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Characteristic Gluing and Conserved Charges in General Relativity


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**Abstract**

Diese Arbeit untersucht das charakteristische Verklebungsproblem in asymptotisch flachen Raumzeiten im Rahmen des Bondi–Sachs-Formalismus sowie dessen Zusammenhang mit Erhaltungsgrößen. Aus den Null-Nebenbedingungen werden zwei Familien radial erhaltener Ladungen hergeleitet und als eichinvariante Obstruktionen für die charakteristische Verklebung identifiziert. Es wird gezeigt, dass sie mit bekannten Ladungen der Allgemeinen Relativitätstheorie, insbesondere ADM- und BMS-Ladungen, in Beziehung stehen. Die radialen und ADM-Ladungen werden zudem explizit für die langsam rotierende Kerr-Raumzeit berechnet. Darüber hinaus werden weitere eichabhängige Obstruktionen diskutiert, die aus den Null-Nebenbedingungen hervorgehen, einschließlich der Rekonstruktion einer vollständig nichtlinearen Erhaltungsgröße, die für das Verklebungsproblem relevant ist.

# Contents

# 1 Introduction

General relativity is one of the most important theories of modern physics. It is the foundation of our current understanding of gravity, black holes and the large scale structure of the universe. Beyond its physical significance, it also provides an interesting mathematical framework, combining differential geometry, analysis and partial differential equations on curved manifolds.

One of the main challenges of general relativity is the analysis of the initial value problem, where one aims to construct a spacetime satisfying the Einstein equations from a given dataset on a hypersurface. Closely related to this is the gluing problem, where two different solutions or initial datasets are attempted to join together in a mathematically consistent way.

This thesis focuses on the characteristic gluing problem in asymptotically flat spacetimes and its relation to conserved quantities in general relativity. In particular, I investigate the obstructions of the gluing, and show how are these related to other well-known conserved charges.

In Section 2 I start with an overview of the main concepts that I will use throughout this thesis [4, 10, 11, 16, 22]. In particular, I introduce the Bondi–Sachs formalism, which provides a geometric framework for describing a general asymptotically flat spacetime. I also discuss the basics of 3+1 decomposition, I compute the extrinsic curvature tensor and I show a linearization procedure used to simplify complicated equations.

Section 3 presents the basics of the characteristic gluing problem [1, 2, 8, 9]. It shows how the gauge-dependent and gauge-invariant obstructions arise, which will play a central role in this thesis.

Section 4 introduces four conserved quantities of asymptotically flat spacetimes: the ADM energy, momentum, angular momentum and center of mass [4, 12, 18, 20]. The section also includes a quick derivation of the ADM mass within the Hamiltonian formulation of general relativity.

Section 5 shows the derivation of two families of conserved charges, $\overset{[1]}{Q}$ and $\overset{[2]}{Q}$, from the null constraint equations [6, 8]. These play an important role in the $C^2$ characteristic gluing, as they make up the gauge-invariant obstructions.

In Section 6 I show that asymptotically the charge $\lim_{r\to\infty} \overset{[2]}{Q}(\lambda = 1)$ is proportional to the ADM energy, and the linearized charge $\overset{[1]}{Q}$ is proportional to the linearized ADM angular momentum.

In Section 7 I calculate the exact values of the previously defined ADM and the radial charges in a slowly rotating limit of the Kerr spacetime [5, 15].

Section 8 gives an overview of the BMS group, which describes the symmetries of asymptotically flat spacetimes at null infinity [19, 21]. In this section I rederive a formula from [10] for the conserved charges associated with a generator of the symmetry group, and then relate it to one of the radial charges.

Section 9 presents the derivation of additional conserved quantities from the null constraint equations [6]. I recover the fully non-linear derivation of a quantity $\overset{(1)}{H}_{uA}$, which is also relevant to the characteristic gluing problem.

For the analytic calculations I used the *Wolfram Mathematica* 14.2 software as an aid. The plots in figure 1 and 2 were also made using *Mathematica*. For the differential geometric and tensor algebra calculations I used the *xAct xTensor*[1] package developed by José M. Martín-García.

Throughout this thesis I adopt the signature convention $(-, +, +, +)$. I work in geometric units, i.e. the gravitational constant and the speed of light are set to unity: $G = c = 1$. I use the Einstein summation convention, Greek indices denote the four-dimensional spacetime components, lowercase Latin indices denote the purely spatial components in the $3 + 1$ decomposition, and uppercase Latin indices denote the angular components on the 2-sphere.

[1] http://www.xact.es/

## 2 Bondi–Sachs and ADM formalism

### 2.1 Bondi metric

In 1960 Hermann Bondi introduced a new formalism that could be used as an effective method to investigate gravitational waves [11, 16]. He presented a general spacetime metric with coordinates based on a set of outgoing null hypersurfaces, along which the outgoing gravitational wave fronts can travel. These null hypersurfaces are given by the

$$u = const \tag{1}$$

condition, where $u = t - r$ is the retarded time. The surfaces being null means that their normal covector $n_\mu = -\partial_\mu u$ has zero length

$$g^{\mu\nu}\ (\partial_\mu u)\ (\partial_\nu u) = 0\ . \tag{2}$$

In $x^\mu = (u,\ r,\ x^A)$ Bondi coordinates we can write the metric as

$$\begin{aligned} g_{\mu\nu}dx^\mu dx^\nu = & -\frac{V}{r}e^{2\beta}du^2 - 2e^{2\beta}dudr \\ & + r^2\gamma_{AB}\Big(dx^A - U^A du\Big)\Big(dx^B - U^B du\Big)\ , \end{aligned} \tag{3}$$

which is called Bondi or alternatively, Bondi–Sachs metric. One can use this metric locally around any null hypersurface with a non-zero divergence scalar [8]. Therefore, it describes a general spacetime, having six degrees of freedom. One of the key characteristics of the metric is that $g_{rr} = g_{rA} = 0$. The conformal 2-metric $\gamma_{AB}$ satisfies the

$$g_{AB} = r^2\gamma_{AB} \quad \text{and} \quad \det(\gamma_{AB}) = \mathring{\mathfrak{h}}(x^A) \tag{4}$$

relations, where $\mathring{\mathfrak{h}}(x^A)$ is the determinant of the unit 2-sphere metric $\mathring{h}_{AB}$. For example, in standard spherical coordinates $(\theta,\ \varphi)$ it is simply $\mathring{h} = d\theta^2 + \sin\theta^2\ d\varphi^2$. The determinant of the full spacetime metric is also proportional to $\mathring{\mathfrak{h}}$ as $\det(g_{\mu\nu}) = -e^{4\beta}r^4\mathring{\mathfrak{h}}(x^A)$.

The non-vanishing components of the inverse metric $g^{\mu\nu}$ are

$$g^{ur} = -e^{-2\beta}\ , \quad g^{rr} = \frac{V}{r}e^{-2\beta}\ , \quad g^{rA} = -U^A e^{-2\beta}\ , \quad g^{AB} = \frac{\gamma^{AB}}{r^2}\ , \tag{5}$$

where $U^A = \gamma^{AB}U_B$.

A spacetime is said to be *asymptotically flat at null infinity* if its metric admits the following large $r$ expansion [10, 16]

$$\begin{aligned} &\gamma_{AB} = \mathring{h}_{AB} + \frac{C_{AB}}{r} + \frac{D_{AB}}{r^2} + \mathcal{O}(r^{-3})\ , \\ &\gamma^{AB} = \mathring{h}^{AB} - \frac{C^{AB}}{r} - \frac{D^{AB} - \mathring{h}^{AC}C^{BD}C_{CD}}{r^2} + \mathcal{O}(r^{-3})\ , \\ &U^A = -\frac{\mathring{D}_B C^{AB}}{2r^2} + \frac{1}{r^3}\Big( -\frac{2}{3}N^A + \frac{1}{16}\mathring{D}_A(C^{BC}C_{BC}) + \frac{1}{2}C^{AB}\mathring{D}^C C_{BC}\Big) + \mathcal{O}(r^{-4})\ , \\ &V = r - 2M + \mathcal{O}(r^{-1})\ , \\ &\beta = -\frac{1}{32}\frac{C^{AB}C_{AB}}{r^2} + \mathcal{O}(r^{-3})\ . \end{aligned} \tag{6}$$

The explicit form of the expansion coefficients are derived from the Einstein equations. The functions $\mathring{h}_{AB}$, $C_{AB}$, $D_{AB}$, $N^A$, $M$ listed above are independent of $r$ and functions only of $(u,\ x^A)$. The covariant derivative $\mathring{D}^A$ is associated with $\mathring{h}_{AB}$, the metric on the unit 2-sphere. The indices $A,\ B,\ ...$ corresponding to the angular coordinates are raised and lowered with $\mathring{h}_{AB}$. The determinant condition in (4) imposes tracelessness on the symmetric $C_{AB}$ Bondi shear

$$\mathring{h}^{AB} C_{AB} = 0 \ \ .$$

The function $M(u,\ x^A)$ is called the Bondi mass aspect, since in Schwarzschild spacetime it reduces to the $m$ mass parameter in the retarded Eddington–Finkelstein metric. The vector $N^A(u,\ x^A)$ is referred to as the Bondi angular momentum aspect. In the non-vacuum case it is governed by the $G_{rA}$ component of the Einstein equations, which is tightly coupled to the angular momentum flux of the matter flow.

A new quantity can be introduced as

$$N_{AB} = \partial_u C_{AB} \ , \tag{7}$$

which is the symmetric Bondi news tensor, and it is related to the energy flux of gravitational waves. Some useful relations can be derived from the $\mathcal{O}(r^{-2})$ order of the Einstein equations. From the $uu$-component, we have an expression for the mass aspect with the news tensor

$$\partial_u M = -\frac{1}{8} N_{AB} N^{AB} + \frac{1}{4} \mathring{D}_A \mathring{D}_B N^{AB} \ . \tag{8}$$

This equation is called the mass loss formula. We obtain a similar relation for the angular momentum aspect from the $uA$-component

$$\begin{aligned} \partial_u N_A = \mathring{D}_A M &+ \frac{1}{4} \mathring{D}_B \mathring{D}_A \mathring{D}_C C^{BC} - \frac{1}{4} \mathring{D}_B \mathring{D}^B \mathring{D}^C C_{AC} \\ &+ \frac{1}{4} \mathring{D}_B (N^{BC} C_{AC}) + \frac{1}{2} \mathring{D}_B N^{BC} C_{AC} \ . \end{aligned} \tag{9}$$

## 2.2 ADM formalism

In the following we will consider a foliation of the four dimensional spacetime [4, 22]. We will consider a family of spacelike hypersurfaces, given by the condition $t = u + r = const$. The normal vector of the hypersurfaces is timelike, and it is defined as

$$n_\mu = -\mathcal{N} \nabla_\mu t \ . \tag{10}$$

Here $\nabla_\mu$ denotes the covariant derivative compatible with the full spacetime metric $g_{\mu\nu}$. Naturally, as $t$ is a scalar, we could simply also write $n_\mu = -\mathcal{N} \partial_\mu t$. The normalization factor $\mathcal{N}$ ensures that our normal has unit length, in other words $n_\mu n^\mu = -1$. Using the normal vector, generally we can decompose the four dimensional metric into two parts

$$g_{\mu\nu} = h_{\mu\nu} - n_\mu n_\nu \ , \tag{11}$$

where $h_{\mu\nu}$ is the metric of the sub-manifold corresponding to the foliating hypersurfaces, called the induced metric.

Now let's split up the timelike vector $(\partial_t)^\mu$ into components perpendicular and tangential to the $t = const$ hypersurface

$$(\partial_t)^\mu = N n^\mu + N^\mu \ , \tag{12}$$

where $N$ is the so-called lapse function and $N^\mu$ is the shift vector. Using these quantities we can write down the 3 + 1 decomposition of the spacetime metric in ADM formalism, developed by Richard Arnowitt, Stanley Deser and Charles Misner in 1962, which reads

$$ds^2 = -N^2 dt^2 + h_{\mu\nu}(dx^\mu + N^\mu dt)(dx^\nu + N^\nu dt) \ . \tag{13}$$

The ADM decomposition enables the definition of a Hamiltonian and allows the theory to be treated dynamically through Hamiltonian equations. Moreover, it serves as a foundation for introducing quantities such as the total energy, momentum etc. of a spacetime, discussed in Section 4.

## 2.3 3+1 decomposition and extrinsic curvature

We will carry out the calculations to obtain the extrinsic curvature tensor $K_{ab}$ of the constant $t$ hypersurfaces. The reason for this is that it will be needed to compute the ADM momentum and angular momentum (which will be presented in the subsequent section 4). Here I present my detailed derivation and results.

First we need to compute $n^\mu$, the future directed unit normal vector of the $\Sigma := \{t = t_0\}$ hypersurfaces, which is constructed with the derivative of the function describing the surface. In our case it is

$$-\partial_\mu t = -\partial_\mu (u + r) = -(1,\ 1,\ 0,\ 0) \ . \tag{14}$$

The normalization factor is needed to ensure the condition $n_\mu n^\mu = \epsilon$, where $\epsilon$ depends on the causality of $n^\mu$, $\epsilon = +1$ corresponds to spacelike normals thus timelike surfaces, $\epsilon = -1$ corresponds to timelike normals thus spacelike surfaces. For our case the latter is suitable. We obtain the normalization factor with:

$$-\mathcal{N}^2 = \underbrace{g^{uu}}_{=0} + 2g^{ur} + g^{rr} = -e^{-2\beta}\Big(\frac{2r - V}{r}\Big) \ . \tag{15}$$

Thus our normalized vector's contravariant and covariant component form can be written as

$$\begin{aligned} n_\mu dx^\mu &= -e^{\beta}\Big(\frac{r}{2r - V}\Big)^{1/2}\Big(du + dr\Big) \ , \\ n^\mu \partial_\mu &= g^{\mu\nu} n_\nu \partial_\mu = e^{-\beta}\Big(\frac{r}{2r - V}\Big)^{1/2}\Big(\partial_u - \frac{V - r}{r}\partial_r\Big) \ . \end{aligned} \tag{16}$$

Now it is possible to construct the first fundamental form $\tilde{h}_{\mu\nu}$ by $\tilde{h}_{\mu\nu} := g_{\mu\nu} + n_\mu n_\nu$:

$$\begin{aligned} \tilde{h}_{\mu\nu} dx^\mu dx^\nu = & \frac{e^{2\beta}(r - V)^2}{r(2r - V)} du^2 - \frac{2e^{2\beta}(r - V)}{2r - V} du dr + \frac{e^{2\beta} r}{2r - V} dr^2 + \\ & r^2 \gamma_{AB}\Big(dx^A - U^A du\Big)\Big(dx^B - U^B du\Big) \ . \end{aligned} \tag{17}$$

The pullback $h_{ab}$ of the first fundamental form onto the surface $\Sigma$ can be obtained by making the substitution $du \to -dr$, since on the surface $dt = 0$. For a general symmetric, rank-2 covariant tensor $\tilde{T}_{\mu\nu}$ the transformation to $T_{ab}$ means

$$\begin{aligned} T_{rr} &\to \tilde{T}_{uu} - 2\tilde{T}_{ur} + \tilde{T}_{rr} \ , \\ T_{rA} &\to \tilde{T}_{rA} - \tilde{T}_{uA} \ , \\ T_{AB} &\to \tilde{T}_{AB} \ , \end{aligned} \tag{18}$$

where the new, three dimensional coordinates along $\Sigma$ are $(r,\ x^A)$ denoted by lowercase Latin indices. Applying these rules on (17) yields to

$$\begin{aligned} h_{ab}dx^a dx^b = &\Big(r^2|U|^2 + \frac{2r - V}{r}e^{2\beta}\Big)dr^2 \\ &+ 2r^2\gamma_{AB}U^A dr dx^B + r^2\gamma_{AB}dx^A dx^B \ , \end{aligned} \tag{19}$$

where $|U|^2 = \gamma_{AB}U^A U^B$. The inverse is

$$h^{ab}\partial_a\partial_b = \frac{r}{2r - V}e^{-2\beta}\Big(\partial_r^2 - 2U^A\partial_A\partial_r + U^A U^B\partial_A\partial_B\Big) + \frac{1}{r^2}\gamma^{AB}\partial_A\partial_B \ . \tag{20}$$

With a quick calculation one can verify that the trace relation $h_{ab}h^{ab} = 3$ holds for (19) and (20).

The extrinsic curvature tensor is a quantity used to describe how a surface embedded into a spacetime bends within the higher dimensional space around it. It is defined as the covariant derivative of the unit normal vector projected onto the surface $\Sigma$ with the projectors $\tilde{h}^\alpha_\beta = g^{\alpha\mu}\tilde{h}_{\mu\beta}$ in the following way [22]

$$\tilde{K}_{\alpha\beta} = \tilde{h}^\mu{}_\alpha\tilde{h}^\nu{}_\beta\nabla_\mu n_\nu \ . \tag{21}$$

For the ADM charges we will use the $K_{ab}$ pullback of $\tilde{K}_{\alpha\beta}$, which is obtained using the same transformation rules in (18), as we did in the induced metric's case. The components after performing an asymptotic series expansion according to (6) are

$$\begin{aligned} K_{rr} &= \frac{\partial_u M}{r} + \mathcal{O}(r^{-2}), \\ K_{rA} &= -\frac{1}{r}\Big[\partial_A M + \frac{1}{2}\mathring{D}^B C_{AB} - \frac{1}{4}\partial_u C_{AC}\mathring{D}_B C^{BC}\Big] + \mathcal{O}(r^{-2}) \ , \\ K_{AB} &= -2r\partial_u C_{AB} - 2M\mathring{h}_{AB} + \frac{1}{4}\Big[\mathring{h}_{AC}\partial_B + \mathring{h}_{BC}\partial_A + \partial_D\mathring{h}_{AB}\Big]\mathring{D}_D C^{CD} \\ &\quad + 2M\partial_u C_{AB} + \mathcal{O}(r^{-1}) \ . \end{aligned} \tag{22}$$

## 2.4 Linearization of the equations

In some parts of this thesis we will work with linearized equations around the Birmingham–Kottler background metric, which is [6, 8]

$$ds^2 = -\Big(\varepsilon - \alpha^2 r^2 - \frac{2m}{r}\Big)du^2 - 2du dr + r^2\mathring{h}_{AB}dx^A dx^B \ , \tag{23}$$

in Bondi coordinates, where $\varepsilon \in \{0,\ \pm 1\}$ and

$$\alpha \in \{0,\ \sqrt{\frac{2|\Lambda|}{6}},\ \sqrt{\frac{2|\Lambda|}{6}}i\ \} \ . \tag{24}$$

With suitably chosen parameters it inculdes the Minkowski (with $m = \alpha = 0$ and $\varepsilon = 1$), de Sitter and anti-de Sitter metrics in itself. The inverse of the metric can be written as

$$g^{\mu\nu}\partial_\mu\partial_\nu = -2\partial_u\partial_r - \Big(\varepsilon - \alpha^2 r^2 - \frac{2M}{r}\Big)(\partial_r)^2 + \frac{1}{r^2}\mathring{h}^{AB}\partial_A\partial_B \ . \tag{25}$$

Then we add a perturbation to the background metric in the following way

$$g_{\mu\nu} \to g_{\mu\nu} + \epsilon h_{\mu\nu} \ , \tag{26}$$

where we require the conditions $h_{rA} = h_{rr} = \mathring{h}^{AB}h_{AB} = 0$ to hold. This way we ensure that the perturbed metric in also in Bondi gauge up to linear order in $\epsilon$. In addition, we can always choose a gauge where $h_{ur} = 0$. An alternative notation of the metric perturbations are

$$\begin{aligned} \delta V + 2V\delta\beta &= -r h_{uu} \ , \\ \delta\beta &= -\frac{h_{ur}}{2} \ , \\ \delta U_A &= -\frac{h_{uA}}{r^2} \ . \end{aligned} \tag{27}$$

From the relations above one can see that from the choice of $h_{ur} = 0$ it follows that $\delta\beta = 0$. In the following we will work in the $\delta\beta = 0$ gauge.

