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Matchings of the Alternating Sign Triangle Graph

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Abstract (deutsch)

Ayyer, Behrend und Fischer führten 2020 einen Graphen ein, dessen Matchings (die fast perfekt sind) mit alternierenden Vorzeichendreiecken (ASTs) auf dieselbe Weise in Verbindung stehen wie perfekte Matchings des Azteken-Graphen mit alternierenden Vorzeichenmatrizen (ASMs). Sie zeigten, dass es gleich viele ASTs wie ASMs gibt, und dass dasselbe für die zugehörigen Matchings gilt. Hier arbeiten wir hin auf eine elementare Abzählung der Matchings, die mit ASTs in Verbindung stehen, vor allem indem wir bekannte Methoden verallgemeinern. Insbesondere geben wir eine Verallgemeinerung von Ciucus Theorem über perfekte Matchings von zellulären Graphen an, wir finden einen gemeinsamen Rahmen für verschiedene bekannte Bijektionen zwischen Matchings und Pfadfamilien, wir verallgemeinern die planare Version des Lindström-Gessel-Viennot Lemmas, wir verknüpfen Operationen auf gerichteten Graphen mit Operationen auf Matrizen, die deren Pfadanzahlen beinhalten, wir behandeln umfassend Anzahlen von Schröder-Pfaden (die auch über der x -Achse starten und enden dürfen), und wir drücken Summen von Minoren als Pfaffsche Formen einzelner Matrizen aus (Okada folgend, aber mit einer Abwandlung ähnlich wie Ishikawa und Wakayama). Als Ergebnis wird die gesuchte Anzahl als Determinante relativ einfacher Matrizen ausgedrückt.

Abstract (english)

In 2020, Ayyer, Behrend, and Fischer introduced a graph whose matchings (which are nearly perfect) are related to alternating sign triangles (ASTs) in the same way that perfect matchings of the Aztec graph are related to alternating sign matrices (ASMs). They showed that ASTs and ASMs are equinumerous, and that the same holds for the associated matchings. Here, we work towards an elementary enumeration of the matchings related to ASTs, mainly by generalizing known techniques. In particular, we give a generalization of Ciucu's theorem about perfect matchings of cellular graphs, we find a common framework for various known bijections between matchings and path families, we generalize the planar version of the Lindström-Gessel-Viennot lemma, we relate operations on directed graphs to operations on matrices containing their path numbers, we broadly treat numbers of Schröder paths (allowed to start and end also above the x -axis), and we express sums of minors as Pfaffians of single matrices (following Okada, but giving a twist to it similar to Ishikawa and Wakayama). As a result, the sought-after number is expressed as the determinant of relatively simple matrices.

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1 Introduction

In this paper, we study Q -partial matchings of a family of graphs we call ‘AST-graphs’, shown in figure 1. Q is the set of vertices marked in purple and ‘ Q -partial’ is the term we introduce to indicate matchings that cover all vertices not in Q , while each vertex in Q may or may not be covered. For the AST-graph of size $n \geq 1$, the set Q consists of $2(n - 1)$ vertices. From the bipartition of the AST-graph, it can easily be seen that every Q -partial matching covers exactly $n - 1$ vertices in Q .

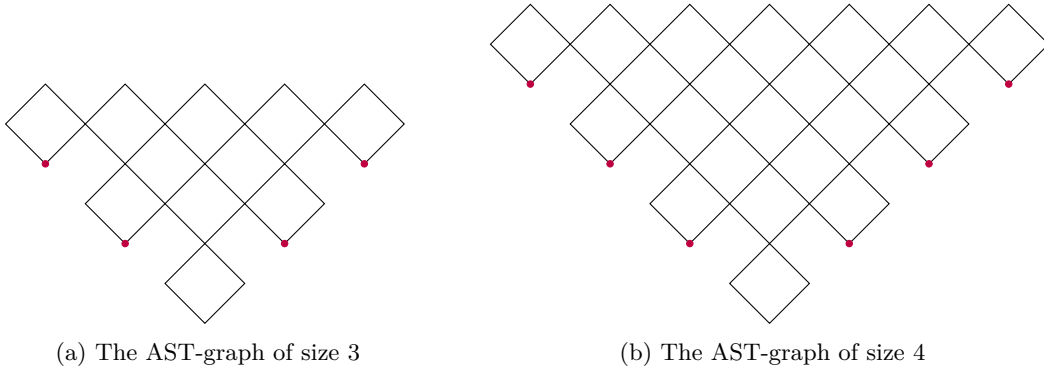


Figure 1: The AST-graph (vertices in Q are marked in purple)

1.1 The problem prompting this work

The mathematical objects mentioned here are rigorously defined and explained in section 2.

The AST-graph (though not called that way) was introduced in a paper by Ayer, Behrend, and Fischer in 2020 [4]. The main results of this paper were several enumeration formulas for certain symmetric types of alternating sign matrices (ASMs). In this context, alternating sign triangles (ASTs) arose as characteristic subarrays of some of these ASMs. Ayer et al. showed that these ASTs are related to Q -partial matchings of the AST-graph in the same way that ASMs are related to perfect matchings of the Aztec graph. (The latter relation has been known since the 1990s, see for instance [18].)

In the same paper, it was shown that the number of ASTs is equal to the number of ASMs and that ASTs and ASMs share certain statistics. From this, Ayer et al. deduced that the number of Q -partial matchings of the AST-graph is equal to the number of perfect matchings of the Aztec graph. They commented this fact in the following way: (Since they neither used the term ‘AST-graph’ nor ‘ Q -partial’ for their objects, we translate a little.)

“A bijection between the set of perfect matchings of this region [the Aztec graph] and the set of [Q -partial matchings of the AST-graph] might give a hint on the bijection between ASMs and ASTs.” [4, page 44 (end of section 5.7)]

This was the starting point for the work that is now presented in this paper.

1.2 The methods used and the results

The ideal goal of this paper would have been to find a bijection between perfect matchings of the Aztec graph and Q -partial matchings of the AST-graph. Since the Aztec graph has been studied

quite a lot in the past, while the AST-graph has not gotten much attention yet, our work focuses on the latter.

Our main method is to look at techniques that are used to enumerate matchings, or further deal with the results coming out of that, and then apply them to the AST-graph. To this end, we often modify or generalize these techniques. In particular, we investigate how they can be adapted for the case of Q -partial matchings in more general settings.

While we do not succeed in finding a bijection, we come relatively close to giving a direct enumeration of Q -partial matchings of the AST-graph that does not use the relation to ASTs (except for a small simplification of the problem in section 3.6). How close we come exactly, we cannot judge before someone finishes it. The enumeration slowly builds up over the whole paper, culminating in the expression of the sought-after number as the Pfaffian or determinant of some relatively simple matrices.

The paper is interspersed with many detours to not strictly necessary generalizations of theorems, applications to other settings, etc. Thereby, it gives general tools for the enumeration of perfect and Q -partial matchings as well as for related problems turning up, and a broad treatment of the specific problem of enumerating Q -partial matchings of the AST-graph.

1.3 Detailed outline of the paper

We here give a very brief overview of the content of each section. Note however that each section has its own introduction, specifying more detailed the contents of the individual subsections. Also the summarizing section 8 should be mentioned, which recapitulates the whole way from the AST-graph to a single determinant with all stages in-between, as well as interesting theorems appearing along the way.

Section 2 fleshes out what is summarized in section 1.1: We give an introduction to ASMs and ASTs and the various statistics they share besides the number. Then we introduce the Aztec graph and the AST-graph and explain the relation between their matchings and ASMs resp. ASTs. Finally, we translate the statistics on ASMs and ASTs to the matchings, proving in particular that Q -partial matchings of the AST-graph and perfect matchings of the Aztec graph not only share their number, but also the distribution of the number of edges tilted in one of the two possible directions.

In section 3, we give a general theorem about C -partial matchings of so-called ‘cellular graphs’: The number of matchings which leave out X is related to the number of matchings which leave out $C \setminus X$. This is a generalization of the work by Mihai Ciucu from 1996 [14]. The proof of this theorem also makes the relation between matchings and ASMs/ASTs, which is presented in section 2, more clear. Finally, we use this theorem to enumerate perfect matchings of the Aztec graph (which was already done by Ciucu) and to simplify the problem of enumerating Q -partial matchings of the AST-graph by reducing it to the ‘half-reduced’ and the ‘reduced’ AST-graph (shown in figures 10a and 10b).

In section 4, we introduce what we call ‘MaPaf-bijections’: These are bijections between matchings of certain graphs and families of disjoint paths. We construct MaPaf-bijections in a quite general setting and show how to derive some well-known MaPaf-bijections, of which we give a thorough overview, from that. Then we apply our MaPaf-bijections to Q -partial matchings of

the (half-)reduced AST-graph. There are several ways of doing that, giving us many different directed graphs in which the families of disjoint paths correspond to Q -partial matchings of the (half-)reduced AST-graph. Some of these directed graphs are shown in the appendix section A.3 (always on the right side).

Section 5 presents the well-known Lindström-Gessel-Viennot lemma, which serves to enumerate path families in terms of the determinant of a matrix containing the numbers of single paths. We develop corollaries of it which are adjusted to our situation, but at the same time quite interesting in themselves. In combination with our MaPaf-bijections, we can thus express the number of Q -partial matchings of the (half-)reduced AST-graph in several ways as a sum of minors of a single matrix. Additionally, we find that the different matrices for which this works can be obtained from each other by elementary matrix row operations, which correspond to certain operations in the related directed graphs.

In section 6, we study Schröder paths, since the paths we are dealing with are of this form. We introduce several ‘Schröder matrices’ containing numbers of certain Schröder paths. In terms of these, we can express some of the matrices of which we want to determine the sum of minors. Furthermore, we present various properties of the Schröder matrices, for instance a generating function, relations and linear recurrences, which can all help to calculate their entries. Interestingly, several of these properties are obtained from the matrix row operations and corresponding graph operations found in section 5.

Section 7 deals with the problem of summing the minors of a matrix. First, a thorough introduction to Pfaffians is given, as all minor summation formulas express the sum of minors as the Pfaffians of some skew-symmetric matrix. While reproducing the formulas by Okada [47], we also develop our own versions which are very similar to work by Ishikawa and Wakayama [29]. Finally, all the different results culminate in several expressions of the number of Q -partial matchings of the (half-)reduced AST-graph in terms of the Pfaffian or determinant of certain matrices, which are given in terms of the Schröder matrices from section 6.

1.4 Convention for n and m , definitions

In this whole paper, unless stated otherwise, n and m (and m') are always considered as being nonnegative integers. In particular, this applies to n being used as the size of ASMs, ASTs, AST-graphs and various other graphs and matrices related to these, as well as m (or m') and n being used as the sizes of matrices or sets of vertices. If the size of an object related to ASMs or ASTs is not mentioned explicitly, it is considered as being of size n (i.e. for instance derived from the reduced AST-graph of size n).

Hence we generally do allow $n = 0$ or $m = 0$. Note however that most statements, although trivially correct, become quite pointless in this case as they then deal with empty matrices (which have determinant 1 and Pfaffian 1), empty graphs (which have a unique empty matching), empty path families (of which there is exactly one) etc. Therefore, we do not mention these cases explicitly in our proofs.

Section B of the appendix refers to the definitions of all mathematical expressions we use in this paper, except for some which are only used in a single section. Therefore, when mentioning a mathematical object, we most often do not specify where it is defined, as this can be looked up there.

2 ASMs and ASTs: Origin of the AST-graph

This section is dedicated to giving a thorough introduction to ASMs, ASTs and the graphs related to them. To start with, section 2.1 deals with ASMs and ASTs, summarizing their origin and certain enumerative results from mathematical literature. In particular, we focus on the statistics which are shared between ASMs and ASTs.

Section 2.2 then introduces the Aztec graph. After explaining the classical and quite direct correspondence of its perfect matchings to domino tilings of the Aztec diamond, we turn our attention to their relation to ASMs: We specify procedures to obtain perfect matchings of the Aztec graph from ASMs and vice versa, and thus derive the number of perfect matchings of the Aztec graph, as is done in the literature.

In section 2.3, we present the AST-graph, which in the same way as the Aztec graph is obtained from ASTs instead of ASMs. In addition to the known fact that there is an equal number of perfect matchings of the Aztec graph and Q -partial matchings of the AST-graph, we can further show that the same holds when only considering matchings with a fixed number of edges tilted in a certain direction. Section 2.4 derives the generating function necessary for that by expressing certain statistics on the matchings neatly in terms of statistics on ASMs and ASTs.

2.1 ASMs and ASTs

Alternating sign matrices are a generalization of permutation matrices, defined as follows:

Definition 2.1. An *alternating sign matrix (ASM)* of size n is an $n \times n$ matrix with entries from the set $\{-1, 0, 1\}$, such that the following two conditions hold:

- In every row and column, the nonzero entries alternate.
- The entries sum up to 1 in each row and column, i.e. the first and last nonzero entry of every row and column is 1.

ASMs were first introduced by Mills, Robbins, and Rumsey in the 1980s (see [46, section 3], [45]). They have been studied quite extensively since and turn out to be in bijection with many other combinatorial objects such as the six-vertex model/square ice model (see [19, section 7], [39]), monotone triangles [45, section 5] and descending plane partitions [24]. Furthermore, ASMs are equinumerous for instance with totally symmetric self-complementary plane partitions (enumerated in [2]) and alternating sign triangles [4, theorem 5.5], but no bijections have been found yet in these cases. ASMs are also related to λ -determinants [49, section 5] and to domino tilings of the Aztec diamond [18, section 3].

There are $\prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}$ ASMs of size n . This was already conjectured by Mills, Robbins, and Rumsey [45] and it was first proved in 1996 by Doron Zeilberger [56]. Clearly, the top row of an ASM only consists of 0's except for a single 1. The refined enumeration of ASMs, which involves the position of this unique 1 in the top row, was also proved by Zeilberger [57]. Additionally, many symmetry classes of ASMs have been studied and enumerated. References in this regard are listed for instance in [4] and [6, section 1.2].

Let us now come to alternating sign triangles, which are defined quite similar to ASMs:

Definition 2.2. An *alternating sign triangle (AST)* of size n is a triangular array as indicated in figure 2 with entries from the set $\{-1, 0, 1\}$, such that the following three conditions hold:

- In every row and column, the nonzero entries alternate.
- The entries sum up to 1 in each row.
- The first nonzero entry of every column is 1.

$$\begin{array}{cccccccc}
 a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} & \cdots & a_{1,2n-2} & a_{1,2n-1} \\
 & a_{2,2} & a_{2,3} & \cdots & a_{2,n} & \cdots & a_{2,2n-2} & \\
 & & \ddots & & \vdots & & \ddots & \\
 & & & & a_{n,n} & & &
 \end{array}$$

Figure 2: Shape and indexing of an AST of size n

It is no coincidence that the conditions for ASTs are essentially a direct analogue of the conditions for ASMs: ASTs first appeared as characteristic subarrays of certain symmetric types of ASMs in “Extreme diagonally and antidiagonally symmetric alternating sign matrices of odd order” by Ayer, Behrend, and Fischer in 2020 [4, section 2.5]. In the same paper, they showed that the number of ASTs of size n equals the number of ASMs of size n and that they also share certain statistics, which we discuss in the following.

Note first by counting the sum of all entries of an AST both row-wise and column-wise, that an AST of size n has to have n columns with sum 1 and $n - 1$ columns with sum 0. A generating function for ASTs which specifies the columns which have sum 1 was given by Fischer in [22, theorem 2]. We generalize the notation from [18] for the number of 1’s and -1 ’s in an ASM as follows:

Definition 2.3. Let A be an ASM or an AST. Then $N_+(A)$, $N_0(A)$, and $N_-(A)$ denote the total number of 1’s, 0’s, and -1 ’s in A respectively.

Note that both ASMs and ASTs of size n have n^2 entries and in each of the n rows, there is one more 1 than there are -1 ’s. Thus we have:

Lemma 2.4. Let A be an ASM or an AST of size n . Then:

$$\begin{aligned}
 N_+(A) + N_0(A) + N_-(A) &= n^2 \\
 N_+(A) - N_-(A) &= n
 \end{aligned}$$

Consequently, it is enough to know any of these three numbers in order to derive the other two. A second important parameter of ASMs and ASTs is their inversion number. This is a generalization of the inversion number known from permutation matrices and there are several ways of defining it. We use the following definition from [4, section 5.4]:

Definition 2.5. Let A be an ASM or an AST. Then the *inversion number* of A is given by

$$\text{inv}(A) := \sum_{\substack{i' < i, j' \leq j \\ \exists a_{i'j}, a_{ij'}}} a_{i'j} a_{ij'}$$

where ‘ $\exists a_{i'j}, a_{ij'}$ ’ is an abuse of notation to say ‘The entries $a_{i'j}$ and $a_{ij'}$ are defined’.

Remark. The choice of $i' < i, j' \leq j$ seems quite arbitrary. Indeed, when varying which of the two inequalities should be strict, we get the following:

$$\sum_{\substack{i' \leq i, j' < j \\ \exists a_{i'j}, a_{ij'}}} a_{i'j} a_{ij'} = \text{inv}(A) \quad (2.1)$$

$$\sum_{\substack{i' < i, j' < j \\ \exists a_{i'j}, a_{ij'}}} a_{i'j} a_{ij'} = \text{inv}(A) + N_-(A) \quad (2.2)$$

$$\sum_{\substack{i' \leq i, j' \leq j \\ \exists a_{i'j}, a_{ij'}}} a_{i'j} a_{ij'} = \text{inv}(A) + N_+(A) \quad (2.3)$$

While the expression (2.2) can be used as an alternative generalization of the inversion number of permutation matrices, the expression (2.3) cannot, as it evaluates to $\text{inv}(A) + n$ for permutation matrices of size n .

Proof of the remark. We prove equation (2.2), the others can be proved similarly:

$$\begin{aligned} \sum_{\substack{i' < i, j' < j \\ \exists a_{i'j}, a_{ij'}}} a_{i'j} a_{ij'} &= \text{inv}(A) - \sum_{\substack{i' < i, j' = j \\ \exists a_{i'j}, a_{ij'}}} a_{i'j} a_{ij'} = \\ &= \text{inv}(A) - \sum_{\substack{i, j \\ \exists a_{ij}}} \left(a_{ij} \sum_{\substack{i' < i \\ \exists a_{i'j}}} a_{i'j} \right) = \\ &= \text{inv}(A) - \sum_{\substack{i, j \\ a_{ij}=1}} \left(\sum_{\substack{i' < i \\ \exists a_{i'j}}} a_{i'j} \right) - \sum_{\substack{i, j \\ a_{ij}=-1}} \left(- \sum_{\substack{i' < i \\ \exists a_{i'j}}} a_{i'j} \right) = \\ &= \text{inv}(A) - \left(\sum_{\substack{i, j \\ a_{ij}=1}} 0 \right) - \left(\sum_{\substack{i, j \\ a_{ij}=-1}} -1 \right) = \\ &= \text{inv}(A) + N_-(A) \end{aligned}$$

□

When introducing ASTs, Ayer, Behrend, and Fischer proved that both N_- and inv have the same distribution on ASMs as they have on ASTs and that furthermore, even their joint distribution is the same (see [4, theorem 5.5] and note that they used μ for N_-):

Theorem 2.6 (Ayer, Behrend, Fischer). *Let m and i be integers. Then the number of ASMs A of size n with $N_-(A) = m$ and $\text{inv}(A) = i$ equals the number of ASTs A of size n with $N_-(A) = m$ and $\text{inv}(A) = i$.*

Remark. By what we have seen above, this theorem still holds true when replacing N_- by N_+ or N_0 and inv by the expressions (2.2) or (2.3).

We already mentioned yet another quite important and well-studied statistic on ASMs, namely the position of the unique 1 in the top row, i.e. the unique r such that $a_{1,r} = 1$. This corresponds to the parameter ρ on ASTs, which was introduced in [4, section 5.6] and is defined as follows (remember that an AST has $2n - 1$ columns, of which exactly n columns have sum 1):

Definition 2.7. Let A be an AST of size n . Let a denote the number of columns of A that are among the first $n - 1$ columns of A , have sum 1, and have a 1 as their last element. Let b denote the number of columns of A that are among the last $n - 1$ columns of A , have sum 1, and have a 0 as their last element. Then we define $\rho(A) := a + b + 1$.

In [22, theorem 1], Fischer proved that the distribution of ρ on ASTs is the same as the distribution of the position of the unique 1 in the top row of ASMs:

Theorem 2.8 (Fischer). *Let $1 \leq r \leq n$. Then the number of ASMs A of size n with $a_{1,r} = 1$ equals the number of ASTs A of size n with $\rho(A) = r$.*

So we have three statistics that are distributed equally on ASMs and ASTs:

- The number of -1 's (equivalently the number of 1 's or 0 's)
- The inversion number (equivalently expressions 2.2 or 2.3)
- The position of the unique 1 in the top row of an ASM / the parameter ρ for an AST

Interestingly, while also the joint distribution of the first two of these statistics is the same for ASMs and ASTs, this does not hold when considering the joint distribution of the third statistic with any of the others [4, section 5.6]. In fact, one can already see that by considering ASMs and ASTs up to size 4.

The equal distribution of the statistics makes it all the more plausible that there is a bijection between ASMs and ASTs waiting to be found. For instance, one could hope for a bijection preserving the numbers of 1 's, -1 's and 0 's as well as the inversion number. Alternatively, one could look for a bijection mapping ASMs with $a_{1,r} = 1$ to ASTs with $\rho(A) = r$. But careful: It is not possible to do both at the same time, since the joint distribution of all three of these statistics is not shared by ASMs and ASTs. This again diminishes the hopes of finding a nice bijection.

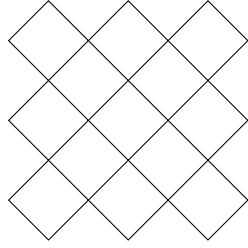
2.2 The Aztec graph

We now consider the *Aztec graph*, which is quite well-known and closely related to ASMs. Instead of a formal definition, consider figures 3a and 3b. We clearly see that the Aztec graph of size n has $2n(n + 1)$ vertices and thus every perfect matching of it consists of $n(n + 1)$ edges.

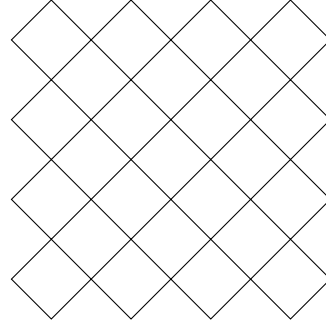
The Aztec graph comes from the *Aztec diamond*, a region in the plane composed of unit squares as shown in figures 3c and 3d. It was introduced by Elkies, Kuperberg, Larsen, and Propp in [18, 19], where they studied its domino tilings. A *domino* in this context consists of two unit squares sharing a side and a *domino tiling* is the partition of a shape into such dominoes. In [18, 19], Elkies et al. gave four different proofs that the Aztec diamond of size n has $2^{\binom{n+1}{2}}$ domino tilings. Since then, the Aztec diamond has been studied quite a lot and numerous other proofs of this so-called 'Aztec Diamond Theorem' have been found.

When rotating the Aztec graph by 45° , we see that it is essentially the dual graph of the Aztec diamond, as shown in figure 3e. Every possible domino on the Aztec diamond corresponds to an edge of the graph and thus domino tilings of the Aztec diamond are equivalent to perfect matchings of the Aztec graph, see figure 3f. Therefore, we can formulate the above-mentioned result by Elkies et al. [18, 19] in the following way:

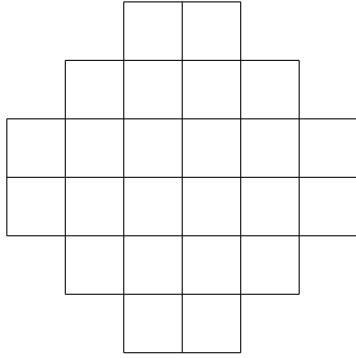
Theorem 2.9 (Elkies, Kuperberg, Larsen, Propp). *The Aztec graph of size n has $2^{\binom{n+1}{2}}$ perfect matchings.*



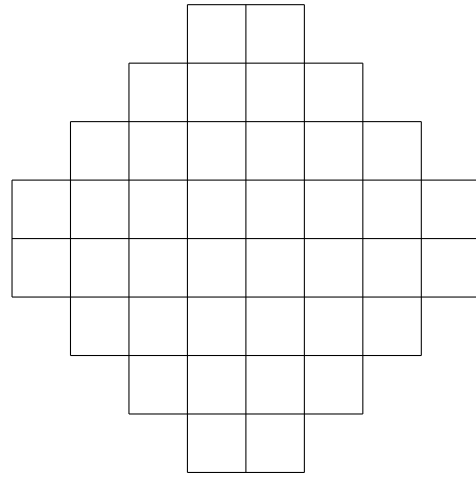
(a) The Aztec graph of size 3



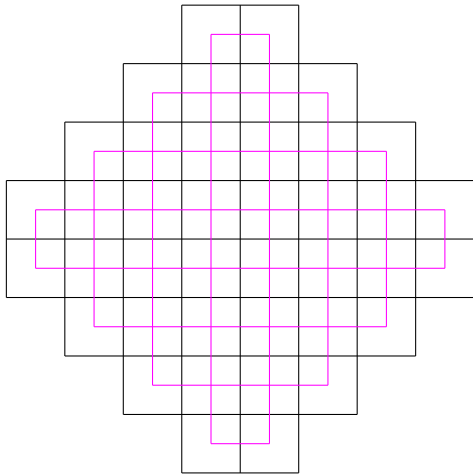
(b) The Aztec graph of size 4



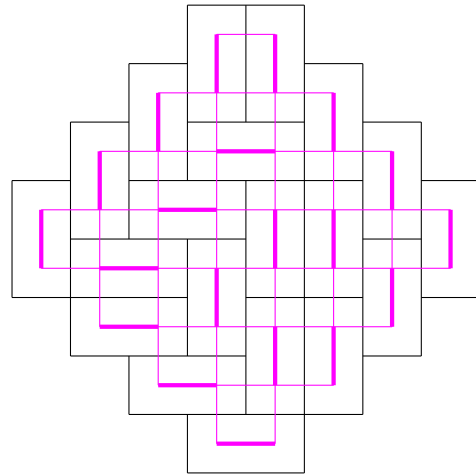
(c) The Aztec diamond of size 3



(d) The Aztec diamond of size 4



(e) The Aztec graph of size 4 (in pink) rotated by 45° and laid over the Aztec diamond of size 4



(f) A domino tiling of the Aztec diamond and the corresponding perfect matching of the Aztec graph

Figure 3: The Aztec graph and the Aztec diamond

Remark. Since the correspondence between domino tilings and perfect matchings is quite well-known, simple and was already mentioned by Elkies et al. [19, section 7], we allow ourselves to translate their results from the language of domino tilings to the language of perfect matchings while citing them, without mentioning it every time.

From a perfect matching of the Aztec graph, we can obtain an ASM of the same size by the following simple procedure. For this, note that the Aztec graph consists of *cells* of four edges each (marked in gray in figure 4b).

Procedure 1 (Perfect matching of the Aztec graph $\rightarrow$ ASM). See figure 4.

- (4a) Start with a perfect matching of the Aztec graph of size n .
- (4b) Write into each cell of the Aztec graph a -1 , 0 , or 1 depending on whether there are 0 , 1 , or 2 edges of the cell in the matching (-1 for 0 edges, 0 for 1 edge, 1 for 2 edges).
- (4d) The resulting array of integers is an ASM of size n .

The fact that the integer array obtained in this way is indeed an ASM was already discovered by Elkies et al. [18, section 3] when working with domino tilings of the Aztec diamond. By using height functions and a second ASM, associated in a slightly different way to the perfect matching, they proved that every ASM A can be obtained by procedure 1 from exactly $2^{N_+(A)}$ perfect matchings of the Aztec graph and thus [18, equation (1)]:

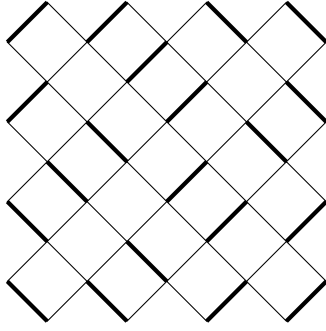
Theorem 2.10 (Elkies, Kuperberg, Larsen, Propp). *The number of perfect matchings of the Aztec graph of size n is*

$$\sum_{A \text{ ASM of size } n} 2^{N_+(A)}$$

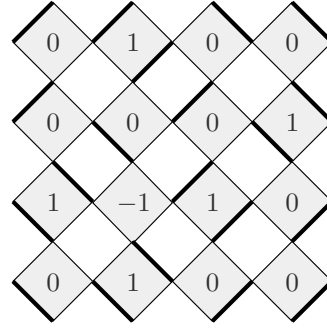
In his paper “Perfect Matchings of Cellular Graphs”, Ciucu gave a simpler proof of theorem 2.10 by essentially reversing procedure 1 [14, lemma 2.2]. He worked in a more general setting of which the Aztec graph is a special case. We now describe a reverse procedure to procedure 1 that follows in the main aspects the approach by Ciucu. Later in section 3, particularly sections 3.2 and 3.3, we apply his ideas in a yet more general setting.

Procedure 2 (ASM $\rightarrow$ Perfect matching of the Aztec graph). See figure 4.

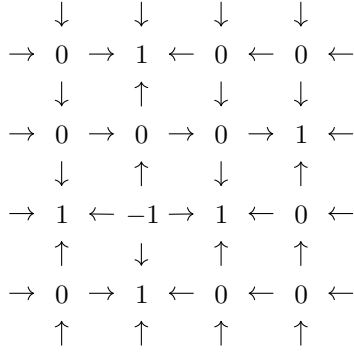
- (4d) Start with an ASM of size n .
- (4c) Insert arrows between neighboring entries of the ASM and at the border of the ASM that point away from -1 's and to 1 's. (I.e. if in a row or column there are no nonzero entries between a 1 and a -1 or the border, all arrows in-between point in the direction of the 1 .)
- (4e) This gives six possible local configurations of a matrix entry with the surrounding arrows. We associate to each such configuration a matching of a single cell (depicted directly below for each one). To the first configuration (with a 1), two possible matchings are associated. The names ‘*br*’ (bottom-right) etc. given to the configurations with a 0 come in use later.
- (4a) Take the union of all the matchings on single cells given by the integer in the cell and the surrounding arrows. In cells with a 1 , choose one of the two possible matchings. The result is a perfect matching of the Aztec graph of size n .



(a) A perfect matching of the Aztec graph



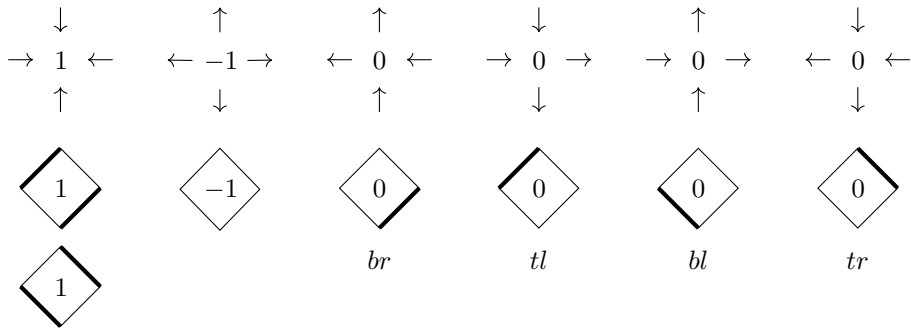
(b) The numbers correspond to the number of edges contained in the matching for each cell (Cells are marked in gray.)



(c) A configuration of arrows pointing away from -1 's and to 1 's

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

(d) An ASM



(e) The correspondence between local arrow configurations and cell matchings

Figure 4: Illustration of procedure 1 (figures 4a - 4b - 4d) and procedure 2 (figures 4d - 4c - 4a)

The two procedures are related to each other in the following way, which proves theorem 2.10:

Lemma 2.11. *Procedures 1 and 2 have the following properties:*

- (i). *The integer array resulting from procedure 1 is indeed an ASM.*
- (ii). *For an ASM A , procedure 2 has $2^{N_+(A)}$ possible outcomes, depending on the choices made in cells with a 1.*
- (iii). *Each set of edges that can result from procedure 2 is indeed a perfect matching of the Aztec graph.*
- (iv). *Procedure 1 and procedure 2 are inverse to each other in the following sense:*
 - (a) *Applying procedure 1 after procedure 2 gives back the same ASM.*
 - (b) *Applying procedure 2 after procedure 1 gives back the same perfect matching when making the right choices in cells with a 1 in procedure 2.*

Proof. Items (ii) and (iv)(a) are clear.

Item (iii): Note that every vertex of the Aztec graph lies exactly on the midpoint of one of the arrows introduced in procedure 2. The local configurations are associated to cell matchings in such a way that a vertex gets matched in a certain cell if and only if the corresponding arrow points to that cell (see figure 4e). Thus, when taking the union of all the matchings of the single cells, indeed every vertex is matched exactly once and we have a perfect matching.

Items (i) and (iv)(b): The above observation implies that we can reformulate procedure 1 in the following way: Write into each cell an integer and put arrows in-between such that each local configuration (consisting of an integer and the surrounding arrows) is given by the cell matching as shown in figure 4e. (So we go from figure 4a to figure 4c instead of 4b.) It can be seen from the possible local configurations that 1's and -1 's have to alternate and thus procedure 1 indeed gives an ASM, proving item (i). Additionally, this formulation makes item (iv)(b) immediately clear.

We do not go into further details here as this is very similar to what we do thoroughly in section 3.3, where following Ciucu's ideas, we prove proposition 3.16 and thus theorem 3.17, which essentially generalize lemma 2.11 and theorem 2.10. $\square$

With the relation to ASMs we now have, there are at least two ways of proceeding to enumerate perfect matchings of the Aztec graph, that is prove theorem 2.9: Firstly, both Elkies et al. and Ciucu were able (in quite similar ways) to derive the recurrence relation $\text{AD}(n) = 2^n \text{AD}(n-1)$ from their work with ASMs, where $\text{AD}(n)$ stands for the number of domino tilings of the Aztec diamond, i.e. perfect matchings of the Aztec graph (see [18, page 121, after equations (3) and (4)] and [14, page 88, after theorem 1.1]). We explain this in section 3.5.

The second and more obvious way is to use theorem 2.10 and show that $\sum_{A \text{ ASM of size } n} 2^{N_+(A)} = 2^{\binom{n+1}{2}}$. This follows directly from the so-called 2-enumeration of ASMs, where every ASM A is weighted with $2^{N_-(A)}$. (Remember that $N_+(A) = N_-(A) + n$ to derive the formula we need.) This 2-enumeration was already calculated by Mills, Robbins, Rumsey when introducing ASMs [45, corollary in section 6]:

Theorem 2.12 (Mills, Robbins, Rumsey). *We have:*

$$\sum_{A \text{ ASM of size } n} 2^{N_-(A)} = 2^{\binom{n}{2}}$$

However, it is probably easier to go in the other direction: Derive the number of perfect matchings of the Aztec graph in another way (from the recurrence relation or by completely different methods) and from this obtain theorem 2.12 as a corollary.

Remark. The arrows introduced in procedure 2 are closely related to the so-called six-vertex models/square ice models used for instance in [39, section 1]. These are orientations of the square grid graph in which each vertex has indegree 2 and outdegree 2. We can obtain them by doing the same as in procedure 2, but reversing all vertical arrows. This gives a bijection between ASMs and six-vertex models, which is used quite often when studying ASMs.

2.3 The AST-graph

In [4, section 5.7], Ayer, Behrend, and Fischer introduced a graph whose matchings are related to ASTs in the same way that perfect matchings of the Aztec graph are related to ASMs, meaning that they are connected by the exactly same procedures 1 and 2. This graph, which we call *AST-graph*, was already shown in figure 1. We denote the set of vertices marked in purple by Q .

In contrast to ASMs, ASTs have columns with sum 0, i.e. in which the last nonzero entry (if it exists) is a -1 . For the matching, this means that the bottom vertex of such a column, which is in Q , is not covered by the matching, see figure 5. (In columns only consisting of 0's, the arrows which are introduced in procedure 2 have to be chosen such that they point downwards.) For a detailed explanation of these facts consider sections 3.2 and 3.3, where this is done thoroughly and in a general setting. Thus, the matchings coming from an AST of size n only cover $n - 1$ of the $2(n - 1)$ vertices from Q . We call such matchings ' Q -partial':

Definition 2.13. Let G be a graph and let Q a subset of the vertices of G . A matching of G is called *Q -partial* if it covers all vertices not in Q (and it may or may not cover vertices from Q).

Note that the AST-graph is bipartite and all vertices in Q lie in the same bipartition class. Thus every Q -partial matching of the AST-graph of size n has to cover exactly $n - 1$ (i.e. half) of the vertices from Q and therefore indeed comes from an AST of size n . As for ASMs, every AST A gives $2^{N_+(A)}$ Q -partial matchings of the AST-graph (compare lemma 2.11 and its proof) and so the analogue of theorem 2.10 holds:

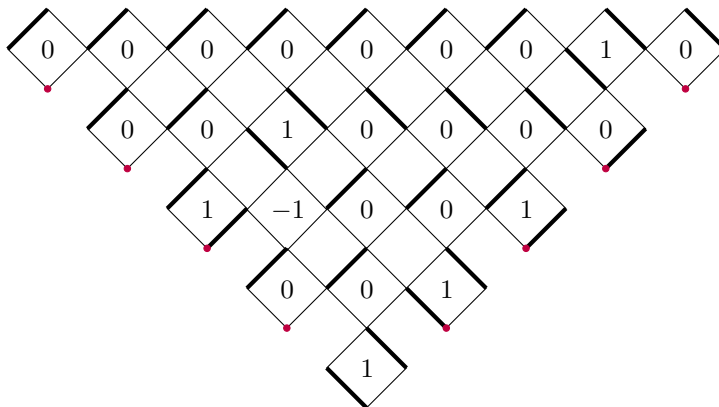


Figure 5: A Q -partial matching of the AST-graph of size 5 and the associated AST

Theorem 2.14 (Ayyer, Behrend, Fischer). *The number of Q -partial matchings of the AST-graph of size n is*

$$\sum_{A \text{ AST of size } n} 2^{N_+(A)}$$

As ASMs and ASTs share their number and the distribution of N_+ (theorem 2.6), we can conclude that the number of perfect matchings of the Aztec graph (theorem 2.9) equals the number of Q -partial matchings of the AST-graph and so we have [4, corollary 5.7]:

Corollary 2.15 (Ayyer, Behrend, Fischer). *There are $2^{\binom{n+1}{2}}$ Q -partial matchings of the AST-graph of size n .*

Note that the equal distribution of N_+ and N_0 on ASMs and ASTs implies that Q -partial matchings of the AST-graph have $n(n+1)$ edges, as is the case for perfect matchings of the Aztec graph. Indeed, one can see that the AST-graph of size n has $2n(n+1) + n - 1$ vertices, of which exactly $n - 1$ are not matched.

Apart from the number, there is a second similarity between perfect matchings of the Aztec graph and Q -partial matchings of the AST-graph that we can deduce from the statistics that ASMs and ASTs share, namely the direction in which the edges are tilted:

Definition 2.16. Let M be either a perfect matching of the Aztec graph or a Q -partial matching of the AST-graph. Let $r(M)$ and $l(M)$ denote half the number of edges in M tilted to the right (i.e. in the southwest-northeast direction) and to the left (i.e. in the southeast-northwest direction) respectively. We denote the generating functions as follows:

$$\begin{aligned} GF_{\text{Aztec}}^{(n)}(x, y) &:= \sum_{\substack{M \text{ perfect matching of} \\ \text{the Aztec graph of size } n}} x^{r(M)} y^{l(M)} \\ GF_{\text{AST}}^{(n)}(x, y) &:= \sum_{\substack{M \text{ } Q\text{-partial matching of} \\ \text{the AST-graph of size } n}} x^{r(M)} y^{l(M)} \end{aligned}$$

Remark. Note that in both cases, the matchings have exactly $n(n+1)$ edges, implying that $l(M) = \binom{n+1}{2} - r(M)$, so $GF_{\text{Aztec}}^{(n)}(x, 1)$ and $GF_{\text{AST}}^{(n)}(x, 1)$ already contain all information.

When introducing the Aztec diamond, Elkies et al. determined this generating function (including even a further statistic) by a method called ‘domino shuffling’, which yields the recurrence relation $GF_{\text{Aztec}}^{(n)}(x, 1) = (1+x)^n \cdot GF_{\text{Aztec}}^{(n-1)}(x, 1)$ and thus: (See [18, theorem in section 1] and [19, section 6, in particular page 227])

Theorem 2.17 (Elkies, Kuperberg, Larsen, Propp). *The generating function of perfect matchings of the Aztec graph is given by*

$$GF_{\text{Aztec}}^{(n)}(x, 1) = (1+x)^{\binom{n+1}{2}}$$

However, in section 2.4 we prove that one can also express the generating function in terms of N_+ and inv :

Proposition 2.18. *The generating functions of the matchings can be written as*

$$\begin{aligned} GF_{\text{Aztec}}^{(n)}(x, 1) &= \sum_{A \text{ ASM of size } n} (1+x)^{N_+(A)} x^{\text{inv}(A)} \\ GF_{\text{AST}}^{(n)}(x, 1) &= \sum_{A \text{ AST of size } n} (1+x)^{N_+(A)} x^{\text{inv}(A)} \end{aligned}$$

By the equal joint distribution of N_+ and inv on ASMs and ASTs (theorem 2.6), this implies that $GF_{\text{AST}}^{(n)}(x, 1) = GF_{\text{Aztec}}^{(n)}(x, 1)$ and so by theorem 2.17 and the fact that $l(M) = \binom{n+1}{2} - r(M)$, we can deduce:

Theorem 2.19. *The generating function of the matchings related to ASMs resp. ASTs is*

$$GF_{\text{AST}}^{(n)}(x, y) = GF_{\text{Aztec}}^{(n)}(x, y) = (x + y)^{\binom{n+1}{2}}$$

In addition, the combination of theorem 2.17 and proposition 2.18 also gives the following interesting fact:

Corollary 2.20. *We have the following identity, which also holds when summing over ASTs instead of ASMs:*

$$\sum_{A \text{ ASM of size } n} (1 + x)^{N_+(A)} x^{\text{inv}(A)} = (1 + x)^{\binom{n+1}{2}}$$

We have now used N_+ , N_- , N_0 and inv to find a quite remarkable similarity between perfect matchings of the Aztec graph and Q -partial matchings of the AST-graph: They have the same number and the same distribution of edges tilted in the two directions. So one would think that there might be a bijection that just translates the edges without rotating them, but there has nothing been found yet.

In section 2.1, we have seen yet another statistic that ASMs and ASTs share: The position of the unique 1 in the top row of an ASM and the parameter ρ for an AST (theorem 2.8). One can easily translate this to the matchings, but the result does not seem to be very interesting.

2.4 Generating function of the matchings

This section is dedicated to proving proposition 2.18. For this, we go back to the local arrow configurations and associated cell matchings introduced in procedure 2, and enumerate them separately:

Definition 2.21. For an ASM or AST A , let $N_0^{br}(A)$, $N_0^{tl}(A)$, $N_0^{bl}(A)$ and $N_0^{tr}(A)$ denote the number of entries 0 in A that, when applying procedure 2, receive the local arrow configuration labeled br , tl , bl and tr respectively in figure 4e.

Remark. ‘ br ’ stands for the bottom-right edge of the cell being in the matching, ‘ tl ’ for the top-left edge, etc.

For ASMs, one can determine these numbers quite easily in terms of inv , which was done by Ayer et al. in [4, section 5.4]. (Note that they used the six-vertex model which we mentioned in the remark at the end of section 2.2, so their vertical arrows are reversed in comparison to ours.) Interestingly, it turns out that $N_0^{br}(A) = N_0^{tl}(A)$ and $N_0^{bl}(A) = N_0^{tr}(A)$:

Lemma 2.22. *For an ASM A of size n , we have:*

$$\begin{aligned} N_0^{br}(A) &= \text{inv}(A) & N_0^{bl}(A) &= \binom{n+1}{2} - N_+(A) - \text{inv}(A) \\ N_0^{tl}(A) &= \text{inv}(A) & N_0^{tr}(A) &= \binom{n+1}{2} - N_+(A) - \text{inv}(A) \end{aligned}$$

Proof. For the proof of the equations on the left, we follow [4, section 5.4]. Since we are only working with ASMs now, let all the indices in sums range a priori from 1 to n , so we do not need to specify ‘ $\exists a_{ij}$ ’ etc.

To determine $N_0^{br}(A)$, note that considering the arrow configurations (procedure 2), a br -zero is characterized by the fact that both above it and left to it, the sum of entries is 1 (i.e. the next nonzero entry above and to the left is 1). When including the entry itself in exactly one of this sums, we ensure that it also is a 0 (and not a -1). Since all such sum are either 0 or 1, we have:

$$N_0^{br}(A) = \sum_{i,j} \left(\sum_{i' < i} a_{i'j} \right) \left(\sum_{j' \leq j} a_{ij'} \right) = \sum_{i' < i, j' \leq j} a_{i'j} a_{ij'} = \text{inv}(A)$$

For $N_0^{tl}(A)$, we can do the same thing by considering the sum of entries below and to the right this time:

$$N_0^{tl}(A) = \sum_{i,j} \left(\sum_{i < i'} a_{i'j} \right) \left(\sum_{j \leq j'} a_{ij'} \right) = \sum_{i < i', j \leq j'} a_{i'j} a_{ij'} = \text{inv}(A)$$

One could find $N_0^{bl}(A)$ and $N_0^{tr}(A)$ by similar considerations, but we use symmetry here: As we have now shown that $N_0^{br}(A) = N_0^{tl}(A)$ holds for every ASM A , also $N_0^{bl}(A) = N_0^{tr}(A)$ has to hold. We know that every perfect matching of the Aztec graph of size n has $n(n+1)$ edges. On the other hand, we can count the number of edges in the matching cell-wise, so we get:

$$\begin{aligned} n(n+1) &= 2N_+(A) + N_0^{br}(A) + N_0^{tl}(A) + N_0^{bl}(A) + N_0^{tr}(A) = \\ &= 2N_+(A) + 2 \text{inv}(A) + 2N_0^{bl}(A) \end{aligned}$$

and thus:

$$N_0^{bl}(A) = \binom{n+1}{2} - N_+(A) - \text{inv}(A) = N_0^{tr}(A)$$

□

For ASTs, these numbers are given in a more complicated way. However, as for ASMs, we have $N_0^{br}(A) + N_0^{tl}(A) = 2 \text{inv}(A)$ and $N_0^{bl}(A) + N_0^{tr}(A) = 2\left(\binom{n+1}{2} - N_+(A) - \text{inv}(A)\right)$, which was already mentioned in [4, section 5.4]. More precisely, the numbers are as follows:

Lemma 2.23. *For an AST A of size n , we have the following:*

$$\begin{aligned} N_0^{br}(A) &= \text{inv}(A) - \theta_r(A) & N_0^{bl}(A) &= \binom{n+1}{2} - N_+(A) - \text{inv}(A) - \theta_l(A) \\ N_0^{tl}(A) &= \text{inv}(A) + \theta_r(A) & N_0^{tr}(A) &= \binom{n+1}{2} - N_+(A) - \text{inv}(A) + \theta_l(A) \end{aligned}$$

where we define

$$\theta_r(A) := \sum_{\substack{j > n \\ \text{column } j \text{ has sum } 1}} (j - n) \quad \theta_l(A) := \sum_{\substack{j < n \\ \text{column } j \text{ has sum } 1}} (n - j)$$

Remark. Note that this implies that we always have $N_0^{br}(A) \leq N_0^{tl}(A)$ and $N_0^{bl}(A) \leq N_0^{tr}(A)$. Since an ASM has exactly $n-1$ columns with sum 1 (excluding the central column n), we could even give bounds for the total difference $(N_0^{tl}(A) + N_0^{tr}(A)) - (N_0^{br}(A) + N_0^{bl}(A)) = 2(\theta_r(A) + \theta_l(A))$.

Before we prove lemma 2.23, we note that together with lemma 2.22, it directly gives proposition 2.18:

Proof of proposition 2.18. Let A be an ASM or an AST. Consider the matchings associated to A by procedure 2. They have an edge tilted to the right for every br -zero and every tl -zero in A . By lemma 2.22 resp. 2.23, there are $N_0^{br}(A) + N_0^{tl}(A) = 2 \operatorname{inv}(A)$ of them. Additionally, each such perfect matching has either 0 or 2 edges tilted to the right for each entry 1 in A , depending on which of the two matchings is chosen in that cell. Thus, when restricting the generating function to the perfect matchings coming from A , the result is $(1+x)^{N_+(A)} x^{\operatorname{inv}(A)}$. (Remember that $r(M)$ denotes *half* the number of edges tilted to the right.) Summing over all ASMs or ASTs gives the result. $\square$

Thus, the only thing left is to prove lemma 2.23:

Proof of lemma 2.23. Since we are working in a triangular array now, we have to be very careful with the index ranges in the sums. A priori, let all the indices in this proof range in the rectangle in which the AST is contained, i.e. row indices (i and i') range from 1 to n and column indices (j and j') range from 1 to $2n-1$. It might be helpful to consider figure 2 while reading the proof.

For $N_0^{br}(A)$, we first do the same as in the proof of lemma 2.22, but we do not get just $\operatorname{inv}(A)$ because of the index issues:

$$\begin{aligned} N_0^{br}(A) &= \sum_{\substack{i,j \\ \exists a_{ij}}} \left(\sum_{\substack{i' < i \\ \exists a_{i'j}}} a_{i'j} \right) \left(\sum_{\substack{j' \leq j \\ \exists a_{ij'}}} a_{ij'} \right) = \\ &= \sum_{i,j} \left(\sum_{\substack{i' < i \\ \exists a_{i'j}}} a_{i'j} \right) \left(\sum_{\substack{j' \leq j \\ \exists a_{ij'}}} a_{ij'} \right) - \sum_{\substack{i,j \\ \nexists a_{ij}}} \left(\sum_{\substack{i' < i \\ \exists a_{i'j}}} a_{i'j} \right) \left(\sum_{\substack{j' \leq j \\ \exists a_{ij'}}} a_{ij'} \right) = \\ &= \operatorname{inv}(A) - \sum_{\substack{i,j \\ \nexists a_{ij}}} \left(\sum_{\substack{i' < i \\ \exists a_{i'j}}} a_{i'j} \right) \left(\sum_{\substack{j' \leq j \\ \exists a_{ij'}}} a_{ij'} \right) \end{aligned}$$

To deal with the part that is subtracted from $\operatorname{inv}(A)$, note that there are two kinds of positions (i, j) in the rectangle such that a_{ij} does not exist: Those left of the AST (characterized by $j < i$) and those right of the AST (characterized by $i + j > 2n$). For the left ones, the sum in the second bracket evaluates to 0 as it has empty scope. For the right ones, the sum in the first bracket evaluates to the sum of column j and the sum in the second bracket evaluates to the sum of row i . Rows always have sum 1, while columns have sum 0 or 1, so we get

$$\sum_{\substack{i,j \\ \nexists a_{ij}}} \left(\sum_{\substack{i' < i \\ \exists a_{i'j}}} a_{i'j} \right) \left(\sum_{\substack{j' \leq j \\ \exists a_{ij'}}} a_{ij'} \right) = \sum_{\substack{i+j > 2n \\ \text{column } j \text{ has sum } 1}} 1 = \sum_{\substack{j > n \\ \text{column } j \text{ has sum } 1}} (j - n) = \theta_r(A)$$

Thus $N_0^{br}(A) = \operatorname{inv}(A) - \theta_r(A)$ is proved. For $N_0^{tl}(A)$, consider the number of local arrow configurations of A where the arrow right of the entry points to the right. These are the ones with a tl -zero, a bl -zero or a -1 (see figure 4e). On the other hand, such entries a_{ij} can be characterized by the fact that the sum of the entries to their right is 1 (i.e. the next nonzero entry to their right is 1), meaning that $\sum_{j' > j, \exists a_{ij'}} a_{ij'} = 1$ holds. Since in the triangular shape of the AST, there clearly

exist $j' - i$ entries to the left of $a_{ij'}$ (i.e. entries a_{ij} with $j < j'$), we have

$$N_0^{tl}(A) + N_0^{bl}(A) + N_-(A) = \sum_{\substack{i,j \\ \exists a_{ij}}} \sum_{\substack{j' > j \\ \exists a_{ij'}}} a_{ij'} = \sum_{\substack{i,j' \\ \exists a_{ij'}}} \sum_{\substack{j < j' \\ \exists a_{ij}}} a_{ij'} = \sum_{\substack{i,j' \\ \exists a_{ij'}}} a_{ij'} \cdot (j' - i)$$

As entries $a_{i'j}$ with $j \leq n$ have $j - i'$ entries below them in the triangular shape while entries $a_{i'j}$ with $j > n$ have $2n - j - i'$ entries below them, counting in the same way local arrow configurations where the arrow above the entry points upwards gives

$$N_0^{br}(A) + N_0^{bl}(A) + N_-(A) = \sum_{\substack{i,j \\ \exists a_{ij}}} \sum_{\substack{i' < i \\ \exists a_{i'j}}} a_{i'j} = \sum_{\substack{i',j \\ \exists a_{i'j}}} \sum_{\substack{i > i' \\ \exists a_{ij}}} a_{i'j} = \sum_{\substack{i',j \\ j \leq n \\ \exists a_{i'j}}} a_{i'j} \cdot (j - i') + \sum_{\substack{i',j \\ j > n \\ \exists a_{i'j}}} a_{i'j} \cdot (2n - j - i')$$

Changing the sum indices to i and j and then subtracting these two equations from each other, we get

$$N_0^{tl}(A) - N_0^{br}(A) = \sum_{\substack{i,j \\ j > n \\ \exists a_{ij}}} a_{ij} \cdot 2(j - n) = \sum_{j > n} 2(j - n) \sum_{\substack{i \\ \exists a_{ij}}} a_{ij} = \sum_{\substack{j > n \\ \text{column } j \text{ has sum } 1}} 2(j - n) = 2\theta_r(A)$$

and thus indeed:

$$N_0^{tl}(A) = N_0^{br}(A) + 2\theta_r(A) = \text{inv}(A) + \theta_r(A)$$

For $N_0^{bl}(A)$ and $N_0^{tr}(A)$, we can again use the vertical symmetry as for the ASM-case: From $N_0^{tl}(A) - N_0^{br}(A) = 2\theta_r(A)$ it follows that also $N_0^{tr}(A) - N_0^{bl}(A) = 2\theta_l(A)$ has to hold for every AST A . Since Q -partial matchings of the AST-graph of size n have $n(n+1)$ edges and we now know $N_0^{br}(A)$ and N_0^{tl} , we can also determine $N_0^{tr}(A) + N_0^{bl}(A)$:

$$N_0^{tr}(A) + N_0^{bl}(A) = n(n+1) - 2N_+(A) - N_0^{br}(A) - N_0^{tl}(A) = n(n+1) - 2N_+(A) - 2 \text{inv}(A)$$

which together with $N_0^{tr}(A) - N_0^{bl}(A) = 2\theta_l(A)$ gives the desired result. $\square$

3 Cellular graphs: Reducing the AST-graph

In 1996, Mihai Ciucu published the paper “Perfect Matchings of Cellular Graphs” [14]. In this paper, he introduced a new class of graphs, called ‘cellular graphs’, and proved a theorem relating the number of perfect matchings of a cellular graph to the number of perfect matchings of a certain subgraph, called the ‘core’ of the graph. He then used this theorem to derive the number of perfect matchings of the Aztec graph in a very elegant way.

