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The gradient flow of the fractional area functional: Well-posedness and nonlocal-to-local convergence

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Abstract

In der vorliegenden Masterarbeit wird der L^2-Gradientenfluss des klassischen Flächenfunktionals sowie des s-fraktionalen Flächenfunktionals untersucht. Das erste zentrale Ziel besteht darin, zu beweisen, dass die Evolutionsgleichungen der Gradientenflüsse dieser Funktionale wohlgestellt sind. Im Rahmen der vorliegenden Arbeit wird zunächst ein Überblick über die klassische Theorie nichtlinearer Halbgruppen für konvexe, untenstetige Funktionale gegeben. Im weiteren Verlauf werden sowohl das klassische als auch das fraktionale Flächenfunktional vorgestellt und analysiert. Besonderes Augenmerk wird auf die strukturellen Eigenschaften des fraktionalen Flächenfunktionals gelegt, darunter Konvexität, untere Halbkontinuität und eine Charakterisierung seines Definitionsbereichs. Des Weiteren wird das abgeschnittene fraktionale Flächenfunktional untersucht, wobei unter geeigneten Annahmen des äußeren Datums die untere Halbkontinuität bewiesen wird. Diese Eigenschaften ermöglichen die Anwendung des Brezis–Komura Theorems, um die Existenz und Eindeutigkeit von Lösungen des entsprechenden Gradientenflusses zu beweisen. Das zweite zentrale Ziel besteht in der Analyse des nichtlokal-zu-lokalen Grenzwerts für den fraktionalen Parameter $s \nearrow 1$. Unter Verwendung von Trunkierungstechniken und der Beziehung zwischen dem fraktionalen und dem klassischen Perimeter wird bewiesen, dass ein geeignet skaliertes fraktionales Flächenfunktional punktweise und im Mosco-Sinne gegen das klassische Flächenfunktional konvergiert. Dies impliziert die gleichmäßige Konvergenz der entsprechenden Gradientenflusslösungen. Zuletzt wird der Gradientenfluss mit Hilfe des WED-Funktionals (Weighted Energy Dissipation) neu formuliert, wodurch eine zeitlich globale Variationsperspektive eingenommen wird. Es werden die Konvergenz zur Lösung des Gradientenflusses und die nichtlokal-zu-lokale Konvergenz des Minimierungsproblems gezeigt.

Abstract

The present thesis investigates the L^2-gradient flow associated with both the classical area functional and its s-fractional counterpart. The first key objective is to ascertain the well-posedness of the gradient flow equations associated with these functionals within the framework of gradient flows in Hilbert spaces. Following a review of the classical nonlinear semigroup theory for convex, lower semicontinuous functionals, the classical area functional and the fractional area functional are introduced and analyzed. Particular attention is given to the structural properties of the fractional area functional, including convexity, lower semicontinuity, and a characterization of its domain. In addition, the truncated, fractional area functional is investigated, and lower semicontinuity under suitable assumptions on the exterior datum is proven. The aforementioned properties permit the application of the Brezis–Komura theorem to prove well-posedness of the corresponding gradient flow. The second key objective is to prove the nonlocal-to-local limit as the fractional parameter $s \nearrow 1$. Utilising truncation techniques and the relationship between the fractional and classical perimeter, it is demonstrated that a suitably rescaled fractional area functional converges to the classical area functional pointwise and in the Mosco sense. This implies the uniform convergence of the corresponding gradient flow solutions. Finally, the gradient flow is reformulated using the weighted energy dissipation (WED) functional, providing a global-in-time variational perspective. The convergence to the gradient flow solutions, and the nonlocal-to-local convergence for the minimization problem are proven.

1 Introduction

In this thesis, the L^2 gradient flow equation for both the classical and the fractional-area functional is considered. The first goal is to establish well-posedness for the flows. We then provide conditions for nonlocal-to-local convergence of solutions. Lastly, a variational, global-in-time reformulation will be considered. The setting is the following: Suppose that $\Omega \subset \mathbb{R}^n$ is an open, bounded domain with Lipschitz boundary. The functionals $\psi_s : L^2(\Omega) \to (-\infty, +\infty]$ denote the area functionals depending on the parameter $s \in (0,1]$ and will be introduced in the forthcoming. We are interested in finding a solution u_s that fulfills the following gradient flow evolution equation

$$\begin{cases} u_s' + \partial\psi_s(u) \ni 0 \quad \text{for a.e. } t \in [0,T] \ , \\ u_s(0) = u_0^s \ , \\ u_s = \varphi \quad \text{on } \mathcal{O}_s \ , \end{cases} \tag{1.0.1}$$

with the initial condition $u_0^s \in \overline{D(\psi_s)} := \{u \in L^2(\Omega) : \psi_s(u) < \infty\}$, the boundary datum φ defined on a set $\mathcal{O}_s \subseteq \mathcal{C}\Omega$ depending on its local or nonlocal nature and the classical convex subdifferential $\partial\psi_s$. In this context, the key ingredient for gradient flows in Hilbert spaces is the Riesz isomorphism. This allows the dual space to be identified with the Hilbert space itself, enabling movement according to the gradient to be understood, as shown in equation Equation (1.0.1). Chapter 2 is dedicated to review the classical theory on gradient flows for convex, nonlinear functions on Hilbert spaces, recalling the important notions from the nonlinear semigroup theory, especially the Brezis–Kōmura theorem which guarantees the existence and well-posedness of solutions. For a more general discussion on gradient systems, see [Mie23].

In Chapter 3, the functionals are introduced, which depend on the fractional parameter $s \in (0,1]$. This parameter distinguishes two fundamentally different cases. $s = 1$ is the realm of the classical, local area functional, which is formally given for continuously differentiable functions by

$$\mathcal{A}(u,\Omega) = \int_\Omega \sqrt{1 + |\nabla u|^2}\, dx \ . \tag{1.0.2}$$

The evolution problems arising from the area functional have been a classical topic, giving rise to the time-dependent minimal surface equation

$$u_t - \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0$$

which corresponds to the nonparametric Plateau problem in the steady-state case. Yet, this problem in general admits no classical solution unless the boundary $\partial\Omega$ is essentially of nonnegative curvature, see [ET74] or [LT78].
The problem is that the functional $\mathcal{A}$, considered for functions in the Sobolev space $W^{1,1}(\Omega)$, is not lower semicontinuous with respect to L^1-convergence. For example, approximating the characteristic function of a set with bounded perimeter leads to a degenerate sequence.
In order to gain lower semicontinuity, the relaxation $\underline{\mathcal{A}} =: \psi_1$ is considered, which naturally leads to the space of functions of bounded variation as $D(\underline{\mathcal{A}}) = \mathrm{BV}(\Omega)$. This is due to the linear growth of the integrand, which features a L^1-bound on the gradient. The theory on the area functional is recalled in Section 3.2, together with the definition of the area functional in terms of measures. Given the lower semicontinuity, the existence of a gradient flow solution follows by application of the classical nonlinear semigroup theory. In case of the local area functional, the boundary condition has to be provided only locally on the boundary $\partial\Omega =: \mathcal{O}_1$. The boundary condition may be imposed in the trace sense for any $\varphi \in L^1(\partial\Omega, \mathcal{H}^{n-1})$. Yet, by relaxation, the boundary condition is incorporated into the definition of the functional, see Lemma 3.2.5.
$s \in (0,1)$ is the realm of the fractional or nonlocal area functional. The s-fractional area functional has been introduced in a paper by Cozzi, Lombardini [CL20] to define a functional closely related to the fractional or nonlocal perimeter. The fractional perimeter has been introduced by Caffarelli, Roquejoffre, Savin [CRS09] in a seminal paper and has been a very active field of research since then. The s-fractional area for $s \in (0,1)$ and $u : \mathbb{R}^n \to \mathbb{R}$ is given by

$$\psi_s(u) := \mathcal{F}_s(u,\Omega) = \iint_{\mathcal{Q}(\Omega)} \mathcal{G}_s\left(\frac{u(x)-u(y)}{|x-y|}\right)\frac{dx\ dy}{|x-y|^{n-1+s}}\ ,$$

where $\mathcal{Q}(\Omega) := (\mathbb{R}^n)^2 \setminus (\mathcal{C}\Omega)^2$ and $\mathcal{C}\Omega := \mathbb{R}^n \setminus \Omega$, the precise definition of $\mathcal{G}_s$ will be recalled in Section 3.4. This provides an initial insight into the conceptual differences between locality and non-locality with regard to the notion of area. While the local area in a region only depends on the features of the function in the region itself, the nonlocal area also takes the features outside of the region into account by an appropriate weight depending on the distance between points x and y, the dimension n, and the degree s. Therefore, the outside datum has to be defined on the whole complement $\mathcal{C}\Omega =: \mathcal{O}_s$ and is a measurable function $\varphi : \mathcal{C}\Omega \to \mathbb{R}$, which will be regarded as a boundary condition throughout this thesis.
The domain of the fractional area functional is the fractional Sobolev space $W^{s,1}(\Omega)$, yet the finiteness of the fractional area $\mathcal{F}_s$ also depends on the outside datum φ. For this reason, also a truncated version of the fractional area functional $\mathcal{F}_s^M$ with $M > 0$ will be introduced in Definition 3.4.2, which is finite for any fractional Sobolev function independent of the exterior datum. In Chapter 4, a careful investigation of the fractional area functionals follows, which will largely depend on the two different Assumptions (A1) and (A2) for the boundary datum φ, corresponding to integrability on $\mathcal{C}\Omega$ and a boundedness condition for a suitable large neighborhood of Ω. While Assumption (A1) is nec-

essary for the fractional area functional $\mathcal{F}_s$ to be proper, the Assumption (A2) enables an energy inequality in terms of trunctations. It is proved here for the first time that under Assumption (A2) the truncated fractional-area functional $\mathcal{F}_s^M$ is lower semi-continuous on $W^{s,1}(\Omega)$. Lastly, the subdifferentials are investigated, yielding equality if φ fulfills both assumptions and exploring the relation to the nonlocal curvature operator $\mathscr{H}$. It is proved that an element in the subdifferential at u is equal to $\mathscr{H}u$ in distributions.
Comparable to the local area functional, the existence and well-posedness of the gradient flow for the fractional area functional is guaranteed by the nonlinear semigroup theory. The essential properties of convexity and lower semicontinuity are proved in Chapter 4.
The general idea of the nonlocal perimeter is to serve as an interpolation between area and perimeter, with nonlocal-to-local convergence to the classical perimeter as $s \to 1$. There have been proofs for different modes of this convergence: The pointwise convergence was proved in [Lom16a]. The Γ-convergence of the perimeter was proved in [APM10]. The goal of Chapter 5 is to prove Mosco convergence of the fractional area functional, based on this Γ-convergence result. For this analysis, we introduce two new Assumptions (A3) and (A4), which can be seen as a decomposition of Assumption (A1). Assumption (A3) implies that the datum φ has bounded variation locally around Ω. Assumption (A4) is only concerned with the integrability of φ at infinity. Together, the two assumptions imply that assumption (A1) holds for a range $[s_0, 1)$. For φ fullfilling the Assumptions (A2) to (A4), we prove the Mosco-convergence, namely

$$\mathfrak{M}\text{-}\lim_{s \nearrow 1}(1-s)\mathcal{F}_s(u,\Omega,\varphi) = \omega_n\big(\underline{\mathcal{A}}(u,\Omega,\varphi^-) - |\Omega|\big)$$

with ω_n the volume of the n dimensional sphere, $|\Omega|$ the Lebesgue measure of Ω, and φ^- the trace of φ from outside. On the way, also the pointwise convergence

$$\lim_{s \nearrow 1}(1-s)\mathcal{F}_s(u,\Omega,\varphi) = \omega_n\big(\underline{\mathcal{A}}(u,\Omega,\varphi^-) - |\Omega|\big)$$

for $u \in \mathrm{BV}(\Omega)$ is proved. Combining this result with the classical theory by Attouch [Att84], the uniform convergence of gradient flow solutions for $s \nearrow 1$ is the final nonlocal-to-local convergence Theorem 5.3.6.

Finally, in Chapter 6, a variational reformulation of the gradient flow is considered. The weighted energy dissipation (WED) has been proposed by Ortiz, Mielke [OM08] and it results in a global-in-time functional. This family of functionals combines the scaled energy and the dissipation by integrating in time with an exponentially decaying weight, defined on the space of trajectories. Applied to the functionals $\mathcal{F}_s, \underline{\mathcal{A}}$ defined on L^2, the WED functional $W_s^\varepsilon : H^1(0,T;L^2(\Omega)) \to (-\infty,+\infty]$ for $\varepsilon > 0$ reads as follows

$$W_s^\varepsilon(u) := \int_0^T e^{-t/\varepsilon}\left(\frac{1}{2}|u'|_{L^2}^2 + \frac{1}{\varepsilon}\psi_s(u)\right) dt \quad . \tag{1.0.3}$$

The general goal is to prove the existence of minimizers u_ε at level $\varepsilon > 0$ and to take the causal limit $\varepsilon \to 0$ to recover a solution for the gradient flow. The general case has been

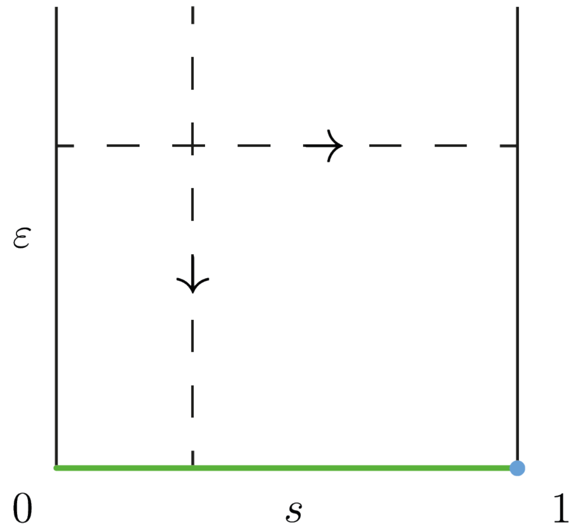

Figure 1.1: The parameter space: In green, the level of the gradient flow and the blue dot marking the point of the gradient flow of the classical, local-area functional

treated by Mielke, Stefanelli [MS11] who proved in the case of a lower semicontinuous, convex, and proper functional that there exists a unique minimizers u_ε to the functional W_ε with an appropriate approximation $u_{0\varepsilon}$ for the initial starting point $u_0 \in D(\psi_s)$ and the minimisers converge uniformly in the Hilbert space to the solution of the gradient flow.
Using this result, the convergence of both the WED functional for the local and nonlocal area functional to the aforementioned gradient flow solutions is obtained. Returning to the initial discussion of the area functional, an alternative approach is also possible. The WED functional of the classical area functional as in Equation (1.0.2) can be defined. Like the area functional itself, it is not lower semicontinuous. A different approach to this obstruction is to relax the WED functional of the classical area functional, which does not a priori coincide with the WED of the relaxation. Yet, as Spadaro, Stefanelli [SS11] have proved, in the case of the local-area functional, both approaches eventually coincide.
Finally, the Γ-convergence of the WED functionals for a level $\varepsilon > 0$

$$\Gamma\text{-}\lim_{s\to 1} W_s^\varepsilon = W_1^\varepsilon$$

is proved. This enables a convergence result for the minimizers u_ε weakly in $H^1(0,T;H)$ for an initial condition $u_0 \in \mathrm{BV}(\Omega)$.
Therefore, in the course of the thesis, a complete picture of the parameter space is obtained, which is symbolized in Figure 1.1.

2 Gradient flows in Hilbert spaces

In this chapter, the classical nonlinear semigroup theory for gradient flows is reviewed, with a focus on providing conditions for well-posedness and convergence of gradient flows. This will provide the framework used for the gradient flow of the area functionals.

2.1 Functions on Hilbert spaces

This chapter recalls the most important properties of functions on Hilbert spaces. The properties introduced here are closely related to the existence of minimizers. This does not come as a surprise, as the gradient flow is designed in such a way that it gradually minimizes the function values. Especially, proving existence via time-discretization and minimizing movements builds on the existence of closeby minimizers combining energy decay and dissipation.

The setting for this chapter is a Hilbert space H with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Consider a function $\psi : H \to (-\infty, +\infty]$ with domain

$$D(\psi) := \{x \in H \mid \psi(x) < \infty\} \; .$$

Throughout this thesis, for a sequence $x_n \in H$ converging strongly in H-norm to $x \in H$, the notation $x_n \to x$ will be adapted, while convergence with respect to the weak topology will be denoted by $x_n \rightharpoonup x$.

Definition 2.1.1 (Convexity)**:** *The function ψ is said to be convex if*

$$\psi\big(\alpha x + (1-\alpha)y\big) \leq \alpha\psi(x) + (1-\alpha)\psi(y) \qquad \forall x, y \in H, \alpha \in [0,1]$$

and strictly convex if the inequality holds strictly for all $\alpha \in (0,1)$.

Furthermore, recall the definition of λ-convexity, which leads to a stricter notion for $\lambda > 0$ and a weaker one for $\lambda < 0$.

Definition 2.1.2 (λ-convexity)**:** *For $\lambda \in \mathbb{R}$, the function ψ is said to be λ-convex if $\psi - \frac{\lambda}{2}\|\cdot\|^2$ is convex.*

Definition 2.1.3 (Lower semicontinuity)**:** *The function ψ is said to be lower semicontinuous with respect to a topology τ if for every sequence $x_n \in H$ converging to $x \in H$ in τ it holds*

$$\psi(x) \leq \liminf_{n\to\infty} \psi(x_n) \; .$$

Definition 2.1.4 (Properness)**:** *The function ψ is said to be proper if $D(\psi) \neq \emptyset$.*

The convex subdifferential extends the idea of regularity given by tangent planes at a point of the graph of the function to Hilbert spaces.

Definition 2.1.5 (Subdifferential)**:** *The element $p \in H$ is said to be in the subdifferential of ψ at $x \in H$ if $x \in D(\psi)$ and*

$$\psi(y) - \psi(x) \geq \langle p, y - x \rangle \qquad \forall y \in H \ .$$

Denote the subdifferential at x by $\partial\psi(x)$ and its domain $D(\partial\psi) \subseteq D(\psi)$ is defined as the set of all points x such that $\partial\psi(x) \neq \emptyset$. An element $p \in \partial\psi(x)$ is called a subgradient of ψ at x.

For general convex functions, the existence of subgradients is a priori implied for points in the interior of the domain. Yet, the Brøndsted–Rockafellar Theorem [BR65] guarantees the density of the domain of the subdifferential in general, that is

$$D(\partial\psi) \subseteq D(\psi) \subseteq \overline{D(\partial\psi)} \ .$$

Lastly, the notion of Γ-convergence and the stronger form of Mosco-convergence are recalled, which are designed in a way to prove the convergence of minimizers.

Definition 2.1.6 (Γ-Convergence)**:** *A sequence of functions $\psi_n : H \to (-\infty, +\infty]$ Γ-converges to $\psi : H \to (-\infty, +\infty]$ with respect to a topology τ if the following to conditions are fullfilled:*

1. *$\Gamma\text{-}\liminf$-inequality: For every sequence $x_n \to x$ in τ it holds*

$$\liminf_{n\to\infty} \psi_n(x_n) \geq \psi(x) \ .$$

   In this case, we write $\Gamma\text{-}\liminf_{n\to\infty} \psi_n \geq \psi$.

2. *$\Gamma\text{-}\limsup$-inequality: For every $x \in H$ there exists a sequence $x_n \to x$ in τ such that*

$$\limsup_{n\to\infty} \psi_n(x_n) \leq \psi(x) \ .$$

   In this case, we write $\Gamma\text{-}\limsup_{n\to\infty} \psi_n \leq \psi$.

If both conditions hold, we write $\psi_n \xrightarrow{\Gamma} \psi$ or $\Gamma\text{-}\lim_{n\to\infty} \psi_n = \psi$ with respect to τ.

Mosco-convergence corresponds to Γ-convergence with respect to both the strong and weak topology.

Definition 2.1.7 (Mosco-Convergence)**:** *A sequence of functions $\psi_n : H \to (-\infty, +\infty]$ Mosco-converges to $\psi : H \to (-\infty, +\infty]$ if it Γ-converges both in the strong and the weak topology. This is*

1. *For every weakly convergent sequence $x_n \rightharpoonup x$ in H it holds*

$$\liminf_{n\to\infty} \psi_n(x_n) \geq \psi(x) \ .$$

2. For every $x \in H$ there exists a strongly convergent sequence $x_n \to x$ in H such that

$$\limsup_{n\to\infty} \psi_n(x_n) \leq \psi(x) .$$

If both conditions hold, we write $\psi_n \xrightarrow{\mathfrak{M}} \psi$ or $\mathfrak{M}\text{-}\lim_{n\to\infty} \psi_n = \psi$ in H.

2.2 Hilbert-space valued paths

Besides the differential of functions on Hilbert spaces, the second essential ingredient for gradient flows is the notion of paths in the Hilbert space and their time derivative. For a more complete overview of the theory of Banach-space-valued paths and their regularity, see Appendix 7.2.

Firstly, we consider strongly differentiable paths

Definition 2.2.1 (Strong differentiability)**:** *Let $E \subset \mathbb{R}$ and $u : E \to H$. We say that u is strongly differentiable at $t_0 \in E$ an accumulation point of E if there exists $x \in H$, such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\left\| \frac{u(t) - u(t_0)}{t - t_0} - x \right\| < \varepsilon$$

for all $t \in E$ with $0 < |t - t_0| \leq \delta$. The element x is called the derivative of u at t_0 and is denoted by $u'(t_0)$.

Furthermore, the notion of weak differentiability comparable to the Sobolev spaces on $\mathbb{R}^n$ is introduced. Here, the integrals are defined in the Bochner sense.

Definition 2.2.2 (Weak differentiability)**:** *Let Y be a Banach space, let $I \subset \mathbb{R}$ be an open interval and let $u : I \to Y$ be locally integrable, i.e. $u \in L^1_{\text{loc}}(I; H)$. We say that u is weakly differentiable if there exists $v \in L^1_{\text{loc}}(I; H)$ such that*

$$\int_I u(t)\varphi'(t)\, dt = -\int_I v(t)\varphi(t)\, dt$$

for every $\varphi \in C^1_c(I)$. The function v is called the weak derivative of u and is denoted by u'.

Then the Bochner–Sobolev space on an interval is defined as follows.

Definition 2.2.3 (Sobolev spaces with values in H)**:** *Let $I \subset \mathbb{R}$ be an open interval. For $1 \leq p \leq \infty$ define*

$$W^{1,p}(I; H) := \left\{ u \in L^p(I; H) \ : \ u \text{ is weakly differentiable and } u' \in L^p(I; H) \right\} ,$$

and in case $p = 2$ write $H^1(I; H) := W^{1,2}(I; H)$. The space $W^{1,p}(I; H)$ is endowed with the norm

$$\|u\|_{W^{1,p}(I;H)} := \|u\|_{L^p(I;H)} + \|u'\|_{L^p(I;H)} .$$

From the discussion in Section 7.2, we may conclude the following.

Lemma 2.2.4: *For an open interval $I = (a, b) \subset \mathbb{R}$ and $u \in W^{1,1}(I; H)$ there exists a representative of u that is continuous on $\overline{I}$ and that is strongly differentiable for almost every $t \in I$. It holds*

$$u(t) - u(s) = \int_s^t u'(\tau)\, d\tau \qquad \forall a \le s \le t \le b \,,$$

where $u'(t)$ is the weak derivative which coincides with the strong one for almost every τ.

2.3 Well-posedness and Convergence of gradient flows

Now, the gradient flow problem in a Hilbert space H and the notions of solutions can be introduced. For a function $\psi : H \to (-\infty, +\infty]$ a final time $T > 0$ and an initial condition $u_0 \in \overline{D(\partial\psi)} = \overline{D(\psi)}$ the gradient flow is the following initial value problem

$$\begin{cases} u' + \partial\psi(u) \ni 0 & \text{a.e. in } (0, T) \,, \\ u(0) = u_0 \,. \end{cases} \tag{2.3.1}$$

A curve $u : [0, T] \to H$ is a **(strong) solution** of Equation (2.3.1) if u is continuous on $[0, T]$ with $u \in \mathrm{AC}_{\mathrm{loc}}(0, T; H)$ and hence strongly differentiable for almost every $t \in (0, T)$, $u(t) \in D(\partial\psi)$ a.e. in $(0, T)$, and the Equation (2.3.1) holds for almost every $t \in (0, T)$.

The well-posedness of solutions has been proved by Brezis and Kōmura. This resulted in the nonlinear semigroup theory, which is the analogon to the Hille–Yosida Theorem for linear operators.

Theorem 2.3.1 (Brezis–Kōmura)**:** *Let $\psi : H \to (-\infty, +\infty]$ be proper, convex, and lower semicontinuous. Then, it holds*

1. *For every $u_0 \in \overline{D(\psi)}$, there exists a unique (strong) solution u to gradient flow.*

2. *For $u_0 \in \overline{D(\psi)}$ it holds that*

   $$\int_0^T t\|u\|^2 \, dt \; < \infty \,,$$

   and, if additionally $u_0 \in D(\psi)$, then $u \in H^1(0, T; H)$.

3. *The operator $u_0 \mapsto u(t) =: S_t u_0$ defines a contraction on $\overline{D(\psi)}$. The family of operators $S_t : \overline{D(\psi)} \to D(\psi)$, $t > 0$ satisfies the semigroup property $S_{t+s} = S_t \circ S_s$ and the contractivity property*

   $$\|S_t x_0 - S_t y_0\| \le \|x_0 - y_0\| \qquad \forall x_0, y_0 \in \overline{D(\psi)} \,.$$

This result can be obtained via time-discretization or regularization. The result and the proof can be found in [ABS21, Theorem 11.7, p.114], compare [Bré73, Theorem 3.6. p.72] for the regularity results of u'.
Attouch [Att84] has treated convergence of minimization problems, subdifferential operators, and semigroups. One is interested in the convergence of solutions of the gradient flow in relation to the convergence of the functions driving the flows. First, the essential notions are recalled.