# 3 Characteristic gluing problem

## 3.1 Initial value formulation of general relativity

The most fundamental equation in the theory of general relativity is the Einstein equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu} \ , \tag{28}$$

which has solutions in the form of $g_{\mu\nu}$ symmetric metric tensors on 4 dimensional Lorentzian manifolds $\mathcal{M}$ [20]. It expresses that matter and energy, encoded in the energy-momentum tensor $T_{\mu\nu}$ generates the curvature of the spacetime, which information is contained in the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$. Generally the equation (28) is a system consisting of 10 non-linear, second order partial differential equations. In vacuum, where $T_{\mu\nu} = 0$ the equation simplifies to

$$R_{\mu\nu} = 0 \ , \tag{29}$$

i. e. the Ricci tensor vanishes. In this thesis we will mostly work in vacuum spacetimes.

It is possible to reformulate the Einstein equations as an initial value problem [1, 13]. The initial values are given on a three dimensional hypersurface $\Sigma$ embedded in the manifold $\mathcal{M}$. The initial data needs to satisfy different constraints, depending on the causality of the surface. For example, if the hypersurface is spacelike, i. e. the induced metric on the surface has a Riemannian signature, then the spacelike initial dataset $(\Sigma,\ g,\ K)$ need to satisfy

$$\begin{aligned} R - K_{\mu\nu} K^{\mu\nu} + (K^{\lambda}{}_{\lambda})^2 = 0 \ , \\ \nabla_{\mu} K^{\mu}{}_{\nu} - \nabla_{\nu} K^{\lambda}{}_{\lambda} = 0 \ , \end{aligned} \tag{30}$$

where $K_{\mu\nu}$ is the extrinsic curvature. The equations above are called the vacuum constraint equations.

If we want to prescribe the initial data on a null hypersurface $\mathcal{N}$, with null vectors pointing in its normal direction at every point, the constraints become transport type equations in the variable $r$, which restrict the components of the metric, Christoffel symbols and curvature.

## 3.2 Characteristic initial data gluing problem

In the general gluing problem we want to construct a solution to the Einstein equations by gluing two spacetimes together [2, 8, 9]. More precisely, in the initial data gluing problem we want to glue two initial datasets of the same type, meaning two spacelike or two characteristic (null) datasets, in order to formulate a solution of the appropriate constraint equations. In the following we will focus on the gluing of characteristic datasets.

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two manifolds of two spacetimes. In each of these spacetimes consider a pair of null hypersurfaces, $\mathcal{N}_1$, $\underline{\mathcal{N}}_1$ and $\mathcal{N}_2$, $\underline{\mathcal{N}}_2$. The intersection of $\mathcal{N}_1$ and $\underline{\mathcal{N}}_1$ is a sphere $S_1$ and similarly, the intersection of $\mathcal{N}_2$ and $\underline{\mathcal{N}}_2$ is the sphere $S_2$ (see figure 1). The $C^k$ sphere data is a set of the metric components and their derivatives up to the $k$th order on a given sphere, where $k \in \mathbb{N}$. We will denote the sphere data with $x_i$ on $S_i$ respectively, $i \in \{1,\ 2\}$. Now the question is whether we can glue

$\mathcal{N}_1$ and $\mathcal{N}_2$ together using an additional null hypersurface $\mathcal{N}$ with a characteristic dataset $x$, such that the null constraint equations are satisfied everywhere along $\mathcal{N}_1 \cup \mathcal{N} \cup \mathcal{N}_2$. We also demand that on the boundaries of $\mathcal{N}$, where the surfaces are attached together, $x$ matches with the characteristic datasets $x_1$ and $x_2$, namely

$$x|_{S_1} = x_1 \qquad \text{and} \qquad x|_{S_2} = x_2 \ . \tag{31}$$

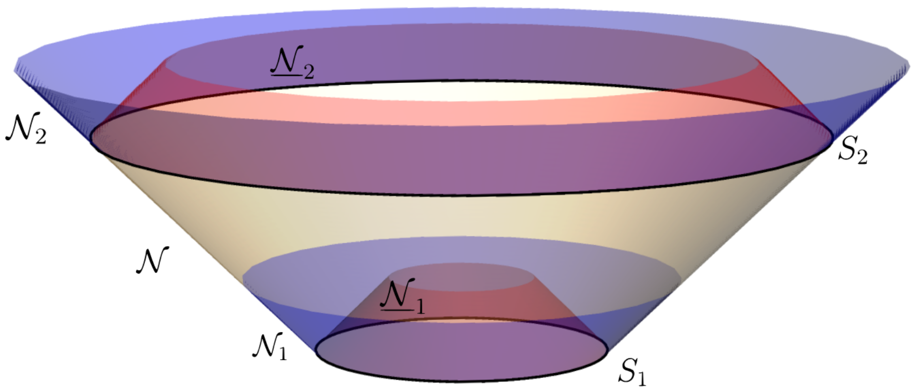


Figure 1: The characteristic gluing of the hypersurfaces $\mathcal{N}_1 \subset \mathcal{M}_1$ and $\mathcal{N}_2 \subset \mathcal{M}_2$ (blue surfaces) with a null surface $\mathcal{N}$ (yellow surface). The spheres $S_i$ (black circles) are the intersections of $\mathcal{N}_i$ and $\underline{\mathcal{N}}_i$ (red surfaces), $i \in \{1,\ 2\}$.

One cannot always find an $x$ dataset that solves the characteristic gluing problem. The first obstructions arise from the null constraint equations, which are the vacuum Einstein equations $R_{\mu\nu} = 0$ combined with the embedding equations of the double null foliation of the spacetime. To work with the equations we switch to a double null coordinate system $(u,\ v,\ x^A)$, where $u$ and $v$ are null coordinates, and $x^A$ are spatial angular coordinates. In this gauge the spacetime metric can be written as

$$g_{\mu\nu} dx^\mu dx^\nu = -4\Omega^2 du dv + \not{g}_{AB}(dx^A - b^A dv)(dx^B - b^B dv) \ , \tag{32}$$

where $\Omega$ is a scalar function called the null-lapse, $\not{g}_{AB}$ is the induced metric on the sphere $S_{u,v}$ determined by the $u = \mathit{const}$ and $v = \mathit{const}$ conditions, and $b^A$ is the shift vector, tangent to $S_{u,v}$. The most important obstruction is coming from the Raychaudhuri equations

$$\begin{aligned} \not{\mathcal{L}}_L \mathrm{tr}\chi + \frac{\Omega}{2}(\mathrm{tr}\chi)^2 - \omega \mathrm{tr}\chi &= -\Omega|\hat{\chi}|^2 \ , \\ \not{\mathcal{L}}_{\underline{L}} \mathrm{tr}\underline{\chi} + \frac{\Omega}{2}(\mathrm{tr}\underline{\chi})^2 - \underline{\omega} \mathrm{tr}\underline{\chi} &= -\Omega|\hat{\underline{\chi}}|^2 \ , \end{aligned} \tag{33}$$

where

$$\chi_{AB} = \mathbf{g}(\nabla^A L,\ \partial_B)\ ,$$
$$\underline{\chi}_{AB} = \mathbf{g}(\nabla^A \underline{L},\ \partial_B)\ ,$$
$$\omega = \mathcal{L}_L \ln\Omega\ ,$$
$$\underline{\omega} = \mathcal{L}_{\underline{L}} \ln\Omega \tag{34}$$

are the Ricci coefficients with $L = \Omega^{-1}(\partial_v + b)$ , $\underline{L} = \Omega^{-1}\partial_u$ , $\not{\mathcal{L}}$ is the Lie-derivative projected onto $S_{u,v}$, the trace is taken with respect to $\not{g}_{AB}$ and the hat symbol on $\hat{\chi}_{AB}$ denotes the traceless part. The Raychaudhuri equations in (33) govern the evolution of the null-expansions $\mathrm{tr}\chi$ and $\mathrm{tr}\underline{\chi}$ along $\mathcal{N}$. Their monotonicity properties constrain the behavior of the expansion thus imposing restrictions on the gluing problem.

Conserved charges arising from the null constraint equations can also act as obstructions. One can differentiate between two types of charges. There exists a finite number of gauge-invariant charges and an infinite number of gauge-dependent charges. The latter can be removed by exploiting the freedom associated with sphere perturbations, while the former are invariant under these perturbations, so they constitute genuine obstructions to characteristic gluing

### 3.3 Perturbative $C^k$ characteristic gluing

In the following we will examine how small perturbations on the $x_2$ sphere dataset change the characteristic gluing problem.

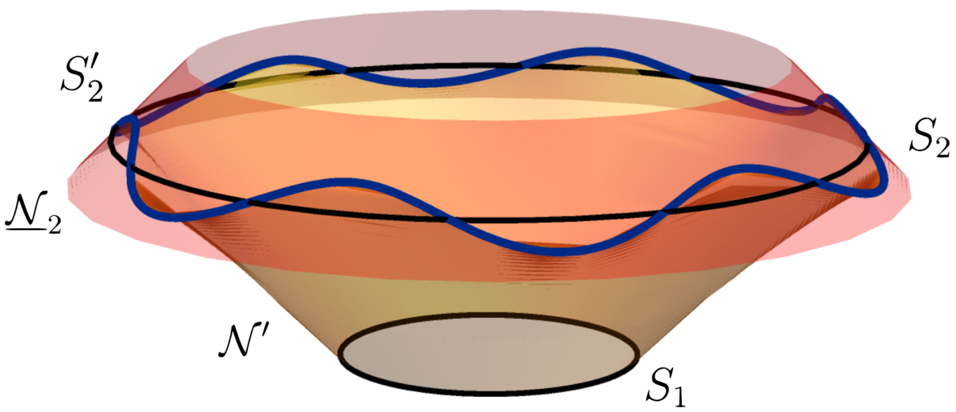


Figure 2: Gluing the perturbed sphere data $x_2'$ on $S_2'$ (blue line) and $x_1$ on $S_1$ (black circle) together with a null hypersurface $\mathcal{N}'$ (yellow surface).

The sphere perturbation of $x_2$, denoted by $x_2'$, is a sphere data on a sphere $S_2' \subset \underline{\mathcal{N}}_2$, where $\underline{\mathcal{N}}_2$ is transversal to $\mathcal{N}_2$, as it is shown in figure 2. In Bondi coordinates the perturbed sphere data consists of the perturbations of the metric functions and their derivatives, namely

$$x_2' = (\partial_u^j h_{AB}|_{S_2},\ \partial_r^j h_{AB}|_{S_2},\ \partial_u^j \delta\beta|_{S_2},\ \partial_u^j \delta U^A|_{S_2},\ \partial_r \delta U^A|_{S_2},\ \delta V|_{S_2})\ , \tag{35}$$

where $0 \leq j \leq k$ for a given $k \in \mathbb{N}$. The transformations that pull back $x'_2$ to $S_2$ under a diffeomorphism of the sphere are called sphere diffeomorphisms.

The fact that we allow perturbations on the sphere data $x_2$ does not change the main objective of the characteristic gluing problem, which is to find a solution

$$x = (\partial_u^\ell \delta V,\ \partial_u^\ell \delta\beta,\ \partial_u^\ell \delta U^A,\ \partial_u^\ell h_{AB}) \tag{36}$$

on a null hypersurface $\mathcal{N}'$ to the null constraint equations along $\mathcal{N}_1\ \cup\ \mathcal{N}'\ \cup\ \mathcal{N}_2$ such that

$$x|_{S_1} = x_1 \qquad \text{and} \qquad x|_{S'_2} = x'_2\ . \tag{37}$$

Thus in the perturbative case the gluing surface connects $S_1$ to the perturbed sphere $S'_2$, and that the solution $x$ needs to admit to the perturbed sphere data $x'_2$ at the boundary of $\mathcal{N}'$. Constructing the perturbed sphere data $x'_1$ of $x_1$ instead of $x'_2$ is entirely equivalent to the scenario introduced above.

In the $C^2$ characteristic gluing, in four dimensions with $\Lambda = 0$ and $m = 0$ the gauge-invariant obstructions form a 10-dimensional space of charges, determined by 10 independent real constants, encoded in one scalar and three vectorial quantities. The charges are written as integrals over the sphere. However these obstructions can vary, depending on the dimensions of the spacetime, whether or not we include a non-zero $\Lambda$ cosmological constant and $m$ mass, and the number or $u$-derivatives of the metric we want to control along the null-surface. This thesis focuses on these charges, showing that they have real physical meaning. This is done by relating them to other well-known charges of general relativity, such as ADM and BMS charges.

# 4 ADM charges

In this section we would like to introduce quantities that describe the total energy, momentum, angular momentum and center of mass of an isolated spacetime [20]. Naturally there are no physical systems that are completely isolated from the rest of the universe, but we find that asymptotic flatness is a sufficient requirement to impose. In asymptotically flat spacetimes there exists a set of coordinates that satisfies the $g_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O}(1/r)$ condition, i.e. they admit to the flat spacetime metric at $r \to \infty$ large distances.

## 4.1 ADM energy

In general relativity the energy of matter is encoded in the energy momentum tensor $T_{\mu\nu}$. The relation $\nabla_\mu T^{\mu\nu} = 0$ expresses some kind of conservation of the energy-momentum flow, but it only stands locally, and does not assume a general energy conservation law over the whole spacetime. The other issue is that it only contains the energy contribution of matter, and leaves out the energy of the gravitational field. So we need a more refined definition of total energy.

There are several approaches of grasping the idea of total energy. One emerged from the Hamiltonian formulation of general relativity, introduced by Arnowitt, Deser and Misner. They gave the component form of energy and momentum on a hypersurface at spatial infinity $i^0$.

The total energy of an asymptotically flat spacetime can be obtained by calculating the value of the Hamiltonian [4]. For the Hamiltonian formulation we will use the ADM 3 + 1 metric given in (13). Our initial vacuum data set $(\mathcal{M}, h, K)$ consists of a three dimensional manifold, a Riemannian induced metric and the extrinsic curvature [12].

The Hamiltonian of a general spacetime can be written as [18]

$$H = \frac{1}{16\pi} \int d^3x \sqrt{\mathfrak{h}} (N\mathcal{H} + N^a \mathcal{H}_a) \ , \tag{38}$$

where

$$\begin{aligned} \mathcal{H} &= -R + \mathfrak{h}^{-1} \pi^{ab} \pi_{ab} - \frac{1}{2} \mathfrak{h}^{-1} \pi^2 \ , \\ \mathcal{H}_a &= -2 h_{ab} \mathcal{D}_c (\mathfrak{h}^{-1/2} \pi^{bc}) \ , \end{aligned} \tag{39}$$

$R$ is the Ricci scalar of $h_{ab}$, $\mathcal{D}_a$ is the covariant derivative compatible with $h_{ab}$, $\mathfrak{h} = \det(h_{ab})$, $\pi^{ab}$ denotes the conjugate momentum and $\pi = h^{ab}\pi_{ab}$ is its trace. The conjugate momentum is defined as the variation of the Einstein–Hilbert action, which is defined as

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-\mathfrak{g}} \ {}^{(4)}R \ , \tag{40}$$

where ${}^{(4)}R$ is the Ricci scalar associated with the four dimensional $g_{\mu\nu}$. We need to carry out the 3+1 decomposition of the action using the twice-contracted Gauss equation $R =^{(4)} R + 2^{(4)} R_{ab} n^a n^b - K^2 + K^{ab} K_{ab}$, which relates the two curvature scalars. Using that ${}^{(4)}R_{ab} n^a n^b = K^2 - K^{ab} K_{ab}$ plus a total divergence term, which we can neglect. Thus the Einstein–Hilbert action becomes

$$S = \frac{1}{16\pi} \int dt \int d^3x \ N \sqrt{\mathfrak{h}} (R + K_{ab} K^{ab} - K^2) \ . \tag{41}$$

With the help of the formula above, we can obtain the conjugate momentum

$$\pi^{ab} \equiv \frac{\delta S}{\delta \dot{h}_{ab}} = \frac{\sqrt{\mathfrak{h}}}{16\pi}(K^{ab} - \mathrm{tr}(K_{ab})h^{ab}) \ . \tag{42}$$

Then the Hamiltonian equations are

$$\dot{H}_{ab} = \frac{\delta H}{\delta \pi^{ab}} \ , \quad \dot{\pi}^{ab} = -\frac{\delta H}{\delta h_{ab}} \ . \tag{43}$$

However, we find that the expression for $H$ in (38) leads to zero in every case. The reason for this is that so far we neglected the boundary terms, that would give us the non-zero results. For any closed $t = const$ surface $H$ remains zero since there aren't any surface terms. Thus the total energy of a closed universe is zero. In the following however we will focus on asymptotically flat hypersurfaces, imposing the fall-off conditions

$$\begin{aligned} h_{ab} &= \delta_{ab} + \mathcal{O}(1/r) \ , \\ \pi^{ab} &= \mathcal{O}(1/r^2) \ , \\ N &= 1 + \mathcal{O}(1/r) \ , \\ N^a &= \mathcal{O}(1/r) \end{aligned} \tag{44}$$

as $r \to \infty$. Now we consider a region of the hypersurface inside a unit 2-sphere, with the unit outward pointing normal $n^a$, and the surface element $dA = r^2 d^2\Omega$. Then the only surface term that is non-zero in the $r \to \infty$ case is coming from the variation of the Ricci scalar and can be written as

$$S = \frac{1}{16\pi}\int_{S^2}(n^b \delta h_{ab}\mathcal{D}^a N - h^{ab}\delta h_{ab} n^c \mathcal{D}_c N) dA \ . \tag{45}$$

Taking the large $r$ limit we get

$$\lim_{r\to\infty} S = \lim_{r\to\infty}\frac{1}{16\pi}\int_{S^2}(\partial^b \delta h_{ab} - \partial_a \delta h^b{}_b) n^a dA = -\delta E_{ADM} \ , \tag{46}$$

where $E_{ADM}$, the ADM energy of an asymptotically flat spacetime is

$$E_{ADM} := \frac{1}{16\pi}\lim_{r\to\infty}\int_{S^2}(\partial^b h_{ab} - \partial_a h^b{}_b) n^a dA \ . \tag{47}$$

Alternatively we can express the ADM energy with the three dimensional Ricci tensor and scalar in the following way [3]

$$E_{ADM} := -\lim_{r\to\infty}\frac{1}{8\pi}\int_{S^2}(R_{ab} - Rh_{ab}) X^b n^a dA \ , \tag{48}$$

where $X = x^a \partial_a$ in Cartesian coordinates $(x^1, x^2, x^3)$.

