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The Connection of Dual and Primal Solutions to Expected
Utility Optimization with Transaction Costs in BMS-setting

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Master thesis:

The Connection of Dual and Primal Solutions to Expected Utility Optimization with Transaction Costs in BMS-setting

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ABSTRACT

Expected utility optimization came up with the growing interest for the stock markets in the 60's, motivated from finance and then modeled in mathematical sense. Roughly speaking it is a search for the trading strategy expecting the biggest return in a mathematically defined financial market.

First the market was still very artificial in the sense that trading was costless and as a result to behave optimally, unlimited trading was necessary. But as it often is in mathematics, if you want to remove certain assumptions it suddenly gets very hard. So it behaved with this *causa* and for a long time the problem with trading accumulating transaction costs was not explicitly solved and the numerical calculations were not really satisfying. Finally just before the millennium turned there was also a turn in the field by two inventive mathematicians who established a different approach to this problem using recent recoveries in another field and it looked promising. But it took almost another 15 years until a solution was actually rolled out in proper mathematics.

We will first give an elegant overview of the solution to the expected utility optimization problem for terminal wealth with proportional transaction costs in the Black-Merton-Scholes model via the recent, the dual ("martingale") approach and with this in mind we analyse the classical, the primal approach and finally come to an explicit formula for an open parameter to finish the primal solution, which was still open. We consider logarithmic utility and also give an outview for the power utility case.

For my loving parents.

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1

1.1 Introduction

The foundation of continuous portfolio selection and optimization was set by Robert C. Merton back in the end of the 60's in the MIT Press. The paper to mention is “Lifetime portfolio Selection under Uncertainty: the Continuous Time Case” [Mer69] and crucially depended on the insights of Itô into stochastic differential equations [Itô65] and Samuelson into approximation [Sam69, Sam70] to develop a continuous-time model consistent with the former existing finite-period ones.

The reality showed that this was necessary, because most well-developed capital markets provided trading opportunities in securities available continuously in time. Merton considers a financial market with two assets (he also gives the results for the m-asset case). One is the safe asset with a deterministic rate of growth (called bond), while the others assets' growth is determined by a stochastic process, namely a Brownian motion¹ with drift (called stock). It's possible to shift money from one asset to the other and in Mertons case without any costs for the transactions and therefore also without any restrictions on how often. Also trading at arbitrary amounts is possible, meaning not necessarily at whole numbers of shares.

The aim is to maximize the expected utility from terminal wealth (with fixed time horizon) of the portfolio with fixed investment at the start. He proves that when assuming the utility function to be “of the form yielding constant relative risk aversion”² (i.e. power and logarithmic utility), it is optimal to keep the ratio of money invested in the two assets constant.

¹For those who are not familiar with Brownian motion resp. the Wiener process and stochastic calculus, W. Schachermayer and J. Teichmann wrote an elegant and easy-to-read essay in German in 2007. See [SchTei07].

²[Mer69, p. 250]

This constant later is known as the famous *Merton proportion* (or more general the mean-variance ratio) which depends on all the model parameters, but not on the wealth itself. When normalizing the bond to $\mathbf{1}$ (discounting s.t. " r " = 0) and assuming logarithmic utility ($\gamma = 0$) this is simply

$$\theta = \frac{\mu}{\sigma^2},$$

where μ and σ are drift and volatility of the stock.

In the 70's and 80's Magill & Constantinides [MagCon76], Constantinides [Con86], Taksar & Klass & Assaf [TakKlaAss88] and Davis & Norman [DavNor90] published important papers based on Mertons work, which examined the question of maximal expected utility of terminal wealth and similar questions, when transactions from one to the other asset are *not free of charge*.

Despite the outstanding results of Merton, his formulation led to a quite unrealistic type of trading behavior. Facing continuously varying security prices the investor will end up in a unrealistic amount of transactions trying to hold his investment at the optimal proportion at every time. To avoid this Magill & Constantinides introduced *transaction costs*, proportional to the amount transacted and they found that with sufficiently small transaction costs "the investor trades in securities when the variation in the underlying security prices forces his portfolio proportions outside a certain region about the optimal proportions in the absence of transaction costs."³ They also proved that this region is independent of the investors wealth and the remaining trading time.

Taksar & Klass & Assaf developed this further (for logarithmic utility) in an abstract approach using the newly developed tools of singular stochastic control and derived an optimality equation for potential strategies and also determined the class of potential strategies, where the optimal lies in.

Constantinides and Davis & Norman further showed that the optimization problem (they additionally considered intermediate consumption) and the search for the optimal strategy results in a complicated free boundary problem, where Davis & Norman proved the existence of a solution requiring some minor conditions and they worked out numerical approximations of the optimal policy.

Constantinides also numerically computed the effect of transaction costs on

³[MagCon76, p. 246f]

the value function for our problem, and observed that transaction costs have a “first-order effect on assets’ demand” and a “second-order effect on equilibrium asset return”.

In 1991 Dumas & Luciano published an analytical solution depending on the numerical solution of equating a specific determinant to zero, for the case of only terminal consumption and also provided some comparative analysis.

Shreve & Soner subsequently removed the conditions Davis & Norman were requiring. They also provided essential work on the liquidity premium, “defined to be the amount of increase in the rate of return for the stock which would be required to compensate the investor for the presence of the transaction costs”⁴ and explained and used viscosity solutions in stochastic control theory.

Following this strain of literature in 2003 Janecek & Shreve proceeded to do some asymptotic analysis and finally get decent results out of this. Also to be mentioned is the group around R. Korn, which was also working further in this direction.

In contrast to this in 1996 Cvitanic & Karatzas published a pioneering paper for the whole field introducing the “dual” or “martingale method” to maximization of the expected utility from terminal wealth with transaction costs. They “characterize[d] the solution to the portfolio optimization problem of maximizing utility from terminal wealth in the model via a transformation to a hedging problem; the optimal portfolio is the one that hedges the inverse of marginal utility, evaluated at the shadow state-price density which solves the corresponding dual problem.”⁵ Inspired by this Wang, Kallsen, Muhle-Karbe, Kramkov, Schachermayer and others worked on this problem.

The final goal of this thesis is to use recent results of the martingale method from [GerMuhSch11] to bring the primal solution of [DumLuc91] into an explicit form, meaning to calculate the open parameters.

The work is structured as following.

In section 2 we try to stay in line with the formulations of Schachermayer et al. We introduce the setting we work in and state the solution Merton gave to the expected utility optimization problem without transaction costs. As it was developed recently we introduce the shadow price process and give a small

⁴[ShrSon94, p. 611f]

⁵[CviKar96, p. 115]

overview of the connection to the stock price process through the function g . This function, which represents the transition to the dual way of attacking the problem is then explicitly calculated and investigated. As this function is leading to the optimal strategy we formulate the resulting statements in an easy to understand manner. Then turning the point of view the involved parameters can be calculated through inversion in terms of fractional Taylor series in λ . Finally we also give explicit representations for the optimal growth rate.

The main part is organized in section 3, where we introduce the value function and modifications of it to be able to give results for the maximal earnings for infinite horizon. Through simplifications using particular properties of the utility function and special considerations we arrive at an easy representation using a newly introduced function h . For h we derive an ODE and state the general solution with which we can once more represent our discounted value function in a different manner. Then through the economic marginal rate of substitution we derive an equality of derivatives of this representation and the previously calculated “shadow price function” g . By solving it we finally get a representation for the missing parameter C_1 in the general solution of h and then we reached the goal and explicitly calculated the maximal possible wealth for the problem. Due to the length of the full representation we state only the result for a specific case.

Section 4 is the appendix. In part *A* we give the (lengthy but exact) general solution for the missing parameter C_1 for the log utility case and in part *B* we come back to the modifications of the terminal value function and develop these for power utility.

2

In this section we will give an overview of the dual approach following [KalMuh10], [GerMuhSch11] and [GerMuhSch12]. We will restrict ourselves to rather give the ideas and the understanding, than reproducing the technical proofs, which can be found in the literature.

2.1 The financial Market and Trading

The financial market is constructed on a finite filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_{0 \leq t \leq T}, \mathbb{P})$ with $T \geq 0$, (Ω finite but assumed rich enough to support a Brownian motion W_t), $\mathcal{F}_t$ the Brownian filtration satisfying the usual conditions (i.e. is right-continuous and $\mathcal{F}_0$ contains the $\mathbb{P}$ -negligible events and $\mathcal{F}_\infty \subseteq \mathcal{F}$; and $\mathbb{P}$ a probability measure on Ω (for simplicity $\mathbb{P}(\omega) > 0$, *for all* $\omega \in \Omega$). Time runs continuously and T determines the time horizon for possible trading, which we will finally extend to ∞ .

We consider a safe and risky asset, which are modeled as two pathwise continuous and adapted price processes denoted by

$$(B_{t(\omega)})_{t=0}^T, (S_{t(\omega)})_{t=0}^T : \Omega \rightarrow \mathbb{R}$$

(dependence on ω will be mostly notationally neglected from here on as we will always keep in mind that all processes are stochastic). We underly the Black-Merton-Scholes model [BlaSch73], meaning the market consists of a riskless *bond* modeled as

$$B_t = e^{rt}$$

with r the riskless interest rate on the bond and a risky *stock* modeled as the

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resulting process of the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

which is with $\mu \geq 0$ and σ , the drift and diffusion, being constants

$$S_t = S_0 e^{(\sigma W_t + (\mu - \frac{\sigma^2}{2})t)} \quad (2.1)$$

with S_0 the initial stock price. We assume w.l.o.g. that $S_0 = 1$.

It should be mentioned that the above market is complete in the sense that all derivatives can be replicated. The positions held in the two assets at time $0 \leq t \leq T$ are denoted by $(\varphi_t^0, \varphi_t^1)$ (closer specified in Definition 1). The trader begins with initial positions $(\varphi_{0-}^0, \varphi_{0-}^1) \geq (0, 0)$ and trading means that he changes the position of one asset to the cost of the other. The first time the trader is allowed to rebalance his portfolio is at time 0.

Proportional *transaction costs* amounting to $0 \leq \lambda \lll 1$ are introduced via the bid price $(1 - \lambda)S_t$ and the ask price S_t of the stock. This means that if we want to increase our position in stock by 1 piece, we have to reduce our position in bond by $\frac{S_t}{B_t}$, but if we want to “sell” one stock we only get $\frac{(1-\lambda)S_t}{B_t}$ numbers of bond. Trading in the assets is possible in arbitrary quantities.

Choosing $(B_{t(\omega)})_{t=0}^T$ as numéraire we may assume that $B_t \equiv 1$ and act here as if this was done beforehand. Furthermore when considering the pair of processes $(\varphi_t^0, \varphi_t^1)_{0 \leq t \leq T}$ as the trading strategy we only allow *self-financing* trading strategies by

Definition 1. A pair of right continuous, predictable, finite variation processes $(\varphi_t^0, \varphi_t^1)_{0 \leq t \leq T}$ starting at $(\varphi_{0-}^0, \varphi_{0-}^1)$ is called λ -self-financing trading strategy if

$\varphi_t^0 = \varphi_t^{0,\uparrow} - \varphi_t^{0,\downarrow}$ and $\varphi_t^1 = \varphi_t^{1,\uparrow} - \varphi_t^{1,\downarrow}$, where $\varphi_t^{0,\uparrow}$, $\varphi_t^{0,\downarrow}$, $\varphi_t^{1,\uparrow}$ and $\varphi_t^{1,\downarrow}$ are the decompositions of φ^0 and φ^1 into the differences of the non-negative increasing processes starting at $\varphi_{0-}^{0,\uparrow} = \varphi_{0-}^{0,\downarrow} = \varphi_{0-}^{1,\uparrow} = \varphi_{0-}^{1,\downarrow} = 0$, and satisfying

$$d\varphi_t^{0,\uparrow} \leq (1 - \lambda)S_t d\varphi_t^{1,\downarrow}, \quad d\varphi_t^{0,\downarrow} \geq S_t d\varphi_t^{1,\uparrow}, \quad 0 \leq t \leq T. \quad (2.2)$$

The trader is therefore committed to the initial capital in bond and stock and cannot add money into the trading after time 0_- , meaning on the other hand that it is possible to follow this trading strategy without any additional

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money.