Definition 2.3.2 (Graph-sense convergence)**:** *In a reflexive Banach space X, a sequence of maximal monotone, set valued operators $A_n : X \rightrightarrows X^*$ is said to converge in graph sense to $A : X \rightrightarrows X$ if for every $y \in A(x)$ there exists a sequences $x_n \to x$ strongly in X and $y_n \to y$ strongly in X^* such that $y_n \in A_n(x_n)$. In this case, we write*

$$A_n \xrightarrow{G} A \qquad \text{as } n \to \infty \; .$$

Furthermore, Attouch proved that Mosco convergence of the functionals is equivalent to graph-sense convergence of the subdifferentials combined with a normalization condition.

Lemma 2.3.3: *In a reflexive Banach space X with $\psi_n, \psi : X \to (-\infty, +\infty]$ of closed, convex, and proper functions, the following are equivalent*

1. $\psi_n \xrightarrow{\mathfrak{M}} \psi$,

2. $$\begin{cases} \partial\psi_n \xrightarrow{G} \partial\psi \; , \\ \\ \exists (x,y) \in \partial\psi \quad \exists (x_n, y_n) \in \partial\psi_n \quad \text{s.t. } x_n \to x \text{ in } X, \; y_n \to y \text{ in } X^* \\ \qquad \text{and } \psi_n(x_n) \to \psi(x). \end{cases}$$

See [Att84, Theorem 3.66, p. 373].
Lastly, the graph-sense convergence of the subdifferentials corresponds to uniform convergence of the gradient flow solutions.

Lemma 2.3.4: *Consider a sequence of subdifferential operators $\partial\psi_n$ that converge to $\partial\psi$ in graph-sense, $\partial\psi_n \xrightarrow{G} \partial\psi$, in the Hilbert space H. Let u_n denote the solution to*

$$\begin{aligned} &u_n' + \partial\psi_n \ni f_n \qquad \text{a.e. in } (0,T), \\ &u_n(0) = x_n \; , \end{aligned}$$

for a given initial condition $x_n \in \overline{D(\psi_n)}$, and, respectively,

$$\begin{aligned} &u' + \partial\psi \ni f \qquad \text{a.e. in } (0,T), \\ &u(0) = x \; , \end{aligned}$$

for a given initial condition $x \in \overline{D(\psi)}$. Then, we have the following:

1. *If $x_n \to x$ strongly in H and $f_n \to f$ in $L^2(0,T;H)$, then $u_n \to u$ uniformly in H,*

and

$$\int_0^T t\|u_n' - u'\|^2 \, dt \ \to 0$$

as $n \to \infty$*, so that* $u_n' \to u'$ *in* $L^2_{\mathrm{loc}}(0, T; H)$*.*

2. *If additionally* $x_n \in D(\psi_n)$*,* $x \in D(\psi)$ *and* $\psi_n(x_n) \to \psi(x)$ *as* $n \to \infty$*, then* $u_n' \to u'$ *strongly in* $L^2(0, T; H)$ *and* $\psi_n(u_n) \to \psi(u)$ *uniformly in* $[0, T]$*.*

See [Att84, Theorem 3.74, p. 388].

3 Introduction of the functionals

This chapter recalls the definitions of the classical perimeter and the classical area functional, as well as their relation. Then the theory on the fractional perimeter and the fractional area functional follows.

3.1 Functions of bounded variation and the classical perimeter

A comprehensive review of the theory of functions of bounded variation, the De Giorgi perimeter, and the area functional can be found in the monographies [EG15] and [Giu84]. Here, we limit ourselves to the essential concepts used in the thesis.

Definition 3.1.1 (Functions of bounded variation and the classical perimeter)**:** *For an open set $\Omega \subset \mathbb{R}^n$ a function $u \in L^1(\Omega)$ has **bounded variation** in Ω if*

$$\sup\left\{ \int_\Omega u \operatorname{div}\varphi \mid \varphi \in \mathcal{C}_c^1(U), |\varphi| \leq 1 \right\} < \infty \,.$$

The space of all functions with bounded variation is denoted by $\mathrm{BV}(\Omega)$. *A Borel set $E \subseteq \mathbb{R}^n$ has **finite perimeter** in Ω if*

$$\chi_E \in \mathrm{BV}(\Omega) \,.$$

For unbounded domains, it is also convenient to introduce some localized concepts.

Definition 3.1.2: *A function $u \in L^1_{\mathrm{loc}}(\Omega)$ has **locally bounded variation** if*

$$u_{|V} \in \mathrm{BV}(V) \textit{ for each open } V \Subset \Omega,$$

*where $V \Subset \Omega$ means that V is compactly contained in Ω. The space of functions with local bounded variation is denoted by $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$. A Borel set $E \subset \mathbb{R}^n$ has **locally finite perimeter** in Ω if*

$$\chi_E \in \mathrm{BV}_{\mathrm{loc}}(\Omega) \,.$$

*A set with locally finite perimeter is called a **Caccioppoli set**.*

Furthermore, recall the Structure theorem for $\mathrm{BV}_{\mathrm{loc}}$ functions, which establishes a notion of derivative for those functions as Radon measures. A proof can be found in [EG15, Thm 5.1, p. 194ff.].

Proposition 3.1.3 (Structure Theorem for $\mathrm{BV}_{\mathrm{loc}}$ functions)**:** *For each $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ there exists a Radon measure μ on Ω and a μ-measurable function $\sigma : \Omega \to \mathbb{R}^n$ such that*

1. *μ-almost everywhere it holds $|\sigma(x)| = 1$,*

2. *for each $\varphi \in \mathcal{C}_c^1(\Omega, \mathbb{R}^n)$, it holds*

$$\int_\Omega u \, \mathrm{div}\varphi \, dx \ = - \int_\Omega \sigma \cdot \varphi \, d\mu \ .$$

It thus follows that

$$\mu(\Omega) = \sup\left\{ \int_\Omega u \, \mathrm{div}\varphi \mid \varphi \in \mathcal{C}_c^1(U), |\varphi| \leq 1 \right\} .$$

For $u \in \mathrm{BV}(\Omega)$, we will denote its derivative measure μ by $\|Du\|$ and $[Du] := \|Du\| \llcorner \sigma$ for the vector-valued measure. The component measures of Du are defined as

$$Du_j := \|Du\| \llcorner \sigma_j \quad \text{for } j = 1, ..., n$$

with $\sigma = (\sigma_1, ..., \sigma_n)$.
It is straightforward to see that the derivative measure is linear.

Lemma 3.1.4: *For $u, v \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ and $\lambda \in \mathbb{R}$ it holds that*

$$[D(u + v)] = [Du] + [Dv] \ ,$$

$$[D(\lambda u)] = \lambda [Du] \ .$$

Proof. Both measures define the same linear functional on $\mathcal{C}_c^1(\Omega)$ and therefore extend uniquely to $\mathcal{C}_c^0(\Omega)$. The equivalence of the measures follows from the uniqueness in the Riesz–Markov–Kakutani representation theorem, see [Sal16, p. 98.]. □

By the Lebesgue Decomposition Theorem and the Radon–Nikodym Theorem, there exists a unique measure ν, singular to the Lebesgue measure, and a density h unique up to a set of Lebesgue measure zero, such that

$$\mu = h \, dx \ + \nu \ .$$

Thus, for every component measure Du_j there exists a density function u_{x_i} and a measure Du_j^s such that

$$Du_j = u_{x_i} \, dx \ + Du_j^s \ .$$

We denote the vector-valued density with respect to the Lebesgue measure by

$$\nabla u := (u_{x_1}, ..., u_{x_n}) \ ,$$

and $[Du]^s := (Du_1^s, ..., Du_n^s)$ the component measures, so that

$$[Du] = \nabla u \, dx \ + [Du]^s \ .$$

Analogously the derivative measure can be decomposed as

$$\|Du\| = |\nabla u| \, dx \, + \|Du\|^s \, .$$

Here, $|\nabla u|$ is indeed the norm of the gradient defined by the densities for the component measures, as the Lebesgue decomposition and the density given by Radon–Nikodym theorem is unique. Analogously, one obtains that $\|Du\|^s = \|[Du]^s\|$, i.e., the total variation of the singular vector-valued measure.
This is consistent with the definition of the gradient in the smooth settings. For a continuously differentiable $u \in \mathrm{BV}(\Omega) \cap \mathcal{C}^1(\Omega)$, by the divergence theorem we get

$$\int_\Omega u \operatorname{div}\varphi \, dx \, = - \int_\Omega \nabla u \cdot \varphi \, dx \, = \int_\Omega \varphi \cdot [Du]$$

for any $\varphi \in \mathcal{C}_c^1(\Omega, \mathbb{R}^n)$. Moreover, the notion also aligns for Sobolev functions $u \in W^{1,1}(\Omega) \subset \mathrm{BV}(\Omega)$.

Definition 3.1.5: *For a set $E \subset \mathbb{R}^n$ that has locally finite perimeter, namely, $\chi_E \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^n)$, the **classical perimeter** $\mathrm{Per}(E, B)$ in some Borel set $B \subseteq \Omega$ is defined as*

$$\mathrm{Per}(E, B) := \|D\chi_E\|(B) \, .$$

3.2 The classical area functional

We provide here a brief review of the classical theory of the area functional and its relaxation. In this section, we restrict the setting to $\Omega \subset \mathbb{R}^n$ open and bounded with Lipschitz boundary. The aim is to calculate the surface area for the graph of a given function, which for a function $u \in \mathcal{C}^1(\Omega)$ is given by

$$\mathcal{A}(u, \Omega) := \int_\Omega \sqrt{1 + |\nabla u|^2} \, dx \ \ .$$

This definition is also well-defined for $u \in W^{1,1}(\Omega)$, yet the functional is not lower semicontinuous on L^1. Therefore, a broader definition is necessary that agrees with $\mathcal{A}$ on smooth functions. The general idea is to consider the relaxation of the area functional. However, there have been different approaches to derive a generalization of the area functional, which ultimately coincide. We give here a minimum of definitions. More information on the different notions of relaxation is provided in Section 7.1.

Definition 3.2.1 (The L^1-relaxation of $\mathcal{A}$)**:** *For $u \in L^1(\Omega)$ we set*

$$\underline{\mathcal{A}}(u, \Omega) := \inf \left\{ \liminf_{n \to \infty} \mathcal{A}(u_n, \Omega) \textit{ for } u_n \in W^{1,1}(\Omega) \textit{ with } u_n \to u \textit{ in } L^1(\Omega) \right\}$$

with

$$D(\underline{\mathcal{A}}) = \mathrm{BV}(\Omega) \, .$$

Just as the area functional $\mathcal{A}$ has the geometric interpretation of measuring the surface area for the graph of a sufficiently regular function, this geometric interpretation also extends to its relaxation as it measures the perimeter in the open cylinder Ω_∞, as defined in Definition 3.1.5 by the theory on functions of bounded variation.

Lemma 3.2.2: *For $u \in \mathrm{BV}(\Omega)$ it holds*

$$\underline{\mathcal{A}}(u, \Omega) = \mathrm{Per}(\mathcal{S}_u, \Omega^\infty)$$

with $\Omega_\infty := \Omega \times \mathbb{R}$ and the subgraph $\mathcal{S}_u := \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid x \in \Omega$ and $t \leq u(x)\}$.

This is proved, for example, in [Giu84, Theorem 14.6, p. 163]. This geometric relation motivates the investigation on how the relaxation acts on both the absolute continuous and the singular part of the derivative measure for a BV function. This has been made precise by Goffman, Serrin [GS64, Theorem 5, p. 174], who have introduced the notion of functions of a Radon measure and their application to the area functional.

Lemma 3.2.3 (Integral formula for the area functional)**:** *For $u \in \mathrm{BV}(\Omega)$ with derivative measure $\|Du\| = |\nabla u|\, dx\; + \|Du\|^s$ it holds*

$$\underline{\mathcal{A}}(u, \Omega) = \int_\Omega \sqrt{1 + |\nabla u|^2}\, dx\; + \|Du\|^s(\Omega)\ . \tag{3.2.1}$$

Next, we consider the setting including a boundary condition $\rho \in L^1(\partial\Omega, \mathcal{H}^{n-1})$, by taking into account the area between the trace of u and the boundary datum. $\mathcal{H}^{n-1}$ denotes the $n-1$ dimensional Hausdorff measure. Due to the assumption of Lipschitz regularity for the boundary of Ω, the trace operator $T : \mathrm{BV}(\Omega) \to L^1(\partial\Omega, \mathcal{H}^{n-1})$ is well-defined, see [EG15, Theorem 5.6, p.204]. In the following, equations at the boundary for BV functions will be meant in the trace sense.

Definition 3.2.4: *For $u \in \mathrm{BV}(\Omega)$ and $\rho \in L^1(\partial\Omega, \mathcal{H}^{n-1})$ the area functional with boundary values ρ is defined as*

$$\underline{\mathcal{A}}(u, \overline{\Omega}, \rho) := \underline{\mathcal{A}}(u, \Omega) + \int_{\partial\Omega} |u - \rho|\, d\mathcal{H}^{n-1}\ \ ,$$

with the boundary values of u in the trace sense. Furthermore for any open $\mathcal{O} \Subset \Omega$ we have

$$\begin{aligned}\underline{\mathcal{A}}(u, \overline{\mathcal{O}}) := \underline{\mathcal{A}}(u, \overline{\mathcal{O}}, u^-) &= \underline{\mathcal{A}}(u, \mathcal{O}) + \int_{\partial\mathcal{O}} |u^+ - u^-|\, d\mathcal{H}^{n-1} \\ &= \underline{\mathcal{A}}(u, \mathcal{O}) + \|Du\|(\partial\Omega)\ ,\end{aligned}$$

where the boundary values of u are considered in a trace sense and u^+, u^- denote the trace of u from inside of Ω and from outside respectively.

A different approach to incorporate boundary values would be the restiction

$$\mathcal{A}_{|D_\rho} := \begin{cases} \mathcal{A}(u,\Omega) & \text{if } u \in D_\rho, \\ \infty & \text{else }, \end{cases}$$

with the set of functions having trace ρ denoted by $D_\rho := \{u \in W^{1,1}(\Omega) : u = \rho \text{ on } \partial\Omega\}$. Yet, $\mathcal{A}_{|D_\rho}$ is not lower semicontinuous, and if one considers the relaxation of $\mathcal{A}_{D_\rho}$, then this yields the previous Definition 3.2.4.

Lemma 3.2.5: *For $u \in L^1(\Omega)$ it holds*

$$\underline{\mathcal{A}}(u,\overline{\Omega},\rho) = \inf\left\{ \liminf_{n\to\infty} \mathcal{A}(u_n,\Omega) \textit{ for } u_n \in D_\rho \textit{ with } u_n \to u \textit{ in } L^1(\Omega) \right\} .$$

As already mentioned, the geometric interpretation of the area functional on the closure is that it also accounts for the area on the boundary with respect to the outside datum. This is backed by the following lemma.

Lemma 3.2.6: *For any open $\mathcal{O} \Subset \Omega$ with Lipschitz boundary it holds*

$$\underline{\mathcal{A}}(u,\overline{\mathcal{O}}) = \operatorname{Per}(\mathcal{S}_u, \overline{\mathcal{O}}^\infty) .$$

Proof. In [Giu84, Remark 2.13, p.38] it is proved that, for $u \in \mathrm{BV}(\Omega)$, one has $\|Du\|(\partial\mathcal{O}) = \int_{\partial\mathcal{O}} |u^+ - u^-| \; d\mathcal{H}^{n-1}$. Then, using Lemma 3.2.2 and the decomposition of the area functional in Equation (3.2.1) it follows that

$$\begin{aligned} \operatorname{Per}(\mathcal{S}_u,\Omega^\infty) = \underline{\mathcal{A}}(u,\Omega) &= \int_{\mathcal{O}} \sqrt{1+|\nabla u|^2}\, dx \; + \|D^s u\|(\mathcal{O}) \\ &= \underline{\mathcal{A}}(u,\mathcal{O}) + \|Du\|(\partial\mathcal{O}) + \; \underline{\mathcal{A}}(u,\Omega\setminus\overline{\mathcal{O}}) \\ &= \underline{\mathcal{A}}(u,\overline{\mathcal{O}}) + \operatorname{Per}(\mathcal{S}_u,(\Omega\setminus\overline{\mathcal{O}})^\infty) . \end{aligned}$$

Subtracting $\operatorname{Per}(\mathcal{S}_u,(\Omega\setminus\overline{\mathcal{O}})^\infty)$ proves the result. □

We derive the continuity of the area functional in the BV norm convergence.

Lemma 3.2.7 (Continuity of the area functional)**:** *For any boundary datum $\varphi \in L^1(\partial\Omega)$, the functional $\underline{\mathcal{A}}(u,\overline{\Omega},\varphi) = \underline{\mathcal{A}}(u,\Omega) + \int_{\partial\Omega} |\varphi - u| \; d\mathcal{H}^{n-1}$ is continuous in* BV(Ω) *with respect to norm convergence.*

Proof. The trace is continuous with respect to the BV-norm, and thus the boundary term depends continuously. Then for $u_k \to u$ in BV(Ω) it follows for the singular part of the derivative measure that

$$\Big|\|Du_k\|^s(\Omega) - \|Du\|^s(\Omega)\Big| \leq \Big|\|[Du_k]^s\|(\Omega) - \|[Du]^s\|(\Omega)\Big| \leq \|D(u_n - u)\|(\Omega) \leq \|u_n - u\|_{\mathrm{BV}} ,$$

and the BV-norm convergence also implies $\nabla u_n \to \nabla u$ in $L^1(\Omega)$ for the absolute continuous part of the derivative measure. Then, by the Lipschitz continuity of $\sqrt{1+t^2}$ it follows

$$\int_\Omega \sqrt{1+|\nabla u_n|^2}\,dx \;\to \int_\Omega \sqrt{1+|\nabla u|^2}\,dx \;.$$

Therefore, using the decomposition for the area functional formula in Equation (3.2.1), it follows

$$\underline{\mathcal{A}}(u_n,\Omega) \to \underline{\mathcal{A}}(u,\Omega) \;. \qquad \square$$

Finally, define

$$\underline{\underline{\mathcal{A}}}(u,\Omega) := \omega_n\big(\underline{\mathcal{A}}(u,\Omega) - |\Omega|\big)$$

and analogously

$$\underline{\underline{\mathcal{A}}}(u,\overline{\Omega},\rho) := \omega_n\big(\underline{\mathcal{A}}(u,\overline{\Omega},\rho) - |\Omega|\big)$$

as renormalized versions, such that $\underline{\underline{\mathcal{A}}}(0,\Omega) = 0$. Here, ω_n denotes the volume of the unit ball in $\mathbb{R}^n$ for $n \geq 1$ and $\omega_0 := 1$. Here and throughout this thesis, for a Borel set $E \subseteq \mathbb{R}^n$ the absolute value $|E|$ will denote its Lebesgue measure. This renormalization is necessary as the constants will show up in the limit of the fractional area functional for $s \nearrow 1$.

3.3 The fractional perimeter

The fractional or nonlocal s-perimeter for $s \in (0,1)$ has been introduced in [CRS09]. Thereafter, the investigations on the existence and regularity of s-minimal sets has attracted strong interest and research. More information are contained in the surveys [DV16], [Buc+16] or [Val12].

The idea of the fractional perimeter is to define a functional for measurable sets $E \subset \mathbb{R}^{n+1}$ measuring the interactions between E and its complement $\mathcal{C}E := \mathbb{R}^{n+1} \setminus E$. The definitions here are for dimension $n+1$ as the nonlocal perimeter will be applied to the case of subgraphs in order to define a functional for functions in $\mathbb{R}^n$. The interaction between points x and y in two measurable and disjoint sets $A, B \subseteq \mathbb{R}^{n+1}$ is given by the hypercritical kernel depending on the distance of points

$$\frac{1}{|x-y|^{n+1+s}} \;.$$

The interaction $\mathcal{L}_s(A,B)$ between A and B is

$$\mathcal{L}_s(A,B) = \int_A \int_B \frac{dx\;dy}{|x-y|^{n+1+s}} \;.$$

The perimeter will be localized in a set $\mathcal{O} \subseteq \mathbb{R}^{n+1}$ in the sense that $E \cap \mathcal{C}\mathcal{O}$ is interpreted as the boundary datum. Interactions are considered between E and $\mathcal{C}E$, when at least one of the points is located in O. Thus, the interactions between $E \cap \mathcal{C}\mathcal{O}$ and $\mathcal{C}E \cap \mathcal{C}\mathcal{O}$ are not considered in the definition of the s-perimeter.

Definition 3.3.1 (The fractional or nonlocal s-perimeter)**:** *For $n \in \mathbb{N}_0$ and $s \in (0,1)$, the **fractional** s**-perimeter** of a measurable set $E \subseteq \mathbb{R}^{n+1}$ in an open set $\mathcal{O} \subseteq \mathbb{R}^{n+1}$ is denoted as $\mathrm{Per}_s(E, \mathcal{O})$ and is defined as*

$$\mathrm{Per}_s(E, \mathcal{O}) := \mathcal{L}_s(E \cap \mathcal{O}, \mathcal{C}E \cap \mathcal{O}) + \mathcal{L}_s(E \cap \mathcal{O}, \mathcal{C}E \cap \mathcal{C}\mathcal{O}) + \mathcal{L}_s(E \cap \mathcal{C}\mathcal{O}, \mathcal{C}E \cap \mathcal{O}) .$$

*E is said to have **finite** s**-perimeter** in Ω if $\mathrm{Per}_s(E, \Omega) < \infty$ and **locally finite** s**-perimeter** in Ω if $\mathrm{Per}_s(E, V) < \infty$ for every $V \Subset \Omega$.*

The set $\mathcal{O}$ localizes the domain of the interactions. It is therefore natural to divide the perimeter into a **local and a nonlocal part** according to the domains of integration. For the local part of the perimeter $\mathrm{Per}_s^{\mathrm{L}}$, both domains lie inside $\mathcal{O}$, whereas for the nonlocal part $\mathrm{Per}_s^{\mathrm{NL}}$, one lies in $\mathcal{C}\mathcal{O}$.

Definition 3.3.2 (Local and nonlocal fractional s-perimeter)**:** *For $E, \mathcal{O} \subseteq \mathbb{R}^{n+1}$ and $s \in (0,1)$ as before, define*

$$\begin{aligned}
\mathrm{Per}_s^{\mathrm{L}}(E, \mathcal{O}) &:= \mathcal{L}_s(E \cap \mathcal{O}, \mathcal{O} \setminus E) = \frac{1}{2}[\chi_E]_{W^{s,1}(\mathcal{O})} , \\
\mathrm{Per}_s^{\mathrm{NL}}(E, \mathcal{O}) &:= \mathcal{L}_s(E \cap \mathcal{O}, \mathcal{C}E \cap \mathcal{C}\mathcal{O}) + \mathcal{L}_s(E \cap \mathcal{C}\mathcal{O}, \mathcal{C}E \cap \mathcal{O}) .
\end{aligned}$$

Here $[\,\cdot\,]_{W^{s,p}(\mathcal{O})}$ denotes the Gagliardo or simply s-fractional semi-norm in $\mathcal{O} \subseteq \mathbb{R}^{n+1}$ which is defined as

$$[u]_{W^{s,p}(\mathcal{O})} := \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{(n+1)+sp}} \, dx \; dy \right)^{1/p}$$

for a measurable function $u : \mathcal{O} \to \mathbb{R}$. The s-fractional Sobolev space $W^{s,p}(\mathcal{O})$ is defined as

$$W^{s,p}(\mathcal{O}) := \{u \in L^p(\mathcal{O}) \mid [u]_{W^{s,p}(\mathcal{O})} < \infty\}$$

with norm $\|u\|_{W^{s,p}(\mathcal{O})} := \|u\|_{L^p(\mathcal{O})} + [u]_{W^{s,p}(\mathcal{O})}$.

The fractional s-perimeter is to be thought of as an interpolation between the measure and the perimeter of a set. For a discussion on the convergence $s \searrow 0$ and in what sense the measure of the set is recovered, see the discussion in [Buc+16, Chapter 6]. For the analysis here, the convergence $s \nearrow 1$ is essential, and the most important properties are recalled. The following pointwise convergence result stems from [Lom16b].

Proposition 3.3.3 (Pointwise convergence of the fractional perimeter)**:** *For an open set $\Omega \subseteq \mathbb{R}^n$ and $E \subset \mathbb{R}^n$. The following holds true*

1. *E has locally finite perimeter in Ω if and only if E has locally finite s-perimeter for all $s \in (0,1)$ and*

$$\liminf_{s \nearrow 1} (1-s)\mathrm{Per}_s^{\mathrm{L}}(E, \mathcal{O}) < \infty \text{ for every } \mathcal{O} \Subset \Omega . \tag{3.3.1}$$

2. *If E has locally finite perimeter in Ω, and $\mathcal{O} \Subset \Omega$ open with Lipschitz boundary,*

then it holds

$$\lim_{s\nearrow 1}(1-s)\mathrm{Per}_s^{\mathrm{L}}(E,\mathcal{O}) = \omega_{n-1}\mathrm{Per}(E,\mathcal{O}) \ , \tag{3.3.2}$$

$$\lim_{s\nearrow 1}(1-s)\mathrm{Per}_s^{\mathrm{NL}}(E,\mathcal{O}) = \omega_{n-1}\mathrm{Per}(E,\partial\mathcal{O}) \ , \tag{3.3.3}$$

$$\lim_{s\nearrow 1}(1-s)\mathrm{Per}_s(E,\mathcal{O}) = \omega_{n-1}\mathrm{Per}(E,\overline{\mathcal{O}}) \ . \tag{3.3.4}$$

The Γ-convergence for $s \nearrow 1$ is discussed in [APM10, Theorem 2, p. 2].

Proposition 3.3.4 (Γ-convergence for the s-perimeter)**:** *For a measurable set E and an open set $\Omega \Subset \mathbb{R}^n$ it holds that*

$$\Gamma\text{-}\liminf_{s\nearrow 1} \mathrm{Per}_s^{\mathrm{L}}(E,\Omega) \geq \omega_{n-1}\mathrm{Per}(E,\Omega) \ ,$$
$$\Gamma\text{-}\limsup_{s\nearrow 1} \mathrm{Per}_s(E,\Omega) \leq \omega_{n-1}\mathrm{Per}(E,\Omega) \ ,$$

with respect to the L^1_{loc} convergence of the characteristic functions of sets in $\mathbb{R}^n$.

