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Spectral distribution of Schrödinger operator with White Noise potential

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Zusammenfassung

Das Ziel dieser Arbeit ist die Analyse des Spektrums des eindimensionalen Schrödinger-Operators mit weißem Rauschen als Potential. Mithilfe der Theorie der symmetrischen Formen und der Operatortheorie wird gezeigt, dass der Sobolev-Raum $H_0^1(a,b)$ ein geeigneter Definitionsbereich ist, um ein Eigenwertproblem mit Dirichlet-Randbedingungen zu formulieren. Wir verwenden die Sturm-Liouville-Theorie, um dieses Problem für eine feste Realisierung ω zu lösen.

Anschließend werden Methoden der stochastischen Analysis angewandt, um nachzuweisen, dass das Spektrum für fast alle Pfade ω ein reines Punktspektrum ohne Häufungspunkte (außer $+\infty$) ist. Unser Hauptsatz zeigt dann, dass die spektrale Verteilungsfunktion für fast alle Pfade ω eine exakte Form annimmt.

Abstract

Our goal is to analyze the spectrum of the one-dimensional Schrödinger operator with White Noise potential. Using symmetric forms and operator theory we will find that the Sobolev space $H_0^1(a,b)$ is a suitable domain to formulate an eigenvalue problem with Dirichlet boundary conditions. We will use Sturm-Liouville theory to solve this problem for a fixed sample path. Then we will use methods from stochastic analysis to establish that the spectrum is pure point with no accumulation point except for $+\infty$ for a. e. ω. Our main theorem will show that the spectral distribution function takes an exact form for a. e. ω.

Inhaltsverzeichnis

1. Introduction

The Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{x},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x},t)\right]\Psi(\mathbf{x},t) \tag{1.1}$$

lies at the heart of quantum mechanics. Applying separation of variables leads to the time-independent Schrödinger equation,

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x})\right]\psi(\mathbf{x}) = E\psi(\mathbf{x}) \tag{1.2}$$

whose study is the focus of this thesis.

While the time-independent Schrödinger equation is well-understood for regular potentials $V(x)$, a fascinating problem arises when the potential is highly irregular. Specifically, when $V(x)$ is a white Gaussian noise. In physical terms, this represents a particle in a completely disordered medium, a model pioneered by Frisch and Lloyd [2] and Halperin [3]. These physicists, using heuristic methods and formal expansions, derived an exact expression for the spectral distribution function $N(\lambda)$.

$$N(\lambda) = \left(\sqrt{2\pi}\int_0^\infty w^{-1/2}\exp\left(-\frac{1}{6}w^3 - 2\lambda w\right)dw\right)^{-1}. \tag{1.3}$$

This formula is remarkable because it provides a closed-form solution for a kind of problem that is typically not solvable analytically. In the physics literature, the expression (1.3) was obtained by analyzing the phase of the wavefunction and assuming it behaves as a Markov process. Specifically, by defining the logarithmic derivative (or phase process) $Z(x) = \psi'(x)/\psi(x)$, one can derive a Fokker-Planck equation for the probability density of Z.

Fukushima and Nakao's rigorous proof [1] confirms that the heuristic counting of nodes in the wavefunction used by physicists is mathematically equivalent to the convergence of the eigenvalue counting function of the operator H defined via symmetric forms. This thesis follows their path: first establishing the operator's existence through the theory of forms, then utilizing the transformation to a stochastic diffusion process to recover the exact form of $N(\lambda)$ given in (1.3).

We will study the operator on the left hand side of the time independent Schrödinger equation (1.2) and, for mathematical convenience, will get rid of physical constants.

We will assume basic knowledge of PDEs, functional analysis as well as stochastic analysis but we will recall certain key theorems which we will use. We will also need to establish some symmetric form theory, Sturm-Liouville theory, pathwise approximation of SDEs, and the theory of infinitesimal generators.

1.1. Problem Formulation

The one-dimensional Schrödinger operator on $\mathbb{R}$ is defined by

$$H = -\Delta + V(x),$$

where $\Delta = \frac{\partial^2}{\partial x^2}$ is the Laplacian and $V : \mathbb{R} \to \mathbb{R}$ is a real-valued potential.

In this thesis we will consider the potential $V(x)$ to be Gaussian White Noise. Formally, White Noise is the derivative of Brownian motion $B(x)$ in the sense of distributions, that is, $V(x) = \dot{B}(x)$, where $B(x)$ is a standard Brownian motion.

We will state the eigenvalue problem of this operator as

$$\begin{aligned} Hu(x) = -\frac{d^2}{dx^2}u(x) + Vu(x) = \lambda u(x), \quad a < x < b, \\ u(a) = u(b) = 0. \end{aligned} \tag{1.4}$$

Let $\mathcal{N}(\lambda, a, b)$ denote the number of eigenvalues of H on (a, b) not exceeding λ. We define the spectral distribution function of H by

$$N(\lambda) = \lim_{l \to \infty} \frac{\mathcal{N}(\lambda, 0, l)}{l}, \quad -\infty < \lambda < \infty.$$

Our main theorem will then prove, that the spectral distribution function almost surely has the exact form

$$N(\lambda) = \left(\sqrt{2\pi} \int_0^\infty u^{-1/2} \exp\left\{ -\frac{1}{6}u^3 - 2\lambda u \right\} du \right)^{-1}.$$

1.2. Outline of the Thesis

The structure of this thesis follows the approach of Fukushima and Nakao [1], while providing additional detail, we will be filling certain technical gaps that were omitted in the original work to make the presentation accessible to a broader mathematical audience. The work is organized as in 4 steps.

In Chapter 2, we address the challenge of defining the one-dimensional Schrödinger operator with a singular White Noise potential. Using the theory of symmetric forms and operator theory, we rigorously formulate the eigenvalue problem on the Sobolev space $H_0^1(a, b)$.

In Chapter 3, we analyze the spectral properties of the operator for a fixed realization of Brownian motion. By applying the spectral theorem and the Sturm-Liouville oscillation theorem (as presented in Teschl [4]), we establish that the spectrum is pure-point and derive a fundamental identity connecting the eigenvalue counting function to the nodes of the eigenfunctions.

In Chapter 4, we transition to the stochastic setting. We utilize the *Wong-Zakai theorem* (following Kunita [8] and Kuo [7]) to bridge the gap between deterministic ODE solutions and stochastic differential equations in the Stratonovich sense. This allows us to treat the phase of the eigenfunctions as a diffusion process.

In Chapter 5, we apply the theory of infinitesimal generators and the Poisson equation for hitting times (referencing Revuz and Yor [6]) to derive the explicit formula for the rescaled eigenvalue counting function in the limit for large volumes $N(\lambda)$, which is introduced as the spectral distribution function. We conclude by verifying that this rigorous derivation recovers the exact Frisch-Lloyd-Halperin formula.

Finally, in Chapter 6 we present the Main Theorem which yields an explicit solution for $N(\lambda)$.

2. Symmetric Forms and Rigorous Problem Formulation

2.1. The Problem with White Gaussian Noise

We begin with the formal eigenvalue problem stated in the introduction:

$$Hu(x) = -\frac{d^2}{dx^2}u(x) + \dot{B}(x)u(x) = \lambda u(x), \quad 0 < x < l, \tag{2.1}$$
$$u(0) = u(l) = 0,$$

where $\dot{B}(x)$ denotes White Gaussian Noise, formally the derivative of Brownian motion.

Let us attempt to interpret equation (2.1) naively. Suppose we take $u \in L^2(0, l)$ and try to compute Hu. The Laplacian term $-u''$ can be defined weakly if $u \in H^1$, but the term $\dot{B}(x)u(x)$ is not well defined, as $\dot{B}(x)$ is not a function in the L^2-sense, as Brownian motion is nowhere differentiable. So Gaussian White Noise can only be defined in the distributional sense and as the product of a distribution with an L^2 function is generally not meaningful, we can't even find a suitable domain for our operator. Therefore we need to take care to formulate the problem in some other way.

Our goal in this chapter is to give rigorous meaning to this eigenvalue problem. The key insight, following Fukushima and Nakao [1], is to integrate against test functions and work with a bilinear form rather than the operator directly.

2.2. The Formal Integration Argument

Before proceeding with the rigorous arguments that Fukushima and Nakao laid out, we should try to make sense of (2.1) formally. For smooth functions u, v vanishing at the endpoints of (a, b), we can write

$$\int_a^b u(x)Hv(x)dx = -\int_a^b u(x)v''(x)dx + \int_a^b u(x)v(x)\dot{B}(x)dx. \tag{2.2}$$

We then integrate the first term by parts and since $u(a) = u(b) = 0$ and $v(a) = v(b) = 0$, we get

$$-\int_a^b u(x)v''(x)dx = \int_a^b u'(x)v'(x)dx.$$

The second term in (2.2) is problematic, since $\dot{B}(x)$ is not a function. But if we interpret $\dot{B}(x)dx$ as the differential $dB(x)$ of Brownian motion, we have

$$\int_a^b u(x)v(x)\dot{B}(x)dx = \int_a^b u(x)v(x)dB(x).$$

In this case, the integral $\int_a^b u(x)v(x)dB(x)$ can be interpreted as a stochastic integral, where we still have the product $u(x)v(x)$ in the integrand. To separate the roles of u and v, we use the product fule on the term $u(x)v(x)B(x)$ (by treating $B(x)$ as a differentiable function formally for now) to get

$$\begin{aligned} d(u(x)v(x)B(x)) &= u(x)v(x)dB(x) + B(x)d(u(x)v(x)) \\ &= u(x)v(x)dB(x) + B(x)(u'(x)v(x) + u(x)v'(x))dx. \end{aligned}$$

Now we integrate from a to b and rearrange to obtain

$$\int_a^b u(x)v(x)dB(x) = [u(x)v(x)B(x)]_a^b - \int_a^b B(x)(u'(x)v(x) + u(x)v'(x))dx.$$

Since $u(a) = u(b) = v(a) = v(b) = 0$, the boundary term vanishes, leading us to

$$\int_a^b u(x)v(x)dB(x) = -\int_a^b B(x)(u'(x)v(x) + u(x)v'(x))dx.$$

We note that we used stochastic calculus in the Stratonovich sense to arrive at our conclusion. Using Itô's stochastic calculus, consider the function $f(x, B(x)) = u(x)v(x)B(x)$ and apply Itô's formula, which yields

$$d(u(x)v(x)B(x)) = \partial_x f(x, B(x))\, dx + \partial_y f(x, B(x))\, dB(x) + \tfrac{1}{2}\partial_{yy} f(x, B(x))\, dx.$$

Since $\partial_{yy} f = 0$, the Itô correction term vanishes. Now we integrate over $[a, b]$ to get

$$\int_a^b u(x)v(x)\, dB(x) = [u(x)v(x)B(x)]_a^b - \int_a^b B(x)(u'(x)v(x) + u(x)v'(x))\, dx,$$

we can apply boundary conditions just as above to eliminate the boundary contribution $[u(x)v(x)B(x)]_a^b$ and observe that the Itô sense produces the same expression as the calulation above.

This formal manipulation should give us the intuition on how to tackle the problem with proper mathematical tools, as it motivates the definition of the Schrödinger operator with White Noise potential through symmetric forms.

