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DIPLOMARBEIT

Titel der Diplomarbeit

Bosons in the lowest Landau-level in an anharmonic trap: Derivation of the mean field energy functional

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Zusammenfassung

Der Gegenstand dieser Diplomarbeit ist ein Vielteilchenmodell zur Beschreibung von kalten Bose-Gasen in einer schnell rotierenden Falle. Das Fallenpotential ist eine Summe eines quadratischen Terms und eines Terms vierter Ordnung in dem Abstand von der Rotationsachse. Die Rotation wird als so schnell angenommen, dass der Zustand des Systems durch eine Einschränkung auf das niedrigste Landau-Niveau beschrieben werden kann. Das Problem bestand nun darin einen Parameterbereich zu identifiziert, wo eine Beschreibung durch ein mean-field Energiefunktional möglich ist. Darüber hinaus wurde die Konvergenz der Grundzustandsenergie des Vielteilchenmodells zu der mean-field Energie bewiesen. Das wesentliche technische Hilfsmittel hierbei war die Anwendung von kohärenten Zuständen.

Für ein harmonisches Fallenpotential wurde das Problem in einer Arbeit aus dem Jahr 2009 von E.H.Lieb, R.Seiringer und J.Yngvason gelöst.

Danksagung

An erster Stelle möchte ich mich bei Prof. Yngvason für die hervorragende Betreuung bedanken. Darüber hinaus gilt mein Dank meiner Frau, Agnes Draxler, die mich immer liebevoll unterstützt hat. Schließlich gebührt meinen Eltern, Gerti und Helmut Draxler, großer Dank, da sie mir ein Studium ohne finanzielle Sorgen ermöglicht haben und mich immer in all meinen Vorhaben tatkräftig unterstützen.

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Chapter 1

Introduction

1.1 Summary

The topic of this work is to derive energy bounds for fast rotating Bose gases in anharmonic traps in a parameter range where all particles can be assumed to be in the lowest Landau level. In a quadratic trapping potential this is the case if the system is rotating close to its critical angular velocity above which the confining potential can not compensate the centrifugal forces and the system blows up. An anharmonic trap can nevertheless confine the system under these conditions. The potential can be considered to model a magneto-optical trap.

The many-body model we shall consider is a modification of the one discussed in [1] and [2]. An upper bound for the corresponding ground state energy can be obtained by variational methods. The behavior of this energy in dependence of the total angular momentum L is in the case of harmonic trapping determined by the so called *Yrast curve* [2], a term originating in nuclear physics. This curve can be subdivided into different parameter regimes according to the value of the filling factor, ν, of the system.

For small filling factor many interesting trial functions, of which the "Laughlin wavefunction" related to the Fractional Quantum Hall Effect is an example, can be used to find upper bounds to this curve. Here the opposite case of large filling factor will be considered. This corresponds to the parameter regime where L is much smaller than N^2, where N is the particle number.

In [1] it is shown, that for quadratic potentials the *Gross-Pitaevskii energy functional* gives in this regime an accurate description of the ground state energy for the N-body Hamiltonian with contact interaction. While the Gross-Pitaevskii energy always gives an upper bound for the energy, a lower bound is a lot more difficult to achieve.

In this thesis we shall derive such a lower bound for the ground state energy in terms of the Gross-Pitaevskii energy in the case of anharmonic trapping. As in [1] the crucial energy bounds are obtained by the use of coherent states.

1.2 Bosons in the lowest Landau Level

1.2.1 Harmonic case

The reason for the appearance of Landau Levels in a system of uncharged particles is that there exists a formal equivalence between the Hamiltonian in a rotating frame and the Hamiltonian for charged particles in a magnetic field. Physically speaking, the Coriolis force in the rotating frame plays the same role as the Lorentz force in the latter case.

This enables us to determine the one-particle ground-states and the kinetic energy in the rotating frame. A general reference for this topic is [3] and [4].

The Hamiltonian for N particles in a reference frame rotating about the x_3-axis with angular velocity $\mathbf{\Omega} = \Omega \mathbf{e}_3$ is given by

$$H = \sum_{j=1}^{N} \left(-\tfrac{1}{2}\Delta_j + V(\mathbf{x}_j) - \mathbf{L}_j \cdot \mathbf{\Omega}\right) + \sum_{1\le i<j\le N} v\left(|\mathbf{x}_i - \mathbf{x}_j|\right), \tag{1.1}$$

where $\mathbf{x}_j \in \mathbb{R}^3$, $\mathbf{L} = -i\mathbf{x} \wedge \nabla$ is the angular momentum operator, V the confining external potential and v an interaction potential assumed to be repulsive. The units are chosen such that $\hbar = m = 1$, where m is the mass. The operator acts on symmetric functions in the Hilbert space $L^2(\mathbb{R}^{3N})$.

We will first consider the special case where V is a *quadratic potential*,

$$V(x) = \tfrac{1}{2}\omega_\perp^2 r^2 + \tfrac{1}{2}\omega_3^2 x_3^2 \tag{1.2}$$

with $r = \sqrt{x_1^2 + x_2^2}$ the radial coordinate in the plane perpendicular to the rotational axis, $\omega_\perp$ the trap frequency in this plane and $\Omega < \omega_\perp$. We can complete the square and write the one-particle Hamiltonian as

$$H_1 = \tfrac{1}{2}\left(i\nabla_\perp + \mathbf{A}(\mathbf{x})\right)^2 + \omega L + \tfrac{1}{2}\left(-\partial_3^2 + \omega_3^2 x_3^2\right) \tag{1.3}$$

with $\omega = (\omega_\perp - \Omega)$, $L = \mathbf{L}\cdot\mathbf{e}_3$ and $\mathbf{A}(\mathbf{x}) = \omega_\perp\,(x_2, -x_1, 0)$. The first and the third term are Hamiltonians for quantum harmonic oscillators. Furthermore all three operators commute with each other.

The spectrum of $\frac{1}{2}(i\nabla_\perp + \mathbf{A}(\mathbf{x}))^2$ is given by the eigenvalues

$$\left(n + \tfrac{1}{2}\right) 2\omega_\perp, \quad n = 0, 1, 2, \ldots \tag{1.4}$$

which are called the *Landau Levels*.

The spectrum of $\frac{1}{2}(-\partial_3^2 + \omega_3^2 x_3^2)$ is

$$\left(m + \tfrac{1}{2}\right)\omega_3 \quad m = 0, 1, 2, \ldots \tag{1.5}$$

and the one of ωL is

$$\ell\omega, \quad \ell = 0 \pm 1, \pm 2, \ldots \tag{1.6}$$

For the rapidly rotating case, $0 < \omega_\perp - \Omega \ll \min(2\omega_\perp, \omega_3)$ we can restrict the possible wave functions to $n = m = 0$. This restriction is what in general is called the "*2D LLL regime*", 2D because of $m = 0$ and LLL because $n = 0$.

For these energy levels we know the corresponding eigenfunctions. The ground state for the last term in (1.3) is

$$\eta_0(x_3) = \text{const.} \exp\left(-\omega_3 \tfrac{x_3^2}{2}\right). \tag{1.7}$$

To describe the ground state of the first term in (1.3), it is convenient to introduce complex notation, $z = x_1 + ix_2$ and to choose units such that $\omega_\perp = 1$. With the Wirtinger-differential operators,

$$\partial = \tfrac{1}{2}\left(\partial_{x_1} - i\partial_{x_2}\right), \ \bar{\partial} = \tfrac{1}{2}\left(\partial_{x_1} + i\partial_{x_2}\right) \tag{1.8}$$

and the creation and annihilation operators,

$$a = \tfrac{1}{2}\left(2\bar{\partial} + z\right), \quad a^\dagger = \tfrac{1}{2}\left(-2\partial + \bar{z}\right) \tag{1.9}$$

satisfying $[a, a^\dagger] = 1$, the first term in (1.3) is given by

$$\tfrac{1}{2}\left(i\nabla_\perp + \mathbf{A}(\mathbf{x})\right)^2 = 2\left(a^\dagger a + \tfrac{1}{2}\right) \tag{1.10}$$

The ground state is now determined by $a\psi = 0$, i.e.

$$\bar{\partial}\psi(z, \bar{z}) = -\tfrac{1}{2} z \psi(z, \bar{z}). \tag{1.11}$$

The solutions of (1.11) are the desired ground state functions for the motion in the plane perpendicular to the axis of rotation,

$$\psi(z, \bar{z}) = \varphi(z) \exp\left(-\tfrac{|z|^2}{2}\right) \tag{1.12}$$

where $\varphi(z)$ is an analytic function, i.e., $\bar{\partial}\varphi(z) = 0$.

Altogether we arrive for the one particle wave function restricted to the LLL at:

$$\Psi(z, \bar{z}, x_3) = \varphi(z) \exp\left(-\tfrac{|z|^2}{2} - \omega_3 \tfrac{x_3^2}{2}\right) \tag{1.13}$$

Since the dependence on x_3 is fixed we may consider Ψ as a function of $(z, \bar{z})$ alone. These states span a whole space, which we will denote by $\mathcal{H}$. In (1.13) wee see that the Gaussian in the x_3-direction is the same for all functions in $\mathcal{H}$, so we can say that $\mathcal{H}$ consists of analytic functions in the complex plane equipped with the following scalar product:

$$\langle \varphi \mid \psi \rangle = \int_{\mathbb{C}} \varphi(z)^* \psi(z) \exp\Big(-|z|^2\Big)\, dz \ , \tag{1.14}$$

where dz means $dx_1 dx_2$. This space is in general known as the *Bargmann space* of analytic functions φ such that $\int |\varphi(z)|^2 \exp\Big(-|z|^2\Big)\, dz < \infty$.

In $\mathcal{H}$ the "kinetic energy" is now the sum of the eigenvalues of the three operators in (1.3). Besides the additive constants $\omega_\perp$ and $\frac{1}{2}\omega_3$ this energy depends only on ℓ and is equal to

$$\omega \ell \ . \tag{1.15}$$

The values of ℓ in $\mathcal{H}$ are only positive, since we are in the lowest Landau level. The corresponding operator for these eigenvalues is L, which has in the lowest Landau level the form

$$L = z\partial_z \ . \tag{1.16}$$

This can be seen by noting that the x_3-component of $\mathbf{L}$ is in general given by
$L = z\partial - \bar{z}\bar{\partial}$ and the fact that eigenfunctions of L in $\mathcal{H}$ have to satisfy $\bar{\partial}\varphi(z) = 0$. Thus the eigenfunctions of L are of the form

$$\varphi_\ell(z) = \text{const.} z^\ell \tag{1.17}$$

The index ℓ is the eigenvalue of L and accounts for the infinite degeneracy of each Landau level.

Now we can expand our model to the full N-body problem, considering the interaction as well.

The Hilbert space of N particles is given by the symmetric tensor product of the one particle spaces

$$\mathcal{H}_N = \otimes_{symm}^N \mathcal{H} \tag{1.18}$$

The wave functions are now symmetric and analytic functions of the coordinates $z_1, \ldots, z_N$ and the "kinetic energy" is given by ω times the total angular momentum. The next step is to introduce an interaction between the particles.

In this work we will make the assumption of pairwise and short range interaction. This allows us to replace $v(|\mathbf{x}_i - \mathbf{x}_j|)$, on the full original space $L^2(\mathbb{R}^{3N})$, by $g\delta(z_i - z_j)$ on $\mathcal{H}_N$, with g being a coupling constant $g > 0$, see [1,2]. In general a delta-function as a potential on dimensions higher than 1 is meaningless but for analytic functions the matrix elements of $\delta(z_i - z_j)$ make perfectly sense.

The precise meaning is as follows: We define the operator δ_{12} on $\mathcal{H}_2$ by

$$\delta_{12}\phi(z_1, z_2) = \frac{1}{2\pi}\phi\left(\frac{1}{2}(z_1 + z_2), \frac{1}{2}(z_1 + z_2)\right). \tag{1.19}$$

We see that δ_{ij} is a bounded operator which takes analytic functions into analytic functions. Due to analyticity the expectation value in a two-particle function $\phi(z_1, z_2)$ is then given by

$$\langle \phi \mid \delta_{12} \mid \phi \rangle = \int_{\mathbb{C}} |\phi(z, z)|^2 \exp\left(-2|z|^2\right) dz. \tag{1.20}$$

We are now able to write the N-particle Hamiltonian on $\mathcal{H}_N$ in the following way:

$$H_{N,\omega,g} = \omega \sum_{i=1}^{N} L_i + g \sum_{1 \le i < j \le N} \delta_{ij} \tag{1.21}$$

In section 2.1.2 we will see which conditions the parameters N, ω and g have to fulfill in order to make sure that the system can be considered to be in the LLL.

Now we want to generalize this model also to anharmonic external potentials.

1.2.2 Anharmonic case

The task now is to derive a Hamiltonian like in (1.21) but with an additional anharmonic term in the external potential. We first note the following:
As we have mentioned in (1.16), the angular momentum operator is $L = z\partial_z$ and the functions in $\mathcal{H}$ are analytic, i.e, $\bar{\partial}\varphi(z) = 0$. Hence by integrating by parts, the expectation value of L is given by

$$\begin{aligned} \langle \varphi \mid L \mid \varphi \rangle &= \int_{\mathbb{C}} \varphi(z)^* z \left(\partial_z \varphi(z)\right) \exp\left(-|z|^2\right) dz \\ &= -\int_{\mathbb{C}} \partial_z \left(\varphi(z)^* z \exp\left(-|z|^2\right)\right) \varphi(z) dz \\ &= \int_{\mathbb{C}} |\varphi(z)|^2 \left(|z|^2 - 1\right) \exp\left(-|z|^2\right) dz \; . \end{aligned} \tag{1.22}$$

We see that a term $\propto L$ in the Hamiltonian corresponds to a term $\propto |z|^2$.

A similar computation for $L^2 = z\partial_z + z^2\partial_z^2$ gives

$$\begin{aligned}
\langle \varphi \mid L^2 \mid \varphi \rangle &= \int_{\mathbb{C}} \varphi(z)^* z^2 \left(\partial_z^2 \varphi(z)\right) \exp\left(-|z|^2\right) dz \\
&\qquad\qquad + \int_{\mathbb{C}} \varphi(z)^* z \left(\partial_z \varphi(z)\right) \exp\left(-|z|^2\right) dz \\
&= \int_{\mathbb{C}} \partial_z^2 \left(\varphi(z)^* z^2 \exp\left(-|z|^2\right)\right) \varphi(z) dz \\
&\qquad\qquad - \int_{\mathbb{C}} \partial_z \left(\varphi(z)^* z \exp\left(-|z|^2\right)\right) \varphi(z) dz \\
&= \int_{\mathbb{C}} |\varphi(z)|^2 \left(1 - 3|z|^2 + |z|^4\right) \exp\left(-|z|^2\right) dz \,, \qquad (1.23)
\end{aligned}$$

which leads us to a term $\propto |z|^4$. Thus by adding a term αL^2, with $\alpha > 0$, to the Hamiltonian in (1.21), we get a system with a harmonic and an anharmonic external potential.