## 4.2 ADM momentum

We can express the total momentum of a system similarly as the total energy, as a surface integral of $\pi_{ab}$

$$(P_{ADM})_a := \lim_{r\to\infty}\frac{1}{8\pi}\int_{S_r}(K_{ab} - \mathrm{tr}K h_{ab}) n^b \ dA \ , \tag{49}$$

where $\mathrm{tr}K = h^{ab}K_{ab}$ .

One can construct a vector $p_\mu$ out of the two ADM charge at spatial infinity, analogously to the four-momentum of a particle [20]

$$p_\mu = -E_{ADM}\; n_\mu + (P_{ADM})_\mu \;, \tag{50}$$

where $n_\mu$ is a future-directed unit normal of the surface. This quantity, often referred to as the ADM energy-momentum, obeys the transformation laws of four-vectors, as one would expect. Using $p_\mu$ we can define the ADM mass as [18]

$$M_{ADM} = \sqrt{E_{ADM}^2 - (P_{ADM})^\mu (P_{ADM})_\mu} \geq 0 \;. \tag{51}$$

### 4.3 ADM angular momentum

The total angular momentum is defined as

$$(L_{ADM})_a = \lim_{r\to\infty} \frac{1}{8\pi} \int_{S_r} (K_{bc} - \mathrm{tr}K\, h_{bc})(\phi_{(a)})^c n^b \; dA \;, \tag{52}$$

where the conformal Killing vector $(\phi_{(a)})^c$ is a rotation field in $\mathbb{R}^3$. In Cartesian coordinates it reads $(\phi_{(a)})_b = \varepsilon_{acb}x^c$, where $\varepsilon_{acb}$ is the antisymmetric Levi-Civita tensor. We can also express the rotation field with the divergence free ($D_A H^A = 0$), $\ell = 1$ magnetic spherical harmonics [3, 9]

$$H_A^{(\ell, m_a)} = \frac{1}{\sqrt{\ell(\ell+1)}} * D_A Y^{(\ell, m_a)} \;, \tag{53}$$

with $Y^{(\ell,m)}$ being the real spherical harmonics, and $*$ denoting the Hodge dual. Using this, the rotation field becomes

$$(\phi_{(a)})_B = \sqrt{\frac{8\pi}{3}} r^2 H_B^{(\ell=1, m_a)} \;. \tag{54}$$

The $m_a \in \{1, -1,\; 0\}$ magnetic numbers correspond to the components $x^a = (r,\; \theta,\; \varphi)$ respectively.

To prove the relation (54), we need to take a look at the real spherical harmonics of $\ell = 1$, defined as

$$\begin{aligned}
Y_{1,1} &= \frac{1}{\sqrt{2}}(\mathcal{Y}_{1,-1} - \mathcal{Y}_{1,1}) = \sqrt{\frac{3}{4\pi}} \sin\theta\cos\varphi \;, \\
Y_{1,-1} &= \frac{i}{\sqrt{2}}(\mathcal{Y}_{1,-1} + \mathcal{Y}_{1,1}) = \sqrt{\frac{3}{4\pi}} \sin\theta\sin\varphi \;, \\
Y_{1,0} &= \mathcal{Y}_{1,0} = \sqrt{\frac{3}{4\pi}} \cos\theta \;,
\end{aligned} \tag{55}$$

where $\mathcal{Y}_{\ell m}$ are the complex-valued spherical harmonics. Comparing them with the transformation rule of $(x^1,\; x^2,\; x^3)$ Cartesian coordinates to $(r,\; \theta,\; \varphi)$ spherical coordinates

$$\begin{aligned}
x^1 &= r\sin\theta\cos\varphi \;, \\
x^2 &= r\sin\theta\sin\varphi \;, \\
x^3 &= r\cos\theta \;,
\end{aligned} \tag{56}$$

with $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$, we can easily point out the relations

$$
\begin{aligned}
\frac{x^1}{r} &= \sqrt{\frac{4\pi}{3}} Y_{1,1} \ , \\
\frac{x^2}{r} &= \sqrt{\frac{4\pi}{3}} Y_{1,-1} \ , \\
\frac{x^3}{r} &= \sqrt{\frac{4\pi}{3}} Y_{1,0} \ .
\end{aligned} \tag{57}
$$

Now let's plug these formulas into the definition of the magnetic spherical harmonic in (53)

$$
H_A^{(1,m_a)} = \sqrt{\frac{3}{8\pi}} \frac{1}{r} \sum_{B=1}^{2} \varepsilon_{AB} D_B x^{a_m} \ . \tag{58}
$$

Next we want complete the $\varepsilon_{AB}$ volume element on the 2-sphere to a three dimensional volume element, which becomes

$$
H_A^{(1,m_a)} = \sqrt{\frac{3}{8\pi}} \frac{1}{r} \Big( \sum_{B=1}^{2} \varepsilon_{rAB} D_B x^{a_m} + \varepsilon_{rAr} \frac{x^b}{r} \partial_b (x^{a_m}) \Big) \ . \tag{59}
$$

Here we split the last index of $\varepsilon_{abc}$ into tangential and normal components to the 2-sphere. In the second term $x^b r^{-1} \partial_b = \partial_r$ is the radial unit vector, perpendicular to the sphere. In (59) only the first term survives as there is a repetition of indices in the second term's Levi-Civita tensor. Now let's write the covariant derivative $D_A$ as a projection of $\nabla_a$ onto the sphere

$$
H_A^{(1,m_a)} = \sqrt{\frac{3}{8\pi}} \frac{1}{r} \varepsilon_{rAb} e^{bc} \nabla_c x^{a_m} \ , \tag{60}
$$

where $(e_1, \ e_2)$ is an arbitrary local orthonormal basis on the 2-sphere. Note that $\nabla_c x^{a_m} = \delta_c^{a_m}$. Now let's carry out the contraction in the $c$ dummy index, and use that $e^{ij}$ is diagonal in $\varepsilon_{rAb} e^{ba_m}$. Thus

$$
H_A^{(1,m_a)} = \sqrt{\frac{3}{8\pi}} \frac{1}{r} \varepsilon_{rAa_m} = \sqrt{\frac{3}{8\pi}} \frac{1}{r^2} x^b \varepsilon_{bAa_m} \ . \tag{61}
$$

In the second equation we used again that $\partial_r = r^{-1} x^b \partial_b$. As a last step we need to perform a cyclic permutation of the indices, and then identify the rotation vector.

$$
H_A^{(1,m_a)} = \sqrt{\frac{3}{8\pi}} \frac{1}{r^2} \varepsilon_{a_m bA} x^b = \sqrt{\frac{3}{8\pi}} \frac{1}{r^2} (\phi_{a_m})_A \ . \tag{62}
$$

And with that, we concluded the proof of the relation (54).

### 4.4 ADM center of mass

The center of mass of an asymptotically flat spacetime is [12]

$$
(C_{ADM})^{(a)} := \frac{1}{16\pi} \lim_{r\to\infty} \frac{1}{8\pi} \int_{S_r} (R_{cb} - R g_{cb}) (Z^{(a)})^c n^b \ dA \ . \tag{63}
$$

The vector $Z^{(a)}$ is a conformal Killing vector on the 2-sphere, in Cartesian coordinates it is $Z^{(a)} = (|x|^2\delta^{ab} - 2x^a x^b)\partial_b$, which generates translations at spatial infinity. Alternatively, it can be expressed with the formula [3]

$$Z^{(a)} = -r^3\sqrt{\frac{8\pi}{3}}E^{(1,m_a)} - r^2(\sqrt{\frac{4\pi}{3}}Y^{(1,m_a)})\partial_r \ , \tag{64}$$

where $Y^{(1,m_a)}$ denotes the real spherical harmonics, and $E^{(1,m_a)}$ the electric spherical harmonics, which are defined with

$$E_A^{(\ell,m)} = -\frac{1}{\sqrt{\ell(\ell+1)}}D_A Y^{(\ell,m)} \ . \tag{65}$$

For (63) we need to choose the $\ell = 1$ mode. Similarly to the angular momentum, the $m_a \in \{1, -1,\ 0\}$ magnetic numbers here also correspond to the components $x^a = (r,\ \theta,\ \varphi)$ respectively.

To prove (64), we'll start off with

$$Z^{(a)} = |x|^2\partial^a - 2x^a x^b\partial_b = r^2\Big(\partial^a - \frac{x^a}{r}\partial_r\Big) - x^a r\partial_r \ , \tag{66}$$

as $x^b\partial_b = r\partial_r$. The expression in the brackets can be rewritten as

$$\partial^a - \frac{x^a}{r}\partial_r = (\nabla_b x^b)^a - \frac{x^a x^b}{r^2}\partial_b = (D_B x^a) \ . \tag{67}$$

From the second step we can see, that our expression is a divergence of which the radial direction has been subtracted, thus we are left with a divergence on the 2-sphere. Since $D_A r = 0$, we can write

$$D_A x^a = rD_A\Big(\frac{x^a}{r}\Big) = r\sqrt{\frac{4\pi}{3}}D_A Y^{(1,m_a)} = -r\sqrt{\frac{4\pi}{3}}E_A^{(1,m_a)} \ . \tag{68}$$

Plugging this result into the first term of (66), then using the relations (57) in the second term, we successfully recovered the formula (64).

# 5 Radial charges

In this section we will introduce two radially conserved charges arising from the null constraint equations. They are especially important regarding the $C^2$ characteristic gluing problem as they form the families of gauge-invariant obstructions.

## 5.1 $\overset{[1]}{Q}$

The $rr$-component of the vacuum Einstein equations in Bondi gauge is

$$\frac{r}{4}G_{rr} = \partial_r\beta - \frac{r}{16}\gamma^{AC}\gamma^{BD}(\partial_r\gamma_{AB})(\partial_r\gamma_{CD}) = 0 \ , \tag{69}$$

and the $rA$-component is

$$2r^2 G_{rA} = \partial_r\Big[r^4 e^{-2\beta}\gamma_{AB}\Big(\partial_r U^B\Big)\Big] - 2r^4\partial_r\Big(\frac{D_A\beta}{r^2}\Big) + r^2\gamma^{EF}D_E(\partial_r\gamma_{AF}) = 0 \ . \tag{70}$$

The term containing the cosmological constant $\Lambda$ is not present in these specific equations as in Bondi gauge $g_{rr} = g_{rA} = 0$. Now let's take the $D_A$ derivative of (69) and multiply it with a factor of $-4r^2$. Then subtract it from (70). This way we obtain

$$\begin{aligned}\partial_r\Big[r^4 e^{-2\beta}\gamma_{AB}\Big(\partial_r U^B\Big)\Big] =& 2r^4\partial_r\Big(\frac{D_A\beta}{r^2}\Big) - r^2\gamma^{EF}D_E(\partial_r\gamma_{AF}) \\ &- 4r^2 D_A\Big[\partial_r\beta - \frac{r}{16}\gamma^{EC}\gamma^{BD}(\partial_r\gamma_{EB})(\partial_r\gamma_{CD})\Big] \ .\end{aligned} \tag{71}$$

After some algebraic manipulation, using that the derivatives commute in $D_A\partial_r\beta = \partial_r D_A\beta$, the equation above simplifies to

$$\begin{aligned}\partial_r\Big[r^4 e^{-2\beta}\gamma_{AB}\Big(\partial_r U^B\Big)\Big] =& -2\partial_r\Big(r^2 D_A\beta\Big) - r^2\gamma^{EF}D_E(\partial_r\gamma_{AF}) \\ &+ \frac{r^3}{4}D_A\Big[\gamma^{EC}\gamma^{BD}(\partial_r\gamma_{EB})(\partial_r\gamma_{CD})\Big] \ .\end{aligned} \tag{72}$$

If we move the first term in the right-hand side to the left-hand side, we get a transport equation of a quantity, which we will denote with $\overset{(*)}{H}_{uA}$:

$$\begin{aligned}\partial_r\Big[\underbrace{r^4 e^{-2\beta}\gamma_{AB}\Big(\partial_r U^B\Big) + 2r^2 D_A\beta}_{\overset{(*)}{H}_{uA}}\Big] =& - r^2\gamma^{EF}D_E(\partial_r\gamma_{AF}) \\ &+ \frac{r^3}{4}D_A\Big[\gamma^{EC}\gamma^{BD}(\partial_r\gamma_{EB})(\partial_r\gamma_{CD})\Big] \ .\end{aligned} \tag{73}$$

Now let's consider the $Y_A = Y_A(u,\ x^A)$ conformal Killing vectors of the 2-sphere. They are solutions of

$$\mathrm{TS}[\mathring{D}_A Y_B] = 0 \ , \tag{74}$$

where $\mathrm{TS}[\,\cdot\,]$ denotes the traceless and symmetric part, i.e. for an arbitrary rank-2 $T_{AB}$ tensor it is

$$\mathrm{TS}[T_{AB}] = \frac{1}{2}\Big(T_{AB} + T_{BA} - \mathring{h}^{CD}T_{CD}\,\mathring{h}_{AB}\Big) \ . \tag{75}$$

The projection of the transport equation (73) onto the conformal Killing vector space, integrated over the 2-sphere $S^2$ gives

$$\begin{aligned}
&\partial_r \int d^2\Omega\ Y^A \overset{(*)}{H}_{uA} \\
&= \int d^2\Omega\ Y^A \left( \frac{r^3}{4} D_A \Big[ \gamma^{EC}\gamma^{BD}(\partial_r \gamma_{EB})(\partial_r \gamma_{CD}) \Big] - r^2 \gamma^{EF} D_E(\partial_r \gamma_{AF}) \right) . \qquad (76)
\end{aligned}$$

After linearization it becomes

$$\partial_r \int d^2\Omega\ Y^A \overset{(*)}{H}_{uA} = -\frac{2}{r} \int d^2\Omega\ Y^A \mathring{D}^B h_{AB} \ . \qquad (77)$$

We can easily show that the right-hand side is zero. After integrating by parts, and dropping the total divergence we have $h_{AB}\mathring{D}^B Y^A$ in the integrand. Using that $Y^A$ satisfies (74) we can write that

$$h_{AB}\Big(\mathring{D}^B Y^A + \mathring{D}^A Y^B - \mathring{D}_C Y^C \mathring{h}^{AB}\Big) = h_{AB}\Big(\mathring{D}^B Y^A + \mathring{D}^A Y^B\Big) = 0 \ , \qquad (78)$$

where the third term in the brackets vanished because $h_{AB}$ is traceless. As $h_{AB}$ is also symmetric, the equation above can only be satisfied if $h_{AB}\mathring{D}^B Y^A = 0$. With that we found a family of charges

$$\overset{[1]}{Q}(Y^A) := \int d^2\Omega\ Y^A \Big[ r^4 e^{-2\beta} \gamma_{AB}(\partial_r U^B) + 2r^2 D_A \beta \Big] \ , \qquad (79)$$

with an arbitrarily chosen $Y^A \in$ CKV parameter, whose linearization is radially conserved in a sense that

$$\partial_r \overset{[1]}{Q} = 0 \qquad (80)$$

along a $u = const$ hypersurface.

The true significance of $\overset{[1]}{Q}$ arises from the characteristic gluing problem. Two datasets $x_1$ and $x_2$ given on the spheres $S_1$ and $S_2$ of two distinct manifolds can only be glued together under certain conditions. There exist a finite number of gauge-independent charges that need to match for the datasets $x_1$ and $x_2$. The charge $\overset{[1]}{Q}$ is a part of this family of obstructions, therefore the gluing is only possible if they match when evaluating them on the two datasets

$$\overset{[1]}{Q}[x_1](Y^A) = \overset{[1]}{Q}[x_2](Y^A) \ . \qquad (81)$$

The charge $\overset{[1]}{Q}$ is gauge-invariant in most cases, except when the surface we integrate over is a round sphere with a non-zero mass parameter $m$. In this case the question is whether we can find transformations $z_1$ and $z_2$ applied on the datasets such that

$$\overset{[1]}{Q}[z_1(x_1)](Y^A) = \overset{[1]}{Q}[z_2(x_2)](Y^A) \ . \qquad (82)$$

## 5.2 $\overset{[2]}{Q}$

In the previous section we introduced a set of charges denoted by $\overset{[1]}{Q}$. Besides this, there exists another family of charges, $\overset{[2]}{Q}$, that are also conserved. In the following we will draft a path to find $\overset{[2]}{Q}$, for a more detailed derivation see also Appendix B.2 of [7].