2.3. Symmetric Forms

As analyzing one-dimensional Schrödinger operator with White Boise potential purely from an operator theory perspective doesn't lead to fruitful results, we have to look at the problem through other perspectives. It turns out, that symmetric forms give us the desired outlook, as we can define the operator rigorously via the representation theorem. First, we will establish some terminology from the theory of symmetric forms on Hilbert spaces. For a detailed treatment see [13] or [15].

Definition 2.3.1 (Symmetric form). *Let $\mathcal{H}$ be a real Hilbert space with inner product $(\cdot,\cdot)$. A symmetric form on $\mathcal{H}$ is a bilinear form $\mathscr{E} : \mathcal{D}[\mathscr{E}] \times \mathcal{D}[\mathscr{E}] \to \mathbb{R}$ defined on a dense subspace $\mathcal{D}[\mathscr{E}] \subset \mathcal{H}$, which is symmetric, in the sense $\mathscr{E}(u,v) = \mathscr{E}(v,u)$ for all $u, v \in \mathcal{D}[\mathscr{E}]$.*

Definition 2.3.2 (The symmetric form associated with white noise). *For a fixed interval (a,b) and a bounded Borel function $h : (a,b) \to \mathbb{R}$, define the symmetric form $\mathscr{E}^h_{(a,b)}$ on $L^2(a,b)$ by*

$$\mathcal{D}[\mathscr{E}^h_{(a,b)}] = H^1_0(a,b), \tag{2.3}$$

$$\mathscr{E}^h_{(a,b)}(u,v) = \int_a^b u'(x)v'(x)dx - \int_a^b (u'(x)v(x) + u(x)v'(x))h(x)dx, \tag{2.4}$$

where $H^1_0(a,b)$ is the Sobolev space containing all absolutely continuous functions $u \in L^2(a,b)$ such that $u' \in L^2(a,b)$ and $u(a) = u(b) = 0$.

When the interval is clear from context, we simply write $\mathscr{E}^h$.

For a fixed ω, we will take $h(x) = B(x,\omega)$. Note that while Brownian motion is not bounded, it is continuous on $[a,b]$ and hence bounded on this compact interval for each ω. Thus $B(\cdot,\omega)$ is a valid choice for h in the above definition for each fixed ω.

The form $\mathscr{E}^h$ has several important properties that we will explore.

Definition 2.3.3 (Lower semibounded form). *A symmetric form $\mathscr{E}$ is called lower semibounded if there exists a constant $\gamma \in \mathbb{R}$ such that*

$$\mathscr{E}(u,u) + \gamma(u,u) \geq 0 \quad \textit{for all } u \in \mathcal{D}[\mathscr{E}].$$

Definition 2.3.4 (Closed form). *A lower semibounded symmetric form $\mathscr{E}$ is called closed if $\mathcal{D}[\mathscr{E}]$ is complete with respect to the norm*

$$\|u\|_{\mathscr{E},\gamma'} = \sqrt{\mathscr{E}(u,u) + \gamma'(u,u)},$$

for some (equivalently, for every) $\gamma' > \gamma$.

Definition 2.3.5 (Complete continuity condition). *A closed symmetric form $\mathscr{E}$ satisfies the complete continuity condition if from any sequence $\{u_n\} \subset \mathcal{D}[\mathscr{E}]$ with $\|u_n\|_{\mathscr{E},\gamma'} \leq C$ for some constant C, one can extract a subsequence that converges strongly in $\mathcal{H}$.*

We note that the complete continuity condition is equivalent to the embedding $\mathcal{D}[\mathscr{E}] \hookrightarrow \mathcal{H}$ being compact, or to the associated operator having compact resolvent.

The importance of these definitions lies in the following fundamental theorem, which connects symmetric forms to self-adjoint operators.

Theorem 2.3.1 (Representation theorem for symmetric forms). *Let $\mathscr{E}$ be a lower semibounded closed symmetric form on a Hilbert space $\mathcal{H}$. Then there exists a unique self-adjoint operator A on $\mathcal{H}$ such that*

1. *$\mathcal{D}(A) \subset \mathcal{D}[\mathscr{E}]$,*

2. *$\mathscr{E}(u, v) = (Au, v)$ for all $u \in \mathcal{D}(A)$ and $v \in \mathcal{D}[\mathscr{E}]$,*

3. *$\mathcal{D}(A)$ is dense in $\mathcal{D}[\mathscr{E}]$ with respect to the norm $\|\cdot\|_{\mathscr{E},\gamma'}$.*

If, in addition, $\mathscr{E}$ satisfies the complete continuity condition, then the spectrum of A consists entirely of eigenvalues $\{\lambda_k\}$ of finite multiplicity, which can be arranged in nondecreasing order

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to +\infty.$$

Proof. Let $\mathscr{E}$ be a closed, lower semibounded symmetric form on a Hilbert space $\mathcal{H}$. Since $\mathscr{E}(u) \geq \gamma\|u\|^2$ for some $\gamma \in \mathbb{R}$, we can choose γ' such that $\gamma + \gamma' > 0$. The expression

$$\mathscr{E}_{\gamma'}(u, v) := \mathscr{E}(u, v) + \gamma'(u, v)$$

defines an inner product on the domain $\mathcal{D}[\mathscr{E}]$. Since $\mathscr{E}$ is closed, the space $\mathcal{D}[\mathscr{E}]$ equipped with the norm $\|u\|_{\mathscr{E},\gamma'} := \sqrt{\mathscr{E}_{\gamma'}(u, u)}$ is a Hilbert space, which we denote $\mathcal{H}_{\mathscr{E}}$.

For any fixed $u \in \mathcal{H}$, the mapping $L_u : v \mapsto (u, v)_{\mathcal{H}}$ is a bounded linear functional on $\mathcal{H}_{\mathscr{E}}$, because

$$|(u, v)_{\mathcal{H}}| \leq \|u\|_{\mathcal{H}}\|v\|_{\mathcal{H}} \leq C\|u\|_{\mathcal{H}}\|v\|_{\mathscr{E},\gamma'}.$$

Applying the Riesz Representation Theorem (see [5]) on $\mathcal{H}_{\mathscr{E}}$, there exists a unique element $w \in \mathcal{D}[\mathscr{E}]$ such that

$$(u, v)_{\mathcal{H}} = \mathscr{E}_{\gamma'}(w, v) = \mathscr{E}(w, v) + \gamma'(w, v) \quad \text{for all } v \in \mathcal{D}[\mathscr{E}].$$

Define the operator $T : \mathcal{H} \to \mathcal{D}[\mathscr{E}] \subset \mathcal{H}$ by $Tu = w$. From the Riesz step, we have

$$(u, v) = \mathscr{E}(Tu, v) + \gamma'(Tu, v).$$

T is bounded, linear, self-adjoint, and injective (since $Tu = 0 \implies (u, v) = 0 \implies u = 0$). Moreover, T maps $\mathcal{H}$ onto the dense subspace $\mathcal{D}[\mathscr{E}]$.

Since T is injective and bounded on $\mathcal{H}$, T^{-1} is well-defined, generally unbounded, and self-adjoint on its domain $\mathcal{D}(T^{-1}) = \text{Range}(T) = \mathcal{D}[\mathscr{E}]$. From the previous step we have

$$T^{-1}u = w \iff u = Tw \quad \text{and} \quad (u, v) = \mathscr{E}(w, v) + \gamma'(w, v)$$

Let $A := T^{-1} - \gamma' I$. The domain of A is $\mathcal{D}(A) = \mathcal{D}(T^{-1}) = \{w \in \mathcal{D}[\mathscr{E}] \mid Tw \in \mathcal{D}[\mathscr{E}]\}$. For $u \in \mathcal{D}(A)$ and $v \in \mathcal{D}[\mathscr{E}]$, let $w = u$. The Riesz relation then becomes

$$(Tu, v) = \mathscr{E}(u, v) + \gamma'(u, v).$$

Substituting $Tu = Au + \gamma' u$:

$$\begin{aligned}(Au + \gamma' u, v) &= \mathscr{E}(u, v) + \gamma'(u, v)\\ (Au, v) + \gamma'(u, v) &= \mathscr{E}(u, v) + \gamma'(u, v)\\ (Au, v) &= \mathscr{E}(u, v).\end{aligned}$$

The operator A is self-adjoint since $A + \gamma' I = T^{-1}$ and T is a bounded self-adjoint operator with bounded inverse. This proves (1) and (2).

If the inclusion $\iota : \mathcal{H}_{\mathscr{E}} \hookrightarrow \mathcal{H}$ is compact (by the complete continuity condition), then the bounded operator $T : \mathcal{H} \to \mathcal{H}$ (which is the product of a bounded map and the compact inclusion ι) is compact. Since T is compact and self-adjoint, the Spectral Theorem for Compact Self-Adjoint Operators (see [5]) implies its spectrum consists of eigenvalues $\mu_k \to 0$. Since $A = T^{-1} - \gamma' I$, the eigenvalues of A are $\lambda_k = \mu_k^{-1} - \gamma'$, which also form a discrete set $\{\lambda_k\}$ with $\lambda_k \to \pm\infty$. Since $\mathscr{E}$ is semibounded, $\lambda_k \to +\infty$ and the spectrum is purely discrete.

□

See [13, Theorem VI.2.1] for a more general version of this theorem.

Now since we are interested in counting the eigenvalues, we need a theorem that relates our symmetric form $\mathscr{E}$ to its eigenvalues. It turns out that they can be characterized by the Minimax principle, which will be crucial for our analysis by enabling us to talk about spectral properties of the symmetric form.

2.4. The Minimax Principle for Symmetric Forms

Theorem 2.4.1 (Minimax Principle for Symmetric Forms)**.** *Let $\mathscr{E}$ be a lower semibounded closed symmetric form on a Hilbert space $\mathcal{H}$ satisfying the complete continuity condition. Let A be the associated self-adjoint operator with eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots$ listed in increasing order. For any k-dimensional subspace $M \subset \mathcal{D}[\mathscr{E}]$, define*

$$\lambda(M) = \sup_{\substack{u \in M \\ \|u\|=1}} \mathscr{E}(u,u).$$

Then the k-th eigenvalue λ_k is given by

$$\lambda_k = \inf\{\lambda(M) : M \subset \mathcal{D}[\mathscr{E}], \dim M = k\}.$$

Proof. Let $\{\phi_j\}_{j=1}^{\infty}$ be the orthonormal eigenvectors of A corresponding to the eigenvalues $\{\lambda_j\}$. For any $u \in \mathcal{D}[\mathscr{E}]$, using the spectral theorem we can write the symmetric form as

$$\mathscr{E}(u,u) = \sum_{j=1}^{\infty} \lambda_j |\langle u, \phi_j \rangle|^2.$$

We consider a k-dimensional subspace $M_k = \operatorname{span}\{\phi_1, \dots, \phi_k\}$. For any $u \in M_k$ with $\|u\| = 1$, we have $u = \sum_{j=1}^{k} c_j \phi_j$ with $\sum |c_j|^2 = 1$. Then we get

$$\mathscr{E}(u,u) = \sum_{j=1}^{k} \lambda_j |c_j|^2 \leq \lambda_k \sum_{j=1}^{k} |c_j|^2 = \lambda_k,$$

since $\lambda_j \leq \lambda_k$ for all $j \leq k$. Now it follows that

$$\lambda(M_k) = \sup_{u \in M_k, \|u\|=1} \mathscr{E}(u,u) \leq \lambda_k.$$

Since $\mathscr{E}(\phi_k, \phi_k) = \lambda_k$, we have $\lambda(M_k) = \lambda_k$. Thus, the infimum over all k-dimensional subspaces $M \subset \mathcal{D}[\mathscr{E}]$ satisfies $\inf_M \lambda(M) \leq \lambda_k$.

