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Abstract

The method of forcing was introduced by Paul Cohen in 1963 to prove the independence of the continuum hypothesis from the standard axioms of set theory, ZFC. Since then, it has proved to be an extremely useful tool in set theory, especially to prove the independence of certain statements. This thesis explores a certain "nice" class of forcing notions, called proper forcings which were first introduced by Saharon Shelah in the 1980s. Proper forcings behave nicely under iterations, and the main focus of this thesis is to examine how their iterations can be used to preserve certain special families of reals with strong combinatorial properties.

The thesis starts with a brief introduction to forcing (Chapter 1). In Chapter 2, we introduce the notion of properness and show that countable support iterations of proper forcings are proper (see 2.3.11). In Chapters 3 and 4, the special combinatorial families of reals are introduced, namely, tight maximal almost disjoint families and tight maximal eventually different families, and we define the notion of strong preservation of tightness for each of these combinatorial families. We show this property also gets preserved under countable support iterations (see 3.2.7 and 4.2.6). Finally in chapter 5, we examine the Miller forcing, and show that it strongly preserves the tightness of ground model tight maximal almost disjoint and tight maximal eventually different families (see 5.2.2, 5.2.3). These methods let us ensure that the value of the almost disjointedess number and the eventual difference number, the cardinal characteristics associated to these families, remains small in generic extensions via countable support iterations of proper forcing, which in turn helps to prove the consistency of certain constellations of cardinal characteristics.

Abstrakt

Die Forcing-Methode wurde 1963 von Paul Cohen eingeführt, um die Unabhängigkeit der Kontinuumshypothese von den Standardaxiomen der Mengenlehre, ZFC, zu beweisen. Seitdem hat sie sich als äußerst nützliches Werkzeug in der Mengenlehre erwiesen, insbesondere um die Unabhängigkeit bestimmter Aussagen zu zeigen. Diese Arbeit untersucht eine bestimmte „schöne" Klasse von Forcings, die "proper" genannt wird und erstmals in den 1980er Jahren von Saharon Shelah eingeführt wurde. Proper Forcings lassen sich gut iterieren, und der Schwerpunkt dieser Arbeit liegt darin, zu untersuchen, wie Iterationen verwendet werden können, um bestimmte spezielle Familien von reellen Zahlen mit starken kombinatorischen Eigenschaften zu erhalten.

Die Arbeit beginnt mit einer kurzen Einführung zu Forcing (Kapitel 1). In Kapitel 2 führen wir den Begriff der "Properness" ein und zeigen, dass Iterationen von proper Forcings mit abzählbarem Träger wiederum proper sind (siehe 2.3.11). In den Kapiteln 3 und 4 werden die speziellen kombinatorischen Familien von reellen Zahlen vorgestellt, nämlich die "tight maximal almost disjoint families" und die "tight maximal eventually different families", und wir definieren, was es heißt, die "Tightness" dieser Familien stark zu erhalten. Wir zeigen, dass diese Eigenschaft auch unter Iterationen mit abzählbarem Träger erhalten bleibt (siehe 3.2.7 und 4.2.6). Schließlich untersuchen wir in Kapitel 5 das Miller-Forcing und zeigen, dass es die Tightness von solchen Familien des Grundmodells stark erhält (siehe 5.2.2, 5.2.3). Mit diesen Methoden können wir sicherstellen, dass die älmost disjointness numberünd die "eventual difference number", also die Kardinalcharakteristika, die diesen Familien zugeordnet sind, in generischen Erweiterungen durch Iterationen abzählbaren Trägers von proper Forcing klein bleiben, was wiederum hilft, die Konsistenz bestimmter Konstellationen von Kardinalcharakteristika zu beweisen.

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Inhaltsverzeichnis

1 What is forcing?

We work in the first-order language $L = \{\in, V\}$ where $\in$ is a binary relation symbol and V is a constant symbol. Now we define the theory "$ZFC + ctm(V)$" as $ZFC + ctm(V) = ZFC \cup \{\phi^V | \phi \in ZFC\} \cup \{V \text{ is countable and transitive}\}$ here ZFC stands for "Zermelo-Fraenkel Set Theory with Axiom of Choice", and denotes the usual axioms of set theory along with choice, and V is transitive means that every element of V is also a subset of V. So basically $ZFC + ctm(V)$ is ZFC along with the assertion that "V is a countable transitive model of ZFC"
In the above definition for a $\phi \in ZFC, \phi^V$ denotes the relativization of ϕ to V, which means that ϕ^V is the formula obtained inductively from ϕ by restricting all the quantifiers occurring in ϕ to V.

1.1 Meta Considerations

In this brief section, we discuss the meta-theoretic aspects of how consistency proofs actually work. This section is based on [12], [2].

Lemma 1.1.1. If ZFC is consistent then so is $ZFC + ctm(V)$

Proof. Assume that $ZFC + ctm(V)$ is inconsistent, then as proofs are finite and conjunctions of finitely many formulae are still a formula, we can assume without loss of generality that there are formulae $\psi, \phi \in ZFC$ such that $\{\psi, \phi^V, V$ is countable and transitive$\}$ proves in the language $\{\in, V\}$ the statement $\forall x(x \neq x)$. Now an easy induction on the complexity of formulae shows that there is a formula $K = K(x)$, in the language $\{\in\}$ such that $\phi^V = K(V)$, so that we have $\{\psi, K(V), V$ is countable and transitive$\}$ proves $\forall x(x \neq x)$. From this, it follows that $\{\psi,\ \exists V(K(V) \wedge V$ is countable and transitive$)\}$ proves, in the language $\{\in\}$, the statement $\forall x(x \neq x)$. See [2] for more on the meta rule we used to make this conclusion. But by the reflection theorem (see [10] for more on the reflection theorem), we know that ZFC proves $\exists V(K(V) \wedge V$ is countable and transitive$)$, from which it follows that ZFC, proves, in the language $\{\in\}$, $\forall x(x \neq x)$, which means ZFC is inconsistent. $\square$

Lemma 1.1.2. Assume ZFC is consistent. Let ψ be a formula in the language $\{\in\}$ and assume that in $ZFC + ctm(V)$ we can prove $\exists N(\chi(N) \wedge (\neg\psi)^N)$, where

χ is some formula in the language $\{\in, V\}$. Moreover assume for each $\phi \in ZFC$, we have $ZFC + ctm(V)$ proves $\forall N(\chi(N) \implies \phi^N)$, then ZFC does not prove ψ.

Remark. Intuitively, this lemma is saying that if in $ZFC + ctm(V)$, we can show the existence of a set N, that is a model for $ZFC \cup \{\neg\psi\}$, then ZFC does not prove ψ.

Proof. Suppose to the contrary that $\phi_1, \phi_2, ..., \phi_n, \phi_{n+1} = \psi$ is a proof of ψ from ZFC. Then via induction we show that for each $k = 1, 2, 3, ..., n+1$, $ZFC+ctm(V)$ proves ϕ_k^N. For $k = 1$, ϕ_k must necessarily be an axiom of ZFC or a logical axiom. We leave the case when ϕ_k $(k = 1)$ is a logical axiom as an exercise. If $\phi_1 \in ZFC$, then we just apply the hypothesis. Now assume the statement is true for all $r < k \leq n + 1$, then we show the statement holds for k as follows: without loss of generality, G assume ϕ_k is not an axiom of ZFC, so that either (a) ϕ_k is a logical axiom, or (b) ϕ_k was obtained from $\phi_{k_1}, \phi_{k_2} = (\phi_{k_1} \implies \phi_k)$ via modus ponens, where $k_1, k_2 < k$, note that we can assume modus ponens is the only rule of inference (see [2]). Now in case(b), we get by the inductive hypothesis that $ZFC + ctm(V)$ proves both $\phi_{k_1}^N$ and $\phi_{k_2}^N = (\phi_{k_1} \implies \phi_k)^N = (\phi_{k_1}^N \implies \phi_k^N)$ and hence we obtain $ZFC + ctm(V)$ proves ϕ_k^N. Case(a) is left as an exercise to the reader. So $ZFC+ctm(V)$ proves both $(\neg\psi)^N$ and ψ^N, which means $ZFC+ctm(V)$ is inconsistent, and by the previous lemma, this contradicts the consistency of ZFC. This proves the lemma.

$\square$

The above two lemmas capture the essence of independence proofs. The method of forcing lets us construct the model "N" from the previous lemma, as an extension of V

1.2 The method of forcing

In this section, we try to give a brief overview of how the method of forcing achieves the construction of the N that appears in Lemma 1.1.2.

Definition 1.2.1.

A *preorder* is a pair $\langle \mathbb{P}, \leq \rangle$ such that $\mathbb{P}$ is a set, $\leq$ is a binary relation on $\mathbb{P}$, and the following are satisfied:

1. $\forall x \in \mathbb{P} \forall y \in \mathbb{P} \forall z \in \mathbb{P}(x \leq y \wedge y \leq z \implies x \leq z)$ (i.e., $\leq$ is transitive)
2. $\forall x \in \mathbb{P}(x \leq x)$ (i.e., $\leq$ is reflexive).

Remark.

1. A preorder might not be antisymmetric, i.e. there might be distinct x, y such that $x \leq y$ and $y \leq x$.

2. If $x \leq y$ we say x extends y or x is stronger than y.

3. From here on, we abuse notation and say "let $\mathbb{P}$ be a preorder" and what we actually mean is $\langle \mathbb{P}, \leq \rangle$ is a preorder, i.e. we only mention the underlying $\mathbb{P}$ when the ordering is clear from the context.

Definition 1.2.2. Let $\mathbb{P}$ be a preorder, and let $p, q \in \mathbb{P}$, then:

1. We say p, q are incompatible if there is no $r \in \mathbb{P}$ such that r extends both p and q. In this case, we write $p \perp q$.

2. We say p, q are compatible if p, q are not incompatible, i.e. there is some $r \in \mathbb{P}$ such that r extends both p and q. In this case, we write $p \not\perp q$.

3. We say $A \subseteq \mathbb{P}$ is an antichain in $\mathbb{P}$ if for each $p, q \in A(p \neq q \implies p \perp q)$, in other words, A consists of pairwise incompatible elements.

4. We say $D \subseteq \mathbb{P}$ is dense in $\mathbb{P}$ if $\forall p \in \mathbb{P} \exists q \in D(q \leq p)$.

5. We say $D \subseteq \mathbb{P}$ is predense in $\mathbb{P}$ if $\forall p \in \mathbb{P} \exists q \in D(p \not\perp q)$

6. We say $D \subseteq \mathbb{P}$ is dense below p if $\forall p_1 \leq p \exists p_2 \in D(p_2 \leq p_1)$

7. We say $D \subseteq \mathbb{P}$ is predense below p if $\forall p_1 \leq p \exists p_2 \in D(p_2 \not\perp p_1)$

Definition 1.2.3. We say a preorder $\mathbb{P}$ is separative iff
$\forall p, q \in \mathbb{P}(p \not\leq q \implies \exists r \leq p(r \perp q))$

Definition 1.2.4. Let $\mathbb{P}$ be a preorder and let $O \subseteq \mathbb{P}$, then we say O is open if $\forall p \in \mathbb{P} \forall q \in O(p \leq q \implies p \in O)$ (this property is also called "Downwards closed")

Definition 1.2.5. Let $\mathbb{P}$ be a preorder, then we say that an element $p \in \mathbb{P}$ is a largest element of $\mathbb{P}$, if $\forall x \in \mathbb{P}(x \leq p)$. Sometimes we denote such a largest element by 1 or $1_{\mathbb{P}}$, even though it might not be unique.

Definition 1.2.6. A *forcing poset/forcing notion* is a triple $\langle \mathbb{P}, \leq, 1 \rangle$ such that $\langle \mathbb{P}, \leq \rangle$ is a separative preorder and $1 \in \mathbb{P}$ is a largest element of $\langle \mathbb{P}, \leq \rangle$.

Less formally, a forcing notion is a separative preorder with a largest element. Most often we just mention the poset $\mathbb{P}$, when $\langle \mathbb{P}, \leq, 1 \rangle$ is understood from context.

Remark. Separative preorders have the nice and useful property that $p \Vdash \check{q} \in \dot{G}$ implies $p \leq q$ (the reverse implication is satisfied by any preorder), which would prove to be useful while proving preservation theorems.

Definition 1.2.7. We say a set x is *transitive* if $\forall y \in x(y \subseteq x)$

Remark. By $ZF-P$, we mean all axioms of ZFC excluding the axiom of powerset and the axiom of choice.

Definition 1.2.8. Let $\mathbb{P}$ be a forcing poset and let $G \subseteq \mathbb{P}$. We say G is a *filter* on $\mathbb{P}$ if the following hold:

1. $1 \in G$
2. $\forall p, q \in \mathbb{P}(q \in G \text{ and } q \leq p \implies p \in G)$ (G is upwards closed)
3. $\forall p, q \in G \exists r \in G(r \leq p, q)$

Moreover, if V is a transitive model of $ZF - P$, we say a filter G on $\mathbb{P}$ is a $(V, \mathbb{P})$ generic filter if "$\forall D \in V(D \subseteq \mathbb{P} \wedge D$ is dense in $\mathbb{P} \implies D \cap G \neq \emptyset)$", in other words G has a nonempty intersection with every dense subset of $\mathbb{P}$ that happens to be in V.

Lemma 1.2.9. Let V be a countable transitive model of $ZF - P$ and let $\mathbb{P} \in V$ be a forcing poset and let $p \in \mathbb{P}$, then there is a $(V, \mathbb{P})$ generic filter $G \subseteq \mathbb{P}$ with $p \in G$. Furthermore if $\mathbb{P}$ is atomless, (this means that for each $r \in \mathbb{P}$, there are elements $p, q \in \mathbb{P}$ with $p, q \leq r \wedge p \perp q$), then any $(V, \mathbb{P})$ generic filter G must lie outside V.

Remark. We use the phrase "Let V be a ctm" to mean that "V is a countable transitive set and V satisfies a large enough fragment of ZFC (finite or infinite), so that the arguments in the proof go through/make sense", or we could just work with our additional constant symbol V which is obviously a ctm by the above convention and satisfies all of ZFC.

Definition 1.2.10. **$\mathbb{P}$-names:** Let $\mathbb{P}$ be a forcing poset, then we say that a set τ is a $\mathbb{P}$-name iff [τ is a relation and $\forall \langle \sigma, p \rangle \in \tau(\sigma$ is a $\mathbb{P}$-name and $p \in \mathbb{P})$]

Note that the above definition is recursive.

Definition 1.2.11. Let τ be a $\mathbb{P}$-name and let $G \subseteq \mathbb{P}$, we define the *evaluation* of τ by G, recursively, as $val(\tau, G) = \tau_G = \{\sigma_G \mid \exists p \in G(\langle \sigma, p \rangle \in \tau)\}$

Remark. The notion of a $\mathbb{P}$-name and the evaluation function $val(\tau, G)$ are both absolute for transitive models of $ZF - P$

Definition 1.2.12. For a transitive model V of $ZF - P$, we denote by $V^{\mathbb{P}}$, the set of $\mathbb{P}$-names that lie in V. More precisely $V^{\mathbb{P}} = \{\tau \in V \mid \tau \text{ is a } \mathbb{P}\text{-name}\}$

Definition 1.2.13. $V[G]$: Let V be a ctm for $ZF - P$, let $\mathbb{P} \in V$ be a forcing poset, and let $G \subseteq \mathbb{P}$ be a $(V, \mathbb{P})$ generic filter, then we define

$$V[G] = \{\tau_G \mid \tau \in V^{\mathbb{P}}\}$$

Remark. The intuition for $V[G]$ is that we are, via the $\mathbb{P}$-names, using the notion of an evaluation, adding all objects that are definable from the generic object G and elements in V

Lemma 1.2.14. Let $V \models ZF - P$ and let $\mathbb{P} \in V$ be a forcing poset, and let G be a $(V, \mathbb{P})$ generic filter on $\mathbb{P}$, then we have

1. V and $V[G]$ have the same ordinals and
2. $|V[G]| = |V|$, so that in particular if V is countable, so is $V[G]$

(For a proof see [10])

Definition 1.2.15. If V is a ctm for $ZF - P$, we define for $x \in V$,
$\check{x} = \{\langle \check{y}, 1_{\mathbb{P}} \rangle \mid y \in x\}$ and call this the *check name/canonical name* for x. We define $\dot{G} = \{\langle \check{p}, p \rangle \mid p \in \mathbb{P}\}$ and call this the canonical name for a generic filter on $\mathbb{P}$.

Remark. An easy computation shows that $\dot{G}_G = G$ and $\check{x}_G = x$ for all $(V, \mathbb{P})$ generic G, which in turn shows that $V \subseteq V[G]$ and $G \in V[G]$. Note that the names $\dot{G}$ and $\check{x}$ are definable in V and hence in V.

With these definitions in mind, it is now a good idea to give an intuition of what these definitions are supposed to do, and how they do it will be the rest of this section. What we have achieved so far using the above definitions is: starting with a model of some set theory (most often something stronger that $ZF - P$) V, and using a forcing poset which belongs to V, construct combinatorial objects called names so that what they name can be determined once we are given a generic filter, by evaluating the name with respect to the given filter G. The set of all evaluations (with respect to G) gives us a bigger universe that contains V as a subset and contains the filter G as a member. So what we have essentially done is add a "generic" object G to V. Then, if we can show that $V[G]$ satisfies ZFC (or a fragment of ZFC), then a witness to the destruction/non-preservation of a property that we want to destroy is usually some object that is definable from the generic G, in $V[G]$

Definition 1.2.16. **(The forcing relation, $\Vdash$)**
Let V be a ctm, and let $\mathbb{P} \in V$, be a forcing poset, and let $\psi - \psi(x_0, ...x_{n-1})$, be a formula in the language of set theory and let $\tau_0, \tau_1, ..., \tau_{n-1} \in V^{\mathbb{P}}$ then for any $p \in \mathbb{P}$, we say "p forces $\psi(\tau_0, ...\tau_{n-1})$ with respect to V and $\mathbb{P}$", (written $p \Vdash_{(V,\mathbb{P})} \psi(\tau_0, ..., \tau_{n-1})$) if:

$$\forall G((G \text{ is } (V,\mathbb{P}) \text{ generic and } p \in G) \implies V[G] \models \psi((\tau_0)_G, ..., (\tau_{n-1})_G))$$

Remark.

1. When the context is clear, we sometimes use $\Vdash$ instead of $\Vdash_{(V,\mathbb{P})}$, and we call this symbol the forcing relation/ forcing relation of $(V, \mathbb{P})$
2. The definition of the forcing relation requires one to come outside of V, since the definition involves the generic G, which generally does not lie in V (See [10]).
3. There is an internal version of the forcing relation, for any forcing poset $\mathbb{P}$, denoted by $\Vdash^*_{\mathbb{P}}$, which relates to $\Vdash$, in the following sense:

   $$V \models (p \Vdash^*_{\mathbb{P}} \psi(\tau_0, ...\tau_{n-1})) \iff p \Vdash_{(V,\mathbb{P})} \psi(\tau_0, \tau_1, ...\tau_{n-1}).$$

   This relation shows that objects defined using $\Vdash^*$, in V, will be members of V, and one can show the following two facts to be true:

Theorem 1.2.17. *Let $V \models ZFC$ be a ctm and $\mathbb{P} \in V$ and let G be a $(V, \mathbb{P})$ generic filter. Then $V[G] \models ZFC$*

Theorem 1.2.18. ***(The forcing theorem)***
Let V be a countable transitive model for ZFC and let $\mathbb{P} \in V$ be a forcing poset and, let G be a $(V, \mathbb{P})$ generic filter, and let $\psi(x_0, ..., x_{n-1})$ be a formula in the language of set theory and assume $\tau_0, ...\tau_{n-1} \in V^{\mathbb{P}}$, then we have,

$$V[G] \models \psi((\tau_0)_G, ...(\tau_{n-1})_G) \iff \exists p \in G(p \Vdash \psi(\tau_0, ...\tau_{n-1})).$$

For proofs of Theorem 1.2.17 and Theorem 1.2.18, see [10].