Let's consider the linear combination $e^{-2\beta}(2G_{ur}+2U^A G_{rA}-Vr^{-1}\,G_{rr})$. In Bondi metric, from the definition of the Einstein tensor it follows that

$$e^{-2\beta}(2G_{ur}+2U^A G_{rA}-Vr^{-1}\,G_{rr})=g^{AB}R_{AB}\ . \tag{83}$$

Combining this relation and the Einstein equations, we obtain the equation

$$\begin{aligned}2\Lambda r^2 =&R[\gamma]-2\gamma^{AB}\Big[D_A D_B\beta+(D_A\beta)(D_B\beta)\Big]+\frac{e^{-2\beta}}{r^2}D_A\Big[\partial_r(r^4U^A)\Big]\\ &-\frac{1}{2}r^4e^{-4\beta}\gamma_{AB}(\partial_r U^A)(\partial_r U^B)-2e^{-2\beta}\partial_r V\ .\end{aligned} \tag{84}$$

Multiplying (73) with $1/r$, and then carrying out the partial $r$-derivative on the left-hand side gives us

$$\begin{aligned}&\frac{r^2}{4}D_A\Big[\gamma^{EC}\gamma^{BD}(\partial_r\gamma_{EB})(\partial_r\gamma_{CD})\Big]-r\,\gamma^{EF}D_E(\partial_r\gamma_{AF})\\ &\qquad=4r^2e^{-2\beta}\gamma_{AB}(\partial_r U^B)+r^3e^{-2\beta}\gamma_{AB}(\partial_r^2U^B)+\frac{1}{r}\partial_r(2r^2D_A\beta)\\ &\qquad\quad-2r^3\gamma_{AB}(\partial_r U^B)e^{-2\beta}\partial_r\beta+r^3e^{-2\beta}\partial_r(\gamma_{AB})(\partial_r U^B)\ .\end{aligned} \tag{85}$$

After that, let's multiply it with $e^{2\beta}$ and then take the $D_A$ derivative. Subtracting the equation we got from (84) leads to

$$\begin{aligned}&2\partial_r\Big[V-\frac{r}{2}D^A\partial_r(r^2U_A)\Big]-2r^2[D_B,\partial_r]U^B-r^3[D_B,\partial_r]\partial_r U^B\\ &-\frac{1}{r}D^A\Big[e^{2\beta}\partial_r(2r^2D_A\beta)\Big]+2\gamma^{AB}e^{2\beta}D_AD_B\beta\\ &=e^{2\beta}\Big[-2\Lambda r^2+R[\gamma]-2\gamma^{AB}(D_A\beta)(D_B\beta)-\frac{r^4}{2}e^{-4\beta}\gamma_{AB}(\partial_r U^A)(\partial_r U^B)\Big]\\ &\quad-2r^3D_B\Big[(\partial_r U^B)\partial_r\beta\Big]+r^3D^A\Big[(\partial_r\gamma_{AB})(\partial_r U^B)\Big]\\ &\quad-D^A\left[\frac{r^2}{4}e^{2\beta}D_A\Big(\gamma^{EC}\gamma^{BD}(\partial_r\gamma_{EB})(\partial_r\gamma_{CD})\Big)\right]+rD^A\Big[e^{2\beta}\gamma^{EF}D_E(\partial_r\gamma_{AF})\Big]\ ,\end{aligned} \tag{86}$$

where the square brackets $[\ \cdot\ ,\ \cdot\ ]$ denote the commutator, i.e. for two arbitrary operators $A$ and $B$ it gives $[A,B]=AB-BA$. We can modify this equation in a way to isolate a partial $r$-derivative of a quantity $\chi$ on the left-hand side, thus obtaining a transport type equation for $\chi$, which reads

$$
\begin{aligned}
&2\partial_r\Big(\underbrace{V-\frac{r}{2}D^A\partial_r(r^2U_A)-re^{2\beta}D^2\beta}_{\chi}\Big)\\
&\;= 2r^2[D_B,\partial_r]U^B+\partial_r\Big[2r(D^Ae^{2\beta})D_A\beta\Big]+2r[D^A,\partial_r](e^{2\beta}D_A\beta)\\
&\quad -2r^2D^A\Big[\partial_r(e^{2\beta}r^{-1})D_A\beta\Big]+r^3[D_B,\partial_r]\partial_rU^B-2\gamma^{AB}e^{2\beta}D_AD_B\beta\\
&\quad +e^{2\beta}\Big[-2\Lambda r^2+R[\gamma]-2\gamma^{AB}(D_A\beta)(D_B\beta)-\frac{r^4}{2}e^{-4\beta}\gamma_{AB}(\partial_rU^A)(\partial_rU^B)\Big]\\
&\quad -2r^3D_B\Big[(\partial_rU^B)\partial_r\beta\Big]+r^3D^A\Big[(\partial_r\gamma_{AB})(\partial_rU^B)\Big]\\
&\quad -D^A\Big[\frac{r^2}{4}e^{2\beta}D_A\Big(\gamma^{EC}\gamma^{BD}(\partial_r\gamma_{EB})(\partial_r\gamma_{CD})\Big]+rD^A\Big[e^{2\beta}\gamma^{EF}D_E(\partial_r\gamma_{AF})\Big]\;, \qquad (87)
\end{aligned}
$$

where $D^2 = D^AD_A$ denotes the Laplace operator on the 2-sphere. This transport equation gives another set of conserved charges

$$\overset{[2]}{Q}(\lambda)=\int d^2\Omega\ \lambda e^{-2\beta}\chi\ , \qquad (88)$$

for which $\partial_r\overset{[2]}{Q}=0$ stands for a $u = const$ surface.

The parameter $\lambda = \lambda(x^A)$ is a function satisfying

$$\mathrm{TS}[D_AD_B\lambda]=0\ . \qquad (89)$$

On compact Einstein manifolds the solutions are constants, except on 2-spheres, where linear combinations of $\ell \le 1$ spherical harmonics also satisfy the equation.

The charge introduced in this section is also part of the family of gauge-invariant obstructions to gluing-up-to-gauge. Therefore, similarly to the previous charge, $\overset{[2]}{Q}$ needs to satisfy the

$$\overset{[2]}{Q}[x_1](\lambda)=\overset{[2]}{Q}[x_2](\lambda) \qquad (90)$$

condition for the sphere data $x_1$ and $x_2$ in order for the gluing to be possible.

In the four dimensional $C^2$ gluing problem, with $m = 0$ these two radial charges comprise the gauge-invariant family of obstructions. The charge $\overset{[1]}{Q}$ brings in 6 parameters, corresponding to rotations of $S^2$ and boosts, and $\overset{[2]}{Q}$ adds an additional 4 parameter corresponding to spacetime translation. That sums up to a 10-dimensional family of gauge-invariant charges.

# 6 Comparison between ADM an radial charges

In this section I would like to show that certain ADM charges are related to the conserved radial charges introduced above. Here I present the main intermediate steps of my calculations and my final conclusions.

## 6.1 ADM energy and $\overset{[2]}{Q}(1)$

If we choose $\lambda = 1$ for the second radial charge (88) with the expression $\chi$ defined in (87), then the charge after asymptotic series expansion according to (6) is proportional to the ADM energy of the spacetime, as

$$\lim_{r\to\infty} \overset{[2]}{Q}(1) = \lim_{r\to\infty} \int d^2\Omega \, \Big( -2M + \mathcal{O}(r^{-1}) \Big) \, . \tag{91}$$

If the Bondi mass aspect $M$ is independent of the $x^A$ angular coordinates, which is a valid assumption in spacetimes such as Schwarzschild, Kerr, Reissner–Nordström etc., then the charge becomes

$$\lim_{r\to\infty} \overset{[2]}{Q}(1) = -8\pi M \, . \tag{92}$$

Now let's compute the ADM energy using the definition in (48). The direction of the $n^a$ unit outward pointing normal vector of the $r = const$ 2-sphere is calculated as $\vec{n} = \partial_a r \, dx^a = (1, \ 0, \ 0)$, and the normalization factor is determined by the $\mathcal{N}^2 = h^{ab} n_a n_b = h^{rr}$ relation. So we have

$$\begin{aligned} n_a dx^a &= \sqrt{\frac{2r - V}{r e^{-2\beta}}} dr \, , \\ n^a \partial_a = h^{ab} n_b \partial_a &= \sqrt{\frac{r e^{-2\beta}}{2r - V}} (\partial_r + U^A \partial_A) \, . \end{aligned}$$

The vector $X^a$ becomes $X = x^a \partial_a \approx r\partial_r$ in the asymptotic limit. Thus the ADM energy after performing the suitable substitutions becomes

$$E_{ADM} = -\lim_{r\to\infty} \frac{1}{8\pi} \int_{S^2} d^2\Omega \; r^3 \sqrt{\frac{r e^{-2\beta}}{2r - V}} \left[ (R_{rr} - R h_{rr}) + U^A (R_{Ar} - R h_{Ar}) \right] \, , \tag{93}$$

where we used that $dA = r^2 d^2\Omega$.

In general Bondi retarded coordinates the integrand after the large $r$ expansion is simply $-2M$, thus the ADM energy charge in leading order is

$$E_{ADM} = \lim_{r\to\infty} \Big( M + \mathcal{O}(r^{-1}) \Big) = M \tag{94}$$

It is a contribution only from the first term in the square brackets, containing purely radial components. The second term with $rA$-components is of $\mathcal{O}(r^{-1})$. The result agrees our expectations that the function $M$ appearing in the asymptotic series of the metric function $V$ can be attributed with the physical meaning of mass, and not only in Schwarzschild spacetime, as the $E_{ADM}$ is a more general notion.

As we can see, the two different charges match up to a constant multiplicative factor of $-8\pi$, therefore we showed that the relation

$$\lim_{r\to\infty} \overset{[2]}{Q}(1) \propto E_{ADM} \tag{95}$$

holds.

### 6.2 ADM angular momentum and $\overset{[1]}{Q}(H^A_{(1,m_a)})$

The first radial charge $\overset{[1]}{Q}(Y^A)$ is defined in (79). After the asymptotic expansion its integrand becomes

$$-Y^A\Big[2r\overset{(2)}{U}_A + 3\overset{(3)}{U}_A + 2C_{AB}\overset{(2)}{U}{}^B - 2\mathring{D}_A\overset{(2)}{\beta}\Big] + \mathcal{O}(1/r)\ , \tag{96}$$

where $Y^A$ is an arbitrary conformal Killing vector, and $\overset{(k)}{U}_A$, $\overset{(k)}{\beta}$ denote the coefficients of the $\mathcal{O}(r^{-k})$ order in the asymptotic series for the Bondi metric functions $U_A$ and $\beta$. After plugging in the formulas for the coefficients given in (6), we obtain

$$\begin{aligned} Y^A\Big[r\,\mathring{D}^B C_{AB} + 2N_A - \frac{1}{4}\mathring{D}_A(C^{BC}C_{BC}) \\ + C_{AB}\mathring{D}_C C^{BC} - \frac{3}{2}\mathring{h}_{AB}C^{BC}\mathring{D}^D C_{CD}\Big] + \mathcal{O}(1/r)\ . \end{aligned} \tag{97}$$

Using that the indices can be moved with the metric $\mathring{h}_{AB}$, the last two terms can be merged together as

$$Y^A\Big[r\,\mathring{D}^B C_{AB} + 2N_A - \frac{1}{4}\mathring{D}_A(C^{BC}C_{BC}) - \frac{1}{2}C_{AB}\mathring{D}_C C^{BC}\Big] + \mathcal{O}(1/r)\ . \tag{98}$$

In the place of $Y^A$ we can insert the magnetic spherical harmonic vector of $\ell = 1$.

Then we would like to compare this result with the ADM angular momentum, defined in (52). The definition contains a $\phi^A$ rotational conformal Killing vector, and the outward pointing unit normal vector $n^a$ is given in (93). Then the expression we need to compute for $L_{ADM}$ is

$$\begin{aligned} L_{ADM} = \lim_{r\to\infty}\int d^2\Omega\, r^2\sqrt{\frac{re^{-2\beta}}{2r-V}}\Big[\Big(K_{rA} - \mathrm{tr}K h_{rA}\Big) \\ + \Big(K_{AB} - \mathrm{tr}K h_{AB}\Big)U^B\Big](\phi_{(a)})^A \end{aligned} \tag{99}$$

We use (21) to compute the extrinsic curvature, and then apply the transformations given in (18) to obtain the pullback onto the $\Sigma$ hypersurface. After having the expressions for the components $K_{rA}$ and $K_{AB}$ let's perform the asymptotic series expansion similarly as we did with the integrand of $\overset{[1]}{Q}$. Then we get

$$\begin{aligned} (\phi_{(a)})^A\Big[\Big(\overset{(2)}{U}_A - \overset{(2)}{U}{}^B\partial_u C_{AB}\Big)r + C_{AB}\overset{(2)}{U}{}^B + \frac{3}{2}\overset{(3)}{U}_A - \overset{(2)}{U}{}^C\partial_C\overset{(2)}{U}_A \\ - \overset{(2)}{U}_B\partial_A\overset{(2)}{U}{}^B + \frac{1}{2}\mathring{D}_A\overset{(1)}{V} + \overset{(2)}{U}_A\overset{(2)}{U}{}^B\mathring{h}^{CD}\partial_B\mathring{h}_{CD} + 2\overset{(2)}{U}_A\partial_B\overset{(2)}{U}{}^B \\ + 2\overset{(2)}{U}_A\partial_u\overset{(2)}{\beta} + 2M\overset{(2)}{U}{}^B\partial_u C_{AB} + U^B\partial_u C_{AB} + \overset{(2)}{U}_A\partial_u\overset{(1)}{V}\Big]\ , \end{aligned} \tag{100}$$

where $\overset{(1)}{V}$ is the coefficient of the $\mathcal{O}(1/r)$ order for the metric function $V$. In the fully non-linear case the charges clearly do not match. However, in the expression above we can find several quadratic terms that would vanish after linearization.

The linearization of the integrand of the ADM angular momentum around a Birmingham–Kottler metric (for more details see section 2.4) is

$$H^A_{(\ell,m_a)}\sqrt{\frac{2\pi}{3}}\left(r^2\partial_r h_{uA} - \frac{r(8-6\varepsilon)h_{uA}}{2-\varepsilon}\right) , \tag{101}$$

where we used that the rotational field $\phi^A$ can be expressed with the magnetic spherical harmonics (see the relation (54)), and that after setting $m = \alpha = 0$ the metric function $V$ simplifies to $V = \varepsilon r$. The choice of $\varepsilon = 1$ corresponds to the Minkowski background metric, in which case we have

$$H^A_{(1,m_a)}\sqrt{\frac{2\pi}{3}}\left(r^2\partial_r h_{uA} - 2rh_{uA}\right) . \tag{102}$$

We can substitute the formula The linearization of the integrand of $\overset{[1]}{Q}$ is

$$-H^A_{(1,m_a)}\left(r^2\partial_r h_{uA} - 2r^2 D_A\delta\beta - 2rh_{uA}\right) . \tag{103}$$

After imposing the gauge condition $\delta\beta = 0$ it becomes

$$-H^A_{(1,m_a)}\left(r^2\partial_r h_{uA} - 2rh_{uA}\right) . \tag{104}$$

By comparing (102) and (104) it is clear that the two linearized integrands and thus the two linearized charges themselves are proportional to each other as

$$\overset{[1]}{Q}(H^A_{(1,m_a)}) = -\sqrt{\frac{3}{2\pi}}L_{ADM} . \tag{105}$$

# 7 Charges in slowly rotating Kerr spacetime

In this section I compute the exact value of the the ADM charges ($E_{ADM}$, $P_{ADM}$, $L_{ADM}$, $C_{ADM}$) and the radial charges ($\overset{[1]}{Q}$, $\overset{[2]}{Q}$) in Lense–Thirring spacetime, which corresponds to a slow rotation limit of Kerr spacetime.

## 7.1 Lense–Thirring metric

The stationary and axis-symmetric vacuum solution of the Einstein equation was found by Roy Kerr in 1963, and it is used to describe the environment of rotating astronomical objects with zero electric charge [17, 18]. It has two free parameters, the mass $M$, and the rotational parameter $a$. The metric of the spacetime in $(t,\ r,\ \theta,\ \phi)$ Boyer–Lindquist coordinates reads as

$$\begin{aligned} g_{\mu\nu}dx^\mu dx^\nu = &-\left(1-\frac{2Mr}{\rho^2}\right)dt^2 - \frac{4Mr}{\rho^2}a\sin^2\theta\,dt\,d\phi + \frac{\rho^2}{\Delta}dr^2 \\ &+ \rho^2\,d\theta^2 + \left(r^2+a^2+\frac{2Mr}{\rho^2}a^2\sin^2\theta\right)\sin^2\theta\,d\phi^2\ , \end{aligned} \tag{106}$$

where we used the notation

$$\begin{aligned} \rho^2 &= r^2 + a^2\cos^2\theta\ , \\ \Delta &= r^2 - 2Mr + a^2\ . \end{aligned} \tag{107}$$

In the center of the Kerr spacetime there is a rotating black hole with two event horizons given by the $\Delta = 0$ condition, which has the solutions $r = M \pm \sqrt{M^2 - a^2}$. The metric has two singularities, namely $\Delta = 0$ and $\rho = 0$, of which only the latter is a real singularity, the other is merely a coordinate singularity. The $\rho = 0$ is the equation of a circle laying in the equatorial plane with a radius of $a$, in Cartesian coordinates it is $x^2 + y^2 = a^2$. Hence the singularity is often referred to as "ring singularity".

If we set $a = 0$, the metric becomes

$$g_{\mu\nu}dx^\mu dx^\nu = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2\ , \tag{108}$$

which is exactly the Schwarzschild metric. It aligns with our initial expectations, that the parameter $a$ is somehow related to the angular momentum, and when it is set to zero, we are left with the spherically symmetric, non-rotating case.

In 1918, way before the Kerr solution, Josef Lense and Hans Thirring introduced a metric in the form [5, 15]

$$\begin{aligned} g_{\mu\nu}dx^\mu dx^\nu = &-\left(1-\frac{2M}{r}\right)dt^2 + r^2\sin^2\theta\left(d\phi - \frac{2Ma}{r^3}dt\right)^2 \\ &+ \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\theta^2\ . \end{aligned} \tag{109}$$

It is an approximate solution of the vacuum Einstein equation, a predecessor of the exact axis-symmetric and stationary solution found by Kerr. Nonetheless, the Lense–Thirring metric is still widely used, as it can be interpreted as the slowly rotating limit of the Kerr metric, where the $a$ rotating parameter is small, therefore we can approximate the metric with $g_{\mu\nu} \approx g_{\mu\nu}(a) + \mathcal{O}(a^2)$.

The slow rotation limit simplifies the Kerr metric considerably, so working with the Lense–Thirring metric is much easier. Furthermore, real rotating stars or planets are not described exactly by the Kerr solution because generally they have additional mass multipoles. Thus Kerr is only valid far from the object, where the spacetime becomes asymptotically Kerr and effectively reduces to the Lense–Thirring form.

Next we would like to transform (109) into a Bondi-like form. First we need to switch to retarded coordinates by carrying out the following transformations:

$$\begin{aligned} dt &\to du + \Big(1 - \frac{2M}{r}\Big)^{-1} dr \ , \\ d\phi &\to d\varphi + \frac{2Ma}{r^3} dr \ . \end{aligned} \tag{110}$$

Thus the metric becomes

$$\begin{aligned} ds^2 = &- \Big(1 - \frac{2M}{r} - \frac{4M^2 a^2 \sin^2\theta}{r^4}\Big) du^2 - 2dudr \\ &- \frac{4Ma\sin^2\theta}{r} dud\varphi + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \ . \end{aligned} \tag{111}$$

Now we have to find out what functions correspond to the various metric functions appearing in (3). Comparing the $g_{ur}$ components in each metric leads us to the equation $-e^{2\beta} = -1$, which is satisfied if and only if

$$\beta = 0 \ . \tag{112}$$

From the $g_{AB}$ angular components, which require the $r^2\gamma_{AB}dx^A dx^B = r^2(d\theta^2 + \sin^2\theta d\varphi^2)$ equality to hold, it follows that

$$\gamma_{AB} = \mathrm{diag}(1,\ \sin^2\theta) \ . \tag{113}$$

The lack of $g_{u\theta}$ component in the Kerr metric suggests that

$$U^\theta = 0 \ , \tag{114}$$

and that leaves the $\varphi$-component to be

$$U^\varphi = \frac{2Ma}{r^3} \ . \tag{115}$$

Finally, using our results the comparison of $g_{uu}$ components tells us that the last Bondi metric function is

$$V = r - 2M \ . \tag{116}$$

## 7.2 Computation of the charges in slowly rotating Kerr spacetime

Now that we have the explicit expressions for the Bondi metric functions, we can calculate the ADM charges in slowly rotating Kerr spacetime. The ADM energy defined in (48) becomes

$$\begin{aligned} E_{ADM} &= \lim_{r\to\infty} M \, \frac{8M^3a^2(4M - r) + 3r^3(2M + r)(4M + r)}{3r^{5/2}(2M + r)^{5/2}} \\ &= \lim_{r\to\infty} \Big(M + \mathcal{O}(1/r)\Big) = M \ . \end{aligned} \tag{117}$$

It nicely matches our expectations, that the parameter $M$ in the Kerr metric describes the total mass.