Let M be an arbitrary k-dimensional subspace of $\mathcal{D}[\mathscr{E}]$. Consider the subspace $V = \operatorname{span}\{\phi_1, \dots, \phi_{k-1}\}^\perp$. Since $\dim M = k$ and codim $V = k - 1$, the intersection $M \cap V$ has dimension at least 1. We can therefore choose a unit vector $w \in M \cap V$. Because $w \perp \phi_j$ for $j < k$, its expansion is $w = \sum_{j=k}^{\infty} c_j \phi_j$. Then we have

$$\mathscr{E}(w, w) = \sum_{j=k}^{\infty} \lambda_j |c_j|^2 \geq \lambda_k \sum_{j=k}^{\infty} |c_j|^2 = \lambda_k,$$

since $\lambda_j \geq \lambda_k$ for all $j \geq k$. Since $w \in M$, the supremum over M must satisfy:

$$\lambda(M) = \sup_{u \in M, \|u\|=1} \mathscr{E}(u, u) \geq \mathscr{E}(w, w) \geq \lambda_k.$$

Because this inequality holds for every k-dimensional subspace M, we conclude the convergence

$$\inf_M \lambda(M) \geq \lambda_k.$$

□

We note that there is an analogue theorem for self-adjoint operators with compact resolvent, where the proof is similar (see e. g. [5]). With this theoretical background, we can now analyze the form $\mathscr{E}^h$ defined in (2.4) to verify that we meet the assumptions of our previously established theorems.

2.5. Properties of the Form $\mathscr{E}^h$

Our first task is to verify that $\mathscr{E}^h$ satisfies the conditions of the Representation Theorem. This is the content of Lemma 1 in Fukushima and Nakao's paper [1].

Lemma 2.5.1 (Properties of $\mathscr{E}^h$)**.** *For any bounded Borel function $h : (a, b) \to \mathbb{R}$, the form $\mathscr{E}^h$ defined in* (2.4) *has the following properties:*

1. *$\mathscr{E}^h$ is a lower semibounded closed symmetric form on $L^2(a, b)$.*

2. *$\mathscr{E}^h$ satisfies the complete continuity condition.*

3. *If $h_n \to h$ uniformly on (a, b), and λ_k^n (resp. λ_k) denotes the k-th eigenvalue of the operator associated with $\mathscr{E}^{h_n}$ (resp. $\mathscr{E}^h$), then:*

$$\lim_{n\to\infty} \lambda_k^n = \lambda_k \quad \textit{for each } k = 1, 2, \dots$$

Proof. The symmetry $\mathscr{E}^h(u,v) = \mathscr{E}^h(v,u)$ is clear from the definition. To prove lower semiboundedness, let $M = \sup_{x\in(a,b)} |h(x)| < \infty$. Then, for any $u \in H_0^1(a,b)$, we get

$$\begin{aligned}\mathscr{E}^h(u,u) &= \int_a^b (u'(x))^2 dx - 2\int_a^b u'(x)u(x)h(x)dx \\ &\geq \int_a^b (u'(x))^2 dx - 2M\int_a^b |u'(x)||u(x)|dx.\end{aligned}$$

Now we use the inequality $2|u'||u| \leq \epsilon(u')^2 + \epsilon^{-1}u^2$ for any $\epsilon > 0$ (which follows from $2ab \leq \epsilon a^2 + \epsilon^{-1}b^2$), then

$$\mathscr{E}^h(u,u) \geq \int_a^b (u'(x))^2 dx - M\left(\epsilon\int_a^b (u'(x))^2 dx + \epsilon^{-1}\int_a^b u(x)^2 dx\right).$$

Let $\epsilon = \frac{1}{2M}$, then we get

$$\mathscr{E}^h(u,u) \geq \frac{1}{2}\int_a^b (u'(x))^2 dx - 2M^2\int_a^b u(x)^2 dx.$$

Now by the Poincaré inequality (see e. g. [5]) for functions in $H_0^1(a,b)$, there exists a constant $C > 0$ such that $\int_a^b u(x)^2 dx \leq C\int_a^b (u'(x))^2 dx$. Therefore we have

$$\mathscr{E}^h(u,u) \geq \left(\frac{1}{2} - 2M^2C\right)\int_a^b (u'(x))^2 dx.$$

If $2M^2C < \frac{1}{2}$, we are done. Otherwise, we can combine with the L^2-term:

$$\mathscr{E}^h(u,u) + (2M^2+1)\int_a^b u(x)^2 dx \geq \frac{1}{2}\int_a^b (u'(x))^2 dx + \int_a^b u(x)^2 dx \geq \frac{1}{2}\|u\|_{H^1}^2,$$

where $\|u\|_{H^1}^2 = \int_a^b (u'(x))^2 dx + \int_a^b u(x)^2 dx$. This shows that $\mathscr{E}^h$ is lower semibounded with $\gamma = 2M^2 + 1$.

The inequality we just proved shows that the $\mathscr{E}^h$-norm

$$\sqrt{\mathscr{E}^h(u,u) + (2M^2+1)(u,u)}$$

is equivalent to the H^1-norm on $H_0^1(a,b)$. Since $H_0^1(a,b)$ is a complete Hilbert space with respect to the H^1-norm, $\mathscr{E}^h$ is closed. For the complete continuity conditon, let $\{u_n\}_{n=1}^\infty$ be bounded in the $\mathscr{E}^h$-norm, i.e., there exists $K > 0$ such that

$$\mathscr{E}^h(u_n,u_n) + \gamma\|u_n\|_{L^2}^2 \leq K \quad \text{for all } n.$$

By the inequality above, this implies

$$\|u_n\|_{H^1}^2 \leq 2K \quad \text{for all } n.$$

Let $x \in [a,b]$ and $n \in \mathbb{N}$, by the fundamental theorem of calculus and Cauchy-Schwarz we get

$$|u_n(x)| = \left|\int_a^x u_n'(t)dt\right| \leq \int_a^x |u_n'(t)|dt \leq \|u_n'\|_{L^2}\sqrt{x-a} \leq \sqrt{2K}\sqrt{b-a}.$$

Then for $x, y \in [a,b]$ and $n \in \mathbb{N}$ it holds that

$$|u_n(x) - u_n(y)| = \left|\int_x^y u_n'(t)dt\right| \leq \|u_n'\|_{L^2}\sqrt{|x-y|} \leq \sqrt{2K}\sqrt{|x-y|}.$$

So for $\epsilon > 0$, take $\delta = \epsilon^2/(2K)$ to guarantee equicontinuity.

By Arzelà-Ascoli there exists a subsequence $\{u_{n_k}\}$ converging uniformly to some $u \in C_0[a,b]$. So we get L^2-convergence

$$\|u_{n_k} - u\|_{L^2}^2 \leq (b-a)\|u_{n_k} - u\|_\infty^2 \to 0,$$

which implies the complete continuity condition.

Next we need to prove the convergence of the eigenvalues. Let $h_n \to h$ uniformly, and set $C_n = \sup_{a<x<b} |h_n(x) - h(x)| \to 0$. For any $u \in H_0^1(a,b)$,

$$\begin{aligned}
|\mathscr{E}^{h_n}(u,u) - \mathscr{E}^h(u,u)| &= \left|2\int_a^b u'(x)u(x)(h_n(x) - h(x))dx\right| \\
&\leq 2C_n \int_a^b |u'(x)||u(x)|dx \\
&\leq C_n\left(\int_a^b (u'(x))^2dx + \int_a^b u(x)^2dx\right) \\
&= C_n\|u\|_{H^1}^2.
\end{aligned}$$

Using our inequality from above, we have

$$\|u\|_{H^1}^2 \leq 2[\mathscr{E}^h(u,u) + (2M^2+1)(u,u)].$$

Therefore we get

$$|\mathscr{E}^{h_n}(u,u) - \mathscr{E}^h(u,u)| \leq 2C_n[\mathscr{E}^h(u,u) + (2M^2+1)(u,u)].$$

This implies that

$$\begin{aligned}
(1-2C_n)[\mathscr{E}^h(u,u) + (2M^2+1)(u,u)] &\leq \mathscr{E}^{h_n}(u,u) + (2M^2+1)(u,u) \\
&\leq (1+2C_n)[\mathscr{E}^h(u,u) + (2M^2+1)(u,u)].
\end{aligned}$$

Applying the minimax principle to both forms, we obtain for each k that

$$(1-2C_n)[\lambda_k + (2M^2+1)] \leq \lambda_k^n + (2M^2+1) \leq (1+2C_n)[\lambda_k + (2M^2+1)].$$

Since $C_n \to 0$, we conclude that $\lambda_k^n \to \lambda_k$. □

We will soon use this lemma for an approximation of Brownian motion B_n, before that we will find the associated operator to the symmetric form $\mathscr{E}^h$ and then give a formulation of the eigenvalue problem.

The following Lemm shows that when h is a smooth function, the form $\mathscr{E}^h$ corresponds to a differential operator. This fact is then used to justify the domain that we used in the definition.

Lemma 2.5.2. *Suppose $h \in C^1[a,b]$ with bounded derivative. Let A be the self-adjoint operator associated with $\mathscr{E}^h$. Then for $u \in \mathcal{D}(A)$, we have:*

$$Au = -u'' + h'(x)u,$$

with domain $\mathcal{D}(A) = \{u \in H_0^1(a,b) : u'$ absolutely continuous, $-u'' + h'u \in L^2(a,b)\}$.

Proof. For smooth $u, v \in C_0^\infty(a,b)$ (compactly supported smooth functions), we can integrate by parts in the definition of $\mathscr{E}^h(u,v)$, yielding

$$\begin{aligned}
\mathscr{E}^h(u,v) &= \int_a^b u'v'dx - \int_a^b (u'v + uv')hdx \\
&= [u'v]_a^b - \int_a^b u''vdx - \int_a^b u'vhdx - \int_a^b uv'hdx \\
&= -\int_a^b u''vdx - \int_a^b u'vhdx - \int_a^b uv'hdx \quad (\text{since } v(a) = v(b) = 0).
\end{aligned}$$

Now we integrate the last term by parts, thus

$$-\int_a^b uv'hdx = -[uvh]_a^b + \int_a^b (uh)'vdx = \int_a^b (u'h + uh')vdx.$$

Then we substitute back in the form equation to get

$$\mathscr{E}^h(u,v) = -\int_a^b u''vdx - \int_a^b u'vhdx + \int_a^b (u'h + uh')vdx = \int_a^b (-u'' + h'u)vdx.$$

So for smooth u, we have $\mathscr{E}^h(u,v) = ((-u'' + h'u), v)_{L^2}$ for all smooth v. By density, this extends to all $v \in H_0^1(a,b)$, showing that $Au = -u'' + h'u$ for u in the domain of A. $\square$

2.6. Rigorous Formulation of the Eigenvalue Problem

We now have all the ingredients to give a rigorous formulation of the eigenvalue problem in terms of symmetric forms.

