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DISSERTATION

Titel der Dissertation

Bowen-York Solutions of the Einstein Constraints with Asymptotically Flat, Hyperboloidal or Cylindrical Ends

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Für meinen Zwillingsbruder und besten Freund Jürgen

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Abstract

This doctoral thesis deals with the construction and investigation of initial data for the Einstein vacuum equations. The analysis crucially depends on the asymptotic properties of the initial data. On the one hand initial data with asymptotically flat ends provide the basic model of isolated systems, such as stars and black holes. On the other hand asymptotically hyperboloidal ends seem to have a few advantages over asymptotically flat ends, e.g. they are well suited to deal with radiation problems. Also initial data with asymptotically cylindrical ends have received increased attention during the last few years. In particular inital data with one asymptotically flat and one asymptotically cylindrical end - called trumpets - are of great interest in the recent literature. The main result of this thesis is a rigorous existence proof for classes of such data, which are locally conformally flat and with extrinsic curvatures of the Bowen-York type. Furthermore, we demonstrate that it is possible to extend the proof to the case where the remaining asymptotically flat end is replaced by an asymptotically hyperboloidal end.

Diese Dissertation beschäftigt sich mit der Konstruktion und Untersuchung von Anfangsdaten für die Einsteinschen Feldgleichungen im Vakuumfall. Eine entscheidende Rolle spielen dabei die asymptotischen Eigenschaften der Anfangsdaten. Auf der einen Seite stellen asymptotisch flache Anfangsdaten das Grundmodell zur Beschreibung von Sternen und Schwarzen Löchern dar. Auf der anderen Seite scheinen asymptotisch hyperboloidale Anfangsdaten am besten geeignet zu sein, um Fragen bezüglich der emittierten Gravitationsstrahlung zu beantworten. In den letzten Jahren haben auch Anfangsdaten mit asymptotisch zylindrischen Enden an Bedeutung gewonnen. Die zentralen Resultate dieser Arbeit sind rigorose Existenzbeweise für sogenannte Trumpet-Konfigurationen im asymptotisch flachen sowie im asymptotisch hyperboloidalen Fall. Die entsprechenden Anfangsdaten zeichnen sich zusätzlich dadurch aus, dass die Metriken lokal konform flach sind und die äußeren Krümmungen aus Bowen-York Tensoren gebildet werden.

Contents

Chapter 1

Introduction

Soon after Einstein introduced his famous field equations of general relativity, explicit solutions, such as the Schwarzschild solution, were found. However, the derivation of these solutions was only possible by imposing restrictive symmetry conditions or other simplifying features. This fact is not suprising, because the Einstein equations, when fully written out as a system of partial differential equations for the spacetime metric, take the form of a rather complicated system of coupled, nonlinear equations, that is, in general, very hard to solve. Given the difficulty of constructing explicit solutions for more general cases, the focus of attention has gradually shifted towards the initial value formulation of the theory. Numerical studies of the field equations are also largely based on this perspective. Especially now, with intense efforts underway to model astrophysical events which produce detectable gravitational radiation, the initial value formulation and numerical efforts to implement it are of major interest to gravitational physicists. The advantage of doing numerical computations is that it is possible to compute specific numbers that can be compared with observations. Nevertheless, it is essential that one lays the foundation in a rigorous mathematical way. One reason is that it is only possible to consider a finite number of solutions by doing numerical studies. Since the space of initial data is infinite dimensional, there is the risk that something may be overlooked. Qualitative mathematical results have no problem with considering an infinite number of initial data sets at the same time. Moreover, proofs usually yield a deeper understanding of what is going on. In the end, combining numerical studies with rigorous mathematical results is the most fruitful approach to acquire extensive knowledge.

Yvonne Choquet-Bruhat showed that finding a solution of the Einstein equations can be formulated as an initial value (or Cauchy) problem. The main

part of her analysis is a proof of the fact that, given suitable initial data on a 3-dimensional manifold Σ, there is a corresponding solution to the field equations. The initial data needed for Cauchy evolutions cannot be freely specified in their entirety. Rather they are subject to certain constraint equations which must be satisfied. The analysis of these equations depends on the topology of Σ. Of particular importance are the cases where Σ is compact, or where the initial data set possesses a finite number of asymptotic ends. Depending on the asymptotic geometry there are several types of asymptotic ends which have been studied to certain different extents, including asymptotically flat ends, asymptotically hyperboloidal ends and asymptotically cylindrical ends. On the one hand initial data with asymptotically flat ends provide the basic model of isolated self-gravitating systems, such as stars and black holes. On the other hand asymptotically hyperboloidal ends seem to have a few advantages over asymptotically flat ends, e.g. they are very well suited to deal with radiation problems. There is a vast number of papers dealing with asymptotically flat ends and asymptotically hyperboloidal ends. Also initial data sets with asymptotically cylindrical ends have received increased attention during the last few years. In the recent numerical relativity literature inital data sets with one asymptotically flat and one asymptotically cylindrical end - called trumpets - are of great interest, because it has been noticed that when initial data with several asymptotically flat ends are evolved using the standard moving-puncture method, the slices lose contact with the extra asymptotically flat ends, and quickly asymptote to cylinders of finite areal radius. This suggests that it is preferable to start with initial data sets that possess a trumpet geometry from the start.
The main goal of this thesis is to give a rigorous existence proof for classes of such data, which are vacuum, locally conformally flat and with extrinsic curvatures of the Bowen-York type. The problem has previously been solved in [47] and [64], based on a method in which the asymptotically cylindrical end is viewed as a certain singular limit of an asymptotically flat end. We show that such a singular limit can be completely avoided by using a suitable method of sub- and supersolutions for the relevant scalar constraint equation. This result has been published and a good deal of the material presented in the following thesis can also be found in [118]. Furthermore, we demonstrate that it is possible to extend this method to the case where the remaining asymptotically flat end is replaced by an asymptotically hyperboloidal end.
We remark that the construction and investigation of vacuum initial data, e.g. certain black hole initial data, is not only important, because the coalescence of black holes is expected to represent one of the strongest sources of gravitational radiation, but also because black holes allow us to study the dynamics of gravity alone. The understanding of highly dynamical, strong

field configurations is still quite limited and many important questions are still unsolved, e.g. the cosmic censorship conjecture.

This thesis is organized as follows:
In Chapter 2 we review the geometric origin of the Einstein constraint equations and introduce strategies for their solution. First, we state the Einstein equations and recall some basic notions from Lorentzian geometry. Then, we show that one can derive the constraints by using the Gauss equation and Codazzi equation respectively. We briefly discuss the relationship between the constraints and the associated Cauchy problem. Next, we turn to the conformal method for constructing solutions of the constraint equations. Finally, we introduce the monotone iteration scheme which we apply extensively in the following chapters.
In Chapter 3 we discuss certain function spaces which we use in our existence and uniqueness results. First, we define weighted Sobolev spaces and state respective basic inequalities. We illustrate how to analyze these spaces in terms of ordinary Sobolev spaces by the method of scaling and discuss relevant mapping properties of some class of elliptic operators. Furthermore, we state a generalized maximum principle. Next, we move on and define weighted Hölder spaces. After proving inequalities between their norms, embeddings and generalized maximum principles, we investigate important isomorphism theorems.
In Chapter 4 we finally construct black hole initial data sets. First, we recall the notion of an asymptotically flat end and show that the Schwarzschild black hole can easily be described by the methods discussed in Chapter 2. We illustrate how the time symmetric one black hole solution can be generalized in a straightforward manner to any finite number of black holes. Then, we turn to the time asymmetric case, i.e. the vector equation of the constraints is no longer trivially satisfied. After introducing Bowen-York tensors which lead to a solution of the vector constraint, we prove the existence of initial data sets with several asymptotically flat ends. Furthermore, we also discuss an interesting result with regard to perturbation theory. Next, we define asymptotically cylindrical ends and direct our attention to initial data sets which exhibit one asymptotically flat and one asymptotically cylindrical end. Finally, we prove the existence of Bowen-York trumpets.
In Chapter 5 we show that our results can be extended to the case where the asymptotically flat end of a trumpet is replaced by an asymptotically hyperboloidal end. First, we recall the notion of asmptotically hyperboloidal geometries. Then, we apply such methods as in the previous chapter to the problem under investigation (asymptotically hyperboloidal Bowen-York trumpets).

In Chapter 6 we summarize the most important results of this thesis and sketch some ideas how one could make further progress.

We close this introduction with a few remarks concerning our notation. We use the following form of the abstract index notation. Indices are used for tensorial objects to clarify the number and type of arguments, but are generally not associated with the components of the tensor with respect to some specific coordinate system. Whenever indices might clutter notation we freely drop the indices. Furthermore, we employ the letter C to denote any constant that can be explicitly computed in terms of known quantities. The exat value denoted by C may therefore change from line to line in a given computation.

Chapter 2

Constraints

In this chapter we give an overview of the Einstein constraint equations. We first discuss the geometric origins of the constraints and the role which the constraint equations play in the initial value formulation of the Einstein equations. Then we move on and discuss the conformal method for constructing solutions of the Einstein constraint equations.

2.1 Origin

Let $(M, g_{\mu\nu})$ be a 4-dimensional spacetime, that is, M is a 4-dimensional manifold, and $g_{\mu\nu}$ is a Lorentzian metric on M with signature $(-,+,+,+)$. The Einstein equations relate the spacetime metric to the matter distribution which is present in the spacetime in the following way

$$G_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{2.1}$$

where $G_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R} g_{\mu\nu}$ is the Einstein tensor, $T_{\mu\nu}$ is the stress-energy tensor of the matter fields and we set $c = G = 1$. $\mathcal{R}_{\mu\nu} = \mathcal{R}_{\mu\sigma\nu}{}^{\sigma}$ is the Ricci tensor and $\mathcal{R} = \mathcal{R}_{\mu\nu} g^{\mu\nu}$ is the Ricci scalar. The Riemann tensor is defined by $\mathcal{R}_{\mu\nu\sigma}{}^{\rho}\omega_\rho = (\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)\,\omega_\sigma$, where ∇_μ is the Levi-Civita connection associated with $g_{\mu\nu}$ and ω_μ is an arbitrary 1-form on M.
Assuming that no matter fields are present, $T_{\mu\nu} \equiv 0$ and we are left with

$$G_{\mu\nu} = 0, \tag{2.2}$$

the so-called vacuum Einstein equations. Given a manifold M, the equations (2.1) and (2.2) form a system of partial differential equations for the metric $g_{\mu\nu}$. A close inspection reveals that the system is quasi-linear and of second order. It does not fall in any of the standard classes, such as hyperbolic, parabolic, or elliptic in an arbitrary coordinate system.

Remark 2.1. *The fact that the (vacuum) Einstein equations are in general not hyperbolic, parabolic, or elliptic should not be surprising. Because of diffeomorphism invariance we know that if g is a solution of the equations and Φ is any diffeomorphism, then $\Phi^* g$ is also a solution. Hence from the PDE point of view solutions cannot be unique.*

We move on to describing how that (2.2) nevertheless have a well posed initial value formulation.

Remark 2.2. *From now on we assume that $T_{\mu\nu} \equiv 0$, i.e. we restrict ourselves to the vacuum case.*

Recall the following definitions:

Definition 2.1. *A hypersurface Σ of M is the image of a 3-dimensional manifold $\hat{\Sigma}$ by an embedding $\Phi : \hat{\Sigma} \to M$:*

$$\Sigma = \Phi(\hat{\Sigma}). \tag{2.3}$$

The hypersurface is said to be

- *spacelike if the induced bilinear form $h := \Phi^* g$ is a Riemannian metric on $\hat{\Sigma}$;*
- *timelike if the induced bilinear form $h := \Phi^* g$ is a Lorentzian metric on $\hat{\Sigma}$;*
- *null if the induced bilinear form $h := \Phi^* g$ is degenerate on $\hat{\Sigma}$.*

Definition 2.2. *A Cauchy surface S is a spacelike hypersurface such that every future and past inextendible causal path*[1] *intersects it once and only once.*

Definition 2.3. *A spacetime $(M, g_{\mu\nu})$ that admits a Cauchy surface S is said to be globally hyperbolic.*

Definition 2.4. *A map $\hat{t} : M \to \mathbb{R}$ is called a time function iff the function $\hat{t}$ is differentiable and the vector field $-\nabla^\mu \hat{t}$ is a timelike, future-directed vector field on M.*

[1] A path $\eta : I \to M$ is causal if its tangent vector V is always either timelike ($g(V, V) < 0$) or null ($g(V, V) = 0$). Such a path is inextendible if there does not exist a path $\tilde{\eta}$ which contains η as proper subset.

Using this terminology, we have the key result:

Theorem 2.1. *Let $(M, g_{\mu\nu})$ be a globally hyperbolic spacetime. Then $(M, g_{\mu\nu})$ can be foliated by a family of spacelike hypersurfaces $(\Sigma_t)_{t\in\mathbb{R}}$, parameterized by a global time function $\hat{t}$, i.e. each hypersurface is a level surface of $\hat{t}$,*

$$\forall t \in \mathbb{R}, \;\; \Sigma_t := \{p \in M : \hat{t}(p) = t\}, \tag{2.4}$$

$$\Sigma_t \cap \Sigma_{t'} = \emptyset \;\; \textit{for} \;\; t \neq t' \tag{2.5}$$

and

$$M = \bigcup_{t\in\mathbb{R}} \Sigma_t. \tag{2.6}$$

Globally hyperbolic spacetimes play a natural role in the initial value formulation. To see how such spacetime solutions of (2.2) are connected with appropriate initial data, it is useful to determine the geometric quantities which are defined on the spacelike hypersurfaces of a chosen foliation, and then calculate what the vacuum Einstein equations (2.2) tell us about these geometric quantities. We focus now on a particular spacelike hypersurface Σ_t. Let n_t denote the field of unit normals to Σ_t. For $p \in \Sigma_t$ we define the projection map by

$$P_t : T_pM \to T_p\Sigma_t, \quad Y \mapsto Y + n_t\, g(Y, n_t). \tag{2.7}$$

If Y, Z are any vector fields tangent to Σ_t, we can decompose the covariant derivative,

$$\nabla_Y Z = (D_t)_Y Z + K_t(Y, Z)n_t, \tag{2.8}$$

where we defined the spatial covariant derivative, D_t, and the extrinsic curvature, K_t, by

$$(D_t)_Y Z := P_t(\nabla_Y Z), \tag{2.9}$$

$$K_t(Y, Z) := -g(\nabla_Y Z, n_t) = g(\nabla_Y n_t, Z). \tag{2.10}$$

From the facts that the commutator of vector fields tangent to Σ_t is a vector field tangent to Σ_t, $[Y, Z] \in T\,\Sigma_t$, and that ∇ has vanishing torsion, it immediately follows that

$$K_t(Y, Z) = K_t(Z, Y). \tag{2.11}$$

Furthermore one can easily prove that D_t is just the Levi-Civita connection compatible with the metric

$$h_t := g|_{T\,\Sigma_t}. \tag{2.12}$$

From now on we remove the "t" subscipts to simplify the presentation. One can also define the associated Riemann tensor by

$$R_{ijk}{}^{l}\omega_l = (D_iD_j - D_jD_i)\,\omega_k, \tag{2.13}$$

where the indices i, j, k, l refer to a coordinate basis of the spatial tangent space and ω_i is an arbitrary 1-form on Σ. Using (2.9) and (2.10) we can calculate components of the spacetime Riemann tensor in terms of quantities defined on the hypersurface. We get

$$\mathcal{R}_{ijkl} = R_{ijkl} + K_{ik}K_{jl} - K_{il}K_{jk} \tag{2.14}$$

and

$$n^{\mu}\mathcal{R}_{\mu ijk} = D_kK_{ij} - D_jK_{ik}. \tag{2.15}$$

(2.14) is often called the Gauss equation, because an analogous computation was done by Gauss for a flat ambient geometry. (2.15) is known as the Codazzi equation. If the spacetime $(M, g_{\mu\nu})$ under consideration satisfies the vacuum Einstein equations (2.2) it follows that

$$2n^{\mu}n^{\nu}G_{\mu\nu} = R[h] - |K|_h^2 + (tr_hK)^2 = 0 \tag{2.16}$$

and

$$n^{\mu}G_{i\mu} = D_jK_i{}^{j} - D_itr_hK = 0, \tag{2.17}$$

where $R[h]$ is the scalar curvature of h_{ij}, $|K|_h^2 := K_{ij}K^{ij}$ and $tr_hK := h^{ij}K_{ij}$. The scalar equation is called the Hamiltonian constraint, while the vector equation is referred to as the momentum constraint. These constraint equations play an important role in the initial value formulation of Einstein's equations.

Definition 2.5. *An initial data set is a triple (Σ, h_{ij}, K_{ij}), where Σ is a 3-manifold, h_{ij} a positive-definite metric on Σ and K_{ij} is a symmetric tensor field on Σ. This initial data set is called a vacuum initial data set, if the constraint equations (2.16)-(2.17), for given (h_{ij}, K_{ij}), are fulfilled.*

It is clear from our discussion above that the constraint equations (2.16)-(2.17) are conditions on the data (Σ, h_{ij}, K_{ij}) which are necessary for the data to arise from a spacelike hypersurface in a spacetime satisfying the vacuum Einstein equations. Of particular importance is the fact that these conditions are also sufficient for the existence of a spacetime satisfying (2.2). To state a theorem concerning this fact we need one further definition.

Definition 2.6. *A globally hyperbolic development of an initial data set (Σ, h_{ij}, K_{ij}) is a globally hyperbolic spacetime $(M, g_{\mu\nu})$, such that there exists an embedding Φ of Σ into M exhibiting the following properties:*

- *The metric h_{ij} is the pullback of $g_{\mu\nu}$ by Φ, $h_{ij} = (\Phi^* g)_{ij}$.*

- *The image under Φ of K_{ij} is the extrinsic curvature of $\Phi(\Sigma)$ as submanifold of $(M, g_{\mu\nu})$.*

A globally hyperbolic development $(M, g_{\mu\nu})$ of (Σ, h_{ij}, K_{ij}) is called maximal if every other globally hyperbolic development of the same set of data is diffeomorphic to a subset of $(M, g_{\mu\nu})$. A globally hyperbolic development $(M, g_{\mu\nu})$ of (Σ, h_{ij}, K_{ij}) is called a globally hyperbolic Einsteinian development if the metric $g_{\mu\nu}$ satisfies the Einstein equations on M.

Using this terminology, the following fundamental theorem can be proven:

Theorem 2.2. *For any smooth vacuum initial data set (Σ, h_{ij}, K_{ij}), there exists a unique (up to diffeomorphism) maximal globally hyperbolic Einsteinian development.*

For more details about the existence and uniqueness of solutions to the Cauchy problem, see [109]. We only mention that the key for proving that the Cauchy problem for (2.2) is well-posed is to show that the system can be, in a certain sense, transformed into a hyperbolic PDE system for which well-posedness is well-established.

Remark 2.3. *The assumption that the initial data are smooth can be relaxed. There are generalizations of (2.2) which require a lower degree of regularity.*

2.2 Conformal method

The object of this section is to present the conformal method for constructing solutions of the constraint equations,

$$R[\bar{h}] - |\bar{K}|^2_{\bar{h}} + (tr_{\bar{h}}\bar{K})^2 = 0 \tag{2.18}$$

and

$$\bar{D}_j \left(\bar{K}^{ij} - \bar{h}^{ij} tr_{\bar{h}} \bar{K} \right) = 0. \tag{2.19}$$

Remark 2.4. *From now on we denote quantities that satisfy the constraints by an overbar.*

As a PDE system for the twelve functions encompassed in $\bar{h}_{ij}$ and $\bar{K}_{ij}$, the four equations (2.18)-(2.19) constitute an underdetermined system. Our goal is to transform the equations into standard elliptic form which can be solved given appropriate boundary conditions. The basic idea is to construct a metric satisfying (2.18) by conformal deformation. Given a non-vanishing function ϕ on Σ, we define the conformally related metric h_{ij} by

$$\bar{h}_{ij} = \phi^4 h_{ij}. \tag{2.20}$$

Let L_h denote the conformal Laplacian associated with the metric h_{ij}, i.e.

$$L_h := -\triangle_h + \frac{1}{8}R[h], \tag{2.21}$$

where

$$\triangle_h := h^{ij} D_i D_j \tag{2.22}$$

is the usual Laplacian for h_{ij}.

Lemma 2.1. *The conformal Laplacian of $\bar{h}_{ij}$ is related to that of h_{ij} by the relation*

$$L_{\bar{h}}\bar{\psi} = \phi^{-5} L_h \psi, \tag{2.23}$$

where $\bar{\psi} = \phi^{-1}\psi$.

Proof. Using (2.20) and the relation for the inverse metrics, $\bar{h}^{ij} = \phi^{-4}h^{ij}$, one easily verifies that the Christoffel symbols fulfill

$$\bar{\Gamma}^i{}_{jk} - \Gamma^i{}_{jk} = \frac{2}{\phi}\left(\delta^i{}_j \partial_k\phi + \delta^i{}_k \partial_j\phi - h^{il}h_{jk}\partial_l\phi\right). \tag{2.24}$$

The lemma follows by employing (2.24) for every covariant derivative which appears in the definition of the conformal Laplacian (2.21), i.e. in the usual Laplacian and in the Ricci scalar associated with the respective metric.

Corollary 2.1. *The scalar curvature of $\bar{h}_{ij}$ fulfils the relation*

$$R[\bar{h}]\phi^5 = \left(-8\triangle_h + R[h]\right)\phi \tag{2.25}$$

Proof. This immediately follows from the previous lemma because setting $\bar{\psi} = 1$ implies

$$\frac{1}{8}R[\bar{h}] = \phi^{-5}L_h\phi. \tag{2.26}$$

The final preliminary result that we will require is the following.

Lemma 2.2. *Let $\bar{A}^{ij}$ be a symmetric, tracefree (i.e. $\bar{h}_{ij}\bar{A}^{ij} = 0$) tensor field on Σ. Let*

$$A^{ij} := \phi^{10}\bar{A}^{ij}. \tag{2.27}$$

Then

$$D_i A^{ij} = \phi^{10}\bar{D}_i\bar{A}^{ij}. \tag{2.28}$$

Proof. We calculate

$$\begin{aligned} D_i A^{ij} &= \bar{D}_i A^{ij} + \left(\Gamma^i{}_{ik} - \bar{\Gamma}^i{}_{ik}\right) A^{kj} + \left(\Gamma^j{}_{ik} - \bar{\Gamma}^j{}_{ik}\right) A^{ik} \\ &= \bar{D}_i \left(\phi^{10} \bar{A}^{ij}\right) - 10\phi^9 \partial_i \phi \, \bar{A}^{ij} \\ &= \phi^{10} \bar{D}_i \bar{A}^{ij}, \end{aligned} \tag{2.29}$$

where we have again used (2.24).