The ADM momentum is given in the definition (49), where $a \in \{r,\ \theta,\ \varphi\}$ . However, in this case the Cartesian coordinates $(x,\ y,\ z)$ would carry the relevant physical meaning. In order to transform the momentum into the new coordinate system we need to carry out the

$$\begin{aligned}
\partial_x &= \sin\theta\cos\varphi\,\partial_r + \frac{\cos\theta\cos\varphi}{r}\,\partial_\theta - \frac{\sin\varphi}{r\sin\theta}\,\partial_\varphi\ , \\
\partial_y &= \sin\theta\sin\varphi\,\partial_r + \frac{\cos\theta\sin\varphi}{r}\,\partial_\theta + \frac{\cos\varphi}{r\sin\theta}\,\partial_\varphi\ , \\
\partial_z &= \cos\theta\,\partial_r + \frac{\sin\theta}{r}\,\partial_\theta
\end{aligned} \tag{118}$$

transformations before performing the integral. The result is that for $x^a = (x,\ y,\ z)$

$$(P_{ADM})_a = 0\ , \tag{119}$$

meaning that the total momentum of the spacetime is zero.

The angular momentum, as one can see in definition (52), contains the conformal Killing vector $Y^A$, which is proportional to the $H^A$ magnetic spherical harmonics with $\ell = 1$. On the 2-sphere they become

$$H_A^{(1,m)} = \frac{1}{\sqrt{2}}(\partial_\varphi Y_{1,m} - (\sin\theta)^{-1}\partial_\theta Y_{1,m}) \tag{120}$$

The various $m \in \{1, -1,\ 0\}$ magnetic numbers correspond to the coordinates $(r,\ \theta,\ \varphi)$ respectively. The final results for the angular momentum are

$$(L_{ADM})_r = 0\ , \qquad (L_{ADM})_\theta = 0\ , \tag{121}$$

and

$$(L_{ADM})_\varphi = \lim_{r\to\infty} a\left(M + \frac{8M^3}{3(2M+r)^2}\right) = \lim_{r\to\infty}\Big(aM + \mathcal{O}(r^{-1})\Big) = aM\ . \tag{122}$$

It shows that the rotational parameter equals with the total angular momentum per unit mass.

The center-of-mass in (63) involves the translation vector $(Z_{(a)})^b$, which is expressed using the electric spherical harmonics. In slowly rotating Kerr spacetime, we get

$$(C_{ADM})_a = 0 \tag{123}$$

for every $a \in \{r,\ \theta,\ \varphi\}$ component.

In the following we will calculate the value of the $\overset{[1]}{Q}$ and $\overset{[2]}{Q}$ radial charges in slowly rotating Kerr spacetime.

The first radial charge in slowly rotating Kerr spacetime for a general $Y^A$ conformal Killing vector is

$$\overset{[1]}{Q}(Y^A) = \int d^2\Omega\ Y^A X_A\ , \tag{124}$$

where $X^A\partial_A = -6Ma\sin^2\theta\,\partial_\varphi$. First let $Y^A$ be the magnetic spherical harmonic $H^A$. Then only the $m=0$ magnetic number leads a non-zero charge, which is

$$\overset{[1]}{Q}(H^A_{(1,0)}) = -4\sqrt{6\pi}\,aM\ , \tag{125}$$

and it is proportional to the $L_{ADM}$ angular momentum.

Now we choose $Y^A$ to be the $\ell = 1$ electric spherical harmonics $E^A$. In that case, for every $m$ number, the charge is

$$\overset{[1]}{Q}(E^A_{(1,m)}) = 0\ . \tag{126}$$

The second radial charge, as we can see in (91), is proportional to the Bondi mass aspect integrated over a 2-sphere in zeroth order. In Kerr spacetime it takes a very similar form

$$\overset{[2]}{Q}(\lambda) = -2\int d^2\Omega\ \lambda M\ . \tag{127}$$

If we choose the $\lambda$ function to be $\lambda = 1$, the charge simply becomes

$$\overset{[2]}{Q}(1) = -8\pi M\ , \tag{128}$$

as the mass aspect is a constant parameter. This result agrees with the leading order of the general case in (92).

If we choose $\lambda$ to be real spherical harmonics (55) with $\ell \geq 1$, the radial charge vanishes for all $m$ magnetic numbers:

$$\overset{[2]}{Q}(Y_{(\ell\geq 1,m)}) = 0\ . \tag{129}$$

This is due to the orthogonality in $\ell$ of the spherical harmonics, because the mass aspect represents the $\ell = 0$ mode.

# 8 BMS symmetry group

## 8.1 BMS group and supertranslations

The BMS group is the asymptotic symmetry group of asymptotically flat spacetimes [10, 19]. It is generated by diffeomorphisms of future null infinity $\mathcal{I}^+$ to itself, that preserve properties such as the Bondi gauge (3) and its asymptotic fall-off conditions.

The original expectation of Bondi, van der Burg, Metzner and Sachs was that studying the asymptotic symmetries would simply recover the isometry group of Minkowski spacetime, known as the Poincaré group. It turned out however that the BMS group is more than that. In fact, the Poincaré translations, rotations and boosts are contained in the BMS group, along with an infinite number of additional generators corresponding to so-called *supertranslations*.

A BMS diffeomorphism is a map $\psi$ that acting on a point in $\mathcal{I}^+$ carries out the transformation $(u,\ x^A) \to (\bar{u},\ \bar{x}^A)$, where the general formula for the coordinates after the transformation can be written as

$$\begin{aligned} \bar{u} &= \frac{u + \alpha(x^A)}{w(x^A)}\ , \\ \bar{x}^A &= \bar{x}^A(x^B)\ . \end{aligned} \tag{130}$$

The function $\alpha(x^A)$ is arbitrary. The infinitesimal form of the transformations

$$\begin{aligned} u &\to u + \xi^u\ , \\ x^A &\to x^A + \xi^A\ , \end{aligned} \tag{131}$$

can be expressed with vectors of a smooth vector field $\vec{\xi}$ on $\mathcal{I}^+$ [10], where

$$\vec{\xi} = \Big(\alpha(x^A) + \frac{1}{2} u D_A Y^A(x^B)\Big)\partial_u + Y^A(x^B)\partial_A\ . \tag{132}$$

The vector $Y^A$ is an element of the conformal Killing vector space (CKV) on the 2-sphere. It can also be written in the form

$$Y^A = D^A\chi + \varepsilon^{AB} D_B\psi\ , \tag{133}$$

where $\chi$ and $\psi$ are linear combinations of $\ell = 1$ spherical harmonics. The first term, with electric parity, corresponds to boosts, and the second, magnetic parity term to rotations, thus these solutions contain the Lorentz group in themselves.

The supertranslations of the BMS group are generated from the vector

$$\vec{\zeta} = f\partial_u - \frac{1}{r} D^A f \partial_A + \frac{1}{2} D^A D_A f \partial_r\ +\ \ldots \tag{134}$$

Here we considered only the leading order terms, and the function $f = f(x^A)$ can be chosen freely. In flat spacetimes, supertranslations can be regarded as a generalization of four translations.

Performing a supertranslation transforms one asymptotically flat solution into another one, which has physically inequivalent geometry. This is a rather unexpected property of a diffeomorphism, but one can see its effect in a simple example. Take a spacetime, where a pulse of a gravitational wave hits the north and south pole of $\mathcal{I}^+$, both at the retarded time of $u = 500$. Let's perform a supertranslation on this system, where we choose the $f$ function to have a value of 500 at the south pole, and 0 at the north pole. That means that in the transformed spacetime the pulse hits the north pole at the same $u = 500$, but the south pole at $u = 1000$, thus the outgoing data was changed by the supertranslation even at a classical level.

We can even extend our $\vec{\xi}$ generator vector field from $\mathcal{I}^+$ to the interior of the solution. In Bondi gauge these extended vectors take the form of

$$\vec{\xi} = f\partial_u + \left(Y^A - \frac{1}{r}D^A f + \frac{1}{2r^2}C^{AB}D_B f + \mathcal{O}(r^{-3})\right)\partial_A - \left(\frac{1}{2}rD_A Y^A - \frac{1}{2}D^A D_A f \right.$$
$$\left. - \frac{1}{2r}U^A D_A f + \frac{1}{4r}D_A(D_B f C^{AB}) + \mathcal{O}(r^{-2})\right)\partial_r \ , \tag{135}$$

where $f(u,\ x^A) = \alpha(x^A) + \frac{1}{2}uD_A Y^A(x^B)$.

## 8.2 Conserved BMS charges

For every $\vec{\xi}$ generator of the BMS group and each cross section of $\mathcal{I}^+$ there exists a $\mathcal{Q}$ charge. These include a finite number of charges corresponding to the Poincaré symmetries, and an infinite number corresponding to supertranslations [10, 21].

Strictly speaking, these charges are not conserved in the classical sense as the outward gravitational radiation violates the conservation. However, there is a way to generalize Noether's theorem, which allows us to define charges associated with the BMS symmetry group.

There are two types of conservation laws. One states the conservation of a quantity between two cross sections of future null infinity, the other relates one charge at future to another at past null infinity. To start with, we'll consider the first case.

For every $\vec{\xi}$ generator one can define a Noether charge two-form in the following way [21]

$$Q_{\mu\nu}(\vec{\xi}) = -\frac{1}{16\pi}\varepsilon_{\mu\nu\sigma\lambda}\nabla^\sigma \xi^\lambda \ , \tag{136}$$

where $\nabla$ is the covariant derivative compatible with the four-dimensional metric. Integrating that over any given cross section cut gives us a charge

$$\mathcal{Q} = \int_\Sigma \mathbf{Q} \ . \tag{137}$$

Additionally, we must impose a gauge condition to ensure that our $\mathcal{Q}$ charge is invariant of our choice of representative $\vec{\xi}$ in the equivalence class of the BMS transformation. To get a unique, well-defined charge we need $\nabla_\mu \xi^\mu = 0$ to hold.

## 8.3 BMS charges in Bondi coordinates

In this section I rederive the formula (3.5) in [10], providing more details on the intermediate steps than the original paper.

Let's compute the charges of a generator, consisting of the sum of two generators, $\vec{\xi} = \vec{\xi_1} + \vec{\xi_2}$, which based on (135) are

$$\vec{\xi_1} = \frac{1}{2}(u - u_0)D_A Y^A \ \partial_u + Y^A\partial_A - \frac{1}{2}rD_A Y^A \partial_r \ ,$$
$$\vec{\xi_2} = (\alpha + \frac{1}{2}u_0 D_A Y^A)\partial_u \ . \tag{138}$$

In the original paper $\vec{\xi_1}$ was defined differently, as it did not have the $\partial_r$ component. However, including that term is justified if we want the generator to be in the form of (135), dropping only the subleading terms in every component. Nevertheless, keeping the $\partial_r$ term did not change the asymptotic final result at $\mathcal{O}(1)$, neither did the inclusion of more subleading terms.

As the next step, we need to substitute into (136) to obtain the Noether charge 2-form. We choose the cross section to be given by the $u = u_0$ and $r = r_0$ conditions, thus only the $\partial_A$ angular directions are tangent to it. After fixing $r$ to a constant value of $r_0$, we will take the $r_0 \to \infty$ limit to reach the asymptotic regions.

First let's focus on the contribution of $\vec{\xi_1}$ to the charge $\mathcal{Q}$. Since on our particular surface $du = dr = 0$, the Noether 2-form (136) pulled back to the cross section simplifies to

$$\begin{aligned}\mathbf{Q}(\vec{\xi}) &= -\frac{1}{16\pi}\varepsilon_{AB\sigma\lambda}\nabla^\sigma\xi^\lambda \,\mathrm{d}x^A \wedge \mathrm{d}x^B \\ &= -\frac{\sqrt{-\mathfrak{g}}}{16\pi}(\nabla^u\xi^r - \nabla^r\xi^u)\,\mathrm{d}x^A \wedge \mathrm{d}x^B\,. \end{aligned} \tag{139}$$

The determinant of the four dimensional Bondi metric is $\sqrt{-\mathfrak{g}} = r^2 e^{2\beta}\sqrt{\gamma}$. To obtain its contribution to the charge we need to carry out the integral over the surface, which after some rather straightforward calculations becomes

$$\mathcal{Q}(\vec{\xi_1}) = \frac{1}{16\pi}\int d^2\Omega\,\left(\frac{1}{2}r^2 D_A Y^A + r^3 D_A Y^A \partial_r\beta - r^4\,e^{-2\beta}\,\gamma_{AB}\,Y^A\,\partial_r U^B\right)\,, \tag{140}$$

where $d^2\Omega = \sqrt{\gamma}\;dx^1\,dx^2$. The first term is a total divergence, hence it can be neglected. Now we need to perform the asymptotic series expansion according to (6) up to the order of $\mathcal{O}(1)$. Then (140) becomes

$$\begin{aligned}\mathcal{Q}(\vec{\xi_1}) =& \frac{1}{16\pi}\int d^2\Omega\,\Big(2N_A Y^A - \frac{1}{8}Y^A\mathring{D}_A(C_{BC}C^{BC}) \\ & - \frac{3}{2}Y^A\mathring{h}_{AB}C^{BC}\mathring{h}^{DE}\mathring{D}_E C_{CD} + C_{AB}Y^B\mathring{D}_C C^{AC} + \,r\,Y^A\mathring{D}^B C_{AB}\Big)\end{aligned} \tag{141}$$

Here the third term can be simplified with some basic index-moving algebra with $\mathring{h}_{AB}$ and its inverse. Then we can unite it with the fourth term, after swapping $A$ and $B$ dummy indices

$$\begin{aligned}-\frac{3}{2}Y^A\mathring{h}_{AB}C^{BC}\mathring{h}^{DE}\mathring{D}_E C_{CD} + C_{AB}Y^B\mathring{D}_C C^{AC} & \\ &= -\frac{3}{2}Y^A C_{AB}\mathring{D}_C C^{BC} + C_{BA}Y^A\mathring{D}_C C^{BC} \\ &= -\frac{1}{2}Y^A C_{AB}\mathring{D}_C C^{BC}\,. \end{aligned} \tag{142}$$

In (141) there is still one term of order $\mathcal{O}(r)$, which would have infinite values in the $r = r_0 \to \infty$ limit, but fortunately it vanishes upon integration. Let's integrate it by parts, which gives us

$$r\int d^2\Omega\;Y^A\mathring{D}^B C_{AB} = r\int d^2\Omega\;\mathring{D}^B(Y^A C_{AB}) - r\int d^2\Omega\;C_{AB}\mathring{D}^B Y^A\,. \tag{143}$$

As we can see the first term is a total divergence, which results in zero. The second term vanishes because $Y^A$ satisfies the conformal Killing vector equation, thus

$C_{AB}\mathring{D}^B Y^A = C_{AB}(\mathring{D}_C Y^C \mathring{h}^{AB} - \mathring{D}^A Y^B)$. The first part in the brackets vanishes because $C_{AB}$ is traceless, so we have

$$C_{AB}\mathring{D}^B Y^A = -C_{AB}\mathring{D}^A Y^B = -C_{BA}\mathring{D}^B Y^A = 0 \; . \tag{144}$$

After the second equality sign we swapped the $A$ and $B$ dummy indices. The expression is zero because of the symmetry of the Bondi shear $C_{AB} = C_{BA}$. Therefore we have

$$\mathcal{Q}(\vec{\xi}_1) = \frac{1}{16\pi}\int d^2\Omega \left(2N_A Y^A - \frac{1}{8}Y^A \mathring{D}_A(C_{BC}C^{BC}) - \frac{1}{2}Y^A C_{AB}\mathring{D}_C C^{BC}\right) . \tag{145}$$

The second vector $\vec{\xi}_2$ is a generator of a BMS supertranslation. Its contribution can be obtained in an analogous way. The charge is

$$\begin{aligned}\mathcal{Q}(\vec{\xi}_2) = -\frac{1}{16\pi}\int d^2\Omega \Big(V - r\partial_r V + 2r^2 U^A D_A \beta + 2rV\partial_r\beta \\ + r^4 e^{-2\beta}\gamma_{AB}U^A\partial_r U^B\Big)\cdot\left(\alpha + \frac{1}{2}u_0 D_A Y^A\right) ,\end{aligned} \tag{146}$$

which in the large $r$ limit, up to the order of $\mathcal{O}(1)$ becomes

$$\mathcal{Q}(\vec{\xi}_2) = \frac{1}{16\pi}\int d^2\Omega \left(2M\alpha + u_0 M\mathring{D}_A Y^A\right) . \tag{147}$$

We can rewrite the second term as

$$u_0 M\mathring{D}_A Y^A = u_0\mathring{D}_A(Y^A M) - u_0 Y^A\mathring{D}_A M \; , \tag{148}$$

where the first term can be dropped as it is a total divergence. Therefore what is left is

$$\mathcal{Q}(\vec{\xi}_2) = \frac{1}{16\pi}\int d^2\Omega \left(2M\alpha + u_0 Y^A\mathring{D}_A M\right) . \tag{149}$$

The charge associated with the generator $\vec{\xi} = \vec{\xi}_1 + \vec{\xi}_2$ is simply the sum of $\mathcal{Q}[\vec{\xi}_1]$ and $\mathcal{Q}[\vec{\xi}_2]$, which is

$$\begin{aligned}\mathcal{Q}(\vec{\xi}) = \frac{1}{16\pi}\int d^2\Omega \Big(2M\alpha + u_0 Y^A\mathring{D}_A M + 2N_A Y^A \\ - \frac{1}{8}Y^A\mathring{D}_A(C_{BC}C^{BC}) - \frac{1}{2}Y^A C_{AB}\mathring{D}_C C^{BC}\Big) .\end{aligned} \tag{150}$$

### 8.4 Comparison between BMS charges and $\overset{[1]}{Q}(Y^A)$

Comparing the asymptotic forms of the BMS charge associated with the generator $\vec{\xi}_1$ in (145) and the first radial charge $\overset{[1]}{Q}(Y^A)$ with an arbitrary $Y^A \in$ CKV in (98), one can see a strong resemblance between the two expressions. The first term in (98) can be dropped using a similar method as we carried out before in (143–144), using the conformal Killing vector equation. Thus we are left with

$$\overset{[1]}{Q}(Y^A) = \int d^2\Omega\; Y^A\left[2N_A - \frac{1}{4}\mathring{D}_A(C^{BC}C_{BC}) - \frac{1}{2}C_{AB}\mathring{D}_C C^{BC}\right] + \mathcal{O}(1/r) \; . \tag{151}$$

Now the only difference left with respect to the BMS charge $\mathcal{Q}(\vec{\xi}_1)$ is the coefficient of the second term. The linearization of the BMS charge gives

$$\mathcal{Q}(\vec{\xi}_1) = \int d^2\Omega \Big( Y^A r^2 \partial_r h_{uA} + r^3 \mathring{D}_A Y^A \partial_r \delta\beta - 2r Y^A h_{uA} \Big) \ . \tag{152}$$

From the series expansion of $\beta$ we know that asymptotically $\delta\beta \propto r^{-2}$, thus $\partial_r \delta\beta = -2\delta\beta/r$. Plugging this into the second term and integrating by parts yields to

$$\mathcal{Q}(\vec{\xi}_1) = \int d^2\Omega \Big( r^2 \partial_r h_{uA} + 2r^2 \mathring{D}_A \delta\beta - 2r h_{uA} \Big) Y^A \ , \tag{153}$$

after dropping the total divergence term. The discrepancy in the coefficient of the middle term remains, however in the $\delta\beta = 0$ gauge the two charges match (cf. (104)).