Chapter 2

The model and the main result

2.1 The model

The discussion in the previous chapter leads us to the following model for N bosons with repulsive, short-range pairwise interaction:

$$H = \omega \sum_{i=1}^{N} L_i + \alpha \sum_{i=1}^{N} L_i^2 + g \sum_{1 \leq i < j \leq N} \delta_{ij} \ . \tag{2.1}$$

This Hamiltonian acts on symmetric functions with N variables $z_i \in \mathbb{C}$, the momentum operators are $L_i = z_i \partial_i$ and ω, α and g are positive. The aim now is to describe its *ground state energy* for large N, which we will denote with $E_0(N, \omega, \alpha, g)$.

For reasons of clarity it is useful to distinguish between two cases, one where the harmonic potential is dominating over the anharmonic potential and vice versa. What is meant is that for the effective radius, R, of the system, one can choose a value R_0, such that for $R \ll R_0$ the choice of $\alpha = 0$ is reasonable and for $R \gg R_0$ the choice of $\omega = 0$ likewise. This distinction of cases will be made in the whole work.

We can estimate R under the assumption that the kinetic energy is of the same order of magnitude as the interaction energy. To find an expression for the interaction energy per particle $E_{\text{int}}^{(1)}$, we note that in a dilute gas $E_{\text{int}}^{(1)}$ is of the order $a\rho$ where a is the scattering length of the three-dimensional interaction potential and ρ is the three-dimensional density [5]. By using the fact that $g \propto a\omega_3^{1/2}$ we get

$$E_{\text{int}}^{(1)} \propto a\rho \propto Na/(R^2 \omega_3^{-1/2}) \propto \frac{Ng}{R^2} \ .$$

For the harmonic case the kinetic energy is given by

$$E_{\text{kin}} = \omega \ell \approx \omega R^2 \ .$$

Using the equal energy assumption, $E_{\text{kin}} \approx E_{\text{int}}$, we see that

$$R \approx (Ng/\omega)^{1/4} \ . \tag{2.2}$$

The same can be done for the anharmonic case, where the kinetic energy is given by

$$E_{\text{kin}} = \alpha \ell^2 \approx \alpha R^4 \ ,$$

which leads us to

$$R \approx (Ng/\alpha)^{1/6} \ . \tag{2.3}$$

At the critical radius R_0, the potentials are of comparable strength,

$$\omega R_0^2 \approx \alpha R_0^4$$

and hence

$$R_0 = \left(\frac{\omega}{\alpha}\right)^{1/2} \ . \tag{2.4}$$

We have now obtained a separation criteria for the two cases:

Harmonic regime, $R \ll R_0$:

$$\frac{Ng\alpha^2}{\omega^3} \ll 1 \tag{2.5}$$

Anharmonic regime, $R \gg R_0$:

$$\frac{Ng\alpha^2}{\omega^3} \gg 1 \tag{2.6}$$

2.1.1 Yrast curve

The Hamiltonian in (2.1) can also be written as

$$H = \omega \hat{L} + \alpha \tilde{L}^2 + g\hat{I}, \tag{2.7}$$

with $\hat{L} = \sum_{i=1}^{N} L_i$, $\tilde{L}^2 = \sum_{i=1}^{N} L_i^2$ and $\hat{I} = \sum_{1 \le i < j \le N} \delta_{ij}$. The three operators commute because of rotation invariance. Hence, for the ground state energy we look at the lower boundary, $I(L_{\text{tot}})$, of the joint spectrum of $\hat{L}$, $\tilde{L}^2$ and $\hat{I}$, i.e.

$I(L_{\text{tot}})$ =lowest eigenvalue of $\left(\alpha \tilde{L}^2 + g\hat{I}\right)$ for a given eigenvalue L_{tot} of $\hat{L}$.

The ground state energy can then be written as

$$E_0(N,\omega,\alpha,g) = \inf_{L_{\text{tot}}} \{\omega L_{\text{tot}} + \Delta(L_{\text{tot}})\} \ , \tag{2.8}$$

where $\Delta(L_{\text{tot}})$ is the spectrum of $\left(\alpha\tilde{L}^2 + g\hat{I}\right)$.

For the harmonic case, i.e. $\alpha = 0$, $I(L_{\text{tot}})$ is known as the *Yrast curve*, see [2]. There is not much known about this curve but it is expected that it is convex and monotonously decreasing. Further it is known that for $L_{\text{tot}} = N(N-1)$ the interaction vanishes in the ground state, i.e., $I = 0$. The ground state function for this energy is the bosonic *Laughlin wave function*, $\phi(z_1,\dots,z_N) = \prod_{i<j}(z_i - z_j)^2 \exp(-\sum_k |z_k|^2)$. The fermionic equivalent of this function plays an important role in the description of the *Fractional Quantum Hall Effect* and is given by $\phi^m(z_1,\dots,z_N) = \prod_{i<j}(z_i - z_j)^m \exp(-\sum_k |z_k|^2)$, where m is an odd integer.

Moreover the different regimes along the Yrast curve can be characterized by the so called *filling factor* [3,4], which is defined for states with angular momentum L_{tot} as

$$\nu = \frac{N^2}{2L_{\text{tot}}}.$$

The bosonic laughlin state has filling factor $\frac{1}{2}$ and the fermionic one $\frac{1}{m}$. The other asymptotic is $\nu \to \infty$, which is the case for $N^2 \gg L_{\text{tot}}$. In all regimes an *upper bound* for E_0 can be found by variational calculation methods. Here we will consider the case where $\nu \to \infty$, i.e., $L_{\text{tot}} \ll N^2$.

2.1.2 Upper bound

For the regime where $L_{\text{tot}} \ll N^2$ it is known that the *Gross-Pitaevskii (GP) energy functional* gives a good approximation (rigorously for the $N \to \infty$ limit) to the ground state energy of H in the case of quadratic potentials, see [1,2].

In our case the GP-functional for functions $\varphi \in \mathcal{H}$ is defined as

$$\mathcal{E}^{\text{GP}}[\varphi] = \omega \langle \varphi \mid L \mid \varphi \rangle + \alpha \langle \varphi \mid L^2 \mid \varphi \rangle + \frac{g}{2}\int_{\mathbb{C}} |\varphi(z)|^4 \exp\left(-2|z|^2\right) dz \ . \tag{2.9}$$

The GP energy is

$$E^{\text{GP}}(N,\omega,\alpha,g) = \inf\left\{\mathcal{E}^{\text{GP}}[\varphi] : \int_{\mathbb{C}} |\varphi(z)|^2 \exp(-|z|^2) dz = N\right\} \ . \tag{2.10}$$

For the harmonic case ($\alpha = 0$), the GP-energy satisfies the scaling relation

$$E^{\text{GP}}(N,\omega,g) = N\omega E^{\text{GP}}\left(1,1,\frac{Ng}{\omega}\right) \tag{2.11}$$

and thus has only one relevant parameter, the *GP parameter*

$$\frac{Ng}{\omega} \ . \tag{2.12}$$

The same for the anharmonic case $(\omega = 0)$,

$$E^{\mathrm{GP}}(N,\alpha,g) = N\alpha E^{\mathrm{GP}}\left(1,1,\frac{Ng}{\alpha}\right) \tag{2.13}$$

leads to

$$\frac{Ng}{\alpha} \ . \tag{2.14}$$

Further, the GP energy is always an *upper bound* for $E_0(N,\omega,\alpha,g)$. This can be seen by using a product ansatz as a trial function,
$\Phi(z_1,\ldots,z_N) = \prod_{i=1}^{N}\varphi(z_i)$, and consider the expectation value of H in this state:

$$\begin{aligned}\langle\Phi \mid H \mid \Phi\rangle = N\Big(\omega\,\langle\varphi \mid L \mid \varphi\rangle + \alpha\,\langle\varphi \mid L^2 \mid \varphi\rangle \\ + \frac{(N-1)g}{2}\int_{\mathbb{C}} |\varphi(z)|^4 \exp\left(-2\,|z|^2\right) dz\Big)\end{aligned} \tag{2.15}$$

As we are interested in an interacting system, we only consider cases where

$$\frac{Ng}{\omega} > c \quad \text{and} \quad \frac{Ng}{\alpha} > c \tag{2.16}$$

for some $c > 0$. The other case, $\frac{Ng}{\omega} \to 0$ or $\frac{Ng}{\alpha} \to 0$, would give a non-interacting gas.

We can estimate the order of magnitude for the GP-energy by taking into account that the energy per particle is $\propto \omega R^2$ or in the anharmonic case $\propto \alpha R^4$, as we mentioned earlier. Together with (2.2) and (2.3) this energy is found to be either

$$E^{(1)} \approx (Ng\omega)^{1/2} \tag{2.17}$$

or

$$E^{(1)} \approx (Ng)^{2/3}\alpha^{1/3} \ . \tag{2.18}$$

The GP-energy in the regime of (2.16) is then of the order

$$E^{\mathrm{GP}}(N,\omega,g) \approx N(Ng\omega)^{1/2} \quad \text{or} \quad E^{\mathrm{GP}}(N,\alpha,g) \approx N(Ng)^{2/3}\alpha^{1/3} \ . \tag{2.19}$$

In addition to (2.16) we can find another condition on the parameters. To this end we note that the condition where mean field can be expected to be a good approximation is that the number of vortices is much smaller than N, i.e.

$$\Omega R^2 \ll N \ . \tag{2.20}$$

In the fast rotating case $\Omega \approx 1$. Hence, with R given in (2.2) we find

$$\frac{g}{N\omega} \ll 1 \; . \tag{2.21}$$

In the anharmonic case we use (2.3) for R in (2.20) and get

$$\frac{g}{N^2\alpha} \ll 1 \; . \tag{2.22}$$

To arrive at the Hamiltonian in (2.1), we made a dimensional reduction from three to two dimensions and additionally restricted the possible wave functions to the LLL. This is only reasonable if the energy per particle is much less than the energy gap $2\omega_\perp$ between Landau Levles and the gap ω_3 for the motion in the x_3-direction. This gives the last condition for the parameters:

$$(Ng\omega)^{1/2} \ll \min\{\omega_\perp, \omega_3\} \quad \text{and} \quad (Ng)^{2/3}\alpha^{1/3} \ll \min\{\omega_\perp, \omega_3\} \tag{2.23}$$

Note that these conditions are not actually necessary for the calculations but they are the basic conditions for the validity of the Hamiltonian in (2.1).

2.2 Lower bound

In this section the missing *lower bound* on $E_0(N, \omega, \alpha, g)$ will be formulated. As mentioned above we distinguish between three parameter regimes of the system, the harmonic and the anharmonic regime as well as the regime where the harmonic and anharmonic terms are of the same order of magnitude. For each regime we will state a lower bound.

2.2.1 Harmonic case

First of all we summarize the conditions on the parameters derived in the previous chapter. The choice of $\alpha = 0$ is justified for

$$\frac{Ng\alpha^2}{\omega^3} \ll 1 \; . \tag{2.24}$$

The Hamiltonian describing the system in this regime is

$$H(N, \omega, g) = \omega \sum_{i=1}^{N} L_i + g \sum_{1 \le i < j \le N} \delta_{ij} \; . \tag{2.25}$$

In addition we assume:

$$\begin{aligned} \frac{Ng}{\omega} &> c \\ \frac{g}{N\omega} &\ll 1 \end{aligned} \tag{2.26}$$

Recall that under these assumptions, the condition for (2.25) describing a system in the LLL becomes $(Ng\omega)^{1/2} \ll \min\{\omega_\perp, \omega_3\}$ and the GP-energy is of the order $N(Ng\omega)^{1/2}$.

For the ground state energy of (2.25) the following lower bound was proved in [1]:

THEOREM 1. *For every $c > 0$ there is a $C < \infty$ such that*

$$E_0(N, \omega, g) \geq E^{\mathrm{GP}}(N, \omega, g)\left(1 - C\left(\frac{g}{N\omega}\right)^{1/10}\right)$$

provided $Ng/\omega > c$.

We now turn to the anharmonic case, where the treatment follows [1] closely.

2.2.2 Anharmonic case

The anharmonic regime is defined by the choice of $\omega = 0$ which is justified for

$$\frac{Ng\alpha^2}{\omega^3} \gg 1 \ . \tag{2.27}$$

The Hamiltonian in this case is

$$H(N, \alpha, g) = \alpha \sum_{i=1}^{N} L_i^2 + g \sum_{1 \leq i < j \leq N} \delta_{ij} \ . \tag{2.28}$$

All the other conditions valid in the regime restrained by (2.27) are

$$\begin{aligned} \frac{Ng}{\alpha} &> c \\ \frac{g}{N^2\alpha} &\ll 1 \ . \end{aligned} \tag{2.29}$$

In this case the LLL condition for (2.28) is $(Ng)^{2/3}\alpha^{1/3} \ll \min\{\omega_\perp, \omega_3\}$ and the GP-energy is of the order $N(Ng)^{2/3}\alpha^{1/3}$.

For (2.28) we have following result:

THEOREM 2. *For every $c > 0$ there is a $C < \infty$ such that*

$$E_0(N, \alpha, g) \geq E^{\mathrm{GP}}(N, \alpha, g)\left(1 - C\left(\frac{g}{N^2\alpha}\right)^{5/69}\right)$$

provided $Ng/\alpha > c$.

The proof is given in Chapter 3.

2.2.3 In-between

In the parameter domain where $R \approx R_0$, the harmonic and anharmonic potential are of the same order of magnitude. In this regime we have

$$\frac{Ng\alpha^2}{\omega^3} \approx 1 \ . \tag{2.30}$$

The Hamiltonian describing this case is

$$H(N, \omega, \alpha, g) = \omega \sum_{i=1}^{N} L_i + \alpha \sum_{i=1}^{N} L_i^2 + g \sum_{1 \leq i < j \leq N} \delta_{ij} \ . \tag{2.31}$$

The parameter conditions listed in (2.26) and (2.29) are all valid in this regime and in (2.19) both expressions for the GP-energy can be used in this case. Further the LLL condition for (2.31) is (2.23).