To apply the previous lemma it is convenient to decompose the tensor field $\bar{K}^{ij}$ into a tracefree part

$$\bar{A}^{ij} := \bar{K}^{ij} - \frac{\tau}{3} \bar{h}^{ij}, \tag{2.30}$$

where

$$\tau := tr_{\bar{h}} \bar{K} \tag{2.31}$$

denotes the trace of K^{ij} and a trace part

$$\frac{\tau}{3} \bar{h}^{ij}. \tag{2.32}$$

The main point of these calculations is the following theorem.

Theorem 2.3. *The fields $(\bar{h}_{ij}, \bar{K}_{ij})$ satisfy the constraint equations (2.18) and (2.19) if and only if the fields $(h_{ij} = \phi^{-4}\bar{h}_{ij}, A_{ij} = \phi^2 \bar{A}_{ij}, \tau, \phi)$ satisfy the relations*

$$D_i A^{ij} = \frac{2}{3} \phi^6 D^j \tau \tag{2.33}$$

$$\triangle_h \phi = \frac{R[h]}{8} \phi + \frac{1}{12} \tau^2 \phi^5 - \frac{1}{8} A_{ij} A^{ij} \phi^{-7}. \tag{2.34}$$

Proof. We use Lemma 2.2 and the momentum constraint (2.19) to show

$$\begin{aligned} D_i A^{ij} &= \phi^{10} \bar{D}_i \bar{A}^{ij} \\ &= \phi^{10} \bar{D}_i \left(\bar{K}^{ij} - \frac{\tau}{3} \bar{h}^{ij}\right) \\ &= \phi^{10} \left(\bar{D}^j \tau - \frac{1}{3} \bar{D}^j \tau\right) \\ &= \frac{2}{3} \phi^{10} \bar{D}^j \tau \\ &= \frac{2}{3} \phi^6 D^j \tau. \end{aligned} \tag{2.35}$$

Using Corollary 2.1 and the Hamiltonian constraint (2.18) we verify

$$\begin{aligned}
\triangle_h \phi &= \frac{R[h]}{8}\phi - \frac{R[\bar{h}]}{8}\phi^5 \\
&= \frac{R[h]}{8}\phi - \frac{1}{8}\left(\bar{K}_{ij}\bar{K}^{ij} - \tau^2\right)\phi^5 \\
&= \frac{R[h]}{8}\phi - \frac{1}{8}\left(\bar{h}_{ij}\bar{h}_{kl}(\bar{A}^{ik} + \frac{\tau}{3}\bar{h}^{ik})(\bar{A}^{jl} + \frac{\tau}{3}\bar{h}^{jl}) - \tau^2\right)\phi^5 \\
&= \frac{R[h]}{8}\phi - \frac{1}{8}\left(\phi^{-12}h_{ij}h_{kl}A^{ik}A^{jl} + \frac{1}{3}\tau^2 - \tau^2\right)\phi^5 \\
&= \frac{R[h]}{8}\phi + \frac{1}{12}\tau^2\phi^5 - \frac{1}{8}A_{ij}A^{ij}\phi^{-7}. \qquad (2.36)
\end{aligned}$$

The semilinear elliptic equation (2.34) for ϕ is known as the Lichnerowicz equation.
It is intuitive that the solution of the constraint equations should not depend on the choice of the specified conformal metric in a conformal class. This fact is the content of the next theorem which can be proven along the lines of Lemma 2.1 and Lemma 2.2.

Theorem 2.4. *Suppose that (A^{ij}, ϕ) is a solution of (2.33) and (2.34) for the metric h_{ij} and a given function τ. Then (2.33) and (2.34) in the metric $h'_{ij} = \theta^4 h_{ij}$, with $\tau' = \tau$, admit the solution $A'^{ij} = \theta^{-10}A^{ij}, \phi' = \theta^{-1}\phi$.*

Equations (2.33) and (2.34) are coupled, but we notice that if we choose τ to be a constant, i.e.

$$D_i\tau = 0, \qquad (2.37)$$

then they decouple partially, because (2.33) becomes independent of ϕ. The condition (2.37) on the extrinsic curvature of Σ defines what is called a constant mean curvature (CMC) hypersurface. A special case of a CMC hypersurface is a so-called maximal hypersurface, having $\tau = 0$. The conformal method works best in the CMC setting where solving the constraints reduces to

- Choose a background metric h_{ij}.
- Find a symmetric, tracefree tensor field A_{ij} that fulfills $D_iA^{ij} = 0$.
- Solve equation (2.34) for given (h_{ij}, A_{ij}) and a choice of constant τ.
- Finally, the vacuum initial data is given by $\bar{h}_{ij} = \phi^4 h_{ij}$ and $\bar{K}_{ij} = \phi^{-2}A_{ij} + \frac{\tau}{3}\bar{h}_{ij}$.

Symmetric 2-tensors that satisfy

$$D_i A^{ij} = 0, \quad h_{ij} A^{ij} = 0, \tag{2.38}$$

are called TT-tensors (Transverse, Tracefree). There is a standard method, which goes back to York, for constructing such tensors [119]. Let S^{ij} be an arbitrary symmetric, tracefree tensor field. Furthermore, let $\mathbb{L}$ denote the conformal Killing operator that maps a vector field Y^i to the symmetric, tracefree tensorfield

$$(\mathbb{L}Y)^{ij} := D^i Y^j + D^j Y^i - \frac{2}{3} h^{ij} D_k Y^k. \tag{2.39}$$

Now we want to find a special vector field Y^i such that

$$A^{ij} = S^{ij} + (\mathbb{L}Y)^{ij} \tag{2.40}$$

is also divergencefree. The condition of being divergencefree leads to an equation for the vector field Y^i:

$$D_i (\mathbb{L}Y)^{ij} = -D_i S^{ij} \tag{2.41}$$

If we define the following second order operator

$$\triangle_{\mathbb{L}} Y^j := D_i (\mathbb{L}Y)^{ij}, \tag{2.42}$$

the equation (2.41) can be written

$$\triangle_{\mathbb{L}} Y^j = -D_i S^{ij}. \tag{2.43}$$

$\triangle_{\mathbb{L}}$ is known as the conformal vector Laplacian and is explicitly given by

$$\triangle_{\mathbb{L}} Y^j = D_i (D^i Y^j + D^j Y^i - \frac{2}{3} h^{ij} D_k Y^k). \tag{2.44}$$

The nice property of this construction is the fact that it leads to an elliptic differential equation, whereas the first order equation $D_i A^{ij} = 0$ is only underdetermined elliptic.

Proposition 2.1. *The operator $\triangle_{\mathbb{L}} : T\Sigma \to T\Sigma$ is elliptic.*

Proof. Recall that the symbol σ of a linear partial differential operator of the form

$$L = \sum_{0 \le l \le m} a^{i_1 \ldots i_l} D_{i_1} \ldots D_{i_l}, \tag{2.45}$$

where the $a^{i_1\dots i_l}$'s are linear maps from fibers of a bundle E to fibers of a bundle F, is defined as the map

$$T^*\Sigma \ni \xi \mapsto \sigma(\xi) := a^{i_1\dots i_m}\xi_{i_1}\dots\xi_{i_m}. \tag{2.46}$$

An operator is said to be elliptic if the symbol is an isomorphism of fibers for all $\xi \neq 0$. In our case the operator $\triangle_{\mathbb{L}}$ acts on vector fields and produces vector fields. Hence the relevant map for the conformal vector Laplacian is given by

$$\sigma(\xi)(Y) = \xi_i(\xi^i Y^j + \xi^j Y^i - \frac{2}{3}h^{ij}\xi_k Y^k)\partial_j. \tag{2.47}$$

To prove the proposition we have to show that this map is bijective. We are done when we can verify that is has a trivial kernel. If $\sigma(\xi)(Y) = 0$, then contracting this expression with ξ_j implies that

$$\xi_j\xi_i(\xi^i Y^j + \xi^j Y^i - \frac{2}{3}h^{ij}\xi_k Y^k) = |\xi|^2\xi_i Y^i\frac{4}{3} = 0, \tag{2.48}$$

hence $\xi_i Y^i = 0$. If we contract instead with Y_j and use the last equality we obtain

$$Y_j\xi_i(\xi^i Y^j + \xi^j Y^i - \frac{2}{3}h^{ij}\xi_k Y^k) = |\xi|^2|Y|^2 = 0, \tag{2.49}$$

hence $Y^i = 0$ and the map has a trivial kernel.

General theorems on elliptic operators can now be invoked to prove existence and uniqueness results. In these theorems global conditions, which the manifold Σ has to meet, play a decisive role. E.g. assuming that Σ is compact one can prove the following theorem.

Theorem 2.5. *Let (Σ, h_{ij}) be a compact manifold. Given any symmetric, tracefree tensor field S^{ij}, there exists a vector field X^i such that $\triangle_{\mathbb{L}}Y^j = -D_iS^{ij}$. The symmetric, tracefree tensor field $A^{ij} := S^{ij} + (\mathbb{L}Y)^{ij}$ then satisfies (2.38).*

The reader is referred to [26] for further details. To finish the presentation of the conformal method we address the question of existence of solutions of the Lichnerowicz equation (2.34). In this chapter we restrict ourselves to the case that Σ is compact. In the following chapters we will discuss more complicated cases. Many methods which we will use are generalizations of techniques which work in the compact case. Therefore the following constructions give a flavor of what steps are involved and how they are proven. We begin with results that do not directly extend to the non compact cases.

Definition 2.7. *The Yamabe number of a metric h_{ij} is defined by*

$$Y(\Sigma, h_{ij}) = \inf_{u \in C_b^\infty, u \not\equiv 0} \frac{\int_\Sigma (|Du|^2 + \frac{1}{8} R[h] u^2)}{(\int_\Sigma u^6)^{1/3}}, \tag{2.50}$$

where C_b^∞ denotes the space of compactly supported smooth functions.

Definition 2.8. *Let (Σ, h_{ij}) be a compact manifold. The metric h_{ij} is said to be in the negative Yamabe class $\mathcal{Y}^-$ if $Y(\Sigma, h_{ij}) < 0$, in the zero Yamabe class $\mathcal{Y}^0$ if $Y(\Sigma, h_{ij}) = 0$, in the positive Yamabe class $\mathcal{Y}^+$ if $Y(\Sigma, h_{ij}) > 0$.*

We now state a version of the Yamabe theorem.

Theorem 2.6. *Let (Σ, h_{ij}) be a compact manifold and assume that h_{ij} is smooth. Then:*

1. *If h_{ij} is in the negative Yamabe class $\mathcal{Y}^-$ it is conformal to a metric with scalar curvature -1.*
2. *If h_{ij} is in the zero Yamabe class $\mathcal{Y}^0$ it is conformal to a metric with scalar curvature 0.*
3. *If h_{ij} is in the positive Yamabe class $\mathcal{Y}^+$ it is conformal to a metric with scalar curvature $+1$.*

The Yamabe theorem is very useful in the compact case because the Lichnerowicz equation is conformally covariant. Hence, we can always first perform a preliminary conformal rescaling on the data and therefore work with data for which the scalar curvature is constant. However not every manifold admits metrics in all Yamabe classes. It usually depends on the topology of the manifold if it can support a metric that belongs to a certain Yamabe class.

Finally, one can use the monotone iteration scheme to prove existence of suitable soltions of (2.34). This scheme uses so-called sub- and supersolutions. Considerations invoking the maximum principle in the theory of elliptic PDEs also play a decisive role.

Proposition 2.2. *Let (Σ, h_{ij}) be compact, suppose that $c > 0$ and let $w \in C^2(\Sigma)$. If*

$$\triangle_h w - c\, w \leq 0, \tag{2.51}$$

then $w \geq 0$.

Proof. Suppose that at some point w possesses a negative minimum. This immediately leads to a contradiction because the term $\triangle_h w$ is non-negative at a minimum.

Furthermore, the operator

$$L = \triangle_h - c \tag{2.52}$$

is elliptic. The symbol of L is given by

$$\sigma_L(\xi) = h^{ij}\xi_i\xi_j. \tag{2.53}$$

One can us this fact to prove existence of solutions of the equation

$$Lw = f, \tag{2.54}$$

assuming that f fulfils certain differentiability properties. Let use write the Lichnerowicz equation (2.34) in the form

$$\triangle_h \phi = f(\phi, x). \tag{2.55}$$

Definition 2.9. *A C^2 function ϕ_+ is called a supersolution of (2.55) if*

$$\triangle_h \phi_+ \leq f(\phi_+, x). \tag{2.56}$$

Definition 2.10. *A C^2 function ϕ_- is called a subsolution of (2.55) if*

$$\triangle_h \phi_- \geq f(\phi_-, x). \tag{2.57}$$

Theorem 2.7. *Let (Σ, h_{ij}) be a compact manifold. Suppose that (2.55) admits a subsolution ϕ_- and a supersolution ϕ_+ satisfying*

$$\phi_- \leq \phi_+ \tag{2.58}$$

If f is differentiable in ϕ, then there exists a C^2 solution ϕ of (2.55) such that

$$\phi_- \leq \phi \leq \phi_+. \tag{2.59}$$

Proof. The idea of the monotone iteration scheme is to construct a pointwise decreasing (or increasing) sequence of functions ϕ_n and show that this sequence converges to a solution of the considered elliptic equation. The sub- and supersolution act as barriers such that

$$\phi_- \leq \phi_n \leq \phi_+. \tag{2.60}$$

To prove theorem (2.7) we set

$$\phi_0 = \phi_+ \tag{2.61}$$

and want to guarantee that

$$\phi_{n+1} \leq \phi_n. \tag{2.62}$$

Let c be a positive constant so that the function

$$F(\phi, x) := f(\phi, x) - c\,\phi \tag{2.63}$$

is monotone decreasing for $\phi_- \leq \phi \leq \phi_+$, i.e. $c \geq \frac{\partial f}{\partial \phi}$. This can clearly be done on a compact manifold. By what has been said we can solve the equation

$$(\triangle_h - c)\,\phi_{n+1} = F(\phi_n, x). \tag{2.64}$$

Cleary (2.60) holds for $n = 0$. Suppose that (2.60) holds for some n, then

$$(\triangle_h - c)\,(\phi_{n+1} - \phi_+) = F(\phi_n, x) - \triangle_h \phi_+ - c\,\phi_+ \geq 0, \tag{2.65}$$

where we have used the monotonicity of F. The maximum principle (2.2) implies

$$\phi_{n+1} \leq \phi_+. \tag{2.66}$$

From the equations satisfied respectively by ϕ_{n+1} and ϕ_- we deduce

$$(\triangle_h - c)\,(\phi_- - \phi_{n+1}) = \triangle_h \phi_- - c\,\phi_- - F(\phi_n, x) \geq 0, \tag{2.67}$$

and (2.60) is established. We also have to show that (2.62) holds. We note that (2.60) implies (2.62) for $n = 0$. To continue the induction, we suppose that (2.62) is true for some $n \geq 0$, then

$$(\triangle_h - c)\,(\phi_{n+2} - \phi_{n+1}) = F(\phi_{n+1}, x) - F(\phi_n, x) \geq 0, \tag{2.68}$$

hence $\phi_{n+2} \leq \phi_{n+1}$. We have proved the existence of a sequence which is monotone decreasing and uniformly bounded. Thus ϕ_n converges at each point to a limit ϕ, with $\phi_- \leq \phi \leq \phi_+$. Continuity of F gives

$$F_n := F(\phi_n, x) \to F_\infty = F(\phi, x), \tag{2.69}$$

again pointwise. One can use elliptic theory to show that $\phi \in C^2$ and

$$(\triangle_h - c)\phi = F(\phi, x) = f(\phi, x) - c\,\phi, \tag{2.70}$$

so that ϕ satisfies

$$\triangle_h \phi = f(\phi, x). \tag{2.71}$$

Note that there is no a priori reason for this solution to be unique, although this may be true in certain circumstances.

In order to apply theorem (2.7) the task consists mostly of constructing appropriate sub- and supersolutions. In the compact case one can often use suitable constants as sub- and supersolutions.

Chapter 3

Mathematical Preparation

In this chapter we define certain function spaces which we will need to state the existence and uniqueness results of the following chapters. We are going to investigate initial data sets which have in common that Σ is non compact.The appropriate analytic setting for studying such initial data sets is weighted function spaces. The beauty of weighted spaces is that they give us analogues of elliptic theory results that are true for compact manifolds. We analyze the relevant mapping properties of some class of elliptic operators which are defined between these spaces.

The theory of elliptic operators on these spaces was introduced by [99], and has been extensively developed [88], [97], [26], [12], [34], [31]. A good starting point for studying usual Sobolev and Hölder spaces is the book [65].

3.1 Weighted Sobolev spaces

We start by defining weighted Sobolev spaces on $\mathbb{R}^3$.

Definition 3.1. *Let* $\sigma = (1+|x|^2)^{1/2}$. *For* $k \in \mathbb{N} \cup \{0\}, 1 \leq p < \infty$ *and* $\delta \in \mathbb{R}$, *the space* $W^{k,p}_\delta(\mathbb{R}^3)$ *is defined as the closure of* $C^\infty_b(\mathbb{R}^3)$ *in the norm*

$$\|w\|^p_{W^{k,p}_\delta(\mathbb{R}^3)} := \sum_{|\alpha|\leq k} \left\|\sigma^{|\alpha|-\delta-3/p} D^\alpha w\right\|_{L^p(\mathbb{R}^3)}, \tag{3.1}$$

where $\|\cdot\|_{L^p(\mathbb{R}^3)}$ *denotes the usual norm of the Lebesgue space* $L^p(\mathbb{R}^3)$.

We denote by $L^p_\delta(\mathbb{R}^3)$ the spaces $W^{0,p}_\delta(\mathbb{R}^3)$. There are basic inequalities for these weighted spaces.

Proposition 3.1. *1. If $p_1 \le p_2, \delta_2 < \delta_1$, and $w \in L^{p_2}_{\delta_2}(\mathbb{R}^3)$, then*

$$\|w\|_{L^{p_1}_{\delta_1}(\mathbb{R}^3)} \le C\,\|w\|_{L^{p_2}_{\delta_2}(\mathbb{R}^3)}\,. \tag{3.2}$$

2. If $kp < 3$ and $p \le q \le 3p/(3-kp)$, then if $w \in W^{k,p}_\delta(\mathbb{R}^3)$,

$$\|w\|_{L^{q}_{\delta}(\mathbb{R}^3)} \le C\,\|w\|_{W^{k,p}_{\delta}(\mathbb{R}^3)}\,. \tag{3.3}$$

3. If $kp > 3$, then if $w \in W^{k,p}_\delta(\mathbb{R}^3)$,

$$\sigma^{-\delta}|w| \le C\,\|w\|_{W^{k,p}_{\delta}(\mathbb{R}^3)}\,, \tag{3.4}$$

and in fact $|w(x)| = o(r^\delta)$ as $r \to \infty$, where $r = |x|$.

Proof. Weighted function spaces can be analyzed in terms of ordinary function spaces by applying a technique of rescaling [12], which we illustrate by proving 3.

Let $W'^{k,p}_\delta(\mathbb{R})$ denote the weighted spaces, where we replace the norm (3.1) with

$$\|w\|^p_{W'^{k,p}_\delta(\mathbb{R}^3)} := \sum_{|\alpha|\le k} \left\| |x|^{|\alpha|-\delta-3/p} D^\alpha w \right\|_{L^p(\mathbb{R}^3)}\,. \tag{3.5}$$

Furthermore let B_R be the ball of radius R centered at the origin, and let A_R be the annulus $A_R = B_{2R} \setminus B_R$. The rescaled function

$$w_R(x) := w(Rx) \tag{3.6}$$

satisfies, by a simple change of variables,

$$\|w_R\|_{W'^{k,p}_\delta(A_1)} = R^\delta\,\|w\|_{W'^{k,p}_\delta(A_R)}\,. \tag{3.7}$$

Supposing that $kp > 3$, we have

$$\begin{aligned}
\sup_{A_R} \sigma^{-\delta}|w| &= \sup_{A_1} R^{-\delta}\sigma^{-\delta}|w_R| \\
&\le CR^{-\delta}\left\|\sigma^{-\delta} w_R\right\|_{W^{k,p}(A_1)} \\
&\le CR^{-\delta}\,\|w_R\|_{W'^{k,p}_\delta(A_1)} \\
&= C\,\|w\|_{W'^{k,p}_\delta(A_R)} \\
&\le C\,\|w\|_{W^{k,p}_\delta(A_R)}\,.
\end{aligned} \tag{3.8}$$

The first line is a trivial change of coordinates. The following inequality is true by the usual Sobolev inequality applied A_1. The next two lines use that on the domain A_1 the standard and the weighted norm are equivalent and the fundamentaly scaling property (3.7). Finally, we have used the equivalence of $\|\cdot\|_{W'^{k,p}_\delta(A_R)}$ and $\|\cdot\|_{W^{k,p}_\delta(A_R)}$. Clearly

$$\sup_{B_1} \sigma^{-\delta}|w| \leq C\,\|w\|_{W^{k,p}_\delta(B_1)}\,. \tag{3.9}$$

Define

$$w_0 = w|_{B_1} \tag{3.10}$$

and

$$w_i = w|_{A_{2^{i-1}}},\ \ i \geq 1. \tag{3.11}$$

Then

$$w = \sum_{i=0}^{\infty} w_i. \tag{3.12}$$

We find that

$$\begin{aligned}
(\sup_{\mathbb{R}^3} \sigma^{-\delta}|w|)^p &\leq \sum_{i=0}^{\infty} (\sup_{\mathbb{R}^3} \sigma^{-\delta}|w_i|)^p \\
&\leq C\sum_{i=0}^{\infty} \|w_i\|^p_{W^{k,p}_\delta(\mathbb{R}^3)} \\
&= C\,\|w\|^p_{W^{k,p}_\delta(\mathbb{R}^3)}\,,
\end{aligned} \tag{3.13}$$

which proves (3.4). Since $\|w\|_{W^{k,p}_\delta(\mathbb{R}^3)} < \infty$ we have $\|w\|_{W^{k,p}_\delta(A_R)} = o(1)$ as $R \to \infty$, which gives $|w(x)| = o(r^\delta)$.

We will also need the following fact [34]:

Lemma 3.1. *If $0 < k - 3/p \leq 1$ and $\delta < 0$ then $W^{k,p}_\delta(\mathbb{R}^3)$ is compactly embedded in the space of continuous and bounded functions, denoted by $C^0_0(\mathbb{R}^3)$.*

Considering the flat Laplacian we omit the subscript from now on. The following theorem plays a crucial role when using the monotone iteration scheme for proving existence results in the setting of weighted Sobolev spaces.

Theorem 3.1. *If $p > 1, \delta \in (-1,0), g \in L^p_{\tilde{\delta}}(\mathbb{R}^3)$ for some $\tilde{\delta} < -2$ and in addition $g \geq 0$, then*

$$\triangle - g : W^{2,p}_\delta(\mathbb{R}^3) \to L^p_{\delta-2}(\mathbb{R}^3) \tag{3.14}$$

is an isomorphism and following inequality holds:

$$\|w\|_{W^{2,p}_\delta(\mathbb{R}^3)} \leq C\,\|(\triangle - g)w\|_{L^p_{\delta-2}(\mathbb{R}^3)} \tag{3.15}$$

One can also prove a generalized maximum principle.