The definition of the first radial charge arose from the transport equation (73). The last term of the equation can be rewritten using the $G_{rr}$ component of the Einstein equations in (69) as

$$\frac{r^3}{4} D_A \Big[ \gamma^{EC} \gamma^{BD} (\partial_r \gamma_{EB})(\partial_r \gamma_{CD}) \Big] = 4r^2 D_A \partial_r \beta = \partial_r \Big( 4r^2 D_A \beta \Big) - 8r D_A \beta \ . \tag{154}$$

This way we expressed another term as a total $r$-derivative, which we can move to the left hand side. Thus we have

$$\partial_r \Big[ \underbrace{r^4 e^{-2\beta} \gamma_{AB} \Big( \partial_r U^B \Big) - 2r^2 D_A \beta}_{\overset{(*)}{\tilde{H}}_{uA}} \Big] = -r^2 \gamma^{EF} D_E (\partial_r \gamma_{AF}) - 8r D_A \beta \ . \tag{155}$$

It's possible to construct a charge out of the new quantity in the square brackets together with a $Y^A \in$ CKV. After linearization the transport equation looks quite similar to the one of the $\overset{[1]}{Q}$ charge

$$\partial_r \int d^2\Omega \ Y^A \overset{(*)}{\tilde{H}}_{uA} = - \int d^2\Omega \ Y^A \Big( \frac{2}{r} \mathring{D}^B h_{AB} + 8r \mathring{D}_A \delta\beta \Big) \ . \tag{156}$$

It has already been shown in Section 5.1 that the first term leads to zero, so we only need to focus on the newly appeared second term. In the $\delta\beta = 0$ gauge it trivially vanishes. If we do not want to impose that gauge condition, we can rewrite the term as

$$-8r \int d^2\Omega \ \mathring{h}^{AB} Y_B \mathring{D}_A \delta\beta = -8r \int d^2\Omega \Big[ \mathring{D}_A \Big( \mathring{h}^{AB} Y_B \delta\beta \Big) - \delta\beta \, \mathring{h}^{AB} \mathring{D}_A Y_B \Big] \ . \tag{157}$$

The first term above is a total divergence and the second term is zero is $Y^A$ is also a Killing vector. Thus the charge defined with $\overset{(*)}{\tilde{H}}_{uA}$ is also radially conserved under certain conditions, and the integrand after the large $r$ expansion gives the same expression as the integrand BMS charge

$$\overset{(*)}{\tilde{H}}_{uA} = 2N_A - \frac{1}{8} \mathring{D}_A (C^{BC} C_{BC}) - \frac{1}{2} C_{AB} \mathring{D}_C C^{BC} + \mathcal{O}(r^{-1}) \ . \tag{158}$$

# 9 Further transport equations

In this section we consider additional null constraint equations of transport type. Our aim is to identify further charges that play a role in characteristic gluing. We will see that these equations give rise to new families of charges, which unlike the previous ones are gauge-dependent obstructions, meaning that they can be adjusted with suitably chosen sphere perturbations.

## 9.1 Equation for $h_{uA}$

The derivations presented here in sections 9.1–9.3 follow Appendix E of [6].

Let's consider the Einstein equations. In vacuum we can assume that

$$\begin{aligned} E_{rr} &= 0 \ , \\ E_{rA} &= 0 \ , \\ \mathrm{TS}[E_{AB}] &= 0 \ , \end{aligned} \tag{159}$$

where the tensor $E_{\mu\nu} = G_{\mu\nu} + \Lambda g_{\mu\nu}$ is the sum of the Einstein tensor $G_{\mu\nu}$ and the term containing the cosmological constant $\Lambda$.

Now let's jump back to equation (73). After linearizing it around a Birmingham–Kottler background metric (23), we have

$$\partial_r\Big(r^4\partial_r(r^{-2}h_{uA}) - \mathring{D}^B h_{AB}\Big) = -2r^4\partial_r\Big(r^{-2}\mathring{D}_A\delta\beta\Big) - \frac{2}{r}\mathring{D}^B h_{AB} \ , \tag{160}$$

but since we chose to work in a gauge where $\delta\beta = 0$, it simplifies to

$$4r^3\partial_r\check{h}_{uA} + r^4\partial_r^2\check{h}_{uA} = r^2\partial_r\Big(r^{-2}\mathring{D}^B h_{AB}\Big) \ . \tag{161}$$

Here we introduced the notation $\check{h}_{\mu\nu} = r^{-2}h_{\mu\nu}$. We can rewrite the equation above as a transport equation of the quantity $\overset{(0)}{H}_{uA}$:

$$\partial_r\Big(\underbrace{3\check{h}_{uA} + r\partial_r\check{h}_{uA} - \frac{1}{r^3}\mathring{D}^B h_{AB}}_{\overset{(0)}{H}_{uA}}\Big) = \frac{1}{r^4}\mathring{D}^B h_{AB} \ . \tag{162}$$

## 9.2 Equation for $\partial_u h_{AB}$

From the $AB$-components of the vacuum Einstein equations we can derive the following

$$\begin{aligned} &\partial_r\Big[r\partial_u\gamma_{AB} - \frac{1}{2}V\partial_r\gamma_{AB} - \frac{1}{2r}V\gamma_{AB}\Big] \\ &\qquad = -\frac{1}{2}\partial_r(r^{-1}V)\gamma_{AB} - \frac{1}{r}\mathrm{TS}\Big[e^{2\beta}R[\gamma]_{AB} - 2e^{\beta}D_AD_Be^{\beta} \\ &\qquad\quad + \gamma_{AC}D_B[\partial_r(r^2U^C)] - \frac{1}{2}r^4e^{-2\beta}\gamma_{AC}\gamma_{BD}(\partial_rU^C)(\partial_rU^D) \\ &\qquad\quad + \frac{r^2}{2}(\partial_r\gamma_{AB})(D_CU^C) + r^2U^CD_C(\partial_r\gamma_{AB}) \\ &\qquad\quad - r^2(\partial_r\gamma_{AC})\gamma_{BE}(D^CU^E - D^EU^C) + \Lambda e^{2\beta}g_{AB}\Big] \ . \end{aligned} \tag{163}$$

The linearization around the Birmingham–Kottler metric is

$$0 = \partial_r\Big(r\partial_u \check{h}_{AB} - \frac{1}{2}V\partial_r \check{h}_{AB} - \frac{1}{2r}V\check{h}_{AB} - r\mathrm{TS}[\mathring{D}_A \check{h}_{uB}]\Big) + \frac{1}{2}\partial_r(r^{-1}V)\check{h}_{AB} - \frac{1}{r}\Big(2\mathring{D}_A\mathring{D}_B\delta\beta + r\mathrm{TS}[\mathring{D}_A\check{h}_{uB}]\Big)\ , \tag{164}$$

where we used that the traceless, symmetric part of $R[\gamma]_{AB}$, the Ricci tensor associated with $\gamma_{AB}$ vanishes, i.e. $\mathrm{TS}[R[\gamma]_{AB}] = 0$. In the gauge where $\delta\beta = 0$, the equation above simplifies to

$$0 = \partial_r\Big(r\partial_u \check{h}_{AB} - \frac{1}{2}V\partial_r \check{h}_{AB} - \frac{1}{2r}V\check{h}_{AB} - r\mathrm{TS}[\mathring{D}_A \check{h}_{uB}]\Big) + \frac{1}{2}\partial_r(r^{-1}V)\check{h}_{AB} - \mathrm{TS}[\mathring{D}_A\check{h}_{uB}]\ . \tag{165}$$

By comparing the $r$-derivatives of the $g_{uu}$ components in the Bondi metric (3) and the Birmingham–Kottler metric (23) we can see the relation $V = r\varepsilon - \alpha^2 r^3 - 2m$, thus $1/2\,\partial_r(r^{-1}V) = mr^{-2} - \alpha^2 r$, that we can apply to the second term of the equation above. Let's take $(2r^2)^{-1} \times \mathrm{TS}[\mathring{D}_B(\,\cdot\,)]$ of (161):

$$\frac{1}{2r^2}\partial_r\Big[r^4\partial_r\Big(r^{-2}\mathrm{TS}[\mathring{D}_B h_{uA}]\Big)\Big] - \frac{1}{2}\partial_r(r^{-2}Ph_{AB}) = 0\ , \tag{166}$$

where we introduced a new operator $Ph_{AB} = \mathrm{TS}[\mathring{D}_A\mathring{D}^C h_{BC}]$. In four dimensions it can be written as $P(h)_{AB} = (1/2\mathring{\Delta} - \varepsilon)h_{AB}$. This formula can be derived by decomposing $P(h)_{AB}$ into scalar, vector and tensor parts. In this case only the scalar and vector parts have non-zero contributions. Subtracting (166) from (165) gives us

$$\partial_r\Big[\underbrace{r\partial_u\check{h}_{AB} - \frac{V}{2}\partial_r\check{h}_{AB} - \frac{V}{2r}\check{h}_{AB} - \frac{1}{2r^2}\partial_r\Big(r^2\mathrm{TS}[\mathring{D}_A h_{uB}]\Big) + \frac{1}{2}P(\check{h})_{AB}}_{q_{AB}}\Big] = \Big(\frac{\alpha^2}{r} - \frac{m}{r^4}\Big)h_{AB}\ , \tag{167}$$

which is a transport equation for the quantity $q_{AB}$. It tells us that if $\alpha = m = 0$, $q_{AB}$ is radially conserved.

## 9.3 Transport equation for $\overset{(1)}{H}_{uA}$

Let's rewrite the right-hand side of the transport equation (162) as $r^{-2}\mathring{D}^B\check{h}_{AB} = \partial_r(-r^{-1}\mathring{D}^B\check{h}_{AB}) + r^{-1}\partial_r\mathring{D}^B\check{h}_{AB}$ to obtain a simpler form:

$$\partial_r\Big(3\check{h}_{uA} + r\partial_r\check{h}_{uA}\Big) = \frac{1}{r}\mathring{D}^B\partial_r\check{h}_{AB}\ . \tag{168}$$

Then let's take the $\partial_u$ derivative, and move every term to the left-hand side:

$$\partial_r\Big(3\partial_u\check{h}_{uA} + r\partial_r\partial_u\check{h}_{uA}\Big) - \frac{1}{r}\mathring{D}^B\partial_r\partial_u\check{h}_{AB} = 0\ . \tag{169}$$

Now let's take the $\mathring{D}^B$ derivative of (165), and multiply it with a factor $Ar^b$, where $A,\ b \in \mathbb{R}$ are constants.

$$0 = Ar^b\,\partial_r\Big(r\partial_u\mathring{D}^B\check{h}_{AB}\Big) - Ar^b\mathring{D}^B\partial_r\Big(\frac{1}{2}(V\partial_r\check{h}_{AB}) - \frac{1}{2r}\mathring{D}^B(V\check{h}_{AB}) - r\mathrm{TS}[\mathring{D}_A\check{h}_{uB}]\Big) + Ar^b\mathring{D}^B\Big(\frac{1}{2}\partial_r(r^{-1}V)\check{h}_{AB} - \mathrm{TS}[\mathring{D}_A\check{h}_{uB}]\Big)\ . \tag{170}$$

Then we add (169) to it. We want to collect all the terms containing $\partial_u \mathring{D}^B \check{h}_{AB}$ into one. This can be executed with the choice of $A = -1/2$ and $b = -2$. After summing the two equations we get:

$$\begin{aligned}
&\partial_r\Big(3\partial_u \check{h}_{uA} + r\partial_r\partial_u \check{h}_{uA} - \frac{1}{2r}\partial_u \mathring{D}^B \check{h}_{AB}\Big) \\
&= \frac{1}{2r^2}\mathring{D}^B\partial_r\Big(\frac{1}{2}V\partial_r \check{h}_{AB} + \frac{1}{2r}V\check{h}_{AB}\Big) - \frac{1}{4r^2}\partial_r(r^{-1}V)\mathring{D}^B\check{h}_{AB} \\
&\quad + \frac{1}{2r^2}\partial_r\Big(r\mathring{D}^B \mathrm{TS}[\mathring{D}_A \check{h}_{uB}]\Big) + \frac{1}{2r^2}\mathring{D}^B \mathrm{TS}[\mathring{D}_A \check{h}_{uB}] \ .
\end{aligned} \tag{171}$$

Now we need to carry out a similar step to collect all the $\mathring{D}^B \mathrm{TS}[\mathring{D}_A \check{h}_{uB}]$ terms together. We will take the last two terms of the equation above and add $Cr^d \times \mathring{D}^B \mathrm{TS}[\mathring{D}_A(*)]$ to it, where we insert the left-hand side of (168) in the place of $*$ . If we set $C = -3/8$ and $d = -1$, we can write

$$\begin{aligned}
&\frac{1}{2r^2}\partial_r\Big(r\mathring{D}^B \mathrm{TS}[\mathring{D}_A \check{h}_{uB}]\Big) + \frac{1}{2r^2}\mathring{D}^B \mathrm{TS}[\mathring{D}_A \check{h}_{uB}] \\
&-\frac{3}{8r}\partial_r\Big(3\mathring{D}^B \mathrm{TS}[\mathring{D}_B \check{h}_{uB}] + r\partial_r \mathring{D}^B \mathrm{TS}[\mathring{D}_A \check{h}_{uA}]\Big) \\
&\qquad = -\frac{3}{8}\partial_r\Big[r^{-8/3}\partial_r\Big(r^{8/3}\mathring{D}^B \mathrm{TS}[\mathring{D}_A \check{h}_{uA}]\Big)\Big] \ .
\end{aligned} \tag{172}$$

Using equation (168) and the definition of the $P$ operator we can rewrite the second line as

$$\partial_r\Big(3\mathring{D}^B \mathrm{TS}[\mathring{D}_B \check{h}_{uB}] + r\partial_r \mathring{D}^B \mathrm{TS}[\mathring{D}_A \check{h}_{uA}]\Big) = \frac{1}{r}\partial_r\Big(\mathring{D}^B P(\check{h})_{AB}\Big) \tag{173}$$

Now let's substitute this back into (171)

$$\begin{aligned}
&\partial_r\Big(3\partial_u \check{h}_{uA} + r\partial_r\partial_u \check{h}_{uA} - \frac{1}{2r}\partial_u \mathring{D}^B \check{h}_{AB}\Big) \\
&\quad = \frac{1}{2r^2}\mathring{D}^B\partial_r\Big(\frac{V}{2}\partial_r \check{h}_{AB} + \frac{V}{2r}\check{h}_{AB}\Big) - \frac{1}{4r^2}\partial_r(r^{-1}V)\mathring{D}^B\check{h}_{AB} \\
&\qquad - \frac{3}{8}\partial_r\Big[r^{-8/3}\partial_r\Big(r^{8/3}\mathring{D}^B \mathrm{TS}[\mathring{D}_A \check{h}_{uA}]\Big)\Big] + \frac{3}{8r^2}\partial_r\Big(\mathring{D}^B P(\check{h})_{AB}\Big)
\end{aligned} \tag{174}$$

After some more algebraic modifications we get

$$\begin{aligned}
&\partial_r\Big(3\partial_u \check{h}_{uA} + r\partial_r\partial_u \check{h}_{uA} - \frac{1}{2r}\partial_u \mathring{D}^B \check{h}_{AB} + \frac{3}{8}r^{-8/3}\partial_r(r^{8/3}\mathring{D}^B \mathrm{TS}[\mathring{D}_A \check{h}_{uB}])\Big) \\
&\qquad = \partial_r\Big(\frac{V}{4r^5}\mathring{D}^B h_{AB} + \frac{V}{4r^4}\mathring{D}^B\partial_r h_{AB} + \frac{3}{8r^4}P(h)_{AB}\Big) \\
&\qquad\quad + \Big(\frac{9V}{4r^6} - \frac{3\partial_r V}{4r^5}\Big)\mathring{D}^B h_{AB} + \frac{3}{4r^5}\mathring{D}^B P(h)_{AB} \ .
\end{aligned} \tag{175}$$

By simply moving terms we can rewrite the equation above into a form of a transport equation, namely separating a $\partial_r$ derivative on one side. We will also use the relation $3Vr^{-1} - \partial_r V = 2\varepsilon$:

$$\begin{aligned}
&\partial_r\Big[3\partial_u \check{h}_{uA} + r\partial_r\partial_u \check{h}_{uA} - \frac{1}{2r}\partial_u \mathring{D}^B \check{h}_{AB} + \frac{3}{8}r^{-8/3}\partial_r(r^{8/3}\mathring{D}^B \mathrm{TS}[\mathring{D}_A \check{h}_{uB}]) \\
&-\frac{V}{4r^5}\mathring{D}^B h_{AB} - \frac{V}{4r^4}\mathring{D}^B\partial_r h_{AB} - \frac{3}{8r^4}P(h)_{AB}\Big] \\
&\qquad = \frac{3}{4r^5}\mathring{D}^B\Big(P + 2\varepsilon\Big)h_{AB} \ .
\end{aligned} \tag{176}$$

This is the transport equation of the quantity

$$\overset{(1)}{H}_{uA} := 3\partial_u \check{h}_{uA} + r\partial_r\partial_u \check{h}_{uA} - \frac{1}{2r}\partial_u \mathring{D}^B \check{h}_{AB} + \frac{3}{8} r^{-8/3}\partial_r (r^{8/3}\mathring{D}^B \mathrm{TS}[\mathring{D}_A \check{h}_{uB}])$$
$$- \frac{V}{4r^5}\mathring{D}^B h_{AB} - \frac{V}{4r^4}\mathring{D}^B \partial_r h_{AB} - \frac{3}{8r^4} P(h)_{AB} \ , \tag{177}$$

i.e. the expression in the square brackets.

Let $\mu^A$ be a vector from the cokernel of the operator $\mathring{\mathrm{div}}(P+2\varepsilon)$ appearing on the right-hand side of the transport equation. Using that we can construct a family of charges

$$\mathcal{Q}_1(\mu^A) = \int_{S^2} d^2\Omega \ \mu^A \overset{(1)}{H}_{uA} \ , \tag{178}$$

which are radially conserved along a $u = const$ hypersurface, meaning that

$$\partial_r \mathcal{Q}_1(\mu^A) = 0 \ . \tag{179}$$

It can be shown (see Appendix C.5.2 of [6]) that $\mathrm{coker}(\mathring{\mathrm{div}}(P+2\varepsilon))$ on the 2-sphere $S^2$ is a 16-dimensional space spanned by the spherical harmonic vectors of the $\ell = 1$ and $\ell = 2$ modes.

The charge $\mathcal{Q}_1(\mu^A)$ corresponds to a set of gauge-dependent family of obstructions. To ensure the continuity of $\partial_u h_{uA}$ along the gluing region and at the boundaries we need to require to

$$\mathcal{Q}_1[x_1] = \mathcal{Q}_1[x_2] \ , \tag{180}$$

where $x_1$ and $x_2$ is the sphere data at which we want to glue two regions together. The charge is gauge-dependent because it is possible to match them with well chosen linearized sphere perturbations and sphere diffeomorphisms.