Remark. Most of the time, we use the internal forcing $\Vdash^*$ and $\Vdash$, interchangeably, and we just notate both by the symbol $\Vdash$.

The results and definitions above capture the essence of forcing. Now we end this chapter with a few more results that might be helpful in the later chapters.

Lemma 1.2.19. If $\psi = \psi(x_1, ..., x_n)$ is absolute between transitive models of set theory, and $\tau_1, \tau_2, ..., \tau_n$ are $\mathbb{P}$-names, then the formula $p \Vdash \psi(\tau_1, \tau_2, ..., \tau_n)$ is absolute between transitive models of set theory.

Lemma 1.2.20. (**Existential completeness/ maximality principle**)
Let V be a ctm for ZFC let $\mathbb{P} \in V$ be a forcing poset and let $p \in \mathbb{P}$, and let $\psi = \psi(x_0, ..., x_{n-1})$ be a formula in the language of set theory and let $\mu_1, ...\mu_{n-1} \in V^{\mathbb{P}}$, then we have have that:

$$p \Vdash \exists x_0 \psi(x_0, \mu_1, ...\mu_{n-1}) \implies \exists \tau \in V^{\mathbb{P}}(p \Vdash \psi(\tau, \mu_1, ...\mu_{n-1}))$$

(See [10] for a proof)

Remark.

1. We really require the ground model V to satisfy the axiom of choice, for this theorem to hold.
2. Existential completeness is very useful in the context of forcing, as it provides "extremely well behaved names" for definable objects, more precisely:

Corollary 1.2.21. Let $\psi = \psi(x_0, ...x_{n-1}, y)$ be a formula in the language of set theory such that $ZFC \vdash \forall x_0, ..., x_{n-1} \exists! y \psi(x_0, ..x_{n-1}, y)$, i.e. ψ defines an n-variable function on the class of all sets, (let us denote this function by $f = f(x_0, ..., x_{n-1})$), then for each $\mu_0, ...\mu_{n-1} \in V^{\mathbb{P}}$, there is a $\tau \in V^{\mathbb{P}}$ such that for each $(V, \mathbb{P})$ generic filter G,

$$V[G] \models \psi((\mu_0)_G, (\mu_1)_G, ..., (\mu_{n-1})_G, (\tau)_G),$$

i.e. for each generic G, we have $f^{V[G]}((\mu_0)_G, ..., (\mu_{n-1})_G) = \tau_G$, i.e. the name τ uniformly evaluates to the object defined by ψ, in all extensions. (This property is what we referred to as "well behaved").

Example: If $\psi = \psi(x, y, z)$ is the formula that defines the ordered pair, i.e. $ZFC \vdash \forall x \forall y(\psi(x, y, z) \iff z = \langle x, y \rangle)$, and if μ_1, μ_2 are names, then it makes sense to use the notation $\langle \mu_1, \mu_2 \rangle$ in forcing formula, and $\langle \mu_1, \mu_2 \rangle$ is actually just a label/notation, for a name τ, provided by the previous corollary, that satisfies $\tau_G = \langle (\mu_1)_G, (\mu_2)_G \rangle$, for all generic filters G.

Definition 1.2.22. For a cardinal λ, we define, $H(\lambda) = \{x \mid |trcl(x)| < \lambda\}$, where $trcl(x)$ is the transitive closure of x (see [10] for a definition)

2 Proper forcing

In this chapter, we define the notion of proper forcing that generalises the notion of ccc forcings as well as σ-closed forcings (see [10] for more on ccc, and σ-closed), and also ends up preserving/not collapsing $\aleph_1$. We will assume the reader is familiar with the definitions of forcing equivalence, complete embeddings, projections, and an iteration of forcing notions. For definitions of these and related notions, see, for example, [1], [9] (which is in turn based on [11]) or [10]. The main result of this section is that "countable support iterations of proper forcing notions are proper"(See 2.3.11). This chapter follows closely [1],[9] and also uses results from [5], [6] and [10].

2.1 Before Properness

In this brief section, we include some definitions, observations and lemmas that capture the minute details required when one develops the theory of proper forcing and its iterations.

Remark. In what follows, we will always assume our forcing posets are separative, so that the useful property $\forall p, q \in \mathbb{P}(q \Vdash \check{p} \in \dot{G} \iff q \leq p)$ holds true. There is no loss of generality, because every preorder with a largest element is forcing equivalent to a separative preorder with a largest element, see [3], [9] for more details. Moreover, for us, a separative preorder might not be antisymmetric, so that iterations of separative preorders are separative preorders (See the approach in [1]).

Notation. We write $\dot{H}(\lambda)$ for a $\mathbb{P}$-name τ that satisfies $\forall G(\tau_G = H(\lambda)^{V[G]})$ and such a name obviously exists via existential completeness from Chapter 1.

Remark. If $\lambda > 2^{|\mathbb{P}|}$, then $\mathbb{P}$ has the $|\mathbb{P}|^+$-cc, so that forcing with $\mathbb{P}$ preserves cardinals and cofinalities $\geq |\mathbb{P}|^+$, so that in particular, $\mathbb{P}$ preserves the cardinal λ and its regularity. Therefore, it makes sense to talk about $H(\lambda)^{V[G]}$ (see [10] for more details and a definition of the notion of κ-cc).

Definition 2.1.1. If $M \in V$, we define for a generic filter G,

$$M[G] = \{\tau_G \mid \tau \in M \cap V^{\mathbb{P}}\}$$

and by $M[\dot{G}]$ we denote a name (provided by existential completeness) that uniformly evaluates to $M[G]$ for all generic filters G.

Remark.

1. In all scenarios we deal with in this thesis, M will be a countable elementary submodel of $H(\lambda)$.

2. If M is an elementary submodel of $H(\lambda)$ and $a_0, ...a_{n-1}$ are elements of M, if $H(\lambda) \models \exists! z\ \psi(a_0, ..., a_{n-1}, z)$, and if $b \in H(\lambda)$ denotes the unique element in $H(\lambda)$ such that $H(\lambda) \models \psi(a_0, ..., a_{n-1}, b)$, then $b \in M$. This fact will be used time and time again, and we refer to this fact via the phrase "b is definable in $H(\lambda)$ from parameters that lie in M and hence $b \in M$".

Lemma 2.1.2. (Assuming choice in the ground model V)
Let $\mathbb{P}$ be a forcing poset and let μ, τ be $\mathbb{P}$-names and let $p \in \mathbb{P}$ and assume that $p \Vdash \mu \in \tau$, then there is a $\nu \in V^{\mathbb{P}}$ such that $p \Vdash \nu = \mu$ and $1 \Vdash (\tau \neq \emptyset \to \nu \in \tau)$.

(See [5] for a proof)

Lemma 2.1.3. Let $\lambda > 2^{|\mathbb{P}|}$ be a regular uncountable cardinal and assume that $\tau \in V^{\mathbb{P}}$ satisfies $1 \Vdash \tau \in \dot{H}(\lambda)$, then there is a $\mu \in V^{\mathbb{P}} \cap H(\lambda)^V$ such that $1 \Vdash \mu = \tau$.

(For a proof, see [6, fact 3.6])

Lemma 2.1.4. Let $\mathbb{P}$ be a forcing notion and let $\lambda > 2^{|\mathbb{P}|}$ be a regular uncountable cardinal, then for each $(V, \mathbb{P})$ generic filter $G \subseteq \mathbb{P}$, we have $(H(\lambda))^V[G] = H(\lambda)^{V[G]}$, in other words $1 \Vdash (H(\lambda)^V)[\dot{G}] = \dot{H}(\lambda)$.

Proof.

Claim. $H(\lambda)^{V[G]} \subseteq H(\lambda)^V[G]$:

Proof. Let $\tau_G \in H(\lambda)^{V[G]}$, now without loss of generality, we can assume $1 \Vdash \tau \in \dot{H}(\lambda)$. To see why, let by the forcing theorem $p \in G$ be such that $p \Vdash \tau \in \dot{H}(\lambda)$, now as ZFC proves $H(\lambda) \neq \emptyset$, this is forced, i.e. $1 \Vdash \dot{H}(\lambda) \neq \emptyset$, therefore applying lemma 2.1.2, we obtain that there is a name ν such that $1 \Vdash \nu \in \dot{H}(\lambda)$ and $p \Vdash \nu = \tau$. Moreover as $p \in G$ we get $\nu_G = \tau_G$. Now by lemma 2.1.3, there is a $\nu \in H(\lambda)^V$ such that $1 \Vdash \nu = \tau$, so that we have $\tau_G = \nu_G \in H(\lambda)^V[G]$. Therefore $H(\lambda)^{V[G]} \subseteq H(\lambda)^V[G]$ □

Claim. $H(\lambda)^V[G] \subseteq H(\lambda)^{V[G]}$:

Proof. let $\tau \in H(\lambda)^V$, now an easy argument shows that $|\tau_G|^{V[G]} \leq |\tau|^{V[G]}$. The argument is as follows: define $h \in V[G]$ by $h = \{\langle\langle\sigma, r\rangle, \sigma_G\rangle \mid \langle\sigma, r\rangle \in \tau\}$. Now $\tau_G \subseteq range(h)$, so that $|\tau_G|^{V[G]} \leq |range(h)|^{V[G]} \leq |\tau|^{V[G]}$). But $|\tau|^{V[G]} \leq |\tau|^V$, where the final inequality follows because forcing has the potential to add functions and hence reduce the cardinality of sets in the ground model, but never increases the cardinality of sets in the ground model. But $|\tau|^V < \lambda$, so we obtain the desired result that $|\tau_G|^{V[G]} < \lambda$ and hence $\tau_G \in H(\lambda)^{V[G]}$. This shows that the claim holds. $\square$

$\square$

Lemma 2.1.5. Let $\mathbb{P}$ be a forcing notion and let $\lambda > 2^{|\mathbb{P}|}$ be a regular uncountable cardinal, then for each $p \in \mathbb{P}$ and each $x \in V$ and each $\tau \in H(\lambda)^V \cap V^{\mathbb{P}}$, we have if $p \Vdash \tau = \check{x}$ then $x \in H(\lambda)^V$.

Proof. Suppose $x \notin H(\lambda)^V$, then we have $|trcl(x)|^V \geq \lambda$, so that $|trcl(x)|^V = |trcl(x)|^{V[G]}$ for all generics G, because $\mathbb{P}$ preserves cardinals greater than $|\mathbb{P}|^+$. Now as $p \Vdash \tau = \check{x}$, choosing a generic filter G that contains p, we get $\tau_G = x$, so that $|trcl(\tau_G)|^{V[G]} = |trcl(x)|^{V[G]} = |trcl(x)|^V \geq \lambda$. Which gives $\tau_G \notin H(\lambda)^{V[G]} = H(\lambda)^V[G]$, which is absurd as $\tau \in H(\lambda)^V$. So we conclude $x \in H(\lambda)^V$. $\square$

Lemma 2.1.6. Let λ be a regular, uncountable cardinal, then $H(\lambda) \models ZFC - P$. (For a proof, see [10])

Remark.

1. If x is a set, we denote the powerset of x by $\mathcal{P}(x)$, i.e. $\mathcal{P}(x) = \{y \mid y \subseteq x\}$

2. For a set x, we define the domain of x as, $dom(x) = \{z \mid \exists y(\langle z, y\rangle \in x)\}$.

Lemma 2.1.7. Assume $\lambda > 2^{|\mathbb{P}|}$ is regular, uncountable and let $M \prec H(\lambda)$ be countable with $\mathbb{P} \in M$ and let G be a $(V, \mathbb{P})$ generic filter on $\mathbb{P}$, then we have that $M[G] \prec H(\lambda)^{V[G]}$, i.e. in other words, $1 \Vdash (M[\dot{G}] \prec \dot{H}(\lambda))$.

Proof. First, we observe: if $p \in \mathbb{P}$ is such that $p \Vdash \psi$ or $p \Vdash \neg\psi$, then we have $((p \Vdash \psi \implies p \Vdash \phi) \implies p \Vdash (\psi \implies \phi))$, i.e. forcing behaves like the satisfaction relation in this case.

Second we observe that as λ is regular, uncountable, and $\lambda > 2^{|\mathbb{P}|}$, the following facts hold for $H(\lambda)$:

1. $\mathcal{P}(\mathbb{P})$, the powerset of $\mathbb{P}$, is a member of $H(\lambda)$.

2. $H(\lambda) \models (ZF - P$+every set can be well ordered), which in particular implies that the forcing theorem is valid for $H(\lambda)^V$.

Remark. Most versions of the axiom of choice, like Zorn's Lemma, the statement that every family of nonempty sets has a choice function, etc., follow from the statement that every set can be well-ordered, even without assuming the powerset axiom (see [10, Theorem I.12.1]).

Using the above two properties, one can show that:

1. $\Vdash_{(H(\lambda)^V,\mathbb{P})}$ satisfies existential completeness/maximality principle.
2. $\{G \mid G \text{ is } (H(\lambda)^V,\mathbb{P}) \text{ generic}\} = \{G \mid G \text{ is } (V,\mathbb{P}) \text{ generic}\}$

Point 2 above will be important to be able to conclude that
$1 \Vdash (M[\dot{G}] \prec \dot{H}(\lambda))$ where here, $\Vdash$ denotes $\Vdash_{(V,\mathbb{P})}$, i.e. the forcing relation of V. We use the famous Tarski-Vaught test from model theory to prove elementarity (see [10] for a proof of the Tarski-Vaught test).

Note: From now on, in this proof, by $\Vdash$ we mean $\Vdash_{(H(\lambda)^V,\mathbb{P})}$.

We are now ready to get to the meat of the proof:
Let $\psi' = \exists y\psi(x_0, x_1, ..., x_{n-1}, y)$ be a formula in the language of set theory, and fix arbitrarily, $\mathbb{P}$-names $\tau_0, \tau_1, ..., \tau_{n-1} \in M$. The first step is to construct a maximal antichain $A \in M$ and a name $\sigma \in M$ such that

$$\forall p \in A(p \Vdash [\exists y\psi(\tau_0, \tau_1, ..., \tau_{n-1}, y) \longrightarrow \psi(\tau_0, \tau_1, ..., \tau_{n-1}, \sigma)])$$

Towards this, we first note that the set $D_\phi = \{p \in \mathbb{P} \mid p \Vdash \phi \vee p \Vdash \neg\phi\}$ is dense in $\mathbb{P}$ and moreover, is definable in $H(\lambda)$ from elements in M, so that $D_\phi \in M$. Now via Zorn's lemma applied in $H(\lambda)$ one can obtain in $H(\lambda)$ an antichain

$$A \subseteq D = \{p \mid p \Vdash \exists y\psi(\tau_0, \tau_1, ..., \tau_{n-1}, y) \vee p \Vdash \neg\exists y\psi(\tau_0, \tau_1, ..., \tau_{n-1}, y)\} \in M,$$

maximal in D. Now using existential completeness one obtains for each $p \in A$ such that $p \Vdash \exists y\psi(\tau_0, \tau_1, ..., \tau_{n-1}, y)$, a $\sigma_p \in H(\lambda) \cap V^{\mathbb{P}}$ such that $p \Vdash \psi(\tau_0, \tau_1, ..., \tau_{n-1}, \sigma_p)$ and for $p \in A$ such that $p \Vdash \neg\exists y\psi(\tau_0, \tau_1, ..., \tau_{n-1}, y)$, let $\sigma_p = \emptyset$. We can moreover assume that $\{\langle p, \sigma_p\rangle \mid p \in A\} \in H(\lambda)$, because $(\text{Axiom of choice})^{H(\lambda)^V}$ holds. Now by $M \prec H(\lambda)$ we can assume the antichain $A \in M$ and also that $\{\langle p, \sigma_p\rangle \mid p \in A\} \in M$
Let

$$\sigma = \{\langle \eta, r\rangle \mid \exists p \in A(r \leq p \wedge \eta \in dom(\sigma_p) \wedge r \Vdash \eta \in \sigma_p)\}$$

then $\sigma \in M$ (because, yet again σ is definable in $H(\lambda)$ from the elements A, $\{\langle p, \sigma_p\rangle \mid p \in A\}$, which lie in M).

Claim. $\forall p \in A(p \Vdash \sigma = \sigma_p)$

Proof. Let $p \in A$ and let as usual G be a $(H(\lambda)^V, \mathbb{P})$ generic filter with $p \in G$. It is easy to see that $\sigma_G \subseteq (\sigma_p)_G$. To show the other direction, let $\eta_G \in (\sigma_P)_G$ (with $\eta \in dom(\sigma_p)$) then let $r \in G$ be such that $r \leq p \wedge r \Vdash \eta \in \sigma_p$, then we have that $\langle \eta, r \rangle \in \sigma$, so that $\eta_G \in \sigma_G$, showing that $(\sigma_p)_G \subseteq \sigma_G$. Thus, the claim follows. □

Now for $p \in A$ such that $p \Vdash \exists y \psi(\tau_0, \tau_1, ..., \tau_{n-1}, y)$, we obtain from the previous claim that $p \Vdash \psi(\tau_0, \tau_1, ..., \tau_{n-1}, \sigma)$, i.e. more precisely, we have just shown that $\forall p \in A(p \Vdash \exists y \psi(\tau_0, \tau_1, ..., \tau_{n-1}, y) \implies p \Vdash \psi(\tau_0, \tau_1, ..., \tau_{n-1}, \sigma))$, so that by the observation in the beginning of the proof, we obtain that

$$\forall p \in A(p \Vdash [\exists y \psi(\tau_0, \tau_1, ..., \tau_{n-1}, y) \longrightarrow \psi(\tau_0, \tau_1, ..., \tau_{n-1}, \sigma)]).$$

Note also that, as $\sigma \in M$, $\sigma_G \in M[G]$ for all generic filters G.
With the above setup, we are now ready to apply the Tarski-Vaught test as follows: let G be arbitrary and let $H(\lambda)^V[G] \models \exists y \psi(\tau_0, \tau_1, ..., \tau_{n-1}, y)$, now as A is a maximal antichain in $\mathbb{P}$ we obtain that $A \cap G \neq \emptyset$, so choose $p_0 \in A \cap G$. For this p_0 we have $p_0 \Vdash [\exists y \psi(\tau_0, \tau_1, ..., \tau_{n-1}, y) \longrightarrow \psi(\tau_0, \tau_1, ..., \tau_{n-1}, \sigma)]$, this means that we have

$$H(\lambda)^V[G] \models [\exists y \psi(\tau_0, \tau_1, ..., \tau_{n-1}, y) \longrightarrow \psi(\tau_0, \tau_1, ..., \tau_{n-1}, \sigma)],$$

so that by the definition of the satisfaction relation one obtains,
$H(\lambda)^V[G] \models \psi(\tau_0, \tau_1, ..., \tau_{n-1}, \sigma)$ i.e. $H(\lambda)^V[G] \models \psi((\tau_0)_G, (\tau_1)_G, ..., (\tau_{n-1})_G, (\sigma)_G)$, but $\sigma_G \in M[G]$, and hence we are done. □

Lemma 2.1.8. Let as usual $\mathbb{P}$ be a forcing poset and let $\tau \in V^{\mathbb{P}}$ and let $\kappa = 2^{max(\{|\mathbb{P}|, |dom(\tau)|\})}$, then we have $1 \Vdash |\mathcal{P}(\tau)| \leq \check{\kappa}$.

Proof. Let G be an arbitrary generic filter. Now, if $J(\tau)$ denotes the set of all nice names of subsets of τ (evaluated in V of course), then we can easily show that $\mathcal{P}(\tau_G) \subseteq \{\nu_G \mid \nu \text{ is a nice name for a subset of } \tau\} = \{\nu_G \mid \nu \in J(\tau)\}$, (See [10, Lemma IV.3.10] for a definition of nice names and more details) and hence $|\mathcal{P}(\tau_G)|^{V[G]} \leq |J(\tau)|^{V[G]} \leq |J(\tau)|^V \leq (2^{|\mathbb{P}|})^{|dom(\tau)|} = 2^{max\{|\mathbb{P}|, |dom(\tau)|\}} = \kappa$, where the last two evaluations/computations are in V. □

Corollary 2.1.9. If $\lambda > 2^{|\mathbb{P}|}$ is a cardinal (in V), then $1 \Vdash_{\mathbb{P}} (|\mathcal{P}(\check{\mathbb{P}})| < \lambda)$.