We here state a theorem that generalizes Ciucu’s theorem in two ways: Firstly, we use a broader definition of a ‘cellular graph’: While Ciucu did mention that his theorem also applies to these graphs, we directly work in that more general setting. Secondly, we generalize his theorem to a statement about C -partial matchings instead of only perfect matchings. Both of these generalizations are quite straightforward, the main proof idea still comes directly from Ciucu’s proof. However, the formulation changes massively to fit the new setting.

In section 3.1, we state our general theorem. We prove it in sections 3.2 and 3.3, where we introduce many new objects: Section 3.2 deals with the ‘cell graph’, which indicates the structure

of a cellular graph, and ‘alternating sign patterns (ASPs)’ on it, which are essentially the analogue of ASMs for the Aztec graph or ASTs for the AST-graph. Section 3.3 then induces orientations of the cell graph, which are necessary to establish the relation between ASPs and C -partial matchings.

After proving our theorem, in section 3.4 we briefly explain Ciucu’s original theorem and proof. In section 3.5, we then show the application to the Aztec graph due to Ciucu, and explain how the procedures in section 2.2 (linking perfect matchings of the Aztec graph to ASMs) are generalized by our treatment of cellular graphs. Finally, section 3.6 presents the application to the AST-graph. This needs our generalized theorem and only gives a small reduction of the problem (resulting in the ‘half-reduced’ and the ‘reduced’ AST-graph), not a whole enumeration as it is the case for the Aztec graph.

Note that most of the content of section 3 has been presented by the author of this paper in the seminar “AG Discrete Mathematics” at Vienna University of Technology (joint seminar with University of Vienna) on June 18, 2024.

3.1 C -partial matchings of cellular graphs

We define cellular graphs as follows:

Definition 3.1. An undirected graph G is called *cellular* if:

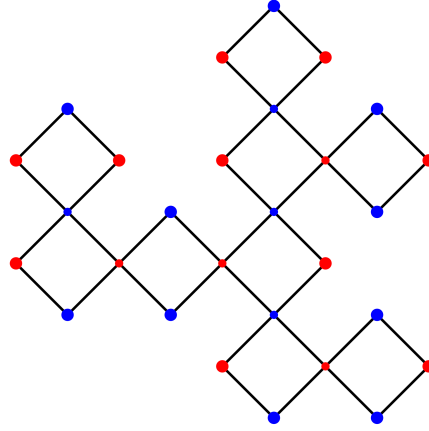
- G is bipartite,
- the edges of G can be partitioned into 4-cycles,
- every vertex of G has degree 2 or 4.

When working with a cellular graph G , we always use a fixed bipartition of the vertices of G and consider the vertices as being colored red and blue accordingly. Also, we use a fixed partition of the edges of G into 4-cycles:

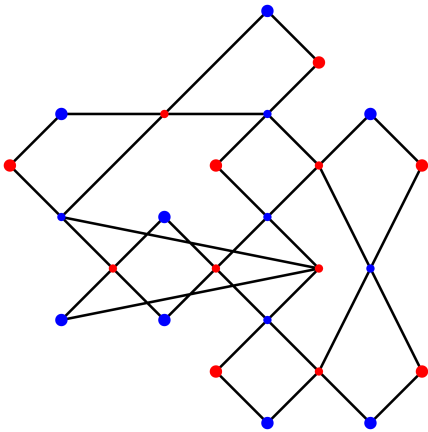
Definition 3.2. Let G be a cellular graph. Then the 4-cycles into which the edges of G are partitioned are called *cells* of G . Vertices with degree 4 are called *inner vertices* of G and vertices with degree 2 are called *corner vertices* or *corners* of G . The set of corner vertices of G is denoted C . (Since we always work with only one cellular graph, it is clear what C refers to.)

Remark. Everything we do in this section would still hold true when replacing ‘4-cycle’ by ‘4-circuit’ (i.e. cycles in which vertices may repeat) in definition 3.1. This would mean that a cell can for example contain only one red vertex which is connected by a double edge to each of the two blue vertices. Although the proofs can be intuitively adapted to that case, certain formulations would get quite tricky, so we stick with 4-cycles. Note however that multiple edges in general are allowed in the definition, but they have to lie in different cells.

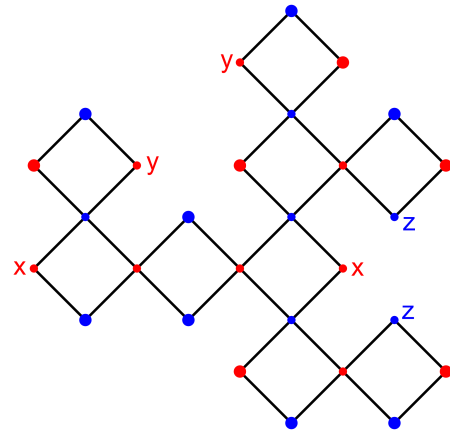
Example 3.3. Figure 6a shows a quite simple cellular graph. The cells are clearly visible lozenges and the corner vertices are displayed by slightly thicker points than the inner vertices. Figure 6b shows a more complicated cellular graph, of which the cell structure is not immediately obvious. However, this graph is just obtained from the first graph by identifying some vertices (which by this procedure turn from corner vertices to inner vertices), as it is shown in figure 6c. There we can read off the cells, which are actually the same for both graphs.



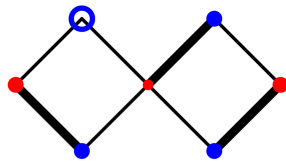
(a) A simple cellular graph



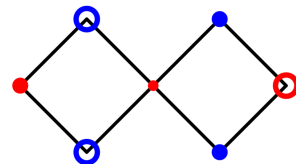
(b) A more complicated cellular graph



(c) The same cellular graph as in figure 6b
(Vertices with the same label are identified.)



(d) A small cellular graph G with a C -partial matching which is a perfect matching of $G \setminus X$, where X contains exactly one corner



(e) A choice of $X \subseteq C$ that makes no perfect matching of $G \setminus X$ possible

Figure 6: Examples of cellular graphs and C -partial matchings

We are interested in C -partial matchings of cellular graphs, i.e. perfect matchings of $G \setminus X$ for some $X \subseteq C$:

Definition 3.4. Let G be a cellular graph and let $X \subseteq C$. Then the number (or total weight in the weighted case) of perfect matchings of $G \setminus X$ is denoted by $m(G \setminus X)$.

Example 3.5. Figures 6d and 6e show a small cellular graph with a C -partial matching resp. a choice of $X \subseteq C$ allowing no perfect matching of $G \setminus X$. Note that in the figures, we mark the corners lying in X by empty dots.

Our main theorem relates the number of C -partial matchings leaving out a certain set of corners of G to the number of C -partial matchings leaving out the complementary set of corners:

Theorem 3.6. Let G be a cellular graph and let $X \subseteq C$ be a subset of its corner vertices. Then:

$$m(G \setminus X) = m(G \setminus (C \setminus X)) \cdot 2^{|C|/4 - |X|/2}$$

Theorem 3.6 is proved in section 3.2. The following two corollaries - trivially equivalent to it - formulate the theorem in a more symmetric and elegant way:

Corollary 3.7. Let G be a cellular graph and let $X \cup Y = C$ with $X \cap Y = \emptyset$ be a partition of its corner vertices. Then:

$$m(G \setminus X) \cdot 2^{|X|/4} = m(G \setminus Y) \cdot 2^{|Y|/4}$$

Corollary 3.8. Let G be a cellular graph and let $X \cup Y = C$ with $X \cap Y = \emptyset$ be a partition of its corner vertices. Assign to each edge the weight $1/\sqrt{2}$. Then:

$$m(G \setminus X) = m(G \setminus Y)$$

Note that for most choices of $X \subseteq C$ or $X \cup Y = C$, the equations in theorem 3.6 and its corollaries just read $0 = 0$. For C -partial matchings to be possible, we have to choose X in such a way that $G \setminus X$ has an equal number of red and blue vertices:

Lemma 3.9. Let G be a cellular graph. Let R and B denote the number of red and blue vertices of G respectively and let r and b denote the number of red and blue corner vertices of G respectively. Then r and b are even and

$$R - B = \frac{r - b}{2}$$

Consequently, if X contains exactly half of the red and half of the blue corner vertices of G , then $G \setminus X$ has an equal number of red and blue vertices.

Remark. Clearly, there are also other choices of X that give an equal number of red and blue vertices in $G \setminus X$. Moreover, this condition alone does not yet guarantee that $G \setminus X$ actually has a perfect matching, as figure 6e shows.

Proof. Let m be the number of cells of G . We have $2m = 2R - r$ by counting the number of pairs (c, v) where c is a cell of G and v is a red vertex in that cell: On the one hand, every cell has exactly two red vertices, so this number is $2m$. On the other hand, each red vertex is in two cells except for corner vertices, which are only in one cell, so this number is also equal to $2R - r$.

Analogously, we have $2m = 2B - b$. So both r and b are even and $2R - r = 2B - b$, which gives the equation in the lemma. Finally, we can write the equation as $R - \frac{r}{2} = B - \frac{b}{2}$, which proves the last part of the lemma. $\square$

A priori, it is not clear whether each cellular graph does actually have C -partial matchings. However, from our proof of theorem 3.6 in the next subsections, we easily obtain the following proposition (which is proved at the end of section 3.2):

Proposition 3.10. *Every cellular graph G has at least $2^{|C|/2}$ C -partial matchings.*

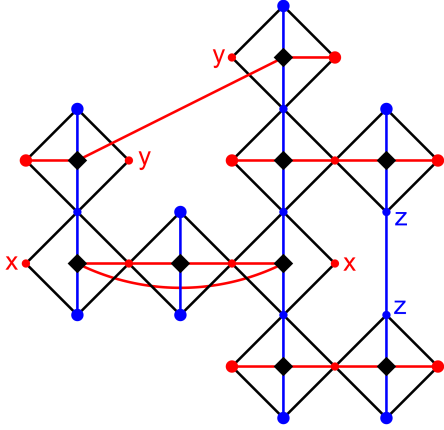
Remark. These $2^{|C|/2}$ C -partial matchings will each cover exactly half of the red and half of the blue corners of G and thus yield an alternative proof for lemma 3.9.

3.2 The cell graph and alternating sign patterns

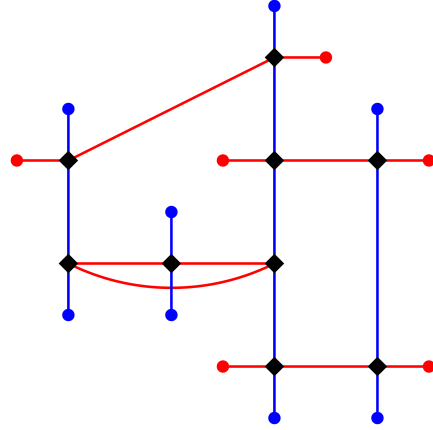
In this section, G is always a cellular graph. For illustrations, we use the graph displayed in figure 6c. We start by introducing a graph that indicates the cell structure of G , i.e. the arrangement of the cells of G :

Definition 3.11 (Figure 7). The *cell graph* $G^\diamond$ is given as follows:

- Every cell of G and every corner of G is a vertex of $G^\diamond$:
 - We still call them cells and corners or corner vertices even when referring to them as vertices of $G^\diamond$. Sometimes we say *cell vertex* to make clear that we are talking about the cell as a vertex of $G^\diamond$.
- Every vertex of G corresponds to an edge of $G^\diamond$:
 - Inner vertices of G correspond to an edge connecting the two cells in which they lie.
 - Corner vertices of G correspond to an edge connecting that corner vertex to the cell in which they lie.
- The corner vertices of G keep their color also as vertices of $G^\diamond$. The edges of $G^\diamond$ inherit the color of the corresponding vertex of G . The cell vertices of $G^\diamond$ do not have any color.



(a) G together with its cell graph $G^\diamond$



(b) The cell graph $G^\diamond$ alone

Figure 7: The cell graph $G^\diamond$ for the graph in figure 6c

Remark. We immediately realize the following:

- $G^\diamond$ is well-defined: Due to their fixed degrees, inner vertices of G lie in exactly two cells and corner vertices of G lie in exactly one cell, as is necessary for the definition of $G^\diamond$.

- Corners of G correspond both to a vertex and to an edge of $G^\diamond$, which together form a leaf.
- G is uniquely determined by $G^\diamond$: To get back G from $G^\diamond$, turn each edge of $G^\diamond$ into a vertex and connect two vertices by an edge if their corresponding edges in $G^\diamond$ have different colors and share an endpoint (which is a cell in which both vertices and the edge connecting them lie). If the corresponding edges share both endpoints, connect the vertices by a double edge.

In order to describe the structure of the cell graph $G^\diamond$, we make the following definition:

Definition 3.12. A *red cycle* of $G^\diamond$ is a cycle in $G^\diamond$ consisting only of red edges and cell vertices. A *red path* of $G^\diamond$ is a path in $G^\diamond$ consisting only of red edges, having red corner vertices as endpoints and only cell vertices in-between. Analogously, *blue cycles* and *blue paths* are defined. By *colored cycle*, we mean ‘red or blue cycle’, and by *colored path*, we mean ‘red or blue path’.

It turns out that the cell graph only consists of colored cycles and paths:

Lemma 3.13. *For $G^\diamond$, we have the following:*

- Every cell vertex of $G^\diamond$ lies on exactly one red and on exactly one blue cycle or path.
- Every corner of $G^\diamond$ lies on exactly one colored path, which has the color of that corner.
- Every edge of $G^\diamond$ is part of exactly one colored cycle or path, which has the color of that edge.

Proof. Since each cell of G contains exactly two red vertices, each cell vertex of $G^\diamond$ is incident to exactly two red edges. Similarly, each red corner vertex of $G^\diamond$ is incident to exactly one red edge and blue corner vertices of $G^\diamond$ are incident to no red edges. Thus, the red edges of $G^\diamond$ fall into cycles and paths, of which the paths have red corners as endpoints all other vertices on these cycles and paths are cell vertices. Consequently, they are indeed red cycles and paths as introduced in definition 3.12, and each cell vertex and each red corner vertex lies on exactly one of them. As the same holds for blue vertices and edges, the lemma is proved. $\square$

With lemma 3.13, we have a quite precise description of the structure of $G^\diamond$. Let us now continue with several definitions:

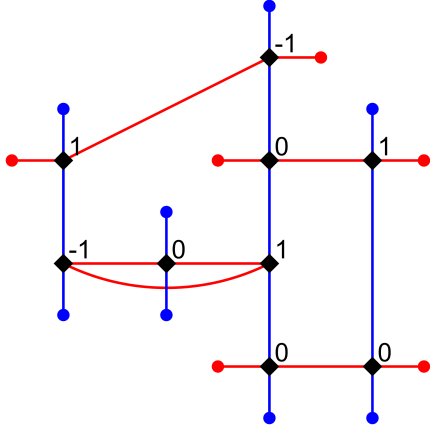
Definition 3.14. We make the following definitions, which are illustrated in figure 8:

- (8a) An *alternating sign pattern (ASP)* P on $G^\diamond$ assigns to every cell vertex of $G^\diamond$ a label from $\{-1, 0, 1\}$ such that along every colored path or cycle, the nonzero labels alternate. For an ASP P , the *inverse ASP* P' is obtained from P by changing the sign of every label in P . Furthermore, we define the following numbers:
 - P_1 resp. P_{-1} denotes the number of labels 1 resp. -1 in P .
 - P_c resp. P_p denotes the number of colored cycles resp. paths of $G^\diamond$ in which all cell vertices are labeled with 0.

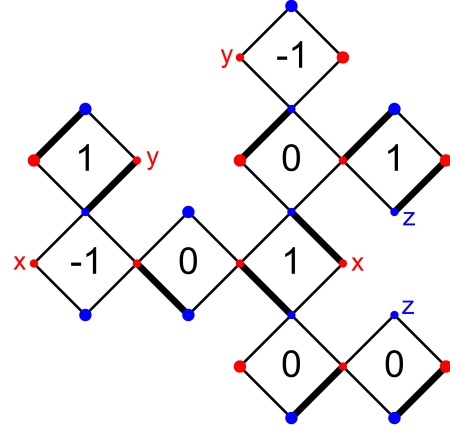
The cells of G are considered as being labeled by P in the same way as the corresponding cell vertices of $G^\diamond$. (When considering the labels, we do not strictly distinguish between these two.)

- (8b) Let P be an ASP on $G^\diamond$. A C -partial matching of G is of *type* P if it contains:
 - two edges of every cell labeled with 1,
 - one edge of every cell labeled with 0,
 - no edge of any cell labeled with -1 .

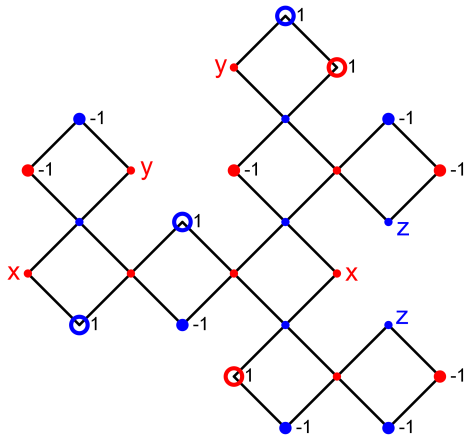
- (8c) For $X \subseteq C$, let p^X denote the following labeling of corners of $G^\diamond$ (or of G):
 - Corners that are in X are assigned the label 1,
 - Corners that are not in X are assigned the label -1 .
- (8d) An ASP P is *compatible* with $X \subseteq C$ if in $P \cup p^X$ (the labeling of all vertices of $G^\diamond$ obtained by putting together P and p^X), still the nonzero labels alternate along all colored cycles and paths.



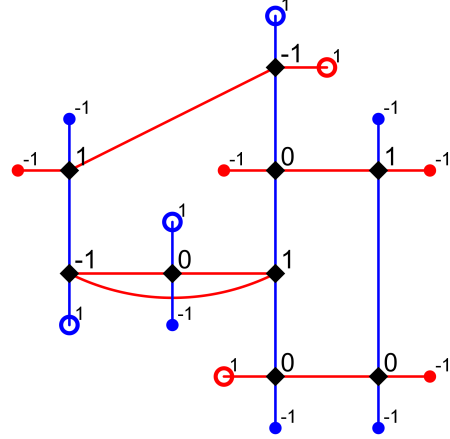
(a) An ASP P on $G^\diamond$ with $P_1 = 3$, $P_{-1} = 2$, $P_c = 0$, $P_p = 1$



(b) A C -partial matching of type P



(c) $X \subseteq C$ (marked by empty dots) and p^X



(d) P is compatible with X

Figure 8: Illustration of definition 3.14

We need the following facts about ASPs:

Lemma 3.15. *Let P be an ASP on $G^\diamond$. Then we have the following:*

- (i). *There are exactly 2^{P_p} subsets $X \subseteq C$ such that P is compatible with X .*
- (ii). *P is compatible with $X \subseteq C$ if and only if its inverse ASP P' is compatible with $C \setminus X$.*
- (iii). *If P is compatible with $X \subseteq C$, then $P_1 - P_{-1} = |C|/4 - |X|/2$.*

Proof. For item (i), note that in order for $P \cup p^X$ to have alternating nonzero labels, p^X (and thus X) is actually forced by P except for the endpoints of colored paths on which all cell vertices are labeled by 0. There, one can choose which of the two corners should be in X and which not, so we have in total 2^{P_p} choices for X .

Item (ii) is trivial as $p^{C \setminus X}$ can be obtained from p^X by changing the sign of every label in p^X .

For item (iii), consider a single colored path or cycle and denote by p_x the number of cell vertices on it with label x and by c_x the number of corners on it (i.e. endpoints of the path) with label x . Then the following equation holds:

$$p_1 - p_{-1} = \frac{c_{-1} - c_1}{2}$$

For cycles, this equation is obvious as 1's and -1's alternate in ASPs. For paths, it is also clear by considering the different possibilities for the labels of the endpoints of the path.

Now we sum this equation over all colored cycles and paths. As every cell vertex lies on exactly one red and one blue cycle or path, it is counted twice on the left side, so we get $2(P_1 - P_{-1})$ there. Every corner is only on one path and thus counted once. The number of corners with label -1 is $C \setminus X$ and the number of corners with label 1 is X , so the right side gives $\frac{|C \setminus X| - |X|}{2}$. Thus:

$$2(P_1 - P_{-1}) = \frac{|C \setminus X| - |X|}{2} = \frac{|C| - 2|X|}{2}$$

□

In section 3.3, we prove the following crucial proposition:

Proposition 3.16. *Let G be a cellular graph. Then we have:*

- (i). *Every C -partial matching of G is a perfect matching of $G \setminus X$ for a unique $X \subseteq C$, and of type P for a unique ASP P on $G^\diamond$. This unique P is compatible with the unique X .*
- (ii). *Let $X \subseteq C$ and let P be an ASP on $G^\diamond$ compatible with X . Then there are $2^{P_c + P_1}$ perfect matchings of $G \setminus X$ of type P .*

Summing over all subsets $X \subseteq C$ resp. all ASPs P on $G^\diamond$ gives the following theorem, allowing us to count C -partial matchings:

Theorem 3.17. *Let G be a cellular graph. Then the following holds:*

- (i). *For any $X \subseteq C$, we have*

$$m(G \setminus X) = \sum_{P \text{ ASP compatible with } X} 2^{P_c + P_1}$$

- (ii). *For any ASP P on $G^\diamond$, G has $2^{P_p + P_c + P_1}$ C -partial matchings of type P .*

Proof. Item (i) is clear from proposition 3.16. For item (ii), note that by item (i) of lemma 3.15, we obtain the number of C -partial matchings of type P as

$$\sum_{\substack{X \subseteq C \\ P \text{ compatible with } X}} 2^{P_c + P_1} = 2^{P_p + P_c + P_1}$$

□

Now we can deduce theorem 3.6 and proposition 3.10:

Proof of theorem 3.6. We use item (i) of theorem 3.17 and lemma 3.15 (first item (iii), then item (ii)). Note that $P_c = P'_c$ and $P_{-1} = P'_1$ where we use item (ii) of lemma 3.15:

$$\begin{aligned}
m(G \setminus X) &\stackrel{(3.17)}{=} \sum_{P \text{ ASP comp. with } X} 2^{P_c + P_1} = \\
&= \sum_{P \text{ ASP comp. with } X} 2^{P_c + P_{-1}} \cdot 2^{P_1 - P_{-1}} = \\
&\stackrel{(3.15)}{=} \sum_{P \text{ ASP comp. with } X} 2^{P_c + P_{-1}} \cdot 2^{|C|/4 - |X|/2} = \\
&\stackrel{(3.15)}{=} 2^{|C|/4 - |X|/2} \cdot \sum_{P' \text{ ASP comp. with } C \setminus X} 2^{P'_c + P'_1} = \\
&\stackrel{(3.17)}{=} 2^{|C|/4 - |X|/2} \cdot m(G \setminus (C \setminus X))
\end{aligned}$$

□

Proof of proposition 3.10. Let P be the trivial ASP assigning 0 to every cell vertex of $G^\diamond$. As each corner of G is the endpoint of exactly one colored path, we have $P_p = |C|/2$ and so by item (ii) of theorem 3.17, G has at least $2^{P_p + P_c + P_1} = 2^{|C|/2 + P_c} \geq 2^{|C|/2}$ C -partial matchings. □

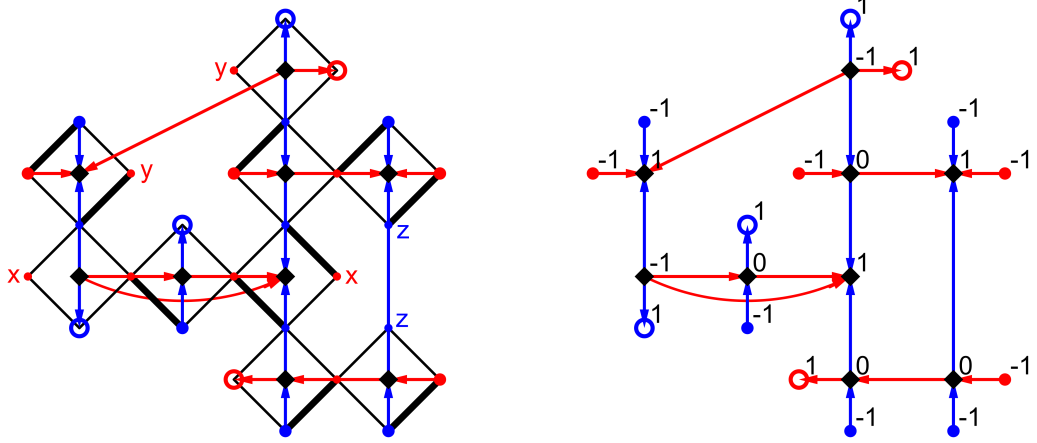
Remark. Note that the bound in proposition 3.10 is never attained (except for the empty graph): If $P_c = 0$ for the trivial ASP, this means that $G^\diamond$ has no colored cycles. Thus, in this case one definitely finds other ASPs (giving further C -partial matchings) by labeling a single arbitrary cell vertex with 1 or -1 and all other cell vertices with 0.

3.3 Orientations of the cell graph

This section is dedicated to proving proposition 3.16. For this, we introduce orientations on the cell graph $G^\diamond$. Note that these are a generalization of the arrows introduced in procedure 2 (section 2.2) for the Aztec graph. Also in this section, G is always a cellular graph.

Definition 3.18. We make the following definitions, illustrated in figure 9:

- (9a) Let M be a C -partial matching of G . Then M *induces* an orientation O_M of the cell graph $G^\diamond$ as follows: Every edge of $G^\diamond$ is directed by O towards the cell in which the corresponding vertex of G is matched by M and away from the cell in which the corresponding vertex of G is not matched by M .
- (9b) Let O be an orientation of $G^\diamond$. Then O *induces* a labeling L_O of the vertices of $G^\diamond$ as follows:
 - Vertices that have all incident edges directed towards them are assigned the label 1.
 - Vertices that have all incident edges directed away from them are assigned the label -1 .
 - Vertices that have two edges directed towards them and two away from them, namely one red and one blue one each, are assigned the label 0.
 - Other vertices are not assigned a label.



(a) A C -partial matching M and the induced orientation O_M of $G^\diamond$

(b) The orientation $O := O_M$ of $G^\diamond$ and the induced labeling L_O of the vertices of $G^\diamond$

Figure 9: Illustration of definition 3.18

Remark. Formulated more precisely, we obtain the orientation O_M as follows: An edge of $G^\diamond$ that corresponds to an inner vertex of G connects the two cells (cell vertices) in which that inner vertex lies. The vertex is matched by M in exactly one of these cells and we direct the edge in $G^\diamond$ towards that cell. An edge of $G^\diamond$ that corresponds to a corner of G connects that corner to the cell (vertex) in which it lies. If the corner is matched in that cell (i.e. if it is matched at all by M), we direct the edge in $G^\diamond$ towards the cell vertex, otherwise away from it.

If we consider the orientation O_M induced by a C -partial matching M and the labeling L_{O_M} induced by O_M , we see that there is a tight correspondence between M and L_{O_M} . This is the main step in proving proposition 3.16:

Lemma 3.19. *Let M be a C -partial matching of G . Then the labeling L_{O_M} of the vertices of $G^\diamond$ has the following form:*

- Cells of which two edges are in M have label 1.
- Cells of which one edge is in M have label 0.
- Cells of which no edge is in M have label -1 .
- Corners which are matched by M have label -1 .
- Corners which are not matched by M have label 1.

Consequently, for any ASP P on $G^\diamond$ and $X \subseteq C$, we have: M is a perfect matching of $G \setminus X$ of type P if and only if $L_{O_M} = P \cup p^X$.

Proof. For illustration, combine figures 9a and 9b. If a cell has two edges in M , this means that O_M directs all vertices towards it and thus it is assigned label 1 by L_{O_M} . Analogously, we see that cells with one resp. no edge in M are assigned label 0 resp. -1 by L_{O_M} . Corners which are matched by M have the incident edge directed away from them by O_M and thus are assigned label -1 by L_{O_M} . Analogously, corner which are not matched by M are assigned label 1 by L_{O_M} .

Comparing this with the definition of ‘type’ and of p^X (definition 3.14), we see that indeed, for any ASP P and $X \subseteq C$, M is a perfect matching of $G \setminus X$ of type P if and only if $L_{O_M} = P \cup p^X$. $\square$

Lemma 3.19 shows that we can actually directly determine L_{O_M} from M . However, the fact that L_{O_M} is induced by an orientation is important, as it shows that L_{O_M} is actually an ASP together with some p^X :

Lemma 3.20. *Let O be an orientation of $G^\diamond$ such that the induced labeling L_O gives a label to each vertex of $G^\diamond$. Then the nonzero labels in L_O alternate along all colored cycles of paths of $G^\diamond$. Consequently, L_O is of the form*

$$L_O = P \cup p^X$$

for a unique ASP P on $G^\diamond$ and a unique $X \subseteq C$. This unique P is compatible with the unique X .

Proof. For illustration, see figure 9b. Consider a colored cycle or path of $G^\diamond$. When going along it, every edge is either directed by O in the direction we are going, or in the other direction. By the definition of L_O , we have the following: In cell vertices with label 0, the orientation of the edges does not change. In cell vertices with label 1, the orientation changes from the direction we are going to the other direction. In cell vertices with label -1 , the orientation changes from the other direction to the direction we are going. In corner vertices, the orientation starts or ends accordingly. As each vertex has a label, these orientation changes imply that the nonzero labels in L_O indeed have to alternate.

Now let P be the labeling we obtain by restricting L_O to the cell vertices and let $X \subseteq C$ be the set of corners labeled with 1 by L_O . Then $L_O = P \cup p^X$. Since nonzero labels in L_O alternate along colored cycles and paths, indeed P is an ASP and it is compatible with p^X . Clearly, P and X are unique. $\square$

In order to find the number of C -partial matchings of $G \setminus X$ of type P , we now just have to count the number of C -partial matchings M such that $L_{O_M} = P \cup p^X$:

Lemma 3.21. *Let $X \subseteq C$ and let P be an ASP on $G^\diamond$ compatible with X . Then we have the following:*

- (i). *There are 2^{P_c} orientations O of $G^\diamond$ such that $L_O = P \cup p^X$.*
- (ii). *Let O be an orientation of $G^\diamond$ with $L_O = P \cup p^X$. Then there are 2^{P_1} C -partial matchings M of G such that $O_M = O$.*

In total, there are $2^{P_c + P_1}$ C -partial matchings M of G such that $L_{O_M} = P \cup p^X$.

Proof. For illustration, we can again use figures 9b and 9a.

To find an orientation O of $G^\diamond$ such that $L_O = P \cup p^X$, note that for vertices having label 1 or -1 assigned by $P \cup p^X$, the orientation of all incident edges is given by definition 3.18. And similar to what we did in the proof of lemma 3.20, when going along a colored path or cycle, the orientation of edges should not change in vertices with label 0 in order for them to have one incoming and one outgoing edge of each color. Thus, by starting at a vertex with nonzero label, the orientation of every edge of a colored path or cycle is given. Also note that we do not run into contradictions as the 1's and -1 's are alternating along every colored path or cycle. Only in a colored cycle that is only labeled with zeros, we can choose the orientation of a first edge and then have to orient the others accordingly. So there are indeed 2^{P_c} orientations O of $G^\diamond$ such that $L_O = P \cup p^X$.

To find a C -partial matching M of G such that $O_M = O$, note that by definition 3.18, each vertex of G has to be matched in the cell to which its corresponding edge of $G^\diamond$ is directed. Cells

with label -1 assigned by L_{O_M} have all edges directed away from them, thus no vertex is matched in them by M . Cells with label 0 have one red and one blue edge directed towards them, so the corresponding two vertices of G have to be matched by M in that cell. Thus the edge of the cell that connects these two vertices has to be in M . Cells with label 1 have all edges directed towards them, thus all four vertices of the cell should be matched by M in the cell. There are two options of how to choose two edges of the cell for M . As there are P_1 cells with label 1 , we indeed have 2^{P_1} C -partial matchings M of G such that $O_M = O$.

Combining items (i) and (ii) directly gives that there are $2^{P_c+P_1}$ C -partial matchings M of G such that $L_{O_M} = P \cup p^X$. $\square$

Now we can finally prove proposition 3.16:

Proof of proposition 3.16. Item (i) follows from lemmata 3.19 and 3.20: We see in lemma 3.19 that L_{O_M} gives a label to each vertex of G° . Thus we can apply lemma 3.20, stating that $L_{O_M} = P \cup p^X$ for a unique ASP P and a unique $X \subseteq C$, and that this P is compatible with this X . Considering again lemma 3.19, this implies that M is a perfect matching of $G \setminus X$ of type P for this unique P and X , which are compatible. Note that the uniqueness both of P and of X is also directly clear by considering the definitions of ‘type’ and p^X .

Item (ii) follows directly from the combination of lemma 3.19 with lemma 3.21. $\square$

3.4 The original theorem by Ciucu

The theorem that Ciucu published in 1996 [14, theorem 1.1] was simpler than our theorem 3.6 in two ways: Firstly, his definition of a cellular graph was more narrow. He only considered cellular graphs that can be neatly embedded in the plane such that the cells are all visible as upright diamonds of the same size. Thus, the graph in figure 6a is a cellular graph according to the definition of Ciucu, the graph in figure 6b is not. However, he mentioned that his theorem still holds when identifying corners with each other [14, remark on page 94 (end of section 2)], which essentially comes down to using our definition of a cellular graph, see figure 6c.

Secondly, Ciucu only considered the case $X = \emptyset$. So he compared the number of perfect matchings of the whole graph G to the number of perfect matchings of $G \setminus C$, which he called the ‘core’ of G . Formulated in our language, the theorem of Ciucu states the following (and we obtain it by setting $X := \emptyset$ in theorem 3.6):

Corollary 3.22 (Ciucu). *Let G be a cellular graph. Then $m(G) = m(G \setminus C) \cdot 2^{|C|/4}$.*

To prove his theorem, Ciucu used the same alternating sign patterns as we did, and so-called ‘hooks’ on cells with a 0 that directly correspond to our orientations of G° . However, due to his simpler definition of cellular graphs, he did not have any colored cycles (except when identifying corners in his remark) but only colored paths. And these colored paths were neatly arranged rows of cells (for one color) and columns of cells (for the other color). Therefore he just talked about horizontal and vertical chains of cells instead of introducing the cell graph G° or dealing with the colors from the bipartition. $|C|/4$ in his setting is just the number of horizontal chains and perfect matchings of G exist if there is an equal number of horizontal and vertical chains (compare lemma 3.9).

Also, Ciucu did not have to introduce labels for the corners and deal with their compatibility, since he fixed $X = \emptyset$, which for us would mean that p^X assigns -1 to every corner: He just said that the ASPs giving perfect matchings of G have to have 1 as the first and last nonzero label in every row and column. The inverse ASPs then have -1 as the first and last nonzero label in every row and column and thus give perfect matchings of $G \setminus C$, where in our language p^X assigns 1 to every corner.

3.5 Application to the Aztec graph

Clearly, the procedures in section 2.2, which relate perfect matchings of the Aztec graph to ASMs, are just a special case of what we have now done in sections 3.2 and 3.3: The Aztec graph is a cellular graph, an ASM is an ASP of a special shape, and the arrows between entries of the ASM (figure 4c) are actually an orientation of the cell graph. Similarly, the AST-graph is a cellular graph and its ASPs are ASTs. Thus, the small argumentative details left open in sections 2.2 and 2.3 should be clear now, in particular the fact from section 2.3 that in the AST case, the related Q -partial matchings cover all vertices except for the bottom vertices of those columns with sum 0 (i.e. with a -1 as the last nonzero entry, if nonzero entries exist).

Let us now apply theorem 3.6 to the Aztec graph: We are interested in perfect matchings, thus we take $X = \emptyset$. Alternatively, we could directly use corollary 3.22, as Ciucu did in [14, section 1]. When we delete all corner vertices of the Aztec graph of size n , we are left with the Aztec graph of size $n - 1$ (see figures 3a and 3b) and the factor we obtain is $2^{|C|/4} = 2^n$. So the number of perfect matchings of the Aztec graph of size n equals 2^n times the number of perfect matchings of the Aztec graph of size $n - 1$. This directly implies that the Aztec graph of size n has $2^{\binom{n+1}{2}}$ perfect matchings.

Another elegant way would be to use corollary 3.8 (with $X = C, Y = \emptyset$). So we give weight $1/\sqrt{2}$ to each edge and obtain that the total weight of perfect matchings is the same for every size of the Aztec graph. Analyzing the Aztec graph of size 1, we see that this total weight is 1. Since perfect matchings of the Aztec graph of size n have $n(n + 1)$ edges, each of them has weight $(\frac{1}{2})^{\binom{n+1}{2}}$ and thus there are $2^{\binom{n+1}{2}}$ of them.

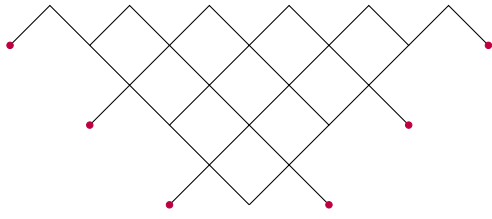
Remark. We could even take the Aztec graph of size 0 to find the total weight of perfect matchings: This is the empty graph, which has one perfect matching, namely the empty matching having weight 1.

3.6 Application to the AST-graph

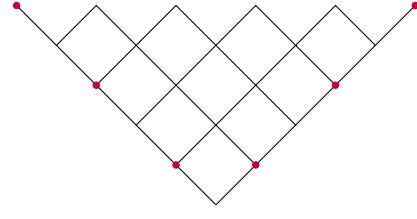
We now apply theorem 3.6 to the AST-graph. This does not solve the enumeration problem as it does for the Aztec graph, but it allows us to reduce the problem to two graphs which we call the *half-reduced* and the *reduced AST-graph*, shown in figures 10a and 10b. Note first that it does not really make difference whether we work with the half-reduced or the reduced AST-graph:

Lemma 3.23. *Q -partial matchings of the reduced AST-graph of size n stand in bijection to Q -partial matchings of the half-reduced AST-graph of size n .*

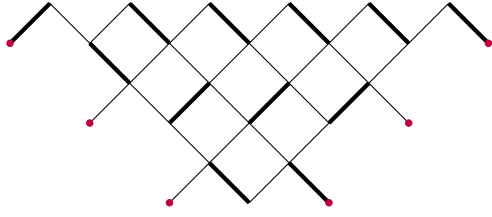
Proof. The bijection is quite easy to describe and illustrated in figures 10c and 10d: To go from the half-reduced to the reduced AST-graph, just restrict the matching to the reduced AST-graph. For the other direction, extend the matching by all edges that can be added. $\square$



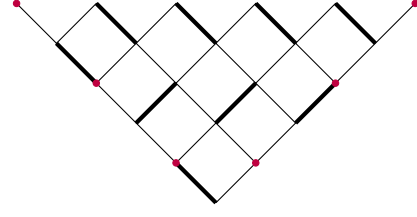
(a) The half-reduced AST-graph of size 3
(vertices in Q are marked in purple)



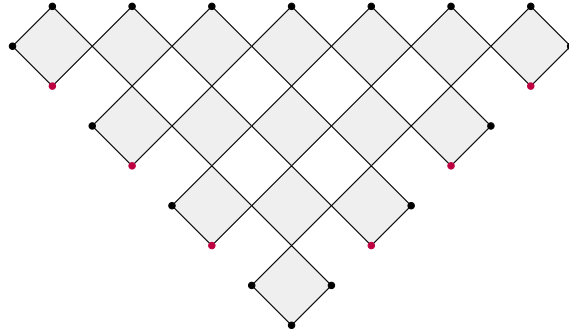
(b) The reduced AST-graph of size 3
(vertices in Q are marked in purple)



(c) A Q -partial matching of the half-reduced
AST-graph



(d) The corresponding Q -partial matching of
the reduced AST-graph



(e) The AST-graph of size 4 as a cellular graph

Figure 10: Reducing the AST-graph

Applying theorem 3.6 to Q -partial matchings of the AST-graph now gives the following:

Theorem 3.24. *For $n \geq 1$, the number of Q -partial matchings of the AST-graph of size n equals 2^n times the number of Q -partial matchings of the half-reduced or the reduced AST-graph of size $n - 1$.*

Proof. Let G be the AST-graph of size n and let G' be the half-reduced AST-graph of size $n - 1$. For this proof, denote the set Q corresponding to G' by Q' . Figure 10e shows that G is a cellular graph by marking the cells and the corners. Note that $Q \subseteq C$ with $|Q| = 2(n - 1)$ and $|C| = 6n - 2$. We see that we can obtain G' by deleting all corners of G that are not in Q , meaning that we have $G' = G \setminus (C \setminus Q)$. Additionally, $Q' = Q$ holds in this setting.

Since every Q -partial matching of G is a perfect matching of $G \setminus X$ for some $X \subseteq Q$ with $|X| = n - 1$ (due to the bipartition), the number of Q -partial matchings of G is the following, where

for (1), we use theorem 3.6, for (2), we rewrite the graph $G \setminus (C \setminus X)$ as $G' \setminus (Q' \setminus X)$, and for (3), we replace $Q' \setminus X$ by Y :

$$\begin{aligned}
\sum_{\substack{X \subseteq Q \\ |X|=n-1}} m(G \setminus X) &\stackrel{(1)}{=} \sum_{\substack{X \subseteq Q \\ |X|=n-1}} m(G \setminus (C \setminus X)) \cdot 2^{|C|/4 - |X|/2} \\
&\stackrel{(2)}{=} \sum_{\substack{X \subseteq Q' \\ |X|=n-1}} m(G' \setminus (Q' \setminus X)) \cdot 2^{\frac{6n-2}{4} - \frac{n-1}{2}} \\
&\stackrel{(3)}{=} \sum_{\substack{Y \subseteq Q' \\ |Y|=n-1}} m(G' \setminus Y) \cdot 2^n
\end{aligned}$$

As every Q -partial matching of G' is a perfect matching of $G' \setminus Y$ for some $Y \subseteq Q'$ with (again due to the bipartition) $|Y| = n - 1$, indeed the result of the above chain of equations is 2^n times the number of Q' -partial matchings of G' , which is the half-reduced AST-graph of size $n - 1$. By lemma 3.23, the theorem still holds when replacing the half-reduced AST-graph by the reduced AST-graph. $\square$

In the rest of the paper, we work towards determining the number of Q -partial matchings of the (half-)reduced AST-graph, since this is a little easier than dealing with the original AST-graph. While the reduced AST-graph is the simpler one, also the half-reduced AST-graph has its advantages, therefore we work with both of them (but mainly with the reduced).

Remark. When following the proofs of theorem 3.24 and lemma 3.23 closely, we see that we can actually obtain a stronger statement involving $m(G \setminus X)$ for a specific X . For this, let G, G', Q and Q' be as in the proof above, let G'' be the reduced AST-graph of size $n - 1$, and denote the set Q corresponding to G'' by Q'' . For $X \subseteq Q$, set $X' := X$. Note that Q'' can be obtained from Q' by shifting each vertex diagonally in along its incident edge (see figures 10a and 10b). Let $X'' \subseteq Q''$ be the set of vertices obtained from $X' \subseteq Q'$ by this procedure. Then, as can easily be seen from the proofs, we have the following:

$$m(G \setminus X) = 2^n \cdot m(G' \setminus (Q' \setminus X')) = 2^n \cdot m(G'' \setminus X'')$$

In an informal way, and by identifying Q'' with $Q' = Q$, we can express this as follows: When going from the AST-graph to the half-reduced AST-graph and then to the reduced AST-graph, we are taking complements on Q two times.

4 MaPaf-bijections: From matchings to path families

For graphs on the square lattice or the honeycomb lattice (i.e. graphs which are a subregion of that lattice), there are well-known bijections between perfect matchings of these graphs and certain families of disjoint paths. Since we work a lot with such bijections in this section, we introduce a name for them:

Definition 4.1. A *MaPaf-bijection* is a bijection between matchings of a graph and certain families of disjoint paths. By *disjoint paths*, we mean pairwise vertex-disjoint paths (including also the starting points and endpoints of the paths). Trivial paths of the form (v) for some vertex v are also considered.

In section 4.1, we lay the foundation for all our MaPaf-bijections: We construct a general MaPaf-bijection which does not require the graphs to lie on the square or the honeycomb lattice. Our only condition is that the graphs are bipartite and have a unique perfect matching when some vertices coming from a single bipartition class are removed. Using this MaPaf-bijection, we then derive MaPaf-bijections for different kinds of Q -partial matchings in the same general setting.

In sections 4.2 and 4.3, we come back to graphs on the square or honeycomb lattice. We show how to derive the known MaPaf-bijections for these graphs from our general construction. We also try to give a short overview about the origin of these bijections and their use in the study of matchings. In section 4.4, we then apply our results from section 4.1 to find MaPaf-bijections for Q -partial matchings of the (half-)reduced AST-graph. It turns out that there are many different such MaPaf-bijections, some of which we organize into five different types of configurations. Finally, section 4.5 gives simple but interesting corollaries for the resulting path families.

4.1 A general MaPaf-bijection

We now construct a MaPaf-bijection in a relatively general setting, only requiring a graph G to fulfill what we call the ‘MaPaf-condition’. The following definition gives the fundamental framework for this section, specifying the MaPaf-condition and constructing the directed graph $\widehat{G}$ which then contains the path families:

Definition 4.2. (See figure 11) The pair (G, A) fulfills the *MaPaf-condition* if G is a bipartite graph and A is a subset of the vertices of G , all belonging to the same bipartition class, such that $G \setminus A$ has a unique perfect matching. If (G, A) fulfills the MaPaf-condition, we fix the following notations:

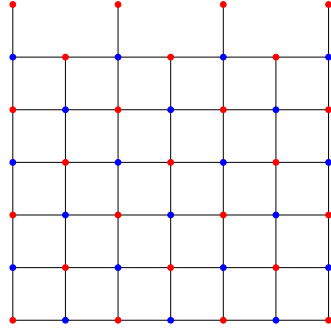
- G_r denotes the set of vertices which are in the same bipartition class of G as A , and G_b denotes the set of vertices in the other bipartition class. If G_r and G_b are not uniquely given (i.e. G has several connected components of which not all contain vertices from A), one has to fix a choice for them. To simplify talking about them, the vertices in G_r are considered as being colored red and the vertices in G_b blue.
- M_0 denotes the unique matching of $G \setminus A$.
- $n_A := |A|$

Furthermore, in this case we define the directed graph $\widehat{G}$ as follows:

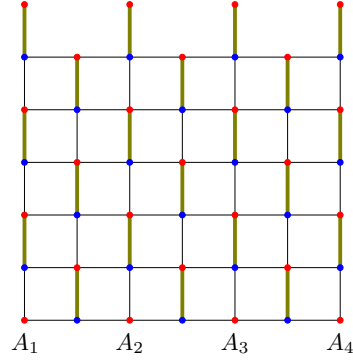
- The vertex set of $\widehat{G}$ is G_r .
- Every edge e of G that is not in M_0 gives a directed edge $\hat{e}$ of $\widehat{G}$ in the following way: Let m_e be the unique edge in the matching M_0 that is incident to the blue endpoint of e . Then (e, m_e) is a path in G starting and ending at a red vertex. The edge $\hat{e}$ in $\widehat{G}$ connects these two red vertices and is oriented in the direction in which the path goes.

Remark.

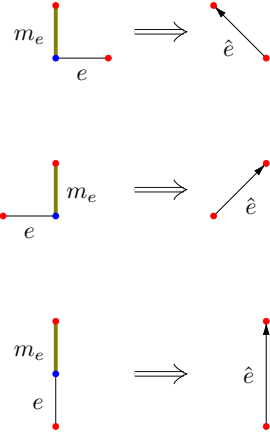
- Note that since $G \setminus A$ has a perfect matching, G contains n_A more red vertices than blue vertices. As all matchings we consider in the following cover all the blue vertices of G , they each have to leave out exactly n_A of the red vertices.
- $n_A = 0$ is allowed, but our MaPaf-bijections (theorem 4.3 and its corollaries) are trivial in that case.



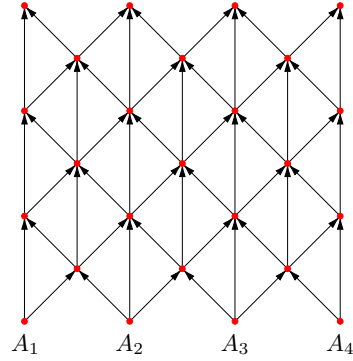
(a) A bipartite graph G



(b) A choice of $A = \{A_1, A_2, A_3, A_4\}$ such that (G, A) fulfills the MaPaf-condition, with the unique perfect matching M_0 marked



(c) From G to $\widehat{G}$



(d) The directed graph $\widehat{G}$

Figure 11: Example of a graph G fulfilling the MaPaf-condition together with a suitable A

The following theorem states our fundamental MaPaf-bijection. The explicit construction of the bijection is given in the proof.

Theorem 4.3 (See figures 12a and 12d). *Let (G, A) fulfill the MaPaf-condition. Then we have the following:*

- (i). $\widehat{G}$ is an acyclic directed graph in which the vertices from A are sources.
- (ii). Let $B \subseteq G_r$ with $|B| = n_A$. Then perfect matchings of $G \setminus B$ stand in bijection to families of n_A disjoint paths from A to B in $\widehat{G}$.

Proof. For item (i), suppose that $\widehat{G}$ has a cycle $\hat{c}$ consisting of edges $(\hat{e}_1, \dots, \hat{e}_k)$. Then $c := (e_1, m_{e_1}, \dots, e_k, m_{e_k})$ is a cycle in G . (Note that the blue vertices in c are all distinct since they are each connected to a different red vertex of c by an edge from the matching M_0 .) But then we could replace $\{m_{e_1}, \dots, m_{e_k}\}$ by $\{e_1, \dots, e_k\}$ in M_0 to obtain a different perfect matching, contradicting

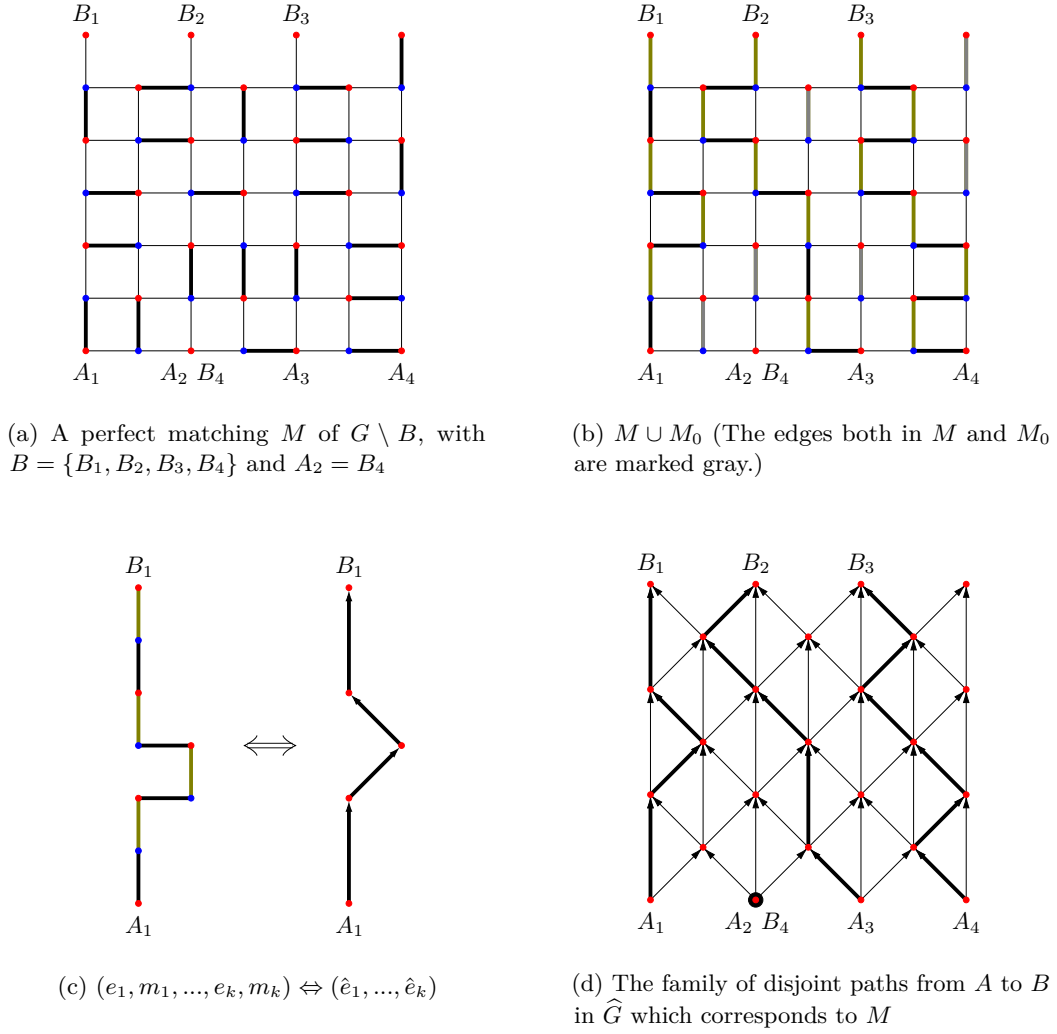


Figure 12: The MaPaf-bijection between perfect matchings of G and families of disjoint paths in $\hat{G}$

to uniqueness of M_0 . So $\hat{G}$ is acyclic. Furthermore, we see that the vertices from A cannot be endpoints of paths of the form (e, m_e) in G since they are not covered by M_0 , so they are indeed sources in $\hat{G}$.

