}\alpha\,(n;k_1,\dots,k_n)$ with the help of this generalised summation operator. Let be ${}^{Q}\alpha\,(1;k_1) := 1$ and set

$${}^{Q}\alpha\,(n;k_1,\dots,k_n) := \sum_{(l_1,\dots,l_{n-1})}^{(k_1,\dots,k_n)} {}^{Q}\alpha(n-1;l_1,\dots,l_{n-1}).$$

If $k_1,\dots,k_n$ are integers such that $k_1 < \cdots < k_n$, then ${}^Q\alpha\,(n;k_1,\dots,k_n)$ clearly is the Q-generating function of monotone triangles with prescribed bottom row $(k_1,\dots,k_n)$. Within this setting, we are able to state the following theorem that provides an explicit formula for the Q-generating function for monotone triangles:

Theorem 15 ([Fis10], Theorem 1)**.** *Let n be a positive integer and $(k_1,k_2,\dots,k_n)\in\mathbb{Z}^n$. Then the function ${}^Q\alpha(n;k_1,\dots,k_n)$ is given by*

$$\prod_{1\le s<t\le n}\left(\mathrm{id}-(2-Q)\,\mathrm{E}_{k_s}+\mathrm{E}_{k_s}\mathrm{E}_{k_t}\right)\prod_{1\le i<j\le n}\frac{k_j-k_i}{j-i}.$$

Before we prove this theorem, we show three further lemmas. The first one deals with $\prod_{1\le i<j\le n}\frac{k_j-k_i}{j-i}$. It is useful to express this product as a determinant. For this, we need a generalised version of the Vandermonde determinant

$$\det_{1\le i,j\le n}\left(X_i^{j-1}\right)=\prod_{1\le i<j\le n}\left(X_j-X_i\right).$$

Lemma 16 ([Kra99], Proposition 1)**.** *Let $X_1,\dots,X_n$ be indeterminants and $p_1,\dots,p_n$ be polynomials such that every p_j is of degree $j-1$ and has leading coefficient a_j. Then*

$$\det_{1\le i,j\le n}\left(p_j\left(X_i\right)\right)=a_1\cdots a_n\prod_{1\le i<j\le n}\left(X_j-X_i\right).$$

This formula can be proved in a similar way as in the case of the standard Vandermonde determinant. These and further variations are discussed in [Kra99].
It follows that

$$\prod_{1\le i<j\le n}\frac{k_j-k_i}{j-i}=\det_{1\le i,j\le n}\binom{k_i}{j-1}\tag{2.3}$$

by setting $p_j := x(x-1)\cdots(x-j+1)$ in Lemma 16 and dividing every column by $(j-1)!$. Hence, Theorem 15 can be rewritten as

$${}^Q\alpha\left(n;k_1,\dots,k_n\right)=Q^{-\binom{n}{2}}\prod_{1\le s<t\le n}{}^Q\mathrm{W}_{k_t,k_s}\det_{1\le i,j\le n}\binom{k_i}{j-1}.\tag{2.4}$$

We need to understand how ${}^Q\mathrm{W}_{x,y}$ and the summation operator interact with each other to prove (2.4). The following lemma provides such an insight:

Lemma 17 ([Fis10], Lemma 1)**.** *If $G(l_1,\dots,l_{n-1})$ is a function on $\mathbb{Z}^{n-1}$ such that for every $i\in\{1,\dots,n-2\}$ one has ${}^Q\mathrm{W}_{l_i,l_{i+1}}\,G(l_1,\dots,l_{n-1})\Big|_{l_i=l_{i+1}}=0$, then*

$$\sum_{(l_1,\dots,l_{n-1})}^{(k_1,\dots,k_n)}\prod_{i=1}^{n-1}\Delta_{l_i}\,G(l_1,\dots,l_{n-1})=\sum_{r=1}^{n}(-1)^{r-1}\prod_{s=1}^{r-1}{}^Q\mathrm{id}_{k_s}\prod_{t=r+1}^{n}{}^Q\mathrm{E}_{k_t}\,G(k_1,\dots,\widehat{k_r},\dots,k_n).$$

Note that $G(k_1,\dots,\widehat{k_r},\dots,k_n):=G(k_1,\dots,k_{r-1},k_{r+1},\dots,k_n)$.

Proof. Lemma 1 in [Fis10] is a more general version of this lemma with respect to two indeterminants P and Q. By setting $P = 1$, we get the statement above. A simplified proof in the special setting is the following:
We proof the lemma by induction on n. In the base case $n = 0$ both sides of the equation vanish. For $n \leq 1$, the definition of the summation operator and the induction hypothesis yield

$$\begin{aligned}
&\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} \prod_{i=1}^{n-1} \Delta_{l_i}\, G(l_1,\ldots,l_{n-1}) \\
&= \Bigg(Q^{-1}\, {}^{Q}\mathrm{E}_{k_n}\, {}^{Q}\mathrm{id}_{k^*_{n-1}} \sum_{l_{n-1}=k^*_{n-1}}^{k_n-1} \Delta_{l_{n-1}} \\
&\qquad \times \sum_{r=1}^{n-1} (-1)^{r-1} \prod_{s=1}^{r-1} {}^{Q}\mathrm{id}_{k_s} \prod_{t=r+1}^{n-1} {}^{Q}\mathrm{E}_{k_t}\, G(k_1,\ldots,\widehat{k_r},\ldots,k_{n-1},l_{n-1}) \Bigg) \Bigg|_{k^*_{n-1}=k_{n-1}} \\
&\qquad - \Delta_{k_{n-1}} \Delta_{k_{n-1}} \sum_{r=1}^{n-2} (-1)^{r-1} \prod_{s=1}^{r-1} {}^{Q}\mathrm{id}_{k_s} \prod_{t=r+1}^{n-2} {}^{Q}\mathrm{E}_{k_t}\, G(k_1,\ldots,\widehat{k_r},\ldots,k_{n-2},k_{n-1},k_{n-1})
\end{aligned}$$

By telescopic cancelling we can split up the first summand and get

$$\begin{aligned}
&\Bigg(Q^{-1}\, {}^{Q}\mathrm{E}_{k_n}\, {}^{Q}\mathrm{id}_{k^*_{n-1}} \sum_{r=1}^{n-1} (-1)^{r-1} \prod_{s=1}^{r-1} {}^{Q}\mathrm{id}_{k_s} \prod_{t=r+1}^{n-1} {}^{Q}\mathrm{E}_{k_t}\, G(k_1,\ldots,\widehat{k_r},\ldots,k_{n-1},k_n) \\
&- Q^{-1}\, {}^{Q}\mathrm{E}_{k_n}\, {}^{Q}\mathrm{id}_{k^*_{n-1}} \sum_{r=1}^{n-1} (-1)^{r-1} \prod_{s=1}^{r-1} {}^{Q}\mathrm{id}_{k_s} \prod_{t=r+1}^{n-1} {}^{Q}\mathrm{E}_{k_t}\, G(k_1,\ldots,\widehat{k_r},\ldots,k_{n-1},k^*_{n-1}) \Bigg) \Bigg|_{k^*_{n-1}=k_{n-1}} \\
&\qquad + \Delta_{k_{n-1}} \Delta_{k_{n-1}} \sum_{r=1}^{n-2} (-1)^{r} \prod_{s=1}^{r-1} {}^{Q}\mathrm{id}_{k_s} \prod_{t=r+1}^{n-2} {}^{Q}\mathrm{E}_{k_t}\, G(k_1,\ldots,\widehat{k_r},\ldots,k_{n-2},k_{n-1},k_{n-1}) \\
&= \Bigg(\sum_{r=1}^{n-1} (-1)^{r-1} \prod_{s=1}^{r-1} {}^{Q}\mathrm{id}_{k_s} \prod_{t=r+1}^{n} {}^{Q}\mathrm{E}_{k_t}\, G(k_1,\ldots,\widehat{k_r},\ldots,k_{n-1},k_n) \\
&\qquad + \sum_{r=1}^{n-1} (-1)^{r} \prod_{s=1}^{r-1} {}^{Q}\mathrm{id}_{k_s} \prod_{t=r+1}^{n-1} {}^{Q}\mathrm{E}_{k_t}\, {}^{Q}\mathrm{id}_{k^*_{n-1}}\, G(k_1,\ldots,\widehat{k_r},\ldots,k_{n-1},k^*_{n-1}) \Bigg) \Bigg|_{k^*_{n-1}=k_{n-1}} \\
&\qquad\qquad + \sum_{r=1}^{n-2} (-1)^{r} \prod_{s=1}^{r-1} {}^{Q}\mathrm{id}_{k_s} \prod_{t=r+1}^{n-2} {}^{Q}\mathrm{E}_{k_t} \\
&\qquad\qquad \times \left({}^{Q}\mathrm{W}_{k_{n-1},k_{n-1}} - {}^{Q}\mathrm{E}_{k_{n-1}}\, {}^{Q}\mathrm{id}_{k_{n-1}} \right) G(k_1,\ldots,\widehat{k_r},\ldots,k_{n-2},k_{n-1},k_{n-1}).
\end{aligned}$$

Now we add the (n-1)$^{\text{st}}$ summand of the second sum to the first one and sum up the rest. We finally obtain

$$\begin{aligned}
&\sum_{r=1}^{n} (-1)^{r-1} \prod_{s=1}^{r-1} {}^{Q}\mathrm{id}_{k_s} \prod_{t=r+1}^{n} {}^{Q}\mathrm{E}_{k_t}\, G(k_1,\ldots,\widehat{k_r},\ldots,k_n) \\
&\qquad + \sum_{r=1}^{n-2} (-1)^{r} \prod_{s=1}^{r-1} {}^{Q}\mathrm{id}_{k_s} \prod_{t=r+1}^{n-2} {}^{Q}\mathrm{E}_{k_t}\, \underbrace{{}^{Q}\mathrm{W}_{k_{n-1},k_{n-1}}\, G(k_1,\ldots,\widehat{k_r},\ldots,k_{n-2},k_{n-1},k_{n-1})}_{=0}.
\end{aligned}$$

$\square$

By specialising the function $G(l_1, \dots, l_n)$ in the previous Lemma 17, we obtain the following result:

Lemma 18 ([Fis10], Lemma 2)**.**

$$\sum_{(l_1,\dots,l_{n-1})}^{(k_1,\dots,k_n)} \prod_{1\le s<t\le n-1} {}^Q\mathrm{W}_{l_t,l_s} \det_{1\le i,j\le n-1} \binom{l_i}{j-1} = Q^{-n+1} \prod_{1\le s<t\le n} {}^Q\mathrm{W}_{k_t,k_s} \det_{1\le i,j\le n} \binom{k_i}{j-1}.$$

Proof. First we set $G(l_1, \dots, l_n) := \prod_{1\le s<t\le n-1} {}^Q\mathrm{W}_{l_t,l_s} \det_{1\le i,j\le n-1} \binom{l_i}{j}$. Note that the operator

$$ {}^Q\mathrm{W}_{l_i,l_{i+1}} \prod_{1\le s<t\le n-1} {}^Q\mathrm{W}_{l_t,l_s}$$

is symmetric in l_i and l_{i+1} and $\det_{1\le i,j\le n-1} \binom{l_i}{j-1}$ is antisymmetric in l_i and l_{i+1}. It follows that

$$S_{l_i,l_{i+1}} \prod_{1\le s<t\le n-1} {}^Q\mathrm{W}_{l_t,l_s} \det_{1\le i,j\le n-1} \binom{l_i}{j-1} = -\prod_{1\le s<t\le n-1} {}^Q\mathrm{W}_{l_t,l_s} \det_{1\le i,j\le n-1} \binom{l_i}{j-1}$$

and therefore

$$(\mathrm{id} + S_{l_i,l_{i+1}}) \prod_{1\le s<t\le n-1} {}^Q\mathrm{W}_{l_t,l_s} \det_{1\le i,j\le n-1} \binom{l_i}{j-1} = 0.$$

The conditions from lemma 17 are fulfilled. It follows that

$$\begin{aligned}
&\sum_{(l_1,\dots,l_{n-1})}^{(k_1,\dots,k_n)} \prod_{1\le s<t\le n-1} {}^Q\mathrm{W}_{l_t,l_s} \det_{1\le i,j\le n-1} \binom{l_i}{j-1} \\
&\qquad = \sum_{(l_1,\dots,l_{n-1})}^{(k_1,\dots,k_n)} \prod_{1\le s<t\le n-1} {}^Q\mathrm{W}_{l_t,l_s} \det_{1\le i,j\le n-1} \left(\Delta_{l_i} \binom{l_i}{j}\right) \\
&\qquad = \sum_{(l_1,\dots,l_{n-1})}^{(k_1,\dots,k_n)} \prod_{i=1}^{n-1} \Delta_{l_i} \prod_{1\le s<t\le n-1} {}^Q\mathrm{W}_{l_t,l_s} \det_{1\le i,j\le n-1} \binom{l_i}{j} \\
&= \sum_{r=1}^{n} (-1)^{r-1} \prod_{s=1}^{r-1} {}^Q\mathrm{id}_{k_s} \prod_{t=r+1}^{n} {}^Q\mathrm{E}_{k_t} \prod_{\substack{1\le s<t\le n-1\\ s,t\ne r}} {}^Q\mathrm{W}_{k_t,k_s} \det_{1\le i,j\le n-1} \binom{l_i}{j} \Bigg|_{(l_1,\dots,l_{n-1})=(k_1,\dots,\widehat{k_r},\dots,k_n)}.
\end{aligned}$$