Definition 2.6.1 (Eigenvalue problem for White Noise potential)**.** *Let $\{B(x,\omega), x \geq 0\}$ be a standard Brownian motion on a probability space $(\Omega, \mathscr{B}, P)$. For a fixed realization $\omega \in \Omega$ and a bounded interval $I = (a,b)$, we define the eigenvalue problem by the symmetric form $\mathscr{E}_I^B$ with domain $\mathcal{D}[\mathscr{E}] = H_0^1(I)$, given by*

$$\mathscr{E}_I^B(u,u) = \int_a^b (u'(x))^2 dx - \int_a^b 2u(x)u'(x)B(x,\omega)dx.$$

This form is almost surely lower semibounded, closed, and satisfies the complete continuity condition. Let A_ω *be the unique self-adjoint operator associated with* $\mathscr{E}_I^B$ *by applying the Representation Theorem. The eigenvalues of* A_ω *are discrete and denoted by* $\lambda_1 \leq \lambda_2 \leq \cdots \to \infty$*. We define the counting function* $\mathcal{N}$ *by*

$$\mathcal{N}^\omega(\lambda, a, b) = |\{\lambda_k \in \sigma(A_\omega) : \lambda_k \leq \lambda\}|\,,$$

representing the number of eigenvalues not exceeding λ.

This definition bypasses the problems of our earlier formulation of the differential equation (2.1) and works directly with the symmetric form.

We note that the Dirichlet boundary conditions $u(a) = u(b) = 0$ are not explicitly added to the eigenvalue problem as external constraints. Instead, they are implicitly imposed by the choice of the form domain $H_0^1(a, b)$.

Moreover, if we approximate Brownian motion B by piecewise linear functions B_n converging uniformly to B , then the eigenvalues λ_k^n of the approximate problems converge to the eigenvalues λ_k of the White Noise problem. This approximation argument will be fundamental for our conclusion.

Definition 2.6.2 (Piecewise linear approximation)**.** *Let* $\{B(x,\omega), x \geq 0\}$ *be a standard Brownian motion defined on a probability space* $(\Omega, \mathscr{B}, P)$ *such that* $B(0,\omega) = 0$ *and* $B(x,\omega)$ *is continuous in* $x \geq 0$ *for any* $\omega \in \Omega$*. Fix an interval* $[a,b] \subset [0,\infty)$*. For each* $n \in \mathbb{N}$*, we define the piecewise linear approximation* $\{B_n(x,\omega), x \in [a,b]\}$ *by*

$$B_n(x,\omega) = B\left(\frac{[2^n x]}{2^n}, \omega\right) + 2^n\left(x - \frac{[2^n x]}{2^n}\right)\left\{B\left(\frac{[2^n x]+1}{2^n}, \omega\right) - B\left(\frac{[2^n x]}{2^n}, \omega\right)\right\},$$

for $x \in [a,b]$*, where* $[2^n x]$ *denotes the floor of* $2^n x$*.*

This definition means that on each subinterval $\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right) \cap [a,b]$, $k = 0, 1, 2, \ldots$, the function $B_n(\cdot,\omega)$ is the linear interpolation between $B\left(\frac{k}{2^n}, \omega\right)$ and $B\left(\frac{k+1}{2^n}, \omega\right)$.

Now using this approximation we notice, by lemma 2.5.1, that the counting functions $\mathcal{N}_n^\omega(\lambda, a, b)$ for the approximate problems converge to the counting function of the white noise problem, namely

$$\lim_{n\to\infty} \mathcal{N}_n^\omega(\lambda, a, b) = \mathcal{N}^\omega(\lambda, a, b). \tag{2.5}$$

With this formulation in place, we can now proceed to analyze the eigenvalues. In the next chapter, we will use Sturm-Liouville theory to relate the eigenvalue counting function to the phase of a solution to an ordinary differential equation.

3. Sturm-Liouville Theory

In the previous chapter, we defined the Schrödinger operator with a White Noise potential as a limit of symmetric forms, precisely we established that for $h \in C^1[a,b]$, the operator A associated with the form $\mathscr{E}^h$ is the Schrödinger operator $Au = -u'' + h'(x)u$ with Dirichlet boundary conditions. We now demonstrate that this problem fits into the classical Sturm-Liouville framework, which allows for a characterization of eigenvalues by the zeros of eigenfunctions.

In this chapter, we introduce the Sturm-Liouville theory, mainly following Teschl [4] to derive the phase equation for our problem. We will introduce the Oscillation theorem, which relates the number of eigenvalues with the winding number of a solution in the phase plane (the oscillations). The explicit solution is given in terms of Prüfer variables, which allow us to count the oscillations. Transforming the eigenvalue problem of our approximated operator will then yield a solution for this approximation.

3.1. Regular Sturm-Liouville Problems

Definition 3.1.1 (Regular Sturm-Liouville Equation)**.** *A regular Sturm-Liouville equation on an interval $[a,b]$ is an equation of the form*

$$-(p(x)u')' + q(x)u = \lambda r(x)u, \tag{3.1}$$

where $p \in C^1([a,b])$, $q, r \in C^0([a,b])$, and $p(x), r(x) > 0$.

For $h \in C^1$, the operator $Au = -u'' + h'(x)u$ fits this form with the following parameters:

$$p(x) = 1, \quad r(x) = 1, \quad q(x) = h'(x).$$

We note that we will especially consider $h'(x) = B_n'$. This works because B_n is only not differentiable on the set of dyadic rational numbers, which means it is differentiable almost everywhere, so (most of) our following results will hold for a. e. ω. The Dirichlet boundary conditions $u(a) = u(b) = 0$ ensure that the spectrum consists of a sequence of simple eigenvalues $\lambda_1 < \lambda_2 < \cdots \to \infty$.

3.2. The Prüfer Transformation

To count how many eigenvalues fall below a certain threshold λ, we look at the oscillation of the solutions. We will change the coordinates of our problems using the Prüfer variables $\rho(x)$ (the amplitude) and $\theta(x)$ (the phase) to describe the state of the system in the phase plane (u, pu').

We define the Prüfer variables by

$$u(x) = \rho(x)\sin(\theta(x)), \tag{3.2}$$

$$p(x)u'(x) = \rho(x)\cos(\theta(x)). \tag{3.3}$$

In this coordinate system, the function $u(x)$ is at zero precisely when the phase $\theta(x)$ is a multiple of π.

By differentiating these expressions and substituting the original differential equation (3.1), we obtain a first-order nonlinear ODE for the phase θ, which is independent of the amplitude ρ by

$$\theta'(x) = \frac{1}{p(x)}\cos^2\theta(x) + (\lambda r(x) - q(x))\sin^2\theta(x). \tag{3.4}$$

3.3. Derivation of the differential equation

We now apply the Prüfer transformation to the piecewise linear approximation $B_n(x, \omega)$, where we fix ω. Substituting our parameters into (3.4), we get

$$\theta'(x) = \cos^2\theta(x) + (\lambda - B_n'(x, \omega))\sin^2\theta(x).$$

Then we rearrange the terms into differential form to get

$$d\theta(x) = -\sin^2\theta(x)B_n'(x, \omega)dx + (\cos^2\theta(x) + \lambda\sin^2\theta(x))dx.$$

We remind ourselves that our choice of θ also depends on λ, n and ω, so we let $Y_n^\lambda(x, \omega) = \theta(x)$, to get the ordinary differential equation

$$dY_n^\lambda = -\sin^2 Y_n^\lambda B_n' dx + (\cos^2 Y_n^\lambda + \lambda\sin^2 Y_n^\lambda)dx, \tag{3.5}$$

$$dY_n^\lambda(0) = 0. \tag{3.6}$$

In the following chapter we will introduce tools from stochastic calculus to show that the we can find a solution to the limit of this ODE as $n \to \infty$.

3.4. The Oscillation Theorem and Eigenvalue Counting

The next step is the Oscillation Theorem, which gives us exactly the distribution function we are looking for in terms of the Prüfer transformation.

Theorem 3.4.1 (Sturm Oscillation)**.** *Let $u(x, \lambda)$ be the solution to the initial value problem with $\theta(a, \lambda) = 0$. The number of eigenvalues $\mathcal{N}(\lambda, a, b)$ of the Sturm-Liouville problem on $[a, b]$ that are less than or equal to λ is given by*

$$\mathcal{N}(\lambda, a, b) = \left\lfloor \frac{\theta(b, \lambda)}{\pi} \right\rfloor. \tag{3.7}$$

Now we notice that for for fixed n and ω we are already able to calculate the spectral distribution function, since we have converted the problem of finding eigenvalues to tracking the growth of the phase process $\theta(x)$. In the following chapter we will argue how the result for the approximating phase processes translates to the limits.

4. Diffusion Limits of the Approximating Phase Processes

4.1. Stochastic Calculus prerequisites

Before diving into the Wong-Zakai Theorem, which connects ODEs with SDEs, we briefly recall some essential tools from Stochastic Calculus.

Definition 4.1.1 (One-Dimensional Itô Process)**.** *A stochastic process $\{X_t\}_{t\geq 0}$ in $\mathbb{R}$ is an Itô process if it satisfies the equation*

$$X_t = X_0 + \int_0^t \mu(X_s, s)ds + \int_0^t \sigma(X_s, s)dB_s,$$

where $\int_0^t \mu(X_s, s)ds$ is a Lebesgue integral, $\int_0^t \sigma(X_s, s)\, dB_s$ is an Itô integral and B_t is a one-dimensional Brownian Motion. For convenience we will denote it as the stochastic differential equation (SDE)

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dB_t,$$

where the coefficients μ and σ are called the drift and diffusion, respectively.

Definition 4.1.2 (Stratonovich Integral)**.** *For a stochastic process $\{X_t\}$ and a Brownian motion $\{B_t\}$, the Stratonovich integral $\int_0^t \sigma(X_s, s) \circ dB_s$, is defined as the limit of the Riemann sums*

$$\int_0^t \sigma(X_s, s) \circ dB_s = \lim_{n\to\infty} \sum_{i=0}^{n-1} \sigma\left(X_{\frac{t_i+t_{i+1}}{2}}, \frac{t_i + t_{i+1}}{2}\right)(B_{t_{i+1}} - B_{t_i}).$$

We note that the integrand is evaluated at the midpoint of each sub-interval, as opposed to the Itô integral, where we evaluate at the starting point. As a consequence, the Stratonovich integral obeys the classical rules of calculus. This makes it the natural choice for modeling physical systems where noise is a limit of a smooth process.

If X_t is an Itô process as defined above, the Stratonovich and Itô integrals satisfy the equation

$$\int_0^t \sigma(X_s, s) \circ dB_s = \int_0^t \sigma(X_s, s)\, dB_s + \frac{1}{2}\int_0^t \sigma(X_s, s)\frac{\partial\sigma}{\partial x}(X_s, s)ds.$$

Looking back at the formal argument (2.2) we can now point out that we should expect the two approaches to yield the same solution, as the function $\sigma(X_t, t) = u(t)v(t)$ does not depend on X_t, and therefore the correction term vanishes.

4.2. Wong-Zakai Theorem

Following the definition of the piecewise linear approximation $B_n(x, \omega)$ on $[a, b]$, we consider the limit of a sequence of ordinary differential equations. These results, which connect deterministic and stochastic behaviour are commonly called Wong-Zakai theorems. We will state and proof a one-dimensional version (see Kuo's book on stochastic integration [7]).