For item (ii), note first that since $n_A = |A| = |B|$, the path families have to use each vertex from A as a starting point for a path and each vertex from B as an endpoint. However, we see that any vertex $v \in A \cap B$ is completely irrelevant both for the matchings and the path families: Neither M_0 nor any of the perfect matchings of $G \setminus B$ cover v . On the other hand, each family of n_A disjoint paths from A to B has to contain the trivial path (v) since this is the only path ending at v as it is a source. So it is enough to construct the bijection for $(G \setminus \{v\}, A \setminus \{v\})$ and it is automatically given for (G, A) as well by just adding (v) to the path families. Therefore, we can assume $A \cap B = \emptyset$ for the rest of this proof.

The bijection is given as follows (see figure 12): Let M be a perfect matching of $G \setminus B$. Consider the union $M \cup M_0$ as a multiset. Then vertices in $A \cup B$ are incident to exactly one edge from $M \cup M_0$, while all other vertices of G are incident to exactly two edges from $M \cup M_0$. Thus, $M \cup M_0$ partitions the vertices of G into cycles and into paths with starting points and endpoints in $A \cup B$.

On both the cycles and the paths, edges from M and M_0 alternate. Similar to our considerations for the acyclicity of $\widehat{G}$, the uniqueness of M_0 implies that the cycles can only have length 2, i.e. consist of two times the same edge (which is contained in both M and M_0).

Consider one of these paths which starts at a vertex from A . Thus it has to start with an edge e_1 from M , followed by an edge from M_0 , which can only be m_{e_1} . Continuing like this, we can write the path as $(e_1, m_{e_1}, \dots, e_k, m_{e_k})$. It indeed has to end with an edge from M_0 and not M since it ends in a red vertex. Consequently, it ends in a vertex from B . Since $n_A = |A| = |B|$, these means that all these paths go from A to B and there are n_A of them. So we indeed obtain a family of n_A disjoint paths from A to B . To prove that this is a bijection, we have to show that each such path family can be obtained from exactly one perfect matching of $G \setminus B$. So consider such a path family and replace each path $(\hat{e}_1, \dots, \hat{e}_k)$ of the family by the corresponding path $(e_1, m_{e_1}, \dots, e_k, m_{e_k})$ in G . We have to show that there is a unique perfect matching of $G \setminus B$ containing all these e_i and apart from them only edges which are also in M_0 .

Let W be the set of all vertices of G covered by one of these e_i or m_{e_i} . Note that $A \cup B \subseteq W$. All the e_i together form a perfect matching of $W \setminus B$. To extend this to a perfect matching of $G \setminus B$, we need to add a perfect matching of $G \setminus W$. However, note that all the m_{e_i} together form a perfect matching of $W \setminus A$. By the uniqueness of M_0 , this implies that $(G \setminus A) \setminus (W \setminus A) = G \setminus W$ has a unique perfect matching, consisting of the remaining edges from M_0 . (In figure 12b, these are the gray edges.) Thus, adding these to the set of e_i 's gives the unique perfect matching of $G \setminus B$ which contains all the e_i . So indeed, we have a bijection. $\square$

Applying theorem 4.3 to Q -partial matchings gives the following:

Corollary 4.4. *Let (G, A) fulfill the MaPaf-condition and let $B, Q \subseteq G_r$ with $B \cap Q = \emptyset$, $|B| \leq n_A \leq |B \cup Q|$. Then Q -partial matchings of $G \setminus B$ stand in bijection to families of n_A disjoint paths from A to $B \cup Q$ in $\widehat{G}$ using all vertices from B as endpoints. In particular, matchings covering exactly $G \setminus (B \cup Q')$ are mapped to path families using exactly $B \cup Q'$ as endpoints.*

Remark. Note that the paths do not have to end in the first vertex of Q they meet, but can pass several vertices from Q until finally using one as their endpoint.

Proof. A Q -partial matching of $G \setminus B$ is a perfect matching of $G \setminus (B \cup Q')$ for some $Q' \subseteq Q$ with $|B \cup Q'| = n_A$ (see the remark after definition 4.2). By theorem 4.3, perfect matchings of $G \setminus (B \cup Q')$ correspond to families of n_A disjoint paths from A to $B \cup Q'$ in G . Thus, Q -partial matchings of $G \setminus B$ correspond to families of n_A disjoint paths from A to $B \cup Q$ using all vertices from B as endpoints. The condition $|B| \leq n_A \leq |B \cup Q|$ is necessary in order to have $n_A = |B \cup Q'|$ for some $\emptyset \subseteq Q' \subseteq Q$. $\square$

Clearly, when setting $Q := \emptyset$ in corollary 4.4, we get back theorem 4.3. However, there are also some other special choices for Q and B giving interesting results:

Corollary 4.5. *Let (G, A) fulfill the MaPaf-condition. Then we have the following:*

- (i). *Let $Q \subseteq G_r$ with $n_A \leq |Q|$. Then Q -partial matchings of G stand in bijection to families of n_A disjoint paths from A to Q in $\widehat{G}$. In particular, matchings covering exactly $G \setminus Q'$ are mapped to path families using exactly Q' as endpoints.*

(ii). Let $Q \subseteq G_r$ with $A \subseteq Q$. Then Q -partial matchings of G stand in bijection to families of arbitrary many disjoint paths from A to $Q \setminus A$ in $\widehat{G}$. In particular, matchings covering exactly $G \setminus Q'$ with $Q' \cap A =: A'$ are mapped to path families using exactly $A \setminus A'$ as starting points and $Q' \setminus A'$ as endpoints.

(iii). Let $B \subseteq G_r$ with $B \cap A = \emptyset$, $|B| \leq n_A$. Then A -partial matchings of $G \setminus B$ stand in bijection to families of $|B|$ disjoint paths from A to B in $\widehat{G}$. In particular, matchings covering exactly $G \setminus (B \cup A')$ are mapped to path families using exactly the vertices from $A \setminus A'$ as starting points.

Proof. Item (i) follows directly by setting $B := \emptyset$ in corollary 4.4.

For item (ii), note that by item (i) we have that matchings covering exactly $G \setminus Q'$ are mapped to families of n_A disjoint paths from A to Q' . Clearly, all these path families contain a trivial path at every vertex in $A' := Q' \cap A$. Taking out these trivial paths indeed gives $n_A - |A'|$ paths from $A \setminus A'$ to $Q' \setminus A'$, thus using all vertices in $A \setminus A'$ as starting points and all vertices in $Q' \setminus A'$ as endpoints. In total, this means that for $A \subseteq Q$, Q -partial matchings of G stand in bijection to families of arbitrary many disjoint paths from A to $Q \setminus A$.

Item (iii) can be derived from corollary 4.4 in a way similar to item (ii), by setting $Q := A$ and taking out the trivial paths at vertices from A' , thus being left with the paths from $A \setminus A'$ to B . $\square$

Remark. Both in theorem 4.3 and in corollaries 4.4 and 4.5, the conditions for the sizes of B and Q can be left out. As one can easily see, if they are outside the given borders, then the MaPaf-bijection reduces to a bijection between empty sets, since the numbers of red and blue vertices do not fit together for the matchings and the numbers of starting points and endpoints do not fit together for the path families.

4.2 MaPaf-bijections for graphs on the square lattice

With the framework we now have, it is quite easy to find MaPaf-bijections for graphs on the square lattice. These bijections are already known and have been derived in various ways, in particular for classical examples such as the Aztec graph. After presenting our approach, we give some insights into the development and application of these MaPaf-bijections.

One simple way of applying theorem 4.3 to a graph on the square lattice is to make sure that the unique matching M_0 consists of edges all parallel to each other. This is actually the case for the graph in figure 11, where M_0 consists of vertical edges, all of them having a blue vertex as the lower endpoint. When choosing B to consist of the four top-most red vertices, theorem 4.3 gives a MaPaf-bijection between perfect matchings of the 6×7 grid graph and families of four disjoint paths going from bottom vertices to top vertices in figure 11d. Obviously, the same can be done for any size of the grid graph. (Note that perfect matchings of the grid graph have been enumerated by completely different means by Temperley and Fisher [53] and Kasteleyn [34].)

We can obtain a similar situation for any graph on the square lattice in the following way: Add a red vertex above each blue vertex that has no red vertex directly above it, and let B be the set of all these new red vertices. Let A be the set of all red vertices having no blue vertex directly below them. Then $G \setminus A$ has a unique matching M_0 which connects each red vertex to the blue vertex directly below it. (To see that this matching is actually unique, note that the top row contains only

red vertices, which thus have to be matched downwards. Ignoring the vertices already matched, in the second row there are now only red vertices left, etc.) So by theorem 4.3, if $|B| = n_A$, then perfect matchings of the original graph $G \setminus B$ stand in bijection to families of n_A disjoint paths from A to B in the directed graph $\widehat{G}$. (And if $|B| \neq n_A$, then $G \setminus B$ cannot have any perfect matching since it has an unequal number of red and blue vertices.)

This MaPaf-bijection for graphs on the square lattice - derived in a way quite similar to ours - was already developed by John, Sachs, and Zernitz in 1987 [33, sections 3 and 4]. Kim, Lee, and Oh later presented this bijection in terms of domino tilings [35, section 2] and applied it in the same paper to various shapes related to the Aztec Diamond. (For the correspondence between domino tilings and perfect matchings consider section 2.2.) Choi, Lee, and Oh then generalized this to dicube tilings of solid figures [11, section 2], i.e. (in the language of perfect matchings) to subregions of the three-dimensional cubic lattice. Indeed, we see that our MaPaf-bijection can easily be generalized to any dimension: Choose a direction parallel to one of the coordinate axes and choose red vertices for B , which are added to the graph, and A such that $G \setminus A$ has a matching M_0 - which is automatically unique - consisting only of edges going from a blue vertex to a red vertex in the chosen axis direction. Figure 5 in [11] gives a nice illustration of the resulting directed graph in the three-dimensional case.

The best-known example of these MaPaf-bijections is probably the one for the Aztec graph. Consulting figure 13, we see that perfect matchings of the Aztec graph of size n correspond to families of n disjoint paths from A_i to B_i in the directed graph depicted in figure 13b. This MaPaf-bijection is presented in different formulations for instance in [32, section 2.1] and [20]. Nice illustrations of the resulting path family for very large n can be found in [7, figures 1 and 2].

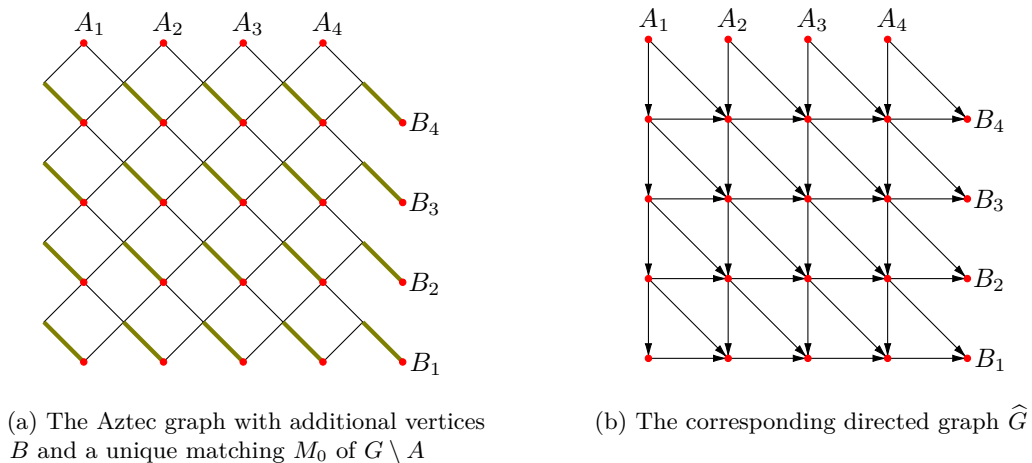


Figure 13: Finding a MaPaf-bijection for the Aztec graph (here of size 4)

We conclude our treatment of graphs on the square lattice by mentioning two other quite nice applications:

In [51], the Aztec graph (rotated such that the edges are vertical and horizontal) is adapted to have a third column of vertices in the middle, of equal length as the other two middle columns (figure 1 in [51]). The consequence is that perfect matchings of this graph correspond to a single path, in fact a Delannoy path. To see this, note that A and B can be chosen to consist of only one

vertex each, which are at the top and the bottom of the middle column. The dual of this adaption of the Aztec graph was later called ‘augmented Aztec diamond’ in [35].

In [41, section 6], the MaPaf-bijection is used very elegantly to show that the number of domino tilings of a so-called ‘mixed Aztec rectangle with holes’ is invariant under color-preserving translation of the whole set of holes. This is because the mixed Aztec rectangle (see figure 14b in [41]) itself has a unique matching, so we only get elements of A and B from the holes (see figures 15 and 16 in [41]). When the holes are shifted, $A \cup B$ is shifted as a whole, not affecting the number of path families from A to B .

4.3 MaPaf-bijections for graphs on the honeycomb lattice

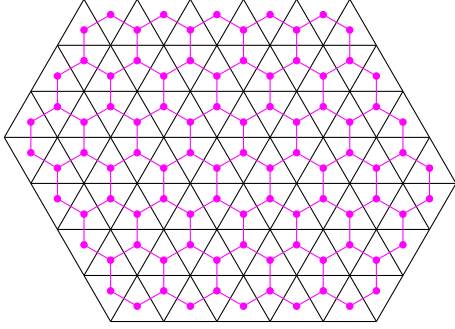
Also for graphs on the honeycomb lattice - maybe even more than for graphs on the square lattice - MaPaf-bijections are present in the literature. Before going into detail about that, we again first give our approach.

The honeycomb lattice is built from regular hexagons. Figure 14a shows in pink a graph on the honeycomb lattice. As one can see, it is the dual of a hexagon divided into equilateral triangles of unit size (in black). For such shapes, rhombus tilings, also called lozenges tilings, are of interest, where a rhombus consists of two triangles sharing a side. Thus, analogously to domino tilings (see section 2.2), rhombus tilings correspond to perfect matchings of their dual graph on the honeycomb lattice. Figures 14c and 14e give an example of a perfect matching and the corresponding rhombus tiling.

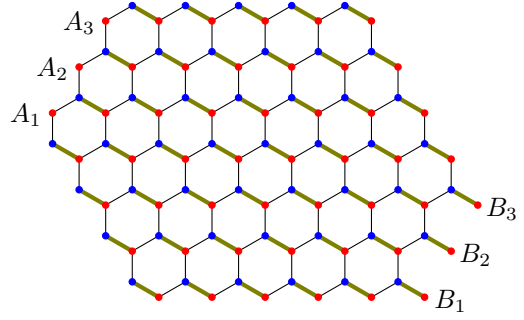
We can easily find a MaPaf-bijection for graphs on the honeycomb lattice by using the same trick as for square lattices, namely making sure that M_0 consists of parallel edges all going from a blue to a red vertex in the same direction. For our example, this is depicted in figure 14b. The resulting bijection is widely known and used. Figure 14d shows the path family corresponding to the perfect matching in figure 14c. However, this MaPaf-bijection is rarely formulated like that in terms of perfect matchings, since it can be done much simpler when working with the rhombus tilings as follows:

Figure 14f shows the path family laid over the rhombus tiling. Note however that the paths are slightly shifted such that the red vertices do not lie in the centers of the small triangles, but on their left sides. Now we see that when starting from an A_i , the paths traverse each rhombus they meet by connecting the midpoint of a side to the midpoint of the opposite (parallel) side. With this rule, the paths can easily be constructed from the rhombus tiling. The concept of such paths was introduced by de Bruijn when studying rhombus tilings of the whole plane [8, section 3].

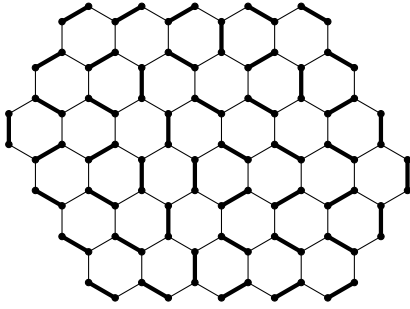
However, the MaPaf-bijection can be formulated yet more simply: Note that the illustration of the rhombus tiling in figure 14e looks like the two-dimensional image of cubes stacked in a box. The paths in figure 14f now run along the top faces and down the side faces of these cubes, never leaving their row of cubes. This interpretation of rhombus tilings as stacks of cubes can for instance be found in [15], where another interesting implication of this point of view is mentioned: One clearly sees that a rhombus tiling of a hexagon with side lengths a, b, c, a, b, c always consists of $a \cdot b$, $a \cdot c$ and $b \cdot c$ rhombuses respectively of the three possible orientations of rhombuses. Furthermore, these stacks of cubes correspond to plane partitions (analogously to Young diagrams for integer partitions), which thus also stand in bijection to perfect matchings and path families.



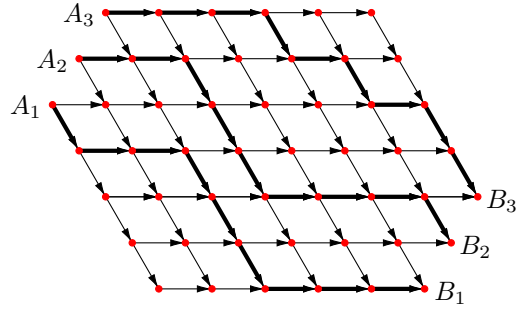
(a) A graph (in pink) on the honeycomb lattice, which is the dual of a triangulated hexagon



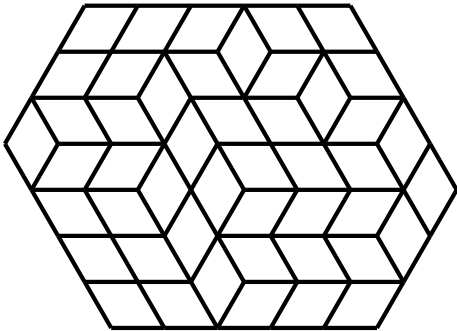
(b) The unique perfect matching M_0 of $G \setminus \{A_1, A_2, A_3\}$



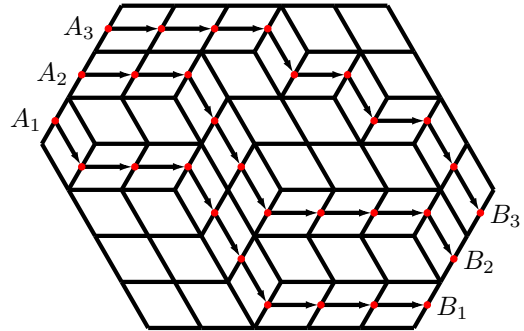
(c) A perfect matching M of $G \setminus B$



(d) The path family corresponding to M



(e) M interpreted as a rhombus tiling of a hexagon



(f) The rhombus tiling together with the path family

Figure 14: A MaPaf-bijection for a graph on the honeycomb lattice

Using the language of perfect matchings, this MaPaf-bijection was for instance presented by Sachs in a way relatively similar to ours [50, section 3]. In [17, section 3], it is formulated quite elegantly using contractions. Indeed, we see that (not only for graphs on the honeycomb lattice, but in the general setting of definition 4.2 and theorem 4.3) by contracting all edges of the unique matching M_0 , the graph G is transformed into the undirected version of $\widehat{G}$ and the perfect matchings turn into the corresponding path families. The MaPaf-bijection is often used for enumerating various versions of rhombus tilings, see for instance [12, section 5], [23, section 2] and [25, section 1].

Finally, let us mention that the MaPaf-bijection derived in this section is actually exactly the same as the one in section 4.2 for square lattices: We can squish a hexagon slightly to deform it into a 2×1 rectangle. Seen that way, the honeycomb lattice is nothing but the square lattice with every second vertical edge deleted. Going back to figure 11b (assuming we delete exactly those vertical edges which are not in M_0), we see that this results in all vertical arrows being deleted in figure 11d. And indeed, the directed graph that is left then has exactly the same inner structure as the directed graph in figure 14d. This connection was used by John, Sachs, and Zernitz in the opposite way: They pulled the square lattice apart to look like a honeycomb lattice with additional long vertical edges. This had advantages for working with the square lattice and additionally implied that the same results also hold for honeycomb lattices [33, section 3].

4.4 MaPaf-bijections for the (half-)reduced AST-graph

We now finally find MaPaf-bijections for the AST-graph, in fact for the reduced and the half-reduced AST-graph coming from section 3.6. In the following, G always denotes one of these two graphs, depending on which we are considering at the moment. Both graphs are depicted in figure 15, with the vertices in $Q = \{Q_1, \dots, Q_{2n}\}$ labeled and all vertices marked red or blue according to the bipartition. Remember that in order for (G, A) to fulfill the MaPaf-condition, A needs to be a set of red vertices such that $G \setminus A$ has a unique perfect matching. It turns out that there are surprisingly many such choices for A , all of which have $|A| = n$ due to the bipartition. For the reduced AST-graph, we introduce several classes of them. The associated illustrations can be found in figure 16 and in the appendix section A.3. We always depict the graph G with the unique matching M_0 marked and diagonally below it the resulting directed graph $\widehat{G}$.

Definition 4.6. We define the following configurations of red vertices $A = \{A_1, \dots, A_n\}$ in the reduced AST-graph of size n :

- *Row-wise configurations (figure 29)* From each row of red vertices, exactly one is chosen for A .
- *Column-wise configurations (figure 30)* In either the left or the right half of the reduced AST-graph, from each column of red vertices, exactly one is chosen for A .
- *Snakes (figure 31)* A snake in the left half of the reduced AST-graph is constructed as follows: A_n is the n -th red vertex in the first row of red vertices. Then, $A_{n-1}, A_{n-2}, \dots, A_1$ are chosen consecutively, such that each A_i is either directly left of A_{i+1} or directly below A_{i+1} , until A_1 is one of $Q_1, \dots, Q_n$. A snake in the right half is constructed in the symmetric way: A_n is the $(n+1)$ -th vertex in the first row of red vertices, then each A_i goes either right or down until A_1 is one of $Q_{n+1}, \dots, Q_{2n}$. To the snake depicted in figure 16a we give a special name: We call it the *first snake*.

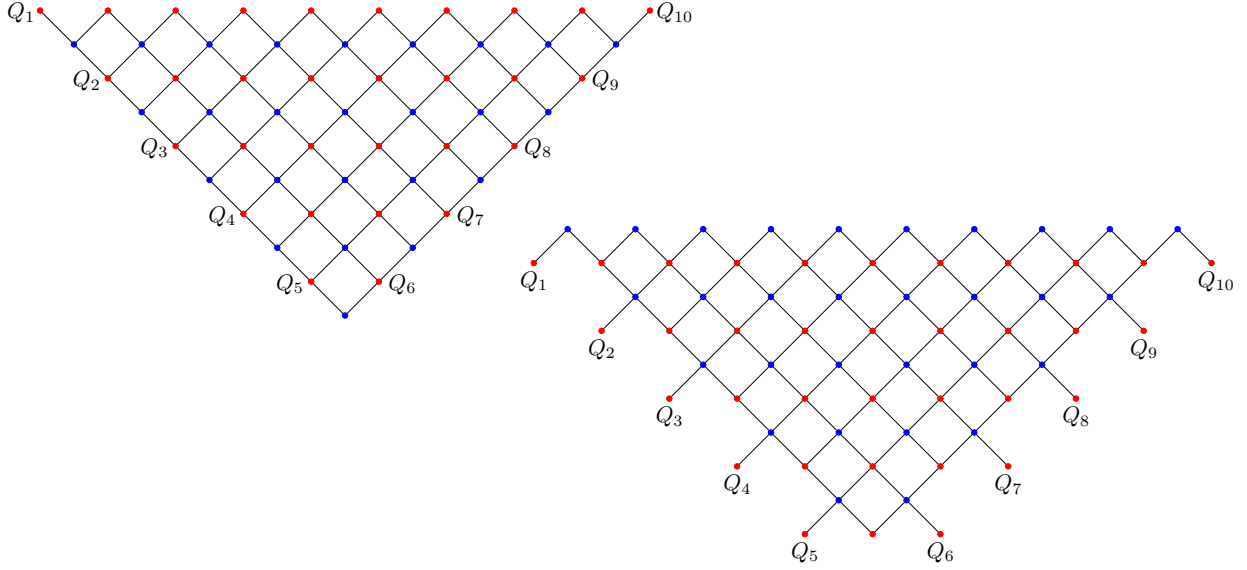


Figure 15: The reduced (left) and half-reduced (right) AST-graph of size 5

- *Lines (figure 32)* These are a special case of row-wise configurations. The k -th line is constructed by choosing A_n as the k -th red vertex in the first row, then taking A_{n-1} directly below it and so on until we have $A_i = Q_k$ for some i . Then the other A_i 's follow the border of the graph diagonally down.
- *Mixed configurations (figure 33)* These are build up from the configurations above in the following way: Take one red vertex for A from each of the first k rows of red vertices. We are left below with a reduced AST-graph of size $n - k$. Decide whether to take the remaining A_i 's from its left or right half. For the right half, take one red vertex for A from each of the first l columns of red vertices in the right half. On the right, the right half of a reduced AST-graph of size $n - k - l$ remains. Choose a snake or line in this right half. Analogously, a mixed configuration using the left half is constructed.

For the half-reduced AST-graph of size n , we only define a single configuration:

- *Symmetric configuration (figure 16c)* This is the configuration obtained by taking the vertices on the vertical symmetry axes for A .

Remark. One can easily see that out of the many configurations for the reduced AST-graph, only the row-wise configurations also work for the half-reduced AST-graph. For the others, there does not seem to be a practical analogue. Therefore we do not work much with the half-reduced AST-graph, but only consider the symmetric configuration (a special case of the row-wise configurations). Its vertical symmetry is a quite useful property that cannot be achieved for the reduced AST-graph, at least not for odd n .

By the following lemma, our MaPaf-bijection can indeed be applied to all these configurations:

Lemma 4.7. *For each of the configurations introduced in definition 4.6, $G \setminus A$ has a unique perfect matching and thus (G, A) fulfills the MaPaf-condition.*

Proof. For the row-wise configurations, note that in $G \setminus A$, the first row of red vertices and the first row of blue vertices contain an equal number of vertices. Thus, all of these vertices have to be

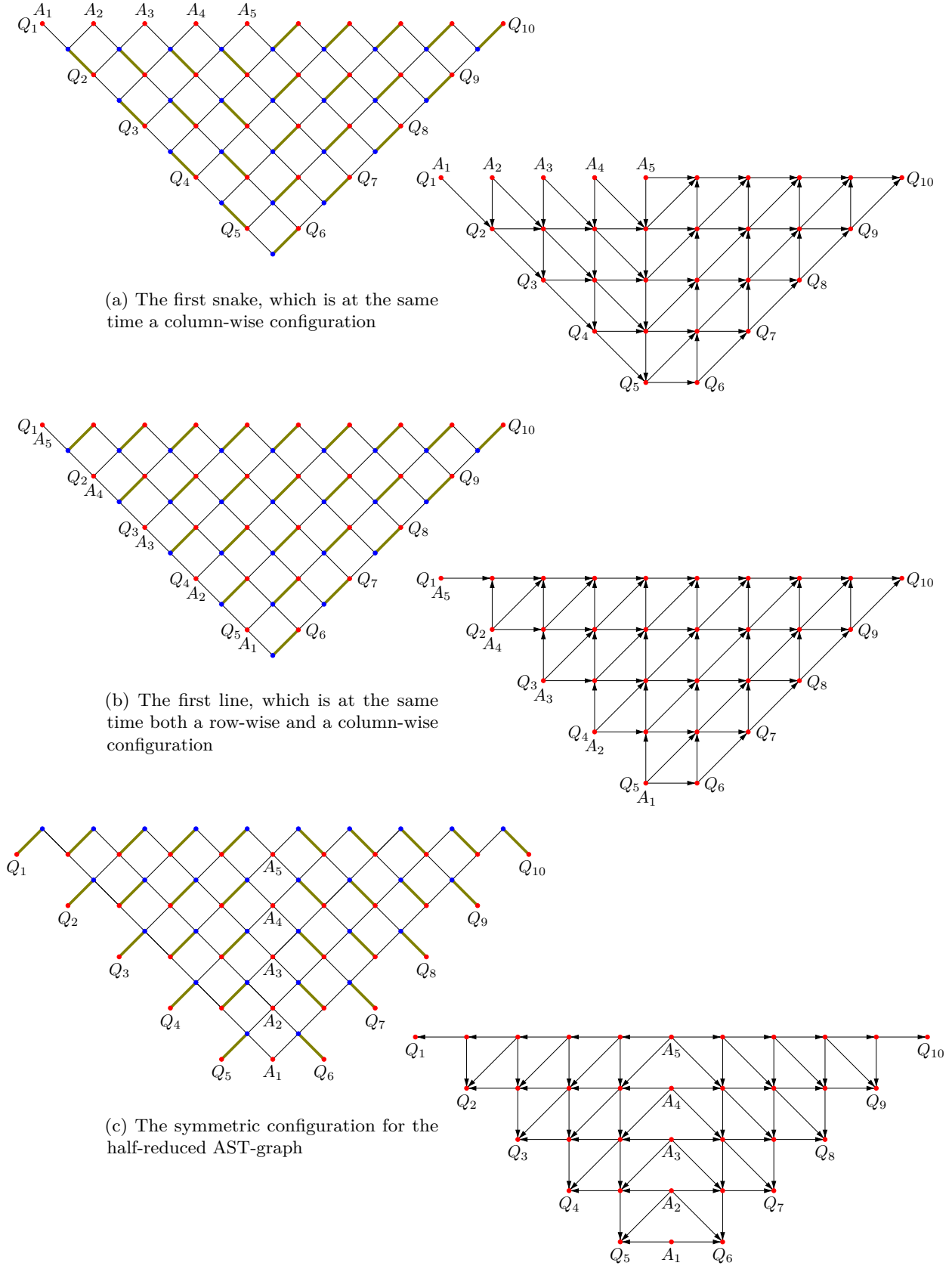


Figure 16: G with M_0 and the corresponding $\widehat{G}$ for some of the configurations introduced in definition 4.6

matched to each other, and there is a unique way of doing so: The leftmost red vertex has to be matched to the leftmost blue vertex etc. Then, the same argument holds for the second row of red vertices together with the second row of blue vertices and so on. Thus, $G \setminus A$ has a unique perfect matching.

Similarly, by considering columns instead of rows, one can show that for the column-wise configurations, $G \setminus A$ has a unique perfect matching. Lines and the symmetric configuration are just special cases of row-wise configurations and thus already covered.

For the remaining configurations, namely snakes and mixed configurations, we only sketch the reason that $G \setminus A$ has a unique perfect matching: Consider a snake in the left half. Then in the right half, the matching is forced (going by columns from the right to the middle). The left half is cut into two parts by the snake. When considering the diagonals parallel to the left border of the graph, we see that each contains exactly one A_i , and thus indeed A_1 lies on the border. By now considering the diagonals normal to that and taking the vertices diagonal by diagonal, we see that the matching is forced in each of the two parts. Note that it is necessary that the A_i actually separate the graph into two parts - it would not be enough to just take one A_i on each diagonal parallel to the border.

For the mixed configurations, the matching is forced in the first k rows as it is for row-wise configurations. In the remaining smaller reduced AST-graph, it is forced in the half not containing any A_i and in the l columns containing one A_i each as for column-wise configurations. And in the last part, which is half of a reduced AST-graph with a snake or line in it, the matching is forced as it is for snakes or lines. $\square$

Now we can apply our MaPaf-bijection for Q -partial matchings. Note however that we do not use the full power of corollary 4.4, but only the case $B = \emptyset$ and thus items (i) and (ii) of corollary 4.5. Using a non-empty B (by adding the vertices of B as auxiliary vertices to G first) would give us even more ways of finding MaPaf-bijections.

Proposition 4.8. *Consider any of the directed graphs $\widehat{G}$ corresponding to a configuration introduced in definition 4.6 for the (half-)reduced AST-graph of size n , in particular any $\widehat{G}$ in figure 16 and in section A.3. Then $\widehat{G}$ is acyclic and we have:*

- (i). *Q -partial matchings of the (half-)reduced AST-graph of size n stand in bijection to families of n disjoint paths from A to Q in $\widehat{G}$. The set of endpoints a path family uses is given by the vertices which are not covered by the corresponding Q -partial matching.*
- (ii). *If $A \subseteq Q$ holds (for instance when considering the first line, see figure 16b), then Q -partial matchings of the (half-)reduced AST-graph of size n also stand in bijection to families of arbitrary many disjoint paths from A to $Q \setminus A$ in $\widehat{G}$. The set of starting points a path family uses is given by the vertices from A which are covered by the corresponding Q -partial matching. The set of endpoints a path family uses is given by the vertices from $Q \setminus A$ which are not covered by the corresponding Q -partial matching.*

Proof. The acyclicity of the directed graphs is given by theorem 4.3 (together with lemma 4.7). Item (i) resp. (ii) follows directly from item (i) resp. (ii) of corollary 4.5. $\square$

Remark. Note that in the case of $A \subseteq Q$, the families of arbitrary many paths can be obtained from the families of n paths for the same $\widehat{G}$ by just taking out the trivial paths, see the proof of corollary 4.5.

4.5 Corollaries for the path families

The correspondence to Q -partial matchings implies quite interesting facts about the path families in the directed graphs $\widehat{G}$. These facts are absolutely easy to find using the matchings, but when just working with $\widehat{G}$, they are not at all obvious. However, once knowing the matchings, it is indeed possible to formulate and prove them without making use of the matchings (but we do not do that here).

We state these properties in general for any directed graph $\widehat{G}$ corresponding to the (half-)reduced AST-graph, i.e. any $\widehat{G}$ coming from a choice of vertices $A \subseteq G$ in the (half-)reduced AST-graph such that $G \setminus A$ has a unique perfect matching (and thus, lemma 4.7 and proposition 4.8 hold for this $\widehat{G}$). Furthermore, we only work with families of n disjoint paths, since the adaption to the families of arbitrary many disjoint paths coming from configurations with $A \subseteq Q$ is very simple. Therefore, for this section only, let ‘path family’ always mean ‘family of n disjoint paths from A to Q ’ and let Q' denote the subset of vertices of $Q = \{Q_1, \dots, Q_{2n}\}$ which are used as endpoints of a path family.

Firstly, we get various bijections between the path families in different graphs $\widehat{G}$:

Corollary 4.9. *There are the following bijections and involutions on path families in the directed graphs $\widehat{G}$ corresponding to the (half-)reduced AST-graph of size n :*

- (i). *For any two of the directed graphs $\widehat{G}$, there is a bijection between the path families in these two graphs. If the directed graphs $\widehat{G}$ both correspond to the reduced AST-graph or both to the half-reduced AST-graph, then the bijection holds the set Q' of endpoints of the paths fixed. Otherwise, the bijection acts as a complement in Q on the set Q' .*
- (ii). *For each directed graph $\widehat{G}$, there is an involution on its path families which vertically reflects the set Q' of endpoints, i.e. which takes $\{Q_{i_1}, \dots, Q_{i_n}\}$ to $\{Q_{2n+1-i_n}, \dots, Q_{2n+1-i_1}\}$.*

Furthermore, all of these bijections and involutions are compatible to each other in the following sense: Different concatenations give the same function as long as they have the same domain and codomain and contain either both an even or both an odd number of involutions.

Proof. The bijections between different path families are induced by the bijections from the path families to Q -partial matchings. Since the set Q' of endpoints of a path family is given by the vertices not covered by the corresponding Q -partial matching, it is held fixed if both directed graphs $\widehat{G}$ correspond to the same graph. If however one corresponds to the reduced and the other to the half-reduced AST-graph, we have to use the bijection between Q -partial matchings of the reduced and the half-reduced AST-graph, given in lemma 3.23. It can easily be seen (compare the remark at the end of section 3.6) that this bijection acts as a complement on the set of vertices not covered by the Q -partial matching. Thus the bijection between the path families acts as a complement on Q' . (Note that Q' here has another meaning than in section 3.6.)

The involutions in item (ii) are induced by vertical reflection of the corresponding Q -partial matching and thus also the set Q' is reflected. Since all these bijections and involutions are directly induced by the Q -partial matchings, it is clear that different concatenations give the same function provided that they perform either both an even or both an odd number of reflections on the Q -partial matchings, i.e. contain either both an even or both an odd number of involutions. $\square$

Our second observation regards the existence of path families having certain sets Q' of endpoints. We only formulate this for the reduced AST-graph, as one can easily find the analogues for the half-reduced AST-graph:

Lemma 4.10. *Let $\widehat{G}$ correspond to the reduced AST-graph of size n . Then the following holds:*

- (i). *If Q' contains both Q_n and Q_{n+1} , then there are no path families in $\widehat{G}$ with endpoints Q' .*
- (ii). *If Q' contains neither Q_1 nor Q_{2n} , then there are no path families in $\widehat{G}$ with endpoints Q' .*
- (iii). *If Q' contains exactly one vertex from each of the pairs $\{Q_1, Q_{2n}\}, \{Q_2, Q_{2n-1}\}, \dots$ and $\{Q_n, Q_{n+1}\}$, then there is exactly one path family in $\widehat{G}$ with endpoints Q' .*

Proof. For item (i), we can easily see that the reduced AST-graph does not have a Q -partial matching which neither covers Q_n nor Q_{n+1} . Thus by proposition 4.8, there are no path families using both Q_n and Q_{n+1} as endpoints. Similarly, item (ii) can be proved.

For item (iii), remember that $G \setminus A$ has a unique perfect matching if A contains exactly one vertex from each row of red vertices (these are the row-wise configurations in definition 4.6). Thus, for the Q' as described, there is exactly one Q -partial matching not covering Q' . Consequently, there is exactly one path family using Q' as endpoints. $\square$

5 LGV lemma: From path families to sums of minors

The Lindström-Gessel-Viennot lemma (LGV lemma) is a powerful tool for the enumeration of families of disjoint paths. Essentially, it gives their number as the determinant of the matrix which for each pair of a starting point and an endpoint of the path family contains the number of paths between them. When the starting points and endpoints of the path family are allowed to vary over given sets, then instead of the determinant one obtains a sum of minors. It is quite common to use the LGV lemma to enumerate the path families which come from some kind of MaPaf-bijection. This is what we also do in this section, finding numerous ways of writing the number of Q -partial matchings of the (half-)reduced AST-graph as a sum of minors of some relatively simple matrix.

Section 5.1 is dedicated to give a preliminary proposition which comes in use later. In section 5.2, we then present the LGV lemma along with two corollaries applying it to planar graphs - one which is very widely used, and a generalization of it which as far as we know is new and comes in handy when dealing with the (half-)reduced AST-graph. Then, we generalize these corollaries to path families with variable sets of starting points and endpoints. This is a quite trivial step, but still worth formulating explicitly since we make use of it a lot. In section 5.3, we then give a very short insight into the applications of the LGV lemma, in particular those related to the MaPaf-bijections from section 4.

In section 5.4, we apply the LGV lemma to the path families corresponding to Q -partial matchings of the (half-)reduced AST-graph as found in section 4.4. This lets us write the sought-after number as a sum of minors for several different matrices. Since these matrices do not directly count the numbers of paths in the directed graphs, but some signs are changed, in section 5.5 the graphs are adapted to get that property back. With this direct correspondence to the graphs, in section 5.6 we show that all the different matrices can actually be obtained from each other by elementary matrix row operations. Section 5.7 finds explicit expressions for some of these operators, which are used further in section 6.

5.1 An identity for sums of minors

Before presenting the LGV lemma, we here introduce an identity for determinants and a generalization of it for sums of minors. This not only allows us to generalize a well-known corollary of the LGV lemma (giving corollary 5.9), but is also important later when operating with sums of minors, see proposition 7.20. We start with several definitions:

Definition 5.1.

- We denote by V_n the $n \times n$ matrix with alternating 1's and -1 's on the main diagonal and by R_n the $n \times n$ matrix with 1's on the anti-diagonal, that is:

$$(V_n)_{ij} := \begin{cases} (-1)^{i-1} & i = j \\ 0 & \text{else} \end{cases} \quad (R_n)_{ij} := \begin{cases} 1 & i + j = n + 1 \\ 0 & \text{else} \end{cases}$$

- For an $m \times n$ matrix M and $I \subseteq \{1, \dots, m\}$, $J \subseteq \{1, \dots, n\}$, we denote by M_I the submatrix of M consisting of the rows with indices in I and by M^J the submatrix of M consisting of the columns with indices in J .

Now we can state the proposition:

Proposition 5.2. *Let X and X' be $n \times m$ resp. $n \times m'$ matrices with $m + m' = n$. Then we have:*

$$\sum_{\substack{I \cup I' = \{1, \dots, n\} \\ I \cap I' = \emptyset \\ |I| = m, |I'| = m'}} \det(X_I) \det(X'_{I'}) = \det(V_n X R_m \quad X')$$

Proof. Set $M := (V_n X R_m \quad X')$ and for $I, I' \subseteq \{1, \dots, n\}$ with $|I| = m, |I'| = m'$, denote the elements of I and I' as $I = \{i_1, \dots, i_m\}$ and $I' = \{i'_1, \dots, i'_{m'}\}$ with $i_1 < \dots < i_m$ and $i'_1 < \dots < i'_{m'}$. Then we have the following:

$$\begin{aligned} \det(X_I) &= \sum_{\sigma \in S_m} \text{sgn}(\sigma) X_{i_{\sigma(1)},1} \cdot \dots \cdot X_{i_{\sigma(m)},m} = \\ &= \sum_{\sigma \in S_m} \text{sgn}(\sigma) \left((-1)^{i_{\sigma(1)}-1} M_{i_{\sigma(1)},m} \right) \cdot \dots \cdot \left((-1)^{i_{\sigma(m)}-1} M_{i_{\sigma(m)},1} \right) = \\ &= \sum_{\sigma \in S_m} \text{sgn}(\sigma) (-1)^{i_1 + \dots + i_m - m} M_{i_{\sigma(m)},1} \cdot \dots \cdot M_{i_{\sigma(1)},m} \\ \det(X'_{I'}) &= \sum_{\sigma' \in S_{m'}} \text{sgn}(\sigma') X'_{i'_{\sigma'(1)},1} \cdot \dots \cdot X'_{i'_{\sigma'(m')},m'} = \\ &= \sum_{\sigma' \in S_{m'}} \text{sgn}(\sigma') M_{i'_{\sigma'(1)},m+1} \cdot \dots \cdot M_{i'_{\sigma'(m')},n} \end{aligned}$$

Thus we have

$$\begin{aligned} \det(X_I) \cdot \det(X'_{I'}) &= \\ &= \sum_{\sigma \in S_m, \sigma' \in S_{m'}} \text{sgn}(\sigma) (-1)^{i_1 + \dots + i_m - m} \text{sgn}(\sigma') M_{i_{\sigma(m)},1} \cdot \dots \cdot M_{i_{\sigma(1)},m} M_{i'_{\sigma'(1)},m+1} \cdot \dots \cdot M_{i'_{\sigma'(m')},n} \\ &= \sum_{\sigma \in S_m, \sigma' \in S_{m'}} \text{sgn}(\sigma) \text{sgn}(\sigma') (-1)^{i_1 + \dots + i_m - m} M_{\tau(1),1} \cdot \dots \cdot M_{\tau(n),n} \end{aligned}$$

where $\tau \in S_n$ in dependence of (σ, σ') is given as follows, using the two-line notation for permutations:

$$\tau := \begin{pmatrix} 1 & \dots & m & m+1 & \dots & n \\ i_{\sigma(m)} & \dots & i_{\sigma(1)} & i'_{\sigma'(1)} & \dots & i_{\sigma'(m')} \end{pmatrix}$$

Clearly, when taking all pairs $(\sigma, \sigma') \in S_m \times S_{m'}$ for all possible partitions $I \cup I' = \{1, \dots, n\}$, $I \cap I' = \emptyset$ with $|I| = m, |I'| = m'$, by this correspondence they stand in bijection to all possible $\tau \in S_n$. Thus, in order to show that the sum of all $\det(X_I)\det(X'_{I'})$ gives indeed $\det(M)$, we only have to show that the sign of τ is $\text{sgn}(\sigma)\text{sgn}(\sigma')(-1)^{i_1+\dots+i_m-m}$. For this, we first take out σ and σ' by transforming τ to

$$\tau' := \begin{pmatrix} 1 & \dots & m & m+1 & \dots & n \\ i_m & \dots & i_1 & i'_1 & \dots & i_{m'} \end{pmatrix}$$

So we have to show that $\text{sgn}(\tau') = (-1)^{i_1+\dots+i_m-m}$. However, the sign of τ' can be determined quite easily: To obtain τ' from the identity permutation, we have to do the following (remember that $i_1 < \dots < i_m$ and $i'_1 < \dots < i'_{m'}$): First move all i_j to the left and all $i'_{j'}$ to the right, without changing their respective internal order, then reverse the order of the i_j .

Moving all i_j to the left and all $i'_{j'}$ to the right can be done by always exchanging an i_j with an $i'_{j'}$, directly left of it, thus moving that i_j to the left by one position. In total, each i_j has to move from position i_j to position j , which is $i_j - j$ moves. Summing over all j , in total we need to do $i_1 + \dots + i_m - 1 - \dots - m = i_1 + \dots + i_m - \frac{(m+1)m}{2}$ moves, so the result is a permutation of sign $(-1)^{i_1+\dots+i_m-\frac{(m+1)m}{2}}$. Reversing the order of $i_1, \dots, i_m$ clearly is an operation of sign $(-1)^{\frac{(m-1)m}{2}}$, so the sign of τ is indeed $(-1)^{i_1+\dots+i_m-\frac{(m+1)m}{2}} \cdot (-1)^{\frac{(m-1)m}{2}} = (-1)^{i_1+\dots+i_m-m}$. $\square$

In order to generalize proposition 5.2 to sums of minors, we introduce a compact notation for them:

Definition 5.3. Let M be an $m \times n$ matrix and $k \geq 0$ an integer. Then we denote by $|M|_k$ the sum of all $k \times k$ minors of M . For $k = 0$, we set $|M|_0 := 1$.

We now obtain the following identity:

Corollary 5.4. Let X and X' be $n \times m$ resp. $n \times m'$ matrices. Then we have:

$$\sum_{\substack{I \cup I' = \{1, \dots, n\} \\ I \cap I' = \emptyset}} |X_I|_{|I|} \cdot |X'_{I'}|_{|I'|} = |V_n X R_m \quad X'|_n$$

Proof. This follows from proposition 5.2 by the following chain of equations. Note that when transforming $(X R_m)^J$ to $X^J R_{|J|}$ for the second line, we would actually have to ‘reflect’ J , that is take $\{m+1-j_m, \dots, m+1-j_1\}$ instead of $\{j_1, \dots, j_m\}$ in the second term. But since we are summing over all $J \subseteq \{1, \dots, m\}$ anyway, this does not make a difference.

$$\begin{aligned} |V_n X R_m \quad X'|_n &= \sum_{\substack{J \subseteq \{1, \dots, m\}, J' \subseteq \{1, \dots, m'\} \\ |J|+|J'|=n}} \det \left((V_n X R_m)^J \quad (X')^{J'} \right) = \\ &= \sum_{\substack{J \subseteq \{1, \dots, m\}, J' \subseteq \{1, \dots, m'\} \\ |J|+|J'|=n}} \det \left(V_n X^J R_{|J|} \quad (X')^{J'} \right) = \end{aligned}$$

$$\begin{aligned}
& \stackrel{(5.2)}{=} \sum_{\substack{J \subseteq \{1, \dots, m\}, J' \subseteq \{1, \dots, m'\} \\ |J| + |J'| = n}} \sum_{\substack{I \cup I' = \{1, \dots, n\} \\ I \cap I' = \emptyset \\ |I| = |J|, |I'| = |J'|}} \det(X_I^J) \det((X')_{I'}^{J'}) = \\
& = \sum_{\substack{I \cup I' = \{1, \dots, n\} \\ I \cap I' = \emptyset}} \sum_{\substack{J \subseteq \{1, \dots, m\} \\ |J| = |I|}} \det(X_I^J) \sum_{\substack{J' \subseteq \{1, \dots, m'\} \\ |J'| = |I'|}} \det((X')_{I'}^{J'}) = \\
& = \sum_{\substack{I \cup I' = \{1, \dots, n\} \\ I \cap I' = \emptyset}} |X_I|_{|I|} \cdot |X'_{I'}|_{|I'|}
\end{aligned}$$

□

5.2 The LGV lemma

Recall our definition of ‘disjoint paths’ as used in the previous section (definition 4.1). Furthermore, we need the following notations:

Definition 5.5. Let G be an acyclic directed graph and let $A = \{A_1, \dots, A_m\}$ and $B = \{B_1, \dots, B_n\}$ be sets of vertices in G . Then we define the following:

- M^{AB} denotes the $m \times n$ matrix in which the entry M_{ij}^{AB} is the number of paths from A_i to B_j in G .
- If $m = n$, then a family $\mathcal{F}$ of paths from A to B induces the permutation $\sigma_{\mathcal{F}} \in S_n$ if it consists of a path from A_i to $B_{\sigma_{\mathcal{F}}(i)}$ for each $i \in \{1, \dots, n\}$.

With these definitions, we can now state the Lindström-Gessel-Viennot lemma. It was first formulated by Lindström in 1973 [42, lemma 1] and then picked up by Gessel and Viennot in their joint works ([27, theorem 1] and [28, section 2]):

Theorem 5.6 (Lindström, Gessel, Viennot). *Let G be an acyclic directed graph and let $A = \{A_1, \dots, A_n\}$ and $B = \{B_1, \dots, B_n\}$ be sets of vertices of G . Then $\det(M^{AB})$ is the number of families of n disjoint paths from A to B in G , where each family $\mathcal{F}$ is weighted with $\text{sgn}(\sigma_{\mathcal{F}})$.*

Remark. Note that the vertex sets A and B may intersect. If we have $A_i = B_j$, by the acyclicity of G there is exactly one path from A_i to B_j , namely the trivial path containing no edge. This path is contained in each family of disjoint paths, since otherwise the path starting in A_i would intersect the path ending in B_j .

Proof. We give here the classical proof of the LGV lemma, which for instance appeared in [28, proof of theorem 1]. We have

$$\det(M^{AB}) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \cdot M_{1, \sigma(1)}^{AB} M_{2, \sigma(2)}^{AB} \dots M_{n, \sigma(n)}^{AB}$$

Note that $M_{1, \sigma(1)}^{AB} M_{2, \sigma(2)}^{AB} \dots M_{n, \sigma(n)}^{AB}$ is the number of path families $\mathcal{F}$ from A to B which induce the permutation σ , without requiring the paths to be disjoint. Thus $\det(M^{AB})$ enumerates all path families $\mathcal{F}$ from A to B using each A_i exactly once as a starting point and each B_j exactly once as an endpoint, where each $\mathcal{F}$ is weighted by $\text{sgn}(\sigma_{\mathcal{F}})$. The idea is now to find an involution on the path families which are not disjoint, which maps families with $\text{sgn}(\sigma_{\mathcal{F}}) = 1$ to families with $\text{sgn}(\sigma_{\mathcal{F}}) = -1$. This then implies that all path families which are not disjoint cancel pairwise in the sum and we are only left with the disjoint path families, which proves the theorem.

So let us define such an involution: Consider a family $\mathcal{F} = \{P_1, \dots, P_n\}$ of n paths from A to B , each P_i starting at A_i , which is not disjoint. Let k be the smallest index such that P_k intersects some other path of $\mathcal{F}$. Let $l > k$ be the smallest index such that P_k intersects P_l . Let v be the first vertex of P_k in which P_k and P_l intersect. Note that v only appears once in each path since G is acyclic. Let $P_k^{\rightarrow v}$ and $P_k^{v \rightarrow}$ be the paths we get by splitting P_k into the path $P_k^{\rightarrow v}$ from A_k to v and the path $P_k^{v \rightarrow}$ from v to $B_{\sigma_{\mathcal{F}}(k)}$. Define $P_l^{\rightarrow v}$ and $P_l^{v \rightarrow}$ in the same way. Now let P'_k be the path we obtain by putting together $P_k^{\rightarrow v}$ and $P_l^{v \rightarrow}$ and let P'_l be the path we obtain by putting together $P_l^{\rightarrow v}$ and $P_k^{v \rightarrow}$. Let $\mathcal{F}'$ be the family of paths we obtain by replacing P_k and P_l in $\mathcal{F}$ by P'_k and P'_l . Clearly, we have $\text{sgn}(\sigma_{\mathcal{F}}) = -\text{sgn}(\sigma_{\mathcal{F}'})$ since $\sigma_{\mathcal{F}'}$ differs from $\sigma_{\mathcal{F}}$ by the transposition $(\sigma_{\mathcal{F}}(k) \ \sigma_{\mathcal{F}}(l))$. Also, we see that $(\mathcal{F}')'$ would again be $\mathcal{F}$, so indeed we have an involution showing that all path families which are not disjoint cancel in the sum. $\square$

Remark. Since $\mathcal{F}$ and $\mathcal{F}'$ consist of the same edges, the LGV lemma can be proved in the exact same way when giving weights to the edges, which by multiplication induce weights on the paths and path families. This is often done in the literature. The entries of the matrix M^{AB} then are defined to be the sums of the weights of the paths, and the determinant of M^{AB} gives the total weight of the path families. Also all of the following corollaries can easily be generalized in this way.

