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DISSERTATION

Compactness of the $\bar{\partial}$-Neumann Operator

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Vorwort

Gegenstand dieser Dissertation ist das gewichtete $\bar{\partial}$-Neumann Problem auf unbeschränkten Gebieten. Während das ungewichtete im Falle beschränkter pseudokonvexer Gebiete gut erforscht und eines der zentralen Themen der komplexen Analysis ist, ist nur wenig über unbeschränkte Gebiete bekannt. Im Allgemeinen macht es keinen Sinn, unbeschränkte Gebiete ohne Gewicht zu betrachten. Auf beschränkten Gebieten sind für die übliche Klasse von Gewichten das gewichtete und das ungewichtete Problem äquivalent. Eng verwandt mit der Frage nach Existenz und Kompaktheit eines gewichteten $\bar{\partial}$-Neumann Operators N_φ wie sie in dieser Dissertation gestellt wird, ist die Frage nach selbigen Eigenschaften des kanonischen Lösungsoperators $\mathcal{S}_\varphi$ zur $\bar{\partial}$-Gleichung. Erste Antworten hierauf finden sich in [14], wo der einfachste Fall eines unbeschränkten Gebietes Ω, nämlich $\Omega = \mathbb{C}^n$, behandelt wird. Für den eindimensionalen Fall konnten diese Ergebnisse in [22] verschärft und unter einer zusätzlichen Annahme an das Gewicht eine Charakterisierung von Kompaktheit von $\mathcal{S}_\varphi$ gefunden werden.

Das erste Kapitel beinhaltet eine generelle Einführung, sowie die notwendigen Definitionen. Insbesondere wird erläutert, wie das gewichtete $\bar{\partial}$-Neumann Problem auf $\mathbb{C}^n$ mit der Spektraltheorie von Schrödinger Operatoren zusammenhängt, und die für uns notwendigen Begriffe der Spektraltheorie werden erklärt.

In Kapitel zwei werden bekannte, in der komplexen Analysis beschränkter Gebiete bereits klassische Methoden entsprechend adaptiert, um den Fall unbeschränkter Gebiete ebenfalls handhaben zu können. Dadurch ergeben sich neue, komplex analytische Beweise der Ergebnisse von [14], welche in diesem Zusammenhang natürlicher wirken als die spektraltheoretischen ebendort. Der Vorteil unserer Methode ist insbesondere, dass sie auf allgemeinere unbeschränkte pseudokonvexe Gebiete übertragen werden kann, was für den spektraltheoretischen Ansatz im mehrdimensionalen nicht gilt. Wir werden das Zusammenspiel von Gewichtsfunktion und der Geometrie des Randes sehen und eine Bedingung für Kompaktheit des $\bar{\partial}$-Neumann Operators angeben, welche eine Kombination einer gewissen Bedingung für den beschränkten Fall und einer für $\mathbb{C}^n$ ist.

Näher eingegangen auf den Fall $\mathbb{C}^n$ wird in Kapitel drei. Es werden neue notwendige Bedingungen für Existenz und Kompaktheit des $\bar{\partial}$-Neumann Operators angegeben. Unter einer zusätzlichen Annahme an die Eigenräume der komplexen Hessematrix der Gewichtsfunktion wird Kompaktheit auch in höheren Dimensionen charakterisiert.
Schließlich werden einige Spezialfälle behandelt. Für den eindimensio-

nalen Fall wird ein Resultat aus der Theorie der Schrödinger Operatoren vorgestellt, um eine neue Charakterisierung von Kompaktheit zu erhalten. Weiters wird die seit längerem vermutete Tatsache gezeigt, dass entkoppelte Gewichte, d.h. Gewichte der Form $\varphi(z) = \sum_{j=1}^{n} \varphi_j(z_j)$ nie einen kompakten Neumann Operator zulassen. Zu guter Letzt werden die vorherigen Ergebnisse auf den Fall radialsymmetrischer Gewichte angewendet.

Kapitel vier widmet sich der Frage nach dem gewichteten Bergmanraum. Hauptresultate sind ein neuer vereinfachter Beweis eines Theorems von I. Shigekawa [28], welches ein Kriterium für einen nichttrivialen Bergmanraum gibt, sowie die Tatsache dass Kompaktheit des gewichteten $\bar{\partial}$-Neumann Operators unter gewissen Voraussetzungen einen unendlichdimensionalen Bergmanraum bedingt.

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Abstract

This dissertation deals with the weighted $\bar{\partial}$-Neumann problem on pseudoconvex unbounded domains. Whereas the unweighted $\bar{\partial}$-Neumann problem on bounded pseudoconvex domains is one of the main aspects of complex analysis, few is known about the unbounded case. In general it does not make sense to consider unbounded domains without weight, since already in very easy cases questions of existence and compactness of the Neumann operator become trivial. Note that for bounded domains and the class of weight functions usually considered, the weighted and unweighted problem are equivalent. Closely related to properties of the $\bar{\partial}$-Neumann operator N_φ are the ones of the canonical solution operator $\mathcal{S}_\varphi$ to the $\bar{\partial}$-equation. First results in this direction can be found in [14], where the case $\Omega = \mathbb{C}^n$ is investigated. In complex dimension one, sharper conditions have been found in [22], assuming an additional property of the weight function. The authors give a characterization of compactness of the canonical solution operator.

The first Chapter contains a general introduction to the problem, as well as the necessary definitions. In particular, the connection between the weighted $\bar{\partial}$-Neumann problem on $\mathbb{C}^n$ and the spectral theory of Schrödinger operators is pointed out. Furthermore, the spectral theoretic notions needed in the sequel are introduced.

In Chapter two, well known classical methods which have been successfully used for bounded domains are adapted to the setting of our problem. Doing this, we obtain new purely complex analytic proofs of the statements given in [14], which in this context seem to be more natural as the spectral theoretic ones therein. The main advantage of our method is that it can be extended to also treat unbounded pseudoconvex domains with boundary. In higher dimensions, this is in fact not true for the spectral theoretic approach. We will see the interplay between weight function and the geometry of the boundary and give a sufficient condition for compactness of the $\bar{\partial}$-Neumann operator, which is a combination of a condition for the bounded case and the one of the case of $\mathbb{C}^n$.

Chapter three is a more accurate investigation of the case $\Omega = \mathbb{C}^n$. New necessary conditions for existence and compactness of the $\bar{\partial}$-Neumann operator are given and under a suitable assumption on the eigenspaces of the complex Hessian of the weight a characterization of compactness in higher dimensions is obtained.
Moreover, some special cases are treated. A Theorem from Schrödinger theory is presented to handle the one dimensional case and give a new characterization of compactness. Furthermore the conjecture that decou-

pled weight functions (i.e. functions of the form $\varphi(z) = \sum_{j=1}^{n} \varphi_j(z_j)$) never admit a compact $\bar{\partial}$-Neumann operator is proved. Eventually, previous results are applied to radially symmetric weights.

The last section is dedicated to the weighted Bergman space $\mathcal{A}^2_\varphi$. Our main results are a new proof of a Theorem due to I. Shigekawa [28], which gives a condition for non-triviality of the weighted Bergman space, as well as connections between compactness of the $\bar{\partial}$-Neumann operator and the dimension of $\mathcal{A}^2_\varphi$. It is shown that in dimension one, compactness of N_φ forces $\mathcal{A}^2_\varphi$ to be infinite dimensional. In higher dimension, the same result is proved under additional assumptions on the weight.

Contents

Chapter 1

Preliminaries

We begin with the basic definitions. For the moment, let us just consider the case $\Omega = \mathbb{C}^n$. A weight function on $\mathbb{C}^n$ is a $\mathcal{C}^2$-function $\varphi : \mathbb{C}^n \to \mathbb{R}^+$. By $L^2(\mathbb{C}^n, e^{-\varphi})$, $n \geq 1$, which we will often write as $L^2(\mathbb{C}^n, \varphi)$ or as L^2_φ if there is no confusion about the domain possible, we denote the space

$$L^2(\mathbb{C}^n, e^{-\varphi}) := \{f : \mathbb{C}^n \to \mathbb{C} \text{ measurable} : \int\limits_{\mathbb{C}^n} |f(z)|^2 e^{-2\varphi(z)} d\lambda(z) < \infty\},$$

where λ is the standard Lebesgue measure. We remark that this space sometimes is defined without the factor 2 in the exponent. L^2_φ is turned into a Hilbert space by declaring the inner product

$$\langle f, g\rangle_\varphi := \int\limits_{\mathbb{C}^n} f(z)\overline{g(z)} e^{-2\varphi(z)} d\lambda(z)$$

and the norm $\|f\|^2_\varphi = \langle f, f\rangle_\varphi$. We denote by $L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ the space of $(0,1)$-forms with coefficients in $L^2(\mathbb{C}^n, \varphi)$, i.e.

$$L^2_{(0,1)}(\mathbb{C}^n, \varphi) = \{f = \sum_{j=1}^n f_j d\overline{z}_j : \; f_j \in L^2_\varphi\},$$

which is a Hilbert space by setting the inner product on $L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ to be the sum of the inner products of the coefficients. An important subspace of $L^2(\mathbb{C}^n, \varphi)$ of particular interest to us is the weighted Bergman space

$$\mathcal{A}^2(\mathbb{C}^n, e^{-\varphi}) := \{f : \mathbb{C}^n \to \mathbb{C} \text{ entire} : \int\limits_{\mathbb{C}^n} |f(z)|^2 e^{-2\varphi(z)} d\lambda(z) < \infty\}, \quad (1.1)$$

abbreviated as $\mathcal{A}^2_\varphi(\mathbb{C}^n)$.
We are only going to deal with plurisubharmonic weight functions. In dimension one, subharmonicity of $\mathcal{C}^2$-functions is equivalent to $\triangle\varphi \geq 0$,

where $\triangle$ denotes the Laplacian. In higher dimension, a function is defined to be plurisubharmonic if it is subharmonic when restricted to an arbitrary complex line. Plurisubharmonicity is a natural assumption on weight functions, in the sense that for each positive weight one can find a plurisubharmonic one defining the same weighted space of holomorphic functions, see [4]. In many cases the weights are even equivalent, see again [4] and the references therein. Equivalence of weight functions means that the norms induced are equivalent.
To define the $\bar{\partial}$-operator, we set

$$\bar{\partial} : \mathcal{C}_0^\infty(\mathbb{C}^n) \to \Lambda_0^{(0,1)}, \;\; \bar{\partial} f := \sum_{j=1}^n \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j,$$

where $\mathcal{C}_0^\infty(\mathbb{C}^n)$ denotes the space of smooth compactly supported functions $f : \mathbb{C}^n \to \mathbb{C}$ and $\Lambda_0^{(0,1)}$ the space of (0,1)-forms with coefficients in $\mathcal{C}_0^\infty$. So $\bar{\partial}$ is a first order differential operator which is densely defined, since $\mathcal{C}_0^\infty$ is dense in $L^2(\mathbb{C}^n, e^{-\varphi})$. By extending its domain to $dom(\bar{\partial}) = \{f \in L^2_\varphi : \bar{\partial} f \in L^2_{(0,1)}(\mathbb{C}^n, \varphi)\}$ it is turned into a closed operator, where in this definition the derivatives are taken in the sense of distributions. This way of extending a differential operator defined on $\mathcal{C}_0^\infty$ is called the maximal closure of that operator. The previously defined space $\mathcal{A}^2_\varphi$ equals the kernel of $\bar{\partial}$, so it is a closed subspace of $L^2(\mathbb{C}^n, \varphi)$ and therefore a Hilbert space itself. For a more detailed introduction to the L^2-theory of the $\bar{\partial}$-Neumann problem see for instance [8].
As a closed, densely defined operator on a Hilbert space, $\bar{\partial}$ possesses an adjoint, denoted by $\bar{\partial}^*_\varphi$, which is closed and densely defined itself. The domain of that adjoint is $dom(\bar{\partial}^*_\varphi) = \{f \in L^2_\varphi : \exists C \; s.t. \; |\langle \bar{\partial} g, f\rangle_\varphi| \le C\|g\|_\varphi \; \forall g \in dom(\bar{\partial})\}$, and we characterize it in our first Proposition. Unlike the case of bounded domains, the boundary condition becomes a simple integrability condition.

Proposition 1.1 *Let $f = \sum f_j d\bar{z}_j \in L^2_{(0,1)}(\mathbb{C}^n, \varphi)$. Then $f \in dom(\bar{\partial}^*_\varphi)$ if and only if*

$$\sum_{j=1}^n \left(\frac{\partial f_j}{\partial z_j} - 2\frac{\partial \varphi}{\partial z_j} f_j \right) \in L^2(\mathbb{C}^n, \varphi).$$

Proof. Suppose first that $\sum_{j=1}^n \left(\frac{\partial f_j}{\partial z_j} - 2\frac{\partial \varphi}{\partial z_j} f_j \right) \in L^2(\mathbb{C}^n, \varphi)$, which equivalently means that $e^{2\varphi} \sum_{j=1}^n \frac{\partial}{\partial z_j} \left(f_j e^{-2\varphi} \right) \in L^2(\mathbb{C}^n, \varphi)$. We have to show that there exists a constant C such that $|\langle \bar{\partial} g, f\rangle_\varphi| \le C\|g\|_\varphi$ for all $g \in dom(\bar{\partial})$. To this end let $(\chi_R)_{R\in\mathbb{N}}$ be a family of radially symmetric smooth cutoff functions which are identically one on $\mathbb{B}_R$, the ball with radius R, such that the support of χ_R is contained in $\mathbb{B}_{R+1}$, $supp(\chi_R) \subset \mathbb{B}_{R+1}$, and such that furthermore all first order derivatives of all functions in this family are uniformly bounded by a constant M. Then for all $g \in \mathcal{C}_0^\infty(\mathbb{C}^n)$:

$$\langle \bar{\partial} g, \chi_R f\rangle_\varphi = \sum_{j=1}^n \langle \frac{\partial g}{\partial \overline{z}_j}, \chi_R f_j\rangle_\varphi = -\int_{\mathbb{C}^n} \sum_{j=1}^n g \frac{\partial}{\partial \overline{z}_j} \left(\chi_R \overline{f}_j e^{-2\varphi}\right) d\lambda,$$

by integration by parts, which in particular means

$$|\langle \bar{\partial} g, f\rangle_\varphi| = \lim_{R\to\infty} |\langle \bar{\partial} g, \chi_R f\rangle_\varphi| = \lim_{R\to\infty} \left| \int_{\mathbb{C}^n} \sum_{j=1}^n g \frac{\partial}{\partial \overline{z}_j} \left(\chi_R \overline{f}_j e^{-2\varphi}\right) \, d\lambda \right|.$$

Now we use the triangle inequality and Cauchy – Schwarz inequality, to get

$$\begin{aligned}
&\lim_{R\to\infty} \left| \int_{\mathbb{C}^n} \sum_{j=1}^n g \frac{\partial}{\partial \overline{z}_j} \left(\chi_R \overline{f}_j e^{-2\varphi}\right) \, d\lambda \right| \\
\leq &\lim_{R\to\infty} \left| \int_{\mathbb{C}^n} \chi_R \, g \sum_{j=1}^n \frac{\partial}{\partial \overline{z}_j} \left(\overline{f}_j e^{-2\varphi}\right) \, d\lambda \right| + \lim_{R\to\infty} \left| \int_{\mathbb{C}^n} \sum_{j=1}^n \overline{f}_j g \frac{\partial \chi_R}{\partial \overline{z}_j} e^{-2\varphi} \, d\lambda \right| \\
\leq &\lim_{R\to\infty} \|\chi_R \, g\|_\varphi \left\| e^{2\varphi} \sum_{j=1}^n \frac{\partial}{\partial z_j} \left(f_j e^{-2\varphi}\right) \right\|_\varphi + M\|g\|_\varphi \|f\|_\varphi \\
= &\|g\|_\varphi \left\| e^{2\varphi} \sum_{j=1}^n \frac{\partial}{\partial z_j} \left(f_j e^{-2\varphi}\right) \right\|_\varphi + M\|g\|_\varphi \|f\|_\varphi .
\end{aligned}$$

Hence by assumption,

$$|\langle \bar{\partial} g, f\rangle_\varphi| \leq \|g\|_\varphi \left\| e^{2\varphi} \sum_{j=1}^n \frac{\partial}{\partial z_j} \left(f_j e^{-2\varphi}\right) \right\|_\varphi + M\|g\|_\varphi \|f\|_\varphi \leq C\|g\|_\varphi$$

for all $g \in \mathcal{C}_0^\infty(\mathbb{C}^n)$, and by density of $\mathcal{C}_0^\infty(\mathbb{C}^n)$ this in fact is true for all $g \in dom(\bar{\partial})$. Conversely, let $f \in dom(\bar{\partial}_\varphi^*)$ and take $g \in \mathcal{C}_0^\infty(\mathbb{C}^n)$. Then $g \in dom(\bar{\partial})$ and

$$\begin{aligned}
\langle g, \bar{\partial}_\varphi^* f\rangle_\varphi = &\langle \bar{\partial} g, f\rangle_\varphi \\
= &\sum_{j=1}^n \langle \frac{\partial g}{\partial \overline{z}_j}, f_j\rangle_\varphi \\
= &-\langle g, \sum_{j=1}^n \frac{\partial}{\partial z_j} \left(f_j e^{-2\varphi}\right)\rangle_{L^2} \\
= &\langle g, -e^{2\varphi} \sum_{j=1}^n \frac{\partial}{\partial z_j} \left(f_j e^{-2\varphi}\right)\rangle_\varphi .
\end{aligned}$$

Since $\mathcal{C}_0^\infty(\mathbb{C}^n)$ is dense in $L^2(\mathbb{C}^n,\varphi)$, we conclude that

$$\bar{\partial}_\varphi^* f = -e^{2\varphi}\sum_{j=1}^n \frac{\partial}{\partial z_j}\left(fe^{-2\varphi}\right),$$

which in particular implies that $e^{2\varphi}\sum_{j=1}^n \frac{\partial}{\partial z_j}\left(f_j e^{-2\varphi}\right) \in L^2(\mathbb{C}^n,\varphi)$.

□

The following Lemma will be important for our considerations.

Lemma 1.2 *Forms with coefficients in $\mathcal{C}_0^\infty$ are dense in $dom(\bar{\partial})\cap dom(\bar{\partial}_\varphi^*)$ in the graph norm $f \mapsto (\|f\|_\varphi^2 + \|\bar{\partial} f\|_\varphi^2 + \|\bar{\partial}_\varphi^* f\|_\varphi^2)^{\frac{1}{2}}$.*

Proof. First we show that compactly supported L^2-forms are dense in the graph norm. So let $\{\chi_R\}_{R\in\mathbb{N}}$ be a family of smooth radially symmetric cutoff functions which are identically one on $\mathbb{B}_R$ and supported in $\mathbb{B}_{R+1}$, such that all first order derivatives of functions in this family are uniformly bounded in R by a constant M.
Let $f \in dom(\bar{\partial})\cap dom(\bar{\partial}_\varphi^*)$. Then $\chi_R f \in dom(\bar{\partial})\cap dom(\bar{\partial}_\varphi^*)$ and $\chi_R f \to f$ in $L^2_{(0,1)}(\mathbb{C}^n,\varphi)$ as $R\to\infty$. As observed in Proposition 1.1, we have

$$\bar{\partial}_\varphi^* f = -e^{2\varphi}\sum_{j=1}^n \frac{\partial}{\partial z_j}\left(f_j e^{-2\varphi}\right),$$

hence

$$\bar{\partial}_\varphi^*(\chi_R f) = -e^{2\varphi}\sum_{j=1}^n \frac{\partial}{\partial z_j}\left(\chi_R f_j e^{-2\varphi}\right).$$

We need to estimate the difference of these expressions

$$\bar{\partial}_\varphi^* f - \bar{\partial}_\varphi^*(\chi_R f) = \bar{\partial}_\varphi^* f - \chi_R \bar{\partial}_\varphi^* f + \sum_{j=1}^n \frac{\partial \chi_R}{\partial z_j} f_j,$$

which by the triangle inequality is

$$\|\bar{\partial}_\varphi^* f - \bar{\partial}_\varphi^*(\chi_R f)\|_\varphi \le \left\|\bar{\partial}_\varphi^* f - \chi_R \bar{\partial}_\varphi^* f\right\|_\varphi + M\sum_{j=1}^n \int\limits_{\mathbb{C}^n\setminus\mathbb{B}_R} |f_j(z)|^2 e^{-2\varphi} d\lambda(z).$$

Now both terms on the right tend to 0 as $R\to\infty$, and one can see similarly that also $\bar{\partial}(\chi_R f)\to\bar{\partial} f$ as $R\to\infty$.
So we have density of compactly supported forms in the graph norm, and density of forms with $\mathcal{C}_0^\infty$-coefficients follows by applying Friedrich's Lem-

ma, see for instance appendix D in [8].