Since the variable k_r does not occur in the summands, we can write $Q^{-1}\,{}^Q\mathrm{W}_{k_t,k_r}$ instead of ${}^Q\mathrm{E}_{k_t}$ and $Q^{-1}\,{}^Q\mathrm{W}_{k_r,k_s}$ instead of ${}^Q\mathrm{id}_{k_s}$. Moreover, if we use the Leibniz formula with respect to the first column to evaluate $\det_{1\le i,j\le n} \binom{k_i}{j-1}$, we get

$$\sum_{r=1}^{n} (-1)^{r-1} \det_{1\le i,j\le n-1} \binom{l_i}{j} \Bigg|_{(l_1,\dots,l_{n-1})=(k_1,\dots,\widehat{k_r},\dots,k_n)}.$$

For this reason, it follows that

$$\begin{aligned}
&\sum_{(l_1,\dots,l_{n-1})}^{(k_1,\dots,k_n)} \prod_{1\le s<t\le n-1} {}^Q\mathrm{W}_{l_t,l_s} \det_{1\le i,j\le n-1}\binom{l_i}{j-1} \\
&\qquad = Q^{n-1}\sum_{r=1}^{n}(-1)^{r-1}\prod_{s=1}^{r-1} {}^Q\mathrm{W}_{k_r,k_s}\prod_{t=r+1}^{n} {}^Q\mathrm{W}_{k_t,k_r} \\
&\qquad\quad \times \prod_{\substack{1\le s<t\le n-1\\ s,t\ne r}} {}^Q\mathrm{W}_{k_t,k_s} \det_{1\le i,j\le n-1}\binom{l_i}{j}\Bigg|_{(l_1,\dots,l_{n-1})=(k_1,\dots,\widehat{k_r},\dots,k_n)} \\
&\qquad = Q^{n-1}\prod_{1\le s<t\le n} {}^Q\mathrm{W}_{k_t,k_s} \det_{1\le i,j\le n}\binom{k_i}{j-1}.
\end{aligned}$$

□

Proof of Theorem 15. We are now able to prove the formula for the Q-generating function for monotone triangles with the help of the preceding Lemma 18. We proceed by induction on n.
The base case $n=1$ is as follows:

$$1 = {}^Q\alpha(1;k_1) = \prod_{1\le s<t\le 1}\left(\mathrm{id} - (2-Q)\,\mathrm{E}_{k_s} + \mathrm{E}_{k_s}\mathrm{E}_{k_t}\right)\prod_{1\le i<j\le 1}\frac{k_j-k_i}{j-i} = 1\cdot 1 = 1.$$

The inductive step $n-1\to n$ yields

$$\begin{aligned}
{}^Q\alpha\left(n;k_1,\dots,k_n\right) &= \sum_{(l_1,\dots,l_{n-1})}^{(k_1,\dots,k_n)} {}^Q\alpha(n-1;l_1,\dots,l_{n-1}) \\
&= Q^{-\binom{n-1}{2}}\sum_{(l_1,\dots,l_{n-1})}^{(k_1,\dots,k_n)}\prod_{1\le s<t\le n-1} {}^Q\mathrm{W}_{l_t,l_s} \det_{1\le i,j\le n-1}\binom{l_i}{j-1} \\
&= Q^{-\binom{n-1}{2}}Q^{-n+1}\prod_{1\le s<t\le n} {}^Q\mathrm{W}_{l_t,l_s} \det_{1\le i,j\le n}\binom{k_i}{j-1} \\
&= Q^{-\binom{n}{2}}\prod_{1\le s<t\le n} {}^Q\mathrm{W}_{l_t,l_s} \det_{1\le i,j\le n}\binom{k_i}{j-1}.
\end{aligned}$$

This completes our proof. □

2.5 Special Values for Q

If we set $Q=1$, we obtain the result of Theorem 10 for the formula of $\alpha\left(n;k_1,\dots,k_n\right)$ and in particular for the number of monotone triangles with prescribed bottom row $(k_1,\dots,k_n)$. Moreover, ${}^1\alpha\left(n;1,\dots,n\right) = A(n)$ is the number of alternating sign matrices of order n. But also other values of Q are of interest.

For $Q = 0$, ${}^{0}\alpha\left(n; 1, \ldots, n\right) = n!$ because it counts the number of permutation matrices of order n. The case $Q = 2$ does not suggest a simple interpretation such as $Q = 0$ or $Q = 1$ but it is astonishingly easy to determine. Already in [MRR83], Mills, Robbins and Rumsey could show by an inductive proof that

$$ {}^{2}\alpha\left(n; 1, \ldots, n\right) = 2^{\binom{n}{2}}. \tag{2.5}$$

This was a corollary of their Theorem 2. It was proved long before the original case of interest $Q = 1$ or even the general Q-generating function. But even the 2-enumeration of general monotone triangles can be written without operators:
For $Q = 2$, the Q-enumeration ${}^{Q}\alpha\left(n; k_1, \ldots, k_n\right)$ simplifies to

$$\prod_{1 \le s < t \le n} \left(\mathrm{id} + \mathrm{E}_{k_s} \mathrm{E}_{k_t}\right) \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i}. \tag{2.6}$$

The operator $\prod_{1 \le s < t \le n} (\mathrm{id} + X_s X_t)$ is symmetric in $X_1, \ldots, X_n$. Therefore, we can use the following result:

Lemma 19 ([Fis07], Lemma 1)**.** *Let $P\left(X_1, \ldots, X_n\right) \in \mathbb{C}\left[X_1, \ldots, X_n\right]$ be a symmetric in polynomial in $\left(X_1, \ldots, X_n\right)$. Then*

$$P\left(\mathrm{E}_{k_1}, \ldots, \mathrm{E}_{k_n}\right) \alpha\left(n; k_1, \ldots, k_n\right) = P\left(\mathrm{id}, \ldots, \mathrm{id}\right) \alpha\left(n; k_1, \ldots, k_n\right).$$

Proof. The shift operators and the variables $X_1, \ldots, X_n$ commute with each other. Thus, it suffices by Theorem 10 and (2.3) to prove that

$$P\left(\mathrm{E}_{k_1}, \ldots, \mathrm{E}_{k_n}\right) \det_{1 \le i,j \le n} \binom{k_i}{j-1} = P\left(\mathrm{id}, \ldots, \mathrm{id}\right) \det_{1 \le i,j \le n} \binom{k_i}{j-1}.$$

Let $(m_1, \ldots, m_n) \neq (0, \ldots, 0)$ be nonnegative integers and let $\mathfrak{S}_n$ denote the symmetric group of order n. Since $\Delta_x = \mathrm{E}_x - \mathrm{id}$, it suffices to show that

$$\sum_{\pi \in \mathfrak{S}_n} \Delta_{k_1}^{m_{\pi(1)}} \cdots \Delta_{k_n}^{m_{\pi(n)}} \det_{1 \le i,j \le n} \binom{k_i}{j-1} = 0.$$

Due to the Leibniz formula for determinants and the fact that $\Delta_x \binom{x}{n} = \binom{x}{n-1}$, this is equivalent to

$$\sum_{\pi, \sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) \binom{k_1}{\sigma(1) - m_{\pi(1)} - 1} \cdots \binom{k_n}{\sigma(n) - m_{\pi(n)} - 1} = 0.$$

We fix $\pi, \sigma \in \mathfrak{S}_n$ and look at the corresponding summand. If there exists an $i \in \{1, \ldots, n\}$ such that $\sigma(i) - m_{\pi(i)} - 1 < 0$, then the summand vanishes. Thus, we only consider $\pi, \sigma \in \mathfrak{S}_n$ such that the corresponding summand does not vanish. Then

$$\{\sigma(i) - m_{\pi(i)} - 1 \mid 1 \le i \le n\} \subseteq \{0, 1, \ldots, n-1\}.$$

Since $(m_1, \ldots, m_n) \neq (0, \ldots, 0)$, there exist $1 \leq i < j \leq n$ such that

$$\sigma(i) - m_{\pi(i)} - 1 = \sigma(j) - m_{\pi(j)} - 1.$$

The set of all pairs (i, j) with this property is *lexicographically ordered*:

$$(i,j) < (i',j') :\Leftrightarrow \begin{cases} i < i' & \text{or} \\ i = i' \wedge j < j'. \end{cases}$$

We fix the minimal pair $(\tilde{\imath}, \tilde{\jmath})$ with respect to this order. The summand corresponding to π and σ is the negative of the summand corresponding to $\pi \circ (\tilde{\imath}, \tilde{\jmath})$ and $\sigma \circ (\tilde{\imath}, \tilde{\jmath})$ where $(\tilde{\imath}, \tilde{\jmath}) \in \mathfrak{S}_n$ is read as a permutation. Thus, we have defined a sign reversing involution on the set of all summands that do not vanish. □

If we apply Lemma 19 to (2.6), we get

$$\prod_{1 \leq s < t \leq n} (\mathrm{id} + \mathrm{E}_{k_s} \mathrm{E}_{k_t}) \prod_{1 \leq i < j \leq n} \frac{k_j - k_i}{j - i} = \prod_{1 \leq s < t \leq n} (\mathrm{id} + \mathrm{id} \cdot \mathrm{id}) \prod_{1 \leq i < j \leq n} \frac{k_j - k_i}{j - i}.$$

This implies the following result:

Theorem 20. *Let $n \in \mathbb{N}$ and $k_1, \ldots, k_n$ be integers. Then*

$$^2\alpha\left(n; k_1, \ldots, k_n\right) = 2^{\binom{n}{2}} \prod_{1 \leq i < j \leq n} \frac{k_j - k_i}{j - i}.$$

By setting $k_1 = 1, \ldots, k_n = n$, we obtain (2.5) in an easy way.

3 Halved Monotone Triangles

3.1 Vertically Symmetric Alternating Sign Matrices

In this section, we take a closer look at vertically symmetric alternating sign matrices (VSASM). These are alternating sign matrices which are invariant under reflection along the vertical axis. Since every alternating sign matrix has exactly one unique 1 in the first row, it is clear that vertically symmetric alternating sign matrices only exist for odd order. Kuperberg proves in [Kup02] an enumeration formula for vertically symmetric alternating sign matrices.

Theorem 21 ([Kup02], Theorem 2)**.** *The number of vertically symmetric alternating sign matrices of order $2n+1$ is given by*

$$(-3)^{n^2} \prod_{\substack{i,j \leq 1 \\ 2|j}}^{2n+1} \frac{3(j-i)+1}{j-i+2n+1}.$$

His proof builds on the correspondence between alternating sign matrices and square ice which is also the base for his proof of the ASM conjecture in [Kup96]. However, we will use the correspondence between alternating sign matrices and monotone triangles to obtain an enumeration formula. Starting with vertically symmetric alternating sign matrices, we will present a generalised notion of monotone triangles.

Example 22.

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}
\longleftrightarrow
\begin{array}{ccccccccccccccccc}
 & & & & & & & & 5 & & & & & & & & \\
 & & & & & & & 4 & & 6 & & & & & & & \\
 & & & & & & 1 & & 5 & & 9 & & & & & & \\
 & & & & & 1 & & 3 & & 7 & & 9 & & & & & \\
 & & & & 1 & & 3 & & 5 & & 7 & & 9 & & & & \\
 & & & 1 & & 2 & & 4 & & 6 & & 8 & & 9 & & & \\
 & & 1 & & 2 & & 4 & & 5 & & 6 & & 8 & & 9 & & \\
 & 1 & & 2 & & 3 & & 4 & & 6 & & 7 & & 8 & & 9 & \\
1 & & 2 & & 3 & & 4 & & 5 & & 6 & & 7 & & 8 & & 9
\end{array}$$

VSASM *monotone triangle*

Due to the vertical symmetry, the middle column of a vertically symmetric alternating sign matrix consists of the vector $(1,-1,1,-1,\ldots,-1,1)^\top$ as we can see in Example 22.

Suppose we have a vertically symmetric alternating sign matrix of order $2n+1$. Then the corresponding monotone triangle shows similar properties:

- There are only the entries $n+1$ in the middle column.
- The entry $a_{i,j}$ and the image $a_{i,i+1-j}$ under the reflection along the vertical symmetry axis sum up to $2n+2$.

We can get rid of this redundant information. One the one hand, we can rotate the VSASM clockwise by 90 degrees and get a horizontally symmetric alternating sign matrix with $(1,-1,1,-1,\dots,-1,1)$ as the $n+1^{\text{st}}$ row. Thus, the n^{th} row of the corresponding partial column sum matrix consists of $(0,1,0,1,\dots,1,0)$. If we transform these first n rows into a monotone triangle, we get a monotone triangle with n rows and bottom row $(2,4,\dots,2n)$. This construction is obviously invertible. That means vertically symmetric alternating sign matrices of order $2n+1$ are in a one-to-one correspondence with monotone triangles with n rows and bottom row $(2,4,\dots,2n)$. Example 22 illustrates this construction.