Clearly, it is inconvenient that the LGV lemma as we stated it above does not directly enumerate the path families, but weights them with $\text{sgn}(\sigma_{\mathcal{F}})$. However, there are settings in which we can get rid of these weights. To be able to describe these setting, we define the following intersection conditions:

Definition 5.7. Let $A = \{A_1, \dots, A_m\}$, $B = \{B_1, \dots, B_n\}$ and $C = \{C_1, \dots, C_k\}$ be sets of vertices in a directed graph G . Then we define the following:

- The *intersection condition* for paths from A to B states that for any indices $i_1 < i_2$ and $j_1 < j_2$, any path from A_{i_1} to B_{j_2} intersects any path from A_{i_2} to B_{j_1} .
- The *reverse intersection condition* for paths from A to B states that for any indices $i_1 < i_2$ and $j_1 < j_2$, any path from A_{i_1} to B_{j_1} intersects any path from A_{i_2} to B_{j_2} .
- The *non-intersection condition* for paths from A to B and C states that no path from A to B intersects any path from A to C , except for the fact that some paths have the same starting point in A .

We can now formulate the following corollary, which was already given by Gessel and Viennot [28, corollary 2]. It is very practical and follows so directly from theorem 5.6 that it is often this corollary which is called ‘LGV lemma’, see for instance [52, theorem 1.2].

Corollary 5.8. Let G be an acyclic directed graph and let $A = \{A_1, \dots, A_n\}$ and $B = \{B_1, \dots, B_n\}$ be sets of vertices of G such that the intersection condition for paths from A to B holds. Then $\det(M^{AB})$ is the number of families of n disjoint paths from A to B in G .

Remark. One very common and simple way of ensuring that the intersection condition holds is to use a planar graph with the vertices $A_1, \dots, A_n, B_n, \dots, B_1$ all lying in that order on the border of the graph. This is already the case in the setting of [27, theorem 1] and mentioned in general for instance in [52, page 101]. In section 5.3, we use that fact that the Aztec graph has A and B arranged like that.

Proof. Clearly, the intersection condition implies that all families $\mathcal{F}$ of disjoint paths have $\sigma_{\mathcal{F}} = \text{id}_n$ and thus $\text{sgn}(\sigma_{\mathcal{F}}) = 1$, so by theorem 5.6, $\det(M^{AB})$ counts the number of families of n disjoint paths from A to B . $\square$

Using proposition 5.2, we can generalize corollary 5.8 in a quite interesting way:

Corollary 5.9. *Let G be an acyclic directed graph and let $A = \{A_1, \dots, A_n\}$ and $B = \{B_1, \dots, B_n\}$ be sets of vertices of G . Suppose there exists an index λ such that:*

- *The reverse intersection condition for paths from A to $\{B_1, \dots, B_\lambda\}$ holds.*
- *The intersection condition for paths from A to $\{B_{\lambda+1}, \dots, B_n\}$ holds.*
- *The non-intersection condition for paths from A to $\{B_1, \dots, B_\lambda\}$ and $\{B_{\lambda+1}, \dots, B_n\}$ holds.*

Let $\widetilde{M}^{AB}$ be the matrix we obtain from M^{AB} by changing the sign of the (i, j) -th matrix entry for every pair (i, j) where i is even and $j \leq \lambda$. Then $\det(\widetilde{M}^{AB})$ is the number of families of n disjoint paths from A to B in G .

Remark. Similarly to before, there is a simple way of ensuring that these three intersection conditions hold: Let G be a planar graph and choose $\{B_1, \dots, B_n\}$ to lie in that order on its border. Choose $A_1, \dots, A_n$ such that they form in that order a line through G which cuts it in two parts, one containing the vertices $\{B_1, \dots, B_\lambda\}$ and one containing the vertices $\{B_{\lambda+1}, \dots, B_n\}$. This is essentially the case for the graphs in lemma 5.12, which prepares for applying this corollary (to be exact, its corollary 5.11) to the (half-)reduced AST-graph.

Proof. For any family of disjoint paths from A to B , let I be the set of indices i such that A_i is connected to a B_j with $j \leq \lambda$, and let I' be the set of indices i' such that $A_{i'}$ is connected to a $B_{j'}$ with $j' \geq \lambda + 1$. The reverse intersection condition for paths from A to $\{B_1, \dots, B_\lambda\}$ implies that the intersection condition for paths from A to $\{B_\lambda, \dots, B_1\}$ holds and thus by corollary 5.8, the number of families of λ disjoint paths from $\{A_i \mid i \in I\}$ to $\{B_1, \dots, B_\lambda\}$ is given by $\det(M^{\{A_i \mid i \in I\}\{B_\lambda, \dots, B_1\}}) = \det(M_I^{A\{B_1, \dots, B_\lambda\}} R_\lambda)$.

Analogously, since the intersection condition for paths from A to $\{B_{\lambda+1}, \dots, B_n\}$ holds, by corollary 5.8 the number of families of $n - \lambda$ disjoint paths from $\{A_{i'} \mid i' \in I'\}$ to $\{B_{\lambda+1}, \dots, B_n\}$ is given by $\det(M^{\{A_{i'} \mid i' \in I'\}\{B_{\lambda+1}, \dots, B_n\}}) = \det(M_{I'}^{A\{B_{\lambda+1}, \dots, B_n\}})$. By the non-intersection condition, the paths from A to $\{B_1, \dots, B_\lambda\}$ do not interfere in any way with the paths from A to $\{B_{\lambda+1}, \dots, B_n\}$. Thus, the total number of families of n disjoint paths from A to B in G is given by multiplying these two determinants and summing over all possible pairs (I, I') . This results in the following, which we modify with the help of proposition 5.2 (note that $R_\lambda R_\lambda = \text{Id}_\lambda$):

$$\begin{aligned}
& \sum_{\substack{I \cup I' = \{1, \dots, n\} \\ I \cap I' = \emptyset \\ |I| = \lambda, |I'| = n - \lambda}} \det(M_I^{A\{B_1, \dots, B_\lambda\}} R_\lambda) \cdot \det(M_{I'}^{A\{B_{\lambda+1}, \dots, B_n\}}) = \\
& = \det(V_n M^{A\{B_1, \dots, B_\lambda\}} R_\lambda R_\lambda \quad M^{A\{B_{\lambda+1}, \dots, B_n\}}) = \\
& = \det(V_n M^{A\{B_1, \dots, B_\lambda\}} \quad M^{A\{B_{\lambda+1}, \dots, B_n\}}) = \\
& = \det(\widetilde{M}^{AB})
\end{aligned}$$

□

Let us now consider the case where the starting points and the endpoints of the path families are allowed to vary over given sets. This setting was treated by Stembridge in [52, sections 3 and 4], where he used combinatorial arguments similar to the proof of theorem 5.6. However, for us another route is more practicable: We here only give the trivial consequence of the LGV lemma,

enumerating these path families as sums of minors. In section 7, we then in detail treat ways of summing minors, in particular using results of Okada [47]. For settings in which corollary 5.8 can be applied, this essentially amounts to the same results as Stembridge obtains, as he himself mentions in his work.

Generalizing theorem 5.6 to variable starting points and endpoints is straightforward, but requires to adapt the definition of the permutation induced by a path family. Since we do not need this setting, we avoid that by instead generalizing corollary 5.8:

Corollary 5.10. *Let G be an acyclic directed graph and let $A = \{A_1, \dots, A_m\}$ and $B = \{B_1, \dots, B_n\}$ be sets of vertices of G such that the intersection condition for paths from A to B holds. Then for any $k \geq 0$, $|M^{AB}|_k$ is the number of families of k disjoint paths from A to B in G . Consequently, $\sum_{k \geq 0} |M^{AB}|_k$ is the number of families of arbitrary many disjoint paths from A to B in G .*

Remark. Since $|M^{AB}|_k = 0$ for $k > m$ or $k > n$, the sum $\sum_{k \geq 0} |M^{AB}|_k$ is well defined.

Proof. By corollary 5.8, for any indices $i_1 < i_2 < \dots < i_k$ and $j_1 < j_2 < \dots < j_k$, the number of families of k disjoint paths from $\{A_{i_1}, \dots, A_{i_k}\}$ to $\{B_{j_1}, \dots, B_{j_k}\}$ is the $k \times k$ minor of M^{AB} corresponding to rows $i_1, \dots, i_k$ and columns $j_1, \dots, j_k$. By summing over all possible choices of the indices, we get the result. Note that also $k = 0$ works: There is a unique empty path family and the corresponding minor is $|M^{AB}|_0 = 1$. $\square$

When only letting the endpoints of the path families vary, we can in the same way generalize corollary 5.9:

Corollary 5.11. *Let G be an acyclic directed graph and let $A = \{A_1, \dots, A_m\}$ and $B = \{B_1, \dots, B_n\}$ with $m \leq n$ be sets of vertices of G . Suppose there exists an index λ such that:*

- *The reverse intersection condition for paths from A to $\{B_1, \dots, B_\lambda\}$ holds.*
- *The intersection condition for paths from A to $\{B_{\lambda+1}, \dots, B_n\}$ holds.*
- *The non-intersection condition for paths from A to $\{B_1, \dots, B_\lambda\}$ and $\{B_{\lambda+1}, \dots, B_n\}$ holds.*

Let $\widetilde{M}^{AB}$ be the matrix we obtain from M^{AB} by changing the sign of the (i, j) -th matrix entry for every pair (i, j) where i is even and $j \leq \lambda$. Then $|\widetilde{M}^{AB}|_m$ is the number of families of m disjoint paths from A to B in G .

Proof. This follows from corollary 5.9 in the same way that corollary 5.10 follows from corollary 5.8. Note however that since in every even row of $\widetilde{M}^{AB}$, the signs of the entries in the first λ columns are changed in comparison to M^{AB} , we always have to use all rows for the minor. Otherwise, the signs in the submatrix are not necessarily changed in every second row, but in arbitrary rows. Therefore, we here only consider $m \times m$ minors and families of m disjoint paths. $\square$

Remark. Note that instead of using corollary 5.9, we could also directly derive corollary 5.11 from corollary 5.8 with the help of corollary 5.4.

5.3 Applications of the LGV lemma

The LGV lemma has a wide range of applications, since a large number of combinatorial objects can in some way be interpreted as a family of disjoint paths. The most classical examples are probably the countless kinds of plane partitions and identities related to them, see for instance [28] and [52,

section 5 and onwards]. Another quite usual thing is to use the LGV lemma in combination with MaPaf-bijections. For examples of this, see the papers [20], [12, section 5] and [23, section 2], which were cited in sections 4.2 and 4.3. It is worth noting that by theorem 4.3, the directed graphs coming from MaPaf-bijections are always acyclic, which is necessary for the application of the LGV lemma.

A quite well-known example is the Aztec graph: We have seen that perfect matchings of the Aztec graph of size n correspond to families of n disjoint paths from $A = \{A_1, \dots, A_n\}$ to $B = \{B_1, \dots, B_n\}$ in the corresponding directed graph shown in figure 13b. Clearly, the intersection condition for paths from A to B holds, as these vertices are arranged on the border of the graph as described in the remark after corollary 5.8. So we can apply corollary 5.8 and obtain that the number of perfect matchings of the Aztec graph is $\det(M^{AB})$, where M_{ij}^{AB} denotes the number of paths from A_i to B_j in that graph. This is an important step for instance in [20] and [37, section 5], where the number of perfect matchings of the Aztec graph is calculated.

While the LGV lemma itself can be applied to the path families coming from the MaPaf-bijection in theorem 4.3, its corollaries for variable starting points and endpoints can be applied after using one of the items of corollary 4.5, which deals with Q -partial matchings. However, one always has to make sure that the corresponding intersection conditions hold. Since it is quite clear how the LGV lemma can be used in these cases, we do not go into further detail here, but directly turn our attention to the (half-)reduced AST-graph.

5.4 Application to the AST-graph

By proposition 4.8, in order to enumerate Q -partial matchings of the (half-)reduced AST-graph, we have to count the number of families of n disjoint paths from A to Q in any of the directed graphs corresponding to the configurations introduced in definition 4.6 (see figure 16 and the appendix section A.3). We now want to use the LGV lemma for this. However, in order to directly obtain the number of path families without any sign, we need some sort of intersection condition to hold. This drastically reduces our choice of configurations: Row-wise, column-wise and mixed configurations do not work in general, while snakes, lines and the symmetric configuration do:

Lemma 5.12. *For the (half-)reduced AST-graph of size n , the following holds:*

- (i). *For each snake and line, as well as for the symmetric configuration, the three intersection conditions stated in corollary 5.11 hold in $\widehat{G}$ if λ is chosen as follows: For lines and for the symmetric configuration, take $\lambda := n$. For snakes, take λ such that $Q_\lambda = A_1$.*
- (ii). *For the first snake (figure 16a), the intersection condition holds in $\widehat{G}$ for paths from A to Q .*
- (iii). *For the first line (figure 16b), the intersection condition holds in $\widehat{G}$ for paths from A to $Q \setminus A$.*

Before proving this lemma, let us fix $\widetilde{M}^{AQ}$ for these configurations:

Definition 5.13. When dealing with any of the configurations mentioned in lemma 5.12, then M^{AQ} (and $M^{A \ Q \setminus A}$ if $A \subseteq Q$) denotes the matrix as given by definition 5.5 with respect to the corresponding directed graph $\widehat{G}$. $\widetilde{M}^{AQ}$ then denotes the matrix obtained from M^{AQ} by changing the sign of the (i, j) -th matrix entry for every pair (i, j) where i is even and $j \leq \lambda$, where λ is chosen as in item (i) of lemma 5.12.

Remark. Note that for the (half-)reduced AST-graph of size n , M^{AQ} and $\widetilde{M}^{AQ}$ are always $n \times 2n$ matrices, while $M^{A \ Q \setminus A}$ is an $n \times n$ matrix.

Proof of lemma 5.12. In the first snake and first line, since the A_i lie in order on the border of the graph, the intersection condition for paths from A to Q resp. $Q \setminus A$ holds trivially, as has already been mentioned in the remark after corollary 5.8. Similarly, for snakes the intersection conditions from corollary 5.9 (which are the same as in corollary 5.11) hold since the configuration is as described in the remark after that corollary: The A_i form a line that cuts the graph in two.

For lines, we essentially have the same configuration, but one has to be a little bit more careful: Take for instance a line in the right half, i.e. the k -th line for some $k \geq n+1$: The part that is not clear is that the intersection condition for paths from A to $\{Q_{n+1}, \dots, Q_{2n}\}$ holds. However, we see that the vertices $A_1 = Q_{n+1}, \dots, A_{k-n-1} = Q_{k-1}$ are each only contained in one single path from A to $\{Q_{n+1}, \dots, Q_{2n}\}$, namely the respective trivial path $A_i = Q_{n+i}$. Thus it is actually enough to show that the intersection condition for paths from $\{A_{k-n}, \dots, A_n\}$ to $\{Q_k, \dots, Q_{2n}\}$ holds, which can be seen clearly as before. For lines in the left half, one can proceed analogously. $\square$

Remark. Note that in item (i) of lemma 5.12, for most configurations one could actually take λ as one less or one more and the intersection conditions would still hold. (For instance, for snakes in the left half, one can also take λ such that $A_1 = Q_{\lambda+1}$.) However, this always results in the same matrix $\widetilde{M}^{AQ}$ as the column in which signs are changed by this is e_1 anyway.

Since the intersection conditions hold, we can use the LGV lemma to enumerate the path families corresponding to Q -partial matchings:

Proposition 5.14. *The number of Q -partial matchings of the (half-)reduced AST-graph of size n can be expressed in any of the following ways:*

$$\begin{aligned} \text{For snakes, lines and the symmetric configuration:} & \quad |\widetilde{M}^{AQ}|_n \\ \text{For the first snake:} & \quad |M^{AQ}|_n \\ \text{For the first line:} & \quad \sum_{l \geq 0} |M^{A \ Q \setminus A}|_l \end{aligned}$$

Remark. Note that actually $M^{AQ} = \widetilde{M}^{AQ}$ holds for the first snake.

Proof. By item (i) of proposition 4.8, the number of Q -partial matchings equals the number of n disjoint paths from A to Q in the considered directed graphs $\widehat{G}$. Lemma 5.12 guarantees that the intersection conditions for using corollary 5.11 resp. (for the first snake) corollary 5.10 are fulfilled, giving $|\widetilde{M}^{AQ}|_n$ resp. $|M^{AQ}|_n$.

For the first line, by item (ii) of proposition 4.8, the sought-after number equals the number of arbitrary many disjoint paths from A to Q . Again by lemma 5.12, we can use corollary 5.10, giving $\sum_{l \geq 0} |M^{A \ Q \setminus A}|_l$. $\square$

As a nice consequence of the enumeration of path families as sums of minors, we can translate corollary 4.9, which gives bijections between the path families in the directed graphs $\widehat{G}$, to the minors of the matrices $\widetilde{M}^{AQ}$ and $M^{A \ Q \setminus A}$:

Corollary 5.15. *We have the following identities:*

- (i). *Let $\widetilde{M}^{AQ}$ correspond to any line, snake or the symmetric configuration and let $I \subseteq \{1, \dots, 2n\}$ with $|I| = n$. Let $I' := \{2n+1-i \mid i \in I\}$ be the reflected set of indices. Then*

$$\det\left((\widetilde{M}^{AQ})^I\right) = \det\left((\widetilde{M}^{AQ})^{I'}\right)$$

and this value is the same for each of the $\widetilde{M}^{AQ}$ for the same n .

- (ii). *Let $M^A Q^{\setminus A}$ correspond to the first line and let $I, J \subseteq \{1, \dots, n\}$ with $|I| = |J|$. Then*

$$\det\left((M^A Q^{\setminus A})_I^J\right) = \det\left((M^A Q^{\setminus A})_{\{1, \dots, n\} \setminus J}^{\{1, \dots, n\} \setminus I}\right)$$

and for each $0 \leq l \leq n$, we have $|M^A Q^{\setminus A}|_l = |M^A Q^{\setminus A}|_{n-l}$.

Remark. In particular, from item (ii) we get $\det(M^A Q^{\setminus A}) = |M^A Q^{\setminus A}|_n = |M^A Q^{\setminus A}|_0 = 1$.

Proof. We here use the term ‘path family’ as it is used in section 4.5. By theorem 5.6, $\det\left((\widetilde{M}^{AQ})^I\right)$ is the number of path families with endpoints $\{Q_i \mid i \in I\}$ in the corresponding graph $\widehat{G}$. Thus, item (i) follows directly from corollary 4.9.

For item (ii), there is more work to do since we have not formulated the bijections for the case of families of arbitrary many paths explicitly in section 4.5. By theorem 5.6, $\det\left((M^A Q^{\setminus A})_I^J\right)$ corresponds to path families with starting points $\{A_i \mid i \in I\}$ and endpoints $\{Q_{n+j} \mid j \in J\}$. By item (ii) of corollary 4.5, these path families correspond to Q -partial matchings leaving out exactly the following set of vertices:

$$(A \setminus \{A_i \mid i \in I\}) \cup \{Q_{n+j} \mid j \in J\} = \{A_i \mid i \notin I\} \cup \{Q_{n+j} \mid j \in J\}$$

By vertically reflecting, there is an equal number of Q -partial matchings leaving out exactly the reflected set of vertices. As Q_{n+i} is the vertical reflection of A_i (see figure 16b), this is:

$$\{A_j \mid j \in J\} \cup \{Q_{n+i} \mid i \notin I\} = (A \setminus \{A_j \mid j \notin J\}) \cup \{Q_{n+i} \mid i \notin I\}$$

Again by item (ii) of corollary 4.5, these correspond to path families with starting points $\{A_j \mid j \notin J\}$ and endpoints $\{Q_{n+i} \mid i \notin I\}$, which by theorem 5.6 are enumerated by $\det\left((M^A Q^{\setminus A})_{\{1, \dots, n\} \setminus J}^{\{1, \dots, n\} \setminus I}\right)$. $|M^A Q^{\setminus A}|_l = |M^A Q^{\setminus A}|_{n-l}$ follows by summing over all $I, J \subseteq \{1, \dots, n\}$ with $|I| = |J| = l$. $\square$

5.5 Adapting the directed graphs to directly match the matrices

This section is a preparation for section 5.6. Remember that $\widetilde{M}^{AQ}$ is obtained from M^{AQ} by changing certain signs, while M^{AQ} is the matrix directly counting the numbers of paths in $\widehat{G}$. We here adapt $\widehat{G}$ slightly - giving $\widetilde{G}$ - such that $\widetilde{M}^{AQ}$ directly counts the total weight of paths in $\widetilde{G}$. Also, we introduce directed graphs $\widetilde{G}$ for ‘half-lines’.

Definition 5.16. Let $\widehat{G}$ by any of the directed graphs coming from MaPaf-bijections for the reduced AST-graph. *Extending* the vertex Q_j means introducing a new vertex $\widetilde{Q}_j$ together with an edge from the old Q_j to the new $\widetilde{Q}_j$. (See for instance figure 17c, where $Q_1, \dots, Q_n$ are extended.) Reversing this procedure is called *shortening* the vertex Q_j .

Remark. Note that for the total weight of paths from any A_i to Q_j , it does not make a difference whether the vertex Q_j is extended or shortened, as long as the edge going to the extended Q_j has weight 1.

We can now define our new graphs $\tilde{G}$:

Definition 5.17. For snakes and lines, the directed graph $\tilde{G}$ is obtained from $\hat{G}$ as follows. In figure 17, examples of $\tilde{G}$ are depicted, with the edges with weight -1 marked by dashed lines.

- *Snakes (17a, 17b):* The snake cuts $\hat{G}$ into two parts. In the left part, all vertical and horizontal edges are given weight -1 , except for the horizontal edges between the n -th and $(n+1)$ -th column of red vertices. Instead, the diagonal edges between these two columns are given weight -1 (if they lie left of the snake).
- *Lines (17c, 17e):* The vertices $Q_1, \dots, Q_n$ in $\hat{G}$ are extended. For each j , the edge going to the extended Q_j is given weight $(-1)^{n-j}$. The line cuts $\hat{G}$ into two parts. In the left part, all vertical and diagonal edges are given weight -1 , except for edges going to some extended Q_j .

The graph $\tilde{G}$ and the matrix $\tilde{M}^{AQ}$ corresponding to the k -th line are denoted $\tilde{G}_k^l$ and $\tilde{M}_k^l$, the ones corresponding to the first snake are denoted $\tilde{G}_1^s$ and $\tilde{M}_1^s$.

Remark. Note that in certain special cases, namely the first line, the first snake and their vertical reflections, one of the two parts into which the graph is cut does not contain any edges. However, it is still clear how to apply the above definitions. For instance, for the first snake (figure 16a), we just have $\tilde{G} = \hat{G}$ and for the $2n$ -th line, all vertical and diagonal edges are given weight -1 .

One can easily see that the matrices $\tilde{M}^{AQ}$ now directly count the total weights of paths in the directed graphs $\tilde{G}$:

Lemma 5.18. *For all snakes and lines in the reduced AST-graph, the total weight of paths from A_i to Q_j in $\tilde{G}$ is given by $\tilde{M}_{ij}^{AQ}$.*

Proof. Remember that by definition 5.13, $\tilde{M}^{AQ}$ is obtained from M^{AQ} by changing the sign of the (i, j) -th matrix entry for every pair (i, j) where i is even and $j \leq \lambda$, where λ is chosen as in item (i) of lemma 5.12. Since clearly the unweighted number of paths from A_i to Q_j does not change between $\hat{G}$ and $\tilde{G}$, we only have to show that in $\tilde{G}$, paths from A_i to Q_j have weight $(-1)^{i-1}$ for $j \leq \lambda$, and weight 1 for $j > \lambda$. For $j > \lambda$, this is clear since these paths only lie on the right side of the snake or line, where all edges still have weight 1. For $j \leq \lambda$, we now only work in the part of $\tilde{G}$ left of the snake or line. Consider figure 17 for illustration.

Note that from each A_i , there goes out a unique path in the left part that only consists of edges with weight 1 (edges going to an extended Q_j are excluded). Call that path starting at A_i *i-path* for now. In some cases, the *i-path* is the trivial path with no edges. Notice that all edges in the left part which do not lie on an *i-path* connect two *i-paths* and have weight -1 (except for edges going to an extended Q_j). Thus, a path switches weight every time it goes from one *i-path* to another. Therefore, a path starting at A_{i_1} and ending on the i_2 -path has weight $(-1)^{i_1-i_2}$.

For snakes, since all Q_j in the left part lie on the 1-path, this means that paths from A_i to Q_j with $j \leq \lambda$ have weight $(-1)^{i-1}$, which is what we want to show. For lines, note that the extended Q_j can only be reached from the $(n-j+1)$ -path, to which it is connected by an edge with weight $(-1)^{n-j}$. Thus, a path from A_i to Q_j with $j \leq \lambda$ has weight $(-1)^{i-(n-j+1)} \cdot (-1)^{n-j} = (-1)^{i-1}$. $\square$

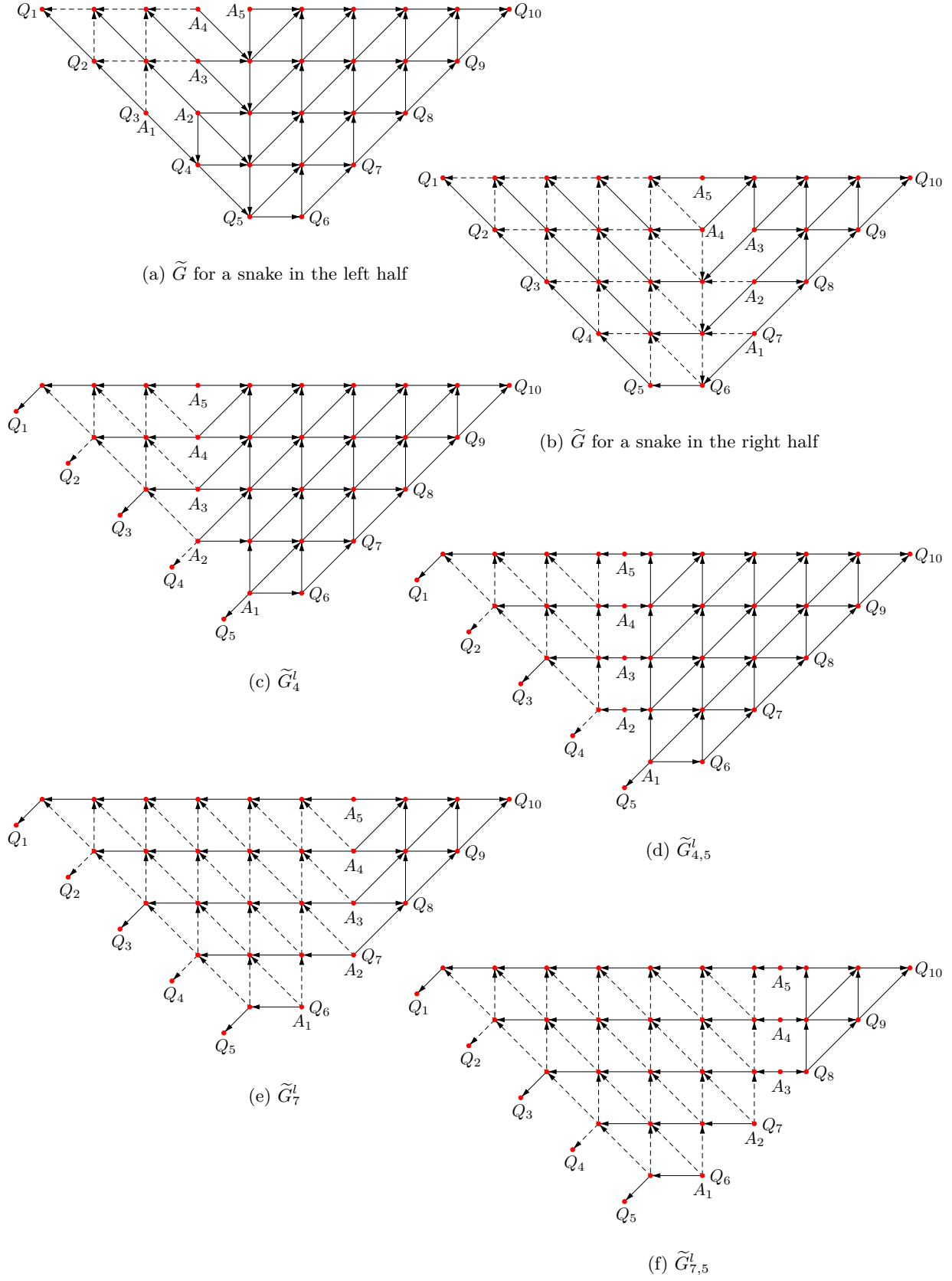


Figure 17: Examples of $\tilde{G}$ (definitions 5.17 and 5.19), edges with weight -1 are marked by dashed lines

We now introduce a new configuration which acts as an in-between step between any two lines. Note that this configuration does not come from any MaPaf-bijection, but is directly defined in terms of $\tilde{G}$:

Definition 5.19. For any index k with $1 \leq k < 2n$, the directed graph $\tilde{G}$ corresponding to the $(k + 1/2)$ -th half-line is denoted $\tilde{G}_{k+1/2}^l$ and obtained from $\tilde{G}_k^l$ by the following steps (see figures 17d and 17f):

- Delete all edges between the k -th and $(k + 1)$ -th column of red vertices.
- Add vertical edges in the k -th column, which go up and have weight -1 .
- For each pair of red vertices in the k -th and $(k + 1)$ -th column which are in the same row, add a new red vertex in between them, from which an edges goes to each of them.
- Let the new red vertices take over the labels A_i from their left neighbours.

The matrix $\tilde{M}^{AQ}$ corresponding to a half-line is denoted $\tilde{M}_{k+1/2}^l$ and defined such that lemma 5.18 holds true also for half-lines.

Remark. Note that analogously to lines, also in $\tilde{G}_{k+1/2}^l$ paths from A_i to Q_j have weight $(-1)^{i-1}$ if $j \leq n$ and weight 1 otherwise. This can be seen in the same way that lemma 5.18 is proved.

5.6 Matrix row operations and graph operations

The objective of this section is to show that all the $n \times 2n$ matrices $\tilde{M}^{AQ}$ corresponding to snakes, lines and half-lines can actually be obtained from each other by elementary matrix row operations. Since the $n \times n$ minors of an $n \times 2n$ matrix use all rows, they are invariant under row operations (but not under column operations), so this implies that $|\tilde{M}^{AQ}|_n$ gives the same value for each of these $\tilde{M}^{AQ}$, which we already know for snakes and lines by proposition 5.14. Let us first introduce a convenient notation for the elementary matrix row operations we use:

Definition 5.20. For $1 \leq a, b \leq n$ with $a \neq b$, we define the following operators:

$$\begin{aligned} O_{a,b}^+ &:= I_n + e_a e_b^\top \\ O_{a,b}^- &:= I_n - e_a e_b^\top \end{aligned}$$

where e_a and e_b are unit vectors in n dimensions. For $a \in \{0, n + 1\}$ or $b \in \{0, n + 1\}$, we set $O_{a,b}^+ := O_{a,b}^- := I_n$.

Remark. Applying the operator $O_{a,b}^+$ or $O_{a,b}^-$ to an $n \times 2n$ matrix M (i.e. multiplying M from the left with $O_{a,b}^+$ or $O_{a,b}^-$) corresponds to adding resp. subtracting row b to/from row a in M .

The following simple properties of these operators come in use later:

Lemma 5.21. For any a, b, c, d with $a \neq b$ and $c \neq d$, we have the following:

$$\begin{aligned} V_n O_{ab}^+ V_n &= \begin{cases} O_{ab}^+ & a \equiv b \pmod{2} \\ O_{ab}^- & a \not\equiv b \pmod{2} \end{cases} & V_n O_{ab}^- V_n &= \begin{cases} O_{ab}^- & a \equiv b \pmod{2} \\ O_{ab}^+ & a \not\equiv b \pmod{2} \end{cases} \\ (O_{a,b}^+)^{-1} &= O_{a,b}^- & (O_{a,b}^-)^{-1} &= O_{a,b}^+ \end{aligned}$$

If $a \neq d$ and $b \neq c$, then $O_{a,b}^\pm$ and $O_{c,d}^\pm$ commute.

Proof. Since multiplying a matrix from the left with V_n changes the signs in every even row and multiplying from the right with V_n changes the signs in every even column, doing both changes the sign of all entries having one even and one odd index, giving the first line of the lemma.

The remaining properties can be seen by considering the operators as elementary matrix row operations as remarked above: The inverse operation of subtracting row b from row a is adding row b to row a , giving the second line of the lemma. For the commutativity, note that if a, b, c, d are all distinct, then $O_{a,b}^+$ and $O_{c,d}^+$ commute since they involve different rows. But also if $a = c$ or $b = d$ they still commute, since it does not matter that both operators add something to the same row or add the same row to something. Only if $a = d$ or $b = c$, the order of operations matters. $\square$

We now translate matrix row operations to changes made in the corresponding graphs $\tilde{G}$. First however, we introduce these changes as local operations in general directed graphs:

Lemma 5.22. *Let G be any directed graph containing vertices $\{A_1, \dots, A_n\}$ and $\{Q_1, \dots, Q_m\}$ and let M be the corresponding $n \times m$ matrix such that M_{ij} is the total weight of paths from A_i to Q_j in G . Then operating on G locally as it is shown in the diagrams in figure 18 (where dashed lines mark edges with weight -1) corresponds to applying the operator specified in the diagram to M , if the following conditions hold for the surrounding graph G :*

- No vertex, edge or label of a vertex in G that is not depicted in the diagram changes by the operation. In particular, the vertices that are introduced or deleted in the diagrams in figure 18c cannot have any incident edges other than the depicted.
- No vertex that is labeled as some A_p before or after the operation has any label from $\{A_1, \dots, A_m\}$ or any incoming edge that is not depicted.

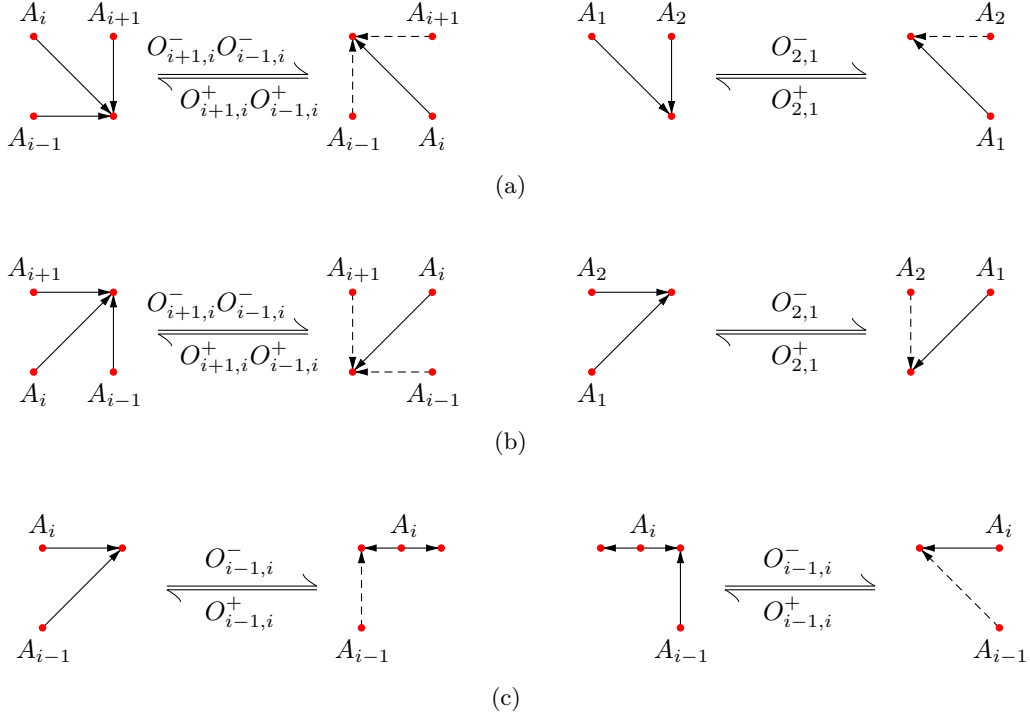


Figure 18: Lemma 5.22

Proof. Applying any of the operators $O_{a,b}^\pm$ to M comes down to applying $O_{a,b}^\pm$ to each column of M separately. Since each column corresponds to an Q_j , it is enough to prove the lemma for a single Q_j . Thus let $m := 1$. When referring to ‘paths from A_p ’ we now mean ‘paths from A_p to Q_1 .’

Note that the conditions make sure our operations only change the numbers of paths from A_p for vertices A_p which are labeled in the diagram: In figures 18a and 18b, each vertex is labeled as an A_p either before or after the operation. Thus, by the second condition of the lemma, no path coming from some other A_p can pass by any of the vertices depicted, and so the numbers of paths from these A_p cannot change by the operation. In figure 18c, paths coming from some non-depicted A_p can pass be the rightmost vertex in the left diagram resp. the leftmost vertex in the right diagram. However, since these vertices only have incoming edges in the diagram, the numbers of paths from A_p cannot change by the operation in this case either. Furthermore, by the same argument we see that also for the A_p labeled in the diagram, the paths from A_p can only be changed in their beginning by the operations: We do not have to worry about the paths going out of the diagram, and then coming back to the diagram and being changed there by the operations.

We now prove the left diagram of figure 18a, all the others can be proved analogously. By what we have already seen, we know that the operation in either direction can only change the values of $M_{i-1,1}$, $M_{i,1}$ and $M_{i+1,1}$. Let x denote the number of paths to Q_1 starting at the top-left vertex of the diagram and going out of the diagram, meaning that their first edge is not depicted in the diagram. By our second condition, this implies that such paths cannot come back to the diagram later and thus the number x is not changed by the operation. Analogously, let y denote the number of paths to Q_1 starting at the bottom-right vertex of the diagram.

Now consider figure 19, where the operation is split up into several steps. In the first step, A_i is moved. However, this does not change $M_{i,1}$ since in both pictures, we clearly have $M_{i,1} = x + y$. Also, $M_{i-1,1}$ and $M_{i+1,1}$ are not changed either. In the second step, the edges coming from A_{i-1} are changed. Thus $M_{i,1}$ and $M_{i+1,1}$ stay the same, while $M_{i-1,1}$ changes: The deletion of the edge going from A_{i-1} to the right corresponds to subtracting y from $M_{i-1,1}$. The addition of the edge going from A_{i-1} up with weight -1 corresponds to adding x to $M_{i-1,1}$, but with negative sign. Thus in total $M_{i-1,1}$ becomes $M_{i-1,1} - y + (-x) = M_{i-1,1} - M_{i,1}$ by the operation, since $M_{i,1} = x + y$. This indeed corresponds to applying the operator $O_{i-1,i}^-$ to M .

Similarly, the third step changing the edges coming from A_{i+1} corresponds to applying the operator $O_{i+1,i}^-$ to M and the fourth step moving A_i does not change any value of M . Analogously, the steps going in the other direction can be explained. Thus the operations on the graphs indeed correspond to applying the operators as given in the diagrams to M . $\square$

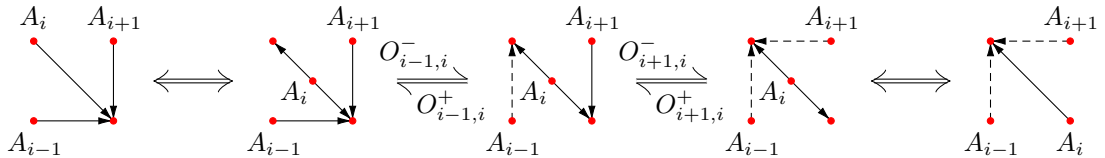


Figure 19: Proof of lemma 5.22

Now note that the operations in figure 18a can be used to go from one snake in the left half of the reduced AST-graph to another: Indeed, when looking at $\tilde{G}$ for such a snake (figure 17a), we see that if A_i can be moved diagonally down to obtain another snake (this is the case if A_{i-1} is below

A_i and A_{i+1} is right of A_i), then $\tilde{G}$ indeed only changes locally, and exactly as depicted in figure 18a. For instance, in figure 17a, the vertices A_4 and A_1 can be moved in that way (where for A_1 we use the right diagram of figure 18a, which is the special case for $i = 1$). In fact, we get:

Lemma 5.23. *Any two snakes in the left half of the reduced AST-graph can be obtained from each other by operations as described in figure 18a of lemma 5.22. Thus, any two matrices $\tilde{M}^{AQ}$ corresponding to snakes in the left half can be obtained from each other by elementary matrix row operations. The same holds for snakes in the right half by using the operations in figure 18b.*

Proof. Clearly, the only snake in the left half which does not contain an A_i that can be moved diagonally down (i.e. an A_i that has A_{i-1} (if existent) below it and A_{i+1} right of it), is the snake that is at the same time the n -th line. Thus, when starting with any snake and repeatedly moving A_i 's diagonally down, we eventually come to the n -th line. Since we can also reverse the operations in order to move A_i 's diagonally up, it follows that any two snakes in the left half can be reached from each other. Snakes in the right half work out analogously. $\square$

Similarly, by using the operations from figure 18c of lemma 5.22, we can go from one (half-)line to another (half-)line, giving the following:

Lemma 5.24. *Any two lines or half-lines can be obtained from each other by operations as described in figure 18c of lemma 5.22. Thus, any two matrices $\tilde{M}_k^l$ can be obtained from each other by elementary matrix row operations.*

Proof. For each integer k with $1 \leq k \leq 2n - 1$, we can go from the k -th line to the $(k + 1/2)$ -th half-line and from the $(k + 1/2)$ -th half-line to the $(k + 1)$ -th line by operations from figure 18c of lemma 5.22. For details, see the proof of proposition 5.30 in the next section. $\square$

Combining lemmata 5.23 and 5.24 now gives the following crucial fact:

Corollary 5.25. *All the matrices $\tilde{M}^{AQ}$ corresponding to snakes or (half-)lines can be obtained from each other by elementary matrix row operations.*

Proof. By lemma 5.23, the fact that all $\tilde{M}^{AQ}$ can be obtained from each other holds for all snakes in the left and in the right half of the reduced AST-graph separately. By lemma 5.24, it also holds for all (half-)lines. Since the n -th line is a snake in the left half and the $(n + 1)$ -th line is a snake in the right half, in total we can obtain every $\tilde{M}^{AQ}$ for a snake or (half-)line from every other such $\tilde{M}^{AQ}$. $\square$

Consequently, proposition 5.14 also holds for half-lines:

Proposition 5.26. *The number of Q -partial matchings of the (half-)reduced AST-graph of size n can be expressed as $|\tilde{M}^{AQ}|_n$ for any $\tilde{M}^{AQ}$ corresponding to a half-line, i.e. any $\tilde{M}^{AQ} := \tilde{M}_{k+1/2}^l$.*

Proof. As the sum of $n \times n$ minors of $n \times 2n$ matrices is invariant under elementary matrix row operations, this follows directly from proposition 5.14 together with corollary 5.25. $\square$

Remark. Note that we could even take the half-lines back to the level of path families now: Let $\hat{G}$ be the directed graph obtained by dropping the weights in the graph $\hat{G}_{k+1/2}^l$ (and optionally shortening the vertices $Q_1, \dots, Q_n$). Then the number of Q -partial matchings of the (half-)reduced AST-graph of size n equals the number of families of n disjoint paths from A to Q in $\hat{G}$.

Proof of the remark. We clearly see that $\widehat{G}$ satisfies the intersection conditions stated in corollary 5.11 for $\lambda := n$ (compare the proof of lemma 5.12). And we already mentioned that exactly the paths from A_i to Q_j in $\widehat{G}_{k+1/2}^l$ for i even and $j \leq n$ have weight -1 (see the remark after definition 5.19). Thus, the matrix $\widetilde{M}^{AQ}$ when applying corollary 5.11 to $\widehat{G}$ is exactly the $\widetilde{M}^{AQ} = \widetilde{M}_{k+1/2}^l$ corresponding to $\widehat{G}_{k+1/2}^l$. Thus, the number of families of n disjoint paths from A to Q in $\widehat{G}$ is $|\widetilde{M}^{AQ}|_n$, which by proposition 5.26 is the number of Q -partial matchings of the (half-)reduced AST-graph of size n . $\square$

There are two reasons we introduced half-lines: Firstly, having them as intermediate steps makes the formulation of the operations between lines easier. And secondly, proposition 5.26 is particularly interesting and useful in the case of the $(n + 1/2)$ -th half-line (see figure 37a): Since $\widehat{G}_{n+1/2}^l$ is vertically symmetric (when ignoring the weights of the edges), also $\widetilde{M}_{n+1/2}^l$ is vertically symmetric except for the signs (see figure 22). Therefore, $\widetilde{M}_{n+1/2}^l$ plays an important role in section 7.4.

5.7 Product expressions for the operators

In this section, we explicitly determine the operators for going from the first snake to the n -th line (which is also a snake), as well as for going from one (half-)line to the next one. Apart from being interesting in themselves, these operators give us a way of determining the matrix entries of the $\widetilde{M}^{AQ}$ in section 6. We first need to introduce some notation for products:

Definition 5.27. When writing a product using the product symbol $\prod$ with the lower bound being greater than the upper bound, this means the factors should be multiplied in reverse order. That is, for $a \leq b$ we have

$$\prod_{i=a}^b x_i = x_a x_{a+1} \cdots x_{b-1} x_b \qquad \prod_{i=b}^a x_i := x_b x_{b-1} \cdots x_{a+1} x_a$$

Remark. Of course, one could avoid using that notation by writing for instance $\prod_{i=0}^{b-a} x_{b-i}$ for $x_b x_{b-1} \cdots x_{a+1} x_a$. However, it seems that using the notation as stated above is less confusing in our case.

Now we can specify the total operator transforming the first snake to the n -th line:

Proposition 5.28. Consider the reduced AST-graph of size n . For $n = 1$, we have $\widetilde{M}_1^s = \widetilde{M}_1^l$. For $n \geq 2$, the operator taking $\widetilde{M}_1^s$ to $\widetilde{M}_n^l$ can be written as a product of operators of the form $O_{i+1,i}^- O_{i-1,i}^-$ in any way that satisfies the following two conditions:

- For any integer $1 \leq i \leq n-1$, the factor $O_{i+1,i}^- O_{i-1,i}^-$ appears exactly $n-i$ times.
- For any integers $2 \leq i \leq n-1$ and $1 \leq r \leq n-i$, the r -th appearance of the factor $O_{i+1,i}^- O_{i-1,i}^-$ lies somewhere between the r -th and the $(r+1)$ -th appearance of the factor $O_{i,i-1}^- O_{i-2,i-1}^-$.

In particular, we have the following two expressions for $n \geq 2$:

$$\begin{aligned} \widetilde{M}_n^l &= \prod_{j=n-1}^1 \prod_{i=1}^j O_{i+1,i}^- O_{i-1,i}^- \widetilde{M}_1^s \\ \widetilde{M}_n^l &= \prod_{j=1}^{n-1} \prod_{i=j}^1 O_{i+1,i}^- O_{i-1,i}^- \widetilde{M}_1^s \end{aligned}$$

Yet other ways of writing the operator for $n \geq 2$ are the following:

$$\begin{aligned}\widetilde{M}_n^l &= \prod_{j=n-1}^1 \left(\prod_{i=1}^j O_{i+1,i}^- \prod_{i=1}^j O_{i-1,i}^- \right) \widetilde{M}_1^s \\ \widetilde{M}_n^l &= \prod_{j=1}^{n-1} \left(\prod_{i=j}^1 O_{i-1,i}^- \prod_{i=j}^1 O_{i+1,i}^- \right) \widetilde{M}_1^s\end{aligned}$$

In order to go from the $(n+1)$ -th line to the vertical reflection of the first snake, one can use the exactly same operators.

Remark. Note that the condition in the proposition is symmetric: It does not matter whether we start counting appearances of $O_{i+1,i}^- O_{i-1,i}^-$ from the left or from the right. Since also $O_{i+1,i}^-$ commutes with $O_{i-1,i}^-$, one can reverse the order of all single operators in an allowed order and get another allowed order.

Proof. The case for $n = 1$ is clear, as in this case $\widetilde{G}_1^s$ and $\widetilde{G}_n^l = \widetilde{G}_1^l$ are essentially the same (except for the fact that in $\widetilde{G}_1^l$, the vertex Q_1 is extended). For $n \geq 2$, we build upon the observations leading to lemma 5.23, as well as its proof:

Clearly, each A_i has to be moved diagonally down $n - i$ times in order to go from the first snake to the n -th line. The only condition on the order of the moves is that they are all legal and leading to snakes again. Thus, A_i may be moved if we have a snake in which A_{i-1} (if existent) is below A_i and A_{i+1} is right of A_i . This is exactly the case if A_{i-1} has already been moved one time more than A_i and A_{i+1} has been moved the same number of times as A_i . Thus, for each $i > 1$, the r -th move of A_i has to happen in-between the r -th and the $(r+1)$ -th move of A_{i-1} . Since by lemma 5.22 (figure 18a), moving A_i in that way corresponds to applying the operator $O_{i+1,i}^- O_{i-1,i}^-$ (also for $i = 1$ as $O_{0,1}^- = I_n$), we get the result. (With this derivation, we actually have to count appearances of $O_{i+1,i}^- O_{i-1,i}^-$ from the right, but we already noted that the condition is symmetric anyway.)

The two orders written down explicitly come from doing the moves in the following order and in its reverse:

$$A_1, A_2, A_1, A_3, A_2, A_1, \dots, A_{n-1}, A_{n-2}, \dots, A_1$$

The last two expressions follow by lemma 5.29. Finally, to see that going from the $(n+1)$ -th line to the vertical reflection of the first snake gives the same operators, note that we just have to move A_i 's diagonally up (from left to right) this time, with exactly the same constraints on the order as before. And by figure 18b of lemma 5.22, these moves give exactly the same operators. $\square$

The following lemma shows that the different product expressions given in proposition 5.28 are not really different because of the commutativity of the $O_{a,b}^-$. Indeed, the operator taking $\widetilde{M}_1^s$ to $\widetilde{M}_n^l$ has to be unique, as we prove by another method in lemma 6.6.

Lemma 5.29. *All the different product expressions for the operator in proposition 5.28 can be obtained from each other by repeatedly switching single neighbouring operators which commute with each other.*

Proof. By lemma 5.21, $O_{i+1,i}^- O_{i-1,i}^-$ commutes with all $O_{i'+1,i'}^- O_{i'-1,i'}^-$ except for $i' = i - 1$ or $i' = i + 1$. And these two operators are exactly the ones in relation to which the position of the

$O_{i+1,i}^- O_{i-1,i}^-$ is prescribed by the proposition. Thus indeed, every allowed order can be reached from every other allowed order by switching neighbouring operators which commute.

The last two expressions can be obtained from the respective above ones by in the inner product moving all the $O_{i+1,i}^-$ to the one side and all the $O_{i-1,i}^-$ to the other side. By careful consideration, we see that in this process, an operator $O_{i+1,i}^-$ only crosses operators $O_{i'-1,i'}^-$ with $i' \leq i$, and with these it indeed commutes by lemma 5.21. $\square$

Remark. When the operators for going from $\widetilde{M}_1^s$ to $\widetilde{M}_n^l$ are considered as ordered in the form $\prod_{j=n-1}^1 \left(\prod_{i=1}^j O_{i+1,i}^- \prod_{i=1}^j O_{i-1,i}^- \right)$, the single $O_{a,b}^-$ can still be translated to operations in the graph performed in that other order: Each A_i is first moved diagonally down with $O_{i-1,1}^-$ by only half a step (introducing a new vertex), and then moved down again with $O_{i+1,1}^-$. For the last expression in proposition 5.28, something similar can be done. We do not go into any more detail here, since this is quite similar to what we are in the following doing with lines and half-lines.

We close this section by specifying the operators taking a (half-)line to the next (half-)line:

Proposition 5.30. *Consider the reduced AST-graph of size n for $n \geq 1$. For any integer k with $2 \leq k \leq 2n-1$, we set $d_k := \lceil |n + 1/2 - k| \rceil$. Then we have:*

$$\begin{aligned} \widetilde{M}_{k+1/2}^l &= \prod_{i=d_k+1}^n O_{i-1,i}^- \widetilde{M}_k^l & \widetilde{M}_{1,5}^l &= \widetilde{M}_1^l \\ \widetilde{M}_k^l &= \prod_{i=n}^{d_k+1} O_{i-1,i}^- \widetilde{M}_{k-1/2}^l & \widetilde{M}_{2n}^l &= \widetilde{M}_{2n-1/2}^l \end{aligned}$$

Proof. This proof heavily depends on the illustrations in section A.4 of the appendix. It is clear that $\widetilde{M}_{1,5}^l = \widetilde{M}_1^l$ and $\widetilde{M}_{2n}^l = \widetilde{M}_{2n-1/2}^l$. So let now $2 \leq k \leq 2n-1$. Note that d_k is chosen such that for the k -th line, A_{d_k} is the first vertex of the line meeting the border when going down from A_n (that is, A_{d_k} is the vertex in which the k -line takes a turn).

For going from the k -th line to the $(k+1/2)$ -th half-line, we use the left diagram in figure 18c of lemma 5.22, see figures 34 ($k=4$) and 35 ($k=7$): When moving $A_n, A_{n-1}, \dots$ in that order to the right, the conditions in lemma 5.22 are indeed satisfied for each of these operations. Thus, the operator $O_{i-1,i}^-$ is applied for (nearly) each A_i that is moved.

However, we need to distinguish two cases: For $k \leq n$ (figure 34), the lowest A_i that is moved is A_{d_k} . But, as can be seen in the figure, this A_i is moved without giving us any operator. Thus the total operator for going from the k -th line to the $(k+1/2)$ -th half-line in that case is indeed $\prod_{i=d_k+1}^n O_{i-1,i}^-$. For $k \geq n+1$ (figure 35), the lowest vertex moved is A_{d_k+1} instead of A_{d_k} , but A_{d_k+1} already gives us an operator, and so also in this case we have $\prod_{i=d_k+1}^n O_{i-1,i}^-$.