$\square$

The complex Laplacian is defined to be

$$\square_\varphi := \bar\partial \bar\partial^*_\varphi + \bar\partial^*_\varphi \bar\partial \tag{1.2}$$

and we again understand the symbol $\square_\varphi$ as the maximal closure of that operator initially defined on (0,1)-forms with coefficients in $\mathcal{C}_0^\infty$, that is, $dom(\square_\varphi) = \{f \in L^2_{(0,1)}(\mathbb{C}^n,\varphi) : \bar\partial f \in L^2_{(0,2)}(\mathbb{C}^n,\varphi),\ \bar\partial^*_\varphi f \in L^2_\varphi,\ \square_\varphi f \in L^2_{(0,1)}(\mathbb{C}^n,\varphi)\}$. By definition, the operator $\square_\varphi$ is selfadjoint and positive, meaning that $\langle \square_\varphi f, f\rangle_\varphi \geq 0$ for all $f \in dom(\square_\varphi)$. Denote the associated Dirichlet form by

$$Q_\varphi(f,g) = \langle \bar\partial f, \bar\partial g\rangle_\varphi + \langle \bar\partial^*_\varphi f, \bar\partial^*_\varphi g\rangle_\varphi \tag{1.3}$$

for all $f, g \in dom(\bar\partial) \cap dom(\bar\partial^*_\varphi)$.
If there exists a bounded linear operator N_φ on $L^2_{(0,1)}(\mathbb{C}^n,\varphi)$ which inverts the unbounded operator $\square_\varphi$, we call N_φ the weighted $\bar\partial$-Neumann operator and say that N_φ exists for that particular weight function. So if N_φ exists we have $\square_\varphi N_\varphi f = f$ for all $f \in L^2_{(0,1)}(\mathbb{C}^n,\varphi)$ and $N_\varphi \square_\varphi f = f$ for all $f \in dom(\square_\varphi)$. Note that it could as well happen that there exists an inverse only on a certain subspace of $L^2_{(0,1)}(\mathbb{C}^n,\varphi)$, but we do not want to consider that case. Besides boundedness, we are also interested in when N_φ is a compact operator. An operator $A: H_1 \to H_2$ between two Hilbert spaces is called compact, if the image of each bounded sequence in H_1 contains a convergent subsequence in H_2. We shall often use the fact that A is compact if and only if this is true for A^*, which is the case if and only if AA^* is compact. Furthermore A is compact if and only if for each $\varepsilon > 0$ there exists a constant C_ε and a compact operator $K_\varepsilon : H_1 \to H_2$ both dependent on ε, such that

$$\|Af\|_{H_2} \leq \varepsilon \|f\|_{H_1} + C_\varepsilon \|K_\varepsilon f\|_{H_2}$$

for all $f \in H_1$. See [29], Chapter 3, for a proof.
Let us also note the following easy but useful reformulations of existence of N_φ.

Proposition 1.3 *The following statements are equivalent:*

1. *The $\bar\partial$-Neumann operator N_φ exists and is bounded.*
2. *There exists $\varepsilon > 0$ such that $\langle \square_\varphi u, u\rangle_\varphi \geq \varepsilon \langle u, u\rangle_\varphi$ for all $u \in dom(\square_\varphi)$, i.e., $\square_\varphi$ is strictly positive.*
3. *The injection $\iota : dom(\bar\partial) \cap dom(\bar\partial^*_\varphi) \hookrightarrow L^2_{(0,1)}(\mathbb{C}^n,\varphi)$ is continuous, where $dom(\bar\partial) \cap dom(\bar\partial^*_\varphi)$ is equipped with the graph norm $f \mapsto (\|\bar\partial f\|^2_\varphi + \|\bar\partial^*_\varphi f\|^2_\varphi)^{\frac{1}{2}}$.*

Proof. (1) $\Longrightarrow$ (2): Suppose that $\Box_\varphi$ is not strictly positive, so there exists a sequence $(u_n)_n \subset dom(\Box_\varphi)$ with $\|u_n\|_\varphi = 1$ and $\Box_\varphi u_n \to 0$ as $n \to \infty$. Clearly, under this assumption $\Box_\varphi$ can not have a bounded inverse.
(2) $\Longrightarrow$ (3): We have $\Lambda_0^{(0,1)} \subset dom(\Box_\varphi)$, so (2) implies $\|u\|_\varphi \leq \varepsilon^{-1}(\|\bar\partial u\|_\varphi^2 + \|\bar\partial_\varphi^* u\|_\varphi^2)^{\frac{1}{2}}$ for all $u \in \Lambda_0^{(0,1)}$. Now by Lemma 1.2 smooth compactly supported forms are dense in $dom(\Box_\varphi)$ in the graph norm, so (3) follows .
(3) $\Longrightarrow$ (1): From $\|u\|_\varphi^2 \leq C(\|\bar\partial u\|_\varphi^2 + \|\bar\partial_\varphi^* u\|_\varphi^2)$ for all $u \in \Lambda_0^{(0,1)}$ we get by substituting $u = N_\varphi v$ that $\|N_\varphi v\|_\varphi^2 \leq C\langle N_\varphi v, v\rangle_\varphi \leq C\|N_\varphi v\|_\varphi \|v\|_\varphi$, which by Lemma 1.2 shows (1).

□

We call the solution u of the equation $\bar\partial u = f$ the canonical one, if it has minimal norm. So the canonical solution is the unique one orthogonal to $\ker(\bar\partial)$. The canonical solution operator to $\bar\partial$, which we denote by $\mathcal{S}_\varphi$, is the operator assigning to $f \in \ker(\bar\partial)$ the canonical solution of $\bar\partial u = f$.
As for bounded domains, $\mathcal{S}_\varphi$ and N_φ are closely related:

Proposition 1.4 *Suppose that the $\bar\partial$-Neumann operator N_φ exists. Then the canonical solution operator $\mathcal{S}_\varphi$ also exists and is given by $\bar\partial_\varphi^* N_\varphi$. Suppose additionally that N_φ is compact. Then so is $\mathcal{S}_\varphi$.*

Proof. If there is a bounded inverse N_φ to the complex Laplacian, also $\mathcal{S}_\varphi := \bar\partial_\varphi^* N_\varphi$ is bounded, since

$$\|\mathcal{S}_\varphi f\|_\varphi^2 = \|\bar\partial_\varphi^* N_\varphi f\|_\varphi^2 \leq \|\bar\partial_\varphi^* N_\varphi f\|_\varphi^2 + \|\bar\partial N_\varphi f\|_\varphi^2 = \langle \Box_\varphi N_\varphi f, N_\varphi f\rangle_\varphi$$

and by Cauchy – Schwarz inequality

$$\langle \Box_\varphi N_\varphi f, N_\varphi f\rangle_\varphi \leq \|f\|_\varphi \|N_\varphi f\|_\varphi \leq \|N_\varphi\| \|f\|_\varphi^2.$$

By definition, $\bar\partial_\varphi^* N_\varphi f$ is perpendicular to $ker(\bar\partial)$, so it remains to show $\bar\partial(\bar\partial_\varphi^* N_\varphi f) = f$ for any f with $\bar\partial f = 0$. From

$$0 = \bar\partial f = \bar\partial(\bar\partial\bar\partial_\varphi^* + \bar\partial_\varphi^*\bar\partial)N_\varphi f = \bar\partial\bar\partial_\varphi^*\bar\partial N_\varphi f$$

it follows that $\bar\partial_\varphi^*\bar\partial N_\varphi f \in \ker(\bar\partial)$. Also, by definition, $\bar\partial_\varphi^*\bar\partial N_\varphi f \in \ker(\bar\partial)^\perp$, so $\bar\partial_\varphi^*\bar\partial N_\varphi f = 0$ and hence

$$\bar\partial\bar\partial_\varphi^* N_\varphi f = (\bar\partial\bar\partial_\varphi^* + \bar\partial_\varphi^*\bar\partial)N_\varphi f = f.$$

When N_φ is compact, then $\mathcal{S}_\varphi$ is also, since

$$\|\bar\partial_\varphi^* N_\varphi f\|_\varphi^2 \leq \langle f, N_\varphi f\rangle_\varphi \leq \varepsilon\|f\|_\varphi^2 + C_\varepsilon \|N_\varphi f\|_\varphi^2$$

for any $\varepsilon > 0$. In the last step we used Cauchy – Schwarz inequality and the so-called s.c. – l.c. inequality (small constant – large constant inequality), which relies on the fact that for each $\varepsilon > 0$ and any real numbers a, b one has

$$|ab| = |\varepsilon a \varepsilon^{-1} b| \leq \varepsilon^2 a^2 + \varepsilon^{-2} b^2.$$

□

Since we are interested in existence and compactness of N_φ, we first notice that equivalent weight functions have the same properties in this regard.

Lemma 1.5 *Let φ_1 and φ_2 be two equivalent weights, i.e., $C^{-1}\|.\|_{\varphi_1} \leq \|.\|_{\varphi_2} \leq C\|.\|_{\varphi_1}$ for some $C > 0$. Suppose that $\mathcal{S}_{\varphi_2}$ exists. Then also $\mathcal{S}_{\varphi_1}$ exists and $\mathcal{S}_{\varphi_1}$ is compact if and only if $\mathcal{S}_{\varphi_2}$ is compact.*
An analog statement is true for the weighted $\bar{\partial}$-Neumann operator.

Proof. Let ι be the identity $\iota : L^2_{(0,1)}(\mathbb{C}^n, \varphi_1) \to L^2_{(0,1)}(\mathbb{C}^n, \varphi_2)$, $\iota f = f$, let j be the identity $j : L^2_{\varphi_2} \to L^2_{\varphi_1}$ and let furthermore P be the orthogonal projection onto $\ker(\bar{\partial})$ in $L^2_{\varphi_1}$. Since the weights are equivalent, ι and j are continuous, so if $\mathcal{S}_{\varphi_2}$ is compact, $j \circ \mathcal{S}_{\varphi_2} \circ \iota$ gives a solution operator on $L^2_{(0,1)}(\mathbb{C}^n, \varphi_1)$ that is compact. Therefore the canonical solution operator $\mathcal{S}_{\varphi_1} = P \circ i^{-1} \circ \mathcal{S}_{\varphi_2} \circ \iota$ is also compact. Since the problem is symmetric in φ_1 and φ_2, we are done.
The assertion for the Neumann operator follows by the identity $N_\varphi = \mathcal{S}_\varphi \mathcal{S}_\varphi^* + \mathcal{S}_\varphi^* \mathcal{S}_\varphi$.

□

In the beginning we assumed our plurisubharmonic weight function to be $\mathcal{C}^2$. The last Lemma in particular tells us that without loss of generality we can restrict our considerations to weight functions that are $\mathcal{C}^\infty$, since having a given weight we can always add an appropriate bounded function to end up with a smooth weight. Note that two weights differing by a bounded function always induce equivalent norms.
The Lemma also shows that existence and compactness of N_φ only depend on the behavior of the weight function at infinity, since given a weight one can arbitrarily change it on a compact set and will not affect existence nor compactness of N_φ.
We also remark that whereas existence and compactness of N_φ are invariant under equivalent weights, regularity is not. We will take a closer look at this in the next Chapter.

In Chapter two we attack questions of existence and compactness of N_φ by adapting methods used for bounded pseudoconvex domains to our problem. The reason why we consider the case $\Omega = \mathbb{C}^n$ separately is that in this setting there is a connection to spectral theory, which we want to point out in the following. We consider the $\bar{\partial}$-complex on L^2

$$L^2(\mathbb{C}^n, \varphi) \xrightarrow{\bar{\partial}} L^2_{(0,1)}(\mathbb{C}^n, \varphi) \xrightarrow{\bar{\partial}} L^2_{(0,2)}(\mathbb{C}^n, \varphi)$$

for a given smooth plurisubharmonic weight $\varphi : \mathbb{C}^n \to \mathbb{R}^+$. For $v \in L^2(\mathbb{C}^n)$ set

$$\overline{D}_1 v = \sum_{k=1}^n \left(\frac{\partial v}{\partial \overline{z}_k} + \frac{\partial \varphi}{\partial \overline{z}_k} v \right) d\overline{z}_k, \tag{1.4}$$

where the derivatives are taken in the sense of distributions, and for $g = \sum g_j d\overline{z}_j \in L^2_{(0,1)}(\mathbb{C}^n)$

$$\overline{D}_1^* g = -\sum_{j=1}^n \left(\frac{\partial \varphi}{\partial z_j} g_j - \frac{\partial g_j}{\partial z_j} \right). \tag{1.5}$$

The equation $\bar{\partial} u = f$ for $u \in L^2(\mathbb{C}^n, \varphi)$ and $f \in L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ holds if and only if $\overline{D}_1 v = g$, where $v = ue^{-\varphi}$ and $g = fe^{-\varphi}$. Note that there is an obvious compatibility condition for this equation: since $\bar{\partial}^2 = 0$, necessarily $\bar{\partial} f = 0$ must hold. Setting

$$\overline{D}_2 g = \sum_{j,k=1}^n \left(\frac{\partial g_j}{\partial \overline{z}_k} + \frac{\partial \varphi}{\partial \overline{z}_k} g_j \right) d\overline{z}_k \wedge d\overline{z}_j, \tag{1.6}$$

we see that this compatibility condition holds if and only if $\overline{D}_2 g = 0$. So existence and compactness of the canonical solution operator to $\bar{\partial}$ is equivalent to existence and compactness of the canonical solution operator $\mathcal{S}_{\overline{D}_1}$ to $\overline{D}_1$, since

$$\mathcal{S}_{\overline{D}_1} = e^{-\varphi} \mathcal{S}_\varphi e^{\varphi}.$$

The $\overline{D}$-Laplacians $\Box_\varphi^{(0,0)}$ and $\Box_\varphi^{(0,1)}$ are defined by $\Box_\varphi^{(0,0)} = \overline{D}_1^* \overline{D}_1$ and $\Box_\varphi^{(0,1)} = \overline{D}_1 \overline{D}_1^* + \overline{D}_2^* \overline{D}_2$, respectively. It follows that for $g \in L^2_{(0,1)}(\mathbb{C}^n)$

$$\begin{aligned} \Box_\varphi^{(0,1)} g = \sum_{k=1}^n \Bigg[\sum_{j=1}^n \Bigg(& 2 \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} g_j - \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_j} g_k - \frac{\partial^2 g_k}{\partial z_j \partial \overline{z}_j} + \frac{\partial g_k}{\partial \overline{z}_j} \frac{\partial \varphi}{\partial z_j} \\ & - \frac{\partial g_k}{\partial z_j} \frac{\partial \varphi}{\partial \overline{z}_j} + \frac{\partial \varphi}{\partial \overline{z}_j} \frac{\partial \varphi}{\partial z_j} g_k \Bigg) \Bigg] d\overline{z}_k, \end{aligned}$$

see [13] for the computation. So by definition, we have the for our considerations important unitary equivalence

$$\Box_\varphi^{(0,1)} = e^{-\varphi} \Box_\varphi e^{\varphi}. \tag{1.7}$$

If one denotes by

$$M_\varphi = \left(\frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \right)_{jk} \tag{1.8}$$

the Levi-matrix of φ and defines its action on (0,1)-forms $g = \sum_{j=1}^n g_j d\overline{z}_j$ as

$$M_\varphi g = \sum_{j,k=1}^n \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} g_k d\overline{z}_j,$$

we can write the expression for $\Box_\varphi^{(0,1)}$ in the more elegant way

$$\Box_\varphi^{(0,1)} = \Box_\varphi^{(0,0)} \otimes I + 2M_\varphi. \tag{1.9}$$

We remark that M_φ sometimes is also called complex Hessian of φ. Analogously to the complex Laplacian, $\Box_\varphi^{(0,1)}$ is by definition a selfadjoint and positive operator. Defining

$$\triangle_\varphi = -\sum_{j=1}^n \left(\left(\frac{\partial}{\partial x_j} + i \frac{\partial \varphi}{\partial y_j} \right)^2 + \left(\frac{\partial}{\partial y_j} - i \frac{\partial \varphi}{\partial x_j} \right)^2 \right), \tag{1.10}$$

we also notice that

$$4\Box_\varphi^{(0,0)} = \triangle_\varphi - \triangle\varphi. \tag{1.11}$$

Let $\mathfrak{L}$ be the space of bounded linear operators on L^2_φ. For a closed, densely defined operator A on L^2_φ we define the resolvent set of A to be

$$\rho(A) := \{\lambda \in \mathbb{C} : (A - \lambda)^{-1} \in \mathfrak{L}\}.$$

To be more precise, $\lambda \in \rho(A)$ if and only if $(A - \lambda) : dom(A) \to L^2_\varphi$ is bijective and its inverse is bounded. By the Closed Graph Theorem, it suffices to check that $A - \lambda$ is bijective. The complement of the resolvent set is called spectrum

$$\sigma(A) = \mathbb{C} \setminus \rho(A) \tag{1.12}$$

of A. In particular, if $A - \lambda$ has nontrivial kernel, then $\lambda \in \sigma(A)$. In this case λ is called eigenvalue of A and all L^2_φ-functions in this kernel of $A - \lambda$ are called eigenvectors to the eigenvalue λ. The essential spectrum of A, $\sigma_{ess}(A)$, is the set of all $\lambda \in \sigma(A)$ such that $A - \lambda$ is not a Fredholm operator. Fredholm operators are operators F such that both $\ker(F)$ and $\ker(F^*)$ are finite dimensional. The discrete spectrum $\sigma_d(A)$ of A is the complement $\sigma_d(A) = \sigma(A) \setminus \sigma_{ess}(A)$. Note that σ_d is not necessarily a closed set. With these definitions, a point in the discrete spectrum corresponds to an isolated eigenvalue with finite multiplicity. A point in the essential spectrum is either an eigenvalue of infinite multiplicity, an accumulation point of eigenvalues or a point in the interior of $\sigma(A)$, the so-called continuous spectrum.

The function

$$R_A : \rho(A) \to \mathfrak{L},\ \lambda \mapsto (A - \lambda)^{-1} \tag{1.13}$$

is the resolvent of A. We say that the operator A has compact resolvent, if there is some $\lambda \in \rho(A)$ such that the resolvent $R_A(\lambda)$ is compact. By the first resolvent identity

$$R_A(\lambda) - R_A(\lambda') = (z - z')R_A(\lambda)R_A(\lambda'))$$

follows that in this case the resolvent $R_A(\lambda)$ is compact for all $\lambda \in \rho(A)$. This is also equivalent to $\sigma_{ess}(A) = \emptyset$.
If A is selfadjoint, then $\sigma(A) \subset \mathbb{R}$. If moreover A is positive, then $\sigma(A) \subset [0, \infty)$. We also note that $A \geq B$ implies $\inf \sigma(A) \geq \inf \sigma(B)$ and the same inequality holds for the bottom of the essential spectra. In other words, if B is invertible, then so is A; if B has compact resolvent, then A has. For proofs and a more detailed introduction to spectral theory, we refer the reader for instance to [29].
Summing up, we have the following equivalences:

Proposition 1.6 *The $\bar{\partial}$-Neumann operator N_φ exists if and only if the unbounded operator $\square_\varphi^{(0,1)}$ has a bounded inverse.*
The $\bar{\partial}$-Neumann operator N_φ is compact if and only if $\square_\varphi^{(0,1)}$ has a bounded inverse and compact resolvent.

Proof. From our considerations above, we have $\square_\varphi^{(0,1)} = e^{-\varphi}\square_\varphi e^{\varphi}$, see (1.7). By definition of the spectrum, the inverse of $\square_\varphi$ is bounded if and only if $0 \notin \sigma(\square_\varphi)$. Now $T : L^2 \to L^2_\varphi$, $Tf = fe^\varphi$ is an isometry, meaning that the spectra of $\square_\varphi$ and $\square_\varphi^{(0,1)}$ coincide. This shows the first assertion.
If N_φ is compact, then it is in particular bounded and $R_{\square_\varphi^{(0,1)}}(0)$ is compact via $R_{\square_\varphi^{(0,1)}}(0) = e^\varphi N_\varphi e^{-\varphi}$. Conversely, compactness of $R_{\square_\varphi^{(0,1)}}(0)$ implies compactness of N_φ.

$\square$

In Chapter 3 we will make use of these equivalences to find spectral theoretic conditions for existence and compactness of N_φ.

Chapter 2

Complex analytic methods

In this Chapter, we adapt methods which were used for bounded domains to the setting of our problem. By doing this in section 2.1, we achieve different proofs of the statements given in [14]. The authors obtain their results for weighted L^2-spaces on $\mathbb{C}^n$ mainly by applying spectral theoretic arguments. The advantage of the approach we use here is that it also extends to unbounded domains with boundary. For the spectral theoretic approach, this is only true in dimension one.

2.1 Weighted L^2-spaces on $\mathbb{C}^n$

The first thing to do for us is to get an analog of the Kohn – Morrey formula in our setting. Note that due to our definition of $L^2(\mathbb{C}^n, \varphi)$, the formula differs from the usual one, as for instance given in [8], by a factor 2.

Theorem 2.1 ***(Kohn – Morrey formula)*** *Let* $f = \sum_{j=1}^n f_j d\overline{z}_j \in dom(\bar{\partial}) \cap dom(\bar{\partial}_\varphi^*)$. *Then*

$$2\sum_{j,k=1}^n \int\limits_{\mathbb{C}^n} \frac{\partial^2\varphi}{\partial z_j \partial \overline{z}_k} f_j \overline{f}_k e^{-2\varphi} d\lambda + \sum_{j,k=1}^n \int\limits_{\mathbb{C}^n} \left| \frac{\partial f_j}{\partial \overline{z}_k} \right|^2 e^{-2\varphi} d\lambda = \|\bar{\partial} f\|_\varphi^2 + \|\bar{\partial}_\varphi^* f\|_\varphi^2. \tag{2.1}$$

Proof. Since compactly supported smooth forms are dense in $dom(\bar{\partial}) \cap dom(\bar{\partial}_\varphi^*)$ in the graph norm by Lemma 1.2, it suffices to show the formula for those. For general forms in $dom(\bar{\partial}) \cap dom(\bar{\partial}_\varphi^*)$, the Theorem follows by approximating it like in Lemma 1.2 and using the Dominated Convergence Theorem.
Now having given a compactly supported form, one can find a ball containing its support and apply the classical Kohn – Morrey formula for bounded domains, see for instance Proposition 4.3.1 in [8]. The boundary term van-

ishes, since the support is compact.

□

The Kohn – Morrey formula analogously holds for (p,q)-forms, see Proposition 4.3.1 in [8]. Combined with Lemma 1.2, we get some interesting Corollaries.

Corollary 2.2 *Let $f \in dom(\bar\partial) \cap dom(\bar\partial^*_\varphi)$. Then there is a sequence $(f_n)_n \subset \Lambda_0^{(0,1)}(\mathbb{C}^n)$ converging to f in $L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ such that $\langle M_\varphi f_n, f_n\rangle_\varphi \to \langle M_\varphi f, f\rangle_\varphi$.*
If moreover $f \in dom(\Box_\varphi)$, then there exists a sequence $(f_n)_n \subset \Lambda_0^{(0,1)}(\mathbb{C}^n)$ converging to f such that $\langle \Box_\varphi f_n, f_n\rangle_\varphi \to \langle \Box_\varphi f, f\rangle_\varphi$.