Proof. Take $\tau = \check{\mathbb{P}}$, in Lemma 2.1.8 and note that $|dom(\tau)| = |\mathbb{P}|$(in V) □

Lemma 2.1.10. Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \gamma, \beta < \gamma \rangle$ be a countable support iteration of forcing notions, and assume $\lambda > 2^{|\mathbb{P}_\gamma|}$ is a regular uncountable cardinal (in V), then we have that

$$\forall \alpha < \gamma(1_\alpha \Vdash |\mathcal{P}(\dot{\mathbb{Q}}_\alpha)| < \check{\lambda})$$

Proof. Let $\alpha < \lambda$, so that $\alpha + 1 \leq \gamma$. Now we know that $\mathbb{P}_{\alpha+1}$ is isomorphic to $\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$, so that $2^{|\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha|} = 2^{|\mathbb{P}_{\alpha+1}|} \leq 2^{\mathbb{P}_\gamma} < \lambda$. Now applying Lemma 2.1.8 with $\tau = (\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha)\check{}$, we get $1_\alpha \Vdash_\alpha (|\mathcal{P}(\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha)| \leq 2^{max(|\mathbb{P}_\alpha|, |\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha|)} \leq 2^{|\mathbb{P}_\gamma|} < \lambda)$. So it is enough to show that $1_\alpha \Vdash_\alpha (|\dot{\mathbb{Q}}_\alpha| \leq |\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha|)$. But this is easy to see as, if G is an arbitrary $(V, \mathbb{P}_\alpha)$-generic filter, then the map f^G with domain $\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$, defined by $f^G(\langle p, \dot{q}\rangle) = (\dot{q})_G$ (Note, $f^G \in V[G]$) satisfies $range(f^G) \supseteq (\dot{\mathbb{Q}}_\alpha)_G$, so that, in $V[G]$, we have $|(\dot{\mathbb{Q}}_\alpha)_G| \leq |\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha|$. This shows $1_\alpha \Vdash_\alpha (|\dot{\mathbb{Q}}_\alpha| \leq |\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha|)$ and hence we are done. □

2.2 Proper Forcing

Definition 2.2.1. Let $\mathbb{P}$ be a forcing poset and let $\lambda > 2^{|\mathbb{P}|}$ be an uncountable regular cardinal. Let $M \prec H(\lambda)$ be countable. Then we say a condition $q \in \mathbb{P}$ is $(M, \mathbb{P})$-generic if

$$\forall D \in M(D \subseteq \mathbb{P} \wedge D \text{ dense in } \mathbb{P} \implies D \cap M \text{ is predense below } q).$$

Remark.

1. Taking D predense or dense open, instead of dense, in the above, gives an equivalent definition.

Definition 2.2.2. We say a forcing poset $\mathbb{P}$ is proper if for all uncountable regular cardinals $\lambda > 2^{|\mathbb{P}|}$ and all countable $M \prec H(\lambda)$ with $\mathbb{P} \in M$ and each $p \in \mathbb{P} \cap M$, there is an $(M, \mathbb{P})$-generic q with $q \leq p$.

Note: An immediate consequence of the definition is that for all uncountable regular cardinals $\lambda > 2^{|\mathbb{P}|}$ and any $A \in H(\lambda)$ the set

$$D(\lambda, A) = \{q \in \mathbb{P} \mid \exists M(M \text{ is countable and } M \prec H(\lambda) \text{ and } A, \mathbb{P} \in M \text{ and } q \text{ is } (M, \mathbb{P})\text{-generic})\}$$

is dense in $\mathbb{P}$

Proof. Let, $\lambda > 2^{|\mathbb{P}|}$ be arbitrary and fix $A \in H(\lambda)$, and let $p \in \mathbb{P}$, now via the downward Löwenheim Skolem Theorem, choose a countable $M \prec H(\lambda)$ with $p, A, \mathbb{P} \in M$ (See [10] for more on the downwards Löwenheim Skolem Theorem), then as $\mathbb{P}$ is proper and $p \in \mathbb{P} \cap M$, we obtain by definition of properness that there is $q \leq p$ such that q is $(M, \mathbb{P})$-generic. Then obviously $q \in D(\lambda, A)$. This shows $D(\lambda, A)$ is dense in $\mathbb{P}$ □

We now give some useful characterisations of $(M, \mathbb{P})$-generic conditions:

Remark. We denote the class of all ordinals by Ord.

Lemma 2.2.3. Let $\lambda > 2^{|\mathbb{P}|}$ be an uncountable, regular cardinal, and let M be a countable elementary submodel of $H(\lambda)$ with $\mathbb{P} \in M$ and let $q \in \mathbb{P}$, then the following are equivalent:

1. q is $(M, \mathbb{P})$-generic
2. $\forall \tau \in V^{\mathbb{P}} \cap M \ ((1 \Vdash \tau \in Ord) \implies q \Vdash \tau \in M)$
3. for each $\tau \in M \cap V^{\mathbb{P}}$, we have that $\forall r \leq q \forall x \in V((r \Vdash \tau = \check{x}) \implies x \in M)$

Proof. $1 \implies 2$
Assume 1 holds for q, now let τ be a name as in 2, and consider the set

$$D = \{p \in \mathbb{P} \mid \exists \alpha \in Ord \ (p \Vdash \tau = \check{\alpha})\}.$$

Note that this set is evaluated in V, but by Lemma 2.1.5, we can easily deduce that

$$V \models \exists \alpha(\alpha \in Ord \wedge p \Vdash \tau = \check{\alpha}) \iff H(\lambda)^V \models \exists \alpha(\alpha \in Ord \wedge p \Vdash \tau = \check{\alpha}).$$

Note that we also need to use the fact that the formula $p \Vdash \tau = \check{\alpha}$ is absolute for transitive models of $ZF - P$ to deduce the statement above properly. This shows $D \in H(\lambda)^V$ and moreover that D satisfies the same definition from the point of view of $H(\lambda)^V$. Now for any $p \in \mathbb{P}$, let G be a generic filter which contains p then as τ_G is an ordinal, let α be an ordinal such that $\tau_G = \alpha = \check{\alpha}_G$. Now the forcing theorem implies there is $r \leq p$ with $r \in G$ such that $r \Vdash \tau = \check{\alpha}$. Then $r \in D$ and $r \leq p$, showing that D is dense in $\mathbb{P}$. Moreover as D is definable in $H(\lambda)$ from the parameters $\tau, \mathbb{P} \in M$, by $M \prec H(\lambda)$, we obtain that $D \in M$. Now note that

$$H(\lambda)^V \models \forall d \in D \exists! \alpha \in Ord(d \Vdash \tau = \check{\alpha}),$$

so that as $H(\lambda)^V \models ZFC - P$, we get that

$$H(\lambda)^V \models \exists f : D \longrightarrow Ord(\forall d \in D(d \Vdash \tau = \check{f(d)})).$$

Or, for obtaining such an f in $H(\lambda)^V$ one could alternatively do the following: in V there is definitely a function f that satisfies $V \models \forall d \in D(d \Vdash \tau = \check{f(d)})$, now by Lemma 2.1.5, $f : D \longrightarrow H(\lambda)^V$, therefore as $D \in H(\lambda)^V$, by regularity of λ (in V of course) we get this f is infact in $H(\lambda)^V$(See [10, Lemma I.13.32] for more details). Now by the fact that $M \prec H(\lambda)^V$, we obtain there is an $f \in M$ such that

$$M \models (f : D \longrightarrow Ord \ \wedge \ \forall d \in D(d \Vdash \tau = \check{f(d)})).$$

Using elementarity once again, one passes this onto $H(\lambda)^V$ and then onto V via absoluteness to obtain that this $f \in M$ satisfies $\forall d \in D(d \Vdash \tau = \check{f(d)})$ not just within M, but also in in V. Now let G be an arbitrary generic filter that contains q, but $D \in M$ and D is dense in $\mathbb{P}$ and q is $(M, \mathbb{P})$-generic implies that $G \cap M \cap D \neq \emptyset$, since $D \cap M$ is predense below q and $q \in G$. Then let $d \in D \cap M \cap G$. Now an easy argument shows that $f(d) \in M$. The easy argument is as follows: $M \models \exists y(\langle d, y\rangle \in f)$, now if $y \in M$ is such an element, then $H(\lambda)^V \models \langle d, y\rangle \in f$ by elementarity, so $f(d) = y$, so that $f(d) \in M$. Now as $d \Vdash \tau = \check{f(d)}$, we have $\tau_G = f(d) \in M$. Hence $\tau_G \in M$. This shows that $q \Vdash \tau \in \check{M}$.

Note: From the next implication onwards and in the rest of this thesis, we will just write $H(\lambda)$ for $H(\lambda)^V$, and we will do this also for other objects defined in V, and it is to be understood that those objects are actually relativizations to V.

$2 \implies 1$

Let D be a dense subset of $\mathbb{P}$ with $D \in M$, we need to show $D \cap M$ is predense below q. To use 2, we first define a name τ which is a name for an ordinal, with $\tau \in M$, as follows:

First note that as λ is regular "$H(\lambda) \models \forall x \exists! y(y$ is a cardinal and $|x| = y)$", moreover one can show that "$\forall x \in H(\lambda)\ (|x|^{H(\lambda)} = |x|)$", i.e. $H(\lambda)$ can compute actual cardinalities. It is easy to show that this is true using [10, Lemma I.13.32]. Using these two facts in the double quotes and the fact that $M \prec H(\lambda)$, one can show that if $a \in M$ then $|a| \in M$. So in particular $|D| \in M$. Now, because $H(\lambda) \models \exists f(f : |D| \longrightarrow D \wedge f$ is onto $D)$, elementarity gives such an f in M. Now using this f we define τ as, $\tau = \bigcup_{r \in \mathbb{P}} A(r) \times \{r\}$ where $A(r) = min\{i \in |D| \mid f(i) \not\perp r\}$. Note that $A(r) \neq \emptyset$ for each $r \in \mathbb{P}$ because D is dense in $\mathbb{P}$ and $range(f) = D$. Now as before, as τ is definable in $H(\lambda)$ via elements in M, and hence $\tau \in M$.

Claim. $1 \Vdash \tau = min(\{i \in |D| \mid f(i) \in \dot{G}\})$.

Proof. To see why the claim is true, let G be an arbitrary $(V, \mathbb{P})$ filter, then put $\alpha = min(\{i \in |D| \mid f(i) \in G\})$, which is well defined due to the density of D. We need to show $\alpha = \tau_G$.

$\tau_G \subseteq \alpha$:

To see this first note that $\forall s \in G\ (f(\alpha) \not\perp s)$(because $f(\alpha) \in G$). Therefore we get that $\forall s \in G(A(s) \leq \alpha)$. So if $\eta = (\check{\eta})_G \in \tau_G$ with $\langle \check{\eta}, s\rangle \in A(s) \times \{s\}$ for some $s \in G$, we have $\eta < A(s) \leq \alpha$ and hence $\eta \in \alpha$. Therefore $\tau_G \subseteq \alpha$.

$\alpha \subseteq \tau_G$:

Let by the forcing theorem $r \in G$ be such that $r \Vdash (\check{\alpha} = \{i \in |D| \mid f(i) \in \dot{G}\})$. Now we show that $A(r) = \alpha$. Firstly note that $f(\alpha) \not\perp r$ (as both are in G), but this implies that $\forall i < \alpha(f(i) \perp r)$, because if not, let $i < \alpha$ be such that $f(i) \not\perp r$, then choose $r_0 \leq r, f(i)$, then $r_0 \Vdash (\check{\alpha} = \{i \in |D| \mid f(i) \in \dot{G}\})$ as well, because stronger conditions force more. Now choosing a $(V, \mathbb{P})$ generic H with $r_0 \in H$ we get the contradiction that $\alpha = min\{i \in |D| \mid f(i) \in H\}$ but $i < \alpha$ and $f(i) \in H$. This shows that $\forall i < \alpha(f(i) \perp r)$, which combined with the fact that $f(\alpha) \not\perp r$ gives that $\alpha = A(r)$, so as $r \in G$ we get $\alpha \subseteq \tau_G$.

This proves the claim. □

From the claim, it also follows that $1 \Vdash \tau \in \ Ord$. Now apply 2 to obtain that $q \Vdash \tau \in \check{M}$ but as $1 \Vdash \tau = min\{i \in |D| \mid f(i) \in G\}$, we obtain $1 \Vdash \tau \in |D|$, so that $q \Vdash \tau \in M \cap |D|$ this gives $q \Vdash f(\tau) \in M \cap D$. Note that we have to use $M \prec H(\lambda)$ to make this conclusion. Now yet again by the claim, we get $q \Vdash f(\tau) \in \dot{G}$. This gives $q \Vdash f(\tau) \in M \cap D \cap \dot{G}$. From this it is easy to conclude q is $(M, \mathbb{P})$-generic as follows: let $p \leq q$, then $p \Vdash f(\tau) \in M \cap D \cap \dot{G}$. Now let G be a generic with $p \in G$ and let $r = f(\tau_G) \in M \cap D \cap G$. Then $r \in M \cap D$ and $r \not\perp p$. This shows that $D \cap M$ is predense below q. Therefore as $D \in M$ was arbitrary, we conclude that q is $(M, \mathbb{P})$-generic.

$1 \implies 3$

As in the proof of $1 \implies 2$, we define $D = \{p \mid \exists x \in V(p \Vdash \tau = \check{x})\}$ and one can show that this set is definable in $H(\lambda)$ from the parameters τ, $\mathbb{P} \in M$, so that it is also in M. Now in order to prove 3, let $x_0 \in V$ and $p \leq q$ be such that $p \Vdash \tau = \check{x}$. This implies $D \neq \emptyset$, from which it follows that $D \cup D^{\perp}$ is dense in $\mathbb{P}$, where $D^{\perp} = \{r \in \mathbb{P} \mid \forall s \in D(s \perp r)\}$ (i.e. elements that are incompatible with every element of D). Using $M \prec H(\lambda)$ one can easily show $D \cup D^{\perp} \in M$. Now, exactly like in the proof of $1 \implies 2$, we get that there is a function $f \in M$ that satisfies $f : D \longrightarrow H(\lambda)$ and $\forall d \in D(d \Vdash \tau = \check{f(d)})$. Then by the $(M, \mathbb{P})$-genericity of q, it follows that $(D \cup D^{\perp}) \cap M \cap G \neq \emptyset$ for any generic G that contains p. This is because any G that contains p, would also contain q and moreover, $(D \cup D^{\perp}) \cap M$ is predense below q. But $G \cap M \cap D^{\perp} = \emptyset$, because $p \in G \cap D$ and any pair of elements in G are compatible. Therefore, it follows that $G \cap M \cap D \neq \emptyset$. So let $r \in G \cap M \cap D$, then by elementarity, $f(r) \in M$, also $r \Vdash \tau = \check{f(r)}$. But as both $p, r \in G$, we get $\tau_G = x$ and $\tau_G = f(r)$ and hence $x = f(r) \in M$.

So we have $1 \iff 2$ and $1 \implies 3$, so all we have left to do is to show that $3 \implies 2$.

$3 \implies 2$

Let τ, q be as in the hypothesis of 2, and assume 3. We need to show that $q \Vdash \tau \in \check{M}$. So let G be an arbitrary generic filter with $q \in G$, we need to show $\tau_G \in M$. Firstly, as τ is forced by 1 to be an ordinal, the forcing theorem implies that there is $r \leq q$ with $r \in G$ and an ordinal α such that $r \Vdash \tau = \check{\alpha}$. Now

3 implies that $\alpha \in M$, so that $\tau_G = \alpha \in M$. This shows 2 holds. $\square$

Lemma 2.2.4. Let $\mathbb{P}$ be a forcing notion and let $\lambda > 2^{|\mathbb{P}|}$ be an uncountable regular cardinal. Let now $M \prec H(\lambda)$ be countable with $\mathbb{P} \in M$ and let $q \in \mathbb{P}$. Then the following are equivalent:

1. q is $(M, \mathbb{P})$-generic
2. for every $D \subseteq \mathbb{P}$ dense in $\mathbb{P}$ with $D \in M$ there is some $\tilde{p} \in V^{\mathbb{P}}$ such that $q \Vdash \tilde{p} \in D \cap M \cap \dot{G}$. Equivalently, this condition can be phrased as follows: $\forall D \in M$ (D is dense in $\mathbb{P} \implies q \Vdash D \cap M \cap \dot{G} \neq \emptyset$)
3. $q \Vdash M[\dot{G}] \cap Ord = M \cap Ord$
4. $q \Vdash M[\dot{G}] \cap V = M \cap V$, which means that for any generic filter G which contains q, $M[G] \cap V = M \cap V = M$

Remark. One can replace “dense” by “dense open” in condition 2 and obtain another equivalent characterisation for $(M, \mathbb{P})$-generic conditions. This will be used in Chapters 3 and 4.

Proof. Note: The equivalence of 1 and 2 was implicit in the proof of the previous lemma, but we write it down here explicitly anyway, for completeness. $1 \implies 2$
Assume 1 holds, we need to show for each $D \in M$ dense in $\mathbb{P}$, there is a $\mathbb{P}$-name $\tilde{p}$ such that $q \Vdash \tilde{p} \in \dot{G} \cap M \cap D$. By definition of $(M, \mathbb{P})$-genericity we obtain that for such a D, $D \cap M$ is predense below q, so that for any generic filter G with $q \in G$, we have $G \cap M \cap D \neq \emptyset$, from which we obtain that $q \Vdash \dot{G} \cap M \cap D \neq \emptyset$, i.e. $q \Vdash \exists x(x \in \dot{G} \cap M \cap D)$. Now use existential completeness, to obtain a $\mathbb{P}$ name $\tilde{p}$ such that $q \Vdash \tilde{p} \in \dot{G} \cap M \cap D$.
$2 \implies 1$
Assume 2, holds and let $D \in M$ be dense in $\mathbb{P}$, we need to show that $D \cap M$ is predense below q. So let $r \leq q$, then by 2, $r \Vdash \tilde{p} \in \dot{G} \cap M \cap D$ for some $\tilde{p}$, so choosing a generic filter G that contains r, we obtain $p = (\tilde{p})_G \in G \cap M \cap D$. For this p we have $r \not\perp p$ and $p \in D \cap M$. This shows that $D \cap M$ is predense below q.
$2 \implies 4$
Fix $(V, \mathbb{P})$ generic G with $q \in G$, we only need to show $M[G] \cap V \subseteq M \cap V = M$ because the the other inclusion follows via the fact that $M \prec H(\lambda)$ as follows: if $x \in M$ then $\check{x}$ is definable in $H(\lambda)$ from x and hence $\check{x} \in M$. So let $\tau \in M \cap V^{\mathbb{P}}$, be such that $\tau_G \in V$ we need to show $\tau_G \in M$. Let $x = \tau_G \in V$ then choose via the forcing theorem yet again $p \in G$ with $p \leq q$ such that $p \Vdash \tau = \check{x}$. But we know by $2 \iff 1$ that q is $(M, \mathbb{P})$-generic, so by the previous lemma, we obtain $x \in M$ so that $\tau_G = x \in M$.
$4 \implies 3$

This is easy, because if G contains q then $M[G] \cap Ord \subseteq M[G] \cap V = M$, therefore $M[G] \cap Ord = M \cap Ord$, so that 3 holds.
$3 \implies 2$
By the equivalence of 2 and 1, it is enough to show that q is $(M, \mathbb{P})$-generic and by the previous lemma, it is enough to show that for any τ which is a name for an ordinal with $\tau \in M$, $q \Vdash \tau \in M$. So let as usual, G be an arbitrary generic filter with $q \in G$, as we are assuming 3 we get $\tau_G \in M[G] \cap Ord = M \cap Ord$ which gives $\tau_G \in M$, therefore $q \Vdash \tau \in M$. $\square$

2.3 Iterating with proper forcings and the first preservation theorem

In this section, our main goal is to show that the countable support iterations of proper forcings are proper. We first prove some lemmas which are required in order to deal with the case of two-step iterations, and then some lemmas needed for the general case of an iteration.