The trader has to be solvent at all times. The subsequent definition assures this and additionally gives us the liquidated wealth for all times, therefore also at the terminal time point, which is exactly what we want to maximize (of course in expectation).

Definition 2. A self-financing trading strategy $(\varphi_t^0, \varphi_t^1)_{0 \leq t \leq T}$ is called λ -admissible, if its liquidated wealth process

$$V_t(\varphi_t^0, \varphi_t^1) := \varphi_t^0 + (\varphi_t^1)^+(1 - \lambda)S_t - (\varphi_t^1)^-S_t, \quad 0 \leq t \leq T$$

is a.s. nonnegative.

So it is not possible for the trader to take positions in our market, where he is actually borrowing money, not even for a short time period.

A short word regarding notation. We agree that from here on we will drop the preceding λ s from λ -self-financing and λ -admissible for convenience, but we are always aware that these definitions depend on the transaction costs.

So we have all ingredients together to define the set of contingent claims super-replicable (essentially meaning reachable) at price 0, which will be the maximization set, where all our potential strategies end-values lie in.

Definition 3. For fixed S and $\lambda > 0$ the set $\mathcal{A}^\lambda$, is the set of $\mathbb{R}^2$ -valued $\mathcal{F}$ -measurable random variables $(\varphi^0, \varphi^1) \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$, such that there exists a self-financing and admissible trading strategy $(\varphi_t^0, \varphi_t^1)_{0 \leq t \leq T}$ starting at $(\varphi_{0-}^0, \varphi_{0-}^1) = (0, 0)$ such that $(\varphi^0, \varphi^1) \leq (\varphi_T^0, \varphi_T^1)$.

And as we are reaching out for the liquidated value V_T we introduce the one dimensional projection of $\mathcal{A}^\lambda$, the cone of claims denoted now only in units of bond reachable at price 0.

Definition 4.

$$\begin{aligned} \mathcal{C}^\lambda &= \{\varphi^0 \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}) : \exists (\varphi_T^0, \varphi_T^1) \in \mathcal{A}^\lambda \text{ s.t. } \varphi_T^0 \geq \varphi^0, \varphi_T^1 \geq 0\} \\ &= \mathcal{A}^\lambda \cap L_0^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2) \end{aligned}$$

This will be the maximization set we use for the problem definition. Now lets go on to the behavior of the agent.

2.2 Expected Utility Optimization

An agent trading in our market is furthermore assumed to have constant relative risk aversion, meaning that as wealth rises he wants to hold his percentage of wealth in the risky asset constant. So if he has double the money he also wants to invest double as much into the stock. In contrast to this an agent having increasing relative risk aversion would want to reduce his percentage in the risky asset as wealth rises and vice versa. Examples for not increasing and decreasing relative risk aversion are $U(x) = x - cx^2$ and $U(x) = -e^{\frac{2}{\sqrt{x}}}$.

Modeling an agent with constant relative risk aversion we introduce a *utility function* $U : D \rightarrow \mathbb{R}$ with general properties as D being either $]0, \infty[$ or $] -\infty, \infty[$ and U is supposed to be strictly concave, increasing and differentiable (the strictness and differentiability avoids us to speak about subgradients and allows for uniqueness of solutions) on D . We also expect U to satisfy the *Inada Conditions*

$$\lim_{x \searrow x_0} U'(x) = \infty, \quad \lim_{x \nearrow \infty} U'(x) = 0,$$

where $x_0 \in \{-\infty, 0\}$ denotes the left boundary of D .

The utility functions meant by constant relative risk aversion are the well-known

$$U_\gamma(x) = \begin{cases} \frac{x^\gamma}{\gamma}, & \gamma < 1, \gamma \neq 0 \text{ (power utility)} \\ \log(x), & \gamma = 0 \text{ (log utility)} \end{cases}$$

In our definition here the utility functions imply constant relative risk aversion equal to $R(x) = 1 - \gamma$, which is the Arrow-Pratt-De Finetti measure of relative risk-aversion

$$R(x) = \frac{-xU''(x)}{U'(x)}.$$

For the main work we will mainly regard a trader modeled by the logarithmic utility function (where $\gamma = 0$), but in the Appendix B we will give an outview for the power utility case.

Economically speaking the agent, starting at initial endowment $x \in]0, \infty[$, is in search for a self-financing and admissible trading strategy, which guarantees him the maximal expected liquidated utility return at the end of the trading horizon T allowed by his relative risk aversion. In our mathematical setting this means that his possible final holdings lie in the set $A^\lambda + (x, 0)$. When we assume the agent to liquidates his portfolio at time T in the sense of selling

all his stocks, meaning his wealth is V_T , then his possible final holdings of bonds lie in the set $C^\lambda + x$ by definition. With regarding logarithmic utility the expected utility maximization problem for finite horizon can be stated as

$$\mathbb{E} \left[\log(V_T) \right] \rightarrow \max!, \quad V_T \in C^\lambda + x. \quad (2.3)$$

There are different mathematical formulations of this problem and further considerations of how to define it best, we will stick with this easy but not completely precise formulation due to the easy accessibility for the reader. We will also take existence and uniqueness of the optimizer as granted in our setting, which is actually a sophisticated endeavor and we refer to [DelSch06] for a treatment of these problems mentioning also the Fundamental Theorem of Asset Pricing in this context for the well-educated reader. For a more general view (in sense of general utility functions and incomplete markets) [KraSch99] is good literature.

2.3 The “Merton”-solution without TC

The key of solving the problem “is to reduce the portfolio choice problem with transaction costs to another portfolio choice problem, without transaction costs, with the bid and ask prices replaced by a single shadow price $\tilde{S}_t$ evolving within the bid-ask spread [(Definition 5)], which coincides with either price at times of trading, and which yields the same optimal policy”¹.

We will use this subsection here to present the frictionless results of Merton (originally presented in [Mer69]) for the BMS setting and log utility to be able to use these for the frictionless shadow price market we are heading to. Merton actually presented his solution for a model with m risky assets, we will be content in our formulation with just one stock. In one aspect we will be a bit more general, as we allow μ and σ to be arbitrary predictable processes².

Theorem 1 (*An adapted Formulation of Mertons Portfolio Opti-*

¹[GerGuaMuhSch11, p. 16]

²In this form presented and proofed by W. Schachermayer in a 2012 lecture at Universität Wien.

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mization). Suppose that the stock-price process satisfies

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t, \quad 0 \leq t \leq T$$

where $(\mu_t)_{0 \leq t \leq T}$ and $(\sigma_t)_{0 \leq t \leq T}$ are predictable, real-valued processes (same as differential form of (2.1) for drift- and diffusion-processes) such that

$$\mathbb{E} \left[\int_0^T \frac{\mu_t^2}{\sigma_t^2} dt \right] < \infty.$$

Define the growth optimal process $\hat{V}_t = (\hat{V}_t)_{0 \leq t \leq T}$, starting at $\hat{V}_0 = 1$, by

$$\frac{d\hat{V}_t}{\hat{V}_t} = \hat{\pi}_t \cdot \frac{dS_t}{S_t},$$

where $\hat{\pi}_t$, the ratio of money invested in stock over the total money invested $\frac{\varphi_t^1 S_t}{\varphi_t^0 + \varphi_t^1 S_t}$, equals the mean-variance ratio (Merton proportion)

$$\hat{\pi}_t = \frac{\mu_t}{\sigma_t^2}. \quad (2.4)$$

Then $\hat{V}_t$ is a well defined $]0, \infty[$ -valued process satisfying

$$\hat{V}_t = e^{\int_0^t \frac{\mu_s}{\sigma_s} dW_s + \int_0^t \frac{\mu_s^2}{2\sigma_s^2} ds}, \quad 0 \leq t \leq T$$

and we then have

$$\mathbb{E} \left[\log(\hat{V}_T) \right] = \int_0^T \frac{\mu_t^2}{2\sigma_t^2} dt$$

and the ratio $\hat{\pi}_t$ **is maximal** in the sense that every competing strategy will result in a smaller or equal expected utility of liquidated wealth.

So we see that trading optimally in the frictionless market means that at every time point the trading agent wants the relative stock holdings to be exactly at the fraction $\frac{\mu_t}{\sigma_t^2}$. It is clear as the stock price is a stochastic process and changes continuously in time that he also has to arrange the portfolio

continuously, but maybe always by very small transactions just to be at the perfect fraction between stock and total wealth. This leads the agent to do small but possibly infinite transactions. Thinking of our original problem facing transaction costs it is clear that this is not possible as the agent would face infinite costs due to his trading. The beautiful way out is that there exists a region around the optimal fraction of the frictionless case (first shown by [MagCon76]). The agent stays with his portfolio as long as his fraction (always evaluated at the current stock price) stays in this region. When the stock price finally moves as far from its former value such that the agents portfolio now is at the lower or higher boundary of the region around the optimal fraction and is about to move outside, he rebalances the portfolio just enough to stay in the region. This region will be called no-trade region.

2.4 The Shadow Price-Process

To get comparable conclusions like above in the case of present transaction costs we connect the following *shadow price-process* to the stock price-process, which is fundamental for the process of finding the optimal strategy. The papers to mention here extraordinarily are [KalMuh10, GerMuhSch11, GerMuhSch12].

First we have to agree upon what the shadow price actually is. The following definition might first be a bit irritating for the reader, but let us just accept it as it is, because this way of defining the shadow price is essential for generating the frictionless market we are searching for.

Definition 5. *The shadow price-process is a continuous semimartingale $(\tilde{S}_t)_{0 \leq t \leq T}$ with $\tilde{S}_0 = S_0$, taking its values in the bid-ask spread $\left([(1-\lambda)S_t, S_t]\right)_{0 \leq t \leq T}$ such that there is an optimal trading strategy $(\hat{\varphi}_t^0, \hat{\varphi}_t^1)_{0 \leq t \leq T}$ for the market without transaction costs, but stock-price $\tilde{S}$, which only trades when $\tilde{S}_t = S_t$ or $\tilde{S}_t = (1-\lambda)S_t$.*

Optimality here meaning in the sense of maximizing (2.3). As the reader might have noticed it is not necessarily true that a process fulfilling the above conditions has to actually exist. In fact it is hard work to show its existence

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and involves a lot of complex and abstract theory. We here take existence as granted in our setting and go on to the crucial dependence of the shadow price market on the original stock price market.

As the primal way to attack the problem essentially is to directly try to solve it. We are now following the dual way, which goes a devious route. We search for a shadow price process $(\tilde{S}_t)_{0 \leq t \leq T}$ resp. a function deducing this process from the original stock process $(S_t)_{0 \leq t \leq T}$. By the following theorem, which is proved with lots of insight of duality theory (see [DelSch06]) we know that the optimal terminal value and also the optimal strategy of the frictionless market with price process $(\tilde{S}_t)_{0 \leq t \leq T}$, which we know from Theorem 1, is the same as the one of the original market with $(S_t)_{0 \leq t \leq T}$. We are not going into the details here of how this is working, because this is a quite delicate topic. We refer to [DelSch06] for the details. But of course we want to use this fundamental result for our setting.