Lemma 3.2. *Under the hypotheses of theorem (3.1), suppose that* $w \in W^{2,p}_{\delta}(\mathbb{R}^3)$*, and moreover*

$$\triangle w - g\, w = f, \tag{3.16}$$

with $f \leq 0$*, then* $w \geq 0$*.*

Details and proofs of more general versions can be found in [32].

3.2 Weighted Hölder spaces

We move on and define weighted Hölder spaces on $\mathbb{R}^3$ and subsets of $\mathbb{R}^3$ respectively. We use definitions which are similar to those employed in [104]. Many of the following results can be found in [118].

Definition 3.2. *Given* $\alpha \in [0,1)$ *and* $\delta_1 \in \mathbb{R}$*, the space* $C^{k,\alpha}_{\delta_1}\left(B_{3/2} \setminus \{0\}\right)$ *is defined to be the set of functions* $w \in C^{k,\alpha}_{loc}\left(B_{3/2} \setminus \{0\}\right)$ *for which the following norm is finite*

$$\begin{aligned} \|w\|_{C^{k,\alpha}_{\delta_1}\left(B_{3/2}\setminus\{0\}\right)} &:= \sum_{j=0}^{k} \sup_{|x|\in(0,\frac{3}{2})} |x|^{-\delta_1+j} \left|D^j w(x)\right| \\ &+ \sup_{|x|\in(0,\frac{3}{2})} \sup_{\substack{|y|\in[\frac{|x|}{2},|x|]\cap(0,\frac{3}{2}) \\ x\neq y}} |x|^{-\delta_1+k+\alpha} \frac{\left|D^k w\,(x) - D^k w\,(y)\right|}{|x-y|^{\alpha}}. \end{aligned} \tag{3.17}$$

For precise definitions of the usual Hölder spaces $C^{k,\alpha}$ we refer to [65]. Loosely speaking, a function in $C^{k,\alpha}_{\delta_1}\left(B_{3/2} \setminus \{0\}\right)$ has continuous derivatives up to order k such that the k^{th} partial derivatives are locally Hölder continuous with exponent α and is of $O(r^{\delta_1})$ for small r. In particular for δ_1 negative this function may diverge at the origin no worse than r^{δ_1}.

Definition 3.3. *For* $k \in \mathbb{N}\cup\{0\}$*,* $\alpha \in [0,1)$ *and* $\delta_2 \in \mathbb{R}$*, the space* $C^{k,\alpha}_{\delta_2}\left(\mathbb{R}^3 \setminus B_1\right)$ *is defined to be the set of functions* $w \in C^{k,\alpha}_{loc}\left(\mathbb{R}^3 \setminus B_1\right)$ *for which the following norm is finite*

$$\begin{aligned} \|w\|_{C^{k,\alpha}_{\delta_2}(\mathbb{R}^3\setminus B_1)} &= \sum_{j=0}^{k} \sup_{|x|\in[1,\infty)} |x|^{\delta_2+j} \left|D^j w(x)\right| \\ &+ \sup_{|x|\in[1,\infty)} \sup_{\substack{|y|\in[\frac{|x|}{2},2|x|\,]\cap\,[1,\infty) \\ x\neq y}} |x|^{\delta_2+k+\alpha} \frac{\left|D^k w\,(x) - D^k w\,(y)\right|}{|x-y|^{\alpha}}. \end{aligned} \tag{3.18}$$

Loosely speaking, a function in $C^{k,\alpha}_{\delta_2}(\mathbb{R}^3 \setminus B_1)$ decays like $r^{-\delta_2}$ for large r, when δ_2 is positive.

We use the norms (3.17), (3.18) to define weighted Hölder spaces on $\mathbb{R}^3 \setminus \{0\}$.

Definition 3.4. *For $k \in \mathbb{N} \cup \{0\}, \alpha \in [0,1)$ and $\delta_1, \delta_2 \in \mathbb{R}$, the space $C^{k,\alpha}_{\delta_1,\delta_2}(\mathbb{R}^3 \setminus \{0\})$ is defined to be the set of functions $w \in C^{k,\alpha}_{loc}(\mathbb{R}^3 \setminus \{0\})$ for which the following norm is finite*

$$\|w\|_{C^{k,\alpha}_{\delta_1,\delta_2}(\mathbb{R}^3\setminus\{0\})} := \|w\|_{C^{k,\alpha}_{\delta_1}\left(B_{3/2}\setminus\{0\}\right)} + \|w\|_{C^{k,\alpha}_{\delta_2}(\mathbb{R}^3\setminus B_1)}. \tag{3.19}$$

The following property is useful for determining wether or not a product of two or more functions belongs to $C^{0,\alpha}_{\delta_1,\delta_2}(\mathbb{R}^3 \setminus \{0\})$.

Lemma 3.3. *Assume that $w \in C^{0,\alpha}_{\delta_1,\delta_2}(\mathbb{R}^3 \setminus \{0\})$ and that $\tilde{w} \in C^{0,\alpha}_{\tilde{\delta_1},\tilde{\delta_2}}(\mathbb{R}^3 \setminus \{0\})$, then $w\tilde{w} \in C^{0,\alpha}_{\delta_1+\tilde{\delta_1},\delta_2+\tilde{\delta_2}}(\mathbb{R}^3 \setminus \{0\})$ and*

$$\|w\tilde{w}\|_{C^{0,\alpha}_{\delta_1+\tilde{\delta_1},\delta_2+\tilde{\delta_2}}} \leq C \|w\|_{C^{0,\alpha}_{\delta_1,\delta_2}} \|\tilde{w}\|_{C^{0,\alpha}_{\tilde{\delta_1},\tilde{\delta_2}}}, \tag{3.20}$$

for some constant $C > 0$ independent of w and $\tilde{w}$.

Proof. This is trivial except for the Hölder conditions. Concerning the latter

$$\frac{|w(x)\tilde{w}(x) - w(y)\tilde{w}(y)|}{|x-y|^\alpha} \leq \frac{|w(x) - w(y)|}{|x-y|^\alpha}|\tilde{w}(x)| + \frac{|\tilde{w}(x) - \tilde{w}(y)|}{|x-y|^\alpha}|w(y)| \tag{3.21}$$

directly implies Lemma 3.3.

We will also need the following fact:

Lemma 3.4. *Assume that $\tilde{k} + \tilde{\alpha} < k + \alpha$ and that $\tilde{\delta_1} < \delta_1, \tilde{\delta_2} < \delta_2$, then the embedding*

$$I : C^{k,\alpha}_{\delta_1,\delta_2}\left(\mathbb{R}^3 \setminus \{0\}\right) \longrightarrow C^{\tilde{k},\tilde{\alpha}}_{\tilde{\delta_1},\tilde{\delta_2}}\left(\mathbb{R}^3 \setminus \{0\}\right) \tag{3.22}$$

is compact.

Proof. This can be proven along the lines of [31].

One can easily derive adapted versions of the maximum principle.

Lemma 3.5. *Let w satisfy the equation*

$$\triangle w - \frac{c}{r^2} w = f, \tag{3.23}$$

with $c \geq 0$ and $f \leq 0$. Suppose that $w \in C^{2,\alpha}_{-1/2,1/2+\epsilon}(\mathbb{R}^3 \setminus \{0\})$, where $0 < \epsilon < \frac{1}{2}$. Then $w \geq 0$.

Proof. We consider the following function

$$v := r^{1/2+\epsilon/2} w. \tag{3.24}$$

This function tends to 0 at the origin and at infinity. Also $v \in C^{2,\alpha}\left(\mathbb{R}^3 \setminus \{0\}\right)$. We then compute

$$\triangle v = r^{1/2+\epsilon/2} \triangle w + \frac{2\left(1/2+\epsilon/2\right)}{r} \nabla_r v + \left(\left(1/2+\epsilon/2\right) - \left(1/2+\epsilon/2\right)^2\right) \frac{1}{r^2} v. \tag{3.25}$$

Suppose that at some point v possesses a negative minimum. Then the r.h. side of (3.25) is negative - a contradiction.

The following lemma is a special case of the more general usual Schauder estimates [65].

Lemma 3.6. *Assume that $\alpha \in (0,1)$ is fixed, and let $w \in C^{2,\alpha}(B_1)$ be a solution of $\triangle w + \tilde{c} w = f$ in B_1, where $f, \tilde{c} \in C^{0,\alpha}(B_1)$, then there exists a constant $C > 0$ such that*

$$\|w\|_{C^{2,\alpha}(B_{1/2})} \leq C \left(\|w\|_{L^\infty(B_1)} + \|f\|_{C^{0,\alpha}(B_1)} \right). \tag{3.26}$$

Using the above lemma we immediately obtain so-called rescaled Schauder estimates [104]. Explicitly we will need:

Corollary 3.1. *Assume that $\alpha \in (0,1)$ is fixed, $f \in C^{k-2,\alpha}_{\delta_1 - 2}(\bar{B}_1 \setminus \{0\})$ and that w solves $\triangle w - \frac{c}{r^2} = f$ in $B_1 \setminus \{0\}$, where c is a constant. Further assume that*

$$\left\| r^{-\delta_1} w \right\|_{L^\infty(B_1)} \leq C \left\| r^{2-\delta_1} f \right\|_{L^\infty(B_1)}, \tag{3.27}$$

for some constant $C > 0$ which does not depend on f, nor on w. Then, there exists $C' > 0$ which does not depend on f, nor on w, such that

$$\|w\|_{C^{k,\alpha}_{\delta_1}(\bar{B}_{1/2} \setminus \{0\})} \leq C' \|f\|_{C^{k-2,\alpha}_{\delta_1 - 2}(\bar{B}_1 \setminus \{0\})}. \tag{3.28}$$

Proof. Fix $x_0 \in B_{1/2} \setminus \{0\}$, set $R = |x_0|/2$ and define the functions

$$v(x) := w(x_0 + Rx) \tag{3.29}$$

and

$$g(x) := R^2 f(x_0 + Rx). \tag{3.30}$$

Then we have

$$\triangle v + \tilde{c} v = g \tag{3.31}$$

in B_1, where $\tilde{c} \in C^{0,\alpha}(B_1)$. Furthermore, the effect of the scaling yields

$$\|g\|_{C^{0,\alpha}(B_1)} \le CR^{\delta_1} \|f\|_{C^{0,\alpha}_{\delta_1-2}(B_1)}, \tag{3.32}$$

as well as

$$\|v\|_{L^\infty(B_1)} \le CR^{\delta_1} \left\|r^{-\delta_1} w\right\|_{L^\infty(B_1)}. \tag{3.33}$$

Using (3.27), the last estimate becomes

$$\|v\|_{L^\infty(B_1)} \le CR^{\delta_1} \|f\|_{C^{0,\alpha}_{\delta_1-2}(B_1)}. \tag{3.34}$$

Now we apply the interior Schauder estimate (Lemma 3.6) and obtain

$$\|v\|_{C^{2,\alpha}(B_{1/2})} \le CR^{\delta_1} \|f\|_{C^{0,\alpha}_{\delta_1-2}(B_1)}. \tag{3.35}$$

Performing the scaling backward, together with a simple covering argument, yields

$$\|w\|_{C^{2,\alpha}_{\delta_1}(\bar{B}_{1/2}\setminus\{0\})} \le C' \|f\|_{C^{0,\alpha}_{\delta_1-2}(\bar{B}_1\setminus\{0\})}. \tag{3.36}$$

The proof of the result is therefore complete when $k = 2$. The general case follows easily by induction.

Now we are able to prove an important isomorphism theorem.

Theorem 3.2. *Let c be a non-negative constant and $0 < \epsilon < \frac{1}{2}$. Then,*

$$\begin{aligned} L : C^{2,\alpha}_{-1/2,1/2+\epsilon}\left(\mathbb{R}^3 \setminus \{0\}\right) &\longrightarrow C^{0,\alpha}_{-5/2,5/2+\epsilon}\left(\mathbb{R}^3 \setminus \{0\}\right) \\ w &\longmapsto \triangle w - \frac{c}{r^2} w \end{aligned} \tag{3.37}$$

is an isomorphism.

Remark 3.1. *The statement is again a specialization of a more general fact, e.g. we have fixed the weights . Nevertheless, it suffices for our intended purposes and the proof can be easily adapted to the analysis of the more general situation.*

Proof. Injectivity follows from Lemma 3.5. To prove surjectivity we closely follow the domain decompositions method used by Pacard and Rivière [104]. We first look at the homogeneous problem. We denote the set of eigenfunctions of the Laplacian on S^2, i.e., the spherical harmonics, by $\phi_j, j \in \mathbb{N}$, that is

$$\triangle_{S^2} \phi_j = -\lambda_j \phi_j \tag{3.38}$$

with

$$\lambda_j = j(j+1). \tag{3.39}$$

If we project the operator on to the eigenspace spanned by ϕ_j, we obtain the operator

$$L_j w := \frac{\partial^2 w}{\partial r^2} + \frac{2}{r}\frac{\partial w}{\partial r} - \frac{\lambda_j + c}{r^2} w. \tag{3.40}$$

We may write the eigenfunction decomposition of w as

$$w(r,\theta,\varphi) = \sum_{j\geq 0} w_j(r)\phi_j(\theta,\varphi). \tag{3.41}$$

Any solution of $L_j w_j = 0$ can be written as a linear combination of two linearly independent functions, which are given by

$$w_j^+ := r^{\gamma_j^+} \tag{3.42}$$

and

$$w_j^- := r^{\gamma_j^-}, \tag{3.43}$$

where we have set

$$\gamma_j^\pm = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + \lambda_j + c}. \tag{3.44}$$

The key observation is that the coefficients $\gamma_j^\pm$ determine all the possible asymptotic behaviors of the solutions of the homogeneous problem. As a next step we recall some properties of so-called Dirichlet-to-Neumann maps. In our case these maps turn out to be fairly simple. First of all we introduce what we call the exterior Dirichlet-to-Neumann map. Let ψ be any function in the space $C^{2,\alpha}(\partial B_1)$. Given this boundary data, we define w_ψ to be the unique solution of $Lw = 0$ in $\mathbb{R}^3 \setminus \bar{B}_1$ which belongs to $C^{2,\alpha}_{\delta_1,1/2+\epsilon}$[1] and which satisfies $w = \psi$ on ∂B_1. It is well known that such a solution always exits. For the proof we refer the reader to [38]. The exterior Dirichlet-to-Neumann map is now defined by

$$S(\psi) := \partial_r w_\psi|_{\partial B_1}. \tag{3.45}$$

In order to have a better understanding of the operator S, let us consider the eigenfunction decomposition of $\psi \in C^{2,\alpha}(\partial B_1)$

$$\psi = \sum_{j\geq 0} \alpha_j \phi_j. \tag{3.46}$$

[1] $C^{2,\alpha}_{\delta_1,1/2+\epsilon}$ denotes the obvious function space when one restricts the domain. To simply the presentation we will omit the domains if there is no danger of missunderstandings.

Then we have the explicit formula

$$w_\psi = \sum_{j\geq 0} \alpha_j r^{\gamma_j^-} \phi_j. \tag{3.47}$$

And thus, we obtain

$$\partial_r w_\psi|_{\partial B_1} = \sum_{j\geq 0} \gamma_j^- \alpha_j \phi_j. \tag{3.48}$$

We now come to the definition of the interior Dirichlet-to-Neumann map. Again, let ψ be any function in $C^{2,\alpha}(\partial B_1)$. This time, we define $\tilde{w}_\psi$ to be the unique solution of $Lw = 0$ in B_1 which satisfies $w = \psi$ on ∂B_1 and belongs to $C^{2,\alpha}_{-1/2,\delta_2}$. Then the interior Dirichlet-to-Neumann map is defined by

$$T(\psi) := \partial_r \tilde{w}_\psi|_{\partial B_1}. \tag{3.49}$$

In our simple case, we even have an explicit representation of T. Let ψ be decomposed in terms of the eigenfunctions again. Then, $\tilde{w}_\psi$ is explicitly given by

$$\tilde{w}_\psi = \sum_{j\geq 0} \alpha_j r^{\gamma_j^+} \phi_j. \tag{3.50}$$

Hence

$$\partial_r \tilde{w}_\psi|_{\partial B_1} = \sum_{j\geq 0} \gamma_j^+ \alpha_j \phi_j. \tag{3.51}$$

We claim that

$$S - T : C^{2,\alpha}(\partial B_1) \longrightarrow C^{1,\alpha}(\partial B_1), \tag{3.52}$$

is an isomorphism. Indeed, in terms of the eigenfunctions ϕ_j the map is given by

$$\sum_{j\geq 0} \alpha_j \phi_j \longrightarrow \sum_{j\geq 0} \left(\gamma_j^- - \gamma_j^+\right) \alpha_j \phi_j. \tag{3.53}$$

Now notice that $\gamma_j^- - \gamma_j^+ \neq 0$. From this our claim easily follows. We proceed to construct a right inverse of L. For all $f \in C^{0,\alpha}_{-5/2,5/2+\epsilon}$, we define w_{ext} to be the unique solution of

$$Lw_{ext} = f \tag{3.54}$$

in $\mathbb{R}^3 \setminus \bar{B}_1$ that vanishes on ∂B_1 and belongs to $C^{2,\alpha}_{\delta_1,1/2+\epsilon}$. The existence of w_{ext} is again guaranteed [38]. We also define w_{int} to be the solution of

$$Lw_{int} = f \tag{3.55}$$

in $B_1 \setminus \{0\}$ that vanishes on ∂B_1 and belongs to $C^{2,\alpha}_{-1/2,\delta_2}$. Now, for existence of w_{int}, we consider the eigenfunction decomposition of f

$$f = \sum_{j \geq 0} f_j \phi_j, \tag{3.56}$$

and look for w_{int} of the form

$$w_{int} = \sum_{j \geq 0} w_j \phi_j. \tag{3.57}$$

Hence we have to solve the ordinary differential equations $L_j w_j = f_j$ in $(0, 1]$, with the boundary condition $w_j(1) = 0$. It is easy to show that the following formula for w_j satisfies the equations. Namely,

$$w_j = -r^{\gamma_j^+} \int_r^1 s^{-2-2\gamma_j^+} \int_0^s t^{2+\gamma_j^+} f_j(t) dt ds. \tag{3.58}$$

Notice that this expression is well defined since $0 < \gamma_j^+$. Furthermore, we can estimate

$$\sup_{(0,1]} \left| r^{1/2} w_j \right| \leq C_j \sup_{(0,1]} \left| r^{5/2} f_j \right|, \tag{3.59}$$

for some constant $C_j > 0$. Hence we can assume that there exists some constant $C_J > 0$, which is independent of f but may a priori depend on J, such that

$$\sup_{B_1 \setminus \{0\}} \left| r^{1/2} \sum_{j=0}^{J} w_j \phi_j \right| \leq C_J \sup_{B_1 \setminus \{0\}} \left| r^{5/2} f \right|. \tag{3.60}$$

It remains to prove that the constant C_J does not depend on J. Once this is known, we can pass to the limit $J \to \infty$ and easily obtain a solution of our problem. Along the lines of [104] one can argue by contradiction and assume that the claim is not true. This assumption easily conflicts with the injectivity of L on certain subspaces of functions. The fact that $w_{int} \in C^{2,\alpha}_{-1/2,\delta_2}$ then follows by Corollary 3.1. Note that our estimates imply

$$\sup_{B_1 \setminus \{0\}} \left| r^{1/2} w_{int} \right| \leq C \sup_{B_1 \setminus \{0\}} \left| r^{5/2} f \right|. \tag{3.61}$$

The reader is referred to [104] for further details. The final step of the construction is looking for a solution of

$$L w_{ker} = 0 \tag{3.62}$$

in $\mathbb{R}^3 \backslash \partial B_1$, which is continuous in $\mathbb{R}^3 \backslash \{0\}$ and possesses the right asymptotic limits. In addition, we want to be able to choose w_{ker} in such a way that the function

$$w := \begin{cases} w_{ext} + w_{ker} & \text{in } \mathbb{R}^3 \setminus \bar{B}_1 \\ \\ w_{int} + w_{ker} & \text{in } B_1 \setminus \{0\} \end{cases} \tag{3.63}$$

is also differentiable in all $\mathbb{R}^3 \setminus \{0\}$. Assuming we have already constructed w_{ker}, we see that

$$Lw = f, \tag{3.64}$$

in $\mathbb{R}^3 \setminus \{0\}$ and that we have found a right inverse for L with the desired properties. In order to build w_{ker}, we must check that we can find a solution $Lw_{ker} = 0$, which is continuous across ∂B_1 and for which the discontinuity of $\partial_r w_{ker}$ through ∂B_1 is equal to $\left(\partial_r w_{ext} - \partial_r w_{int}\right)|_{\partial B_1}$. Remember that solutions of $Lw_{ker} = 0$ are parameterized by their values on ∂B_1. So we are done when we are able to show that there exists a function $\psi \in C^{2,\alpha}\left(\partial B_1\right)$ which is a solution of the equation

$$\left(S - T\right)\left(\psi\right) = -\left(\partial_r w_{ext} - \partial_r w_{int}\right)|_{\partial B_1}. \tag{3.65}$$

But we already know that the

$$S - T : C^{2,\alpha}\left(\partial B_1\right) \longrightarrow C^{1,\alpha}\left(\partial B_1\right) \tag{3.66}$$

is an isomorphism. The desired properties of w_{ker} then follow from the constructions of the Dirichlet-to-Neumann maps. The proof of surjectivity of L is therefore complete.

There is a generalization of Theorem 3.2.

Theorem 3.3. *Let $\tilde{c}$ be a non-negative smooth bounded function in $\mathbb{R}^3 \setminus \{0\}$ and $0 < \epsilon < \frac{1}{2}$. Then,*

$$\begin{aligned} \tilde{L} : C^{2,\alpha}_{-1/2+\epsilon,1/2+\epsilon}\left(\mathbb{R}^3 \setminus \{0\}\right) &\longrightarrow C^{0,\alpha}_{-5/2+\epsilon,5/2+\epsilon}\left(\mathbb{R}^3 \setminus \{0\}\right) \\ w &\longmapsto \triangle w - \frac{\tilde{c}}{r^2} w \end{aligned} \tag{3.67}$$

is an isomorphism.