3.4 The fractional area functional

The classical setting exhibits a direct relation between the area functional and the perimeter of the subgraph, as displayed by Lemma 3.2.2. Ideally, one would like to develop a comparable theory for the fractional setting as well, to recover usefull properties of the fractional perimeter. Yet, this approach does not yield a useful definition, as nonlocality prevents the fractional perimeter of the subgraph to be finite even for fairly regular functions. For example, Lombardini [Lom16a] has proved that the s-perimeter of a subgraph $\mathcal{S}_u$ in the unbounded cylinder Ω^∞ is infinite for any $u \in L^\infty(\mathbb{R}^n)$, independent of the regularity of u and Ω.

This has led to the investigation of the local part of the s-fractional perimeter and the development of the local fractional area functional, which remains directly related to the local part of the perimeter of the subgraph and the regularity of the function and is given by

$$\mathcal{A}_s(u,\Omega) = \int_\Omega\int_\Omega \mathcal{G}_s\left(\frac{u(x)-u(y)}{|x-y|}\right)\frac{dx\ dy}{|x-y|^{n-1+s}}$$

and

$$\mathrm{Per}_s^{\mathrm{L}}(\mathcal{S}_u,\Omega^\infty) = \mathcal{A}_s(u,\Omega) + \kappa^1_{s,\Omega}$$

with the definition of $\kappa^1_{s,\Omega}$ independent of u and precisely recalled in Equation (3.4.4).

The following definitions are used for evaluating and weighting the interactions for $t \in \mathbb{R}$

$$
\begin{aligned}
g_s(t) &:= \frac{1}{(1+t^2)^{\frac{n+1+s}{2}}} \ , & G_s(t) &:= \int_0^t g_s(\tau)\,\mathrm{d}\tau \ , \\
\overline{G}_s(t) &:= \int_{-\infty}^t g_s(\tau)\,\mathrm{d}\tau \ , & \mathcal{G}_s(t) &:= \int_0^t G_s(\tau)\,\mathrm{d}\tau \ . \qquad (3.4.1)
\end{aligned}
$$

The integral representation of the functional $\mathcal{A}_s(u,\Omega)$ is then extended to complement $\mathcal{C}\Omega$.

Definition 3.4.1 (The s-fractional area functional)**:** *For $\Omega \subset \mathbb{R}^n$ bounded and open, $s \in (0,1)$, and a measurable function $u : \mathbb{R}^n \to \mathbb{R}$, the* ***local s-fractional area functional*** *is defined by*

$$\mathcal{A}_s(u,\Omega) = \int_\Omega \int_\Omega \mathcal{G}_s\left(\frac{u(x)-u(y)}{|x-y|}\right) \frac{dx\ dy}{|x-y|^{n-1+s}}$$

and the ***nonlocal s-fractional area functional*** *by*

$$\mathcal{N}_s(u,\Omega) = 2\int_\Omega \int_{\mathcal{C}\Omega} \mathcal{G}_s\left(\frac{u(x)-u(y)}{|x-y|}\right) \frac{dx\ dy}{|x-y|^{n-1+s}} \ .$$

Altogether, the ***s-fractional area functional*** *is defined as the sum*

$$\mathcal{F}_s(u,\Omega) = \mathcal{A}_s(u,\Omega) + \mathcal{N}_s(u,\Omega) = \iint_{\mathcal{Q}(\Omega)} \mathcal{G}_s\left(\frac{u(x)-u(y)}{|x-y|}\right) \frac{dx\ dy}{|x-y|^{n-1+s}} \ ,$$

where $\mathcal{Q}(\Omega) := \mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2$.

Notice that in order to consider nonlocality, it is required that the functions are not only defined in Ω but in the whole of $\mathbb{R}^n$. While the functional $\mathcal{N}_s$ obtained in this manner will not have the straightforward connection to the nonlocal part of the fractional perimeter of the subgraph, it is finite under limited growth conditions in $\mathcal{C}\Omega$. The precise condition will be reviewed in the upcoming Chapter 4. To define a nonlocal functional which is finite, independent of the growth of φ, a truncated version of the nonlocal part is introduced, canceling interactions which arise above a certain height M.

Definition 3.4.2 (The truncated s-fractional area functional)**:** *For $\Omega \subset \mathbb{R}^n$ bounded and open, $s \in (0,1)$, $M > 0$, and a measurable function $u : \mathbb{R}^n \to \mathbb{R}$, the* ***truncated nonlocal s-fractional area functional*** *is defined by*

$$\mathcal{N}_s^M(u,\Omega) = \int_\Omega \left\{ \int_{\mathcal{C}\Omega} \left(\int_{\frac{-M-u(y)}{|x-y|}}^{\frac{u(x)-u(y)}{|x-y|}} \overline{G}_s(t)\ dt \ + \int_{\frac{u(x)-u(y)}{|x-y|}}^{\frac{M-u(y)}{|x-y|}} \overline{G}_s(-t)\ dt \right) \frac{dy}{|x-y|^{n-1+s}} \right\} dx \ . \qquad (3.4.2)$$

Then, analogously, the ***truncated s-fractional area functional*** *is the sum*

$$\mathcal{F}_s^M(u,\Omega) = \mathcal{A}_s(u,\Omega) + \mathcal{N}_s^M(u,\Omega) \ .$$

The truncation can be reformulated as in [CL20, Lemma 2.4]. For a measurable function $u : \mathbb{R}^n \to \mathbb{R}$ it holds

$$\mathcal{N}_s^M(u,\Omega) = \int_\Omega \int_{\mathcal{C}\Omega} \left\{ 2\mathcal{G}_s\left(\frac{u(x)-u(y)}{|x-y|}\right) - \mathcal{G}_s\left(\frac{-M-u(y)}{|x-y|}\right) \right. \\ \left. - \mathcal{G}_s\left(\frac{M-u(y)}{|x-y|}\right)\right\} \frac{dy\ dx}{|x-y|^{n-1+s}} - M\Lambda_s \int_\Omega \int_{\mathcal{C}\Omega} \frac{dx\ dy}{|x-y|^{n+s}} \tag{3.4.3}$$

with Λ_s as in Equation (4.0.1).
As mentioned above, the function u has to be defined in the whole of $\mathbb{R}^n$. Yet, the setting that is considered in this thesis will take the exterior datum as a boundary condition. Therefore, the notion of the exterior datum will be incorporated into the functionals to obtain a simpler function space to work on.

Definition 3.4.3: *For Ω, s, M as in Definition 3.4.2, and a measurable outside datum $\varphi : \mathcal{C}\Omega \to \mathbb{R}$ define for a measurable function $u : \Omega \to \mathbb{R}^n$ the **nonlocal s-fractional area functional for the datum φ** by*

$$\mathcal{N}_s(u,\Omega,\varphi) := 2\int_\Omega \int_{\mathcal{C}\Omega} \mathcal{G}_s\left(\frac{u(x)-\varphi(y)}{|x-y|}\right)\frac{dx\ dy}{|x-y|^{n-1+s}}\ ,$$

*and the **truncated, nonlocal s-fractional area functional for the datum φ** by*

$$\mathcal{N}_s^M(u,\Omega,\varphi) = \int_\Omega \left\{ \int_{\mathcal{C}\Omega} \left(\int_{\frac{-M-\varphi(y)}{|x-y|}}^{\frac{u(x)-\varphi(y)}{|x-y|}} \overline{G}_s(t)\ dt\ + \int_{\frac{u(x)-\varphi(y)}{|x-y|}}^{\frac{M-\varphi(y)}{|x-y|}} \overline{G}_s(-t)\ dt \right) \frac{dy}{|x-y|^{n-1+s}} \right\} dx\ .$$

The functionals $\mathcal{F}_s(\,\cdot\,,\Omega,\varphi)$ and $\mathcal{F}_s^M(\,\cdot\,,\Omega,\varphi)$ are defined analogously.

Lastly, the nonlocal (s,q)-energy is introduced. This is a nonlocal extension of the Gagliardo semi-norm, combined with a nonlocal part. A treatment of this energy can be found in [Buc+21].

Definition 3.4.4: *For $u : \mathbb{R}^n \to \mathbb{R}$ measurable, $s \in (0,1)$, and $p \geq 1$ the nonlocal (s,q)-energy is defined as*

$$\mathcal{E}_s^p(u) := \frac{1}{2p} \iint_{\mathcal{Q}(\Omega)} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\ dx\ dy\ .$$

Now, the connection between the fractional perimeter and the fractional area function can be made precise in the following way.

Lemma 3.4.5 (Connections between fractional perimeter and area functional)**:** *For an open, bounded domain $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary and $u : \mathbb{R}^n \to \mathbb{R}$ we have that*

1. $$\mathrm{Per}_s^{\mathrm{L}}(\mathcal{S}_u, \Omega^\infty) = \mathcal{A}_s(u,\Omega) + \kappa_{s,\Omega}^1\ , \tag{3.4.4}$$

   where

   $$\kappa_{s,\Omega}^1 := \mathrm{Per}_s^{\mathrm{L}}(\{X_{n+1} < 0\}, \Omega^\infty)\ .$$

2. *If futhermore* $\|u\|_{L^\infty(\Omega)} \leq M$ *for some* $M > 0$ *then*

$$\mathrm{Per}_s(\mathcal{S}_u, \Omega^M) = \mathcal{F}_s^M(u, \Omega) + \kappa^2_{\Omega,M,s} \,, \tag{3.4.5}$$

   where

$$\kappa^2_{\Omega,M,s} := \mathrm{Per}_s^{\mathrm{L}}\big(\{X_{n+1} < 0\}, \Omega^\infty\big) - \mathrm{Per}_s^{\mathrm{L}}\big(\{X_{n+1} < 0\}, \Omega^\infty \setminus \Omega^M\big) \,.$$

with $\{X_{n+1} < 0\} := \{x \in \mathbb{R}^{n+1} \mid x_{n+1} < 0\}$.
See [CL20, Lemma 2.10 and 2.12] for the proofs.

4 Properties of the fractional area functional

The most important properties of the fractional area will be presented in this chapter, following closely the analysis in Cozzi, Lombardini [CL20]. From this point on, throughout the next chapters, the domain $\Omega \subset \mathbb{R}^n$ is assumed to be open, bounded, with Lipschitz boundary, and all functions considered are assumed to be measurable.
Before we come to the properties of the s-fractional area functional, we recall the most important analytic properties of g_s, G_s, $\overline{G}_s$, and $\mathcal{G}_s$ to be used throughout the following analysis in this chapter.

Lemma 4.0.1: *For g_s, G_s, $\overline{G}_s$, and $\mathcal{G}_s$ defined as in Equation (3.4.1), the following properties hold.*

1. *The function $g_s : \mathbb{R} \to [0,1]$ is smooth, and even.*
2. *The function $G_s : \mathbb{R} \to \mathbb{R}$ is smooth, odd, and strictly increasing. It holds $G(0) = 0$, and*
$$|G(t)| < \frac{1}{2} \int_{\mathbb{R}} g_s(\tau)\, d\tau \ =: \frac{\Lambda_s}{2} < \infty \qquad \text{for all } t \in \mathbb{R}. \tag{4.0.1}$$
Furthermore, it holds
$$\overline{G}_s(t) = \frac{\Lambda_s}{2} + G_s(t) \qquad \text{for all } t \in \mathbb{R},$$
which implies that $\overline{G}_s$ is also smooth, strictly increasing, and $0 < \overline{G}_s < \Lambda_s$.
3. *The function $\mathcal{G}_s : \mathbb{R} \to [0,\infty)$ is smooth, even, and strictly convex. It holds $\mathcal{G}_s(0) = 0$, and that $\mathcal{G}_s$ is Lipschitz continuous with Lipschitz constant $\frac{\Lambda_s}{2}$, namely*
$$|\mathcal{G}_s(t_1) - \mathcal{G}_s(t_2)| \leq \left| \int_{t_1}^{t_2} G_s(\tau)\, d\tau \right| \leq \frac{\Lambda_s}{2} |t_1 - t_2| \qquad \text{for all } t_1, t_2 \in \mathbb{R}. \tag{4.0.2}$$
Furthermore, it holds
$$\frac{\Lambda_s}{2}|t| - \lambda_s \leq \mathcal{G}_s(t) \leq \frac{\Lambda_s}{2}|t| \qquad \text{for all } t \in \mathbb{R}, \tag{4.0.3}$$
where
$$\lambda_s = \int_0^\infty t g_s(t)\, dt \ < \infty\ . \tag{4.0.4}$$

For the proof see [CL20, Lemma 2.1].

Remark 4.0.2: *It holds*

$$\Lambda_s = B\left(\frac{1}{2}, \frac{n+s}{2}\right) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{n+2}{2})}{\Gamma(\frac{n+s+1}{2})} = \sqrt{\pi}\frac{\Gamma(\frac{n+s}{2})}{\Gamma(\frac{n+s+1}{2})} \ ,$$

where B denotes the Beta function, and Γ denotes the Gamma function. This implies

$$\Lambda_s < \infty \qquad \text{for all } s \in [0,1],$$

and

$$\Lambda_1 \omega_{n-1} = \frac{\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2}+1)} \frac{\sqrt{\pi}\ \Gamma(\frac{n+1}{2})}{\Gamma(\frac{n+2}{2})} = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} = \omega_n \ . \tag{4.0.5}$$

See [Gra+07, Equation 3.251, p. 324] for the formula of the integral.

4.1 The domain and continuity

The goal is to characterize the domain of the s-fractional area functional. As mentioned in Section 3.4, the finiteness of $\mathcal{A}_s$ depends only on the regularity of the function in Ω.

Lemma 4.1.1 (Finiteness of the local s-fractional area functional)**:** *For a function $u : \Omega \to \mathbb{R}^n$, it holds*

$$\mathcal{A}_s(u,\Omega) < \infty \quad \Leftrightarrow \quad u \in W^{s,1}(\Omega) \ . \tag{4.1.1}$$

More precisely

$$\frac{c_\star}{2}\left([u]_{W^{s,1}(\Omega)} - c_s(\Omega)\right) \le \mathcal{A}_s(u,\Omega) \le \frac{\Lambda_s}{2}[u]_{W^{s,1}(\Omega)} \ , \tag{4.1.2}$$

where Λ_s as in Equation (4.0.1)*,*

$$c_\star^s := \inf_{t\in[0,1]} g_s(t) = 2^{-\frac{n+1+s}{2}} \ ,$$

and

$$c_s(\Omega) := \frac{\mathcal{H}^{n-1}(S^{n-1})}{1-s}|\Omega|\mathrm{diam}(\Omega)^{1-s} \ .$$

We denote by $S^n \subset \mathbb{R}^{n+1}$ the n-sphere, and by $\mathcal{H}^n(S^n)$ its area.

See [CL20, Lemma 2.3] for a proof.

Furthermore, for the nonlocal part, the finiteness depends additionally on φ.

Lemma 4.1.2 (Finiteness of the nonlocal s-fractional area functional)**:** *For an exterior datum $\varphi : \mathcal{C}\Omega \to \mathbb{R}$, and a function $u \in W^{s,1}(\Omega)$, it holds*

$$\mathcal{N}_s(u,\Omega,\varphi) < \infty \quad \Leftrightarrow \quad \int_\Omega \left(\int_{\mathcal{C}\Omega} \frac{|\varphi(y)|}{|x-y|^{n+s}}\, dy\right) := \sigma_s(\varphi)\, dx \ < \infty \ , \tag{4.1.3}$$

more precisely

$$|\mathcal{N}_s(u,\Omega,\varphi)| \le \Lambda_s\big(C_2(n,\Omega,s)\|u\|_{W^{s,1}(\Omega)} + \sigma_s(\varphi)\big) ,$$

where $C_2(n,\Omega,s)$ *as in Lemma 7.3.3.*

Proof. For $u \in W^{s,1}(\Omega)$, assume $\sigma_s(\varphi) < \infty$. Then, with the upper bound on $\mathcal{G}_s$ in Equation (4.0.3) it follows

$$\begin{aligned}
|\mathcal{N}_s(u,\Omega,\varphi)| &= 2\int_\Omega\int_{\mathcal{C}\Omega} \mathcal{G}_s\left(\frac{u(x)-\varphi(y)}{|x-y|}\right) \frac{dy\ dx}{|x-y|^{n-1+s}} \\
&\le \Lambda_s \int_\Omega\int_{\mathcal{C}\Omega} \frac{|u(x)|+|\varphi(y)|}{|x-y|^{n+s}}\, dy\ dx \\
&= \Lambda_s\left(\int_\Omega |u(x)| \int_{\mathcal{C}\Omega} \frac{dy}{|x-y|^{n+s}}\, dx + \int_\Omega\int_{\mathcal{C}\Omega} \frac{|\varphi(y)|}{|x-y|^{n+s}}\, dx\ dy\right) \\
&\le \Lambda_s\Big(C_2(n,\Omega,s)\|u\|_{W^{s,1}(\Omega)} + \sigma_s(\varphi)\Big) < \infty ,
\end{aligned}$$

The estimate with the constant $C_2(n,\Omega,s)$ in Lemma 7.3.3 has been used to conclude the last line. To prove the converse, we use the lower bound on $\mathcal{G}_s$ from Equation (4.0.3).

$$\begin{aligned}
|\mathcal{N}_s(0,\Omega,\varphi)| &= 2\int_\Omega\int_{\mathcal{C}\Omega} \mathcal{G}_s\left(\frac{|\varphi(y)|}{|x-y}\right) \frac{dx\ dy}{|x-y|^{n-1+s}} \\
&\ge \Lambda_s \int_\Omega\int_{\mathcal{C}\Omega} \frac{|\varphi(y)|-\lambda_s}{|x-y|^{n+s}}\, dx\ dy \ .
\end{aligned}$$

From

$$\Lambda_s \int_\Omega\int_{\mathcal{C}\Omega} \frac{\lambda_s}{|x-y|^{n+s}}\, dx\ dy \ = \Lambda_s\lambda_s \mathrm{Per}_s(\Omega,\mathbb{R}^n) < \infty ,$$

the reverse implication follows. ☐

For the truncation, the mere regularity in Ω is enough to obtain finiteness.

Lemma 4.1.3 (Finiteness of the truncated, nonlocal s-fractional area functional)**:** *For an exterior datum* $\varphi : \mathcal{C}\Omega \to \mathbb{R}$ *and* $u \in W^{s,1}(\Omega)$ *it holds that*

$$\mathcal{N}_s^M(u,\Omega,\varphi) < \infty .$$

More precisely

$$\mathcal{N}_s^M(u,\Omega,\varphi) \le C(n,\Omega,s)\Lambda_s\big(\|u\|_{W^{s,1}(\Omega)} + M\big) . \tag{4.1.4}$$

See [CL20, Lemma 2.4] for a proof. Therefore, the following assumption regarding the integrability of the exterior datum φ is introduced to guarantee the properness of the fractional area functional.

Assumption (A1): For a fixed $s \in (0,1)$, the exterior datum $\varphi : \mathcal{C}\Omega \to \mathbb{R}$ fulfills the integrability condition

$$\int_\Omega \left(\int_{\mathcal{C}\Omega} \frac{|\varphi(y)|}{|x-y|^{n+s}} \, dy \right) dx \; =: \sigma_s(\varphi) < \infty \, .$$

After establishing these conditions for the finiteness of the local and nonlocal parts, we investigate the difference between the fractional area of two functions with the same fixed exterior datum.

Lemma 4.1.4: *For $u, v \in W^{s,1}(\Omega)$, an exterior datum $\varphi : \mathcal{C}\Omega \to \mathbb{R}$, and $M, M' > 0$, it holds*

$$\begin{aligned}
&\mathcal{N}_s^M(u,\Omega,\varphi) - \mathcal{N}_s^M(v,\Omega,\varphi) \\
&\quad = \; 2 \int_\Omega \int_{\mathcal{C}\Omega} \left\{ \mathcal{G}_s\left(\frac{u(x)-\varphi(y)}{|x-y|} \right) \; - \mathcal{G}_s\left(\frac{v(x)-\varphi(y)}{|x-y|} \right) \right\} \frac{dx \; dy}{|x-y|^{n-1+s}} \\
&\quad = \mathcal{N}_s^{M'}(u,\Omega,\varphi) - \mathcal{N}_s^{M'}(v,\Omega,\varphi) \, ,
\end{aligned} \tag{4.1.5}$$

and

$$\mathcal{F}_s^M(u,\Omega,\varphi) - \mathcal{F}_s^M(v,\Omega,\varphi) = \mathcal{F}_s^{M'}(u,\Omega,\varphi) - \mathcal{F}_s^{M'}(v,\Omega,\varphi) \, . \tag{4.1.6}$$

If, additionally, φ fulfills Assumption (A1), it holds

$$\begin{aligned}
&\mathcal{N}_s(u,\Omega,\varphi) - \mathcal{N}_s(v,\Omega,\varphi) \\
&\quad = \; 2 \int_\Omega \int_{\mathcal{C}\Omega} \left\{ \mathcal{G}_s\left(\frac{u(x)-\varphi(y)}{|x-y|} \right) \; - \mathcal{G}_s\left(\frac{v(x)-\varphi(y)}{|x-y|} \right) \right\} \frac{dx \; dy}{|x-y|^{n-1+s}} \\
&\quad = \mathcal{N}_s^{M}(u,\Omega,\varphi) - \mathcal{N}_s^{M}(v,\Omega,\varphi) \, ,
\end{aligned} \tag{4.1.7}$$

and

$$\mathcal{F}_s^M(u,\Omega,\varphi) - \mathcal{F}_s^M(v,\Omega,\varphi) = \mathcal{F}_s(u,\Omega,\varphi) - \mathcal{F}_s(v,\Omega,\varphi) \, . \tag{4.1.8}$$

Proof. Lemma 4.1.3 implies that $\mathcal{N}_s^M(v,\Omega,\varphi)$ and $\mathcal{N}_s^M(u,\Omega,\varphi)$ are finite for any $u, v \in W^{s,1}(\Omega)$, and any M. From the reformulation of the truncated, nonlocal part in Equation (3.4.3), we deduce

$$\begin{aligned}
&\mathcal{N}_s^M(u,\Omega,\varphi) - \mathcal{N}_s^M(v,\Omega,\varphi) \\
&\quad = 2 \int_\Omega \int_{\mathcal{C}\Omega} \left\{ \mathcal{G}_s\left(\frac{u(x)-\varphi(y)}{|x-y|} \right) - \mathcal{G}_s\left(\frac{v(x)-\varphi(y)}{|x-y|} \right) \right\} \frac{dx \; dy}{|x-y|^{n+s}}
\end{aligned} \tag{4.1.9}$$

for any $M > 0$, since all terms that do not contain u or v cancel. The right-hand side is independent of M, and thus Equation (4.1.5) follows. If φ fulfills Assumption (A1), then $\mathcal{N}_s(v,\Omega,\varphi)$ and $\mathcal{N}_s(u,\Omega,\varphi)$ are finite, and it holds

$$\mathcal{N}_s(u,\Omega,\varphi) - \mathcal{N}_s(v,\Omega,\varphi) = 2 \int_\Omega \int_{\mathcal{C}\Omega} \left\{ \mathcal{G}_s\left(\frac{u(x)-\varphi(y)}{|x-y|} \right) - \mathcal{G}_s\left(\frac{v(x)-\varphi(y)}{|x-y|} \right) \right\} \frac{dx \; dy}{|x-y|^{n+s}} \, ,$$

which is equal to the right-hand side in Equation (4.1.9). Thus, Equation (4.1.7) follows.

Then, for the s-fractional area functionals, it holds that

$$\begin{aligned}\mathcal{F}_s^M(u,\Omega,\varphi)-\mathcal{F}_s^M(v,\Omega,\varphi) &= \big(\mathcal{A}_s(u,\Omega)-\mathcal{A}_s(v,\Omega)\big)+\big(\mathcal{N}_s^M(u,\Omega,\varphi)-\mathcal{N}_s^M(v,\Omega,\varphi)\big)\\ &= \big(\mathcal{A}_s(u,\Omega)-\mathcal{A}_s(v,\Omega)\big)+\big(\mathcal{N}_s^{M'}(u,\Omega,\varphi)-\mathcal{N}_s^{M'}(v,\Omega,\varphi)\big)\\ &= \mathcal{F}_s^{M'}(u,\Omega,\varphi)-\mathcal{F}_s^{M'}(v,\Omega,\varphi)\ ,\end{aligned}$$

and, if φ fulfills Assumption (A1), also

$$\begin{aligned}\mathcal{F}_s^M(u,\Omega,\varphi)-\mathcal{F}_s^M(v,\Omega,\varphi) &= \big(\mathcal{A}_s(u,\Omega)-\mathcal{A}_s(v,\Omega)\big)+\big(\mathcal{N}_s^M(u,\Omega,\varphi)-\mathcal{N}_s^M(v,\Omega,\varphi)\big)\\ &= \big(\mathcal{A}_s(u,\Omega)-\mathcal{A}_s(v,\Omega)\big)+\big(\mathcal{N}_s(u,\Omega,\varphi)-\mathcal{N}_s(v,\Omega,\varphi)\big)\\ &= \mathcal{F}_s(u,\Omega,\varphi)-\mathcal{F}_s(v,\Omega,\varphi)\ .\end{aligned}$$

□

The Lipschitz continuity of $\mathcal{G}_s$ can be propagated to the fractional area functionals in the following sense.

Lemma 4.1.5: *For $u,v\in W^{s,1}(\Omega)$, and an exterior datum $\varphi:\mathcal{C}\Omega\to\mathbb{R}$, and $M>0$, it holds*

$$\begin{aligned}&|\mathcal{A}_s(u,\Omega)-\mathcal{A}_s(v,\Omega)|\le\frac{\Lambda_s}{2}\|u-v\|_{W^{s,1}}\ ,\\ &|\mathcal{N}_s^M(u,\Omega,\varphi)-\mathcal{N}_s^M(v,\Omega,\varphi)|\le C_2(n,\Omega,s)\frac{\Lambda_s}{2}\|u-v\|_{W^{s,1}}\ ,\\ &|\mathcal{F}_s^M(u,\Omega,\varphi)-\mathcal{F}_s^M(v,\Omega,\varphi)|\le\big(C_2(n,\Omega,s)+1\big)\frac{\Lambda_s}{2}\|u-v\|_{W^{s,1}}\ .\end{aligned}$$

If φ fulfills Assumption (A1), it also holds

$$\begin{aligned}&|\mathcal{N}_s(u,\Omega,\varphi)-\mathcal{N}_s(v,\Omega,\varphi)|\le C_2(n,\Omega,s)\frac{\Lambda_s}{2}\|u-v\|_{W^{s,1}}\ ,\\ &|\mathcal{F}_s(u,\Omega,\varphi)-\mathcal{F}_s(v,\Omega,\varphi)|\le\big(C_2(n,\Omega,s)+1\big)\frac{\Lambda_s}{2}\|u-v\|_{W^{s,1}}\ .\end{aligned}$$

The constant $C_2(n,\Omega,s)$ defined as in Lemma 7.3.3.