The following lower bound for (2.31) holds:

THEOREM 3. *For every $c, \tilde{c} > 0$ there are $C, \tilde{C} < \infty$ such that*

$$E_0(N, \omega, \alpha, g) \geq E^{\mathrm{GP}}(N, \omega, \alpha, g) \left(1 - C\left(\frac{g}{N\omega}\right)^{1/36} - \tilde{C}\left(\frac{g}{N^2\alpha}\right)^{2/27}\right)$$

provided $Ng/\omega > c$ and $Ng/\alpha > \tilde{c}$.

The proof is given in Chapter 3.

Chapter 3

The proof

3.1 Proof of Theorem 2

3.1.1 Coherent states

This whole chapter is devoted to the proofs of Theorem 2 and 3. We start with the former. The proof is based on the use of *Fock space* and *coherent states*. In this subsection those aspects of coherent states which are necessary for the proof are summarized. A general reference for this topic is [6].

The Fock space consists of the direct sum over all particle number sectors. We define the Fock space over the one particle space $\mathcal{H}$ as

$$\mathcal{F}(\mathcal{H}) = \oplus_{N=0}^{\infty}\mathcal{H}_N \tag{3.1}$$

where $\mathcal{H}_0 \cong \mathbb{C}$ and $\mathcal{H}_N$ as in (1.18). A basis vector in $\mathcal{H}_0$ is called "Fock vacuum" and will be denoted by $|0\rangle$. Elements in $\mathcal{F}(\mathcal{H})$ are sequences

$$\Phi = (\phi_0, \phi_1, \phi_2, \ldots), \tag{3.2}$$

with $\phi_N \in \mathcal{H}_N$ and $\sum_N \|\phi_N\|^2 < \infty$. The N-particle space is spanned by ϕ_N of the form $|\varphi_1, \ldots, \varphi_N\rangle$ which is shorthand for the bosonic tensorproduct of the one particle states φ_i, i.e., $|\varphi_1\rangle \otimes_{\text{symm.}} |\varphi_2\rangle \otimes_{\text{symm.}} \cdots \otimes_{\text{symm.}} |\varphi_N\rangle$.

On Fock space we have creation and annihilation operators, $a^\dagger(\varphi)$ and $a(\varphi)$, for $\varphi \in \mathcal{H}$. They are defined as:

$$a^\dagger(\varphi) : \mathcal{H}_N \longrightarrow \mathcal{H}_{N+1} \tag{3.3}$$

$$a^\dagger(\varphi)\,|\varphi_1, \ldots, \varphi_N\rangle \equiv |\varphi, \varphi_1, \ldots, \varphi_N\rangle$$

$$a(\varphi) : \mathcal{H}_N \longrightarrow \mathcal{H}_{N-1} \tag{3.4}$$

$$a(\varphi)\,|\varphi_1, \ldots, \varphi_N\rangle \equiv \sum_{i=1}^{N} \langle \varphi \mid \varphi_i\rangle\,|\varphi_1, \ldots, \varphi_{i-1}, \varphi_{i+1}, \ldots, \varphi_N\rangle$$

In particular for $\langle \varphi_i \mid \varphi_j \rangle = \delta_{ij}$, $a(\varphi_i)$ acts like

$$a(\varphi_i)\,|\varphi_1, \ldots, \varphi_N\rangle = |\varphi_1, \ldots, \varphi_{i-1}, \varphi_{i+1}, \ldots, \varphi_N\rangle\,. \tag{3.5}$$

These operators satisfy

$$[a(\varphi), a^\dagger(\psi)] = \langle \varphi \mid \psi \rangle\,. \tag{3.6}$$

Moreover an operator k on $\mathcal{H}$ can be extended to an operator K on $\mathcal{F}(\mathcal{H})$ by

$$K|_{\mathcal{H}_N} := \sum_{i=1}^{N} \underbrace{\mathbb{I} \otimes \ldots \otimes \mathbb{I}}_{i-1} \otimes k \otimes \underbrace{\mathbb{I} \otimes \ldots \otimes \mathbb{I}}_{N-i}. \tag{3.7}$$

This extension can be written with creation and annihilation operators. Let $\{|\varphi_i\rangle\}$, $i = 1, 2, \ldots$, be a basis in $\mathcal{H}$ and $a^\dagger(\varphi_i) \equiv a_i^\dagger$, $a(\varphi_i) \equiv a_i$. The second quantization of k is then given by

$$K = \sum_{i,j} a_i^\dagger \langle \varphi_i \mid k \mid \varphi_j \rangle\, a_j. \tag{3.8}$$

In the case of $|\varphi_i\rangle$ being eigenstates of k, K has a diagonal representation,

$$K = \sum_i k_i a_i^\dagger a_i\ . \tag{3.9}$$

For a two-body interaction operator w on $\mathcal{H}_2$, the analogous representation is

$$W = \frac{1}{2} \sum_{i,j,k,l} a_i^\dagger a_j^\dagger \langle \varphi_i \otimes \varphi_j\, |w|\, \varphi_k \otimes \varphi_l \rangle\, a_k a_l\ . \tag{3.10}$$

From now on we will choose the normalized eigenvectors of the angular momentum operator L, i.e.

$$\varphi_j(z) = (\pi j!)^{-1/2} z^j \tag{3.11}$$

as a basis in $\mathcal{H}$ and refer to them as *modes*. The extended version of the Hamiltonian (2.28) to the whole of Fock space is given by

$$\tilde{\mathbb{H}} = \alpha \sum_{i \geq 0} i^2 a_i^\dagger a_i + \frac{g}{2} \sum_{ijk} \langle \varphi_i \otimes \varphi_j \mid \delta \mid \varphi_k \otimes \varphi_{i+j-k} \rangle\, a_i^\dagger a_j^\dagger a_k a_{i+j-k}\ , \tag{3.12}$$

where δ is shorthand for δ_{12}. All the other matrix elements vanish because of conservation of angular momentum.

Now we will discuss several aspects of *coherent states* that will be used in this work.

The coherent state for a mode j and complex number ζ_j is defined by

$$|\zeta_j\rangle = \exp\left[-|\zeta_j|^2/2 + \zeta_j a_j^\dagger\right]|0\rangle\,. \tag{3.13}$$

More generally for a vector $\vec{\zeta} = (\zeta_0, \ldots, \zeta_J) \in \mathbb{C}^{J+1}$ we define

$$\left|\vec{\zeta}\right\rangle = \prod_{j=0}^{J} \exp\left[-|\zeta_j|^2/2 + \zeta_j a_j^\dagger\right]|0\rangle\,. \tag{3.14}$$

The coherent states form an overcomplete basis, which means that

$$\int \left|\vec{\zeta}\right\rangle\left\langle\vec{\zeta}\right| \frac{d\vec{\zeta}}{\pi^{J+1}} = \mathbb{I} \quad \text{and} \quad \left\langle\vec{\zeta}|\vec{\zeta}'\right\rangle = \exp\left(-\left|\vec{\zeta} - \vec{\zeta}'\right|^2\right). \tag{3.15}$$

For each operator $\mathbb{H}$ on $\mathcal{F}(\mathcal{H})$ there is a *lower* and *upper* symbol of $\mathbb{H}$. The *lower symbol* of an operator $\mathbb{H}$ on $\mathcal{F}(\mathcal{H})$ is defined by the expectation value in the coherent state, i.e.

$$h_\ell(\vec{\zeta}) = \left\langle\vec{\zeta}\right|\mathbb{H}\left|\vec{\zeta}\right\rangle. \tag{3.16}$$

The *upper symbol* is defined by

$$\mathbb{H} = \int \frac{d\vec{\zeta}}{\pi^{J+1}} \left|\vec{\zeta}\right\rangle\left\langle\vec{\zeta}\right| \otimes h_u(\vec{\zeta})\,, \tag{3.17}$$

where $\left|\vec{\zeta}\right\rangle\left\langle\vec{\zeta}\right|$ is the projection on the coherent state $\left|\vec{\zeta}\right\rangle$. Between the lower symbol and the upper symbol there exists a useful relation, namely

$$h_u(\vec{\zeta}) = \exp\left(-\sum_{i=0}^{J} \partial_{\zeta_i}\partial_{\zeta_i^*}\right) h_\ell(\vec{\zeta})\,. \tag{3.18}$$

The upper symbol will be relevant for the lower bound but to calculate the upper symbol out of the definition (3.17) is difficult. What we will do instead is to use the relation (3.18) since the lower symbol is much more easy to compute.

For example let us consider the choice of $J = 0$ and the operator given in (3.12). We see that the operator $\tilde{\mathbb{H}}$ in (3.12) contains four relevant operators for computing the expectation value in (3.16) (since $a_i = \mathbb{I}\otimes\ldots\otimes a_i\otimes\mathbb{I}\otimes\mathbb{I}\ldots$). The expectation values of the four operators are:

$$\begin{aligned} \langle\zeta|\, a_0^\dagger a_0\,|\zeta\rangle &= |\zeta|^2 & \qquad \langle\zeta|\, a_0^\dagger a_0^\dagger a_0 a_0\,|\zeta\rangle &= |\zeta|^4 \\ \langle\zeta|\, a_0\,|\zeta\rangle &= \zeta & \langle\zeta|\, a_0^\dagger\,|\zeta\rangle &= \zeta^* \end{aligned} \tag{3.19}$$

This is easy to see by using $\left[a_0, a_0^\dagger\right] = \mathbb{I}$, (3.15) and the definition of $|\zeta\rangle$ (3.13).

Another way to calculate these expectation values is to normal order all the $a_i^\dagger$ and a_i and afterwards replace them by the complex numbers ζ_i^* and ζ_i.

With the help of (3.18) it is now easy to compute the *upper symbol.* For the special case of $J = 0$ we get

$$a_0^\dagger a_0 \longrightarrow |\zeta|^2 - 1 \qquad\qquad a_0^\dagger a_0^\dagger a_0 a_0 \longrightarrow |\zeta|^4 - 4\,|\zeta|^2 + 2\;. \tag{3.20}$$

What we see here is that the upper symbol consists of additional negative terms. These negative terms will force us to add an additional small positive term to the Hamiltonian, in order to have control over the resulting errors in the energy. This will be done in Step two.

But before starting to calculate we want to make clear why we are using coherent states. We note that, with $\Phi \in \mathcal{F}(\mathcal{H})$, $\|\Phi\| = 1$ and $\mathcal{F}(\mathcal{H}^>)$ in (3.35),

$$\begin{aligned}
\inf_{\Phi} \langle \Phi \mid \mathbb{H} \mid \Phi\rangle &=^{(3.17)} \inf_{\Phi} \left\langle \Phi \mid \int \frac{d\,\vec{\zeta}}{\pi^{J+1}} \left|\vec{\zeta}\right\rangle \left\langle \vec{\zeta}\right| \otimes h_u(\vec{\zeta}\,) \mid \Phi \right\rangle \\
&\geq \inf_{\Phi} \inf_{\vec{\zeta}} \left\langle \Phi \mid \int \frac{d\,\vec{\zeta}'}{\pi^{J+1}} \left|\vec{\zeta}'\right\rangle \left\langle \vec{\zeta}'\right| \otimes h_u(\vec{\zeta}\,) \mid \Phi \right\rangle \\
&=^{(3.15)} \inf_{\substack{\Psi \in \mathcal{F}(\mathcal{H}^>) \\ \|\Psi\|=1}} \inf_{\vec{\zeta}} \left\langle \Psi \mid h_u(\vec{\zeta}\,) \mid \Psi \right\rangle .
\end{aligned} \tag{3.21}$$

What we see is that

$$\inf \operatorname{spec} \mathbb{H} \geq \inf_{\vec{\zeta}} \inf \operatorname{spec}\, h_u(\vec{\zeta}\,)\;. \tag{3.22}$$

The use of this inequality is the central tool of the proof, since we can find a lower bound for the right side of (3.22).

With this background we can start the proof, which we will seperate into 8 steps to make it more transparent.

3.1.2 Step one

The operator $\tilde{\mathbb{H}}$ in (3.12) acts on the whole Fock space, which is the direct sum over all particle sectors as mentioned above. We are interested in a lower bound for the ground state energy of $\tilde{\mathbb{H}}$ in the *sector of particle number* N, i.e.

$$\inf_{\substack{\Phi\in\mathcal{H}_N \\ \|\Phi\|=1}} \left\langle \Phi \mid \tilde{\mathbb{H}} \mid \Phi \right\rangle . \tag{3.23}$$

To insist on fixed particle number is however not convenient. It is easier to look for a lower bound *irrespective of particle number* and stay on the whole Fock space. For this reason we will modify $\tilde{\mathbb{H}}$ by adding a term

$$\mu \left(\sum_{i\geq 0} a_i^\dagger a_i - N \right)^2 , \tag{3.24}$$

with $\mu > 0$. Obviously the energy in (3.23) doesn't change under this modification, since this term vanishes in the sector of particle number N. Since we are looking for a lower bound, we can then use the fact that

$$\inf_{\substack{\Phi\in\mathcal{H}_N \\ \|\Phi\|=1}} \left\langle \Phi \mid \tilde{\mathbb{H}} \mid \Phi \right\rangle \geq \inf_{\substack{\Phi\in\mathcal{F}(\mathcal{H}) \\ \|\Phi\|=1}} \left\langle \Phi \mid \tilde{\mathbb{H}} + \mu \left(\sum_{i\geq 0} a_i^\dagger a_i - N \right)^2 \mid \Phi \right\rangle . \tag{3.25}$$

The value of μ will be chosen in such a way that the ratio of the left hand side and the right hand side of (3.25) is close to one. The modification of the operator in (3.12) is now given by

$$\begin{aligned} \hat{\mathbb{H}} = {} & \alpha \sum_{i\geq 0} i^2 a_i^\dagger a_i \\ & + \frac{g}{2} \sum_{ijk} \langle \varphi_i \otimes \varphi_j \mid \delta \mid \varphi_k \otimes \varphi_{i+j-k} \rangle \, a_i^\dagger a_j^\dagger a_k a_{i+j-k} \\ & + \mu \left(\sum_{i\geq 0} a_i^\dagger a_i - N \right)^2 . \end{aligned} \tag{3.26}$$