Proof. Suppose given a sequence of forms $(f_n)_n$ with coefficients in $\mathcal{C}_0^\infty$ that converges to zero in the graph norm $f \mapsto (\|f\|^2_\varphi + \|\bar\partial f\|^2_\varphi + \|\bar\partial^*_\varphi f\|^2_\varphi)^{\frac{1}{2}}$. Then by Theorem 2.1 also $\langle M_\varphi f_n, f_n\rangle_\varphi$ converges to zero, since both terms on the left hand side are positive.
If $f \in dom(\Box_\varphi)$, then $f \in dom(\bar\partial) \cap dom(\bar\partial^*_\varphi)$. In this case,

$$\langle \Box_\varphi f, f\rangle_\varphi = \|\bar\partial f\|^2_\varphi + \|\bar\partial^*_\varphi f\|^2_\varphi,$$

thus by Lemma 1.2, there exists the claimed sequence with $\langle \Box_\varphi f_n, f_n\rangle_\varphi \to \langle \Box_\varphi f, f\rangle_\varphi$.

□

Corollary 2.3 *Suppose that $f \in dom(\bar\partial) \cap dom(\bar\partial^*_\varphi)$. Then there is a sequence $(f_n)_n \subset \Lambda_0^{(0,1)}(\mathbb{C}^n)$ converging to f in $L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ such that $M_\varphi f_n \to M_\varphi f$ in $L^2_{(0,1)}(\mathbb{C}^n, \varphi)$.*
If moreover $f \in dom(\Box_\varphi)$, then there exists a sequence $(f_n)_n \subset \Lambda_0^{(0,1)}(\mathbb{C}^n)$ converging to f such that $\Box_\varphi f_n \to \Box_\varphi f$ in L^2_φ.

Proof. Since both claims can be proved similarly, we will only show that for any $f \in dom(\Box_\varphi)$ there exists a sequence $(f_n)_n \subset \Lambda_0^{(0,1)}(\mathbb{C}^n)$ such that $\Box_\varphi f_n \to \Box_\varphi f$. For a linear operator A which is positive with respect to an inner product $\langle .,.\rangle$, it is easy to see that $\langle Af, f\rangle = 0$ if and only if $Af = 0$, since such operators always possess a square root, and therefore $\langle Af, f\rangle = \|A^{\frac{1}{2}} f\|$. Thus it suffices to show that we can find a sequence $(f_n)_n$ such that

$$\langle \Box_\varphi (f - f_n), f - f_n\rangle_\varphi \to 0,$$

since $\Box_\varphi$ is cloded and moreover $dom(A) \subset dom(A^{\frac{1}{2}})$. See for instance [29], Satz 8.22 for a proof.
If $f_n \to f$ in $L^2_{(0,1)}(\mathbb{C}^n, \varphi)$, then by Cauchy – Schwarz inequality

$$|\langle \Box_\varphi f, f - f_n\rangle_\varphi| \leq \|\Box_\varphi f\|_\varphi \|f - f_n\|_\varphi \to 0.$$

Thus it remains to show that also

$$\langle \Box_\varphi f_n, f - f_n \rangle_\varphi \to 0.$$

But, by the previous Corollary, there is a sequence such that $\langle \Box_\varphi f_n, f_n \rangle_\varphi \to \langle \Box_\varphi f, f \rangle_\varphi$, so

$$\begin{aligned}
\langle \Box_\varphi f_n, f - f_n \rangle_\varphi &= \langle \Box_\varphi f_n, f \rangle_\varphi - \langle \Box_\varphi f_n, f_n \rangle_\varphi \\
&= \langle \Box_\varphi f_n, f \rangle_\varphi - \langle \Box_\varphi f, f \rangle_\varphi + \langle \Box_\varphi f, f \rangle_\varphi - \langle \Box_\varphi f_n, f_n \rangle_\varphi \\
&= \langle f_n - f, \Box_\varphi f \rangle_\varphi + \langle \Box_\varphi f, f \rangle_\varphi - \langle \Box_\varphi f_n, f_n \rangle_\varphi \\
&\to 0.
\end{aligned}$$

□

Another interesting combination of Theorem 2.1 with Proposition 1.1 is the following Lemma:

Lemma 2.4 *Suppose that the greatest eigenvalue λ_n of M_φ is bounded. Then $\mathcal{A}^2_{(0,1)}(\mathbb{C}^n, \varphi) \subset dom(\bar{\partial}^*_\varphi)$.*

Proof. Let $f \in \mathcal{A}^2_{(0,1)}(\mathbb{C}^n, \varphi)$ and choose a family of smooth cut-off functions $\{\chi_R\}_{R\in\mathbb{N}}$ which are identically one on $\mathbb{B}_{(0,R)}$ and supported in $\mathbb{B}_{(0,R+1)}$, such that all the first order derivatives of χ_R are uniformly bounded in R by a constant C. Then $\chi_R f \in \Lambda_0^{(0,1)}(\mathbb{C}^n) \subset dom(\bar{\partial}) \cap dom(\bar{\partial}^*_\varphi)$, and by Theorem 2.1

$$\|\bar{\partial}(\chi_R f)\|^2_\varphi + \|\bar{\partial}^*_\varphi(\chi_R f)\|^2_\varphi = 2\chi_R^2 \langle M_\varphi f, f \rangle_\varphi + \sum_{j,k} \int_{\mathbb{C}^n} \left| \frac{\partial(\chi_R f_j)}{\partial \bar{z}_k} \right|^2 e^{-2\varphi} d\lambda.$$

By the Leibniz rule, the right hand side equals

$$2\chi_R^2 \langle M_\varphi f, f \rangle_\varphi + \sum_{j,k} \int_{\mathbb{B}_{R+1} \setminus \mathbb{B}_R} \left| \frac{f_j \partial(\chi_R)}{\partial \bar{z}_k} \right|^2 e^{-2\varphi} d\lambda.$$

Now since $f_j \in L^2_\varphi$ and the first order derivatives of χ_R are uniformly bounded, the second term on the right hand side tends to zero as $R \to \infty$. The same argument also applies to the first term on the left. Hence taking the limit, we get

$$\|\bar{\partial}^*_\varphi f\|^2_\varphi \leq 2 \sup \lambda_n(z) \|f\|^2_\varphi,$$

since on the one hand $2\chi_R^2 \langle M_\varphi f, f \rangle_\varphi \leq 2 \sup \lambda_n \|f\|^2_\varphi$, and on the other hand $\|\bar{\partial}^*_\varphi(\chi_R f)\|^2_\varphi \to \|\bar{\partial}^*_\varphi f\|^2_\varphi$ for $R \to \infty$, as observed in the proof of Lemma 1.2.

□

The Kohn – Morrey formula also immediately gives us our first result on existence of the weighted $\bar{\partial}$-Neumann operator. But note that the following criterion is not sharp, as will become clear in Chapter 3, section 3.3.1.

Theorem 2.5 *Suppose that φ is a plurisubharmonic $\mathcal{C}^2$-weight function and suppose that on the complement of a compact set K it holds that $\lambda_1 > \varepsilon$ for some $\varepsilon > 0$, where λ_1 denotes the least eigenvalue λ_1 of the Levi-matrix M_φ . Then N_φ exists and is bounded.*

Proof. Let K be such a compact set with $\lambda_1 > \varepsilon$ on $\mathbb{C}^n \setminus K$. Let $\mathbb{B}$ be a ball with radius R containing K and let χ be a radially symmetric function with compact support, which is identically one on $\mathbb{B}$. Adding $\kappa|z|^2\chi(|z|^2)$ to the weight function φ will give an equivalent weight ψ, since the difference $\varphi - \psi$ is bounded. By construction, the least eigenvalue of ψ is bigger than κ on K. Since it has compact support, the Hessian of $\kappa|z|^2\chi(|z|^2)$ must have negative eigenvalues on $\mathbb{C}^n \setminus K$ – but choosing κ small enough, i.e. for instance $\kappa < \varepsilon(4nR\sup_{|\alpha|\leq 2}|D^\alpha\chi|)^{-1}$, the Hessian of φ will compensate this on $\mathbb{C}^n \setminus K$, such that by switching to an equivalent weight, we may without loss of generality assume that the least eigenvalue of M_φ is uniformly bounded from below by some $\delta > 0$ for all $z \in \mathbb{C}^n$. Now from the Kohn – Morrey formula (Theorem 2.1) it follows that

$$\delta\|f\|_\varphi^2 \leq \|\bar{\partial}f\|_\varphi^2 + \|\bar{\partial}_\varphi^* f\|_\varphi^2.$$

So the complex Laplacian is strictly positive and therefore has a bounded inverse by Proposition 1.3.

□

Analogously it follows, that if the sum of any q eigenvalues of M_φ is greater than $\varepsilon > 0$, then the weighted $\bar{\partial}$-Neumann operator exists on the level of (0,q)-forms.

Example. Suppose that the weight function is $\varphi(z) = |z|^2$. Then $M_\varphi = I$ for all $z \in \mathbb{C}^n$, all eigenvalues are identically 1, and by Theorem 2.1

$$\langle \Box_\varphi f, f\rangle_\varphi \geq \|f\|_\varphi.$$

So the weighted $\bar{\partial}$-Neumann operator N_φ exists and is bounded by Theorem 2.5.

To obtain conditions for compactness of N_φ similar to the bounded case, we first give the following definition:

Definition 2.6 *We say that the norm $|.|$ on $L^2_\varphi(\mathbb{C}^n)$ is strictly weaker than the norm $\|.\|$ if the identity is a compact operator from $(L^2_\varphi(\mathbb{C}^n), \|.\|)$ to $(L^2_\varphi(\mathbb{C}^n), |.|)$.*

With this we may carry over the following well known characterization of compactness to our setting.

Proposition 2.7 *Suppose that φ is a plurisubharmonic weight function such that a bounded $\bar\partial$-Neumann operator N_φ exists (i.e. for instance if it satisfies the assumption of Theorem 2.5) and let $\|.\|_X$ be a norm strictly weaker than $\|.\|_\varphi$. Then the following are equivalent:*

1. *The $\bar\partial$-Neumann operator on the level of $(0,1)$-forms $N_{1,\varphi}$ is compact.*
2. *The embedding of the space $dom(\bar\partial) \cap dom(\bar\partial^*_\varphi)$ provided with the graph norm $u \mapsto (\|u\|^2_\varphi + \|\bar\partial u\|^2_\varphi + \|\bar\partial^*_\varphi u\|^2_\varphi)^{\frac{1}{2}}$ into $L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ is compact.*
3. *For each $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that*
$$\|u\|_\varphi \leq \varepsilon(\|\bar\partial u\|^2_\varphi + \|\bar\partial^*_\varphi u\|^2_\varphi)^{\frac{1}{2}} + C_\varepsilon \|u\|_X$$
for all $u \in dom(\bar\partial) \cap dom(\bar\partial^*_\varphi)$.
4. *The operators $\bar\partial^*_\varphi N_{1,\varphi}$ and $\bar\partial^*_\varphi N_{2,\varphi}$ are both compact, where $N_{2,\varphi}$ denotes the $\bar\partial$-Neumann operator on the level of $(0,2)$-forms.*

Combined with Lemma 1.2, the proof is the same as for bounded pseudoconvex domains, see [5].

Weighted Sobolev spaces

In order to find such weaker norms, we need to define weighted Sobolev spaces. There are several more or less obvious ways to do this, but as is turns out not each of them is suitable to our problem. Having in mind our characterization of compactness in Proposition 2.7, we first try what seems to be the most natural:

Definition 2.8 *Let $H^k(\mathbb{C}^n)$ be the classical Sobolev space on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ with norm $\|f\|^2_k = \sum_{|\alpha|\leq k} \|D^\alpha f\|^2$, where $\alpha = (\alpha_1, \dots, \alpha_{2n})$ is a multiindex and $|\alpha| = \alpha_1 + \cdots + \alpha_{2n}$. Then we define for $k \in \mathbb{N}$ the weighted Sobolev space*
$$e^\varphi H^k(\mathbb{C}^n) := \{f \in L^2_\varphi(\mathbb{C}^n) : \|fe^{-\varphi}\|_k < \infty\}$$
and set $\|f\|_{e^\varphi H^k} := \|fe^{-\varphi}\|_k$.

Then for all k, $e^\varphi H^k$ is a Hilbert space by definition, with Hilbert structure induced by the isometry $T : H^k(\mathbb{C}^n) \to e^\varphi H^k(\mathbb{C}^n)$, $Tf = e^\varphi f$. In particular, T is an isometry between $L^2(\mathbb{C}^n)$ and $L^2(\mathbb{C}^n, \varphi)$. This would be a convenient definition of a weighted Sobolev space, since T would immediately allow to carry over properties from the classical Sobolev spaces to the weighted ones, but the next Lemma shows that it unfortunately is of no use for us.

Lemma 2.9 *For every weight function φ, the injection*

$$\iota : e^{\varphi} H^k(\mathbb{C}^n) \hookrightarrow L^2(\mathbb{C}^n, \varphi)$$

is not compact.

Proof. We have the following diagram:

$$\begin{array}{ccc} H^k(\mathbb{C}^n) & \xrightarrow{T} & e^{\varphi}H^k(\mathbb{C}^n) \\ \iota_1 \big\downarrow & & \big\downarrow \iota_2 \\ L^2(\mathbb{C}^n) & \xrightarrow{T} & L^2(\mathbb{C}^n, \varphi), \end{array}$$

where ι_1 and ι_2 are the inclusions. So if ι_2 is a compact embedding, then the embedding $H^k(\mathbb{C}^n) \hookrightarrow L^2(\mathbb{C}^n)$ is compact. This is not the case, as one can take any $f(z) \in H^k(\mathbb{C}^n)$ with norm equal to one and compact support, and transverses of it, i.e., functions of the form $f_n(z) = f(z - z_n)$. If one chooses z_n such that the supports of f_n are pairwise disjoint, the obtained sequence is bounded and can never have a convergent subsequence in $L^2(\mathbb{C}^n)$, since all f_n are pairwise orthogonal.

□

A better try is the following.

Definition 2.10 *We define for $k \in \mathbb{N}$ the weighted Sobolev space*

$$H^k_{\varphi}(\mathbb{C}^n) := \{f \in L^2_{\varphi}(\mathbb{C}^n) : D^{\alpha} f \in L^2_{\varphi}(\mathbb{C}^n) \quad \forall |\alpha| \leq k\}$$

and set $\|f\|^2_{k,\varphi} := \sum_{|\alpha|\leq k} \|D^{\alpha} f\|^2_{\varphi}$.

This definition already appeared before in the literature, also in connection with the question whether the injection $H^k_{\varphi} \hookrightarrow L^2_{\varphi}$ is compact. Embedding theorems of this kind have been successfully used in Probability Theory as well as in PDE Theory. The reader can find more about this way of defining a Sobolev space in [1] or [21] and the references therein. Unfortunately, we are not aware of any sufficient condition on φ, which would be applicable in our context. But let us nevertheless mention an interesting necessary one taken from [2], that is, $e^{-2\varphi} \in L^1(\mathbb{C}^n)$.
The only working definition of a weighted Sobolev space we know of is the following, which already appeared in [12]:

Definition 2.11 *Let for $X_j = \frac{\partial}{\partial x_j} - 2\frac{\partial \varphi}{\partial x_j}$ and $Y_j = \frac{\partial}{\partial y_j} - 2\frac{\partial \varphi}{\partial y_j}$*

$$H^1_{\varphi,\nabla\varphi}(\mathbb{C}^n) := \{f \in L^2_{\varphi} : X_j f \in L^2_{\varphi}, Y_j f \in L^2_{\varphi} \quad \forall 1 \leq j \leq n\}$$

and set

$$\|f\|^2_{1,\varphi,\nabla\varphi} := \|f\|^2_{\varphi} + \sum_{1\leq j\leq n} (\|X_j f\|^2_{\varphi} + \|Y_j f\|^2_{\varphi}).$$

Similarly define $H^1_{\varphi,\nabla\varphi}(\Omega) := \{f \in L^2_{\varphi}(\Omega) : X_j f \in L^2_{\varphi}(\Omega), Y_j f \in L^2_{\varphi}(\Omega)\ \forall j\}$.

Although this seems at the first glimpse very similar, it is not the same as defining $\|f\|_{1,\varphi,\nabla\varphi} := \|fe^{-2\varphi}\|_1$. In fact, $e^{2\varphi}H^k(\mathbb{C}^n)$ is in general not even embedded in $L^2(\mathbb{C}^n,\varphi)$. The space $H^1_{\varphi,\nabla\varphi}$ can be characterized as follows.

Lemma 2.12 *Let $D_j : dom(D_j) \to L^2_\varphi$, $D_j f = \frac{\partial f}{\partial z_j}$, let $\overline{D}_j$ be the complex conjugate of D_j and let D_j^* be the Hilbert space adjoint of D_j in L^2_φ. Then*

$$H^1_{\varphi,\nabla\varphi} = \cap_{j=1}^n (dom(D_j^*) \cap dom(\overline{D}_j^*)).$$

Proof. By a similar computation as in the proof of Proposition 1.1, the adjoint of D_j is given by $D_j^* f = -e^{2\varphi}\frac{\partial}{\partial \overline{z}_j}(e^{-2\varphi}f)$. Now it is easy to see that

$$2D_j^* f = 2\left(\frac{\partial f}{\partial \overline{z}_j} - 2f\frac{\partial \varphi}{\partial \overline{z}_j}\right) = X_j f + iY_j f$$

and similarly

$$2\overline{D}_j^* f = 2\left(\frac{\partial f}{\partial z_j} - 2\frac{\partial \varphi}{\partial z_j} f\right) = X_j f - iY_j f.$$

So if $X_j f \in L^2_\varphi$ and $Y_j f \in L^2_\varphi$ for all j, then $f \in dom(D_j^*) \cap dom(\overline{D}_j^*)$ for all j and vice versa.

□

Note that by definition, $H^1_{\varphi,\nabla\varphi}(\mathbb{C}^n) \hookrightarrow L^2_\varphi(\mathbb{C}^n)$ continuously. Therefore one has the converse inclusion for the dual spaces $(L^2_\varphi)^* \cong L^2_\varphi \hookrightarrow (H^1_{\varphi,\nabla\varphi})^*$. But $H^1_{\varphi,\nabla\varphi}$ is a Hilbert space with inner product

$$\langle f,g\rangle_{H^1_{\varphi,\nabla\varphi}} = \langle f,g\rangle_\varphi + \sum_{j=1}^n \langle X_j f, X_j g\rangle_\varphi + \langle Y_j f, Y_j g\rangle_\varphi,$$

so by the Riesz Theorem each continuous linear functional ϕ on $H^1_{\varphi,\nabla\varphi}$ can be represented by

$$\phi(f) = \langle f,y\rangle_{H^1_{\varphi,\nabla\varphi}}$$

for an uniquely determined $y \in H^1_{\varphi,\nabla\varphi}$. If we denote the L^2_φ-adjoint of X_j by X_j^*, we can express $\phi(f)$ as

$$\phi(f) = \langle f, y + \sum_{j=1}^n (X_j^* X_j + Y_j^* Y_j) y\rangle_\varphi \tag{2.2}$$

via integrating by parts, since $\mathcal{C}_0^\infty$ is dense in $H^1_{\varphi,\nabla\varphi}$. X_j and Y_j can be viewed as unbounded operators on L^2_φ. From this point of view, for each continuous linear functional ϕ there exists an unique $y \in dom(X_j) \cap dom(Y_j)$

such that (2.2), but note that $(X_j^* X_j + Y_j^* Y_j)y$ not necessarily belongs to L^2_φ anymore. The dual norm is defined by

$$\|\phi\|_{-1,\varphi,\nabla\varphi} = \sup_{\|f\|_{1,\varphi,\nabla\varphi}=1} |\phi(f)|, \tag{2.3}$$

where we denote the dual space $(H^1_{\varphi,\nabla\varphi})^*$ by $H^{-1}_{\varphi,\nabla\varphi}$. Coming back to the injection $H^1_{\varphi,\nabla\varphi} \hookrightarrow L^2_\varphi$ and its dual, each $y \in L^2_\varphi$ defines a continuous linear functional on $H^1_{\varphi,\nabla\varphi}$ in the natural way by $\phi_y(f) = \langle f, y\rangle_\varphi$. Therefore for functionals ϕ_y, which we now identify with y, it holds that

$$\|y\|_{-1,\varphi,\nabla\varphi} = \sup_{f\in\mathcal{C}_0^\infty,\ \|f\|_\varphi=1} |\langle f, y\rangle_\varphi|. \tag{2.4}$$

By density of $\mathcal{C}_0^\infty$, (2.4) holds for any $\phi \in H^{-1}_{\varphi,\nabla\varphi}$. Now we need to know when $\|.\|_{-1,\varphi,\nabla\varphi}$ in fact defines a norm strictly weaker than the weighted L^2-norm $\|.\|_\varphi$. A condition is given by the following Proposition from [12], which is the analog of the Rellich – Kondrachov Theorem in our setting:

Proposition 2.13 *Suppose that there is a $\theta \in (0,2)$ such that*

$$\lim_{|z|\to\infty} (\theta|\nabla\varphi(z)|^2 + \triangle\varphi(z)) = \infty.$$

Then the embedding of $H^1_{\varphi,\nabla\varphi}(\mathbb{C}^n)$ into $L^2(\mathbb{C}^n,\varphi)$ is compact.

Proof. For the proof we just refer to [12], since we will prove a more general result (Proposition 2.23) using the same idea later.