Proof. First of all we want to point out that the different weights (compare this theorem with the previous theorem) at the origin are only chosen because of the applications that we have in mind. In fact the operators L and $\tilde{L}$ are isomorphisms between the same weighted function spaces. For our proofs

to work, it is only important that the weights which belong to the function spaces in the domain of the operators fulfill certain assumptions, namely $\delta_1, -\delta_2 \in (\gamma_0^-, \gamma_0^+)$ and that $-\delta_1 < \delta_2$. Most likely proofs exist without the need of the second assumption.
Again a similar argument like in the previous theorem can be used to show that the operator is injective. To prove surjectivity we can use the fact that we have already shown that when $\tilde{c}$ is a non-negative constant the mapping is surjective. This is enough to define a global subsolution and also a global supersolution which in turn implies surjectivity of the more general mapping. Let f be any function that belongs to $C^{0,\alpha}_{-5/2+\epsilon,5/2+\epsilon}(\mathbb{R}^3 \setminus \{0\})$. We show that there exist $w_-, w_+ \in C^{2,\alpha}_{-1/2+\epsilon,1/2+\epsilon}(\mathbb{R}^3 \setminus \{0\})$ such that

$$\tilde{L}w_- \geq f, \tag{3.68}$$

$$\tilde{L}w_+ \leq f \tag{3.69}$$

and

$$w_- \leq w_+ \tag{3.70}$$

pointwise. Let c be a constant with the following property:

$$0 \leq c \leq \sup_{\mathbb{R}^3} \tilde{c}(r, \theta, \varphi). \tag{3.71}$$

Now define w_- to be the unique solution to

$$\left(\triangle - \frac{c}{r^2}\right) w = f_1, \tag{3.72}$$

which belongs to $C^{2,\alpha}_{-1/2+\epsilon,1/2+\epsilon}(\mathbb{R}^3 \setminus \{0\})$ and where f_1 denotes a function with the following properties:

(1) $f_1 \in C^{0,\alpha}_{-5/2+\epsilon,5/2+\epsilon}(\mathbb{R}^3 \setminus \{0\})$,

(2) $f_1 \geq f$,

(2) $f_1 \geq 0$.

Similarly we define w_+ to be the unique solution (see Theorem 3.2) to

$$\left(\triangle - \frac{c}{r^2}\right) w = f_2, \tag{3.73}$$

which belongs to $C^{2,\alpha}_{-1/2+\epsilon,1/2+\epsilon}(\mathbb{R}^3 \setminus \{0\})$ and where f_2 denotes a function with the following properties:

(1) $f_2 \in C^{0,\alpha}_{-5/2+\epsilon,5/2+\epsilon}\left(\mathbb{R}^3 \setminus \{0\}\right)$,

(2) $f_2 \leq f$,

(2) $f_1 \leq 0$.

Because of the previous theorem we know that solutions to (3.72) and (3.73) always exist. One can easily show that these solutions possess the desired properties, e.g. $w_- \leq w_+$ follows because our assumptions imply that $w_- \leq 0$ and $w_+ \geq 0$. Now the existence of global functions w_- and w_+ that belong to the right space is always enough to construct a right inverse for $\tilde{L}$ by standard methods. For example the solvability of the Dirichlet problem in arbitrary compact domains can be proven by using the method of continuity (see [65] or [81]) w_- and w_+ imply bounds which then can be used to extend the solution on $\mathbb{R}^3 \setminus \{0\}$.

In Chapter 5 we are going to study initial data sets with a so-called asymptotically hyperboloidal end. In this case Σ will be given by $B_1 \setminus \{0\}$, where B_1 again denotes the unit ball in $\mathbb{R}^3$. As background metric we will use

$$\hat{h}_{ij}\mathrm{d}x^i\mathrm{d}x^j = 4\left(1-r^2\right)^{-2}\left(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\right) =: \rho^{-2}\left(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\right). \qquad (3.74)$$

Next we define suitable Hölder spaces which can be used to express boundary conditions for $r \to 1$, or rather $\rho \to 0$. Here and subsequently we assume $0 < r' < 1$.

Definition 3.5. *For $k \in \mathbb{N}\cup\{0\}$, $\alpha \in [0,1)$ and $\delta_2 \in \mathbb{R}$, the space $\tilde{C}^{k,\alpha}_{\delta_2}\left(B_1 \setminus B_{r'}\right)$ is defined to be the set of functions $w \in C^{k,\alpha}_{loc}\left(B_1 \setminus B_{r'}\right)$ for which the following norm is finite*

$$\begin{aligned}\|w\|_{\tilde{C}^{k,\alpha}_{\delta_2}(B_1\setminus B_{r'})} :=& \sum_{j=0}^{k} \sup_{\rho(x)\in(0,\rho']} \rho(x)^{-\delta_2+j}\left|D^j w(x)\right| \\ &+ \sup_{\rho(x)\in(0,\rho']} \sup_{\substack{y\in B(x,\frac{\rho(x)}{2}),\rho(y)\in(0,\rho'] \\ x\neq y}} \rho(x)^{-\delta_2+k+\alpha}\frac{\left|D^k w\left(x\right) - D^k w\left(y\right)\right|}{|x-y|^{\alpha}}, \end{aligned} \qquad (3.75)$$

where $\rho = (1-r^2)/\sqrt{2}$.

We use the norms (3.17), (3.75) to define weighted Hölder spaces on $B_1 \setminus \{0\}$.

Definition 3.6. *For $k \in \mathbb{N}\cup\{0\}, \alpha \in [0,1)$ and $\delta_1, \delta_2 \in \mathbb{R}$, the space $\tilde{C}^{k,\alpha}_{\delta_1,\delta_2}\left(B_1 \setminus \{0\}\right)$ is defined to be the set of functions $w \in C^{k,\alpha}_{loc}\left(B_1 \setminus \{0\}\right)$ for which the following norm is finite*

$$\|w\|_{\tilde{C}^{k,\alpha}_{\delta_1,\delta_2}(B_1\setminus\{0\})} := \|w\|_{C^{k,\alpha}_{\delta_1}(B_{r'}\setminus\{0\})} + \|w\|_{\tilde{C}^{k,\alpha}_{\delta_2}(B_1\setminus B_{r'})}. \qquad (3.76)$$

Remark 3.2. *When one replaces* $C^{k,\alpha}_{\delta_1,\delta_2}(\mathbb{R}^3 \setminus \{0\})$ *by* $\tilde{C}^{k,\alpha}_{\delta_1,\delta_2}(B_1 \setminus \{0\})$ *Lemma 3.3 and Lemma 3.4 still hold and can be proven in a similar way.*

Lemma 3.7. *Let* w *satisfy the equation*

$$\triangle_{\hat{h}} w - \frac{c}{r^2} w = f, \tag{3.77}$$

with $c \geq 0$ *and* $f \leq 0$. *Suppose that* $w \in C^{2,\alpha}_{-1/2,1}(B_1 \setminus \{0\})$. *Then* $w \geq 0$.

Proof. Define $v = r^{1/2+\epsilon}\rho^{-1/2} w$ and show that the assumption of a negative minimum yields a contradiction.

Theorem 3.4. *Let* $\tilde{c}$ *be a non-negative smooth bounded function defined on* $B_1 \setminus \{0\}$. *Then,*

$$\begin{aligned} \hat{L} : C^{2,\alpha}_{-1/2,1}(B_1 \setminus \{0\}) &\longrightarrow C^{0,\alpha}_{-5/2,1}(B_1 \setminus \{0\}) \\ w &\longmapsto \triangle_{\hat{h}} w - \frac{\tilde{c}}{r^2} w \end{aligned} \tag{3.78}$$

is an isomorphism.

Proof. Injectivity follows from Lemma 3.7. To prove surjectivity we first of all consider the simpler case where $\tilde{c}$ is a non-negative constant. In this case we can explicitly show the construction of a right inverse of $\triangle_{\hat{h}} - \frac{\tilde{c}}{r^2}$. We build the inverse by solving the equation

$$\hat{L} w = f \text{ in } B_1 \setminus \{0\}, \tag{3.79}$$

where f denotes any function in the space $C^{0,\alpha}_{-5/2,1}$, in three steps:

(i) We solve the Dirichlet problem near the boundary $r = 1$, i.e. in $B_1 \setminus \bar{B}_{r'}$, where $0 < r' < 1$.

(ii) We solve the Dirichlet problem in $B_{r'} \setminus \{0\}$.

(iii) We show that the sum of these solutions can be modified to a true solution using Dirichlet to Neumann maps on the boundary $\partial B_{r'}$.

Step (i): Let $f \in C^{0,\alpha}_{-5/2,1}(B_1 \setminus \{0\})$. Then there exists $0 < r' < 1$ such that there is a unique solution $w_{ext} \in C^{2,\alpha}_{\delta_1,1}$ to

$$\hat{L} w_{ext} = f \text{ in } B_1 \setminus \bar{B}_{r'} \tag{3.80}$$

that vanishes on $\partial B_{r'}$. For the proof we refer the reader to [2].

Remark 3.3. *In terms of ρ the operator takes the following form*

$$\left(\triangle_{\hat{h}} - \frac{\tilde{c}}{r^2}\right) w = \rho^2 \partial_\rho^2 w - \rho \partial_\rho w - \tilde{c}\, w + \tilde{L} w, \tag{3.81}$$

where $\forall \alpha \in \mathbb{R}$

$$\tilde{L} \rho^\alpha = O\left(\rho^{\alpha+\epsilon}\right). \tag{3.82}$$

Using this expression one can calculate the corresponding indicial roots $\mu_\pm$ that are given by $\mu_\pm = 1 \pm \sqrt{1+\tilde{c}}$. The fact $\mu_- < 1\,(=\delta_2) < \mu_+$ plays a decisive role in our claim. Details can be found in [2].

Step (ii): Let $f \in C^{0,\alpha}_{-5/2,1}(B_1 \setminus \{0\})$. Then there exists a unique solution $w_{int} \in C^{2,\alpha}_{-1/2,\delta_2}$ to

$$\hat{L} w_{int} = f \text{ in } B_{r'} \setminus \{0\} \tag{3.83}$$

that vanishes on $\partial B_{r'}$. One knows that this fact is true for operators of the form $L = \triangle - \frac{\tilde{c}}{r^2}$ (recall Theorem 3.2). Now (3.83) can be written as an operator equation, where the operator constitutes a compact pertubation of L. Hence the claim easily follows.
Step (iii): In order to glue the exterior and interior solutions together, we must add correction terms that belong to the kernel of $\hat{L}$. We define the following mappings.

Definition 3.7. *Let ψ be any function that belongs to $C^{2,\alpha}(\partial B_{r'})$. Then there exists a function $w_\psi \in C^{2,\alpha}_{\delta_1,1}$ which is the unique solution of $\hat{L} w = 0$ in $B_1 \setminus \bar{B}_{r'}$ subject to the boundary condition $w = \psi$ on $\partial B_{r'}$. We define the exterior Dirichlet-to-Neumann map by*

$$S(\psi) := \partial_r w_\psi|_{\partial B_{r'}}. \tag{3.84}$$

Definition 3.8. *Let ψ be any function that belongs to $C^{2,\alpha}(\partial B_{r'})$. Then there exists a function $\tilde{w}_\psi \in C^{2,\alpha}_{-1/2,\delta_2}$ which is the unique solution of $\hat{L} w = 0$ in $B_{r'} \setminus \{0\}$ subject to the boundary condition $w = \psi$ on $\partial B_{r'}$. We define the interior Dirichlet-to-Neumann map by*

$$T(\psi) := \partial_r \tilde{w}_\psi|_{\partial B_{r'}}. \tag{3.85}$$

The Dirichlet data are mapped to the Neumann data of the solutions to the corresponding Dirichlet problems. The construction will require one further supplementary fact.

$$S - T : C^{2,\alpha}\left(\partial B_{r'}\right) \longrightarrow C^{1,\alpha}\left(\partial B_{r'}\right) \tag{3.86}$$

is an isomorphism. Along the lines of [85] and [98] one can show that (3.84) and (3.85) are well defined linear bounded pseudodifferential operators that are elliptic. The proof uses microlocal factorization near the boundary $\partial B_{r'}$. Their principal symbols are given by $\sqrt{\hat{h}^{ij}\xi_i\xi_j}|_{\partial B_{r'}}$ and $-\sqrt{\hat{h}^{ij}\xi_i\xi_j}|_{\partial B_{r'}}$ respectively, hence the difference is also elliptic. Every elliptic operator on any vector bundle on a compact manifold is Fredholm. In our case the index has to be zero, because the vector bundle is trivial [30]. Thus we only need to check that the mapping is injective. But this follows easily by using elliptic regularity for solutions of the underlying problem (for more details consult [104]). The final step is looking for a function w_{ker} defined in $B_1 \setminus \partial B_{r'}$ so that

$$w := \begin{cases} w_{ext} + w_{ker} & \text{in } B_1 \setminus \bar{B}_{r'} \\ \\ w_{int} + w_{ker} & \text{in } B_{r'} \setminus \{0\} \end{cases} \tag{3.87}$$

satisfies

$$\hat{L}w = f \tag{3.88}$$

in $B_1 \setminus \{0\}$. For this to be true we need to choose w_{ker} to be continuous across $\partial B_{r'}$, to satisfy $\hat{L}w_{ker} = 0$ away from $\partial B_{r'}$, and such that the jump of $\partial_r w_{ker}$ across $\partial B_{r'}$ equals $-(\partial_r w_{ext} - \partial_r w_{int})$. Remember that solutions of $\hat{L}w_{ker} = 0$ are parameterized by their values on $\partial B_{r'}$. So we are done when we are able to show that there exists a function $\psi \in C^{2,\alpha}(\partial B_{r'})$ which is a solution of the equation

$$(S - T)(\psi) = -(\partial_r w_{ext} - \partial_r w_{int})|_{\partial B_{r'}}. \tag{3.89}$$

But we already know that the map

$$S - T : C^{2,\alpha}(\partial B_{r'}) \longrightarrow C^{1,\alpha}(\partial B_{r'}) \tag{3.90}$$

is an isomorphism. The desired properties of w_{ker} then follow from the constructions of the Dirichlet-to-Neumann maps. The proof of surjectivity of the case where $\tilde{c}$ is a non-negative constant is therefore complete.The result for the more general case follows because the existence of a right inverse in the simpler setting can be used to define a global subsolution and a global supersolution. This in turn implies surjectivity of the more general mapping.

Chapter 4

Asymptotically Flat Initial Data

In this chapter we begin our discussion of initial data sets which can be used to model an isolated system, e.g. a black hole. These initial data sets are widely used in the field of numerical relativity and allow one to deduce statements about the behavior of one or more black holes together with various physical properties such as mass, momentum, emitted radiation etc. that can be attributed to them. Since it is hopeless to try to analyze the behaviour of a certain physical system along with the rest of the universe, we need to idealize the physical situation in an appropriate way. We consider the following idealization. We imagine the system as being alone in an otherwise empty universe. In this case Σ is naturally divided in two regions. One compact region B which contains the sources, and its complement $\Sigma \setminus B$ which is unbounded. Depending on the topology of Σ the latter region usually has more than one asymptotic end. Furthermore one has to impose fall-off conditions to guarantee that the resulting spacetime looks like Minkowski spacetime at large distances.

Definition 4.1. *A Riemannian 3-manifold (Σ, h_{ij}) possesses an asymptotically flat end if there exists a subset S of Σ and a diffeomorphism $\Phi : \Sigma \setminus S \to E_R$, where $E_R = \mathbb{R}^3 \setminus \bar{B}_R$. and furthermore, in the coordinate chart given by this diffeomorphism, the following conditions have to be satisfied, for some $k > 1$ and $\alpha > 0$:*

$$h_{ij} - \delta_{ij} = O_k(r^{-\alpha}). \tag{4.1}$$

We write $f = O_k(r^\beta)$ if f satisfies

$$\partial_{k_1} \dots \partial_{k_l} f = O(r^{\beta - l}), \quad 0 \le l \le k. \tag{4.2}$$

Definition 4.2. *A Riemannian 3-manifold (Σ, h_{ij}) is called asymptotically flat if there is a compact subset $B \subset \Sigma$, so that $\Sigma \setminus B$ consists of a finite number of asymptotically flat ends.*

Definition 4.3. *A (vacuum) initial data set $(\Sigma, \bar{h}_{ij}, \bar{K}_{ij})$ is called asymptotically flat if $(\Sigma, \bar{h}_{ij})$ is an asymptotically flat manifold and in addition the extrinsic curvature $\bar{K}_{ij}$ satifies, for some $k > 1$ and $\alpha > 0$,*

$$\bar{K}_{ij} = O_{k-1}(r^{-1-\alpha}) \tag{4.3}$$

at every asymptotically flat end.

Remark 4.1. *The above decay conditions can also be implemented by using certain weighted Sobolev and Hölder spaces defined on Σ. The constraint equations can be conveniently treated in Sobolev and Hölder spaces and we will make use of both.*

With regard to the conformal method for constructing vacuum initial data sets there are at least two ways to guarantee that the physical manifold $(\Sigma, \bar{h}_{ij})$ is asymptotically flat. One can pick an asymptotically flat background metric h_{ij} on Σ and impose the boundary conditions that $\phi \to 1$ as $r \to \infty$. Sometimes, a convenient alternative is to take h_{ij} to be defined on a compact manifold $\tilde{\Sigma}$ and to obtain $(\Sigma, \bar{h}_{ij})$ by conformal decompactification from $(\tilde{\Sigma}, h_{ij})$. Assuming that we want to construct a physical manifold with only one asymptotic end the procedure is essentially just the stereographic projection of $\tilde{\Sigma}$. Working with $\tilde{\Sigma}$ has the technical advantage that it is usually simpler to prove existence of solutions for an elliptic equation on a compact manifold. Because we have the machinery of weighted function spaces at our disposal we will always start with a decompactified asymptotic end. Nevertheless, we will see that even then compactified asymptotic ends play a decisive role.
We know that the constraints (see Theorem 2.4) in their conformal formulation are invariant under conformal rescalings. In the case of a compact manifold Σ a convenient first step before studying the solution of the Lichnerowicz equation is to use the Yamabe theorem (see Theorem 2.6) which says that each compact manifold is conformal to a manifold with constant scalar curvature. There is no known analogous theorem for asymptotically flat manifolds. However, an interesting theorem was proved by Brill and Cantor [27], with a correction first noted by Maxwell [93] (also see [60]).

Theorem 4.1. *Let (Σ, h_{ij}) be an asymptotically flat manifold. It is conformal to a manifold with zero scalar curvature if and only if its Yamabe number $Y(\Sigma, h_{ij})$ is positive.*

Remark 4.2. *Because of the conditions of asymptotic flatness, it follows that every initial manifold with constant mean curvature has to be maximal, i.e.* $tr_{\bar{h}}\bar{K} \equiv 0$*. As a consequence of (2.18) the physical metric* $\bar{h}_{ij}$ *that solves the constraints together with the symmetric tensor field* $\bar{K}_{ij}$ *has non-negative scalar curvature* $R[\bar{h}]$*.*

Remark 4.3. *Associated to each asymptotically flat end there is an asymptotic Poincaré symmetry. The corresponding conserved quantities may be computed by integrals over spheres, which are given by the following expressions,*

$$E = \frac{1}{16\pi} \lim_{r\to\infty} \sum_{i,j=1}^{3} \int_{S(r)} \left(\frac{\partial \bar{h}_{ij}}{\partial x^j} - \frac{\partial \bar{h}_{jj}}{\partial x^i} \right) dS_i, \tag{4.4}$$

$$p_i = \frac{1}{8\pi} \lim_{r\to\infty} \sum_{j=1}^{3} \int_{S(r)} \bar{K}_{ij} dS_j, \tag{4.5}$$

where (4.4) is called the ADM energy and (4.5) is called the ADM momentum. In a similar way one can define an expression for the ADM angular momentum,

$$j_i = \frac{1}{8\pi} \lim_{r\to\infty} \sum_{j,k,l=1}^{3} \int_{S(r)} \epsilon_{ijk} x^j \bar{K}_{kl} dS_l. \tag{4.6}$$

For more details the reader is referred to [8].

4.1 Time symmetric initial data

The simplest solutions of the Einstein equations which describe the metric outside a spherically symmetric star or a single, static black hole are the famous Schwarzschild solutions. The Schwarzschild metric can be written as

$$g = -\left(1 - \frac{2m}{R}\right) dt^2 + \left(1 - \frac{2m}{R}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2). \tag{4.7}$$

We can also choose an isotropic radial coordinate. Then the metric is given by

$$g = -\left(\frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}}\right)^2 dt^2 + \left(1 + \frac{m}{2r}\right)^4 (dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)). \tag{4.8}$$

In both expressions, (4.7) and (4.8), m represents the mass of the star or the black hole. Note that the 3-geometry of the $t = const$ slices associated with (4.8) is conformally flat. These slices also have vanishing extrinsic curvature, i.e., $\bar{K}_{ij} = 0$. To see this, we recall some basic facts.

Definition 4.4. *A submanifold Σ of a manifold $(M, g_{\mu\nu})$ is called totally geodesic if any geodesic on the submanifold Σ with its induced metric h_{ij} is also a geodesic of $(M, g_{\mu\nu})$.*

Theorem 4.2. *A vanishing extrinsic curvature is equivalent to the property that the submanifold is totally geodesic.*

Proof. Let us consider a submanifold Σ whose extrinsic curvature is constant zero. From (2.8) we then have $\nabla_Y Z = D_Y Z$ for all vector fields Y, Z tangent to the submanifold. In particular, if $\eta : I \to \Sigma$ is a curve with tangent vector v, then

$$\nabla_v v = D_v v \tag{4.9}$$

and η is a geodesic in the submanifold iff it is a geodesic in M.

Theorem 4.3. *Let I be an isometry of $(M, g_{\mu\nu})$. Assume that the hypersurface Σ belongs to the fixed-point set of I, then Σ is totally geodesic and its extrinsic curvature vanishes.*

For the proof we refer the reader to [82].

Remark 4.4. *Totally geodesic initial data hypersurfaces are also called time symmetric because one can show that the maximal development, M, allows an isometry fixing Σ pointwise and exchanging the components of $M \setminus \Sigma$.*

Let us return to the $t = const$ slices of (4.8). The $t = 0$ slice of the Schwarzschild manifold is for example a fixed-point set of the isometry

$$I(t, r, \theta, \varphi) := (-t, r, \theta, \varphi). \tag{4.10}$$

Furthermore, the slices are asymptotically flat for $r \to \infty$, which can easily be seen from (4.8).

The expression for the Schwarzschild metric in isotropic coordinates is easily generated by the methods discussed. We choose the trivial solution of the momentum constraint, i.e $\bar{K}_{ij} = 0$, and only have to solve the Hamiltonian constraint. Using a flat conformal metric, i.e. $h_{ij} = \delta_{ij}$, (2.34) reduces to the ordinary Laplace equation in flat space for the single function ϕ:

$$\triangle \phi = 0. \tag{4.11}$$

For the solution ϕ to yield an asymptotically flat physical 3-metric $\bar{h}_{ij}$, we have the boundary condition that $\phi(r \to \infty) \to 1$. We cannot take $\Sigma = \mathbb{R}^3$, since then the only solution to the Laplace equation in $\mathbb{R}^3$ which asymptotically approaches 1 is identically 1.