Example 23.

$$\begin{pmatrix} 0&0&0&0&1&0&0&0&0\\ 0&0&0&1&-1&1&0&0&0\\ 1&0&0&-1&1&-1&0&0&1\\ 0&0&1&0&-1&0&1&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&1&-1&1&-1&1&-1&1&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&1&0&-1&0&1&0&0\\ 0&0&0&0&1&0&0&0&0 \end{pmatrix} \overset{\circlearrowright}{\longleftrightarrow} \begin{pmatrix} 0&0&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&1&0&-1&0&1&0&0&0\\ 0&0&0&1&0&0&-1&1&0\\ 1&-1&1&-1&1&-1&1&-1&1\\ 0&0&0&1&0&0&-1&1&0\\ 0&1&0&-1&0&1&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0 \end{pmatrix}$$

$$\Big\updownarrow \textit{partial column sum}$$

$$\begin{array}{ccccccc} & & & 7 & & & \\ & & 4 & & 7 & & \\ & 2 & & 6 & & 7 & \\ 2 & & 4 & & 6 & & 8 \end{array} \longleftrightarrow \left(\begin{array}{ccccccccc} 0&0&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&1&0&-1&0&1&0&0&0\\ 0&0&0&1&0&0&-1&1&0\\ \hline 1&-1&1&-1&1&-1&1&-1&1\\ 0&0&0&1&0&0&-1&1&0\\ 0&1&0&-1&0&1&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0 \end{array}\right)$$

On the other hand, we can cut off the middle column and the right half of the monotone triangle (see example 22). The remaining left part contains all the information to restore the original alternating sign matrix. This triangular array of intergers is called *halved monotone triangle*.

Definition 24. A *halved monotone triangle (HMT)* is a triangular array of integers $(a_{i,j})_{\substack{1 \le i \le n, \\ 1 \le j \le \lceil \frac{i}{2} \rceil}}$ with n rows of one of the following shapes:

$$\begin{array}{ccccccccc}
 & & & & & & a_{1,1} & & \\
 & & & & & a_{2,1} & & & \\
 & & & & a_{3,1} & & a_{3,2} & & \\
 & & & a_{4,1} & & a_{4,2} & & & \\
 & & \dots & & \dots & & \dots & & \\
 & a_{n-1,1} & & \dots & & a_{n-1,\frac{n-1}{2}} & & & \\
a_{n,1} & & \dots & & \dots & & a_{n,\frac{n+1}{2}} & &
\end{array}$$

case n odd

$$\begin{array}{ccccccc}
 & & & & & & a_{1,1} \\
 & & & & & a_{2,1} & \\
 & & & & a_{3,1} & & a_{3,2} \\
 & & & \dots & & \dots & \\
 & a_{n-1,1} & & & \dots & & a_{n-1,\frac{n}{2}} \\
a_{n,1} & & & \dots & & a_{n,\frac{n}{2}} &
\end{array}$$

case n even

The entries

- strictly increase along rows, i.e.
$$a_{i,j} < a_{i,j+1},$$
- weakly increase along northeast and southeast direction, i.e.
$$a_{i+1,j} \le a_{i,j} \le a_{i+1,j+1},$$
- do not exceed a given parameter x, i.e.
$$a_{i,j} \le x.$$

We denote the set of halved monotone triangles with n rows, prescribed bottom row $(k_1, \dots, k_{\lceil \frac{n}{2} \rceil})$ and no entry larger than x by $\mathcal{HMT}\left(n, x; k_1, \dots, k_{\lceil \frac{n}{2} \rceil}\right)$.

By introducing the notion of halved monotone triangles, we generalise the problem of enumerating vertically symmetric alternating sign matrices. Vertically symmetric alternating sign matrices of order $2n+1$ are in a one-to-one correspondence with halved monotone triangles with $2n$ rows, prescribed bottom row $(1, 2, \dots, n)$ and no entry larger than n.

Like in section 2.2, let us introduce a function γ to count the number of halved monotone triangles. For given $(k_1, \dots, k_n, x) \in \mathbb{Z}^{n+1}$ first set $\gamma(1; k_1) := 1$. Then, using the ordinary summation operator, we define inductively:

$$\gamma(n, x; k_1, \dots, k_{\frac{n+1}{2}}) := \sum_{(l_1, \dots, l_{\frac{n-1}{2}})}^{(k_1, \dots, k_{\frac{n+1}{2}})} \gamma(n-1; l_1, \dots, l_{\frac{n-1}{2}}) \qquad \text{for odd } n \text{ and}$$

$$\gamma(n, x; k_1, \dots, k_{\frac{n}{2}}) := \sum_{(l_1, \dots, l_{\frac{n}{2}})}^{(k_1, \dots, k_{\frac{n}{2}}, x)} \gamma(n-1; l_1, \dots, l_{\frac{n}{2}}) \qquad \text{for even } n.$$

If $k_1 < \dots < k_{\lceil \frac{n}{2} \rceil} \le x$, then $\gamma(n, x; k_1, \dots, k_{\lceil \frac{n}{2} \rceil})$ enumerates the number of halved monotone triangles with n rows, prescribed bottom row $(k_1, \dots, k_{\lceil \frac{n}{2} \rceil})$ and no entry larger than x.

Example 25. *We want to derive the formula for $\gamma(3,x;k_1,k_2)$ like in example 9:*

- $\gamma(1,x;k_1)=1$.
- $\gamma(2,x;k_1)=\sum\limits_{(l_1)}^{(k_1,x)}\gamma(1,x;l_1)=x-k_1+1$.
- $\gamma(3,x;k_1,k_2)=\sum\limits_{(l_1)}^{(k_1,x)}\gamma(2,x;l_1)=\sum_{l_1=k_1}^{k_2}(x-l_1+1)$
  $=(k_2-k_1+1)(x+1)-\left(\binom{k_2+1}{2}-\binom{k_1}{2}\right)$.

It follows that

$$\gamma(3,x;k_1,k_2)=1+x-\frac{3}{2}k_1-xk_1+\frac{1}{2}k_1^2+\frac{1}{2}k_2+xk_2-\frac{1}{2}k_2^2.$$

We compute $\gamma(3,4;1,3)=9$. The are indeed nine corresponding HMTs:

$$\begin{smallmatrix}&&1\\&1&\\1&&3\end{smallmatrix},\quad \begin{smallmatrix}&&2\\&1&\\1&&3\end{smallmatrix},\quad \begin{smallmatrix}&&3\\&1&\\1&&3\end{smallmatrix},\quad \begin{smallmatrix}&&4\\&1&\\1&&3\end{smallmatrix},\quad \begin{smallmatrix}&&2\\&2&\\1&&3\end{smallmatrix},\quad \begin{smallmatrix}&&3\\&2&\\1&&3\end{smallmatrix},\quad \begin{smallmatrix}&&4\\&2&\\1&&3\end{smallmatrix},\quad \begin{smallmatrix}&&3\\&3&\\1&&3\end{smallmatrix},\quad \begin{smallmatrix}&&4\\&3&\\1&&3\end{smallmatrix}.$$

Fischer proved in [Fis09] the following formula for $\gamma(n,x;k_1,\dots,k_{\lceil\frac{n}{2}\rceil})$.

Theorem 26 ([Fis09], Theorem 1)**.** *If n is odd, then*

$$\gamma(n,x;k_1,\dots,k_{\frac{n+1}{2}})=\prod_{1\le s<t\le\frac{n+1}{2}}(\mathrm{id}-\mathrm{E}_{k_s}+\mathrm{E}_{k_s}\mathrm{E}_{k_t})(\mathrm{E}_{k_s}^{-1}+\mathrm{E}_{k_t}^{-1}-\mathrm{id})$$
$$\times\prod_{1\le i<j\le\frac{n+1}{2}}\frac{(k_j-k_i)(2x+1-k_j-k_i)}{(j-i)(j+i-1)}$$

and if n is even, then

$$\gamma(n,x;k_1,\dots,k_{\frac{n}{2}})=\prod_{1\le s<t\le\frac{n}{2}}(\mathrm{id}-\mathrm{E}_{k_s}+\mathrm{E}_{k_s}\mathrm{E}_{k_t})(\mathrm{E}_{k_s}^{-1}+\mathrm{E}_{k_t}^{-1}-\mathrm{id})$$
$$\times\prod_{1\le i<j\le\frac{n}{2}}\frac{(k_j-k_i)(2x+2-k_j-k_i)}{(j-i)(j+i)}\prod_{i=1}^{\frac{n}{2}}\frac{x+1-k_i}{i}.$$

Later on, Theorem 28 will provide a generalised formula with respect to a special Q-weight. Then Theorem 26 will follow as a simple corollary by setting $Q=1$.

3.2 The Q-Enumeration of Halved Monotone Triangles

We have seen in theorem 13 that an entry -1 in an alternating sign matrix corresponds to an entry $a_{i,j}$ in the associated monotone triangle with $i\neq1$ which is strictly bigger than its southwest and strictly smaller than its southeast neighbour, i.e. $a_{i+1,j}<a_{i,j}<a_{i+1,j+1}$. This has lead us to an Q-enumeration of monotone triangles. We would similarly like to enumerate vertically symmetric alternating sign matrices with respect to the entries -1 and therefore establish a Q-enumeration of halved monotone triangles.

We need to clarify how to define the Q-weight of a halved monotone triangle. Given a halved monotone triangle, we associate the indeterminant Q to an entry with a southeast and a southwest neighbour if it lies strictly in between them. If it has only a southwest neighbour, we associate Q if the entry is strictly bigger than this neighbour. We then define the Q-*weight* of the halved monotone triangle as the product of all indeterminants Q associated to some entries of the halved monotone triangle.
Our goal in this chapter is a formula for the generating function of halved monotone triangles with respect to their Q-weight, i.e. if $Q(\gamma)$ denotes the Q-weight of a halved monotone triangle γ, we would like to get a formula for

$$\sum_{\gamma \in \mathcal{HMT}(n,x;k_1,\ldots,k_{\lceil \frac{n}{2} \rceil})} Q(\gamma).$$

Example 27. *We have seen in example 25 that $\gamma(3,4;1,3) = 9$. We want to compute the Q-generating function:*

γ	$\begin{smallmatrix} & & 1 \\ & 1 & \\ 1 & & 3 \end{smallmatrix}$,	$\begin{smallmatrix} & & 2 \\ & 1 & \\ 1 & & 3 \end{smallmatrix}$,	$\begin{smallmatrix} & & 3 \\ & 1 & \\ 1 & & 3 \end{smallmatrix}$,	$\begin{smallmatrix} & & 4 \\ & 1 & \\ 1 & & 3 \end{smallmatrix}$,	$\begin{smallmatrix} & & 2 \\ & 2 & \\ 1 & & 3 \end{smallmatrix}$,	$\begin{smallmatrix} & & 3 \\ & 2 & \\ 1 & & 3 \end{smallmatrix}$,	$\begin{smallmatrix} & & 4 \\ & 2 & \\ 1 & & 3 \end{smallmatrix}$,	$\begin{smallmatrix} & & 3 \\ & 3 & \\ 1 & & 3 \end{smallmatrix}$,	$\begin{smallmatrix} & & 4 \\ & 3 & \\ 1 & & 3 \end{smallmatrix}$.
$Q(\gamma)$	1	Q	Q	Q	Q	Q^2	Q^2	1	Q

The Q-generating function is $2 + 5Q + 2Q^2$.

For given integers $(k_1, \ldots, k_n, x) \in \mathbb{Z}^{n+1}$ we set ${}^{Q}\gamma(1; k_1) := 1$ and define recursively

$$ {}^{Q}\gamma(n,x;k_1,\ldots,k_{\frac{n+1}{2}}) := \sum_{(l_1,\ldots,l_{\frac{n-1}{2}})}^{(k_1,\ldots,k_{\frac{n+1}{2}})} {}^{Q}\gamma(n-1;l_1,\ldots,l_{\frac{n-1}{2}}) \qquad \text{for odd } n \text{ and}$$

$$ {}^{Q}\gamma(n,x;k_1,\ldots,k_{\frac{n}{2}}) := \sum_{(l_1,\ldots,l_{\frac{n}{2}})}^{(k_1,\ldots,k_{\frac{n}{2}},x)} {}^{Q}\gamma(n-1;l_1,\ldots,l_{\frac{n}{2}}) \qquad \text{for even } n.$$

Note that we use the generalised summation operator for this definition and throughout this section. If $k_1 < \cdots < k_{\lceil \frac{n}{2} \rceil} \le x$, then ${}^{Q}\gamma(n,x;k_1,\ldots,k_{\lceil \frac{n}{2} \rceil})$ gives the Q-generating function for halved monotone triangles with n rows, prescribed bottom row $(k_1,\ldots,k_{\lceil \frac{n}{2} \rceil})$ and no entry larger than x. The following theorem provides an explicit formula:

Theorem 28. *Let n be a positive integer and $(k_1, k_2, \ldots, k_n, x) \in \mathbb{Z}^{n+1}$. Then the function ${}^{Q}\gamma(n,x;k_1,\ldots,k_{\lceil \frac{n}{2} \rceil})$ is given by*

$$\prod_{1 \le s < t \le \frac{n+1}{2}} \left(\mathrm{id} - (2-Q)\,\mathrm{E}_{k_s} + \mathrm{E}_{k_s}\mathrm{E}_{k_t}\right)\left(\mathrm{E}_{k_s}^{-1} + \mathrm{E}_{k_t}^{-1} - (2-Q)\,\mathrm{id}\right)$$
$$\times \prod_{1 \le i < j \le \frac{n+1}{2}} \frac{(k_j - k_i)(2x+1-k_j-k_i)}{(j-i)(j+i-1)}$$

if $n \in \mathbb{N}$ is odd and by

$$\prod_{r=1}^{\frac{n}{2}} \left((Q-1)\,\mathrm{id} + \mathrm{E}_{k_r}^{-1} \right) \prod_{1 \le s < t \le \frac{n}{2}} (\mathrm{id} - (2-Q)\,\mathrm{E}_{k_s} + \mathrm{E}_{k_s}\mathrm{E}_{k_t}) \left(\mathrm{E}_{k_s}^{-1} + \mathrm{E}_{k_t}^{-1} - (2-Q)\,\mathrm{id} \right)$$
$$\times \prod_{1 \le i < j \le \frac{n}{2}} \frac{(k_j - k_i)(2x - k_j - k_i)}{(j-i)(j+i)} \prod_{i=1}^{\frac{n}{2}} \frac{x - k_i}{i}$$

if $n \in \mathbb{N}$ is even.