Theorem 4.2.1 (One-dimensional Wong-Zakai Theorem)**.** *Let $\{B_n(t)\}$ be a linear approximating sequence of a Brownian motion $B(t)$. Assume that $\sigma(t, x)$ and $f(t, x)$ are Lipschitz-continuous and satisfy the linear growth conditions*

$$|f(t, x)| \leq K(1 + |x|)$$

$$|\sigma(t, x)| \leq K(1 + |x|)$$

and that $\frac{\partial \sigma}{\partial x}$ exists and satisfies the same conditions. Let $X_t^{(n)}$ be the solution of the ordinary differential equation

$$dX_t^{(n)} = \sigma(t, X_t^{(n)})dB_n(t) + f(t, X_t^{(n)})dt, \quad X_a^{(n)} = \xi.$$

Then $X_t^{(n)} \to X_t$ almost surely as $n \to \infty$ for each $t \in [a, b]$, where X_t is the unique continuous solution of the Itô stochastic differential equation

$$dX_t = \sigma(t, X_t)dB(t) + \left[f(t, X_t) + \frac{1}{2}\sigma(t, X_t)\frac{\partial \sigma}{\partial x}(t, X_t)\right]dt, \quad X_a = \xi.$$

Equivalently, in Stratonovich form we have

$$dX_t = \sigma(t, X_t) \circ dB(t) + f(t, X_t)dt.$$

Proof. Let $B_n(t)$ be the piecewise linear approximation of $B(t)$. For each n, let $X_n(t)$ be the solution to the ODE

$$\dot{X}_n(t) = \sigma(t, X_n(t))\dot{B}_n(t) + f(t, X_n(t)).$$

We define the function $G(t, x) = \int_0^x \frac{1}{\sigma(t,y)}dy$. Since σ is Lipschitz and satisfies linear growth, G is well-defined and $\frac{\partial G}{\partial x}(t, x) = 1/\sigma(t, x)$. Applying the chain rule to $G(t, X_n(t))$ we get

$$\begin{aligned} \frac{d}{dt}G(t, X_n(t)) &= \frac{\partial G}{\partial t}(t, X_n(t)) + \frac{\partial G}{\partial x}(t, X_n(t))\,\dot{X}_n(t) \\ &= \frac{\partial G}{\partial t}(t, X_n(t)) + \frac{1}{\sigma(t, X_n(t))}\left[\sigma(t, X_n(t))\dot{B}_n(t) + f(t, X_n(t))\right]. \end{aligned}$$

Simplifying the right-hand side, we arrive at

$$\frac{d}{dt}G(t, X_n(t)) = \frac{\partial G}{\partial t}(t, X_n(t)) + \dot{B}_n(t) + \frac{f(t, X_n(t))}{\sigma(t, X_n(t))}.$$

Now we integrate from a to t to get

$$G(t, X_n(t)) = G(a, X_n(a)) + B_n(t) - B_n(a) + \int_a^t \left[\frac{\partial G}{\partial t}(s, X_n(s)) + \frac{f(s, X_n(s))}{\sigma(s, X_n(s))}\right] ds.$$

As $n \to \infty$, $B_n(t) \to B(t)$ uniformly almost surely and, X_n converges to a limit process X, and the integral converges, so

$$G(t, X_t) = G(a, X_a) + B_t - B_a + \int_a^t \left[\frac{\partial G}{\partial t}(s, X_s) + \frac{f(s, X_s)}{\sigma(s, X_s)}\right] ds.$$

To find the Itô SDE for X_t, we note that since X_t is an Itô process, it must be of the form

$$X_t = X_a + \int_a^t \theta(s)\, dB(s) + \int_a^t g(s)\, ds,$$

where θ and g are to be determined. We apply Itô's Lemma to $G(t, X_t)$ to get

$$\begin{aligned} G(t, X_t) = G(a, X_a) &+ \int_a^t \frac{\theta(s)}{\sigma(s, X_s)}\, dB(s) \\ &+ \int_a^t \left[\frac{\partial G}{\partial t}(s, X_s) + \frac{g(s)}{\sigma(s, X_s)} - \frac{1}{2}\frac{1}{\sigma(s, X_s)^2}\frac{\partial \sigma}{\partial x}(s, X_s)\,\theta(s)^2\right] ds. \end{aligned}$$

Comparing with (4.2), we match the drift and diffusion parts. From the dB term we get

$$\frac{\theta(s)}{\sigma(s, X_s)} = 1 \implies \theta(s) = \sigma(s, X_s).$$

From the dt term, the $\frac{\partial G}{\partial t}$ contributions cancel from both sides, and substituting $\theta(s) = \sigma(s, X_s)$ yields

$$\frac{g(s)}{\sigma(s, X_s)} - \frac{1}{2\,\sigma(s, X_s)^2}\frac{\partial \sigma}{\partial x}(s, X_s)\,\sigma(s, X_s)^2 = \frac{f(s, X_s)}{\sigma(s, X_s)},$$

which gives $g(s) = f(s, X_s) + \frac{1}{2}\,\sigma(s, X_s)\,\frac{\partial \sigma}{\partial x}(s, X_s)$. Distributing $\sigma(t, X_t)$, we obtain the Itô SDE

$$dX_t = \sigma(t, X_t)\, dB_t + \left[f(t, X_t) + \frac{1}{2}\sigma(t, X_t)\frac{\partial \sigma}{\partial x}(t, X_t)\right] dt. \qquad \square$$

We note that in the paper on spectra of Schrödinger operators Fukushima and Nakao invoke Kunitas Theorem, which is weaker, as it only guarantees convergence in probability, and here we have almost sure convergence. This comes as a convenience, as we don't have to consider subsequences to achieve convergence.

4.2.1. Convergence to the Stochastic Phase Process

We briefly remind ourselves of the sequence of ODEs we want to tackle

$$dY_n^\lambda = -\sin^2 Y_n^\lambda B_n' dx + (\cos^2 Y_n^\lambda + \lambda \sin^2 Y_n^\lambda) dx, \tag{4.1}$$

$$dY_n^\lambda(0) = 0, \tag{4.2}$$

and note that since B_n is a linear approximation of Brownian motion, the coefficients are Lipschitz and meet the linear growth conditions and $-\sin^2(x)$ is differentiable and meets the same conditions we can now apply the Wong-Zakai Theorem. We get the following solution in terms of Stratonovich

$$dY^\lambda(x) = \Big(\cos^2 Y^\lambda(x) + \lambda \sin^2 Y^\lambda(x)\Big)\, dx - \sin^2 Y^\lambda(x) \circ dB(x),$$

and the corresponding Itô SDE is given by

$$dY^\lambda(x) = \Big(\cos^2 Y^\lambda(x) + \lambda \sin^2 Y^\lambda(x) + \sin^3 Y^\lambda(x) \cos Y^\lambda(x)\Big)\, dx \tag{4.3}$$

$$- \sin^2 Y^\lambda(x) dB(x). \tag{4.4}$$

We will need this form later to directly calculate the spectral distribution.

4.3. Continuity Lemmas

At this stage, we must address a measure-theoretic difficulty. While the Wong-Zakai theorem ensures that the sequence of approximating processes converges pathwise, $Y_n^\lambda \to Y^\lambda$, almost surely for a fixed value of λ, this does not immediately imply that the identity

$$\mathcal{N}(\lambda, 0, l) = \left\lfloor \frac{Y^\lambda(l)}{\pi} \right\rfloor$$

holds for all $\lambda \in \mathbb{R}$ simultaneously. This would require the intersection of an uncountable collection of sets of measure one, which is not guaranteed to be a set of measure one. In the worst case, the union of the exceptional null sets could span the entire sample space.

To resolve this, we follow the construction from Fukushima and Nakao ([1]) by first establishing the almost sure convergence for all $\lambda \in \mathbb{Q}$ and rational $l > 0$. Let $\Omega_{\lambda,l}$ be the set such that $Y_n^\lambda \to Y^\lambda$ for a fixed λ and rational $l > 0$, then for any fixed pair (λ, l). We define $\Omega_1 = \bigcap_{\lambda \in \mathbb{Q}} \bigcap_{l \in \mathbb{Q}^+} \Omega_{\lambda,l}$. Since this is a countable intersection of sets of measure one, $P(\Omega_1) = 1$. First we will show that the identity from above holds for all such sets, then we will extend to all real numbers by continuity to uniquely determine the behavior of the process for all λ. This will ensure that the spectral distribution function is well-defined. This motivates the following lemma.

Lemma 4.3.1. *For almost every $\omega \in \Omega_1$, the following inequality holds for every $\lambda \in \mathbb{Q}$ and every $l \in \mathbb{Q}^+$:*

$$0 \leq \frac{Y^\lambda(l,\omega)}{\pi} - \mathcal{N}^\omega(\lambda, 0, l) \leq 1.$$

Proof. For a fixed ω, for every $\lambda \in \mathbb{Q}$ and every $l \in \mathbb{Q}^+$, by Sturm-Liouville oscillation theorem we have, for any linear approximating sequence $\{B_n\}$ of B, the result that

$$\mathcal{N}_n^\omega(\lambda, 0, l) = \left\lfloor \frac{Y_n^\lambda(l,\omega)}{\pi} \right\rfloor,$$

which is equivalent to

$$0 \leq \frac{Y_n^\lambda(l,\omega)}{\pi} - \mathcal{N}_n^\omega(\lambda, 0, l) < 1.$$

By the Wong-Zakai theorem, $Y_n^\lambda \to Y^\lambda$ almost surely as $n \to \infty$ and the uniform convergence of paths conserves of the eigenvalue count. We note that the strict inequality may become non-strict in the limit if $Y^\lambda(l,\omega)$ is a multiple of π, therefore

$$0 \leq \frac{Y^\lambda(l,\omega)}{\pi} - \mathcal{N}^\omega(\lambda, 0, l) \leq 1.$$

□

While we will get continuity in l by the continuity of sample paths of the limit process Y^λ, the extension to all $\lambda \in \mathbb{R}$ will be achieved by Kolmogorov's continuity criterion and the following lemma, we state the theorem first.

Theorem 4.3.1 (Kolmogorov's Continuity Criterion)**.** *Let $\{X_t\}_{t\geq 0}$ be a real-valued stochastic process. Suppose there exist constants $\alpha, \epsilon, C > 0$ such that for every $t \geq 0$ and $h > 0$ we have*

$$E\left[|X_{t+h} - X_t|^\alpha\right] \leq Ch^{1+\epsilon}.$$

Then there exists a modification $\tilde{X}$ of X which is almost surely continuous.

The proof of this standard result can be found in most introductory texts on stochastic processes (see e. g. Øksendal [9] or Revuz/Yor [6])

Lemma 4.3.2. *Let a, b and c be bounded Lipschitz continuous functions on $\mathbb{R}$ and $X^\lambda(t)$ be the solution of the stochastic differential equation*

$$dX^\lambda(t) = a(X^\lambda(t))\, dB(t) + \big(b(X^\lambda(t)) + \lambda c(X^\lambda(t))\big)\, dt, \qquad X^\lambda(0) = 0.$$

Then, for each $T > 0$ and $K > 0$,

$$\mathbb{E}[|X^\lambda(t) - X^{\lambda'}(t)|^2] < C(\lambda - \lambda')^2 e^{Ct}, \qquad t \in [0,T],\ |\lambda|, |\lambda'| \leq K,\ \lambda, \lambda' \in \mathbb{Q},$$

for some constant $C > 0$.