Remark 4.5. *$\phi = 1$ and therefore $\bar{h}_{ij} = h_{ij}$, where h_{ij} is a flat metric, corresponds to initial data for Minkowski spacetime.*

We must allow ϕ to blow up at some points, which we then remove from the manifold. In this way we let the solution tell us what topology to choose in order to have an everywhere regular solution. If the resulting physical manifold would turn out to be incomplete, we had to find a suitable completion. But, as a matter of fact, one can show that for our chosen solutions ϕ punctured $\mathbb{R}^3$ is complete in the physical metric. The most general solution of (4.11) on $\Sigma = \mathbb{R}^3 \setminus \{0\}$ is given by

$$\phi(r, \theta, \varphi) = 1 + \frac{m}{2r}, \quad m \in \mathbb{R}_+. \tag{4.12}$$

Higher multipole moments are not possible, because then ϕ would necessarily have zeros on Σ. In this case removing $\phi^{-1}(0)$ from Σ does not work since these points are at finite distance. The resulting space would not be (geodesically) complete. Also m has to be positive. From $\bar{h}_{ij} = \phi^4 \delta_{ij}$ we obtain

$$\bar{h} = \left(1 + \frac{m}{2r}\right)^4 (dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\varphi^2)). \tag{4.13}$$

This is exactly the form of the $t = \mathit{const}$ slices of (4.8). One can now compute the contained energy by using (4.4) and gets $E = m$. Thus the integration constant m can be interpreted as the mass of the source as measured at $r \to \infty$. But there is also another asymptotic end. It is easy to verify that the following diffeomorphism I is an isometry:

$$I(r, \theta, \varphi) = \left(\frac{m^2}{4r}, \theta, \varphi\right). \tag{4.14}$$

The map I is referred to as inversion through the sphere $r = m/2$. It maps every point inside $r = m/2$ onto a point outside $r = m/2$. The geometry close to the origin is therefore identical to the geometry near infinity, which suggests that we can think of the geometry as two asymptotically flat sheets, that are connected by a throat.

Remark 4.6. *One can use the spatial coordinate transformation*

$$R = r\left(1 + \frac{m}{2r}\right)^2 \tag{4.15}$$

to obtain the usual form (4.7).

Remark 4.7. *Black holes are defined as regions of spacetime out of which no null geodesics can escape to infinity [74]. The event horizon, or surface of the black hole, is the boundary between those events which can emit light rays to infinity and those which cannot. Unfortunately, its global properties make it very difficult to locate, since, in principle, knowledge of the entire future spacetime is required. In practice one therefore locates apparent horizons. The formal definition of an apparent horizon is the following:*

Definition 4.5. *Let $(\Sigma, \bar{h}_{ij}, \bar{K}_{ij})$ be a given initial data set and let $\sigma \subset \Sigma$ be an embedded 2-surface with outward pointing normal l^i. σ is called an apparent horizon if and only if the following relation between $\bar{K}_{ij}$ and k_{ij}, the extrinsic curvature of σ in Σ, is satisfied:*

$$q^{ij} k_{ij} = -q^{ij} \bar{K}_{ij}, \tag{4.16}$$

where $q_{ij} := \bar{h}_{ij} - l_i l_j$ is the induced Riemannian metric on σ.

Note that for time symmetric initial data, i.e. $\bar{K}_{ij} = 0$, the condition (4.16) states the k_{ij} is tracefree.

The famous singularity theorems state that under certain conditions the existence of an apparent horizon implies that the evolving spacetime will be singular. Assuming that the Cosmic Censor Conjecture is true, one infers the existence of an event horizon and hence a black hole. This is the reason that by black hole initial data sets one understands initial data sets which contain apparent horizons.

Remark 4.8. *For the initial data corresponding to the Schwarzschild solution one can also try to locate the apparent horizon using equation (4.16). Because the data is time symmetric, one just has to find a 2-surface which has vanishing mean curvature. It is easy to see that the sphere $r = m/2$ is fixed-point set of the isometry (4.14) and hence has vanishing mean curvature. Therefore it is an apparent horizon. In this case the apparent horizon is equal to the well-known event horizon of Schwarzschild spacetime.*

Note that equation (4.11) is linear. As an immediate consequence, time symmetric initial data containing multiple black holes can be constructed by simply adding several terms of the form (4.12). More precisely, for two black holes in an asymptotically flat spacetime, we take $\Sigma = \mathbb{R}^3 \setminus \{x_1, x_2\}$ and choose the solution

$$\phi = 1 + \frac{m_1}{2r_1} + \frac{m_2}{2r_2}, \tag{4.17}$$

where

$$r_1 = |x - x_1|, \quad r_2 = |x - x_2|, \tag{4.18}$$

and $m_1, m_2 \in \mathbb{R}_+$. The physical 3-metric $\bar{h}_{ij}$ is then given by

$$\bar{h} = \left(1 + \frac{m_1}{2r_1} + \frac{m_2}{2r_2}\right)^4 (dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)). \tag{4.19}$$

The manifold has now three asymptotic ends. First of all it is obvious that $\bar{h}_{ij}$ is asymptotically flat for $r = |x| \to \infty$. In addition to that there are also asymptotic ends for $r_1 \to 0$ and $r_2 \to 0$. We proceed by showing this for the case $r_1 \to 0$. Let $(r_1, \theta_1, \varphi_1)$ be spherical polar coordinates centered at x_1. Then we introduce the inverted radial coordinate $\tilde{r}_1$:

$$\tilde{r}_1 := \frac{{m_1}^2}{4r_1}. \tag{4.20}$$

In the limit $\tilde{r}_1 \to \infty$, which is the same as $r_1 \to 0$, the metric is given by

$$\bar{h} = \left(1 + \frac{m_1\,(1 + m_2/2r_{12})}{2\tilde{r}_1} + O\left(\frac{1}{{\tilde{r}_1}^2}\right)\right)^4 (d\tilde{r}_1^2 + {\tilde{r}_1}^2(d\theta_1^2 + \sin^2\theta_1\, d\varphi_1^2)), \tag{4.21}$$

where

$$r_{12} = |x_1 - x_2|. \tag{4.22}$$

From this expression it is easy to see that the 3-manifold has an asymptotically flat end at $r_1 \to 0$. The proof for the case $r_2 \to 0$ works in the same way. The method can be generalized in a straightforward manner to any number n of black holes. Now the manifold Σ has $n + 1$ asymptotically flat ends. Such initial data sets have been studied by Brill and Lindquist [22]. We summarize:

Definition 4.6. *Let $n \in \mathbb{N}$ and $m_i \in \mathbb{R}^+$. An initial data set $\left(\Sigma, \bar{h}_{ij}, \bar{K}_{ij}\right)$ is called a Brill-Lindquist initial data set, if it can be expressed in the following form*

$$\Sigma = \mathbb{R}^3 \setminus \{x_1, \ldots, x_n\}, \tag{4.23}$$

$$\bar{h}_{ij} = \phi^4 \delta_{ij}, \tag{4.24}$$

where $\phi = 1 + \sum_{i=1}^n \frac{m_i}{2r_i}$, and

$$\bar{K}_{ij} = 0. \tag{4.25}$$

4.2 Time asymmetric initial data

In order to study more interesting initial data sets, from a mathematical and physical point of view, one must drop the condition of time symmetry, that is, the extrinsic curvature $\bar{K}_{ij}$ must be nonzero initially. Still we may assume that the mean curvature, i.e. $\tau := tr_{\bar{h}}\bar{K}$, is a constant. Because of the conditions of asymptotic flatness, it follows that $\tau \equiv 0$. If we use the conformal method in this setting, the first step is to find a symmetric, tracefree tensor field A_{ij} that fulfills $D_i A^{ij} = 0$ for a given background metric. In the case where $\Sigma = \mathbb{R}^3 \setminus \{0\}$ equipped with the standard flat metric, i.e. $h_{ij} = \delta_{ij}$, a 7-parameter class of TT tensors satisfying also the necessary conditions at both asymptotic ends (compare with (4.3)) are the following:

$$K_A^{ij} = \frac{A}{r^3}\left(3n^i n^j - \delta^{ij}\right), \tag{4.26}$$

$$K_S^{ij} = \frac{3}{r^3}\left(n^i \epsilon^{jkl} S_k n_l + n^j \epsilon^{ikl} S_k n_l\right) \tag{4.27}$$

and

$$K_P^{ij} = \frac{3}{2r^2}\left(P^i n^j + P^j n^i - (\delta^{ij} - n^i n^j) P^k n_k\right), \tag{4.28}$$

where (A, S_i, P_i) are constants and $n^i = x^i/r$ with x^i being usual Cartesian coordinates. These are members of the Bowen-York class of TT tensors [20]. Bowen-York tensors have a natural place in the context of the conformal geometry of conformally flat 3-spaces too [16]. The constants S_i, P_i have the respective interpretation of total spin and total linear momentum of the configuration, measured at spatial infinity, which are conserved under time evolution. Using (4.5) and (4.6) one can assure oneself that the components of the physical linear momentum $p_i = P_i$ and the components of the physical angular momentum $j_i = S_i$. Furthermore K_A^{ij} is the unique spherically symmetric TT tensor on flat space.

Remark 4.9. *In Chapter 2 we defined the conformal Killing operator $\mathbb{L}$ (2.39) and discussed its role in finding suitable TT tensors. With the help of the conformal Killing operator K_A^{ij},K_S^{ij} and K_P^{ij} can be written as*

$$K_A^{ij} = (\mathbb{L}Y_A)^{ij}, \tag{4.29}$$

$$K_S^{ij} = (\mathbb{L}Y_S)^{ij} \tag{4.30}$$

and

$$K_P^{ij} = (\mathbb{L}Y_P)^{ij}, \tag{4.31}$$

where Y_A,Y_S and Y_P are certain vector fields on $\Sigma = \mathbb{R}^3 \setminus \{0\}$.

The generalization to any finite number of punctures is straightforward because the constraint $D_i A^{ij} = 0$ is linear.

4.3 Bowen-York wormholes

The final step in the construction of initial data sets via the conformal method is to find a suitable solution of (2.34), which for our assumptions reduces to

$$\triangle\phi + \frac{1}{8}A_{ij}A^{ij}\phi^{-7} = 0. \tag{4.32}$$

(4.32) is no longer a simple Laplace equation but a non-linear elliptic equation. Although there is no hope to get any analytical solution one can prove that a suitable positive function ϕ exists that also satifies the desired boundary conditions. One way to facilitate the proof is to first split off the monopole parts of ϕ, which blow up at the punctures [21]. Based on the time symmetric Brill-Lindquist initial data, we make the following ansatz

$$\phi = \phi_0 + \psi, \tag{4.33}$$

where ϕ_0 denotes the conformal factor of the corresponding Brill-Lindquist solution with the same topology. This translates the original equation into an equation for ψ, namely

$$\triangle\psi = -\frac{1}{8}(1 + {\phi_0}^{-1}\psi)^{-7}{\phi_0}^{-7}A_{ij}A^{ij} =: f(x, \psi). \tag{4.34}$$

To complete the definition of the problem, we have to specify boundary conditions for ψ. For asymptotic flatness at $r \to \infty$ we require $\psi = O(r^{-\alpha})$, for $\alpha > 0$, at large distances to the punctures. The key observation is that there exists a regular solution ψ on the entire $\mathbb{R}^3$ that fulfills this requirement (see Theorem 4.4).

Remark 4.10. *By using the Brill-Lindquist conformal factor ϕ_0 we enforce the existence of multiple asymptotically flat ends. Furthermore by choosing a regular extension of ψ we also make a choice about the inner boundaries.*

Remark 4.11. *Our proofs will employ weighted Sobolev spaces instead of weighted Hölder spaces which will be important in the following sections. Furthermore by explicitly handling the singular terms it is enough to use function spaces with only one weight at infinity.*

Theorem 4.4. *Let ϕ_0 be some conformal factor associated with a Brill-Lindquist initial data set. Furthermore assume that ${\phi_0}^{-7}A_{ij}A^{ij} \in L^2_{-5/2}(\mathbb{R}^3)$. Then there exists a non-negative solution $\psi \in W^{2,2}_{-1/2}(\mathbb{R}^3)$ of (4.34) on $\mathbb{R}^3$.*

Proof. The equation admits suitable sub- and supersolutions. Namely, take $\psi_- = 0$ and define ψ_+ to be the unique solution of

$$\triangle\psi = f(x, \psi_-) = -\frac{1}{8}{\phi_0}^{-7} K_{ij}K^{ij}, \tag{4.35}$$

which belongs to $W^{2,2}_{-1/2}(\mathbb{R}^3)$. From the fact, that the Laplacian

$$\triangle : W^{2,2}_{-1/2}\left(\mathbb{R}^3\right) \longrightarrow L^2_{-5/2}\left(\mathbb{R}^3\right) \tag{4.36}$$

is an isomorphism (see Theorem 3.1) this solution always exists and by the generalized maximum principle (3.2) fulfills

$$\psi_+ \geq \psi_-. \tag{4.37}$$

Furthermore, let $g \in L^2_{-5/2}(\mathbb{R}^3)$ be a positive function such that at each point

$$g \geq \frac{\partial f}{\partial \psi}, \quad \text{for all } \psi_- \leq \psi \leq \psi_+. \tag{4.38}$$

We construct a solution by induction, starting from ψ_-. We solve

$$(\triangle - g)\psi_1 = f(x, \psi_-) - g\psi_- \tag{4.39}$$

Eq. (4.39) has a unique solution $\psi_1 \in W^{2,2}_{-1/2}(\mathbb{R}^3)$, since the right hand side of the equation is in $L^2_{-5/2}(\mathbb{R}^3)$ and the generalized operator $\triangle - g$ is an isomorphism between these function spaces too (Theorem 3.1). From the equations satisfied respectively by ψ_1 and ψ_- we deduce:

$$(\triangle - g)(\psi_1 - \psi_-) = f(x, \psi_-) \leq 0. \tag{4.40}$$

Hence, from the generalized maximum principle,

$$\psi_1 \geq \psi_-. \tag{4.41}$$

Also,

$$(\triangle - g)(\psi_+ - \psi_1) = \triangle\psi_+ - f(x, \psi_-) - g(\psi_1 - \psi_-) - g(\psi_+ - \psi_1), \tag{4.42}$$

and

$$\triangle\psi_+ - f(x, \psi_-) - g(\psi_1 - \psi_-) - g(\psi_+ - \psi_1) \leq f(x, \psi_+) - f(x, \psi_-) - g(\psi_+ - \psi_-). \tag{4.43}$$

Thus we get

$$(\triangle - g)(\psi_+ - \psi_1) \leq (\phi_+ - \phi_-) \int_0^1 \{ \frac{\partial f}{\partial \psi}(x, \psi_- + t(\psi_+ - \psi_-)) - g \} dt. \quad (4.44)$$

By the assumption made on g it follows that

$$(\triangle - g)(\psi_+ - \psi_1) \leq 0, \quad (4.45)$$

hence

$$\psi_+ \geq \psi_1. \quad (4.46)$$

We use the following induction formula:

$$(\triangle - g)\psi_n = f(x, \psi_{n-1}) - g\psi_{n-1}. \quad (4.47)$$

We suppose that ψ_m exists for $0 \leq m \leq n-1$ with $\psi_0 = \psi_-$ and $\psi_m \in W^{2,2}_{-1/2}(\mathbb{R}^3)$ for $0 \leq m \leq n-1$ and that for these

$$\psi_- \leq \psi_{m-1} \leq \psi_m \leq \psi_+. \quad (4.48)$$

It follows that $\psi_n \in W^{2,2}_{-1/2}(\mathbb{R}^3)$ exists and by completely analogous arguments we find that

$$\psi_{n-1} \leq \psi_n \leq \psi_+. \quad (4.49)$$

We have shown that the sequence of continuous functions ψ_n is pointwise increasing and bounded. It converges at each point to a limit $\psi(x)$, with $\psi_- \leq \psi \leq \psi_+$. The linear elliptic inequality

$$\|\psi_{n+1}\|_{W^{2,2}_{-1/2}(\mathbb{R}^3)} \leq C\|f(\cdot, \psi_n) - g\psi_n\|_{L^2_{-5/2}(\mathbb{R}^3)} \quad (4.50)$$

shows that the sequence ψ_n is uniformly bounded in the $W^{2,2}_{-1/2}(\mathbb{R}^3)$ norm. Since $W^{2,2}_{-1/2}(\mathbb{R}^3)$ is compactly embedded in $C^0_0(\mathbb{R}^3)$ (see Lemma 3.1), there is a subsequence which converges to a function $\psi \in C^0_0(\mathbb{R}^3)$, identical to the previously found ψ. The functions $f(\cdot, \psi_n)$ converge to $f(\cdot, \psi)$ in the $L^2_{-5/2}(\mathbb{R}^3)$ norm because of the following inequality

$$\|f(\cdot, \psi) - f(\cdot, \psi_n)\|_{L^2_{-5/2}(\mathbb{R}^3)} \leq C\|\psi - \psi_n\|_{C^0_0}\|g\|_{L^2_{-5/2}(\mathbb{R}^3)}, \quad (4.51)$$

which follows from the mean value theorem. Hence, from the elliptic inequality, it follows that ψ_n converges to ψ in the $W^{2,2}_{-1/2}(\mathbb{R}^3)$ norm and that ψ satisfies the given equation. By construction ψ in non-negative.

Another particularly with regard to perturbation theory interesting result is the following.

Theorem 4.5. *Let $\left(\bar{\Sigma}, \bar{h}^0_{ij}, \bar{K}^0_{ij}\right)$ be some Brill-Lindquist initial data set. Then there is a small $\epsilon' > 0$ such that for $-\epsilon' < \epsilon < \epsilon'$ there exist families of initial data sets $\left(\bar{\Sigma}, \bar{h}_{ij}(\epsilon), \bar{K}_{ij}(\epsilon)\right)$ which are differentiable in ϵ, for $\epsilon = 0$ equal $\left(\bar{\Sigma}, \bar{h}^0_{ij}, \bar{K}^0_{ij}\right)$, i.e. $\bar{h}_{ij}(0) = \bar{h}^0_{ij}$ and $\bar{K}_{ij}(0) = \bar{K}^0_{ij}$, for nonvanishing ϵ have the same asymptotic geometry as the Brill-Lindquist initial data set and the extrinsic curvatures are of the Bowen-York type, i.e. $\bar{K}_{ij}(\epsilon) = \phi^{-2}\epsilon A_{ij}$, where ϕ is a non-vanishing function on $\bar{\Sigma}$ and A_{ij} is a linear combination of Bowen-York tensors.*

Again we use the conformal method to prove Theorem 4.5. We assume that $\bar{h}_{ij}(\epsilon)$ is still conformally flat, i.e. $\bar{h}_{ij}(\epsilon) = \phi^4(\epsilon)\delta_{ij}$. The relevant Lichnerowicz equation is given by

$$\triangle\phi(\epsilon) + \frac{\epsilon^2}{8}A_{ij}A^{ij}\phi^{-7}(\epsilon) = 0. \tag{4.52}$$

The idea is to perturb equation (4.52) around $\epsilon = 0$ and find a solution of the form

$$\phi = \phi_0 + \psi, \tag{4.53}$$

where ϕ_0 is the conformal factor of the given Brill-Lindquist initial data set. Inserting (4.53) in (4.52) and using

$$\triangle\phi_0 = 0 \tag{4.54}$$

we obtain

$$\triangle\psi + \frac{\epsilon^2}{8}(1 + {\phi_0}^{-1}\psi)^{-7}{\phi_0}^{-7}A_{ij}A^{ij} = 0 \tag{4.55}$$

Then, Theorem 4.5 is a direct consequence of the following existence theorem.

Theorem 4.6. *Let ϕ_0 be some Brill-Lindquist conformal factor and $A^{ij} = K^{ij}_A + K^{ij}_S + K^{ij}_P$, where K^{ij}_A, K^{ij}_S and K^{ij}_P are given by (4.26),(4.27) and (4.28) with (A, S_i, P_i) arbitrary constants, and with the implicit understanding that the singularites of A^{ij} match the singularities of ϕ_0. Then, there is $\epsilon' > 0$ such that for all $\epsilon \in (-\epsilon', \epsilon')$ there exists a unique $\psi(\epsilon)$ which belongs to $W^{2,2}_{-1/2}(\mathbb{R}^3)$ and which is a solution of*

$$\triangle\psi + \frac{\epsilon^2}{8}(1 + {\phi_0}^{-1}\psi)^{-7}{\phi_0}^{-7}A_{ij}A^{ij} = 0. \tag{4.56}$$

Furthermore $\psi(\epsilon)$ is continuously differentiable in ϵ.