Before we start proving Theorem 28, we use Lemma 16 to rewrite the appearing operands as determinants.

Lemma 29. *If $n \in \mathbb{N}$ is odd, then*

$$\prod_{1 \le i < j \le \frac{n+1}{2}} \frac{(k_j - k_i)(2x + 1 - k_j - k_i)}{(j-i)(j+i-1)} = (-1)^{\binom{\frac{n+1}{2}}{2}} \det_{1 \le i,j \le \frac{n+1}{2}} \binom{k_i + j - x - 2}{2j-2}$$

and if $n \in \mathbb{N}$ is even, then

$$\prod_{1 \le i < j \le \frac{n}{2}} \frac{(k_j - k_i)(2x - k_j - k_i)}{(j-i)(j+i)} \prod_{i=1}^{\frac{n}{2}} \frac{x - k_i}{i} = (-1)^{\binom{\frac{n}{2}+1}{2}} \det_{1 \le i,j \le \frac{n}{2}} \binom{k_i + j - x - 1}{2j-1}$$

and

$$\prod_{1 \le i < j \le \frac{n}{2}} \frac{(k_j - k_i)(2x + 2 - k_j - k_i)}{(j-i)(j+i)} \prod_{i=1}^{\frac{n}{2}} \frac{x + 1 - k_i}{i} = (-1)^{\binom{\frac{n}{2}+1}{2}} \det_{1 \le i,j \le \frac{n}{2}} \binom{k_i + j - x - 2}{2j-1}.$$

Proof. First we prove by induction on j that

$$\binom{k_i + j - 1}{2j-1} = \frac{k_i}{(2j-1)!} \prod_{r=1}^{j-1} \left(k_i^2 - r^2 \right). \tag{3.1}$$

The base case $j = 1$ is clear because $k_i = \binom{k_i}{1} = \frac{k_i}{1!} \cdot 1$. The inductive step $j \to j+1$ is the following:

$$\binom{k_i + (j+1) - 1}{2(j+1) - 1} = \binom{k_i + j}{2j+1} = \frac{(k_i + j)\left((k_i + j - 1) \cdots (k_i - j + 1)\right)(k_i - j)}{(2j+1)(2j)(2j-1)!}$$
$$= \frac{(k_i + j)(k_i - j)}{(2j+1)(2j)} \cdot \frac{k_i}{(2j-1)!} \prod_{r=1}^{j-1} \left(k_i^2 - r^2 \right) = \frac{k_i}{(2j+1)!} \prod_{r=1}^{j} \left(k_i^2 - r^2 \right).$$

From equation (3.1) it follows that

$$\det_{1 \le i,j \le n} \binom{k_i + j - 1}{2j-1} = \prod_{j=1}^{n} \frac{1}{(2j-1)!} \prod_{i=1}^{n} k_i \det_{1 \le i,j \le n} \left(\prod_{r=1}^{j-1} \left(k_i^2 - r^2 \right) \right).$$

By using the generalised Vandermonde determinant evaluation in Lemma 16 and the fact that

$$\prod_{j=1}^{n} \frac{1}{j} \prod_{1\le i<j\le n} \frac{1}{(j-i)(j+i)} = \prod_{j=1}^{n} \frac{1}{j} \prod_{j=2}^{n} \prod_{i=1}^{j-1} \frac{1}{(j-i)(j+i)}$$
$$= \prod_{j=1}^{n} \frac{1}{j} \prod_{j=2}^{n} \frac{1}{1\cdots(j-1)(j+1)\cdots(2j-1)} = \prod_{j=1}^{n} \frac{1}{(2j-1)!}$$

we finally get

$$\det_{1\le i,j\le n} \binom{k_i+j-1}{2j-1} = \prod_{1\le i<j\le n} \frac{(k_j-k_i)(k_j+k_i)}{(j-i)(j+i)} \prod_{i=1}^{n} \frac{k_i}{i}.$$

If we replace k_i by $k_i - x$ or $k_i - x - 1$, we get the two formulae in the even case of the lemma.
For the odd case, we can similarly prove that

$$\binom{k_i+j-\frac{3}{2}}{2j-2} = \frac{1}{(2j-2)!} \prod_{r=1}^{j-1} \left(k_i^2 - (r-\frac{1}{2})^2\right)$$

and consequently by Lemma 16 it follows that

$$\det_{1\le i,j\le n} \binom{k_i+j-\frac{3}{2}}{2j-2} = \prod_{j=1}^{n} \frac{1}{(2j-2)!} \prod_{1\le i<j\le n} \left(k_j^2 - k_i^2\right).$$

Since $\prod_{j=1}^{n} \frac{1}{(2j-2)!} = \prod_{1\le i<j\le n} \frac{1}{(j-i)(j+i-1)}$, it follows that

$$\det_{1\le i,j\le n} \binom{k_i+j-\frac{3}{2}}{2j-2} = \prod_{1\le i<j\le n} \frac{(k_j-k_i)(k_j+k_i)}{(j-i)(j+i-1)}.$$

The assertion in the lemma is now obtained by the substitution $k_i \mapsto k_i - x - \frac{1}{2}$. □

Using the operators and Lemma 29 for the determinants, we can formulate Theorem 28 in a more abbreviative way, namely if n is odd, then

$$\begin{aligned} {}^{Q}\gamma\left(n, x; k_1, \ldots, k_{\frac{n+1}{2}}\right) &= Q^{-\binom{\frac{n+1}{2}}{2}} \prod_{1\le s<t\le \frac{n+1}{2}} {}^{Q}\mathrm{W}_{k_t,k_s}\, {}^{Q}\mathrm{I}_{k_s,k_t} \\ &\quad \times (-1)^{\binom{\frac{n+1}{2}}{2}} \det_{1\le i,j\le \frac{n+1}{2}} \binom{k_i+j-x-2}{2j-2} \end{aligned} \tag{3.2}$$

and if n is even, then

$$\begin{aligned} {}^{Q}\gamma\left(n, x; k_1, \ldots, k_{\frac{n}{2}}\right) &= Q^{-\binom{\frac{n}{2}}{2}} \prod_{r=1}^{\frac{n}{2}} {}^{Q}\mathrm{E}_{k_r}^{-1} \prod_{1\le s<t\le \frac{n}{2}} {}^{Q}\mathrm{W}_{k_t,k_s}\, {}^{Q}\mathrm{I}_{k_s,k_t} \\ &\quad \times (-1)^{\binom{\frac{n}{2}+1}{2}} \det_{1\le i,j\le \frac{n}{2}} \binom{k_i+j-x-1}{2j-1}. \end{aligned} \tag{3.3}$$

Our goal is to prove the formulae for the Q-generating functions by induction. Therefore, it is necessary to understand how the summation operator interacts with the operators used in Theorem 28. The next two lemmas – for the odd and the even case – provide sufficient insight. They are a generalisation of [Rie14, Lemmas 5.2.2 & 5.2.3].

Lemma 30. *Let $n \in \mathbb{N}$ be odd. Then*

$$\sum_{(l_1,\dots,l_{\frac{n-1}{2}})}^{(k_1,\dots,k_{\frac{n+1}{2}})} \prod_{r=1}^{\frac{n-1}{2}} {}^Q\mathrm{E}_{l_r}^{-1} \prod_{1\le s<t\le \frac{n-1}{2}} {}^Q\mathrm{W}_{l_t,l_s}\, {}^Q\mathrm{I}_{l_s,l_t} \det_{1\le i,j\le \frac{n-1}{2}} \binom{l_i+j-x-1}{2j-1}$$
$$= Q^{-\frac{n-1}{2}} \prod_{1\le s<t\le \frac{n+1}{2}} {}^Q\mathrm{W}_{k_t,k_s}\, {}^Q\mathrm{I}_{k_s,k_t} \det_{1\le i,j\le \frac{n+1}{2}} \binom{k_i+j-x-2}{2j-2}.$$

Proof. In order to prove this statement, we want to use Lemma 17. Therefore, we set $G(l_1,\dots,l_{\frac{n-1}{2}}) := \prod_{r=1}^{\frac{n-1}{2}} {}^Q\mathrm{E}_{l_r}^{-1} \prod_{1\le s<t\le\frac{n-1}{2}} {}^Q\mathrm{W}_{l_t,l_s}\, {}^Q\mathrm{I}_{l_s,l_t} \det_{1\le i,j\le\frac{n-1}{2}} \binom{l_i+j-x-1}{2j}$.
Note that the operator

$${}^Q\mathrm{W}_{l_i,l_{i+1}} \prod_{r=1}^{\frac{n-1}{2}} {}^Q\mathrm{E}_{l_r}^{-1} \prod_{1\le s<t\le \frac{n-1}{2}} {}^Q\mathrm{W}_{l_t,l_s}\, {}^Q\mathrm{I}_{l_s,l_t}$$

is symmetric in l_i and l_{i+1}. Therefore the swap operator $S_{l_i,l_{i+1}}$ applied to the operator ${}^Q\mathrm{W}_{l_i,l_{i+1}}\, G(l_1,\dots,l_{\frac{n-1}{2}})$ simply interchanges the row i and $i+1$ in the determinant. Thus, it holds that

$$S_{l_i,l_{i+1}}\, {}^Q\mathrm{W}_{l_i,l_{i+1}}\, G(l_1,\dots,l_{\frac{n-1}{2}}) = -\, {}^Q\mathrm{W}_{l_i,l_{i+1}}\, G(l_1,\dots,l_{\frac{n-1}{2}})$$

and so the function G fulfills the condition of Lemma 17.
If we keep in mind that $\Delta_x \binom{x}{n} = \binom{x}{n-1}$ for $n \ge 1$, then Lemma 17 yields

$$\sum_{(l_1,\dots,l_{\frac{n-1}{2}})}^{(k_1,\dots,k_{\frac{n+1}{2}})} \prod_{r=1}^{\frac{n-1}{2}} {}^Q\mathrm{E}_{l_r}^{-1} \prod_{1\le s<t\le \frac{n-1}{2}} {}^Q\mathrm{W}_{l_t,l_s}\, {}^Q\mathrm{I}_{l_s,l_t} \det_{1\le i,j\le \frac{n-1}{2}} \binom{l_i+j-x-1}{2j-1}$$
$$= \sum_{(l_1,\dots,l_{\frac{n-1}{2}})}^{(k_1,\dots,k_{\frac{n+1}{2}})} \prod_{r=1}^{\frac{n-1}{2}} {}^Q\mathrm{E}_{l_r}^{-1} \prod_{1\le s<t\le \frac{n-1}{2}} {}^Q\mathrm{W}_{l_t,l_s}\, {}^Q\mathrm{I}_{l_s,l_t} \det_{1\le i,j\le \frac{n-1}{2}} \left(\Delta_{l_i} \binom{l_i+j-x-1}{2j}\right)$$
$$= \sum_{(l_1,\dots,l_{\frac{n-1}{2}})}^{(k_1,\dots,k_{\frac{n+1}{2}})} \prod_{i=1}^{\frac{n-1}{2}} \Delta_{l_i}\, G(l_1,\dots,l_{\frac{n-1}{2}})$$
$$= \sum_{r=1}^{\frac{n+1}{2}} (-1)^{r-1} \prod_{s=1}^{r-1} {}^Q\mathrm{id}_{k_s} \prod_{t=r+1}^{\frac{n+1}{2}} {}^Q\mathrm{E}_{k_t}\, G(k_1,\dots,\widehat{k_r},\dots,k_n)$$

$$= \sum_{r=1}^{\frac{n+1}{2}} (-1)^{r-1} \prod_{s=1}^{r-1} {}^Q\mathrm{id}_{k_s} \prod_{t=r+1}^{\frac{n+1}{2}} {}^Q\mathrm{E}_{k_t} \prod_{\substack{u=1\\u\neq r}}^{\frac{n+1}{2}} {}^Q\mathrm{E}_{k_u}^{-1}$$
$$\times \prod_{\substack{1\leq s<t\leq \frac{n-1}{2}\\ s,t\neq r}} {}^Q\mathrm{W}_{k_t,k_s}\, {}^Q\mathrm{I}_{k_s,k_t} \det_{1\leq i,j\leq \frac{n-1}{2}} \binom{l_i+j-x-1}{2j}\Bigg|_{(l_1,\dots,l_{\frac{n-1}{2}})=(k_1,\dots,\widehat{k_r},\dots,k_{\frac{n+1}{2}})}.$$