Proof. We define the process

$$Y_t := X^\lambda(t) - X^{\lambda'}(t), \quad Y_0 = 0.$$

Then, by definition of $X^\lambda(t)$ we get

$$\begin{aligned} dY_t &= \big[a(X^\lambda(t)) - a(X^{\lambda'}(t))\big]dB(t) \\ &\quad + \big[b(X^\lambda(t)) - b(X^{\lambda'}(t))\big]dt \\ &\quad + \big[\lambda c(X^\lambda(t)) - \lambda' c(X^{\lambda'}(t))\big]dt. \end{aligned}$$

Now we apply Itô's formula to $f(Y_t) = Y_t^2$. Since $f'(y) = 2y$ and $f''(y) = 2$, we have

$$\begin{aligned} d(Y_t^2) &= 2Y_t dY_t + \big[a(X^\lambda(t)) - a(X^{\lambda'}(t))\big]^2 (dB(t))^2 \\ &= \big[a(X^\lambda(t)) - a(X^{\lambda'}(t))\big]^2 dt. \end{aligned}$$

We substitute into the formula for dY_t to get

$$\begin{aligned} d(Y_t^2) &= 2Y_t\big[a(X^\lambda(t)) - a(X^{\lambda'}(t))\big]dB(t) \\ &\quad + 2Y_t\big[b(X^\lambda(t)) - b(X^{\lambda'}(t))\big]dt \\ &\quad + 2Y_t\big[\lambda c(X^\lambda(t)) - \lambda' c(X^{\lambda'}(t))\big]dt \\ &\quad + \big[a(X^\lambda(t)) - a(X^{\lambda'}(t))\big]^2\, dt. \end{aligned}$$

Then we take the expectation, and note that the stochastic integral term vanishes since the integrand is bounded, so we compute

$$\begin{aligned} \mathbb{E}[Y_t^2] &= \mathbb{E}\left[\int_0^t 2Y_s\big[b(X^\lambda(s)) - b(X^{\lambda'}(s))\big]ds\right] \\ &\quad + \mathbb{E}\left[\int_0^t 2Y_s\big[\lambda c(X^\lambda(s)) - \lambda' c(X^{\lambda'}(s))\big]ds\right] \\ &\quad + \mathbb{E}\left[\int_0^t \big[a(X^\lambda(s)) - a(X^{\lambda'}(s))\big]^2 ds\right]. \end{aligned}$$

Now rewrite the term involving c by adding and substracting $\lambda c(X^{\lambda'}(s))$:

$$\begin{aligned} \lambda c(X^\lambda(s)) - \lambda' c(X^{\lambda'}(s)) &= \lambda c(X^\lambda(s)) - \lambda c(X^{\lambda'}(s)) + \lambda c(X^{\lambda'}(s)) - \lambda' c(X^{\lambda'}(s)) \\ &= \lambda\big[c(X^\lambda(s)) - c(X^{\lambda'}(s))\big] + (\lambda - \lambda')c(X^{\lambda'}(s)). \end{aligned}$$

Therefore we get

$$\begin{aligned} \mathbb{E}[Y_t^2] &= \mathbb{E}\left[\int_0^t 2Y_s\big[b(X^\lambda(s)) - b(X^{\lambda'}(s))\big]ds\right] \\ &\quad + \mathbb{E}\left[\int_0^t 2Y_s\lambda\big[c(X^\lambda(s)) - c(X^{\lambda'}(s))\big]ds\right] \\ &\quad + \mathbb{E}\left[\int_0^t 2Y_s(\lambda - \lambda')c(X^{\lambda'}(s))ds\right] \\ &\quad + \mathbb{E}\left[\int_0^t \big[a(X^\lambda(s)) - a(X^{\lambda'}(s))\big]^2 ds\right]. \end{aligned}$$

Since a, b and c are bounded Lipschitz continuous functions we can find a common Lipschitz constant L and a common bound M (by taking the maximum of the Lipschitz-constants and bounds, respectively) to get

$$\begin{aligned}\mathbb{E}[Y_t^2] \le 2L &\int_0^t \mathbb{E}[|Y_s|^2]ds \\ &+ 2\lambda L \int_0^t \mathbb{E}[|Y_s|^2]ds \\ &+ 2M|\lambda - \lambda'| \int_0^t \mathbb{E}[|Y_s|]ds \\ &+ L^2 \int_0^t \mathbb{E}[|Y_s|^2]ds.\end{aligned}$$

Now we let $f(t) = \mathbb{E}[Y_t^2]$, by Cauchy-Schwarz $\mathbb{E}[|Y_s|] \le \sqrt{f(s)}$, so we obtain

$$f(t) \le \left(L^2 + 2L(1 + |\lambda|)\right) \int_0^t f(s)ds + 2M|\lambda - \lambda'| \int_0^t \sqrt{f(s)}ds.$$

Then we define $C_0 = \max\{C_1 + C_2|\lambda - \lambda'|, C_2|\lambda - \lambda'|T\}$ for $t \in [0, T]$ and use the inequality $\sqrt{f(s)} \le 1 + f(s)$ for all $s \ge 0$ to get

$$f(t) \le C_0|\lambda - \lambda'| + C_0 \int_0^t f(s)ds, \quad t \in [0, T].$$

Applying Gronwall's inequality (see e. g. [4, 9]) yields

$$f(t) \le C|\lambda - \lambda'|e^{Ct}, \quad t \in [0, T].$$

Finally, since $|\lambda - \lambda'| \le (\lambda - \lambda')^2 + 1$ for $|\lambda|, |\lambda'| \le K$, we can absorb constants to obtain the desired result

$$\mathbb{E}[|X^\lambda(t) - X^{\lambda'}(t)|^2] \le C(\lambda - \lambda')^2 e^{Ct}$$

for some constant $C > 0$ independent of λ, λ', and t.

□

Proposition 4.3.1. *For each fixed $l > 0$, the stochastic process $\{Y^\lambda(l, \omega) : \lambda \in \mathbb{R}\}$ admits a modification which is almost surely continuous in λ.*

Proof. Let $l > 0$ be fixed. Since Y^λ is the solution of a SDE where the coefficients are Lipschitz continuous, we can apply the Lemma, therefore we have found an upper bound for the process, which is

$$\mathbb{E}\left[|Y^\lambda(l) - Y^{\lambda'}(l)|^2\right] \le C_l|\lambda - \lambda'|^2, \quad \forall \lambda, \lambda' \in \mathbb{Q} \cap [-K, K].$$

Now by letting $\alpha = 2$ and $\epsilon = 1$ the hypotheses of the Kolmogorov Continuity Criterion are satisfied. Therefore, there exists a set Ω_2 with $P(\Omega_2) = 1$ such that for every $\omega \in \Omega_2$, the mapping $\lambda \mapsto Y^\lambda(l, \omega)$ is continuous for all $\lambda \in \mathbb{R}$. □

5. Infinitesimal Generators

To compute the explicit solution for our spectral distribution function we need to introduce the infinitesimal generator. We will see that this definition leads us to Dynkin's theorem, which has many applications, but in particular it will give us the key to calculate expectations of hitting time problems. During this section we will mainly follow Øksendal [9] and Revuz/Yor [6].

Definition 5.0.1. *Let $\{X_t\}_{t\geq 0}$ be a time-homogeneous Itô process (the drift and diffusion don't depend on t) on $\mathbb{R}^d$. The infinitesimal generator $\mathcal{A}$ is defined by:*

$$\mathcal{A}f(x) = \lim_{t\downarrow 0} \frac{\mathbb{E}^x[f(X_t)] - f(x)}{t}$$

for f in the domain $\mathcal{D}(\mathcal{A})$, where the limit exists.

If we let $\{X_t\}_{t\geq 0}$ be a time-homogeneous Markov process on $\mathbb{R}$ and f a function in $C^2(\mathbb{R}^d)$ and then define

$$M_t^f = f(X_t) - f(X_0) - \int_0^t \mathcal{A}f(X_s)ds,$$

it can also be shown, that the operator $\mathcal{A}$ is an infinitesimal generator if and only if M_t^f is a martingale with respect to the natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$ for all f in the domain $\mathcal{D}(\mathcal{A})$. For our purposes the former definition will suffice.

The next lemma will help us to establish a relation between the coefficients of an Itô process and its infinitesimal generator. We will state the one-dimensional case, as it applies to our goals, see [9] for the higher-dimensional case, the proof is similar.

Lemma 5.0.1. *Let $Y_t = Y_t^x$ be an Itô process in $\mathbb{R}$ of the form*

$$Y_t^x = x + \int_0^t \mu(s,\omega)ds + \int_0^t \sigma(s,\omega)dB_s$$

where B is a Brownian motion. Let $f \in C_0^2(\mathbb{R})$ and let τ be a stopping time with $\mathbb{E}[\tau] < \infty$. Assume that μ and σ are bounded when Y_t is in the support of f. Then

$$\mathbb{E}[f(Y_\tau)] = f(x) + \mathbb{E}\left[\int_0^\tau \left(\mu(s)f'(Y_s) + \frac{1}{2}\sigma^2(s)f''(Y_s)\right)ds\right]$$

Proof. First we apply Itô's formula to $f(Y_t)$ to get

$$df(Y_t) = f'(Y_t)dY_t + \frac{1}{2}f''(Y_t)(dY_t)^2.$$

Now we substitute $dY_t = \mu dt + \sigma dB_t$ into this equation, resulting in

$$df(Y_t) = \left(\mu f'(Y_t) + \frac{1}{2}\sigma^2 f''(Y_t)\right) dt + \sigma f'(Y_t) dB_t.$$

Then we integrate this formula and take expectations to arrive at

$$\mathbb{E}[f(Y_\tau)] = f(x) + \mathbb{E}\left[\int_0^\tau \left(\mu f' + \frac{1}{2}\sigma^2 f''\right) ds\right] + \mathbb{E}\left[\int_0^\tau \sigma f' dB_s\right].$$

Now we note that the expectation of the stochastic integral is zero, concluding our proof. □

Theorem 5.0.1. *Let X_t be the Itô diffusion*

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t.$$

If $f \in C_0^2(\mathbb{R})$, then f belongs to the domain $\mathcal{D}_\mathcal{A}$ of the infinitesimal generator and

$$\mathcal{A}f(x) = \mu(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x)$$

Proof. We apply the definition of the infinitesimal generator and our lemma for $\tau = t$ to get

$$\mathbb{E}^x[f(X_t)] = f(x) + \mathbb{E}^x\left[\int_0^t \left(\mu(X_s)f'(X_s) + \frac{1}{2}\sigma^2(X_s)f''(X_s)\right) ds\right].$$

Now we divede by t and take the limit as $t \to 0$, yielding

$$\frac{\mathbb{E}^x[f(X_t)] - f(x)}{t} = \frac{1}{t}\mathbb{E}^x\left[\int_0^t \left(\mu(X_s)f'(X_s) + \frac{1}{2}\sigma^2(X_s)f''(X_s)\right) ds\right].$$

By the mean value theorem for integrals and continuity of the coefficients we get

$$\lim_{t\downarrow 0} \frac{1}{t}\mathbb{E}^x\left[\int_0^t \left(\mu(X_s)f'(X_s) + \frac{1}{2}\sigma^2(X_s)f''(X_s)\right) ds\right] = \mu(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x).$$

Therefore we have

$$\mathcal{A}f(x) = \mu(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x).$$

□

Now we have all the tools to formulate and prove Dynkin's formula in the one-dimensional case, for a more general formulation see e. g. Dynkin's book on Markov processes [10].