Going from half-lines to lines works the same way, this time using the right diagram in figure 18c of lemma 5.22, see figures 36 ($k=5$) and 37 ($k=6$). The A_i are now moved starting from the bottom and going up until A_n . For $k \leq n$ (figure 36), the first vertex moved is A_{d_k+1} . For $k \geq n+1$ (figure 37), the first vertex moved is A_{d_k} , but A_{d_k+1} is the first vertex also giving an operator when moved. So we indeed have $\prod_{i=n}^{d_k+1} O_{i-1,i}^-$ in both cases. $\square$

6 Schröder numbers: Filling the matrices with values

All the paths in the different directed graphs $\widehat{G}$ and $\widetilde{G}$ coming from snakes or (half-)lines can be seen as Schröder paths, where we weaken the usual conditions to allow the paths to start and end also above the x -axis. The purpose of this section is twofold: On the one hand, our study of Schröder paths gives us ways of determining the entries of some of the matrices $\widetilde{M}^{AQ}$ counting the paths in $\widetilde{G}$. On the other hand, the operations on these graphs and matrices, which we introduced in sections 5.6 and 5.7, give us back some interesting and useful properties of the Schröder paths.

In section 6.1, we introduce four matrices, which we call ‘Schröder matrices’, counting the numbers of Schröder paths in certain settings. Section 6.2 then expresses some of the matrices $\widetilde{M}^{AQ}$ in terms of these Schröder matrices, as well as (following from that) some of the operators linking them to each other.

Section 6.3 collects several properties of the Schröder matrices: From their definition, we can easily find linear recurrences for their entries. And from the operators in sections 5.6 and 5.7, we can deduce relations in-between them, as well as express them directly in terms of the elementary operators introduced in definition 5.20. Finally, section 6.4 provides a general generating function for the numbers of Schröder paths.

6.1 Schröder paths and Schröder matrices

Schröder paths are a quite well-known object in discrete mathematics. We can define them as follows:

Definition 6.1. A *Schröder path* is a path between lattice points in the cartesian plane, only using steps from the set $\{(1, 1), (1, -1), (2, 0)\}$ and never going below the x -axis. For nonnegative integers m, a, b , let $S(m, a, b)$ denote the number of Schröder paths starting at $(0, a)$ and ending at (m, b) .

Remark. Normally, Schröder paths are required to start and end on the x -axis. The number of Schröder paths from $(0, 0)$ to $(2m, 0)$ is then usually denoted S_m and called *m-th Schröder number*. However, we here need this more general definition.

In order to express the matrices we are dealing with in terms of numbers of Schröder paths, we introduce certain matrices of Schröder numbers:

Definition 6.2. Consider lattice points $D_i, D'_i, H_i, V_i, V'_i$ for $i \in \{1, \dots, n\}$ each, arranged as shown in figure 20, where all indicated edges are steps from the set $\{(1, 1), (1, -1), (2, 0)\}$, i.e. the set of steps allowed for Schröder paths. Then we define the $n \times n$ matrices S_n^{DH} , S_n^{VH} , $S_n^{V'H}$ and $S_n^{DD'}$, which we refer to as *Schröder matrices*, as follows:

$$\begin{aligned} S_n^{DH}(i, j) &:= \text{Number of Schröder paths from } D_i \text{ to } H_j \\ S_n^{VH}(i, j) &:= \text{Number of Schröder paths from } V_i \text{ to } H_j \\ S_n^{V'H}(i, j) &:= \text{Number of Schröder paths from } V'_i \text{ to } H_j \\ S_n^{DD'}(i, j) &:= \text{Number of Schröder paths from } D_i \text{ to } D'_j \end{aligned}$$

We also write $S_{i,j}^{DH}$ for $S_n^{DH}(i, j)$, and similarly $S_{i,j}^{VH}$ and $S_{i,j}^{V'H}$, but not $S_{i,j}^{DD'}$.

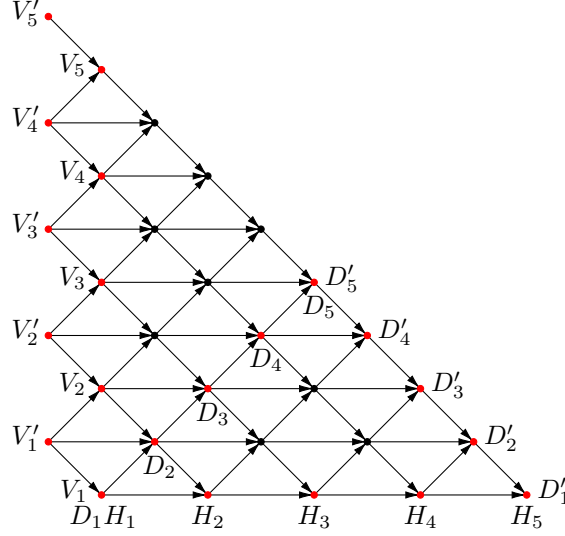


Figure 20: The arrangement of vertices for definition 6.2, here for $n = 5$

Remark. Note that there would be many more ways to choose vertices in a Schröder lattice, define Schröder matrices and derive interesting properties of them. Indeed, the term ‘Schröder matrix’ is sometimes used for other - similar - matrices. However, in order to keep our work within reasonable limits, we constrict ourselves to these four, of which three are practicable to express the $\widetilde{M}^{AQ}$ we are working with, and S_n^{VH} is included to have at least a little bit of completeness. Only at some points we mention a fifth Schröder matrix $S_n^{V'D}$ (defined analogously), but we do not go into detail about it.

Each entry of one of these Schröder matrices is the number of Schröder paths from $(0, a)$ to (m, b) for the correctly chosen m, a, b . Consulting figure 20, it is easy to read off the right values. We obtain the following:

Lemma 6.3. *For $0 \leq i, j \leq n$, we have:*

$$\begin{aligned} S_n^{DH}(i, j) &= S(2j - i - 1, i - 1, 0) \\ S_n^{VH}(i, j) &= S(2j - 2, 2i - 2, 0) \\ S_n^{V'H}(i, j) &= S(2j - 1, 2i - 1, 0) \\ S_n^{DD'}(i, j) &= S(2n - i - j, i - 1, j - 1) \end{aligned}$$

In sections 6.3 and 6.4, we give numerous ways of calculating the entries of the Schröder matrices. Here, we only show some of these matrices in figure 21. Incidentally, this also illustrates the following facts, which can easily be derived from definition 6.2 or lemma 6.3:

Remark.

- The entries of S_n^{DH} , S_n^{VH} and $S_n^{V'H}$ do not depend on n . This is the reason we often write them as $S_{i,j}^{DH}$ etc.
- The rows of S_n^{VH} resp. $S_n^{V'H}$ are parts of the odd resp. even rows of S_{2n}^{DH} , shifted such that the 1's lie again on the main diagonal.
- $S_n^{DD'}$ is a symmetric matrix.
- For $n < m$, the bottom-right triangles of size n in $S_n^{DD'}$ and $S_m^{DD'}$ coincide.

$$\begin{aligned}
S_9^{DH} &= \begin{pmatrix} 1 & 2 & 6 & 22 & 90 & 394 & 1806 & 8558 & 41586 \\ & 1 & 4 & 16 & 68 & 304 & 1412 & 6752 & 33028 \\ & & 1 & 6 & 30 & 146 & 714 & 3534 & 17718 \\ & & & 1 & 8 & 48 & 264 & 1408 & 7432 \\ & & & & 1 & 10 & 70 & 430 & 2490 \\ & & & & & 1 & 12 & 96 & 652 \\ & & & & & & 1 & 14 & 126 \\ & & & & & & & 1 & 16 \\ & & & & & & & & 1 \end{pmatrix} \\
S_6^{VH} &= \begin{pmatrix} 1 & 2 & 6 & 22 & 90 & 394 \\ & 1 & 6 & 30 & 146 & 714 \\ & & 1 & 10 & 70 & 430 \\ & & & 1 & 14 & 126 \\ & & & & 1 & 18 \\ & & & & & 1 \end{pmatrix} \quad S_6^{V'H} = \begin{pmatrix} 1 & 4 & 16 & 68 & 304 & 1412 \\ & 1 & 8 & 48 & 264 & 1408 \\ & & 1 & 12 & 96 & 652 \\ & & & 1 & 16 & 160 \\ & & & & 1 & 20 \\ & & & & & 1 \end{pmatrix} \\
S_6^{DD'} &= \begin{pmatrix} 394 & 304 & 146 & 48 & 10 & 1 \\ 304 & 236 & 116 & 40 & 9 & 1 \\ 146 & 116 & 62 & 25 & 7 & 1 \\ 48 & 40 & 25 & 13 & 5 & 1 \\ 10 & 9 & 7 & 5 & 3 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad S_8^{DD'} = \begin{pmatrix} 8558 & 6752 & 3534 & 1408 & 430 & 96 & 14 & 1 \\ 6752 & 5340 & 2820 & 1144 & 360 & 84 & 13 & 1 \\ 3534 & 2820 & 1538 & 664 & 230 & 61 & 11 & 1 \\ 1408 & 1144 & 664 & 320 & 129 & 41 & 9 & 1 \\ 430 & 360 & 230 & 129 & 63 & 25 & 7 & 1 \\ 96 & 84 & 61 & 41 & 25 & 13 & 5 & 1 \\ 14 & 13 & 11 & 9 & 7 & 5 & 3 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}
\end{aligned}$$

Figure 21: Examples of Schröder matrices

Proof of the fourth item of the remark. Note that the starting points and endpoints of the paths corresponding to these entries lie such that they are actually ‘Delannoy paths’: Schröder paths that may cross the x -axis. This is the case because the x -axis is far enough away that the paths cannot cross it anyway, and so the exact distance from the x -axis does not make a difference. $\square$

6.2 Application to the AST-graph

In addition to V_n and R_n already defined, we need two more auxiliary matrices:

Definition 6.4. We define the $n \times n$ matrices T_n^+ and T_n^- as follows:

$$T_n^+ := \begin{cases} 1 & i = j \\ 1 & i + 1 = j \\ 0 & \text{else} \end{cases} \quad T_n^- := \begin{cases} 1 & i = j \\ -1 & i + 1 = j \\ 0 & \text{else} \end{cases}$$

We can now express some of the matrices $\widetilde{M}^{AQ}$ from section 5 in terms of Schröder matrices:

Proposition 6.5. *The matrices $\widetilde{M}^{AQ}$ corresponding to the reduced AST-graph of size n can be written as follows (see figure 22 for explicit examples):*

$$\begin{aligned}
\widetilde{M}_1^s &= (T_n^- S_n^{DH} \quad T_n^- S_n^{DD'} T_n^+ S_n^{DH}) & \widetilde{M}_{n+1/2}^l &= (V_n S_n^{DH} R_n \quad S_n^{DH}) \\
\widetilde{M}_1^l &= (V_n R_n \quad S_n^{V'H}) & \widetilde{M}_n^l &= (V_n T_n^- S_n^{DH} R_n \quad T_n^+ S_n^{DH}) \\
\widetilde{M}_{2n}^l &= (V_n S_n^{V'H} R_n \quad I_n) & \widetilde{M}_{n+1}^l &= (V_n T_n^+ S_n^{DH} R_n \quad T_n^- S_n^{DH})
\end{aligned}$$

For the first line, we additionally have $M^{AQ \setminus A} = S_n^{V'H}$.

Proof. For $\widetilde{M}_1^s$, consider figure 16a illustrating the first snake. Suppose there would be edges from A_i to A_{i+1} each, then the numbers of paths from A_i ’s to Q_j ’s for $j \in \{1, \dots, n\}$ would be given by S_n^{DH} . Thus, the actual numbers of paths from A_i ’s to Q_j ’s are given by the differences $S_{i,j}^{DH} - S_{i+1,j}^{DH} = (T_n^- S_n^{DH})_{ij}$, so the left half of $\widetilde{M}_1^s$ is $T_n^- S_n^{DH}$.

$$\begin{aligned}
\widetilde{M}_1^s &= \begin{pmatrix} 1 & 1 & 2 & 6 & 22 & 22 & 82 & 306 & 1231 & 5281 \\ & 1 & 3 & 10 & 38 & 38 & 142 & 531 & 2139 & 9184 \\ & & 1 & 5 & 22 & 22 & 83 & 313 & 1268 & 5464 \\ & & & 1 & 7 & 7 & 27 & 104 & 428 & 1864 \\ & & & & 1 & 1 & 4 & 16 & 68 & 304 \end{pmatrix} \\
\widetilde{M}_{n+1/2}^l &= \begin{pmatrix} 90 & 22 & 6 & 2 & 1 & 1 & 2 & 6 & 22 & 90 \\ -68 & -16 & -4 & -1 & & & 1 & 4 & 16 & 68 \\ 30 & 6 & 1 & & & & & 1 & 6 & 30 \\ -8 & -1 & & & & & & & 1 & 8 \\ 1 & & & & & & & & & 1 \end{pmatrix} \\
\widetilde{M}_1^l &= \begin{pmatrix} & & & -1 & 1 & 1 & 4 & 16 & 68 & 304 \\ & & & & & & 1 & 8 & 48 & 264 \\ & & 1 & & & & & 1 & 12 & 96 \\ & -1 & & & & & & & 1 & 16 \\ 1 & & & & & & & & & 1 \end{pmatrix} \\
\widetilde{M}_{2n}^l &= \begin{pmatrix} 304 & 68 & 16 & 4 & 1 & 1 & & & & \\ -264 & -48 & -8 & -1 & & & 1 & & & \\ 96 & 12 & 1 & & & & & 1 & & \\ -16 & -1 & & & & & & & 1 & \\ 1 & & & & & & & & & 1 \end{pmatrix}
\end{aligned}$$

Figure 22: Some of the matrices in proposition 6.5 for $n = 5$

In order to determine the right half of $\widetilde{M}_1^s$, we introduce auxiliary vertices $B_1, \dots, B_n$ going in a line straight up from $B_1 := Q_n$ to $B_n := A_n$. If we allow the edges going from a B_{l+1} to a B_l to be contained in paths from A_i to B_k , but not in paths from B_k to Q_j , then each path from A_i to Q_j has a unique decomposition into a path from A_i to B_k and a path from B_k to Q_j , for a unique k . The numbers of paths from A_i 's to B_k 's are given by $T_n^- S_n^{DD'}$ by a similar consideration to before. On the other side, if we look at the two possibilities for the first step of a path from B_k to Q_{n+j} , we see that the number of these paths is given by $S_{k,j}^{DH} + S_{k+1,j}^{DH} = (T_n^+ S_n^{DH})_{k,j}$ and thus the numbers of paths from B_k 's to Q_{n+j} 's are given by $T_n^+ S_n^{DH}$. So the numbers of paths from A_i 's to Q_{n+j} 's are given by the matrix product $T_n^- S_n^{DD'} T_n^+ S_n^{DH}$, as stated in the proposition.

Let us now come to the matrices $\widetilde{M}_k^l$ corresponding to (half-)lines. The right halves of these matrices are quite easy to read off the corresponding figures (figures 16b, 32b and 37a), using the auxiliary matrices T_n^- and T_n^+ in the same ways as before. Since the k -th and the $(2n+1-k)$ -th (half-)line are vertically symmetric to each other, the left half of $\widetilde{M}_k^l$ is obtained by taking the right half of the matrix $\widetilde{M}_{2n+1-k}^l$, reversing the order of its columns and switching all signs in the even rows (as $\lambda = n$ for all (half-)lines, see lemma 5.12 and the proof of the remark after proposition 5.26). On the matrix level, this corresponds to multiplying with R_n from the right and V_n from the left, thus we obtain the matrices as given in the proposition.

Finally, again for the first line, note that $M^A Q \setminus A$ is the right half of M^{AQ} , which is the same as the right half of $\widetilde{M}^{AQ} = \widetilde{M}_1^l$. So by the expression for $\widetilde{M}_1^l$, we indeed have $M^A Q \setminus A = S_n^{V'H}$. $\square$

Remark. For the symmetric configuration of the half-reduced AST-graph (figure 16c), one can also determine $\widetilde{M}^{AQ}$. In addition to the methods above, the following trick is needed (in order to use the Schröder matrices we defined): Reverse the orientation of all edges, express the numbers of paths from Q_i 's to A_j 's in a matrix, then transpose this matrix to switch back the roles of Q_i 's and A_j 's. The right half of $\widetilde{M}^{AQ}$ then turns out to be $(S_n^{V'D} T_n^-)^T$ (where $S^{V'D}$ is defined analogously to the other Schröder matrices in definition 6.2), or equivalently $(S_n^{V'H} (S_n^{DH})^{-1})^T$, since we can easily read off the relation $S_n^{V'H} = S_n^{V'D} T_n^- S_n^{DH}$ from figure 20.

Remember that all the matrices $\widetilde{M}^{AQ}$ coming from snakes and (half-)lines can be obtained from each other by elementary matrix row operations (corollary 5.25). This has some interesting consequences:

Lemma 6.6. *We have the following:*

- (i). S_n^{DH} , S_n^{VH} , $S_n^{V'H}$ and $S_n^{DD'}$ are all regular.
- (ii). For any matrix $\widetilde{M}^{AQ}$ corresponding to a snake or a (half-)line, both the left half and the right half of $\widetilde{M}^{AQ}$ is regular.
- (iii). For any two matrices $\widetilde{M}^{AQ}$ corresponding each to a snake or a (half-)line, there is a unique regular $n \times n$ matrix which takes one $\widetilde{M}^{AQ}$ to the other by multiplication from the left.

Proof. Clearly, S_n^{DH} , S_n^{VH} and $S_n^{V'H}$ are regular as they are upper triangular matrices with determinant 1. Also, all our auxiliary matrices V_n, R_n, T_n^+, T_n^- are regular. Thus, both the left and the right half of $\widetilde{M}_1^l = (V_n R_n \ S_n^{V'H})$ are regular. By corollary 5.25, this implies item (ii). Therefore, also $S_n^{DD'}$ has to be regular as it is contained in $\widetilde{M}_1^s$ (proposition 6.5), so item (i) is proved. Finally, item (iii) is clear: Since for instance the left halves of two matrices $\widetilde{M}^{AQ}$ are regular, the matrix taking one to the other is unique (and exists because of corollary 5.25). $\square$

We can now also express some of the operators described in section 5.7 in terms of Schröder matrices:

Corollary 6.7. *For the matrices $\widetilde{M}^{AQ}$ corresponding to the reduced AST-graph of size n , the following relations hold:*

$$\begin{aligned}\widetilde{M}_{n+1/2}^l &= S_n^{DH} \widetilde{M}_{2n}^l \\ \widetilde{M}_{n+1/2}^l &= V_n S_n^{DH} V_n \widetilde{M}_1^l \\ \widetilde{M}_1^l &= S_n^{V'H} \widetilde{M}_{2n}^l \\ \widetilde{M}_1^s &= T_n^- S_n^{DD'} T_n^+ \widetilde{M}_{n+1/2}^l = T_n^- S_n^{DD'} \widetilde{M}_n^l\end{aligned}$$

Proof. Consider the expressions for $\widetilde{M}_{n+1/2}^l$ and $\widetilde{M}_{2n}^l$ given in proposition 6.5. Looking at their right halves, which are regular by lemma 6.6, we see that S_n^{DH} is the only option for the unique matrix taking $\widetilde{M}_{2n}^l$ to $\widetilde{M}_{n+1/2}^l$. In the same way we can prove the other equations (for the second one, we should use the left halves of the matrices). $\square$

6.3 Properties of Schröder matrices

In this section, we collect several useful or interesting properties of the Schröder matrices. Firstly, their entries can easily be calculated recursively. Indeed, although it is not necessarily visible at first glance, the following recurrence relations yield enough information to completely fill all Schröder matrices:

Proposition 6.8. *For all $i, j, n \geq 1$, the following holds, where matrix entries with one of indices being less than 1 are considered to be 0:*

$$S_{ij}^{DH} = \begin{cases} 1 & i = j = 1 \\ S_{i+1,j}^{DH} + S_{i,j-1}^{DH} + S_{i-1,j-1}^{DH} & \text{else} \end{cases}$$

$$\begin{aligned}
S_{ij}^{VH} &= \begin{cases} 1 & i = j = 1 \\ 2 & i = 1, j = 2 \\ 3S_{1,j-1}^{VH} + S_{2,j-1}^{VH} - S_{1,j-2}^{VH} & i = 1, j \geq 3 \\ S_{i-1,j-1}^{VH} + 4S_{i,j-1}^{VH} + S_{i+1,j-1}^{VH} - S_{i,j-2}^{VH} & i \geq 2 \end{cases} \\
S_{ij}^{V'H} &= \begin{cases} 1 & i = j = 1 \\ S_{i-1,j-1}^{V'H} + 4S_{i,j-1}^{V'H} + S_{i+1,j-1}^{V'H} - S_{i,j-2}^{V'H} & \text{else} \end{cases} \\
S_n^{DD'}(i,j) &= \begin{cases} 1 & i = j = n \\ S_n^{DD'}(i+1,j) + S_{n-1}^{DD'}(i,j) + S_{n-1}^{DD'}(i-1,j) & \text{else} \end{cases}
\end{aligned}$$

Proof. For S_n^{DH} and $S_n^{DD'}$, the recurrence is quite easy to see when consulting figure 20: The number of paths is just split up depending on whether the first step of the path is $(1,1)$, $(2,0)$ or $(-1,1)$. For S_n^{VH} and $S_n^{V'H}$, the situation is more difficult since the starting steps $(1,1)$ and $(1,-1)$ lead us to the other matrix respectively. For $i, j \geq 1$, we obtain:

$$S_{ij}^{VH} = \begin{cases} 1 & i = j = 1 \\ S_{i,j-1}^{V'H} + S_{i,j-1}^{VH} + S_{i-1,j-1}^{V'H} & \text{else} \end{cases} \quad (6.1)$$

$$S_{ij}^{V'H} = S_{i+1,j}^{VH} + S_{i,j-1}^{V'H} + S_{i,j}^{VH} \quad (6.2)$$

Transforming this to the recurrence relations in the proposition takes some work. We first rewrite equation (6.2) with k in the place of j and then sum it over k (for k going from 1 to j):

$$\begin{aligned}
S_{i,k}^{V'H} - S_{i,k-1}^{V'H} &= S_{i+1,k}^{VH} + S_{i,k}^{VH} \\
S_{i,j}^{V'H} &= \sum_{k=1}^j S_{i+1,k}^{VH} + \sum_{k=1}^j S_{i,k}^{VH}
\end{aligned}$$

Note that this equation correctly gives $0 = 0$ for $j \leq 0$ (but not for $i = 0$). Thus we can plug it into equation (6.1) for $i \neq 1$, and then evaluate $S_{ij}^{VH} - S_{i,j-1}^{VH}$ (which eliminates the sums) in order to obtain the recurrence for $i \neq 1$. Note that despite of the case distinction in the second line, the last line also holds for $j = 1$ since all terms on the right side are 0 in this case anyway:

$$\begin{aligned}
S_{ij}^{VH} &= \left(\sum_{k=1}^{j-1} S_{i+1,k}^{VH} + \sum_{k=1}^{j-1} S_{i,k}^{VH} \right) + S_{i,j-1}^{VH} + \left(\sum_{k=1}^{j-1} S_{i,k}^{VH} + \sum_{k=1}^{j-1} S_{i-1,k}^{VH} \right) \\
S_{ij}^{VH} - S_{i,j-1}^{VH} &= \begin{cases} S_{i,j-1}^{VH} - S_{i,j-2}^{VH} & j = 1 \\ \left(S_{i+1,j-1}^{VH} + S_{i,j-1}^{VH} \right) + S_{i,j-1}^{VH} - S_{i,j-2}^{VH} + \left(S_{i,j-1}^{VH} + S_{i-1,j-1}^{VH} \right) & j \geq 2 \end{cases} \\
S_{ij}^{VH} &= S_{i+1,j-1}^{VH} + 4S_{i,j-1}^{VH} + S_{i-1,j-1}^{VH} - S_{i,j-2}^{VH}
\end{aligned}$$

Now for $i = 1$, equation (6.1) gives $S_{1,j}^{V'H} = S_{1,j-1}^{V'H} + S_{1,j-1}^{VH}$, holding for $j \geq 2$. We again plug in our sum expression of $S_{i,j}^{V'H}$ and evaluate $S_{1,j}^{V'H} - S_{1,j-1}^{V'H}$, which is possible for $j-1 \geq 2$, i.e. $j \geq 3$:

$$S_{1,j}^{VH} = \left(\sum_{k=1}^{j-1} S_{2,k}^{VH} + \sum_{k=1}^{j-1} S_{1,k}^{VH} \right) + S_{1,j-1}^{VH}$$

$$\begin{aligned}
S_{1,j}^{VH} - S_{1,j-1}^{VH} &= (S_{2,j-1}^{VH} + S_{1,j-1}^{VH}) + S_{1,j-1}^{VH} - S_{1,j-2}^{VH} \\
S_{1,j}^{VH} &= 3S_{1,j-1}^{VH} + S_{2,j-1}^{VH} - S_{1,j-2}^{VH}
\end{aligned}$$

Thus we have the recurrence relation for $S_{i,j}^{VH}$. For $S_{i,j}^{V'H}$, it is derived analogously, with certain differences in the details: Rewriting equation (6.1) with k in the place of j (for $k \geq 2$) and then summing over k (for k going from 2 to j) gives a sum expression for $S_{i,j}^{V'H}$, which is clearly also valid for $j = 1$ and which we plug into equation (6.2):

$$\begin{aligned}
S_{i,k}^{VH} - S_{i,k-1}^{VH} &= S_{i,k-1}^{V'H} + S_{i-1,k-1}^{V'H} \\
S_{i,j}^{VH} - S_{i,1}^{VH} &= \sum_{k=2}^j S_{i,k-1}^{V'H} + \sum_{k=2}^j S_{i-1,k-1}^{V'H} \\
S_{i,j}^{VH} &= S_{i,1}^{VH} + \sum_{k=1}^{j-1} S_{i,k}^{V'H} + \sum_{k=1}^{j-1} S_{i-1,k}^{V'H} \\
S_{i,j}^{V'H} &= \left(S_{i+1,1}^{VH} + \sum_{k=1}^{j-1} S_{i+1,k}^{V'H} + \sum_{k=1}^{j-1} S_{i,k}^{V'H} \right) + S_{i,j-1}^{V'H} + \left(S_{i,1}^{VH} + \sum_{k=1}^{j-1} S_{i,k}^{V'H} + \sum_{k=1}^{j-1} S_{i-1,k}^{V'H} \right)
\end{aligned}$$

Note that $S_{i,1}^{V'H} = \begin{cases} 1 & i = 1 \\ 0 & \text{else} \end{cases}$ holds. Thus the above sum expression is correct also for $j = 0$ (giving $0 = 0$), except if $i = 1$. So for $(i, j) \neq (1, 1)$, we can evaluate $S_{i,j}^{V'H} - S_{i,j-1}^{V'H}$ to get rid of the sums and obtain the recurrence relation for $S_{i,j}^{V'H}$: (Again, despite of the case distinction, the last line also holds for $j = 1$.)

$$\begin{aligned}
S_{i,j}^{V'H} - S_{i,j-1}^{V'H} &= \begin{cases} S_{i,j-1}^{V'H} - S_{i,j-2}^{V'H} & j = 1 \\ \left(S_{i+1,j-1}^{V'H} + S_{i,j-1}^{V'H} \right) + S_{i,j-1}^{V'H} - S_{i,j-2}^{V'H} + (S_{i,j-1}^{V'H} + S_{i-1,j-1}^{V'H}) & j \geq 2 \end{cases} \\
S_{i,j}^{V'H} &= S_{i+1,j-1}^{V'H} + 4S_{i,j-1}^{V'H} + S_{i-1,j-1}^{V'H} - S_{i,j-2}^{V'H}
\end{aligned}$$

□

Equation (6.2) in this proof is quite useful, as it gives the following relation:

Lemma 6.9. *We have $T_n^+ S_n^{VH} = S_n^{V'H} T_n^-$.*

Proof. Equation (6.2) can be written as $S_{i,j}^{V'H} - S_{i,j-1}^{V'H} = S_{i,j}^{VH} + S_{i+1,j}^{VH}$, which is the same as $(S_n^{V'H} T_n^-)_{ij} = (T_n^+ S_n^{VH})_{ij}$. □

Before continuing with the properties of the Schröder matrices, we need some facts about T_n^+ and T_n^- . For this, remember our convention for products in reverse order (definition 5.27). This convention is also the reason we have to exclude the case $n = 1$ here:

Lemma 6.10. *For $n \geq 2$, the matrices T_n^+ and T_n^- satisfy the following relations:*

$$\begin{aligned}
T_n^+ &= V_n T_n^- V_n & T_n^- &= V_n T_n^+ V_n \\
T_n^+ &= \prod_{i=n}^2 O_{i-1,i}^+ & T_n^- &= \prod_{i=n}^2 O_{i-1,i}^- \\
(T_n^+)^{-1} &= \prod_{i=2}^n O_{i-1,i}^- & (T_n^-)^{-1} &= \prod_{i=2}^n O_{i-1,i}^+
\end{aligned}$$

Proof. The first line of the lemma is proved analogously to the first line of lemma 5.21. The second line is a simple calculation: For each k , the matrix $\prod_{i=k}^2 O_{i-1,i}^+$ is similar to T_n^+ , but of the $n-1$ entries next to the main diagonal, it only contains the first $k-1$. The same holds for T_n^- . The third line then follows from the second one by $(O_{i-1,i}^+)^{-1} = O_{i-1,i}^-$ (lemma 5.21).

Note that the first line can alternatively be deduced from the second line, using the fact that $V_n O_{i-1,i}^- V_n = O_{i-1,i}^+$ (again lemma 5.21) and $V_n^2 = I_n$:

$$T_n^+ = \prod_{i=n}^2 O_{i-1,i}^+ = \prod_{i=n}^2 V_n O_{i-1,i}^- V_n = V_n \left(\prod_{i=n}^2 O_{i-1,i}^- \right) V_n = V_n T_n^- V_n$$

□

From our application of the Schröder matrices to the AST-graph, in particular from the matrix row operations connected to graph operations, we can deduce certain relations between S_n^{DH} , $S_n^{V'H}$, $S_n^{DD'}$ and - thanks to lemma 6.9 - also S_n^{VH} :

Corollary 6.11. *For the Schröder matrices, the following relations hold:*

$$\begin{aligned} V_n S_n^{V'H} &= (S_n^{DH})^{-1} V_n S_n^{DH} & (V_n S_n^{V'H})^2 &= I_n \\ V_n S_n^{VH} &= (S_n^{DH} T_n^-)^{-1} V_n S_n^{DH} T_n^- & (V_n S_n^{VH})^2 &= I_n \\ S_n^{DD'} T_n^+ V_n &= S_n^{DH} R_n (S_n^{DH})^{-1} & (S_n^{DD'} T_n^+ V_n)^2 &= I_n \end{aligned}$$

Remark.

- Remember that S_n^{VH} and $S_n^{V'H}$ can be directly read off S_{2n}^{DH} (see the remark after lemma 6.3). Here however, they are expressed in terms of S_n^{DH} for the same n .
- If we multiply the first identity by $(S_n^{DH})^{-1}$ from the right, using the relation $S_n^{V'H} = S_n^{V'D} T_n^- S_n^{DH}$ which we already mentioned in the remark after the proof of proposition 6.5, we obtain $V_n S_n^{V'D} T_n^- = (S_n^{DH})^{-1} V_n$. Thus, also $S_n^{V'D}$ has some quite interesting relations, but we do not deal with this further.

Proof. Take the equation $\widetilde{M}_{n+1/2}^l = V_n S_n^{DH} V_n \widetilde{M}_1^l$ from corollary 6.7 and consider the right halves of the matrices as given in proposition 6.5. Thus we get:

$$\begin{aligned} S_n^{DH} &= V_n S_n^{DH} V_n S_n^{V'H} \\ (S_n^{DH})^{-1} V_n S_n^{DH} &= V_n S_n^{V'H} \end{aligned}$$

Using lemma 6.9, we can plug in $S_n^{V'H} = T_n^+ S_n^{VH} (T_n^-)^{-1}$ to obtain the second identity (note that $V_n T_n^+ = T_n^- V_n$ by lemma 6.10):

$$\begin{aligned} (S_n^{DH})^{-1} V_n S_n^{DH} &= V_n T_n^+ S_n^{VH} (T_n^-)^{-1} \\ (S_n^{DH})^{-1} V_n S_n^{DH} &= T_n^- V_n S_n^{VH} (T_n^-)^{-1} \\ (S_n^{DH} T_n^-)^{-1} V_n S_n^{DH} T_n^- &= V_n S_n^{VH} \end{aligned}$$

Finally, from the left half of $\widetilde{M}_1^s = T_n^- S_n^{DD'} T_n^+ \widetilde{M}_{n+1/2}^l$ in corollary 6.7, together with proposition 6.5, we get the following, by multiplying with $(T_n^-)^{-1}$ from the left and $R_n (S_n^{DH})^{-1} = (S_n^{DH} R_n)^{-1}$

from the right:

$$\begin{aligned} T_n^- S_n^{DH} &= T_n^- S_n^{DD'} T_n^+ V_n S_n^{DH} R_n \\ S_n^{DH} R_n (S_n^{DH})^{-1} &= S_n^{DD'} T_n^+ V_n \end{aligned}$$

The three other equations now follow: We have

$$(V_n S_n^{V'H})^2 = (S_n^{DH})^{-1} V_n S_n^{DH} (S_n^{DH})^{-1} V_n S_n^{DH} = I_n$$

and similarly for $(V_n S_n^{VH})^2$ and $(S_n^{DD'} T_n^+ V_n)^2$, as $V_n^2 = \text{Id}_n$ and $R_n^2 = \text{Id}_n$. $\square$

We close this section by using the operators found in section 5.7 in order to express the Schröder matrices in terms of the elementary matrices from definition 5.20:

Proposition 6.12. *For $n \geq 2$, we have:*

$$\begin{aligned} S_n^{DH} &= \prod_{j=2}^n \left(\prod_{i=j}^n O_{i-1,i}^+ \prod_{i=n}^j O_{i-1,i}^+ \right) \\ S_n^{V'H} &= \prod_{j=n}^2 \left(\prod_{i=j}^n O_{i-1,i}^+ \prod_{i=n}^j O_{i-1,i}^+ \right) \prod_{j=2}^n \left(\prod_{i=j}^n O_{i-1,i}^+ \prod_{i=n}^j O_{i-1,i}^+ \right) \\ S_n^{VH} &= \prod_{i=2}^n O_{i-1,i}^- \prod_{j=n}^2 \left(\prod_{i=j}^n O_{i-1,i}^+ \prod_{i=n}^j O_{i-1,i}^+ \right) \prod_{j=2}^n \left(\prod_{i=j}^n O_{i-1,i}^+ \prod_{i=n}^j O_{i-1,i}^+ \right) \prod_{i=n}^2 O_{i-1,i}^- \\ S_n^{DD'} &= \prod_{i=2}^n O_{i-1,i}^+ \prod_{j=1}^{n-1} \left(\prod_{i=j}^1 O_{i-1,i}^+ \prod_{i=j}^1 O_{i+1,i}^+ \right) = \prod_{i=2}^n O_{i-1,i}^+ \prod_{j=n-1}^1 \left(\prod_{i=1}^j O_{i+1,i}^+ \prod_{i=1}^j O_{i-1,i}^+ \right) \end{aligned}$$

Remark. The reason we have to restrict to $n \geq 2$ here is that because of definition 5.27, the products $\prod_{j=2}^n$ etc. are not empty for $n = 1$. However, for $n = 1$, we actually would want all products to be empty and thus have $S_1^{DH} = S_1^{V'H} = S_1^{VH} = S_1^{DD'} = \text{Id}_1$.

Proof. By corollary 6.7 and lemma 6.6, $V_n S_n^{DH} V_n$ is the unique matrix taking $\widetilde{M}_1^l$ to $\widetilde{M}_{n+1/2}^l$. On the other hand, proposition 5.30 gives $\widetilde{M}_{1,5}^l = \widetilde{M}_1^l$ and for $2 \leq k \leq n$ (since $d_k = n + 1 - k$ in that case):

$$\widetilde{M}_{k+1/2}^l = \prod_{i=n+2-k}^n O_{i-1,i}^- \prod_{i=n}^{n+2-k} O_{i-1,i}^- \widetilde{M}_{k-1/2}^l$$

Combining these to go from $\widetilde{M}_1^l$ to $\widetilde{M}_{n+1/2}^l$, we can express $V_n S_n^{DH} V_n$:

$$V_n S_n^{DH} V_n = \prod_{k=n}^2 \left(\prod_{i=n+2-k}^n O_{i-1,i}^- \prod_{i=n}^{n+2-k} O_{i-1,i}^- \right) = \prod_{j=2}^n \left(\prod_{i=j}^n O_{i-1,i}^- \prod_{i=n}^j O_{i-1,i}^- \right)$$

We get rid of the V_n by the same trick as at the end of the proof of lemma 6.10: Replace $O_{i-1,i}^-$ by $V_n O_{i-1,i}^+ V_n$ (lemma 5.21) and use $V_n^2 = I_n$. Thus we get the following, which indeed gives the

expression for S_n^{DH} after canceling the V_n :

$$V_n S_n^{DH} V_n = V_n \prod_{j=2}^n \left(\prod_{i=j}^n O_{i-1,i}^+ \prod_{i=n}^j O_{i-1,i}^+ \right) V_n$$

For $S_n^{V'H}$, one could again combine proposition 5.30 with corollary 6.7. However, the more direct way is to apply corollary 6.11 stating that $S_n^{V'H} = V_n (S_n^{DH})^{-1} V_n S_n^{DH}$. From our expression of $V_n S_n^{DH} V_n$ in terms of $O_{i-1,i}^-$ above, we obtain $(V_n S_n^{DH} V_n)^{-1}$, which is $V_n (S_n^{DH})^{-1} V_n$, by reversing the order of all operators and changing them from $O_{i-1,i}^-$ to $(O_{i-1,i}^-)^{-1} = O_{i-1,i}^+$ (lemma 5.21). Thus we have:

$$\begin{aligned} V_n (S_n^{DH})^{-1} V_n &= \prod_{j=n}^2 \left(\prod_{i=j}^n O_{i-1,i}^+ \prod_{i=n}^j O_{i-1,i}^+ \right) \\ S_n^{V'H} &= \prod_{j=n}^2 \left(\prod_{i=j}^n O_{i-1,i}^+ \prod_{i=n}^j O_{i-1,i}^+ \right) \prod_{j=2}^n \left(\prod_{i=j}^n O_{i-1,i}^+ \prod_{i=n}^j O_{i-1,i}^+ \right) \end{aligned}$$

The expression for $S_n^{V'H}$ now follows directly from $S_n^{V'H} = (T_n^+)^{-1} S_n^{V'H} T_n^-$ (lemma 6.9) together with the expressions of $(T_n^+)^{-1}$ and T_n^- in terms of $O_{i-1,i}^-$ from lemma 6.10.

For $S_n^{DD'}$, we use the fact that $\widetilde{M}_1^s = T_n^- S_n^{DD'} \widetilde{M}_n^l$ by corollary 6.7. Thus, we can express $(T_n^- S_n^{DD'})^{-1}$ using the last two lines of proposition 5.28, and then take inverses by reversing the order of the operators and changing them from $O_{i-1,i}^-$ to $O_{i-1,i}^+$ etc. This gives:

$$\begin{aligned} (T_n^- S_n^{DD'})^{-1} &= \prod_{j=n-1}^1 \left(\prod_{i=1}^j O_{i+1,i}^- \prod_{i=1}^j O_{i-1,i}^- \right) = \prod_{j=1}^{n-1} \left(\prod_{i=j}^1 O_{i-1,i}^- \prod_{i=j}^1 O_{i+1,i}^- \right) \\ T_n^- S_n^{DD'} &= \prod_{j=1}^{n-1} \left(\prod_{i=j}^1 O_{i-1,i}^+ \prod_{i=j}^1 O_{i+1,i}^+ \right) = \prod_{j=n-1}^1 \left(\prod_{i=1}^j O_{i+1,i}^+ \prod_{i=1}^j O_{i-1,i}^+ \right) \end{aligned}$$

Together with the expression of $(T_n^-)^{-1}$ from lemma 6.10, this gives the result. $\square$

Remark.

- By proposition 5.28, we can actually find many more equivalent ways of writing $S_n^{DD'}$, differing from the given ones just by the way the operators are ordered.
- Except for its third equation on the left, the whole of corollary 6.11 can be derived from proposition 6.12.

Proof of the second item of the remark. $V_n S_n^{V'H} = (S_n^{DH})^{-1} V_n S_n^{DH}$ can be derived directly from the expressions of $S_n^{V'H}$ and S_n^{DH} in terms of the $O_{i-1,i}^+$ by reversing what is done in the proof of proposition 6.12 to obtain the expression of $S_n^{V'H}$. Analogously, $V_n S_n^{V'H} = (S_n^{DH} T_n^-)^{-1} V_n S_n^{DH} T_n^-$ follows with lemma 6.10.

The equations on the right of corollary 6.11, which can be written as $(S_n^{V'H})^{-1} = V_n S_n^{V'H} V_n$ etc., are just inherited from that same property $(O_{i-1,i}^+)^{-1} = V_n O_{i-1,i}^+ V_n$ (lemma 5.21) for the single operators: The critical point is that the expression of $S_n^{V'H}$ in terms of the $O_{i-1,i}^+$ is symmetric, that is when reversing the order of the $O_{i-1,i}^+$, we get back the same order. Since we obtain the

inverse of $S_n^{V'H}$ by reversing the order of the $O_{i-1,i}^+$ and changing them to $V_n O_{i-1,i}^+ V_n$, where all V_n cancel except for the outmost ones, we indeed have $(S_n^{V'H})^{-1} = V_n S_n^{V'H} V_n$.

The exact same holds for S_n^{VH} . Finally, for $S_n^{DD'}$, we have to consider $S_n^{DD'} T_n^+$ (where $T_n^+ = \prod_{i=n}^2 O_{i-1,i}^+$ according to lemma 6.10). This is not symmetric to itself, but the two different expressions we are given for it by proposition 6.12 are symmetric to each other and thus again $(S_n^{DD'} T_n^+)^{-1} = V_n S_n^{DD'} T_n^+ V_n$. $\square$

6.4 A generating function for Schröder paths

In order to provide yet another way of determining Schröder numbers, this section derives a general generating function for them.

Definition 6.13. We define the generating function for Schröder paths as follows:

$$GF(x, y, z) := \sum_{m, a, b \geq 0} S(m, a, b) x^m y^a z^b$$

We obtain the following result:

Theorem 6.14. *The generating function of Schröder paths is given by*

$$GF(x, y, z) = \frac{(1 + yz)(1 - x^2) - (1 - yz)\sqrt{1 - 6x^2 + x^4} - 2xy - 2xz}{2(x - f(y))(x - f(z))(1 - yz)}$$

where $f(y) := y - x^2 y - xy^2 = y(1 - x^2 - xy)$. Consequently, we have:

$$GF(x, y) := GF(x, y, 0) = \frac{1 - x^2 - \sqrt{1 - 6x^2 + x^4} - 2xy}{2x(x - f(y))}$$

$$GF(x) := GF(x, 0, 0) = \frac{1 - x^2 - \sqrt{1 - 6x^2 + x^4}}{2x^2}$$

Remark.

- In the classical case of Schröder paths going from $(0, 0)$ to $(2m, 0)$, the generation function is well-known to be $\frac{1 - x - \sqrt{1 - 6x + x^2}}{2x}$, see for instance the short summary in [16, section 1]. Indeed, $GF(x)$ coincides with this, as it is obtained from it by plugging in x^2 for x , which corrects the shift from $(2m, 0)$ as endpoint to $(m, 0)$ as endpoint.
- $GF(x, y, z)$ is symmetric in y and z , as one would expect since the Schröder paths can be reflected vertically and thus $S(m, a, b) = S(m, b, a)$.

Proof. Combinatorial considerations lead to the following three equations:

$$GF(x) = 1 + x^2 GF(x) + x GF(x) x GF(x) \quad (6.3)$$

$$GF(x, y) = 1 + x^2 GF(x, y) + xy GF(x, y) + xy^{-1} (GF(x, y) - GF(x)) \quad (6.4)$$

$$GF(x, y, z) = \frac{1}{1 - yz} + x^2 GF(x, y, z) + xz GF(x, y, z) + xz^{-1} (GF(x, y, z) - GF(x, y)) \quad (6.5)$$

In order to prove these, we use classical methods coined among others by Banderier and Flajolet [5], described very nicely in [40].

Firstly, for the derivation of the well-known equation (6.3), compare [40, example 3.7], where the exact same principle is used for Dyck paths. In this one paragraph, let ‘Schröder path’ mean ‘Schröder path starting and ending on the x -axis’. We can divide these Schröder paths into three categories, depending on their first step:

- Schröder paths that have no first step: This is only the empty Schröder path from $(0, 0)$ to $(0, 0)$ with generating function 1.
- Schröder paths with $(2, 0)$ as the first step: The remaining path after that first step can be any Schröder path. The first step increases the length by 2, thus the generating function of these Schröder paths is $x^2 \cdot GF(x)$.
- Schröder paths with $(1, 1)$ as the first step: Such paths have the following structure: The first step is followed by a Schröder path raised by 1, i.e. starting at height $y = 1$ and never going below it. Then, at some point we have a step $(1, -1)$ taking the path down again to the x -axis. After that, we have a normal Schröder path. Since the two single steps increase the length of the path by 1 each, the generating function of these Schröder paths is $x \cdot GF(x) \cdot x \cdot GF(x)$.

Summing up these generating functions has to give $GF(x)$, so we obtain equation (6.3).

Let us now come back to the general Schröder paths in order to derive equation (6.5). Also here, the principle we use is quite standard, see for instance [5, section 2.2, in particular equations (14) and (16)]. We consider the last step of each path, so we obtain $GF(x, y, z)$ as the sum of the generating functions of:

- Schröder paths without steps: $1 + yz + y^2z^2 + \dots = \frac{1}{1-yz}$ (One trivial path for each height.)
- Schröder paths ending with the step $(2, 0)$: $GF(x, y, z) \cdot x^2$
- Schröder paths ending with the step $(1, 1)$: $GF(x, y, z) \cdot xz$
- Schröder paths ending with the step $(1, -1)$: $(GF(x, y, z) - GF(x, y, 0)) \cdot xz^{-1}$
(Here, the Schröder path before the last step $(1, -1)$ may not end on the x -axis, therefore we subtract $GF(x, y, 0)$.)

Equation (6.4) is derived in the exact symmetric way by considering the first step of each path. The only difference is that this time, the paths without any steps are just the trivial path at $(0, 0)$.

Now that we have these three equations, we can easily derive the generating functions. Equation (6.3) gives

$$GF(x) = \frac{1 - x^2 \pm \sqrt{1 - 6x^2 + x^4}}{2x^2}$$

Since the Taylor expansion of the square root is $\sqrt{1 - 6x^2 + x^4} = 1 - 3x^2 - 4x^4 + \dots$, the $\pm$ has to be a minus, otherwise a term $\frac{1+x}{2x^2} = \frac{1}{x^2}$ would remain in the generating function, which would then not be a power series.

Having $GF(x)$, we can now use equation (6.4) to find $GF(x, y)$:

$$\begin{aligned} GF(x, y) &= \frac{1 - xy^{-1}GF(x)}{1 - x^2 - xy - xy^{-1}} = \frac{1 - xy^{-1}GF(x)}{y^{-1}f(y) - xy^{-1}} = \\ &\quad [Expand\ by\ -2xy] \\ &= \frac{2x^2GF(x) - 2xy}{2x(x - f(y))} = \frac{1 - x^2 - \sqrt{1 - 6x^2 + x^4} - 2xy}{2x(x - f(y))} \end{aligned}$$

From this, we can deduce $GF(x, y, z)$ with equation (6.5):

$$\begin{aligned}
GF(x, y, z) &= \frac{\frac{1}{1-yz} - xz^{-1}GF(x, y)}{z^{-1}f(z) - xz^{-1}} = \\
&= [Expand \text{ by } -2(x - f(y))z(1 - yz)] \\
&= \frac{2(x - f(y))(1 - yz)xGF(x, y) - 2(x - f(y))z}{2(x - f(y))(x - f(z))(1 - yz)} = \\
&= \frac{(1 - yz)(1 - x^2 - \sqrt{1 - 6x^2 + x^4} - 2xy) - 2xz + 2yz - 2x^2yz - 2xy^2z}{2(x - f(y))(x - f(z))(1 - yz)} = \\
&= \frac{(1 - yz)(1 - x^2 - \sqrt{1 - 6x^2 + x^4}) - 2xy + 2xy^2z - 2xz + 2yz - 2x^2yz - 2xy^2z}{2(x - f(y))(x - f(z))(1 - yz)} = \\
&= \frac{(1 - yz)(1 - x^2) - (1 - yz)\sqrt{1 - 6x^2 + x^4} - 2xy - 2xz + 2yz(1 - x^2)}{2(x - f(y))(x - f(z))(1 - yz)} = \\
&= \frac{(1 + yz)(1 - x^2) - (1 - yz)\sqrt{1 - 6x^2 + x^4} - 2xy - 2xz}{2(x - f(y))(x - f(z))(1 - yz)}
\end{aligned}$$

□

Remark. Note that $GF(x)$ could also be derived directly from equation (6.4). This is the so-called ‘kernel method’, which is used even if $GF(x, y)$ itself is not of any interest. It is described in [40, section 4.2] and [5, section 2.2], the former of which also mentions exactly our setting, that is Schröder paths starting and ending on the x -axis.

7 Minor summation formulas: From sums of minors to determinants

The goal of this section is to simplify the sums of minors obtained in section 5. However, most minor summation formulas involve the so-called ‘Pfaffian’: This is a polynomial in the entries of a skew-symmetric matrix, and its definition and properties are comparable to those of the determinant. We give a thorough introduction to Pfaffians from an elementary linear algebra viewpoint in section 7.1. In the appendix section A.1, the most important property of the Pfaffian is proved, namely that it squares to the determinant.

The problem of summing minors falls into two parts, which we treat in sections 7.2 and 7.3 respectively: Summing the minors which use all columns (or rows) of a matrix, and summing all minors of a matrix. In both cases, we present a summation formula by Okada [47], and an equivalent formula which only slightly differs from a special case of formulas by Ishikawa and Wakayama [29]. All formulas express the sum of minors as the Pfaffian of another matrix. In Okada’s formulas, this other matrix is quite small, but involves some computations to determine. In our alternative formulas, it is bigger, but its entries are directly given by the original matrix.

In section 7.4, we then apply the minor summation formulas to the matrices whose sums of minors give the number of Q -partial matchings of the (half-)reduced AST-graph, in particular to those which we can explicitly express in terms of Schröder matrices (see section 6). Thus, our

sought-after number is expressed as the Pfaffian of some relatively simple matrices. In the case of the $(n + 1/2)$ -th half-line, the special structure allows us to further work on these matrices, finally expressing the result as the determinant of two considerably smaller matrices.

7.1 Introduction to Pfaffians

Determinants of skew-symmetric matrices are the square of another polynomial, which is called ‘Pfaffian’. These polynomials were first treated in works of Pfaff, Jacobi [31] and Cayley [10] in the context of linear differential equations, see [36, section 6] for a historical summary. Apart from differential geometry, Pfaffians come in use for instance in combinatorics: They can be applied for the enumeration of perfect matchings in a much more direct way than we do it here. Consider [34] for this method developed by Kasteleyn.

The different definitions of the Pfaffian and its basic properties can for instance be found in the following papers: [26, appendix D], [48, section 2], [52, section 2], [9, section A.1.3], [54, section 4.3.3]. Since these facts are all quite well-known, in the following we mostly do not cite them specifically. To start with, let us recall the definition of skew-symmetry:

Definition 7.1. A *skew-symmetric* matrix is a matrix A with $A^\top = -A$, that is $A_{ij} = -A_{ji}$ for all indices $1 \leq i, j \leq n$.

$$A = \begin{pmatrix} 0 & A_{12} & A_{13} & A_{14} & \cdots & A_{1n} \\ -A_{12} & 0 & A_{23} & A_{24} & \cdots & A_{2n} \\ -A_{13} & -A_{23} & 0 & A_{34} & \cdots & A_{3n} \\ -A_{14} & -A_{24} & -A_{34} & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ -A_{1n} & -A_{2n} & -A_{3n} & \cdots & \cdots & 0 \end{pmatrix}$$

Figure 23: A general $n \times n$ skew-symmetric matrix

Furthermore, we need to following definition from [52, section 2]:

Definition 7.2. Let K_n denote the complete graph on n vertices $\{1, 2, \dots, n\}$ and let M be a matching of K_n (not necessarily perfect). Then we define:

- Two edges in M *cross* if they can be written as $\{i, k\}, \{j, l\} \in M$ with $i < j < k < l$. A *crossing* of M is a pair of edges in M which cross.
- The *sign* of M is

$$\text{sgn}(M) := (-1)^{\text{number of crossings of } M}$$

Now we can define the Pfaffian in the following three ways, which have certain similarities with the definition of the determinant. Note that while the definitions summing over permutations are quite common, summing over perfect matchings is more rare, and there are many different ways of specifying the sign there. The following version is taken from [52, section 2].

Definition 7.3. Let A be an $n \times n$ skew-symmetric matrix. Then the *Pfaffian* of A is

$$\text{pf}(A) := \sum_{M \text{ perfect matching of } K_n} \text{sgn}(M) \prod_{\substack{i < j \\ \{i,j\} \in M}} A_{ij}$$

For even n , we can alternatively write:

$$\begin{aligned} \text{pf}(A) &:= \frac{1}{(n/2)! 2^{n/2}} \sum_{\sigma \in S_n} \text{sgn}(\sigma) A_{\sigma(1)\sigma(2)} A_{\sigma(3)\sigma(4)} \dots A_{\sigma(n-1)\sigma(n)} \\ \text{pf}(A) &:= \sum_{\substack{\sigma \in S_n \\ \sigma(1) < \sigma(2), \sigma(3) < \sigma(4), \dots, \sigma(n-1) < \sigma(n) \\ \sigma(1) < \sigma(3) < \dots < \sigma(n-1)}} \text{sgn}(\sigma) A_{\sigma(1)\sigma(2)} A_{\sigma(3)\sigma(4)} \dots A_{\sigma(n-1)\sigma(n)} \end{aligned}$$

Remark. For odd n , we have $\text{pf}(A) = 0$ as K_n does not have any perfect matching. However, since the definitions summing over permutations are more common, in the literature the Pfaffian is most often not defined for odd n .