## 9.4 Transport equations for $\overset{(i)}{H}_{uA}$

It is possible to define a more general $\overset{(i)}{H}_{uA}$ for every integer $i \geq 1$. From these quantities we can introduce more gauge-dependent charges that are obstructions for the continuous gluing of $\partial_u^i h_{uA}$. We will look for the formulas for the quantities in a form of $\overset{(i)}{H}_{uA} = 3\partial_u^i \check{h}_{uA} + r\partial_r \partial_u^i \check{h}_{uA} +$ other terms containing $r$, $h_{uA}$, $h_{AB}$ and their higher $\partial_u$ and $\partial_r$ derivatives. Here I will simply present the results from [8], but the interested reader can find more details on the derivation of the expressions appearing in this section in Appendix A of the same paper.

The $\overset{(i)}{H}_{uA}$ quantities are defined with a recursion formula, where the first $i = 0$ element is given in (162), the next $i = 1$ is

$$\overset{(1)}{H}_{uA} = \partial_u \overset{(0)}{H}_{uA} - \mathring{D}^B \hat{q}^{(-4)}_{AB} \ , \tag{181}$$

where

$$\hat{q}^{(k)}_{AB} = \frac{r^{k+1}}{k+2}\partial_u h_{AB} - \frac{(k+1)r^{k+2+\frac{2}{k+1}}}{k(k+2)(k+3)}\partial_r \left( r^{-\frac{2}{k+1}} \mathrm{TS}[\mathring{D}_A(h_{uB})] \right)$$
$$+ \frac{r^{k+2}}{k+2}\tilde{q}_{AB} - \frac{2r^k(k-1)}{4k(k+3)} P(h)_{AB} + \frac{r^k}{2}\left(\varepsilon - \alpha^2 r^2 - \frac{2m}{r}\right) h_{AB} \ , \tag{182}$$

for $k \in \mathbb{Z} \setminus \{-3; -2;\ 0\}$, and

$$\tilde{q}_{AB} = -\frac{1}{2} V \partial_r \check{h}_{AB} - \frac{1}{2r} V \check{h}_{AB} + \frac{1}{2} P(\check{h})_{AB} \ . \tag{183}$$

We define every proceeding $i \geq 2$ element with the recursion formula

$$\begin{aligned}
\overset{(i)}{H}_{uA} =& \partial_u \overset{(i-1)}{H}{}_{uA} - \mathring{D}^B \overset{(i-1)}{\chi}(\mathring{\Delta}, P) \hat{q}^{(-i-3)}_{AB} - m^{i-1} \mathring{D}^B \overset{(i-1)}{\chi}{}_{[m]} \hat{q}^{-(4+2(i-1))}_{AB} \\
& - \sum_{j,\ell}^{(i-1)_*} m^j \alpha^{2\ell} \mathring{D}^B \overset{(i-1)}{\chi}{}_{j,\ell}(\mathring{\Delta}, P) \hat{q}^{-(i+3)-j+2\ell} \\
& - \alpha^2 \mathcal{K}_{[\alpha]}\Big( - (i+3) \Big) \tilde{\mathcal{K}}\Big( - (i+2), \mathring{\Delta}, \mathring{\operatorname{div}}_{(2)} C \Big) \overset{(i-2)}{H}{}_{uA} \ ,
\end{aligned} \tag{184}$$

where the sum is over every $j,\ \ell \in \mathbb{N}$ that is $1 \leq j \leq i-1\ ,\ \ j+\ell \leq i\ ,\ \ 0 \leq 2\ell \leq i+j$, and

$$\begin{aligned}
\overset{(i)}{\chi}(\mathring{\Delta}, P) &= \prod_{j=1}^{i} \mathcal{K}\Big( - (j+3), \mathring{\Delta}, P \Big) \ , \qquad \overset{(0)}{\chi}(\mathring{\Delta}, P) = 1 \ , \\
\overset{(i)}{\chi}{}_{[m]} &= \prod_{j=1}^{i} \mathcal{K}_{[m]}(-(4+2(j-1))) \ , \qquad \overset{(0)}{\chi}{}_{[m]} = 0 \ ,
\end{aligned} \tag{185}$$

with $\mathring{\Delta}$ being the Laplace operator of $\mathring{h}$, and

$$\begin{aligned}
\mathcal{K}\Big(k, \mathring{\Delta}, P\Big) &= -\frac{1}{4+2k} \Big( \frac{4P}{k(k+3)} + 2\mathring{\mathcal{R}} + \mathring{\Delta} + (k+1)(k+2)\varepsilon \Big) \ , \\
\mathcal{K}_{[m]}(k) &= \frac{(k+1)^2}{k+2} \\
\mathcal{K}_{[\alpha]}(k) &= -\frac{(k+4)(k+1)}{2(k-2)} \ .
\end{aligned} \tag{186}$$

The action of the operator $\mathring{\mathcal{R}}$ on a covariant 2-tensor is $\mathring{\mathcal{R}}(h)_{AB} = \mathrm{TS}[\mathring{R}_A{}^C{}_B{}^D h_{CD}]$ with $\mathring{R}_{ABCD}$ being the Riemann tensor of $\mathring{h}$. Since $\mathring{h}$ is a space form, the previous definition simplifies to $\mathring{\mathcal{R}}(h)_{AB} = -\varepsilon h_{AB}$.

The transport equation for a general $\overset{(i)}{H}_{uA}$ can be written as

$$\begin{aligned}
\partial_r \overset{(i)}{H}_{uA} =& \mathring{D}^B \overset{(i)}{\chi}(\mathring{\Delta}, P) r^{-(i+4)} h_{AB} + m^i \mathring{D}^B \overset{(i)}{\chi}{}_{[m]} r^{-(4+2i)} h_{AB} \\
& + \sum_{j,\ell}^{i_*} m^j \alpha^{2\ell} \mathring{D}^B \overset{(i)}{\chi}{}_{j,\ell}(\mathring{\Delta}, P) r^{-(i+4)-j+2\ell} h_{AB} \ .
\end{aligned} \tag{187}$$

With a straightforward substitution of $\alpha = m = 0$ into (181) and (187) I will show that the equation (176) what we derived in section 9.3 fits into the general definitions of section 9.4 with $i = 1$. It's easy to check that $\overset{(1)}{\chi}(\mathring{\Delta}, P) = \mathcal{K}(-4, \mathring{\Delta}, P) = (P + 2\mathring{\mathcal{R}} + \mathring{\Delta} + 6\varepsilon)/4$, thus (187) becomes

$$\partial_r \overset{(1)}{H}_{uA} = \frac{1}{4r^5} \mathring{D}^B (P + 2\mathring{\mathcal{R}} + \mathring{\Delta} + 6\varepsilon) h_{AB} = \frac{3}{4r^5} \mathring{D}^B (P + 2\varepsilon) h_{AB} \ , \tag{188}$$

which agrees with the right-hand side of (176). In the last step we used that $\mathring{\mathcal{R}}(h)_{AB} = -\varepsilon h_{AB}$ and $\mathring{\Delta} h_{AB} = 2(P+\varepsilon)h_{AB}$. For the left-hand side we will need the expression in (182) for $k=-4$

$$\hat{q}^{(-4)}_{AB} = -\frac{1}{2r^3}\partial_u h_{AB} - \frac{3}{8}r^{-8/3}\partial_r\Big(r^{8/3}\mathrm{TS}[\mathring{D}_A(\check{h}_{uB})]\Big) + \frac{1}{4r^2}V\partial_r\check{h}_{AB} + \frac{1}{4r^3}V\check{h}_{AB} - \frac{1}{4r^2}P(\check{h})_{AB} + \frac{5}{8r^4}P(h)_{AB} + \frac{1}{2r^4}\varepsilon h_{AB}\ . \tag{189}$$

Putting everything together according to (181), where $\overset{(0)}{H}_{uA}$ is defined in (162)

$$\begin{aligned}\overset{(1)}{H}_{uA} = {}& 3\partial_u\check{h}_{uA} + r\partial_r\partial_u\check{h}_{uA} - \frac{1}{2r^3}\partial_u\mathring{D}^B h_{AB} + \frac{3}{8}r^{-8/3}\partial_r\Big(r^{8/3}\mathring{D}^B\mathrm{TS}[\mathring{D}_A(\check{h}_{uB})]\Big)\\ &- \frac{1}{4r^2}\mathring{D}^B(V\partial_r\check{h}_{AB}) - \frac{1}{4r^3}\mathring{D}^B(V\check{h}_{AB}) - \frac{3}{8r^4}\mathring{D}^B P(h)_{AB}\\ &- \frac{1}{2r^4}\mathring{D}^B(\varepsilon h_{AB})\ ,\end{aligned} \tag{190}$$

which we can rewrite, keeping in mind that $\varepsilon = V/r$ when $\alpha = m = 0$, so we can pull it out from the $\mathring{D}^B$ derivatives:

$$\begin{aligned}\overset{(1)}{H}_{uA} = {}& 3\partial_u\check{h}_{uA} + r\partial_r\partial_u\check{h}_{uA} - \frac{1}{2r^3}\partial_u\mathring{D}^B h_{AB} + \frac{3}{8}r^{-8/3}\partial_r\Big(r^{8/3}\mathring{D}^B\mathrm{TS}[\mathring{D}_A(\check{h}_{uB})]\Big)\\ &- \frac{V}{4r^4}\mathring{D}^B\partial_r h_{AB} - \frac{V}{4r^5}\mathring{D}^B h_{AB} - \frac{3}{8r^4}\mathring{D}^B P(h)_{AB}\end{aligned} \tag{191}$$

This way we recovered the exactly the same equation as (176), therefore the conserved quantity $\overset{(1)}{H}_{uA}$ fits in the general recursive definitions introduced in this section.

### 9.5 Non-linear equation for $\overset{(1)}{H}_{uA}$

In the previous sections we considered only the linearized equations, which resulted in a clearer and simpler formulation. However, linearization inherently includes some loss of information. Furthermore, we worked in the $\delta\beta = 0$ gauge, which also led to dropping terms containing the metric function $\beta$. In this section I aim to recover the fully non-linear version of the transport equation (176). This can be achieved by taking similar steps to the linearized derivation.

We will start by rewriting the equation (73) into a form analogous to (168). After rearranging the terms we get

$$\partial_r\Big[r^4 e^{-2\beta}\gamma_{AB}\Big(\partial_r U^B\Big) + 2r^2 D_A\beta\Big] + r^2\Big[\gamma^{EF}D_E(\partial_r\gamma_{AF})\Big] - \frac{r^3}{4}D_A\Big[\gamma^{EC}\gamma^{BD}(\partial_r\gamma_{EB})(\partial_r\gamma_{CD})\Big] = 0\ . \tag{192}$$

Let's carry out the $\partial_r$ derivative in the first term using the product rule

$$\begin{aligned}&4r^3 e^{-2\beta}\gamma_{AB}\partial_r U^B - 2\beta\partial_r\beta r^4 e^{-2\beta}\gamma_{AB}\Big(\partial_r U^B\Big) + r^4 e^{-2\beta}\partial_r\gamma_{AB}\partial_r U^B\\ &+ r^4 e^{-2\beta}\gamma_{AB}\Big(\partial_r^2 U^B\Big) + 4rD_A\beta + 2r^2 D_A\partial_r\beta + r^2 D^B(\partial_r\gamma_{AB})\\ &- \frac{r^3}{4}D_A\Big[\gamma^{EC}\gamma^{BD}(\partial_r\gamma_{EB})(\partial_r\gamma_{CD})\Big] = 0\ .\end{aligned} \tag{193}$$

Let's divide it by $r^3$ and rewrite it as

$$\partial_r\Big[-\frac{4}{r}D_A\beta+3e^{-2\beta}U_A+re^{-2\beta}\gamma_{AB}\partial_rU^B\Big]-3U^B\partial_r(e^{-2\beta}\gamma_{AB})$$
$$\frac{1}{r}\gamma^{BC}D_C\partial_r(\gamma_{AB})+\frac{6}{r}D_A\partial_r\beta-\frac{1}{4}D_A\Big[\gamma^{EC}\gamma^{BD}(\partial_r\gamma_{EB})(\partial_r\gamma_{CD})\Big]=0\ . \tag{194}$$

The last two terms the second line can be simplified using the $rr$-component of the vacuum Einstein equations (see e.g. (4.4) of [7]). From that we obtain the relation

$$\partial_r\beta=\frac{r}{16}\gamma^{EC}\gamma^{BD}(\partial_r\gamma_{EB})(\partial_r\gamma_{CD})\ , \tag{195}$$

so the equation simplifies to

$$\partial_r\Big[-\frac{4}{r}D_A\beta+3e^{-2\beta}U_A+re^{-2\beta}\gamma_{AB}\partial_rU^B\Big]-3U^B\partial_r(e^{-2\beta}\gamma_{AB})$$
$$\frac{1}{r}\gamma^{BC}D_C\partial_r(\gamma_{AB})-\frac{1}{8}D_A\Big[\gamma^{EC}\gamma^{BD}(\partial_r\gamma_{EB})(\partial_r\gamma_{CD})\Big]=0\ . \tag{196}$$

Now we have an equation exactly in the form as its linear analogue. From that we can identify the non-linear $\overset{(0)}{H}_{uA}$ charge, which is the expression inside the $\partial_r$ derivative

$$\overset{(0)}{H}_{uA}:=-\frac{4}{r}D_A\beta+3e^{-2\beta}U_A+re^{-2\beta}\gamma_{AB}\partial_rU^B \tag{197}$$

Let's take $u$-derivative of the equation

$$\partial_r\Big[\partial_u\overset{(0)}{H}_{uA}\Big]-3\partial_u\Big(U^B\partial_r(e^{-2\beta}\gamma_{AB})\Big)$$
$$\frac{1}{r}\partial_u\Big(\gamma^{BC}D_C\partial_r(\gamma_{AB})\Big)-\frac{1}{8}D_A\partial_u\Big[\gamma^{EC}\gamma^{BD}(\partial_r\gamma_{EB})(\partial_r\gamma_{CD})\Big]=0\ . \tag{198}$$

Now let's jump back to (163), and take the $D^A$ derivative of it, divided by $2r^2$. After that rename the indices by swapping $A\leftrightarrow B$. Symmetric tensors such as $\gamma_{AB}$ and the Ricci tensor remain unchanged. We get:

$$\begin{aligned}0=&\frac{1}{2r^2}\partial_rD^B(r\partial_u\gamma_{AB})-\frac{1}{4r^2}D^B\partial_r(V\partial_r\gamma_{AB})-\frac{1}{4r^3}D^B\partial_r(V\gamma_{AB})\\&+\frac{1}{4r^2}D^B\partial_r(r^{-1}V)\gamma_{AB}+\frac{1}{2r^3}D^B\mathrm{TS}\Big[X_{AB}\Big]\ ,\end{aligned} \tag{199}$$

where for brevity we introduced the notation

$$\begin{aligned}X_{AB}:=&e^{2\beta}R[\gamma]_{AB}-2e^{\beta}D_AD_Be^{\beta}\\&+\gamma_{AC}D_B[\partial_r(r^2U^C)]-\frac{1}{2}r^4e^{-2\beta}\gamma_{AC}\gamma_{BD}(\partial_rU^C)(\partial_rU^D)\\&+\frac{r^2}{2}(\partial_r\gamma_{AB})(D_CU^C)+r^2U^CD_C(\partial_r\gamma_{AB})\\&-r^2(\partial_r\gamma_{AC})\gamma_{BE}(D^CU^E-D^EU^C)+r^2\Lambda e^{2\beta}\gamma_{AB}\ .\end{aligned} \tag{200}$$

Putting the two equations together we get

$$\begin{aligned}\partial_r\Big[-\partial_u\overset{(0)}{H}_{uA}\Big]=&-\frac{1}{r}\partial_u\Big(\gamma^{BC}D_C\partial_r(\gamma_{AB})\Big)+\frac{1}{2r^2}\partial_rD^B(r\partial_u\gamma_{AB})\\&+3\partial_u\Big(U^B\partial_r(e^{-2\beta}\gamma_{AB})\Big)+\frac{1}{8}D_A\partial_u\Big[\gamma^{EC}\gamma^{BD}(\partial_r\gamma_{EB})(\partial_r\gamma_{CD})\Big]\\&-\frac{1}{4r^2}D^B\partial_r(V\partial_r\gamma_{AB})-\frac{1}{4r^3}D^B\partial_r(V\gamma_{AB})+\frac{1}{4r^2}D^B\partial_r(r^{-1}V)\gamma_{AB}\\&+\frac{1}{2r^3}D^B\mathrm{TS}\Big[X_{AB}\Big]\ .\end{aligned} \tag{201}$$

We can merge the first two terms on the right-hand side into an $r$-derivative:

$$-\frac{1}{r}\partial_u D^B(\partial_r\gamma_{AB}) + \frac{1}{2r^2}\partial_r D^B(r\partial_u\gamma_{AB}) = -\frac{1}{2}\partial_r\Big(\frac{1}{r}D^B\partial_u\gamma_{AB}\Big) \tag{202}$$

and move it to the left-hand side

$$\begin{aligned}\partial_r\Big[-\partial_u\overset{(0)}{H}_{uA} + \frac{1}{2r}D^B\partial_u\gamma_{AB}\Big] =& \\ &+ 3\partial_u\Big(U^B\partial_r(e^{-2\beta}\gamma_{AB})\Big) + \frac{1}{8}D_A\partial_u\Big[\gamma^{EC}\gamma^{BD}(\partial_r\gamma_{EB})(\partial_r\gamma_{CD})\Big] \\ &- \frac{1}{4r^2}D^B\partial_r(V\partial_r\gamma_{AB}) - \frac{1}{4r^3}D^B\partial_r(V\gamma_{AB}) + \frac{1}{4r^2}D^B\partial_r(r^{-1}V)\gamma_{AB} \\ &+ \frac{1}{2r^3}D^B\mathrm{TS}\Big[X_{AB}\Big]\ .\end{aligned} \tag{203}$$

Now we need to go back to (193). After dividing it by $r^3$, let's multiply it with $e^{2\beta}$. After doing the same steps as before, we get

$$\begin{aligned}&\partial_r\Big[3U_A + r\partial_r U_A\Big] - rU^B\partial_r^2\gamma_{AB} - 4U^B\partial_r\gamma_{AB} - r\partial_r U^B\partial_r\gamma_{AB} - 2r\beta\partial_r\beta\gamma_{AB}\partial_r U^B \\ &+ \frac{4e^{2\beta}}{r^2}D_A\beta - \frac{2e^{2\beta}}{r}D_B\partial_r\beta + \frac{e^{2\beta}}{r}D^B(\partial_r\gamma_{AB})\Big] = 0\ ,\end{aligned} \tag{204}$$

Let's apply the operators $D^B\mathrm{TS}[D_A(\ \cdot\ )]$ on it, then multiply it with 3/8

$$\begin{aligned}&\frac{3}{8r}\partial_r\Big[3D^B\mathrm{TS}[D_AU_B] + r\partial_r D^B\mathrm{TS}[D_AU_B]\Big] \\ &= \frac{3}{8r}D^B\mathrm{TS}\Big[D_A\Big(rU^C\partial_r^2\gamma_{BC} + 4U^C\partial_r\gamma_{BC} + r\partial_r U^C\partial_r\gamma_{BC} \\ &+ 2r\beta\partial_r\beta\gamma_{BC}\partial_r U^C - \frac{4e^{2\beta}}{r^2}D_B\beta + \frac{2e^{2\beta}}{r}D_B\partial_r\beta - \frac{e^{2\beta}}{r}D^C(\partial_r\gamma_{BC})\Big)\Big]\ ,\end{aligned} \tag{205}$$