$\square$

Note that the assumption of Proposition 2.13 is in particular fulfilled if the greatest eigenvalue $\lambda_n(z)$ of the Levi-matrix M_φ tends to infinity as $|z| \to \infty$, since $\triangle\varphi = 4\,\mathrm{tr}(M_\varphi)$, and the trace equals to the sum of the eigenvalues, which are all positive. But clearly, this is not necessary, as for the weight $\varphi(z) = |z|^2$ for instance all eigenvalues of M_φ are bounded by one and the assumption is nevertheless fulfilled. This example also shows that having a compact injection of our weighted Sobolev space for some φ does not imply compactness of N_φ. Nevertheless, the form domain of the operator $\square_\varphi^{(0,1)}$ as defined in (3.5) looks very similar to $H^1_{\varphi,\nabla\varphi}$ and compactness of the injection of the form domain into L^2 is sufficient for $\square_\varphi^{(0,1)}$ to have compact resolvent.

Compactness of N_φ

In the sequel we give a new independent proof of the main result on compactness in [14]. The idea we use is the one of Catlin's proof in [6], showing

that property (P) implies compactness of the $\bar\partial$-Neumann operator.
Let us recall that a bounded domain is said to satisfy property (P), if for each positive number M there is a plurisubharmonic function $\lambda \in \mathcal{C}^\infty(\overline{\Omega})$ with $0 \leq \lambda \leq 1$, such that

$$\sum_{j,k=1}^{n} \frac{\partial^2 \lambda}{\partial z_j \partial \overline{z}_k}(z) t_j \bar{t}_k \geq M|t|^2$$

for all $z \in \partial\Omega$ and all $t \in \mathbb{C}^n$. This notion is due to Catlin and was introduced in [6] for the first time.
It was also shown in [6] that each bounded domain of finite type satisfies property (P).

Theorem 2.14 ***(Gårding's inequality)*** *Let Ω be a smooth bounded domain. Then for any form u compactly supported in Ω with coefficients in $H^1_{\varphi,\nabla\varphi}(\Omega)$ it holds*

$$\|u\|^2_{1,\varphi,\nabla\varphi} \leq C(\Omega,\varphi)\left(\|\bar\partial u\|^2_\varphi + \|\bar\partial^*_\varphi u\|^2_\varphi + \|u\|^2_\varphi\right).$$

Proof. The operator $-\Box_\varphi$ is strictly elliptic since its principal part equals the Laplacian. Now $-\Box_\varphi = -(\bar\partial \oplus \bar\partial^*_\varphi)^* \circ (\bar\partial \oplus \bar\partial^*_\varphi)$, so from general PDE theory follows that the system $\bar\partial \oplus \bar\partial^*_\varphi$ is elliptic. This is, because a differential operator P of order s is elliptic if and only if $(-1)^s P^* \circ P$ is strictly elliptic. So because of ellipticity, one has on each smooth bounded domain Ω the classical Gårding inequality

$$\|u\|^2_1 \leq C(\Omega)\left(\|\bar\partial u\|^2 + \|\bar\partial^*_\varphi u\|^2 + \|u\|^2\right).$$

But our weight φ is smooth on $\overline{\Omega}$, so φ and all first order derivatives are uniformly bounded, therefore

$$\|u\|^2_{1,\varphi,\nabla\varphi} \lesssim \|u\|^2_{1,\varphi} + \|u\|^2_\varphi \leq \|u\|^2_1 + \|u\|^2.$$

Applying the classical Gårding inequality, we get

$$\|u\|^2_{1,\varphi,\nabla\varphi} \lesssim \|\bar\partial u\|^2 + \|\bar\partial^*_\varphi u\|^2 + \|u\|^2$$

and by boundedness of φ also

$$\|\bar\partial u\|^2 + \|\bar\partial^*_\varphi u\|^2 + \|u\|^2 \lesssim \|\bar\partial u\|^2_\varphi + \|\bar\partial^*_\varphi u\|^2_\varphi + \|u\|^2_\varphi,$$

which proves the Theorem.

□

Theorem 2.15 *Let φ be a plurisubharmonic smooth weight function. Suppose that the least eigenvalue $\lambda_1(z)$ of the Levi-matrix M_φ tends to infinity at infinity. Then N_φ is compact.*

Proof. By assumption also $\triangle\varphi \to \infty$ as $z \to \infty$, since $\triangle\varphi = 4\,\mathrm{tr}(M_\varphi)$. So by Proposition 2.13, it suffices to show a compactness estimate like in Proposition 2.7 to prove the Theorem.
Let $\varepsilon > 0$ be given and choose an integer M such that $M^{-1} < \frac{\varepsilon}{2}$. By assumption, there exists R such that $\lambda(z) > M$ whenever $|z| > R$. Now let χ be a smooth cutoff function which is identically one on $\mathbb{B}(R, 0)$. Hence for each $f \in dom(\bar\partial) \cap dom(\bar\partial^*_\varphi)$ we estimate:

$$\begin{aligned}
M\|f\|_\varphi^2 \leq & \sum_{j,k=1}^{n} \int\limits_{\mathbb{C}^n\setminus\mathbb{B}(R,0)} \frac{\partial^2\varphi}{\partial z_j \partial \overline{z}_k} f_j \overline{f}_k e^{-2\varphi} d\lambda + M\|\chi f\|_\varphi^2 \\
\leq & Q_\varphi(f,f) + M\langle \chi f, f\rangle_\varphi \\
\leq & Q_\varphi(f,f) + M\|\chi f\|_{1,\varphi,\nabla\varphi}\|f\|_{-1,\varphi,\nabla\varphi}
\end{aligned}$$

by definition of the dual norm (2.4). Now using s.c. - l.c. inequality, we estimate the last line by

$$Q_\varphi(f,f) + Ma\|\chi f\|^2_{1,\varphi,\nabla\varphi} + a^{-1}M\|f\|^2_{-1,\varphi,\nabla\varphi} = (^*),$$

where a is to be chosen a bit later. Applying Gårdings inequality 2.14 to the second term yields

$$(^*) \leq Q_\varphi(f,f) + MaC_{\chi,\varphi}Q_\varphi(f,f) + a^{-1}M\|f\|^2_{-1,\varphi,\nabla\varphi},$$

where the constant $C_{\chi,\varphi}$ is dependent on the support and the derivatives of χ. Dividing by M, we get

$$\|f\|^2_\varphi \leq \frac{1}{M}Q_\varphi(f,f) + aC_{\chi,\varphi}Q_\varphi(f,f) + a^{-1}\|f\|^2_{-1,\varphi,\nabla\varphi},$$

so if we choose a such that $aC_{\chi,\varphi} < \frac{\varepsilon}{2}$, we get the desired estimate.

□

Remark. Requiring that the least eigenvalue of M_φ tends to infinity, is analogous to property (P) introduced by Catlin in [6] in case of bounded pseudoconvex domains. Therefore the proof is essentially the same. Note that we do not need assumptions like boundedness of the weight function, since we need not to switch back from weighted norms to unweighted ones.

Regularity

We conclude this section with the following remark on regularity. Since there is no boundary in our situation, we are dealing with a purely elliptic problem and get regularity in the interior more or less for free.

Theorem 2.16 *Denote by $H^m_{loc}(\mathbb{C}^n)$ the space of functions which locally belong to the classical unweighted Sobolev space $H^m(\mathbb{C}^n)$.*
Suppose that $\bar\partial u = f$ and suppose $f \in H^m_{loc\,(0,1)}(\mathbb{C}^n) \cap L^2_{(0,1)}(\mathbb{C}^n, \varphi)$. Then $u \in H^{m+1}_{loc}(\mathbb{C}^n) \cap L^2(\mathbb{C}^n, \varphi)$. In particular, if there exists a canonical solution operator $\mathcal{S}_\varphi$ to $\bar\partial$, it is always globally regular in the sense that it maps $\mathcal{C}^\infty_{(0,1)}(\mathbb{C}^n) \cap L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ into $\mathcal{C}^\infty(\mathbb{C}^n) \cap L^2(\mathbb{C}^n, \varphi)$.
Suppose that $\Box_\varphi v = g$ and suppose $g \in H^m_{loc\,(0,1)}(\mathbb{C}^n) \cap L^2_{(0,1)}(\mathbb{C}^n, \varphi)$. Then $v \in H^{m+2}_{loc\,(0,1)}(\mathbb{C}^n) \cap L^2_{(0,1)}(\mathbb{C}^n, \varphi)$. In particular, if there exists a weighted $\bar\partial$-Neumann operator N_φ, it is always globally regular in the sense that it maps $\mathcal{C}^\infty_{(0,1)}(\mathbb{C}^n) \cap L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ into itself.

Proof. Since $\bar\partial u = f$, one has $\partial\bar\partial u = \partial f$. Now $\partial f \in H^{m-1}_{loc\,(0,2)}(\mathbb{C}^n)$ and $\partial\bar\partial$ acts componentwise as the Laplacian, so the first part of the Theorem follows from regularity properties of the elliptic operator $\triangle$, since from $\triangle g = h$ and $h \in H^m_{loc}(\mathbb{C}^n)$ follows $g \in H^{m+2}_{loc}(\mathbb{C}^n)$. This is in fact true for any elliptic operator of second order, thus in particular for $\Box_\varphi$. The reader can find more on elliptic regularity for instance in [9], Chapter 6.3.

$\square$

Remark. This should be compared with regularity properties of the $\bar\partial$-Neumann operator on bounded pseudoconvex domains. If we consider a smooth bounded pseudoconvex domain Ω in $\mathbb{C}^n$, then the spaces $L^2(\Omega)$ and $L^2(\Omega, \phi_t)$ coincide, when $\phi_t = t|z|^2$, since all these weights are equivalent. Lemma 1.5 tells us that either N_{ϕ_t} is compact for all $t \geq 0$ or it fails to be compact for all $t \geq 0$. However, for each domain of this form a sufficiently large t can be found such that N_{ϕ_t} becomes exactly regular, meaning that N_{ϕ_t} maps the Sobolev space $H^k_{(p,q)}(\Omega)$ boundedly into itself for every $k \geq 0$, while nonetheless there are domains for which the unweighted $\bar\partial$-Neumann operator N is not exactly regular at all. See for instance [8], Chapter 6.1 on this topic.

Although $\Box_\varphi$ is strictly elliptic, the question whether $\mathcal{S}_\varphi$ is exactly regular is harder to answer. This is because our domain is not bounded and neither are the coefficients of $\Box_\varphi$. Only in a very special case the question is easy - this is, when $\mathcal{A}^2_\varphi = \{0\}$. In this case, there is only one solution operator to $\bar\partial$, namely the canonical one. So if f is a (0,1)-form with coefficients belonging to H^k_φ (see Definition 2.10) and u is a solution to $\bar\partial u = f$, it follows that $\bar\partial D^\alpha u = D^\alpha f$, since $\bar\partial$ commutes with $\frac{\partial}{\partial x_j}$ and $\frac{\partial}{\partial y_j}$. Now $\mathcal{S}_\varphi$ is continuous, therefore $\|D^\alpha u\|_\varphi \leq \|\mathcal{S}_\varphi\| \|D^\alpha f\|_\varphi$, meaning that $u \in H^k_\varphi$. Thus in this case, $\mathcal{S}_\varphi$ is a bounded operator from $H^k_\varphi \to H^k_\varphi$.

2.2 Unbounded domains with boundary

Now let us consider unbounded pseudoconvex domains $\Omega \subset \mathbb{C}^n$, and denote the boundary of Ω by $\partial\Omega$. Suppose that Ω is given by a defining function $r(z)$, and in the sequel we always will assume $|\nabla r(z)|^2 = 1$ for $z \in \partial\Omega$. Analogously to the previous section we define the spaces $L^2(\Omega)$ and $L^2_\varphi(\Omega)$ and similarly for forms, but since we are interested in when the $\bar\partial$-Neumann operator is compact we shall restrict ourselves to the case with weight. The next Lemma clarifies the reason why we do this. But let us first give the following Definition, see for instance [1] for more details on this notion.

Definition 2.17 *We say that a domain Ω quasibounded if and only if*

$$\lim_{z\in\Omega,\ |z|\to\infty} dist(z, \partial\Omega) = 0.$$

Equivalently, Ω is quasibounded if and only if there does not exist an $r > 0$ and a sequence of congruent pairwise disjoint balls with radius r contained in Ω.

Example. An example of an unbounded but quasibounded domain is $\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : \ |z_1^2 + z_2^2| < 1\}$. The domain contains all points of the form $(z, \pm iz)$, so it is definitely unbounded. To see that it is quasibounded, let us introduce the new coordinates $w_1 = z_1 + iz_2$ and $w_2 = z_1 - iz_2$, so $\Omega = \{|w_1 w_2| < 1\}$. Written in this form, quasiboundedness of Ω is obvious. This latter domain $\{|zw| < 1\} \subset \mathbb{C}^2$ in fact is the prime example of a quasibounded domain.

A modified bounded version of this latter domain is $\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1^2 + z_2^2| + |z_1|^2 + |z_2|^2 < 1\}$, the so-called minimal ball. This domain is interesting since an explicit formula of the Bergman kernel is known, although it is neither Reinhardt nor homogeneous, see [27].

Lemma 2.18 *If Ω is unbounded but not quasibounded, then the unweighted $\bar\partial$-Neumann operator is – if it exists – not compact on $L^2(\Omega)$.*

Proof. If $\mathbb{B}(z_l, r)$ is a sequence of disjoint balls contained in Ω, take a $(0,1)$-form v with coefficients in $\mathcal{C}_0^\infty$ supported in $\mathbb{B}(z_0, r)$ such that $\|v\|_{L^2} = 1$. Without loss of generality we can assume $z_0 = 0$ and clearly, $v \in dom(\Box)$. Now take transverses $v_n(z) = v(z - z_n)$ of v. They have disjoint support and $\|v_n\|_{L^2} = 1$. $\Box v_n$ is a bounded sequence, but the functions $N\Box v_n = v_n$ are pairwise orthogonal hence they can not contain a convergent subsequence. Thus N is not compact.

$\Box$

Remark. If there is a sequence $\mathbb{B}(z_l, r_l)$ of disjoint balls contained in Ω, such that $r_l \to \infty$, then by a similar argument it follows that the unweighted $\bar\partial$-Neumann operator on Ω is not bounded.

So consider the space $L^2_\varphi(\Omega)$ with $\varphi : \Omega \to \mathbb{R}^+$ smooth and plurisubharmonic in Ω as well as continuous on $\overline{\Omega}$, such that $\varphi(z) \to \infty$ for $|z| \to \infty$. From now on in this section, a weight function on Ω is a function having these properties. $\bar\partial$ can be extended to an unbounded closed operator on $L^2_\varphi(\Omega)$ and therefore possesses an adjoint $\bar\partial^*_\varphi$. The following Proposition characterizes the domain of $\bar\partial^*_\varphi$, and the criterion is exactly the combination of the $\bar\partial$-Neumann conditions and the integrability condition from Proposition 1.1.

Proposition 2.19 *Let $f = \sum f_j d\bar z_j \in L^2_{(0,1)}(\Omega, \varphi)$. Then $f \in dom(\bar\partial^*_\varphi)$ if and only if $\sum_{j=1}^n f_j \frac{\partial r}{\partial z_j} = 0$ on $\partial\Omega$ as well as*

$$\sum_{j=1}^n \left(\frac{\partial f_j}{\partial z_j} - 2\frac{\partial \varphi}{\partial z_j} f_j \right) \in L^2(\mathbb{C}^n, \varphi).$$

Proof. The proof is similar to the one of Proposition 1.1. Given an f fulfilling the conditions and using the same notation as in the proof of 1.1 we have

$$\begin{aligned}\langle \chi_R f, \bar\partial g\rangle_\varphi &= -\int\limits_{\mathbb{C}^n} \sum_{j=1}^n \frac{\partial}{\partial z_j} \left(\chi_R f_j e^{-2\varphi}\right) \overline{g}\, d\lambda + \int\limits_{\partial\Omega} \chi_R \sum_{j=1}^n f_j \frac{\partial r}{\partial z_j} e^{-2\varphi} \overline{g}\, d\sigma \\ &= -\int\limits_{\mathbb{C}^n} \sum_{j=1}^n \frac{\partial}{\partial z_j} \left(\chi_R f_j e^{-2\varphi}\right) \overline{g}\, d\lambda\end{aligned}$$

for all $g \in dom(\bar\partial)$ by assumption. So $f \in dom(\bar\partial^*_\varphi)$ by the same arguments as in the proof of Proposition 1.1.
Conversely, for $f \in dom(\bar\partial^*_\varphi)$ and any $g \in \mathcal{C}^\infty_0$,

$$\langle \bar\partial^*_\varphi f, g\rangle_\varphi = \langle f, \bar\partial g\rangle_\varphi = \sum_{j=1}^n \int\limits_\Omega f_j \frac{\partial \overline{g}}{\partial z_j} e^{-2\varphi} d\lambda.$$

Again in the same way as in the proof of Proposition 1.1 we obtain

$$\bar\partial^*_\varphi f = -e^{2\varphi} \sum_{j=1}^n \frac{\partial}{\partial z_j} \left(f e^{-2\varphi}\right),$$

hence $e^{2\varphi}\sum_{j=1}^n \frac{\partial}{\partial z_j}\left(f_j e^{-2\varphi}\right) \in L^2_\varphi(\Omega)$. Now doing the same calculation for general $g \in dom(\bar\partial)$, we get from integrating by parts

$$\langle \bar\partial^*_\varphi f, g\rangle_\varphi = \langle f, \bar\partial g\rangle_\varphi = \langle -e^{2\varphi}\sum_{j=1}^n \frac{\partial}{\partial z_j}\left(f e^{-2\varphi}\right), g\rangle_\varphi + \int\limits_{\partial\Omega} g\sum_{j=1}^n f_j \frac{\partial r}{\partial z_j} e^{-2\varphi} d\sigma.$$

Thus comparing the two expressions for $\bar\partial^*_\varphi f$ we see that the boundary integral has to vanish for all g, which is the case if and only if $\sum_{j=1}^n f_j \frac{\partial r}{\partial z_j} = 0$ on $\partial\Omega$.

□

Lemma 2.20 *Suppose that $\partial\Omega$ is $\mathcal{C}^{k+1}$. Then for any $f \in dom(\bar\partial) \cap dom(\bar\partial^*_\varphi)$ there is a sequence $(f^{(n)})_n \subset \mathcal{C}^k(\overline{\Omega})$ such that $f^{(n)} \to f$ in the graph norm $f \mapsto (\|f\|^2_\varphi + \|\bar\partial f\|^2_\varphi + \|\bar\partial^*_\varphi f\|^2_\varphi)^{\frac{1}{2}}$ and $f^{(n)}$ vanishes on $\Omega \setminus \mathbb{B}(0,n)$ as well as $\sum_{j=1}^n f_j^{(n)} \frac{\partial r}{\partial z_j}$ on $\partial\Omega$.*

Proof. Keeping the notation from Lemma 1.2, we see by a similar proof that $\chi_R f \to f$ in the graph norm. Now by Lemma 4.3.2 in [8], for each R fixed $\chi_R f$ can be approximated by a sequence with the claimed properties. Thus the Lemma follows by choosing a diagonal sequence.

□

As in the previous section, we have the following

Theorem 2.21 ***(Kohn – Morrey formula)*** *Let Ω be of class $\mathcal{C}^2$ and let r be a defining function of Ω. Then for any $f = \sum_{j=1}^n f_j d\bar z_j \in dom(\bar\partial) \cap dom(\bar\partial^*_\varphi)$*

$$2\sum_{j,k}\int\limits_\Omega \frac{\partial^2\varphi}{\partial z_j \partial \bar z_k} f_j \overline{f}_k e^{-2\varphi} d\lambda + \sum_{j,k}\int\limits_\Omega \left|\frac{\partial f_j}{\partial \bar z_k}\right|^2 e^{-2\varphi} d\lambda + \sum_{j,k}\int\limits_{\partial\Omega} \frac{\partial^2 r}{\partial z_j \partial \bar z_k} f_j \overline{f}_k e^{-2\varphi} d\sigma$$
$$= \|\bar\partial f\|^2_\varphi + \|\bar\partial^*_\varphi f\|^2_\varphi,$$

where σ denotes the surface measure.

Proof. The proof is exactly the same as in the case of bounded Ω, using Lemma 2.20.

□

Having this, one can show analogously to the previous section:

Theorem 2.22 *Suppose that Ω is pseudoconvex and of class $\mathcal{C}^2$, and let φ be a weight function on Ω. Suppose that*

$$\liminf_{z\in\Omega,\ |z|\to\infty} \lambda_1(z) > \varepsilon > 0,$$

where λ_1 denotes the least eigenvalue of the Levi-matrix M_φ. Then N_φ exists and is bounded.

To get a first result on compactness, let us now impose the following condition on the weight function. Suppose that

$$\lambda_1 \to \infty \text{ for } |z| \to \infty \text{ as well as } \lambda_1 \to \infty \text{ for } z \to \partial\Omega. \tag{2.5}$$

Under this assumption, we can carry over the results from the previous section.

Proposition 2.23 *Suppose that the weight function satisfies*

$$\lim_{|z|\to\infty}(\theta|\nabla\varphi(z)|^2 + \triangle\varphi(z)) = +\infty \text{ as well as}$$
$$\lim_{z\to\partial\Omega}(\theta|\nabla\varphi(z)|^2 + \triangle\varphi(z)) = +\infty$$

for some $\theta \in (0,2)$. Then the embedding of $H^1_{\varphi,\nabla\varphi}(\Omega)$ into $L^2(\Omega,\varphi)$ is compact.