Theorem 2. *The unique optimizer of the expected utility optimization problem of (2.3) with price process $(S_t)_{0 \leq t \leq T}$ and transaction costs $\lambda > 0$ is the same optimizer as of the expected utility optimization problem with price process $(\tilde{S}_t)_{0 \leq t \leq T}$ and without transaction costs for the corresponding function g calculated in this section, connecting the two processes through $\tilde{S}_t = g(S_t)$.*

2.5 Assumptions

In the papers mentioned in the beginning of this section it is verified that the following assumptions are eligible and are leading to the desired solutions.

Assumption 1. *When the shadow price-process $(\tilde{S}_t)_{0 \leq t \leq T}$ ranges in the interior $] (1 - \lambda)S_t, S_t[$ of the bid-ask interval $[(1 - \lambda)S_t, S_t]$ the process $\tilde{S}_t$ is a deterministic smooth function of S_t :*

$$\tilde{S}_t = g(S_t).$$

More precisely we suppose that there is a family of (deterministic, smooth) functions $g_i(\cdot)$, depending on the parameter i , such that, whenever we have

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random times $\varrho \leq \tau$ such that $\tilde{S}_t \in](1 - \lambda)S_t, S_t[$, for all $t \in]\varrho, \tau[$, then there is a fixed parameter i (depending on the random interval $] \varrho, \tau[$) such that

$$\tilde{S}_t = g_i(S_t) \quad \varrho \leq t \leq \tau. \quad (2.5)$$

The goal clearly is to determine the function g_i . The parameter i is fixed while the shadow-price ranges in the interior of the random interval. Only when $\tilde{S}_t$ reaches the bounds $(1 - \lambda)S_t$ or S_t we allow the parameter i to vary.

Assumption 2. *An agent, who can invest in a frictionless way in the market with stock-price $\tilde{S}_t$ does not want to change his position in bond and stock as long as $\tilde{S}_t$ ranges in the interior $] (1 - \lambda)S_t, S_t[$ of the bid-ask interval (therefore also called the no-trade region abbrev. *NTR*).*

Note that this is only here an assumption, but from a different point of view this is deducable (compare [MagCon76]).

In [GerMuhSch11, Prop. 2.1] it is shown that the shadow price is representable as:

$$\frac{d\tilde{S}_t}{\tilde{S}_t} = \tilde{\mu}_t dt + \tilde{\sigma}_t dW_t, \quad (2.6)$$

where $(\tilde{\mu}_t)_{0 \leq t \leq T}$ and $(\tilde{\sigma}_t)_{0 \leq t \leq T}$ now are general predictable processes, which will be determined in the following calculation.

2.6 Derivation of the g -function

As we assumed the smoothness of g_i we can apply Itô to (2.5) (dropping the subscript i):

$$g(S_t) = g(S_0) + \int_0^t g'(S_u) dS_u + \int_0^t \frac{g''(S_u)}{2} d\langle S, S \rangle_u$$

resp. in differential notation

$$d\tilde{S}_t = dg(S_t) = g'(S_t) dS_t + \frac{g''(S_t)}{2} d\langle S, S \rangle_t.$$

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Inserting the differential form of Black-Merton-Scholes (2.1) for the stock price with constant drift and diffusion, using the quadratic variation of this stock $d\langle S, S \rangle_t = \sigma^2 S_t^2 dt$ and rearranging we get

$$\frac{dg(S_t)}{g(S_t)} = \left(\frac{\mu S_t g'(S_t) + \frac{\sigma^2}{2} S_t^2 g''(S_t)}{g(S_t)} \right) dt + \left(\frac{\sigma S_t g'(S_t)}{g(S_t)} \right) dW_t$$

and therefore

$$\tilde{\mu}_t = \frac{\mu S_t g'(S_t) + \frac{\sigma^2}{2} S_t^2 g''(S_t)}{g(S_t)} \quad \text{and} \quad \tilde{\sigma}_t = \frac{\sigma S_t g'(S_t)}{g(S_t)}. \quad (2.7)$$

So the mean-variance ratio of the shadow price-process is

$$\frac{\tilde{\mu}_t}{\tilde{\sigma}_t^2} = \frac{\mu S_t g'(S_t) + \frac{\sigma^2}{2} S_t^2 g''(S_t)}{\sigma^2 S_t^2 (g'(S_t))^2}.$$

Consider now the fraction $\tilde{\pi}_t$ of money invested in the stock over the total money invested at time t , if we evaluate the stock at the shadow price $\tilde{S}$ (recall φ_t^0, φ_t^1 the holdings in bond and stock at time t)

$$\tilde{\pi}_t = \frac{\varphi_t^1 \tilde{S}_t}{\varphi_t^0 + \varphi_t^1 \tilde{S}_t}. \quad (2.8)$$

From Assumption 2 we follow that φ_t^0, φ_t^1 and therefore also the fraction

$$\frac{\varphi_t^0}{\varphi_t^1} =: c$$

are constant as long as we are in the NTR, therefore meant for t being in some interval $]t_0, t_1[$. Then (2.8) can be reformulated as

$$\tilde{\pi}_t = \frac{g(S_t)}{c + g(S_t)}.$$

The crucial argument now is coming from Merton's Rule (2.4), that to trade optimally in the frictionless market (like the shadow price-market) the ratio of money invested $\tilde{\pi}$ must equal the mean-variance ratio $\frac{\tilde{\mu}_t}{\tilde{\sigma}_t^2}$. So we have

$$\frac{g(S_t)}{c + g(S_t)} = \frac{\mu S_t g'(S_t) + \frac{\sigma^2}{2} S_t^2 g''(S_t)}{\sigma^2 (S_t)^2 (g'(S_t))^2}$$

for a fixed i depending on the stock-price S_t through its current bid-ask spread. Rearranging and substituting S_t by the variable $s \in \mathbb{R}_+$ we get the ODE

$$g''(s) = \frac{2g'(s)^2}{c + g(s)} - \frac{\mu}{\sigma^2} \frac{2g'(s)}{s}, \quad s > 0, \quad (2.9)$$

which admits a closed form solution. We choose the boundary conditions at $s = 1$ as

$$g(1) = 1, \quad g'(1) = 1 \quad (2.10)$$

because of the following heuristic argumentation ([GerMuhSch11]):

At the beginning of the time interval $[t_0, t_1]$ the shadow price is at the boundary of the NTR. We can w.l.o.g. assume that this is the upper bound where $\tilde{S}_{t_0} = S_{t_0}^1$, which is the same as $g(1) = 1$ with the normalization of the stock to 1. This means that the agent just bought stock. Additionally we assume that the stock price has a positive excursion at time t_0 and we match the shadow price in first order to this: $g'(1) = 1$. This is also known as smooth pasting.

We introduce the constant $\theta := \frac{\mu}{\sigma^2}$, because (2.9) only depends on this mean-variance ratio of drift and diffusion constants of the Black-Merton-Scholes model. The general form of the solution for g then is

$$g(s) = g_{c,\theta}(s) = \begin{cases} \frac{-cs + (2\theta - 1 + 2c\theta)s^{2\theta}}{s - (2 - 2\theta - c(2\theta - 1))s^{2\theta}}, & \theta \in \mathbb{R}_+ \setminus \{\frac{1}{2}\} \\ \frac{c+1+c\log(s)}{c+1-\log(s)}, & \theta = \frac{1}{2} \end{cases} \quad (2.11)$$

with parameter c for the asset holdings still free. Note that for the moment we are so far, that if one knows the optimal strategy, the shadow price is directly determined by the stock price, but as we will see, we can turn this around later.

2.7 Behavior of g and parameter values

Concerning the behavior of the agent with his portfolio selection we distinguish the regimes $\theta \in]0, 1[$, $\theta = 1$ and $\theta > 1$.

The case $\theta = \frac{1}{2}$ is only different in its mathematical expression in logarithms rather than powers, but perfectly interpolates when $\theta \rightarrow \frac{1}{2}$.

The case $\theta = 1$ is degenerate. In the frictionless market by (2.4) we know

that the optimal fraction of wealth is $\equiv 1$, which economically means at time zero the agent fully invests the initial endowment x into stock resp. $\hat{\varphi}_t^0 \equiv 0$ and $\hat{\varphi}_t^1 \equiv x$ for $t > 0$ (and S_0 normalized to 0) and never trades again until liquidating at time T . In the presence of transaction costs, from an economic point of view, it is now rather obvious that this strategy still is optimal and the agent is not willing to trade at any point and pay for it. In fact by defining the shadow price process as $\tilde{S}_t = S_t$ the above strategy is the optimal one for our problem (2.3). For an explicit proof of this statement and also the handling of the case $\theta > 1$ (which involves a short position in bond and some modifications) we refer to [GerMuhSch11].

We now make the following standing assumption, which ensures that the holdings in bond and stock remain positive at all times.

Assumption 3.

$$0 < \theta = \frac{\mu}{\sigma^2} < 1. \quad (2.12)$$

One may discuss the function $g_{c,\theta}(s)$ and find out that it is concave in a neighborhood of 1, strictly increasing from 1 up to a positive pole, having a unique reflection point between 1 and the pole and then eventually is convex. We are still free to complement the boundary conditions (2.10) of the ODE (2.9) at the other endpoint, the “selling boundary” $\bar{s}$, where $\tilde{S}_t = (1 - \lambda)S_t^1$. Meaning that we are in search of $\bar{s}(c) > 1$ and a $0 < \lambda < 1$ (we will turn this logic around later, for now we are in search of a unknown λ) fulfilling the conditions

$$g_{c,\theta}(\bar{s}) = (1 - \lambda)\bar{s}, \quad g'_{c,\theta}(\bar{s}) = (1 - \lambda). \quad (2.13)$$

This is qualitatively illustrated in Figure 2.1 on page 21, where the blue line describes the unique line through the origin touching $g_{c,\theta}(s)$ at $\bar{s}$ and because of (2.13) its slope is $(1 - \lambda)$.

Summarizing we recall that we translated the NTR from the shadow price-process of being in the bid-ask spread to the stock price s staying in $]1, \bar{s}[$ and in this regime deterministically determining the value of $\tilde{S}_t$ by $g_{c,\theta}(s)$. We are now able to find expressions for $\bar{s}$ and λ in terms of the parameter c . For the calculations of $\bar{s}$, $g(\bar{s})$, λ and the valid regimes of c we refer to [GerMuhSch11]

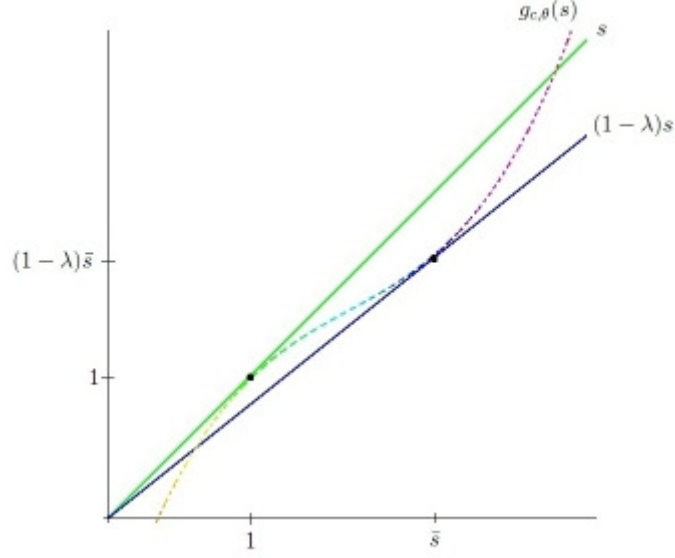


Figure 2.1: Qualitative behavior of $g_{c,\theta}(s)$ with smooth pasting conditions.

and state here the results important for us, which are

$$\bar{s} = \bar{s}_\theta(c) = \begin{cases} \left(\frac{c}{(2\theta-1+2c\theta)(2-2\theta-c(2\theta-1))} \right)^{\frac{1}{2\theta-1}}, & \theta \in]0, 1[\setminus \{\frac{1}{2}\} \\ \exp\left(\frac{c^2-1}{c}\right), & \theta = \frac{1}{2} \end{cases} \quad (2.14)$$

and

$$\lambda = \lambda_\theta(c) = \begin{cases} \frac{(1-2(c+1)\theta) \bar{s}(c)^{2\theta} + c\bar{s}(c)}{\bar{s}(c) ((2(c+1)\theta - c - 2) \bar{s}(c)^{2\theta} + \bar{s}(c))}, & \theta \in]0, 1[\setminus \{\frac{1}{2}\} \\ 1 - c^2 \exp\left(\frac{1-c^2}{c}\right), & \theta = \frac{1}{2} \end{cases}. \quad (2.15)$$

We point out that we did all the argumentation here in a fixed NTR and did not worry much about that this region changes (and therefore the parameter i from (2.5) changes) whenever the shadow price is forced to the boundary of the NTR, trading takes place and therefore the positions in the assets and c changes. For our purposes of understanding the problem and its solution heuristically we will neglect to construct the shadow price-process as a full stochastic process in all technical details, where all the tradings affect the process at stochastic time points, but rather give the results in easy understandable, but not rigorous formulations. The deeply interested reader is

invited to the rigorous constructions, proofs and existence considerations in [KalMuh10, GerMuhSch11, CziMuhSch12].