Proof. First, for the local part, using the Lipschitz continuity of $\mathcal{G}_s$, it holds

$$\begin{aligned}|\mathcal{A}_s(u,\Omega)-\mathcal{A}_s(v,\Omega)| &\le\int_\Omega\int_\Omega\left|\mathcal{G}_s\left(\frac{u(x)-u(y)}{|x-y|}\right)-\mathcal{G}_s\left(\frac{v(x)-v(y)}{|x-y|}\right)\right|\frac{dx\ dy}{|x-y|^{n+s}}\\ &\le\frac{\Lambda_s}{2}\int_\Omega\int_\Omega\frac{\big|\big((u(x)-v(x)\big)-\big(u(y)-v(y)\big)\big|}{|x-y|^{n+s}}\,dx\ dy\\ &\le\frac{\Lambda_s}{2}\|u-v\|_{W^{s,1}(\Omega)}\ .\end{aligned}$$

And for the truncated nonlocal part, it holds

$$\begin{aligned}|\mathcal{N}_s^M(u,\Omega,\varphi)-\mathcal{N}_s^M(v,\Omega,\varphi)| &\le \int_\Omega\int_{\mathcal{C}\Omega}\left|\mathcal{G}_s\left(\frac{u(x)-\varphi(y)}{|x-y|}\right)-\mathcal{G}_s\left(\frac{v(x)-\varphi(y)}{|x-y|}\right)\right|\frac{dx\ dy}{|x-y|^{n+s}}\\ &\le \frac{\Lambda_s}{2}\int_\Omega|u(x)-v(x)|\int_{\mathcal{C}\Omega}\frac{dy}{|x-y|^{n+s}}\,dx\\ &\le C_2(n,\Omega,s)\frac{\Lambda_s}{2}\|u-v\|_{W^{s,1}(\Omega)}\ .\end{aligned}$$

If additionally Assumption (A1) holds, then using Lemma 4.1.4, the same Lipschitz property holds for $\mathcal{N}_s$. For $\mathcal{F}_s$ and $\mathcal{F}_s^M$ the the Lipschitz continuity follows by spliting each into its local and nonlocal part. □

4.2 Fractional area estimates by truncation

If the exterior datum is bounded in a suitable large neighborhood of Ω, it is possible to obtain a functional inequality for the fractional area in terms of the truncation. This has been used by Cozzi, Lombardini [CL20] in order to prove L^∞ bounds for minimizers. First, we define the cut-off at height $N>0$ for $u:\Omega\to\mathbb{R}$ as $u^{[N]}=\max(-N,u^{(N)})$, where $u^{(N)}:=\min(N,u)$, and the δ-neighborhood Ω_δ of Ω by

$$\Omega_\delta=\{x\in\mathbb{R}^n\mid d(x,\Omega)<\delta\}\ . \tag{4.2.1}$$

We have the following.

Lemma 4.2.1: *There exists a positive constant Θ_s depending on n and s such that the following holds true: For any domain $\Omega\subset\mathbb{R}^n$ open, bounded and with Lipschitz boundary, $M\ge 0$, an exterior datum $\varphi:\mathcal{C}\Omega\to\mathbb{R}$ bounded from above on the annulus $P:=\Omega_{\Theta_s\mathrm{diam}(\Omega)}\setminus\Omega$, any $u:\Omega\to\mathbb{R}$, and any*

$$N\ge\mathrm{diam}(\Omega)+\sup_P\varphi\ ,$$

it holds

$$\begin{aligned}\mathcal{A}_s(u^{(N)},\Omega)&\le\mathcal{A}_s(u,\Omega)\ ,\\ \mathcal{N}_s(u^{(N)},\Omega,\varphi)&\le\mathcal{N}_s(u,\Omega,\varphi)\ ,\\ \mathcal{N}_s^M(u^{(N)},\Omega,\varphi)&\le\mathcal{N}_s^M(u,\Omega,\varphi)\ .\end{aligned}$$

For a proof, see [CL20, Prop. 3.5].

Remark 4.2.2: *There exists a constant $C(n)$ depending only on n such that for all $s\in(0,1)$*

$$\Theta_s\le\left(\frac{C(n)}{s}\right)^{1/s}$$

Proof. From the proof in [CL20, Prop. 3.5] it follows that Θ_s has to be chosen such that

$$\Theta_s \geq \left(\frac{C_2}{C_1}\right)^{1/s}$$

with

$$C_1 = \frac{G_s(1/2)|D_1|}{(2R_0)^n} \ ,$$

and

$$C_2 = \frac{\mathcal{H}^{n-1}(S^{n-1})}{s}\frac{\Lambda_s}{2} \ ,$$

where $D_1 \subseteq \Omega_{R_0}$ and $R_0 := \operatorname{diam}(\Omega)$. It follows

$$|D_1| \leq |\Omega_{R_0}| \leq |B_{3R_0}| = (3R_0)^n\omega_n \ .$$

Combining the estimates, we can conclude

$$\left(\frac{C_2}{C_1}\right)^{1/s} \leq \left(\frac{\mathcal{H}^{n-1}(S^{n-1})\Lambda_s}{2\omega_n s G_s(1/2)}\right)^{1/s} \leq \left(\frac{C(n)}{s}\right)^{1/s} \qquad \square$$

Lemma 4.2.3: *If Θ_s, Ω, P, M as above, and an exterior datum $\varphi : \mathcal{C}\Omega \to \mathbb{R}$ bounded from above and below in P, then for any $u : \Omega \to \mathbb{R}$, and any*

$$N \geq \operatorname{diam}(\Omega) + \sup_P |\varphi| \ ,$$

it holds

$$\begin{aligned} \mathcal{A}_s(u^{[N]}, \Omega) &\leq \mathcal{A}_s(u, \Omega) \ , \\ \mathcal{N}_s(u^{[N]}, \Omega, \varphi) &\leq \mathcal{N}_s(u, \Omega, \varphi) \ , \\ \mathcal{N}_s^M(u^{[N]}, \Omega, \varphi) &\leq \mathcal{N}_s^M(u, \Omega, \varphi) \ . \end{aligned}$$

Proof. Since $\mathcal{G}_s$ is even, it holds

$$\mathcal{G}_s\left(\frac{\varphi(y) - u(x)}{|x-y|}\right) = \mathcal{G}_s\left(\frac{u(x) - \varphi(y)}{|x-y|}\right) \ ,$$

and thus $\mathcal{N}_s(u, \Omega, \varphi) = \mathcal{N}_s(-u, \Omega, -\varphi)$. For the truncation by switching the direction of integration, it holds

$$\int_{\frac{-M+\varphi(y)}{|x-y|}}^{\frac{-u(x)+\varphi(y)}{|x-y|}} \overline{G}_s(t)\,dt + \int_{\frac{-u(x)+\varphi(y)}{|x-y|}}^{\frac{M+\varphi(y)}{|x-y|}} \overline{G}_s(-t)\,dt = \int_{\frac{u(x)-\varphi(y)}{|x-y|}}^{\frac{M-\varphi(y)}{|x-y|}} \overline{G}_s(-t)\,dt + \int_{\frac{-M-\varphi(y)}{|x-y|}}^{\frac{u(x)-\varphi(y)}{|x-y|}} \overline{G}_s(t)\,dt \ ,$$

and thus $\mathcal{N}_s^M(-u,\Omega,-\varphi) = \mathcal{N}_s^M(u,\Omega,\varphi)$. Using Lemma 4.2.1, it follows

$$\begin{aligned}\mathcal{N}_s^M(u,\Omega,\varphi) &\geq \mathcal{N}_s^M(u^{(N)},\Omega,\varphi) = \mathcal{N}_s^M(-u^{(N)},\Omega,-\varphi)\\ &\geq \mathcal{N}_s^M(\big(-u^{(N)}\big)^{(N)},\Omega,-\varphi) = \mathcal{N}_s^M(-u^{[N]},\Omega,-\varphi)\\ &= \mathcal{N}_s^M(u^{[N]},\Omega,\varphi)\ .\end{aligned}$$

The same argument can be conducted for $\mathcal{A}_s$ and $\mathcal{N}_s$. □

This lemma justifies the introduction of the following assumption regarding the boundedness of the exterior datum φ around Ω.

Assumption (A2): For a fixed $s \in (0,1)$, the exterior datum $\varphi : \mathcal{C}\Omega \to \mathbb{R}$ is bounded on $\Omega_{\Theta_s \mathrm{diam}(\Omega)} \setminus \Omega$ with Θ_s as in Lemma 4.2.1.

For any function $u : \Omega \to \mathbb{R}$, the truncations $u^{[N]}$ converge everywhere pointwise as $N \to \infty$. Moreover, $u^{[N]}$ are dominated by u, so that L^1 convergence follows if $u \in L^1(\Omega)$. Also, stronger modes of convergence can be proved if u is more regular. The following statements are relevant for our discussion here.

Lemma 4.2.4: *For an open set $\Omega \subseteq \mathbb{R}^n$ and $u \in W^{s,1}(\Omega)$ it holds that $u^{[N]} \to u$ in $W^{s,1}(\Omega)$ as $N \to \infty$. Thus, $W^{s,1}(\Omega) \cap L^\infty(\Omega)$ is dense in $W^{s,1}(\Omega)$.*

Proof. For $u \in W^{s,1}(\Omega)$ by dominated convergence $u^{[N]} \to u$ in $L^1(\Omega)$ as $N \nearrow \infty$. Furthermore, since

$$|u^{[N]}(x) - u^{[N]}(y)| \leq |u(x) - u(y)|\ ,$$

it follows $u^{[N]} \in W^{s,1}(\Omega)$ and again by dominated convergence

$$\lim_{N\nearrow\infty} [u - u^{[N]}]_{W^{s,1}} = \int_\Omega \int_\Omega \frac{\big|u(x) - u(y) - \big(u^{[N]}(x) - u^{[N]}(y)\big)\big|}{|x-y|^{n+s}}\, dx\ \to 0$$

as $N \to \infty$ by dominated convergence, since the integrand is dominated by $\frac{2|u(x)-u(y)|}{|x-y|^{n+s}}$. □

The same holds true for BV functions.

Lemma 4.2.5 (Density of truncations)**:** *For an open set $\Omega \subseteq \mathbb{R}^n$ and $u \in \mathrm{BV}(\Omega)$ it holds that $u^{[N]} \to u$ in $\mathrm{BV}(\Omega)$ as $N \to \infty$. Thus, $\mathrm{BV}(\Omega) \cap L^\infty(\Omega)$ is dense in $\mathrm{BV}(\Omega)$.*

Proof. For $u \in \mathrm{BV}(\Omega)$ by dominated convergence $u^{[N]} \to u$ in $L^1(\Omega)$ as $N \nearrow \infty$, and using the co-area formula, see ,i.e., [EG15, Theorem 5.9, p. 212], it holds

$$\|Du\|(\Omega) = \int_{-\infty}^{\infty} \mathrm{Per}(\{u > t\}, U)\, dt\ \geq \int_{-N}^{N} \mathrm{Per}(\{u > t\}, U)\, dt\ = \|Du^{[N]}\|(\Omega)\ .$$

Thus, $u^{[N]} \in \mathrm{BV}(\Omega)$. Using the co-area formula again, it holds

$$\begin{aligned}\|D(u-u^{[N]})\|(\Omega) &= \int_{-\infty}^{\infty} \mathrm{Per}(\{u-u^{[N]}>t\},U)\,dt \\ &\geq \int_{-\infty}^{-N} \mathrm{Per}(\{u>t\},U)\,dt + \int_{N}^{\infty} \mathrm{Per}(\{u>t\},U)\,dt \to 0\end{aligned}$$

as $N \to \infty$ by monotone convergence. Thus, $\|u-u^{[N]}\|_{\mathrm{BV}(\Omega)} \to 0$ as $N \to \infty$. □

4.3 Lower semicontinuity

The next step is to prove the lower semicontinuity of the s-fractional area functional. The s-fractional area functional and the truncated functional differ from each other because the latter relies on cancellation to remain finite, regardless of the datum φ. This means that the integrand cannot be assumed to be nonnegative, so Fatou's Lemma cannot be used to interchange the limit and the integral. Therefore, the lower semicontinuity can be proved a priori only for the nontruncated functionals. Nonetheless, in the following, we provide a new proof of the fact that under Assumption (A2) the lower semicontinuity can be derived for the truncated functionals $\mathcal{F}_s^M$ on L^p as well.

Theorem 4.3.1 (Lower Semicontiniuty)**:** *For an exterior datum $\varphi : \mathcal{C}\Omega \to \mathbb{R}^n$, the s-fractional area functionals $\mathcal{A}_s(\,\cdot\,,\Omega)$, $\mathcal{N}_s(\,\cdot\,,\Omega,\varphi)$ and $\mathcal{F}_s(\,\cdot\,,\Omega,\varphi)$ are lower semicontinuous with respect to the strong topology on $L^p(\Omega)$.*

Proof. Let $u_n \to u$ in $L^p(\Omega)$ be given. W.l.o.g, assume $\liminf_{n\to\infty} \mathcal{A}_s(u_n,\Omega) < \infty$ as the claim is trivially satisfied otherwise. By extracting subsequences without relabeling assume that $\liminf_{n\to\infty} \mathcal{A}_s(u_n,\Omega) = \lim_{n\to\infty} \mathcal{A}_s(u_n,\Omega)$ holds and that u_n converges pointwise to u almost everywhere. Then, using Fatou's lemma, the positivity and the continuity of $\mathcal{G}_s$,we obtain

$$\begin{aligned}\liminf_{n\to\infty} \mathcal{A}_s(u_n,\Omega) &= \liminf_{n\to\infty} \int_\Omega\int_\Omega \mathcal{G}_s\left(\frac{u_n(x)-u_n(y)}{|x-y|}\right)\frac{dy\ dx}{|x-y|^{n-1+s}} \\ &\geq \int_\Omega\int_\Omega \liminf_{n\to\infty} \mathcal{G}_s\left(\frac{u_n(x)-u_n(y)}{|x-y|}\right)\frac{dy\ dx}{|x-y|^{n-1+s}} \\ &= \int_\Omega\int_\Omega \lim_{n\to\infty} \mathcal{G}_s\left(\frac{u_n(x)-u_n(y)}{|x-y|}\right)\frac{dy\ dx}{|x-y|^{n-1+s}} \\ &= \int_\Omega\int_\Omega \mathcal{G}_s\left(\frac{u(x)-u(y)}{|x-y|}\right)\frac{dy\ dx}{|x-y|^{n-1+s}} = \mathcal{A}_s(u,\Omega)\ .\end{aligned}$$

The analogous argument applies to $\mathcal{N}_s$ so that the lower semicontinuity is obtained. □

The lower semicontinuity for $\mathcal{N}_s^M$ may only be obtained for functions bounded by M due to the positivity of the integrand.

Lemma 4.3.2: *For $M > 0$, an exterior datum $\varphi : \mathcal{C}\Omega \to \mathbb{R}^n$, and a sequence $u_n \to u$ in $L^p(\Omega)$ such that $\|u_n\|_{L^\infty(\Omega)} \leq M$ it holds*

$$\liminf_{n\to\infty} \mathcal{N}_s^M(u_n, \Omega, \varphi) \geq \mathcal{N}_s^M(u, \Omega, \varphi) \ .$$

Proof. From $\|u_n\|_{L^\infty(\Omega)} \leq M$ it follows that

$$\frac{u(x) - \varphi(y)}{|x-y|} \geq \frac{-M - \varphi(y)}{|x-y|}$$

and

$$\frac{M - \varphi(y)}{|x-y|} \geq \frac{u(x) - \varphi(y)}{|x-y|} \ .$$

Thus, it holds that

$$\int_{\frac{-M-u(y)}{|x-y|}}^{\frac{u(x)-u(y)}{|x-y|}} \overline{G}(t)\, dt \ + \int_{\frac{u(x)-u(y)}{|x-y|}}^{\frac{M-u(y)}{|x-y|}} \overline{G}(-t)\, dt \ \geq 0$$

for all $x \in \Omega$, $y \in \mathcal{C}\Omega$ since $\overline{G} \geq 0$. By Fatou's Lemma, we have

$$\liminf_{n\to\infty} \mathcal{N}_s^M(u_n, \Omega, \varphi) \geq \mathcal{N}_s^M(u, \Omega, \varphi) \ . \qquad \square$$

For this reason, it is natural to consider the subset of $L^p(\Omega)$ functions bounded by $M > 0$ and the restricted functional.

Definition 4.3.3: *For an open set $\Omega \subset \mathbb{R}^n$ and any function space of measurable functions $V(\Omega)$ define the subsets*

$$\mathfrak{B}V(\Omega) := \{u \in V(\Omega) \mid \|u\|_{L^\infty(\Omega)} < \infty\} \ ,$$
$$\mathfrak{B}_M V(\Omega) := \{u \in V(\Omega) \mid \|u\|_{L^\infty(\Omega)} \leq M\} \ .$$

Lemma 4.3.4: *The L^∞-norm is lower semicontinuous with respect to the L^p-norm on an open set $\Omega \subseteq \mathbb{R}^n$. Therefore the sublevel-sets $\mathfrak{B}_M L^p$ are closed, convex subsets of $L^p(\Omega)$ for $p \in [1, \infty]$ and $M > 0$.*

Proof. For a sequence $u_n \xrightarrow{L^p} u$ with $\|u_n\|_{L^\infty(\Omega)} \leq M$, $x \in \Omega$, and $r > 0$, such that the ball $B_r(x) \subset \Omega$, it holds

$$\begin{aligned}\frac{1}{|B_r(x)|} \int_{B_r(x)} |u|^p dx &\leq \frac{1}{|B_r(x)|} \left(\|u_n - u\|^p_{L^p(\Omega)} + \int_{B_r(x)} |u_n|^p dx \right) \\ &\leq \frac{1}{|B_r(x)|} \|u_n - u\|^p_{L^p(\Omega)} + \|u_n\|^p_{L^\infty(\Omega)} \ .\end{aligned}$$

Then, let $n \to \infty$ such that $\|u_n - u\|^p_{L^p(\Omega)} \to 0$ and we can deduce

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} |u|^p dx \leq M^p \ .$$

Thus, by Lebesgue Differentiation Theorem, the left-hand side converges to $|u(x)|^p$ for almost every $x \in \Omega$ as $r \to 0$ and so $\|u\|_{L^\infty(\Omega)} \leq M$. The convexity is a direct consequence of the triangle inequality. □

For a Banach space X, and a subset $A \subseteq X$ the indicator function ι_A is defined as

$$\iota_A(x) := \begin{cases} 0, & \text{if } x \in A, \\ +\infty, & \text{otherwise.} \end{cases}$$

Definition 4.3.5: *For any functional Φ on $L^p(\Omega)$ and $M > 0$ define its restriction $\mathfrak{R}_M \Phi$ as*

$$\mathfrak{R}_M \Phi := \Phi + \iota_{\mathfrak{B}_M L^p(\Omega)} \ .$$

Then, Lemma 4.3.2 may be formulated with the help of the restriction in the following way.

Theorem 4.3.6: *For an exterior datum $\varphi : \mathcal{C}\Omega \to \mathbb{R}^n$, the restriction of the truncated, nonlocal s-fractional area functional $\mathfrak{R}_M \mathcal{N}_s^M(\,\cdot\,, \Omega, \varphi)$ is lower semicontinuous on $L^p(\Omega)$*

Proof. As $\mathfrak{R}_M \mathcal{N}_s^M$ is finite only in $\mathfrak{B}_M L^p(\Omega)$, where it equals $\mathcal{N}_s^M$, its lower semicontinuity can be derived by Theorem 4.3.1 as $\mathfrak{B}_M L^p(\Omega)$ is closed in $L^p(\Omega)$. □

Yet, the equality for the differences in Lemma 4.1.4 indicates that the lower semicontinuity does not actually depend on M.

Lemma 4.3.7: *For any $M, N > 0$, an exterior datum $\varphi : \mathcal{C}\Omega \to \mathbb{R}$ and $u_n, u \in \mathfrak{B}_N L^p(\Omega)$ such that $u_n \to u$ in L^p, we have*

$$\liminf_{n\to\infty} \mathcal{N}_s^M(u_n, \Omega, \varphi) \geq \mathcal{N}_s^M(u, \Omega, \varphi) \ .$$

Proof. By Lemma 4.3.2, $\mathcal{N}_s^N$ is lower semicontinuous for functions bounded by N and with Lemma 4.1.4, it holds

$$\liminf_{n\to\infty} \mathcal{N}_s^M(u_n, \Omega, \varphi) - \mathcal{N}_s^M(u, \Omega, \varphi) = \liminf_{n\to\infty} \big(\mathcal{N}_s^N(u_n, \Omega, \varphi) - \mathcal{N}_s^N(u, \Omega, \varphi)\big) \geq 0 \ .$$

□

Under Assumption (A2), this result can be extended further with the help of Lemma 4.2.3.

Theorem 4.3.8: *For $s \in (0,1)$, an exterior datum $\varphi : \mathcal{C}\Omega \to \mathbb{R}$ that fullfills the Assumption (A2) for s, and $M > 0$, it holds that $\mathcal{N}_s^M(\,\cdot\,, \Omega, \varphi)$ is lower semicontinuous on $L^p(\Omega)$.*

Proof. According to Assumption (A2), φ is bounded on $P := \Omega_{\Theta_s \mathrm{diam}(\Omega)} \setminus \Omega$. For $u_n \to u$ in $L^p(\Omega)$ and $\varepsilon > 0$, by the convergence of the truncations $u^{[N]}$ in $W^{s,1}(\Omega)$, see Lemma 4.2.4, and Assumption (A2) choose $N > 0$ large enough such that

$$\|u^{[N]} - u\|_{W^{s,1}(\Omega)} < \varepsilon \quad \text{and} \quad N \geq \mathrm{diam}(\Omega) + \sup_P |\varphi| \, .$$

Using the Lipschitz continuity of the fractional area from Lemma 4.1.5, it follows

$$|\mathcal{N}_s^M(u^{[N]}, \Omega, \varphi) - \mathcal{N}_s^M(u, \Omega, \varphi)| \leq C\|u^{[N]} - u\|_{W^{s,1}(\Omega)} \leq C\varepsilon \, .$$

By the lower-semicontinuity result in Lemma 4.3.7, applied to the truncated functions $u_n^{[N]}$ and $u^{[N]}$, together with the Lemma 4.2.3, it holds

$$\begin{aligned} 0 &\leq \liminf_{n\to\infty} \big(\mathcal{N}_s^M(u_n^{[N]}, \Omega, \varphi) - \mathcal{N}_s^M(u^{[N]}, \Omega, \varphi)\big) \\ &\leq \liminf_{n\to\infty} \big(\mathcal{N}_s^M(u_n, \Omega, \varphi) - \mathcal{N}_s^M(u^{[N]}, \Omega, \varphi)\big) \\ &\leq \liminf_{n\to\infty} \big(\mathcal{N}_s^M(u_n, \Omega, \varphi) - \mathcal{N}_s^M(u, \Omega, \varphi)\big) + \big|\mathcal{N}_s^M(u^{[N]}, \Omega, \varphi) - \mathcal{N}_s^M(u, \Omega, \varphi)\big| \, . \end{aligned}$$

Thus,

$$\mathcal{N}_s^M(u, \Omega, \varphi) - C\varepsilon \leq \liminf_{n\to\infty} \mathcal{N}_s^M(u_n, \Omega, \varphi)$$

and as ε is abitrary, the lower semicontinuity of $\mathcal{N}_s^M(\,\cdot\,, \Omega, \varphi)$ follows. □

4.4 Convexity

Theorem 4.4.1 (Convexity)**:** *For an exterior datum $\varphi : \mathcal{C}\Omega \to \mathbb{R}^n$, the following holds:*

1. *The functional $\mathcal{A}_s(\,\cdot\,, \Omega)$ is convex on $W^{s,1}(\Omega)$.*
2. *The functionals $\mathcal{N}_s^M(\,\cdot\,, \Omega, \varphi)$ and $\mathcal{F}_s^M(\,\cdot\,, \Omega, \varphi)$ are strictly convex on $W^{s,1}(\Omega)$.*
3. *If Assumption (A1) holds then, the functionals $\mathcal{N}_s(\,\cdot\,, \Omega, \varphi)$ and $\mathcal{F}_s(\,\cdot\,, \Omega, \varphi)$ are strictly convex on $W^{s,1}(\Omega)$.*

Proof. 1. The convexity of $\mathcal{A}_s$ follows directly from the convexity of $\mathcal{G}_s$. Namely, for $t \in [0,1]$ and $u, v \in W^{s,1}(\Omega)$ we have

$$\begin{aligned} &\mathcal{A}_s(tu + (1-t)v, \Omega) \\ &\quad = \int_\Omega \int_\Omega \mathcal{G}_s\left(\frac{\big(tu + (1-t)v\big)(x) - \big(tu + (1-t)v\big)(y)}{|x-y|}\right) \frac{dx\ dy}{|x-y|^{n+s}} \\ &\quad \leq \int_\Omega \int_\Omega \left(t\,\mathcal{G}_s\left(\frac{u(x) - u(y)}{|x-y|}\right) + (1-t)\,\mathcal{G}_s\left(\frac{v(x) - v(y)}{|x-y|}\right)\right) \frac{dx\ dy}{|x-y|^{n+s}} \\ &\quad = t\,\mathcal{A}_s(u, \Omega) + (1-t)\,\mathcal{A}_s(v, \Omega) \, . \end{aligned}$$

2. In general, the convexity for $\mathcal{N}_s$ can be derived analogously by just changing the domains of integration. If the exterior datum $\varphi : \mathcal{C}\Omega \to \mathbb{R}$ is fixed and fulfills Assumption (A1), then $\mathcal{N}_s$ is finite for $u, v \in W^{s,1}(\Omega)$ and strict convexity is obtained, as horizontal shifts are not admissible. Namely, we have

$$\begin{aligned}&\mathcal{N}_s(tu+(1-t)v,\Omega,\varphi) - t\mathcal{N}_s(u,\Omega,\varphi) - (1-t)\mathcal{N}_s(v,\Omega,\varphi)\\ &\quad= 2\int_\Omega\int_{\mathcal{C}\Omega}\bigg[\mathcal{G}_s\Big(t\frac{u(x)-\varphi(y)}{|x-y|} + (1-t)\frac{v(x)-\varphi(y)}{|x-y|}\Big) - t\mathcal{G}_s\Big(\frac{u(x)-\varphi(y)}{|x-y|}\Big)\\ &\qquad\qquad - (1-t)\mathcal{G}_s\Big(\frac{v(x)-\varphi(y)}{|x-y|}\Big)\bigg]\frac{dy\ dx}{|x-y|^{n-1+s}}\ .\end{aligned}$$

By the strict convexity of $\mathcal{G}_s$ for the integrand to be 0 it is required that

$$\frac{u(x)-\varphi(y)}{|x-y|} = \frac{v(x)-\varphi(y)}{|x-y|} \text{ for almost all } (x,y)\in\Omega\times\mathcal{C}\Omega\ .$$

Which implies $u = v$ almost everywhere in Ω.