3.1.3 Step two

Due to the use of coherent states there will occur some negative terms in the calculations. To control these terms in the lower bound, we will add another small positive term to H on the N particle space in (2.28). This bigger Hamiltonian, H', is constructed in the following way. Let us consider the function

$$\chi_J(L) = \begin{cases} 0 & \text{for } L \leq J \\ \exp[-\frac{1}{8}(\sqrt{L} - \sqrt{J+1})^2] & \text{for } L \geq J+1 \end{cases} \tag{3.27}$$

for integers J. What we now show is that for an integer $J_0 \geq 1$ there exists a J with $J_0 \leq J < 2J_0$ such that

$$\left\langle \sum_{i=1}^{N} \chi_J(L_i) \right\rangle \leq 2^{5/2}(1 + 4\sqrt{\pi}) \frac{E_0(N, \alpha, g)}{\alpha J^{5/2}} \tag{3.28}$$

where $\langle . \rangle$ denotes the expectation value in the ground state of H. To show this we make the following estimation:

$$\begin{aligned}
\sum_{J\geq 0} \chi_J(L) &= \sum_{J=0}^{L-1} \exp[-\frac{1}{8}(\sqrt{L}-\sqrt{J+1})^2] \leq 1 + \int_0^{L-2} \exp[-\frac{1}{8}(\sqrt{L}-\sqrt{J+1})^2] dJ \\
&= 1 + 2\int_1^{\sqrt{L-1}} \exp[-\frac{1}{8}(\sqrt{L}-s)^2] s ds \\
&\leq 1 + 2\int_0^{\sqrt{L}} \exp[-\frac{1}{8}(\sqrt{L}-s)^2] s ds = 1 - 2\int_0^{\sqrt{L}} \exp[-\frac{1}{8}u^2](u-\sqrt{L}) du \\
&\leq 1 - 2\int_0^{\infty} \exp[-\frac{1}{8}u^2] u du + \sqrt{L}\int_{-\infty}^{\infty} \exp[-\frac{1}{8}u^2] du \\
&\leq 1 + 4\sqrt{\pi L}
\end{aligned} \tag{3.29}$$

Because of the fact that $\chi_J(0) = 0$ and $L \geq 0$, we even get

$$\sum_J \chi_J(L) \leq (1 + 4\sqrt{\pi})\sqrt{L} \,.$$

This leads us to

$$\frac{1}{J_0}\sum_{J=J_0}^{2J_0-1} \chi_J(L) \leq (1+4\sqrt{\pi})\frac{\sqrt{L}}{J_0} \leq 2^{5/2}(1+4\sqrt{\pi})\frac{L^2}{J^{5/2}}, \tag{3.30}$$

where we have used that $L \geq J_0$ and $J_0 \geq J/2$. The expectation value of $\sum_{i=1}^N L_i^2$ in the ground state is bounded above by $\frac{E_0(N,\alpha,g)}{\alpha}$ and hence

$$\left\langle \sum_{i=1}^N \frac{1}{J_0} \sum_{J=J_0}^{2J_0-1} \chi_J(L_i) \right\rangle \leq 2^{5/2}(1+4\sqrt{\pi})\frac{E_0(N,\alpha,g)}{\alpha J^{5/2}}. \tag{3.31}$$

Obviously the bound in (3.28) holds on average for $J_0 \leq J < 2J_0$ and therefore holds at least for one such J.

Altogether we can now modify the Hamiltonian by picking a small parameter $\eta > 0$ and construct

$$H' = H + \eta J^{5/2}\alpha \sum_{i=1}^N \chi_J(L_i) \,. \tag{3.32}$$

By using (3.28) we see that the ground state energy E_0' of H' is bounded above by $E_0(1+b\eta)$ with $b = 2^{5/2}(1+4\sqrt{\pi})$. This means that E_0 is bounded below by

$$E_0(N,\alpha,g) \geq \frac{E_0'}{(1+b\eta)}. \tag{3.33}$$

The task now will be to derive a lower bound on E_0'. To this end it will be necessary to determine suitably the new parameters μ, J and η in order to get a good bound.

3.1.4 Step three

The starting point is the second quantized version of the new Hamiltonian in (3.32) modified by the additional μ-term introduced in Step 1,

$$\begin{aligned}\mathbb{H} = {} & \alpha \sum_{i\geq 0} \left[i^2 + \eta J^{5/2} \chi_J(i) \right] a_i^\dagger a_i \\ & + \frac{g}{2} \sum_{ijk} \langle \varphi_i \otimes \varphi_j \mid \delta \mid \varphi_k \otimes \varphi_{i+j-k} \rangle \, a_i^\dagger a_j^\dagger a_k a_{i+j-k} \\ & + \mu \left(\textstyle\sum_{i\geq 0} a_i^\dagger a_i - N \right)^2 . \end{aligned} \tag{3.34}$$

We pick a $J \ll N$ and split the one particle space $\mathcal{H}$ into two subspaces, one generated by the modes φ_j with $j \leq J$, which we denote with $\mathcal{H}^<$, and its orthogonal complement $\mathcal{H}^>$. This leads to a factorization of the Fock space over $\mathcal{H}$, i.e.

$$\mathcal{F}(\mathcal{H}) = \mathcal{F}(\mathcal{H}^<) \otimes \mathcal{F}(\mathcal{H}^>). \tag{3.35}$$

We now introduce coherent states for the modes up to J. To compute the lower symbol we thus normal order and replace a_i by ζ_i and $a_i^\dagger$ by ζ_i^* for all $i \leq J$. Here we will not choose $J = \infty$ but $J \ll N$. The reason becomes clear at the end of this subsection. Obviously for such a choice, there will be some $a_i^\dagger$ and a_i left in the Hamiltonian, which means that $h_\ell(\vec{\zeta})$ and $h_u(\vec{\zeta})$ are operators acting on $\mathcal{F}(\mathcal{H}^>)$.

We now compute the upper symbol by using (3.18). At first we note that the lower symbol of $\mathbb{H}$ is a polynomial in the ζ_i^* and ζ_i of degree four. Hence only the first three terms in the Taylor expansion of the exponential in (3.18) contribute. Thus we can write

$$h_u(\vec{\zeta}) = \exp\left(-\sum_{i=0}^{J} \partial_{\zeta_i} \partial_{\zeta_i^*} \right) h_\ell(\vec{\zeta}) = h_\ell(\vec{\zeta}) + U_1(\vec{\zeta}) + U_2(\vec{\zeta}) \,, \tag{3.36}$$

with

$$U_1(\vec{\zeta}) = -\sum_{i=0}^{J} \partial_{\zeta_i} \partial_{\zeta_i^*} h_\ell(\vec{\zeta}) \quad \text{and} \quad U_2(\vec{\zeta}) = \frac{1}{2} \left(\sum_{i=0}^{J} \partial_{\zeta_i} \partial_{\zeta_i^*} \right)^2 h_\ell(\vec{\zeta}) \,. \tag{3.37}$$

The lower symbol of $\mathbb{H}$ is given by

$$\begin{aligned} h_\ell(\vec{\zeta}) = {} & \alpha \sum_{i\leq J} i^2 \zeta_i^* \zeta_i + \alpha \sum_{i>J} i^2 a_i^\dagger a_i + \eta \alpha J^{5/2} \sum_{i>J} \chi_J(i) a_i^\dagger a_i \\ & + \left\langle \vec{\zeta} \right| \frac{g}{2} \sum_{ijk} \langle \varphi_i \otimes \varphi_j \mid \delta \mid \varphi_k \otimes \varphi_{i+j-k} \rangle \, a_i^\dagger a_j^\dagger a_k a_{i+j-k} \left| \vec{\zeta} \right\rangle \\ & + \mu \left\langle \vec{\zeta} \right| \left(\textstyle\sum_{i\geq 0} a_i^\dagger a_i - N \right)^2 \left| \vec{\zeta} \right\rangle \,. \end{aligned} \tag{3.38}$$

We now compute $U_1(\overrightarrow{\zeta})$ term by term. The second and the third term of (3.38) vanish under the differentiation in (3.37) while the first one is

$$-\alpha \sum_{i \leq J} i^2 \, . \tag{3.39}$$

The interaction term includes couplings between the modes. Various terms occur and we have to make a case differentiation for the indices i, j, k and $i + j - k$.

The cases where either only one of the indices is $> J$ or only one of them is $< J$ can be ignored, since they vanish under the differentiation in (3.37). The two cases where $i \leq J, j \leq J, k > J, i + j - k > J$ and $i > J, j > J, k \leq J, i + j - k \leq J$ vanish as well under (3.37). What is left are the two cases where all modes, the initial and final one, are smaller or bigger than J as well as the four cases where the interaction takes place between modes $\leq J$ and $> J$. These cases are:

1. $i \leq J,\ j \leq J,\ k \leq J,\ i + j - k \leq J$
2. $i > J,\ j \leq J,\ k > J,\ i + j - k \leq J$
3. $i > J,\ j \leq J,\ k \leq J,\ i + j - k > J$
4. $i \leq J,\ j > J,\ k > J,\ i + j - k \leq J$
5. $i \leq J,\ j > J,\ k \leq J,\ i + j - k > J$
6. $i > J,\ j > J,\ k > J,\ i + j - k > J$

In case 1, equation (3.37) gives

$$\begin{aligned} &-\frac{g}{2} \sum_{\ell \leq J} \partial_{\zeta_\ell} \partial_{\zeta_\ell^*} \left(\sum_{\substack{i,j,k \leq J \\ i+j-k \leq J}} \langle \varphi_i \otimes \varphi_j \mid \delta \mid \varphi_k \otimes \varphi_{i+j-k} \rangle \, \zeta_i^* \zeta_j^* \zeta_k \zeta_{i+j-k} \right) = \\ &= -2g \sum_{i,k \leq J} \langle \varphi_i \otimes \varphi_j \mid \delta \mid \varphi_i \otimes \varphi_j \rangle \, \zeta_j^* \zeta_j = -2g \sum_{0 \leq i \leq J} \langle \phi_\zeta \otimes \varphi_i \mid \delta \mid \phi_\zeta \otimes \varphi_i \rangle \, , \end{aligned} \tag{3.40}$$

where $|\phi_\zeta\rangle = \sum_{j=0}^{J} \zeta_j \, |\phi_j\rangle$ with $\|\phi_\zeta\|^2 = \langle \phi_\zeta \mid \phi_\zeta \rangle = \sum_{i=0}^{J} |\zeta_j|^2$. The result of case 2 under (3.37) is

$$\begin{aligned} &-\frac{g}{2} \sum_{\ell \leq J} \partial_{\zeta_\ell} \partial_{\zeta_\ell^*} \left(\sum_{\substack{i,k > J \\ j,i+j-k \leq J}} \langle \varphi_i \otimes \varphi_j \mid \delta \mid \varphi_k \otimes \varphi_{i+j-k} \rangle \, a_i^\dagger \zeta_j^* a_k \zeta_{i+j-k} \right) = \\ &= -\frac{g}{2} \sum_{\substack{0 \leq i \leq J \\ k > J}} \langle \varphi_k \otimes \varphi_i \mid \delta \mid \varphi_k \otimes \varphi_i \rangle \, a_k^\dagger a_k \, . \end{aligned} \tag{3.41}$$

All the cases from 2-5 give the same result since all of them have two indices smaller than J and two indices bigger than J. Hence we can sum over them and get

$$-2g \sum_{\substack{0 \le i \le J \\ k > J}} \langle \varphi_k \otimes \varphi_i \mid \delta \mid \varphi_k \otimes \varphi_i \rangle \, a_k^\dagger a_k \, . \tag{3.42}$$

The terms for the last case vanish, since all indices are $> J$.

The only term left over in $U_1(\vec{\zeta})$ is the μ-term given by

$$-\sum_{\ell=0}^{J} \partial_{\zeta_\ell} \partial_{\zeta_\ell^*} \mu \left\langle \vec{\zeta} \middle| \sum_{ij} \left(a_i^\dagger a_i a_j^\dagger a_j - 2N \sum_i a_i^\dagger a_i + N^2 \right) \middle| \vec{\zeta} \right\rangle \, . \tag{3.43}$$

We first have to calculate the expectation value. As mentioned above, for all indices $\le J$ we have to normal order the creation and annihilation operators by using $[a_i, a_j^\dagger] = \delta_{ij}$ and then replace them by the complex numbers. We get

$$\begin{aligned} -\sum_{\ell=0}^{J} \partial_{\zeta_\ell} \partial_{\zeta_\ell^*} \mu \Bigg(& \sum_{i,j \le J} \zeta_i^* \zeta_j^* \zeta_i \zeta_j + \sum_{i \le J} \zeta_i^* \zeta_i - 2N \sum_{i \le J} \zeta_i^* \zeta_i - 2N \sum_{i > J} a_i^\dagger a_i \\ & + \sum_{i,j > J} a_i^\dagger a_i a_j^\dagger a_j + \sum_{i \le J, j > J} \zeta_i^* \zeta_i a_j^\dagger a_j \\ & + \sum_{i > J, j \le J} a_i^\dagger a_i \zeta_j^* \zeta_j + N^2 \Bigg). \end{aligned} \tag{3.44}$$

Carrying out the differentiation we arrive at

$$-\mu \left[(2 \left\| \Phi_\zeta \right\|^2 - 2N + 2N^> + 1)(J+1) + 2 \left\| \Phi_\zeta \right\|^2 \right] \, , \tag{3.45}$$

where $N^> = \sum_{i > J} a_i^* a_i$. Finally, by adding (3.39), (3.40), (3.42) and (3.45), we arrive at the following result for $U_1(\vec{\zeta})$:

$$\begin{aligned} U_1(\vec{\zeta}) = & -\alpha \sum_{i \le J} i^2 - 2g \sum_{0 \le i \le J} \langle \phi_\zeta \otimes \varphi_i \mid \delta \mid \phi_\zeta \otimes \varphi_i \rangle \\ & - 2g \sum_{\substack{0 \le i \le J \\ k > J}} \langle \varphi_k \otimes \varphi_i \mid \delta \mid \varphi_k \otimes \varphi_i \rangle \, a_k^\dagger a_k \\ & - \mu \left[(2 \left\| \Phi_\zeta \right\|^2 - 2N + 2N^> + 1)(J+1) + 2 \left\| \Phi_\zeta \right\|^2 \right] \end{aligned} \tag{3.46}$$

The next step is to calculate $U_2(\vec{\zeta})$.

By (3.37) we see that $U_2(\vec{\zeta}) = -\frac{1}{2} \sum_{i=0}^{J} \partial_{\zeta_i} \partial_{\zeta_i^*} U_1(\vec{\zeta})$. Hence the only relevant terms in $U_1(\vec{\zeta})$ are those where some ζ_i^* and ζ_i are left. This is only the case for the second term and the μ-term in (3.46).