Proof. In our proof the application of the Implicit Function Theorem is the crucial factor. Therefore our first task is a restatement of the problem which is suitable for that. We define the map $\mathcal{F}$ by

$$\mathcal{F}(\epsilon, \psi) := \triangle\psi + \frac{\epsilon^2}{8}(1 + {\phi_0}^{-1}\psi)^{-7}{\phi_0}^{-7}A_{ij}A^{ij}. \tag{4.57}$$

Evidently $\mathcal{F}(0, 0) = 0$. We have to choose appropriate Banach spaces X,Y and Z, together with neighborhoods $U \subset X$ and $V \subset Y$, such that

$$\mathcal{F} : U \times V \to Z \tag{4.58}$$

is continuously differentiable. To handle the required fall off behavior at infinity we use weighted Sobolev spaces. We take $X = \mathbb{R}$, $Y = W^{2,2}_{-1/2}(\mathbb{R}^3)$ and $Z = L^2_{-5/2}(\mathbb{R}^3)$. We also choose

$$U = \mathbb{R}. \tag{4.59}$$

When we take a closer look at the second term of (4.57), we see that V has to be chosen in such a way that $\phi_0 + \psi > 0$. Because of the form of ϕ_0 (see Definition 4.6 this condition is cleary satisfied, if we are able to guarantee that

$$|\psi| < 1. \tag{4.60}$$

We define

$$V := \{\psi \in W^{2,2}_{-1/2}(\mathbb{R}^3) : \|\psi\|_{W^{2,2}_{-1/2}(\mathbb{R}^3)} < \xi\}, \tag{4.61}$$

where ξ is a small enough constant to ensure that $|\psi| < 1$ Using (3.4) one can easily convince oneself that this is possible. From now on we assume that ξ is chosen in such a manner and remains fixed for the rest of the proof. Let us first show that the mapping $\mathcal{F}$ is well defined. Using the triangle inequality we estimate

$$\begin{aligned}\|\triangle\psi + \frac{\epsilon^2}{8}(1 + {\phi_0}^{-1}\psi)^{-7}{\phi_0}^{-7}A_{ij}A^{ij}\|_{L^2_{-5/2}(\mathbb{R}^3)} &\leq \|\triangle\psi\|_{L^2_{-5/2}(\mathbb{R}^3)} \\ &+ \|\frac{\epsilon^2}{8}(1 + {\phi_0}^{-1}\psi)^{-7}{\phi_0}^{-7}A_{ij}A^{ij}\|_{L^2_{-5/2}(\mathbb{R}^3)}.\end{aligned} \tag{4.62}$$

From Theorem 3.1 it follows that the first term of (4.62) is bounded. For the second term we define

$$F(\psi)(x) := f(x, \psi(x)), \tag{4.63}$$

where the function $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is given by

$$f(x, y) = (1 + {\phi_0}^{-1}(x)y)^{-7}{\phi_0}^{-7}(x)A_{ij}(x)A^{ij}(x). \tag{4.64}$$

Then

$$\frac{\epsilon^2}{8}(1+{\phi_0}^{-1}\psi)^{-7}{\phi_0}^{-7}A_{ij}A^{ij} = \frac{\epsilon^2}{8}F(\psi). \tag{4.65}$$

Furthermore it is convenient to decompose the function f. Define

$$g(x,y) := (1+{\phi_0}^{-1}(x)y)^{-7} \tag{4.66}$$

and

$$h(x) := {\phi_0}^{-7}(x)A_{ij}(x)A^{ij}(x). \tag{4.67}$$

Then

$$f(x,y) = g(x,y)h(x). \tag{4.68}$$

We have the following estimate:

$$\left\|\frac{\epsilon^2}{8}F(\psi)\right\|_{L^2_{-5/2}(\mathbb{R}^3)} = \left\|\frac{\epsilon^2}{8}g(\cdot,\psi)h\right\|_{L^2_{-5/2}(\mathbb{R}^3)} \leq \left\|\frac{\epsilon^2}{8}g(\cdot,\psi)\right\|_{L^\infty(\mathbb{R}^3)} \|h\|_{L^2_{-5/2}(\mathbb{R}^3)}\,. \tag{4.69}$$

Assuming that A_{ij} is given by a linear combination of Bowen-York tensors, it is easy to check that $h \in L^2_{-5/2}(\mathbb{R}^3)$ and

$$\left\|\frac{\epsilon^2}{8}g(\cdot,\psi)\right\|_{L^\infty(\mathbb{R}^3)} < \infty \tag{4.70}$$

for $\psi \in V$. Hence $\mathcal{F} : \mathbb{R}\times V \to L^2_{-5/2}(\mathbb{R}^3)$ is a well defined map. Our next task is to prove that $\mathcal{F}$ is also C^1 between the mentioned Sobolev spaces. More specifically, we have to show that there exist linear operators

$$D_1\mathcal{F}(\epsilon,\psi) : \mathbb{R} \to L^2_{-5/2}(\mathbb{R}^3) \tag{4.71}$$

and

$$D_2\mathcal{F}(\epsilon,\psi) : W^{2,2}_{-1/2}(\mathbb{R}^3) \to L^2_{-5/2}(\mathbb{R}^3) \tag{4.72}$$

such that

$$\lim_{\gamma\to 0}\frac{\|\mathcal{F}(\epsilon+\gamma,\psi)-\mathcal{F}(\epsilon,\psi)-D_1\mathcal{F}(\epsilon,\psi)\gamma\|_{L^2_{-5/2}}}{|\gamma|} = 0 \tag{4.73}$$

and

$$\lim_{\|\delta\psi\|_{W^{2,2}_{-1/2}(\mathbb{R}^3)}\to 0}\frac{\|\mathcal{F}(\epsilon,\psi+\delta\psi)-\mathcal{F}(\epsilon,\psi)-D_2\mathcal{F}(\epsilon,\psi)\delta\psi\|_{L^2_{-5/2}}}{\|\delta\psi\|_{W^{2,2}_{-1/2}}} = 0 \tag{4.74}$$

respectively. In addition these operators have to fulfill the following items:

(i) (4.71) and (4.72) are bounded, i.e.

$$\|D_1\mathcal{F}(\epsilon,\psi)\gamma\|_{L^2_{-5/2}(\mathbb{R}^3)} \leq C_1|\gamma| \tag{4.75}$$

and

$$\|D_2\mathcal{F}(\epsilon,\psi)\delta\psi\|_{L^2_{-5/2}(\mathbb{R}^3)} \leq C_2\|\delta\psi\|_{W^{2,2}_{-1/2}(\mathbb{R}^3)}, \tag{4.76}$$

where the constants C_1 and C_2 do not depend on γ and $\delta\psi$ respectively.

(ii) (4.71) and (4.72) are continuous in (ϵ,ψ) with respect to the operator norms, namely, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|\epsilon_1 - \epsilon_2| < \delta \Rightarrow \|D_1\mathcal{F}(\epsilon_1,\psi) - D_1\mathcal{F}(\epsilon_2,\psi)\|_{\mathcal{L}(\mathbb{R},L^2_{-5/2}(\mathbb{R}^3))} < \varepsilon \tag{4.77}$$

and

$$\begin{aligned}&\|\psi_1 - \psi_2\|_{W^{2,2}_{-1/2}(\mathbb{R}^3)} < \delta \Rightarrow \\ &\qquad \|D_2\mathcal{F}(\epsilon,\psi_1) - D_2\mathcal{F}(\epsilon,\psi_2)\|_{\mathcal{L}(W^{2,2}_{-1/2}(\mathbb{R}^3),L^2_{-5/2}(\mathbb{R}^3))} < \varepsilon.\end{aligned} \tag{4.78}$$

It is easy to see that

$$D_1\mathcal{F}(\epsilon,\psi)\gamma := \frac{\epsilon}{4}(1+{\phi_0}^{-1}\psi)^{-7}{\phi_0}^{-7}A_{ij}A^{ij}\gamma \tag{4.79}$$

satisfies (4.73),(4.75) and (4.77). Hence, (4.57) is continuously differentiable in ϵ. To prove the necessary conditions for the second argument it is clearly enough to investigate the nonlinear term $F(\psi)$. We start with the following expression:

$$\begin{aligned}&F(\psi+\delta\psi)(x) - F(\psi)(x) - D_2f(x,\psi(x))\delta\psi(x) = \\ &\qquad \int_{\psi(x)}^{\psi(x)+\delta\psi(x)} (D_2g(x,y) - D_2g(x,\psi(x)))h(x)dy,\end{aligned} \tag{4.80}$$

where D_2 denotes the derivative with respect to the second argument. Let $\varepsilon > 0$ be given. By the uniform equicontinuity of $(D_2g(x,\cdot))_{x\in\mathbb{R}^3}$ on the subset $\{y \in \mathbb{R} : |y| < 1\}$, it follows that there is $\delta > 0$ such that $|D_2g(x,y_1) - D_2g(x,y_2)| < \varepsilon$ for every $x \in \mathbb{R}^3$ whenever $|y_1 - y_2| < \delta$. Now, if $\|\delta\psi\|_{W^{2,2}_{-1/2}}$ is small enough, then $|\delta\psi(x)| < \delta$ uniformly in $x \in \mathbb{R}^3$ and so we get

$$|F(\psi+\delta\psi)(x) - F(\psi)(x) - D_2f(x,\psi(x))\delta\psi(x)| < \varepsilon\,|h(x)|\,|\delta\psi(x)|\,. \tag{4.81}$$

(4.81) yields

$$\|F(\psi+\delta\psi) - F(\psi) - D_2f(\cdot,\psi)\delta\psi\|_{L^2_{-5/2}(\mathbb{R}^3)} < \varepsilon\,\|\delta\psi\|_{L^\infty(\mathbb{R}^3)}\,\|h\|_{L^2_{-5/2}(\mathbb{R}^3)}\,. \tag{4.82}$$

Furthermore

$$\|\delta\psi\|_{L^\infty(\mathbb{R}^3)} \leq C \|\delta\psi\|_{W^{2,2}_{-1/2}(\mathbb{R}^3)}\,, \tag{4.83}$$

which easily follows from (3.4). This basically shows that F is differentiable and its differential is the mapping defined by putting

$$DF(\psi)(\delta\psi) = D_2 f(\cdot,\psi)\delta\psi. \tag{4.84}$$

It is easy to show that $DF(\psi)$ is bounded. We estimate

$$\|DF(\psi)(\delta\psi)\|_{L^2_{-5/2}(\mathbb{R}^3)} \leq C \|D_2 g(\cdot,\psi)\|_{L^\infty(\mathbb{R}^3)} \|h\|_{L^2_{-5/2}(\mathbb{R}^3)} \|\delta\psi\|_{W^{2,2}_{-1/2}(\mathbb{R}^3)}\,, \tag{4.85}$$

where each term on the r.h. side is finite. It remains to prove that the operator $\psi \mapsto DF(\psi)$ is continuous. This easily follows from

$$\begin{aligned} \|(DF(\psi_1) - DF(\psi_2))(\delta\psi)\|_{L^2_{-5/2}(\mathbb{R}^3)} &= \|(D_2 g(\cdot,\psi_1) - D_2 g(\cdot,\psi_2))h\delta\psi\|_{L^2_{-5/2}(\mathbb{R}^3)} \\ &\leq C \|D_2 g(\cdot,\psi_1) - D_2 g(\cdot,\psi_2)\|_{L^\infty(\mathbb{R}^3)} \|h\|_{L^2_{-5/2}(\mathbb{R}^3)} \|\delta\psi\|_{W^{2,2}_{-1/2}(\mathbb{R}^3)} \end{aligned} \tag{4.86}$$

and the continuity of the mapping

$$\psi \in V \subset W^{2,2}_{-1/2}\left(\mathbb{R}^3\right) \mapsto D_2 g(\cdot,\psi) \in L^\infty\left(\mathbb{R}^3\right). \tag{4.87}$$

The satisfy all assumptions of the Implicit Function Theorem we have to show one more thing, namely that $D_2\mathcal{F}(\epsilon_0 = 0, \psi_0 = 0) \in \mathcal{L}(W^{2,2}_{-1/2}(\mathbb{R}^3), L^2_{-5/2}(\mathbb{R}^3))$ is injective and surjective. But $D_2\mathcal{F}(0,0)\delta\psi = \triangle\delta\psi$, so Theorem 3.1 immediately reveals that this is true. Theorem 4.6 now follows from the Implicit Function Theorem.

Theorem 4.7 (Implicit Function Theorem)**.** *Let X, Y and Z be Banach spaces. Suppose $F : X \times Y \to Z$ is continuously differentiable in a neighbourhood of the point (x_0, y_0) and $F(x_0, y_0) = 0$. Further suppose that the map $D_y F(x_0, y_0) \in \mathcal{L}(Y, Z)$ is injective and surjective. Then there exists open sets $U \subseteq X$ and $V \subseteq Y$, with $x_0 \in U$ and $y_0 \in V$, such that for each $x \in U$ there exists a unique $y(x) \in B$ with $F(x, y(x)) = 0$. Furthermore $y(x)$ is continuously differentiable in x.*

Remark 4.12. *Given $A^{ij} = K^{ij}_A + K^{ij}_S + K^{ij}_P$, it is easy to see that $\bar{K}_{ij} = \phi^{-2}A_{ij}$ satifies the asymptotic conditions (4.3) for $r \to \infty$. To assure oneself that the resulting initial data sets are asymptotically flat at all ends, it remains to be shown that at every puncture, for some $k > 1$ and $\alpha > 0$, $\bar{K}_{ij} = O_{k-1}(\tilde{r}^{-1-\alpha})$, where $\tilde{r}$ denotes an inverted radial coordinate (compare with (4.20)). Taking the transformation of ϕ into account, one finds that the momentum terms of order r^{-2} are mapped to $O(\tilde{r}^{-4})$ while the spin terms and the spherically symmetric terms of order r^{-3} are mapped to $O(\tilde{r}^{-3})$. Hence the asymptotic conditions are fulfilled at the punctures.*

If an asymptotically flat initial data set has more than two asymptotically flat ends, one can define the following interaction energy [22].

Definition 4.7. *Let $n, k \in \mathbb{N}$ and $(\Sigma, \bar{h}_{ij}, \bar{K}_{ij})$ be an asymptotically flat initial data set with $n+1 \geq 3$ asymptotically flat ends. Furthermore let E_k, $0 \leq k \leq n$ denote the ADM energies (see (4.4)) of the data at the end k, then the interaction energy at the end k is defined by*

$$IE_k = E_k - \sum_{\substack{k'=0 \\ k' \neq k}}^{n} E_{k'}. \tag{4.88}$$

Remark 4.13. *The interaction energy IE_k is a geometric quantity and its definition does not involve any approximation. The problem is how to calculate IE_k in terms of physically relevant parameters.*

For simplicity we will fix $n = 2$. In this case the data can in certain circumstances be interpreted as initial data with two black holes.

Theorem 4.8. *Given the initial data set $(\Sigma = \mathbb{R}^3 \backslash \{x_1, x_2\}, \bar{h}_{ij} = \phi^4 \delta_{ij}, \bar{K}_{ij} = 0$, where $\phi = 1 + \sum_{i=1}^{2} \frac{m_i}{2r_i}$ with $m_i \in \mathbb{R}^+$ and $r_i = |x - x_i|$, the following expansion for the interaction energy IE_0 at the end $r = |x| \to \infty$ holds*

$$IE_0 = -\frac{E_1 E_2}{r_{12}} + O({r_{12}}^{-2}), \tag{4.89}$$

when $r_{12} = |x_1 - x_2|$ is large compared to m_i.

The leading order term is just the Newtonian expression for the binding energy of two point-particles. Nevertheless r_{12} is not an invariantly defined geometric distance measure. As such one might use the length of the shortest geodesic joining the two apparent horizons. Unfortunately these horizons are not easy to locate analytically.

Proof. Comparing $\bar{h}_{ij}$ with the one-hole metric (4.13) we immediately find the ADM energy at $r \to \infty$

$$E_0 = m_1 + m_2. \tag{4.90}$$

By introducing inverted coordinates and subsequent comparison with (4.13) we can write down the corresponding ADM energies for $r_1 \to 0$ and $r_2 \to 0$ respectively. We obtain

$$E_1 := E\,(r_1 \to 0) = m_1 \left(1 + \frac{m_2}{2r_{12}}\right) \tag{4.91}$$

and

$$E_2 := E\left(r_2 \to 0\right) = m_2 \left(1 + \frac{m_1}{2r_{12}}\right). \tag{4.92}$$

Assuming well separated holes, i.e. $m_i/r_{12} \ll 1$, we can calculate $IE_0 = E_0 - E_1 - E_2$ as function of E_1 and E_2 and get

$$IE_0 = -\frac{E_1 E_2}{r_{12}} + O({r_{12}}^{-2}). \tag{4.93}$$

Remark 4.14. *One can also compute the interaction energy for two well separated black holes with finite linear momenta and spins respectively. Using Bowen-York tensors to equip the sources with finite spins Dain [49] obtains*

$$IE_0 = -\frac{E_1 E_2}{r_{12}} + \frac{-S_1 \cdot S_2 + 3(S_1 \cdot \hat{n})(S_2 \cdot \hat{n})}{{r_{12}}^3} + O({r_{12}}^{-4}), \tag{4.94}$$

where $\hat{n}$ is the unit vector connecting the punctures. The spin-spin contribution agrees with the result obtained by Wald [115].

For two black holes with finite linear momenta the calculation has been made by the author following the method due to Dain. One finds that the total energy agree with the expression obtained in post-Newtonian calculations to quadratic order in the momenta. For the details the reader is referred to [117].

4.4 Bowen-York trumpets

The use of nontrivial topology can be regarded as a mere trick to conveniently construct black hole initial data. It comes at the expense of using initial data sets, where large parts have no physical relevance in typical numerical evolutions, e.g. the entire Kruskal extension to the Schwarzschild spacetime. In addition when the wormhole data (discussed in the previous section) are evolved using the standard moving-puncture method the numerical slices lose contact with the extra asymptotically flat wormhole ends and quickly asymptote to cylinders of finite areal radius. This suggests that it is preferable to start with initial data sets that possess a trumpet geometry from the start, i.e. all but one asymptotically flat ends are replaced by asymptotically cylindrical ends. These data leave out most of the unphysical region too.

Definition 4.8. *A Riemannian 3-manifold (Σ, h_{ij}) possesses an asymptotically cylindrical end if there exists a subset S of Σ and a diffeomorphism $\Phi : \Sigma \setminus S \to \mathbb{R}^+ \times N$, where N is a compact manifold without boundary, and furthermore, in the coordinate chart given by this diffeomorphism, the following conditions have to be satisfied, for some $\alpha > 0$ and some cylindrical metric $\tilde{h} = dt^2 + g_\Xi$ with g_Ξ a Riemannian metric on N :*

$$h_{ij} - \tilde{h}_{ij} = O(e^{-\alpha t}). \tag{4.95}$$

Remark 4.15. *We refrain from giving a definition - so far lacking in the literature - of an asymptotically cylindrical initial data set $(\Sigma, \bar{h}_{ij}, \bar{K}_{ij})$, i.e. specifying general conditions on $\bar{K}_{ij}$.*

In [47] and [64] the problem of proving existence of initial data sets, which are vacuum, maximal, locally conformally flat, exhibit one asymptotically flat and one asymptotically cylindrical end and with extrinsic curvatures of the Bowen-York type, has been solved. The proof uses a method in which the asymptotically cylindrical end is viewed as a certain singular limit of an asymptotically flat end. We show that such a singular limit can be completely avoided by using a suitable method of sub- and supersolutions for the relevant scalar constraint equation. In the following chapter we will see that it is possible to extend this method also to the case where the remaining asymptotically flat end is replaced by an asymptotically hyperboloidal end.

A good deal of the following material can also be found in [118]. We are interested in the case where Σ is $\mathbb{R}^3 \setminus \{0\}$ equipped with the flat background metric, i.e., $h_{ij} = \delta_{ij}$ in standard coordinates x^i. In the setting of maximal hypersurfaces the Lichnerowicz equation again reduces to

$$\triangle\phi + \frac{1}{8} A_{ij} A_{ij} \phi^{-7} = 0. \tag{4.96}$$

As for boundary conditions we require standard conditions of asymptotic flatness near infinity. At the other (small r) end we want the physical metric $\bar{h}_{ij}$ to approach a cylindrical metric. Note that, with $t = -\log r$, the manifold $(\mathbb{R}^3 \setminus \{0\}, r^{-2}\delta_{ij})$ is actually cylindrical with g_Ξ the standard metric on S^2. Thus the means, in our setting, to make the origin an asymptotically cylindrical end will be to demand that ϕ blows up like $r^{-\frac{1}{2}}$ for small r. An obvious way to guarantee this, as a simple scaling argument shows, is to require that A_{ij} in (4.96) blows up like r^{-3}. The main result, here stated informally, is the following: Given $A^{ij} = K^{ij}_A + K^{ij}_S + K^{ij}_P$ with (A, S_i) not all zero, there is a unique solution of (4.96) subject to appropriate boundary conditions so that $(\mathbb{R}^3 \setminus \{0\}, \phi^4\delta_{ij}, \phi^{-2}A_{ij})$ is an initial data set with one asymptotically flat and one asymptotically cylindrical end. Interestingly, in the case of nonzero S, the metric g_Ξ on S^2, to which $\bar{h}_{ij}$ tends near the origin, is a non-constant multiple of the standard one, which is determined by a nonlinear equation on S^2.

When S_i, P_i are zero, the solution to the above problem is known explicitly [9] and by virtue of its spherical symmetry comes, of course, from a specific maximal slice of a Schwarzschild spacetime. One is given A. One finds $m = \sqrt{\frac{4|A|}{3\sqrt{3}}}$ with $m > 0$ and ϕ_A, which solves Eq.(4.96) given by

$$\begin{aligned}\phi_A &= \left[\frac{4R}{2R + m\,(4R^2 + 4m\,R + 3m^2)^{1/2}}\right]^{1/2} \\ &\quad \times \left[\frac{8R + 6m + 3\,(8R^2 + 8m\,R + 6m^2)^{1/2}}{\left(4 + 3\sqrt{2}\right)(2R - 3m)}\right]^{1/2\sqrt{2}}, \qquad (4.97)\end{aligned}$$

where the areal coordinate R is implicitly defined as a function of r as the inverse of the map $r(R)$ given by

$$\begin{aligned}r &= \left[\frac{2R + m + (4R^2 + 4m\,R + 3m^2)^{1/2}}{4}\right] \\ &\quad \times \left[\frac{(4 + 3\sqrt{2})(2R - 3m)}{8R + 6m + 3(8R^2 + 8m\,R + 6M^2)^{1/2}}\right]^{1/\sqrt{2}}. \qquad (4.98)\end{aligned}$$

Note that the map (4.98) is an orientation-preserving diffeomorphism from $(\frac{3m}{2}, \infty)$ to $(0, \infty)$. Using this expression one easily obtains the asymptotic limits of the conformal factor

$$\phi_A \to \sqrt{\frac{3m}{2r}}\,(1 + O(r)) \text{ as } r \to 0 \qquad (4.99)$$

$$\phi_A \to 1 + \frac{m}{2r} + O(r^{-2}) \text{ as } r \to \infty. \qquad (4.100)$$

Thus there is an asymptotically flat end for large r. Furthermore (4.99) ensures that small r is an asymptotically cylindrical end where $r = e^{-\frac{2t}{3m}}$ and $g_\Xi = (\frac{3m}{2})^2 d\Omega^2$, with $d\Omega^2$ the standard metric on S^2.

Remark 4.16. *Depending on the sign of A in (4.26) the initial data set* $(\mathbb{R}^3 \setminus \{0\}, {\phi_A}^4 \delta_{ij}, {\phi_A}^{-2} K_A^{ij})$ *corresponds to a slice in the extended Schwarzschild spacetime which, when $A > 0$, runs from spacelike infinity to future timelike infinity, while if $A < 0$ it goes to past timelike infinity.*

Our existence and uniqueness results are consequences of the following theorem.

Theorem 4.9. *Consider the equation*

$$\triangle\phi = f(x,\phi) \tag{4.101}$$

for a scalar function ϕ on $\mathbb{R}^3 \setminus \{0\}$. Let $f \in C^1(\mathbb{R}^3 \setminus \{0\} \times \mathbb{R}^+)$ be non-decreasing in the second argument and non-positive. Assume that there exist positive functions ϕ_+ and ϕ_-, such that (here and subsequently we assume that $0 < \epsilon < \frac{1}{2}$)

(i) $\phi_\pm - 1 \in C^{2,\alpha}_{-1/2,1/2+\epsilon}\left(\mathbb{R}^3 \setminus \{0\}\right), 0 < \alpha < 1,$

(ii) $\phi_- \leq \phi_+$

and

(iii) $\triangle\phi_+ \leq f(x,\phi_+)\ ,\ \triangle\phi_- \geq f(x,\phi_-).$

For all functions ϕ with $\phi - 1 \in C^{0,\alpha}_{-1/2,1/2+\epsilon}\left(\mathbb{R}^3 \setminus \{0\}\right), 0 \leq \alpha < 1$, that are pointwise bounded between ϕ_- and ϕ_+, i.e., $\phi_- \leq \phi \leq \phi_+$, we have that

(iv) $f(x,\phi) \in C^{0,\alpha}_{-5/2,5/2+\epsilon}\left(\mathbb{R}^3 \setminus \{0\}\right)$

and

(v) $\frac{\partial f}{\partial \phi}(x,\phi) \in C^{0,\alpha}_{-2,2}\left(\mathbb{R}^3 \setminus \{0\}\right).$

Then there exists a unique solution ϕ' of (4.101) such that

(a) $\phi' - 1 \in C^{2,\alpha}_{-1/2,1/2+\epsilon}\left(\mathbb{R}^3 \setminus \{0\}\right), 0 < \alpha < 1,$

and

(b) $\phi_- \leq \phi' \leq \phi_+.$

Remark 4.17. *The standard way to treat the pole singularity occurring in wormhole puncture data is not generally applicable to trumpet puncture data. The catch is that the Schwarzschild solution does not drop out trivially when making the ansatz*

$$\phi = \phi_A + \psi. \tag{4.102}$$

Hence we have to use weighted function spaces with at least two weights.