The fundamental properties of the Pfaffian, namely that the above definitions coincide and that the Pfaffian squares to the determinant, are given by the following proposition:

Proposition 7.4. *Let A be an $n \times n$ skew-symmetric matrix. Then:*

- (i). *For even n , the three definitions of the Pfaffian stated in definition 7.3 are equivalent.*
- (ii). *For all n , we have $\det(A) = \text{pf}(A)^2$.*

Proposition 7.4 is proved in the appendix section A.1. Here, we concentrate on further properties of the Pfaffian, which we need in the following subsections. To begin with, item (ii) of proposition 7.4 allows us to deduce the following very useful lemma:

Lemma 7.5. *Let A be an $n \times n$ skew-symmetric matrix and let B be any $n \times n$ square matrix. Then $\text{pf}(BAB^\top) = \det(B) \cdot \text{pf}(A)$.*

Remark. Note that since A is skew-symmetric, also $BAB^\top$ is skew-symmetric as we have $(BAB^\top)^\top = BA^\top B^\top = -BAB^\top$.

Proof. From proposition 7.4, we get $\text{pf}(BAB^\top)^2 = \det(BAB^\top) = \det(B)^2 \det(A) = \det(B)^2 \text{pf}(A)^2$. Since this is a polynomial equation in the entries of the two matrices, it implies that either $\text{pf}(BAB^\top) = +\det(B)\text{pf}(A)$ or $\text{pf}(BAB^\top) = -\det(B)\text{pf}(A)$ holds in general. By letting B be the identity matrix, we see that it is the former. $\square$

Remember that determinants are invariant under certain elementary row operations and that they are linear and alternating in their rows (and similarly for columns). For Pfaffians, something analogous holds. However, this is rarely treated in the literature, only briefly mentioned in [48, section 2.1] and [3, exercises in section III.5]. Therefore we develop our own way of formulating these properties by introducing a notion of ‘row-column pairs’:

Definition 7.6. Let A be an $n \times n$ skew-symmetric matrix. Then the i -th *row-column pair* of A is the i -th row of A together with the i -th column of A . An *elementary row-column pair operation* is an elementary operation which is performed both on the rows and on the corresponding columns (in any order).

Remark. Note that the i -th row of a skew-symmetric matrix determines the i -th column and vice versa, since they are just the negative transposes of each other.

Example 7.7. In order to see how the elementary row-column pair operations work, consider the skew-symmetric matrix A below.

- B is the matrix obtained by multiplying the third row-column pair of A by 10.
- C is the matrix obtained by exchanging the second and the fourth row-column pair of A .
- D is the matrix obtained by adding the second row-column pair of A multiplied by 10 to the fourth row-column pair.

$$A := \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 5 & 6 & 7 \\ -2 & -5 & 0 & 8 & 9 \\ -3 & -6 & -8 & 0 & 10 \\ -4 & -7 & -9 & -10 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 & 20 & 3 & 4 \\ -1 & 0 & 50 & 6 & 7 \\ -20 & -50 & 0 & 80 & 90 \\ -3 & -6 & -80 & 0 & 10 \\ -4 & -7 & -90 & -10 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 3 & 2 & 1 & 4 \\ -3 & 0 & -8 & -6 & 10 \\ -2 & 8 & 0 & -5 & 9 \\ -1 & 6 & 5 & 0 & 7 \\ -4 & -10 & -9 & -7 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 0 & 1 & 2 & 13 & 4 \\ -1 & 0 & 5 & 6 & 7 \\ -2 & -5 & 0 & -42 & 9 \\ -13 & -6 & 42 & 0 & 80 \\ -4 & -7 & -9 & -80 & 0 \end{pmatrix}$$

Elementary row-column pair operations have the following properties:

Lemma 7.8. *Let A be an $n \times n$ skew-symmetric matrix. Then the elementary row-column pair operations are independent of whether the row operation or the column operation is done first. The result of a row-column pair operation on A is again skew-symmetric.*

Furthermore, let $1 \leq i, j \leq n$ be distinct indices and let λ be a scalar. Then we have the following:

- (i). *If the i -th row-column pair of A is multiplied by λ , then the Pfaffian of A is multiplied by λ .*
- (ii). *If each entry of A is multiplied by λ , then the Pfaffian of A is multiplied by $\lambda^{n/2}$.*
- (iii). *If a multiple of the i -th row-column-pair of A is added to the j -th row-column pair, then the Pfaffian of A does not change.*
- (iv). *If the i -th row-column-pair of A is exchanged with the j -th row-column pair, then the Pfaffian of A is multiplied by -1 .*

Proof. Let E be the elementary matrix corresponding to some elementary row operation. Thus, multiplying from the left by E gives the row operation while multiplying from the right by $E^\top$ gives the column operation. So the corresponding row-column operation transforms A to $EAE^\top$. This shows that it does not make a difference whether we first do the row or the column operation and that the result is again skew-symmetric (compare the remark after lemma 7.5). Furthermore, by lemma 7.5, we have $\text{pf}(EAE^\top) = \det(E) \cdot \text{pf}(A)$, which proves items (i), (iii), and (iv), as the corresponding elementary matrices clearly have determinants λ , 1, and -1 respectively. Item (ii) can directly be seen from any of the definitions of the Pfaffian. $\square$

Similar to determinants being linear in each row, Pfaffians are linear in each row-column pair:

Lemma 7.9. *Let A be an $n \times n$ skew-symmetric matrix and let $1 \leq k \leq n$. Then $\text{pf}(A)$ is linear in the k -th row-column pair. Formulated precisely, this means:*

- (i). *Let B be an $n \times n$ skew-symmetric matrix differing from A only in the k -th row-column pair. Let C be the $n \times n$ skew-symmetric matrix we obtain by adding A and B ‘along the k -th row-column pair’, that is*

$$C_{ij} := \begin{cases} A_{ij} = B_{ij} & i, j \neq k \\ A_{ij} + B_{ij} & i = k \vee j = k \end{cases}$$

Then $\text{pf}(C) = \text{pf}(A) + \text{pf}(B)$.

- (ii). *Let D be the matrix obtained from A by multiplying the k -th row-column pair by λ . Then $\text{pf}(D) = \lambda \cdot \text{pf}(A)$.*

Remark. Note that while determinants are even multilinear in their rows or columns, for the Pfaffian we do not have any multilinearity since the different row-column pairs intersect each other.

Proof. For item (i), we consider the definition of the Pfaffian:

$$\text{pf}(C) = \sum_{M \text{ perfect matching of } K_n} \text{sgn}(M) \prod_{\substack{i < j \\ \{i, j\} \in M}} C_{ij}$$

Now note that for each perfect matching M , there is exactly one edge $\{i, j\} \in M$ ($i < j$) with $i = k \vee j = k$. Thus, in the product $\prod_{i < j, \{i, j\} \in M} C_{ij}$, for all C_{ij} we have $C_{ij} = A_{ij} = B_{ij}$, except for one C_{ij} which satisfies $C_{ij} = A_{ij} + B_{ij}$. So we have

$$\prod_{\substack{i < j \\ \{i, j\} \in M}} C_{ij} = \prod_{\substack{i < j \\ \{i, j\} \in M}} A_{ij} + \prod_{\substack{i < j \\ \{i, j\} \in M}} B_{ij}$$

and by summing over all perfect matchings M of K_n we indeed get $\text{pf}(C) = \text{pf}(A) + \text{pf}(B)$.

Item (ii) can easily be proved by similar means. However, it has already been stated and proved as item (i) of lemma 7.8, so we leave it out here. $\square$

In addition to the properties mentioned so far, also the Laplace expansion of determinants has an equivalent, which is very useful for calculating Pfaffians:

Proposition 7.10. *Let A be an $n \times n$ skew-symmetric matrix. For $1 \leq i, j \leq n$, denote by $\widehat{A^{ij}}$ the $(n-2) \times (n-2)$ skew-symmetric matrix obtained by deleting the i -th row, i -th column, j -th row and j -th column of A . Then we have:*

$$\text{For any } 1 \leq i \leq n: \text{ pf}(A) = \sum_{j \neq i} (-1)^{i+j+[i < j]} A_{ij} \text{ pf}(\widehat{A^{ij}})$$

$$\text{For any } 1 \leq j \leq n: \text{ pf}(A) = \sum_{i \neq j} (-1)^{i+j+[i < j]} A_{ij} \text{ pf}(\widehat{A^{ij}})$$

where we use the Iverson bracket $[P] = \begin{cases} 1 & \text{if } P \text{ is true} \\ 0 & \text{otherwise} \end{cases}$.

To prove this expansion formula, we first need a small lemma about crossings of perfect matchings:

Lemma 7.11. *Let M be a perfect matching of K_n and let $\{i, j\} \in M$. Then*

$$\text{sgn}(M) = \text{sgn}(M \setminus \{\{i, j\}\}) \cdot (-1)^{i+j+1}$$

Proof. Let w.l.o.g. $i < j$ and define $I := \{i+1, i+2, \dots, j-1\}$. Then the edge $\{i, j\}$ crosses exactly those edges of M which have one vertex in I and one outside of I . Since the vertices in I which are not matched outside of I are matched to each other in pairs, the number of edges which $\{i, j\}$ crosses has the same parity as $|I|$. Since $|I| = |j-i| - 1 \equiv i+j+1 \pmod{2}$, the sign of M indeed changes by $(-1)^{i+j+1}$ when taking away $\{i, j\}$. $\square$

Now we can easily prove proposition 7.10:

Proof of proposition 7.10. We prove the first equation. The second one can be proved in the same way, since it does not make a difference whether we sum over j or over i . So let $1 \leq i \leq n$. Then:

$$\begin{aligned} \text{pf}(A) &= \sum_{M \text{ perf. mat. of } K_n} \text{sgn}(M) \prod_{\substack{k < l \\ \{k, l\} \in M}} A_{kl} = \\ &= \sum_{j \neq i} \sum_{M \text{ perf. mat. of } K_n} \text{sgn}(M) \prod_{\substack{k < l \\ \{k, l\} \in M}} A_{kl} = \\ &= \sum_{j \neq i} \sum_{M' \text{ perf. mat. of } K_n \setminus \{i, j\}} \text{sgn}(M' \cup \{\{i, j\}\}) (-1)^{[i < j] + 1} A_{ij} \prod_{\substack{k < l \\ \{k, l\} \in M'}} A_{kl} = \\ &= \sum_{j \neq i} \sum_{M' \text{ perf. mat. of } K_n \setminus \{i, j\}} \text{sgn}(M') (-1)^{i+j+1} (-1)^{[i < j] + 1} A_{ij} \prod_{\substack{k < l \\ \{k, l\} \in M'}} A_{kl} = \\ &= \sum_{j \neq i} (-1)^{i+j+[i < j]} A_{ij} \text{pf}(A^{\widehat{ij}}) \end{aligned}$$

Two steps in this merit some explanation: Firstly, $(-1)^{[i < j] + 1} A_{ij}$ appears in the third line because $\prod_{k < l, \{k, l\} \in M} A_{kl}$ contains A_{ij} if $i < j$ and $A_{ji} = -A_{ij}$ otherwise. Secondly, $\text{sgn}(M' \cup \{\{i, j\}\}) = (-1)^{i+j+1} \text{sgn}(M')$ is the application of lemma 7.11. $\square$

The following lemma is an easy corollary of lemma 7.5. The block diagonalization in the first line is given in [1, equation (4) in section 29]:

Lemma 7.12. *Let Q and Q' be $m \times n$ matrices and let A and B be $m \times m$ resp. $n \times n$ matrices with A being regular. Then the following holds:*

$$\begin{pmatrix} A & 0 \\ 0 & B + Q^\top A^{-1} Q' \end{pmatrix} = \begin{pmatrix} I_m & 0 \\ Q^\top A^{-1} & I_n \end{pmatrix} \begin{pmatrix} A & Q' \\ -Q^\top & B \end{pmatrix} \begin{pmatrix} I_m & -A^{-1} Q' \\ 0 & I_n \end{pmatrix}$$

If additionally A and B are skew-symmetric, then we have:

$$\text{pf} \begin{pmatrix} A & Q \\ -Q^\top & B \end{pmatrix} = \text{pf}(A) \text{pf}(B + Q^\top A^{-1} Q)$$

Proof. The first equation can simply be checked by evaluating the product. For the second equation, set $Q' := Q$. Then $B + Q^\top A^{-1}Q$ is skew-symmetric and we have the following, where we use lemma 7.5 for the penultimate line. Note that the very first step follows simply from the definition of the Pfaffian:

$$\begin{aligned}
\text{pf}(A)\text{pf}(B + Q^\top A^{-1}Q) &= \text{pf}\begin{pmatrix} A & 0 \\ 0 & B + Q^\top A^{-1}Q \end{pmatrix} = \\
&= \text{pf}\left(\begin{pmatrix} I_m & 0 \\ Q^\top A^{-1} & I_n \end{pmatrix} \begin{pmatrix} A & Q \\ -Q^\top & B \end{pmatrix} \begin{pmatrix} I_m & -A^{-1}Q \\ 0 & I_n \end{pmatrix}\right) = \\
&= \text{pf}\left(\begin{pmatrix} I_m & 0 \\ Q^\top A^{-1} & I_n \end{pmatrix} \begin{pmatrix} A & Q \\ -Q^\top & B \end{pmatrix} \begin{pmatrix} I_m & 0 \\ Q^\top A^{-1} & I_n \end{pmatrix}^\top\right) = \\
&= \det\begin{pmatrix} I_m & 0 \\ Q^\top A^{-1} & I_n \end{pmatrix} \text{pf}\begin{pmatrix} A & Q \\ -Q^\top & B \end{pmatrix} = \\
&= \text{pf}\begin{pmatrix} A & Q \\ -Q^\top & B \end{pmatrix}
\end{aligned}$$

□

Finally, we introduce a matrix H_n , which plays an important role in this section:

Definition 7.13. We define the $n \times n$ skew-symmetric matrix $H_n := (H_{ij})_{1 \leq i, j \leq n}$ by

$$H_{ij} := \begin{cases} (-1)^{i+j-1} & i < j \\ 0 & i = j \\ (-1)^{i+j} & i > j \end{cases}$$

$$\text{i.e. } H_n := \begin{pmatrix} 0 & 1 & -1 & \dots & \dots & (-1)^n \\ -1 & 0 & 1 & \dots & \dots & -(-1)^n \\ 1 & -1 & 0 & \dots & \dots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 & 1 \\ -(-1)^n & (-1)^n & \dots & \dots & -1 & 0 \end{pmatrix}$$

Furthermore, we define the $n \times n$ skew-symmetric matrix $\mathbb{1}_n^s$ to be the matrix consisting of 1's above and -1 's below the main diagonal, that is

$$\mathbb{1}_n^s(i, j) := \begin{cases} 1 & i < j \\ 0 & i = j \\ -1 & i > j \end{cases}$$

We immediately see the following relations, which are used in the next sections:

Lemma 7.14. *Let n be even. Then the following holds:*

$$\begin{aligned}
H_n \mathbb{1}_n^s &= -I_n & \mathbb{1}_n^s H_n &= -I_n \\
V_n \mathbb{1}_n^s V_n &= -H_n & V_n H_n V_n &= -\mathbb{1}_n^s \\
\text{pf}(\mathbb{1}_n^s) &= 1 & \text{pf}(H_n) &= 1
\end{aligned}$$

Proof. The first identities can be checked by straightforward calculation. For the Pfaffians, note that deleting any two row-column pairs of $\mathbb{1}_n^s$ gives $\mathbb{1}_{n-2}^s$. Thus, proposition 7.10 with $i := 1$ gives

$$\text{pf}(\mathbb{1}_n^s) = \sum_{j=2}^n (-1)^{1+j+1} \cdot 1 \cdot \text{pf}(\mathbb{1}_{n-2}^s) = \text{pf}(\mathbb{1}_{n-2}^s) \sum_{j=2}^n (-1)^j = \text{pf}(\mathbb{1}_{n-2}^s)$$

Since $\mathbb{1}_0^s$ is the empty matrix, we have $\text{pf}(\mathbb{1}_0^s) = 1$ and so by induction we obtain $\text{pf}(\mathbb{1}_n^s) = 1$ for even n . For the Pfaffian of H_n , we use the fact that $H_n = -V_n \mathbb{1}_n^s V_n$ together with item (ii) of lemma 7.8 and lemma 7.5:

$$\text{pf}(H_n) = (-1)^{n/2} \text{pf}(V_n \mathbb{1}_n^s V_n) = (-1)^{n/2} \det(V_n) \text{pf}(\mathbb{1}_n^s) = (-1)^{n/2} \cdot (-1)^{n/2} \cdot 1 = 1$$

□

Remark. Alternatively, one could derive $\text{pf}(H_n)$ from $\text{pf}(\mathbb{1}_n^s)$ by the fact that $\text{pf}(A^{-1}) = (-1)^{n/2} \text{pf}(A)$ holds for regular skew-symmetric $n \times n$ matrices A , which is proved in [43, theorem 3.1].

7.2 Sum of all $n \times n$ minors of an $m \times n$ matrix

We start by giving a very brief overview of the development of minor summation formulas, both for the case of summing minors using all columns/rows and for the case of summing all minors:

In 1989, Soichi Okada presented formulas which express the sum of minors as a Pfaffian of another matrix [47, theorems 3 and 4]. Stembridge derived the same formulas directly in the context of path families [52, theorems 3.1 and 4.1], noting that instead of his combinatorial proof one could apply the LGV lemma together with Okada's formulas. Ishikawa and Wakayama then generalized Okada's formulas by giving weights (which depend on another Pfaffian) to the minors [30, theorems 1 and 3, corollary 1]. Later, they reformulated their formulas a little and provided combinatorial proofs similar to Stembridge's approach [29, theorems 3.2 and 3.5, section 4].

In this section and the next one, we present the formulas by Okada, as well as equivalent formulas which are very similar to special cases of the formulas presented in [29]. Remember that when dealing with $m \times n$ matrices, we always have $m, n \geq 0$ (see section 1.4). Furthermore, we use the following notations:

Definition 7.15. Let A be an $m \times n$ matrix. Then:

- $A|$ resp. $|A$ denotes the $m \times (n+1)$ matrix obtained from A by adding a column of zeros on the right resp. left side.
- $\underline{A}$ resp. $\overline{A}$ denotes the $(m+1) \times n$ matrix obtained from A by adding a row of zeros at the bottom resp. top.
- ${}_{-1}A$ denotes the $(m+1) \times (n+1)$ matrix obtained from $|A$ by changing the bottom-left entry from 0 to -1 , that is

$${}_{-1}A := \begin{pmatrix} 0 & & & \\ \vdots & & A & \\ 0 & & & \\ -1 & 0 & \cdots & 0 \end{pmatrix}$$

Also, remember the notation for sums of minors introduced in definition 5.3. In this section, we start by dealing with the case of summing the minors which use all columns of a matrix, that is the $n \times n$ minors of an $m \times n$ matrix. The result from Okada [47, theorem 3] can be written down as follows:

Theorem 7.16 (Okada). *Let $M = (m_1 \ m_2 \ \dots \ m_n)$ be an $m \times n$ matrix with column vectors $m_1, m_2, \dots, m_n$.*

- The $n \times n$ skew-symmetric matrix $T_0(M)$ is defined by

$$T_0(M) := (|m_i \ m_j|_2)_{1 \leq i, j \leq n} = \begin{pmatrix} 0 & |m_1 \ m_2|_2 & |m_1 \ m_3|_2 & \dots & |m_1 \ m_n|_2 \\ -|m_1 \ m_2|_2 & 0 & |m_2 \ m_3|_2 & \dots & |m_2 \ m_n|_2 \\ -|m_1 \ m_3|_2 & -|m_2 \ m_3|_2 & 0 & \dots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -|m_1 \ m_n|_2 & -|m_2 \ m_n|_2 & \dots & \dots & 0 \end{pmatrix}$$

- The $(n+1) \times (n+1)$ skew-symmetric matrix $T_1(M)$ is defined by

$$T_1(M) := \begin{pmatrix} 0 & |m_1|_1 & \dots & |m_n|_1 \\ -|m_1|_1 & & & \\ \vdots & & T_0(M) & \\ -|m_n|_1 & & & \end{pmatrix} = \begin{pmatrix} 0 & |m_1|_1 & |m_2|_1 & |m_3|_1 & \dots & |m_n|_1 \\ -|m_1|_1 & 0 & |m_1 \ m_2|_2 & |m_1 \ m_3|_2 & \dots & |m_1 \ m_n|_2 \\ -|m_2|_1 & -|m_1 \ m_2|_2 & 0 & |m_2 \ m_3|_2 & \dots & |m_2 \ m_n|_2 \\ -|m_3|_1 & -|m_1 \ m_3|_2 & -|m_2 \ m_3|_2 & 0 & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -|m_n|_1 & -|m_1 \ m_n|_2 & -|m_2 \ m_n|_2 & \dots & \dots & 0 \end{pmatrix} = T_0({}_{-1}M)$$

$$\text{Then } |M|_n = \begin{cases} \text{pf}(T_0(M)) & n \text{ even} \\ \text{pf}(T_1(M)) & n \text{ odd} \end{cases}$$

Proof. The following proof is essentially the one given by Okada in [47, proof of theorem 3].

The case for odd n can be easily derived from the case for even n : By expanding each $(n+1) \times (n+1)$ minor of ${}_{-1}M$ along the first column, we see that $|{}_{-1}M|_{n+1} = (-1)^{n+1}|M|_n = |M|_n$ for odd n . Thus $|M|_n = |{}_{-1}M|_{n+1} = \text{pf}(T_0({}_{-1}M)) = \text{pf}(T_1(M))$ in this case.

For even n , note first that both $|M|_n$ and $\text{pf}(T_0(M))$ are alternating multilinear maps in the columns of M (i.e. multilinear maps that evaluate to 0 if any two columns of M equal each other). For $|M|_n$, this is obvious: Each $n \times n$ minor of M uses all columns of M and is thus an alternating multilinear map in the columns of M . For $\text{pf}(T_0(M))$, we see that the i -th row-column pair of $T_0(M)$ depends linearly on m_i , since all its entries are of the form $|m_i \ m_j|_2$ or $|m_j \ m_i|_2$, which depend linearly on m_i . Since the entries of $T_0(M)$ which are not in the i -th row-column pair are

independent of m_i , indeed $\text{pf}(T_0(M))$ linearly depends on m_i by lemma 7.9. Since two columns of M equaling each other result in two columns (and rows) of $T_0(M)$ equaling each other and thus $\text{pf}(T_0(M)) = 0$ as $\det(T_0(M)) = 0$, indeed $\text{pf}(T_0(M))$ is an alternating multilinear map in $m_1, \dots, m_n$.

Since both $|M|_n$ and $\text{pf}(T_0(M))$ are alternating multilinear maps in the columns of M , it is enough to verify $|M|_n = \text{pf}(T_0(M))$ for matrices of the form $M = (e_{i_1} \ e_{i_2} \ \dots \ e_{i_n})$ satisfying $1 \leq i_1 < i_2 < \dots < i_n \leq m$. For such M , we clearly have $|M|_n = 1$. Furthermore, we have

$$|e_{i_k} \ e_{i_l}|_2 = \begin{cases} 1 & k < l \\ 0 & k = l \\ -1 & k > l \end{cases}$$

and thus $T_0(M) = \mathbb{1}_n^s$. By lemma 7.14, we indeed have $\text{pf}(T_0(M)) = \text{pf}(\mathbb{1}_n^s) = 1 = |M|_n$ for such matrices M . $\square$

We now want to rewrite Okada's formula. The key for this is to notice that $T_0(M)$ is just a simple matrix product. This fact was already used by Ishikawa and Wakayama in their more general setting [30, equations 2.15 and 3.6] (using row vectors instead of column vectors and an arbitrary B instead of $\mathbb{1}_n^s$):

Lemma 7.17. *For any $m \geq 1$, we have the following:*

- (i). *If a and b are vectors in m dimensions, then $|a \ b|_2 = a^\top \mathbb{1}_m^s b$.*
- (ii). *If M is an $m \times n$ matrix, then $T_0(M) = M^\top \mathbb{1}_m^s M$.*

Proof. For item (i), we have

$$\begin{aligned} |a \ b|_2 &= \sum_{i < j} \det \begin{pmatrix} a_i & b_i \\ a_j & b_j \end{pmatrix} = \\ &= \sum_{i < j} a_i b_j - a_j b_i = \\ &= \sum_{i, j} \mathbb{1}_m^s(i, j) a_i b_j = \\ &= a^\top \mathbb{1}_m^s b \end{aligned}$$

For item (ii), if $m_1, \dots, m_n$ are the column vectors of M , we have $T_0(M)_{ij} = |m_i \ m_j|_2 = m_i^\top \mathbb{1}_m^s m_j$ and thus $T_0(M) = M^\top \mathbb{1}_m^s M$. $\square$

Now that we can express $T_0(M)$ as a matrix product, we can use lemma 7.12 to rewrite theorem 7.16. The advantage of the matrices $H_{0/1}(M)$ we now introduce is that one does not have to do any calculations in order to determine their entries, contrary to $T_{0/1}(M)$. In return, they are much bigger, which makes it questionable whether using them in total is a computational advantage. In any way, it is quite nice to have a result where the matrix M is directly visible, and therefore we present it. Note that up to small modifications, this theorem is a special case of [29, theorem 3.2] (take $A := \mathbb{1}_m^s$):

Theorem 7.18. *Let M be an $m \times n$ matrix.*

- *The $(m+n) \times (m+n)$ skew-symmetric matrix $H_0(M)$ is defined by*

$$H_0(M) := \begin{pmatrix} H_m & M \\ -M^\top & 0 \end{pmatrix}$$

- *The $(m+n+1) \times (m+n+1)$ skew-symmetric matrix $H_1(M)$ is defined by*

$$H_1(M) := H_0(\underline{M}) = \begin{pmatrix} H_{m+1} & \underline{M} \\ -M^\top | & 0 \end{pmatrix} = \begin{pmatrix} & & & & -(-1)^m & & & \\ & & & & (-1)^m & & & \\ & & H_m & & \vdots & & M & \\ & & & & 1 & & & \\ (-1)^m & -(-1)^m & \cdots & -1 & 0 & 0 & \cdots & 0 \\ & & & & 0 & & & \\ -M^\top & & & & \vdots & & 0 & \\ & & & & 0 & & & \end{pmatrix}$$

$$\text{Then } |M|_n = (-1)^{\lfloor n/2 \rfloor} \begin{cases} \text{pf}(H_0(M)) & m+n \text{ even} \\ \text{pf}(H_1(M)) & m+n \text{ odd} \end{cases}$$

Remark. For $m < n$, the statement reads $0 = 0$ as M then has no $n \times n$ minors and $H_{0/1}(M)$ is clearly singular. In theorem 7.16, this is a little bit less obvious, but also $T_{0/1}(M)$ is singular in that case.

The following proposition shows that theorems 7.16 and 7.18 are equivalent:

Proposition 7.19. *Let M be an $m \times n$ matrix. Then the following holds:*

$$\begin{aligned} \text{pf}(H_0(M)) &= (-1)^{\lfloor n/2 \rfloor} \begin{cases} \text{pf}(T_0(M)) & m \text{ even} \\ \text{pf}(T_1(M)) & m \text{ odd} \end{cases} \\ \text{pf}(H_1(M)) &= (-1)^{\lfloor n/2 \rfloor} \begin{cases} \text{pf}(T_0(M)) & m \text{ odd} \\ \text{pf}(T_1(M)) & m \text{ even} \end{cases} \end{aligned}$$

Proof. The second equation follows from the first equation: We have $H_1(M) = H_0(\underline{M})$ by definition and $T_0(M) = T_0(\underline{M})$ as an additional row of zeros does not change the 2×2 minors between any two columns of M . Thus we get

$$\begin{aligned} \text{pf}(H_1(M)) &= \text{pf}(H_0(\underline{M})) = (-1)^{\lfloor n/2 \rfloor} \begin{cases} \text{pf}(T_0(\underline{M})) & m+1 \text{ even} \\ \text{pf}(T_1(\underline{M})) & m+1 \text{ odd} \end{cases} = \\ &= (-1)^{\lfloor n/2 \rfloor} \begin{cases} \text{pf}(T_0(M)) & m \text{ odd} \\ \text{pf}(T_1(M)) & m \text{ even} \end{cases} \end{aligned}$$

For the first equation, note that it reads $0 = 0$ if $m \not\equiv n \pmod{2}$, as in this case all the skew-symmetric matrices involved are of odd order. So we only have to consider the cases $m \equiv n \equiv 0$ and $m \equiv n \equiv 1$.

Let first m and n be even. By lemmata 7.17 and 7.14, we get $T_0(M) = M^\top \mathbb{1}_m^s M = -M^\top H_m^{-1} M$. Using the second equation in lemma 7.12 with $A := H_m$, $B := 0$ and $Q := M$, and then lemmata 7.14 and 7.8 gives

$$\text{pf}(H_0(M)) = \text{pf}(H_m) \cdot \text{pf}(M^\top H_m^{-1} M) = 1 \cdot \text{pf}(-T_0(M)) = (-1)^{\lfloor n/2 \rfloor} \text{pf}(T_0(M))$$

Now if m and n are odd, note that $H_0(M)$ can be obtained from $H_0(-_1 M)$ by deleting the $(m+1)$ -th and $(m+2)$ -th row-column pair. Since in the $(m+2)$ -th column of $H_0(-_1 M)$, the -1 in the $(m+1)$ -th row is the only nonzero entry, applying proposition 7.10 with $j := m+2$ gives

$$\text{pf}(H_0(-_1 M)) = (-1)^{m+1+m+2+1} (-1) \text{pf}(H_0(M)) = -\text{pf}(H_0(M))$$

Thus, as $T_1(M) = T_0(-_1 M)$, by applying the case for m and n even to $-_1 M$ we get

$$\text{pf}(H_0(M)) = -\text{pf}(H_0(-_1 M)) = -(-1)^{\lfloor (n+1)/2 \rfloor} \text{pf}(T_0(-_1 M)) = (-1)^{\lfloor n/2 \rfloor} \text{pf}(T_1(M))$$

□

Even though theorem 7.18 is proved by the equivalence to theorem 7.16, it is noteworthy that it can also be proved directly in a way similar to Okada's proof of 7.16. We only sketch that here:

Sketch of proof of theorem 7.18. As adding a row of zeros to M does not change its $n \times n$ minors, we have $|M|_n = |(\underline{M})|_n$ and so by $H_1(M) = H_0(\underline{M})$, the case for $m+n$ odd follows by applying the case for $m+n$ even to $\underline{M}$: $|M|_n = |(\underline{M})|_n = (-1)^{\lfloor n/2 \rfloor} \text{pf}(H_0(\underline{M})) = (-1)^{\lfloor n/2 \rfloor} \text{pf}(H_1(M))$.

In order to prove the case for $m+n$ even, one can use the concept of multilinearity as in Okada's proof of theorem 7.16: Clearly $|M|_n$ and $\text{pf}(H_0(M))$ are alternating multilinear maps in the columns of M . For $\text{pf}(H_0(M))$, this is due to the fact that the $m+i$ -th row-column pair of $H_0(M)$ is linear in the i -th column of M , while all other entries of $H_0(M)$ are independent of it.

So one only has to consider matrices M of the form $M = (e_{i_1} \ e_{i_2} \ \dots \ e_{i_n})$. It is useful to multiply every second of the first m row-column pairs of $H_0(M)$ by -1 . This multiplies the Pfaffian by $(-1)^{\lfloor m/2 \rfloor}$ (lemma 7.8), transforms the unit vectors e_{i_k} to $(-1)^{i_k+1} e_{i_k}$ and H_m to $-\mathbb{1}_m^s$. Then, one can easily prove the theorem by induction on n , using the expansion formula for the Pfaffian (proposition 7.10). □

$$\text{pf}(H_0(M)) = \begin{vmatrix} 0 & 1 & -1 & 1 & -1 & 1 & M & M & M & M \\ -1 & 0 & 1 & -1 & 1 & -1 & M & M & M & M \\ 1 & -1 & 0 & 1 & -1 & 1 & M & M & M & M \\ -1 & 1 & -1 & 0 & 1 & -1 & M & M & M & M \\ 1 & -1 & 1 & -1 & 0 & 1 & M & M & M & M \\ -1 & 1 & -1 & 1 & -1 & 0 & M & M & M & M \\ -M^\top & -M^\top & -M^\top & -M^\top & -M^\top & -M^\top & 0 & 0 & 0 & 0 \\ -M^\top & -M^\top & -M^\top & -M^\top & -M^\top & -M^\top & 0 & 0 & 0 & 0 \\ -M^\top & -M^\top & -M^\top & -M^\top & -M^\top & -M^\top & 0 & 0 & 0 & 0 \\ -M^\top & -M^\top & -M^\top & -M^\top & -M^\top & -M^\top & 0 & 0 & 0 & 0 \end{vmatrix}$$

Figure 24: Illustration of theorem 7.18: The summand of $\text{pf}(H_0(M))$ using the marked entries corresponds to the minor of M using rows 1, 3, 5, 6

Remark. Even more so, theorem 7.18 can be proved in a much more direct way. As this is very tedious to write down with all the correct signs, we only hint at it with figure 24. Note that it is very similar to the direct proof of proposition 7.20 in the following section, see the remark after the proof of proposition 7.20.

7.3 Sum of all minors of a matrix

We now consider the sum of all minors of a matrix. The following proposition, which is deduced directly from corollary 5.4, reduces the problem of evaluating this sum to the case just treated in section 7.2:

Proposition 7.20. *Let M be an $m \times n$ matrix. Then we have:*

$$\sum_{k \geq 0} |M|_k = |V_m R_m \quad M|_m$$

$$\sum_{k \geq 0} |M|_k = \left| \begin{smallmatrix} R_n V_n \\ M \end{smallmatrix} \right|_n$$

Remark. The matrix products $V_m R_m$ and $R_n V_n$ have alternating 1's and -1 's on the anti-diagonal (and 0's elsewhere): $V_m R_m$ starts with a 1 in the first row, $R_n V_n$ starts with a 1 in the first column.

Proof. By setting $X := I_m$ and $X' := M$ in corollary 5.4, we obtain the first equation since for any I , we have $|(I_m)_I|_{|I|} = 1$ and thus are left with the following:

$$|V_m R_m \quad M|_m = \sum_{I' \subseteq \{1, \dots, m\}} |M_{I'}|_{|I'|} = \sum_{k \geq 0} |M|_k$$

The second equation follows by applying the first equation to $M^\top$ and then transposing the whole equation. $\square$

Remark. From the proof of proposition 5.2, which underlies corollary 5.4, we can derive a more direct way of proving proposition 7.20, illustrated in figure 25. For details (for instance regarding the signs), consider the proofs of proposition 5.2 and corollary 5.4.

$$(V_m R_m \quad M) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & M & M & M & M \\ 0 & 0 & 0 & 0 & -1 & 0 & M & M & M & M \\ 0 & 0 & 0 & 1 & 0 & 0 & M & M & M & M \\ 0 & 0 & -1 & 0 & 0 & 0 & M & M & M & M \\ 0 & 1 & 0 & 0 & 0 & 0 & M & M & M & M \\ -1 & 0 & 0 & 0 & 0 & 0 & M & M & M & M \end{pmatrix}$$

Figure 25: Illustration of proposition 7.20: The minors of $(V_m R_m \quad M)$ using the marked entries correspond to the 2×2 minors of M using rows 3 and 5

The rest of this section is built up in the same way as section 7.2. Proposition 7.20 allows us to deduce the following theorem of Okada [47, theorem 4], which gives a summation formula for all minors of a matrix:

Theorem 7.21 (Okada). *Let $M = (m_1 \ m_2 \ \dots \ m_n)$ be an $m \times n$ matrix with column vectors $m_1, m_2, \dots, m_n$.*

- *The $(n+1) \times (n+1)$ skew-symmetric matrix $TH_1(M)$ is defined by*

$$TH_1(M) := T_1(M) + H_{n+1} = \begin{pmatrix} 0 & |m_1|_1 + 1 & |m_2|_1 - 1 \dots \dots \dots |m_n|_1 - (-1)^n \\ -|m_1|_1 - 1 & 0 & |m_1 \ m_2|_2 + 1 \dots \dots \dots |m_1 \ m_n|_2 + (-1)^n \\ -|m_2|_1 + 1 & -|m_1 \ m_2|_2 - 1 & 0 \dots \dots \dots \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ -|m_n|_1 + (-1)^n & -|m_1 \ m_n|_2 - (-1)^n \dots \dots \dots \dots \dots 0 \end{pmatrix}$$

- *The $(n+2) \times (n+2)$ skew-symmetric matrix $TH_2(M)$ is defined by*

$$TH_2(M) := \begin{pmatrix} & (-1)^n & \\ & -(-1)^n & \\ & \vdots & \\ & 1 & \\ -(-1)^n & (-1)^n & \dots & -1 & 0 \end{pmatrix} = TH_1(M|) = T_1(M|) + H_{n+2}$$

$$\text{Then } \sum_{k \geq 0} |M|_k = \begin{cases} \text{pf}(TH_1(M)) & n \text{ odd} \\ \text{pf}(TH_2(M)) & n \text{ even} \end{cases}$$

Since we use proposition 7.20, our proof of theorem 7.21 looks quite different from Okada's proof [47, proof of theorem 4]. Our main step is the following lemma, which we prove in the appendix section A.2 as it is rather lengthy, but not at all difficult:

Lemma 7.22. *Let M be an $m \times n$ matrix. Then the following holds:*

$$\text{pf}(T_1 \left(\begin{smallmatrix} R_n V_n \\ M \end{smallmatrix} \right)) = \text{pf}(TH_1(M))$$

Remark. For even n , this equation just reads $0 = 0$.

With this, theorem 7.21 follows directly:

Proof of theorem 7.21. Let n be odd. Using proposition 7.20, theorem 7.16 and lemma 7.22 gives

$$\sum_{k \geq 0} |M|_k = \left| \begin{smallmatrix} R_n V_n \\ M \end{smallmatrix} \right|_n = \text{pf}(T_1 \left(\begin{smallmatrix} R_n V_n \\ M \end{smallmatrix} \right)) = \text{pf}(TH_1(M))$$

The case for even n follows from the case for odd n : Clearly, we have $\sum_{k \geq 0} |M|_k = \sum_{k \geq 0} |(M|)|_k$ and by definition $TH_2(M) = TH_1(M|)$, so applying the case for odd n to $M|$ gives the following for even n :

$$\sum_{k \geq 0} |M|_k = \sum_{k \geq 0} |(M|)|_k = \text{pf}(TH_1(M|)) = \text{pf}(TH_2(M))$$

□

Again, we can use lemma 7.12 to rewrite theorem 7.21 such that the matrices we get are a lot bigger, but directly contain the entries of M . This gives the following theorem, which looks quite similar to [29, theorem 3.5] when taking $A := \mathbb{1}_m^s$ and $B := \mathbb{1}_n^s$. However, this correspondence is not as direct as between theorem 7.18 and [29, theorem 3.2], since [29, theorem 3.5] only sums the minors of even dimension, not all minors.

Theorem 7.23. *Let M be an $m \times n$ matrix.*

- *The $(m+n+1) \times (m+n+1)$ skew-symmetric matrix $HH_1(M)$ is defined by*

$$HH_1(M) := \begin{pmatrix} & & & & -(-1)^m & & & & \\ & & & & (-1)^m & & & & \\ & H_m & & & \vdots & & & M & \\ & & & & 1 & & & & \\ (-1)^m & -(-1)^m & \cdots & -1 & 0 & -(-1)^n & (-1)^n & \cdots & 1 \\ & & & & (-1)^n & & & & \\ & -M^\top & & & -(-1)^n & & & -H_n & \\ & & & & \vdots & & & & \\ & & & & -1 & & & & \end{pmatrix}$$

- *The $(m+n+2) \times (m+n+2)$ skew-symmetric matrix $HH_2(M)$ is defined by*

$$HH_2(M) := \begin{pmatrix} & & & & -(-1)^m & 0 & & & & \\ & & & & (-1)^m & 0 & & & & \\ & H_m & & & \vdots & \vdots & & & M & \\ & & & & 1 & 0 & & & & \\ (-1)^m & -(-1)^m & \cdots & -1 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -1 & 0 & -(-1)^n & (-1)^n & \cdots & 1 \\ & & & & 0 & (-1)^n & & & & \\ & -M^\top & & & 0 & -(-1)^n & & & -H_n & \\ & & & & \vdots & \vdots & & & & \\ & & & & 0 & -1 & & & & \end{pmatrix}$$

$$\text{Then } \sum_{k \geq 0} |M|_k = (-1)^{\lfloor n/2 \rfloor} \begin{cases} \text{pf}(HH_1(M)) & m+n \text{ odd} \\ \text{pf}(HH_2(M)) & m+n \text{ even} \end{cases}$$

For the proof we need the following lemma. It is proved using the expansion of the Pfaffian and elementary row-column operations. However, since writing down the matrices takes up quite some space, the proof is moved to the appendix section A.2.

Lemma 7.24. *Let M be an $m \times n$ matrix. Then the following holds:*

$$\begin{aligned} \text{pf}(HH_1(\underline{M})) &= \text{pf}(HH_2(M)) \\ \text{pf}(HH_2(|M|)) &= (-1)^n \text{pf}(HH_1(M)) \end{aligned}$$

Now we can prove theorem 7.23:

Proof of theorem 7.23. Let us rewrite the theorem using four different cases numbered (1) to (4):

$$\sum_{k \geq 0} |M|_k = (-1)^{\lfloor n/2 \rfloor} \begin{cases} \text{pf}(HH_2(M)) & (1): m \text{ odd}, n \text{ odd} \\ \text{pf}(HH_1(M)) & (2): m \text{ odd}, n \text{ even} \\ \text{pf}(HH_2(M)) & (3): m \text{ even}, n \text{ even} \\ \text{pf}(HH_1(M)) & (4): m \text{ even}, n \text{ odd} \end{cases}$$

Note that as $(-1)^{\lfloor (n+1)/2 \rfloor} (-1)^{\lfloor n/2 \rfloor} = (-1)^n$, lemma 7.24 implies

$$\begin{aligned} (-1)^{\lfloor n/2 \rfloor} \text{pf}(HH_1(\underline{M})) &= (-1)^{\lfloor n/2 \rfloor} \text{pf}(HH_2(M)) \\ (-1)^{\lfloor (n+1)/2 \rfloor} \text{pf}(HH_2(|M|)) &= (-1)^{\lfloor n/2 \rfloor} \text{pf}(HH_1(M)) \end{aligned}$$

Since adding a row or a column of zeros to M does not change the sum of minors, we see now that all four cases are equivalent to each other:

- Case (1) resp. (3) is proved by applying case (4) resp. (2) to the $(m+1) \times n$ matrix $\underline{M}$:

$$\sum_{k \geq 0} |M|_k = \sum_{k \geq 0} |(\underline{M})|_k = (-1)^{\lfloor n/2 \rfloor} \text{pf}(HH_1(\underline{M})) = (-1)^{\lfloor n/2 \rfloor} \text{pf}(HH_2(M))$$

- Case (2) resp. (4) is proved by applying case (1) resp. (3) to the $m \times (n+1)$ matrix $|M|$:

$$\sum_{k \geq 0} |M|_k = \sum_{k \geq 0} |(|M|)|_k = (-1)^{\lfloor (n+1)/2 \rfloor} \text{pf}(HH_2(|M|)) = (-1)^{\lfloor n/2 \rfloor} \text{pf}(HH_1(M))$$

Thus we have $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1)$.

So we only have to prove one of the four cases. We choose case (1), so let m and n be odd. Note that when we multiply the $(m+2)$ -th row-column pair of $HH_2(M)$ by -1 , by lemma 7.8 the Pfaffian is multiplied by -1 and we obtain the matrix

$$\begin{pmatrix} H_{m+1} & -_1M \\ -(-_1M)^\top & -H_{n+1} \end{pmatrix}$$

As $\text{pf}(H_{m+1}) = 1$ (lemma 7.14), applying the second equation in lemma 7.12 with $A := H_{m+1}$, $B := -H_{n+1}$ and $Q := -_1M$ to that matrix gives

$$\text{pf}(HH_2(M)) = -\text{pf} \begin{pmatrix} H_{m+1} & -_1M \\ -(-_1M)^\top & -H_{n+1} \end{pmatrix} = -\text{pf}(-H_{n+1} + (-_1M)^\top H_{m+1}^{-1} (-_1M))$$

By lemmata 7.17 and 7.14, we have $T_0(-_1M) = (-_1M)^\top \mathbb{1}_{m+1}^s (-_1M) = -(-_1M)^\top H_{m+1}^{-1} (-_1M)$ and thus

$$\begin{aligned} \text{pf}(HH_2(M)) &= -\text{pf}(-H_{n+1} + (-_1M)^\top H_{m+1}^{-1} (-_1M)) = \\ &= -\text{pf}(-H_{n+1} - T_0(-_1M)) = \\ &= -\text{pf}(-H_{n+1} - T_1(M)) = \\ &= -\text{pf}(-TH_1(M)) = \end{aligned}$$

$$\begin{aligned}
&= -(-1)^{(n+1)/2} \text{pf}(TH_1(M)) = \\
&= (-1)^{\lfloor n/2 \rfloor} \sum_{k \geq 0} |M|_k
\end{aligned}$$

where we use theorem 7.21 in the very last step. $\square$

7.4 Application to the AST-graph

We can now combine our different minor summation formulas with propositions 5.14 and 5.26 in order to finally find relatively simple expressions for the number of Q -partial matchings of the AST-graph. This gives the following:

Proposition 7.25. *The number of Q -partial matchings of the (half-)reduced AST-graph of size n is given by the following, where $\widetilde{M}^{AQ}$ can correspond to any snake, (half-)line or the symmetric configuration, in particular to those for which $\widetilde{M}^{AQ}$ is given explicitly in proposition 6.5:*

$$|\widetilde{M}^{AQ}|_n = \begin{cases} \text{pf}(T_0((\widetilde{M}^{AQ})^\top)) = (-1)^{\lfloor n/2 \rfloor} \text{pf}(H_0((\widetilde{M}^{AQ})^\top)) & n \text{ even} \\ \text{pf}(T_1((\widetilde{M}^{AQ})^\top)) = (-1)^{\lfloor n/2 \rfloor} \text{pf}(H_1((\widetilde{M}^{AQ})^\top)) & n \text{ odd} \end{cases}$$

Furthermore, the same number is also given by

$$\sum_{k \geq 0} |S_n^{V'H}|_k = \begin{cases} \text{pf}(TH_1(S_n^{V'H})) & n \text{ odd} \\ \text{pf}(TH_2(S_n^{V'H})) & n \text{ even} \end{cases} = (-1)^{\lfloor n/2 \rfloor} \text{pf}(HH_2(S_n^{V'H}))$$

Remark.

- As can be seen from the respective definitions, the matrices appearing in this proposition have the following sizes:

$$\begin{aligned}
T_0((\widetilde{M}^{AQ})^\top) &: n \times n & H_0((\widetilde{M}^{AQ})^\top) &: 3n \times 3n \\
T_1((\widetilde{M}^{AQ})^\top) &: (n+1) \times (n+1) & H_1((\widetilde{M}^{AQ})^\top) &: (3n+1) \times (3n+1) \\
TH_1(S_n^{V'H}) &: (n+1) \times (n+1) & HH_2(S_n^{V'H}) &: (2n+2) \times (2n+2) \\
TH_2(S_n^{V'H}) &: (n+2) \times (n+2) & &
\end{aligned}$$

- Since squaring the Pfaffian gives the determinant (proposition 7.4) and the number of Q -partial matchings has to be nonnegative, we can also replace each Pfaffian in the proposition by the square root of the determinant, and simply leave out all $(-1)^{\lfloor n/2 \rfloor}$.

Proof. For the first expression, we simply combine propositions 5.14 and 5.26 with the minor summation formulas theorem 7.16 and theorem 7.18. Note that we have to take the transpose of $\widetilde{M}^{AQ}$ in order to sum its $n \times n$ minors. The second expression is obtained by considering the first line, for which we have $M^A Q \setminus A = S_n^{V'H}$ by proposition 6.5. Thus, combining proposition 5.14 with the minor summation formulas theorem 7.21 and theorem 7.23 gives the result.

It is noteworthy that applying proposition 7.20 to $\sum_{k \geq 0} |S_n^{V'H}|_k$ would actually again give $\sum_{k \geq 0} |S_n^{V'H}|_k = |V_n R_n S_n^{V'H}|_n = |\widetilde{M}_1^l|_n$. The reason for this is that corollary 5.11 and proposition 7.20 both follow quite directly from corollary 5.4. $\square$

$$HH_2(S_5^{V'H}) = \begin{pmatrix} 0 & 1 & -1 & 1 & -1 & 1 & 0 & 1 & 4 & 16 & 68 & 304 \\ -1 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 1 & 8 & 48 & 264 \\ 1 & 1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 1 & 12 & 96 \\ -1 & 1 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 16 \\ 1 & -1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & -1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 1 & -1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & -1 & 1 \\ -4 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 1 & -1 \\ -16 & -8 & -1 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & -1 & 1 \\ -68 & -48 & -12 & -1 & 0 & 0 & 1 & 1 & -1 & 1 & 0 & -1 \\ -304 & -264 & -96 & -16 & -1 & 0 & -1 & -1 & 1 & -1 & 1 & 0 \end{pmatrix}$$

Figure 26: One of the matrices appearing in proposition 7.25, here for $n = 5$

As can be seen in the remark above, the smallest matrices of which the Pfaffian gives us the sought-after number in proposition 7.25 are of size n resp. $n + 1$. We can take one of the $\widetilde{M}^{AQ}$ which we explicitly know in terms of Schröder matrices (proposition 6.5), but we still need to do some calculations, expressible for instance as matrix multiplications (see lemma 7.17), in order to determine the matrices on the left side in the remark. The matrices on the right side however are directly given in terms of $\widetilde{M}^{AQ}$ or $S_n^{V'H}$. This makes $HH_2(S_n^{V'H})$ quite interesting, since it is only of size $2n + 2$ and the entries of $S_n^{V'H}$ can be determined by their recurrence relation or any other method given in sections 6.3 and 6.4. Figure 26 shows this matrix for the case $n = 5$.

Finally, there is one more trick we can apply: Remember that $\widetilde{M}_{n+1/2}^l = (V_n S_n^{DH} R_n \ S_n^{DH})$ is nearly symmetric (see figures 37a and 22). This allows us to further modify $T_{0/1}((\widetilde{M}_{n+1/2}^l)^\top)$, bringing our minimal sizes for the matrices considerably down: We obtain a size of $\lceil n/2 \rceil$ in total, and a size of $3\lceil n/2 \rceil$ for matrices given directly by S_n^{DH} without further calculation. However, we loose the skew-symmetry in this process, forcing us to switch to determinants instead of Pfaffians. Note that as polynomials, determinants have double the degree of Pfaffians. Still, finding these smaller matrices is quite interesting. To get there, we need further definitions:

Definition 7.26. Consider the matrix S_n^{DH} . Then:

- For $1 \leq i \leq n$, we denote by s_i the i -th row vector of S_n^{DH} and set $s_0 := (0 \ 0 \ \dots \ 0)$ (a row vector of length n). By s'_i , we denote the row vector obtained by reversing s_i and, if i is even, changing signs. Thus we have $s'_i := -(-1)^i s_i R_n$.
- By $S_n^{D^1H}$ resp. $S_n^{D^2H}$, we denote the $\lceil n/2 \rceil \times n$ resp. $\lfloor n/2 \rfloor \times n$ matrix consisting of the odd resp. even rows of S_n^{DH} . Furthermore, $S_n^{D^0H}$ denotes the $(\lfloor n/2 \rfloor + 1) \times (n + 1)$ matrix we obtain by adding s_0 as a first row to $S_n^{D^2H}$ and then e_1 as a first column.
- $\mathbb{1}_n := (1)_{1 \leq i, j \leq n}$ denotes the constant $n \times n$ matrix consisting only of 1's.

Remark.

- The idea behind the notation for $S_n^{D^0H}$ is that in definition 6.2, one imagines additional vertices D_0 and H_0 , connected to each other by a single path and not connected to any other vertices.
- Note that if n is even, then $S_n^{D^1H}$ and $S_n^{D^2H}$ have equal dimensions $(n/2) \times n$. If n is odd, then $S_n^{D^0H}$ and $S_n^{D^1H}$ (the matrix obtained from $S_n^{D^1H}$ by adding a first column of 0's, see definition 7.15) have equal dimensions $((n + 1)/2) \times (n + 1)$.

Now we can start working with $\widetilde{M}_{n+1/2}^l$. Note that we use the Kronecker delta $\delta_{i,j} := \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$.