Theorem 5.0.2 (Dynkin's Formula)**.** *Let $f \in C_0^2(\mathbb{R}^n)$. Suppose τ is a stopping time with $\mathbb{E}^x[\tau] < \infty$. Then*

$$\mathbb{E}^x[f(X_\tau)] = f(x) + \mathbb{E}^x\left[\int_0^\tau Af(X_s)ds\right]$$

where $\mathcal{A}$ is the infinitesimal generator of the diffusion process X_t.

Proof. First, for any fixed $t > 0$, we apply the lemma to the stopping time $\tau \wedge t$

$$\mathbb{E}[f(X_{\tau\wedge t})] = f(x) + \mathbb{E}\left[\int_0^{\tau\wedge t} \left(\mu(X_s)f'(X_s) + \frac{1}{2}\sigma^2(X_s)f''(X_s)\right) ds\right]$$

Since $\mathbb{E}^x[\tau] < \infty$, we have $\tau \wedge t \to \tau$ almost surely as $t \to \infty$. We can use dominated convergence to get

$$\lim_{t\to\infty} \mathbb{E}^x[f(X_{\tau\wedge t})] = \mathbb{E}^x[f(X_\tau)].$$

For the integral term, since $\mathcal{A}f$ is bounded (as $f \in C_0^2(\mathbb{R}^n)$ and the coefficients are Lipschitz continuous), and $\mathbb{E}^x[\tau] < \infty$, we can apply dominated convergence to get

$$\lim_{t\to\infty} \mathbb{E}^x\left[\int_0^{\tau\wedge t} \mathcal{A}f(X_s)ds\right] = \mathbb{E}^x\left[\int_0^{\tau} \mathcal{A}f(X_s)ds\right]$$

Taking the limit as $t \to \infty$ in the equation from Step 1:

$$\mathbb{E}^x[f(X_\tau)] = f(x) + \mathbb{E}^x\left[\int_0^{\tau} \mathcal{A}f(X_s)ds\right]$$

This completes the proof of Dynkin's formula. □

5.1. Scale Functions and Speed Measures

We will now introduce the scale function s and speed measure m . These two functions will be defined such that the infinitesimal generator $\mathcal{A}$ of an Itô diffusion can be written in the canonical form

$$\mathcal{A} = \frac{d}{dm}\frac{d}{ds}$$

which will be the key to find closed form expressions of hitting probabilities and expected hitting times. In this chapter we will mainly follow [6, 16, 17].

We will be analysing one–dimensional diffusion processes using the representation of their generators in *scale–speed* form. This representation reduces a general diffusion to an equivalent diffusion in natural scale and yields closed–form expressions for hitting probabilities, Green functions, and expected exit times. In this chapter we will mainly follow [6, 16, 17].

Definition 5.1.1. *Let $X = (X_t)_{t\geq 0}$ be a one–dimensional Îto diffusion with infinitesimal generator*

$$\mathcal{A}f(x) = \frac{1}{2}\sigma^2(x)f''(x) + \mu(x)f'(x)$$

We define the scale function $s(x)$ as the solution to the ordinary differential equation

$$\mathcal{A}s(x) = 0.$$

We can now solve
$$\frac{1}{2}\sigma^2(x)s''(x) + \mu(x)s'(x) = 0$$
and get the formula for the first derivative of the scale function, the scale density
$$s'(x) = \exp\left(-\int_{x_0}^{x} \frac{2\mu(u)}{\sigma^2(u)}du\right).$$
We note that the lower integration limit x_0 is arbitrary. We integrate again to get a formula for the scale function
$$s(x) = \int_{x_1}^{x} s'(x)du,$$
where the lower integration bound x_1 again can be chosen arbitrarily. The function s is determined only up to an affine transformation $s \mapsto As + B$, but all probabilistic quantities which we will derive in this section are invariant under this normalization. Now if we apply Itô's formula we compute

$$\begin{aligned}
dY_t &= s'(X_t)dX_t + \frac{1}{2}s''(X_t)(dX_t)^2 \\
dY_t &= s'(X_t)\left[\mu(X_t)dt + \sigma(X_t)dB_t\right] + \frac{1}{2}s''(X_t)\sigma^2(X_t)dt \\
dY_t &= \left[\mu(X_t)s'(X_t) + \frac{1}{2}\sigma^2(X_t)s''(X_t)\right]dt + s'(X_t)\sigma(X_t)dB_t.
\end{aligned}$$

But the dt coefficient is exactly $\mathcal{A}f(x)$, so
$$dY_t = s'(X_t)\sigma(X_t)dB_t,$$
We observe that the transformation $Y_t = s(X_t)$ eliminates the drift, so Y is a diffusion in natural scale.

Once we have removed the drift, the generator of Y is of the form
$$\widetilde{\mathcal{A}}f = \frac{1}{2}\widetilde{\sigma}^2(y)f''(y).$$
Now we transform back to the x variable to get
$$\mathcal{A}f(x) = \frac{2}{\sigma^2(x)s'(x)}\frac{d}{dx}\left(\frac{1}{s'(x)}f'(x)\right).$$

Definition 5.1.2. *We define the speed density as*
$$m(x) = \frac{2}{\sigma^2(x)s'(x)}.$$
The corresponding measure $m(x)dx$ is called the speed measure.

We notice, that by this definition and application of the chain rule, the infinitesimal generator has the canonical form $\mathcal{A} = \frac{d}{dm}\frac{d}{ds}$. Our intuition should be that s determines the geometric structure of the state space, while m describes how much time the diffusion spends near a given point.
In the Appendix we compute the scale density and speed density for the diffusion operator $\mathcal{A}$ which corresponds to Y^λ.

5.2. Derivation Of First Hitting Time Formula

Definition 5.2.1. *Let X_t be an Itô diffusion process on an interval (a, b), for a fixed boundary point b, define the first hitting time by*

$$\tau_b = \inf\{t > 0 : X_t = b\}.$$

Definition 5.2.2. *We define the mean hitting time (or expected exit time) function by*

$$u(x) = \mathbb{E}_x[\tau_b]$$

where $\mathbb{E}_x[\cdot]$ denotes expectation given $X_0 = x$.

We now want our function $u(x)$ be expressed in terms of Dynkin's formula, this will lead us to an boundary value problem which we will solve. We set the right hand side of Dynkin's formula equal to u(x) to get

$$u(x) = \mathbb{E}_x[\tau_b] = \mathbb{E}_x\left[\int_0^{\tau_b} ds\right] = \mathbb{E}_x[f(X_{\tau_b})] - f(x),$$

so we note that f satisfies the equation

$$\mathcal{A}f(x) = 1$$

In the case we are interested in, b is an absorbing boundary, which means that $X_{\tau_b} = b$ almost surely, so $\mathbb{E}_x[f(X_{\tau_b})] = f(b)$, and the above equation simplifies to

$$u(x) = f(b) - f(x).$$

5.2.1. Boundary Value Problem for $u(x)$

We now apply the infinitesimal generator to this equation to get

$$\mathcal{A}u(x) = \mathcal{A}[f(b) - f(x)] = -\mathcal{A}f(x) = -1$$

since $f(b)$ is constant and $\mathcal{A}$ applied to a constant is zero.
This is also called the Poisson equation and leads us to the fact that expectation of the first hitting time $u(x) = \mathbb{E}[\tau_b | X_0 = x]$ satisfies the boundary value problem

$$\mathcal{A}u(x) = -1 \quad \text{for } a < x < b$$
$$\frac{du}{ds}(0) = 0, \quad u(\pi) = 0.$$

with boundary conditions dictated by Feller's boundary classification (see e.g. [11, 12]). By our previous arguments (see Appendix), the scale function for the solution to our SDE $Y^\lambda(x)$ satisfies

$$ds(z) = \exp\left(2\cot^3 z + 2\lambda \cot z\right)\sin^2 z dz.$$

Now we calculate the limits on the boundary to see that

$$\lim_{z\to 0^+} s(z) = -\infty,$$

$$s(\pi) < \infty.$$

By Feller's boundary classification this means that 0 is an entrance boundary and π is an exit boundary. This is expected from the physical interpretation of this problem, but also gives mathematical rigour for choosing the Neumann-Dirichlet boundary conditions for the mean first hitting time problem

$$u(x) = \mathbb{E}[\tau_\pi | Y_0^\lambda = x].$$

Definition 5.2.3. *At this point we define the mean first hitting time $\mathbb{E}[\tau^\lambda]$ as the expectation of the first hitting time τ_π of the solution of the Itô SDE* (4.4) *starting at* 0*, namely*

$$\mathbb{E}[\tau^\lambda] = u(0) = \mathbb{E}[\tau_\pi | Y_0^\lambda = 0].$$

Proposition 5.2.1 (The Boundary Value Problem)**.** *Let $Y^\lambda(x)$ be the phase process defined by the Itô SDE* (4.4) *with infinitesimal generator $\mathcal{A}$. The mean time $u(x) = \mathbb{E}_x[\tau_\pi]$ for the process to hit the exit boundary π starting from $x \in (0,\pi)$ is the unique solution to the Poisson equation*

$$\mathcal{A}u(x) = -1, \quad x \in (0,\pi) \tag{5.1}$$

with boundary conditions

$$\frac{du}{ds}(0) = 0, \quad u(\pi) = 0. \tag{5.2}$$

Solving this equation yields the formula for the mean first hitting time

$$\mathbb{E}[\tau^\lambda] = \int_0^\pi (s(\pi) - s(z))dm(z). \tag{5.3}$$

Proof. The argument above covers the first part of the proposition, what is left to show is the mean first hitting time formula (5.3). To do this we first integrate $\frac{d}{dm}\frac{d}{ds}u(x) = -1$ twice to get

$$u(x) = -\int_0^x m(z)ds(z) + C_1(s(x) - s(0)) + C_2$$

Invoking the Neumann boundary condition yields

$$\frac{du}{ds}(0) = -m(0) + C_1 = 0 \Rightarrow C_1 = m(0).$$

After applying the Dirichlet boundary condition we get

$$u(\pi) = -\int_0^\pi m(z)ds(z) + m(0)(s(\pi) - s(0)) + C_2 = 0$$

$$\Rightarrow C_2 = \int_0^\pi m(z)ds(z) - m(0)(s(\pi) - s(0))$$

Then we substitute back into the formula for $u(x)$ to get

$$u(x) = -\int_0^x m(z)ds(z) + m(0)(s(x) - s(0)) + \int_0^\pi m(z)ds(z) - m(0)(s(\pi) - s(0))$$

$$u(x) = \int_x^\pi m(z)ds(z) + m(0)(s(x) - s(\pi))$$

Since $m(0) = 0$ the second term vanishes. In particular, if we set the starting point to zero we get the following formula

$$u(0) = \mathbb{E}[\tau^\lambda] = \int_0^\pi m(z)ds(z).$$

Integrating by parts once yields

$$u(0) = [m(z)s(z)]_0^\pi - \int_0^\pi s(z)dm(z),$$

where the first term vanishes,

$$u(x) = \int_x^\pi m(z)ds(z)$$

Finally, after adding and subtracting $s(\pi)m(\pi) = 0$ we arrive at the mean first hitting time formula

$$\mathbb{E}[\tau^\lambda] = \int_0^\pi (s(\pi) - s(z))dm(z). \tag{5.4}$$

□

For the next Proposition we need the scale- and speed-density of the diffusion process Y^λ, the calculations can be found in the Appendix.