Lemma 2.3.1. Assume $\mathbb{P} * \dot{\mathbb{Q}}$ is a two-step iteration and let $\lambda > 2^{|\mathbb{P}*\dot{\mathbb{Q}}|}$ be an uncountable regular cardinal, so that by 2.1.10, $1_{\mathbb{P}} \Vdash_{\mathbb{P}} 2^{|\dot{\mathbb{Q}}|} < \lambda$. Let $M \prec H(\lambda)$ be countable and such that $\mathbb{P} * \dot{\mathbb{Q}} \in M$, let G be $(V, \mathbb{P})$ generic and $H \subseteq (\dot{\mathbb{Q}})_G$ be $(V[G], (\dot{\mathbb{Q}})_G)$ generic, then we have that $M[G][H] = M[G * H]$, where $G * H = \{\langle p, \dot{q}\rangle \mid p \in G \wedge (\dot{q})_G \in H\}$, is a generic filter on $\mathbb{P} * \dot{\mathbb{Q}}$. (See [9] or [10] for more details.)

Proof. Put $K = G * H$ and first of all we note that the assumption that $\mathbb{P} * \dot{\mathbb{Q}} \in M$ implies that $\mathbb{P} \in M$ and that there is a name $\tau \in M$ such that $1 \Vdash \tau = \dot{\mathbb{Q}}$, so that without loss of generality, we can assume $\dot{\mathbb{Q}} \in M$. To see why this is true: the fact that $\mathbb{P} \in M$ follows from elementarity of M and the fact that $\mathbb{P}$ is definable in $H(\lambda)$ from $\mathbb{P} * \dot{\mathbb{Q}} \in M$. The fact that one can find such a τ can be directly verified by putting $\tau = \{\langle \dot{q}, p\rangle \mid \langle p, \dot{q}\rangle \in \mathbb{P} * \dot{\mathbb{Q}}\}$ which is obviously in M again via the same definability and elementarity argument.

Claim. $M[G][H] \subseteq M[K]$:

Proof. Note first that as $M \prec H(\lambda)$ and as the canonical name for a generic filter on $\mathbb{P} * \dot{\mathbb{Q}}$, $\dot{K}$, is definable from $\mathbb{P} * \dot{\mathbb{Q}} \in M$ in $H(\lambda)$, we obtain $\dot{K} \in M$ so that $K \in M[K]$ now $G = \{p \in \mathbb{P} \mid \langle p, 1_{\dot{\mathbb{Q}}}\rangle \in K\}$, therefore $G \in M[K]$, since $M[K] \prec H(\lambda)^{V[K]}$. Now as for any $\tau \in M$, τ_G is definable from $\tau, G \in M[K]$ in $H(\lambda)^{V[K]}$, so that by $M[K] \prec H(\lambda)^{V[K]}$ by obtain $\tau_G \in M[K]$, so that $M[G] \subseteq M[K]$. Now note that $H = \{(\dot{q})_G \mid \exists p \in G(\langle p, \dot{q}\rangle \in K)\} \in M[K]$, by the same definability and elementarity argument we have used again and again. Now for any $\dot{\mathbb{Q}}_G$ ($\in M[G]$)-name $\tau \in M[G]$ we have $\tau, H \in M[K]$ and hence by the same arguments, we obtain $\tau_H \in M[K]$. Therefore we conclude $M[G][H] \subseteq M[K]$. $\square$

Claim. $M[K] \subseteq M[G][H]$:

Proof. $G, H \in M[G][H]$ so that $K = G * H \in M[G][H]$, since $M[G] \prec H(\lambda)^{V[G]}$ and hence $M[G][H] \prec H(\lambda)^{V[G][H]} = H(\lambda)^{V[K]}$ (by repeated application of lemma 2.1.7), so that K is definable in $H(\lambda)^{V[K]}$, from elements in $M[G][H]$. Therefore, arguing as in the previous claim, we can easily show that $M[K] \subseteq M[G][H]$. $\square$

$\square$

Lemma 2.3.2. Let, as in the previous lemma, $\mathbb{P} * \dot{\mathbb{Q}}$ be a two-step iteration and assume $\lambda > 2^{|\mathbb{P}*\dot{\mathbb{Q}}|}$ is an uncountable regular cardinal (so that as in the previous lemma, $1 \Vdash_{\mathbb{P}} 2^{|\dot{\mathbb{Q}}|} < \lambda$), and let $M \prec H(\lambda)$, then we have that, for any $\langle p, \dot{q} \rangle \in \mathbb{P} * \dot{\mathbb{Q}}$,

$$\langle p, \dot{q} \rangle \text{ is } (M, \mathbb{P} * \dot{\mathbb{Q}})\text{-generic iff } p \text{ is } (M, \mathbb{P})\text{-generic and } p \Vdash \dot{q} \text{ is } (M[\dot{G}], \dot{\mathbb{Q}})\text{-generic.}$$

Proof. ($\Longrightarrow$):
We use Lemma 2.2.4 (3) to prove this, as follows:
Let G be an arbitrary $(V, \mathbb{P})$ generic filter that contains p and then choose a $(V[G], \dot{\mathbb{Q}}_G)$ generic filter H with $\dot{q}_G \in H$ and put $K = G * H$ so that $\langle p, \dot{q} \rangle \in K$ and hence $M[K] \cap Ord = M \cap Ord$, showing that $M[G] \cap Ord = M \cap Ord$ (since $M[G] \subseteq M[K]$). Therefore we have $p \Vdash M[\dot{G}] \cap Ord = M \cap Ord$. So by Lemma 2.2.4 (3), we obtain that p is $(M, \mathbb{P})$-generic. Similarly, to show $p \Vdash$ ($\dot{q}$ is $(M[\dot{G}], \dot{\mathbb{Q}})$-generic), we only need to show that for each $(V, \mathbb{P})$ generic G with $p \in G$, $\dot{q}_G$ *is* $(M[G], \dot{\mathbb{Q}}_{\mathbb{G}})$-generic, which is, yet again by Lemma 2.2.4 (3), equivalent to showing $\dot{q}_G \Vdash_{(V[G], \dot{Q}_G)} (M[G][\dot{H}] \cap Ord = M[G] \cap Ord)$, which is definitely true, by the previous lemma (which says that $M[G][H] = M[K]$).
($\Longleftarrow$)
Yet again by Lemma 2.2.4 (3), we only need to show that for any $(V, \mathbb{P} * \dot{\mathbb{Q}})$ generic K with $\langle p, \dot{q} \rangle \in K$, $M[K] \cap Ord = M \cap Ord$. So fix such a K and now put as usual $G = \{p \mid \langle p, 1_{\dot{\mathbb{Q}}} \rangle \in K\}$ and let $H = \{\dot{q}_G \mid \exists p \in G(\langle p, \dot{q} \rangle \in K)\}$, then as $p \in G$ and $\dot{q}_G \in H$, by hypothesis, we obtain, first that $M[G] \cap Ord = M \cap Ord$ and then we obtain $M[G][H] \cap Ord = M[G] \cap Ord$, so that $M[K] \cap Ord = M \cap Ord$ (since $M[K] = M[G][H]$)

$\square$

Corollary 2.3.3. Let $\mathbb{P}$ be proper and assume that $1_{\mathbb{P}} \Vdash$ ($\dot{\mathbb{Q}}$ is proper), then the two-step iteration $\mathbb{P} * \dot{\mathbb{Q}}$ is proper.

Proof. Let as in the definition of properness $\lambda > 2^{|\mathbb{P}*\dot{\mathbb{Q}}|}$ be an arbitrary uncountable regular cardinal and let $M \prec H(\lambda)$ be countable. Let $\langle p, \dot{q}\rangle \in \mathbb{P}*\dot{\mathbb{Q}} \cap M$ be arbitrary (so by elementarity, $p \in M$ and $\dot{q} \in M$). We need to find an $(M, \mathbb{P} * \dot{\mathbb{Q}})$-generic condition below $\langle p, \dot{q}\rangle$. Firstly note that $2^{|\mathbb{P}|} < \lambda$, so that as $\mathbb{P}$ is proper, choose $p' \in \mathbb{P}$ such that p' is $(M, \mathbb{P})$-generic and $p' \leq p$. Now it is easily seen that:

$$p' \Vdash (\dot{\mathbb{Q}} \text{ is proper } \wedge 2^{|\dot{\mathbb{Q}}|} < \lambda \wedge M[\dot{G}] \prec \dot{H}(\lambda) \wedge M[\dot{G}] \text{ is countable } \wedge \dot{q} \in \dot{\mathbb{Q}} \cap M[\dot{G}]).$$

Note that we have applied Lemma 2.1.10 get $p' \Vdash (2^{|\dot{\mathbb{Q}}|} < \lambda)$.
Now by the definition of properness applied inside the above forcing formula, and then applying existential completeness/maximality principle, we get a name $\dot{q}' \in V^{\mathbb{P}}$ such that $p' \Vdash (\dot{q}' \in \dot{\mathbb{Q}} \wedge \dot{q}'$ is $(M[\dot{G}], \dot{\mathbb{Q}})$-generic $\wedge\ \dot{q}' \leq_{\dot{\mathbb{Q}}} \dot{q})$. So that by Lemma 2.3.2, we have that $\langle p', \dot{q}'\rangle$ is $(M, \mathbb{P} * \dot{\mathbb{Q}})$-generic. Moreover, it is clear that $\langle p', \dot{q}'\rangle \leq \langle p, q\rangle$, in the $\mathbb{P} * \dot{\mathbb{Q}}$ ordering. Thus, we are done. □

Remark. From now onwards, when we use the phrase “let λ be a sufficiently large regular cardinal”, we usually mean λ is strictly bigger than the cardinality of the power set of the forcing poset we are trying to prove to be proper.

Notation: If we have an iteration of forcing notions, $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \gamma \wedge \beta < \gamma\rangle$, for $\alpha \leq \gamma$, we denote by $\dot{G}_\alpha$, the canonical name for a generic filter on $\mathbb{P}_\alpha$

Lemma 2.3.4. Properness extension lemma (PEL) for two-step iterations:
Let $\mathbb{P}$ be a proper forcing poset and assume $1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\dot{\mathbb{Q}}$ is a proper forcing poset), let $\mathbb{R} = \mathbb{P} * \dot{\mathbb{Q}}$ and let $\pi : \mathbb{R} \longrightarrow \mathbb{P}$, be the projection to the first coordinate, defined by $\pi(\langle p, \dot{q}\rangle) = p$. Let now λ be a sufficiently large regular cardinal and assume $M \prec H(\lambda)$ is countable with $\mathbb{R} \in M$. Moreover, assume the following:
$p_0 \in \mathbb{P}$ is $(M, \mathbb{P})$-generic and $\tilde{r} \in V^{\mathbb{P}}$ satisfies $p_0 \Vdash_{\mathbb{P}} (\tilde{r} \in M \cap \mathbb{R}$ and $\pi(\tilde{r}) \in \dot{G}_0)$, then there is a $\tilde{p_1} \in V^{\mathbb{P}}$ such that $\langle p_0, \tilde{p_1}\rangle$ is $(M, \mathbb{P} * \dot{\mathbb{Q}})$-generic and $\langle p_0, \tilde{p_1}\rangle \Vdash_{\mathbb{P}*\dot{\mathbb{Q}}} e(\tilde{r}) \in \dot{G}$, where $\dot{G}_0$ and $\dot{G}$ denotes the canonical names for a generic filter on $\mathbb{P}$ and on $\mathbb{R}$ respectively, and $e : \mathbb{P} \longrightarrow \mathbb{R}$, denotes the complete embedding defined by $e(p) = \langle p, 1_{\dot{\mathbb{Q}}}\rangle$ and $e(\tilde{r})$ denotes the translation of the $\mathbb{P}$-name $\tilde{r}$ into an $\mathbb{R}$-name via e (see [10] for more on these complete embeddings and translation of names via complete embeddings).

Lemma 2.3.5. Properness extension lemma (PEL) general version for γ step countable support iterations:
Let γ be an ordinal and suppose that $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \gamma \wedge \beta < \gamma\rangle$ is a countable support iterations of forcing notions such that for each $\beta < \gamma$, we have

$$1_\beta \Vdash_\beta (\dot{\mathbb{Q}}_\beta \text{ is proper}),$$

let $M \prec H(\lambda)$ be countable (for λ sufficiently large and regular) with $\mathbb{P}_\gamma \in M$. Then for any $\gamma_0 \in M \cap \gamma$ and any $(M, \mathbb{P}_{\gamma_0})$-generic $q_0 \in \mathbb{P}_{\gamma_0}$ and any $\tilde{p_0} \in V^{\mathbb{P}_{\gamma_0}}$ with $q_0 \Vdash_{\gamma_0} (\tilde{p_0} \in \check{\mathbb{P}_\gamma} \cap \check{M} \wedge \tilde{p_0} \restriction \gamma_0 \in \dot{G_{\gamma_0}})$, there is $q \in \mathbb{P}_\gamma$ such that $(q \restriction \gamma_0 = q_0$ and q is $(M, \mathbb{P}_\gamma)$-generic and $q \Vdash e_\gamma(\tilde{p_0}) \in \dot{G_\gamma})$, where $e_\gamma : \mathbb{P}_{\gamma_0} \longrightarrow \mathbb{P}_\gamma$ is as usual the canonical complete embedding (See [10] for more on these canonical complete embeddings).

Before we prove these lemmas, we need some more results to tackle some of the minute details in their proofs.

Lemma 2.3.6. Let $e : \mathbb{P} \longrightarrow \mathbb{Q}$ be a complete embedding, where $\mathbb{P}$ and $\mathbb{Q}$ are forcing posets and let $\psi = \psi(x_0, ..., x_{n-1})$ be a formula with free variables among $x_0, ..., x_{n-1}$ such that ψ is upwards absolute between transitive models of set theory, then for any $\mathbb{P}$-names $\tau_0, ..., \tau_{n-1}$ and any $p \in \mathbb{P}$ we have:
$p \Vdash_{\mathbb{P}} \psi(\tau_0, ..., \tau_{n-1}) \implies e(p) \Vdash_{\mathbb{Q}} \psi(e(\tau_0), ..., e(\tau_{n-1}))$

Proof. Use upwards absoluteness of ψ, the forcing theorem, the fact that $(e(\tau))_G = \tau_{(e^{-1}(G))}$ and the fact that $V[e^{-1}(G)] \subseteq V[G]$, for any generic filter $G \subseteq \mathbb{Q}$ □

Lemma 2.3.7. Let $e : \mathbb{P} \longrightarrow \mathbb{Q}$ be a complete embedding, where $\mathbb{P}$ and $\mathbb{Q}$ are forcing posets, then for any $x \in V$ we have $e((\check{x})^{\mathbb{P}}) = (\check{x})^{\mathbb{Q}}$, i.e. complete embeddings preserve check names. Here $(\check{x})^{\mathbb{P}}$ and $(\check{x})^{\mathbb{Q}}$, denotes taking check with respect to $\mathbb{P}$ and check with respect to $\mathbb{Q}$ respectively.

Proof. An easy induction, using the definition of translation of names via complete embeddings. □

Lemma 2.3.8. Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \gamma \wedge \beta < \gamma \rangle$ be a countable support iteration of forcing notions, let $\gamma_0 < \gamma$ and $q_0 \in \mathbb{P}_{\gamma_0}$ and $q \in \mathbb{P}_\gamma$ be such that $q \restriction \gamma_0 \leq_{\mathbb{P}_{\gamma_0}} q_0$, and let $e : \mathbb{P}_{\gamma_0} \longrightarrow \mathbb{P}_\gamma$ denote the canonical complete embedding. Now let $\tilde{p_0} \in V^{\mathbb{P}_{\gamma_0}}$ be such that $q_0 \Vdash_{\gamma_0} \tilde{p_0} \in \check{\mathbb{P}_\gamma}$, then we have the following equivalence:

$$q \Vdash_\gamma e(\tilde{p_0}) \in \dot{G_\gamma} \text{ iff } \forall q' \leq q \forall p \in \mathbb{P}_\gamma (q' \Vdash_\gamma ((e(\tilde{p_0}) = \check{p})) \implies q' \leq p)$$

(i.e. iff for any element q' in $\mathbb{P}_\gamma$ below q that decides $\tilde{p_0}$ as an element $p \in \mathbb{P}_\gamma$, we have $q' \leq p$) (here $\dot{G_\gamma}$ is the canonical name for a generic filter of $\mathbb{P}_\gamma$).

Proof. $(\Longrightarrow)$: Follows by separativity of $\mathbb{P}_\gamma$.
$(\Longleftarrow)$:
We need to show that $q \Vdash e(\tilde{p_0}) \in \dot{G_\gamma}$. So let $G \subseteq \mathbb{P}_\gamma$ be a generic filter with $q \in G$, we need to show that $(e(\tilde{p_0}))_G \in G$. Towards this, first we note that, from the hypothesis we have $q_0 \Vdash_{\gamma_0} \tilde{p_0} \in \check{\mathbb{P}_\gamma}$, from this and the fact that $q \leq_\gamma e(q_0)$ $(q \restriction \gamma_0 \leq_{\mathbb{P}_{\gamma_0}} q_0)$, so we obtain that $q \Vdash_\gamma e(\tilde{p_0}) \in \check{\mathbb{P}_\gamma}$ (because of Lemma 2.3.6 and Lemma 2.3.7). Now via the forcing theorem let $q' \in G$ with $q' \leq q$ and $p \in \mathbb{P}_\gamma$

be such that $q' \Vdash e(\tilde{p_0}) = \check{p}$ (here $p = (e(\tilde{p_0}))_G$), then by hypothesis we have, $q' \leq p = (e(\tilde{p_0}))_G$ which gives $(e(\tilde{p_0}))_G \in G$ (since $q' \in G$). Thus proved.

$\square$

Lemma 2.3.9. Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \gamma \wedge \beta < \gamma \rangle$ be a countable support iteration of forcing notions, let $q \in \mathbb{P}_\gamma$ let $\gamma' < \gamma$ and let $(\gamma_n)_{n\in\omega}$ (each $\gamma_n < \gamma$) be a strictly increasing sequence of ordinals with $\gamma_0 = \gamma'$. Assume moreover that $\tilde{p} \in V^{\mathbb{P}_{\gamma'}}$ satisfies:

1. $\forall n \in \omega (q \restriction \gamma_n \Vdash_{\gamma_n} e_n(\tilde{p}) \restriction \gamma_n \in \dot{G_{\gamma_n}})$, and
2. $q \restriction \gamma' \Vdash (\tilde{p} \in \mathbb{P}_\gamma \wedge \mathit{supp}(\tilde{p}) \subseteq \bigcup_{n\in\omega} \gamma_n)$

then we have that $q \Vdash_\gamma e(\tilde{p}) \in \dot{G}_\gamma$.

Here, $e_n : \mathbb{P}_{\gamma_0} \longrightarrow \mathbb{P}_{\gamma_n}$ is, as usual, the canonical complete embedding (with $e_0 = id_{\mathbb{P}_{\gamma'}}$ = the identity map on $\mathbb{P}_{\gamma'}$) and $e : \mathbb{P}_{\gamma_0} \longrightarrow \mathbb{P}_\gamma$ is also the canonical embedding and *supp*, denotes the support of a condition in the iteration, i.e. more precisely, if $p \in \mathbb{P}_\gamma$, then we define $\mathit{supp}(p) = \{\eta \in \gamma \mid p(\eta) \neq 1_{\dot{\mathbb{Q}}_\eta}\}$.