2.8 Statements

We will stay here with easy formulations and are satisfied with knowing that what we constructed actually goes in line with the problem and leads to the following propositions.

Proposition 1 (Compare [GerMuhSch11, Lemma 4.3]). *For $s \in [1, \bar{s}]$ and $\bar{s}$ and λ as in (2.14) and (2.15) and the standing Assumption 3, it holds that $g_{c,\theta}(s)$ of (2.11) is strictly increasing, takes its values in $[1, (1-\lambda)\bar{s}]$ and solves the ODE (2.9) satisfying both boundary conditions (2.10) and (2.13).*

This provides us the safety that the function g is actually what we expected it to be and determines the shadow price process. Furthermore it solves our problem telling us the optimal behavior.

Proposition 2 (Compare [GerMuhSch11, Theorem 5.1]). *For the same assumptions as in the above proposition, it holds that the process $(\hat{\varphi}_t^0, \hat{\varphi}_t^1)_{0 \leq t \leq T}$ of holdings in bond and stock (with $(\hat{\varphi}_{0-}^0, \hat{\varphi}_{0-}^1) = (x, 0)$) optimizing*

$$\mathbb{E} \left[\log(\hat{\varphi}^0 + \hat{\varphi}^1 \cdot \tilde{S})_T \right] \rightarrow \max!$$

(trading in the frictionless market with price process $\tilde{S}$) can be explicitly uniquely determined.

It starts at initial endowment $(\hat{\varphi}_{0-}^0, \hat{\varphi}_{0-}^1) = (x, 0)$, then immediately trades to be at $(\hat{\varphi}_0^0, \hat{\varphi}_0^1) = (\frac{c}{c+1}x, \frac{1}{c+1}x)$ and furthermore only trades whenever the shadow price equals the bid or the ask price of the stock, in fixed proportions of the current holdings explicitly calculated in [GerMuhSch11, Theorem 5.1].

The corresponding optimal fraction of wealth invested into the stock is given by

$$\tilde{\pi} = \frac{\hat{\varphi}_t^1 \tilde{S}_t}{\hat{\varphi}_t^0 + \hat{\varphi}_t^1 \tilde{S}_t} = \frac{1}{1 + c/g_{c,\theta}(s)}$$

and oscillates between $\frac{1}{1+c}$ and $\frac{1}{1+c/(1-\lambda)\bar{s}}$ as g oscillates between 1 and $(1-\lambda)\bar{s}$.

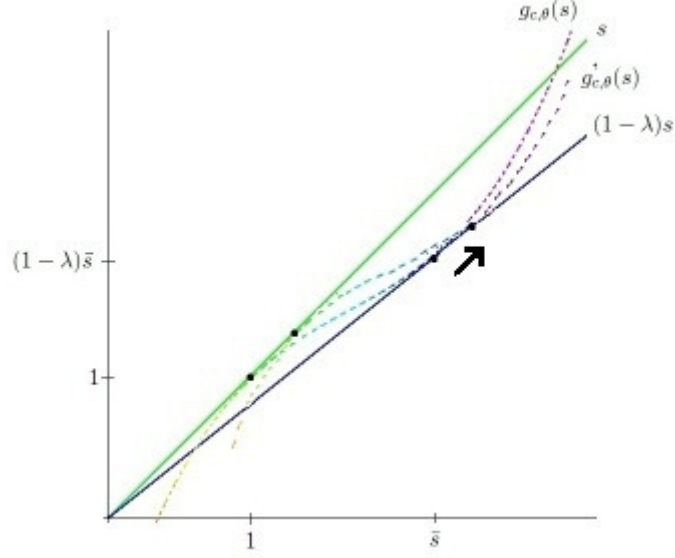


Figure 2.2: Adapted $g_{c,\theta}(s)$ after rebalancing a threatened drop out of the no-trade region.

The optimal behavior for the agent now is as we indicated in section 2.3. From the initial holdings just in bonds he buys enough stock to be in the middle of the no-trade region for the current stock price. If we look at Figure 2.1 on page 21 this means that his g function lies between the two black dots (i.e. in $]1, (1-\lambda)\bar{s}[$). Let's now assume the stock price changes such that the value of g rises up to the boundary value $(1-\lambda)\bar{s}$. The agent now trades just to stay in the no-trade region. If the stock price turns at the boundary and decreases the picture doesn't change and the trader waits for the next touching of the boundary. On the other hand if the stock price rises farther threatening to force the trader out of his region, he continuously trades to hold his fraction inside the boundaries and with this the index i of the function g changes as described in section 2.5. The transformation of the function is actually a shifting movement up between the two lines s and $(1-\lambda)s$ always keeping two touching points. When the stock price finally changes its route and the trader doesn't need to trade any more to stay in the no-trade region this results in fixing the new graph of g which has now higher touching points but mainly a similar curvature in between. This is illustrated in Figure 2.2 on page 23

2.9 Asymptotic Expansions

So now we know for a given c the optimal fraction of wealth to hold and also the optimal strategy to maintain at this fraction. The question however is the other way around. We are not given c and want to calculate λ and by this the optimal strategy, but naturally we know from the market how much the transactions cost, so we actually start with a given λ , which is the free variable parametrization the solution. So following our problem we need representations of c , $\bar{s}_\theta(c)$ and $g_{c,\theta}(s)$ in terms of λ . Unfortunately to achieve this we have to leave the pleasant situation of closed form solutions and have to invert (2.15) locally around $\lambda = 0$ in fractional Taylor series. As this series only converges in a neighborhood of $\lambda = 0$, we only have a result in the context of “ $\lambda > 0$ being sufficiently small”. We will state the result and refer to [GerMuhSch11] for the calculations.

Proposition 3 (Compare [GerMuhSch11, Prop. 6.1]). *For fixed $0 < \theta < 1$ and sufficiently small $\lambda > 0$ the functions $c(\lambda)$ and $\bar{s}(\lambda)$ have the following converging fractional Taylor series representations*

$$c(\lambda) = \frac{1-\theta}{\theta} + \frac{1-\theta}{2\theta} \left(\frac{6}{\theta(1-\theta)} \right)^{\frac{1}{3}} \lambda^{\frac{1}{3}} + \frac{(1-\theta)^2}{4\theta} \left(\frac{6}{\theta(1-\theta)} \right)^{\frac{2}{3}} \lambda^{\frac{2}{3}} + O(\lambda),$$

$$\bar{s}(\lambda) = 1 + \left(\frac{6}{\theta(1-\theta)} \right)^{\frac{1}{3}} \lambda^{\frac{1}{3}} + \frac{1}{2} \left(\frac{6}{\theta(1-\theta)} \right)^{\frac{2}{3}} \lambda^{\frac{2}{3}} + O(\lambda)$$

with the function $g_{c,\theta}(s)$ from (2.11) defined on $[1, \bar{s}(\lambda)]$ then satisfying the ODE (2.9), the lower boundary condition (2.10) and the upper boundary condition (2.13) which spells in λ -dependence as:

$$g(\bar{s}(\lambda)) = (1-\lambda)\bar{s}(\lambda), \quad g'(\bar{s}(\lambda)) = (1-\lambda). \quad (2.16)$$

Combining these representations and the boundaries $\frac{1}{1+c}$ and $\frac{1}{1+c/(1-\lambda)\bar{s}}$ it is also possible to asymptotically determine the boundaries of the NTR in terms of λ . The expansions are stated in [GerMuhSch11, Cor. 6.2]. Moreover it is possible to calculate the growth rate of the optimal portfolio.

2.10 Optimal Growth Rate

The optimal growth rate for infinite horizon for the bid-ask process is

$$\delta = \limsup_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\log(1 + \hat{\varphi}_T^0 + \hat{\varphi}_T^1 \tilde{S}_T) \right]$$

with initial endowment normalized w.l.o.g. to $x = 1$ and $(\hat{\varphi}_T^0, \hat{\varphi}_T^1)$ denoting the log-optimal portfolio for the shadow price $\tilde{S}$ from proposition 2 resp. [GerMuhSch11, Theorem 5.1].

As we mentioned before we want to avoid to go into the details of the construction of the shadow price as an entire process and its existence, but unfortunately here we have to go over its technical definition shortly to be able to use results in the correct way. The following is in the same manner as in [GerMuhSch11, Section 2].

We define the running minimum process of S as

$$m_t = \inf_{\varrho_0 \leq u \leq t} S_u, \quad 0 \leq t \leq \eta_1,$$

where $\varrho_0 = 0$ and the stopping time η_1 is defined as

$$\eta_1 = \inf \{t \geq \varrho_0 : \frac{S_t}{m_t} \geq \bar{s}\}.$$

Next define the running maximum process of S after time η_1 as

$$M_t = \sup_{\eta_1 \leq u \leq t} S_u, \quad \eta_1 \leq t \leq \varrho_1,$$

where the stopping time ϱ_1 is defined as

$$\varrho_1 = \inf \{t \geq \eta_1 : \frac{S_t}{M_t} \leq \frac{1}{\bar{s}}\}.$$

We continue with these definitions for $t \geq \varrho_1$, $t \geq \eta_2$ and so on.

This builds sequences of a.s. finite stopping times $(\varrho_n)_{n=0}^\infty$, $(\eta_n)_{n=1}^\infty$ increasing a.s. to infinity, such that the processes $(m_t)_{t \geq 0}$ (resp. $(M_t)_{t \geq 0}$) are the relative running minima (resp. maxima) of S defined on the stochastic intervals $([\varrho_n, \eta_{n+1}])_{n=0}^\infty$ (resp. $([\eta_n, \varrho_n])_{n=1}^\infty$).

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Note that

$$\bar{s}m_{\varrho_n} = M_{\varrho_n} = \bar{s}S_{\varrho_n}, \quad \bar{s}m_{\eta_n} = M_{\eta_n} = S_{\eta_n} \quad \text{for } n \in \mathbb{N}.$$

We may therefore continuously extend the processes m and M to $\mathbb{R}_+$ by letting

$$\begin{aligned} M_t &:= \bar{s}m_t, & \text{for } t \in \bigcup_{n=0}^{\infty} [[\varrho_n, \eta_{n+1}]], \\ m_t &:= \frac{M_t}{\bar{s}}, & \text{for } t \in \bigcup_{n=1}^{\infty} [[\eta_n, \varrho_n]]. \end{aligned}$$

We then have $\bar{s}m_t = M_t$ for $t \geq 0$ as well as $m_t \leq S_t \leq M_t$ and hence

$$m_t \leq S_t \leq \bar{s}m_t, \quad \text{for } t \geq 0.$$

By construction m and M are of finite variation and only decrease (resp. increase) on the predictable set $\{m_t = S_t\}$ (resp. $\{M_t = S_t\} = \{m_t = \frac{S_t}{\bar{s}}\}$).