Proof. Before starting, we point out the following fact, which is easily verified: Let w be any function which belongs to $C^{0,\alpha}_{-2,2}\left(\mathbb{R}^3 \setminus \{0\}\right)$ and let the function $g \in C^{0,\alpha}_{-2,2}\left(\mathbb{R}^3 \setminus \{0\}\right)$ be given by

$$g\left(x\right) := \frac{c}{r^2}, \tag{4.103}$$

where c is a positive number. Then one can always choose the constant c large enough so that $g \geq |w|$ pointwise. We choose c large enough so that

$$\frac{c}{r^2} \geq |\frac{\partial f}{\partial \phi}| \quad \text{in}\, \mathbb{R}^3 \setminus \{0\} \quad \text{for all}\ \phi_- \leq \phi \leq \phi_+. \tag{4.104}$$

We will now construct a pointwise increasing sequence of functions, starting from ϕ_-. We solve

$$(\triangle - g)\phi_1 = f(x, \phi_-) - g\phi_- \iff (\triangle - g)u_1 = f\left(x, \phi_-\right) - g\left(\phi_- - 1\right) \tag{4.105}$$

with $\phi_1 = 1 + u_1$. Eq.(4.105) has a unique solution $u_1 \in C^{2,\alpha}_{-1/2,1/2+\epsilon}\left(\mathbb{R}^3 \setminus \{0\}\right)$, since the right hand side of the equation for u_1 is in $C^{0,\alpha}_{-5/2,5/2+\epsilon}\left(\mathbb{R}^3 \setminus \{0\}\right)$ and the operator is an isomorphism between these function spaces (see Theorem 3.2). From the equations satisfied respectively by ϕ_1 and ϕ_- we deduce the following inequality:

$$\left(\triangle - g\right)\left(\phi_1 - \phi_-\right) = f\left(x, \phi_-\right) - \triangle\phi_- - g\phi_- + g\phi_- \leq 0. \tag{4.106}$$

Hence, from Lemma 3.5,

$$\phi_1 \geq \phi_-. \tag{4.107}$$

Also,

$$\begin{aligned}(\triangle - g)(\phi_+ - \phi_1) &= \triangle\phi_+ - f(x, \phi_-) - g(\phi_+ - \phi_-) \leq \\ &\leq f\left(x, \phi_+\right) - f\left(x, \phi_-\right) - g\left(\phi_+ - \phi_-\right) = \\ &= \left(\phi_+ - \phi_-\right)\int_0^1 \{\frac{\partial f}{\partial \phi}\left(x, \phi_- + t\left(\phi_+ - \phi_-\right)\right) - g\}dt.\end{aligned} \tag{4.108}$$

In particular

$$\triangle\left(\phi_+ - \phi_1\right) - g\left(\phi_+ - \phi_1\right) \leq 0\,, \tag{4.109}$$

whence $\phi_1 \leq \phi_+$. We define

$$u_n \equiv \phi_n - 1 \tag{4.110}$$

and use the following induction formula:

$$(\triangle - g)\phi_n = f(x,\phi_{n-1}) - g\phi_{n-1} \iff (\triangle - g)u_n = f(x,\phi_{n-1}) - gu_{n-1}. \tag{4.111}$$

We suppose that ϕ_m exists for $0 \leq m \leq n-1$ with $\phi_0 = \phi_-$ and $u_m \in C^{2,\alpha}_{-1/2,1/2+\epsilon}(\mathbb{R}^3 \setminus \{0\})$ for $0 \leq m \leq n-1$ and that for these

$$\phi_- \leq \phi_{m-1} \leq \phi_m \leq \phi_+. \tag{4.112}$$

Theorem 3.2 shows that $u_n \in C^{2,\alpha}_{-1/2,1/2+\epsilon}(\mathbb{R}^3 \setminus \{0\})$ exists. By arguments completely analogous to that for ϕ_1 we find that

$$\phi_{n-1} \leq \phi_n \leq \phi_+. \tag{4.113}$$

Thus ϕ_n converges at each point to a limit $\phi(x) = 1 + u(x)$, with $\phi_- \leq \phi \leq \phi_+$. It turns out that ϕ is a solution of our original equation. The sequence $\{u_n\}$ is uniformly bounded in the $C^{2,\alpha}_{-1/2,1/2+\epsilon}(\mathbb{R}^3 \setminus \{0\})$ norm and $C^{2,\alpha}_{-1/2,1/2+\epsilon}(\mathbb{R}^3 \setminus \{0\})$ is compactly embedded in $C^{2,\alpha'}_{-1/2-\epsilon,1/2}(\mathbb{R}^3 \setminus \{0\})$, where $\alpha' < \alpha$. Hence there is a subsequence, still denoted u_n, which converges in the $C^{2,\alpha'}_{-1/2-\epsilon,1/2}(\mathbb{R}^3 \setminus \{0\})$ norm to a function $u \in C^{2,\alpha'}_{-1/2-\epsilon,1/2}(\mathbb{R}^3 \setminus \{0\})$, identical to the previously defined u. Later we will show that $u \in C^{2,\alpha}_{-1/2,1/2+\epsilon}(\mathbb{R}^3 \setminus \{0\})$. The functions $f(\cdot,\phi_n)$ converge to $f(\cdot,\phi)$ in the $C^{0,\alpha}_{-5/2-\epsilon,5/2}(\mathbb{R}^3 \setminus \{0\})$ norm because of the following inequality

$$\|f(\cdot,\phi) - f(\cdot,\phi_n)\|_{C^{0,\alpha}_{-5/2-\epsilon,5/2}} \leq C\|\phi - \phi_n\|_{C^{0,\alpha}_{-1/2-\epsilon,1/2}}, \tag{4.114}$$

which results form combining the mean value theorem with Lemma 3.3. By continuity of the respective maps we have that $(\triangle - g)u_n$ converges to $(\triangle - g)u$ and $F(\cdot,u_n)$ converges to $F(\cdot,u)$, where $F(\cdot,u) = f(\cdot,u+1) - gu$, both in the $C^{0,\alpha}_{-5/2-\epsilon,5/2}$ norm (see (4.114) for the latter fact). But

$$\begin{aligned}\lim_{n\to\infty} \|(\triangle - g)u_n - F(\cdot,u_n)\|_{C^{0,\alpha}_{-5/2-\epsilon,5/2}} &= \\ = \lim_{n\to\infty} \|F(\cdot,u_{n-1}) - F(\cdot,u_n)\|_{C^{0,\alpha}_{-5/2-\epsilon,5/2}} &= 0. \end{aligned} \tag{4.115}$$

Since $\phi_- \leq \phi \leq \phi_+$, we also have that $u = \phi - 1 \in C^0_{-1/2,1/2+\epsilon}(\mathbb{R}^3 \setminus \{0\})$. Furthermore the equation $(\triangle - g)u = F(x,u)$ shows that

$$\triangle u = f(x,\phi). \tag{4.116}$$

Hence

$$u(x) = -\frac{1}{4\pi}\int \frac{f(x',\phi)}{|x-x'|}dx', \tag{4.117}$$

where

$$f(\cdot,\phi) \in C^0_{-5/2,5/2+\epsilon}\left(\mathbb{R}^3 \setminus \{0\}\right). \tag{4.118}$$

By directly estimating the Poisson integral (4.117) we get

$$\nabla u \in C^0_{-3/2,3/2+\epsilon}\left(\mathbb{R}^3 \setminus \{0\}\right) \tag{4.119}$$

which implies

$$u \in C^{0,\alpha}_{-1/2,1/2+\epsilon}\left(\mathbb{R}^3 \setminus \{0\}\right), \tag{4.120}$$

where we are using the elementary fact that w is an element of some $C^{0,\alpha}_{\delta_1}(\mathbb{R}^3 \setminus B_{3/2})$, when locally $r(x)^{-\delta_1}|w(x)| \leq C$ and $r(x)^{-\delta_1+1}|\nabla w(x)| \leq C$ and a function w is an element of $C^{0,\alpha}_{\delta_2}(\mathbb{R}^3 \setminus B_1)$, when locally $r(x)^{\delta_2}|w(x)| \leq C$ and $r(x)^{\delta_2+1}|\nabla w(x)| \leq C$. Using (4.120), the definition of f, and Lemma 3.3 gives

$$F(\cdot,u) \in C^{0,\alpha}_{-5/2,5/2+\epsilon}\left(\mathbb{R}^3 \setminus \{0\}\right). \tag{4.121}$$

From the isomorphism property of $\triangle - g$ and the bounded inverse theorem of linear functional analysis we infer that

$$\|w\|_{C^{2,\alpha}_{-1/2,1/2+\epsilon}} \leq C\|(\triangle - g)w\|_{C^{0,\alpha}_{-5/2,5/2+\epsilon}}. \tag{4.122}$$

Thus

$$u = \phi - 1 \in C^{2,\alpha}_{-1/2,1/2+\epsilon}\left(\mathbb{R}^3 \setminus \{0\}\right). \tag{4.123}$$

As for uniqueness, suppose there are two solutions, ϕ_1 and ϕ_2, which fulfill

$$\triangle\phi_1 = f(x,\phi_1), \quad \triangle\phi_2 = f(x,\phi_2) \tag{4.124}$$

subject to our boundary conditions. Let us consider the following function

$$\sigma := r^{1/2+\epsilon/2}(\phi_1 - \phi_2). \tag{4.125}$$

The function σ tends to 0 at the origin and at infinity. We now compute $\triangle\sigma$ (see Lemma 3.5). Assuming σ to have a positive maximum or a negative minimum and using the non-decreasing nature of $f(\cdot,\phi)$, this yields a contradiction.

Theorem 4.9 can now be used to prove the following theorems.

Theorem 4.10. *Let A^{ij} be given by $A^{ij} = K^{ij}_A + K^{ij}_S$, defined respectively in (4.26) and (4.27), with (A, S_i) not all zero. Then, there exists a unique positive ϕ with the property that $\phi - 1 \in C^{2,\alpha}_{-1/2,1/2+\epsilon}(\mathbb{R}^3 \setminus \{0\})$ and which is a solution of*

$$\triangle\phi = -\frac{1}{8}A_{ij}A^{ij}\phi^{-7} =: f(\cdot,\phi). \tag{4.126}$$

Proof. First note that

$$A_{ij}A^{ij} = |A|^2 = |K_A|^2 + |K_S|^2 = 6\frac{A^2 + 3S^2 \sin^2\theta}{r^6}, \tag{4.127}$$

where θ is the angle between the angular momentum vector $\vec{S} = (S^x, S^y, S^z)$ and the radial vector $\vec{R} = (x, y, z)$. Alternatively, if we rotate the coordinates so that the angular momentum is along the z axis, θ is the standard inclination angle. Clearly f is in $C^1(\mathbb{R}^3 \setminus \{0\} \times \mathbb{R}^+)$, is non-decreasing in the second argument and non-positive. Recall that ϕ_A referred to in the introduction in Eqs.(4.97, 4.98) satisfies $\triangle\phi_A = -\frac{1}{8}|K_A|^2\phi_A^{-7}$. Now choose A' large enough so that $A'^2 > A^2 + 3S^2$. This guarantees $|K_{A'}|^2 \geq |A|^2$. It follows that $\phi_{A'}$ furnishes a supersolution, again called ϕ_+. Next we observe that $\phi_+ - 1$ belongs to $C^{2,\alpha}_{-1/2,1/2+\epsilon}(\mathbb{R}^3 \setminus \{0\})$. Furthermore

$$A_{ij}A^{ij}\phi_+^{-7} = 6\frac{A^2 + 3S^2 \sin^2\theta}{r^6}\phi_+^{-7} \in C^{0,\alpha}_{-5/2,5/2+\epsilon}\left(\mathbb{R}^3 \setminus \{0\}\right). \tag{4.128}$$

Here we are making crucial use of the asymptotic behaviour of ϕ_+ near the origin, in particular that $r^{\frac{1}{2}}\phi_+$ tends to a positive constant as r goes to zero. Next we define ϕ_- by setting $\phi_- := u_- + 1$, where u_- fulfills

$$\triangle u_- = -\frac{1}{8}K_{ij}K^{ij}\phi_+^{-7}. \tag{4.129}$$

From the Poisson integral or Theorem 3.2, there exists u_- satisfying (4.129) in $C^{2,\alpha}_{-1/2,1/2+\epsilon}(\mathbb{R}^3 \setminus \{0\})$ which, by Lemma 3.5 is non-negative. Now calculate

$$\triangle(\phi_+ - \phi_-) = \triangle\phi_+ - f(x, \phi_+) \leq 0. \tag{4.130}$$

Hence, from Lemma 3.5, we have $\phi_- \leq \phi_+$. From the non-decreasing property of the function f it follows that ϕ_- furnishes a subsolution. In addition the Poisson integral shows that, like ϕ_+, the function $r^{\frac{1}{2}}\phi_-$ has a positive limit at the origin and

$$A_{ij}A^{ij}\phi_-^{-7} = 6\frac{A^2 + 3S^2 \sin^2\theta}{r^6}\phi_-^{-7} \in C^{0,\alpha}_{-5/2,5/2+\epsilon}\left(\mathbb{R}^3 \setminus \{0\}\right) \tag{4.131}$$

and

$$\frac{\partial f}{\partial \phi}(x, \phi_-) \propto A_{ij}A^{ij}\phi_-^{-8} = 6\frac{A^2 + 3S^2 \sin^2\theta}{r^6}\phi_-^{-8} \in C^{0,\alpha}_{-2,2}\left(\mathbb{R}^3 \setminus \{0\}\right). \tag{4.132}$$

Similarly we check the validity of assumptions (iv) and (v) for all ϕ's with $\phi_- \leq \phi \leq \phi_+$. This ends the proof of the Theorem.

Remark 4.18. *The pair of sub-/supersolutions that we have used is convenient because it traps the asymptotic behavior of the solution.*

More generally we have the following result:

Theorem 4.11. *Let A^{ij} be given by $A^{ij} = K_A^{ij} + K_S^{ij} + K_P^{ij}$, defined respectively in (4.26,4.27,4.28), with (A, S_i) not all zero. Then, there exists a unique positive ϕ with the property that $\phi - 1 \in C^{2,\alpha}_{-1/2,1/2+\epsilon}(\mathbb{R}^3 \setminus \{0\})$ and which is a solution of*

$$\triangle\phi = -\frac{1}{8}A_{ij}A^{ij}\phi^{-7} =: f(\cdot, \phi). \tag{4.133}$$

Proof. Here it is simplest to use, similar as in [64], as supersolution the conformal factor coming from the extreme Reissner Nordström trumpet initial data, namely

$$\phi_+ = \sqrt{1 + \frac{Q}{r}}. \tag{4.134}$$

Then, choosing

$$Q^2 \geq 3A + 9S + 7P^2, \tag{4.135}$$

we have that

$$\triangle\phi_+ = -\frac{1}{4}\frac{Q^2}{r^4}\phi_+^{-3} \leq -\frac{1}{8}|A|^2\phi_+^{-7}. \tag{4.136}$$

Clearly there again holds that

$$\phi_+ - 1 \in C^{2,\alpha}_{-1/2,1/2+\epsilon}\left(\mathbb{R}^3 \setminus \{0\}\right). \tag{4.137}$$

The rest of the argument follows as in the proof of the previous theorem.

The importance of the previous results lies in the fact that now we are able to show the following lemma.

Lemma 4.1. *Let ϕ be a solution of (4.133), such that $\phi - 1 \in C^{2,\alpha}_{-1/2,1/2+\epsilon}(\mathbb{R}^3 \setminus \{0\})$. Define*

$$\bar{h}_{ij} = \phi^4\delta_{ij}, \quad \bar{K}_{ij} = \phi^{-2}A_{ij}. \tag{4.138}$$

The resulting initial data set $\left(\mathbb{R}^3 \setminus \{0\}, \bar{h}_{ij}, \bar{K}_{ij}\right)$ has an asymptotically flat and an asymptotically cylindrical end.

Proof. First of all we show that $u = \phi - 1 \in C^{2,\alpha}_{-1/2,1/2+\epsilon}(\mathbb{R}^3 \setminus \{0\})$ can be uniquely decomposed as

$$u = \frac{d(\theta, \varphi)}{\sqrt{r}}\chi(r) + h, \tag{4.139}$$

where h diverges more slowly at the origin than $r^{-\frac{1}{2}}$ and $\chi(r)$ is a bump function with support in $|x| \leq 2$ and which equals 1 on $|x| \leq 1$. The quantity d is a C^∞ function on the 2-sphere, which is subject to the following nonlinear equation

$$\left(\triangle_{S^2} - \frac{1}{4}\right) d = -\frac{1}{8}\{6A^2 + 18S^2 \sin^2\theta\} d^{-7}. \tag{4.140}$$

The origin of the nonlinear equation for d can be found [71], but will of course be implicit in the ensuing proof. For an existence proof for a unique d, which is also positive, the reader is referred to [64]. Given d, define h as the solution to equation

$$\triangle h = \triangle \left(u - \frac{d}{\sqrt{r}}\chi(r)\right). \tag{4.141}$$

The function h is clearly in $C^{2,\alpha}_{-1/2,1/2+\epsilon}(\mathbb{R}^3 \setminus \{0\})$.
But we want to show more. Using (4.127) and (4.141), we calculate

$$\begin{aligned}
\triangle h &= -\frac{1}{8}A_{ij}A^{ij}(u+1)^{-7} - \triangle\left(\frac{d}{\sqrt{r}}\right)\chi(r) \\
&\quad -2\vec{\nabla}\left(\frac{d}{\sqrt{r}}\right)\cdot\vec{\nabla}\chi(r) - \frac{d}{\sqrt{r}}\triangle\chi(r) \\
&= \frac{1}{8}A_{ij}A^{ij}\left(\left(\frac{d}{\sqrt{r}}\right)^{-7}\chi(r) - \left(\frac{d}{\sqrt{r}}\chi(r) + h + 1\right)^{-7}\right) \\
&\quad -2\vec{\nabla}\left(\frac{d}{\sqrt{r}}\right)\cdot\vec{\nabla}\chi(r) - \frac{d}{\sqrt{r}}\triangle\chi(r),
\end{aligned} \tag{4.142}$$

where in the second equality sign we have used that the curly bracket on the r.h. side of (4.140) is equal to $A_{ij}A^{ij}r^6$. With the help of the function

$$\rho(r) := \chi(r)^{-1/7}, \tag{4.143}$$

we can write (4.142) as

$$\begin{aligned}
\triangle h &= \frac{1}{8}A_{ij}A^{ij}\left(h + 1 + (\chi(r) - \rho(r))\frac{d}{\sqrt{r}}\right)\sum_{i=0}^{6}\left(\frac{d}{\sqrt{r}}\rho(r)\right)^{i-7}\phi^{-1-i} \\
&\quad -2\vec{\nabla}\left(\frac{d}{\sqrt{r}}\right)\cdot\vec{\nabla}\chi(r) - \frac{d}{\sqrt{r}}\triangle\chi(r),
\end{aligned} \tag{4.144}$$

where we have used the following elementary identity

$$\frac{1}{a^p} - \frac{1}{b^p} = (b-a)\sum_{i=0}^{p-1} a^{i-p}b^{-1-i}, \tag{4.145}$$

which is true for real numbers a and b. We define

$$c(r,\theta,\varphi) := \frac{r^2}{8} A_{ij}A^{ij} \left(\sum_{i=0}^{6} \left(\frac{d}{\sqrt{r}} \rho(r) \right)^{i-7} \phi^{-1-i} \right) \tag{4.146}$$

and furthermore

$$f(r,\theta,\varphi) := \frac{c(r,\theta,\varphi)}{r^2} \left(1 + (\chi(r) - \rho(r)) \frac{d}{\sqrt{r}} \right) - 2\vec{\nabla} \left(\frac{d}{\sqrt{r}} \right) \cdot \vec{\nabla} \chi(r) - \frac{d}{\sqrt{r}} \triangle \chi(r). \tag{4.147}$$

Due to the known behavior of ϕ the function $c(r,\theta,\varphi)$ is everywhere bounded and non-negative. Last but not least we define the operator

$$\tilde{L} := \triangle - \frac{c(\theta,\varphi,r)}{r^2}. \tag{4.148}$$

With the help of these definitions, we arrive at

$$\tilde{L}h = f, \tag{4.149}$$

which the function h has to fulfill. Taking a closer look shows that $f \in C^{0,\alpha}_{-5/2+\epsilon,5/2+\epsilon}(\mathbb{R}^3 \setminus \{0\})$. But from Theorem 3.3 the map

$$\tilde{L} : C^{2,\alpha}_{-1/2+\epsilon,1/2+\epsilon} \left(\mathbb{R}^3 \setminus \{0\} \right) \longrightarrow C^{0,\alpha}_{-5/2+\epsilon,5/2+\epsilon} \left(\mathbb{R}^3 \setminus \{0\} \right) \tag{4.150}$$

is an isomorphism. It is obvious that the resulting initial data set has an asymptotically flat end as $r \to \infty$. To show that it also has an asymptotically cylindrical end, one uses the cylindrical metric

$$\tilde{h}_{ij} = \frac{d^4}{r^2} \delta_{ij}, \tag{4.151}$$

where d denotes the solution of (4.140).

Chapter 5

Asymptotically Hyperboloidal Initial Data

This chapter deals with solutions of the Einstein constraint equations which are asymptotically hyperboloidal. Mainly we are interested in the case where the asymptotically flat end of a trumpet is replaced by an asymptotically hyperboloidal end. After recalling the notion of asymptotically hyperboloidal geometries we state and prove an existence and uniqueness theorem for the relevant Lichnerovicz equation with Bowen-York sources.

The intuitive idea of an asymptotically hyperboloidal geometry is that of a Riemannian metric on a non compact manifold Σ that approaches a metric of constant negative curvature as one approaches each connected component of infinity in Σ. The prototype for asymptotically hyperboloidal manifolds is a constant negative curvature hyperboloid in Minkowski spacetime endowed with the induced metric. It turns out that in asymptotically flat spacetimes the geometry induced on spacelike hypersurfaces which hit null infinity is so-called conformally compact. Thus one may define asymptotically hyperboloidal manifolds precisely via conformal compactifications.