Proof. We use the previous lemma to prove this, so let $q' \leq q$ and let $p \in \mathbb{P}_\lambda$ be such that $q' \Vdash_\gamma e(\tilde{p}) = \check{p}$, we need to show that $q' \leq p$

Claim. Enough to show: $\forall n \in \omega (q' \restriction \gamma_n \leq_{\gamma_n} p \restriction \gamma_n)$:

Proof. To see why this is true, first note that by hypothesis (2), we get that $\mathit{supp}(p) \subseteq \bigcup_{n\in\omega} \gamma_n$, so we only have to show $q' \restriction \eta \Vdash_\eta q'(\eta) \leq_{\dot{\mathbb{Q}}_\eta} p(\eta)$ for each $\eta \in \bigcup_{n\in\omega} \gamma_n$. But if $\eta \in \bigcup_{n\in\omega} \gamma_n$ there is an $n \in \omega$ such that $\eta \in \gamma_n$ (i.e. $\eta < \gamma_n$), therefore, from the fact that $q' \restriction \gamma_n \leq_{\gamma_n} p \restriction \gamma_n$, we obtain by definition of $\leq_{\gamma_n}$ that $q' \restriction \eta \Vdash_\eta q'(\eta) \leq_{\dot{\mathbb{Q}}_\eta} p(\eta)$. Thus, the claim is proved.

$\square$

Claim. $\forall n \in \omega (q' \restriction \gamma_n \leq_{\gamma_n} p \restriction \gamma_n)$:

Proof. We show this using hypothesis (1) as follows:
Let $f_n : \mathbb{P}_{\gamma_n} \longrightarrow \mathbb{P}_\gamma$ denote the canonical complete embedding, then f_n, e_n and e commute, more precisely $e = f_n \circ e_n$, so that an easy induction shows this remains valid for translations of names, i.e. if $\tau \in V^{\mathbb{P}_{\gamma'}}$, then $e(\tau) = f_n(e_n(\tau))$, so that applying Lemma 2.3.6 to (1), yields $f_n(q \restriction \gamma_n) \Vdash_\gamma e(\tilde{p}) \restriction \gamma_n \in f_n(\dot{G_{\gamma_n}})$.

Remark. In condition (1), what we really mean when we write the statement $q \restriction \gamma_n \Vdash_{\gamma_n} e_n(\tilde{p}) \restriction \gamma_n \in \dot{G_{\gamma_n}}$ is that $q \restriction \gamma_n \Vdash_{\gamma_n} \exists z\ (\psi(e_n(\tilde{p}), \check{\gamma_n}, z) \wedge z \in \dot{G_{\gamma_n}})$, where, $\psi(x, y, z) \iff x \restriction y = z$. Then it is easy to see that ψ is absolute between transitive models of set theory so that $\exists z \psi(x, y, z)$ is upwards absolute between transitive models of set theory and hence we basically apply Lemma 2.3.6 with $e = f_n$, to conclude $f_n(q \restriction \gamma_n) \Vdash_\gamma e(\tilde{p}) \restriction \gamma_n \in f_n(\dot{G_{\gamma_n}})$.

Now as $q \leq_\gamma f_n(q \restriction \gamma_n)$ we get $q \Vdash_\gamma e(\tilde{p}) \restriction \gamma_n \in f_n(\dot{G_{\gamma_n}})$) and as $q' \leq_\gamma q$, we get $q' \Vdash_\gamma e(\tilde{p}) \restriction \gamma_n \in f_n(\dot{G_{\gamma_n}})$, which gives $q' \Vdash_\gamma \check{p} \restriction \gamma_n \in f_n(\dot{G_{\gamma_n}})$ (since $q \Vdash_\gamma e(\tilde{p}) = \check{p}$). Now let G be an arbitrary $(V, \mathbb{P}_\gamma)$ generic filter with $q' \in G$ then we have $p \restriction \gamma_n \in (f_n(\dot{G}_{\gamma_n}))_G = (\dot{G}_{\gamma_n})_{(f_n)^{-1}(G)} = (f_n)^{-1}(G)$ which gives $f_n(p \restriction \gamma_n) \in G$, this shows that $q' \Vdash_\gamma (f_n(p \restriction \gamma_n))\check{} \in \dot{G}_\gamma$, now separativity of $\mathbb{P}_\gamma$ implies that $q' \leq_\gamma f_n(p \restriction \gamma_n))$, applying the canonical projection onto $\mathbb{P}_{\gamma_n}$ to both sides, we get $q' \restriction \gamma_n \leq_{\gamma_n} p \restriction \gamma_n$. But as $n \in \omega$ was arbitrary, the claim follows. □

□

Now we are ready to prove both the properness extension lemmas (the PELs):

Proof. **Proof of Lemma 2.3.4**

Remark. Since for any poset $\mathbb{L}$ and any $\mathbb{L}$-name τ, $\tau_G = val(\tau, G)$ is defined by a formula (obtained via recursion of course) of the form $\psi(x, y, z)$, where $\psi(\tau, G, z) \iff z = \tau_G$, and as ZFC proves

$$\forall \tau \forall G((\tau \text{ is an } \mathbb{L}\text{-name and } G \subseteq \mathbb{L}) \implies \exists! z \psi(\tau, G, z)),$$

this big statement is forced by $1_{\mathbb{L}}$. Therefore if τ is a name forced by some condition $q \in \mathbb{L}$ to be an $\mathbb{L}$-name (so that τ is a name for a name according to q!) we can use existential completeness to obtain a name μ such that for all generics G with $q \in G$, $\mu_G = (\tau_G)_G$, as follows: $q \Vdash \tau \in V^{\mathbb{L}} \wedge \dot{G} \subseteq \mathbb{L}$ so that $q \Vdash \exists! z \psi(\tau, \dot{G}, z)$, so now existential completeness gives that there is μ an $\mathbb{L}$-name such that $q \Vdash \psi(\tau, \dot{G}, \mu)$. So that for any G with $q \in G$, $V[G] \models \psi(\tau_G, G, \mu_G)$. Now by absoluteness of the formula $\psi(x, y, z)$ we get $\psi(\tau_G, G, \mu_G)$, so that $(\tau_G)_G = \mu_G$. Such a name μ will be denoted by $val(\tau, \dot{G})$ (val for "evaluation").

Now we are ready to prove our theorem as follows:

As $p_0 \Vdash \tilde{r} \in M \cap \mathbb{R}$, so if G is an arbitrary generic filter on $\mathbb{P}$ with $p_0 \in G$ then we have that $\tilde{r}_G = \langle r_1, \tilde{r_2} \rangle \in M \cap R$ so that $\tilde{r_2} \in M$ and hence $(\tilde{r_2})_G \in M[G]$. Moreover as $p_0 \Vdash \pi(\tilde{r}) \in \dot{G}_0$, we have $r_1 \in G$, so that as $r_1 \Vdash \tilde{r_2} \in \dot{\mathbb{Q}}$, we obtain $(\tilde{r_2})_G \in \dot{\mathbb{Q}}_G$, thus we have shown that $(\tilde{r_2})_G \in M[G] \cap \dot{\mathbb{Q}}_G$, but we also have $M[G] \prec H(\lambda)^{V[G]}$ and that $(\dot{\mathbb{Q}}_G \text{ is proper})^{V[G]}$, therefore we obtain an $(M[G], \dot{\mathbb{Q}}_G)$-generic condition, say $p_1 \in \dot{\mathbb{Q}}_G$ below $(\tilde{r_2})_G$, in the $(\leq_{\dot{\mathbb{Q}}})_G$ ordering. Essentially, what we have shown is that:

$$p_0 \Vdash \exists p_1 (p_1 \text{ is } (M[\dot{G}], \dot{Q})\text{-generic} \wedge p_1 \leq_{\dot{\mathbb{Q}}} val(\pi'(\tilde{r}), \dot{G})).$$

(where π' denotes the projection onto the second coordinate). Note moreover that $p_0 \Vdash \pi'(\tilde{r}) \in V^{\mathbb{P}}$, i.e. $\pi'(\tilde{r})$ is a name for a name according to p_0. Now use existential completeness to obtain a $\mathbb{P}$ name $\tilde{p_1}$ such that

$$p_0 \Vdash (\tilde{p_1} \text{ is } (M[\dot{G}], \dot{Q})\text{-generic and } \tilde{p_1} \leq_{\dot{\mathbb{Q}}} val(\pi'(\tilde{r}), \dot{G})).$$

Then Lemma 2.3.2 implies that $\langle p_0, \tilde{p_1}\rangle$ is $(M, \mathbb{R})$-generic. Now we only need to show that $\langle p_0, \tilde{p_1}\rangle \Vdash_{\mathbb{R}} e(\tilde{r}) \in \dot{G}$, so towards this let $\langle q_0, \tilde{q_1}\rangle \in \mathbb{R}$ be below $\langle p_0, \tilde{p_1}\rangle$ in the $\mathbb{R}$ ordering and let $\langle r_0, \tilde{r_1}\rangle \in \mathbb{R}$ be such that $\langle q_0, \tilde{q_1}\rangle \Vdash_{\mathbb{R}} e(\tilde{r}) = \langle r_0, \tilde{r_1}\rangle$ (so that $\langle q_0, \tilde{q_1}\rangle \Vdash_{\mathbb{R}} \pi(e(\tilde{r})) = \check{r_0}$), we need to show $\langle q_0, \tilde{q_1}\rangle \leq_{\mathbb{R}} \langle r_0, \tilde{r_1}\rangle$. Note that as $q_0 \leq p_0$ in the $\mathbb{P}$-ordering, we have that $q_0 \Vdash_{\mathbb{P}} \pi(\tilde{r}) \in \dot{G_0}$. Also we have $q_0 \Vdash_{\mathbb{P}} \pi(\tilde{r}) = \check{r_0}$, because, choosing an arbitrary $(V, \mathbb{P})$ generic G with $q_0 \in G$ and a $(V[G], \dot{\mathbb{Q}}_G)$ generic H with $(\tilde{q_1})_G \in H$, then we get putting $K = G * H$, $[\pi(e(\tilde{r}))]_K = (\check{r_0})_K = r_0$, but also $[\pi(e(\tilde{r}))]_K = \pi([e(\tilde{r})]_K) = \pi((\tilde{r})_{e^{-1}(K)}) = \pi(\tilde{r}_G) = (\pi(\tilde{r}))_G$. Therefore $(\pi(\tilde{r}))_G = r_0$ for each $(V, \mathbb{P})$ generic G with $q_0 \in G$, so that $q_0 \Vdash_{\mathbb{P}} \pi(\tilde{r}) = \check{r_0}$. Thus we obtain $q_0 \Vdash_{\mathbb{P}} \check{r_0} \in \dot{G_0}$ and hence as $\mathbb{P}$ is separative, we get $q_0 \leq_{\mathbb{P}} r_0$. Moreover, we already know that $p_0 \Vdash (\tilde{p_1}$ is $(M[\dot{G}], \dot{Q})$-generic and $\tilde{p_1} \leq_{\dot{\mathbb{Q}}} val(\pi'(\tilde{r}), \dot{G}))$, from which it follows that $q_0 \Vdash (\tilde{p_1}$ is $(M[\dot{G}], \dot{Q})$-generic and $\tilde{p_1} \leq_{\dot{\mathbb{Q}}} val(\pi'(\tilde{r}), \dot{G}))$, so in particular $q_0 \Vdash (\tilde{p_1} \leq_{\dot{\mathbb{Q}}} val(\pi'(\tilde{r}), \dot{G}))$, but we also have $q_0 \Vdash \tilde{q_1} \leq_{\dot{\mathbb{Q}}} \tilde{p_1}$, which gives $q_0 \Vdash \tilde{q_1} \leq_{\dot{\mathbb{Q}}} val(\pi'(\tilde{r}), \dot{G})$. Therefore if we are able to show that $q_0 \Vdash val(\pi'(\tilde{r}), \dot{G}) = \tilde{r_1}$ then we would have $q_0 \Vdash \tilde{q_1} \leq_{\dot{\mathbb{Q}}} \tilde{r_1}$ and we would be done. So, this is exactly what we do:

Claim. $q_0 \Vdash val(\pi'(\tilde{r}), \dot{G}) = \tilde{r_1}$

Proof. Let G be a $(V, \mathbb{P})$ generic filter with $q_0 \in G$, choose now a $(V[G], \dot{\mathbb{Q}}_G)$ generic filter H with $(\dot{q_1})_G$, then put $K = G * H$, so that $\langle q_0, \tilde{q_1}\rangle \in K$, then $(e(\tilde{r}))_K = \langle r_0, \tilde{r_1}\rangle$, so that $\tilde{r}_{(e^{-1}(K))} = \tilde{r}_G = \langle r_0, \tilde{r_1}\rangle$. From this it follows that $(\pi'(\tilde{r}_G))_G = (\tilde{r_1})_G$, so that $((\pi'(\tilde{r}))_G)_G = (\tilde{r_1})_G$, i.e. $(val(\pi'(\tilde{r}), \dot{G}))_G = (\tilde{r_1})_G$. Therefore, the claim is proved, and hence we are done. □

□

Note: Looking closely at the above proof, specifically the part where we concluded $q_0 \Vdash_{\mathbb{P}} \pi(\tilde{r}) = \check{r_0}$, from the statement $\langle q_0, \tilde{q_1}\rangle \Vdash_{\mathbb{R}} e(\tilde{r}) = \langle r_0, \tilde{r_1}\rangle$, one can generalise the argument with care to obtain the following general lemma which might be useful later on:

Lemma 2.3.10. Assume $\langle \mathbb{P}_\alpha, \dot{Q}_\beta \mid \alpha \leq \gamma \wedge \beta < \gamma\}$ is a countable support iteration of forcing notions, and assume $\psi = \psi(x_0, ..., x_{n-1})$ is downward absolute and let $\alpha_0 \leq \alpha_1 \leq \gamma$, $\tau_0, ...\tau_{n-1} \in V^{\mathbb{P}_{\alpha_0}}$, and let $p \in \mathbb{P}_{\alpha_1}$, then we have:

$$p \Vdash_{\alpha_1} \psi(e(\tau_0), ..., e(\tau_{n-1})) \implies p \restriction \alpha_0 \Vdash_{\alpha_0} \psi(\tau_0, ...\tau_{n-1}).$$

Where as usual, $e : \mathbb{P}_{\alpha_0} \longrightarrow \mathbb{P}_{\alpha_1}$, denotes the canonical complete embedding.

Proof. **Proof of Lemma 2.3.5**

Notation: For any $\gamma_0 \leq \alpha \leq \alpha' \leq \gamma$, we denote by e_α the canonical complete embedding from $\mathbb{P}_{\gamma_0}$ into $\mathbb{P}_\alpha$, where as usual if $\alpha = \gamma_0$, then e_α is just the identity, and we denote by $e_{\alpha',\alpha}$, the canonical embedding from $\mathbb{P}_\alpha$ into $\mathbb{P}_{\alpha'}$

Fix $M \prec H(\lambda)$ (with λ sufficiently large) and let $q_0 \in \mathbb{P}_{\gamma_0}$ and $\gamma_0 \in M \cap \gamma$ and $\tilde{p_0} \in V^{\mathbb{P}_{\gamma_0}}$ be as in the statement of the lemma (i.e. q_0 is $(M, \mathbb{P}_{\gamma_0})$-generic and $q_0 \Vdash_{\gamma_0} (\ \tilde{p_0} \in \check{\mathbb{P}_\gamma} \cap \check{M} \wedge \tilde{p_0} \restriction \gamma_0 \in \dot{G}_{\gamma_0}))$

The proof is by induction, so assume the lemma holds for all $\alpha < \gamma$.

CASE 1 : γ is a successor

Let $\gamma = \gamma' + 1$

Case 1(a): $\gamma' = \gamma_0$

Notation: In the proof of Case 1(a), in order to conserve space, we write e for the complete embedding from $\mathbb{P}_{\gamma_0}$ into $\mathbb{P}_\gamma$, and we write e' for the complete embedding from $\mathbb{P}_{\gamma_0}$ into $\mathbb{P}_{\gamma_0} * \dot{\mathbb{Q}}_{\gamma_0}$.

By the definition of a countable support iteration, we know that that the map $T : \mathbb{P}_{\gamma_0+\mathbb{1}}(= \mathbb{P}_\gamma) \longrightarrow \mathbb{P}_{\gamma_0} * \dot{\mathbb{Q}}_{\gamma_0}$ given by $T(p) = \langle p \restriction \gamma_0, p(\gamma_0)\rangle$ is an isomorphism. (See [1] for more details). Note that one might be tempted to apply Lemma 2.3.4 directly in this case, but to be completely rigorous, one has to use the isomorphism T given above, as $\mathbb{P}_{\gamma_0+\mathbb{1}}$ is only isomorphic to $\mathbb{P}_{\gamma_0} * \dot{\mathbb{Q}}_{\gamma_0}$, but is not equal to it. From the hypothesis that $q_0 \Vdash_{\gamma_0} (\tilde{p_0} \in \check{\mathbb{P}_\gamma} \cap \check{M} \wedge \tilde{p_0} \restriction \gamma_0 \in \dot{G}_{\gamma_0})$ we get $q_0 \Vdash_{\gamma_0} (T(\tilde{p_0}) \in \mathbb{P}_{\gamma_0} * \dot{\mathbb{Q}}_{\gamma_0} \cap M \wedge \pi(T(\tilde{p_0})) \in \dot{G}_{\gamma_0})$. Now we can apply the properness extension lemma for the two-step case (i.e. Lemma 2.3.4) to obtain $\tilde{q_1} \in V^{\mathbb{P}_{\gamma_0}}$ such that $\langle q_0, \tilde{q_1}\rangle$ is $(M, \mathbb{P}_{\gamma_0} * \dot{\mathbb{Q}}_{\gamma_0})$-generic and $\langle q_0, \tilde{q_1}\rangle \Vdash e'(T(\tilde{p_0})) \in G_{\mathbb{P}_{\gamma_0} * \dot{\mathbb{Q}}_{\gamma_0}}$. Now put $q = q_0 \bigcup \{\langle \gamma_0, \tilde{q_1}\rangle\} = T^{-1}(\langle q_0, \tilde{q_1}\rangle)$, then obviously q is $(M, \mathbb{P}_\gamma)$-generic and $q \restriction \gamma_0 = q_0$. Therefore, it is enough to show:

Claim. $q \Vdash_\gamma e(\tilde{p_0}) \in \dot{G}_\gamma$

Proof. Let as usual G be a $(V, \mathbb{P}_\gamma)$ generic filter containing q, now $T(G)$ is a generic filter on $\mathbb{P}_{\gamma_0} * \dot{\mathbb{Q}}_{\gamma_0}$ containing $\langle q_0, \tilde{q_1}\rangle$ so that we have $(e'(T(\tilde{p_0})))_{T(G)} \in T(G)$, but a simple computation as in some of the previous proofs gives that $(e'(T(\tilde{p_0})))_{T(G)} = T((\tilde{p_0})_{(e')^{-1}(T(G))})$, so as T is a bijection, we obtain that $(\tilde{p_0})_{(e')^{-1}(T(G))} \in G$. But note that $e = T^{-1} \circ e'$, so we get $(\tilde{p_0})_{(e')^{-1}(T(G))} = (\tilde{p_0})_{e^{-1}(G)} = (e(\tilde{p_0}))_G \in G$. Thus, the claim is proved, and we are done. $\square$

Case 1(b): $\gamma_0 < \gamma'$

We reduce this case to case 1(a), using the induction hypothesis as follows:

Firstly note that $q_0 \Vdash_{\gamma_0} (\tilde{p_0} \restriction \gamma' \in \mathbb{P}_{\gamma'} \cap M \wedge (\tilde{p_0} \restriction \gamma') \restriction \gamma_0 \in \dot{G}_{\gamma_0})$. This is justified because, $\gamma \in M$ and $M \prec H(\lambda)$ and $\gamma = \gamma' + 1$, so that γ' is definable from γ in $H(\lambda)$ and hence $\gamma' \in M$ and also $q_0 \Vdash_{\gamma_0} (\tilde{p_0} \in M \implies \tilde{p_0} \restriction \gamma' \in M)$. Now using the induction hypothesis we obtain that there is an $(M, \mathbb{P}_{\gamma'})$-generic q' such that $q' \restriction \gamma_0 = q_0$ and $q' \Vdash_{\gamma'} (e_{\gamma'}(\tilde{p_0} \restriction \gamma') \in \dot{G}_{\gamma'})$. The fact that $e_{\gamma'}(q_0) \geq q'$ and an easy computation shows that $q' \Vdash_{\gamma'} (e_{\gamma'}(\tilde{p_0} \restriction \gamma') = e_{\gamma'}(\tilde{p_0}) \restriction \gamma')$, so that we obtain, $q' \Vdash_{\gamma'} e_{\gamma'}(\tilde{p_0}) \restriction \gamma' \in \dot{G}_{\gamma'}$, but we also have $q' \Vdash_{\gamma'} e_{\gamma'}(\tilde{p_0}) \in \mathbb{P}_{\gamma} \cap M$ (from Lemma 2.3.6). Now we have achieved the reduction to Case 1(a) (with the name $e_{\gamma'}(\tilde{p_0})$ doing the role of $\tilde{p_0}$). Now by Case 1(a), we know there is $q \in \mathbb{P}_{\gamma}$ which is $(M, \mathbb{P}_{\gamma})$-generic with $q \restriction \gamma' = q'$ (so that $q \restriction \gamma_0 = q' \restriction \gamma_0 = q_0$) such that $q \Vdash_{\gamma} e_{\gamma,\gamma'}(e_{\gamma'}(\tilde{p_0})) \in \dot{G}_{\gamma}$ which gives $q \Vdash_{\gamma} e_{\gamma}(\tilde{p_0}) \in \dot{G}_{\gamma}$ (since $e_{\gamma,\gamma'} \circ e_{\gamma'} = e_{\gamma}$), and hence we are done.