We thus have that the process

$$X_t = \frac{S_t}{m_t} = \frac{\bar{s}S_t}{M_t}$$

takes values in $[1, \bar{s}]$, is reflected at the boundaries and also admits a stochastic differential representation. This is crucial for the main result of [GerMuhSch11, Section 2], that the actual shadow price process $\tilde{S}_t = m_t g(\frac{S_t}{m_t})$ admits a stochastic differential representation:

Proposition 4 (Compare [GerMuhSch11, Proposition 2.1]). *For fixed $\theta = \frac{\mu}{\sigma^2} \in]0, 1[$ and $S_0 = 1$ in the Black-Scholes model (2.1) and $c(\lambda)$, $\bar{s}(\lambda)$ from proposition 3 and $g_{c,\theta}(s)$ from (2.11) define the continuous process*

$$\tilde{S}_t = m_t g\left(\frac{S_t}{m_t}\right), \quad t \geq 0.$$

Then $\tilde{S}$ is an Itô process (a stochastic process admitting a stochastic differential representation) starting at $\tilde{S}_0 = S_0 = 1$ and satisfying

$$d\tilde{S}_t = g'\left(\frac{S_t}{m_t}\right)dS_t + \frac{1}{2m_t}g''\left(\frac{S_t}{m_t}\right)d\langle S, S \rangle_t.$$

Moreover $\tilde{S}$ takes values in the bid-ask spread $[(1 - \lambda)S_t, S_t]$.

The process $\frac{S_t}{m_t}$ actually is a “doubly reflected geometric Brownian motion with drift on the interval $[1, \bar{s}]$ ”³ and therefore by an ergodic theorem for positively recurrent one-dimensional diffusions (cf. e.g. [BorSal02, Sections II.36, II.37]) one gets an explicit representation of its distribution $\nu(ds)$ (see [GerMuhSch11, Prop. 5.4])

$$\nu(ds) = \begin{cases} \frac{2\theta-1}{\bar{s}^{2\theta-1}-1} s^{2\theta-2} \mathbf{1}_{[1, \bar{s}]}(s) ds, & \theta \in]0, 1[\setminus \{\frac{1}{2}\} \\ \frac{1}{\log(\bar{s})} s^{-1} \mathbf{1}_{[1, \bar{s}]}(s) ds, & \theta = \frac{1}{2} \end{cases}.$$

The optimal growth rate for infinite horizon for the bid-ask process actually is

$$\delta = \limsup_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\log(1 + \hat{\varphi}_T^0 + \hat{\varphi}_T^1 \tilde{S}_T) \right] = \limsup_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T \frac{\tilde{\mu}^2(\frac{S_t}{m_t})}{2\tilde{\sigma}^2(\frac{S_t}{m_t})} dt \right] \quad (2.17)$$

with $\tilde{\mu}$ and $\tilde{\sigma}$ from (2.7) and the equality following from [KarLehShrXu91, Ex. 6.4: ”solution to the log-optimal portfolio problem for Itô processes with bounded drift and diffusion coefficients”].

Using this and the stationary distribution $\nu(ds)$ of $\frac{S_t}{m_t}$ it is possible to calculate

$$\delta = \int_1^{\bar{s}} \frac{\tilde{\mu}_t^2(s)}{2\tilde{\sigma}_t^2(s)} \nu(ds) = \begin{cases} \frac{(2\theta-1)\sigma^2\bar{s}}{2(1+c)(\bar{s}+(-2-c+2\theta(1+c))\bar{s}^{2\theta})}, & \theta \in]0, 1[\setminus \{\frac{1}{2}\} \\ \frac{\sigma^2}{2(1+c)(1+c-\log(\bar{s}))}, & \theta = \frac{1}{2} \end{cases} \quad (2.18)$$

and by proposition 3 as $\lambda \rightarrow 0$ the optimal growth rate has the asymptotics

$$\delta = \frac{\mu^2}{2\sigma^2} - \left(\frac{3\sigma^3}{\sqrt{128}} \theta^2 (1 - \theta)^2 \right)^{\frac{2}{3}} \lambda^{\frac{2}{3}} + O(\lambda^{\frac{4}{3}})$$

as seen in [GerMuhSch11, Prop. 6.3].

³[GerMuhSch11, p. 22]

3

As we have seen how to attack the problem using the “martingale method” and come to an explicit solution in terms of asymptotic expansions of the transaction costs λ in proposition 3, we now look at the classical approach with longer history, which is also called the primal approach. In this section we develop the classical theory of [DumLuc91] and [TakKlaAss88] with notation and knowledge of this work here.

The aim is that where the primal approach stops and wants to solve a free boundary differential equation we want to go on with knowledge of the dual solution and to bring the primal approach to an explicit end.

3.1 Value Functions

For our purposes we have to adopt a little bit notation and understanding in contrast to the primal papers. What we definitely agree on is that we are trying to explicitly calculate the value function, which is the function giving the maximal wealth possible of our problem. We will drag this function through a few modifications, but let’s start with the following definition.

For a fixed level $\lambda > 0$ of (sufficiently small) transaction costs, a fixed $\gamma < 1$ for the utility function, a fixed horizon T and an initial endowment $(\varphi_{0-}^0, \varphi_{0-}^1) \in \mathbb{R}_+^2$ in bond and stock¹ we define the *terminal value function*

$$u(\varphi^0, \varphi^1, s, T) = \sup_{(\varphi_T^0, \varphi_T^1) \in \mathcal{A}_*^\lambda} \left\{ \mathbb{E} \left[U(\varphi_T^0 + \varphi_T^1 S_T) \middle| S_0 = s \right] \right\},$$

where the maximization area $\mathcal{A}_*^\lambda$ is meant to be $\mathcal{A}^\lambda$, but not reachable just from 0 initial endowment, but from $(\varphi_{0-}^0, \varphi_{0-}^1) \in \mathbb{R}_+^2$.

¹in [DumLuc91] and [TakKlaAss88] assumptions are also made translating to $\theta = \frac{\mu}{\sigma^2} \in]0, 1[$, which guarantees that $\varphi^0 \geq 0, \varphi^1 \geq 0$ and no short-selling is allowed (see also (2.12)).

3.2. SIMPLIFICATIONS

The expression $\varphi_T^0 + \varphi_T^1 S_T$ inside the expectation above corresponds to the assumption that we can liquidate the portfolio at time T for the price S_T per stock. To justify this as the liquidation price instead of $(1 - \lambda)S_T$ we mention that when we pass to the (properly scaled) limit $T \rightarrow \infty$, there will be no difference.

For $t \in [0, T]$ we now define the *intermediate value function* as

$$u(\varphi^0, \varphi^1, s, t, T) = \sup_{(\varphi_T^0, \varphi_T^1) \in \mathcal{A}_{*t}^\lambda} \left\{ \mathbb{E} \left[U(\varphi_T^0 + \varphi_T^1 S_T) \middle| S_t = s \right] \right\}, \quad (3.1)$$

where now the maximization area $\mathcal{A}_{*t}^\lambda$ consist of holdings obtainable by trading during the period $[t, T]$, starting at time t_- with holdings $(\varphi_{t_-}^0, \varphi_{t_-}^1) = (\varphi^0, \varphi^1)$.

We want to take the limit $T \rightarrow \infty$ in 3.1 to get a *value function* $u(\varphi^0, \varphi^1, s, t)$ which is now independent of the time horizon. If we do this blindly, we will typically find $u(\varphi^0, \varphi^1, s, t) \equiv \infty$ of course because it is describing the maximal wealth possible, which gives us no information. The authors of [DumLuc91] and [TakKlaAss88] found that the optimal growth rate is exponential with the horizon and that the proper scaling, we were talking about before, to get a finite limit, is to discount the value of the portfolio $\varphi_T^0 + \varphi_T^1 S_T$ continuously with a constant $\delta > 0$, so by the factor $e^{-\delta T}$.

From the dual approach we know this optimal growth rate δ ((2.18)), but we put this aside for the moment and behave as if this is still a free parameter to be determined.

So finally the *discounted value function* $u(\varphi^0, \varphi^1, s, t)$ is defined for starting at $t > 0$ and infinite trading horizon as

$$u^\infty(\varphi^0, \varphi^1, s, t) = \lim_{T \rightarrow \infty} \sup_{(\varphi_T^0, \varphi_T^1) \in \mathcal{A}_{*t}^\lambda} \left\{ \mathbb{E} \left[U(e^{-\delta T}(\varphi_T^0 + \varphi_T^1 S_T)) \middle| S_t = s \right] \right\}$$

3.2 Simplifications

As the following simplifications works differently for log- and power-utility it is once more mentioned that we work here in the logarithmic utility setting, but refer for the following derivations and the solution for the discounted value function in the power-utility-case to the Appendix B.

We will first find a connection between the discounted value function at different times, then use a property and finally make a useful observation to

3.2. SIMPLIFICATIONS

get to a much easier representation.

It holds that

$$\begin{aligned}
u^\infty(\varphi^0, \varphi^1, s, t) &= \lim_{T \rightarrow \infty} \sup_{(\varphi_T^0, \varphi_T^1) \in \mathcal{A}_{*t}^\lambda} \left\{ \mathbb{E} \left[U(e^{-\delta T}(\varphi_T^0 + \varphi_T^1 S_T)) \middle| S_t = s \right] \right\} \\
&= \lim_{T \rightarrow \infty} \sup_{(\varphi_T^0, \varphi_T^1) \in \mathcal{A}_{*t}^\lambda} \left\{ \mathbb{E} \left[\log(\varphi_T^0 + \varphi_T^1 S_T) \middle| S_t = s \right] \right\} - \delta T \\
&= \lim_{T \rightarrow \infty} \sup_{(\varphi_T^0, \varphi_T^1) \in \mathcal{A}_{*t}^\lambda} \left\{ \mathbb{E} \left[\log(\varphi_{T-t}^0 + \varphi_{T-t}^1 S_{T-t}) \middle| S_0 = s \right] \right\} - \delta(T-t) - \delta t \\
&= u^\infty(\varphi^0, \varphi^1, s, 0) - \delta t
\end{aligned}$$

where we substituted $T \rightarrow T - t$ in the conditional expectation in the third line and added $\delta t - \delta t = 0$.

PROPERTY: For the discounted value function u^∞ the following holds by the scaling property of the logarithm ($\log(ax) = \log(a) + \log(x)$) for $t \geq 0$:

$$\begin{aligned}
u^\infty(a \cdot \varphi^0, a \cdot \varphi^1, s, t) &= \lim_{T \rightarrow \infty} \sup_{(\varphi_T^0, \varphi_T^1) \in \mathcal{A}_{*t}^\lambda} \left\{ \mathbb{E} \left[\log(a \cdot \varphi_T^0 + a \cdot \varphi_T^1 S_T) \middle| S_t = s \right] \right\} - \delta T \\
&= \lim_{T \rightarrow \infty} \sup_{(\varphi_T^0, \varphi_T^1) \in \mathcal{A}_{*t}^\lambda} \left\{ \mathbb{E} \left[\log(\varphi_{T-t}^0 + \varphi_{T-t}^1 S_{T-t}) \middle| S_0 = s \right] \right\} - \delta T + \log(a) \\
&= u^\infty(\varphi^0, \varphi^1, s, t) + \log(a)
\end{aligned}$$

Or the way we want to use it is to normalize the bond-part of the portfolio and represent the stock-part as a fraction of the holdings in the two assets:

$$u^\infty(\varphi^0, \varphi^1, s, t) = u^\infty(\varphi^0, \varphi^1, s, 0) - \delta t = u^\infty(1, \frac{\varphi^1}{\varphi^0}, s, 0) - \delta t + \log(\varphi^0).$$

OBSERVATION: Note that the variables φ^1 (quantity of stock) and s (value of stock) only enter in the discounted value function u^∞ as the product $\varphi^1 s$, because we are always interested in the actual value of the stock-holdings, not just the quantity. Note also that the final u^∞ we are calculating will be independent of t . So we may introduce a function $h : \mathbb{R}_+ \rightarrow \mathbb{R}$ to simplify the representation of this value function and rewrite

3.3. THE ODE FOR H AND DISCOUNTED VALUE FUNCTION REPRESENTATION

$$u^\infty(\varphi^0, \varphi^1, s, t) = h\left(\frac{\varphi^1 s}{\varphi^0}\right) - \delta t + \log(\varphi^0),$$

where $h(\frac{\varphi^1 s}{\varphi^0}) = u^\infty(1, \frac{\varphi^1}{\varphi^0}, s, 0)$ is independent of t .