Definition 5.1. *Let (Σ, h_{ij}) be an oriented, compact C^∞ Riemannian manifold with nonempty boundary $\partial\Sigma$ and interior $int\,(\Sigma)$. Assume that $\rho \in C^\infty(\Sigma)$ is a defining function for $\partial\Sigma$, i.e. $\rho > 0$ on $int\,(\Sigma)$ while $\rho = 0$ but $d\rho \neq 0$ everywhere on $\partial\Sigma$. Then the manifold $\left(int\,(\Sigma), \bar{h}_{ij}\right)$, where $\bar{h}_{ij} = \rho^{-2} h_{ij}$, is said to be conformally compact.*

Definition 5.2. *A conformally compact manifold is called asymptotically hyperboloidal if one has the further condition $|d\rho|_h = 1$ on $\partial\Sigma$. Each connected component of $\partial\Sigma$ is then called an asymptotically hyperboloidal end.*

Remark 5.1. *By considering the conformal transformation of the Riemann tensor one easily finds that the sectional curvature $\kappa_{\bar{h}}$ of a conformally compact manifold satisfies $\kappa_{\bar{h}}(p) \to -|d\rho|^2_{\bar{h}}(q)$ as $p \to q \in \partial\Sigma$. In particular the sectional curvature converges to -1 at an asymptotically hyperboloidal end. This motivates the used terminology.*

Remark 5.2. *The basic example of an asymptotically hyperboloidal manifold is given by the Poincare Ball model of n-dimensional hyperbolic geometry $\mathbb{H}^n$, i.e. the n-dimensional unit ball B_1^n endowed with the metric $\bar{h}_{ij} = \rho^{-2}\delta_{ij}$ where δ_{ij} denotes the Euclidean metric on $\mathbb{R}^n$ and $\rho(x) = (1-|x|^2)/2$.*

We use the following definition of an initial data set with an asymptotically hyperboloidal end.

Definition 5.3. *An (vacuum) initial data set $\left(\Sigma, \bar{h}_{ij}, \bar{K}_{ij}\right)$ has an asymptotically hyperboloidal end if*

(1) $\left(\Sigma, \bar{h}_{ij}\right)$ features an asymptotivally hyperboloidal end in the sense of (5.2)

(2) $\tau = tr_{\bar{h}}\bar{K}$ is bounded away from zero asymptotically (i.e. outside some ball)

(3) $\bar{K}^{ij} - \frac{\tau}{3}\bar{h}^{ij}$ is at least of order ρ^3 asymptotically, where ρ is a boundary defining function.

5.1 Bowen-York trumpets

In our following existence proof for initial data sets with one hyperboloidal and one cylindrical asymptotic end (asymptotically hyperboloidal trumpets) we will start with the asymptotically hyperboloidal background geometry given by the Poincare Ball model of 3-dimensional hyperbolic geometry $\mathbb{H}^3$, punctered at $r = |x| = 0$. By demanding that the conformal factor ϕ blows up like $r^{-\frac{1}{2}}$ for small r we guarantee that the manifold cylindrical metric will possess an asymptotically cylindrical end too. From now on let Σ be $B_1 \setminus \{0\}$, where B_1 denotes the unit ball in $\mathbb{R}^3$ and the metric $\hat{h}_{ij}$ be given by

$$\hat{h}_{ij}\mathrm{d}x^i\mathrm{d}x^j = 4\left(1-r^2\right)^{-2}\left(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\right) =: \rho^{-2}\left(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\right). \tag{5.1}$$

Remark 5.3. *Note that $\hat{h}_{ij}$ as the associated metric of the Poincare Ball model of $\mathbb{H}^3$ is conformally flat.*

The next step to complete our background data is to find a symmetric, tracefree tensor field $\hat{A}_{ij}$ that fulfills

$$\hat{D}_i \hat{A}^{ij} = 0, \tag{5.2}$$

where $\hat{D}_i$ is the covariant derivate associated with $\hat{h}_{ij}$. In the case of a usual trumpet with one asymptotically flat end, we used certain Bowen-York tensors satisfying the necessary conditions at both ends. In particular

$$K_A^{ij} = \frac{A}{r^3}\left(3n^i n^j - \delta^{ij}\right), \tag{5.3}$$

$$K_S^{ij} = \frac{3}{r^3}\left(n^i \epsilon^{jkl} S_k n_l + n^j \epsilon^{ikl} S_k n_l\right). \tag{5.4}$$

These tensors are transverse and tracefree with respect to the flat metric. Because of the particular properties of TT tensors under conformal transformations (compare Lemma 2.2), conformal rescalings of (5.3) and (5.4) can be used. Namely

$$\hat{A}^{ij} = \rho^5 \left(K_A^{ij} + K_S^{ij}\right) \tag{5.5}$$

are solutions of (5.2). It turns out that these rescalings are compatible with the required boundary conditions too. We state our main result:

Theorem 5.1. *Let $\hat{A}^{ij}$ be given by $\hat{A}^{ij} = \rho^5 \left(K_A^{ij} + K_S^{ij}\right)$, with A unequal zero. Then, there exists a unique positive ϕ with the property that $\phi - 1 \in C^{2,\alpha}_{-1/2,1}\left(B_1 \setminus \{0\}\right)$ and which is a solution of*

$$\triangle_{\hat{h}} \phi = \frac{R[\hat{h}]}{8}\phi + \frac{3}{4}\phi^5 - \frac{1}{8}\hat{A}_{ij}\hat{A}^{ij}\phi^{-7} =: \hat{f}(x,\phi). \tag{5.6}$$

Remark 5.4. *Without loss of generality we have chosen $\tau = 3$.*

We further remark that in the statement of Theorem 5.1 we require that A is not zero. We restrict ourselves to this case, because the existence of suitable sub- and supersolutions needed in our proof is now easier to show. Most likely proofs exist without this assumption. Let us show the restriced but nevertheless interesting case.

Proof. We first start with suitable sub- and supersolutions of (5.6). More precisely we show that there exist positive functions ϕ_- and ϕ_+, such that

(i) $\phi_\pm - 1 \in C^{2,\alpha}_{-1/2,1}\left(B_1 \setminus \{0\}\right), 0 < \alpha < 1$

(ii) $\phi_- \leq \phi_+$

(iii) $\triangle_{\hat{h}}\phi_+ \leq \hat{f}(x,\phi_+)$, $\triangle_{\hat{h}}\phi_- \geq \hat{f}(x,\phi_-)$.

We further demand that for all functions ϕ with $\phi-1 \in C^{2,\alpha}_{-1/2,1}(B_1 \setminus \{0\})$, $0 \leq \alpha < 1$, that are pointwise bounded between ϕ_- and ϕ_+, we have that

(iv) $\hat{f}(x,\phi) \in C^{0,\alpha}_{-5/2,1}(B_1 \setminus \{0\})$

and

(v) $\frac{\partial \hat{f}}{\partial \phi}(x,\phi) \in C^{0,\alpha}_{-2,0}(B_1 \setminus \{0\})$.

We recall the fact that the family of Schwarzschild spacetimes contains CMC slices which inherit the underlying spherical symmetry and are of trumpet type. In addition to maximal trumpets with one asymptotically flat and one asymptotically cylindrical end there are more general ones which approach null infinity and possess one asymptotically hyperboloidal and one asymptotically cylindrical end. These trumpets will provide us with suitable sub- and supersolutions. For more details the reader is referred to [90]. The induced spatial metric can be written in the following form

$$\tilde{h}_{ij}\mathrm{d}x^i\mathrm{d}x^j = f^{-2}\mathrm{d}R^2 + R^2\mathrm{d}\Omega^2, \tag{5.7}$$

where the function $f = f(R)$ is

$$f^2 = 1 - \frac{2m}{R} + \left(R - \frac{A}{R^2}\right)^2, \tag{5.8}$$

R denotes the areal radius, the constant m the mass of the spacetime and where without loss of generality we again confined ourselves to the case $\tau = 3$.

Remark 5.5. *The constant A is actually the same as in (5.3). For certain A dependent on the value of m the geometry has a cylindrical end. In other words by choosing the constant A in equation (5.6) we fix the mass of the initial data set in the spherically symmetric case. From now on we assume that the constants A and m are chosen in such a way that there is a cylindrical end.*

As with any spherically symmetric metric, a radial coordinate transformation $R \to r = r(R)$ can be used to make the metric conformally flat. r is determined by the ordinary differential equation

$$\frac{\mathrm{d}r}{\mathrm{d}R} = \frac{r}{Rf}, \tag{5.9}$$

or alternatively

$$r\left(R\right)=C\exp\left(\int\frac{1}{f}\frac{\mathrm{d}R}{R}\right). \tag{5.10}$$

The constant of integration can be fixed by requiring that $r\to 1$ as $R\to\infty$. For large R we get

$$r\left(R\right)=1-\frac{1}{R}+\frac{1}{2R^2}+O\left(\frac{1}{R^4}\right). \tag{5.11}$$

The background metric $\hat{h}_{ij}=\rho^{-2}\delta_{ij}$ is also conformaly flat, hence these metrics can be conformally related, i.e.

$$\tilde{h}_{ij}=\phi_A^4\hat{h}_{ij} \tag{5.12}$$

and the corresponding conformal factors will play the role of seed sub- and supersolutions respectively. Using (5.12) and the asymptotic expression (5.11) one easily finds

$$\phi_A=\frac{d_A}{\sqrt{r}}\left(1+O\left(r\right)\right)\ \text{ as } r\to 0 \tag{5.13}$$

and

$$\phi_A=1+O\left(\rho\right)\ \text{ as } \rho\to 0, \tag{5.14}$$

where d_A denotes a positive constant which depends on A.

Remark 5.6. *We will discuss the boundary behaviour of solutions of the more general equation later in this chapter.*

Although there is no explicit expression of ϕ_A one can use the method of implicit differentiation together with (5.13) and (5.14) to show

$$\phi_A-1\in C^{2,\alpha}_{-1/2,1}\left(B_1\setminus\{0\}\right),\ 0<\alpha<1. \tag{5.15}$$

It is clear that ϕ_A solves the associated scalar constraint equation

$$\triangle_{\hat{h}}\phi_A=\frac{3}{4}\left(\phi_A^5-\phi_A\right)-\frac{1}{8}|\hat{K}_A|^2\phi_A^{-7}, \tag{5.16}$$

where

$$|\hat{K}_A|^2=\rho^6|K_A|^2=\frac{3A^2}{32}\frac{\left(1-r^2\right)^6}{r^6}. \tag{5.17}$$

Remark 5.7. *In the more general case there is also term including the spin of the configuration, namely*

$$\hat{A}_{ij}\hat{A}^{ij}=|\hat{A}|^2=|\hat{K}_A|^2+|\hat{K}_S|^2=\frac{3}{32}\left(A^2+3S^2\sin^2\theta\right)\frac{\left(1-r^2\right)^6}{r^6}, \tag{5.18}$$

where θ is the angle between the angular momentum vector $\vec{S}=(S^x,S^y,S^z)$ and the radial vector $\vec{R}=(x,y,z)$.

We go on and show how one can choose functions $\phi_{A'}$ and $\phi_{A''}$ which also fulfill (iii). Choose A' large enough so that $A'^2 \geq A^2 + 3S^2$. This guarantees $|\hat{K}_{A'}|^2 \geq |\hat{A}|^2$. It follows that $\phi_{A'}$ furnishes a supersolution, i.e.

$$\triangle_{\hat{h}} \phi_{A'} \leq \hat{f}\left(x, \phi_{A'}\right). \tag{5.19}$$

From now on ϕ_+ denotes such a function. Similarly choose A'' small enough so that $A''^2 \leq A^2$. This guarantees $|\hat{K}_{A''}|^2 \leq |\hat{A}|^2$ and it follows that $\phi_{A''}$ plays the role of a subsolution, i.e.

$$\triangle_{\hat{h}} \phi_{A''} \geq \hat{f}\left(x, \phi_{A''}\right), \tag{5.20}$$

called ϕ_-. Next we check the validity of (ii). The difference $\phi_+ - \phi_-$ fulfills the following equation

$$\triangle_{\hat{h}}\left(\phi_+ - \phi_-\right) = \frac{3}{4}\left(\phi_+^5 - \phi_+\right) - \frac{3}{4}\left(\phi_-^5 - \phi_-\right) - \frac{1}{8}|\hat{K}_+|^2 \phi_+^{-7} + \frac{1}{8}|\hat{K}_-|^2 \phi_-^{-7}. \tag{5.21}$$

Furthermore

$$\triangle_{\hat{h}}\left(\phi_+ - \phi_-\right) + \frac{3}{4}\left(\phi_+ - \phi_-\right) = \rho^{5/2} \triangle\left(\rho^{-1/2}\left(\phi_+ - \phi_-\right)\right), \tag{5.22}$$

where $\triangle$ denotes the flat Laplacian. Let us consider the following function

$$\chi := r^{1/2+\epsilon} \rho^{-1/2}\left(\phi_+ - \phi_-\right), \tag{5.23}$$

with $\epsilon > 0$. The function χ tends to 0 at the origin and at $r = 1$. Also $\chi \in C^{2,\alpha}\left(B_1 \setminus \{0\}\right)$. By computing $\triangle \chi$ and assuming the opposite to be true it is easy to see that χ can never have a negative minimum on Σ (see Lemma 3.5). Hence

$$\phi_- \leq \phi_+. \tag{5.24}$$

As for verifying (iv) and (v), we examine the non-linear function $\hat{f}\left(x, \phi\right)$. We have the explicit expressions

$$\hat{f}(x, \phi) = \frac{3}{4}\left(\phi^5 - \phi\right) - \frac{1}{8}|\hat{A}|^2 \phi^{-7} \tag{5.25}$$

and

$$\frac{\partial \hat{f}}{\partial \phi}(x, \phi) = \frac{3}{4}\left(5\phi^4 - 1\right) + \frac{7}{8}|\hat{A}|^2 \phi^{-8}. \tag{5.26}$$

By using the asymptotic limits (5.13) and (5.14) together with the form of $|\hat{A}|^2$, we easily check that $\hat{f}(x, \phi) \in C^{0,\alpha}_{-5/2,1}\left(B_1 \setminus \{0\}\right)$ and $\frac{\partial \hat{f}}{\partial \phi}(x, \phi) \in$

$C^{0;\alpha}_{-2,0}\left(B_1 \setminus \{0\}\right)$ for all ϕ's with $\phi_- \leq \phi \leq \phi_+$.

We construct a solution by induction, starting from ϕ_-. We define $\phi_1 = 1+u_1$ by solving the equation

$$\left(\triangle_{\hat{h}} - \frac{c}{r^2}\right) u_1 = \frac{3}{4}\left(\phi_-^5 - \phi_-\right) - \frac{1}{8}|\hat{A}|^2\phi_-^{-7} - \frac{c}{r^2}\left(\phi_- - 1\right), \tag{5.27}$$

where c is a positive number. Clearly

$$\frac{c}{r^2} \in C^{0,\alpha}_{-2,0}\left(B_1 \setminus \{0\}\right). \tag{5.28}$$

Theorem 3.4 shows that (5.27) has a unique solution $u_1 \in C^{2,\alpha}_{-1/2,1}\left(B_1 \setminus \{0\}\right)$. By choosing the constant c large enough and using arguments like in the proof of [76] we find that

$$\phi_- \leq \phi_1 \leq \phi_+. \tag{5.29}$$

Now we can use ϕ_1 to define a function ϕ_2 and so forth. In this way we obtain a sequence of functions $\{\phi_n\}$ by resolution of linear problems. Furthermore every function ϕ_n of this sequence is bounded between ϕ_- and ϕ_+. Standard arguments (compare with the usual trumpet case) can be used to prove convergence in the appropriate function space. Finally it turns out that the limit is a solution of (5.6).

We note that our method using a sub- and supersolution of (5.6) can be used to prove existence but says nothing about uniqueness. For a given subsolution ϕ_-, we do get a unique limit of the constructed sequence, which solves the equation. But there will generally be a lot of possible subsolutions (e.g. just choose different values of A'') and each could a priori lead to a different solution. Nevertheless we can easily show the uniqueness. It is a consequence of the equation satisfied by the difference of two solutions. Suppose there are two functions, ϕ_1 and ϕ_2, which fulfill

$$\triangle_{\hat{h}}\phi_1 = \hat{f}\left(x, \phi_1\right), \quad \triangle_{\hat{h}}\phi_2 = \hat{f}\left(x, \phi_2\right) \tag{5.30}$$

subject to our boundary conditions. We assume that $\phi_1 - \phi_2$ is not identically zero. This implies that the function

$$\chi := r^{1/2+\epsilon}\rho^{-1/2}\left(\phi_1 - \phi_2\right), \tag{5.31}$$

which tends to 0 at the origin (provided that $\epsilon > 0$) and at $r = 1$, must have a positive maximum, a negative minimum or both. Neither of these is compatible with the equation

$$\triangle_{\hat{h}}\left(\phi_1 - \phi_2\right) = \frac{3}{4}\left(\phi_1^5 - \phi_1\right) - \frac{3}{4}\left(\phi_2^5 - \phi_2\right) - \frac{1}{8}|\hat{A}|^2\phi_1^{-7} + \frac{1}{8}|\hat{A}|^2\phi_2^{-7}. \tag{5.32}$$

Let us prove that no positive maximum can exist. At the point of the maximum we get

$$\triangle_{\hat{h}}\left(\phi_1-\phi_2\right)+\frac{3}{4}\left(\phi_1-\phi_2\right)>0 \tag{5.33}$$

which by the conformal covariance implies

$$\triangle\left(\rho^{-1/2}\left(\phi_1-\phi_2\right)\right)>0. \tag{5.34}$$

Given that ϵ is small enough ($0<\epsilon<1/2$) this it at odds with the constraint $\triangle\chi\leq 0$ at the maximum, which one easily verifies by computing the Laplacian. Similar arguments can be used to derive contradictions in the other cases. Since we have ruled out every possible case, it follows

$$\phi_1\equiv\phi_2. \tag{5.35}$$

The importance of the previous result lies in the following lemma

Lemma 5.1. *Let ϕ be a solution of (5.6), such that $\phi-1\in C^{2,\alpha}_{-1/2,1}\left(B_1\setminus\{0\}\right)$. Define*

$$\bar{h}_{ij}=\phi^4\hat{h}_{ij},\quad \bar{K}_{ij}=\phi^{-2}\hat{A}_{ij}+\phi^4\hat{h}_{ij}. \tag{5.36}$$

The resulting initial data set $\left(B_1\setminus\{0\},\bar{h}_{ij},\bar{K}_{ij}\right)$ has an asymptotically hyperboloidal and an asymptotically cylindrical end.

Proof. Because of our background metric, it is obvious that the resulting initial data set has an asymptotically hyperboloidal end at $r=1$. To show that it also has an asymptotically cylindrical end, we first show that $u=\phi-1\in C^{2,\alpha}_{-1/2,1}\left(B_1\setminus\{0\}\right)$ can be uniquely decomposed as

$$u=\frac{d\left(\theta,\varphi\right)}{\sqrt{r}}\chi\left(r\right)+h, \tag{5.37}$$

where

$$h\in C^{2,\alpha}_{-1/2+\epsilon,1}\left(B_1\setminus\{0\}\right), \tag{5.38}$$

with $\epsilon>0$, i.e. h diverges more slowly at the origin than $r^{-1/2}$, $\chi\left(r\right)$ is a bump function with support near the origin and where d is a C^∞ function on the 2-sphere. Define d as the solution to

$$\left(\triangle_{S^2}-\frac{1}{4}\right)d=\frac{3}{4}\sqrt{2}^4d^5-\frac{\sqrt{2}^{-8}}{8}\{6A^2+18S^2\sin^2\theta\}d^{-7}, \tag{5.39}$$

where $\triangle_{S^2}$ is the Laplacian on S^2. For (5.39) we can easily find constant sub- and supersolutions that are positive, called d_- and d_+ respectively. So the existence of a positive d is guaranteed.

Remark 5.8. *Equation (5.39) also determines the boundary behaviour of* $\phi_{A'}$ *and* $\phi_{A''}$*.*

The properties of the function h are proved by inserting the decomposition into (5.6) and using Theorem 3.4 in a completely analogous manner like in the usual trumpet case. See also [118]. To complete the proof and show that there is also an asymptotically cylindrical end, one uses the cylindrical metric

$$h'_{ij} = \frac{d^4}{r^2}\delta_{ij}, \tag{5.40}$$

where d denotes the solution of (5.39).

Chapter 6

Summary and Outlook

In this thesis we proved the existence of initial data sets which possess one asymptotically flat end and one asymptotically cylindrical end - called trumpets in the recent numerical relativity literature. Furthermore, we extended the result to the case where the asymptotically flat end of the trumpet is replaced by an asymptotically hyperboloidal end. Our method directly used a suitable adapted version of the monotone iteration scheme presented in [76]. To ensure the desired asymptotic behavior of the solutions, we employed sub- and supersolutions with certain asymptotic limits and adapted background metrics. More precisely, we demanded that the sub- and supersolutions belong to suitable adapted weighted Hölder spaces. As supersolutions of the respective Lichnerowicz equations we applied conformal factors associated with trumpet slices of spherically symmetric Schwarzschild and (extreme) Reissner Nordström spacetimes. Nevertheless, the resulting initial data sets are in general not spherically symmetric. The reason is that Bowen-York sources which are not spherically symmetric lead to initial data sets with nonvanishing linear momenta and angular momenta respectively. In the case of a trumpet with nonvanishing spin, the metric on S^2, to which the physical metric tends at the asymptotically cylindrical end, is a non-constant multiple of the standard one, which is determined by a nonlinear equation on S^2.
In addition to trumpet initial data sets we considered wormhole-like configurations, i.e. initial data sets with several asymptotically flat ends. There were two reasons for this: First, it is easier to handle the singularities of the conformal factor (assuming a flat background metric) for wormholes than for trumpets. In this case it is possible to explicitly split off the parts of the conformal factor, which blow up at the punctures, and thus it is enough to employ function spaces with only one weight at infinity. It is reasonable to illustrate the necessary steps of the proof in the simpler setting prior to moving to the more difficult problem. Second, our study of initial data sets,

which only possess asymptotically flat ends, includes an interesting result with regard to perturbation theory. By using the implicit function theorem, we show that in a neighbourhood of the time symmetric solution the associated conformal factor is continuously differentiable in the momenta of the source.

We conclude this work with a few remarks concerning possible generalizations of the results obtained so far. First, one might try to extend Theorem 5.1 to include linear momentum. In the case of an asymptotically flat trumpet this posed no real problem, only another supersolution of the respective Lichnerowicz equation was needed. Most likely a conformal factor associated with a CMC trumpet slice of an extreme Reissner Nordström spacetime should do the job. One could also try to construct a global supersolution by matching suitable barriers that are defined on each asymptotic end. However, the global supersolution obtained in this way will in general only be continuous rather than $C^{2,\alpha}$. Nevertheless, there is a version of the monotone iteration scheme that works in the setting of weak sub- and supersolitions. The reader is referred to [41]. Second, one might try to extend the trumpet data to multiple black holes with different spin- and boost-parameters for each black hole. The employed weighted function spaces can easily be adapted to dictate the behavior at any finite number of punctures, see [104]. One can thus hope that it will be rather straightforward to obtain a rigorous existence result for initial data sets with several trumpets. Finally, we conclude this work with a remark concerning the limitations of our assumptions. In our analysis we always assumed conformal flatness of the (background) metric and we required that the trace of the extrinsic curvature is a constant (CMC condition). These conditions allowed explicit solutions of the momentum constraint which simplified the process of constructing interesting initial data sets. But this does not mean that these assumptions are adequate for the generation of all desired black hole initial data sets. It is well-known that there is no slicing of Kerr spacetimes, that is conformally flat. Furthermore, Bowen-York initial data contain spurios radiation [73]. Therefore, in addition to conformally flat initial data, one should consider more general solutions of the Einstein constraint equations.

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