CASE 2: γ is a limit ordinal

Claim. There is a strictly increasing sequence $(\gamma_n)_{n\in\omega}$ satisfying the following:

1. $\forall n \in \omega(\gamma_n \in M \cap \gamma)$ and
2. $\gamma_0 = \gamma_0$ and
3. $(\gamma_n)_{n\in\omega}$ is cofinal in $\gamma \cap M$, which means that $\forall x \in \gamma \cap M \exists n \in \omega(x < \gamma_n)$

Proof. To see why this is true, firstly note that

$$H(\lambda) \models \gamma \text{ is a limit ordinal,}$$

so that,

$$H(\lambda) \models \forall x \in \gamma \exists y \in \gamma(x < y),$$

now as $\gamma \in M$ and as $M \prec H(\lambda)$ we obtain,

$$M \models \forall x \in \gamma \exists y \in \gamma(x < y),$$

writing out the definition of "$\models$" this means that $\forall x \in M \cap \exists y \in M \cap \gamma(x < y)$. Now enumerate $M \cap \gamma = \{\eta_i \mid i < \omega\}$ and define recursively our required sequence $(\gamma_n)_{n\in\omega}$ by letting $\gamma_0 = \gamma_0$, and assuming γ_n has been defined satisfying $\gamma_n > \eta_n$, define $\gamma_{n+1} = max(\{\gamma_n, \eta_{n+1}\}) + 1$. Then it is clear that $\gamma_{n+1} \in \gamma \cap M$, by elementarity of M and the fact that γ is a limit ordinal. It is easy to verify that $(\gamma_n)_{n\in\omega}(\in V)$ is as required. □

Now let (in V) $(D_n)_{n\in\omega}$ be an enumeration of all dense subsets of $\mathbb{P}_{\gamma}$ that are in M.

Claim. Enough to construct sequences $(q_n)_{n\in\omega}$ and $(\tilde{p_n})_{n\in\omega}$ such that the following hold:

0. $\forall n \in \omega\ (q_n \in \mathbb{P}_{\gamma_n} \wedge \tilde{p_n} \in V^{\mathbb{P}_{\gamma_n}})$

1. $q_0 = q_0$ (the given $(M, \mathbb{P}_{\gamma_0})$-generic condition) and
$\forall n \in \omega\ (q_n$ is $(M, \mathbb{P}_{\gamma_n})$-generic and $q_{n+1} \restriction \gamma_n = q_n)$

2. $\tilde{p_0} = \tilde{p_0}$, is the given $\mathbb{P}_{\gamma_0}$-name, and for each $n \geq 1$, we have:
$q_n \Vdash_{\gamma_n} (\tilde{p_n} \in \mathbb{P}_\gamma \cap M \wedge \tilde{p_n} \restriction \gamma_n \in \dot{G}_{\gamma_n} \wedge \tilde{p_n} \check{\leq}_\gamma e_{n,n-1}(\tilde{p_{n-1}}) \wedge \tilde{p_n} \in \check{D_{n-1}})$.

Note: Here for notational simplicity we use the symbol $e_{n,n-1}$ for the canonical complete embedding from $\mathbb{P}_{\gamma_{n-1}}$ into $\mathbb{P}_{\gamma_n}$, instead of using $e_{\gamma_n,\gamma_{n-1}}$, we do the same for e_{γ_m,γ_n}, i.e. we use $e_{m,n}$ instead of e_{γ_m,γ_n}. Moreover since, for $\gamma_0 \leq \alpha \leq \gamma$, our notation entails that $e_{\alpha,\gamma_0} = e_\alpha$, so that we might use these symbols interchangeably.

Proof. Assume we have such sequences as given in the claim above, then put $q = \bigcup_{n\in\omega}(q_n)$. Here, outside $\bigcup_{n\in\omega}(\gamma_n)$ we just define q to be the name for the biggest element, more precisely if $\eta \notin \bigcup_{n\in\omega}(\gamma_n)$, then we define $q(\eta) = 1_{\dot{\mathbb{Q}}_\eta}$. One can easily show, using the definition of a countable support iteration, $q \in \mathbb{P}_\gamma$. Moreover, it is easy to see that, for each $n \in \omega$, $q \restriction \gamma_n = q_n$.
Now we observe that it is enough to show that $\forall n \in \omega(q \Vdash_\gamma e_{\gamma,\gamma_n}(\tilde{p_n}) \in \dot{G}_\gamma)$, this is because, then we would have that $q \Vdash_\gamma e_{\gamma,\gamma_n}(\tilde{p_n}) \in \check{M} \cap \check{D}_{n-1} \cap \dot{G}_\gamma$ (because $q \leq_\gamma e_{\gamma,\gamma_n}(q_n)$, condition (2) in the claim and Lemma 2.3.6), so that by Lemma 2.2.4 part 2, we get that q is $(M, \mathbb{P}_\gamma)$-generic. Moreover, taking $n = 0$ in $q \Vdash_\gamma (e_{\gamma,\gamma_n}(\tilde{p_n}) \in \dot{G}_\gamma)$, we get $q \Vdash_\gamma (e_\gamma(\tilde{p_0}) \in \dot{G}_\gamma)$. Moreover, the construction of q as the union of the q_n's entails that $q \restriction \gamma_0 = q_0$. Now we observe that to prove $q \Vdash_\gamma e_{\gamma,\gamma_n}(\tilde{p_n}) \in \dot{G}_\gamma$, it is enough to prove $\forall m \geq n(q_m \Vdash_{\gamma_m} \tilde{p_m} \leq_\gamma e_{m,n}(\tilde{p_n}))$, because if this were true then we will have, for each $m \geq n$, $q_m \Vdash_{\gamma_m} (\tilde{p_m} \restriction \gamma_m \leq_{\gamma_m} e_{m,n}(\tilde{p_n}) \restriction \gamma_m)$, which gives $q_m \Vdash_{\gamma_m} (e_{m,n}(\tilde{p_n}) \restriction \gamma_m \in \dot{G}_{\gamma_m})$ (because, by (2), $q_m \Vdash_{\gamma_m} (\tilde{p_m} \restriction \gamma_m \in \dot{G}_{\gamma_m})$). Moreover we also have $q_n \Vdash_{\gamma_n} supp(\tilde{p_n}) \subseteq \bigcup_{m\geq n}(\gamma_m)$, this is because, if $p \in \mathbb{P}_\gamma \cap M$ then $supp(p)$ being definable in $H(\lambda)$ from parameters in M, will infact be a member of M but we also have that $supp(p)$ is countable, thus it follows that $supp(p) \subseteq M$ (see [10, Lemma III.8.1] for more details). From this, it follows that $q_n \Vdash_{\gamma_n} (supp(\tilde{p_n}) \subseteq \gamma \cap M \subseteq \bigcup_{n\in\omega}(\gamma_n) = \bigcup_{m\geq n}(\gamma_m))$. Thus we can now apply Lemma 2.3.9 to conclude that $q \Vdash_\gamma e_{\gamma,\gamma_n}(\tilde{p_n}) \in \dot{G}_\gamma$. Now we finally show that $\forall m \geq n(q_m \Vdash_{\gamma_m} \tilde{p_m} \leq_\gamma e_{m,n}(\tilde{p_n}))$ as follows: for $m = n+1$, this follows directly by condition (2) of the claim. Now assume $q_{n+k} \Vdash_{\gamma_{n+k}} \tilde{p}_{n+k} \leq_\gamma e_{n+k,n}(\tilde{p_n})$, then we obtain using Lemma 2.3.6 and the facts, $q_{n+k+1} \leq_{\gamma_{n+k+1}} e_{n+k+1,n+k}(q_{n+k})$ and $e_{n+k+1,n+k} \circ e_{n+k,n} = e_{n+k+1,n}$, that

$$q_{n+k+1} \Vdash_{\gamma_{n+k+1}} e_{n+k+1,n+k}(\tilde{p}_{n+k}) \leq_\gamma e_{n+k+1,n}(\tilde{p_n}),$$

now combining this with the fact that

$$q_{n+k+1} \Vdash_{\gamma_{n+k+1}} [\tilde{p}_{n+k+1} \leq_\gamma e_{n+k+1,n+k}(\tilde{p}_{n+k})],$$

we get that, $q_{n+k+1} \Vdash_{\gamma_{n+k+1}} \tilde{p}_{n+k+1} \leq_\gamma e_{n+k+1,n}(\tilde{p}_n)$, thus by induction on the natural numbers, we conclude that $\forall m \geq n(q_m \Vdash_{\gamma_m} \tilde{p_m} \leq_\gamma e_{m,n}(\tilde{p_n}))$, and hence we are done. □

Constructing such sequences $\tilde{p_n}$ and q_n :
This is yet again via induction on the natural numbers, we let first $q_0 = q_0 \in \mathbb{P}_{\gamma_0}$ and $\tilde{p_0} = \tilde{p_0} \in V^{\mathbb{P}_{\gamma_0}}$, now assuming q_n and $\tilde{p_n}$ have been constructed, we construct $\tilde{p}_{n+1}$ and q_{n+1}:

Construction of $\tilde{p}_{n+1}$:

Claim. $q_n \Vdash_{\gamma_n} \exists x\ (x \in M \cap D_n \wedge x \leq_\gamma \tilde{p}_n \wedge x \restriction \gamma_n \in \dot{G}_{\gamma_n})$

Proof. To see why this is true let $G_n \subseteq \mathbb{P}_{\gamma_n}$ be $(V, \mathbb{P}_{\gamma_n})$ generic with $q_n \in G_n$, then from the induction hypothesis we know that $q_n \Vdash_{\gamma_n} \tilde{p_n} \in \mathbb{P}_\gamma \cap M$, so that $p_n = (\tilde{p_n})_{G_n} \in \mathbb{P}_\gamma \cap M$, now we show that the set $B = \{p \restriction \gamma_n \mid p \in D_n \wedge p \leq_\gamma p_n\}$ is dense below $p_n \restriction \gamma_n$. To see this, let $r \in \mathbb{P}_{\gamma_n}$ be such that $r \leq p_n \restriction_{\gamma_n}$, note that then we have $r \cup^* p_n \in \mathbb{P}_\gamma$ (here $r \cup^* p_n = r \cup p_n \restriction_{[\gamma_n, \gamma]}$, see [9] for more details), so that as D_n is dense in $\mathbb{P}_\gamma$, choose $p \in D_n$ with $p \leq_\gamma r \cup^* p_n$, then it is easy to see that $p \leq_\gamma p_n$ and moreover $p \in D_n$ and $p \restriction \gamma_n \leq_{\gamma_n} r$. This shows that B is dense below $p_n \restriction \gamma_n$ in $\mathbb{P}_{\gamma_n}$, so that $B' = B\ \cup \{r \in \mathbb{P}_{\gamma_n} \mid r \perp (p_n \restriction \gamma_n)\}$ is dense in $\mathbb{P}_{\gamma_n}$. Moreover $B' \in M$, because, $p_n, \gamma_n, \mathbb{P}_{\gamma_n}, D_n \in M$, note that $\mathbb{P}_{\gamma_n} \in M$, because $\gamma_n \in M$, $\mathbb{P}_\gamma \in M$ and $\mathbb{P}_{\gamma_n} = \{p \restriction \gamma_n \mid p \in \mathbb{P}_\gamma\}$. Therefore as q_n is $(M, \mathbb{P}_{\gamma_n})$-generic, we get that $B' \cap M \cap G_n \neq \emptyset$, so let $x \in B' \cap M \cap G_n$. Then it follows that $x \not\perp (p_n \restriction \gamma_n)$, because of our inductive hypothesis that $q_n \Vdash_{\gamma_n} \tilde{p_n} \restriction \gamma_n \in \dot{G}_{\gamma_n}$ (so that both x, $p_n \restriction \gamma_n \in G_n$), so it follows that $x \in G_n \cap B \cap M$. Now by definition of B, we have that "$\exists p \in \mathbb{P}_\gamma(p \in D_n \wedge p \leq_\gamma p_n \wedge p \restriction \gamma_n = x)$", now this statement in the double quotes obviously holds in $H(\lambda)$ and hence by elementarity also holds in M, so that we can assume without loss of generality, that $p \in M$. Essentially we have shown that $\exists p(p \in (\check{M} \cap \check{D}_n)_{G_n} \wedge p\ (\check{\leq}_\gamma)_{G_n}\ (\tilde{p_n})_{G_n} \wedge p \restriction \gamma_n \in (\dot{G}_{\gamma_n})_{G_n})$, and hence we obtain via absoluteness that $V[G_n] \models \exists x(x \in \check{M} \cap \check{D}_n \wedge x \check{\leq}_\gamma \tilde{p_n} \wedge x \restriction \gamma_n \in \dot{G}_{\gamma_n})$. Thus, we conclude that $q_n \Vdash_{\gamma_n} \exists x\ (x \in M \cap D_n \wedge x \leq_\gamma \tilde{p}_n \wedge x \restriction \gamma_n \in \dot{G}_{\gamma_n})$ □

Now use existential completeness to obtain a name $\dot{p}_{n+1} \in V^{\mathbb{P}_{\gamma_n}}$ such that $q_n \Vdash_{\gamma_n} (\dot{p}_{n+1} \in M \cap D_n \wedge \dot{p}_{n+1} \leq_\gamma \tilde{p}_n \wedge \dot{p}_{n+1} \restriction \gamma_n \in \dot{G}_{\gamma_n})$ and now finally put $\tilde{p}_{n+1} = e_{\gamma_{n+1},\gamma_n}(\dot{p}_{n+1}) \in V^{\mathbb{P}_{\gamma_{n+1}}}$.

Construction of q_{n+1} :

The construction of the $\mathbb{P}_{\gamma_n}$-name $\dot{p}_{n+1}$ in the previous paragraph, entails that $q_n \Vdash_{\gamma_n} (\dot{p}_{n+1} \restriction \gamma_{n+1} \in \mathbb{P}_{\gamma_{n+1}} \cap M \wedge (\dot{p}_{n+1} \restriction \gamma_{n+1}) \restriction \gamma_n \in \dot{G}_{\gamma_n})$, now as $\gamma_{n+1} < \gamma$ and $\gamma_{n+1} \in M$, the inductive hypothesis gives that there is an $(M, \mathbb{P}_{\gamma_{n+1}})$-generic q_{n+1} with $q_{n+1} \restriction \gamma_n = q_n$ and $q_{n+1} \Vdash_{\gamma_{n+1}} (e_{\gamma_{n+1},\gamma_n}(\dot{p}_{n+1} \restriction \gamma_{n+1}) \in \dot{G}_{\gamma_{n+1}})$. With this, we are done with the construction of q_{n+1}.

Claim. $\tilde{p}_{n+1}$ and q_{n+1} are as required.

Proof. We need to show:

$$q_{n+1} \Vdash_{\gamma_{n+1}} (\tilde{p}_{n+1} \in \mathbb{P}_\gamma \cap M \wedge \tilde{p}_{n+1} \restriction \gamma_{n+1} \in \dot{G}_{\gamma_{n+1}} \wedge \tilde{p}_{n+1} \leq_\gamma e_{n+1,n}(\tilde{p}_n) \wedge \tilde{p}_{n+1} \in \check{D}_n)$$

This basically follows from the following facts (combined with Lemma 2.3.6):

- $q_{n+1} \leq_{\gamma_{n+1}} e_{n+1,n}(q_n)$ (since $q_{n+1} \restriction \gamma_n = q_n$)
- $q_{n+1} \Vdash_{\gamma_{n+1}} (e_{n+1,n}(\dot{p}_{n+1} \restriction \gamma_{n+1}) = \underbrace{(e_{n+1,n}(\dot{p}_{n+1}))}_{\tilde{p}_{n+1}} \restriction \gamma_{n+1})$ and

  $q_{n+1} \Vdash_{\gamma_{n+1}} (e_{n+1,n}(\dot{p}_{n+1} \restriction \gamma_{n+1}) \in \dot{G}_{\gamma_{n+1}})$
  (from which it follows that $q_{n+1} \Vdash_{\gamma_{n+1}} \tilde{p}_{n+1} \restriction \gamma_{n+1} \in \dot{G}_{\gamma_{n+1}}$)
- $q_n \Vdash_{\gamma_n} \dot{p}_{n+1} \leq_\gamma \tilde{p}_n$
  (from which it follows that $q_{n+1} \Vdash_{\gamma_{n+1}} (\tilde{p}_{n+1} \leq_\gamma e_{n+1,n}(\tilde{p}_n))$)
- $q_n \Vdash_{\gamma_n} \dot{p}_{n+1} \in M \cap D_n$
  (from which it follows that $q_{n+1} \Vdash_{\gamma_{n+1}} (\tilde{p}_{n+1} \in \mathbb{P}_\gamma \cap M \wedge \tilde{p}_{n+1} \in \check{D}_n)$).

$\square$

Thus, by induction on the natural numbers, we are done.

$\square$

Corollary 2.3.11. (Preservation Theorem 1)
let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \gamma, \beta < \gamma \rangle$ be a countable support iteration of forcing posets, such that $\forall \beta < \gamma \ (1_{\mathbb{P}_\beta} \Vdash_\beta (\dot{\mathbb{Q}}_\beta \text{ is proper}))$, then $\mathbb{P}_\gamma$ is proper.

Remark. From here on, we refer to a forcing iteration that satisfies the conditions in the statement of Corollary 2.3.11 as "a countable support iteration of proper forcings".

Proof. Fix first, a λ sufficiently large and let $M \prec H(\lambda)$ be arbitrary, countable, and let $p \in \mathbb{P}_\gamma \cap M$ also be arbitrary. We will find an $(M, \mathbb{P}_\gamma)$-generic condition $q \in \mathbb{P}_\gamma$ with $q \leq_\gamma p$, hence showing $\mathbb{P}_\gamma$ is proper.