Lemma 7.27. *The entries of $T_0\left((\widetilde{M}_{n+1/2}^l)^\top\right)$ and $T_1\left((\widetilde{M}_{n+1/2}^l)^\top\right)$ are given as follows:*

$$T_0\left((\widetilde{M}_{n+1/2}^l)^\top\right)_{ij} = \begin{cases} 0 & i \equiv j \pmod{2} \\ 2s_i(\mathbb{1}_n^s - (-1)^i \mathbb{1}_n) s_j^\top & i \not\equiv j \pmod{2} \end{cases}$$

$$T_1\left((\widetilde{M}_{n+1/2}^l)^\top\right)_{ij} = \begin{cases} 0 & i \equiv j \pmod{2} \\ 2(\delta_{i-1,0} \ s_{i-1})(\mathbb{1}_n^s - (-1)^{i-1} |\overline{\mathbb{1}_n}|)(\delta_{j-1,0} \ s_{j-1})^\top & i \not\equiv j \pmod{2} \end{cases}$$

Proof. Since $\widetilde{M}_{n+1/2}^l = (V_n S_n^{DH} R_n \ S_n^{DH})$ (proposition 6.5), the i -th row of $\widetilde{M}_{n+1/2}^l$ is given by $(s'_i \ s_i)$ and the entries of $T_0\left((\widetilde{M}_{n+1/2}^l)^\top\right)$ are given by the sums of 2×2 minors of any two rows of $\widetilde{M}_{n+1/2}^l$. By applying lemma 7.17 and using the fact that $\mathbb{1}_{2n}^s = \begin{pmatrix} \mathbb{1}_n^s & \mathbb{1}_n \\ -\mathbb{1}_n & \mathbb{1}_n^s \end{pmatrix}$, as well as $R_n \mathbb{1}_n^s R_n = -\mathbb{1}_n^s$ and $R_n \mathbb{1}_n = \mathbb{1}_n = \mathbb{1}_n R_n$, we get the following:

$$\begin{aligned} T_0\left((\widetilde{M}_{n+1/2}^l)^\top\right)_{ij} &= \begin{vmatrix} (s'_i)^\top & (s'_j)^\top \\ (s_i)^\top & (s_j)^\top \end{vmatrix}_2 = \\ &= (s'_i \ s_i) \mathbb{1}_{2n}^s (s'_j \ s_j)^\top = \\ &= s'_i \mathbb{1}_n^s (s'_j)^\top + s'_i \mathbb{1}_n s_j^\top - s_i \mathbb{1}_n (s'_j)^\top + s_i \mathbb{1}_n^s s_j^\top = \\ &= (-1)^{i+j} s_i R_n \mathbb{1}_n^s R_n s_j^\top - (-1)^i s_i R_n \mathbb{1}_n s_j^\top + (-1)^j s_i \mathbb{1}_n R_n s_j^\top + s_i \mathbb{1}_n^s s_j^\top = \\ &= -(-1)^{i+j} s_i \mathbb{1}_n^s s_j^\top - (-1)^i s_i \mathbb{1}_n s_j^\top + (-1)^j s_i \mathbb{1}_n s_j^\top + s_i \mathbb{1}_n^s s_j^\top = \\ &= \begin{cases} 0 & i \equiv j \pmod{2} \\ 2s_i(\mathbb{1}_n^s - (-1)^i \mathbb{1}_n) s_j^\top & i \not\equiv j \pmod{2} \end{cases} \end{aligned}$$

For $T_1\left((\widetilde{M}_{n+1/2}^l)^\top\right)$, we have to distinguish several cases: For $i, j \neq 1$, we have

$$\begin{aligned} T_1\left((\widetilde{M}_{n+1/2}^l)^\top\right)_{ij} &= T_0\left((\widetilde{M}_{n+1/2}^l)^\top\right)_{i-1,j-1} = \\ &= \begin{cases} 0 & i-1 \equiv j-1 \pmod{2} \\ 2s_{i-1}(\mathbb{1}_n^s - (-1)^{i-1} \mathbb{1}_n) s_{j-1}^\top & i-1 \not\equiv j-1 \pmod{2} \end{cases} = \\ &= \begin{cases} 0 & i \equiv j \pmod{2} \\ 2(0 \ s_{i-1})(\mathbb{1}_{n+1}^s - (-1)^{i-1} |\overline{\mathbb{1}_n}|)(0 \ s_{j-1})^\top & i \not\equiv j \pmod{2} \end{cases} \end{aligned}$$

For $i = 1$ and $j \neq 1$, we have

$$\begin{aligned} T_1\left((\widetilde{M}_{n+1/2}^l)^\top\right)_{1,j} &= |s'_{j-1} \ s_{j-1}|_1 = \\ &= \begin{cases} 0 & j-1 \equiv 0 \pmod{2} \\ 2(1 \ 1 \ \dots \ 1) s_{j-1}^\top & j-1 \equiv 1 \pmod{2} \end{cases} = \\ &= \begin{cases} 0 & 1 \equiv j \pmod{2} \\ 2(1 \ s_0)(\mathbb{1}_{n+1}^s - (-1)^{i-1} |\overline{\mathbb{1}_n}|)(0 \ s_{j-1})^\top & 1 \not\equiv j \pmod{2} \end{cases} \end{aligned}$$

Analogously, for $j = 1$ and $i \neq 1$ we get

$$T_1\left((\widetilde{M}_{n+1/2}^l)^\top\right)_{i,1} = \begin{cases} 0 & i \equiv 1 \pmod{2} \\ 2(0 \ s_{i-1})(\mathbb{1}_{n+1}^s - (-1)^{i-1}|\overline{\mathbb{1}}_n)(1 \ s_0)^\top & i \not\equiv 1 \pmod{2} \end{cases}$$

And finally, $i = j = 1$ gives $T_1\left((\widetilde{M}_{n+1/2}^l)^\top\right)_{1,1} = 0$. So indeed, for all i, j we have

$$T_1\left((\widetilde{M}_{n+1/2}^l)^\top\right)_{ij} = \begin{cases} 0 & i \equiv j \pmod{2} \\ 2(\delta_{i-1,0} \ s_{i-1})(\mathbb{1}_n^s - (-1)^{i-1}|\overline{\mathbb{1}}_n)(\delta_{j-1,0} \ s_{j-1})^\top & i \not\equiv j \pmod{2} \end{cases}$$

□

Due to the special structures of $T_0\left((\widetilde{M}_{n+1/2}^l)^\top\right)$ and $T_1\left((\widetilde{M}_{n+1/2}^l)^\top\right)$ given in lemma 7.27, we now can write their Pfaffian as the determinant of a square matrix of only half the size:

Corollary 7.28. *The number of Q -partial matchings of the reduced AST-graph of size n can be expressed as the determinant of an $(n/2) \times (n/2)$ (for even n) resp. $((n+1)/2) \times ((n+1)/2)$ (for odd n) matrix as follows:*

$$|\widetilde{M}_{n+1/2}^l|_n = \begin{cases} 2^{n/2} \cdot |\det(S_n^{D^1H}(\mathbb{1}_n^s + \mathbb{1}_n)(S_n^{D^2H})^\top)| & n \text{ even} \\ 2^{(n+1)/2} \cdot |\det(|S_n^{D^1H}|(\mathbb{1}_{n+1}^s + |\overline{\mathbb{1}}_n|)(S_n^{D^0H})^\top)| & n \text{ odd} \end{cases}$$

Proof. Set $X := S_n^{D^1H}(\mathbb{1}_n^s + \mathbb{1}_n)(S_n^{D^2H})^\top$ for now. From lemma 7.27, we see that the matrix consisting of the entries in odd rows and even columns of $T_0\left((\widetilde{M}_{n+1/2}^l)^\top\right)$ is exactly $2X$. Since $T_0\left((\widetilde{M}_{n+1/2}^l)^\top\right)$ is skew-symmetric, the entries in even rows and odd columns form the matrix $-2X^\top$. Thus, when exchanging row-column-pairs in $T_0\left((\widetilde{M}_{n+1/2}^l)^\top\right)$ such that all odd row-column pairs come first and all even row-column pairs afterwards, and pulling out the factor 2, we obtain the matrix $2\begin{pmatrix} 0 & X \\ -X^\top & 0 \end{pmatrix}$.

Note that we can just consider the absolute value of the Pfaffian as the number of Q -partial matchings is definitely nonnegative. Combining the above information with lemma 7.8 and proposition 7.4, we indeed obtain the following:

$$\begin{aligned} |\text{pf}(T_0\left((\widetilde{M}_{n+1/2}^l)^\top\right))| &= 2^{n/2} \cdot |\text{pf}\begin{pmatrix} 0 & X \\ -X^\top & 0 \end{pmatrix}| = \\ &= 2^{n/2} \cdot \sqrt{\det\begin{pmatrix} 0 & X \\ -X^\top & 0 \end{pmatrix}} = \\ &= 2^{n/2} \cdot \sqrt{|\det(X) \cdot \det(-X)|} = \\ &= 2^{n/2} \cdot |\det(X)| \end{aligned}$$

By proposition 7.25, the result follows. The case for odd n is proved analogously: $T_1\left((\widetilde{M}_{n+1/2}^l)^\top\right)$ is an $(n+1) \times (n+1)$ matrix and by lemma 7.27, the matrix consisting of its even rows and odd columns is $2(|S_n^{D^1H}|(\mathbb{1}_{n+1}^s + |\overline{\mathbb{1}}_n|)(S_n^{D^0H})^\top)$, since the rows of $|S_n^{D^1H}|$ and $S_n^{D^0H}$ are $(\delta_{i-1,0} \ s_{i-1})$ for even i and odd i with $1 \leq i \leq n+1$ respectively. □

$$S_6^{D^1H}(\mathbb{1}_6^s + \mathbb{1}_6)(S_6^{D^2H})^\top = \begin{pmatrix} 204372 & 31784 & 636 \\ 70004 & 10920 & 220 \\ 3716 & 584 & 12 \end{pmatrix}$$

$$(|S_7^{D^1H})(\mathbb{1}_8^s + |\overline{\mathbb{1}}_7)(S_7^{D^0H})^\top = \begin{pmatrix} -2321 & 4208804 & 780488 & 34668 \\ -897 & 1594964 & 296040 & 13180 \\ -81 & 133620 & 24872 & 1116 \\ -1 & 1412 & 264 & 12 \end{pmatrix}$$

Figure 27: The matrices appearing in corollary 7.28 for $n = 6$ and $n = 7$

The matrices in corollary 7.28 are surprisingly small, only of size $\lceil n/2 \rceil$. Figure 27 shows examples of them for $n = 6$ and $n = 7$. In contrast, the matrices in the following corollary 7.29 have the threefold size, but their entries are given directly by S_n^{DH} . Also for these, we give examples for $n = 6$ and $n = 7$ in figure 28.

Corollary 7.29. *The number of Q -partial matchings of the reduced AST-graph of size n can be expressed as the determinant of an $(3n/2) \times (3n/2)$ (for even n) resp. $(3(n+1)/2) \times (3(n+1)/2)$ (for odd n) matrix as follows:*

$$|\widetilde{M}_{n+1/2}^l|_n = \begin{cases} 2^{n/2} \cdot \left| \det \begin{pmatrix} V_n \mathbb{1}_n V_n - H_n & (S_n^{D^2H})^\top \\ S_n^{D^1H} & 0 \end{pmatrix} \right| & n \text{ even} \\ 2^{(n+1)/2} \cdot \left| \det \begin{pmatrix} e_1 e_1^\top - H_{n+1} & (S_n^{D^0H})^\top \\ |S_n^{D^1H} & 0 \end{pmatrix} \right| & n \text{ odd} \end{cases}$$

Proof. Note that $\mathbb{1}_n^s + \mathbb{1}_n$ is a regular matrix with determinant 1 (as it is an upper triangular matrix with 1's on the main diagonal). Thus, for even n , setting $A := (\mathbb{1}_n^s + \mathbb{1}_n)^{-1}$, $Q := (S_n^{D^1H})^\top$, $Q' := (S_n^{D^2H})^\top$ and $B := 0$ in the first equation in lemma 7.12, and then applying the determinant to that equation gives

$$\begin{pmatrix} (\mathbb{1}_n^s + \mathbb{1}_n)^{-1} & 0 \\ 0 & S_n^{D^1H}(\mathbb{1}_n^s + \mathbb{1}_n)(S_n^{D^2H})^\top \end{pmatrix} =$$

$$= \begin{pmatrix} I_n & 0 \\ S_n^{D^1H}(\mathbb{1}_n^s + \mathbb{1}_n) & I_{n/2} \end{pmatrix} \begin{pmatrix} (\mathbb{1}_n^s + \mathbb{1}_n)^{-1} & (S_n^{D^2H})^\top \\ -S_n^{D^1H} & 0 \end{pmatrix} \begin{pmatrix} I_n & -(\mathbb{1}_n^s + \mathbb{1}_n)(S_n^{D^2H})^\top \\ 0 & I_{n/2} \end{pmatrix}$$

$$\det(S_n^{D^1H}(\mathbb{1}_n^s + \mathbb{1}_n)(S_n^{D^2H})^\top) = \det \begin{pmatrix} (\mathbb{1}_n^s + \mathbb{1}_n)^{-1} & (S_n^{D^2H})^\top \\ -S_n^{D^1H} & 0 \end{pmatrix}$$

Thus, for even n , we only have to show that $(\mathbb{1}_n^s + \mathbb{1}_n)^{-1} = V_n \mathbb{1}_n V_n - H_n$, then the result follows by corollary 7.28. Indeed, simply calculating the products (using the fact that n is even) gives:

$$\begin{aligned} (\mathbb{1}_n^s + \mathbb{1}_n)(V_n \mathbb{1}_n V_n - H_n) &= \mathbb{1}_n^s V_n \mathbb{1}_n V_n - \mathbb{1}_n^s H_n + \mathbb{1}_n V_n \mathbb{1}_n V_n - \mathbb{1}_n H_n = \\ &= -\mathbb{1}_n V_n + I_n + 0 + \mathbb{1}_n V_n = \\ &= I_n \end{aligned}$$

The case for odd n works out analogously, as in this case $(\mathbb{1}_{n+1}^s + |\overline{\mathbb{1}}_n)^{-1} = e_1 e_1^\top - H_{n+1}$ can be checked in an equally simple way. ($e_1 e_1^\top$ is the matrix having a single 1 in the top-left corner.) $\square$

$$\begin{pmatrix} V_6 \mathbb{1}_6 V_6 - H_6 & (S_6^{D^2 H})^\top \\ S_6^{D^1 H} & 0 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 2 & -2 & 2 & -2 & 0 & 0 & 0 \\ 0 & 1 & -2 & 2 & -2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 2 & -2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 2 & 16 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 68 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 304 & 48 & 1 \\ 1 & 2 & 6 & 22 & 90 & 394 & 0 & 0 & 0 \\ 0 & 0 & 1 & 6 & 30 & 146 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 10 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} e_1 e_1^\top - H_8 & (S_7^{D^0 H})^\top \\ |S_7^{D^1 H} & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & -1 & 1 & -1 & 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & -1 & 1 & -1 & 1 & 0 & 4 & 0 & 0 \\ -1 & 1 & -1 & 1 & 0 & -1 & 1 & -1 & 0 & 16 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 & 0 & -1 & 1 & 0 & 68 & 8 & 0 \\ -1 & 1 & -1 & 1 & -1 & 1 & 0 & -1 & 0 & 304 & 48 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & 0 & 0 & 1412 & 264 & 12 \\ 0 & 1 & 2 & 6 & 22 & 90 & 394 & 1806 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 6 & 30 & 146 & 714 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 70 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Figure 28: The matrices appearing in corollary 7.29 for $n = 6$ and $n = 7$

8 Summary and outlook

Over this whole paper, we have worked on enumerating Q -partial matchings of the AST-graph, by transforming the AST-graph, bringing the matchings in bijection with other combinatorial objects, enumerating and modifying these further, etc. For this, we have developed generalizations of quite a few different mathematical theorems. However, remember that the number of Q -partial matchings of the AST-graph is actually known, thanks to another method (see corollary 2.15). Consequently, we also know the numbers of all the objects we derived from the AST-graph.

Section 8.1 specifies all these numbers in form of corollaries, and in this way gives a concise overview of the various stages we went through when going from Q -partial matchings of the AST-graph to determinants of certain matrices. Note that by our work, these corollaries are equivalent to each other. In section 8.2, we then summarize all the interesting general theorems we found and used in this paper. Finally, section 8.3 briefly lists the most obvious points where further study could be taken up.

In order to find the definitions of the objects that appear in this section, use section B. Also, remember that unless stated otherwise, we always have $n, m, m' \geq 0$ (see section 1.4).

8.1 From Q -partial matchings to a determinant

In the following, for each section of the paper we give a very brief description of how the enumeration of Q -partial matchings is advanced by it, as well as one or two enumerative corollaries following directly from that. All of these corollaries (except for corollary 8.2, which includes an additional parameter r) are thus equivalent to the fact that the AST-graph of size n has $2^{\binom{n+1}{2}}$ Q -partial matchings, which is also our starting point. In fact, it suffices to choose a corollary and for each n prove a single instance of it (i.e. only one statement, one configuration, one equation etc.) in order for all statements to follow.

Section 2: Ayer, Behrend and Fischer [4] showed that ASMs and ASTs have the same number and share certain statistics (theorem 2.6). Thus there is an equal number of perfect matchings of

the Aztec graph and Q -partial matchings of the AST-graph, of which the former is known, and so (corollary 2.15):

Corollary 8.1 (Ayyer, Behrend, Fischer). *There are $2^{\binom{n+1}{2}}$ Q -partial matchings of the AST-graph of size n .*

We can even refine that enumeration, finding that perfect matchings of the Aztec graph and Q -partial matchings of the AST-graph still share their number when fixing the number of edges tilted in a certain direction (theorem 2.19):

Corollary 8.2. *For each $0 \leq r \leq \binom{n+1}{2} = n(n+1)/2$, there are $\binom{n(n+1)/2}{r}$ Q -partial matchings of the AST-graph of size n containing exactly $2r$ edges tilted to the right.*

Section 3: By a generalization of Ciucu's work [14], we can go from the AST-graph to what we call the 'half-reduced' AST-graph of one size smaller, losing exactly a factor of 2^n (theorem 3.24). Additionally, Q -partial matchings of the half-reduced AST-graph trivially stand in bijection to Q -partial matchings of the 'reduced' AST-graph (lemma 3.23). Thus, as $2^{\binom{n+1}{2}} = 2^n \cdot 2^{\binom{n}{2}}$:

Corollary 8.3. *Both the half-reduced and the reduced AST-graph of size n have $2^{\binom{n+1}{2}}$ Q -partial matchings.*

Section 4: Applying our general 'MaPaf-bijection' in different ways to the (half-)reduced AST-graph gives bijections of its Q -partial matchings to certain path families (proposition 4.8), and so we get:

Corollary 8.4. *For any of the directed graphs $\widehat{G}$ corresponding to one of the configurations specified in definition 4.6, there are $2^{\binom{n+1}{2}}$ families of n disjoint paths from A to Q in $\widehat{G}$. Additionally, for the configurations with $A \subseteq Q$, there are $2^{\binom{n+1}{2}}$ families of arbitrary many disjoint paths from A to $Q \setminus A$ in $\widehat{G}$.*

Section 5: With the help of the LGV lemma, for some of the configurations, we can enumerate these path families in terms of minors of certain matrices (proposition 5.14). By operations transforming the graphs and at the same time their corresponding matrices into each other, we also get 'half-lines', a configuration type not coming from MaPaf-bijections, but giving the same sum of minors (proposition 5.26), so we have:

Corollary 8.5. *For any matrix $\widetilde{M}^{AQ}$ corresponding to a snake, a (half-)line or the symmetric configuration, we have $|\widetilde{M}^{AQ}|_n = 2^{\binom{n+1}{2}}$.*

For the matrix $M^{A \ Q \setminus A}$ corresponding to the first line, we also have $\sum_{k \geq 0} |M^{A \ Q \setminus A}|_k = 2^{\binom{n+1}{2}}$.

Section 6: For some of these matrices, we can express $\widetilde{M}^{AQ}$ in terms of Schröder matrices (proposition 6.5). This gives (among other expressions):

Corollary 8.6. *We have the following sums of minors:*

$$\begin{aligned} |(V_n R_n \ S_n^{V'H})|_n &= 2^{\binom{n+1}{2}} \\ |(V_n S_n^{DH} R_n \ S_n^{DH})|_n &= 2^{\binom{n+1}{2}} \\ \sum_{k \geq 0} |S_n^{V'H}|_k &= 2^{\binom{n+1}{2}} \end{aligned}$$

Section 7: Applying different minor summation formulas, we can write the sums of minors in terms of single Pfaffians (proposition 7.25), resulting in:

Corollary 8.7. *For any matrix $\widetilde{M}^{AQ}$ corresponding to a snake, a (half-)line or the symmetric configuration (in particular for those given explicitly in proposition 6.5), we have:*

$$\begin{array}{ll}
\text{If } n \text{ is even :} & \text{If } n \text{ is odd :} \\
\text{pf}\left(T_0\left((\widetilde{M}^{AQ})^\top\right)\right) = 2^{\binom{n+1}{2}} & \text{pf}\left(T_1\left((\widetilde{M}^{AQ})^\top\right)\right) = 2^{\binom{n+1}{2}} \\
\text{pf}\left(H_0\left((\widetilde{M}^{AQ})^\top\right)\right) = (-1)^{\lfloor n/2 \rfloor} 2^{\binom{n+1}{2}} & \text{pf}\left(H_1\left((\widetilde{M}^{AQ})^\top\right)\right) = (-1)^{\lfloor n/2 \rfloor} 2^{\binom{n+1}{2}} \\
\text{pf}\left(TH_2(S_n^{V'H})\right) = 2^{\binom{n+1}{2}} & \text{pf}\left(TH_1(S_n^{V'H})\right) = 2^{\binom{n+1}{2}} \\
\text{pf}\left(HH_2(S_n^{V'H})\right) = (-1)^{\lfloor n/2 \rfloor} 2^{\binom{n+1}{2}} & \text{pf}\left(HH_2(S_n^{V'H})\right) = (-1)^{\lfloor n/2 \rfloor} 2^{\binom{n+1}{2}}
\end{array}$$

Finally, since one of these $\widetilde{M}^{AQ}$, namely $\widetilde{M}_{n+1/2}^l = (V_n S_n^{DH} R_n \ S_n^{DH})$, is vertically symmetric when ignoring signs, $T_{0/1}\left((\widetilde{M}^{AQ})^\top\right)$ in this case can further be modified and made smaller (corollaries 7.28 and 7.29), giving:

Corollary 8.8. *If n is even, we have:*

$$\begin{aligned}
|\det(S_n^{D^1H}(\mathbf{1}_n^s + \mathbf{1}_n)(S_n^{D^2H})^\top)| &= 2^{n^2/2} \\
|\det\begin{pmatrix} V_n \mathbf{1}_n V_n - H_n & (S_n^{D^2H})^\top \\ S_n^{D^1H} & 0 \end{pmatrix}| &= 2^{n^2/2}
\end{aligned}$$

If n is odd, we have:

$$\begin{aligned}
|\det(|S_n^{D^1H})(\mathbf{1}_{n+1}^s + |\overline{\mathbf{1}}_n)(S_n^{D^0H})^\top)| &= 2^{(n^2-1)/2} \\
|\det\begin{pmatrix} e_1 e_1^\top - H_{n+1} & (S_n^{D^0H})^\top \\ |S_n^{D^1H} & 0 \end{pmatrix}| &= 2^{(n^2-1)/2}
\end{aligned}$$

8.2 Interesting theorems found on the way

We here present the general theorems we have developed in this paper. Since this is only a brief summary, we shorten and combine most of our findings. The full statements are referred to at the beginning of each theorem.

Inspired by Ayyer et al. doing the same for ASMs [4], we are able to express the numbers of 0's in an AST receiving a certain local arrow configuration in terms of N_+ , inv and a new parameter θ_r resp. θ_l :

Theorem 8.9 (Lemma 2.23). *For an AST A of size n , we have the following:*

$$\begin{aligned}
N_0^{br}(A) &= \text{inv}(A) - \theta_r(A) & N_0^{bl}(A) &= \binom{n+1}{2} - N_+(A) - \text{inv}(A) - \theta_l(A) \\
N_0^{tl}(A) &= \text{inv}(A) + \theta_r(A) & N_0^{tr}(A) &= \binom{n+1}{2} - N_+(A) - \text{inv}(A) + \theta_l(A)
\end{aligned}$$

Broadening Ciucu's notion of a 'cellular graph' and considering C -partial matchings instead of only perfect matchings, we obtain the following generalization of his work [14]:

Theorem 8.10 (Proposition 3.16, corollaries 3.7, 3.8). *Let G be a cellular graph.*

- (i). *Let $X \subseteq C$ be a subset of the corner vertices of G and let P be an ASP on G° compatible with X . Then there are $2^{P_c + P_1}$ perfect matchings of $G \setminus X$ of type P . Each C -partial matching of G is of this form, for unique P and X .*
- (ii). *Let $X \cup Y = C$ with $X \cap Y = \emptyset$ be a partition of the corner vertices of G . Then*

$$m(G \setminus X) \cdot 2^{|X|/4} = m(G \setminus Y) \cdot 2^{|Y|/4}$$

Furthermore, if each edge is assigned the weight $1/\sqrt{2}$, then $m(G \setminus X) = m(G \setminus Y)$.

We collect the various known bijections between perfect matchings of bipartite graphs and path families (for instance the bijections for the Aztec graph or for rhombus tilings of a hexagon) into a common framework, which relies on a slightly adapted graph having a unique perfect matching. We also apply this to Q -partial matchings in various settings:

Theorem 8.11 (Theorem 4.3, corollaries 4.4, 4.5). *Let (G, A) fulfill the MaPaf-condition. Then $\widehat{G}$ is an acyclic directed graph in which the vertices from A are sources. We have the following:*

- (i). *Let $B \subseteq G_r$ with $|B| = n_A$. Then perfect matchings of $G \setminus B$ stand in bijection to families of n_A disjoint paths from A to B in $\widehat{G}$.*
- (ii). *Let $B, Q \subseteq G_r$ with $B \cap Q = \emptyset$ and $|B| \leq n_A \leq |B \cup Q|$. Then Q -partial matchings of $G \setminus B$ stand in bijection to families of n_A disjoint paths from A to $B \cup Q$ in $\widehat{G}$ using all vertices from B as endpoints.*
- (iii). *Let $Q \subseteq G_r$ with $A \subseteq Q$. Then Q -partial matchings of G stand in bijection to families of arbitrary many disjoint paths from A to $Q \setminus A$ in $\widehat{G}$.*

For the Q -partial matchings, the vertices which are not covered by a matching stand in a simple relation to the starting points and endpoints the corresponding path family uses.

The following theorem about sums of minors, which is proved by elementary linear algebra, is the crucial underlying fact both for theorem 8.13 and the second equation in theorem 8.15:

Theorem 8.12 (Proposition 5.2, corollary 5.4). *Let X and X' be $n \times m$ resp. $n \times m'$ matrices. Then we have the following:*

$$\sum_{\substack{I \cup I' = \{1, \dots, n\} \\ I \cap I' = \emptyset}} |X_I|_{|I|} \cdot |X'_{I'}|_{|I'|} = |V_n X R_m \quad X'|_n$$

In particular, if $m + m' = n$ holds, we have:

$$\sum_{\substack{I \cup I' = \{1, \dots, n\} \\ I \cap I' = \emptyset \\ |I| = m, |I'| = m'}} \det(X_I) \det(X'_{I'}) = \det(V_n X R_m \quad X')$$

Thanks to theorem 8.12, we can give a generalization of the usual ‘planar’ version of the LGV lemma, which now splits the endpoints of the paths into two sets and has intersection conditions for each of them separately:

Theorem 8.13 (Corollaries 5.9, 5.11). *Let G be an acyclic directed graph and let $A = \{A_1, \dots, A_m\}$ and $B = \{B_1, \dots, B_n\}$ with $m \leq n$ be sets of vertices of G . Suppose there exists an index λ such that:*

- *The reverse intersection condition for paths from A to $\{B_1, \dots, B_\lambda\}$ holds.*
- *The intersection condition for paths from A to $\{B_{\lambda+1}, \dots, B_n\}$ holds.*
- *The non-intersection condition for paths from A to $\{B_1, \dots, B_\lambda\}$ and $\{B_{\lambda+1}, \dots, B_n\}$ holds.*

Let $\widetilde{M}^{AB}$ be the matrix we obtain from M^{AB} by changing the sign of the (i, j) -th matrix entry for every pair (i, j) where i is even and $j \leq \lambda$. Then $|\widetilde{M}^{AB}|_m$ is the number of families of m disjoint paths from A to B in G . If $m = n$, then $|\widetilde{M}^{AB}|_m$ reduces to $\det(\widetilde{M}^{AB})$.

We consider the numbers of Schröder paths with arbitrary starting points and endpoints, and define different ‘Schröder matrices’ collecting these numbers. By standard methods, we find linear recurrences and a generating function for them. From our work with the reduced AST-graph, we can additionally derive relations between the Schröder matrices, as well as express them as products of elementary matrices:

Theorem 8.14 (Lemma 6.9, corollary 6.11, proposition 6.12, theorem 6.14). *The entries of the Schröder matrices S_n^{DH} , S_n^{VH} , $S_n^{V'H}$ and $S_n^{DD'}$ can be determined by simple linear recurrences given in proposition 6.8. Moreover, they satisfy the following relations:*

$$\begin{aligned} T_n^+ S_n^{VH} &= S_n^{V'H} T_n^- \\ V_n S_n^{V'H} &= (S_n^{DH})^{-1} V_n S_n^{DH} & (V_n S_n^{V'H})^2 &= I_n \\ V_n S_n^{VH} &= (S_n^{DH} T_n^-)^{-1} V_n S_n^{DH} T_n^- & (V_n S_n^{VH})^2 &= I_n \\ S_n^{DD'} T_n^+ V_n &= S_n^{DH} R_n (S_n^{DH})^{-1} & (S_n^{DD'} T_n^+ V_n)^2 &= I_n \end{aligned}$$

Schröder matrices can be expressed in terms of elementary matrices. For instance, we have the following expressions for $n \geq 2$:

$$\begin{aligned} S_n^{DH} &= \prod_{j=2}^n \left(\prod_{i=j}^n O_{i-1,i}^+ \prod_{i=n}^j O_{i-1,i}^+ \right) \\ S_n^{DD'} &= \prod_{i=2}^n O_{i-1,i}^+ \prod_{j=1}^{n-1} \left(\prod_{i=j}^1 O_{i-1,i}^+ \prod_{i=j}^1 O_{i+1,i}^+ \right) \end{aligned}$$

Finally, the generating function of Schröder paths is given by

$$GF(x, y, z) = \frac{(1 + yz)(1 - x^2) - (1 - yz)\sqrt{1 - 6x^2 + x^4} - 2xy - 2xz}{2(x - f(y))(x - f(z))(1 - yz)}$$

where $f(y) := y - x^2y - xy^2 = y(1 - x^2 - xy)$.

Inspired by and in some parts derived from Okada's formulas [47], we give minor summation formulas resulting in the Pfaffians of matrices which (up to at most two additional rows and columns) have the form $\begin{pmatrix} H_m & M \\ -M^\top & 0 \end{pmatrix}$ resp. $\begin{pmatrix} H_m & M \\ -M^\top & -H_n \end{pmatrix}$, similar to special cases of [29]:

Theorem 8.15 (Proposition 7.20, theorems 7.18, 7.23). *Let M be an $m \times n$ matrix. Then:*

$$|M|_n = (-1)^{\lfloor n/2 \rfloor} \begin{cases} \text{pf}(H_0(M)) & m+n \text{ even} \\ \text{pf}(H_1(M)) & m+n \text{ odd} \end{cases}$$

$$\sum_{k \geq 0} |M|_k = |V_m R_m \quad M|_m = \left| \begin{smallmatrix} R_n V_n \\ M \end{smallmatrix} \right|_n$$

$$\sum_{k \geq 0} |M|_k = (-1)^{\lfloor n/2 \rfloor} \begin{cases} \text{pf}(H H_1(M)) & m+n \text{ odd} \\ \text{pf}(H H_2(M)) & m+n \text{ even} \end{cases}$$

8.3 Open questions

Remember that the original goal of the work leading to this paper was to find a bijection between perfect matchings of the Aztec graph and Q -partial matchings of the AST-graph, ideally giving hints towards a bijection between ASMs and ASTs. This problem is still wide open. We have here mainly studied the AST-graph and its Q -partial matchings, hopefully making them more accessible for future research.

Obviously, our attempt at giving an elementary enumeration of the Q -partial matchings of the AST-graph is incomplete, it remains to evaluate any of the Pfaffians or determinants in section 7.4. It might be possible to do that directly or by finding a reduction of such a Pfaffian/determinant to a smaller one of the same form. This is the most urgent question left open by our work.

Regarding the search for a bijection between perfect matchings of the Aztec graph and Q -partial matchings of the AST-graph, our paper gives at least two points where future research could start: Firstly, remember that we have shown that for each number of edges tilted in one of the two possible directions, still the numbers of matchings coincide (theorem 2.19). So it might be possible to find a bijection under which this statistic is invariant.

Secondly, we have found lots of sets of path families standing in bijection to Q -partial matchings of the (half-)reduced AST-graph. And it is known that also perfect matchings of the Aztec graph stand in bijection to a certain set of path families (see section 4.2). Thus, it could be a promising idea to try and find a bijection between these two sets of path families. Analyzing the bijections between the different sets of path families for the (half-)reduced AST-graph (corollary 4.9) might help in finding bijective operations on path families. In general, further studying the results for path families in section 4.5 could be interesting. Also, developing MaPaf-bijections directly for the AST-graph (not the (half-)reduced one) might be an idea.

Additionally, we should mention that there are several techniques for enumerating matchings - in particular perfect matchings of the Aztec graph - that we have not explored in this paper and that might yield interesting results for the AST-graph. These are for instance the graphical condensation method [38], Ciucu's method for graphs with reflective symmetry [13], height functions [18, section 2], or domino shuffling ([19, section 6], see also [21]).

Finally, note that our research regarding Schröder numbers and Schröder matrices in section 6 can definitely be expanded, with still much to discover. Our choice of Schröder matrices is quite arbitrary, the relations between them and expressions for them quite intriguing: Can they be proved more directly instead of using our correspondence between path operations and matrix operations (which by the way could also be studied further by itself)? Do similar facts hold for other Schröder matrices which are still to be defined? Is there a sensible choice of ‘fundamental’ Schröder matrices, in terms of which all others can be expressed?

A Appendix

A.1 Proof of proposition 7.4

In order to show item (i) of proposition 7.4, we need the following definitions:

Definition A.1. Let n be even and let $\sigma \in S_n$. Then the matching of K_n induced by σ is

$$M_\sigma := \{ \{ \sigma(i), \sigma(i+1) \} \mid 1 \leq i \leq n, i \text{ odd} \}$$

Furthermore, we set

$$\iota(\sigma) := |\{ i \mid 1 \leq i \leq n, i \text{ odd}, \sigma(i) > \sigma(i+1) \}|$$

With these definitions, we have the following:

Lemma A.2. Let n be even and let $\sigma \in S_n$. Then

$$(-1)^{\iota(\sigma)} \text{sgn}(\sigma) = \text{sgn}(M_\sigma)$$

Proof. Clearly, the statement holds if σ is the identity permutation, as each factor equals 1 in that case. Now consider a transposition of the form $(k \ k+1)$. We show that if the statement holds for σ , it also holds for $\sigma' := (k \ k+1) \circ \sigma$. By induction, this then implies that the statement holds for all $\sigma \in S_n$.

We have $\text{sgn}(\sigma') = -\text{sgn}(\sigma)$ and $\iota(\sigma') = \iota(\sigma)$, unless $\{k, k+1\}$ is an edge in M_σ , in which case we have $\iota(\sigma') = \iota(\sigma) \pm 1$. Thus

$$(-1)^{\iota(\sigma')} \text{sgn}(\sigma') = \begin{cases} (-1)^{\iota(\sigma)} \text{sgn}(\sigma) & \{k \ k+1\} \in M_\sigma \\ -(-1)^{\iota(\sigma)} \text{sgn}(\sigma) & \text{otherwise} \end{cases}$$

For $\text{sgn}(M_{\sigma'})$ and $\text{sgn}(M_\sigma)$, the same relation holds: If $\{k \ k+1\}$ is an edge in M_σ , then $M_{\sigma'} = M_\sigma$ and thus $\text{sgn}(M_{\sigma'}) = \text{sgn}(M_\sigma)$. Otherwise, the perfect matching $M_{\sigma'}$ is obtained from M_σ by exchanging the roles of k and $k+1$: Edges $\{k, a\}$ and $\{k+1, b\}$ are transformed to edges $\{k+1, a\}$ and $\{k, b\}$. Let e be any edge not among these. Then $\{k, a\}$ crosses e if and only if $\{k+1, a\}$ crosses e . The same holds for $\{k+1, b\}$ and $\{k, b\}$. Thus, $M_{\sigma'}$ and M_σ have exactly the same crossings, with one exception: One can easily see that $\{k, a\}$ and $\{k+1, b\}$ cross if and only if $\{k+1, a\}$ and $\{k, b\}$ do not cross. Thus indeed $\text{sgn}(M_{\sigma'}) = -\text{sgn}(M_\sigma)$ in the case where $\{k, k+1\} \notin M$ and so $(-1)^{\iota(\sigma')} \text{sgn}(\sigma') = \text{sgn}(M_{\sigma'})$ follows from $(-1)^{\iota(\sigma)} \text{sgn}(\sigma) = \text{sgn}(M_\sigma)$. $\square$

Lemma A.2 now directly implies that all three definitions of the Pfaffian are equivalent:

Proof of item (i) of proposition 7.4. Note that for any $\sigma \in S_n$, we have

$$A_{\sigma(1)\sigma(2)}A_{\sigma(3)\sigma(4)}\dots A_{\sigma(n-1)\sigma(n)} = (-1)^{\iota(\sigma)} \prod_{\substack{i < j \\ \{i,j\} \in M_\sigma}} A_{ij}$$

This is because for each odd i , the product on the right side contains $A_{\sigma(i)\sigma(i+1)}$ if $\sigma(i) < \sigma(i+1)$ and $A_{\sigma(i+1)\sigma(i)} = -A_{\sigma(i)\sigma(i+1)}$ if $\sigma(i+1) < \sigma(i)$, in which case i contributes to $\iota(\sigma)$, correcting the sign. Thus, using lemma A.2, we obtain

$$\begin{aligned} & \sum_{\substack{\sigma \in S_n \\ \sigma(1) < \sigma(2), \sigma(3) < \sigma(4), \dots, \sigma(n-1) < \sigma(n) \\ \sigma(1) < \sigma(3) < \dots < \sigma(n-1)}} \text{sgn}(\sigma) A_{\sigma(1)\sigma(2)} A_{\sigma(3)\sigma(4)} \dots A_{\sigma(n-1)\sigma(n)} = \\ &= \sum_{\substack{\sigma \in S_n \\ \sigma(1) < \sigma(2), \sigma(3) < \sigma(4), \dots, \sigma(n-1) < \sigma(n) \\ \sigma(1) < \sigma(3) < \dots < \sigma(n-1)}} \text{sgn}(M_\sigma) \prod_{\substack{i < j \\ \{i,j\} \in M_\sigma}} A_{ij} = \\ &= \sum_{M \text{ perf. mat. of } K_n} \sum_{\substack{\sigma \in S_n \\ \sigma(1) < \sigma(2), \sigma(3) < \sigma(4), \dots, \sigma(n-1) < \sigma(n) \\ \sigma(1) < \sigma(3) < \dots < \sigma(n-1) \\ M_\sigma = M}} \text{sgn}(M) \prod_{\substack{i < j \\ \{i,j\} \in M}} A_{ij} = \\ &= \sum_{M \text{ perf. mat. of } K_n} \text{sgn}(M) \prod_{\substack{i < j \\ \{i,j\} \in M}} A_{ij} \end{aligned}$$

since (for the last step) there is exactly one permutation $\sigma \in S_n$ satisfying the ordering conditions in the sum and inducing M , i.e. satisfying $M_\sigma = M$. Thus, the third definition in 7.3 is equivalent to the first one. Analogously, we can show that the second definition is equivalent to the first one: This time, the sum we get is of the form $\sum_{\sigma \in S_n, M_\sigma = M}$. As there are clearly $(n/2)! 2^{n/2}$ permutations $\sigma \in S_n$ inducing M for each perfect matching M of K_n , the sum cancels with the term $\frac{1}{(n/2)! 2^{n/2}}$ in the definition. $\square$

Item (ii) of proposition 7.4, that is the fact that the Pfaffian squares to the determinant, was first proved by Cayley [10]. Veltmann [55] and Mertens [44] were the first to give combinatorial proofs (as noted in [36, section 6]). We here loosely follow Stembridge's proof [52, proof of proposition 2.2], which uses the same general graph-theoretic idea as Veltmann and Mertens did. In particular, lemma A.3 and its proof, as well as item (iii) of lemma A.6 and its proof idea (using conjugations with $(j \ j+1)$ to reduce to simple permutations) are taken from Stembridge.

For the remainder of this proof, from the different definitions of the Pfaffian, we use the one which sums over perfect matchings. The following lemma is the fundamental step for proving item (ii) of proposition 7.4, and at the same time quite interesting in itself:

Lemma A.3. *Let A be an $n \times n$ skew-symmetric matrix. Then*

$$\det(A) = \sum_{\substack{\sigma \in S_n \\ \sigma \text{ consists only of} \\ \text{cycles of even length}}} \text{sgn}(\sigma) A_{1\sigma(1)} A_{2\sigma(2)} \dots A_{n\sigma(n)}$$

Proof. We have

$$\det(A) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) A_{1\sigma(1)} A_{2\sigma(2)} \dots A_{n\sigma(n)}$$

Consider some $\sigma \in S_n$ which has a fixed point $\sigma(i) = i$. Since $A_{ii} = 0$, for such a σ we have $\text{sgn}(\sigma) A_{1\sigma(1)} A_{2\sigma(2)} \dots A_{n\sigma(n)} = 0$. Now consider some $\sigma \in S_n$ which has no fixed point, but a cycle $c = (i_1 i_2 \dots i_k)$ of odd length $k \geq 3$. Let σ' be the permutation we obtain by reversing c in σ . This does not change the sign of σ , so $\text{sgn}(\sigma) = \text{sgn}(\sigma')$. However, we have

$$A_{i_1 i_2} A_{i_2 i_3} \dots A_{i_{k-1} i_k} A_{i_k i_1} = (-1)^k A_{i_1 i_k} A_{i_k i_{k-1}} \dots A_{i_3 i_2} A_{i_2 i_1}$$

and so $\text{sgn}(\sigma) A_{1\sigma(1)} A_{2\sigma(2)} \dots A_{n\sigma(n)} = -\text{sgn}(\sigma') A_{1\sigma'(1)} A_{2\sigma'(2)} \dots A_{n\sigma'(n)}$. This implies that the summands for all $\sigma \in S_n$ which have no fixed point, but contain a cycle of odd length, cancel pairwise in the sum: Consider for instance the involution reversing the cycle which contains the smallest index. Thus we are left with those permutations consisting only of cycles of even length. $\square$

Remark. Note that lemma A.3 directly implies $\det(A) = 0 = \text{pf}(A)^2$ for odd n . Thus, for the rest of this section, we can consider n as being even. The following lemmata and definitions are valid also for odd n , but quite uninteresting since they then just deal with empty sets or empty sums.

We first need a lemma about the crossings of perfect matchings:

Lemma A.4. *Let L be a subset of the vertices of K_n . Let M and M' be two perfect matchings of $K_n \setminus L$ and let N and N' be two perfect matchings of L . Then*

$$\text{sgn}(M \cup N) \cdot \text{sgn}(M' \cup N') = \text{sgn}(M \cup N') \cdot \text{sgn}(M' \cup N)$$

Proof. We show that $\text{sgn}(M \cup N) = \text{sgn}(M) \cdot \nu$, where ν only depends on N , but not on the choice of M . This then implies

$$\frac{\text{sgn}(M \cup N)}{\text{sgn}(M' \cup N)} = \frac{\text{sgn}(M)}{\text{sgn}(M')} = \frac{\text{sgn}(M \cup N')}{\text{sgn}(M' \cup N')}$$

So consider $\text{sgn}(M \cup N)$. By lemma 7.11, for an edge $\{i, j\} \in N$ with $i < j$ we have $\text{sgn}(M \cup N) = \text{sgn}(M \cup N \setminus \{\{i, j\}\}) \cdot (-1)^{i+j+1}$. Now transform $K_n \setminus \{i, j\}$ to K_{n-2} by ‘pushing together’ the vertices to get rid of the ‘holes’ at i and j . So $i+1$ becomes i , $i+2$ becomes $i+1$, ..., $j+1$ becomes $j-1$ etc. Let M_1 and N_1 be the matchings of K_{n-2} obtained from M and $N \setminus \{\{i, j\}\}$ by this procedure. This clearly does not change the sign of the matchings, so we have $\text{sgn}(M \cup N) = \text{sgn}(M_1 \cup N_1) \cdot \nu_1$ with $\nu_1 := (-1)^{i+j+1}$. Now we can iterate this process with another edge in N_1 . We get

$$\text{sgn}(M_1 \cup N_1) \cdot \nu_1 = \text{sgn}(M_2 \cup N_2) \cdot \nu_2 \cdot \nu_1 = \dots = \text{sgn}(M_k) \cdot \nu_k \cdot \dots \cdot \nu_1$$

where $k := |L|/2$, so the original N consisted of k edges and thus $N_k = \emptyset$. Now note that $\nu := \nu_k \cdot \dots \cdot \nu_1$ does only depend on N : With any M , we could have done this process in the exact same way. Also, M_k has the same structure as M , just pushed together, which does not affect the crossings. So indeed, we have

$$\text{sgn}(M \cup N) = \text{sgn}(M_k) \cdot \nu_k \cdot \dots \cdot \nu_1 = \text{sgn}(M) \cdot \nu$$

$\square$

We also need one more definition:

Definition A.5. We set

$$CE_n := \{C \mid C \text{ is a multiset of edges of } K_n \text{ that partitions } K_n \text{ into cycles of even length}\}$$

Furthermore, we say that a permutation $\sigma \in S_n$ induces an element $C \in CE_n$ if we obtain C from σ by considering σ as a set of cycles on K_n and forgetting about the orientations of the cycles. A pair (M_1, M_2) of perfect matchings of K_n induces $C \in CE_n$ if $M_1 \cup M_2 = C$.

Remark. The reason that we work with multisets here is that we want to allow 2-cycles. Thus, the union $M_1 \cup M_2$ also has to be understood as a multiset, though we do not mention that explicitly.

Now we can state the following lemma, which is essential for proving that the Pfaffian squares to the determinant:

Lemma A.6. Let $C \in CE_n$ and let k be the number of cycles in C which have length > 2 . Then:

- (i). There are 2^k permutations $\sigma \in S_n$ which induce C . For each of these, $\text{sgn}(\sigma) \cdot (-1)^{|\{i \mid i > \sigma(i)\}|}$ gives the same value.
- (ii). There are 2^k pairs (M_1, M_2) of perfect matchings of K_n which induce C . For each of these, $\text{sgn}(M_1) \cdot \text{sgn}(M_2)$ gives the same value.
- (iii). Let $\sigma \in S_n$ be a permutation and let (M_1, M_2) be a pair of perfect matchings of K_n , such that σ and (M_1, M_2) induce the same $C \in CE_n$. Then $\text{sgn}(\sigma) \cdot (-1)^{|\{i \mid i > \sigma(i)\}|} = \text{sgn}(M_1) \cdot \text{sgn}(M_2)$.

Proof. For item (i), we see that in order to obtain a σ which induces C , one just has to choose an orientation for each cycle in C . Since in cycles of length 2, both orientations give the same permutation, there are 2^k different σ inducing C . All of these have the same sign, since this only depends on the cycle type of the permutation. But also $(-1)^{|\{i \mid i > \sigma(i)\}|}$ is the same for all these possible σ : Let σ' be the permutation we obtain by changing the orientation of a single cycle c in σ . Then we have $|\{i \in c \mid i > \sigma(i)\}| + |\{i \in c \mid i > \sigma'(i)\}| = |c|$, since each pair (a, b) of neighbouring elements in c either contributes to the first or to the second of these sets, depending on whether we have $a > b$ or $a < b$. As $|c|$ is even, we obtain $|\{i \mid i > \sigma(i)\}| \equiv |\{i \mid i > \sigma'(i)\}| \pmod{2}$, and thus indeed $\text{sgn}(\sigma) \cdot (-1)^{|\{i \mid i > \sigma(i)\}|} = \text{sgn}(\sigma') \cdot (-1)^{|\{i \mid i > \sigma'(i)\}|}$.

For item (ii), note that each cycle in C falls into two distinct sets of edges by putting the edges alternately in one set and in the other. To find M_1 and M_2 , in each of the cycles we have to choose which of these two sets should belong to M_1 and which to M_2 . In cycles of length 2, this does not make a difference, in the other cycles it does. So there are 2^k different pairs (M_1, M_2) of perfect matchings inducing C . Now let c be a cycle in C , contributing the set c_1 of edges to M_1 and c_2 to M_2 . Exchanging c_1 and c_2 gives the perfect matchings $M'_1 = (M_1 \setminus c_1) \cup c_2$ and $M'_2 = (M_2 \setminus c_2) \cup c_1$. By lemma A.4, we have

$$\begin{aligned} \text{sgn}(M'_1) \cdot \text{sgn}(M'_2) &= \text{sgn}((M_1 \setminus c_1) \cup c_2) \cdot \text{sgn}((M_2 \setminus c_2) \cup c_1) = \\ &= \text{sgn}((M_1 \setminus c_1) \cup c_1) \cdot \text{sgn}((M_2 \setminus c_2) \cup c_2) = \\ &= \text{sgn}(M_1) \cdot \text{sgn}(M_2) \end{aligned}$$

and so indeed, each of the pairs (M_1, M_2) inducing C gives the same value of $\text{sgn}(M_1) \cdot \text{sgn}(M_2)$.

For item (iii), we proceed as follows: By item (ii), it is enough to show that for each $\sigma \in S_n$, there is some pair (M_1, M_2) inducing the same $C \in CE_n$ as σ and satisfying $\text{sgn}(\sigma) \cdot (-1)^{|\{i \mid i > \sigma(i)\}|} = \text{sgn}(M_1) \cdot \text{sgn}(M_2)$. We first do this for those $\sigma \in S_n$ in which each cycle consists of consecutive numbers. Then we prove that it continues to hold when conjugating σ with a transposition of the form $(j \ j+1)$. Since by such operations we can obtain each $\sigma \in S_n$ having the same cycle type as the original permutation, we reach all $\sigma \in S_n$ and thus prove item (iii).

So let $1 \leq i_1 < i_2 < \dots < i_l \leq n$ be even integers and consider the permutation $\sigma \in S_n$ which in cycle notation can be written as

$$\sigma := (1 \ 2 \dots i_1)(i_1 + 1 \ i_1 + 2 \dots i_2)(i_2 + 1 \ i_2 + 2 \dots i_3) \dots (i_{l-1} + 1 \ i_{l-1} + 2 \dots i_l)$$

Then we have $\text{sgn}(\sigma) \cdot (-1)^{|\{i \mid i > \sigma(i)\}|} = (-1)^l \cdot (-1)^l = 1$, since there are l cycles of even length, so $\text{sgn}(\sigma) = (-1)^l$, and $|\{i \mid i > \sigma(i)\}| = |\{i_1, i_2, \dots, i_l\}| = l$. And we also have $\text{sgn}(M_1) \cdot \text{sgn}(M_2) = 1$ for any (M_1, M_2) inducing the same $C \in CE_n$ as σ , as all such M_1 and M_2 obviously contain no crossings at all. Thus $\text{sgn}(\sigma) \cdot (-1)^{|\{i \mid i > \sigma(i)\}|} = \text{sgn}(M_1) \cdot \text{sgn}(M_2)$ indeed holds for such σ .