Proof. Using the same notation as in the formulation of Proposition 2.13, we have for the vector fields X_j and their formal adjoints $X_j^* = -\frac{\partial}{\partial x_j}$ for any $f \in \mathcal{C}_0^\infty(\Omega)$

$$(X_j + X_j^*)f = -2\frac{\partial\varphi}{\partial x_j}\, f \text{ and } [X_j, X_j^*]f = -2\frac{\partial^2\varphi}{\partial x_j^2}\, f,$$

as well as

$$\langle[X_j, X_j^*]f, f\rangle_\varphi = \|X_j^* f\|_\varphi^2 - \|X_j f\|_\varphi^2,$$
$$\|(X_j + X_j^*)f\|_\varphi^2 \leq (1 + 1/\epsilon)\|X_j f\|_\varphi^2 + (1+\epsilon)\|X_j^* f\|_\varphi^2,$$

for each $\varepsilon > 0$. Similar relations hold for the vector fields Y_j. It follows that

$$\langle 2|\nabla\varphi(z)|^2 + (1+\epsilon)\triangle\varphi(z)f, f\rangle_\varphi \leq (2 + \epsilon + 1/\epsilon)\sum_{j=1}^n(\|X_j f\|_\varphi^2 + \|Y_j f\|_\varphi^2),$$

and since $\mathcal{C}_0^\infty(\Omega)$ is dense in $H^1_{\varphi,\nabla\varphi}(\Omega)$ by definition, this inequality is valid for all $f \in H^1_{\varphi,\nabla\varphi}(\Omega)$.
If $(f_k)_k$ is a sequence in $H^1_{\varphi,\nabla\varphi}(\Omega)$ converging weakly to 0, then $(f_k)_k$ is also bounded and our assumption implies that for any $N \in \mathbb{N}$ we can find a smooth bounded domain $\Omega_N \subset \Omega$ such that

$$\Psi(z) = 2|\nabla\varphi(z)|^2 + (1+\epsilon)\triangle\varphi(z) > N$$

on $\Omega \setminus \Omega_N$. Therefore we obtain

$$\begin{aligned}\int_\Omega |f_k|^2 e^{-2\varphi}\, d\lambda &\leq \int_{\Omega_N} |f_k|^2 e^{-2\varphi}\, d\lambda + \int_{\Omega\setminus\Omega_N} \frac{\Psi|f_k|^2}{N} e^{-2\varphi}\, d\lambda \\ &\leq \|f_k\|^2_{L^2(\Omega_N)} + \frac{C_\theta}{N}\|f_k\|^2_{1,\varphi,\nabla\varphi}.\end{aligned}$$

Now the classical Rellich – Kondrachov Theorem asserts that the injection $H^1(\Omega_N) \hookrightarrow L^2(\Omega_N)$ is compact. Combined with our assumption, this shows that a subsequence of $(f_k)_k$ tends to 0 in $L^2(\mathbb{C}^n, \varphi)$, which proves the Proposition.

□

Remark. Note that we do not need to assume φ to be plurisubharmonic in Proposition 2.23. If it is, one can drop the θ in the formulation of the Proposition.

Using Lemma 2.20, again the same proof as in [5] shows that Proposition 2.7 analogously holds in the setting of section 2.2.

Proposition 2.24 *Suppose that φ is a plurisubharmonic weight function such that a bounded $\bar\partial$-Neumann operator N_φ exists and let $\|.\|_X$ be a norm on $L^2_\varphi(\Omega)$ strictly weaker than $\|.\|_\varphi$. Then the following are equivalent:*

1. *The $\bar\partial$-Neumann operator on the level of $(0,1)$-forms $N_{1,\varphi}$ on $L^2_{(0,1)}(\Omega, \varphi)$ is compact.*
2. *The embedding of the space $dom(\bar\partial) \cap dom(\bar\partial^*_\varphi)$ provided with the graph norm $u \mapsto (\|u\|^2_\varphi + \|\bar\partial u\|^2_\varphi + \|\bar\partial^*_\varphi u\|^2_\varphi)^{\frac{1}{2}}$ into $L^2_{(0,1)}(\Omega, \varphi)$ is compact.*
3. *For each $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that*
$$\|u\|_\varphi \le \varepsilon(\|\bar\partial u\|^2_\varphi + \|\bar\partial^*_\varphi u\|^2_\varphi)^{\frac{1}{2}} + C_\varepsilon \|u\|_X$$
for all $u \in dom(\bar\partial) \cap dom(\bar\partial^*_\varphi)$.
4. *The operators $\bar\partial^*_\varphi N_{1,\varphi}$ and $\bar\partial^*_\varphi N_{2,\varphi}$ are both compact, where $N_{2,\varphi}$ denotes the $\bar\partial$-Neumann operator on the level of $(0,2)$-forms.*

Eventually we use Theorem 2.14 to repeat the proof of Theorem 2.15 in this setting and get

Theorem 2.25 *Suppose that the weight function φ on Ω fulfills condition (2.5). Then N_φ is compact.*

Example. Suppose that $\Omega \subset \mathbb{C}$ is the upper halfspace, given by $\Omega = \{z : \mathfrak{Im} z > 0\}$. Let $\varphi_M = e^{-My}$, where $y = \mathfrak{Im} z$. Then, clearly, $0 \le \varphi_M \le 1$ and φ_M is subharmonic since $\triangle\varphi_M = M^2 e^{-My}$. In particular $\triangle\varphi_M = M^2$ on $\partial\Omega$. If we set
$$\varphi = \sum_{j=1}^{\infty} \frac{1}{2^j} \varphi_{2^j}, \tag{2.6}$$
then φ equals a smooth subharmonic function in Ω such that $\triangle\varphi \to \infty$ as $z \to \partial\Omega$. This consideration shows that given a weight ψ on Ω such that the

least eigenvalue λ_1 of M_ψ fulfills $\lambda_1 \to \infty$ for $|z| \to \infty$, one can always find a weight inducing an equivalent norm and satisfying condition (2.5).
In a bit more generality, suppose that $\Omega \subset \mathbb{C}^n$ admits a global strictly plurisubharmonic defining function $r(z)$. Ω therefore is strictly pseudoconvex and we again set $\varphi_M = e^{Mr(z)}$. As before, $0 \le \varphi_M \le 1$ and φ_M is strictly plurisubharmonic:

$$\frac{\partial^2 \varphi_M}{\partial z_j \partial \bar{z}_k}(z) = M \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}(z) e^{Mr(z)} + M^2 \frac{\partial r}{\partial z_j}(z) \frac{\partial r}{\partial \bar{z}_k}(z) e^{Mr(z)}.$$

Since we assumed strict plurisubharmonicity, for all $z \in \partial\Omega$ the least eigenvalue of $\left(\frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}\right)_{jk}$ is strictly positive. But although it could possibly tend to 0, (2.6) will nevertheless give us a bounded function such that the least eigenvalue of the Hessian of φ explodes at every boundary point, meaning that we proved the following

Lemma 2.26 *Let $\Omega \subset \mathbb{C}^n$ be a domain with a smooth strictly plurisubharmonic global defining function $r(z)$. Then there is a bounded smooth plurisubharmonic function φ such that all eigenvalues of the Levi-matrix of φ tend to infinity as $z \to \partial\Omega$.*

In particular we have the following Proposition

Proposition 2.27 *Let $\Omega \subset \mathbb{C}^n$ be a domain with a smooth strictly plurisubharmonic global defining function $r(z)$. Then the $\bar{\partial}$-Neumann N_φ operator is compact, if the least eigenvalue of the Levi-matrix of the weight tends to infinity at infinity.*

A similar construction also works under the weaker assumption that $\partial\Omega$ just satisfies property (P). Here we can not find a bounded function with properties like in Lemma 2.26, but nevertheless it is possible to construct for any given weight with $\lambda_1 \to \infty$ for $|z| \to \infty$ an equivalent one which fulfills condition (2.5).

Theorem 2.28 *Suppose that Ω is smooth, pseudoconvex and suppose that $\partial\Omega$ satisfies property (P) introduced by Catlin [6]. Then the weighted $\bar{\partial}$-Neumann operator N_φ is compact provided that the least eigenvalue $\lambda_1(z)$ of the complex Hessian of the weight function tends to infinity at infinity.*

Proof. First choose an arbitrary integer M. By assumption, there is R_0 such that $\lambda_1(z) > 2^M$ for $|z| > R_0$. Now let $U_1 \subset \partial\Omega$ be an open set in the induced topology containing $\partial\Omega \cap \mathbb{B}_{R_0}$. There is a smooth bounded pseudoconvex domain $\Omega_1 \subset \Omega$, such that $\partial\Omega_1 \cap \partial\Omega = U_1$ and $\partial\Omega_1 \setminus \partial\Omega$ is strictly pseudoconvex, see [24] for the construction. By assumption, Ω_1 has property (P), so after choosing M we can find $\varphi_1 \in \mathcal{C}^\infty(\overline{\Omega}_1)$ with $0 \le \varphi_1 \le 1$

and the least eigenvalue of the complex Hessian of φ_1 greater than 2^M on $\partial\Omega_1$. φ_1 is smooth on a closed set with smooth boundary, hence we can extend it smoothly to a bigger one. So extend φ_1 to a function $\psi_1 \in \mathcal{C}_0^\infty(\overline{\Omega})$, such that $0 \le \psi_1 \le 2$ and ψ vanishes outside a ball with radius R_1. We can choose R_1 so big that the least eigenvalue of the complex Hessian of ψ_1 is bounded from below by -2^M and $\lambda(z) > 2^{M+1}$ for $|z| > R_1$. Next we find an open set $U_2 \subset \partial\Omega$, such that $\partial\Omega \cap \mathbb{B}_{R_1} \subset U_2$. Again, we complete U_2 to the boundary of a bounded smooth pseudoconvex domain Ω_2 by adding something strictly pseudoconvex. Ω_2 has property (P). Therefore we find φ_2 with $0 \le \varphi_2 \le 1$ and the least eigenvalue of the complex Hessian of φ_2 greater than 2^{M+1}. Extend φ_2 to a function $0 \le \psi_2 \le 2 \in \mathcal{C}_0^\infty(\overline{\Omega})$ with support in $\Omega \cap \mathbb{B}_{R_2}$ and Hessian bounded from below by -2^{M+1} and $\lambda(z) > 2^{M+2}$ for $|z| > R_2$.
Inductively, we construct functions ψ_j and by construction,

$$\psi = \varphi + \sum \frac{1}{2^j}\psi_j$$

is an equivalent weight function satisfying (2.5). Therefore N_φ is compact by Theorem 2.25.

□

Remark. The condition of Theorem 2.28 is exactly the combination of Property (P) and the condition implying compactness of the $\bar{\partial}$-Neumann operator on $\mathbb{C}^n$.

Remark. A so-called model domain is a domain of the form $\Omega_p = \{(z', z_n) \in \mathbb{C}^n |\ \mathfrak{Im} z_n > p(z')\}$, where $p(z')$ is a plurisubharmonic function. Each model domain in $\mathbb{C}^2$ is of finite type, so this provides a class of domains for which the Theorem can be applied.

Example. Consider for instance the domain $\Omega = \{\mathfrak{Im} z_2 > (\mathfrak{Re} z_1)^2(\mathfrak{Im} z_1)^2\}$ with weight $\varphi(z) = \|z\|^4$. Clearly the least eigenvalue of $\varphi(z)$ tends to infinity at infinity, and since Ω is a model domain in $\mathbb{C}^2$, this is already sufficient for compactness of N_φ by Theorem 2.28.
If one considers this domain without weight, then a bounded $\bar{\partial}$-Neumann operator does not exist. Since $(\mathfrak{Re} z_1)^2(\mathfrak{Im} z_1)^2 \le (|z_1|^2 + |z_2|^2)^2$, it is easy to see that one can find a sequence of disjoint balls $\big(\mathbb{B}(z_l, l)\big)_l$ contained in Ω.

To conclude with, let us give the following remark.

Remark. Suppose that $\Omega_2 \subset \Omega_1$, let φ_1 be a weight function on Ω_1 and let φ_2 be the restriction of φ_1 to Ω_2. Then $L^2_{\varphi_2}(\Omega_2)$ is continuously embedded in $L^2_{\varphi_1}(\Omega_1)$. But it is worth emphasizing that N_{φ_2} is not the restriction of N_{φ_1} to $L^2_{\varphi_2}(\Omega_2)$. This is because $ker_{\Omega_2}(\bar{\partial})$ is not embedded in $ker_{\Omega_1}(\bar{\partial})$.

In particular, compactness of N_{φ_1} does not imply compactness of N_{φ_2}.
The considerations in this section also point out the interplay between the weight function and the geometry of the boundary. Roughly speaking, where the boundary is good, conditions on the weight can be weakened. Conversely, at points where the boundary is bad, the weight has to help. In particular analytic discs in the boundary do not necessarily obstruct compactness in our case.

Chapter 3

Approach via spectral theory

3.1 Preliminaries

In this third Chapter, we obtain our results on existence and compactness of the weighted $\bar{\partial}$-Neumann operator N_φ on weighted L^2-spaces on $\mathbb{C}^n$ by analyzing the spectrum of the operator

$$\Box_\varphi^{(0,1)} = \Box_\varphi^{(0,0)} \otimes I + 2M_\varphi$$

from (1.9) and using Proposition 1.6. So we start by collecting some facts from spectral theory which are important for us.
Equation (1.9) is the form of $\Box_\varphi^{(0,1)}$ for general dimension — if $n = 1$ it becomes easier, since then $4M_\varphi = \triangle\varphi$. In this case, the operator takes the form

$$4\Box_\varphi^{(0,1)} = \triangle_\varphi + \triangle\varphi.$$

This means that in complex dimension one, $\Box_\varphi^{(0,1)}$ is a so-called Schrödinger operator (on $\mathbb{R}^2$), at least up to the constant 4. A Schrödinger operator H on $\mathbb{R}^n$ is an elliptic 2nd order partial differential operator of the form

$$H = \triangle_A + V(x), \tag{3.1}$$

where the term $\triangle_A$ is called magnetic Laplacian and the real-valued function $V(x)$ electric potential. In our setting, by Lemma 1.5, we can assume $V(x)$ to be smooth. Since we are furthermore only dealing with subharmonic weight functions, we also get $V(x) \geq 0$. A magnetic Laplacian is a differential operator of the form

$$\triangle_A = -\sum_{j=1}^{n} \left(\frac{\partial}{\partial x_j} + iA_j \right)^2, \tag{3.2}$$

with corresponding magnetic potential $\vec{A} = (A_1, \dots, A_n)$. $\vec{A}$ is often identified with the 1-form $A = A_1 dx_1 + \dots A_n dx_n$, and one calls the skew symmetric matrix $B = \operatorname{curl} \vec{A}$ the magnetic field. So we have

$$B_{jk} = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k}. \tag{3.3}$$

Each Schrödinger operator H defines an associated quadratic form. Setting $\nabla_A = \nabla - i\vec{A}$, we define

$$Q_{A,V}(u) = \int_{\mathbb{R}^n} |\nabla_A u(x)|^2 + V(x)|u(x)|^2 d\lambda(x). \tag{3.4}$$

If $Q_{A,V}(u) \geq 0$, then the form domain is

$$D_{A,V} = \{u \in L^2(\mathbb{R}^n) : \nabla_A u \in L^2(\mathbb{R}^n),\ V^{\frac{1}{2}} u \in L^2(\mathbb{R}^n)\}. \tag{3.5}$$

Under this assumption, H can always be extended to a closed operator that is essentially selfadjoint, see [26]. A compact injection of $D_{A,V}$ into $L^2(\mathbb{R}^n)$ is sufficient for H to have compact resolvent.
Schrödinger operators also possess an important gauge invariance, that is, given two magnetic potentials with $\operatorname{curl} \vec{A_1} = \operatorname{curl} \vec{A_2}$, the operators $\triangle_{A_1}$ and $\triangle_{A_2}$ are unitarily equivalent. In particular both operators have the same spectral properties, which reflects the fact that the physics of a system described by a Schrödinger equation only depends on the actual magnetic field and not on the choice of the potential. Applied to our case and going back to the definition of $\triangle_\varphi$, see (1.10), we have $\triangle_\varphi = \triangle_A$, where

$$\vec{A} = (\varphi_{y_1}, -\varphi_{x_1}, \dots, \varphi_{y_n}, -\varphi_{x_n}).$$

So also the magnetic potential and hence the magnetic field can be assumed to be smooth in our case. Gauge invariance means that we do not change the situation when adding something (pluri-)harmonic to our chosen weight function φ, since on the one hand $\triangle_\varphi$ and $\triangle_{\varphi+h}$ are unitarily equivalent for every pluriharmonic function h, and on the other hand $M_\varphi = M_{\varphi+h}$. We always have restricted ourselves to positive weight functions, but this means that we can also construct weights that go fast to $-\infty$ in some direction and nevertheless have the same properties concerning existence and compactness of N_φ as positive weights.
A consequence of Kato's inequality (see e.g. [7]) is the diamagnetic property of Schrödinger operators. This property means that for any locally integrable magnetic potential $\vec{A}$, one has $\triangle \leq \triangle_A$. We will make use of this several times in our estimates.
For the operators we consider, the magnetic field is very closely related to

the Levi-matrix of the weight function. If we compute B_{jk} by (3.3), we get

$$B = \begin{pmatrix} 0 & \varphi_{x_1x_1} + \varphi_{y_1y_1} & \varphi_{x_2y_1} - \varphi_{x_1y_2} & \varphi_{x_1x_2} + \varphi_{y_1y_2} & \cdots \\ -\varphi_{x_1x_1} - \varphi_{y_1y_1} & 0 & -\varphi_{x_1x_2} - \varphi_{y_1y_2} & -\varphi_{x_2y_1} + \varphi_{x_1y_2} & \cdots \\ -\varphi_{x_2y_1} + \varphi_{x_1y_2} & \varphi_{x_1x_2} + \varphi_{y_1y_2} & 0 & \varphi_{x_1x_2} + \varphi_{y_1y_2} & \cdots \\ -\varphi_{x_1x_2} - \varphi_{y_1y_2} & \varphi_{x_2y_1} - \varphi_{x_1y_2} & -\varphi_{x_1x_2} - \varphi_{y_1y_2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

while on the other hand $4(M_\varphi)_{jk} = \left(\varphi_{x_jx_k} + \varphi_{y_jy_k}\right) + i\left(\varphi_{x_jy_k} - \varphi_{x_ky_j}\right)$, since by definition $(M_\varphi)_{jk} = \frac{\partial^2\varphi}{\partial z_j \partial \bar{z}_k}$. So we can express the magnetic field as

$$B = 4\begin{pmatrix} -\mathfrak{Im}(M_\varphi)_{11} & \mathfrak{Re}(M_\varphi)_{11} & -\mathfrak{Im}(M_\varphi)_{12} & \mathfrak{Re}(M_\varphi)_{12} & \cdots \\ -\mathfrak{Re}(M_\varphi)_{11} & \mathfrak{Im}(M_\varphi)_{11} & -\mathfrak{Re}(M_\varphi)_{12} & \mathfrak{Im}(M_\varphi)_{12} & \cdots \\ -\mathfrak{Im}(M_\varphi)_{21} & \mathfrak{Re}(M_\varphi)_{21} & -\mathfrak{Im}(M_\varphi)_{22} & \mathfrak{Re}(M_\varphi)_{22} & \cdots \\ -\mathfrak{Re}(M_\varphi)_{21} & \mathfrak{Im}(M_\varphi)_{21} & -\mathfrak{Re}(M_\varphi)_{22} & \mathfrak{Im}(M_\varphi)_{22} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

From this we observe the following Lemma:

Lemma 3.1 *Let $\triangle_\varphi$ be as in (1.10) and let λ_n be the greatest eigenvalue of the complex Hessian of the weight function $\varphi(z)$. Then $|B_{jk}(z)| \leq 4\lambda_n(z)$ for all $1 \leq j, k \leq n$.*

Proof. Clearly, $|\langle M_\varphi(z)v, w\rangle| \leq \lambda_n(z)\|v\|\|w\|$ at every point and for all vectors $v, w \in \mathbb{C}^n$. Choosing appropriate unit vectors to filter out the single elements of M_φ, one gets

$$\left|\frac{\partial^2\varphi}{\partial z_j \partial \bar{z}_k}(z)\right| \leq \lambda_n$$

and therefore the same inequality for $\mathfrak{Re}(M_\varphi)_{jk}$ and $\mathfrak{Im}(M_\varphi)_{jk}$.

$\square$

So if there is an open set U such that $\triangle\varphi = 0$ on U, then also the magnetic field B vanishes identically on U.
By Proposition 1.6, the weighted $\bar{\partial}$-Neumann operator N_φ is bounded if and only if $\Box_\varphi^{(0,1)}$ is strictly positive. Furthermore, N_φ is compact if and only if $\Box_\varphi^{(0,1)}$ has compact resolvent, which equivalently means that the essential spectrum $\sigma_{ess}(\Box_\varphi^{(0,1)})$ is empty. The foregoing Lemma is important, since we will compare operators to get information about the bottom of their spectra. As observed above, our electric potential $\triangle\varphi$ is positive. This means in particular, that $\Box_\varphi^{(0,1)}$ and $\Box_\varphi^{(0,0)}$ are positive operators, as one can see after doing integration by parts. One could also have observed this

directly from the definitions in Chapter one, namely $\Box_\varphi^{(0,0)} = \overline{D}_1\overline{D}_1^*$ and $\Box_\varphi^{(0,1)} = \overline{D}_1\overline{D}_1^* + \overline{D}_2^*\overline{D}_2$.
From positivity of $\Box_\varphi^{(0,0)} = \triangle_\varphi - \triangle\varphi$, we conclude $\triangle\varphi \le \triangle_\varphi$. So we have the following comparison

$$\triangle_\varphi \le \triangle_\varphi + \triangle\varphi \le 2\triangle_\varphi \tag{3.6}$$

for all plurisubharmonic functions φ, which implies the following Proposition taken from [14].

Proposition 3.2 *Let $n = 1$. Then $\Box_\varphi^{(0,1)}$ has bounded inverse if and only if $\triangle_\varphi$ is strictly positive. Furthermore, $\Box_\varphi^{(0,1)}$ has compact resolvent if and only if $\triangle_\varphi$ has.*

Proof. Since for $n = 1$ we have $\Box_\varphi^{(0,1)} = \triangle_\varphi + \triangle\varphi$, the Proposition is immediate from (3.6).