Proposition 5.2.2. *The mean first hitting formula has the explicit solution*

$$\mathbb{E}[\tau^\lambda] = \sqrt{2\pi}\int_0^\infty w^{-1/2}\exp\left(-\frac{1}{6}w^3 - 2\lambda w\right)dw. \tag{5.5}$$

Proof. To evalute the integral (5.4), using the scale function and speed density, which correspond to the generator of the diffusion process Y^λ, we have to expand the first hitting time formula to a double integral, since we only know s', to get

$$\mathbb{E}[\tau^\lambda] = \int_0^\pi m(z)\left(\int_z^\pi s'(y)dy\right)dz,$$

before substituting in we reduce the scale and speed density formulas by substituting $u = \cot z$ in both formulas, so

$$\begin{aligned} s'(z) &= (1+u^2)\exp\left(\frac{2}{3}u^3 + 2\lambda u\right) \\ m(z) &= 2(1+u^2)^{-1}\exp\left(-\frac{2}{3}u^3 - 2\lambda u\right). \end{aligned}$$

Now we substitute into the expanded first hitting time formula (5.4), yielding

$$\mathbb{E}[\tau^\lambda] = 2\int_{-\infty}^\infty\int_{-\infty}^u \exp\left(-\frac{2}{3}(u^3 - v^3) - 2\lambda(u-v)\right)dvdu.$$

To evaluate this integral, we make another change of variables, setting $w = u - v$ and $s = u + v$, now we compute

$$\begin{aligned} \mathbb{E}[\tau^\lambda] &= 2\int_{-\infty}^\infty\int_{-\infty}^u \exp\left(-\frac{2}{3}(u^3 - v^3) - 2\lambda(u-v)\right)dv\,du \\ &= 2\cdot\frac{1}{2}\int_0^\infty \exp\left(-\frac{1}{6}w^3 - 2\lambda w\right)\left(\int_{-\infty}^\infty \exp\left(-\frac{w}{2}s^2\right)ds\right)dw \end{aligned}$$

The expression $\int_{-\infty}^\infty \exp\left(-\frac{w}{2}s^2\right)ds$ is a Gaussian integral with solution $\sqrt{\frac{2\pi}{w}}$, therefore we get our final result

$$\mathbb{E}[\tau^\lambda] = \sqrt{2\pi}\int_0^\infty w^{-1/2}\exp\left(-\frac{1}{6}w^3 - 2\lambda w\right)dw. \tag{5.6}$$

$\square$

6. The Spectral Distribution Function

We are now ready to state and prove the main theorem of Fukushima and Nakao's paper ([1]).

Theorem 6.0.1. *There exists a subset Ω_0 of the probability space Ω with $P(\Omega_0) = 1$ such that for every $\omega \in \Omega_0$, the spectral distribution function $N(\lambda)$ is given by*

$$N(\lambda) = \lim_{l\to\infty} \frac{\mathcal{N}^{\omega}(\lambda, 0, l)}{l} = \frac{1}{\mathbb{E}[\tau^{\lambda}]} \quad \textit{for every } \lambda, \tag{6.1}$$

where τ^{λ} is the first hitting time of π for the diffusion process $Y^{\lambda}(x)$ defined by the stochastic differential equation

$$dY^{\lambda}(x) = -\sin^2 Y^{\lambda}(x) \circ dB(x) + \Big(\cos^2 Y^{\lambda}(x) + \lambda \sin^2 Y^{\lambda}(x)\Big)\, dx, \quad Y^{\lambda}(0) = 0.$$

More explicitly, the mean first hitting time is given by

$$\mathbb{E}[\tau^{\lambda}] = \sqrt{2\pi} \int_0^{\infty} u^{-\frac{1}{2}} \exp\left\{-\frac{1}{6}u^3 - 2\lambda u\right\} du. \tag{6.2}$$

Consequently, the spectral distribution function has the form

$$N(\lambda) = \left(\sqrt{2\pi} \int_0^{\infty} u^{-\frac{1}{2}} \exp\left\{-\frac{1}{6}u^3 - 2\lambda u\right\} du\right)^{-1}.$$

Proof. Using Proposition 5.2.1 and Proposition (5.2.2) we already have shown the mean first hitting time formula (6.2). We will now prove the spectral distribution function formula (6.1).

First, we know from Sturm-Liouville theory that

$$\lim_{l\to\infty} \frac{\mathcal{N}^{\omega}(\lambda, 0, l)}{l} = \lim_{l\to\infty} \frac{Y^{\lambda}(l, \omega)}{\pi l}$$

To show that

$$\lim_{l\to\infty} \frac{Y^{\lambda}(l, \omega)}{\pi l} = \frac{1}{\mathbb{E}[\tau^{\lambda}]},$$

we let $\tau_n^{\lambda}(\omega) = \inf\{x > 0 : Y^{\lambda}(x, \omega) = n\pi\}$ for $n = 1, 2, \dots$ be a sequence of hitting times. We split up τ_n into a sum of independent, identically distributed random variables, yielding

$$\tau_n^{\lambda} = \tau_1^{\lambda} + (\tau_2^{\lambda} - \tau_1^{\lambda}) + \cdots + (\tau_n^{\lambda} - \tau_{n-1}^{\lambda}).$$

By the Strong Law of Large Numbers, we have $\lim_{n\to\infty} \frac{\tau_n}{n} = E[\tau_1^\lambda] = E[\tau^\lambda]$ almost surely. Now, for every $\omega \in \Omega$, and for any $l > 0$, there exists an integer n such that

$$\tau_n(\omega) \leq l < \tau_{n+1}(\omega).$$

By the definition of the hitting times, we have the inequality

$$n\pi \leq Y^\lambda(l) < (n+1)\pi$$

Dividing by the first inequality, we get that

$$\frac{n}{\tau_{n+1}(\omega)} < \frac{Y^\lambda(l,\omega)}{\pi l} \leq \frac{n+1}{\tau_n(\omega)},$$

and as $l \to \infty$, both the upper and lower bounds converge to $1/E[\tau^\lambda]$.

Finally, it remains to show that for every $\lambda \in \mathbb{R}$, the spectral distribution function $N(\lambda)$ exists and coincides with the formula $1/\mathbb{E}[\tau^\lambda]$.

The monotonicity of the limit $N(\lambda) = \lim_{l\to\infty} \mathcal{N}^\omega(\lambda, 0, l)/l$ in λ is a consequence of the Minimax Principle. It also ensures that the counting function is monotone in l and that the limit is independent of the specific boundary conditions chosen at the endpoints.

Now we have to make sure that this limit holds for all $\lambda \in \mathbb{R}$ simultaneously, so we define the set $\Omega_0 = \Omega_1 \cap \Omega_2$, where we recall that $\Omega_1 = \bigcap_{\lambda\in\mathbb{Q}} \bigcap_{l\in\mathbb{Q}^+} \Omega_{\lambda,l}$, and Ω_2 is the set we defined using Kolmogorovs Continuity Criterion 4.3.1 to be the collection of $\omega \in \Omega$ such that the mapping $\lambda \mapsto Y^\lambda(l,\omega)$ is continuous for all $\lambda \in \mathbb{R}$. As intersection of sets of measure one, $P(\Omega_0) = 1$ and we can use the properties of both sets in our following arguments.

First, by Lemma 4.3.2, the equality holds for all rational λ. For any irrational $\lambda \in \mathbb{R}$, we choose sequences of rationals $\{\lambda'_n\}$ and $\{\lambda''_n\}$ such that $\lambda'_n \uparrow \lambda$ and $\lambda''_n \downarrow \lambda$. Because the phase process Y^λ and the counting function $\mathcal{N}^\omega$ are monotone in λ, we have, for any $l > 0$

$$\frac{Y^{\lambda'_n}(l,\omega)}{\pi l} \leq \frac{Y^\lambda(l,\omega)}{\pi l} \leq \frac{Y^{\lambda''_n}(l,\omega)}{\pi l}. \tag{6.3}$$

Taking the limit as $l \to \infty$, the outer terms converge to $1/\mathbb{E}[\tau^{\lambda'_n}]$ and $1/\mathbb{E}[\tau^{\lambda''_n}]$ respectively for all $\omega \in \Omega_0$.

Furthermore, by the Kolmogorov Continuity Criterion 4.3.1, on Ω_0 the mean hitting time $\mathbb{E}[\tau^\lambda]$ is a continuous function of λ. Thus, as $n \to \infty$, both $1/\mathbb{E}[\tau^{\lambda'_n}]$ and $1/\mathbb{E}[\tau^{\lambda''_n}]$ converge to $1/\mathbb{E}[\tau^\lambda]$. By the Sandwich Theorem, the identity holds for irrational λ as well. We therefore conclude that for all $\omega \in \Omega_0$ and for every $\lambda \in \mathbb{R}$ it holds that

$$N(\lambda) = \frac{1}{\mathbb{E}[\tau^\lambda]} = \left(\sqrt{2\pi}\int_0^\infty u^{-1/2} \exp\left\{-\frac{1}{6}u^3 - 2\lambda u\right\} du\right)^{-1}. \tag{6.4}$$

$\square$

We remark that, although we considered a Schrödinger operator with a stochastic potential, the resulting spectral distribution function $N(\lambda)$ is deterministic. The behavior of the corresponding eigenfunctions has been studied, for instance, by Dumaz and Labbé ([18]), who provided a comprehensive proof that Anderson localization holds for this specific model. They established that, almost surely, the spectral measure is pure point and that the corresponding eigenfunctions are exponentially localized.

A. Appendix

A.1. Derivation of Scale Function and Speed Measure

We now apply the definition of the infinitesimal generator $\mathcal{A}$ which corresponds to Y^λ in order to compute $s'(x)$ and $m(x)$.

$$\mathcal{A}f(z) = \frac{1}{2}\sin^4 z f''(z) + \Big(\cos^2 z + \lambda \sin^2 z + \sin^3 z \cos z\Big) f'(z).$$

By definition of the Îto diffusion the coefficients are

$$\sigma^2(z) = \sin^4 z, \qquad \mu(z) = \cos^2 z + \lambda \sin^2 z + \sin^3 z \cos z.$$

Scale Density

We compute the exponent of the scale density

$$\frac{2\mu(z)}{\sigma^2(z)} = \frac{2(\cos^2 z + \lambda \sin^2 z + \sin^3 z \cos z)}{\sin^4 z} = 2\csc^4 z + 2(\lambda - 1)\csc^2 z + 2\cot z.$$

Integrating term by term using,

$$\int \csc^2 z dz = -\cot z, \qquad \int \cot z dz = \ln(\sin z),$$

and using

$$\int \csc^4 z dz = \frac{1}{3}\cot z - \frac{1}{3}\cot^3 z,$$

we obtain

$$\int^z \frac{2\mu(u)}{\sigma^2(u)} du = 2\ln(\sin z) + \Big(-2\lambda + \frac{2}{3}\Big)\cot z \frac{2}{3}\cot^3 z + C.$$

Therefore, up to normalization,

$$s'(z) = \sin^{-2} z \exp\bigg(\Big(2\lambda - \frac{2}{3}\Big)\cot z + \frac{2}{3}\cot^3 z\bigg).$$

Speed Density

Using the definition of the speed density we get

$$m(z) = \frac{2}{\sin^4 z s'(z)} = 2\sin^{-2} z \exp\bigg(\Big(\frac{2}{3} - 2\lambda\Big)\cot z - \frac{2}{3}\cot^3 z\bigg)$$

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