Take $\gamma' = 0$, $\tilde{p}_0 = \check{p}$ where the check is taken with respect to the poset $\mathbb{P}_0 = \{\emptyset\}$, and let $q_0 = \emptyset$ in the general version of the properness extension lemma (i.e. Lemma 2.3.5). Then by trivial reasons q_0 is $(M, \mathbb{P}_{\gamma'})$-generic, so that we can apply Lemma 2.3.5 to obtain an $(M, \mathbb{P}_\gamma)$-generic condition q with $q \restriction \gamma' = q \restriction 0 = q_0 = \emptyset$ such that $q \Vdash_\gamma (e_{\gamma,0}(\check{p}) \in \dot{G}_\gamma)$, but as check names are preserved under complete embeddings this gives, $q \Vdash_\gamma \check{p} \in \dot{G}_\gamma$ (where now the check is with respect to the poset $\mathbb{P}_\gamma$), but now by the fact that $\mathbb{P}_\gamma$ is separative we obtain that $q \leq_\gamma p$. Thus, we are done. $\square$

3 Tight MAD Families and Preservation

3.1 Tight MAD families

In this chapter, we introduce the notion of a "tight MAD family", which is a strengthening of "MAD" families which is easier to preserve while forcing. We also introduce the notion of "Strong preservation of tightness" and prove that the countable support iteration of proper forcings that strongly preserve the tightness of a tight MAD family $\mathcal{A}$ (in the ground model), strongly preserves the tightness of $\mathcal{A}$ (see 3.2.7). This preservation theorem lets us preserve tight MAD families from the ground model, which in turn lets us control the value of the almost disjointness number (which is an important cardinal characteristic of the continuum) in forcing extensions. This chapter is based entirely on [7].

Remark. For a set A, and a cardinal κ, we denote by $[A]^\kappa$, the set of all subsets of A of size exactly κ and by $[A]^{\leq\kappa}$, we denote the set of all subsets of A of size $\leq \kappa$. We might use $[A]^{\leq\omega}$ and $[A]^{\leq\aleph_0}$ interchangeably.

Definition 3.1.1. We say a family $\mathcal{A}$ of infinite subsets of ω, is an almost disjoint family if $\forall x, y \in \mathcal{A}(|x \cap y| < \aleph_0)$

Definition 3.1.2. We say an infinite family $\mathcal{A}$ of infinite subsets of ω is a maximal almost disjoint family (MAD family) if the following hold:

1. $\mathcal{A}$ is almost disjoint and
2. $\forall b \in [\omega]^\omega$ ($\mathcal{A} \cup \{b\}$ is not an almost disjoint family).

Remark. Condition 2, in the above definition, is equivalent to the statement:

$$\forall b \in [\omega]^\omega \exists a \in \mathcal{A} \ (|a \cap b| = \aleph_0).$$

This condition captures maximality of the collection $\mathcal{A}$, i.e. it essentially says that $\mathcal{A}$ cannot be extended further without breaking almost disjointness.

Associated with these special combinatorial families of reals is a cardinal characteristic, $\mathfrak{a}$, which is defined as follows:

Definition 3.1.3.

$$\mathfrak{a} = min(\{|\mathcal{A}| \mid \mathcal{A}(\subseteq [\omega]^{\omega}) \text{ is a maximal almost disjoint family}\})$$

Note: It is well known that $\aleph_1 \leq \mathfrak{a} \leq 2^{\aleph_0}$ (See [8] for more details)

Definition 3.1.4. For two sets a, b we write $a \subseteq^* b$ (and say a is almost contained in b) if $a \setminus b$ is finite.

Definition 3.1.5. Let $\mathcal{A} \subseteq [\omega]^{\omega}$ be an almost disjoint family, we define the "ideal generated by $\mathcal{A}$" as

$$I(\mathcal{A}) = \{X \subseteq \omega \mid \exists \mathcal{F} \in [\mathcal{A}]^{<\omega} \ (X \subseteq^* \bigcup \mathcal{F}\ )\}.$$

We define, the positive sets with respect to $\mathcal{A}$, as

$$I(\mathcal{A})^+ = \mathcal{P}(\omega) \setminus I(\mathcal{A}) = [\omega]^{\omega} \setminus I(\mathcal{A}) = \ \{X \in [\omega]^{\omega} \mid X \notin I(\mathcal{A})\}$$

Some observations:

- $\forall X \ (X \in I(\mathcal{A}) \implies \mathcal{A} \cup \{X\}$ is not almost disjoint)
- $\emptyset \in I(\mathcal{A})$, $\forall a, b(b \in I(\mathcal{A}) \wedge a \subseteq b \implies a \in I(\mathcal{A}))$(closure under subsets) and moreover, $\forall a, b(a, b \in I(\mathcal{A}) \implies a \cup b \in I(\mathcal{A}))$(closure under finite unions), i.e. in other words, $I(\mathcal{A})$ is an ideal.

Now we give a characterisation of maximal almost disjoint families via the ideal $I(\mathcal{A})$. This characterisation would help us make sense of the next definition of "tight almost disjoint" families.

Lemma 3.1.6. Let $\mathcal{A}$ be an almost disjoint family; then the following are equivalent:

1. $\mathcal{A}$ is maximal (i.e. MAD)
2. $\forall n \in \omega \forall A_1, A_2, ..., A_n \in I(\mathcal{A})^+ \exists B \in I(\mathcal{A}) \ (\forall 1 \leq i \leq n \ (|A_i \cap B| = \omega))$

Proof.
$(\implies)$
Since $\mathcal{A}$ is maximal, there is, for each $i \in \{1, 2, 3, ..., n\}$, a $B_i \in \mathcal{A}$ such that $A_i \cap B_i$ is infinite. Now put $B = B_1 \cup B_2 \cup .... \cup B_n \in I(\mathcal{A})$, so that,
$\forall 1 \leq i \leq n \ (|A_i \cap B| = \omega)$.
$(\impliedby)$
By the first part of the observations above, we only need to show that

$$\forall X \in I(\mathcal{A})^+ \exists B \in \mathcal{A} \ (|X \cap B| = \omega).$$

By hypothesis, we first obtain for $X \in I(\mathcal{A})^+$ a $B \in I(\mathcal{A})$ such that $X \cap B$ is infinite. Now by the definition of $I(\mathcal{A})$ choose $A_1, A_2, ..., A_n \in \mathcal{A}$ such that $B \subseteq^* A_1 \cup .... \cup A_n$, from this, we obtain that $X \cap B \subseteq^* A_1 \cup .... \cup A_n$ but we also have that $X \cap B$ is infinite and hence $(B \cap X) \cap (A_1 \cup A_2 \cup .... \cup A_n)$ is infinite and hence for some $1 \leq i \leq n$ ($B \cap X \cap A_i$) is infinite, from which it follows that $|X \cap A_i| = \aleph_0$. Thus $\mathcal{A}$ is maximal.

□

Definition 3.1.7. We say an almost disjoint family $\mathcal{A} \subseteq [\omega]^\omega$ is "tight" if

$$\forall \mathcal{F} \in [I(\mathcal{A})^+]^{\leq \aleph_0} \exists B \in I(\mathcal{A}) \ (\forall X \in \mathcal{F}(\ |X \ \cap B| = \aleph_0))$$

Note: From Lemma 3.1.6, it is clear how "tightness" is a natural strengthening of maximality for almost disjoint families, and it is also clear that all tight almost disjoint families are maximal.

3.2 Preserving tight MAD families

Definition 3.2.1. Let $\mathcal{A}$ be a MAD family and assume that $\mathbb{P}$ is a forcing notion, then we say that $\mathcal{A}$ is $\mathbb{P}$-indestructible if

$$1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\check{\mathcal{A}} \text{ is a MAD family}).$$

Definition 3.2.2. let $\mathcal{A}$ be a tight almost disjoint family and let $\mathbb{P}$ be a forcing poset, let $M \prec H(\lambda)$ be countable with $\mathbb{P}, \mathcal{A} \in M$. Let $B \in I(\mathcal{A})$ be such that $\forall x \in I(\mathcal{A})^+ \cap M(|x \cap B| = \aleph_0)$. Then, we say a condition $q \in \mathbb{P}$ is $(M, \mathbb{P}, \mathcal{A}, B)$-generic if

$$q \text{ is } (M, \mathbb{P})\text{-generic and } q \Vdash_{\mathbb{P}} \forall x \in M[\dot{G}] \cap I(\mathcal{A})^+ \ (|x \cap \check{B}| = \aleph_0)$$

Definition 3.2.3. Let $\mathcal{A}$ be a tight almost disjoint family, then, we say a proper forcing $\mathbb{P}$ "strongly preserves the tightness of $\mathcal{A}$" if, for each regular $\lambda > 2^{|\mathbb{P}|}$ and each $M \prec H(\lambda)$ countable with $\mathbb{P}, \mathcal{A} \in M$, and each $B \in I(\mathcal{A})$ such that $\forall x \in I(\mathcal{A})^+ \cap M(|x \cap B| = \aleph_0)$, and each $p \in \mathbb{P} \cap M$, there is an $(M, \mathbb{P}, \mathcal{A}, B)$-generic q with $q \leq p$

Some observations:

1. If $\mathcal{A}$ is a tight almost disjoint family and if $\mathbb{P}$ is a forcing poset that strongly preserves the tightness of $\mathcal{A}$, then for all sufficiently large regular λ (i.e. $\forall \lambda > 2^{|\mathbb{P}|}$), and each $A \in H(\lambda)$, the set

$$D(\lambda, A) = \{q \in \mathbb{P} \mid \exists M \exists B (B \in I(\mathcal{A}) \wedge M \prec H(\lambda) \wedge |M| \leq \aleph_0 \wedge \mathcal{A}, \mathbb{P}, A \in M \wedge \forall x \in M \cap I(\mathcal{A})^+ (|x \cap B| = \aleph_0) \wedge q \text{ is } (M, \mathbb{P}, \mathcal{A}, B)\text{-generic}\}$$

is dense in $\mathbb{P}$

2. Let $\mathcal{A}$ be a tight almost disjoint family and assume that $\mathbb{P}$ is a forcing poset that strongly preserves the tightness of $\mathcal{A}$, then we have that

$$1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\mathcal{A} \text{ is tight almost disjoint}),$$

so that, in particular, $\mathcal{A}$ is $\mathbb{P}$-indestructible.

Proof.

1. Use the downward Löwenheim Skolem Theorem

2. Let G be an arbitrary $(V, \mathbb{P})$ generic filter,
Now in $V[G]$:
Let $X \in [I(\mathcal{A})^+]^{\leq\omega}$ be arbitrary, we need to find a $B \in I(\mathcal{A})$ that satisfies $\forall y \in X(|y \cap A| = \aleph_0)$. Towards this, choose $f : \omega \longrightarrow I(\mathcal{A})^+$ such that $range(f) = X$
Now we come out of $V[G]$:
Note that it is easily seen that

$$f,\ (I(\mathcal{A})^+)^{V[G]} = (I(\mathcal{A})^+)^{H(\lambda)^{V[G]}},\ X \in H(\lambda)^{V[G]} = H(\lambda)^V[G]$$

(by the largeness and regularity of λ). So we have

$$H(\lambda)^{V[G]} \models (\tilde{f} : \omega \longrightarrow I(\mathcal{A})^+ \wedge range(\tilde{f}) = \tilde{X}),$$

where $\tilde{X},\ \tilde{f} \in H(\lambda)^V \cap V^{\mathbb{P}}$ satisfies $\tilde{X}_G = X$, $\tilde{f}_G = f$. Now the forcing theorem applied to the forcing relation of $H(\lambda)^V$ (which we denote by $\Vdash'$) gives a $p \in G$ such that $p \Vdash' (\tilde{f} : \omega \longrightarrow I(\mathcal{A})^+ \wedge range(\tilde{f}) = \tilde{X})$ but as the formula "$x : \omega \longrightarrow I(\mathcal{A})^+ \wedge range(x) = y$" is absolute between $H(\lambda)^{V[G]}$ and $V[G]$, one can easily see via the forcing theorem that $p \Vdash \tilde{f} : \omega \longrightarrow I(\mathcal{A})^+$. Now via the existential completeness of the forcing relation for $H(\lambda)^V$ ($\Vdash'$), obtain in $H(\lambda)^V$ a sequence of $\mathbb{P}$-names $\langle \tau_n \mid n \in \omega \rangle$ ($\in H(\lambda)^V$) such that $\forall n \in \omega(p \Vdash' \langle n, \tau_n \rangle \in \tilde{f})$, then again by absoluteness of the formula "$\langle x, y \rangle \in z$", we obtain that $\forall n \in \omega(p \Vdash \langle n, \tau_n \rangle \in \tilde{f})$. Now put $C = \{\tau_n \mid n \in \omega\} \in H(\lambda)^V$ and consider the dense subset of $\mathbb{P}$, $D(C)$ (by part(1) we know $D(C)$ is dense in $\mathbb{P}$). Choose a $q \in D(C) \cap G$ with $q \leq p$, and let $M \prec H(\lambda), B \in I(\mathcal{A})$ witness $q \in D(C)$, then by definition of $D(C)$, we have

$$q \Vdash (\forall x \in M[\dot{G}] \cap I(\mathcal{A})^+(|x \cap B| = \aleph_0)).$$

Now back in $V[G]$:

Let $x \in X$ so that chose $n \in \omega$ such that $f(n) = x$. Now we have $(\tau_n)_G = f(n) = x$. But because $C \in M$ and C is countable, we have $C \subseteq M$ and hence $\tau_n \in M$. Therefore we obtain $x = (\tau_n)_G = f(n) \in I(\mathcal{A})^+ \cap M[G]$ so that as $q \in G$ and $q \Vdash_\gamma (\forall x \in M[\dot{G}] \cap I(\mathcal{A})^+(|x \cap B| = \aleph_0))$, it follows that $|x \cap B| = \aleph_0$.
Now outside $V[G]$:
What we have essentially shown is that:
$V[G] \models \forall x \in X(|x \cap B| = \aleph_0)$, but as $X \in ([I(\mathcal{A})^+]^{\leq\omega})^{V[G]}$ was arbitrary what we have really shown is that

$$V[G] \models \forall X \in [I(\mathcal{A})^+]^{\leq\omega} \exists B \in I(\mathcal{A})(\forall x \in X(|x \cap B| = \aleph_0).$$

Thus as G was arbitrary, we conclude that $1_{\mathbb{P}} \Vdash (\mathcal{A}$ is tight almost disjoint).

□

Now we show that the property of strong preservation of tightness of $\mathcal{A}$ gets preserved under countable support iterations. Parts of the methods are essentially the same as in the case of properness, and hence we do not work these proofs out in as much detail as in Chapter 2. Moreover if we have a complete embedding $e : \mathbb{P} \longrightarrow \mathbb{Q}$, then for a $\mathbb{P}$-name τ, we abuse notation and sometimes write τ instead of $e(\tau)$ in $\mathbb{Q}$-forcing formulas. First, a lemma that deals with the case of two-step iterations:

Lemma 3.2.4. Let $\mathcal{A}$ be a tight almost disjoint family (in V of course) then:

1. Assume that $\mathbb{P}$ is a proper forcing that strongly preserves the tightness of $\mathcal{A}$ and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a forcing poset such that

   $$1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\dot{\mathbb{Q}} \text{ is a proper forcing that strongly preserves the tightness of } \mathcal{A}).$$

   Then we have that the two-step product $\mathbb{P} * \dot{\mathbb{Q}}$ strongly preserves the tightness of $\mathcal{A}$.

2. If $M \prec H(\lambda)$ (for sufficiently large λ) is countable with $\mathcal{A}$, $\mathbb{P} * \dot{\mathbb{Q}} \in M$ and $B \in I(\mathcal{A})$ satisfies $\forall x \in M \cap I(\mathcal{A})^+(|x \cap B| = \aleph_0)$, then for each $p \in \mathbb{P}$ and each $\dot{q} \in V^{\mathbb{P}}$, we have,

   $$\langle p, \dot{q}\rangle \text{ is } (M, \mathbb{P} * \dot{\mathbb{Q}}, \mathcal{A}, B)\text{-generic} \iff p \text{ is } (M, \mathbb{P}, \mathcal{A}, B)\text{-generic and } p \Vdash_{\mathbb{P}} (\dot{q} \text{ is } (M[\dot{G}], \dot{\mathbb{Q}}, \mathcal{A}, B)\text{-generic})$$

Proof.
First, we prove 2:
Let λ be a a sufficiently large regular cardinal and assume $M \prec H(\lambda)$ is countable with $\mathcal{A}, \mathbb{P} \in M$ and let $B \in I(\mathcal{A})$ satisfiy, $\forall x \in M \cap I(\mathcal{A})^+(|x \cap B| = \aleph_0)$ and let $p \in \mathbb{P}$ and $\dot{q} \in V^{\mathbb{P}}$.

$(\implies)$
This follows easily from definitions and the fact that for any $(V, \mathbb{P})$ generic G, with $p \in G$ and any $(V[G], (\dot{\mathbb{Q}})_G)$, generic H with $(\dot{q})_G \in H$, $\langle p, \dot{q}\rangle \in K = G * H$ and $M[K] = M[G][H]$.
$(\impliedby)$
Definitely $\langle p, \dot{q}\rangle$ is $(M, \mathbb{P} * \dot{\mathbb{Q}})$-generic. So we only need to show that $\langle p, \dot{q}\rangle \Vdash_{\mathbb{P}*\dot{\mathbb{Q}}} (\forall x \in M[\dot{K}] \cap I(\mathcal{A})^+ \ (|x \cap B| = \aleph_0))$, where as usual $\dot{K}$ is the canonical name for a generic filter on $\mathbb{P}*\dot{\mathbb{Q}}$. So let $K \subseteq \mathbb{P}*\dot{\mathbb{Q}}$ be a generic filter with $\langle p, \dot{q}\rangle \in K$, and decompose K as $K = G * H$ (as we have done time and time again in Chapter 2) such that $p \in G$ and $(\dot{q})_G \in H$. So that as p is $(M, \mathbb{P}, \mathcal{A}, B)$-generic we obtain that $\forall x \in M[G] \cap I(\mathcal{A})^+(|x \cap B| = \aleph_0\ )$. Now we also have that $M[G] \prec H(\lambda)^{V[G]}$ is countable and $(\dot{\mathbb{Q}})_G, \mathcal{A}, \dot{q}_G \in M[G]$, so that as $\dot{q}_G$ is $(M[G], \dot{\mathbb{Q}}_G, \mathcal{A}, B)$-generic and $\dot{q}_0 \in H$, which gives that $\forall x \in M[G][H] \cap I(\mathcal{A})^+\ (\ |x \cap B| = \aleph_0\ )$, but $M[K] = M[G][H]$ so that essentially what we have is $\forall x \in M[K] \cap I(\mathcal{A})^+\ (|x \cap B| = \aleph_0)$, and hence we conclude that $\langle p, \dot{q}\rangle \Vdash_{\mathbb{P}*\dot{\mathbb{Q}}} (\forall x \in M[\dot{K}] \cap I(\mathcal{A})^+(|x \cap B| = \aleph_0\ ))$.

Now we prove 1, using 2:
$\langle p_0, \dot{q}_0\rangle \in \mathbb{P} * \dot{\mathbb{Q}} \cap M$ be an arbitrary condition. We need to find a condition $\langle p, \dot{q}\rangle \leq \langle p_0, \dot{q}_0\rangle$ which is $(M, \mathbb{P} * \dot{\mathbb{Q}}, \mathcal{A}, B)$-generic. Towards this, firstly, as $\mathbb{P}$ strongly preserves the tightness of $\mathcal{A}$, pick $p \in \mathbb{P}$ below p_0 that is $(M, \mathbb{P}, \mathcal{A}, B)$-generic. Now note that we have $p_0 \Vdash_{\mathbb{P}} (\dot{q}_0 \in M[\dot{G}] \cap \dot{\mathbb{Q}})$ and hence we obtain

$$\begin{aligned} p \Vdash_{\mathbb{P}} (\dot{q}_0 \in M[\dot{G}] \cap \dot{\mathbb{Q}} \wedge \forall x \in M[\dot{G}] \cap I(\mathcal{A})^+(|x \cap B| = \aleph_0) \\ \wedge \dot{\mathbb{Q}} \text{ strongly preserves the tightness of } \mathcal{A}), \end{aligned}$$

from which it follows that $p \Vdash_{\mathbb{P}} \exists q \leq \dot{q}_0 (q \text{ is } (M[\dot{G}], \dot{\mathbb{Q}}, \mathcal{A}, B)\text{-generic})$. Using existential completeness, we then obtain a name $\dot{q} \in V^{\mathbb{P}}$ such that $p \Vdash_{\mathbb{P}} (\dot{q} \leq \dot{q}_0 \wedge \dot{q} \text{ is } (M[\dot{G}], \dot{\mathbb{Q}}, \mathcal{A}, B)\text{-generic})$, so that $\langle p, \dot{q}\rangle \leq \langle p_0, \dot{q}_0\rangle$ and $\langle p, \dot{q}\rangle$ is $(M, \mathbb{P} * \dot{\mathbb{Q}}, \mathcal{A}, B)$-generic. Thus, as $\langle p_0, \dot{q}_0\rangle$ was arbitrary, we conclude that $\mathbb{P} * \dot{\mathbb{Q}}$ strongly preserves the tightness of $\mathcal{A}$.