Introducing a new variable $y = \frac{\varphi^1 s}{\varphi^0}$ we have

$$u^\infty(\varphi^0, \varphi^1, s, t) = h(y) - \delta t + \log(\varphi^0).$$

3.3 The ODE for h and discounted Value Function Representation

We will now derive the modified Hamilton-Jacobi-Bellman equation from [DumLuc91] for logarithmic utility, but in a more martingale theory way.

As we already know from Assumption 2 in the no-trade region the optimal strategy is to hold $(\varphi_t^0, \varphi_t^1)$ constant (see [MagCon76] for proof). In this region we use Itô's formula (see e.g. [Pha09, 1.2.3]) in differential form and the Black-Merton-Scholes representation of the stock S_t (2.1) to derive

$$\begin{aligned} du^\infty(\varphi^0, \varphi^1, S_t, t) &= u_t^\infty dt + u_s^\infty dS_t + \frac{u_{ss}^\infty}{2} d\langle S, S \rangle_t \\ &= -\delta dt + h'(y) \frac{\varphi^1}{\varphi^0} (\mu S_t dt + \sigma S_t dW_t) + h''(y) \left(\frac{\varphi^1}{\varphi^0}\right)^2 \frac{\sigma^2 S_t^2}{2} dt \end{aligned}$$

Because W_t is a Brownian motion, we use the formal identity $\mathbb{E}[dW_t] = 0$ ([Bjo04, Prop. 4.4]) to take the expectation (also remember that $S_t = s$)

$$\mathbb{E}[du^\infty(\varphi_t^0, \varphi_t^1, S_t, t)] = \mathbb{E}\left[\left(-\delta + h'(y) \frac{\varphi^1}{\varphi^0} \mu S_t + h''(y) \left(\frac{\varphi^1}{\varphi^0}\right)^2 \frac{\sigma^2 S_t^2}{2}\right) dt\right].$$

In general stochastic optimization theory in [Pha09] it is shown that $(u^\infty(\varphi^0, \varphi^1, S_t, t))_{t \geq 0}$ is a super-martingale for any self-financing trading strategy and becomes a (local) martingale if we plug in the optimal strategy and in the no-trade region it is even constant. So in this region the term in the big round brackets must be constant and non-positive and zero for the optimal strategy. And therefore

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we expect the function h for the optimal strategy to satisfy the ODE

$$-\delta + h'(y)y\mu + h''(y)\frac{y^2\sigma^2}{2} = 0 \quad (3.2)$$

in the no-trade region i.e. $y \in NTR$.

This is an elementary ODE, which can be transferred into a second order linear ODE by the simple logarithmic transformation $z = \log(y)$ and has the solution

$$h(y) = \begin{cases} C_1 y^{2\theta-1} + C_2 + \frac{\delta}{\mu - \frac{\sigma^2}{2}} \log(y), & \theta = \frac{\mu}{\sigma^2} \in]0, 1[\setminus \{\frac{1}{2}\} \\ C_1 \log(y) + C_2 + \frac{\delta}{\sigma^2} \log(y)^2, & \theta = \frac{1}{2} \end{cases}$$

where C_1 and C_2 are still free constants due to open boundary values for h and $y \in NTR$ and we are only interested in $\theta \in]0, 1[$ (see (2.12)).

Now we plug in the result and get

$$u^\infty(\varphi^0, \varphi^1, s, t) = \begin{cases} C_1 y^{2\theta-1} + C_2 + \frac{\delta}{\mu - \frac{\sigma^2}{2}} \log(y) - \delta t + \log(\varphi^0), & \theta \in]0, 1[\setminus \{\frac{1}{2}\} \\ C_1 \log(y) + C_2 + \frac{\delta}{\sigma^2} \log(y)^2 - \delta t + \log(\varphi^0), & \theta = \frac{1}{2} \end{cases}$$

and for $t = 0$ and $C_2 = 0$ (as an additive constant doesn't matter for the discounted value function)

$$u^\infty(\varphi^0, \varphi^1, s) := u^\infty(\varphi^0, \varphi^1, s, 0) = \begin{cases} C_1 y^{2\theta-1} + \frac{\delta}{\mu - \frac{\sigma^2}{2}} \log(y) + \log(\varphi^0), & \theta \in]0, 1[\setminus \{\frac{1}{2}\} \\ C_1 \log(y) + \frac{\delta}{\sigma^2} \log(y)^2 + \log(\varphi^0), & \theta = \frac{1}{2} \end{cases} \quad (3.3)$$

for $y \in NTR$.

The further method in [DumLuc91] and [TakKlaAss88] is to handle this as a free boundary problem with 4 free parameters C_1, δ, l, u where $[l, u] := NTR$ (we already know 3 of them; δ by Section 2.10 and the explicit asymptotic expansions of the boundaries in [GerMuhSch11, Cor. 6.2]). They eventually are able to do some numerical calculations and interpret these. For these interpretations we want to refer to the papers. Definitely they were far away from achieving solutions as the authors of [GerMuhSch11] did in terms of fractional Taylor series in $\lambda^{\frac{1}{3}}$.

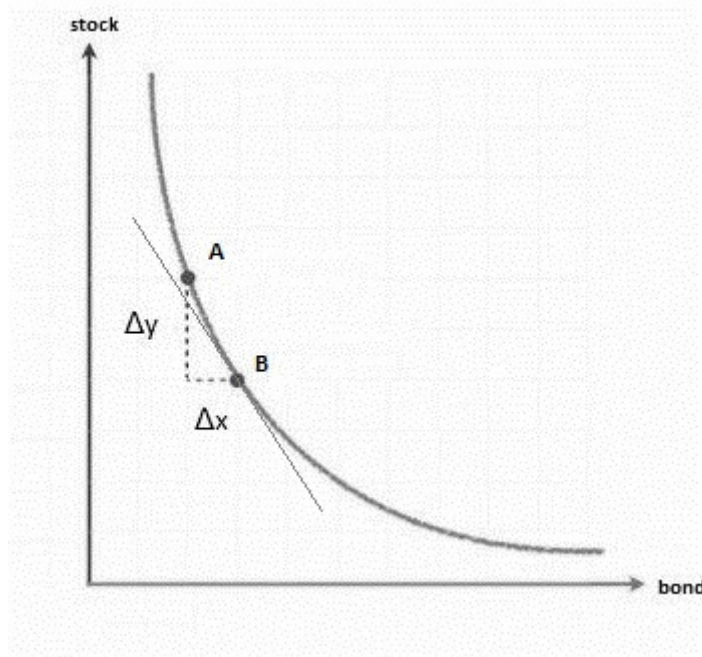


Figure 3.1: The discrete and continuous marginal rate of substitution.

3.4 Marginal Rate of Substitution

What we are doing here now is to solve the primal approach with the solution of the martingale method resp. the dual approach, in particular working on the ODE (2.9) for the shadow price process.

The way to link the two approaches is through the economic idea of the *marginal rate of substitution*. We try to give the reader a short understanding of this by Figure 3.1 on page 33. The axis describe holdings in bond and stock and the curve sketches the indifference curve where the utility $U(\varphi^0 + \varphi^1 S)$ for the agent stays constant independent of his portfolio on this curve. Lets now fix the two portfolios A and B and assume the agent is moving from A to B , then his discrete marginal rate of substitution is $MRS = \frac{\Delta x}{-\Delta y} = -\frac{\Delta \varphi^0}{\Delta \varphi^1}$. For the continuous case the slope of the tangent line in point A is $MRS = -\frac{d\varphi^0}{d\varphi^1}$, which is interpreted as the rate, where the agent is willing to give up φ^1 in favor of φ^0 (the same holds true the other way around), while maintaining the same level of utility.

The idea is to look at the marginal changes in the positions in the assets. From (2.2) (remember that the $\uparrow, \downarrow$ -processes are the non-positive increasing

3.5. THE CONNECTION TO G

decompositions of φ^0 and φ^1) we have (dropping the time point t)

$$\frac{\begin{matrix} + \begin{cases} d\varphi^{0,\uparrow} & \leq (1-\lambda)S\varphi^{1,\downarrow} \\ -d\varphi^{0,\downarrow} & \leq -S\varphi^{1,\uparrow} \end{cases} \end{matrix}}{d\varphi^0 \leq -\left(S_t\varphi^{1,\uparrow} - (1-\lambda)S\varphi^{1,\downarrow}\right)}$$

We can w.l.o.g. suppose that we fix the start $(\varphi^0, \varphi^1, s)$ in the NTR i.e. that $y = \frac{\varphi^1 s}{\varphi^0} \in NTR$. The question is what is the frictionless price $\tilde{s} = \tilde{s}(\varphi^0, \varphi^1, s)$ such that the trading agent is indifferent (of first order) in buying and selling stock for the price $\tilde{s}$ instead of S .

The answer is the above inequality for the frictionless price $\tilde{s}$, which has to hold now as an equality (because of the frictionless market)

$$d\varphi^0 = -\left(\tilde{s}\varphi^{1,\uparrow} - \tilde{s}\varphi^{1,\downarrow}\right) = -\tilde{s}d\varphi^1$$

Solving for $\tilde{s}$:

$$\tilde{s} = -\frac{d\varphi^0}{d\varphi^1}$$

which is the marginal rate of substitution.