3. For the truncation, in Equation (3.4.3) we have seen that

$$\begin{aligned}\mathcal{N}_s^M(u,\Omega,\varphi) = &\int_\Omega\int_{\mathcal{C}\Omega}\bigg\{2\mathcal{G}_s\Big(\frac{u(x)-\varphi(y)}{|x-y|}\Big) - \mathcal{G}_s\Big(\frac{-M-\varphi(y)}{|x-y|}\Big)\\ &- \mathcal{G}_s\Big(\frac{M-\varphi(y)}{|x-y|}\Big)\bigg\}\frac{dy\ dx}{|x-y|^{n-1+s}} - M\Lambda_s\int_\Omega\int_{\mathcal{C}\Omega}\frac{dx\ dy}{|x-y|^{n+s}}\ .\end{aligned}$$

So, when we consider the following difference

$$\mathcal{N}_s^M(tu+(1-t)v,\Omega,\varphi) - t\mathcal{N}_s^M(u,\Omega,\varphi) - (1-t)\mathcal{N}_s^M(v,\Omega,\varphi)\ ,$$

the parts that do not depend on u and v cancel. One obtains

$$\begin{aligned}&\mathcal{N}_s^M(tu+(1-t)v,\Omega,\varphi) - t\mathcal{N}_s^M(u,\Omega,\varphi) - (1-t)\mathcal{N}_s^M(v,\Omega,\varphi)\\ &\quad= 2\int_\Omega\int_{\mathcal{C}\Omega}\bigg[\mathcal{G}_s\Big(t\frac{u(x)-\varphi(y)}{|x-y|} + (1-t)\frac{v(x)-\varphi(y)}{|x-y|}\Big) - t\mathcal{G}_s\Big(\frac{u(x)-\varphi(y)}{|x-y|}\Big)\\ &\qquad\qquad - (1-t)\mathcal{G}_s\Big(\frac{v(x)-\varphi(y)}{|x-y|}\Big)\bigg]\frac{dy\ dx}{|x-y|^{n-1+s}}\ .\end{aligned}$$

Then, the strict convexity for $\mathcal{N}_s^M$ follows equivalently as in the previous step. □

4.5 The subdifferential of the fractional area functional

We now aim at characterizing the L^2-subdifferential of the functionals $\mathcal{F}_s$ and $\mathcal{F}_s^M$. For a fixed exterior datum $\varphi : \mathcal{C}\Omega \to \mathbb{R}$ and $u \in L^2(\Omega)$, a function $v \in L^2(\Omega)$ is in the subdifferential of $\mathcal{F}_s$ at u, namely $v \in \partial\mathcal{F}_s(u,\Omega,\varphi) \subseteq L^2(\Omega)$, if the following two conditions hold

$$u \in D(\mathcal{F}_s) = W^{s,1}(\Omega) \,, \tag{4.5.1}$$

$$\mathcal{F}_s(w,\Omega,\varphi) - \mathcal{F}_s(u,\Omega,\varphi) \geq \langle v, w-u\rangle_{L^2} \text{ for all } w \in L^2(\Omega) \,. \tag{4.5.2}$$

From Lemma 4.1.4, we can conclude the following equality of the subdifferentials.

Theorem 4.5.1: *For a fixed exterior datum $\varphi : \mathcal{C}\Omega \to \mathbb{R}$ and $M, M' > 0$ it holds*

$$\partial\mathcal{F}_s^M(u,\Omega,\varphi) = \partial\mathcal{F}_s^{M'}(u,\Omega,\varphi)$$

for all $u \in L^2(\Omega)$. Furthermore, if φ fulfills Assumption (A1), then

$$\partial\mathcal{F}_s(u,\Omega,\varphi) = \partial\mathcal{F}_s^M(u,\Omega,\varphi)$$

for all $u \in L^2(\Omega)$.

Proof. Assume that $h \in \partial\mathcal{F}_s^M(u,\Omega,\varphi)$ for $u \in W^{s,1}(\Omega)$, it then follows from Equation (4.1.6) for any $v \in W^{s,1}(\Omega)$ that

$$\langle h, v-u\rangle_{L^2} \leq \mathcal{F}_s^M(v,\Omega,\varphi) - \mathcal{F}_s^M(u,\Omega,\varphi) = \mathcal{F}_s^{M'}(u,\Omega,\varphi) - \mathcal{F}_s^{M'}(v,\Omega,\varphi) \,.$$

From the discussion in Section 4.1 it follows that

$$D(\mathcal{F}_s^M(\,\cdot\,,\Omega,\varphi)) = W^{s,1}(\Omega) = D(\mathcal{F}_s^{M'}(\,\cdot\,,\Omega,\varphi)) \,.$$

This implies that u fulfills Equation (4.5.1) and h Equation (4.5.2) respectively, for $\mathcal{F}_s^{M'}$. Thus, $h \in \partial\mathcal{F}_s^{M'}(u,\Omega,\varphi)$. Switching the roles of M and M' yields the equality.
$\partial\mathcal{F}_s(\,\cdot\,,\Omega,\varphi) = \partial\mathcal{F}_s^M(\,\cdot\,,\Omega,\varphi)$ follows under Assumption (A1) with Equation (4.1.8) in the same way. $\square$

Cozzi, Lombardini [CL20] have proved that the Gâteaux derivative of the nonlocal area functional considered on $W^{s,1}(\Omega)$ is given by the operator $\mathscr{H}$. This operator is closely related to the s-mean curvature of the subgraph, see [CL20] and the references therein for more information. In the following, the operator $\mathscr{H}$ is recalled, and the relation to the subdifferential is explored.

Definition 4.5.2 (Pointwise Definition of $\mathscr{H}$)**:** *For $u : \mathbb{R}^n \to \mathbb{R}$ measurable we define the integro-differential operator $\mathscr{H}$ acting on u at a point $x \in \mathbb{R}^n$ by the principal value integral*

$$\mathscr{H}u(x) := 2\mathrm{P.V.}\int_{\mathbb{R}^n} G_s\left(\frac{u(x)-u(y)}{|x-y|}\right)\frac{dx}{|x-y|^{n+s}} = \mathrm{P.V.}\int_{\mathbb{R}^n} \frac{\delta_s(u,x;\xi)}{|\xi|^{n+s}}\,d\xi \;,$$

where

$$\delta_s(u,x;\xi) := G_s\left(\frac{u(x)-u(x+\xi)}{|\xi|}\right) - G_s\left(\frac{u(x-\xi)-u(x)}{|\xi|}\right) .$$

The Cauchy principal value (P.V.*) is defined as the limit $\varepsilon \searrow 0$ of the integral on the domain $\mathbb{R}^n \setminus B_\varepsilon(x)$ for the first, and on the domain $\mathbb{R}^n \setminus B_\varepsilon(0)$ for the latter.*

This pointwise limit is well-defined in case u is regular enough around x.

Lemma 4.5.3: *For $u : \mathbb{R}^n \to \mathbb{R}$ measurable such that $u \in \mathcal{C}^{1,\gamma}(B_R(x))$ for some $x \in \mathbb{R}^n$, $R > 0$, and $\gamma \in (s, 1]$, $\mathscr{H}u(x)$ is finite with*

$$\mathscr{H}u(x) = \int_{\mathbb{R}^n} \frac{\delta_s(u, x; \xi)}{|\xi|^{n+s}}\, d\xi \ \leq \omega_{n-1} \left(\frac{R^{\gamma-s}}{\gamma - s} \|u\|_{\mathcal{C}^{1,\gamma}(B_R(x))} + \frac{R^{-s}}{s} \Lambda_s \right) .$$

Notice that the integral is defined in the usual Lebesgue sense, not as a principal value.

Proof. In [CL20, Lemma 2.14.] it has been proved, that

$$\frac{|\delta_s(u, x; \xi)|}{|\xi|^{n+s}} \leq 2^\gamma \|u\|_{\mathcal{C}^{1,\gamma}(B_R(x))} |\xi|^{-n-s+\gamma}$$

for all $\xi \in B_R(0) \backslash 0$, which is integrable on $B_R(0) \backslash \{0\}$. Furthermore, the bound $|G_s| \leq \frac{\Lambda}{2}$ implies

$$\frac{|\delta_s(u, x; \xi)|}{|\xi|^{n+s}} \leq \frac{\Lambda}{|\xi|^{n+s}} \ ,$$

and thus integrability on $\mathbb{R}^n \setminus B_R(0)$. Combining the estimates and using dominated convergence yields the result. The bound is obtained by integrating the bounds on the respective domains. $\square$

If u is not regular enough, the pointwise quantity is generally not well-defined, but $\mathscr{H}$ can nevertheless be understood as an operator $\mathscr{H}$ defined in the following weak sense.

Definition 4.5.4 (The weak defintion of $\mathscr{H}$)**:** *For $u, v : \mathbb{R}^n \to \mathbb{R}$ measurable, we define the operator $\mathscr{H}$ weakly for u by its action on v, namely*

$$\langle \mathscr{H}u, v \rangle := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G_s\left(\frac{u(x) - u(y)}{|x - y|} \right) \big(v(x) - v(y)\big) \frac{dy\ dx}{|x - y|^{n+s}} \ .$$

Then, for any u and for $v \in W^{s,1}(\mathbb{R}^n)$ it holds

$$\langle \mathscr{H}u, v \rangle \leq \frac{\Lambda_s}{2} \|v\|_{W^{s,1}(\mathbb{R}^n)} \ ,$$

and thus the weak notion defines a linear map $\langle \mathscr{H}u, \cdot \rangle \in (W^{s,1}(\mathbb{R}^n))^*$.

Proof. The boundedness directly follows from the bound $G_s \leq \frac{\Lambda_s}{2}$ and the regularity assumed for v. $\square$

It is also possible to consider the functional on $W^{s,1}(\Omega)$ for $\Omega \subset \mathbb{R}^n$ open, bounded and with Lipschitz boundary, since in this case $\|\cdot\|_{W^{s,1}(\mathbb{R}^n)}$ is an equivalent norm on $W^{s,1}(\Omega)$, see [CL20, Corollary A.3]. Thus, also consider the following definition

Definition 4.5.5: *For $u : \mathbb{R}^n \to \mathbb{R}$ measurable, we define $\mathscr{H}$ weakly as $\langle \mathscr{H}u, \cdot \rangle \in (W^{s,1}(\Omega))^*$ by*

$$\langle \mathscr{H}u, v\rangle := \int_\Omega \int_\Omega G_s\Big(\frac{u(x)-u(y)}{|x-y|}\Big)\big(v(x)-v(y)\big)\frac{dy\ dx}{|x-y|^{n+s}} + 2\int_\Omega v(x)\int_{\mathcal{C}\Omega} G_s\Big(\frac{u(x)-u(y)}{|x-y|}\Big)\frac{dy\ dx}{|x-y|^{n+s}}\ .$$

This agrees with the previous notion of $\mathscr{H}$ in Definition 4.5.4 by viewing $u \in W^{s,1}(\Omega)$ as extended by 0 in the complement $\mathcal{C}\Omega$. Also define the operator $\mathscr{H}^\varphi$ for $u : \Omega \to \mathbb{R}$, where the values on $\mathcal{C}\Omega$ are given by an exterior datum φ.

Definition 4.5.6: *For an exterior datum $\varphi : \mathcal{C}\Omega \to \mathbb{R}$, and $u : \Omega \to \mathbb{R}$, both measurable, define $\mathscr{H}^\varphi$ weakly as $\langle \mathscr{H}^\varphi u, \cdot\rangle \in (W^{s,1}(\Omega))^*$ by*

$$\langle \mathscr{H}^\varphi u, v\rangle := \langle \mathscr{H}u_\varphi, v\rangle, \quad \textit{where} \qquad u_\varphi(x) := \begin{cases} u(x) & \textit{if } x \in \Omega, \\ \varphi(x) & \textit{if } x \in \mathcal{C}\Omega. \end{cases}$$

Now, the connection to the s-fractional area functional is recalled.

Lemma 4.5.7: *For an exterior datum $\varphi : \mathcal{C}\Omega \to \mathbb{R}$, $M > 0$, and $u \in W^{s,1}(\Omega)$, it holds*

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{F}_s^M(u+\varepsilon v, \Omega, \varphi) = \langle \mathscr{H}^\varphi u, v\rangle \quad \textit{for all } v \in W^{s,1}(\Omega)\ ,$$

and, analogously, if φ fulfills Assumption (A1), then it holds

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{F}_s(u+\varepsilon v, \Omega, \varphi) = \langle \mathscr{H}^\varphi u, v\rangle \quad \textit{for all } v \in W^{s,1}(\Omega)\ .$$

This implies, that

$$D\mathcal{F}_s^M(u, \Omega, \varphi) = \langle \mathscr{H}^\varphi u, \cdot\rangle\ ,$$

and under Assumption (A1) also

$$D\mathcal{F}_s(u, \Omega, \varphi) = \langle \mathscr{H}^\varphi u,\ \cdot\ \rangle\ .$$

Where D denotes the Gâteaux derivative in $W^{s,1}(\Omega)$.

Proof. Cozzi, Lombardini [CL20, p. 2.16] have proved, that for any $\tilde{u}, \tilde{v} : \mathbb{R}^n \to \mathbb{R}$, such that both $\tilde{u}_{|\Omega}, \tilde{v}_{|\Omega} \in W^{s,1}(\Omega)$, $\tilde{u} \equiv \varphi$ and $\tilde{v} \equiv 0$ on $\mathcal{C}\Omega$, it holds

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{F}_s^M(u+\varepsilon v, \Omega) = \langle \mathscr{H}u, v\rangle\ .$$

As the exterior datum is incorporated into the functional, this translates for the setting used in this thesis to

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{F}_s^M(u+\varepsilon v, \Omega, \varphi) = \langle \mathscr{H}^\varphi u, v\rangle$$

for any $u, v \in W^{s,1}(\Omega)$. Furthermore, with Assumption (A1) it holds

$$\frac{\mathcal{F}_s(u+\varepsilon v, \Omega, \varphi) - \mathcal{F}_s(u, \Omega, \varphi)}{\varepsilon} = \frac{\mathcal{F}_s^M(u+\varepsilon v, \Omega, \varphi) - \mathcal{F}_s^M(u, \Omega, \varphi)}{\varepsilon}$$

by Equation (4.1.8). Thus, the Gâteaux derivatives are equal under Assumption (A1). $\square$

Theorem 4.5.8 (L^2 subgradient and $\mathscr{H}$)**:** *For an exterior datum $\varphi : \mathcal{C}\Omega \to \mathbb{R}$, $M > 0$, $u \in W^{s,1}(\Omega)$, and $h \in \partial\mathcal{F}_s^M(u, \Omega, \varphi)$ it follows*

$$h = \langle \mathscr{H}^\varphi u, \,\cdot\, \rangle \quad \textit{in } \mathcal{D}'(\Omega)\,. \tag{4.5.3}$$

In particular, the subdifferential is single-valued.

Proof. Assume that $h \in \partial\mathcal{F}_s(u, \Omega, \varphi) \subseteq L^2(\Omega)$ for $u \in L^2(\Omega)$, then Equation (4.5.1) implies $u \in D(\mathcal{F}_s) = W^{s,1}(\Omega)$ and it follows for any $v \in L^2(\Omega) \cap W^{s,1}(\Omega)$ that

$$\langle h, v\rangle_{L^2(\Omega)} = \frac{1}{\varepsilon}\langle h, (u+\varepsilon v) - u\rangle_{L^2(\Omega)} \leq \frac{\mathcal{F}_s(u+\varepsilon v, \Omega, \varphi) - \mathcal{F}_s(u, \Omega, \varphi)}{\varepsilon}\,.$$

Letting $\varepsilon \to 0$ it follows from Lemma 4.5.7 that $\langle h, v\rangle_{L^2(\Omega)} \leq \langle \mathscr{H}^\varphi u, v\rangle$. Switching v with $-v$ reverses the inequality and altogether yields

$$\langle h, v\rangle_{L^2(\Omega)} = \langle \mathscr{H}^\varphi u, v\rangle \qquad \forall v \in L^2(\Omega) \cap W^{s,1}(\Omega)\,.$$

$\partial\mathcal{F}_s^M(\,\cdot\,, \Omega, \varphi)$ being single-valued, follows from the density of $\mathcal{C}_c^\infty(\Omega) \subseteq L^2(\Omega) \cap W^{s,1}(\Omega)$ in $L^2(\Omega)$. Namely, for $h, h' \in \partial\mathcal{F}_s(u, \Omega, \varphi) \subseteq L^2(\Omega)$ it follows from the above calculation, that

$$\langle h, v\rangle_{L^2(\Omega)} = \langle \mathscr{H}^\varphi u, v\rangle = \langle h', v\rangle_{L^2(\Omega)} \qquad \forall v \in L^2(\Omega) \cap W^{s,1}(\Omega)\,.$$

Thus, by the fundamental lemma of the calculus of variations, $h = h'$ almost everywhere in Ω. $\square$

The proof shows that the Equation (4.5.3) does not only hold in distribution, but the linear maps yield the same result applied to any function in $W^{s,1}(\Omega) \cap L^2(\Omega)$.

4.6 Well-posedness of the gradient flow for the area functionals

By directly applying the abstract theory to the classical and the s-fractional area functional, we obtain the following.

Theorem 4.6.1 (Well-posedness for the gradient flows)**:** *For $s \in (0, 1)$, a local boundary datum $\rho \in L^1(\partial\Omega, \mathcal{H}^{n-1})$ and a nonlocal exterior datum $\varphi : \mathcal{C}\Omega \to \mathbb{R}$, initial conditions $u_1^0, u_s^0 \in L^2(\Omega)$, $s \in (0, 1)$, and $M > 0$ the following well-posedness results hold:*

1. *There exists a unique solution* $u_1 \in AC_{\text{loc}}(0,T;L^2(\Omega))$ *to the gradient flow of the relaxed area functional* $\underline{\underline{\mathcal{A}}}(\,\cdot\,,\overline{\Omega},\rho)$*, which depends continuously on the initial condition* u_1^0.

2. *If* φ *fulfills Assumption (A1) for* s*, then there exists an unique solution* $u_s \in AC_{\text{loc}}(0,T;L^2(\Omega))$ *to the gradient flow of the fractional area functional* $(1-s)\mathcal{F}_s(\,\cdot\,,\Omega,\varphi)$*, which depends continuously on the initial condition* u_s^0.

3. *If* φ *fulfills Assumption (A2) for* s*, then there exists an unique solution* $\tilde{u}_s \in AC_{\text{loc}}(0,T;L^2(\Omega))$ *to the gradient flow of the truncated, fractional area functional* $(1-s)\mathcal{F}_s^M(\,\cdot\,,\Omega,\varphi)$*, which depends continuously on the initial condition* u_s^0.

4. *Without any assumptions on the exterior datum* φ *but with* $\|u_0\|_\infty \leq M$*, there exists a unique solution* $u_s^M \in AC_{\text{loc}}(0,T;L^2(\Omega))$ *to the gradient flow of the restriced, truncated, fractional area functional* $(1-s)\mathfrak{R}_M\mathcal{F}_s^M(\,\cdot\,,\Omega,\varphi)$*, which depends continuously on the initial condition* u_s^0.

The existence of the gradient flow solutions directly follows from the Brezis–Kōmura Theorem 2.3.1 and the properties of the functionals, discussed in this chapter.
Some notes on the well-posedness result

1. If an exterior datum φ fulfills both Assumptions (A1) and (A2) the corresponding solutions u_s and $\tilde{u}_s$ agree, since the subdifferentials agree, see Theorem 4.5.1. Furthermore, if an exterior datum φ fulfills only Assumption (A2), the solutions $\tilde{u}_s$ agree for any $M > 0$, because the subdifferentials agree, see Theorem 4.5.1. For this reason, the solution does not depend on the parameter M.

2. Notice that $\partial\underline{\underline{\mathcal{A}}} = \omega_{n-1}\partial\underline{\mathcal{A}}$, which yields that the gradient flow solution for the rescaled version $\underline{\underline{\mathcal{A}}}$ is just a solution for $\underline{\mathcal{A}}$ in rescaled time. The same holds for the rescaling factor $(1-s)$, which is added for the nonlocal-to-local convergence and will be discussed in the next chapter.

3. As discussed in Theorem 2.3.1, if it is assumed that $u_s^0 \in W^{s,1}(\Omega)$ for $s \in (0,1)$ and $u_1^0 \in \mathrm{BV}(\Omega)$, then the solutions are in $H^1(0,T;L^2(\Omega))$ for all $s \in (0,1]$.

5 Nonlocal-to-local convergence

We now focus on nonlocal-to-local convergence, meaning that we consider the limit $s \nearrow 1$ and establish conditions under which convergence of the gradient flows can be proved. As Lemma 2.3.4 suggests, the aim is to prove the Mosco convergence, as well as the pointwise convergence of the s-fractional area functional to the classical area functional.

5.1 Gamma-convergence of the fractional perimeter with fixed outside datum

We start from a Γ-convergence result for the s-fractional perimeter of sets with a fixed boundary datum. This will be later utilized by connecting the truncated, fractional area functional, and the fractional perimeter of the subgraph.
Consider $\Omega \Subset \mathcal{U} \subseteq \mathbb{R}^n$ such that Ω has Lipschitz boundary and the outside datum $E_0 \subseteq \mathbb{R}^n$ has finite perimeter in $\mathcal{U}$. Furthermore, consider the collection of sets $\mathcal{C} := \{E \subseteq \mathbb{R}^n \text{ measurable s.t. } E \setminus \Omega = E_0 \setminus \Omega\}$ equipped with the L^1_{loc} convergence.

Theorem 5.1.1: *On $\mathcal{C}$ it holds that*

$$\Gamma\text{-}\lim_{s \nearrow 1}(1-s)\mathrm{Per}_s(\,\cdot\,,\Omega) = \omega_{n-1}\mathrm{Per}(\,\cdot\,,\overline{\Omega})\ .$$

Proof.

1. The Γ- lim sup - inequality:
As the pointwise convergence results in Proposition 3.3.3 suggests, the constant sequence can be used to recover the perimeter in the closure $\overline{\Omega}$. For $E \in \mathcal{C}$, such that $\mathrm{Per}(E,\overline{\Omega}) < \infty$, then E has finite perimeter in $\mathcal{U}$, because

$$\mathrm{Per}(E,\mathcal{U}) = \mathrm{Per}(E,\mathcal{U}\setminus\overline{\Omega}) + \mathrm{Per}(E,\mathcal{U}\cap\overline{\Omega}) \le \mathrm{Per}(E_0,\mathcal{U}) + \mathrm{Per}(E,\overline{\Omega}) < \infty\ .$$

Then, Equation (3.3.4) implies

$$\lim_{s \nearrow 1}(1-s)\mathrm{Per}_s(E,\Omega) = \omega_{n-1}\mathrm{Per}(E,\overline{\Omega})\ .$$

2. The Γ- lim inf - inequality:
First, choose $\Omega \Subset \mathcal{O} \Subset \mathcal{U}$ such that $\mathcal{O}$ has Lipschitz boundary and $\mathrm{Per}(E_0,\partial\mathcal{O}) = 0$. $\mathcal{O}$ can be chosen in this way, since one can find a neighborhood of Ω for a small enough radius with Lipschitz boundary, see for example [Lom16b, Proposition B.1]. Furthermore, since $\mathrm{Per}(E_0,\mathcal{U}) < \infty$, there are at most countably many radii

$\delta > 0$, such that $\mathrm{Per}(E_0, \partial\Omega_\delta) > 0$ where Ω_δ is the δ-neighborhood of Ω as in Equation (4.2.1).