We observe, similar to the proof of Lemma 18, that k_r does not occur in the determinant $\det_{1\leq i,j\leq \frac{n-1}{2}} \binom{l_i+j-x-1}{2j}\Big|_{(l_1,\dots,l_{\frac{n-1}{2}})=(k_1,\dots,\widehat{k_r},\dots,k_{\frac{n+1}{2}})}$. Therefore the operators ${}^Q\mathrm{W}_{k_t,k_r}$, ${}^Q\mathrm{W}_{k_r,k_s}$, ${}^Q\mathrm{I}_{k_s,k_r}$ and ${}^Q\mathrm{I}_{k_r,k_t}$ applied to that determinant simplify as follows: $Q\,{}^Q\mathrm{E}_{k_t}$, $Q\,{}^Q\mathrm{id}_{k_s}$, ${}^Q\mathrm{E}_{k_s}^{-1}$ and ${}^Q\mathrm{E}_{k_t}^{-1}$.
Thus, we have

$$\sum_{r=1}^{\frac{n+1}{2}} (-1)^{r-1} \prod_{s=1}^{r-1} {}^Q\mathrm{id}_{k_s} \prod_{t=r+1}^{\frac{n+1}{2}} {}^Q\mathrm{E}_{k_t} \prod_{\substack{u=1\\u\neq r}}^{\frac{n+1}{2}} {}^Q\mathrm{E}_{k_u}^{-1}$$
$$\times \prod_{\substack{1\leq s<t\leq \frac{n-1}{2}\\ s,t\neq r}} {}^Q\mathrm{W}_{k_t,k_s}\, {}^Q\mathrm{I}_{k_s,k_t} \det_{1\leq i,j\leq \frac{n-1}{2}} \binom{l_i+j-x-1}{2j}\Bigg|_{(l_1,\dots,l_{\frac{n-1}{2}})=(k_1,\dots,\widehat{k_r},\dots,k_{\frac{n+1}{2}})}$$
$$= Q^{-\frac{n-1}{2}} \sum_{r=1}^{\frac{n+1}{2}} (-1)^{r-1} \prod_{1\leq s<t\leq \frac{n-1}{2}} {}^Q\mathrm{W}_{k_t,k_s}\, {}^Q\mathrm{I}_{k_s,k_t}$$
$$\times \det_{1\leq i,j\leq \frac{n-1}{2}} \binom{l_i+j-x-1}{2j}\Bigg|_{(l_1,\dots,l_{\frac{n-1}{2}})=(k_1,\dots,\widehat{k_r},\dots,k_{\frac{n+1}{2}})}.$$

If we expand the determinant $\det_{1\leq i,j\leq \frac{n+1}{2}} \binom{k_i+j-x-2}{2j-2}$ with respect to the first column, we get

$$\sum_{r=1}^{\frac{n+1}{2}} (-1)^{r-1} \det_{1\leq i,j\leq \frac{n-1}{2}} \binom{l_i+j-x-1}{2j}\Bigg|_{(l_1,\dots,l_{\frac{n-1}{2}})=(k_1,\dots,\widehat{k_r},\dots,k_{\frac{n+1}{2}})}.$$

This completes our proof. □

Lemma 31. *Let $n\in\mathbb{N}$ be even. Then*

$$\sum_{(l_1,\dots,l_{\frac{n}{2}})}^{(k_1,\dots,k_{\frac{n}{2}},x)} \prod_{1\leq s<t\leq \frac{n}{2}} {}^Q\mathrm{W}_{l_t,l_s}\, {}^Q\mathrm{I}_{l_s,l_t} (-1)^{\binom{\frac{n}{2}}{2}} \det_{1\leq i,j\leq \frac{n}{2}} \binom{l_i+j-x-2}{2j-2}$$
$$= \prod_{r=1}^{\frac{n}{2}} {}^Q\mathrm{E}_{k_r}^{-1} \prod_{1\leq s<t\leq \frac{n}{2}} {}^Q\mathrm{W}_{k_t,k_s}\, {}^Q\mathrm{I}_{k_s,k_t} (-1)^{\binom{\frac{n}{2}+1}{2}} \det_{1\leq i,j\leq \frac{n}{2}} \binom{k_i+j-x-1}{2j-1}.$$

Proof. We set $G_x(l_1,\dots,l_{\frac{n}{2}}) := \prod_{1\le s<t\le \frac{n}{2}} {}^Q\mathrm{W}_{l_t,l_s}\,{}^Q\mathrm{I}_{l_s,l_t} \det_{1\le i,j\le \frac{n}{2}} \binom{l_i+j-x-2}{2j-1}$. Similarly to the proof of Lemma 30, we see that this function fulfills the condition required by Lemma 17 since

$$ {}^Q\mathrm{W}_{l_i,l_{i+1}} \prod_{1\le s<t\le \frac{n}{2}} {}^Q\mathrm{W}_{l_t,l_s}\,{}^Q\mathrm{I}_{l_s,l_t} $$

is symmetric in l_i and l_{i+1}.
It follows by Lemma 17 that

$$ \begin{aligned} \sum_{(l_1,\dots,l_{\frac{n}{2}})}^{(k_1,\dots,k_{\frac{n}{2}},x)} \prod_{1\le s<t\le \frac{n}{2}} &{}^Q\mathrm{W}_{l_t,l_s}\,{}^Q\mathrm{I}_{l_s,l_t}(-1)^{\binom{\frac{n}{2}}{2}} \det_{1\le i,j\le \frac{n}{2}} \binom{l_i+j-x-2}{2j-2} \\ &= (-1)^{\binom{\frac{n}{2}}{2}} \sum_{(l_1,\dots,l_{\frac{n}{2}})}^{(k_1,\dots,k_{\frac{n}{2}},x)} \prod_{i=1}^{\frac{n}{2}} \Delta_{l_i} G_x(l_1,\dots,l_{\frac{n}{2}}) \\ &= \sum_{r=1}^{\frac{n}{2}} (-1)^{r-1+\binom{\frac{n}{2}}{2}} \prod_{s=1}^{r-1} {}^Q\mathrm{id}_{k_s} \prod_{t=r+1}^{\frac{n}{2}} {}^Q\mathrm{E}_{k_t}\,{}^Q\mathrm{E}_x\, G_x(k_1,\dots,\widehat{k_r},\dots,k_{\frac{n}{2}},x) \\ &\qquad + (-1)^{\frac{n}{2}+\binom{\frac{n}{2}}{2}} \prod_{s=1}^{\frac{n}{2}} {}^Q\mathrm{id}_{k_s}\, G_x(k_1,\dots,k_{\frac{n}{2}}). \end{aligned} $$

If we plug in the definition of G_x and extract the factor $\prod_{r=1}^{\frac{n}{2}} \mathrm{E}_{k_r}^{-1}$, the second summand becomes

$$ \prod_{s=1}^{\frac{n}{2}} {}^Q\mathrm{id}_{k_s} \prod_{r=1}^{\frac{n}{2}} \mathrm{E}_{k_r}^{-1} \prod_{1\le s<t\le \frac{n}{2}} {}^Q\mathrm{W}_{l_t,l_s}\,{}^Q\mathrm{I}_{l_s,l_t}(-1)^{\binom{\frac{n}{2}+1}{2}} \det_{1\le i,j\le \frac{n}{2}} \binom{k_i+j-x-1}{2j-1}. $$

Because of ${}^Q\mathrm{id}_y\,\mathrm{E}_y^{-1} = ((Q-1)\,\mathrm{E}_y + \mathrm{id})\,\mathrm{E}_y^{-1} = (Q-1)\,\mathrm{id} + \mathrm{E}_y^{-1} = {}^Q\mathrm{E}_y^{-1}$, this is exactly the right-hand side of the statement.
We complete our proof by showing that the first summand is equal to zero, so

$$ \sum_{r=1}^{\frac{n}{2}} (-1)^{r-1+\binom{\frac{n}{2}}{2}} \prod_{s=1}^{r-1} {}^Q\mathrm{id}_{k_s} \prod_{t=r+1}^{\frac{n}{2}} {}^Q\mathrm{E}_{k_t}\,{}^Q\mathrm{E}_x\, G_x(k_1,\dots,\widehat{k_r},\dots,k_{\frac{n}{2}},x) $$

should vanish. This is done similarly to the proof of Lemma 5.2.3 in [Rie14].
First we split the summand up into a sum of two parts by using ${}^Q\mathrm{E}_x = (Q-1)\,\mathrm{id} + \mathrm{E}_x$:

$$ (Q-1)\sum_{r=1}^{\frac{n}{2}} (-1)^{r-1+\binom{\frac{n}{2}}{2}} \prod_{s=1}^{r-1} {}^Q\mathrm{id}_{k_s} \prod_{t=r+1}^{\frac{n}{2}} {}^Q\mathrm{E}_{k_t}\, G_x(k_1,\dots,\widehat{k_r},\dots,k_{\frac{n}{2}},x) \tag{3.4} $$

$$ +\sum_{r=1}^{\frac{n}{2}} (-1)^{r-1+\binom{\frac{n}{2}}{2}} \prod_{s=1}^{r-1} {}^Q\mathrm{id}_{k_s} \prod_{t=r+1}^{\frac{n}{2}} {}^Q\mathrm{E}_{k_t}\, G_x(k_1,\dots,\widehat{k_r},\dots,k_{\frac{n}{2}},x+1). \tag{3.5} $$

Considering the term (3.5), it suffices to show that $G_x(l_1,\dots,l_{\frac{n}{2}-1},x+1) = 0$. Due to the definition of the function G_x and Lemma 29, this is true if

$$ \prod_{s=1}^{\frac{n}{2}-1} {}^Q\mathrm{W}_{l_{\frac{n}{2}},l_s}\,{}^Q\mathrm{I}_{l_s,l_{\frac{n}{2}}} \left(x+1-l_{\frac{n}{2}}\right) \prod_{i=1}^{\frac{n}{2}-1} \left(l_{\frac{n}{2}}-l_i\right)\left(2x+2-l_{\frac{n}{2}}-l_i\right)\Bigg|_{l_{\frac{n}{2}}=x+1} = 0. $$

Moreover, it holds for the operators that

$$\begin{aligned}
&{}^{Q}\mathrm{W}_{y,z}\,{}^{Q}\mathrm{I}_{z,y}\\
&\quad= Q(\mathrm{id}-(2-Q)\,\mathrm{E}_z+\mathrm{E}_z\,\mathrm{E}_y)(\mathrm{E}_z^{-1}+\mathrm{E}_y^{-1}-\mathrm{id})\\
&\quad= Q\Big(\mathrm{E}_z^{-1}+\mathrm{E}_y^{-1}-\mathrm{id}\\
&\qquad-(2-Q)\,\mathrm{id}+-(2-Q)\,\mathrm{E}_z\,\mathrm{E}_y^{-1}+(2-Q)^2\,\mathrm{E}_z\\
&\qquad\mathrm{E}_y+\mathrm{E}_z-(2-Q)\,\mathrm{E}_z\,\mathrm{E}_y\Big)\\
&\quad= Q\left((5-4Q+Q^2)\,\mathrm{E}_z+\mathrm{E}_z^{-1}-2(2-Q)\,\mathrm{id}+(\mathrm{id}-(2-Q)\,\mathrm{E}_z)(\mathrm{E}_y+\mathrm{E}_y^{-1})\right).
\end{aligned} \tag{3.6}$$

Therefore it is enough to show that for all $M\in\mathbb{N}_0$ it holds that

$$\left(\mathrm{E}_y+\mathrm{E}_y^{-1}\right)^M(x+1-y)\prod_{i=1}^{\frac{n}{2}-1}(y-l_i)\,(2x+2-y-l_i)\Big|_{y=x+1}=0.$$