Changing the position φ^1 of holdings in stock to $\varphi^1 + d\varphi^1$, for some tiny $d\varphi^1$, the discounted value changes (of first order) by the quantity $u_{\varphi^1}^\infty d\varphi^1$, similarly for φ^0 (u^∞ of (3.3)). So holding the utility constant the equation $u_{\varphi^1}^\infty d\varphi^1 + u_{\varphi^0}^\infty d\varphi^0 = du^\infty \stackrel{!}{=} 0$ has to hold true, which is the same as $\frac{u_{\varphi^1}^\infty}{u_{\varphi^0}^\infty} = -\frac{d\varphi^0}{d\varphi^1}$ and leads to

$$\tilde{s} = \frac{u_{\varphi^1}^\infty}{u_{\varphi^0}^\infty}(\varphi^0, \varphi^1, s) \tag{3.4}$$

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In other words the shadow price $\tilde{s}$ is given by the marginal rate of substitution between φ^1 and φ^0 . In the martingale approach it is assumed that the shadow price $\tilde{S}$ is a deterministic, smooth function $g(\cdot)$ of the stock price S , where $g(\cdot)$ was solved for the real variable $s \in \mathbb{R}_+$. We finally conclude that the two approaches come to the same result in the following (informal) equation over the two representations for the shadow price $\tilde{s}$:

3.5. THE CONNECTION TO G

$$\frac{u_{\varphi^1}^\infty}{u_{\varphi^0}^\infty} = \tilde{s} \stackrel{!}{=} g(s) \quad (3.5)$$

Now we recall $g_{c,\theta}(s)$ (where s denotes the variable and $c = \frac{\varphi^0}{\varphi^1}$ and $\theta = \frac{\mu}{\sigma^2}$ are treated as fixed parameters), remembering that we are in the NTR and that we have boundary conditions (2.10) at $s = 1$ and (2.13) at $s = \bar{s}$:

$$g(s) = g_{c,\theta}(s) = \begin{cases} \frac{-cs + (2\theta - 1 + 2c\theta)s^{2\theta}}{s - (2 - 2\theta - c(2\theta - 1))s^{2\theta}}, & \theta \in]0, 1[\setminus \{\frac{1}{2}\} \\ \frac{c+1+c\log(s)}{c+1-\log(s)}, & \theta = \frac{1}{2} \end{cases}$$

For the marginal rate of substitution we calculate by differentiating (3.3)

$$u_{\varphi^0}^\infty = \begin{cases} \frac{1}{\varphi^0} \left(-C_1(2\theta - 1)y^{2\theta-1} - \frac{\delta}{\mu - \frac{\sigma^2}{2}} + 1 \right), & \theta \in]0, 1[\setminus \{\frac{1}{2}\} \\ \frac{1}{\varphi^0} \left(-C_1 - 2\frac{\delta}{\sigma^2} \log(y) + 1 \right), & \theta = \frac{1}{2} \end{cases}$$

and

$$u_{\varphi^1}^\infty = \begin{cases} \frac{1}{\varphi^1} \left(C_1(2\theta - 1)y^{2\theta-1} + \frac{\delta}{\mu - \frac{\sigma^2}{2}} \right), & \theta \in]0, 1[\setminus \{\frac{1}{2}\} \\ \frac{1}{\varphi^1} \left(C_1 + 2\frac{\delta}{\sigma^2} \log(y) \right), & \theta = \frac{1}{2} \end{cases}$$

and then we plug into (3.4):

$$\tilde{s}_{\theta,\delta,C_1}(y) = \begin{cases} \frac{\varphi^0}{\varphi^1} \frac{\frac{-\delta}{\mu - \frac{\sigma^2}{2}} + (2\theta - 1)C_1 y^{2\theta-1}}{\left(1 - \frac{\delta}{\mu - \frac{\sigma^2}{2}}\right) - (2\theta - 1)C_1 y^{2\theta-1}}, & \theta \in]0, 1[\setminus \{\frac{1}{2}\} \\ \frac{\varphi^0}{\varphi^1} \frac{C_1 + 2\frac{\delta}{\sigma^2} \log(y)}{(1 - C_1) - 2\frac{\delta}{\sigma^2} \log(y)}, & \theta = \frac{1}{2} \end{cases}.$$

We substitute $\frac{\varphi^0}{\varphi^1}$ by the variable c and y by $\frac{s}{c}$

$$\tilde{s}_{\theta,c,\delta,C_1}(s) = \begin{cases} c \frac{\frac{-\delta}{\mu - \frac{\sigma^2}{2}} + (2\theta - 1)C_1 (\frac{s}{c})^{2\theta-1}}{\left(1 - \frac{\delta}{\mu - \frac{\sigma^2}{2}}\right) - (2\theta - 1)C_1 (\frac{s}{c})^{2\theta-1}}, & \theta \in]0, 1[\setminus \{\frac{1}{2}\} \\ c \frac{C_1 + 2\frac{\delta}{\sigma^2} \log(\frac{s}{c})}{(1 - C_1) - 2\frac{\delta}{\sigma^2} \log(\frac{s}{c})}, & \theta = \frac{1}{2} \end{cases}$$

3.6. RESULT FOR C_1

and remember from (2.18) that

$$\delta_{\bar{s}, \theta, c} = \begin{cases} \frac{(2\theta-1)\sigma^2 \bar{s}}{2(1+c)(\bar{s}+(-2-c+2\theta(1+c))\bar{s}^{2\theta})}, & \theta \in]0, 1[\setminus \{\frac{1}{2}\} \\ \frac{\sigma^2}{2(1+c)(1+c-\log(\bar{s}))}, & \theta = \frac{1}{2} \end{cases} \quad (3.6)$$

and from (2.14)

$$\bar{s}_{\theta, c} = \begin{cases} \left(\frac{c}{(2\theta-1+2c\theta)(2-2\theta-c(2\theta-1))} \right)^{\frac{1}{2\theta-1}}, & \theta \in]0, 1[\setminus \{\frac{1}{2}\} \\ \exp\left(\frac{c^2-1}{c}\right), & \theta = \frac{1}{2} \end{cases}. \quad (3.7)$$

So we are prepared to calculate C_1 by (3.5).

3.6 Result for C_1

We solve $\tilde{s}_c(s) = g_c(s)$ for C_1 in mathematica and only state here the result for the case $\theta = \frac{1}{2}$, the result of the more general case can be looked up in the Appendix A as screenshot from mathematica (although it is lengthy, it is a closed formula):

$$C_1 = \begin{cases} \text{Appendix A}, & \theta \in]0, 1[\setminus \{\frac{1}{2}\} \\ \frac{c+1+c\log(c)}{1+c^2}, & \theta = \frac{1}{2} \end{cases}.$$

So we came to an explicit expression for the missing parameter C_1 and therefore also for the final expression for the discounted value function (3.3)(remember $y = \frac{\varphi^1 s}{\varphi^0} = \frac{s}{c}$).

For the trading beginning at time 0 ($t = 0$) and Black-Merton-Scholes model parameters fulfilling $\theta = \frac{1}{2}$ this is (still using $c = \frac{\varphi^0}{\varphi^1}$ for notational purposes):

$$u^\infty(\varphi^0, \varphi^1, s) = \frac{c+1+c\log(c)}{1+c^2} \log\left(\frac{s}{c}\right) + \frac{\delta}{\sigma^2} \log\left(\frac{s}{c}\right)^2 + \log(\varphi^0), \quad \theta = \frac{1}{2}. \quad (3.8)$$

When we insert (3.7) into (3.6) for $\theta = \frac{1}{2}$ we get

$$\delta = \frac{c\sigma^2}{2(1+c)^2}, \quad \theta = \frac{1}{2}$$

3.7. CONCLUSIO

and can therefore state the following fully finished expression only depending on φ^0, φ^1, s

$$u^\infty(\varphi^0, \varphi^1, s) = \frac{\frac{\varphi^0}{\varphi^1} + 1 + \frac{\varphi^0}{\varphi^1} \log(\frac{\varphi^0}{\varphi^1})}{1 + \frac{\varphi^0}{\varphi^1}} \log(\frac{\varphi^1 s}{\varphi^0}) + \frac{\frac{\varphi^0}{\varphi^1}}{2(1 + \frac{\varphi^0}{\varphi^1})^2} \log(\frac{\varphi^1 s}{\varphi^0})^2 + \log(\varphi^0), \quad \theta = \frac{1}{2}. \quad (3.9)$$

3.7 Conclusio

We now came to an explicit expression for the discounted value function in terms of initial capital (φ^0, φ^1) and stock price $s = S_0$ at time zero.

Let us recapitulate the whole progress. We began with fixing the underlying financial market with a stochastic stock price process. Aligned to this we developed an inherent market which is free of transaction costs, but has a far more complicated price process, namely the shadow price process, for which we first only know that it stays in the bid-ask spread of the original stock price. We use existing knowledge that the optimal strategy of this proper shadow price market is also the optimal strategy of the original market with transaction costs λ and additionally find the function g which determines the shadow price by knowing the stock price in this proper way, meaning that we use Mertons results, namely his optimal fraction of wealth invested into the stock, for this frictionless market. With this it is possible to prove proposition 2, which gives us the optimal strategy in the market with transaction costs and the according optimal fraction of wealth invested. Furthermore it is possible to calculate the optimal growth rate for the trading period being unrestricted. All this is verified in recent literature and is here developed superficially to be able to use it.

In the main part of the work we introduced the value function as it is done in the primal approach, which is the function giving the maximal expected terminal value of the portfolio depending on the model parameters and under our assumptions on the trader and remember that we have a market, which charges the trader transaction costs λ . This enters here in the maximization set $\mathcal{A}_*^\lambda$ consisting of holdings which are obtainable only by λ -self-financing and λ -admissible trading starting from the given initial capital.

We develop this value function further to an unrestricted trading horizon

3.7. CONCLUSIO

to be able to say something about the long-time behavior of the optimal strategy. Over a few simplifications and considerations we are actually able to have a solution of this in terms of a solution of an ODE, but still including a free parameter C_1 making a difference in this long-time behavior. By using the economical idea of the marginal rate of substitution we are able to link derivatives of the value function to the already familiar function g from the martingale approach. Through this it is finally possible to calculate C_1 and therefore come to the explicit expression (3.9) which gives us the maximal expected liquidation value exponentially discounted by the optimal growth rate. This value, which is to be understood as “liquidated” in the sense that all stock holdings are rated as if they were converted in bonds at all times.

With our trick of using the result of the function g and therefore the insights into the shadow price market and its information about the optimal strategy of the stock price market we were able to explicitly determine the long-time behavior of the “value” of the portfolio when trading optimally.

4

4.1 Appendix A: Log-utility solution of C_1 for

$$\theta \in]0, 1[\setminus \left\{ \frac{1}{2} \right\}$$

Here we state a screenshot of the resulting expression from mathematica, which is already simplified.