Now we can make use of the Γ-convergence result in Proposition 3.3.4, where the neighborhood $P := \mathcal{O} \setminus \overline{\Omega}$ will ensure that the outside datum is fixed through the limiting process. Consider the integrals according to where the interactions occur. Then, for the local part, it holds

$$\begin{aligned} \mathrm{Per}_s^{\mathrm{L}}(E, \mathcal{O}) = {}& \mathrm{Per}_s^{\mathrm{L}}(E, \Omega) + \mathrm{Per}_s^{\mathrm{L}}(E, P) \\ &+ \mathcal{L}_s(E \cap \Omega, \mathcal{C}E \cap P) + \mathcal{L}_s(\mathcal{C}E \cap \Omega, E \cap P) \, . \end{aligned}$$

For the non-local part, it holds

$$\begin{aligned} \mathrm{Per}_s^{\mathrm{NL}}(E, \mathcal{O}) = {}& \mathrm{Per}_s^{\mathrm{NL}}(E, \Omega) \\ &+ \mathcal{L}_s(E \cap P, \mathcal{C}E \cap \mathcal{C}\mathcal{O}) + \mathcal{L}_s(\mathcal{C}E \cap P, E \cap \mathcal{C}\mathcal{O}) \\ &- \mathcal{L}_s(E \cap \Omega, \mathcal{C}E \cap P) - \mathcal{L}_s(\mathcal{C}E \cap \Omega, E \cap P) \, . \end{aligned}$$

Thus, altogether

$$\begin{aligned} \mathrm{Per}_s(E, \mathcal{O}) = {}& \mathrm{Per}_s(E, \Omega) \\ &+ \mathrm{Per}_s^{\mathrm{L}}(E, P) \\ &+ \mathcal{L}_s(E \cap P, \mathcal{C}E \cap \mathcal{C}\mathcal{O}) + \mathcal{L}_s(\mathcal{C}E \cap P, E \cap \mathcal{C}\mathcal{O}) \, . \end{aligned} \tag{5.1.1}$$

Take now a sequence $E_k \in \mathcal{C}$ such $E_k \to E$ in L^1_{loc} and such that

$$\liminf_{k\to\infty} (1 - s_k)\mathrm{Per}_{s_k}(E_k, \Omega) < \infty \, .$$

Investigating the behavior of the constants in Equation (5.1.1), we find with the help of the pointwise convergence results in Equation (3.3.2) that

$$(1 - s_k)\mathrm{Per}_s^{\mathrm{L}}(E_k, P) = (1 - s_k)\mathrm{Per}_s^{\mathrm{L}}(E_0, P) \to \mathrm{Per}(E_0, P) \, ,$$

and

$$\begin{aligned} &(1 - s_k)\Big(\mathcal{L}_{s_k}(E_k \cap P, \mathcal{C}E_k \cap \mathcal{C}\mathcal{O}) + \mathcal{L}_s(\mathcal{C}E_k \cap P, E_k \cap \mathcal{C}\mathcal{O})\Big) \\ &\quad = (1 - s_k)\Big(\mathcal{L}_{s_k}(E_0 \cap P, \mathcal{C}E_0 \cap \mathcal{C}\mathcal{O}) + \mathcal{L}_s(\mathcal{C}E_0 \cap P, E_0 \cap \mathcal{C}\mathcal{O})\Big) \\ &\quad \leq (1 - s_k)\mathrm{Per}_{s_k}^{\mathrm{NL}}(E_0, \mathcal{O}) \to \omega_{n-1}\mathrm{Per}(E_0, \partial\mathcal{O}) = 0 \, . \end{aligned}$$

Therefore, by the Γ-convergence of $\mathrm{Per}_s(\,\cdot\,,\mathcal{O})$ in Proposition 3.3.4 it holds

$$\begin{aligned}\liminf_{k\to\infty}(1-s_k)\mathrm{Per}_{s_k}(E_k,\overline{\Omega}) + \omega_{n-1}\mathrm{Per}(E_0,P) &= \\ \liminf_{k\to\infty}(1-s_k)\mathrm{Per}_{s_k}(E_k,\mathcal{O}) &\geq \omega_{n-1}\mathrm{Per}(E,\mathcal{O}) \\ &= \omega_{n-1}\mathrm{Per}(E_0,P) + \omega_{n-1}\mathrm{Per}(E,\overline{\Omega})\,.\end{aligned}$$

Substracting $\omega_{n-1}\mathrm{Per}(E_0,P)$ on both sides yields the Γ-$\liminf$-inequality. □

5.2 Convergence of the truncated, fractional area functional

Motivated by the regularity of the exterior datum E_0 in an open set $\mathcal{U}$ around Ω in the previous section, we introduce the following assumption for the exterior datum.

Assumption (A3): The outside datum $\varphi : \mathcal{C}\Omega \to \mathbb{R}$ has bounded variation locally around Ω, in the sense that

$$\exists \mathcal{O} \subseteq \mathbb{R}^n \text{ open: } \Omega \Subset \mathcal{O} \text{ and } \varphi_{|\mathcal{O}\setminus\overline{\Omega}} \in \mathrm{BV}(\mathcal{O}\setminus\overline{\Omega})\,. \tag{5.2.1}$$

Denote by $\varphi^- \in L^1(\partial\Omega, \mathcal{H}^{n-1})$ the trace from outside of Ω on $\partial\Omega$.

Before we come to the convergence results, we want to revisit Assumption (A1). The assumption balances two different goals: the growth at infinity and the integrability of the hypercritical kernel, when x and y are close. As we have introduced Assumption (A3) regarding the behavior around Ω, we want to introduce a second assumption solely concerning the growth at infinity.

Assumption (A4): There exists $s_0 \in (0,1)$ such that the outside datum $\varphi : \mathcal{C}\Omega \to \mathbb{R}$ fulfills the following condition

$$\int_{\mathcal{C}\Omega} \frac{|\varphi(y)|}{(1+|y|)^{n+s_0}}\,dy\ < \infty\,. \tag{5.2.2}$$

Note that, if Assumption (A4) holds for any $s_0 \in (0,1)$, then it holds for all $s \in [s_0,1)$, since

$$\int_{\mathcal{C}\Omega} \frac{|\varphi(y)|}{(1+|y|)^{n+s}}\,dy\ \leq \int_{\mathcal{C}\Omega} \frac{|\varphi(y)|}{(1+|y|)^{n+s_0}}\,dy\ < \infty\,.$$

In that sense, the assumption is tailored in a way to consider convergence $s \nearrow 1$. We want to show that Assumptions (A3) and (A4) imply Assumption (A1).

Lemma 5.2.1: *For $\Omega \subseteq \mathbb{R}^n$ open, bounded and with Lipschitz boundary, if $\varphi : \mathcal{C}\Omega \to \mathbb{R}$ fulfills Assumptions (A3) and (A4), with $s_0 \in (0,1)$ as in Assumption (A4), then Assumption (A1) holds for all $s \in [s_0,1)$.*
Furthermore, it holds $\lim_{s\nearrow 1}(1-s)\sigma_s(\varphi) < \infty$ for $\sigma_s(\varphi)$ as in Assumption (A1).

Proof. Denote $\mathcal{O} \subseteq \mathbb{R}^n$ as in Equation (5.2.1) and choose $0 < \delta < \mathrm{dist}(\mathcal{C}\mathcal{O},\Omega)$, such that

Ω_δ has Lipschitz boundary. Then define

$$\tilde{\varphi} : \mathbb{R}^n \to \mathbb{R}, \quad \tilde{\varphi}(x) := \begin{cases} \varphi(x) & \text{if } x \in \mathcal{C}\Omega, \\ 0 & \text{else.} \end{cases}$$

Assumption (A3) implies $\tilde{\varphi} \in \mathrm{BV}(\mathcal{O})$ and hence $\tilde{\varphi} \in W^{s,1}(\Omega_\delta)$ for all $s \in (0,1)$. There exists $C_\delta > 1$ such that

$$C_\delta |x-y| \geq 1 + |y| \quad \text{for every } (x,y) \in \Omega \times \mathcal{C}\Omega_\delta .$$

Hence, we can estimate

$$\begin{aligned} \sigma_s(\varphi) &= \int_{\mathcal{C}\Omega} \int_\Omega \frac{|\varphi(y)|}{|x-y|^{n+s}}\, dx\ dy \\ &= \int_{\Omega_\delta \setminus \Omega} \int_\Omega \frac{|\varphi(y)|}{|x-y|^{n+s}}\, dx\ dy + \int_{\mathcal{C}\Omega_\delta} \int_\Omega \frac{|\varphi(y)|}{|x-y|^{n+s}}\, dx\ dy \\ &\leq \left[\tilde{\varphi}\right]_{W^{s,1}(\Omega_\delta)} + C_\delta^{n+s} |\Omega| \int_{\mathcal{C}\Omega} \frac{|\varphi(y)|}{(1+|y|)^{n+s}}\, dy \ < \infty \, . \end{aligned}$$

Thus, φ fulfills Assumption (A1) for all $s \in [s_0, 1)$. Furthermore, it holds

$$\begin{aligned} (1-s)\sigma_s(\varphi) &\leq (1-s)\left[\tilde{\varphi}\right]_{W^{s,1}(\Omega_\delta)} + (1-s) C_\delta^{n+s} |\Omega| \int_{\mathcal{C}\Omega} \frac{|\varphi(y)|}{(1+|y|)^{n+s}}\, dy \\ &\to \omega_{n-1} \|u\|_{\mathrm{BV}(\Omega_\delta)} < \infty \quad \text{as } s \nearrow 1. \end{aligned}$$

See [Lom16b, Section 2.2.3] for the convergence and the constant ω_{n-1}. □

We turn now to the nonlocal-to-local convergence for bounded functions with the help of the truncated functional.

Theorem 5.2.2 (Γ-$\liminf$ for the truncation)**:** *For $\Omega \subseteq \mathbb{R}^n$ open, bounded and with Lipschitz boundary, an exterior datum $\varphi : \mathcal{C}\Omega \to \mathbb{R}$ that fulfills Assumption (A3), $M > 0$, $s_k \nearrow 1$, and $u_k, u \in L^1(\Omega)$ with $\|u_k\|_{L^\infty(\Omega)} \leq M$ such that $u_k \rightharpoonup u$ in $L^p(\Omega)$, it holds*

$$\liminf_{k \to 1} (1-s_k) \mathcal{F}^M_{s_k}(u_k, \Omega, \varphi) \geq \underline{\underline{\mathcal{A}}}\big(u, \overline{\Omega}, (\varphi^-)^{[M]}\big) \, , \tag{5.2.3}$$

where $(\varphi^-)^{[M]}$ denotes the truncation of the trace from outside at height M.

Recall that $\underline{\underline{\mathcal{A}}}(u,\Omega) := \omega_n\big(\underline{\mathcal{A}}(u,\Omega) - |\Omega|\big)$ is the renomalization of the area functional and define the subgraph for a function $u : \Omega \to \mathbb{R}$ together with an outside datum $\varphi : \mathcal{C}\Omega \to \mathbb{R}$ by

$$\mathcal{S}_{u,\varphi} := \{(x,t) \in \mathbb{R}^{n+1} \mid u(x) > t \text{ if } x \in \Omega \text{ and } \varphi(x) > t \text{ if } x \in \mathcal{C}\Omega\} \, .$$

Proof. By Equation (3.4.5) it holds

$$\mathcal{F}^M_s(u, \Omega, \varphi) = \mathrm{Per}_s(\mathcal{S}_{u,\varphi}, \Omega^M) - \kappa^2_{\Omega,M,s} \, .$$

For the constant $\kappa^2_{\Omega,M,s}$ it holds

$$
\begin{aligned}
&\lim_{k\to\infty}(1-s_k)\kappa^2_{\Omega,M,s_k}\\
&\quad= \lim_{k\to\infty}(1-s_k)\Big(\mathrm{Per}^{\mathrm{L}}_{s_k}(\{X_{n+1}<0\},\Omega^\infty)-\mathrm{Per}^{\mathrm{L}}_{s_k}(\{X_{n+1}<0\},\Omega^\infty\setminus\Omega^M)\Big)\\
&\quad= \omega_n\mathrm{Per}(\{X_{n+1}<0\},\Omega^\infty)+\omega_{n-1}\mathrm{Per}(\{X_{n+1}<0\},\Omega^\infty\setminus\Omega^M)\\
&\quad= \omega_n|\Omega|\ ,
\end{aligned}
\tag{5.2.4}
$$

where the pointwise convergence of the local part in Equation (3.3.2) has been used. Thus, using the previous result in Theorem 5.1.1 for $u_k\to u$ in $L^1_{\mathrm{loc}}(\Omega)$ and, respectively $\mathcal{S}_{u_k,\varphi}\to\mathcal{S}_{u,\varphi}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, if follows

$$
\begin{aligned}
\liminf_{k\to\infty}(1-s_k)\mathcal{F}^M_{s_k}(u_k,\Omega,\varphi) &= \liminf_{k\to\infty}(1-s_k)\Big(\mathrm{Per}_{s_k}(\mathcal{S}_{u_k,\varphi},\Omega^M)-\kappa^2_{\Omega,M,s_k}\Big)\\
&\geq \omega_n\mathrm{Per}(\mathcal{S}_{u,\varphi},\overline{\Omega^M})-\omega_n|\Omega|\\
&= \omega_n\underline{\mathcal{A}}\big(u,\overline{\Omega},(\varphi^-)^{[M]}\big)-\omega_n|\Omega|\\
&= \underline{\underline{\mathcal{A}}}\big(u,\overline{\Omega},(\varphi^-)^{[M]}\big)\ .
\end{aligned}
$$

Now if $u_k\rightharpoonup u$ in $L^p(\Omega)$ and we assume w.l.o.g. that $\sup_{k\in\mathbb{N}}(1-s_k)\mathcal{F}^M_s(u_k,\Omega,\varphi)=C<\infty$. Using Equation (4.1.2) we get

$$
\begin{aligned}
\sup_{k\in\mathbb{N}}(1-s_k)\mathcal{F}_{s_k}(u_k,\Omega,\varphi) &\geq \sup_{k\in\mathbb{N}}(1-s_k)\mathcal{A}_{s_k}(u_k,\Omega,\varphi)\\
&= \sup_{k\in\mathbb{N}}(1-s_k)\frac{c^s_\star}{2}\Big([u_{s_k}]_{W^{s,1}(\Omega)}-c_{s_k}(\Omega)\Big)
\end{aligned}
$$

with $c^s_\star=\inf_{t\in[0,1]}g_s(t)=2^{-\frac{n+1+s}{2}}$ and $c_s(\Omega)=\frac{\mathcal{H}^{n-1}(S^{n-1})}{1-s}|\Omega|\mathrm{diam}(\Omega)^{1-s}$. Investigating the behaviour of the constants as $s\nearrow 1$

$$
\lim_{s\nearrow1}c^s_\star=2^{-\frac{n+2}{2}}\ ,
$$

and

$$
\lim_{s\nearrow1}(1-s)c_s(\Omega)=\mathcal{H}^{n-1}(S^{n-1})|\Omega|\ . \tag{5.2.5}
$$

Altogether, we obtain a uniform bound

$$
\sup_{k\in\mathbb{N}}\Big(\|u_k\|_{L^1(\Omega)}+(1-s_k)[u_k]_{W^{s_k,1}(\Omega)}\Big)<\infty\ .
$$

Using the equi-coercivity result [Buc+21, A.4], we can conclude $u_k\to u$ in $L^1_{\mathrm{loc}}(\Omega)$ holds, and thus using the first part, the inequality is proved for the weak convergence. □

Theorem 5.2.3 (Pointwise convergence of the truncation)**:** *For $M>0$, $u\in\mathrm{BV}(\Omega)$*

such that $\|u\|_{L^\infty(\Omega)} \le M$, *and* $s_k \nearrow 1$ *it holds*

$$\lim_{k\to\infty}(1-s_k)\mathcal{F}^M_{s_k}(u,\Omega,\varphi) = \underline{\underline{\mathcal{A}}}\big(u,\overline{\Omega},(\varphi^-)^{[M]}\big) \ . \tag{5.2.6}$$

Proof. Let $u \in \mathrm{BV}(\Omega)$, which directly implies $u \in W^{s,1}(\Omega)$ for every $s \in (0,1)$. Using Equation (3.3.4) for the convergence of the s-perimeter to the perimeter in $\overline{\Omega}$ with the convergence of the constants Equation (5.2.4) it holds

$$\begin{aligned}\lim_{k\to\infty}(1-s_k)\mathcal{F}^M_{s_k}(u,\Omega,\varphi) &= \lim_{k\to\infty}(1-s_k)(\mathrm{Per}_{s_k}(\mathcal{S}_{u,\varphi},\Omega^M) - \kappa^2_{\Omega,M,s_k})\\ &= \omega_n \mathrm{Per}(\mathcal{S}_{u,\varphi},\overline{\Omega^M}) - \omega_n|\Omega|\\ &= \omega_n\big(\underline{\mathcal{A}}(u,\overline{\Omega},(\varphi^-)^{[M]}) - |\Omega|\big) \ . \end{aligned} \tag{5.2.7}$$

□

5.3 Mosco convergence of the fractional area functional

Our next goal is to generalize the above convergence results to the nontruncated, s-fractional area functional and unbounded functions. First, the difference between truncation and nontruncation is considered.

Lemma 5.3.1: *For* $\Omega \subseteq \mathbb{R}^n$ *open, bounded with Lipschitz boundary, an outside datum* $\varphi : \mathcal{C}\Omega \to \mathbb{R}$ *that fulfills Assumptions (A3) and (A4), and* $u_s \in W^{s,1}(\Omega)$, *it holds*

$$\lim_{n\to\infty}(1-s)\bigg(\mathcal{F}_s(u_s,\Omega,\varphi) - \mathcal{F}^M_s(u_s,\Omega,\varphi)\bigg) = \omega_n \int_{\partial\Omega} \max\big(0,\varphi^- - M, -\varphi^- - M\big)\, d\mathcal{H}^{n-1} \ . \tag{5.3.1}$$

Proof. The difference between truncation and nontruncation can be rewritten with the help of the representation Equation (3.4.3) as

$$\begin{aligned}&\mathcal{F}_s(u_n,\Omega,\varphi) - \mathcal{F}^M_s(u_n,\Omega,\varphi)\\ &\quad = \mathcal{N}_s(u_n,\Omega,\varphi) - \mathcal{N}^M_s(u_n,\Omega,\varphi)\\ &\quad = \underbrace{\int_\Omega\bigg\{\int_{\mathcal{C}\Omega}\bigg\{\mathcal{G}_s\bigg(\frac{M-\varphi(y)}{|x-y|}\bigg) + \mathcal{G}_s\bigg(\frac{-M-\varphi(y)}{|x-y|}\bigg)\bigg\}\frac{dy}{|x-y|^{n-1+s}}\bigg\}\,dx}_{=:A_s}\\ &\qquad\quad \underbrace{-\, M\Lambda_s\int_\Omega\int_{\mathcal{C}\Omega}\frac{dx\ dy}{|x-y|^{n+s}}}_{=:B_s} \ .\end{aligned}$$

Note that A_s and B_s are independent of u_s. The limit $s \nearrow 1$ is now investigated, using the bounds on $\mathcal{G}_s$ from Equation (4.0.3).

1. **Convergence of B_s:** Using the pointwise convergence results for the fractional

perimeter, it holds that

$$\lim_{s\nearrow 1}(1-s)B_s = \omega_{n-1}M\Lambda_1 \mathrm{Per}(\Omega,\mathbb{R}^n) = \omega_n M\mathcal{H}^{n-1}(\partial\Omega)\ ,$$

where the regularity assumptions on the domain Ω have been used and Equation (4.0.5).

2. **Upper Bound for A_s:** Using the estimates on $\mathcal{G}_s$ it follows

$$A_n \le \int_\Omega \int_{\mathcal{C}\Omega} \frac{\Lambda_s}{2} \frac{|M-\varphi(y)|+|M+\varphi(y)|}{|x-y|^{n+s}}\, dy\ dx\ = \frac{\Lambda_s}{2}\Big(\mathcal{E}^1_s(\overline{\varphi})+\mathcal{E}^1_s(\underline{\varphi})\Big)\ ,$$

where

$$\underline{\varphi}(x) = \begin{cases} -M & \text{for } x\in\Omega\ ,\\ \varphi(x) & \text{on } x\in\mathcal{C}\Omega\ ,\end{cases} \qquad \overline{\varphi}(x) = \begin{cases} M & \text{on } x\in\Omega\ ,\\ \varphi(x) & \text{on } x\in\mathcal{C}\Omega\ .\end{cases}$$

Then, $\underline{\varphi}_{|\mathcal{O}}, \overline{\varphi}_{|\mathcal{O}} \in \mathrm{BV}(\mathcal{O})$ from Assumption (A3), and the Lipschitz boundary of Ω, see ,i.e., [Giu84, Remark 2.12, p. 38]. Using a convergence result for the energies $\mathcal{E}^1_s$ in [Buc+21, Theorem A.2.], it follows that

$$\begin{aligned}\lim_{s\nearrow 1}(1-s)A_s &\le \frac{\omega_{n-1}\Lambda_1}{2}\Big(\|D\overline{\varphi}\|(\partial\Omega)+\|D\underline{\varphi}\|(\partial\Omega)\Big)\\ &= \frac{\omega_n}{2}\int_{\partial\Omega} |\varphi^- - M| + |\varphi^- + M|\, d\mathcal{H}^{n-1}\\ &= \omega_n M\mathcal{H}^{n-1}(\partial\Omega) + \omega_n \int_{\partial\Omega} \max\big(0,\varphi^- - M, -\varphi^- - M\big)\, d\mathcal{H}^{n-1}\ ,\end{aligned}$$

where we have used Assumption (A4) on φ.

3. **Lower Bound for A_s:** Using the lower bounds on $\mathcal{G}_s$ one gets a lower bound for A_s. Yet, as also a constant λ_s-term is obtained, the domain of integration has to be reduced beforehand by considering the δ neighborhood Ω_δ of Ω and the surrounding $\mathcal{S}_\delta := \Omega_\delta \setminus \overline{\Omega}$. By virtue of the estimates in Equation (4.0.3), it follows

$$\begin{aligned}A_s &\ge \int_\Omega \bigg\{ \int_{\mathcal{S}_\delta} \bigg\{ \mathcal{G}_s\Big(\frac{M-\varphi(y)}{|x-y|}\Big) + \mathcal{G}_s\Big(\frac{-M-\varphi(y)}{|x-y|}\Big)\bigg\} \frac{dy}{|x-y|^{n-1+s}} \bigg\}\, dx\\ &\ge \frac{\Lambda_s}{2}\int_\Omega\int_{\mathcal{S}_\delta} \frac{|M-\varphi(y)|+|M+\varphi(y)|}{|x-y|^{n+s}}\, dy\ dx\ - \underbrace{\lambda_s \int_\Omega\int_{\mathcal{S}_\delta} \frac{dy\ dx}{|x-y|^{n-1+s}}}_{D_s}\ .\end{aligned}$$

In order to show that the non-local part away from the boundary of Ω does not contribute in the limit $s \nearrow 1$, the term is split as follows

$$\begin{aligned}\frac{\Lambda_s}{2}\int_\Omega\int_{\mathcal{S}_\delta}\frac{|M-\varphi(y)|+|M+\varphi(y)|}{|x-y|^{n+s}}\,dy\;dx\\ &= \underbrace{\frac{\Lambda_s}{2}\int_\Omega\int_{\mathcal{C}\Omega}\frac{|M-\varphi(y)|+|M+\varphi(y)|}{|x-y|^{n+s}}\,dy\;dx}_{E_s}\\ &\quad-\underbrace{\frac{\Lambda_s}{2}\int_\Omega\int_{\mathcal{C}\Omega_\delta}\frac{|M-\varphi(y)|+|M+\varphi(y)|}{|x-y|^{n+s}}\,dy\;dx}_{F_s}\ .\end{aligned}$$

We estimate each part as follows:

- For D_s it holds

  $$D_s \leq \lambda_s\omega_{n-1}\frac{\min\{|\Omega|,|\mathcal{S}_\delta|\}\mathrm{diam}(\Omega_\delta)^{1-s}}{1-s}\ ,$$

  and thus

  $$\lim_{s\nearrow 1}(1-s)D_s \leq \lambda_1\omega_{n-1}|\mathcal{S}_\delta| \to 0 \text{ when } \delta\to 0\ .$$

- For E_s, as in the case of A_s it holds

  $$\lim_{s\nearrow 1}(1-s)E_s = \omega_n M\mathcal{H}^{n-1}(\partial\Omega)+\omega_n\int_{\partial\Omega}\max\bigl(0,\varphi^- - M,-\varphi^- - M\bigr)\,d\mathcal{H}^{n-1}\ .$$

- For F_s it holds

  $$F_s \leq \frac{\Lambda_s}{2}\int_\Omega\int_{\mathcal{C}B_\delta(x)}\frac{2(M+L)}{|z|^{n+s}}\,dz\;dx\ = \Lambda_s(M+L)|\Omega|\frac{\delta^{-s}}{s}\ .$$

  Thus, for any $\delta > 0$

  $$\lim_{s\nearrow 1}(1-s)F_s = 0\ .$$

Altogether,

$$\begin{aligned}\liminf_{s\nearrow 1}(1-s)A_s \geq\ &\omega_n M\mathcal{H}^{n-1}(\partial\Omega)-\lambda_1\omega_{n-1}|\mathcal{S}_\delta|\\ &+\omega_n\int_{\partial\Omega}\max\bigl(0,\varphi^- - M,-\varphi^- - M\bigr)\,d\mathcal{H}^{n-1}\ .\end{aligned}$$

Since $\delta > 0$ may be taken to be arbitrarily small, the following conclusion holds

$$\liminf_{s\nearrow 1}(1-s)A_s \geq \omega_n M\mathcal{H}^{n-1}(\partial\Omega)+\omega_n\int_{\partial\Omega}\max\bigl(0,\varphi^- - M,-\varphi^- - M\bigr)\,d\mathcal{H}^{n-1}\ .$$

Combining all the estimates for A_s and B_s it holds that

$$\lim_{s\nearrow 1}(1-s)\Big(\mathcal{F}_s(u_s,\Omega,\varphi)-\mathcal{F}_s^M(u_s,\Omega,\varphi)\Big)=\omega_n\int_{\partial\Omega}\max\big(0,\varphi^- -M,-\varphi^- -M\big)\,d\mathcal{H}^{n-1}\ . \ \square$$

This can now be used to prove the Γ-$\liminf$-inequality.

Lemma 5.3.2 (Γ-$\liminf$-inequality for $\mathcal{F}_s$)**:** *For $\Omega \subseteq \mathbb{R}^n$ open, bounded, and with Lipschitz boundary, an outside datum $\varphi : \mathcal{C}\Omega \to \mathbb{R}$ that fulfills both Assumptions (A2) to (A4) for some $s_0 \in (0,1)$, $s_k \nearrow 1$, and $u_k \rightharpoonup u$ in $L^p(\Omega)$, it holds*

$$\liminf_{k\to\infty}(1-s_k)\mathcal{F}_{s_k}(u_k,\Omega,\varphi) \geq \underline{\underline{\mathcal{A}}}(u,\overline{\Omega},\varphi^-)\ .$$

Notice that choosing $\varphi \in L^\infty(\mathcal{C}\Omega)\cap \mathrm{BV}_{\mathrm{loc}}(\mathcal{C}\Omega)$ implies that Assumptions (A2) to (A4) are fulfilled. From our discussion in Remark 4.2.2 it follows that Assumption (A2) holds for all $s \in [s_0, 1)$.