By expanding $\left(\mathrm{E}_y+\mathrm{E}_y^{-1}\right)^M$ using the binomial formula, this is equivalent to

$$\begin{aligned}
&\sum_{m=0}^{M}\binom{M}{m}\mathrm{E}_y^{2m-M}(x+1-y)\prod_{i=1}^{\frac{n}{2}-1}(y-l_i)\,(2x+2-y-l_i)\Big|_{y=x+1}\\
&\quad=\sum_{m=0}^{M}\binom{M}{m}(M-2m)\prod_{i=1}^{\frac{n}{2}-1}(x+1+2m-M-l_i)\,(x+1-2m+M-l_i)\\
&\qquad\qquad=\sum_{m=0}^{M}\binom{M}{m}(M-2m)\prod_{i=1}^{\frac{n}{2}-1}\left((x+1-l_i)^2-(M-2m)^2\right)=0.
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{m=0}^{M}\binom{M}{m}(M-2m)^{2N+1}&=\sum_{m=0}^{M}\binom{M}{M-m}(M-2(M-m))^{2N+1}\\
&=-\sum_{m=0}^{M}\binom{M}{m}(M-2m)^{2N+1},
\end{aligned}$$

it is true for any $M,N\in\mathbb{N}_0$ that

$$\sum_{m=0}^{M}\binom{M}{m}(M-2m)^{2N+1}=0,$$

which finally proves that the term (3.5) is zero.
For the proof that the term (3.4) is equal to zero, too, we rewrite it by extracting the factor $\prod_{r=1}^{\frac{n}{2}}\mathrm{E}_{k_r}^{-1}$ from G_x and therefore it suffices to show that

$$\prod_{s=1}^{\frac{n}{2}-1}{}^{Q}\mathrm{W}_{l_{\frac{n}{2}},l_s}\,{}^{Q}\mathrm{I}_{l_s,l_{\frac{n}{2}}}\left(x-l_{\frac{n}{2}}\right)\prod_{i=1}^{\frac{n}{2}-1}\left(l_{\frac{n}{2}}-l_i\right)\left(2x-l_{\frac{n}{2}}-l_i\right)\Big|_{l_{\frac{n}{2}}=x}=0. \tag{3.7}$$

If we replace x bx $x+1$, we get exactly the case which we have just proved. So equation (3.7) holds as well and our proof of the lemma is completed. □

By now, we have seen how the generalised summation operator is defined and what happens if it is applied to some special classes of operator formulae. This enables us to finally proof our main theorem:

Proof of Theorem 28. We prove the main theorem by induction on $n \in \mathbb{N}$ distinguishing the two cases whether n is odd or even.
The base case $n = 1$ is

$$\begin{aligned}1 &= {}^{Q}\gamma\,(1, x; k_1)\\ &= Q^{-\binom{1}{2}} \prod_{1\le s<t\le 1} {}^{Q}\mathrm{W}_{k_t,k_s}\,{}^{Q}\mathrm{I}_{k_s,k_t} \prod_{1\le i<j\le 1} \frac{(k_j-k_i)(2x+1-k_j-k_i)}{(j-i)(j+i-1)}\\ &\hspace{20em}= Q^0\cdot 1\cdot 1 = 1\end{aligned}$$

and for $n = 2$ it is

$$\begin{aligned}1 + Q\,(x-k_1) &= {}^{Q}\gamma\,(2, x; k_1)\\ &= Q^{-\binom{1}{2}}\,{}^{Q}\mathrm{E}_{k_1}^{-1}\left(\prod_{1\le s<t\le 1} {}^{Q}\mathrm{W}_{k_t,k_s}\,{}^{Q}\mathrm{I}_{k_s,k_t} \prod_{1\le i<j\le 1} \frac{(k_j-k_i)(2x-k_j-k_i)}{(j-i)(j+i)}\cdot\frac{x-k_1}{1}\right)\\ &= (Q\,\mathrm{id}-\delta_{k_1})\,(x-k_1) = 1 + Q\,(x-k_1).\end{aligned}$$

For the inductive step $n-1 \to n$ for odd n, it holds that

$$\begin{aligned}{}^{Q}\gamma\left(n, x; k_1, \ldots, k_{\frac{n+1}{2}}\right) &= \sum_{(l_1,\ldots,l_{\frac{n-1}{2}})}^{(k_1,\ldots,k_{\frac{n+1}{2}})} {}^{Q}\gamma\left(n-1, x; l_1, \ldots, l_{\frac{n-1}{2}}\right)\\ = \sum_{(l_1,\ldots,l_{\frac{n-1}{2}})}^{(k_1,\ldots,k_{\frac{n+1}{2}})} Q^{-\binom{\frac{n-1}{2}}{2}}&(-1)^{\binom{\frac{n+1}{2}}{2}} \prod_{r=1}^{\frac{n-1}{2}} {}^{Q}\mathrm{E}_{l_r}^{-1} \prod_{1\le s<t\le\frac{n-1}{2}} {}^{Q}\mathrm{W}_{l_t,l_s}\,{}^{Q}\mathrm{I}_{l_s,l_t} \det_{1\le i,j\le\frac{n-1}{2}} \binom{l_i+j-x-1}{2j-1}\\ &= Q^{-\binom{\frac{n-1}{2}}{2}} Q^{-\frac{n-1}{2}} \prod_{1\le s<t\le\frac{n+1}{2}} {}^{Q}\mathrm{W}_{k_t,k_s}\,{}^{Q}\mathrm{I}_{k_s,k_t}(-1)^{\binom{\frac{n+1}{2}}{2}} \det_{1\le i,j\le\frac{n+1}{2}} \binom{k_i+j-x-2}{2j-2}\\ &= Q^{-\binom{\frac{n+1}{2}}{2}} \prod_{1\le s<t\le\frac{n+1}{2}} {}^{Q}\mathrm{W}_{k_t,k_s}\,{}^{Q}\mathrm{I}_{k_s,k_t}(-1)^{\binom{\frac{n+1}{2}}{2}} \det_{1\le i,j\le\frac{n+1}{2}} \binom{k_i+j-x-2}{2j-2}.\end{aligned}$$

The inductive step $n-1 \to n$ for even n is

$$\begin{aligned}{}^{Q}\gamma\left(n, x; k_1, \ldots, k_{\frac{n}{2}}\right) &= \sum_{(l_1,\ldots,l_{\frac{n}{2}})}^{(k_1,\ldots,k_{\frac{n}{2}},x)} {}^{Q}\gamma\left(n-1, x; l_1, \ldots, l_{\frac{n}{2}}\right)\\ &= \sum_{(l_1,\ldots,l_{\frac{n}{2}})}^{(k_1,\ldots,k_{\frac{n}{2}},x)} Q^{-\binom{\frac{n}{2}}{2}} \prod_{1\le s<t\le\frac{n}{2}} {}^{Q}\mathrm{W}_{l_t,l_s}\,{}^{Q}\mathrm{I}_{l_s,l_t}(-1)^{\binom{\frac{n}{2}}{2}} \det_{1\le i,j\le\frac{n}{2}} \binom{l_i+j-x-2}{2j-2}\\ &= Q^{-\binom{\frac{n}{2}}{2}} \prod_{r=1}^{\frac{n}{2}} {}^{Q}\mathrm{E}_{k_r}^{-1} \prod_{1\le s<t\le\frac{n}{2}} {}^{Q}\mathrm{W}_{l_t,l_s}\,{}^{Q}\mathrm{I}_{l_s,l_t}(-1)^{\binom{\frac{n}{2}+1}{2}} \det_{1\le i,j\le\frac{n}{2}} \binom{k_i+j-x-1}{2j-1}.\end{aligned}$$

We have shown the identities (3.2) and (3.3). Thus, we have finally proved Theorem 28. □

3.3 The 2-Enumeration of Halved Monotone Triangles

We have seen in Section 2.5 that the 2-enumeration of monotone triangles can be written without operators. The same can be observed for the 2-enumeration of halved monotone triangles in the odd case:

Theorem 32. *If $n \in \mathbb{N}$ is odd and $k_1, k_2, \dots, k_{\frac{n+1}{2}}$ and x are integers, then*

$$ {}^2\gamma(n, x; k_1, \dots, k_{\frac{n+1}{2}}) = 4^{\binom{\frac{n+1}{2}}{2}} \prod_{1 \le i < j \le \frac{n+1}{2}} \frac{(k_j - k_i)(2x + 1 - k_j - k_i)}{(j - i)(j + i - 1)}. $$

Proof. Let $n > 0$ be odd. We evaluate the Q-enumeration ${}^Q\gamma(n, x; k_1, \dots, k_{\frac{n+1}{2}})$ at $Q = 2$:

$$ \begin{aligned} &{}^2\gamma(n, x; k_1, \dots, k_{\frac{n+1}{2}}) \\ &= \prod_{1 \le s < t \le \frac{n+1}{2}} (\mathrm{id} + \mathrm{E}_{k_s} \mathrm{E}_{k_t}) \left(\mathrm{E}_{k_s}^{-1} + \mathrm{E}_{k_t}^{-1} \right) \prod_{1 \le i < j \le \frac{n+1}{2}} \frac{(k_j - k_i)(2x + 1 - k_j - k_i)}{(j - i)(j + i - 1)} \\ &\quad = \prod_{1 \le s < t \le \frac{n+1}{2}} \left(\mathrm{E}_{k_s} + \mathrm{E}_{k_s}^{-1} + \mathrm{E}_{k_t} + \mathrm{E}_{k_t}^{-1} \right) \prod_{1 \le i < j \le \frac{n+1}{2}} \frac{(k_j - k_i)(2x + 1 - k_j - k_i)}{(j - i)(j + i - 1)}. \end{aligned} $$

By Lemma 29, we see that Theorem 32 is equivalent to the following statement:

$$ \begin{aligned} &\prod_{1 \le s < t \le \frac{n+1}{2}} \left(\mathrm{E}_{k_s} + \mathrm{E}_{k_s}^{-1} + \mathrm{E}_{k_t} + \mathrm{E}_{k_t}^{-1} - 4\,\mathrm{id} \right) \det_{1 \le i,j \le \frac{n+1}{2}} \binom{k_i + j - x - 2}{2j - 2} \\ &\qquad = \prod_{1 \le s < t \le \frac{n+1}{2}} (\Delta_{k_s} - \delta_{k_s} + \Delta_{k_t} - \delta_{k_t}) \det_{1 \le i,j \le \frac{n+1}{2}} \binom{k_i + j - x - 2}{2j - 2} = 0. \end{aligned} $$

Similarly to the proof of Lemma 19, it suffices to show that for nonnegative integers $(m_1, \dots, m_{\frac{n+1}{2}})$ with $m_i \neq 0$ for at least one $i \in \{1, \dots, \frac{n+1}{2}\}$ it holds that

$$ \sum_{\pi \in \mathfrak{S}_{\frac{n+1}{2}}} \prod_{i=1}^{\frac{n+1}{2}} (\Delta_{k_i} - \delta_{k_i})^{m_{\pi(i)}} \det_{1 \le i,j \le \frac{n+1}{2}} \binom{k_i + j - x - 2}{2j - 2} = 0. \tag{3.8} $$

The Binomial Theorem yields

$$ (\Delta_{k_i} - \delta_{k_i})^{m_{\pi(i)}} = \sum_{a_i = 0}^{m_{\pi(i)}} \binom{m_{\pi(i)}}{a_i} (-1)^{a_i} \Delta_{k_i}^{m_{\pi(i)} - a_i} \delta_{k_i}^{a_i}. $$

Thus, we are able to rewrite (3.8) as

$$\sum_{\pi\in\mathfrak{S}_{\frac{n+1}{2}}}\sum_{a_1=0}^{m_{\pi(1)}}\cdots\sum_{a_{\frac{n+1}{2}}=0}^{m_{\pi\left(\frac{n+1}{2}\right)}}(-1)^{a_1+\cdots+a_{\frac{n+1}{2}}}\Delta_{k_1}^{m_{\pi(1)}-a_1}\cdots\Delta_{k_{\frac{n+1}{2}}}^{m_{\pi\left(\frac{n+1}{2}\right)}-a_{\frac{n+1}{2}}}$$
$$\times\,\delta_{k_1}^{a_1}\cdots\delta_{k_{\frac{n+1}{2}}}^{a_{\frac{n+1}{2}}}\det_{1\le i,j\le\frac{n+1}{2}}\binom{k_i+j-x-2}{2j-2}=0.$$

If we keep in mind that $\Delta_x\binom{x}{n}=\binom{x}{n-1}$ and $\delta_x\binom{x}{n}=\binom{x-1}{n-1}$ and if we use the Leibniz formula for the determinant, we see that this is equivalent to

$$\begin{aligned}&\sum_{\pi,\sigma\in\mathfrak{S}_{\frac{n+1}{2}}}\operatorname{sgn}(\sigma)\sum_{a_1=0}^{m_{\pi(1)}}\cdots\sum_{a_{\frac{n+1}{2}}=0}^{m_{\pi\left(\frac{n+1}{2}\right)}}(-1)^{a_1+\cdots+a_{\frac{n+1}{2}}}\\&\times\binom{k_1+\sigma(1)-a_1-x-2}{2\sigma(1)-m_{\pi(1)}-2}\cdots\binom{k_{\frac{n+1}{2}}+\sigma(\frac{n+1}{2})-a_{\frac{n+1}{2}}-x-2}{2\sigma(\frac{n+1}{2})-m_{\pi\left(\frac{n+1}{2}\right)}-2}=0.\end{aligned}\tag{3.9}$$