□

For $n \ge 2$, we do not have such a direct connection to a Schrödinger operator. Nevertheless, since both $\Box_\varphi^{(0,0)}$ and M_φ are positive, we can estimate $\Box_\varphi^{(0,1)}$ from below by

$$\Box_\varphi^{(0,1)} \ge 2M_\varphi \ge 2\lambda_1 \otimes I, \tag{3.7}$$

where λ_1 denotes the least eigenvalue of M_φ. The second obvious estimate

$$\Box_\varphi^{(0,1)} \ge \Box_\varphi^{(0,0)} \otimes I$$

unfortunately is of no use for our purposes, as will become clear later (Theorem 3.8). Since on the other hand $\triangle\varphi = 4\,\mathrm{tr}(M_\varphi)$, we have $M_\varphi \le \triangle\varphi \otimes I$ and get an estimate from above via

$$\Box_\varphi^{(0,1)} \le \Box_\varphi^{(0,0)} \otimes I + 2\triangle\varphi \otimes I = (\triangle_\varphi + \triangle\varphi) \otimes I. \tag{3.8}$$

This gives rise to necessary conditions for boundedness and compactness of N_φ. These conditions also essentially already appeared before in [14], at least for compactness.

Proposition 3.3 *Let $n \ge 1$. If $\Box_\varphi^{(0,1)}$ is strictly positive, then so is $\triangle_\varphi$. If moreover $\Box_\varphi^{(0,1)}$ has compact resolvent, the same is true for $\triangle_\varphi$.*

Proof. By our above considerations we have $\Box_\varphi^{(0,1)} \le 2\triangle_\varphi \otimes I$, so the Proposition follows.

□

For the following special case it was shown in [14], that conversely compactness of the resolvent of $\triangle_\varphi$ implies same for $\square_\varphi^{(0,1)}$. So we state it here as an equivalence:

Proposition 3.4 *Suppose that there exists some $t > 0$ such that $M_\varphi \geq t\triangle\varphi \otimes I$. Then $\square_\varphi^{(0,1)}$ has compact resolvent if and only if $\triangle_\varphi$ has.*

Remark. Note that $M_\varphi \geq t\triangle\varphi \otimes I$ for some $t > 0$ is true if and only if all eigenvalues of M_φ are comparable, meaning $\lambda_1 \leq \lambda_n \leq C\lambda_1$ for some $C > 0$, since $\triangle\varphi = 4\,\mathrm{tr}(M_\varphi)$.

Let us conclude this introductory part by introducing a powerful tool for determining the bottom of the (essential) spectrum. For a proof we refer to [29], Satz 8.27.

Theorem 3.5 ***(Max-Min Principle)*** *Let A be a selfadjoint operator that is semibounded from below and let $dom(A)$ be the domain of A. Setting*

$$\lambda_n(A) = \sup_{\psi_1,\ldots,\psi_{n-1}} \inf \left\{ \frac{\langle A\phi, \phi\rangle}{\langle \phi, \phi\rangle} \,:\, \phi \in dom(A) \cap [\mathrm{span}(\psi_1, \ldots, \psi_{n-1})]^\perp \right\},$$

$\lambda_n(A)$ either is the n-th eigenvalue of A when ordering the eigenvalues in increasing order and counting the multiplicity, or λ_n is the bottom of the essential spectrum. In this latter case, we have $\lambda_k(A) = \lambda_n(A)$, for all $k \geq n$.

The Theorem also holds if one takes the infimum only over a subset which is dense in $dom(A)$.

3.2 Necessary and sufficient conditions

Let us start with taking a closer look at our operators.

Lemma 3.6 *Let $f = \sum_{j=1}^n f^{(j)} e^{-\varphi} d\bar{z}_j \in L^2_{(0,1)}$. Then*

$$(4\square_\varphi^{(0,0)} \otimes I)f = -\sum_{j=1}^n \left[\left(f^{(j)}_{x_j x_j} + f^{(j)}_{y_j y_j} \right) e^{-\varphi} + 2(\varphi_{x_j} - i\varphi_{y_j}) \left(f^{(j)}_{x_j} + i f^{(j)}_{y_j} \right) e^{-\varphi} \right] d\bar{z}_j.$$

Proof. The proof is a straight forward computation:

$$\begin{aligned}
\triangle_\varphi(f^{(j)}e^{-\varphi}) &= -\sum_{j=1}^{n}\left[\left(\frac{\partial}{\partial x_j}+i\varphi_{y_j}\right)^2-\left(\frac{\partial}{\partial y_j}-i\varphi_{x_j}\right)^2\right](f^{(j)}e^{-\varphi})\\
&= -\sum_{j=1}^{n}\left(\frac{\partial}{\partial x_j}+i\varphi_{y_j}\right)\left(f^{(j)}_{x_j}e^{-\varphi}-f^{(j)}\varphi_{x_j}e^{-\varphi}+if^{(j)}\varphi_{y_j}e^{-\varphi}\right)\\
&\quad-\sum_{j=1}^{n}\left(\frac{\partial}{\partial y_j}-i\varphi_{x_j}\right)\left(f^{(j)}_{y_j}e^{-\varphi}-f^{(j)}\varphi_{y_j}e^{-\varphi}-if^{(j)}\varphi_{x_j}e^{-\varphi}\right)\\
&= -\sum_{j=1}^{n}\left(f^{(j)}_{x_jx_j}+f^{(j)}_{y_jy_j}\right)e^{-\varphi}+2(\varphi_{x_j}-i\varphi_{y_j})\left(f^{(j)}_{x_j}+if^{(j)}_{y_j}\right)e^{-\varphi}\\
&\quad+\triangle\varphi f^{(j)}e^{-\varphi}
\end{aligned}$$

$\square$

Lemma 3.7 *The kernel of the Schrödinger operator $H = \triangle_\varphi - \triangle\varphi$ consists exactly of those L^2-functions of the form $fe^{-\varphi}$, where f is holomorphic. In particular, if $\dim \mathcal{A}^2_\varphi = 0$, then $\square^{(0,0)}_\varphi$ and $\square^{(0,1)}_\varphi$ are injective.*

Proof. We have $H = 4\square^{(0,0)}_\varphi = 4\overline{D}_1^*\overline{D}_1$. So if $Hf = 0$ for some $f \in dom(H)$, then $\overline{D}_1^*\overline{D}_1 f = 0$, hence also $\|\overline{D}_1 f\|^2 = 0$ and the kernels of H and $\overline{D}_1$ coincide. But since $\overline{D}_1 = e^{-\varphi}\bar{\partial}e^{\varphi}$, the statement is obviously true for $\overline{D}_1$, so the Lemma is proved.

$\square$

Remark. Let $\varphi : \mathbb{C} \to \mathbb{R}$ be a subharmonic weight function. According to Lemma 3.6, a function $u = fe^{-\varphi}$ belongs to $\ker H$ if and only if

$$-\frac{\partial^2 f}{\partial z\partial \overline{z}} + 2\frac{\partial \varphi}{\partial z}\frac{\partial f}{\partial \overline{z}} = 0.$$

So the last Lemma tells us that there are no functions $f \in L^2_\varphi$ which obey this differential equation and are not holomorphic.

Theorem 3.8 *Let $\varphi : \mathbb{C}^n \to \mathbb{R}^+$ be a plurisubharmonic weight function and let $n \geq 2$. Then the Schrödinger operator $H = 4\square^{(0,0)}_\varphi$ has no compact resolvent. Indeed one has $0 \in \sigma_{ess}(\square^{(0,0)}_\varphi)$.*

Proof. Let us recall that $\square^{(0,0)}_\varphi = \overline{D}_1^*\overline{D}_1$, where $\overline{D}_1$ was defined in (1.4). We also pointed out the unitary equivalence $\square_\varphi = e^{\varphi}\square^{(0,1)}_\varphi e^{-\varphi}$ in the Preliminaries, see (1.7). From the Kohn – Morrey formula (Theorem 2.1), we know that

$$\langle \Box_\varphi f, f\rangle_\varphi = 2\sum_{j,k=1}^{n} \int_{\mathbb{C}^n} \frac{\partial^2\varphi}{\partial z_j \partial \overline{z}_k} f_j \overline{f}_k e^{-2\varphi} d\lambda + \sum_{j,k=1}^{n} \int_{\mathbb{C}^n} \left| \frac{\partial f_j}{\partial \overline{z}_k} \right|^2 e^{-2\varphi} d\lambda$$

for all $f \in dom(\Box_\varphi)$. Choosing forms $f = f_j d\overline{z}_j \in dom(\Box_\varphi)$ and comparing with the identity $\Box_\varphi^{(0,1)} = \Box_\varphi^{(0,0)} \otimes I + 2M_\varphi$, see (1.9), we can read off that

$$\langle e^\varphi \Box_\varphi^{(0,0)} e^{-\varphi}(f_j d\overline{z}_j), f_j d\overline{z}_j\rangle_\varphi = \sum_{k=1}^{n} \int_{\mathbb{C}^n} \left| \frac{\partial f_j}{\partial \overline{z}_k} \right|^2 e^{-2\varphi} d\lambda.$$

So in order to show $0 \in \sigma_{ess}(\Box_\varphi^{(0,0)})$, it suffices to construct a normed sequence $(v_j)_j \subset \mathcal{C}_0^\infty$, such that $v_j \perp v_k$ for $j \neq k$ and

$$\sum_{k=1}^{n} \int_{\mathbb{C}^n} \left| \frac{\partial v_j}{\partial \overline{z}_k} \right|^2 e^{-2\varphi} d\lambda \to 0$$

for $j \to \infty$. The Theorem then follows from the Max-Min Principle 3.5, since $\inf \sigma_{ess}(\Box_\varphi^{(0,0)}) > \varepsilon$ for some $\varepsilon > 0$ implies that there can only be a finite dimensional subspace F of $L^2(\mathbb{C}^n)$ such that

$$\langle \Box_\varphi^{(0,0)} u, u\rangle \leq \varepsilon \langle u, u\rangle$$

for $u \in F$. So let χ_j be a smooth radially symmetric cutoff function with support in the unit ball which is identically one on the ball with center 0 and radius $1 - \frac{1}{j}$. We also can assume $\sup |\nabla \chi_j(z)| \leq 2j$. Define

$$u_j = \frac{\chi_j(z)}{\|\chi_j(z)\|_\varphi}.$$

Then by definition $\|u_j\|_\varphi = 1$ and we need to estimate the expression

$$\sum_{k=1}^{n} \int_{\mathbb{C}^n} \left| \frac{\partial \chi_j(z)}{\partial \overline{z}_k} \right|^2 e^{-2\varphi(z)} d\lambda(z).$$

To this end we first estimate the exponential by 1 and then switch to polar coordinates centered at the origin to obtain

$$\sum_{k=1}^{n} \int_{1-\frac{1}{j}}^{1} \int_{S^{n-1}} \left| \frac{\partial \chi_j(r)}{\partial \overline{z}_k} \right|^2 r^{2n-1} dr d\sigma \leq nj^2 \left| S^{n-1} \right| \int_{1-\frac{1}{j}}^{1} r^{2n-1} dr,$$

where S^{n-1} denotes the unit sphere in $\mathbb{C}^n$ and $d\sigma$ the surface element. Setting $t = 1 - r$, the last integral becomes

$$-nj^2 \left| S^{n-1} \right| \int_0^{\frac{1}{j}} t^{2n-1} dt \leq -nj^2 \left| S^{n-1} \right| \int_0^{\frac{1}{j}} \left(\frac{1}{j} \right)^{2n-1} dt.$$

Since we assumed $n \geq 2$ and transforming back to the original coordinates, we conclude

$$\sum_{k=1}^{n} \int\limits_{\mathbb{C}^n} \left| \frac{\partial \chi_j(z)}{\partial \overline{z}_k} \right|^2 e^{-2\varphi(z)} d\lambda(z) \leq \frac{C(n)}{j},$$

where the constant $C(n)$ only depends on the dimension. Summing up this means

$$\sum_{k=1}^{n} \int\limits_{\mathbb{C}^n} \left| \frac{\partial u_j}{\partial \overline{z}_k} \right|^2 e^{-2\varphi} d\lambda \leq \frac{C(n)}{j} \left(\int\limits_{\mathbb{B}(z_j, 1-\frac{1}{j})} e^{-2\varphi(z)} d\lambda(z) \right)^{-1}.$$

Now

$$\int\limits_{\mathbb{B}(z_j, 1-\frac{1}{j})} e^{-2\varphi(z)} d\lambda(z) \rightarrow \int\limits_{\mathbb{B}(z_j, 1)} e^{-2\varphi(z)} d\lambda(z) > 0$$

for $j \to \infty$ by the Dominated Convergence Theorem, so we constructed a normed sequence $(u_j)_j$ such that $\langle \square_{\varphi}^{(0,0)} u_j, u_j \rangle \to 0$. This construction in fact works in every ball with fixed center, so if we take a discrete sequence $(z_j)_j$ in $\mathbb{C}^n$ such that $|z_j - z_k| > 2$ for $j \neq k$ and set

$$u_j^{(k)} = \frac{\chi_j(z - z_k)}{\|\chi_j(z - z_k)\|_{\varphi}},$$

it is possible to choose a diagonal sequence $(v_j)_j = u_j^{(j)}$ such that the v_j are pairwise orthogonal, $supp(v_j) \subset \mathbb{B}(z_j, 1)$ and

$$\sum_{k=1}^{n} \int\limits_{\mathbb{C}^n} \left| \frac{\partial v_j}{\partial \overline{z}_k} \right|^2 e^{-2\varphi} d\lambda \leq \frac{2C(n)}{j}.$$

$\square$

Remark. As we will see later in Theorem 3.20, an immediate consequence of this Theorem is that decoupled weights can never admit a compact $\bar{\partial}$-Neumann operator. We will see in section 3.3.2, that the Theorem in fact also holds for $n = 1$.
This Theorem also shows that existence and compactness of the weighted $\bar{\partial}$-Neumann operator only depends on the Levi-matrix of the weight function and particularly on its eigenvalues and eigenspaces. This is, because $0 \in \sigma_{ess}(\square_{\varphi}^{(0,0)})$ implies that $\inf \sigma(\square_{\varphi}^{(0,1)}) \geq \inf \sigma(M_{\varphi})$ is the best bound from below that one can in general expect, and analogously for the essential spectra.
Since $H = \triangle_{\varphi} - \triangle\varphi$ is semibounded from below, it is essentially selfadjoint. This is worth mentioning, since $\triangle\varphi$ is allowed to tend to infinity at

any speed. In general, nothing is known about essential selfadjointness of Schrödinger operators whose electric potentials tend to minus infinity.

Corollary 3.9 *Let λ_n denote the greatest eigenvalue of the Levi-matrix M_φ of the weight function. Suppose that there is an $r > 0$ and a sequence $(z_j)_j$ which is discrete in $\mathbb{C}^n$ such that*

$$\sup_{z\in\mathbb{B}(z_j,r)} \lambda_n(z) \to 0$$

for $j \to \infty$. Then $\Box_\varphi^{(0,1)}$ has no bounded inverse.

Proof. From the construction in the proof of Theorem 3.8, we know that there is a sequence $(u_j)_j \subset \mathcal{C}_0^\infty$, such that $supp(u_j) \subset \mathbb{B}(z_j, r)$, $\|u_j\| = 1$ and moreover $\langle \Box_\varphi^{(0,0)} u_j, u_j\rangle \to 0$ as $j \to \infty$. Passing to a subsequence, we also can assume that the supports of the functions u_j are pairwise disjoint. Taking such a sequence $(u_j)_j$ and setting $v_j = u_j d\bar{z}_1$ gives a bounded sequence with $\langle (\Box_\varphi^{(0,0)} \otimes I)v_j, v_j\rangle \to 0$.
Since

$$\langle M_\varphi v_j, v_j\rangle \le \sup_{z\in supp(v_j)} \lambda_n(z)\ \ \|v_j\|^2,$$

our assumption on λ_n implies that also $\langle M_\varphi v_j, v_j\rangle \to 0$, thus the inverse of $\Box_\varphi^{(0,1)}$ can not be bounded by Proposition 1.3.

□

Remark. One should note that the condition of the Corollary holds if and only if

$$\sup_{z\in\mathbb{B}(z_j,r)} \triangle\varphi(z) \to 0$$

for $j \to \infty$.

Corollary 3.10 *Let λ_1 denote the least eigenvalue of the Levi-matrix M_φ. Suppose that there is an $r > 0$ and a sequence $(z_j)_j$ discrete in $\mathbb{C}^n$ such that*

$$\sup_{z\in\mathbb{B}(z_j,r)} \lambda_1(z) \to 0$$

for $j \to \infty$. Suppose furthermore that it is possible to choose an eigenvector to λ_1 that is holomorphically dependent on z on $\cup\mathbb{B}(z_j, r)$. Then $\Box_\varphi^{(0,1)}$ has no bounded inverse.

Proof. Let $f = \sum f_k d\bar{z}_k$ be such that $M_\varphi f = \lambda_1 f$ and the functions f_k are holomorphic on $\cup\mathbb{B}(z_j, r)$. To prove the Corollary, we recognize that a similar construction to the one given in the proof of Theorem 3.8 works if we set $u_j = \frac{f\chi_j}{\|f\chi_j\|}$ for some holomorphic function f, where we kept the same notation as in the proof of the Theorem. This is because

$$\frac{\partial(f\chi_j)}{\partial \overline{z}_k} = f\frac{\partial \chi_j}{\partial \overline{z}_k}$$

for f holomorphic. So setting $v_j^{(k)}(z) = \frac{f_k(z)\chi_j(z-z_j)}{\|f_k(z)\chi_j(z-z_j)\|_\varphi}$, we obtain a bounded sequence $v_j = \sum v_j^{(k)} d\overline{z}_k$ such that $\langle (e^{\varphi}\Box_\varphi^{(0,0)} \otimes I)e^{-\varphi}v_j, v_j\rangle$ tends to zero. By assumption, we also have $\langle M_\varphi v_j, v_j\rangle_\varphi \to 0$, hence $\Box_\varphi^{(0,1)}$ has no bounded inverse by Proposition 1.3.

$\square$

Corollary 3.11 *Suppose that there are $C, r > 0$ and a discrete sequence $(z_j)_j \subset \mathbb{C}^n$ such that for the greatest eigenvalue $\lambda_n(z)$ of the Levi-matrix M_φ it holds*

$$\sup_{z\in\mathbb{B}(z_j,r)} \lambda_n(z) \leq C$$

for all j. Then the weighted $\bar{\partial}$-Neumann operator N_φ is not compact.
If an eigenvector to the least eigenvalue λ_1 of M_φ can be chosen which depends holomorphically on z for all $z \in \cup\mathbb{B}(z_j, r)$, then already

$$\sup_{z\in\mathbb{B}(z_j,r)} \lambda_1(z) \leq C$$

for all j obstructs compactness of N_φ.

Proof. From the proofs of Theorem 3.8 and Corollary 3.9 we know that there is a normed sequence $(v_j)_j$ such that $\langle (\Box_\varphi^{(0,0)} \otimes I)v_j, v_j\rangle \to 0$ and $supp(v_j) \subset \mathbb{B}(z_j, r)$. So if $\lambda_n(z)$ is bounded on the union of these balls, we immediately get $\inf \sigma_{ess}(\Box_\varphi^{(0,1)}) \leq 2C$ by the Max-Min Principle 3.5, meaning that N_φ is not compact.
If we have an holomorphic dependence of the eigenvector corresponding to the least eigenvalue, an analog argument as in Corollary 3.10 combined with the Max-Min Principle finishes the proof.

$\square$

Remark. Suppose that we are given two weight functions such that $\varphi_1(z) \leq \varphi_2(z)$. Then compactness of N_{φ_1} does not imply compactness of N_{φ_2}, although heuristically it should be easier for an operator to be compact, the heavier the weight at infinity is. In fact Corollary 3.11 implies that for each weight function φ_1 such that N_{φ_1} is compact another weight φ_2 can be found such that $\varphi_1(z) \leq \varphi_2(z)$ and N_{φ_2} is not compact, simply by adding a positive function such that the new weight φ_2 is constant on the

union of a sequence of balls with fixed radius. In this case, N_{φ_2} will not even be bounded.
Anticipating our result about decoupled weights (Theorem 3.20), we consider the example $\varphi_1 = \|z\|^4$ and $\varphi_2 = |z_1|^4 + |z_2|^4$ in $\mathbb{C}^2$. Here we even have $\frac{1}{2}\varphi_1(z) \leq \varphi_2(z) \leq \varphi_1(z)$, but N_{φ_1} is compact (Theorem 2.15), while N_{φ_2} is not, as we will see below by Theorem 3.20.
So in order to have a bounded weighted $\bar{\partial}$-Neumann operator, the growth of the weight must be faster than linear in every direction. Else one could add something bounded to make it locally constant, see Lemma 1.5. Nevertheless the other results show, that existence and compactness of N_φ seems to depend more on the second derivatives of the weight than on the weight itself.

A sharper necessary condition than the one of Corollary 3.11 can be obtained by applying a result of Iwatsuka [18]:

Proposition 3.12 *Let φ be a smooth plurisubharmonic weight function and suppose that N_φ is compact. Then for each fixed $r > 0$*

$$\int_{\mathbb{B}(z,r)} \sum_{j<k} |B_{jk}|^2 d\lambda \to \infty \quad \textit{as} \quad |z| \to \infty.$$

Proof. By Iwatsuka's result, compactness of the resolvent of a Schrödinger operator $H = \triangle_A + V$ implies

$$\int_{B(z,r)} \left(\sum_{j<k} |B_{jk}|^2 + V \right) d\lambda \to \infty \quad \text{as} \quad |z| \to \infty,$$

for each $r > 0$, see [18]. Now using Proposition 3.3, compactness of N_φ implies compactness of the resolvent of $\triangle_\varphi$, so the Proposition is proved.