Proof. Let $u_k \to u$ strongly in L^1_{loc} and preliminarily assume that u_n and u are uniformly bounded by $N < \infty$. Consider the difference of the s-fractional area to the truncation and compute

$$\begin{aligned}&\liminf_{k\to\infty}(1-s_k)\mathcal{F}_{s_k}(u_k,\Omega,\varphi)\\ &\geq \liminf_{k\to\infty}(1-s_k)\mathcal{F}^M_{s_k}(u_k,\Omega,\varphi)+\liminf_{k\to\infty}(1-s_k)\Big(\mathcal{F}_{s_k}(u_k,\Omega,\varphi)-\mathcal{F}^M_{s_k}(u_k,\Omega,\varphi)\Big)\ .\end{aligned}$$

If $M > N$ then by Equation (5.2.3) it follows that

$$\liminf_{k\to\infty}(1-s_k)\mathcal{F}^M_{s_k}(u_k,\Omega) \geq \underline{\underline{\mathcal{A}}}\big(u,\overline{\Omega},(\varphi^-)^{[M]}\big)\ .$$

Combined with the previous Lemma 5.3.1 and Equation (3.2.1), it follows

$$\begin{aligned}\liminf_{k\to\infty}(1-s_k)\mathcal{F}_{s_k}(u_k,\Omega,\varphi) \geq\ & \omega_n\underline{\mathcal{A}}(u,\overline{\Omega})-\omega_n|\Omega|+\omega_n\int_{\partial\Omega}|u-(\varphi^-)^{[N]}|\,d\mathcal{H}^{n-1}\\ &+\omega_n\int_{\partial\Omega}\max\big(0,\varphi^- - M,-\varphi^- - M\big)\,d\mathcal{H}^{n-1}\\ =\ &\omega_n\underline{\mathcal{A}}(u,\overline{\Omega})-\omega_n|\Omega|+\omega_n\int_{\partial\Omega}|u-\varphi^-|\,d\mathcal{H}^{n-1}\\ =\ &\underline{\underline{\mathcal{A}}}(u,\overline{\Omega},\varphi^-)\ ,\end{aligned} \tag{5.3.2}$$

where we have used that the boundedness of u_k in Ω by N implies the same bound for the trace on $\partial\Omega$. And thus the two integrals on the boundary $\partial\Omega$ perfectly add up. Since the datum φ fulfills the Assumption (A2), the inequalities for the cut-off in Lemma 4.2.3 can be used to simplify to the bounded case

$$\liminf_{k\to\infty}(1-s_k)\mathcal{F}_{s_k}(u_k,\Omega,\varphi) \geq \liminf_{k\to\infty}(1-s_k)\mathcal{F}_{s_k}(u_k^{[N]},\Omega,\varphi) \geq \underline{\underline{\mathcal{A}}}(u^{[N]},\overline{\Omega},\varphi^-)\ . \tag{5.3.3}$$

If $N \to \infty$, then by continuity of $\underline{\mathcal{A}}$ Lemma 3.2.7 and $u^{[N]} \to u$ in BV, it holds

$$\liminf_{k\to\infty}(1-s_k)\mathcal{F}_{s_k}(u_k,\Omega,\varphi) \geq \liminf_{N\to\infty} \underline{\underline{\mathcal{A}}}(u^{[N]},\overline{\Omega},\varphi^-) = \underline{\underline{\mathcal{A}}}(u,\overline{\Omega},\varphi^-) \ .$$

In order to relax the condition on strong L^1-convergence $u_k \to u$, the same argument as in the proof of Theorem 5.2.2 can be used. For a weakly convergent sequence $u_k \rightharpoonup u$ by the uniform bound on the local, s-fractional area, the L^1_{loc}-convergence $u_k \to u$ may be deduced, and a reduction to the above setting follows. □

In order to complete the proof of Mosco convergence, the pointwise convergence for $s \nearrow 1$ of the s-fractional area functional is now proved with the help of truncations. Therefore, we recall dependence on s of the continuous embedding $\mathrm{BV}(\Omega) \hookrightarrow W^{s,1}(\Omega)$.

Lemma 5.3.3: *For $s \in (0,1)$, and $u \in \mathrm{BV}(\Omega)$ it holds*

$$\|u\|_{W^{s,1}(\Omega)} \leq \frac{C(n,\Omega)}{1-s}\|u\|_{\mathrm{BV}(\Omega)} \ . \tag{5.3.4}$$

Proof. In [Lom16a, Proposition 2.1] it has been proved that

$$[u]_{W^{s,1}(\Omega)} \leq C_3(n,\Omega,s)\|u\|_{\mathrm{BV}(\Omega)} \ ,$$

where $C_3(n,\Omega,s)$ as in [NPV11, Proposition 2.2]. From the calculations in [NPV11, Proposition 2.2] it follows that

$$C(n,\Omega,s) = C(n,\Omega)\int_{B_1(0)} \frac{1}{|z|^{n+(s-1)}}\,dx \ = C(n,\Omega)\int_0^1 r^{-1+(1-s)}\,dr \ = \frac{C(n,\Omega)}{1-s} \ ,$$

where $C(n,\Omega)$ is a constant depending only on n and Ω. □

This enables us to prove the pointwise convergence result in a general setting.

Theorem 5.3.4 (Pointwise convergence of $\mathcal{F}_s$)**:** *For $\Omega \subseteq \mathbb{R}^n$ open, bounded, and with Lipschitz boundary, an outside datum $\varphi : \mathcal{C}\Omega \to \mathbb{R}$ that fulfills Assumptions (A3) and (A4), and $u \in \mathrm{BV}(\Omega)$, it holds*

$$\lim_{s\to\infty}(1-s)\mathcal{F}_s(u,\Omega,\varphi) = \underline{\underline{\mathcal{A}}}(u,\overline{\Omega},\varphi^-) \ .$$

Proof. In the following, let $s \in [s_0,1)$, where $s_0 > 0$ as inAssumption (A4), which implies Assumption (A1) holds for all s. Consider the s-fractional area for truncations of u, therefore depending on both $s \in (0,1)$ and the height of truncation $N > 0$. We first prove that the convergence $N \nearrow \infty$ to $\mathcal{F}_s(u,\Omega,\varphi)$ is uniform in s due to the Lipschitz continuity of $\mathcal{F}_s$, see Lemma 4.1.5, and the uniform bound on the rescaled norm Lemma 5.3.3. We have

$$\begin{aligned}(1-s)|\mathcal{F}_s(u,\Omega,\varphi)-\mathcal{F}_s(u^{[N]},\Omega,\varphi)| &\leq \frac{\Lambda_s C_2(n,s,\Omega)}{2}(1-s)\|u-u^{[N]}\|_{W^{s,1}(\Omega)}\\ &\leq \frac{C(n,\Omega)}{s^2}\|u-u^{[N]}\|_{\mathrm{BV}}\ ,\end{aligned}$$

where the estimates and constants from Lemmas 5.3.3 and 7.3.1 have been used. Thus, for $N \nearrow \infty$, the s-fractional area functional of the truncations converges uniformly in s. Furthermore, for $M > N > 0$ with the help of the convergence results, both pointwise of the truncations in Theorem 5.2.3 and of the difference between $\mathcal{F}_s$ and $\mathcal{F}_s^M$ in Lemma 5.3.1

$$\begin{aligned}&\lim(1-s)\mathcal{F}_s(u^{[N]},\Omega,\varphi)\\ &\quad= \lim_{s\nearrow 1}(1-s)\mathcal{F}_s^M(u^{[N]},\Omega,\varphi)+\lim_{s\nearrow 1}(1-s)\Big(\mathcal{F}_s(u^{[N]},\Omega,\varphi)-\mathcal{F}_s^M(u^{[N]},\Omega,\varphi)\Big)\\ &\quad= \underline{\underline{\mathcal{A}}}(u^{[N]},\Omega,(\varphi^-)^{[N]})+\int_{\partial\Omega}\max\big(0,\varphi^- - M,-\varphi^- - M\big)\,d\mathcal{H}^{n-1}\\ &\quad= \underline{\underline{\mathcal{A}}}(u^{[N]},\Omega,\varphi^-)\ .\end{aligned}$$

Thus, by the Moore–Osgood theorem [Zak04, p. 221], the double limit for $s \nearrow 1$ and $N \nearrow \infty$ exists and is equal. Together with the continuity of the area, see Lemma 3.2.7, it holds

$$\begin{aligned}\lim_{s\nearrow 1}\mathcal{F}_s(u,\Omega,\varphi) &= \lim_{s\nearrow 1}\lim_{N\nearrow\infty}\mathcal{F}_s(u^{[N]},\Omega,\varphi)\\ &= \lim_{N\nearrow\infty}\lim_{s\nearrow 1}\mathcal{F}_s(u^{[N]},\Omega,\varphi) = \lim_{N\nearrow\infty}\underline{\underline{\mathcal{A}}}(u^{[N]},\Omega,\varphi^-) = \underline{\underline{\mathcal{A}}}(u,\Omega,\varphi^-)\ . \qquad \square\end{aligned}$$

Combining the pointwise limit with the Γ-$\liminf$ result 5.3.2 lets us conclude Mosco convergence.

Theorem 5.3.5 (Nonlocal-to-local Mosco-convergence)**:** *For $\Omega \subset \mathbb{R}^n$ open, bounded, and with Lipschitz boundary, an outside datum $\varphi : \mathcal{C}\Omega \to \mathbb{R}$ that fulfills Assumptions (A2) to (A4), it holds*

$$\mathfrak{M}\text{-}\lim_{s\nearrow 1}(1-s)\mathcal{F}_s(\,\cdot\,,\Omega,\varphi) = \underline{\underline{\mathcal{A}}}(\,\cdot\,,\overline{\Omega},\varphi^-)$$

in $L^p(\Omega)$.

Proof. The Γ-$\liminf$-inequality has been proved in Lemma 5.3.2, and from Theorem 5.3.4 it follows that the constant sequence can be chosen as the recovery sequence to prove the Γ-$\limsup$-inequality. $\square$

Now it is possible to derive the convergence of the gradient flow solutions according to Lemma 2.3.4

Theorem 5.3.6 (Convergence of the gradient flow solutions)**:** *For $\Omega \subseteq \mathbb{R}^n$ open, bounded, and with Lipschitz boundary, an outside datum $\varphi : \mathcal{C}\Omega \to \mathbb{R}$ that fulfills Assumptions (A2) to (A4), denote the solutions to the gradient flows for the s-fractional area functional $(1-s)\mathcal{F}_s$ by $u_s(t)$ and to the classical area functional $\underline{\underline{\mathcal{A}}}(u, \overline{\Omega}, \varphi^-)$ by $u_1(t)$, with initial data $u_s^0 \in L^2(\Omega)$ and $u_1^0 \in L^2(\Omega)$, respectively.*
Then the following holds true:

1. *If $u_s^0 \to u_1^0$ in $L^2(\Omega)$ then $u_s(t) \to u_1(t)$ uniformly in $L^2(\Omega)$ and $u_s' \to u_1'$ in $L^2_{\mathrm{loc}}(0, T; H)$.*
2. *If also $u_s^0 \in W^{s,1}(\Omega)$, $u_1^0 \in \mathrm{BV}(\Omega)$, and $(1-s)\mathcal{F}_s(u_s^0, \Omega, \varphi) \to \underline{\underline{\mathcal{A}}}(u_1^0, \overline{\Omega}, \varphi^-)$ then $u_s' \to u_1'$ strongly in $L^2(0, T; H)$ and $(1-s)\mathcal{F}_s(u_s, \Omega, \varphi) \to \underline{\underline{\mathcal{A}}}(u_1, \overline{\Omega}, \varphi^-)$ uniformly in $[0, T]$*

Proof. Combine the previous Mosco convergence Theorem 5.3.5 with the general convergence result for semi-groups in Lemma 2.3.4. □

6 The WED Functional

In this chapter, a class of global-in-time functionals is considered, the so-called weighted energy dissipation (WED) functionals. For any trajectory, they account for both the amount of dissipation by integrating $\|u'\|_H$, as well as the energy ψ together with an exponentially decaying weight. The minimization of these functionals balances both objectives, and the parameter $\varepsilon > 0$ defines the corresponding relative trade-off.

6.1 Introduction and the causal limit

For the general setting of the WED Functional denote by H a real Hilbert space, again with inner product $\langle \cdot, \cdot \rangle_H$ and norm $\| \cdot \|_H$, and assume the following

$$\psi : H \to (-\infty, +\infty] \text{ proper, weakly lower semicontinuous, } \lambda\text{-convex and bounded from below,} \tag{6.1.1}$$

$$f \in L^2(0, T; H) \;, \tag{6.1.2}$$

$$u_0 \in D(\psi) \;. \tag{6.1.3}$$

Then the WED functional is defined as follows.

Definition 6.1.1 (The Weighted Energy Dissipation functionals)**:** *The weighted energy dissipation (WED) functional is given by*

$$W^\varepsilon(u) := \begin{cases} \displaystyle\int_0^T e^{-t/\varepsilon} \left[\frac{1}{2} \|u'\|_H^2 + \frac{1}{\varepsilon} \big(\psi(u) - \langle f, u \rangle_H \big) \right] dt \;, & \textit{if } \psi \circ u \in L^1(0,T), \\ +\infty, & \textit{otherwise.} \end{cases} \tag{6.1.4}$$

for $u \in H^1(0, T; H)$.

Without loss of generality, we can assume that $\min \psi = \psi(0) = 0$. This can be achieved by replacing ψ, f and u_0 by

$$\tilde{\psi}(x) := \psi(x + v) - \langle \eta, x \rangle_H - \psi(x), \quad \tilde{f} := f - \eta, \quad u_0 := u_0 - v$$

for a fixed $v \in D(\partial\psi)$ and $\eta \in \partial\psi(v)$.
Mielke, Stefanelli [MS11] have proved the existence of unique minimizers u_ε of $W^\varepsilon$for ε small enough in $K(u_{0\varepsilon}) := \{v \in H^1(0, T; H) \;:\; v(0) = u_{0\varepsilon}\}$ where $u_{0\varepsilon}$ are appropriate approximations of u_0. For more information on the approximation, see [MS11, Section 2.5]. Furthermore, they have proved that the minimizers u_ε at level ε solve an elliptic-

in-time regularization of the gradient flow. Namely $u_\varepsilon \in H^2(0,T;H)$ and it fulfills the following Euler-Lagrange equation

$$\begin{aligned} -\varepsilon u'' + u' + \partial\psi &\ni f \quad \text{a.e. in } (0,T)\,, \\ u(0) &= u_{0\varepsilon}\,, \\ u'(T) &= 0\,. \end{aligned} \tag{6.1.5}$$

First, the existence of minimizers of the WED functional is recalled, where the proof of lower semicontinuity and the existence of minimizers via the Direct Method are added in detail.

Theorem 6.1.2 (Properties of the WED-functional)**:** *Let H, ψ and W^ε be as above and define $K(w_0) := \{v \in H^1(0,T;H) \ : \ v(0) = w_0\}$ for $w_0 \in H$. Then the following properties hold true*

1. *The functionals W^ε on $H^1(0,T;H)$ are weakly lower semicontinuous and proper for every $\varepsilon > 0$.*
2. *For $w_0 \in H$, and ε small enough, the functionals W^ε are κ_ε-convex in $K(w_0)$ with respect to the metric of $H^1(0,T;H)$ for*

$$\kappa_\varepsilon := \varepsilon^2 e^{-T/\varepsilon}\,.$$

   In particular, W_ε is uniformly convex on $K(w_0)$.

3. *The WED functional is weakly coercive on $K(w_0) \subset H^1(0,T;H)$ for any $w_0 \in H$.*
4. *The WED functionals W^ε admit a unique minimizer u_ε on $K(w_0)$.*

Proof. 1. Let $u_n, u \in H^1(0,T;H)$ s.t. $u_n \rightharpoonup u$ in $H^1(0,T;H)$. Then, select subsequences without relabeling, such that the lim inf is obtained and such that $u_n(t) \rightharpoonup u(t)$ in H for almost every t. Then, using the lower-semicontinuity of ψ, lower semicontinuity of the $H^1(0,T;H)$ norm with weight, and Fatou's lemma, we may deduce

$$\begin{aligned} \liminf_{n\to\infty} W^\varepsilon(u_n) &\geq \liminf_{n\to\infty} \int_0^T \frac{e^{-t/\varepsilon}}{2}\|u_n'\|_H^2\,dt + \int_0^T \liminf_{n\to\infty} \frac{e^{-t/\varepsilon}}{\varepsilon}\big(\psi(u_n) - \langle f, u_n\rangle_H\big)\,dt \\ &\geq \int_0^T e^{-t/\varepsilon}\left(\frac{1}{2}\|u'\|_H^2 + \frac{1}{\varepsilon}\big(\psi(u) - \langle f, u_n\rangle_H\big)\right)dt \\ &= W^\varepsilon(u)\,. \end{aligned}$$

2. The proof for κ_ε-convexity of W^ε with $\kappa_\varepsilon \geq 0$ for ε small enough is contained in [MS11, Proposition 2.1].

3. The WED functional on $K(w_0)$ is coercive with respect to weak convergence on $H^1(0,T;H)$. Recall first, that by Theorem 7.2.1 it holds

$$\|u(t)\|_H \leq \|w_0\|_H + \| \int_0^T u' \, dt \,\|_H \leq \|w_0\|_H + \sqrt{T}\|u'\|_{L^2(0,T;H)} \,. \tag{6.1.6}$$

Therefore, it also holds $\|u\|^2_{L^2(0,T;H)} \leq C(\|w_0\|^2_H + \|u'\|^2_{L^2(0,T;H)})$ for some constant C. Then, we have

$$\begin{aligned} K > W^\varepsilon(u) &= \int_0^T \frac{e^{-t/\varepsilon}}{2}\|u'\|^2_H \, dt + \int_0^T \frac{e^{-t/\varepsilon}}{\varepsilon}\psi(u) \, dt - \int_0^T \frac{e^{-t/\varepsilon}}{\varepsilon}\langle f, u\rangle_H \, dt \\ &\geq \tilde{C}\|u'\|^2_{L^2(0,T;H)} - \frac{1}{\varepsilon}\|u\|_{L^2(0,T;H)}\|f\|_{L^2(0,T;H)} \\ &\geq \tilde{C}\|u'\|^2_{L^2(0,T;H)} - \frac{\tilde{C}}{2C}\|u\|^2_{L^2(0,T;H)} - \frac{C}{2\tilde{C}\varepsilon^2}\|f\|^2_{L^2(0,T;H)} \\ &= \frac{\tilde{C}}{2}\|u'\|^2_{L^2(0,T;H)} - \frac{\tilde{C}}{2}\|w_0\|^2_H - \frac{C}{2\tilde{C}\varepsilon^2}\|f\|^2_{L^2(0,T;H)} \,, \end{aligned}$$

where we have used Young's inequality and that $\psi \geq 0$. Thus, a bound on $\|u'\|_{L^2(0,T;H)}$ is obtained and, together with Equation (6.1.6), we can derive a bound on $\|u\|_{H^1(0,T;H)}$. Thus, by reflexivity and weak compactness of bounded sets, the coercivity is implied.

4. The existence of a minimizer u_ε follows via the Direct Method of the calculus of variations, as the WED functional is weakly lower semicontinuous and weakly coercive on $H^1(0,T;H)$. The minimizer is unique due to the strict convexity of the WED functional. □

Definition 6.1.3 (The WED Functional for the area functionals)**:** *For $s \in (0,1)$, a local boundary datum $\rho \in L^1(\partial\Omega, \mathcal{H}^{n-1})$, and a nonlocal exterior datum $\varphi : \mathcal{C}\Omega \to \mathbb{R}$ which fulfills Assumption (A1), denote by W^ε_s the WED functional for the rescaled s-fractional area functional $(1-s)\mathcal{F}_s$ at level $\varepsilon > 0$, namely*

$$W^\varepsilon_s(u) = \int_0^T e^{-t/\varepsilon}\left(\frac{1}{2}\|u'\|^2_{L^2(\Omega)} + \frac{(1-s)}{\varepsilon}\mathcal{F}_s(u,\Omega,\varphi)\right) dt$$

for $u \in H^1(0,T;L^2(\Omega))$ with

$$D(W^\varepsilon_s) = H^1(0,T;L^2(\Omega)) \cap L^1(0,T;W^{s,1}(\Omega)) \,.$$

Moreover, let W^ε_1 be the WED functional for the standard area functional $\underline{\underline{\mathcal{A}}}$ at level $\varepsilon > 0$, namely

$$W^\varepsilon_1(u) = \int_0^T e^{-t/\varepsilon}\left(\frac{1}{2}\|u'\|^2_{L^2(\Omega)} + \frac{1}{\varepsilon}\underline{\underline{\mathcal{A}}}(u,\overline{\Omega},\rho)\right) dt$$

for $u \in H^1(0,T;L^2(\Omega))$ *with*

$$D(W_1^\varepsilon) = H^1(0,T;L^2(\Omega)) \cap L^1(0,T;\mathrm{BV}(\Omega)) \ .$$

The following convergence statement for $\varepsilon \to 0$ stems from [MS11, Theorem 1.1].

Lemma 6.1.4: *In the setting* (6.1.1) *to* (6.1.3)*, denote by* $u_\varepsilon \in H^1(0,T;H)$ *the unique minimizers for the WED functional* W^ε *on* $K(u_{0\varepsilon})$*, where* $u_{0\varepsilon}$ *is an appropriate approximation of an initial condition* $u_0 \in D(\psi)$*, and* $u \in H^1(0,T;H)$ *be the solution to the gradient flow for* ψ *then*

$$u_\varepsilon \to u \text{ uniformly in } H.$$

This directly applies to our setting, and the convergence $\varepsilon \to 0$ may be concluded.

Theorem 6.1.5: *Let* $s \in (0,1)$*,* $u_0^s \in W^{s,1}(\Omega)$*, and* $u_0^1 \in \mathrm{BV}(\Omega)$

1. *Let* u_s *be the solution to the* $\mathcal{F}_s$ *gradient flow starting from* u_0^s *as in Theorem 4.6.1, and* u_s^ε *be the unique minimizer for the WED functional* W_s^ε *on* $K(u_{0\varepsilon}^s)$*. It holds*

$$u_s^\varepsilon \to u_s \text{ uniformly in } L^2(\Omega,\varphi).$$

2. *Let* $u(t)$ *be the solution to the* $\underline{\mathcal{A}}$ *gradient flow starting from* u_0^1 *as in Theorem 4.6.1, and* u_1^ε *the unique minimizer for the WED functional* W_1^ε *on* $K(u_{0\varepsilon}^1)$*. It holds*

$$u_1^\varepsilon \to u_1 \text{ uniformly in } L^2(\Omega,\varphi).$$

6.2 The nonlocal-to-local convergence for the WED

We now fix a level $\varepsilon > 0$ and consider the limit $s \nearrow 1$ for the WED functionals. We will prove Γ-convergence of the WED functionals and convergence of the minimizers.

Lemma 6.2.1: *Let* W_s^ε *and* W_1^ε *be as in Definition 6.1.3 with an exterior datum* $\varphi : \mathcal{C}\Omega \to \mathbb{R}$ *that fulfills Assumptions (A2) to (A4),* $s_k \nearrow 1$*, and* $u_k \rightharpoonup u$ *in* $L^2(0,T;L^2(\Omega))$*, then*

$$\liminf_{k\to\infty} W_{s_k}^\varepsilon(u_k) \geq W_1^\varepsilon(u) \ .$$

Before we come to the proof, we want to point out that the coercivity in Theorem 6.1.2 for the WED of the fractional and classical area does not depend on $s \in (0,1]$, as the functionals are equi-bounded from below by 0. Thus, the functionals are equi-coercive.

Proof. Let $u_k \rightharpoonup u$ in $L^2(0,T;L^2(\Omega))$, and w.l.o.g. assume $\liminf_{k\to\infty} W_{s_k}^\varepsilon(u_k) < \infty$. By equi-coercivity we can assume that $u_k \rightharpoonup u$ in $H^1(0,T;L^2(\Omega))$. Furthermore, passing to not relabled subsequences, for all $t \in [0,T]$ it holds that $u_k(t) \rightharpoonup u(t)$ in $L^2(\Omega)$. Then

$$\begin{aligned}\liminf_{n\to\infty} W^\varepsilon_{s_k}(u_k) &\geq \liminf_{k\to\infty} \int_0^T \frac{e^{-t/\varepsilon}}{2}\|u_k'\|^2_{L^2(\Omega)}\,dt + \int_0^T \frac{e^{-t/\varepsilon}}{\varepsilon} \liminf_{k\to\infty}\big[(1-s_k)\mathcal{F}_{s_k}(u_k,\Omega,\varphi)\big]\,dt \\ &\geq \int_0^T e^{-t/\varepsilon}\left(\frac{1}{2}\|u'\|^2_{L^2(\Omega)} + \frac{1}{\varepsilon}\underline{\mathcal{A}}(u,\overline{\Omega},\varphi^-)\right)dt \\ &= W^\varepsilon_1(u)\ ,\end{aligned}$$

where the pointwise Γ-$\liminf$ inequality in Lemma 5.3.2

$$\liminf_{k\to\infty}(1-s_k)\mathcal{F}_{s_k}(u_k(t),\Omega,\varphi) \geq \underline{\mathcal{A}}(u(t),\overline{\Omega},\varphi^-)\ ,$$

the lower-semicontinuity of the $L^2(0,T;L^2(\Omega))$ norm with weight

$$\liminf_{k\to\infty}\int_0^T e^{-t/\varepsilon}\|u_k'\|^2_{L^2(\Omega)}\,dt \geq \int_0^T e^{-t/\varepsilon}\|u'\|^2_{L^2(\Omega)}\,dt\ ,$$

and Fatou's lemma have been used. □

Lemma 6.2.2: *Let W^ε_s and W^ε_1 be as in Definition 6.1.3, an exterior datum $\varphi : \mathcal{C}\Omega \to \mathbb{R}$ fulfilling Assumptions (A3) and (A4), and $s_k \nearrow 1$, then for any $u \in D(W^\varepsilon_1)$ it holds*

$$\limsup_{k\to\infty} W^\varepsilon_{s_k}(u) = W^\varepsilon_1(u)\ .$$

Proof. Let $u \in D(W^\varepsilon_s) = H^1(0,T;L^2(\Omega)) \cap L^1(0,T;\mathrm{BV}(\Omega))$, then we have proved the pointwise convergence in Theorem 5.3.4. Furthermore, from Lemma 4.1.5 it follows

$$(1-s)\big|\mathcal{F}_s(u,\Omega,\varphi)\big| \leq (1-s)\big|\mathcal{F}_s(0,\Omega,\varphi)\big| + (1-s)\frac{C(n,\Omega)}{s^2}\|u-v\|_{W^{s,1}}\ .$$

Then, from Lemma 5.3.3 it follows

$$(1-s)\|u\|_{W^{s,1}(\Omega)} \leq C(n,\Omega)\|u\|_{\mathrm{BV}(\Omega)}$$

and we only have to bound the first term

$$\big|\mathcal{F}_s(0,\Omega,\varphi)\big| \leq \int_\Omega\int_{\mathcal{C}\Omega}\frac{|\varphi(y)|}{|x-y|^{n+s}}\,dx\ dy = \sigma_s\ ,$$

where σ_s as in Assumption (A1). We have seen in Lemma 5.2.1, that from Assumptions (A3) and (A4) it follows that $\sigma_s < \infty$ for all $s \in [s_0,1)$ with s_0 as in Equation (5.2.2) and also $\lim_{s\nearrow 1}(1-s)\sigma_s < \infty$. This implies

$$(1-s_k)\big|\mathcal{F}_{s_k}(u,\Omega,\varphi)\big| \leq C(n,\Omega)\|u\|_{\mathrm{BV}(\Omega)}$$

for all $s_k \in [s_0,1)$, which is integrable on $[0,T]$ by assumption. Then, by dominated convergence, it follows

$$\lim_{k \nearrow 1} W_{s_k}^{\varepsilon}(u) = W_1^{\varepsilon}(u) \ . \qquad \square$$

Theorem 6.2.3 (Mosco-convergence of the WED Functionals)**:** *Let W_s^ε and W_1^ε be as in Definition 6.1.3, an exterior datum $\varphi : \mathcal{C}\Omega \to \mathbb{R}$ fulfilling Assumptions (A2) to (A4), then*

$$\mathfrak{M}\text{-}\lim_{s \nearrow 1} W_s^\varepsilon = W_1^\varepsilon \tag{6.2.1}$$

in $H^1(0,T;L^2(\Omega))$

Combining the equicoercivity result and the Mosco-convergence of the WED functionals, we obtain the following.