We want to simplify this expression. Therefore, we observe that

$$\begin{aligned}\sum_{a_i=0}^{m_{\pi(i)}}\binom{k_i+\sigma(i)-a_i-x-2}{2\sigma(i)-m_{\pi(i)}-2}&=\sum_{a_i=0}^{m_{\pi(i)}}\Delta_{k_i}\binom{k_i+\sigma(i)-a_i-x-2}{2\sigma(i)-m_{\pi(i)}-1}\\&=\binom{k_i+\sigma(i)-x-1}{2\sigma(i)-m_{\pi(i)}-1}-\binom{k_i+\sigma(i)-m_{\pi(i)}-x-2}{2\sigma(i)-m_{\pi(i)}-1}.\end{aligned}\tag{3.10}$$

Furthermore, we define a function $\varepsilon_{x_1,\ldots,x_n}$ for an arbitrary number of integers $x_1,\ldots,x_n$ such that

$$\varepsilon_{x_1,\ldots,x_n}:=\begin{cases}0 & \text{if at least one } x_i \text{ is odd,}\\ 1 & \text{if all } x_i \text{ are even.}\end{cases}$$

It is easy to see that

$$\sum_{a_1=0}^{m_{\pi(1)}}\cdots\sum_{a_{\frac{n+1}{2}}=0}^{m_{\pi\left(\frac{n+1}{2}\right)}}(-1)^{a_1+\cdots+a_{\frac{n+1}{2}}}=\varepsilon_{m_{\pi(1)},\ldots,m_{\pi\left(\frac{n+1}{2}\right)}}.\tag{3.11}$$

Combining (3.10) and (3.11), we transform (3.9) into

$$\sum_{\pi,\sigma\in\mathfrak{S}_{\frac{n+1}{2}}}\operatorname{sgn}(\sigma)\varepsilon_{m_{\pi(1)},\ldots,m_{\pi\left(\frac{n+1}{2}\right)}}$$
$$\times\prod_{i=1}^{\frac{n+1}{2}}\left(\binom{k_i+\sigma(i)-x-1}{2\sigma(i)-m_{\pi(i)}-1}-\binom{k_i+\sigma(i)-m_{\pi(i)}-x-2}{2\sigma(i)-m_{\pi(i)}-1}\right)=0.$$

If at least one of the $m_1,\ldots,m_{\frac{n+1}{2}}$ is odd, then the statement above is clearly true. Therefore we assume that m_i is even for all $i\in\{1,\ldots,\frac{n+1}{2}\}$. We still have to show that

$$\sum_{\pi,\sigma\in\mathfrak{S}_{\frac{n+1}{2}}}\operatorname{sgn}(\sigma)\prod_{i=1}^{\frac{n+1}{2}}\left(\binom{k_i+\sigma(i)-x-1}{2\sigma(i)-m_{\pi(i)}-1}-\binom{k_i+\sigma(i)-m_{\pi(i)}-x-2}{2\sigma(i)-m_{\pi(i)}-1}\right)=0.$$

For fixed permutations $\pi, \sigma \in \mathfrak{S}_{\frac{n+1}{2}}$, we see that the corresponding summand vanishes if $2\sigma(i) - m_{\pi(i)} - 1 \leq 0$. Therefore, we assume that $2\sigma(i) - m_{\pi(i)} \geq 2$. For every fixed permutation σ, there exists an $j \in \{1, \dots, \frac{n+1}{2}\}$ such that $\sigma(j) = 1$. Hence, if we fix the permutation π, the integer $m_{\pi(j)}$ must be equal to 0 because $2\sigma(j) - 2 \geq m_{\pi(j)} \geq 0$. Since π ranges over the whole symmetric group of order $\frac{n+1}{2}$, it holds that $m_{\pi(j)} = 0$ for every $\pi \in \mathfrak{S}_{\frac{n+1}{2}}$. Consequently, all integers $m_1, \dots, m_{\frac{n+1}{2}}$ are zero. But this contradicts our assumption that $m_i \neq 0$ at least one $i \in \{1, \dots, \frac{n+1}{2}\}$. This completes our proof. $\square$

3.4 Halved Truncated Trees

Let us have a closer look at the correspondence between alternating sign matrices and monotone triangles. How does the corresponding monotone triangle reflect the position of the unique entry 1 in the first and last row and in the first and last column of the alternating sign matrix? We take the alternating sign matrix of Example 5:

Example 33.

Top row

Right column

Left column

$$\begin{pmatrix} 0 & 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & \boxed{1} \\ \boxed{1} & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 & 0 \end{pmatrix} \longleftrightarrow \begin{array}{ccccccccccc} & & & & & \boxed{4} & & & & & \\ & & & & 2 & & 5 & & & & \\ & & & 2 & & 4 & & 5 & & & \\ & & 2 & & 3 & & 5 & & \boxed{6} & & \\ & \boxed{1} & & 2 & \boxed{} & 4 & & 5 & & 6 & \\ 1 & & 2 & & 3 & & 4 & & 5 & & 6 \end{array}$$

Bottom row

Consider an alternating sign matrix A of order n and its corresponding monotone triangle α. It is clear by the correspondence of alternating sign matrices and monotone triangles that the unique entry 1 in the top row of A translates into the top entry of α. The bottom row of the monotone triangle is given by $(1, 2, \dots, n)$ and the penultimate row by a strongly increasing sequence of $n-1$ integers between 1 and n. The missing entry from $\{1, 2, \dots, n\}$ in the penultimate row of α corresponds to the unique 1 in the bottom row of A. If the unique 1 in the left-most column of A is in row i, then the entry $a_{i,1}$ in the monotone triangle is 1 whereas $a_{i-1,1}$ is greater than 1. If the unique 1 in the right-most column of the alternating sign matrix is in row j, then the entry $a_{j,j}$ of α is n but $a_{j-1,j-1}$ is smaller than n.

In order to control the position of the unique 1 in the left-most column of vertically symmetric alternating sign matrices, it is useful to consider halved monotone triangles with truncated diagonals. That means we cut off bottom entries of northeast diagonals. This idea motivates the following definition:

Definition 34. Let d be a nonnegative and n a positive integer such that $d \leq n$ and let $\mathbf{s} = (s_1, \dots, s_d)$ be a weakly decreasing sequence of nonnegative integers. A *halved*

($\mathbf{s}$)-tree[2] is an array of integers in the shape of a halved monotone triangle with n rows such that for each $1 \leq i \leq d$ the s_i bottom entries of the i^{th} northeast diagonal (counted from the left) are deleted like in Figure 3.2.

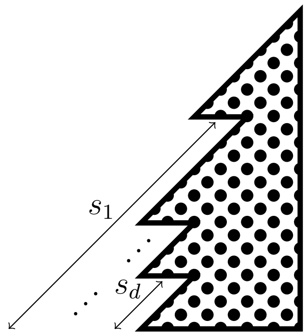

Figure 3.2: Shape of a halved $(\mathbf{s})$-tree

The entries
- strictly increase along rows,
- weakly increase along northeast and southeast direction (if entries are existent) and
- do not exceed a certain parameter x.

We say that a halved $(\mathbf{s})$-tree has bottom row $\left(k_1, \ldots, k_{\lceil \frac{n}{2} \rceil}\right)$ if for any $1 \leq i \leq \lceil \frac{n}{2} \rceil$ the bottom entry in the i^{th} northeast diagonal is k_i.

Notice that a halved $(0, \ldots, 0)$-tree simply is a halved monotone triangle. Thus, the notion of a halved tree generalises the notion of a halved monotone triangle.
We assign a Q-weight to any halved tree the same ways as we have done for halved monotone triangles: We associate Q to an entry with a southeast and a southwest neighbour if it lies strictly in between them. If there is no southeast neighbour, we associate Q if the entry is strictly bigger than its southwest neighbour. The *Q-weight* of a halved tree is the product of all indeterminants Q associated to its entries. The *Q-generating function* of a given set of halved trees is the sum of the Q-weights of all the halved trees inside this set.

Example 35. *Let us consider halved (3,1)-tree with five rows, no entry larger than 6 and prescribed bottom row (1,4,5). There are the following 18 trees:*

halved tree	$\begin{array}{cc} & 1 \\ 1 & \\ & 4 \\ 4 & \\ & 5 \end{array}$,	$\begin{array}{cc} & 2 \\ 1 & \\ & 4 \\ 4 & \\ & 5 \end{array}$,	$\begin{array}{cc} & 3 \\ 1 & \\ & 4 \\ 4 & \\ & 5 \end{array}$,	$\begin{array}{cc} & 4 \\ 1 & \\ & 4 \\ 4 & \\ & 5 \end{array}$,	$\begin{array}{cc} & 5 \\ 1 & \\ & 4 \\ 4 & \\ & 5 \end{array}$,	$\begin{array}{cc} & 6 \\ 1 & \\ & 4 \\ 4 & \\ & 5 \end{array}$,	$\begin{array}{cc} & 1 \\ 1 & \\ & 5 \\ 4 & \\ & 5 \end{array}$,	$\begin{array}{cc} & 2 \\ 1 & \\ & 5 \\ 4 & \\ & 5 \end{array}$,	$\begin{array}{cc} & 3 \\ 1 & \\ & 5 \\ 4 & \\ & 5 \end{array}$,
Q-weight	1	Q	Q	Q	Q	Q	Q	Q^2	Q^2

[2]This notion is not connected to the graph theoretical notion of a tree. It rather refers to the schematic drawings of (Christmas) trees.

halved tree	$\begin{array}{cc} & \textcircled{4} \\ 1 & \\ & \textcircled{5} \\ 4 & \\ & 5 \end{array},$	$\begin{array}{cc} & \textcircled{5} \\ 1 & \\ & \textcircled{5} \\ 4 & \\ & 5 \end{array},$	$\begin{array}{cc} & \textcircled{6} \\ 1 & \\ & \textcircled{5} \\ 4 & \\ & 5 \end{array},$	$\begin{array}{cc} & 1 \\ 1 & \\ & \textcircled{6} \\ 4 & \\ & 5 \end{array},$	$\begin{array}{cc} & \textcircled{2} \\ 1 & \\ & \textcircled{6} \\ 4 & \\ & 5 \end{array},$	$\begin{array}{cc} & \textcircled{3} \\ 1 & \\ & \textcircled{6} \\ 4 & \\ & 5 \end{array},$	$\begin{array}{cc} & \textcircled{4} \\ 1 & \\ & \textcircled{6} \\ 4 & \\ & 5 \end{array},$	$\begin{array}{cc} & \textcircled{5} \\ 1 & \\ & \textcircled{6} \\ 4 & \\ & 5 \end{array},$	$\begin{array}{cc} & \textcircled{6} \\ 1 & \\ & \textcircled{6} \\ 4 & \\ & 5 \end{array}.$
Q-weight	Q^2	Q^2	Q^2	Q	Q^2	Q^2	Q^2	Q^2	Q^2

The Q-generating function is $1+7Q+10Q^2$. Moreover, it is obviously independent of k_3 (as long as $k_2 \leq k_3 \leq x$).

Theorem 36. *The Q-generating function of a halved $(s_1,\ldots,s_d)$-tree with n rows, no entry larger than x and bottom row $\left(k_1,\ldots,k_{\lceil \frac{n}{2}\rceil}\right)$ is obtained by applying the operator $(-\,{}^{Q}\Delta_{k_1})^{s_1}\cdots(-\,{}^{Q}\Delta_{k_d})^{s_d}$ to the Q-generating function ${}^{Q}\gamma\left(n,x;k_1,\ldots,k_{\lceil \frac{n}{2}\rceil}\right)$ of the underlying halved monotone triangle.*

For the proof of Theorem 36, we make use of an alternative version of the summation operator. This version was presented in [Fis16]. It is based on the following observation:

$$\sum_{\substack{(l_i,l_{i+1})\in\mathbb{Z}^2,\\ l_i<l_{i+1},\\ k_{i-1}\leq l_i\leq k_i\leq l_{i+1}\leq k_{i+1}}} F(l_i,l_{i+1}) = \left(\operatorname{Strict}_{k_i^{(1)},k_i^{(2)}} \sum_{l_{i-1}=k_{i-1}}^{k_i^{(1)}} \sum_{l_i=k_i^{(2)}}^{k_{i+1}} F(l_i,l_{i+1})\right)\Bigg|_{k_i^{(1)}=k_i^{(2)}=k_i}.$$

This extends to

$$\begin{aligned}
&\sum_{\substack{(l_1,\ldots,l_{n-1})\in\mathbb{Z}^{n-1},\\ l_1<l_2<\cdots<l_{n-1},\\ k_1\leq l_1\leq k_2\leq\cdots\leq k_{n-1}\leq l_{n-1}\leq k_n}} F(l_1,\ldots,l_{n-1})\\
&\qquad = \Bigg(\operatorname{Strict}_{k_1^{(1)},k_1^{(2)}}\cdots\operatorname{Strict}_{k_n^{(1)},k_n^{(2)}}\\
&\qquad\quad \times \sum_{l_1=k_1^{(2)}}^{k_2^{(1)}} \sum_{l_2=k_2^{(2)}}^{k_3^{(1)}} \cdots \sum_{l_{n-1}=k_{n-1}^{(2)}}^{k_n^{(1)}} F(l_1,\ldots,l_{n-1})\Bigg)\Bigg|_{k_i^{(1)}=k_i^{(2)}=k_i}.
\end{aligned}$$