Carrying out the differentiation we get

$$U_2(\vec{\zeta}) = g \sum_{0\leq i,j\leq J} \langle \varphi_i \otimes \varphi_j \mid \delta \mid \varphi_i \otimes \varphi_j \rangle + \mu(J+1)(J+2) \, . \tag{3.47}$$

It is now the task to bound the terms in $U_1(\vec{\zeta})$ and $U_2(\vec{\zeta})$ from below.

Obviously $U_2(\vec{\zeta}) \geq 0$ and since we are looking for a lower bound on $h_u(\vec{\zeta})$ we can neglect $U_2(\vec{\zeta})$ from now on. To bound the various terms in $U_1(\vec{\zeta})$ from below, we can use the fact that

$$\sum_{i\leq J} |\varphi_i(z)|^2 = \sum_{i\leq J} |z|^i \, (\pi i!)^{-1} \leq \pi^{-1} e^{|z|} \, , \tag{3.48}$$

simply by using the definition (3.11). With this inequality we are able to bound the interaction terms in $U_1(\vec{\zeta})$. We note that

$$\begin{aligned}
\sum_{0\leq i\leq J} \langle \phi_\zeta \otimes \varphi_i | \, \delta \, | \phi_\zeta \otimes \varphi_i \rangle &= \sum_{i,j\leq J} \zeta_j^* \zeta_j \, \langle \varphi_j | \, \delta \, | \varphi_j \rangle \, \langle \varphi_i | \, \delta \, | \varphi_i \rangle \\
&= \sum_{i,j\leq J} \zeta_j^* \zeta_j \int_{\mathbb{C}} |\varphi_j|^2 \, e^{-2|z|^2} dz \int_{\mathbb{C}} |\varphi_i|^2 \, e^{-2\left|z'\right|^2} dz' \\
&\leq \sum_{j\leq J} \zeta_j^* \zeta_j \pi^{-2} \int_{\mathbb{C}} e^{-|z|^2} dz \int_{\mathbb{C}} e^{-\left|z'\right|^2} dz' \\
&= \|\phi_\zeta\|^2 \, \pi^{-1} \, ,
\end{aligned} \tag{3.49}$$

where we have used (1.20) and (3.48). The same can be done for the second interaction term in (3.46),

$$\sum_{i\leq J, k>J} \langle \varphi_k \otimes \varphi_i | \, \delta \, | \varphi_k \otimes \varphi_i \rangle \, a_k^\dagger a_k \leq N^{>} \pi^{-1} \, . \tag{3.50}$$

The μ-term in $U_1(\vec{\zeta})$ can be bounded simply by neglecting the positive term and the $\mu(J+1)$-term is compensated by the last term in $U_2(\vec{\zeta})$.

Altogether $U_1(\vec{\zeta})$ is bounded from below by

$$\begin{aligned}
U_1(\vec{\zeta}) \geq & -\alpha \frac{J(J+1)(2J+1)}{6} - \left(\tfrac{2g}{\pi} + 2\mu(J+2)\right) \|\phi_\zeta\|^2 \\
& - 2N^{>} \left[\mu(J+1) + \tfrac{g}{\pi}\right] \, .
\end{aligned} \tag{3.51}$$

At this point it becomes clear why we could not choose $J \to \infty$, since we would not get a finite lower bound.

It has to be required that the term with highest order in J is much smaller than the GP-energy, $E^{GP}(N, \alpha, g) \approx N(Ng)^{2/3}\alpha^{1/3}$, i.e.

$$\alpha J^3 \ll N(Ng)^{2/3}\alpha^{1/3} \Longrightarrow J \ll N(\frac{g}{N^2\alpha})^{2/9} \, . \tag{3.52}$$

What we have achieved until now is a lower bound on $U_1(\vec{\zeta})$ and a precise restriction for J. What is left to do is to bound the remaining lower symbol $h_\ell(\vec{\zeta})$.

3.1.5 Step four

As mentioned right before, we need to bound the terms in $h_\ell(\vec{\zeta})$, given in (3.38). There are two terms in (3.38) which require some effort. The one is the η-dependent term and the other the interaction term. The interaction term will cost the most efforts and will be treated in Step 5. In this subsection we find an *upper* bound for the η-dependent term.

We know by (3.22) that $\mathbb{H}$ and by (3.25) even more H' in (3.32) is bounded below by the minimum over $\vec{\zeta}$ of the ground state energy of $h_u(\vec{\zeta})$. We will denote the minimizing vector by $\vec{\zeta_0}$ and the corresponding expectation value in the ground state of $h_u(\vec{\zeta_0})$ by $\langle . \rangle_0$. The η-dependent term is an operator on $\mathcal{F}(\mathcal{H}^>)$, so we are looking for an upper bound in the state $\langle . \rangle_0$.

To achieve this we use the fact that H' is bounded above by $E_0(N,\alpha,g)(1+b\eta) \leq 1.5E^{\mathrm{GP}}(N,\alpha,g)$. What is left is to find a simple lower bound for all the terms in $h_u(\vec{\zeta})$, except the η-term.

We will see that for small $g/(N^2\alpha)$,

$$\eta \left\langle \sum_{i>J} \chi_J(i) a_i^\dagger a_i \right\rangle_0 \leq 2\frac{E^{GP}(N,\alpha,g)}{\alpha J^{5/2}} \ . \tag{3.53}$$

To show this, we start with finding a lower bound on $h_\ell(\vec{\zeta})$ in (3.38). Since the interaction term is always positive, we can neglect it. Of course this loss would cause a massive error in the final result and therefore we will reinsert it at a later stage. However, to show (3.53) it is sufficient since we are looking for any lower bound and not for the best one. The first term in (3.38) can be neglected as well, while for the second term we simply use

$$\alpha \sum_{i>J} i^2 a_i^\dagger a_i \geq \alpha(J+1)^2 N^> \ . \tag{3.54}$$

The μ-term consists of the terms inside the brackets of (3.44). Hence

$$\begin{aligned} h_\ell(\vec{\zeta}) \geq \eta\alpha J^{5/2} \sum_{i>J} \chi_J(i) a_i^\dagger a_i + \alpha(J+1)^2 N^> \\ + \mu\Big[(\|\phi_\zeta\|^2 - N)^2 + 2N^>(\|\phi_\zeta\|^2 - N) \\ + (N^>)^2 + \|\phi_\zeta\|^2 \Big] \ . \end{aligned} \tag{3.55}$$

Next we will use (3.51) to bound $h_u(\vec{\zeta})$ from below,

$$\begin{aligned} h_u(\vec{\zeta}) \geq{}& \eta\alpha J^{3/2} \sum_{i>J} \chi_J(i) a_i^\dagger a_i + \mu \left(\|\phi_\zeta\|^2 - N \right)^2 \\ &+ N^> \left[(J+1)(\alpha(J+1) - 2\mu) - \tfrac{2g}{\pi} - 2\mu N \right] \\ &- N \left(\tfrac{2g}{\pi} + 2\mu(J+2) \right) - \alpha \frac{J(J+1)(2J+1)}{6} \\ &- \left(\tfrac{2g}{\pi} + 2\mu(J+2) \right) \left(\|\phi_\zeta\|^2 - N \right) . \end{aligned} \tag{3.56}$$

We would like to drop all the $\|\phi_\zeta\|^2$- and $N^>$-terms. The former ones can be dealt with by applying

$$a^2 - 2ab \geq -b^2 \tag{3.57}$$

to the second and last term in (3.56). This leads us to

$$\begin{aligned} h_u(\vec{\zeta}) \geq{}& \eta\alpha J^{5/2} \sum_{i>J} \chi_J(i) a_i^\dagger a_i \\ &+ N^> \left[(J+1)(\alpha(J+1) - 2\mu) - \tfrac{2g}{\pi} - 2\mu N \right] \\ &- N \left(\tfrac{2g}{\pi} + 2\mu(J+2) \right) \\ &- \alpha \frac{J(J+1)(2J+1)}{6} - \frac{\left(\tfrac{2g}{\pi} + 2\mu(J+2) \right)^2}{4\mu} . \end{aligned} \tag{3.58}$$

To get rid of the $N^>$-terms, we have to choose μ in such a way that the second line in (3.58) is always positive, i.e.

$$(J+1)(\alpha(J+1) - 2\mu) - \tfrac{2g}{\pi} - 2\mu N > 0 . \tag{3.59}$$

First we choose

$$\mu \ll \alpha J \quad \text{and} \quad g \ll \alpha J^2 . \tag{3.60}$$

Moreover we make the choice of

$$\mu = \frac{\alpha J^2}{4N} . \tag{3.61}$$

Hence (3.59) is fulfilled and we arrive, for small $g/(N^2\alpha)$, at

$$\begin{aligned} h_u(\vec{\zeta}) \geq{}& \eta\alpha J^{5/2} \sum_{i>J} \chi_J(i) a_i^\dagger a_i \\ &- N \left(\tfrac{2g}{\pi} + 2\mu(J+2) \right) \\ &- \alpha \frac{J(J+1)(2J+1)}{6} - \frac{\left(\tfrac{2g}{\pi} + 2\mu(J+2) \right)^2}{4\mu} . \end{aligned} \tag{3.62}$$

For our choice of μ, (3.61), g, (3.60) and J, (3.52), we see that all terms in the last two lines of (3.62) are much smaller than $E^{\mathrm{GP}}(N,\alpha,g)$, provided $g/N^2\alpha \ll 1$. Hence we get

$$h_u(\vec{\zeta}) \geq \eta\alpha J^{5/2} \sum_{i>J} \chi_J(i) a_i^\dagger a_i - 0.5 E^{\mathrm{GP}}(N,\alpha,g) \tag{3.63}$$

and since we know that the ground state energy of $h_u(\vec{\zeta_0})$ is bounded above by $E_0(N,\alpha,g)(1+b\eta) \leq 1.5E^{\mathrm{GP}}(N,\alpha,g)$, as mentioned above, we finally arrive at

$$1.5E^{\mathrm{GP}}(N,\alpha,g) \geq \eta \left\langle \sum_{i>J} \chi_J(i) a_i^\dagger a_i \right\rangle_0 - \tfrac{1}{2} E^{\mathrm{GP}}(N,\alpha,g)\ . \tag{3.64}$$

This proves inequality (3.53).

What we have achieved in this subsection is an upper bound for the η-term in the lower symbol and we have fixed the value of μ. What is left over is the bound for the interaction term in the lower symbol as well as the definition of the parameters η and J.

3.1.6 Step five

Until now we have found bounds for all terms in the upper symbol, except for the interaction terms. The upcoming subsections are devoted to this missing bound and we will see that this one is the hardest to get.

Let us have a look at the interaction term in $h_\ell(\vec{\zeta})$,

$$\frac{g}{2} \left\langle \vec{\zeta} \right| \sum_{ijk} \langle \varphi_i \otimes \varphi_j \mid \delta \mid \varphi_k \otimes \varphi_{i+j-k} \rangle\, a_i^\dagger a_j^\dagger a_k a_{i+j-k} \left| \vec{\zeta} \right\rangle\ . \tag{3.65}$$

This expression is a polynomial in the $a_i^\dagger$ and a_i of degree four. What we want to do, is to reduce this polynomial to degree one, i.e., to get an expression linear in the $a_i^\dagger$ and a_i. We can do this by using the positivity of δ and neglect the interaction between the modes $> J$.

This loss of interaction energy obviously causes an error in the ground state energy. However, in the end we are interested in a lower bound for the ground state energy in the sector of particle number N, i.e., $\|\phi_{\zeta_0}\|^2 \approx N$. For a suitable choice of μ, η and J we try to be close to this case and therefore can accept the missing energy terms.

There are only five possible cases left for the interaction between modes:

$$
\begin{aligned}
&1.\ i \le J,\ j \le J,\ k \le J,\ i+j-k \le J \\
&2.\ i > J,\ j \le J,\ k \le J,\ i+j-k \le J \\
&3.\ i \le J,\ j > J,\ k \le J,\ i+j-k \le J \\
&4.\ i \le J,\ j \le J,\ k \le J,\ i+j-k > J \\
&5.\ i \le J,\ j \le J,\ k > J,\ i+j-k \le J
\end{aligned}
$$

In the 1. case we have

$$
\begin{aligned}
&\frac{g}{2} \sum_{i,j,k \le J} \langle \varphi_i \otimes \varphi_j \mid \delta \mid \varphi_k \otimes \varphi_{i+j-k} \rangle \, \zeta_i^* \zeta_j^* \zeta_k \zeta_{i+j-k} = \\
&= \frac{g}{2} \langle \phi_\zeta \otimes \phi_\zeta \mid \delta \mid \phi_\zeta \otimes \phi_\zeta \rangle = \frac{g}{2} \int |\phi_\zeta|^4 \, e^{-2|z|^2} dz \, . \qquad (3.66)
\end{aligned}
$$

The sum of the second an the third case is given by

$$
g \sum_{i>J} \langle \varphi_i \otimes \phi_\zeta \mid \delta \mid \phi_\zeta \otimes \phi_\zeta \rangle \, a_i^* \qquad (3.67)
$$

and the sum of the fourth and fifth case is

$$
g \sum_{k>J} \langle \phi_\zeta \otimes \phi_\zeta \mid \delta \mid \phi_\zeta \otimes \varphi_k \rangle \, a_k \, . \qquad (3.68)
$$

Hence, by putting together (3.66), (3.67) and (3.68) we get

$$
\begin{aligned}
&\frac{g}{2} \left\langle \vec{\zeta} \middle| \sum_{ijk} \langle \varphi_i \otimes \varphi_j \mid \delta \mid \varphi_k \otimes \varphi_{i+j-k} \rangle \, a_i^\dagger a_j^\dagger a_k a_{i+j-k} \middle| \vec{\zeta} \right\rangle \\
&\ge \frac{g}{2} \int |\phi_\zeta|^4 \, e^{-2|z|^2} dz + g \sum_{k>J} \langle \phi_\zeta \otimes \phi_\zeta \mid \delta \mid \phi_\zeta \otimes \varphi_k \rangle \, a_k + \text{h.c.} \, . \qquad (3.69)
\end{aligned}
$$