Let's rewrite the first term in the second line of the expression $X_{AB}$ in (200)

$$\gamma_{BC}D_A\partial_r(r^2U^C) = rD_AU_B + r\partial_r(rD_AU_B) - r^2(\partial_r\gamma_{BC})D_AU^C \tag{206}$$

Then take the first two terms and add the left hand side of (205) to it, which can be written in a more compact form

$$\begin{aligned}&-\frac{1}{2r^2}D^B\mathrm{TS}\Big[D_AU_B + \partial_r(rD_AU_B)\Big] \\ &+\frac{3}{8r}\partial_r\Big[3D^B\mathrm{TS}[D_AU_B] + r\partial_r D^B\mathrm{TS}[D_AU_B]\Big] \\ &\qquad = \frac{3}{8}\partial_r\Big(r^{-8/3}\partial_r(r^{8/3}D^B\mathrm{TS}[D_AU_B])\Big)\end{aligned} \tag{207}$$

Now we can put it back to the main equation

$$\begin{aligned}\partial_r\Big[-\partial_u\overset{(0)}{H}_{uA} + \frac{1}{2r}D^B\partial_u\gamma_{AB} + \frac{3}{8}r^{-8/3}\partial_r(r^{8/3}D^B\mathrm{TS}[D_AU_B])\Big] =& \\ &+ 3\partial_u\Big(U^B\partial_r(e^{-2\beta}\gamma_{AB})\Big) + \frac{1}{8}D_A\partial_u\Big[\gamma^{EC}\gamma^{BD}(\partial_r\gamma_{EB})(\partial_r\gamma_{CD})\Big] \\ &- \frac{1}{4r^2}D^B\partial_r(V\partial_r\gamma_{AB}) - \frac{1}{4r^3}D^B\partial_r(V\gamma_{AB}) + \frac{1}{4r^2}D^B\partial_r(r^{-1}V)\gamma_{AB} \\ &+ D^B\mathrm{TS}\Big[Y_{AB}\Big]\ .\end{aligned} \tag{208}$$

where $Y_{AB}$ denotes

$$
\begin{aligned}
Y_{AB} =& \frac{3}{8r} D_A\Big( rU^C\partial_r^2\gamma_{BC} + 4U^C\partial_r\gamma_{BC} + r\partial_r U^C\partial_r\gamma_{BC} + 2r\beta\partial_r\beta\gamma_{BC}\partial_r U^C \\
& - \frac{4e^{2\beta}}{r^2} D_B\beta + \frac{2e^{2\beta}}{r} D_B\partial_r\beta - \frac{e^{2\beta}}{r} D^C(\partial_r\gamma_{BC})\Big) \\
& - \frac{1}{2r^3}\Big( e^{2\beta}R[\gamma]_{AB} - 2e^{\beta}D_AD_Be^{\beta} - 2r^2(\partial_r\gamma_{BC})D_AU^C \\
& - \frac{1}{2}r^4e^{-2\beta}\gamma_{AC}\gamma_{BD}(\partial_rU^C)(\partial_rU^D) + \frac{r^2}{2}(\partial_r\gamma_{AB})(D_CU^C) \\
& + r^2U^CD_C(\partial_r\gamma_{AB}) - r^2(\partial_r\gamma_{BC})D^CU_A + r^2\Lambda e^{2\beta}\gamma_{AB}\Big) \ . \qquad (209)
\end{aligned}
$$

The first term of the third line of (208) is

$$
\frac{1}{4r^2}D^B\partial_r(V\partial_r\gamma_{AB}) = \partial_r\Big(\frac{1}{4r^2}D^B(V\partial_r\gamma_{AB})\Big) + \frac{1}{2r^3}D^B(V\partial_r\gamma_{AB}) \ , \qquad (210)
$$

therefore

$$
\begin{aligned}
&\partial_r\Big[ -\partial_u \overset{(0)}{H}_{uA} + \frac{1}{2r}D^B\partial_u\gamma_{AB} + \frac{3}{8}r^{-8/3}\partial_r(r^{8/3}D^B\mathrm{TS}[D_AU_B]) \\
&\quad + \frac{1}{4r^2}D^B\partial_r(V\partial_r\gamma_{AB})\Big] = \\
&\qquad + 3\partial_u\Big(U^B\partial_r(e^{-2\beta}\gamma_{AB})\Big) + \frac{1}{8}D_A\partial_u\Big[\gamma^{EC}\gamma^{BD}(\partial_r\gamma_{EB})(\partial_r\gamma_{CD})\Big] \\
&\qquad - \frac{3}{4r^3}D^B(V\partial_r\gamma_{AB}) - \frac{1}{4r^4}D_A\partial_rV + D^B\mathrm{TS}\Big[Y_{AB}\Big] \ . \qquad (211)
\end{aligned}
$$

Now let's consider the second term in the second line of $Y_{AB}$ in (209). We want to include the term in the charge, so we separate a total $r$-derivative part of it. To do that, we rewrite the term as

$$
\begin{aligned}
&\frac{3}{4r^2}D^B\mathrm{TS}\Big[e^{2\beta}D_B\partial_r\beta\Big] = \partial_r\Big(\frac{3}{4r^2}D^B\mathrm{TS}\Big[D_A(e^{2\beta}D_B\beta)\Big]\Big) \\
&-\frac{3}{2r^3}D^B\mathrm{TS}\Big[D_A(e^{2\beta}D_B\beta)\Big] - \frac{3}{4r^2}D^B\mathrm{TS}\Big[D_A(\partial_r(e^{2\beta})D_B\beta)\Big] \ , \qquad (212)
\end{aligned}
$$

of which the first term is

$$
\begin{aligned}
&\partial_r\left(\frac{3}{4r^2}D^B\mathrm{TS}\Big[D_A(e^{2\beta}D_B\beta)\Big]\right) = \partial_r\left(\frac{3e^{2\beta}}{4r^2}D^B\mathrm{TS}\Big[D_AD_B\beta\Big]\right. \\
&\qquad \left. + \frac{3e^{2\beta}}{2r^2}D^B\mathrm{TS}\Big[D_A\beta D_B\beta\Big] + \frac{3}{2r^2}\Big[(D^B\beta)(D_Ae^{\beta}D_Be^{\beta})\Big]\right) \qquad (213)
\end{aligned}
$$

Thus the final equation becomes

$$
\begin{aligned}
&\partial_r\Big( -\partial_u \overset{(0)}{H}_{uA} + \frac{1}{2r}D^B\partial_u\gamma_{AB} + \frac{3}{8}r^{-8/3}\partial_r(r^{8/3}D^B\mathrm{TS}[D_AU_B]) \\
&\quad + \frac{1}{4r^2}D^B\partial_r(V\partial_r\gamma_{AB}) - \frac{3e^{2\beta}}{4r^2}D^B\mathrm{TS}\Big[D_AD_B\beta\Big]\Big) = \\
&\qquad + 3\partial_u\Big(U^B\partial_r(e^{-2\beta}\gamma_{AB})\Big) + \frac{1}{8}D_A\partial_u\Big[\gamma^{EC}\gamma^{BD}(\partial_r\gamma_{EB})(\partial_r\gamma_{CD})\Big] \\
&\qquad - \frac{3}{4r^3}D^B(V\partial_r\gamma_{AB}) - \frac{1}{4r^4}D_A\partial_rV + \partial_r\Big(\frac{3}{2r^2}(e^{2\beta}D^B\mathrm{TS}\Big[D_A\beta D_B\beta\Big] \\
&\qquad + (D^B\beta)(D_Ae^{\beta}D_Be^{\beta}))\Big) + D^B\mathrm{TS}\Big[Z_{AB}\Big] \ , \qquad (214)
\end{aligned}
$$

with

$$
\begin{aligned}
Z_{AB} =& \frac{3}{8r} D_A\Big( rU^C \partial_r^2 \gamma_{BC} + 4U^C \partial_r \gamma_{BC} + r\partial_r U^C \partial_r \gamma_{BC} + 2r\beta \partial_r \beta \gamma_{BC} \partial_r U^C \\
& - \frac{17e^{2\beta}}{4r^2} D_B \beta - \frac{1}{2r^2}\partial_r(e^{2\beta}) D_B\beta - \frac{e^{2\beta}}{r} D^C(\partial_r \gamma_{BC})\Big) - \frac{1}{2r^3}\Big( e^{2\beta} R[\gamma]_{AB} \\
& - 2e^{\beta} D_A D_B e^{\beta} - 2r^2(\partial_r \gamma_{BC}) D_A U^C - \frac{1}{2} r^4 e^{-2\beta} \gamma_{AC}\gamma_{BD}(\partial_r U^C)(\partial_r U^D) \\
& + \frac{r^2}{2}(\partial_r \gamma_{AB})(D_C U^C) + r^2 U^C D_C(\partial_r \gamma_{AB}) - r^2(\partial_r \gamma_{BC}) D^C U_A \\
& + r^2 \Lambda e^{2\beta} \gamma_{AB}\Big) \ .
\end{aligned}
\tag{215}
$$

After the long derivation we managed to obtain the non-linear form of the charge, which is

$$
\begin{aligned}
\overset{(1)}{H}_{uA} :=& - \partial_u \overset{(0)}{H}_{uA} + \frac{1}{2r} D^B \partial_u \gamma_{AB} + \frac{3}{8} r^{-8/3} \partial_r (r^{8/3} D^B \mathrm{TS}[D_A U_B]) \\
& + \frac{1}{4r^2} D^B \partial_r (V \partial_r \gamma_{AB}) - \frac{3e^{2\beta}}{4r^2} D^B \mathrm{TS}\Big[ D_A D_B \beta \Big]
\end{aligned}
\tag{216}
$$

After an asymptotic series expansion it becomes

$$
\begin{aligned}
\overset{(1)}{H}_{uA} = \Bigg[ & \frac{1}{16}[\mathring{D}_A, \mathring{D}_C]\mathring{D}_B C^{BC} + \frac{1}{2}\mathring{D}_A(C^{BC}N_{BC}) - \frac{1}{4}\mathring{D}^B C_{AB} - \frac{1}{2}\mathring{D}_B(C^{BC}N_{AC}) \\
& - \frac{1}{16}\mathring{D}^C \mathring{D}_C \mathring{D}^B C_{AB} - \frac{1}{2} C_{AB}\mathring{D}_C N^{BC} \Bigg] r^{-3} + \mathcal{O}(r^{-4}) \ ,
\end{aligned}
\tag{217}
$$

where $N_{AB} = \partial_u C_{AB}$ is the Bondi news tensor. In the first term a commutator of the covariant derivatives appear, which would give us the curvature tensor associated with $\mathring{h}_{AB}$

$$
[\mathring{D}_A, \mathring{D}_C]\mathring{D}_B C^{BC} = \mathring{R}^C{}_{DAC}\mathring{D}_B C^{BD} = -\mathring{R}_{DA}\mathring{D}_B C^{BD} = -\mathring{D}^B C_{AB} \ , \tag{218}
$$

where in the last line we used that on the 2-sphere $\mathring{R}_{DA} = \mathring{h}_{DA}$.

Using the identities (C7a-b) from [10]

$$
\begin{aligned}
(\mathring{D}_A C^{BC})N_{BC} &= (\mathring{D}_B C_{AC})N^{BC} + N_{AB}(\mathring{D}_C C^{BC}) \ , \\
(\mathring{D}_A N_{BC})C^{BC} &= (\mathring{D}_B N_{AC})C^{BC} + C_{AB}(\mathring{D}_C N^{BC}) \ ,
\end{aligned}
\tag{219}
$$

after rewriting the first term in (217) as $\mathring{D}_A(C^{BC}N_{BC}){=}(\mathring{D}_A C^{BC})N_{BC}{+}C^{BC}(\mathring{D}_A N_{BC})$, the charge simplifies to

$$
\overset{(1)}{H}_{uA} = \left[ \frac{1}{2} N^{BC}\mathring{D}_B C_{AC} - \frac{1}{16}\Big(\mathring{D}^C\mathring{D}_C + 5\Big)\mathring{D}^B C_{AB} \right] r^{-3} + \mathcal{O}(r^{-4}) \ . \tag{220}
$$

Here we separated the expression into two parts. The first is a non-linear contribution, containing also the news tensor, and the second, linear part is expressed as a differential operator $(\mathring{D}^2 + 5)$ acting on the divergence $\mathring{D}^B C_{AB}$. This second term vanishes when integrating its product with a vector spherical harmonic of $\ell = 1, 2$ mode. The divergence annihilates the $\ell = 1$ modes in $C_{AB}$, and the operator $(\mathring{D}^2{+}5)$ annihilates the $\ell = 2$ modes in $\mathring{D}^B C_{AB}$.

To prove this, first let's recall the eigenvalue equation of the spherical harmonics

$$\mathring{D}^2 Y_{\ell,m} = -\ell(\ell+1) Y_{\ell,m} \ . \tag{221}$$

Then take the $\mathring{D}_A$ derivative of both sides. On the left hand side, after commuting the covariant derivatives in order to move $\mathring{D}_A$ to the right end, we have

$$\mathring{D}_A \mathring{D}^2 \mathring{D}_A Y_{\ell,m} = \mathring{D}^2 \mathring{D}_A Y_{\ell,m} - \mathring{h}^{BC} \mathring{R}_{DCAB} \mathring{D}^D Y_{\ell,m} = (\mathring{D}^2 - 1) \mathring{D}_A Y_{\ell,m} \ . \tag{222}$$

Returning to the eigenvalue equation, after adding $6\mathring{D}_A Y_{\ell,m}$ to both sides we obtain

$$(\mathring{D}^2 + 5) \mathring{D}_A Y_{\ell,m} = (6 - \ell(\ell+1)) \mathring{D}_A Y_{\ell,m} \ . \tag{223}$$

With that we derived an equation of the differential operator appearing in the charge acting on the electric type vector spherical harmonics. The right hand side of the equation is zero if

$$6 - \ell(\ell+1) = 0 \qquad \rightarrow \qquad \ell = 2 \ , \tag{224}$$

which means that the operator $(\mathring{D}^2 + 5)$ annihilates the $\ell = 2$ modes. Thus, integrating the second term of (220) multiplied with an $\ell = 2$ spherical harmonic is zero due to orthogonality.

Finally, let's integrate $\overset{(1)}{H}_{uA}$ multiplied with an $\ell = 1, 2$ spherical harmonic vector $\mu^A$ over $S^2$. The only part that leads to a non-trivial result is

$$\mathcal{Q}_1(\mu^A) = \frac{1}{2r^3} \int d^2\Omega \ \mu^A \Big( N^{BC} \mathring{D}_B C_{AC} \Big) + \mathcal{O}(r^{-4}) \ . \tag{225}$$

With that we obtained the asymptotic, gauge-dependent charge associated with the $C^3$ gluing problem.

# 10 Summary and outlook

In this thesis I investigated the characteristic gluing problem in asymptotically flat spacetimes and its relation to conserved quantities in general relativity. The main goal was to examine the obstructions that arise in the gluing problem and to interpret them as physically meaningful charges. In this final chapter I summarize the main results of the thesis and discuss briefly what might be the next steps in this research project.

In Sections 2–5 I laid down the basic framework and formalisms used in the subsequent calculations. Most importantly I introduced four well-known ADM charges, $E_{ADM}$, $P_{ADM}$, $L_{ADM}$ and $C_{ADM}$ with concrete physical meaning, and two families of radial charges, $\overset{[1]}{Q}$ and $\overset{[2]}{Q}$, which arose as obstructions of the characteristic gluing.

I showed in Section 6, that it is possible to relate the radial charges to the ADM charges thus equipping them with physical meaning. Asymptotically the charge $\lim_{r\to\infty} \overset{[2]}{Q}(\lambda = 1)$ is proportional to $E_{ADM}$, therefore it can be interpreted as an energy-type charge. The linearized radial charge $\overset{[1]}{Q}(H^A_{(\ell,m_a)})$ is proportional to $L_{ADM}$, thus it quantifies the total angular momentum of a spacetime at the linearized level.

In Section 7 I calculated the exact values of the charges in Lense–Thirring spacetime. I found that

$$\begin{aligned}
&E_{ADM} = M \\
&(P_{ADM})_x = 0\ , \quad (P_{ADM})_y = 0\ , \quad (P_{ADM})_z = 0\ , \\
&(L_{ADM})_r = 0\ , \quad (L_{ADM})_\theta = 0\ , \quad (L_{ADM})_\varphi = aM\ , \\
&(C_{ADM})_r = 0\ , \quad (C_{ADM})_\theta = 0\ , \quad (C_{ADM})_\varphi = 0\ ,
\end{aligned} \tag{226}$$

and for the radial charges

$$\begin{aligned}
\overset{[1]}{Q}(H^A_{(1,0)}) &= -4\sqrt{6\pi}\ aM\ , \qquad \overset{[1]}{Q}(E^A_{(1,m)}) = 0\ , \\
\overset{[2]}{Q}(1) &= -8\pi M, \qquad \overset{[2]}{Q}(Y_{(\ell\geq 1,m)}) = 0\ .
\end{aligned} \tag{227}$$

In Section 8, after discussing the basics of the BMS symmetry group and the associated charges, I rederived the formula (3.5) of [10], providing more details on the intermediate steps. I found that the linearized charge associated with the generators of BMS symmetries is also proportional to the radial charge $\overset{[1]}{Q}(Y^A)$ in the $\delta\beta = 0$ gauge.

In Section 9 I present the derivation of additional conserved quantities arising from the null constraint equations. Most of these derivations (in Sections 9.1–9.3) are based on existing results from the literature. The main result of the section is the non-linear derivation in Section 9.5. With that I obtained the non-linear form of the quantity $\overset{(1)}{H}_{uA}$, recovering the information that has been disregarded by the linearization process.

Despite this thesis being concluded, there are still many interesting aspects of this research topic. One of the next challenges, that I already started working on, is investigating the physical meaning of the non-linear $\overset{(1)}{H}_{uA}$ charge. It could be related to subleading BMS charges or Newman-Penrose charges, which are the conserved charges of asymptotically flat spacetimes with linearized gravity, even if a non-zero flux is present at null-infinity [14]. Furthermore, it might be possible that with different arguments the radial charges in the asymptotic large $r$ limit could be also proportional to the ADM momentum and center of mass, namely $\lim_{r\to\infty} \overset{[2]}{Q}(Y^{(1,m_a)}) \propto (P_{ADM})^a$ and $\lim_{r\to\infty} \overset{[1]}{Q}(E^{(1,m_a)}) \propto (C_{ADM})^a$. These relations are suspected based on the analysis of e.g. [3], where similar obstructions were studied.

The characteristic gluing problem remains closely connected to fundamental questions about gravitational radiation, asymptotic symmetries and the global structure of solutions to the Einstein equations. Understanding these conserved quantities more deeply may therefore provide further insight into both the mathematical structure and the physical interpretation of general relativity.

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