Now let $\sigma \in S_n$ satisfy $\text{sgn}(\sigma) \cdot (-1)^{|\{i \mid i > \sigma(i)\}|} = \text{sgn}(M_1) \cdot \text{sgn}(M_2)$ for some pair (M_1, M_2) inducing the same $C \in CE_n$ as σ , and let σ' be the permutation obtained from σ by conjugating with the transposition $(j \ j+1)$. Note that in the cycle notation of σ' , j and $j+1$ are just switched in comparison to σ , and that we can obtain a pair (M'_1, M'_2) corresponding to σ' by just switching j and $j+1$ both in M_1 and in M_2 . If $(j \ j+1)$ is itself a cycle of σ , then $\sigma' = \sigma$ and $(M_1, M_2) = (M'_1, M'_2)$, so we do not have to prove anything. For the other cases, we say that j and $j+1$ are *neighbouring* if either $\sigma(j) = j+1$ or $\sigma(j+1) = j$. Let us now see what happens to the different parameters:

- We have $\text{sgn}(\sigma') = \text{sgn}(\sigma)$ since the sign only depends on the cycle type of a permutation.
- We have $|\{i \mid i > \sigma'(i)\}| = |\{i \mid i > \sigma(i)\}|$ except when j and $j+1$ are neighbouring. This is because replacing j by $j+1$ and vice versa does not change whether they are greater or smaller than other numbers, except for when they are compared to each other. If j and $j+1$ are neighbouring, we have $|\{i \mid i > \sigma'(i)\}| = |\{i \mid i > \sigma(i)\}| \pm 1$ and thus

$$(-1)^{|\{i \mid i > \sigma'(i)\}|} = \begin{cases} -(-1)^{|\{i \mid i > \sigma(i)\}|} & j \text{ and } j+1 \text{ are neighbouring} \\ (-1)^{|\{i \mid i > \sigma(i)\}|} & \text{otherwise} \end{cases}$$

- If $\{j, j+1\}$ is an edge of M_1 , then we have $M'_1 = M_1$. Otherwise, we have $\text{sgn}(M'_1) = -\text{sgn}(M_1)$ since the two edges going to j and $j+1$ either cross or uncross by switching j with $j+1$, while nothing else changes. Obviously, the same holds for M_2 . If j and $j+1$ are neighbouring in σ , then exactly one of M_1 and M_2 contains the edge $\{j, j+1\}$, otherwise none does. Thus we have

$$\text{sgn}(M'_1) \cdot \text{sgn}(M'_2) = \begin{cases} -\text{sgn}(M_1) \cdot \text{sgn}(M_2) & j \text{ and } j+1 \text{ are neighbouring} \\ \text{sgn}(M_1) \cdot \text{sgn}(M_2) & \text{otherwise} \end{cases}$$

So in total, from $\text{sgn}(\sigma) \cdot (-1)^{|\{i \mid i > \sigma(i)\}|} = \text{sgn}(M_1) \cdot \text{sgn}(M_2)$, indeed the same follows for σ' together with (M'_1, M'_2) , which completes the proof. $\square$

Clearly, combining the three items of lemma A.6 directly gives the following:

Corollary A.7. *Let $C \in CE_n$. Then we have:*

$$\sum_{\substack{\sigma \in S_n \\ \sigma \text{ induces } C}} \text{sgn}(\sigma) \cdot (-1)^{|\{i \mid i > \sigma(i)\}|} = \sum_{\substack{M_1, M_2 \text{ perfect matchings of } K_n \\ (M_1, M_2) \text{ induces } C}} \text{sgn}(M_1) \cdot \text{sgn}(M_2)$$

With this, we can finally prove item (ii) of proposition 7.4:

Proof of item (ii) of proposition 7.4. Note that we have

$$A_{1\sigma(1)} A_{2\sigma(2)} \dots A_{n\sigma(n)} = (-1)^{|\{i \mid i > \sigma(i)\}|} \prod_{\substack{i < j \\ \{i, j\} \in C \\ C \in CE_n \text{ induced by } \sigma}} A_{ij}$$

Thus, using lemma A.3 and corollary A.7 gives the following: (Remember that $M_1 \cup M_2$ has to be interpreted as a multiset when taking the product over $\{i, j\} \in M_1 \cup M_2$, see the remark after definition A.5.)

$$\begin{aligned} \det(A) &= \sum_{\substack{\sigma \in S_n \\ \sigma \text{ consists only of} \\ \text{cycles of even length}}} \text{sgn}(\sigma) A_{1\sigma(1)} A_{2\sigma(2)} \dots A_{n\sigma(n)} = \\ &= \sum_{\substack{\sigma \in S_n \\ \sigma \text{ consists only of} \\ \text{cycles of even length}}} \text{sgn}(\sigma) \cdot (-1)^{|\{i \mid i > \sigma(i)\}|} \prod_{\substack{i < j \\ \{i, j\} \in C \\ C \in CE_n \text{ induced by } \sigma}} A_{ij} = \\ &= \sum_{C \in CE_n} \prod_{\substack{i < j \\ \{i, j\} \in C}} A_{ij} \sum_{\substack{\sigma \in S_n \\ \sigma \text{ induces } C}} \text{sgn}(\sigma) \cdot (-1)^{|\{i \mid i > \sigma(i)\}|} = \\ &= \sum_{C \in CE_n} \prod_{\substack{i < j \\ \{i, j\} \in C}} A_{ij} \sum_{\substack{M_1, M_2 \text{ perf. mat. of } K_n \\ (M_1, M_2) \text{ induces } C}} \text{sgn}(M_1) \cdot \text{sgn}(M_2) = \\ &= \sum_{M_1, M_2 \text{ perf. mat. of } K_n} \text{sgn}(M_1) \cdot \text{sgn}(M_2) \prod_{\substack{i < j \\ \{i, j\} \in M_1 \cup M_2}} A_{ij} = \\ &= \left(\sum_{M_1 \text{ perf. mat. of } K_n} \text{sgn}(M_1) \prod_{\substack{i < j \\ \{i, j\} \in M_1}} A_{ij} \right) \left(\sum_{M_2 \text{ perf. mat. of } K_n} \text{sgn}(M_2) \prod_{\substack{i < j \\ \{i, j\} \in M_2}} A_{ij} \right) = \\ &= \text{pf}(A)^2 \end{aligned}$$

□

A.2 Proofs of lemmata 7.22 and 7.24

Proof of lemma 7.22. Let N be the $(n+m) \times n$ matrix $N := \begin{pmatrix} R_n V_n \\ M \end{pmatrix}$. So we want to show that $\text{pf}(T_1(N)) = \text{pf}(TH_1(M))$. As the i -th column of $R_n V_n$ is $(-1)^{i-1} e_{n+1-i}$, the i -th column of N is $n_i := \begin{pmatrix} (-1)^{i-1} e_{n+1-i} \\ m_i \end{pmatrix}$. Thus we have the following:

$$|(-1)^{i-1} e_{n+1-i} \quad (-1)^{j-1} e_{n+1-j}|_2 = (-1)^{i+j} |e_{n+1-i} \quad e_{n+1-j}|_2 = \begin{cases} (-1)^{i+j+1} & i < j \\ 0 & i = j \\ (-1)^{i+j} & i > j \end{cases} = H_{ij}$$

$$|n_i|_1 = |m_i|_1 + (-1)^{i-1}$$

$$|n_i \quad n_j|_2 = |m_i \quad m_j|_2 + (-1)^{i-1} |m_j|_1 - (-1)^{j-1} |m_i|_1 + H_{ij}$$

With this, we can write down the entries of $T_1(N)$:

$$T_1(N)_{ij} = \begin{cases} 0 & i = 1, j = 1 \\ |n_{j-1}|_1 = |m_{j-1}|_1 + (-1)^j & i = 1, j \neq 1 \\ -|n_{i-1}|_1 = -|m_{i-1}|_1 - (-1)^i & i \neq 1, j = 1 \\ |n_{i-1} \quad n_{j-1}|_2 = |m_{i-1} \quad m_{j-1}|_2 + (-1)^i |m_{j-1}|_1 - (-1)^j |m_{i-1}|_1 + H_{i-1, j-1} & i \neq 1, j \neq 1 \end{cases}$$

Now let $T_1(N)'$ be the matrix obtained from $T_1(N)$ by subtracting the first row-column pair multiplied by $(-1)^l$ from the l -th row-column pair for each $2 \leq l \leq n$. Then the first row-column pair of $T_1(N)'$ is the same as for $T_1(N)$ and for $i, j \neq 1$ we have:

$$\begin{aligned} T_1(N)'_{ij} &= T_1(N)_{ij} - (-1)^i T_1(N)_{1,j} - (-1)^j T_1(N)_{i,1} = \\ &= \left(|m_{i-1} \quad m_{j-1}|_2 + (-1)^i |m_{j-1}|_1 - (-1)^j |m_{i-1}|_1 + H_{i-1, j-1} \right) + \\ &\quad - (-1)^i \left(|m_{j-1}|_1 + (-1)^j \right) - (-1)^j \left(-|m_{i-1}|_1 - (-1)^i \right) = \\ &= |m_{i-1} \quad m_{j-1}|_2 + H_{i-1, j-1} \end{aligned}$$

Thus, $T_1(N)'$ looks as follows:

$$T_1(N)'_{ij} = \begin{cases} 0 = 0 + H_{0,0} & i = 1, j = 1 \\ |m_{j-1}|_1 + (-1)^j = |m_{j-1}|_1 + H_{1,j} & i = 1, j \neq 1 \\ -|m_{i-1}|_1 - (-1)^i = -|m_{i-1}|_1 + H_{i,1} & i \neq 1, j = 1 \\ |m_{i-1} \quad m_{j-1}|_2 + H_{i-1, j-1} = |m_{i-1} \quad m_{j-1}|_2 + H_{ij} & i \neq 1, j \neq 1 \end{cases}$$

So we have $T_1(N)' = T_1(M) + H_{n+1} = TH_1(M)$ and since this matrix was obtained from $T_1(N)$ by elementary row-column operations, by lemma 7.8 we indeed have

$$\text{pf}(T_1(N)) = \text{pf}(T_1(N)') = \text{pf}(TH_1(M))$$

□

Proof of lemma 7.24. $\text{pf}(HH_1(\underline{M})) = \text{pf}(HH_2(M))$ is proved by the following chain of equations:

$$\begin{aligned} \text{pf}(HH_1(\underline{M})) &= \text{pf} \begin{pmatrix} & & & (-1)^m & & & & & & \\ & & & -(-1)^m & & & & & & \\ & H_{m+1} & & \vdots & & & \underline{M} & & & \\ & & & 1 & & & & & & \\ -(-1)^m & (-1)^m & \cdots & 0 & & -(-1)^n & (-1)^n & \cdots & 1 & \\ & & & (-1)^n & & & & & & \\ -M^\top & & & -(-1)^n & & & -H_n & & & \\ & & & \vdots & & & & & & \\ & & & -1 & & & & & & \end{pmatrix} = \\ &= \text{pf} \begin{pmatrix} & & & -(-1)^m & & (-1)^m & & & & \\ & & & (-1)^m & & -(-1)^m & & & & \\ & H_m & & \vdots & & \vdots & & & M & \\ & & & 1 & & -1 & & & & \\ (-1)^m & -(-1)^m & \cdots & 0 & & 1 & & 0 & 0 & \cdots & 0 \\ -(-1)^m & (-1)^m & \cdots & -1 & & 0 & & -(-1)^n & (-1)^n & \cdots & 1 \\ & & & 0 & & (-1)^n & & & & & \\ -M^\top & & & 0 & & -(-1)^n & & & -H_n & & \\ & & & \vdots & & \vdots & & & & & \\ & & & 0 & & -1 & & & & & \end{pmatrix} = \end{aligned}$$

[Lemma 7.8: Add the $(m+1)$ -th row-column pair to the $(m+2)$ -th row-column pair]

$$\begin{aligned} &= \text{pf} \begin{pmatrix} & & & -(-1)^m & & 0 & & & & \\ & & & (-1)^m & & 0 & & & & \\ & H_m & & \vdots & & \vdots & & & M & \\ & & & 1 & & 0 & & & & \\ (-1)^m & -(-1)^m & \cdots & 0 & & 1 & & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & -1 & & 0 & & -(-1)^n & (-1)^n & \cdots & 1 \\ & & & 0 & & (-1)^n & & & & & \\ -M^\top & & & 0 & & -(-1)^n & & & -H_n & & \\ & & & \vdots & & \vdots & & & & & \\ & & & 0 & & -1 & & & & & \end{pmatrix} = \\ &= \text{pf}(HH_2(M)) \end{aligned}$$

$\text{pf}(HH_2(|M)) = (-1)^n \text{pf}(HH_1(M))$ is proved by the following chain of equations:

$$\text{pf}(HH_2(|M)) = \text{pf} \begin{pmatrix} & & & -(-1)^m & & 0 & & & & \\ & & & (-1)^m & & 0 & & & & \\ & H_m & & \vdots & & \vdots & & & |M & \\ & & & 1 & & 0 & & & & \\ (-1)^m & -(-1)^m & \cdots & 0 & & 1 & & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & -1 & & 0 & & (-1)^n & -(-1)^n & \cdots & 1 \\ & & & 0 & & -(-1)^n & & & & & \\ -\overline{M^\top} & & & 0 & & (-1)^n & & & -H_{n+1} & & \\ & & & \vdots & & \vdots & & & & & \\ & & & 0 & & -1 & & & & & \end{pmatrix} =$$

$$= \text{pf} \begin{pmatrix} & & & & -(-1)^m & 0 & 0 & & & & & \\ & H_m & & & (-1)^m & 0 & 0 & & M & & & \\ & & & & \vdots & \vdots & \vdots & & & & & \\ & & & & 1 & 0 & 0 & & & & & \\ (-1)^m & -(-1)^m & \cdots & -1 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -1 & 0 & (-1)^n & -(-1)^n & (-1)^n & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 0 & -(-1)^n & 0 & -1 & 1 & \cdots & (-1)^n \\ & & & & 0 & (-1)^n & 1 & & & & & \\ & -M^\top & & & 0 & -(-1)^n & -1 & & -H_n & & & \\ & & & & \vdots & \vdots & \vdots & & & & & \\ & & & & 0 & -1 & -(-1)^n & & & & & \end{pmatrix} =$$

[Lemma 7.8: Subtract the $(m+2)$ -th row-column pair multiplied by $(-1)^n$ from the $(m+3)$ -th row-column pair]

$$= \text{pf} \begin{pmatrix} & & & & -(-1)^m & 0 & 0 & & & & & \\ & H_m & & & (-1)^m & 0 & 0 & & M & & & \\ & & & & \vdots & \vdots & \vdots & & & & & \\ & & & & 1 & 0 & 0 & & & & & \\ (-1)^m & -(-1)^m & \cdots & -1 & 0 & 1 & -(-1)^n & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -1 & 0 & (-1)^n & -(-1)^n & (-1)^n & \cdots & 1 \\ 0 & 0 & \cdots & 0 & (-1)^n & -(-1)^n & 0 & 0 & 0 & \cdots & 0 \\ & & & & 0 & (-1)^n & 0 & & & & & \\ & -M^\top & & & 0 & -(-1)^n & 0 & & -H_n & & & \\ & & & & \vdots & \vdots & \vdots & & & & & \\ & & & & 0 & -1 & 0 & & & & & \end{pmatrix} =$$

[Proposition 7.10: Expand the Pfaffian along the $(m+3)$ -th row (deleted entries marked by stars)]

$$= (-1)^{m+3+m+1} (-1)^n \text{pf} \begin{pmatrix} & & & & \star & 0 & \star & & & & & \\ & H_m & & & \star & 0 & \star & & M & & & \\ & & & & \vdots & \vdots & \vdots & & & & & \\ & & & & \star & 0 & \star & & & & & \\ \star & \star & \cdots & \star & \star & \star & \star & \star & \star & \cdots & \star \\ 0 & 0 & \cdots & 0 & \star & 0 & \star & -(-1)^n & (-1)^n & \cdots & 1 \\ \star & \star & \cdots & \star & \star & \star & \star & \star & \star & \cdots & \star \\ & & & & \star & (-1)^n & \star & & & & & \\ -M^\top & & & & \star & -(-1)^n & \star & & -H_n & & & \\ & & & & \vdots & \vdots & \vdots & & & & & \\ & & & & \star & -1 & \star & & & & & \end{pmatrix} +$$

$$+ (-1)^{m+3+m+2} (-1)^{n+1} \text{pf} \begin{pmatrix} & & & & -(-1)^m & \star & \star & & & & & \\ & H_m & & & (-1)^m & \star & \star & & M & & & \\ & & & & \vdots & \vdots & \vdots & & & & & \\ & & & & 1 & \star & \star & & & & & \\ (-1)^m & -(-1)^m & \cdots & -1 & 0 & \star & \star & 0 & 0 & \cdots & 0 \\ \star & \star & \cdots & \star & \star & \star & \star & \star & \star & \cdots & \star \\ \star & \star & \cdots & \star & \star & \star & \star & \star & \star & \cdots & \star \\ & & & & 0 & \star & \star & & & & & \\ & -M^\top & & & 0 & \star & \star & & -H_n & & & \\ & & & & \vdots & \vdots & \vdots & & & & & \\ & & & & 0 & \star & \star & & & & & \end{pmatrix} =$$

[Lemma 7.9: Add the matrices along the $(m+1)$ -th row-column pair]

☐

On the following pages, we depict examples of A , M_0 and the resulting directed graph $\widehat{G}$ for each of the configurations introduced in definition 4.6 for the reduced AST-graph. We always take $n = 5$ here.

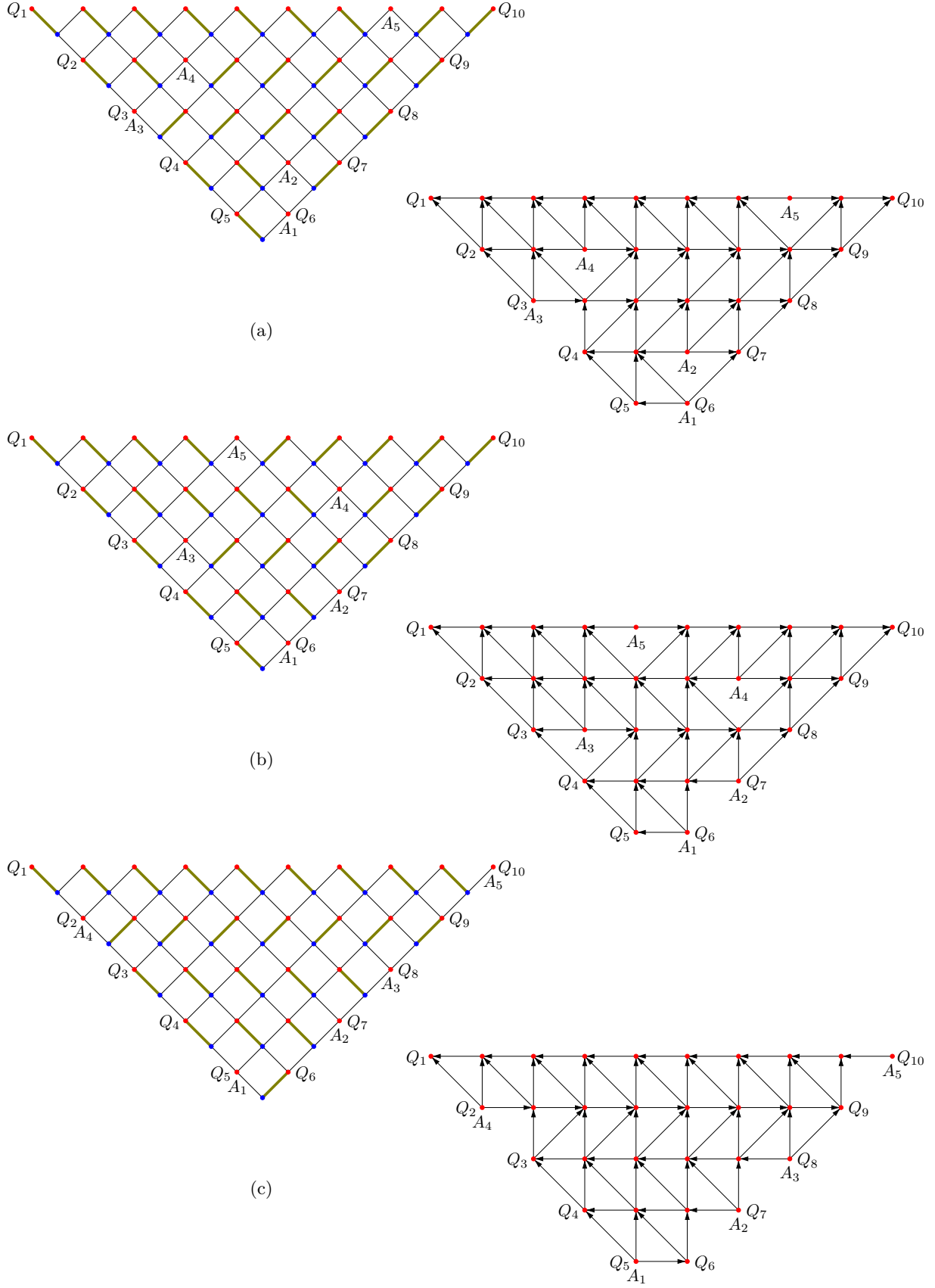


Figure 29: Row-wise configurations

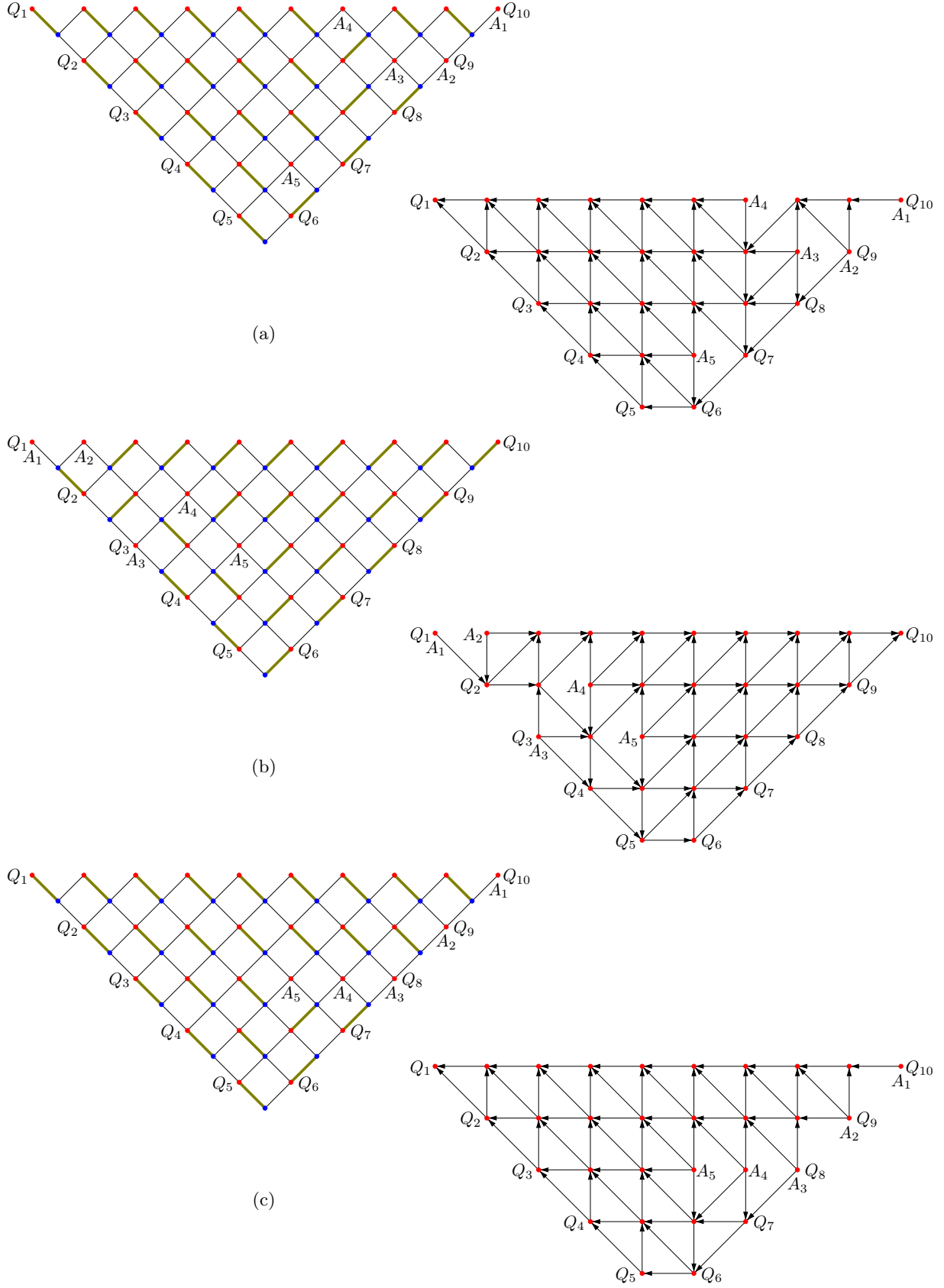


Figure 30: Column-wise configurations

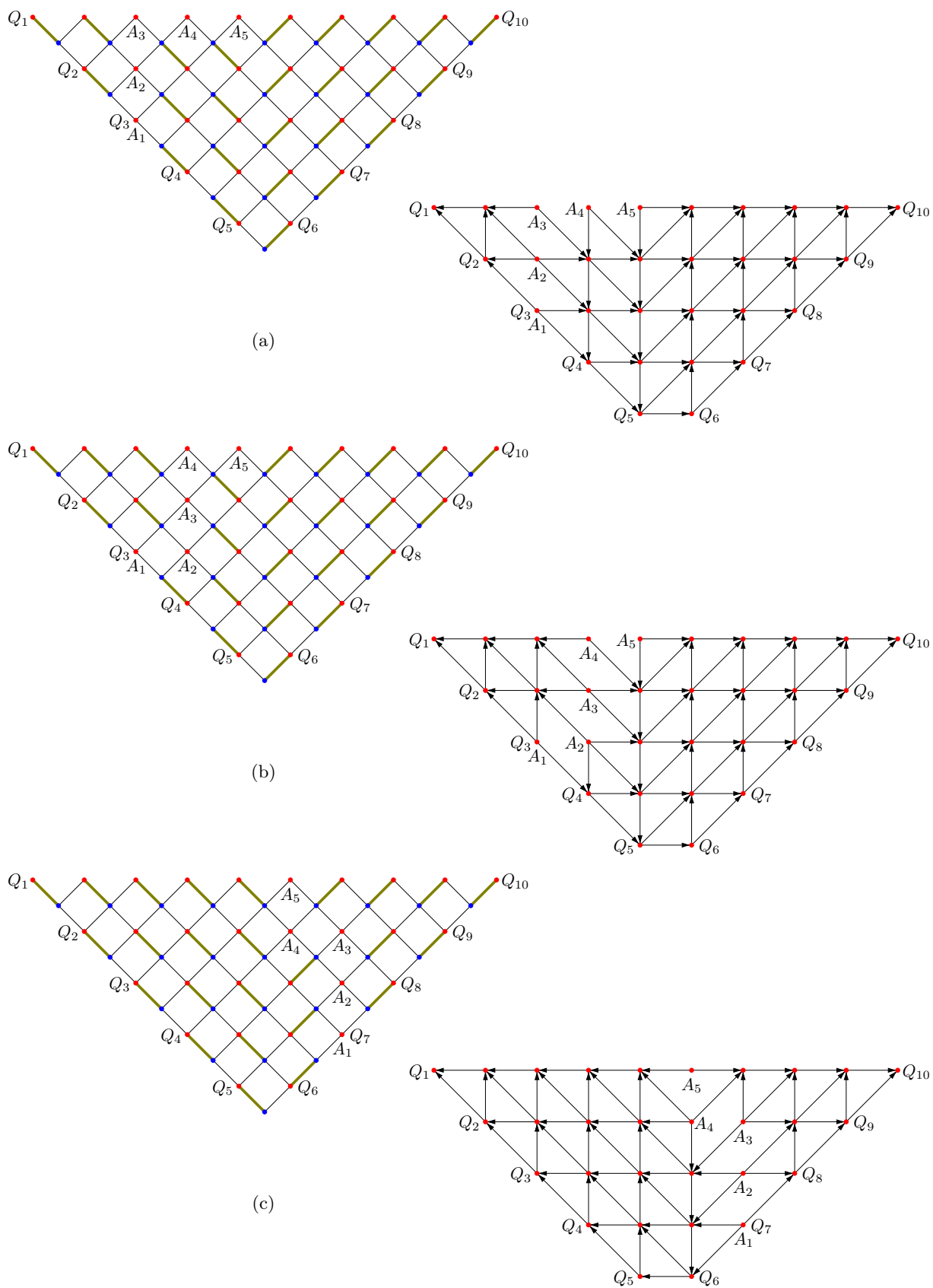


Figure 31: Snakes

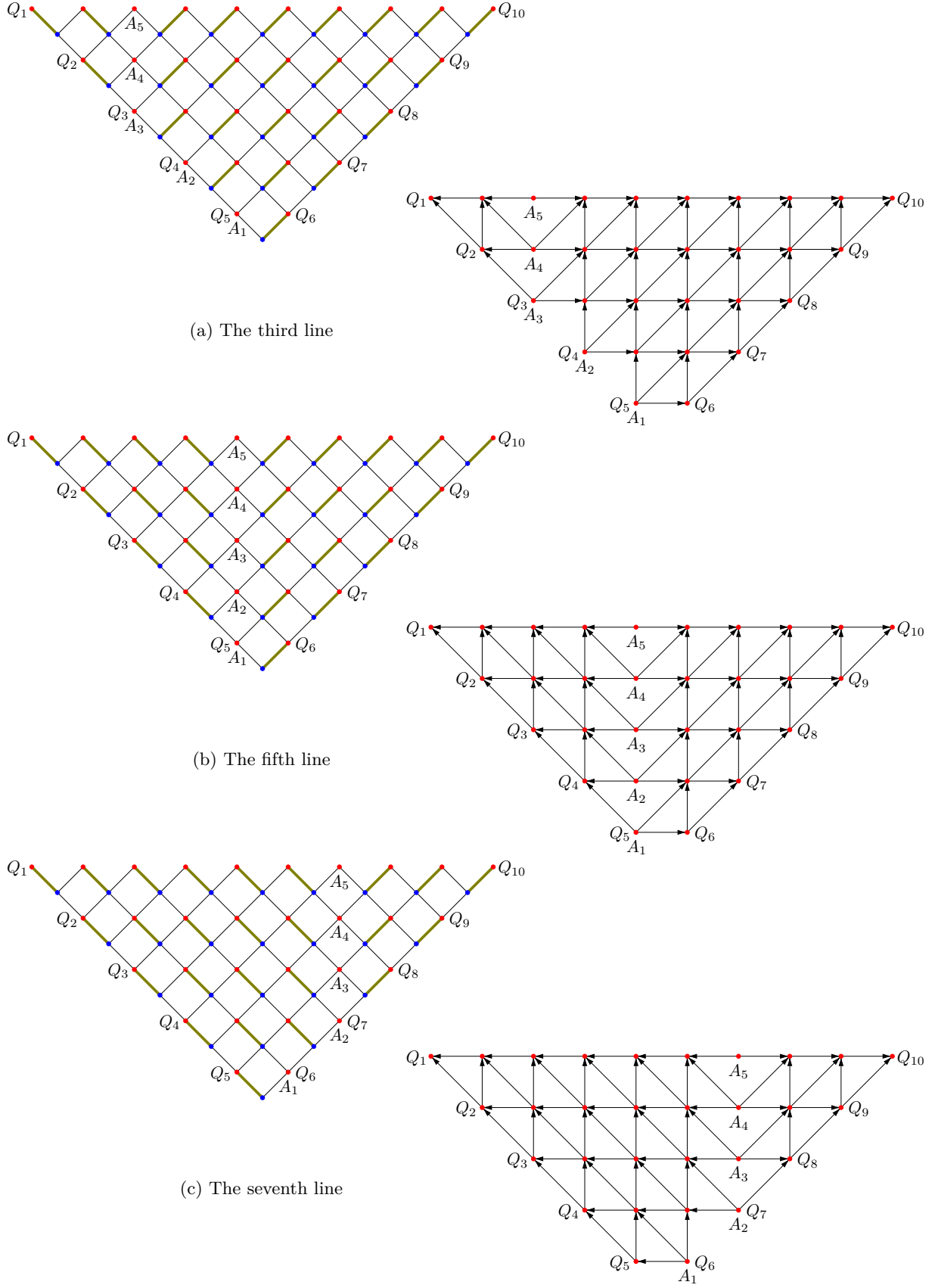


Figure 32: Lines

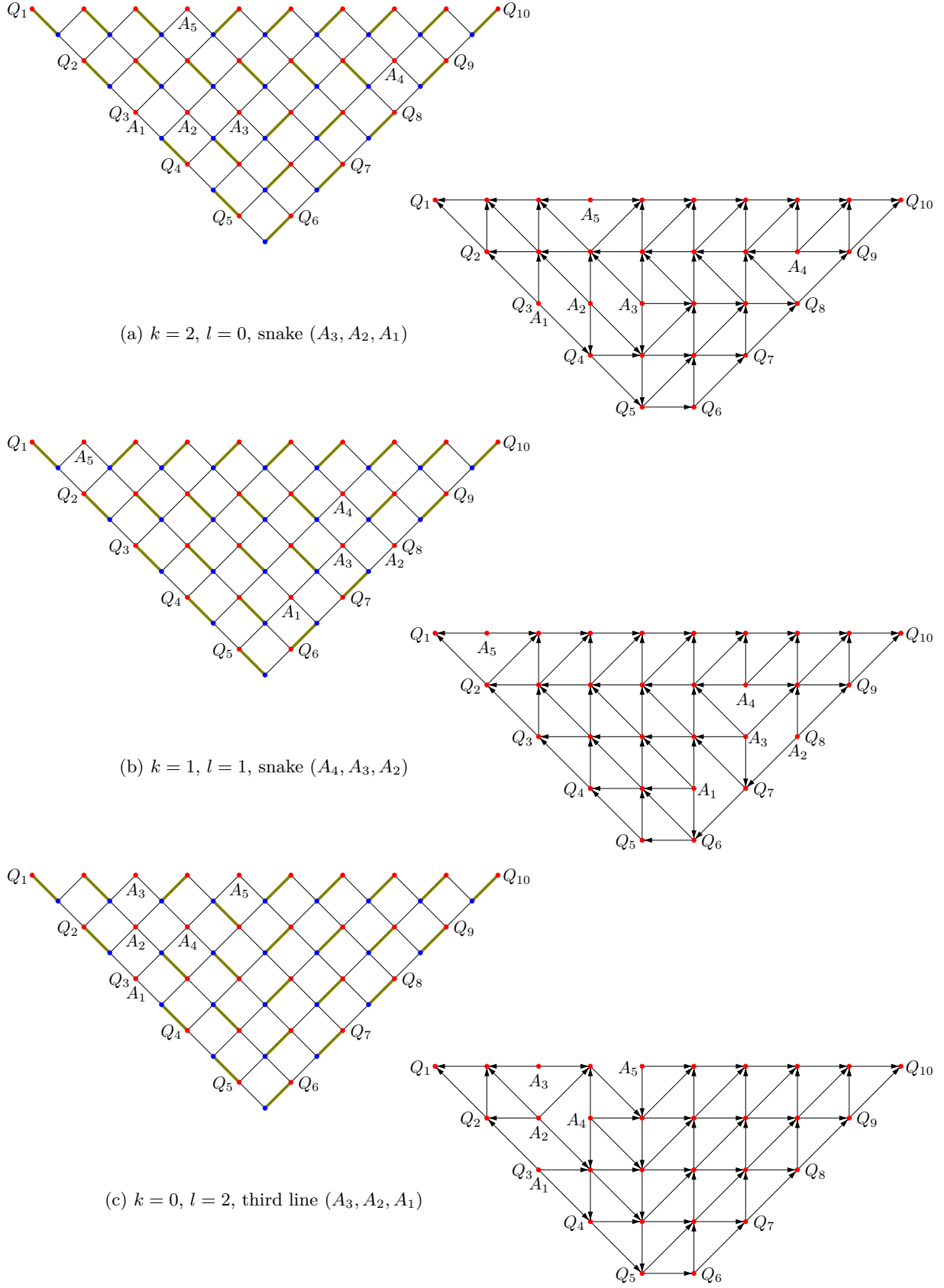


Figure 33: Mixed configurations

A.4 Figures for the proof of proposition 5.30

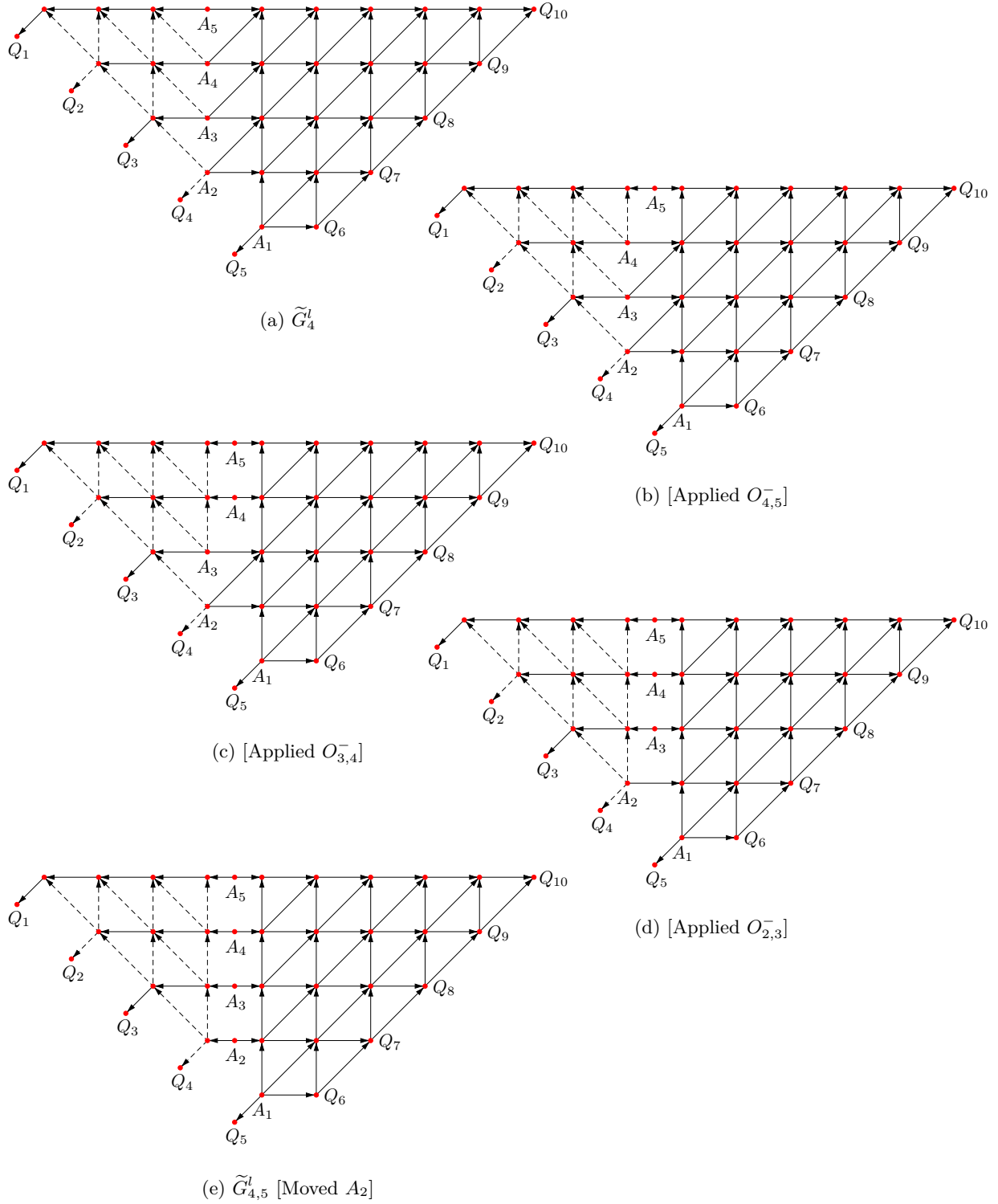


Figure 34: From $\tilde{G}_k^l$ to $\tilde{G}_{k+1/2}^l$ for $1 \leq k \leq n$, here for $k = 4$ ($d_k = 2$)

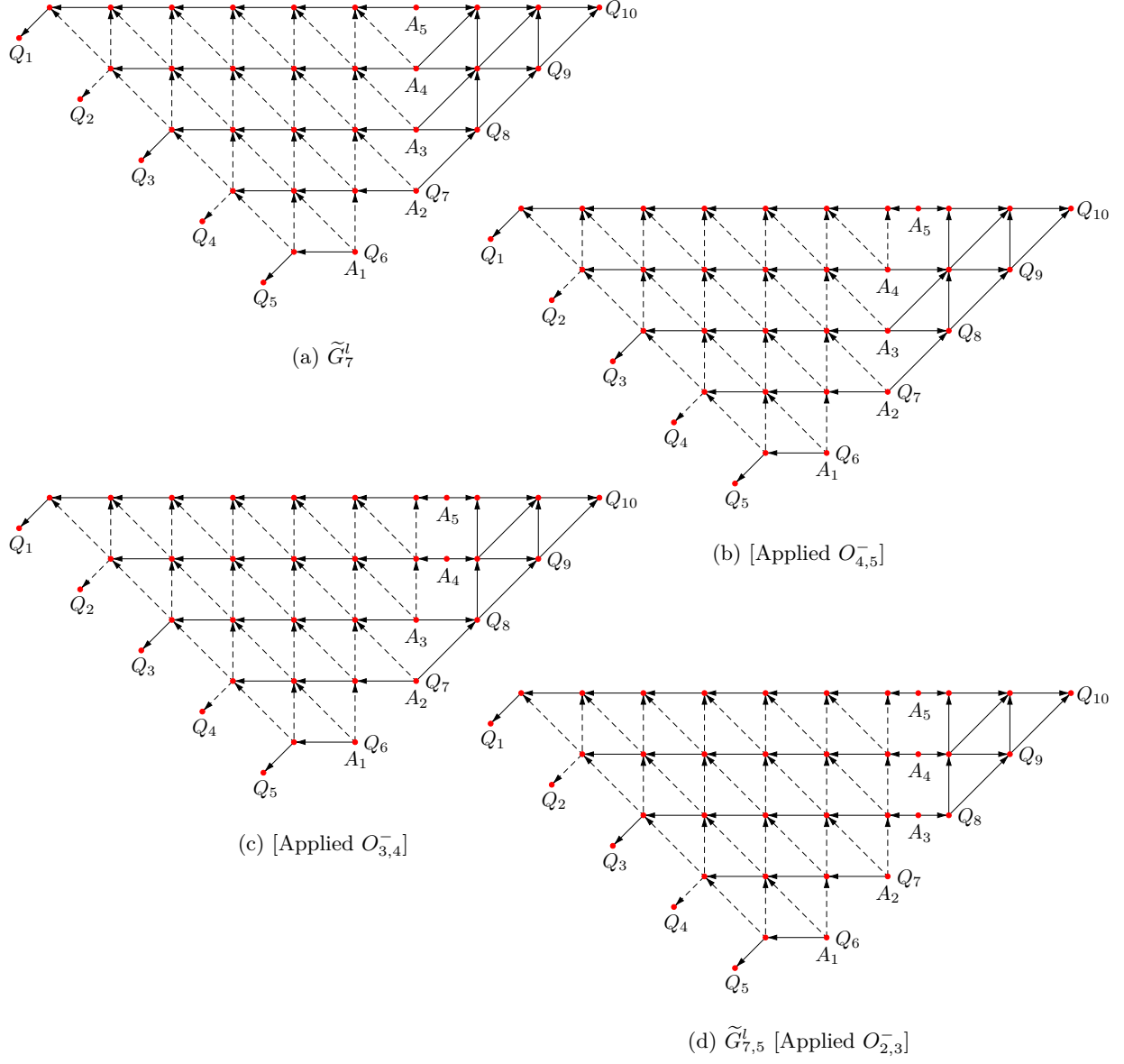


Figure 35: From $\tilde{G}_k^l$ to $\tilde{G}_{k+1/2}^l$ for $n+1 \leq k < 2n$, here for $k=7$ ($d_k=2$)

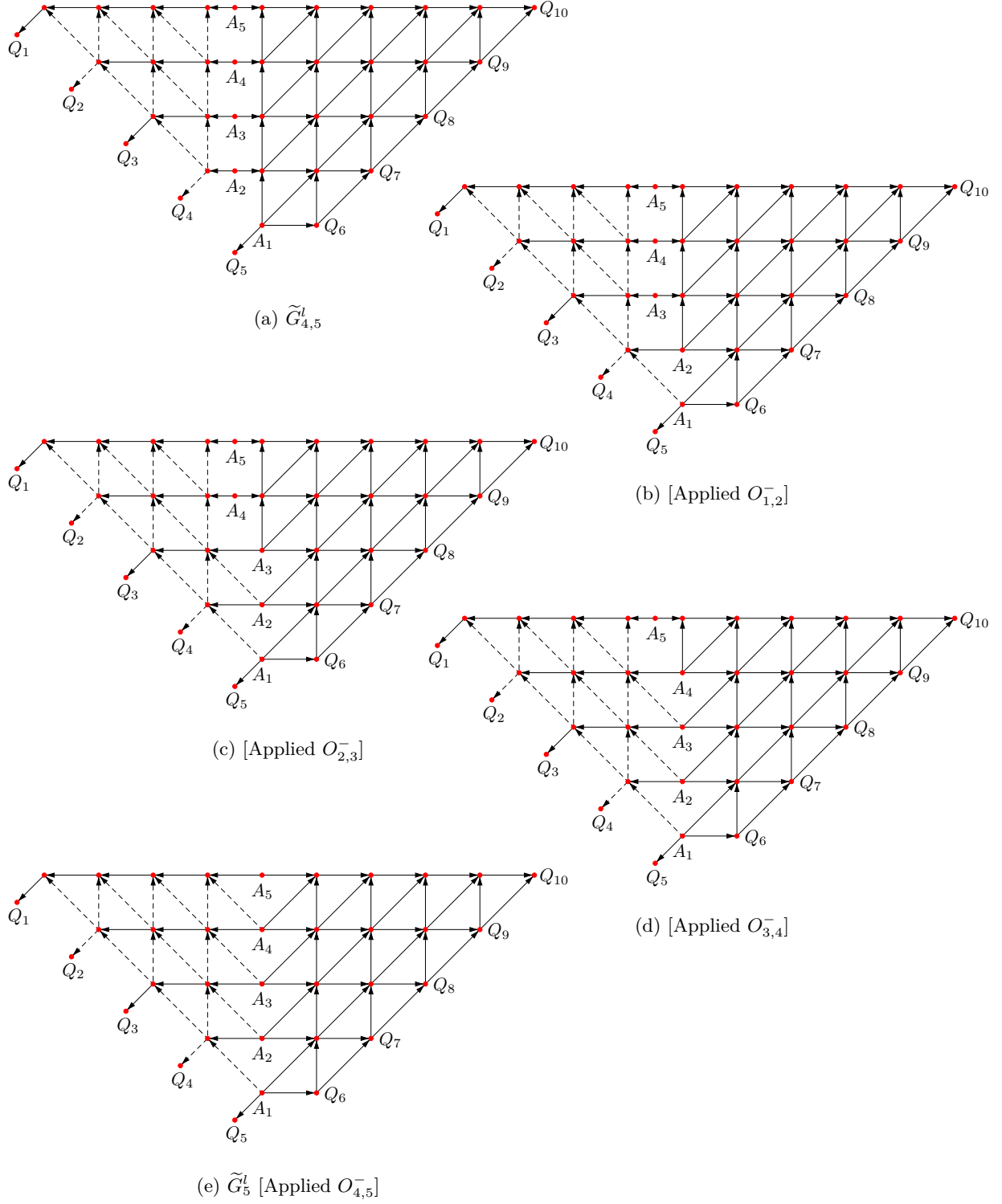


Figure 36: From $\tilde{G}_{k-1/2}^l$ to $\tilde{G}_k^l$ for $1 < k \leq n$, here for $k = 5$ ($d_k = 1$)

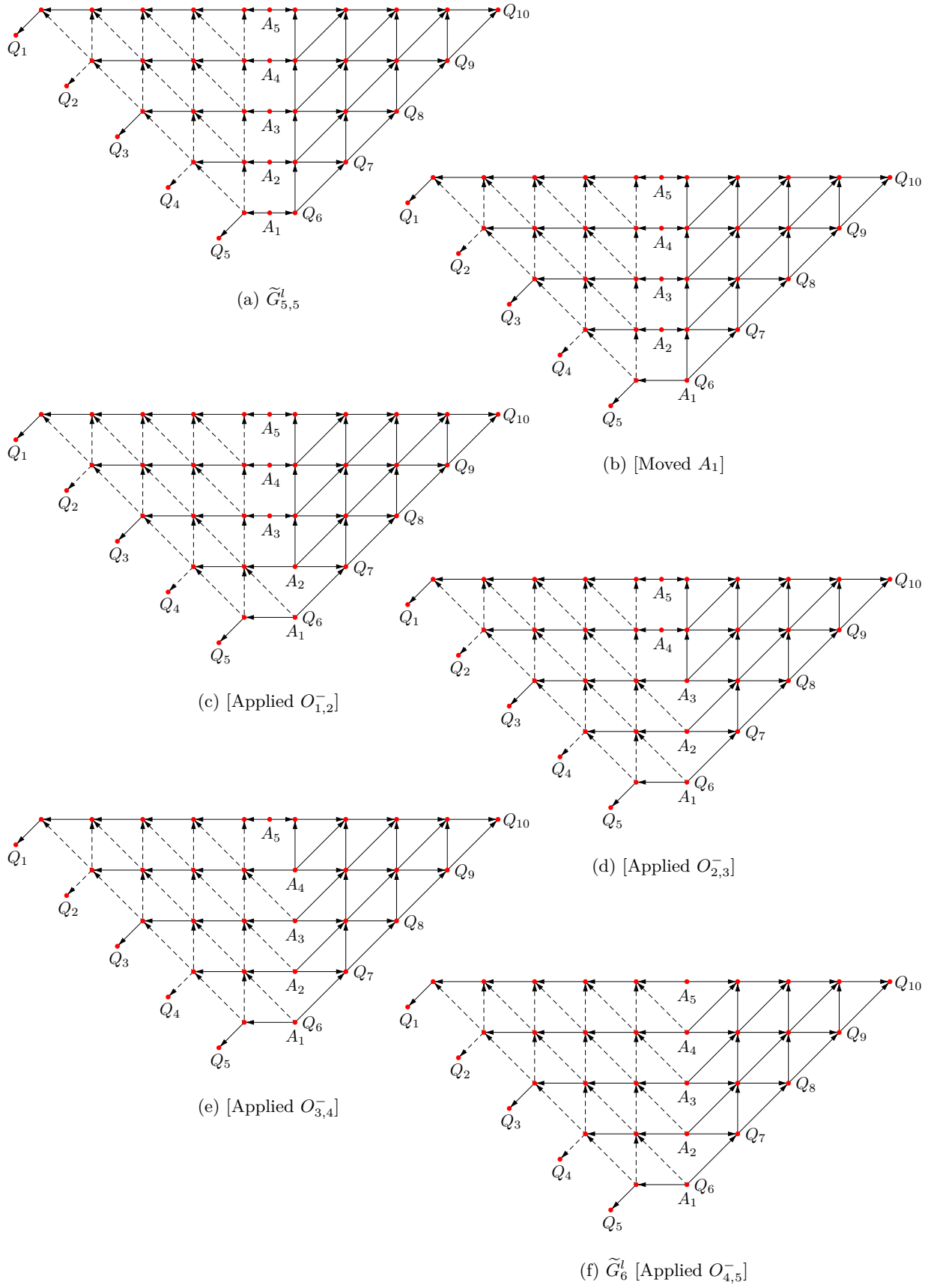


Figure 37: From $\tilde{G}_{k-1/2}^l$ to $\tilde{G}_k^l$ for $n+1 \leq k \leq 2n$, here for $k=6$ ($d_k=1$)

B Definitions

We here list the definitions appearing in this paper so that they can be found easily. Note that we do not claim completeness: Expressions that are only used locally, for instance in a single subsection, are mostly left out in order to keep the tables within a reasonable length. Section B.1 lists mathematical expressions, ordered alphabetically as accurately as it is possible, while section B.2 lists the names introduced for mathematical objects.

B.1 Mathematical expressions

$\mathbb{1}_n$	definition 7.26
$\mathbb{1}_n^s$	definition 7.13
$\widehat{A^{ij}}$	proposition 7.10
$ A, A , \underline{A}, \overline{A}$	definition 7.15
$_{-1}A$	definition 7.15
C	definition 3.2
G°	definition 3.11
$\widehat{G}$	definition 4.2
$\widetilde{G}, \widetilde{G}_k^l, \widetilde{G}_1^s$	definitions 5.17, 5.19
G_b, G_r	definition 4.2
$GF_{\text{Aztec}}^{(n)}(x, y), GF_{\text{AST}}^{(n)}(x, y)$	definition 2.16
$GF(x, y, z), GF(x, y), GF(x)$	definition 6.13, theorem 6.14
H_n, H_{ij}	definition 7.13
$H_0(\cdot), H_1(\cdot)$	theorem 7.18
$HH_1(\cdot), HH_2(\cdot)$	theorem 7.23
$\text{inv}(A)$	definition 2.5
K_n	definition 7.2
$l(M)$	definition 2.16
M_0	definition 4.2
M^{AB}, M^{AQ}	definitions 5.5, 5.13
$\widetilde{M}^{AB}, \widetilde{M}^{AQ}$	corollaries 5.9, 5.11, definitions 5.13, 5.19
$\widetilde{M}_k^l, \widetilde{M}_1^s$	definitions 5.17, 5.19
M_I, M^J	definition 5.1
$m(G)$	definition 3.4
$N_+(A), N_-(A), N_0(A)$	definition 2.3
$N_0^{br}(A), N_0^{tl}(A), N_0^{bl}(A), N_0^{tr}(A)$	definition 2.21
n_A	definition 4.2
$O_{a,b}^+, O_{a,b}^-$	definition 5.20
P'	definition 3.14
P_1, P_{-1}, P_c, P_p	definition 3.14

p^X	definition 3.14
$\text{pf}(\cdot)$	definition 7.3
Q	figures 1, 10a, 10b
Q_i	figure 15
R_n	definition 5.1
$r(M)$	definition 2.16
$\rho(A)$	definition 2.7
$S(m, a, b)$	definition 6.1
$S_n^{DH}, S_n^{VH}, S_n^{V'H}, S_n^{DD'}$	definition 6.2
$S_n^{D^0H}, S_n^{D^1H}, S_n^{D^2H}$	definition 7.26
s_i, s'_i	definition 7.26
$\text{sgn}(M)$	definition 7.2
$\sigma_{\mathcal{F}}$	definition 5.5
T_n^+, T_n^-	definition 6.4
$T_0(\cdot), T_1(\cdot)$	theorem 7.16
$TH_1(\cdot), TH_2(\cdot)$	theorem 7.21
$\theta_r(A), \theta_l(A)$	lemma 2.23
V_n	definition 5.1
$ \cdot _n$	definition 5.3
$\prod_{i=a}^b$	definition 5.27

B.2 Names of mathematical objects

Alternating sign matrix (ASM)	definition 2.1
Alternating sign pattern (ASP)	definition 3.14
Alternating sign triangle (AST)	definition 2.2
AST-graph	figure 1
Aztec diamond	figures 3c, 3d
Aztec graph	figures 3a, 3b
Blue cycle, blue path	definition 3.12
Cell (Aztec graph)	figure 4b
Cell, cell vertex (cellular graph)	definitions 3.2, 3.11
Cell graph	definition 3.11
Cellular graph	definition 3.1
Colored cycle, colored path	definition 3.12
Column-wise configuration	definition 4.6
Compatible ASP	definition 3.14
Corner, corner vertex	definitions 3.2, 3.11
Crossing	definition 7.2

Disjoint paths	definition 4.1
Elementary row-column pair operation	definition 7.6
Extending a vertex Q_j	definition 5.16
First snake	definition 4.6
Generating function of perfect matchings of the Aztec graph	definition 2.16
Generating function of Q -partial matchings of the AST-graph	definition 2.16
Generating function of Schröder paths	definition 6.13
Half-line	definition 5.19
Half-reduced AST-graph	figure 10a
Inner vertex	definition 3.2
Intersection condition	definition 5.7
Inverse ASP	definition 3.14
Inversion number	definition 2.5
Line	definition 4.6
MaPaf-bijection	definition 4.1
MaPaf-condition	definition 4.2
Mixed configuration	definition 4.6
Non-intersection condition	definition 5.7
Permutation induced by a path family	definition 5.5
Pfaffian	definition 7.3
Q -partial matching	definition 2.13
Red cycle, red path	definition 3.12
Reduced AST-graph	figure 10b
Reverse intersection condition	definition 5.7
Row-column pair	definition 7.6
Row-wise configuration	definition 4.6
Schröder matrix	definition 6.2
Schröder path	definition 6.1
Shortening a vertex Q_j	definition 5.16
Sign of a matching	definition 7.2
Skew-symmetric	definition 7.1
Snake	definition 4.6
Symmetric configuration	definition 4.6
Type of a matching	definition 3.14

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