The corresponding Q-version is

$$\begin{aligned}
&\sum_{\substack{(l_1,\ldots,l_{n-1})\in\mathbb{Z}^{n-1},\\ l_1<l_2<\cdots<l_{n-1},\\ k_1\leq l_1\leq k_2\leq\cdots\leq k_{n-1}\leq l_{n-1}\leq k_n}} Q^{[k_1<l_1<k_2]+\cdots+[k_{n-1}<l_{n-1}<k_n]} F(l_1,\ldots,l_{n-1})\\
&\qquad = \Bigg(Q^{-1}\,{}^{Q}\operatorname{Strict}_{k_1^{(1)},k_1^{(2)}}\cdots{}^{Q}\operatorname{Strict}_{k_n^{(1)},k_n^{(2)}} \qquad (3.12)\\
&\qquad\quad \times \sum_{l_1=k_1^{(2)}}^{k_2^{(1)}} \sum_{l_2=k_2^{(2)}}^{k_3^{(1)}} \cdots \sum_{l_{n-1}=k_{n-1}^{(2)}}^{k_n^{(1)}} F(l_1,\ldots,l_{n-1})\Bigg)\Bigg|_{k_i^{(1)}=k_i^{(2)}=k_i}.
\end{aligned}$$

Proof of Theorem 36. The proof builds on three important observations. First, we apply $-\Delta_{k_1}$ to the ordinary summation operator. This was done in [Fis11]. For any function F on $\mathbb{Z}^{n-1}$ and for any integers $k_1 < \cdots < k_n$ it holds that

$$
\begin{aligned}
&-\Delta_{k_1} \sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} F(l_1, l_2, \ldots, l_{n-1}) \\
&\quad = \sum_{\substack{(l_1,\ldots,l_{n-1})\in\mathbb{Z}^{n-1},\\ l_1<l_2<\cdots<l_{n-1},\\ k_1\le l_1\le k_2\le\cdots\le k_{n-1}\le l_{n-1}\le k_n}} F(l_1, l_2, \ldots, l_{n-1}) - \sum_{\substack{(l_1,\ldots,l_{n-1})\in\mathbb{Z}^{n-1},\\ l_1<l_2<\cdots<l_{n-1},\\ k_1+1\le l_1\le k_2\le\cdots\le k_{n-1}\le l_{n-1}\le k_n}} F(l_1, l_2, \ldots, l_{n-1}) \\
&\quad = \sum_{\substack{(l_2,\ldots,l_{n-1})\in\mathbb{Z}^{n-2},\\ l_2<l_3<\cdots<l_{n-1},\\ k_2\le l_2\le k_3\le\cdots\le k_{n-1}\le l_{n-1}\le k_n}} F(k_1, l_2 \ldots, l_{n-1}).
\end{aligned}
$$

Consequently, applying $-\Delta_{k_1}$ has the effect of truncating the leftmost entry in the bottom row of the (halved) monotone triangle and setting k_1 as the leftmost entry in the penultimate row.

Second, we want to extend this observation to the Q-enumeration by applying the operator $-{}^{Q}\Delta_{k_1} = -Q^{-1}((1-Q^{-1})\,\mathrm{E}_{k_1} + Q^{-1})^{-1}\,\Delta_{k_1}$ to the Q-generating function of a halved monotone triangle. This is indeed well defined: By using the formula for the geometric series, we derive

$$
\begin{aligned}
{}^{Q}\Delta_x &= Q^{-1}((1-Q^{-1})\,\mathrm{E}_x + Q^{-1})^{-1}\,\Delta_x = Q^{-1}(\mathrm{id} + (1-Q^{-1})\,\Delta_x)^{-1}\,\Delta_x \\
&= Q^{-1}\,\Delta_x \sum_{i=0}^{\infty}(Q^{-1}-1)^i\,\Delta_x^i = Q^{-1}\sum_{i=0}^{\infty}(Q^{-1}-1)^i\,\Delta_x^{i+1}\,. \qquad (3.13)
\end{aligned}
$$

The Q-generating function of a halved monotone triangle is polynomial. Let $p(x)$ be a polynomial of degree d. Then $\Delta_x^D\, p(x) = 0$ for $D > d$. Therefore, the sum in (3.13) becomes finite if ${}^{Q}\Delta_x$ is applied to $p(x)$.

Third, we see that in (3.12) the operator ${}^{Q}\mathrm{Strict}_{k_1^{(1)},k_1^{(2)}}$ is applied to a function which is independent of $k_1^{(1)}$. In this case, ${}^{Q}\mathrm{Strict}_{k_1^{(1)},k_1^{(2)}}$ can be simplified in the following way:

$$
Q\,\mathrm{E}_{k_1^{(2)}} - \Delta_{k_1^{(2)}} = (Q-1)\,\mathrm{E}_{k_1^{(2)}} + \mathrm{id}\,.
$$

Finally, we combine these oberservations. If F is a polynomial function on $\mathbb{Z}^{n-1}$, then it follows that

$$
\begin{aligned}
&- {}^Q\Delta_{k_1}\Bigg(Q^{-1}\, {}^Q\mathrm{Strict}_{k_1^{(1)},k_1^{(2)}}\, {}^Q\mathrm{Strict}_{k_2^{(1)},k_2^{(2)}} \cdots {}^Q\mathrm{Strict}_{k_n^{(1)},k_n^{(2)}} \\
&\qquad\qquad \times \sum_{l_1=k_1^{(2)}}^{k_2^{(1)}} \sum_{l_2=k_2^{(2)}}^{k_3^{(1)}} \cdots \sum_{l_{n-1}=k_{n-1}^{(2)}}^{k_n^{(1)}} F(l_1,l_2,\ldots,l_{n-1})\Bigg)\Bigg|_{k_i^{(1)}=k_i^{(2)}=k_i} \\
&= -Q^{-1}((1-Q^{-1})\,\mathrm{E}_{k_1}+Q^{-1})^{-1}\,\Delta_{k_1}\Bigg(Q^{-1}((Q-1)\,\mathrm{E}_{k_1^{(2)}}+\mathrm{id}) \\
&\times {}^Q\mathrm{Strict}_{k_2^{(1)},k_2^{(2)}} \cdots {}^Q\mathrm{Strict}_{k_n^{(1)},k_n^{(2)}} \sum_{l_1=k_1^{(2)}}^{k_2^{(1)}} \sum_{l_2=k_2^{(2)}}^{k_3^{(1)}} \cdots \sum_{l_{n-1}=k_{n-1}^{(2)}}^{k_n^{(1)}} F(l_1,l_2,\ldots,l_{n-1})\Bigg)\Bigg|_{k_i^{(1)}=k_i^{(2)}=k_i} \\
&= -((1-Q^{-1})\,\mathrm{E}_{k_1}+Q^{-1})^{-1}\,\Delta_{k_1}\Bigg(Q^{-1}((1-Q^{-1})\,\mathrm{E}_{k_1^{(2)}}+Q^{-1}) \\
&\times {}^Q\mathrm{Strict}_{k_2^{(1)},k_2^{(2)}} \cdots {}^Q\mathrm{Strict}_{k_n^{(1)},k_n^{(2)}} \sum_{l_1=k_1^{(2)}}^{k_2^{(1)}} \sum_{l_2=k_2^{(2)}}^{k_3^{(1)}} \cdots \sum_{l_{n-1}=k_{n-1}^{(2)}}^{k_n^{(1)}} F(l_1,l_2,\ldots,l_{n-1})\Bigg)\Bigg|_{k_i^{(1)}=k_i^{(2)}=k_i} \\
&= -\,\Delta_{k_1}\Bigg(Q^{-1}\, {}^Q\mathrm{Strict}_{k_2^{(1)},k_2^{(2)}} \cdots {}^Q\mathrm{Strict}_{k_n^{(1)},k_n^{(2)}} \\
&\qquad\qquad \times \sum_{l_1=k_1}^{k_2^{(1)}} \sum_{l_2=k_2^{(2)}}^{k_3^{(1)}} \cdots \sum_{l_{n-1}=k_{n-1}^{(2)}}^{k_n^{(1)}} F(l_1,l_2,\ldots,l_{n-1})\Bigg)\Bigg|_{k_i^{(1)}=k_i^{(2)}=k_i} \\
&= \Bigg(Q^{-1}\, {}^Q\mathrm{Strict}_{k_2^{(1)},k_2^{(2)}} \cdots {}^Q\mathrm{Strict}_{k_n^{(1)},k_n^{(2)}} \\
&\qquad\qquad\qquad\qquad \times \sum_{l_2=k_2^{(2)}}^{k_3^{(1)}} \cdots \sum_{l_{n-1}=k_{n-1}^{(2)}}^{k_n^{(1)}} F(k_1,l_2,\ldots,l_{n-1})\Bigg)\Bigg|_{k_i^{(1)}=k_i^{(2)}=k_i}.
\end{aligned}
$$

This is equal to

$$
\sum_{\substack{(l_2,\ldots,l_{n-1})\in\mathbb{Z}^{n-2},\\ l_2<l_3<\cdots<l_{n-1},\\ k_2\le l_2\le k_3\le\cdots\le k_{n-1}\le l_{n-1}\le k_n}} Q^{[k_2<l_2<k_3]+\cdots+[k_{n-1}<l_{n-1}<k_n]} F(k_1,l_2\ldots,l_{n-1}).
$$

If we apply $(-\,{}^Q\Delta_{k_1})^{s_1}\cdots(-\,{}^Q\Delta_{k_d})^{s_d}$, we truncate the s_i bottom entries from the i^{th} northeast diagonal for each $1\le i\le d$. Hence,

$$
(-\,{}^Q\Delta_{k_1})^{s_1}\cdots(-\,{}^Q\Delta_{k_d})^{s_d}\,{}^Q\gamma\left(n,x;k_1,\ldots,k_{\lceil\frac{n}{2}\rceil}\right)
$$

is the Q-generating function of halved $(s_1, \ldots, s_d)$-trees with n rows, no entry larger than x and bottom row $\left(k_1, \ldots, k_{\lceil \frac{n}{2} \rceil}\right)$. $\square$

To complete this thesis, we present an example of a Q-generating function of halved trees using the previous theorem.

Example 37. *According to Theorem 36, the Q-generating function corresponding to halved (3,1)-tree with five rows is*

$$(-\,{}^{Q}\Delta_{k_1})^3(-\,{}^{Q}\Delta_{k_2})Q^{-\binom{3}{2}} \prod_{1 \le s < t \le 3} {}^{Q}\mathrm{W}_{k_t,k_s}\,{}^{Q}\mathrm{I}_{k_s,k_t} \prod_{1 \le i < j \le 3} \frac{(k_j - k_i)(2x + 1 - k_j - k_i)}{(j-i)(j+i-1)}.$$

This is equal to

$$1 + 2Qx + Q^2x^2 - Qk_1 - Q^2xk_1 - Qk_2 - Q^2xk_2 + Q^2k_1k_2.$$

Note that this Q-generating function is independent of k_3. If we set $(x, k_1, k_2, k_3) = (6, 1, 4, 5)$, we get $1 + 7Q + 10Q^2$ like in Example 35.

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Appendix

Abstract

In the early 1980s, W. Mills, D. Robbins and H. Rumsey introduced alternating sign matrices. These are square matrices with entries from the set $\{-1, 0, +1\}$ such that the entries in every row and in every column add up to 1 and that the nonzero elements alternate in sign in each row and in each column. Shortly after, R. Stanley suggested to systematically study the different symmetry classes of alternating sign matrices. In 2009, I. Fischer defined halved monotone triangles as a generalisation of vertically symmetric alternating sign matrices. We present a weighted operator formula for the number of halved monotone triangles which generalises the generating function for vertically symmetric alternating sign matrices with respect to the number of entries equal to -1. In addition, we study the 2-enumeration of halved monotone triangles and present a refined enumeration formula for so-called halved trees, a further generalisation of halved monotone triangles.

Zusammenfassung

Zu Beginn der 1980er Jahre führten W. Mills, D. Robbins und H. Rumsey den Begriff der alternierenden Vorzeichenmatrizen ein. Diese sind quadratische Matrizen mit den Einträgen -1, 0 oder 1 und der Eigenschaft, dass jede Zeilen- und Spaltensumme 1 beträgt und sich in jeder Zeile und in jeder Spalte die Vorzeichen der Einträge ungleich Null abwechseln. Bald darauf schlug R. Stanley vor, die verschiedenen Symmetrieklassen der alternierenden Vorzeichenmatrizen zu untersuchen. Im Jahre 2009 definierte I. Fischer halbierte monotone Dreiecke als eine Verallgemeinerung vertikalsymmetrischer alternierender Vorzeichenmatrizen. Wir geben eine gewichtete Operatorformel für die Anzahl halbierter monotoner Dreiecke an, die eine weitere Verallgemeinerung der erzeugenden Funktion vertikalsymmetrischer alternierender Vorzeichenmatrizen bezüglich der Anzahl der -1er darstellt. Weiters untersuchen wir Spezialfälle dieser Formel und betrachten sogenannte halbierte Bäume, eine Verallgemeinerung halbierter monotoner Dreiecke.