Theorem 6.2.4: *For an initial condition $u_0 \in D(\underline{\mathcal{A}}) = \mathrm{BV}(\Omega)$ and W_s^ε and W_1^ε as in Definition 6.1.3, an exterior datum $\varphi : \mathcal{C}\Omega \to \mathbb{R}$ fulfilling Assumptions (A2) to (A4), and unique minimizers u_s^ε and u_1^ε on $K(u_{0\varepsilon}^s)$, it holds*

$$W_s^\varepsilon(u_s^\varepsilon) \to W_1^\varepsilon(u_1^\varepsilon) \quad \text{as } s \nearrow 1 \ ,$$

and furthermore

$$u_s^\varepsilon \to u_1^\varepsilon \quad \text{weakly in } H^1(0,T;L^2(\Omega)) \ .$$

Proof. This follows from the Fundamental Theorem of Γ-convergence, see for example [Bra06, p.115] and the uniqueness of the minimizers of the WED functionals. □

7 Appendix

7.1 Definitions of the relaxed area functional

In the literature, different relaxations of the area functional have been presented. In [Ser61] and [GS64], the L^1_{loc} relaxation is considered, and in [Giu84], the area functional is defined in analogy to the definition of the total variation for BV functions. The definitions are stated here, and it is proved that they coincide.

Lemma 7.1.1 (Relaxations of the area functional $\mathcal{A}$)**:** *For an open, bounded set Ω with Lipschitz boundary and $u \in L^1(\Omega)$ define*

$$\mathcal{A}^S(u,\Omega) = \inf\left\{ \liminf_{n\to\infty} \mathcal{A}(u_n,\Omega_n) \text{ for } u_n \in W^{1,1}(\Omega_n) \text{ with } u_n \to u \text{ in } L^1_{\text{loc}}(\Omega) \right\} \tag{7.1.1}$$

and

$$\mathcal{A}^G(u,\Omega) = \sup\left\{ \int_\Omega g_{n+1} + u \operatorname{div}((g_1,...,g_n))\, dx \;\; \text{for } g \in C^1_c(\Omega,\mathbb{R}^{n+1}) \text{ s.t. } |g| \leq 1 \right\} \tag{7.1.2}$$

Then for $u \in L^1(\Omega)$

$$\underline{\mathcal{A}}(u,\Omega) = \mathcal{A}^S(u,\Omega) = \mathcal{A}^G(u,\Omega)$$

And

$$D(\underline{\mathcal{A}}) = \mathrm{BV}(\Omega)$$

Proof.

1. $\underline{\mathcal{A}}(u,\Omega) \geq \mathcal{A}^S(u,\Omega)$ is straightforward as L^1 convergence implies L^1_{loc} convergence.

2. Next we want to show that $\underline{\mathcal{A}}(u,\Omega) \leq \mathcal{A}^S(u,\Omega)$ by controlling an approximating sequence of functions near the boundary using the Gagliardo–Nirenberg–Sobolev inequality. Therefore, let $u \in L^1(\Omega)$ with $\mathcal{A}^S(u,\Omega) < \infty$. Then, there exists a sequence $u_i \in W^{1,1}(\Omega)$ such that $u_n \to u$ in $L^1_{\text{loc}}(\Omega)$ and $\lim_{i\to\infty} \mathcal{A}(u_i,\Omega) = \mathcal{A}^S(u,\Omega)$. This gives a uniform bound for the first weak derivative using the area functional

$$\sup_i \|\nabla u_i\|_{L^1(\Omega)} \leq \sup_i \int_\Omega \sqrt{1 + \|\nabla u_i\|^2}\, dx \; = \sup_i \mathcal{A}(u_i,\Omega) \leq C < \infty$$

and by Gagliardo–Nirenberg–Sobolev inequality it follows

$$\sup_j \|u\|_{L^{p^*}} \leq C$$

with $p^* := \frac{n}{n-1}$ being the Sobolev exponent. For $\delta > 0$ we can choose $K \subset \Omega$ compact such that $|\Omega \setminus K| < \delta$ and $\int_{\Omega\setminus K} |u|\, dx \ < \delta$. Then, using the Hölder inequality for all $i \in \mathbb{N}$ it holds

$$\int_{\Omega\setminus K} |u - u_i|\, dx \ \leq \int_{\Omega\setminus K} |u|\, dx \ + \int_{\Omega\setminus K} |u_i|\, dx \ \leq \delta + |\Omega \setminus K|^{1/n} \|u_i\|_{L^{p^*}} \leq C\delta^{1/n} \ .$$

Then, as $u_i \to u$ in $L^1(K)$ and δ arbitrary it follows that $u_i \to u$ in $L^1(\Omega)$. Thus, $\underline{\mathcal{A}}(u,\Omega) \leq \mathcal{A}^S(u,\Omega)$.

3. $\mathcal{A}^S(u,\Omega) \geq \mathcal{A}^G(u,\Omega)$ may be deduced by the L^1_{loc}-lower semicontinuity of $\mathcal{A}^G$. To prove the lower semicontinuity, consider $u_m \to u$ in $L^1_{\mathrm{loc}}(\Omega)$ and, without relabeling, choose a subsequence converging pointwise almost everywhere on $\mathrm{supp}(g)$. It holds

$$\int_\Omega g_{n+1} + \sum_{i=1}^n u \frac{\partial g_i}{\partial x_i}\, dx \ = \int_\Omega g_{n+1} + \lim_{m\to\infty} u_m \sum_{i=1}^n \frac{\partial g_i}{\partial x_i}\, dx \ \leq \liminf_{n\to\infty} \mathcal{A}^G(u_m,\Omega) \ . \tag{7.1.3}$$

By taking the supremum over g, the lower semi-continuity follows. Then, as $\mathcal{A}^G$ agrees with $\mathcal{A}$ on $W^{1,1}(\Omega)$ and the relaxation being the supremum over all lower semicontinuous functions that are smaller than $\mathcal{A}$, it follows that $\mathcal{A}^S(u,\Omega) \geq \mathcal{A}^G(u,\Omega)$.

4. Lastly, we will approximate $u \in \mathrm{BV}(\Omega) = D(\mathcal{A}^G)$ smoothly by convolution. Therefore, let $\varepsilon > 0$, $\Omega_i := \{x \in \Omega \mid \mathrm{dist}(x,\partial\Omega) > \frac{1}{i+m}\}$ and choose m large enough such that both $|\Omega \setminus \Omega_0| < \varepsilon$ and $\|Du\|(\Omega \setminus \Omega_0) < \varepsilon$. Then, define $A_i := \Omega_{i+1} \setminus \Omega_{i-1}$ with $\Omega_{-1} = \emptyset$ and denote by $\{\varphi_i\}_{i=0}^\infty$ a partition of unity subordinated to A_i with $\varphi_i \in \mathcal{C}_c^\infty(A_i)$.

   Now, let η be a positive symmetric mollifier and choose ε_i such that the following holds for the mollifier $\eta_i := \eta_{\varepsilon_i}$

$$\mathrm{supp}(\eta_i * (u\varphi)) \subseteq \Omega_{i+2} \setminus \Omega_{i-2} \ , \tag{7.1.4}$$

$$\int_\Omega |\eta_i * (u\varphi) - (u\varphi_i)|\, dx \ < \varepsilon 2^{-i} \ , \tag{7.1.5}$$

$$\int_\Omega |\eta_i * (u\nabla\varphi_i) - (u\nabla\varphi_i)|\, dx \ < \varepsilon 2^{-i} \ . \tag{7.1.6}$$

Define $K := \frac{1}{\sqrt{1+|\nabla u_\varepsilon|^2}}$ and $\rho_j := K\sum_{i=1}^j \varphi_i$. Then, using the integration by parts formula, the properties of mollifiers, and the local finiteness of the partition of unity, it holds

$$\begin{aligned}
&\int_\Omega \nabla u_\varepsilon \cdot (\rho_j \nabla u_\varepsilon)\, dx \\
&\quad= \int_\Omega -\nabla u_\varepsilon \cdot (-\rho_j \nabla u_\varepsilon)\, dx \\
&\quad= \int_\Omega u_\varepsilon \operatorname{div}(-\rho_j \nabla u_\varepsilon)\, dx \\
&\quad= \int_\Omega \sum_{i=1}^\infty \big(\eta_i * (\varphi_i u)\big) \operatorname{div}(-\rho_j \nabla u_\varepsilon)\, dx \\
&\quad= \int_\Omega \sum_{i=1}^\infty \varphi_i u \operatorname{div}\big(\eta_i * (-\rho_j \nabla u_\varepsilon)\big)\, dx \\
&\quad= \int_\Omega \sum_{i=1}^\infty u \operatorname{div}\big(\varphi_i \eta_i * (-\rho_j \nabla u_\varepsilon)\big) - \sum_{i=1}^\infty u \nabla\varphi_i \cdot \big(\eta_i * (-\rho_j \nabla u_\varepsilon)\big)\, dx \\
&\quad= \int_\Omega u \operatorname{div}\big(\sum_{i=1}^\infty \varphi_i \eta_i * (-\rho_j \nabla u_\varepsilon)\big) + \sum_{i=1}^\infty (\eta_i * u\nabla\varphi_i) \cdot \rho_j \nabla u_\varepsilon\, dx\ . \qquad (7.1.7)
\end{aligned}$$

Then for the second part we can estimate

$$\begin{aligned}
&\left| \int_\Omega \sum_{i=1}^\infty (\eta_i * u\nabla\varphi_i) \cdot \rho_j \nabla u_\varepsilon\, dx \right| \\
&\quad\le \int_\Omega \left| \sum_{i=1}^\infty \big[(\eta_i * u\nabla\varphi_i - u\nabla\varphi_i) + u\nabla\varphi\big] \cdot \rho_j \nabla u_\varepsilon\, dx \right| dx \\
&\quad\le \int_\Omega \underbrace{|\rho_j \nabla u_\varepsilon|}_{\le 1} \left(\sum_{i=1}^\infty |\eta_i * u\nabla\varphi_i - u\nabla\varphi_i| + \underbrace{|u\, \nabla(\sum_{i=1}^\infty \varphi_i)}_{=0} \right) dx \\
&\quad\le \sum_{i=1}^\infty \int_\Omega \underbrace{|\eta_i * u\nabla\varphi_i - u\nabla\varphi_i|}_{\le \varepsilon 2^-}\, dx\ \le \varepsilon\ . \qquad (7.1.8)
\end{aligned}$$

Then, combining Equations (7.1.7) and (7.1.8)and letting $j \to \infty$ it holds

$$\begin{aligned}
\mathcal{A}(u_\varepsilon, \Omega) &= \lim_{j\to\infty} \int_\Omega 1 + \nabla u_\varepsilon \cdot (\rho_j \nabla u_\varepsilon)\, dx \\
&= \lim_{j\to\infty} \int_\Omega \big(\sum_{i=1}^\infty \varphi_i \eta_i * \rho_j\big) + u \operatorname{div}\big(\sum_{i=1}^\infty \varphi_i \eta_i * (\rho_j \nabla u_\varepsilon)\big)\, dx\ + 2\varepsilon \qquad (7.1.9) \\
&\le \mathcal{A}^G(u, \Omega) + 2\varepsilon\ .
\end{aligned}$$

Where we have used that

$$\left|\sum_{i=1}^{\infty} \varphi_i \eta_i * \left(\rho_j\left(\frac{\partial u_\varepsilon}{\partial x_1}, ..., \frac{\partial u_\varepsilon}{\partial x_n}, 1\right)\right)\right| \leq K \left|\left(\frac{\partial u_\varepsilon}{\partial x_1}, ..., \frac{\partial u_\varepsilon}{\partial x_n}, 1\right)\right| \leq 1 \ .$$

Thus, as $u_\varepsilon \to u$ in $L^1(\Omega)$, it follows

$$\underline{\mathcal{A}}(u, \Omega) \leq \mathcal{A}^G(u, \Omega) \ . \qquad \square$$

7.2 Absolute continuous and weak differentiable paths in Banach spaces

The theory on paths in Banach spaces strongly builds on the theory of Bochner integrable functions. A quick recap will be provided, see Diestel, Uhl [DU91] and Showalter [Sho13] and the references therein for the corresponding proofs.

Given a measure space (S, Σ, μ) and a Banach space X, a function $f : S \to X$ is strongly μ-measurable if it can be approximated by X-valued step functions such that for μ almost every point x, they converge to $f(x)$ in X. A function such that $\langle f, \varphi\rangle$ is μ-measurable for every $\varphi \in X^*$ is called weakly μ-measurable. By Pettis' theorem, f is strongly μ-measurable if and only if f is almost-separable-valued and weakly μ-measurable.

By Bochner's theorem, a function f is μ integrable if and only if f is strongly μ measurable and $s \mapsto \|f(s)\|_X$ is integrable. Then, for $1 \leq p < \infty$ the Bochner L^p space $L^p(S; X)$ consists of (all equivalent classes of) all strongly μ-measurable functions such that

$$\|f\|^p_{L^p(S;X)} := \int_\Omega \|f\|^p_X \, d\mu \ < \infty$$

and

$$\|f\|_{L^\infty(S;X)} := \operatorname*{ess\,sup}_\Omega \|u\|_X < \infty \ .$$

The Bochner L^p spaces are Banach spaces for $1 \leq p \leq \infty$ and, if X is separable, so is $L^p(S; X)$ for $1 \leq p < \infty$. The reflexivity for $1 < p < \infty$ and the Riesz representation theorem leading to the equality $(L^p(S; X))^* = L^q(S; X^*)$ with $1/p + 1/q = 1$ depends on the Radon Nikodým property of X^*. Indeed, if the space X is reflexive and μ is σ-finite, then indeed the spaces can be identified, and thus reflexivity of the Bochner spaces holds. This is known as Phillips' Theorem. Comparable versions of Fatou's Lemma and Lebesgue's dominated convergence exist.

We now turn to Banach-space valued paths, when S is an interval $I \subseteq \mathbb{R}$ with the classical Lebesgue measure. Firstly, a version of the Lebesgue point theorem extends to the case of Banach spaces. For $f \in L^1(I; X)$ then for almost ervery $s \in I$ it holds

$$\lim_{h \searrow 0} \frac{1}{h} \int_s^{s+h} \|f(t) - f(s)\|_X \, dt \ = 0$$

and thus
$$\lim_{h \searrow 0} \frac{1}{h} \int_s^{s+h} f(t)\, dt \; = f(s)$$
Using the Lebesgue point theorem, one obtains the following.

Theorem 7.2.1: *Given an open interval $I \subseteq \mathbb{R}$ and $u : I \to X$ locally integrable function, u is weakly differentiable if and only if there exists a representatitve of u and a locally integrable function $v : I \to X$ such that*
$$u(t) = u(t_0) + \int_{t_0}^{t} v(s)\, ds$$
for all $t, t_0 \in \overline{I}$. Then u is strongly differentiable for a.e. $t \in I$ and its strong derivative coincides with $v(t)$.

A proof can be found in [Eva10, p. 302]. This directly implies continuity, and furthermore, it can be used to prove an analog of Morrey's inequality for Banach space-valued Sobolev paths and their Hölder regularity.

Lemma 7.2.2: *For and open intervall $I \subseteq \mathbb{R}$, $1 \leq p \leq \infty$ and $u \in W^{1,p}(I, X)$, it holds*
$$\|u(t) - u(s)\|_X \leq |s - t|^{1-1/p} \|u'\|_{L^p(S;X)}$$
, and thus
$$W^{1,p}(I, X) \hookrightarrow \mathcal{C}^{0,1-1/p}(\overline{I}, X)\ .$$
Next, recall the definition of absolute continuous curves.

Definition 7.2.3 (Absolute continuity)**:** *For $I \subset \mathbb{R}$ be an open interval and let $u : I \to X$ it is absolutely continuous on I if one of the following equivalent properties holds:*

1. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that for every finite family of pairwise disjoint intervals $\{(a_k, b_k)\}_{k=1}^m \subset I$ with*
$$\sum_{k=1}^{m} |b_k - a_k| < \delta$$
it holds
$$\sum_{k=1}^{m} \|u(b_k) - u(a_k)\|_X < \varepsilon.$$
2. *There exists $g \in L^1(I)$ such that*
$$\|u(t) - u(s)\|_X \leq \int_s^t g(r)\, dr$$
for all $s, t \in I$ with $s \leq t$.

We denote the space of absolutely continuous curves from I to X by $AC(I, X)$.

A proof for the equivalence of the definitions can be found in [ABS21].
In general for $I \subseteq \mathbb{R}$ open interval and $u \in W^{1,1}(I, X)$ one has

$$\|u(s) - u(t)\|_X = \| \int_s^t v(\tau)\, d\tau \,\|_X \leq \int_s^t \|v(\tau)\|_X \, d\tau \; = \|v(\tau)\|_{L^1(I,X)}$$

and thus

$$W^{1,1}(I; X) \subseteq \mathrm{AC}(I, X)\ . \tag{7.2.1}$$

In $\mathbb{R}$, every absolutely continuous curve is differentiable almost everywhere, but in a general Banach space, this might fail. Yet, the result is indeed true for reflexive space X and known as Kōmura's Theorem.

Theorem 7.2.4 (Kōmura)**:** *For an interval $I = [a, b] \subset \mathbb{R}$ and $u : I \to X$ be absolutely continuous, then u is differentiable for almost every $t \in I$ and there exists $v \in L^1(I; X)$ such that*

$$u(t) - u(a) = \int_a^t v(s)\, ds$$

for a.e. $t \in I$. In particular,

$$u'(t) = v(t)$$

for almost every $t \in I$.

For bounded intervals, this implies the following.

Lemma 7.2.5: *For a reflexive Banach space X and an bounded intervall $K \subseteq \mathbb{R}$ it holds*

$$AC(K, X) = W^{1,1}(K; X)\ .$$

7.3 A Hardy-type inequality

We recall here a Hardy-type inequality, some direct implications, and especially derive the dependency of the constants on s.

Lemma 7.3.1 (Hardy-type ineqaulity)**:** *Let $s \in (0, 1)$, and $\Omega \subseteq \mathbb{R}^n$ open, bounded, and with Lipschitz boundary. Then, there exists a constant $C_1(n, \Omega, s) > 0$, depending on n, Ω, and s, such that it holds*

$$\int_\Omega \frac{|u(x)|}{dist(x, \partial\Omega)^s}\, dx \;\leq C_1(n, \Omega, s)\|u\|_{W^{s,1}(\Omega)} \tag{7.3.1}$$

for every $u \in W^{s,1}(\Omega)$.

See [CL20, Proposition A.2] for a proof.

Lemma 7.3.2: *For the constant $C_1(n, \Omega, s)$ it holds*

$$C_1(n, \Omega, s) \leq \frac{\tilde{C}(n, \Omega)}{s}\ ,$$

where $\tilde{C}(n,\Omega)$ is a constant that depends only on n and Ω.

Proof. The proof of Lemma 7.3.1 in [CL20] reduced the case of Ω to the Hardy inequality for the halfplane. This is done by covering the boundary of Ω with a finite number N of balls and defining bi-Lipschitz homeomorphisms $\{T_j\}_{j=0}^N$ transposing to a cylinder at the origin. The constant that is generated through this process depends only on N, T_j, and Ω, and can be chosen uniformly in s. We therefore denote it by $\tilde{C}(\Omega)$. The important estimate is the Hardy inequality for the halfplane. The optimal constant $\mathcal{D}_{n,1,s}$ has been calculated in [FS09], and is given by

$$\mathcal{D}_{n,1,s} = 2\pi^{(n-1)/2}\frac{\Gamma(\frac{1+s}{2})}{\Gamma(\frac{N+s}{2})}\int_0^1 |1-r^{s-1}|\frac{dr}{(1-r)^{1+s}} \ .$$

By the classical binomial series, for $|r|<1$ it holds

$$(1-r)^{-1-s} = \sum_{n=0}^{\infty} a_n r^n,$$

where

$$a_n = \frac{\Gamma(n+1+s)}{\Gamma(1+s)\Gamma(n+1)} \ .$$

By monotone convergence, it holds

$$\int_0^1 \frac{r^{s-1}-1}{(1-r)^{n+1}}\,dr \ = \sum_{n=0}^{\infty} a_n \int_0^1 (r^{s-1}-1)r^n\,dr \ = \sum_{n=0}^{\infty} a_n\left(\frac{1}{n+s}-\frac{1}{n+1}\right) .$$

a_n satisfies the following relation

$$a_n = \frac{\Gamma(n+1+s)}{\Gamma(1+s)\Gamma(n+1)} = \frac{(n+s)\,\Gamma(n+s)}{n\,\Gamma(1+s)\Gamma(n)} = \frac{n+s}{n}a_{n-1} \ .$$

Thus, it follows for $N\in\mathbb{N}$

$$\begin{aligned}
\sum_{n=0}^{N} a_n\left(\frac{1}{n+s}-\frac{1}{n+1}\right) &= \sum_{n=0}^{N}\frac{a_n}{n+s} - \sum_{n=0}^{N}\frac{a_n}{n+1}) \\
&= \sum_{n=0}^{N}\frac{a_n}{n+s} - \sum_{n=0}^{N}\frac{a_n}{n+1} \\
&= \sum_{n=0}^{N}\frac{a_n}{n+s} - \sum_{n=0}^{N}\frac{a_{n+1}}{n+1+s} \\
&= \frac{a_0}{s} + \sum_{n=1}^{N}\frac{a_n}{n+s} - \sum_{n=1}^{N}\frac{a_n}{n+s} - \frac{a_{N+1}}{N+1+s} \to \frac{1}{s}
\end{aligned}$$

as $N \to \infty$. Thus, we obtain

$$\begin{aligned}\mathcal{D}_{n,1,s} &= 2\pi^{(n-1)/2}\frac{\Gamma(\frac{1+s}{2})}{\Gamma(\frac{n+s}{2})}\frac{1}{s}\\ &\leq \frac{\tilde{C}(n)}{s}\end{aligned}$$

for $s \in (0,1)$. Therefore, we can conclude

$$C_1(n,\Omega,s) \leq \tilde{C}(\Omega)\mathcal{D}_{n,1,s} \leq \frac{\tilde{C}(\Omega)\tilde{C}(n)}{s}$$

□

Lemma 7.3.3: *For $u \in W^{s,1}(\Omega)$, $s \in (0,1)$, $\Omega \subseteq \mathbb{R}^n$ open, bounded, and with Lipschitz boundary, it holds*

$$\int_\Omega |u(x)| \int_{\mathcal{C}} \frac{dy}{|x-y|^{n+s}}\,dx \leq C_2(n,\Omega,s)\|u\|_{W^{s,1}(\Omega)}\ ,$$

with

$$C_2(n,\Omega,s) = \frac{\mathcal{H}^{n-1}(S^{n-1})}{s}C_1(n,\Omega,s) \leq \frac{\tilde{C}(n,\Omega)}{s^2}$$

where $C_1(n,\Omega,s)$ as in Lemma 7.3.1.

Proof. It holds

$$\int_{\mathcal{C}\Omega}\frac{dy}{|x-y|^{n+s}} \leq \int_{\mathcal{C}B_r(x)}\frac{dy}{|y|^{n+s}} = \int_r^\infty \frac{\mathcal{H}^{n-1}(S^{n-1})}{r^{1+s}}\,dr = \frac{\mathcal{H}^{n-1}(S^{n-1})}{sr^s}$$

with $r := \operatorname{dist}(x,\partial\Omega) > 0$. Thus, applying Lemma 7.3.2 we may deduce

$$\begin{aligned}\int_\Omega |u(x)| \int_{\mathcal{C}} \frac{dy}{|x-y|^{n+s}}\,dx &\leq \frac{\mathcal{H}^{n-1}(S^{n-1})}{s}\int_\Omega \frac{|u(x)|}{\operatorname{dist}(x,\partial\Omega)^s}\,dx\\ &\leq \frac{\mathcal{H}^{n-1}(S^{n-1})}{s}C_1(n,\Omega,s)\|u\|_{W^{s,1}(\Omega)}\end{aligned}$$

□

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