We can now bound $h_\ell(\vec{\zeta})$ from below by noting that

$$
\alpha \left\langle \phi_\zeta \mid L^2 \mid \phi_\zeta \right\rangle = \alpha \sum_i \zeta_i^* \left\langle \varphi_i \mid L^2 \mid \varphi_i \right\rangle \zeta_i = \alpha \sum_i i^2 \zeta_i^* \zeta_i \, , \qquad (3.70)
$$

for $\varphi_i(z)$ given in (3.11) and using (3.69). What we get is

$$
\begin{aligned}
h_\ell(\vec{\zeta}) \ge\ & \mathcal{E}^{\mathrm{GP}}(\phi_\zeta) + \alpha \sum_{i>J} i^2 a_i^* a_i \\
& + \mu \Big[\left(\|\phi_\zeta\|^2 - N \right)^2 + 2N^> \left(\|\phi_\zeta\|^2 - N \right) \\
& \quad + (N^>)^2 + \|\phi_\zeta\|^2 \Big] \\
& + g \sum_{k>J} \langle \phi_\zeta \otimes \phi_\zeta \mid \delta \mid \phi_\zeta \otimes \varphi_k \rangle \, a_k + \text{h.c.} \, . \qquad (3.71)
\end{aligned}
$$

For the upper symbol $h_u(\overrightarrow{\zeta})$ we can proceed as in the previous subsection ((3.55), (3.58) and (3.62)) and arrive at

$$\begin{aligned}
h_u(\overrightarrow{\zeta}) \geq \mathcal{E}^{\mathrm{GP}}(\phi_\zeta) &+ \frac{\mu}{2}\left(\|\phi_\zeta\|^2 - N\right)^2 \\
&+ g\sum_{k>J} \langle \phi_\zeta \otimes \phi_\zeta \mid \delta \mid \phi_\zeta \otimes \varphi_k \rangle\, a_k + \text{h.c.} \\
&- N\left(\tfrac{2g}{\pi} + 2\mu(J+2)\right) - \alpha \frac{J(J+1)(2J+1)}{6} \\
&- \frac{\left(\tfrac{2g}{\pi} + 2\mu(J+2)\right)^2}{4\mu} \, . \qquad (3.72)
\end{aligned}$$

For our choice of μ and J all the terms in the last two lines are much smaller than $E^{\mathrm{GP}}(N, \alpha, g)$. The term $\frac{\mu}{2}(\|\phi_\zeta\|^2 - N)^2$ will not be neglected because it allows us to check later on whether our choice of the parameters imply $\|\phi_{\zeta_0}\|^2 \approx N$ or not. Or in other words, whether

$$E^{\mathrm{GP}}(\|\phi_{\zeta_0}\|^2, \alpha, g) \approx E^{\mathrm{GP}}(N, \alpha, g) \qquad (3.73)$$

or not.

The question may arise why the η-term has been neglected. The answer is that on the one hand we have already used it in step four in order to choose μ and J and on the other hand it will be revived anyway to bound the remaining interaction terms. This bound will be completed in the last subsections.

3.1.7 Step six

What is left over to do is to treat the two remaining interaction terms in the upper symbol, which are

$$g\sum_{k>J} \langle \phi_\zeta \otimes \phi_\zeta \mid \delta \mid \phi_\zeta \otimes \varphi_k \rangle\, a_k + \text{h.c.} \, . \qquad (3.74)$$

At the end of the last subsection we briefly mentioned that the bound for the η-dependent term will be used to bound the interaction terms. This will be done now and the connection between the η-term and these interaction terms is the following inequality:

For any sequence of complex numbers c_k and positive numbers e_k we have

$$\sum_k \left(c_k a_k + c_k^* a_k^\dagger\right) \leq \sum_k \frac{|c_k|^2}{e_k} + \sum_k e_k a_k^\dagger a_k \, . \qquad (3.75)$$

In our case we will choose

$$c_k = \begin{cases} 0 & \text{if } x \le J \\ \langle \phi_\zeta \otimes \phi_\zeta \mid \delta \mid \phi_\zeta \otimes \varphi_k \rangle & \text{if } x > J \end{cases} \tag{3.76}$$

and

$$e_k = \kappa^{-1} \chi_J(k) \,, \tag{3.77}$$

for $k > J$ with $\kappa > 0$ and $\chi_J(k)$ given in (3.27). For the first term on the r.h.s of (3.75) we note that

$$\begin{aligned} |c_k|^2 &= \langle \phi_\zeta \otimes \phi_\zeta \mid \delta \mid \phi_\zeta \otimes \varphi_k \rangle \langle \varphi_k \otimes \phi_\zeta \,|\delta|\, \phi_\zeta \otimes \phi_\zeta \rangle \\ &= \langle \phi_\zeta \mid \langle \phi_\zeta \mid \delta \mid \phi_\zeta \rangle \mid \varphi_k \rangle \langle \varphi_k \mid \langle \phi_\zeta \mid \delta \mid \phi_\zeta \rangle \mid \phi_\zeta \rangle \\ &= \left\langle \phi_\zeta \mid |\phi_\zeta(z)|^2 e^{-|z|^2} \mid \varphi_k \right\rangle \left\langle \varphi_k \mid |\phi_\zeta(z)|^2 e^{-|z|^2} \mid \phi_\zeta \right\rangle \,. \end{aligned} \tag{3.78}$$

By defining

$$\chi_J^{-1}(L) \equiv \sum_{k=J+1}^{\infty} \frac{1}{\chi_J(k)} \left| \varphi_k \right\rangle \left\langle \varphi_k \right| \tag{3.79}$$

and

$$\rho_\zeta \equiv |\phi_\zeta(z)|^2 e^{-|z|^2} \,, \tag{3.80}$$

we get

$$\sum_k \frac{|c_k|^2}{e_k} = \kappa \left\langle \phi_\zeta \mid \rho_\zeta \chi_J^{-1}(L) \rho_\zeta \mid \phi_\zeta \right\rangle \,. \tag{3.81}$$

Next we will bound this term from above. For this reason we make use of several inequalities from functional- and Fourier-analysis. It is this bound which demands the most efforts to be proved and the whole last subsection will be devoted to this task.

We shall show that

$$\left\langle \phi_\zeta \mid \rho_\zeta \chi_J^{-1}(L) \rho_\zeta \mid \phi_\zeta \right\rangle \le 6 \left[\int_{\mathbb{C}} |\phi_\zeta(z)|^4 e^{-2|z|^2} dz \right]^{3/2} \,. \tag{3.82}$$

By using this inequality, we see that

$$\left(c_k a_k + c_k^* a_k^\dagger \right) \le 6\kappa \left[\int_{\mathbb{C}} |\phi_\zeta(z)|^4 e^{-2|z|^2} dz \right]^{3/2} + \sum_k e_k a_k^\dagger a_k \,. \tag{3.83}$$

Obviously the second term on the r.h.s. of (3.83) must be bounded from above, in order to get a finite bound. This is exactly the point where we make use of the $\chi_J(L)$-term, which we have bounded already in step four. There we have shown in equation (3.53) that

$$\left\langle \sum_k e_k a_k^\dagger a_k \right\rangle_0 \le 2 \frac{E^{\mathrm{GP}}(N, \alpha, g)}{\kappa \eta \alpha J^{5/2}} \,. \tag{3.84}$$

Recall the definition of e_k in (3.77) and that the expectation value is taken in the ground state of $h_u(\vec{\zeta_0})$.

We use (3.84) in (3.83) and get a bound depending on the parameter κ. In order to find the optimal bound in dependence of this parameter, we look for the minimum of

$$f(\kappa) = 6\kappa \left[\int_{\mathbb{C}} |\phi_\zeta(z)|^4 \, e^{-2|z|^2} dz \right]^{3/2} + \frac{E^{\mathrm{GP}}(N, \alpha, g)}{\kappa \eta \alpha J^{5/2}} \,, \tag{3.85}$$

which is attained at

$$\kappa = \left(\tfrac{1}{3}\right)^{1/2} \left[\int_{\mathbb{C}} |\phi_\zeta(z)|^4 \, e^{-2|z|^2} dz \right]^{-3/4} \left(\frac{E^{GP}(N, \alpha, g)}{\eta \alpha J^{5/2}} \right)^{1/2} . \tag{3.86}$$

This optimization leads us to

$$-\left(c_k a_k + c_k^* a_k^\dagger \right) \geq -4\sqrt{3} \left[\int_{\mathbb{C}} |\phi_\zeta(z)|^4 \, e^{-2|z|^2} dz \right]^{3/4} \left(\frac{E^{GP}(N, \alpha, g)}{\eta \alpha J^{5/2}} \right)^{1/2} . \tag{3.87}$$

Furthermore we use Young's inequality to split the product in (3.87) into a sum. Young's inequality is a special case of the inequality of weighted arithmetic and geometric means, which says that for nonnegative real numbers a, b, q and d such that $1 = 1/p + 1/q$ we have

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \,. \tag{3.88}$$

There also exists a scaled version, which is derived from (3.88) by defining $a \equiv (\beta p)^{1/p} x$ and $b \equiv (\beta p)^{-1/p} y$,

$$xy \leq \beta x^p + \frac{(p\beta)^{1-q}}{q} y^q \,, \tag{3.89}$$

with $\beta > 0$ and $x, y \in \mathbb{R}$. In our case we will choose

$$\beta \longrightarrow \tfrac{1}{8\sqrt{3}} \, \beta \quad \text{and} \quad p = 4/3 \,,\ q = 4 \tag{3.90}$$

and arrive at

$$\begin{aligned} -\left(c_k a_k + c_k^* a_k^\dagger \right) &\geq -\tfrac{\beta}{2} \int_{\mathbb{C}} |\phi_{\zeta_0}(z)|^4 \, e^{-2|z|^2} dz - 9 \left(\frac{6}{\beta} \right)^3 \left(\frac{E^{\mathrm{GP}}(N, \alpha, g)}{\eta \alpha J^{5/2}} \right)^2 \\ &\geq -\mathcal{E}^{\mathrm{GP}}(\phi_{\zeta_0}) - 9 \left(\frac{6}{\beta} \right)^3 \left(\frac{E^{\mathrm{GP}}(N, \alpha, g)}{\eta \alpha J^{5/2}} \right)^2 . \end{aligned} \tag{3.91}$$

We have bounded now all the terms in the upper symbol and hence we are almost completed. The only thing left to do is to determine the remaining parameters η, β and J and of course to proof the inequality in (3.82). This will be done in the last two steps.

3.1.8 Step seven

The discussion in the last subsection has led us to the final lower bound for the ground state energy of the upper symbol,

$$\begin{aligned}h_u(\vec{\zeta_0}) \geq (1-\beta)\mathcal{E}^{\mathrm{GP}}(\phi_{\zeta_0}) + \frac{\mu}{2}\left(\|\phi_{\zeta_0}\|^2 - N\right)^2 \\ - N\left(\tfrac{2g}{\pi} + 2\mu(J+2)\right) - \frac{\left(\tfrac{2g}{\pi} + 2\mu(J+2)\right)^2}{4\mu} \\ - \alpha\frac{J(J+1)(2J+1)}{6} - g\,9\left(\frac{6}{\beta}\right)^3\left(\frac{E^{\mathrm{GP}}(N,\alpha,g)}{\eta\alpha J^{5/2}}\right)^2 .\end{aligned} \tag{3.92}$$

What remains to do is to choose the parameters η, β and J. They should be chosen in such a way that all the error terms are of the order $(\frac{g}{N^2\alpha})^b E^{\mathrm{GP}}(N,\alpha,g)$ with some positive exponent b and E^{GP} given in (2.19).

The biggest term of the second, third and fourth error term is αJ^3, so we require

$$\alpha J^3 = \left(\frac{g}{N^2\alpha}\right)^b E^{\mathrm{GP}}(N,\alpha,g) . \tag{3.93}$$

Further we set $\eta = \beta$ and choose

$$\eta = \left(\frac{g}{N^2\alpha}\right)^b . \tag{3.94}$$

For the last error term we require

$$g\beta^{-3}\eta^{-2}\alpha^{-2}J^{-5}E^{\mathrm{GP}}(N,\alpha,g) = \left(\frac{g}{N^2\alpha}\right)^b . \tag{3.95}$$

For the last parameter J we required in (3.52) that $J \ll N(\frac{g}{N^2\alpha})^{2/9}$. This restriction suggests the choice of

$$J = N\left(\frac{g}{N^2\alpha}\right)^u . \tag{3.96}$$

The equations (3.93), (3.94), (3.95) and (3.96) are leading us to a system of equations for the powers u and b,

$$\begin{aligned}\alpha N^3\left(\frac{g}{N^2\alpha}\right)^{3u} &= \left(\frac{g}{N^2\alpha}\right)^b N\,(Ng)^{2/3}\,\alpha^{1/3} \\ g\alpha^{-2}N^{-5}\left(\frac{g}{N^2\alpha}\right)^{-5u} N\,(Ng)^{2/3}\,\alpha^{1/3} &= \left(\frac{g}{N^2\alpha}\right)^{6b} ,\end{aligned} \tag{3.97}$$

where we have used (2.19) for $E^{\mathrm{GP}}(N,\alpha,g)$.

The solution is

$$b = \frac{5}{69} \qquad u = \frac{17}{69} . \tag{3.98}$$

Hence, the parameters are finaly given by

$$\begin{aligned} \eta &= \left(\frac{g}{N^2\alpha}\right)^{5/69} \quad , \quad \mu = \frac{\alpha J^2}{4N} \\ J &\approx N\left(\frac{g}{N^2\alpha}\right)^{17/69} \quad , \quad J_0 = \left\lfloor N\left(\frac{g}{N^2\alpha}\right)^{17/69}\right\rfloor , \end{aligned} \tag{3.99}$$

with $J_0 \leq J < 2J$. Note that J_0 is a big number for $gN/\alpha \geq 1$. Furthermore, we see that the requirement for J being much smaller than $N\left(\frac{g}{N^2\alpha}\right)^{2/9}$ is fulfilled within our choice of u since $17/69 > 2/9$.

Next we need to check whether our choice of the parameters implies

$$E^{\mathrm{GP}}(\|\phi_{\zeta_0}\|^2, \alpha, g) \approx E^{\mathrm{GP}}(N, \alpha, g) \tag{3.100}$$

or not. For this reason we estimate the second term in (3.92). This can be done similar to the procedure in step four, where we have used a lower and an upper bound for $h_u(\vec{\zeta_0})$ in order to bound the η-dependent term.