□

Corollary 3.13 *By Proposition 3.12 and Lemma 3.1, compactness of the weighted $\bar{\partial}$-Neumann operator N_φ necessarily implies for each fixed $r > 0$*

$$\int_{\mathbb{B}(z,r)} \lambda_n(w)^2 \, d\lambda(w) \to \infty \quad \textit{as} \quad |z| \to \infty,$$

where λ_n denotes the biggest eigenvalue of M_φ. This is equivalent to

$$\int_{\mathbb{B}(z,r)} (\triangle\varphi)^2 \, d\lambda \to \infty \quad \textit{as} \quad |z| \to \infty.$$

Remark. Part of the last Corollary already was shown before in [14]. It was proved there that in complex dimension one, compactness of the weighted $\bar{\partial}$-Neumann operator N_φ implies for each fixed $r > 0$

$$\int_{\mathbb{B}(z,r)} (\triangle\varphi)^2 \, d\lambda \to \infty \quad \text{as} \quad |z| \to \infty.$$

For a large class of weight functions (more precisely the Muckenhaupt class) it was also shown there that in dimension one a sufficient condition for compactness is

$$\int_{\mathbb{B}(z,r)} (\triangle\varphi) \, d\lambda \to \infty \quad \text{as} \quad |z| \to \infty. \tag{3.9}$$

Later, J. Marzo and J. Ortega-Cerdá proved in [22], that condition (3.9) in fact characterizes compactness of N_φ in dimension one, if one additionally assumes $\mu = \triangle\varphi \, d\lambda$ to be a doubling measure. Doubling means, that there is a constant $C > 0$ such that $\mu(\mathbb{B}_{2r}) \leq C\mu(\mathbb{B}_r)$, for every ball $\mathbb{B}_r$ with radius r.
In general dimension, a sufficient condition for compactness is $\lambda_1 \to \infty$ as $|z| \to \infty$, where λ_1 denotes the least eigenvalue of the Levi-matrix M_φ. Under the following additional assumption, this in fact characterizes compactness of N_φ.

Theorem 3.14 *Suppose that $n \geq 2$. Suppose furthermore that an eigenvector corresponding to the least eigenvalue λ_1 of M_φ can be chosen which is holomorphically dependent on z. Then the weighted $\bar{\partial}$-Neumann operator N_φ is compact if and only if $\lambda_1 \to \infty$ as $|z| \to \infty$.*

Proof. By Theorem 2.15 or Theorem 5.2 in [14], the condition $\lambda_1 \to \infty$ as $|z| \to \infty$ is sufficient for compactness.
Assume conversely that N_φ is compact and there exists $f = \sum f_k d\bar{z}_k$, such that $M_\varphi f = \lambda_1 f$ and the functions f_j are holomorphic for all j. The first step in the proof is to recognize that under this additional assumption the inequality

$$\langle \Box_\varphi^{(0,1)} u, u \rangle \geq \langle 2\lambda_1 u, u \rangle$$

for all $u \in dom(\Box_\varphi^{(0,1)})$ is optimal. Suppose that it is not — then there exists a measurable function Λ and an open set U such that $\Lambda > 2\lambda_1$ almost everywhere on U and $\langle \Box_\varphi^{(0,1)} u, u \rangle \geq \langle \Lambda u, u \rangle$ for all $u \in dom(\Box_\varphi^{(0,1)})$. So we can find a ball $\mathbb{B}$ contained in U and since f has holomorphic coefficients construct by the same method as in the proof of Theorem 3.8 a normed sequence $u_j = \chi_j f e^{-\varphi}$, such that $supp(u_j) \subset U$ and $\langle \Box_\varphi^{(0,0)} u_j, u_j \rangle \to 0$. Since on the other hand $M_\varphi u_j = \lambda_1 u_j$, we get $\langle \Box_\varphi^{(0,1)} u_j, u_j \rangle \to \langle 2\lambda_1 u_j, u_j \rangle$ by

equation (1.9), a contradiction to $\Lambda > 2\lambda_1$ almost everywhere.
Compactness of N_φ implies $\sigma_{ess}(\Box_\varphi^{(0,1)}) = \emptyset$ or equivalently $\inf \sigma_{ess}(\Box_\varphi^{(0,1)}) > N$ for each $N \in \mathbb{N}$. So by the Max-Min Principle 3.5, the subspace F of $\Lambda_0^{(0,1)}(\mathbb{C}^n)$, such that

$$\langle \Box_\varphi^{(0,1)} u, u\rangle \leq 2N\langle u, u\rangle$$

must be finite dimensional, else we would have $\inf \sigma_{ess}(\Box_\varphi^{(0,1)}) \leq 2N$. If $v_1, \dots, v_k$ is a basis of F, then there is a compact set K such that $supp(v_j) \subset K$ for all j. Hence, $\langle \Box_\varphi^{(0,1)} u, u\rangle \geq 2N\langle u, u\rangle$ for all $u \in \Lambda_0^{(0,1)}(\mathbb{C}^n \setminus K)$ and since the inequality $\langle \Box_\varphi^{(0,1)} u, u\rangle \geq \langle 2\lambda_1 u, u\rangle$ is optimal, we have $\lambda_1(z) \geq N$ for all $z \in \mathbb{C}^n \setminus K$.

$\Box$

Example. Let $n = 1$ and let $\varphi(z) = |z|^2$. Then $4M_\varphi = I$, so N_φ exists by Theorem 2.5. But the weighted $\bar\partial$-Neumann operator is not compact, since $\triangle\varphi$ is bounded, see Corollary 3.13. To compute the essential spectrum of $\Box_\varphi$ we use Theorem 1.5 from [15], which gives a characterization of the essential spectrum of a large class of magnetic Schrödinger operators, including all operators such that $\vec{A}$ and V are polynomials. According to this Theorem the essential spectrum of $\Box_\varphi^{(0,1)}$ and thus also of the complex Laplacian equals to the spectrum of the operator

$$H = \left(\frac{\partial}{\partial x} + \frac{y}{2}\right)^2 + \left(\frac{\partial}{\partial y} + \frac{x}{2}\right)^2 + 1.$$

It is a classical result from spectral theory (see e.g. [3]) that the spectrum of the Schrödinger operator H_0 with constant magnetic field

$$H_0 = \left(\frac{\partial}{\partial x} + \frac{y}{2}\right)^2 + \left(\frac{\partial}{\partial y} + \frac{x}{2}\right)^2$$

is $2\mathbb{Z}+1$, each point of $\sigma(H_0)$ being an eigenvalue of infinite multiplicity. So the essential spectrum of the complex Laplacian equals $\sigma_{ess}(\Box_\varphi^{(0,1)}) = 2\mathbb{Z}$.

Example. Let us consider the weight $\varphi(z, w) = |z|^2|w|^2$ on $\mathbb{C}^2$. φ is plurisubharmonic, with eigenvalues $\lambda_1(z, w) = 0$ and $\lambda_2(z, w) = |z|^2 + |w|^2$ of M_φ. A possible choice of an eigenvector to the eigenvalue zero is $(-z, w)$ which is holomorphic, so by Corollary 3.10, N_φ is not bounded.

Example. Let $\varphi(z, w) = \frac{1}{2}|z|^2|w|^2 + \frac{1}{4}(|z|^2 + |w|^2)^2$. Then the eigenvalues of M_φ are $\lambda_1 = |z|^2 - |z||w| + |w|^2$ and $\lambda_2 = |z|^2 + |z||w| + |w|^2$. The lower eigenvalue λ_1 tends to infinity for $(z, w) \to \infty$, so N_φ is compact. But the corresponding eigenvectors are $(\overline{z}w, \pm|z||w|)$, so it is not possible to choose one which is holomorphically dependent on (z, w).

3.3 Some special cases

3.3.1 The one dimensional case

In complex dimension one, we know by Proposition 3.2 that boundedness and compactness of the weighted $\bar{\partial}$-Neumann operator are equivalent to certain properties of a Schrödinger operator, since

$$\triangle_\varphi \leq \triangle_\varphi + \triangle\varphi \leq 2\triangle_\varphi.$$

It can be completely characterized when a Schrödinger operator is strictly positive or has compact resolvent respectively, see [20]. Unfortunately, this characterization is very complicated and uses quantities which in practice are hard to handle. Nevertheless we are in the easier situation that we can assume the electric potential to be zero. In the sequel, we define the necessary notions.
Let $\Omega \subset \mathbb{R}^2$ be open, and let $F \subset \Omega$ be a compact set. The capacity of F with respect to Ω is

$$\mathrm{cap}_\Omega(F) = \inf\left\{\int_{\mathbb{R}^2} |\nabla u(x)|^2 d\lambda : u \in Lip_0(\Omega),\ u \equiv 1 \text{ on } F\right\},$$

where $Lip_0(\Omega)$ is the space of Lipschitz functions with compact support in Ω. In this section, Q_d always denotes a square with sidelength d and edges parallel to the coordinate axes. If we drop the subscript, $\mathrm{cap}(F)$ is the capacity of F with respect to Q_{2d}, where Q_d is the smallest square containing the compact set F and the center of Q_{2d} is the center of Q_d. Note that in the definition of capacity one could equivalently take the infimum over $u \in \mathcal{C}_0^\infty,\ 0 \leq u \leq 1$, see [19], section 3.
The Molchanov functional is

$$M_\gamma(Q_d, V) = \inf_F \left\{\int_{Q_d \setminus F} V(x) d\lambda(x) : \mathrm{cap}(F) \leq \gamma\, \mathrm{cap}(Q_d)\right\}, \tag{3.10}$$

where $0 < \gamma < 1$. Due to properties of the capacity the infimum will not change if we restrict it to compact sets F which are the closures of smooth open subsets of Q_d. Finally, the local energy of the magnetic field (in Q_d) is defined to be

$$\mu_0(Q_d) = \inf_{\|u\|=1} \{\langle \nabla_\varphi u, \nabla_\varphi u\rangle_{Q_d},\ u \in \mathcal{C}^\infty(Q_d) \cap L^2(Q_d)\}. \tag{3.11}$$

Remember, $\nabla_\varphi = \nabla - i\vec{A}$, where $\vec{A} = (\varphi_{y_1}, -\varphi_{x_1}, \ldots, \varphi_{y_n}, -\varphi_{x_n})$. This latter quantity equals the bottom of the spectrum of the Schrödinger operator $\triangle_\varphi$ in Q_d with Neumann boundary conditions. Now the condition for strict positivity given in [20] is uniform positivity of a function of $\mu_0(Q_d)$

and $M_\gamma(Q_d, V)$ for all squares Q_d and each $d > 0$ fixed. Here, γ itself is a function of $\mu_0(Q_d)$, and the condition on the electric potential V is weaker at the places where the local energy of the magnetic field is larger.
In our situation, it reads as follows.

Theorem 3.15 *Let $n = 1$. The following are equivalent:*

1. *The weighted $\bar\partial$-Neumann operator N_φ is bounded.*
2. *For all $d > 0$, there is $\varepsilon > 0$ such that*
$$\mu_0(Q_d) \geq \varepsilon$$
for all squares Q_d.

The result for compactness of N_φ is

Theorem 3.16 *Let $n = 1$. The following are equivalent:*

1. *The weighted $\bar\partial$-Neumann operator N_φ is compact.*
2. *For all $d > 0$, we have*
$$\mu_0(Q_d) \to \infty \quad \text{as} \quad Q_d \to \infty,$$
where $Q_d \to \infty$ means that the center of Q_d tends to infinity.

For a proof, see Theorem 1.2 and Theorem 1.8 in [20]. We now derive some sufficient conditions from these two Theorems. Let us call a compact set F negligible in the sense of Molchanov, if $\operatorname{cap}(F) \leq \gamma \operatorname{cap}(Q_d)$ for some $0 < \gamma < 1$. The following two results were originally proved by Molchanov for γ sufficiently small, but in fact it follows from the results in [20] that the conditions are equivalent for all $0 < \gamma < 1$.

Proposition 3.17 *Let $\varphi(z) : \mathbb{C} \to \mathbb{R}^+$ be a subharmonic smooth function. Suppose there exists a constant d such that for all squares Q_d with sidelength d and all compact sets $F \subset Q_d$ which are negligible in the sense of Molchanov it holds*
$$\int\limits_{Q_d \setminus F} \triangle\varphi \, d\lambda \geq \varepsilon$$
for some $\varepsilon > 0$. Then the weighted $\bar\partial$-Neumann operator is bounded.

Proof. By Theorem 1.8 in [20], our assumption on $\triangle\varphi$ implies strict positivity of $H = -\triangle + \triangle\varphi$. By the diamagnetic property of Schrödinger operators we have
$$-\triangle + \triangle\varphi \leq \triangle_\varphi + \triangle\varphi,$$
thus $0 \notin \sigma(\Box_\varphi^{(0,1)})$. Boundedness of N_φ now follows by Proposition 1.6.

□

Remark. Note that the Proposition in particular shows that

$$\liminf_{|z|\to\infty} \lambda_1(z) = 0,$$

where λ_1 denotes the least eigenvalue of M_φ, is not necessarily an obstruction to boundedness of N_φ. This should be compared with Theorem 2.5.

Proposition 3.18 *Let $\varphi : \mathbb{C} \to \mathbb{R}^+$ be subharmonic and smooth. If we have*

$$\inf_F \int\limits_{Q_d \setminus F} \triangle\varphi \, d\lambda \to \infty \quad \text{as} \quad Q_d \to \infty,$$

where Q_d is a square with sidelength d and $F \subset Q_d$ a compact set negligible in the sense of Molchanov, then N_φ exists and is compact.

Proof. Since by Theorem 1.2 in [20] the assumption is sufficient for discreteness of the spectrum of $H = -\triangle + \triangle\varphi$, the Proposition follows by the diamagnetic property of Schrödinger operators and Proposition 1.6.

□

Conditions involving capacity are in practice hard to handle. It was proved in [19] (Section 6.1), that it is possible to replace the capacity by the Lebesgue measure λ in order to get sufficient conditions:

Proposition 3.19 *Suppose that $\varphi : \mathbb{C} \to \mathbb{R}^+$ is a subharmonic smooth function and suppose there are $\varepsilon, \gamma > 0$ such that*

1. *For any discrete sequence $(z_l)_l \subset \mathbb{C}$ and any compact sets $F_l \subset \mathbb{B}(z_l, r)$ with $\lambda(F_l) \leq \gamma r^2$*

   $$\inf_l \int\limits_{\mathbb{B}(z_l,r)\setminus F_l} \triangle\varphi \, d\lambda > \varepsilon > 0.$$

   Then the weighted $\bar\partial$-Neumann operator N_φ is bounded.

2. *For any discrete sequence $(z_l)_l \subset \mathbb{C}$ and any compact sets $F_l \subset \mathbb{B}(z_l, r)$ with $\lambda(F_l) \leq \gamma r^2$*

   $$\int\limits_{\mathbb{B}(z_l,r)\setminus F_l} \triangle\varphi \, d\lambda \to \infty \quad \text{as} \quad l \to \infty.$$

   Then the weighted $\bar\partial$-Neumann operator N_φ is compact.

Remark. To get other necessary conditions, one can play with the definition of the local energy of the magnetic field (3.11) and try some specific functions. For instance choosing $u \equiv 1$, we get $\|u\|_{L^2(Q_d)} = d^2$ and $|\nabla_\varphi u|^2 = |\nabla\varphi|^2$, so in particular

$$\mu_0(Q_d) \leq d^{-2} \int\limits_{Q_d} |\nabla\varphi|^2 \, d\lambda.$$

Thus boundedness of the derivatives of φ obstructs compactness of N_φ and similarly, if $|\nabla\varphi(z)| \to 0$ as $z \to \infty$, then N_φ is not bounded.
An interesting question is whether the infimum in the definition of μ_0 actually is a minimum, since this would give an explicit expression of μ_0. If this is the case and if u is a minimum, then

$$\left.\frac{\partial}{\partial\tau}\right|_{\tau=0} \int\limits_{Q_d} |\nabla_\varphi(u+\tau v)|^2 \, d\lambda = 0$$

for all $v \in \mathcal{C}^\infty \cap L^2(Q_d)$. Computing the derivative, this is equivalent to

$$\mathfrak{Re} \int\limits_{Q_d} \nabla_\varphi u \overline{\nabla_\varphi v} \, d\lambda = 0,$$

which in particular implies $\nabla_\varphi u \equiv 0$, after choosing $v = u$. So if there is a minimizing function u, the weighted $\bar\partial$-Neumann operator is not bounded by Theorem 3.15.

Remark. All equivalences and necessary conditions presented in this section can be carried over to the case $n \geq 2$ as necessary ones. As observed in equation (3.8),

$$\square_\varphi^{(0,1)} \leq (\triangle_\varphi + \triangle\varphi) \otimes I \leq 2\triangle_\varphi \otimes I.$$

Thus strict positivity of $\square_\varphi^{(0,1)}$ implies the same for $\triangle_\varphi$, and similar for compactness of the resolvent.

3.3.2 Decoupled weights and Pauli operators

The main result of this section is the following Theorem.

Theorem 3.20 *Let $n \geq 2$ and let $\varphi : \mathbb{C}^n \to \mathbb{R}^+$ be a plurisubharmonic decoupled weight function, i.e., of the form*

$$\varphi(z) = \sum_{j=1}^n \varphi_j(z_j).$$

Then $\square_\varphi^{(0,1)}$ does not have a compact resolvent, in particular the weighted $\bar\partial$-Neumann operator N_φ is not compact.

Proof. To prove the Theorem, we will make use of the connection between this problem and Pauli operators. In real dimension two, a Pauli operator $P_\pm$ is a Schrödinger operator of the form

$$P_\pm = -\left(\frac{\partial}{\partial x} + iA_1\right)^2 - \left(\frac{\partial}{\partial y} + iA_2\right)^2 \pm B(x, y), \tag{3.12}$$

where $B = \operatorname{curl} \vec{A}$. So the electric potential of a Pauli operator equals the magnetic field. In our case, we have $B(x, y) \geq 0$, which means $0 \leq P_- \leq P_+$. Now if the weight function is decoupled, the Levi-matrix is diagonal, with entries $\frac{1}{4}\triangle\varphi_j$ in the main diagonal. Hence $\Box_\varphi^{(0,1)}$ acts diagonally on forms and this action can be expressed in terms of Pauli operators:

$$\Box_\varphi^{(0,1)}\left(\sum_{k=1}^n u_k d\overline{z}_k\right) = \frac{1}{4}\sum_{k=1}^n H_k u_k d\overline{z}_k = \frac{1}{4}\sum_{k=1}^n \left(\sum_{j\neq k} P_-^{(j)} + P_+^{(k)}\right) u_k d\overline{z}_k,$$

where $P_\pm^{(l)}$ is the Pauli operator in the variables (x_l, y_l),

$$P_\pm^{(l)} = -\left(\frac{\partial}{\partial x_l} - i\varphi_{y_l}\right)^2 - \left(\frac{\partial}{\partial y_l} + i\varphi_{x_l}\right)^2 \pm \triangle\varphi_l(x_l, y_l).$$

So suppose now that $\Box_\varphi^{(0,1)}$ has compact resolvent. Then each H_k has compact resolvent, hence same for the operators $P_\pm^{(j)}$ for all j, since they act separately in the variables z_j. Consequently also $H = \sum_{j=1}^n P_-^{(j)} = \triangle_\varphi - \triangle\varphi$ must have empty essential spectrum, but the latter is not the case according to Theorem 3.8. So there must be a j such that $P_-^{(j)}$ has non-compact resolvent.

$\Box$

Remark. This connection to Pauli operators was pointed out in [14] for the first time, where it was already shown that for decoupled weights compactness of the canonical solution operator $\mathcal{S}_\varphi$ to $\bar{\partial}$ fails under weak additional assumptions. This was done by using a result from [16] on the resolvent of the Dirac operator and the fact that if $B \geq 0$, non-compactness of the resolvent of the Dirac operator obstructs compactness of the resolvent of P_-. In [16] it is conjectured, that the Dirac operator actually never has compact resolvent. From this point of view, the next Corollary strengthens this conjecture.

Corollary 3.21 *Let $\varphi : \mathbb{C} \to \mathbb{R}^+$ be subharmonic. Then the Pauli operator $P_- = (-i\partial_x - \varphi_y)^2 + (-i\partial_y + \varphi_x)^2 - \triangle\varphi(x)$ has no compact resolvent.*

Proof. Define $\psi : \mathbb{C}^2 \to \mathbb{R}^+$, $\psi(z) = \varphi(z_1) + \varphi(z_2)$. Then ψ is a plurisubharmonic decoupled weight function. Using the same notation as in the proof of Theorem 3.20, $P_-^{(1)}$ and $P_-^{(2)}$ are the same operators, but acting in different variables. By Theorem 3.20, $H = P_-^{(1)} + P_-^{(2)}$ has non-empty essential spectrum, so there is a j such that $P_-^{(j)}$ has non-compact resolvent, which proves the Corollary.

□

Remark. This Corollary shows that Theorem 3.8 in fact also holds in complex dimension one.

Corollary 3.22 *Suppose that $\varphi(z)$ is plurisubharmonic and of the form*

$$\varphi(z) = \varphi_1(z_1) + \varphi_2(z_2) + \varphi_3(z_3, \dots, z_n).$$

Then N_φ is not compact.

Proof. If $\varphi(z)$ is of the assumed form, then

$$\Box_\varphi^{(0,1)}(u_1 d\bar{z}_1) = \frac{1}{4}\left(P_+^{(1)} + P_-^{(2)} + \triangle_{\varphi_3} - \triangle\varphi_3\right) u_1 d\bar{z}_1.$$

Hence if one chooses u_1 to be only dependent on z_2, it is easily seen that $\Box_\varphi^{(0,1)}$ cannot have compact resolvent, since this is true for $P_-^{(2)}$ by Corollary 3.21.