$$\begin{aligned} & \left\{ \left\{ c_1 + - \left(s^{-2\theta} \left(\frac{s}{c} \right)^{1-2\theta} \left(c^2 (-1+2\theta) \left(\left(-\frac{c}{(-2-c+2\theta+2c\theta)(-1+2(1+c)\theta)} \right)^{\frac{1}{1-2\theta}} \right)^{2\theta} (s-2s^{2\theta}\theta) (2\mu-\sigma^2) + 2s^{2\theta}(-1+2\theta) \right. \right. \right. \right. \\ & \quad \left(-2 \left(\left(-\frac{c}{(-2-c+2\theta+2c\theta)(-1+2(1+c)\theta)} \right)^{\frac{1}{1-2\theta}} \right)^{2\theta} + 2\theta \left(\left(-\frac{c}{(-2-c+2\theta+2c\theta)(-1+2(1+c)\theta)} \right)^{\frac{1}{1-2\theta}} \right)^{2\theta} + \right. \\ & \quad \left. \left(-\frac{c}{(-2-c+2\theta+2c\theta)(-1+2(1+c)\theta)} \right)^{\frac{1}{1-2\theta}} \right) \mu + \left(-\left(\left(-\frac{c}{(-2-c+2\theta+2c\theta)(-1+2(1+c)\theta)} \right)^{\frac{1}{1-2\theta}} \right)^{2\theta} + \theta \right. \\ & \quad \left. \left(\left(\left(-\frac{c}{(-2-c+2\theta+2c\theta)(-1+2(1+c)\theta)} \right)^{\frac{1}{1-2\theta}} \right)^{2\theta} + \left(-\frac{c}{(-2-c+2\theta+2c\theta)(-1+2(1+c)\theta)} \right)^{\frac{1}{1-2\theta}} \right) \right) \sigma^2 \right) + \\ & \quad c \left(s \left(-2 \left(\left(-\frac{c}{(-2-c+2\theta+2c\theta)(-1+2(1+c)\theta)} \right)^{\frac{1}{1-2\theta}} \right)^{2\theta} + 2\theta \left(\left(-\frac{c}{(-2-c+2\theta+2c\theta)(-1+2(1+c)\theta)} \right)^{\frac{1}{1-2\theta}} \right)^{2\theta} + \right. \right. \\ & \quad \left. \left(-\frac{c}{(-2-c+2\theta+2c\theta)(-1+2(1+c)\theta)} \right)^{\frac{1}{1-2\theta}} \right) (2\mu-\sigma^2) + \\ & \quad s^{2\theta} \left(-2 \left(\left(\left(-\frac{c}{(-2-c+2\theta+2c\theta)(-1+2(1+c)\theta)} \right)^{\frac{1}{1-2\theta}} \right)^{2\theta} + \theta \theta^2 \left(\left(-\frac{c}{(-2-c+2\theta+2c\theta)(-1+2(1+c)\theta)} \right)^{\frac{1}{1-2\theta}} \right)^{2\theta} \right) + \right. \\ & \quad \left. 2\theta \left(-4 \left(\left(-\frac{c}{(-2-c+2\theta+2c\theta)(-1+2(1+c)\theta)} \right)^{\frac{1}{1-2\theta}} \right)^{2\theta} + \left(-\frac{c}{(-2-c+2\theta+2c\theta)(-1+2(1+c)\theta)} \right)^{\frac{1}{1-2\theta}} \right) \right) \\ & \quad \mu + \left(\left(\left(-\frac{c}{(-2-c+2\theta+2c\theta)(-1+2(1+c)\theta)} \right)^{\frac{1}{1-2\theta}} \right)^{2\theta} + \left(-\frac{c}{(-2-c+2\theta+2c\theta)(-1+2(1+c)\theta)} \right)^{\frac{1}{1-2\theta}} + \right. \\ & \quad \left. 4\theta^2 \left(2 \left(\left(-\frac{c}{(-2-c+2\theta+2c\theta)(-1+2(1+c)\theta)} \right)^{\frac{1}{1-2\theta}} \right)^{2\theta} + \left(-\frac{c}{(-2-c+2\theta+2c\theta)(-1+2(1+c)\theta)} \right)^{\frac{1}{1-2\theta}} \right) - \right. \\ & \quad \left. 2\theta \left(4 \left(\left(-\frac{c}{(-2-c+2\theta+2c\theta)(-1+2(1+c)\theta)} \right)^{\frac{1}{1-2\theta}} \right)^{2\theta} + \left(-\frac{c}{(-2-c+2\theta+2c\theta)(-1+2(1+c)\theta)} \right)^{\frac{1}{1-2\theta}} \right) \right) \sigma^2 \right) \right) \right) \Bigg/ \\ & \quad \left((1+c)^2 (1-2\theta)^2 \left(-2 \left(\left(-\frac{c}{(-2-c+2\theta+2c\theta)(-1+2(1+c)\theta)} \right)^{\frac{1}{1-2\theta}} \right)^{2\theta} + 2\theta \left(\left(-\frac{c}{(-2-c+2\theta+2c\theta)(-1+2(1+c)\theta)} \right)^{\frac{1}{1-2\theta}} \right)^{2\theta} + \right. \right. \\ & \quad \left. \left. c(-1+2\theta) \left(\left(-\frac{c}{(-2-c+2\theta+2c\theta)(-1+2(1+c)\theta)} \right)^{\frac{1}{1-2\theta}} \right)^{2\theta} + \right. \right. \\ & \quad \left. \left. \left(-\frac{c}{(-2-c+2\theta+2c\theta)(-1+2(1+c)\theta)} \right)^{\frac{1}{1-2\theta}} \right) (2\mu-\sigma^2) \right) \right) \Bigg\} \end{aligned}$$

4.2 Appendix B: Power-utility outview

In Section 3.2 we again fixed the utility function to be the log-utility in order to make the simplifications for the discounted value function. Here we will regard the power utility (where $\gamma \neq 0$) and develop the analogous simplification for this case.

We will first find the connection between the discounted value function at different times, then also use a property of the power utility and finally make the analogous observation as above to get to the much easier representation.

It holds that

$$\begin{aligned}
 u^\infty(\varphi^0, \varphi^1, s, t) &= \lim_{T \rightarrow \infty} \sup_{(\varphi_T^0, \varphi_T^1) \in \mathcal{A}_{*t}^\lambda} \left\{ \mathbb{E} \left[U(e^{-\delta T}(\varphi_T^0 + \varphi_T^1 S_T)) \middle| S_t = s \right] \right\} \\
 &= \lim_{T \rightarrow \infty} \sup_{(\varphi_T^0, \varphi_T^1) \in \mathcal{A}_{*t}^\lambda} \left\{ \mathbb{E} \left[U(\varphi_T^0 + \varphi_T^1 S_T) \middle| S_t = s \right] \right\} \cdot e^{-\gamma \delta T} \\
 &= \lim_{T \rightarrow \infty} \sup_{(\varphi_T^0, \varphi_T^1) \in \mathcal{A}_{*t}^\lambda} \left\{ \mathbb{E} \left[U(\varphi_{T-t}^0 + \varphi_{T-t}^1 S_{T-t}) \middle| S_0 = s \right] \right\} \cdot e^{-\gamma \delta (T-t)} \cdot e^{-\gamma \delta t} \\
 &= u^\infty(\varphi^0, \varphi^1, s, 0) \cdot e^{-\gamma \delta t}
 \end{aligned}$$

where we shifted the time in the conditional expectation in the third line for a $-t$.

PROPERTY: The discounted value function for $\gamma \neq 0$ is *homogeneous of degree γ* , which means that

$$u^\infty(a \cdot \varphi^0, a \cdot \varphi^1, s, t) = a^\gamma \cdot u^\infty(\varphi^0, \varphi^1, s, t) \quad \text{for } t \geq 0.$$

Or the way we want to use it is to normalize the bond-part of the portfolio and represent the stock-part as a fraction of the holdings in the two assets:

$$u^\infty(\varphi^0, \varphi^1, s, t) = u^\infty(\varphi^0, \varphi^1, s, 0) \cdot e^{-\gamma \delta t} = (\varphi^0)^\gamma \cdot u^\infty(1, \frac{\varphi^1}{\varphi^0}, s, 0) \cdot e^{-\gamma \delta t}.$$

OBSERVATION: Note again that the variables φ^1 (quantity of stock) and s (value of stock) only enter in the discounted value function u^∞ as the product φs . So we may similarly introduce the function h to simplify the representation of our value function (remember that γ is fixed) and rewrite

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$$u^\infty(\varphi^0, \varphi^1, s, t) = (\varphi^0)^\gamma \cdot h\left(\frac{\varphi^1 s}{\varphi^0}\right) \cdot e^{-\gamma \delta t},$$

where $h(\frac{\varphi^1 s}{\varphi^0}) = u^\infty(1, \frac{\varphi^1}{\varphi^0}, s, 0)$ independent of t .

Introducing the new variable $y = \frac{\varphi^1 s}{\varphi^0}$ we have

$$u^\infty(\varphi^0, \varphi^1, s, t) = (\varphi^0)^\gamma \cdot h(y) \cdot e^{-\gamma \delta t},$$

which is similar as [DumLuc91] simplify in section IV.

We will now derive the modified Hamilton-Jacobi-Bellman equation from [DumLuc91] for power utility, but in a more martingale theory way.

As we already know from Assumption 2 in the no-trade region the optimal strategy is to hold $(\varphi_t^0, \varphi_t^1)$ constant (see [MagCon76] for proof). In this region we use Itô's formula (see e.g. [Pha09, 1.2.3]) in differential form and the Black-Merton-Scholes representation of the stock S_t (2.1) to derive

$$\begin{aligned} du^\infty(\varphi_t^0, \varphi_t^1, S_t, t) &= u_t^\infty dt + u_s^\infty dS_t + \frac{u_{ss}^\infty}{2} d\langle S, S \rangle_t \\ &= (\varphi^0)^\gamma h(y) e^{-\gamma \delta t} (-\gamma \delta) dt + (\varphi^0)^\gamma h'(y) \frac{\varphi^1}{\varphi^0} e^{-\gamma \delta t} (\mu S_t dt + \sigma S_t dW_t) + (\varphi^0)^\gamma h''(y) \left(\frac{\varphi^1}{\varphi^0}\right)^2 e^{-\gamma \delta t} \left(\frac{\sigma^2 S_t^2}{2} dt\right) \end{aligned}$$

Because W_t is a Brownian motion, we again use the formal identity $\mathbb{E}[dW_t] = 0$ ([Bjo04, Prop. 4.4]) to take the expectation (also remember that $S_t = s$)

$$\begin{aligned} \mathbb{E}[du^\infty(\varphi_t^0, \varphi_t^1, S_t, t)] &= \mathbb{E}\left[(\varphi^0)^\gamma e^{-\gamma \delta t} \left(-h(y)\gamma\delta + h'(y)\frac{\varphi^1}{\varphi^0}(\mu S_t) + h''(y)\left(\frac{\varphi^1}{\varphi^0}\right)^2 \frac{\sigma^2 S_t^2}{2}\right) dt\right] \\ &= (\varphi^0)^\gamma e^{-\gamma \delta t} \mathbb{E}\left[\left(-h(y)\gamma\delta + h'(y)y\mu + h''(y)\frac{y^2\sigma^2}{2}\right) dt\right] \end{aligned}$$

Similarly as above in general stochastic optimization theory in [Pha09] it is shown that $(u^\infty(\varphi^0, \varphi^1, S_t, t))_{t \geq 0}$ is a super-martingale for any self-financing trading strategy and becomes a (local) martingale if we plug in the optimal strategy and in the no-trade region it is even constant. So in this region the term in the big round brackets must be constant and non-positive and zero for the optimal strategy. And therefore we expect the function h for the optimal strategy to satisfy the ODE

$$-h(y)\gamma\delta + h'(y)y\mu + h''(y)\frac{y^2\sigma^2}{2} = 0$$

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which is the same ODE as in [DumLuc91].

The general solution to this ODE is

$$h(y) = y^{\kappa_+} C_1 + y^{\kappa_-} C_2$$

for $\kappa_{\pm} = \frac{-2\sqrt{2}\mu + \sqrt{2}\sigma^2 \pm \sqrt{2}\sqrt{4\mu^2 + 8\gamma\delta\sigma^2 - 4\mu\sigma^2 + \sigma^4}}{2\sqrt{2}\sigma^2}$, where C_1 and C_2 are still free constants.

Now we plug in our result and get

$$u^{\infty}(\varphi^0, \varphi^1, s, t) = (\varphi^0)^{\gamma} \cdot (y^{\kappa_+} C_1 + y^{\kappa_-} C_2) \cdot e^{-\gamma\delta t}$$

and respectively

$$u^{\infty}(\varphi^0, \varphi^1, s) := u^{\infty}(\varphi^0, \varphi^1, s, 0) = (\varphi^0)^{\gamma} \cdot (y^{\kappa_+} C_1 + y^{\kappa_-} C_2)$$

because $u^{\infty}(\varphi^0, \varphi^1, s, t) \cdot e^{\gamma\delta t} = u^{\infty}(\varphi^0, \varphi^1, s, 0)$.

Here we see that C_2 is not an additive constant so the way of the log case will not work here.

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DEUTSCHER ABSTRACT

Grob gesprochen geht es in der Optimierung der Erwartungsnutzenfunktion um die Suche nach der Handelsstrategie, die uns den größten Ertrag erwarten lässt, in unserem mathematischen Markt-Modell basierend auf einem stochastischen Prozess.

Im Gegensatz zu den klassischen Modellen, wo handeln kostenlos war, wurde diese Theorie kürzlich auch für Handeln mit proportionalen Kosten entwickelt, wo einiges mehr an mathematischer Theorie nötig war.

Durch einige Erwägungen und technische Transformierungen ist es möglich jüngste Ergebnisse mit den Primalen zu verknüpfen und im Speziellen ist es möglich eine explizite Darstellung für die optimale abgezinste Nutzenfunktion zu erlangen, die das Langzeitverhalten der optimalen Strategie beschreibt.

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