Therefore we note that within our choice of μ, η and J all the error terms in (3.92) are of the order $(g/N^2\alpha)^{5/69}E^{\mathrm{GP}}$, which means that they are much smaller than E^{GP}. Hence, we can bound them below by $-E^{\mathrm{GP}}(N,\alpha,g)/2$. On the other side we know that the ground state energy of $h_u(\vec{\zeta_0})$ is bounded above by $E^{\mathrm{GP}}(N,\alpha,g)(1+b\eta) \leq 1.5\, E^{\mathrm{GP}}(N,\alpha,g)$. Thus the second term in (3.92) can be estimated by

$$\frac{\mu}{2}\left(\|\phi_{\zeta_0}\|^2 - N\right)^2 \leq 2E^{\mathrm{GP}}(N,\alpha,g) + (\beta-1)\mathcal{E}^{\mathrm{GP}}(\phi_{\zeta_0}) \leq 2E^{\mathrm{GP}}(N,\alpha,g) \tag{3.101}$$

and hence

$$\left|\|\phi_{\zeta_0}\|^2 - N\right| \leq \left(\frac{4E^{\mathrm{GP}}(N,\alpha,g)}{\mu}\right)^{1/2} \approx N\left(\frac{g}{N^2\alpha}\right)^{2/23} \ll N\,. \tag{3.102}$$

What we see is that our choice was suitable and we get $\|\phi_{\zeta_0}\|^2 \approx N$, which implies (3.100). The term $\frac{\mu}{2}(\|\phi_{\zeta_0}\|^2 - N)^2$ will not be used anymore and we can neglect it in (3.92) in order to arrive at the final lower bound for the ground state energy of the upper symbol,

$$h_u(\vec{\zeta_0}) \geq E^{\mathrm{GP}}(N,\alpha,g)\left(1 - C\left(\frac{g}{N^2\alpha}\right)^{5/69}\right) \tag{3.103}$$

for some $C > 0$. This provides the desired lower bound for the ground state energy of H,

$$E_0(N,\alpha,g) \geq E^{\mathrm{GP}}(N,\alpha,g)\left(1 - C\left(\frac{g}{N^2\alpha}\right)^{5/69}\right)\,. \tag{3.104}$$

3.1.9 Step eight

In this subsection we proof the inequality in (3.82). The inequality is

$$\left\langle \phi_\zeta \mid \rho_\zeta \chi_J^{-1}(L) \rho_\zeta \mid \phi_\zeta \right\rangle \leq 6 \left[\int_{\mathbb{C}} \left|\phi_\zeta(z)\right|^4 e^{-2|z|^2} dz \right]^{3/2} . \tag{3.105}$$

For this reason we define a p-norm as

$$\|\phi\|_p = \left(\int_{\mathbb{C}} |\phi(z)|^p \, e^{-(p/2)|z|^2} dz \right)^{1/p} \tag{3.106}$$

for $1 \leq p < \infty$.

The first step is to bound the l.h.s of (3.105) by the operator norm of $\sqrt{\rho_\zeta} \chi_J^{-1}(L) \sqrt{\rho_\zeta}$. By recalling (3.80) we see that

$$\begin{aligned} \left\langle \phi_\zeta \mid \rho_\zeta \chi_J^{-1}(L) \rho_\zeta \mid \phi_\zeta \right\rangle &= \left\langle \phi_\zeta \mid |\phi(z)|^2 \, e^{-|z|^2} \chi_J^{-1}(L) \, |\phi(z)|^2 \, e^{-|z|^2} \mid \phi_\zeta \right\rangle \\ &\leq \left\langle \phi_\zeta \mid |\phi(z)|^2 \, e^{-|z|^2} \mid \phi_\zeta \right\rangle \sup_z \left\| \sqrt{\rho_\zeta} \chi_J^{-1}(L) \sqrt{\rho_\zeta} \right\| \\ &= \|\phi_\zeta\|_4^2 \left\| \sqrt{\rho_\zeta} \chi_J^{-1}(L) \sqrt{\rho_\zeta} \right\| \end{aligned} \tag{3.107}$$

where the latter norm in the last line is an operator norm on $L^2(\mathbb{C}, e^{-|z|^2} dz)$, which is equal to the operator norm of $\sqrt{\chi_J^{-1}(L)} \rho_\zeta \sqrt{\chi_J^{-1}(L)}$ on $\mathcal{H}$.

Next, we derive a pointwise bound on $\rho_\zeta = |\phi_\zeta(z)|^2 \, e^{-|z|^2}$. Therefore we make use of the projection operator $P_J(z, z') = \sum_{j=0}^{J} \varphi_j(z) \varphi_j(z')^*$, which projects states on the subspace of $\mathcal{H}$ consisting of modes with $j \leq J$. With this operator we have

$$|\phi_\zeta(z)| = \left| \int_{\mathbb{C}} P_J(z, z') \phi_\zeta(z') e^{-\left|z'\right|^2} dz' \right| . \tag{3.108}$$

The right side can be bounded by Hölder's inequality. Choosing $p = 1$ and $q = 4/3$ respectively, we have

$$|\phi_\zeta(z)| \leq \|\phi_\zeta\|_4 \, \|P_J(z, \cdot)\|_{4/3} . \tag{3.109}$$

To bound the 4/3-norm we use again Hölder's inequality and find that

$$\|P_J(z, \cdot)\|_{4/3} \leq \|P_J(z, \cdot)\|_2^{(4-3p)/(4-2p)} \, \|P_J(z, \cdot)\|_p^{p/(4-2p)} \tag{3.110}$$

for any $1 \leq p \leq 4/3$. To see this note that

$$\begin{aligned} \|P_J(z, \cdot)\|_{4/3} &= \left(\int P^{4/3} e^{-(2/3)|z|^2} dz \right)^{3/4} = \left(\int P^a P^b e^{-(a+b)/2|z|^2} dz \right)^{3/4} \\ &= \left(\left\| P^a P^b \right\|_1 \right)^{3/4} . \end{aligned} \tag{3.111}$$

The 1-norm can be bounded again by Hölder,

$$\left\| P^a P^b \right\| \leq \| P^a \|_{p'} \ \left\| P^b \right\|_{q'} \ . \tag{3.112}$$

Thus the parameters $a, b, p^{'}$ and $q^{'}$ have to fulfill the following equations,

$$\begin{aligned} a + b &= 4/3 \\ 1/p^{'} + 1/q^{'} &= 1 \\ p^{'} a &= 2 \\ q^{'} b &= p \end{aligned} \tag{3.113}$$

for which we find

$$a = \frac{2(4-3p)}{3(2-p)} \qquad b = \frac{2p}{3(2-p)} \ . \tag{3.114}$$

This leads us to (3.110). The reason for choosing the L^2 norm of $P_J(z, \cdot)$ in (3.110) is that it has a simple expression,

$$\| P_J(z, \cdot) \|_2^2 = \sum_{j=0}^{J} |\varphi_j(z)|^2 \ . \tag{3.115}$$

All the other norms with $p \neq 2$ have no simple expression. But what we can use is the following inequality:

$$\| P_J(z, \cdot) \|_p \leq \frac{1}{\sin(\pi/p)} \| P_\infty(z, \cdot) \|_p \tag{3.116}$$

This is a general result out of harmonic analysis, for a detailed explanation see [7, Ch.II, Sec.1] and [8]. By recalling (3.11) we see that $P_\infty(z, \cdot) = \pi^{-1} e^{-z z^{'*}}$ and hence

$$\| P_\infty(z, \cdot) \|_p \leq \frac{(2\pi/p)^{1/p}}{\pi} e^{|z|^2/2} \ . \tag{3.117}$$

We use this bound in (3.116) and get

$$\| P_J(z, \cdot) \|_p \leq \left[\frac{(2\pi/p)^{1/p}}{\pi \sin(\pi/p)} \right] e^{|z|^2/2} \ . \tag{3.118}$$

Putting together (3.109), (3.110) and (3.118), we can bound ρ_ζ by

$$\rho_\zeta \leq \left[\frac{(2\pi/p)^{1/p}}{\pi \sin(\pi/p)} \right]^{2p/(4-2p)} \| \phi_\zeta \|_4^2 \left(\sum_{j=0}^{J} |\varphi_j(z)|^2 \, e^{-|z|^2} \right)^{(4-3p)/(4-2p)} \ . \tag{3.119}$$

As we are free to choose p as long as $1 \leq p \leq 4/3$, we make the choice of $p = 5/6$. For this choice we get

$$\rho_\zeta \leq 4.02 \, \|\phi_\zeta\|_4^2 \left(\sum_{j=0}^{J} |\varphi_j(z)|^2 \, e^{-|z|^2} \right)^{1/4} . \tag{3.120}$$

This estimation we insert into (3.107) and arrive at

$$\begin{aligned} \|\phi_\zeta\|_4^2 \left\| \sqrt{\chi_J^{-1}(L)} \rho_\zeta \sqrt{\chi_J^{-1}(L)} \right\| &= \|\phi_\zeta\|_4^2 \left\langle \psi \mid \sqrt{\chi_J^{-1}(L)} \rho_\zeta \sqrt{\chi_J^{-1}(L)} \mid \psi \right\rangle \\ &\leq \|\phi_\zeta\|_4^2 \, 4.02 \, \|\phi_\zeta\|_4^2 \sum_{k \geq J+1} |d_k| \exp\left[\tfrac{1}{8} \left(\sqrt{k} - \sqrt{J+1} \right)^2 \right] \\ &\quad \times \int_{\mathbb{C}} |\varphi_k(z)|^2 \left(\sum_{j=0}^{J} |\varphi_j(z)|^2 \, e^{-|z|^2} \right)^{1/4} e^{-|z|^2} dz \, , \end{aligned} \tag{3.121}$$

for any $|\psi\rangle = \sum_{k>J} d_k \, |\varphi_k\rangle$.

To bound the integral we can use Jensen's inequality for the concave function $t \to t^{1/4}$. Jensen's inequality says that for any concave function τ and real valued function λ the following inequality holds:

$$\int \tau \circ \lambda d\mu \leq \tau \left(\int \lambda d\mu \right) \tag{3.122}$$

By using Jensen's inequality and noting that $|\varphi_k|^2 \, e^{-|z|^2} \leq (|\varphi_k|^2 \, e^{-|z|^2})^{1/4}$, we see that

$$\begin{aligned} &\int_{\mathbb{C}} |\varphi_k(z)|^2 \left(\sum_{j=0}^{J} |\varphi_j(z)|^2 \, e^{-|z|^2} \right)^{1/4} e^{-|z|^2} dz \\ &\leq \left(\int_{\mathbb{C}} |\varphi_k(z)|^2 \left(\sum_{j=0}^{J} |\varphi_j(z)|^2 \, e^{-|z|^2} \right) e^{-|z|^2} dz \right)^{1/4} . \end{aligned} \tag{3.123}$$

Now we can bound the inner bracket in (3.123) by using Stirling's formula, which says that $j! \geq (j/e)^j \sqrt{1+j}$ for any $j \geq 0$, see [9].

By using this we see that

$$|\varphi_j(z)|^2 \, e^{-|z|^2} \leq \tfrac{1}{\pi\sqrt{1+j}} e^{-(|z|-\sqrt{j})^2} . \tag{3.124}$$

The same can be done for the outer bracket in (3.123),

$$|\varphi_k(z)|^2\, e^{-|z|^2} \le \tfrac{3^{3/2}}{\pi\sqrt{8e}}\tfrac{1}{|z|} e^{-(|z|-\sqrt{k})^2}\ , \tag{3.125}$$

with $k \ge 1$. Thus we have

$$\begin{aligned}
&\int_{\mathbb{C}} |\varphi_j(z)|^2\, |\varphi_k(z)|^2\, e^{-2|z|^2} dz \\
&\le \tfrac{3^{3/2}}{\pi\sqrt{2e}}\tfrac{1}{\sqrt{1+j}} \int_0^\infty e^{-(r-\sqrt{j})^2} e^{-(r-\sqrt{k})^2} dr \\
&\le \tfrac{3^{3/2}}{\sqrt{4\pi e}}\tfrac{1}{\sqrt{1+j}} e^{-(\sqrt{j}-\sqrt{k})^2/2}\ .
\end{aligned} \tag{3.126}$$

The sum over j from this expression can be bounded above by the integral

$$\begin{aligned}
&\sum_{j=0}^{J} \tfrac{1}{\sqrt{1+j}} e^{-(\sqrt{j}-\sqrt{k})^2/2} \\
&\le \int_0^{J+1} e^{-(\sqrt{s}-\sqrt{k})^2/2} \tfrac{ds}{\sqrt{s}} \le 2\int_{-\infty}^0 e^{-(t+\sqrt{J+1}-\sqrt{k})^2/2} dt \\
&\le 2\int_{-\infty}^{\infty} e^{-(t+\sqrt{J+1}-\sqrt{k})^2/2} e^{-t(\sqrt{k}-\sqrt{J+1})^2} dt \\
&= \sqrt{8\pi} e^{-(\sqrt{k}-\sqrt{J+1})^2/2}\ .
\end{aligned} \tag{3.127}$$

If we insert this in (3.121), we finaly arrive at (3.105).

3.2 Proof of Theorem 3

The proof of Theorem 3 is similar to the one of Theorem 2. Therefore we will not be that explicit but rather mark out the points where differences occur.

First of all we summarise the parameter conditions which are valid in this case. We have

$$\frac{Ng\alpha^2}{\omega^3} \approx 1 , \tag{3.128}$$

which means the same as

$$N(Ng\omega)^{1/2} \approx N(Ng)^{2/3}\alpha^{1/3} . \tag{3.129}$$

Further we assume

$$\frac{g}{N\omega} \ll 1 \quad , \quad \frac{g}{N^2\alpha} \ll 1 \tag{3.130}$$

as well as

$$\frac{gN}{\omega} \geq 1 \quad , \quad \frac{gN}{\alpha} \geq 1 . \tag{3.131}$$

The starting point is the following Hamiltonian,

$$H(N,\omega,\alpha,g) = \omega \sum_{i=1}^{N} L_i + \alpha \sum_{i=1}^{N} L_i^2 + g \sum_{1\leq i<j\leq N} \delta_{ij} . \tag{3.132}$$

We modify this Hamiltonian by adding the $\chi_J(L)$ function given in (3.27),

$$H' = H + \left(\eta J^{3/2}\omega + \beta J^{5/2}\alpha\right) \sum_{i=1}^{N} \chi_J(L_i) \tag{3.133}$$

with two small parameters η, $\beta > 0$. The two last terms in (3.133) are bounded by

$$\eta\omega J^{3/2} \left\langle \sum_{i=1}^{N} \chi_J(L_i) \right\rangle \leq \eta 2^{3/2} \left(1 + 4\sqrt{\pi}\right) E_0(N,\omega,\alpha,g) \tag{3.134}$$

and

$$\beta\alpha J^{5/2} \left\langle \sum_{i=1}^{N} \chi_J(L_i) \right\rangle \leq \beta 2^{5/2} \left(1 + 4\sqrt{\pi}\right) E_0(N,\omega,\alpha,g) \tag{3.135}$$

both in the ground state of H.