□

Example. Consider the weight $\varphi_1 = |z_1|^4 + |z_2|^4$. This is decoupled, so the weighted $\bar{\partial}$-Neumann operator is not compact by Theorem 3.20. By Corollary 3.10, for the weight $\varphi_2 = |z_1|^2|z_2|^2$ the $\bar{\partial}$-Neumann operator is not even bounded. But if one considers $\varphi = \varphi_1 + \varepsilon\varphi_2$ for some $\varepsilon > 0$, then the corresponding Neumann operator N_φ is compact, as the following calculations shows.
The Levi-matrix of φ is

$$M_\varphi = \begin{pmatrix} 4|z_1|^2 + \varepsilon|z_2|^2 & \varepsilon z_1 \bar{z}_2 \\ \varepsilon \bar{z}_1 z_2 & 4|z_2|^2 + \varepsilon|z_1|^2 \end{pmatrix},$$

therefore

$$\det(M_\varphi - \lambda I) = \lambda^2 - (4+\varepsilon)\left(|z_1|^2 + |z_2|^2\right)\lambda + 16|z_1|^2|z_2|^2 + 4\varepsilon\left(|z_1|^4 + |z_1|^4\right).$$

Thus the lower eigenvalue of the Levi-matrix is

$$\lambda_1 = \left(2 + \frac{\varepsilon}{2}\right)\left(|z_1|^2 + |z_2|^2\right) - \sqrt{\left(2 - \frac{\varepsilon}{2}\right)^2 (|z_1|^2 + |z_2|^2)^2 - (16 - 8\varepsilon)|z_1|^2|z_2|^2}.$$

So we can estimate λ_1 from below by

$$\lambda_1 \geq \left(2 + \frac{\varepsilon}{2}\right)(|z_1|^2 + |z_2|^2) - \left(2 - \frac{\varepsilon}{2}\right)(|z_1|^2 + |z_2|^2) = \varepsilon(|z_1|^2 + |z_2|^2),$$

which tends to infinity for $|z| \to \infty$. Therefore N_φ is compact by Theorem 2.15.

Remark. The previous example $\varphi(z) = |z_1|^4 + |z_2|^4$ also shows, that a condition in the spirit of Theorem 2.15 on an eigenvalue which is not the lowest one can in general not be sufficient for compactness. In fact we have $\lambda_2(z) = \max(4|z_1|^2, 4|z_2|^2))$, so $\lambda_2 \to \infty$ as $|z| \to \infty$, but since φ is decoupled, N_φ is not compact.

3.3.3 Radially symmetric weights

Theorem 3.23 *Let $n \geq 2$ and $\varphi : \mathbb{C}^n \to \mathbb{R}^+$ be a smooth plurisubharmonic weight function that is radially symmetric, i.e. $\varphi(z) = h(s)$, where $s = |z_1|^2 + \cdots + |z_n|^2$. Let furthermore λ_1 be the least eigenvalue of the Levi-matrix of φ and suppose that $h''(s) \geq 0$. Then the weighted $\bar{\partial}$-Neumann operator is compact if and only if*

$$\lambda_1(z) \to \infty \quad \textit{as} \quad |z| \to \infty.$$

The proof is based on the following Lemma of F. Haslinger (personal communication):

Lemma 3.24 *Let $\varphi : \mathbb{C}^n \to \mathbb{R}^+$ be a plurisubharmonic weight function of the form $\varphi(z) = h(s)$, where $s = |z_1|^2 + \cdots + |z_n|^2$. Then the eigenvalues of M_φ are*

$$\lambda = h'(s) \quad \textit{and} \quad \mu = h'(s) + sh''(s),$$

where λ has multiplicity $n-1$. There exists an holomorphic eigenvector to λ and an antiholomorphic one to μ.

Proof. Suppose that $\varphi(z) = h(s)$. By the chain rule,

$$(M_\varphi)_{jk} = \delta_{jk} h'(s) + \bar{z}_j z_k h''(s).$$

It is easily seen that the matrix $M = (\bar{z}_j z_k)_{jk}$ has rank one, so in particular $\lambda = h'(s)$ is a solution of $\det(M_\varphi - \lambda I) = 0$ with multiplicity $n-1$. Now by elementary linear algebra, $\operatorname{tr}(M_\varphi) = sh''(s) + nh'(s)$ equals the sum of all eigenvalues. So the n-th eigenvalue is

$$\mu = \operatorname{tr}(M_\varphi) - (n-1)h'(s) = sh''(s) + h'(s).$$

As one checks immediately, an eigenvector to μ is $(\bar{z}_1, \dots, \bar{z}_n)$. A possible holomorphic eigenvector to λ is $(a_1, \ldots, a_n)$, where $a_1 = z_2 \cdots z_n$ and $a_k = -\frac{1}{n-1}\prod_{j\neq k} z_j$ for $k > 1$.

$\square$

Proof of the Theorem. If $h''(s) \geq 0$, then $\lambda(z) \leq \mu(z)$ for all $z \in \mathbb{C}^n$ by the Lemma. So it is possible to choose an eigenvector to the lower eigenvalue, which is holomorphically dependent on z. Thus Theorem 3.14 can be applied.

$\square$

Remark. Note that if $\varphi(z) = h(s)$ is a radially symmetric weight function such that $h''(s) \leq 0$, then $h'(s)$ is monotonically decreasing. Therefore λ, which in this case is the bigger eigenvalue, is bounded. Hence N_φ can not be compact, by Corollary 3.11.

Example. This for instance shows that radially symmetric plurisubharmonic weight functions which are polynomials in $|z|$ with degree greater than or equal to 3 always admit a compact weighted $\bar{\partial}$-Neumann operator. This is since due to our definition of a weight function, the leading coefficient of this polynomial has to be positive, so the same is true for the second derivative of $h(s)$ — at least on the complement of a compact set.

3.4 On the dimension of the space $\mathcal{A}^2_\varphi(\mathbb{C}^n)$

In this section we study the relationship of certain growth properties of the least eigenvalue $\lambda_1(z)$ of the Levi-matrix M_φ of the weight function $\varphi(z)$ with the dimension of the Bergman space $\mathcal{A}^2_\varphi$. Moreover, we will see that compactness of the weighted $\bar{\partial}$-Neumann operator in some cases also has effects on this dimension.
First, we prepare the following Lemma taken from [28].

Lemma 3.25 *Let $g(z) = \log(1 + K|z - \xi|^2)$ and $K > 0$. Then we have for all $w \in \mathbb{C}^n$*

$$\frac{K}{(1 + K|z - \xi|^2)^2}|w|^2 \leq \sum_{j,k=1}^{n} \frac{\partial^2 g}{\partial z_j \partial \overline{z}_k} w_j \overline{w}_k \leq \frac{K}{1 + K|z - \xi|^2}|w|^2$$

Proof. Without loss of generality we set $\xi = 0$. Differentiating, we find

$$\begin{aligned}\frac{\partial^2 g}{\partial z_j \partial \overline{z}_k}(z) =& - \frac{K^2 \overline{z}_j z_k}{(1 + |z|^2)^2} + \frac{K\delta_{jk}}{1 + K|z|^2} \\ =& \frac{K}{(1 + K|z|^2)^2}\left((1 + K|z|^2)\delta_{jk} - K\overline{z}_j z_k\right)\end{aligned}$$

and hence

$$\sum_{j,k} \frac{\partial^2 g}{\partial z_j \partial \overline{z}_k} w_j \overline{w}_k = \frac{K}{(1+K|z|^2)^2}\big((1+K|z|^2)|w|^2 - K|\langle w, z\rangle|^2\big).$$

This last line makes the statement of the Lemma immediate.

□

The following result is due to I. Shigekawa [28]. Its proof has been remarkably simplified in [25]. With the proof presented here, we even can enhance the result a bit, since the density statement was originally only made for $\liminf \lambda(z) > 0$.

Theorem 3.26 *Let φ be a subharmonic $\mathcal{C}^2$-weight function. Suppose that*

$$\lim_{|z|\to\infty} |z|^2 \lambda_1(z) = \infty \tag{3.13}$$

holds for the least eigenvalue $\lambda_1(z)$ of the Levi-matrix M_φ. Then the weighted Bergman space $\mathcal{A}^2_\varphi$ is dense in $\mathcal{H}(\mathbb{C}^n)$, the space of holomorphic functions on $\mathbb{C}^n$. In particular $\dim \mathcal{A}^2_\varphi = \infty$.

Proof. By assumption on $\lambda(z)$, there exist $R > 0$ and $s > -2$ such that $\lambda(z) \geq |z|^s$ for all $|z| \geq R$. So we can find $\varepsilon > 0$ such that $\lambda(z) - \frac{1}{|z|^{2-\varepsilon}} > 0$ on the complement of a compact set. According to Lemma 3.25

$$\sum_{j,k=1}^{n} \frac{\partial^2(\log(1+|z|^2))}{\partial z_j \partial \overline{z}_k} w_j \overline{w}_k \geq \frac{1}{(1+|z|^2)^2}|w|^2,$$

so we can choose an integer N large enough such that

$$\Psi(z) = \varphi(z) - \frac{1}{\varepsilon^2}|z|^\varepsilon + N\log(1+|z|^2)$$

is a plurisubharmonic function. By Theorem 4.4.4. in [17], the space

$$\overline{\mathcal{A}} = \{f : f \text{ hol. and } \exists \text{K s.t. } \int_{\mathbb{C}^n} |f|^2(1+|z|^2)^{-K} e^{-\Psi(z)} < \infty\}$$

is dense in $\mathcal{H}(\mathbb{C}^n)$. Hence it suffices to show $\overline{\mathcal{A}} \subseteq \mathcal{A}^2_\varphi$. To this end choose any $f \in \bar{\mathcal{A}}$ and note that

$$\begin{aligned}
\int_{\mathbb{C}^n} |f|^2 e^{-\varphi(z)} d\lambda &= \int_{\mathbb{C}^n} |f|^2 e^{-\Psi - \frac{1}{\varepsilon^2}|z|^\varepsilon + N\log(1+|z|^2)} d\lambda \\
&= \int_{\mathbb{C}^n} |f|^2 (1+|z|^2)^{-K} e^{-\Psi} (1+|z|^2)^{K+N} e^{-|z|^\varepsilon} d\lambda \\
&\leq \sup_{z\in\mathbb{C}^n} \{(1+|z|^2)^{K+N} e^{-|z|^\varepsilon}\} \int_{\mathbb{C}^n} |f|^2 (1+|z|^2)^{-K} e^{-\Psi} d\lambda \\
&< \infty,
\end{aligned}$$

since $(1+|z|^2)^{K+N}e^{-|z|^\varepsilon}$ is a bounded function.

□

Example. Theorem 3.26 for instance shows that the so-called Fock space

$$\mathcal{F}(\mathbb{C}^n) = \{f \text{ holomorphic: } \int |f(z)|^2 e^{-2|z|^2} d\lambda < \infty\}$$

is dense in the space of holomorphic functions $\mathcal{H}(\mathbb{C}^n)$ in the topology of uniform convergence on compact sets.

Remark. This condition is not sharp. In order to see this, consider the function $f(z) = |z|^2(1+\cos(|z|))$, where $z \in \mathbb{C}$. One can find a function φ, such that $\triangle\varphi = f$. This function f is subharmonic by construction and a reasonable weight, in the sense that it tends to infinity at infinity. By adding an appropriate constant, we can assume φ to be positive and our considerations are not influenced by the fact that φ is not smooth in the origin. By construction, (3.13) does not hold for φ since

$$\liminf_{z\to\infty} |z|^2 \triangle\varphi = 0.$$

Nevertheless, the space $\mathcal{A}^2_\varphi$ is of infinite dimension, since it is easily seen that

$$\int_{\mathbb{B}(w,\pi)} \triangle\varphi \to \infty \quad \text{for} \quad |w| \to \infty$$

and we therefore can apply Corollary 3.29.

Remark. Another consequence of Theorem 3.26 is that there can not be a bounded $\mathcal{C}^2$-function φ, such that $\triangle\varphi(z) > \varepsilon$ uniformly on $\mathbb{C}^n$. If there was such a function, then $\mathcal{A}^2_\varphi$ would be of infinite dimension for any weight function ψ, since $\varphi + \psi$ would give a weight equivalent to φ, which satisfies (3.13).

Nevertheless, let us give the construction of a bounded continuous weight function φ such that $\triangle\varphi \equiv 1$. To this end consider a locally finite covering of $\mathbb{C}^n$ by balls $\mathbb{B}(z_l, 1)$ of radius 1, and a subordinate smooth partition of unity $\{f_l\}_{l\in I}$. Solving Poisson's equation one can find φ_l such that $\varphi_l = 0$ on $\partial\mathbb{B}(z_l, 1)$ and $\triangle\varphi_l = f_l$. By the Maximum Principle we will have $\varphi_l \leq 0$ and because of ellipticity of Poisson's equation $\inf \varphi_l \geq -C\|f_l\|_q$ for some $q > n/2$, see for instance [11], Theorem 8.17. This means that the family $\{\varphi_l\}_{l\in I}$ is uniformly bounded from above and below and hence the same for $\sum_{l\in I} \varphi_l$. By adding an appropriate constant φ will be positive, such that it is a weight function in the sense of our definition. But note that φ is not continuously differentiable.

An interesting question would be whether the above condition (3.13) is sharp in the sense that if $\mathcal{A}^2_\varphi$ is of infinite dimension for some weight function, one can find an equivalent weight that fulfills it.

Example. To give a negative example, we consider $\varphi(z) = |\sum_{j=1}^n z_j^2|^2$ in $\mathbb{C}^n$. The Levi-matrix of this weight is $(M_\varphi)_{jk} = (4z_j\overline{z}_k)_{jk}$, thus the least eigenvalue of M_φ is identically 0. In particular condition (3.13) is not satisfied. We show that $\mathcal{A}^2_\varphi$ is trivial.
To begin with, let us restrict ourselves to $n = 2$. By introducing the new coordinates $z = z_1 + iz_2$ and $w = z_1 - iz_2$, the weight takes the form $\varphi(z, w) = |zw|^2$. Now to prove that $\mathcal{A}^2_\varphi$ is trivial, we first show $1 \notin \mathcal{A}^2_\varphi$. Transforming to polar coordinates in both variables, we need to evaluate

$$\int\limits_{\mathbb{C}^2} e^{-|zw|^2} d\lambda(z, w) = 4\pi^2 \int\limits_{r_1, r_2 \geq 0} r_1 r_2 e^{-r_1^2 r_2^2}\, d\lambda(r_1, r_2).$$

But

$$\int\limits_{r_1, r_2 \geq 0} r_1 r_2 e^{-r_1^2 r_2^2}\, d\lambda(r_1, r_2) = \int_0^\infty \frac{1}{2r_1} dr_1 = \infty,$$

so indeed $1 \notin \mathcal{A}^2_\varphi$. Now suppose that there is an $f \in \mathcal{A}^2_\varphi$. Then we obviously need to have $f(z, 0) \to 0$ and $f(0, w) \to 0$, which implies by the identity Theorem that $f(z, 0) = f(0, w) \equiv 0$, hence $f(z, w) = zwg(z, w)$ for some holomorphic function g. But now again $zw \notin \mathcal{A}^2_\varphi$, since using the same notation as before

$$\begin{aligned}
\int\limits_{\mathbb{C}^2} |zw|^2 e^{-|zw|^2} d\lambda(z, w) &= 4\pi^2 \int\limits_{r_1, r_2 \geq 0} r_1^3 r_2^3\, e^{-r_1^2 r_2^2}\, d\lambda(r_1, dr_2) \\
&= \pi^2 \int_{u, v \geq 0} uv\, e^{-uv}\, d\lambda(u, v) \\
&= \pi^2 \int_0^\infty \frac{1}{u}\, d\lambda(u) = \infty.
\end{aligned}$$

This means that also $g(z, 0)$ and $g(0, w)$ have to tend to zero for $|w| \to \infty$ or $|z| \to \infty$, in other words $f(z, w) = z^2 w^2 h(z, w)$. Repeating this argument, we see that f has to vanish to infinite order on the complex lines $z = 0$ and $w = 0$, hence we may conclude by the identity Theorem that f is identically zero.
In dimension three, we have $|z_1^2 + z_2^2 + z_3^2|^2 \leq 2|z_1^2 + z_2^2|^2 + 2|z_3^2|^2$. From our considerations above, it is easy to see that for all holomorphic functions f

$$\int\limits_{\mathbb{C}} \int\limits_{\mathbb{C}^2} |f(z_1, z_2, z_3)|^2 e^{-2|z_1 z_2|^2} d\lambda(z_1, z_2) e^{-2|z_3|^2} d\lambda(z_3) \leq \int\limits_{\mathbb{C}^3} |f|^2 e^{-\varphi} d\lambda < \infty$$

forces $f(z_1, z_2, z_3)$ to be zero for all fixed z_3, hence f to be identically zero. Eventually, a similar argument works in any higher dimension.

Theorem 3.27 *Let $\varphi : \mathbb{C}^n \to \mathbb{R}^+$ be a weight function such that $\triangle\varphi < -\varepsilon$ for some $\varepsilon > 0$. Then there exists no holomorphic function which is squareintegrable with respect to this weight.*

Proof. Since $-\triangle\varphi > \varepsilon$, we have

$$-\int\limits_{Q_d} \triangle\varphi > \kappa$$

for another positive constant κ and each cube Q_d with sidelength d. Consequently, the Schrödinger operator $-\triangle - \triangle\varphi$ is strictly positive, see [23] or Proposition 3.19, and therefore also $\triangle_\varphi - \triangle\varphi$ by the diamagnetic property of Schrödinger operators. But as seen in Lemma 3.7, the latter operator annihilates exactly those L^2-functions of the form $fe^{-\varphi}$, where f is holomorphic. Hence no such function f can exist.

□

There is also an interesting connection between the dimension of the Bergman space $\mathcal{A}^2_\varphi(\mathbb{C}^n)$ and compactness of the weighted $\bar\partial$-Neumann operator N_φ, as the next two Theorems point out.

Theorem 3.28 *Let $\varphi(z) : \mathbb{C} \to \mathbb{R}$ be a subharmonic function and suppose that the weighted $\bar\partial$-Neumann operator N_φ is compact. Then the space $\mathcal{A}^2_\varphi$ is of infinite dimension.*

Proof. If N_φ is compact then $H_1 = \triangle_\varphi + \triangle\varphi$ has compact resolvent, in other words its essential spectrum is empty. On the other hand by Corollary 3.21, the Pauli operator $H_2 = \triangle_\varphi - \triangle\varphi$ is not with compact resolvent. In real dimension two, it is known that the spectra of H_1 and H_2 coincide everywhere but possibly in zero, see [7], Theorem 6.4. So our assumption reduces the essential spectrum of H_2 to zero, and since we assumed H_1 to have compact resolvent, 0 must correspond to an eigenvalue of infinite multiplicity of H_2. This is, because it can not be a limit point of eigenvalues nor a point in the continuous spectrum — if it is an accumulation point of eigenvalues, same was true for H_1, and similarly if 0 is in the continuous spectrum of H_2, H_1 needs to have non empty continuous spectrum, too.
Therefore, the kernel of H_2 is infinite dimensional and by Lemma 3.7, each function u in this kernel is of the form $u = fe^{-\varphi}$, where f is holomorphic. So there is an infinite dimensional subspace of holomorphic functions in L^2_φ.

□

We have the following Corollary as a direct consequence of Theorem 3.28, which in complex dimension one gives an alternative condition to (3.13):

Corollary 3.29 *Let $\varphi(z) : \mathbb{C} \to \mathbb{R}$ be a subharmonic function such that*

$$\int_{B(z,r)} (\triangle\varphi) \to \infty \quad as \quad |z| \to \infty \tag{3.14}$$

for some fixed $r > 0$. Then the space $\mathcal{A}^2_\varphi$ is of infinite dimension.

Proof. By Lemma 2.5 in [14], condition 3.14 is sufficient for compactness of N_φ in dimension one. So the Corollary follows by Theorem 3.28.

□

Theorem 3.30 *Suppose that $n \geq 2$ and suppose that an eigenvector corresponding to the least eigenvalue λ_1 of M_φ can be chosen which depends holomorphically on z. Then compactness of the weighted $\bar{\partial}$-Neumann operator N_φ implies $\dim \mathcal{A}^2_\varphi = \infty$.*

Proof. Under the assumptions of the Theorem, we can apply Theorem 3.14 and get $\lambda_1 \to \infty$ as $|z| \to \infty$. So the assertion follows by Theorem 3.26.

□

Remark. For bounded convex domains Ω it is known that compactness of the $\bar{\partial}$-Neumann operator on $L^2(\Omega)$ is equivalent to compactness of its restriction to $\mathcal{A}^2(\Omega)$, see [10]. In the case of weighted L^2-spaces on $\mathbb{C}^n$ such a statement can not be true without additional assumptions, as the following example shows. If one considers $\varphi(z,w) = |z|^2|w|^2$ on $\mathbb{C}^2$, then the restriction of N_φ to $\mathcal{A}^2_\varphi(\mathbb{C}^n)$ is trivially compact, since $\mathcal{A}^2_\varphi(\mathbb{C}^n) = \{0\}$, but on the other hand N_φ is not even continuous on $L^2_\varphi(\mathbb{C}^n)$.
Nevertheless show the results of [22] that in complex dimension one, compactness of the canonical solution operator $\mathcal{S}_\varphi$ on $L^2_\varphi(\mathbb{C}^n)$ is equivalent to compactness on $\mathcal{A}^2_\varphi(\mathbb{C}^n)$, under the assumption that $\mu = \triangle\varphi \, d\lambda$ is a doubling measure. Note that in this case $\mathcal{A}^2_\varphi(\mathbb{C}^n)$ always is of infinite dimension, by Theorem 3.28. In fact, the assumption on $\triangle\varphi$ to be doubling implies certain growth properties of $\triangle\varphi$.

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Appendix A

Curriculum vitae

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