Hence we see

$$E_0(N,\omega,\alpha,g) \geq \frac{E_0^{'}}{(1+b\eta+\tilde{b}\beta)}\ , \tag{3.136}$$

with $b = 2^{3/2}(1+4\sqrt{\pi})$ and $\tilde{b} = 2^{5/2}(1+4\sqrt{\pi})$.

In Step 1 we added the μ-term to $H^{'}$ in order to treat the problem irrespective of particle number. Now we add the same term to $H^{'}$ and afterwards we extend this new Hamiltonian to Fock space,

$$\begin{aligned}\mathbb{H} = &\sum_{i\geq 0}\left[\omega\, i + \alpha\, i^2 + \left(\omega\eta J^{3/2} + \alpha\beta J^{5/2}\right)\chi_J(i)\right] a_i^\dagger a_i \\ &+ \frac{g}{2}\sum_{ijk}\langle \varphi_i\otimes\varphi_j \mid \delta \mid \varphi_k\otimes\varphi_{i+j-k}\rangle\, a_i^\dagger a_j^\dagger a_k a_{i+j-k} \\ &+ \mu\left(\textstyle\sum_{i\geq 0} a_i^\dagger a_i - N\right)^2 \ .\end{aligned} \tag{3.137}$$

Next we compute the upper symbol given by

$$h_u(\overrightarrow{\zeta}) = \exp\left(-\sum_{i=0}^{J}\partial_{\zeta_i}\partial_{\zeta_i^*}\right) h_\ell(\overrightarrow{\zeta}) = h_\ell(\overrightarrow{\zeta}) + U_1(\overrightarrow{\zeta}) + U_2(\overrightarrow{\zeta})\ . \tag{3.138}$$

With the results from subsection 3.1.4. we have

$$\begin{aligned}U_1(\overrightarrow{\zeta}) = &-\omega\sum_{i\leq J} i - \alpha\sum_{i\leq J} i^2 - 2g\sum_{0\leq i\leq J}\langle \phi_\zeta\otimes\varphi_i \mid \delta \mid \phi_\zeta\otimes\varphi_i\rangle \\ &- 2g\sum_{\substack{0\leq i\leq J\\ k>J}}\langle \varphi_k\otimes\varphi_i \mid \delta \mid \varphi_k\otimes\varphi_i\rangle\, a_k^\dagger a_k \\ &- \mu\left[(2\left\|\Phi_\zeta\right\|^2 - 2N + 2N^{>} + 1)(J+1) + 2\left\|\Phi_\zeta\right\|^2\right]\ ,\end{aligned} \tag{3.139}$$

$$U_2(\overrightarrow{\zeta}) = g\sum_{0\leq i,j\leq J}\langle \varphi_i\otimes\varphi_j \mid \delta \mid \varphi_i\otimes\varphi_j\rangle + \mu(J+1)(J+2) \tag{3.140}$$

and

$$\begin{aligned}h_\ell(\overrightarrow{\zeta}) = &\ \omega\sum_{i\leq J} i\zeta_i^*\zeta_i + \alpha\sum_{i\leq J} i^2\zeta_i^*\zeta_i + \mu\left\langle\overrightarrow{\zeta}\right|\left(\textstyle\sum_{i\geq 0} a_i^\dagger a_i - N\right)^2\left|\overrightarrow{\zeta}\right\rangle \\ &+ \left\langle\overrightarrow{\zeta}\right|\frac{g}{2}\sum_{ijk}\langle \varphi_i\otimes\varphi_j \mid \delta \mid \varphi_k\otimes\varphi_{i+j-k}\rangle\, a_i^\dagger a_j^\dagger a_k a_{i+j-k}\left|\overrightarrow{\zeta}\right\rangle \\ &+ \sum_{i\leq J}\left(\omega i + \alpha i^2 + \left(\eta\omega J^{3/2} + \beta\alpha J^{5/2}\right)\chi_J(i)\right) a_i^\dagger a_i\ .\end{aligned} \tag{3.141}$$

For a lower bound on $h_u(\overrightarrow{\zeta})$ we can drop U_2 since $U_2 > 0$. To bound $U_1(\overrightarrow{\zeta})$ we can use the bound in (3.51),

$$U_1(\overrightarrow{\zeta}) \geq -\frac{\omega}{2}J(J+1) - \alpha\frac{J(J+1)(2J+1)}{6} - \left(\tfrac{2g}{\pi} + 2\mu(J+2)\right)\|\phi_\zeta\|^2 \\ - 2N^{>}\left[\mu(J+1) + \tfrac{g}{\pi}\right] \ . \tag{3.142}$$

To keep this bound finite we require

$$J \ll N\left(\frac{g}{N^2\alpha}\right)^{2/9} \quad \text{and} \quad J \ll N\left(\frac{g}{N\omega}\right)^{1/4} \ . \tag{3.143}$$

The next step is to find an upper bound for the $\chi_J(i)$-term. The crucial estimation was a lower bound on $h_u(\overrightarrow{\zeta})$ given in (3.58). This bound is now given by

$$\begin{aligned} h_u(\overrightarrow{\zeta}) \geq\ & \left(\eta\omega J^{3/2} + \beta\alpha J^{5/2}\right)\sum_{i>J}\chi_J(i)a_i^\dagger a_i \\ & + N^{>}\left[(J+1)(\omega-\mu) - \frac{g}{\pi} - \mu N\right] \\ & + N^{>}\left[(J+1)(\alpha(J+1)-\mu) - \frac{g}{\pi} - \mu N\right] \\ & - N\left(\tfrac{2g}{\pi} + 2\mu(J+2)\right) \\ & - \frac{\omega}{2}J(J+1) - \alpha\frac{J(J+1)(2J+1)}{6} - \frac{\left(\frac{2g}{\pi} + 2\mu(J+2)\right)^2}{4\mu} \ . \end{aligned} \tag{3.144}$$

In order to keep the second and third line positive we choose

$$\begin{aligned} \mu \ll \omega \quad &, \quad g \leq \omega J \\ \mu \ll \alpha J \quad &, \quad g \leq \alpha J^2 \ . \end{aligned} \tag{3.145}$$

A possible choice for μ, satisfying these conditions, is

$$\mu = \frac{\omega J}{4N} \ . \tag{3.146}$$

Within this choice all the error terms in (3.144) are much smaller than $E^{\mathrm{GP}}(N,\omega,\alpha,g)$ if $g \ll N\omega$ and $g \ll N^2\alpha$. Hence we can bound these terms from below by $-E^{\mathrm{GP}}/2$.

For an upper bound on $h_u(\overrightarrow{\zeta_0})$ we use (3.134) and (3.135) to arrive at

$$\begin{aligned} 1.5E^{\mathrm{GP}}(N,\omega,\alpha,g) &\geq E_0(N,\omega,\alpha,g)(1 + b\eta + \tilde{b}\beta) \\ &\geq \left(\eta\omega J^{3/2} + \beta\alpha J^{5/2}\right)\left\langle\sum_{i>J}\chi_J(i)a_i^\dagger a_i\right\rangle_0 - \frac{1}{2}E^{\mathrm{GP}}(N,\omega,\alpha,g) \ . \end{aligned} \tag{3.147}$$

Thus

$$\left\langle \sum_{i>J} \chi_J(i) a_i^\dagger a_i \right\rangle_0 \leq 2 \frac{E^{\mathrm{GP}}(N,\omega,\alpha,g)}{\left(\eta\omega J^{3/2} + \beta\alpha J^{5/2}\right)} \ . \tag{3.148}$$

After bounding the $\chi_J(i)$-term, we now take the interaction terms in $h_\ell(\overrightarrow{\zeta_0})$ into account.

To reduce the order of the $a_i^\dagger$ and a_i in the interaction terms we neglected all interactions between modes $> J$. This procedure led us to (3.72), which is in this case

$$\begin{aligned} h_u(\overrightarrow{\zeta}) \geq\ & \mathcal{E}^{\mathrm{GP}}(\phi_\zeta) + \frac{\mu}{2}\left(\|\phi_\zeta\|^2 - N\right)^2 \\ & + g \sum_{k>J} \langle \phi_\zeta \otimes \phi_\zeta \mid \delta \mid \phi_\zeta \otimes \varphi_k \rangle \, a_k + \mathrm{h.c.} \\ & - N\left(\tfrac{2g}{\pi} + 2\mu(J+2)\right) - \frac{\omega}{2} J(J+1) \\ & - \alpha \frac{J(J+1)(2J+1)}{6} - \frac{\left(\frac{2g}{\pi} + 2\mu(J+2)\right)^2}{4\mu} \ . \end{aligned} \tag{3.149}$$

For the remaining interaction terms we used the inequality in (3.75) which was

$$\sum_k (c_k a_k + c_k^* a_k^\dagger) \leq \sum_k \frac{|c_k|^2}{e_k} + \sum_k e_k a_k^\dagger a_k \ . \tag{3.150}$$

To bound the first term on the r.h.s we used the inequality given in (3.82). The second term is equivalent to the $\chi_J(i)$-term for which we can use the bound in (3.148). Hence, for the interaction terms, we finally arrive at

$$\begin{aligned} & g \sum_{k>J} \langle \phi_\zeta \otimes \phi_\zeta \mid \delta \mid \phi_\zeta \otimes \varphi_k \rangle \, a_k + \mathrm{h.c.} \\ & \geq -\frac{\gamma}{2} \int_{\mathbb{C}} |\phi_{\zeta_0}(z)|^4 \, e^{-2|z|^2} dz - 9 \left(\frac{6}{\gamma}\right)^3 \left(\frac{E^{\mathrm{GP}}(N,\omega,\alpha,g)}{\eta\omega J^{3/2} + \beta\alpha J^{5/2}} \right) \ , \end{aligned} \tag{3.151}$$

for arbitrary $\gamma > 0$.

Putting together (3.149) and (3.151), we arrive at the final bound for the ground state expectation value of the upper symbol:

$$\begin{aligned} h_u(\overrightarrow{\zeta_0}) \geq\ & (1-\gamma)\mathcal{E}^{\mathrm{GP}}(\phi_{\zeta_0}) + \frac{\mu}{2}\left(\|\phi_{\zeta_0}\|^2 - N\right)^2 \\ & - N\left(\tfrac{2g}{\pi} + 2\mu(J+2)\right) - \frac{\left(\frac{2g}{\pi} + 2\mu(J+2)\right)^2}{4\mu} \\ & - \frac{\omega}{2} J(J+1) - \alpha \frac{J(J+1)(2J+1)}{6} \\ & - 9g \left(\frac{6}{\gamma}\right)^3 \left(\frac{E^{\mathrm{GP}}(N,\omega,\alpha,g)}{\eta\omega J^{3/2} + \beta\alpha J^{5/2}} \right)^2 \ . \end{aligned} \tag{3.152}$$

Now we have to choose the parameters η, β, γ and J in such a way that firstly all the error terms are much smaller than the GP-energy, secondly the chosen values have to imply $\|\phi_{\zeta_0}\|^2 \approx N$ and finaly J has to satisfy (3.143). Apart from these conditions the choice is completely free. We proceed as follows:

Let us define

$$\eta = \left(\frac{g}{N\omega}\right)^u \quad , \quad \beta = \left(\frac{g}{N^2\alpha}\right)^v$$
$$\gamma = \eta \quad , \quad J = N\left(\frac{g}{N\omega}\right)^a \approx N\left(\frac{g}{N^2\alpha}\right)^{2/3a} , \tag{3.153}$$

for which we have to find suitable positive exponents. First of all we consider the last term in (3.152). What we want is that

$$\frac{1}{\left(\eta\omega J^{3/2} g^{-1/2}\gamma^{3/2} E_{\mathrm{GP}}^{-1/2} + \beta\alpha J^{5/2} g^{-1/2}\gamma^{3/2} E_{\mathrm{GP}}^{-1/2}\right)^2} \approx \left(\frac{g}{N^2\alpha}\right)^v . \tag{3.154}$$

For this reason we require that the powers of the two terms in the denominator are equal, which leads us to a first condition for the parameter,

$$u - a - 3/2v + 1/2 = 0 . \tag{3.155}$$

If we assume (3.155), then (3.154) additionaly requires that $-2\times$ the power of one of the two terms in the denominator is equal to v,

$$-2v - 2u + 5/18 = v . \tag{3.156}$$

Since we require $u, v > 0$ it follows from (3.155) and (3.156) that $13/36 < a < 23/36$. We will choose a to be

$$a = \frac{5}{12} \tag{3.157}$$

for which the solution of the two equations (3.155) and (3.156) is given by

$$u = 1/36 \quad \text{and} \quad v = 2/27 . \tag{3.158}$$

Within this choice of the parameters all the error terms in (3.152) are much smaller than $E^{\mathrm{GP}}(N, \omega, \alpha, g)$ and in particular they are of the order

$$\left(\frac{g}{N\omega}\right)^{1/36} \quad \text{or} \quad \left(\frac{g}{N^2\alpha}\right)^{2/27} . \tag{3.159}$$

To check whether our choice for the parameters implies $\|\phi_{\zeta_0}\|^2 \approx N$ or not, we proceed as in (3.147) and therefore see that

$$\left|\|\phi_{\zeta_0}\|^2 - N\right| \leq \left(\frac{2E^{\mathrm{GP}}(N, \omega, \alpha, g)}{\mu}\right)^{1/2} \approx N\left(\frac{g}{N\omega}\right)^{1/24} . \tag{3.160}$$

Hence, we have

$$E^{\mathrm{GP}}(\|\phi_{\zeta_0}\|^2, \omega, \alpha, g) \approx E^{\mathrm{GP}}(N, \omega, \alpha, g) \tag{3.161}$$

and we arrive at the final result:

$$E_0(N, \omega, \alpha, g) \geq E^{\mathrm{GP}}(N, \omega, \alpha, g) \left[1 - C\left(\frac{g}{N\omega}\right)^{1/36} - \tilde{C}\left(\frac{g}{N^2\alpha}\right)^{2/27}\right] \tag{3.162}$$

for any $C, \tilde{C} > 0$.

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