□

Lemma 3.2.5. Strong tightness extension lemma (for two-step iterations) Let $\mathcal{A}$ be a tight almost disjoint family and let $\mathbb{P}$ be a forcing notion that is proper and strongly preserves the tightness of $\mathcal{A}$, and assume that $\dot{\mathbb{Q}}$ is a name for a forcing notion such that

$$1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\dot{\mathbb{Q}} \text{ is proper and strongly preserves the tightness of } \mathcal{A}).$$

Let $M \prec H(\lambda)$ (for sufficiently large regular λ) be countable with $\mathbb{P} * \dot{\mathbb{Q}}\ \mathcal{A} \in M$ and let $B \in I(\mathcal{A})$ as usual satisfy $\forall x \in M \cap I(\mathcal{A})^+(|x \cap B| = \aleph_0)$. Let for simplicity of notation, $\mathbb{R} = \mathbb{P} * \dot{\mathbb{Q}}$.
Let $p_0 \in \mathbb{P}$ be an $(M, \mathbb{P}, \mathcal{A}, B)$-generic condition and let $\tilde{r} \in V^{\mathbb{P}}$ satisfy

$$p_0 \Vdash_{\mathbb{P}} \ (\tilde{r} \in \mathbb{R} \cap M \wedge \pi(\tilde{r}) \in \dot{G}_0),$$

then there is a $\mathbb{P}$-name $\tilde{p_1}$ such that the following hold:

1. $\langle p_o, \tilde{p_1} \rangle$ is $(M, \mathbb{R}, \mathcal{A}, B)$-generic and
2. $\langle p_o, \tilde{p_1} \rangle \Vdash_{\mathbb{R}} (\tilde{r} \in \dot{G})$. Here, as usual, $\dot{G}_0, \dot{G}$ denote the canonical names for generic filters on $\mathbb{P}$, and $\mathbb{R}$ respectively.

Note: We have abused notation here and written $\tilde{r}$ instead of $e(\tilde{r})$, in an $\mathbb{R}$-forcing formula (where $e : \mathbb{P} \longrightarrow \mathbb{R}$ is the canonical complete embedding as usual).

Proof. This is proved exactly like the "properness extension lemma" for two-step iterations (Lemma 2.3.4). □

Lemma 3.2.6. Strong tightness extension lemma (general version)
Let $\mathcal{A}$ be a tight almost disjoint family, and let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \gamma, \beta < \gamma \rangle$ be a countable support iteration of proper forcings such that

$$\forall \beta < \gamma \ (1_{\mathbb{P}_\beta} \Vdash_\beta (\dot{\mathbb{Q}}_\beta \text{ strongly preserves the tightness of } \mathcal{A})),$$

let $B \in I(\mathcal{A})$ and let $M \prec H(\lambda)$ (λ large enough and regular i.e. $\lambda > 2^{|\mathbb{P}_\gamma|}$) be countable with $\mathcal{A}, \mathbb{P}_\gamma \in M$ and assume that $\forall x \in M \cap I(\mathcal{A})^+(|x \cap B| = \aleph_0)$, then we have that:
For each $\gamma_0 \in \gamma \cap M$ and each $\mathbb{P}_{\gamma_0}$-name $\tilde{p}_0 \in V^{\mathbb{P}_{\gamma_0}}$ and each $(M, \mathbb{P}_{\gamma_0}, \mathcal{A}, B)$-generic condition $q_0 \in \mathbb{P}_{\gamma_0}$, such that $q_0 \Vdash_{\gamma_0} (\tilde{p}_0 \in \mathbb{P}_\gamma \cap M \wedge \tilde{p}_0 \restriction \gamma_0 \in \dot{G}_{\gamma_0})$ there is an $(M, \mathbb{P}_\gamma, \mathcal{A}, B)$-generic condition $q \in \mathbb{P}_\gamma$, satisfying $q \restriction \gamma_0 = q_0$ and $q \Vdash_\gamma \tilde{p}_0 \in \dot{G}_\gamma$. (Note we have abused notation here and written $\tilde{p}_0$ instead of $e_{\gamma,\gamma_0}(\tilde{p}_0)$)

Proof. The proof is via induction on the length of the iteration γ.
Fix $M, \gamma_0, \tilde{p}_0, q_0$ as in the statement of the lemma

The case when γ is a successor is dealt with using Lemma 3.2.5 exactly as in the proof of CASE 1 of Lemma 2.3.5.

Assume now that γ is a limit ordinal:

Choose a strictly increasing sequence $(\gamma_n)_{n\in\omega} \subseteq M \cap \gamma$ such that $M \cap \gamma \subseteq \bigcup_{n\in\omega}(\gamma_n)$ satisfying $\gamma_0 = \gamma_0$. Fix an enumeration $(D_n)_{n\in\omega}$ of all dense "open" subsets of $\mathbb{P}_\gamma$ that are in M and also fix an enumeration $(\dot{Z}_n)_{n\in\omega}$ of all $\mathbb{P}_\gamma$-names for elements of $I(\mathcal{A})^+$ which lie in M

$$\text{(i.e. more precisely, } \{\dot{Z}_n \mid n \in \omega\} = \{\tau \in V^{\mathbb{P}_\gamma} \cap M \mid 1_{\mathbb{P}_\gamma} \Vdash_\gamma \tau \in I(\mathcal{A})^+\})$$

such that each name occurs infinitely often in the enumeration (more precisely $\{k \in \omega \mid \dot{Z}_k = \dot{Z}_n\}$ is infinite for each $n \in \omega$).

Claim. Enough to inductively construct sequences $(q_n)_{n\in\omega}, (\tilde{p}_n)_{n\in\omega}, (\dot{m}_n)_{n\in\omega}$ with the following properties (for each $n \in \mathbb{N}$):

1. $q_0 = q_0$ and $\tilde{p}_0 = \tilde{p}_0$
2. $q_n \in \mathbb{P}_{\gamma_n}$ is an $(M, \mathbb{P}_{\gamma_n}, \mathcal{A}, B)$-generic condition
3. $q_{n+1} \restriction \gamma_n = q_n$
4. $\tilde{p}_n \in V^{\mathbb{P}_{\gamma_n}}$ satisfies $q_n \Vdash_{\gamma_n} (\tilde{p}_n \in \mathbb{P}_\gamma \cap M \wedge \tilde{p}_n \restriction \gamma_n \in \dot{G}_{\gamma_n})$
5. $q_{n+1} \Vdash_{\gamma_{n+1}} (\tilde{p}_{n+1} \leq_\gamma \tilde{p}_n \wedge \tilde{p}_{n+1} \in D_n)$
6. $\dot{m}_n \in V^{\mathbb{P}_{\gamma_n}}$ and $q_n \Vdash_{\gamma_n} (\dot{m}_n \in \check{\omega} \wedge \tilde{p}_n \Vdash_\gamma \dot{m}_n \in \dot{Z}_n \cap B \setminus n))$

Proof. Assuming we have such sequences, put $q = \bigcup_{n\in\omega}(q_n)$ it follows from the conditions (1) to (5) that q is $(M, \mathbb{P}_\gamma)$-generic and $q \restriction \gamma_0 = q_0$ and $q \Vdash_\gamma (\tilde{p}_0 \in \dot{G}_\gamma)$ (exactly like in the proof of CASE 2 of Lemma 2.3.5). So that we only need to show that q satsifies:

$$q \Vdash_\gamma (\forall x \in M[\dot{G}_\gamma] \cap I(\mathcal{A})^+(|x \cap B| = \aleph_0))$$

To see why this is true, firstly let G be an arbitrary generic filter that contains q and let $x \in M[G] \cap I(\mathcal{A})^+$, we need to show that $x \cap B$ is infinite. Now choose a name $\dot{Z}_n$ such that $x = (\dot{Z}_n)_G$ (This is possible by the fact that $M \prec H(\lambda)$), and choose a strictly increasing sequence of natural numbers $(n_k)_{k\in\omega}$ such that $\forall k \in \omega(\dot{Z}_{n_k} = \dot{Z}_n)$ (this is possible because we have assumed each name appears infinitely often in the enumeration). Now by (6), and the fact that $q \leq_\gamma e_{\gamma,\gamma_n}(q_n)$, we get that $q \Vdash_\gamma (\tilde{p}_n \Vdash_\gamma \dot{m}_n \in \dot{Z}_n \cap B \setminus n)$, for all $n \in \omega$, so that in particular, for each $k \in \omega$, we have,

$$q \Vdash_\gamma (\tilde{p}_{n_k} \Vdash_\gamma \dot{m}_{n_k} \in \dot{Z}_n \cap B \setminus n_k)$$

(here we used the fact that the names $\dot{Z}_n$ and $\dot{Z}_{n_k}$ are equal). Moreover, exactly as in the proof of (CASE 2) Lemma 2.3.5, we can show that $q \Vdash_\gamma \tilde{p}_n \in \dot{G}_\gamma$ for each $n \in \omega$ so that in particular $q \Vdash \tilde{p}_{n_k} \in \dot{G}_\gamma$, so that we obtain $(\tilde{p}_{n_k})_G \in G$ and hence from this it follows that $(\dot{m}_{n_k})_G \in (\dot{Z}_n)_G \cap B \setminus n_k = x \cap B \setminus n_k$, showing that $|x \cap B| = \aleph_0$. Therefore we conclude that $q \Vdash_\gamma (\forall x \in M[\dot{G}_\gamma] \cap I(\mathcal{A})^+ (|x \cap B| = \aleph_0))$, showing that q is $(M, \mathbb{P}_\gamma, \mathcal{A}, B)$-generic. Thus, the claim is proved. $\square$

Construction of such $q_n, \tilde{p}_n, \dot{m}_n$:
We prove this via induction. So assume we have already constructed such $q_n, \tilde{p}_n, \dot{m}_n$. Choose now an arbitrary $(V, \mathbb{P}_{\gamma_n})$ generic filter G_n with $q_n \in G_n$.
We first show that

$$V[G_n] \models \exists r \exists m_{n+1} (m_{n+1} \in \omega \wedge r \in \mathbb{D}_n \cap M \wedge r \restriction \gamma_n \in G_n \wedge r \leq_\gamma (\tilde{p}_n)_{G_n} \\ \wedge r \Vdash_\gamma (m_{n+1} \in \dot{Z}_{n+1} \cap B \setminus n+1))$$

To see why this is true, first note that the inductive hypothesis gives

$$q_n \Vdash_{\gamma_n} (\tilde{p}_n \in \mathbb{P}_\gamma \cap M \wedge \tilde{p}_n \restriction \gamma_n \in \dot{G}_{\gamma_n}),$$

letting $p_n = (\tilde{p}_n)_{G_n}$ we get, $p_n \in \mathbb{P}_\gamma \cap M$ and $p_n \restriction \gamma_n \in G_n$, now as in the proof of CASE (2) of Lemma 2.3.5, it is easily seen that the set $\{r \restriction \gamma_n \mid r \in D_n \wedge r \leq_\gamma p_n\}$ is dense below $p_n \restriction \gamma_n$ so that by the $(M, \mathbb{P}_{\gamma_n})$-genericity of q_n, we obtain that $M \cap \{r \restriction \gamma_n \mid r \in D_n \wedge r \leq_\gamma p_n \} \cap G_n \neq \emptyset$ (a more detailed explanation for this is available in the proof of CASE (2) of Lemma 2.3.5). Now by $M \prec H(\lambda)$, we obtain $r \in D_n \cap M$ with $r \leq_\gamma p_n$ and $r \restriction \gamma_n \in G_n$. Now let

$$W = \{m \in \omega \mid \exists r' \in \mathbb{P}_\gamma (r' \leq_\gamma r \wedge r' \restriction \gamma_n \in G_n \wedge r' \Vdash_\gamma (m \in \dot{Z}_{n+1} \setminus n+1))\}$$

Now we claim that $W \in I(\mathcal{A})^+ \cap M[G_n]$. To see why this claim is true, first note that W is definable in $H(\lambda)^{V[G_n]}$, from elements in $M[G_n]$, so that, as $M[G_n] \prec H(\lambda)^{V[G_n]}$, we obtain that $W \in M[G_n]$. To see why $W \in I(\mathcal{A})^+$, we consider the canonical projection $\pi : \mathbb{P}_\gamma \longrightarrow \mathbb{P}_{\gamma_n}$, given by $p \longrightarrow p \restriction \gamma_n$ and consider a $(V[G_n], \mathbb{P}_\gamma / G_n)$ generic H with $r \in H$ (note that this is possible because $\pi(r) = r \restriction \gamma_n \in G_n$, so that $r \in \pi^{-1}(G_n) = \mathbb{P}_\gamma / G_n$), but then, we know that this H is then also $(V, \mathbb{P}_\gamma)$ generic, so that as $1 \Vdash_\gamma \dot{Z}_{n+1} \setminus n+1 \in I(\mathcal{A})^+$, we obtain $(\dot{Z}_{n+1})_H \setminus n+1 \in I(\mathcal{A})^+ \cap V[H]$. Here we have used the fact that if some set is in $I(\mathcal{A})^+$ then removing finitely many elements from this set does not affect its membership in $I(\mathcal{A})^+$, to conclude that $1 \Vdash_\gamma \dot{Z}_{n+1} \setminus n+1 \in I(\mathcal{A})^+$. With this it is easy to see that, it is enough to show that $(\dot{Z}_{n+1})_H \setminus n+1 \subseteq W$, because if W were almost coverable by finitely many elements from $\mathcal{A}$, then so would $(\dot{Z}_{n+1})_H \setminus n+1$, thus giving rise to a contradiction. So we show precisely this, i.e. we show $(\dot{Z}_{n+1})_H \setminus n+1 \subseteq W$. So let $m \in (\dot{Z}_{n+1})_H \setminus n+1$, so via the forcing theorem, choose $r' \leq_\gamma r$ such that $r' \Vdash_\gamma (m \in \dot{Z}_{n+1} \setminus n+1)$ and $r' \in H$ so then as $H \subseteq \mathbb{P}_\gamma / G_n$, we also have that $r' \restriction \gamma_n \in G_n$, so that by definition of W, we have $m \in W$. Thus we conclude that $W \in M[G_n] \cap I(\mathcal{A})^+$. Now as q_n is $(M, \mathbb{P}_{\gamma_n}, \mathcal{A}, B)$-generic and $q_n \in G_n$, we obtain that $W \cap B$ is infinite, so there is

$m_{n+1} \in W \cap B \setminus (n+1)$, so that, yet again, by definition of W, there is $r' \leq r$ with $r' \restriction \gamma_n \in G_n$ and $r' \Vdash_\gamma (m_{n+1} \in \dot{Z}_{n+1} \setminus n+1)$. But as $m_{n+1} \in B$, we get the stronger statement $r' \Vdash_\gamma (m_{n+1} \in \dot{Z}_{n+1} \cap B \setminus n+1)$. Now as D_n is open and $r \in D_n$ and $r' \leq r$ we obtain $r' \in D_n$. Note that we have just shown that

$$\exists r'(r' \in D_n \wedge r' \restriction \gamma_n \in G_n \wedge r' \leq r \wedge r' \Vdash_\gamma (m_{n+1} \in \dot{Z}_{n+1} \cap B \setminus n+1)),$$

but it is easily seen that this statement also holds relativized to $H(\lambda)^{V[G_n]}$, i.e.

$$H(\lambda)^{V[G_n]} \models \exists r'(r' \in D_n \wedge r' \restriction \gamma_n \in G_n \wedge r' \leq r \wedge r' \Vdash_\gamma (m_{n+1} \in \dot{Z}_{n+1} \cap B \setminus n+1)),$$

so that we have

$$M[G_n] \models \exists r'(r' \in D_n \wedge r' \restriction \gamma_n \in G_n \wedge r' \leq r \wedge r' \Vdash_\gamma (m_{n+1} \in \dot{Z}_{n+1} \cap B \setminus n+1)).$$

So that there is $r' \in M[G_n]$ such that

$$(r' \in D_n \wedge r' \restriction \gamma_n \in G_n \wedge r' \leq r \wedge r' \Vdash_\gamma (m_{n+1} \in \dot{Z}_{n+1} \cap B \setminus n+1)).$$

But then this r' is also in M, because G_n contains the $(M, \mathbb{P}_{\gamma_n})$-generic condition q_n and hence $r' \in M[G_n] \cap D_n \subseteq M[G_n] \cap V = M \cap V = M$ (See part 4 of Lemma 2.2.4). So we conclude that

$$\exists r' \in M(r' \in D_n \wedge r' \restriction \gamma_n \in G_n \wedge r' \leq r \leq p_n = (\tilde{p}_n)_{G_n} \wedge r' \Vdash_\gamma (m_{n+1} \in \dot{Z}_{n+1} \cap B \setminus n+1)).$$

So essentially we have shown that:

$$\begin{aligned} V[G_n] \models \exists r \exists m_{n+1}(\ & m_{n+1} \in \omega \wedge r \in D_n \cap M \wedge r \restriction \gamma_n \in G_n \wedge r \leq_\gamma (\tilde{p}_n)_{G_n} \wedge \\ & r \Vdash_\gamma (m_{n+1} \in \dot{Z}_{n+1} \cap B \setminus n+1)), \end{aligned}$$

so that as G_n with $q_n \in G_n$ was arbitrary, we obtain

$$\begin{aligned} q_n \Vdash_{\gamma_n} \exists r \exists m_{n+1}(& m_{n+1} \in \check{\omega} \wedge r \in \mathbb{D}_n \cap M \wedge r \restriction \gamma_n \in \dot{G}_{\gamma_n} \wedge r \leq_\gamma \tilde{p}_n \wedge \\ & r \Vdash_\gamma (m_{n+1} \in \dot{Z}_{n+1} \cap B \setminus n+1)). \end{aligned}$$

Now using existential completeness of $\Vdash_{\gamma_n}$, let $\tilde{p}_{n+1}$ and $\dot{m}_{n+1}$ be names that are forced by q_n to have all the properties of r and m_{n+1}, respectively, so that we have

$$\begin{aligned} q_n \Vdash_{\gamma_n} (& \tilde{p}_{n+1} \in M \wedge \tilde{p}_{n+1} \in D_n \wedge \tilde{p}_{n+1} \restriction \gamma_n \in \dot{G}_{\gamma_n} \wedge \tilde{p}_{n+1} \leq \tilde{p}_n \wedge \\ & \tilde{p}_{n+1} \Vdash_\gamma (\dot{m}_{n+1} \in \dot{Z}_{n+1} \cap B \setminus n+1)). \end{aligned}$$

Now as $\gamma_{n+1} < \gamma$, we use the inductive hypothesis applied to γ_n, γ_{n+1} and the name $\tilde{p}_{n+1} \restriction \gamma_{n+1}$ to obtain an $(M, \mathbb{P}_{\gamma_{n+1}}, \mathcal{A}, B)$-generic condition $q_{n+1} \in \mathbb{P}_{\gamma_{n+1}}$ satisfying $q_{n+1} \restriction \gamma_n = q_n$ and $q_{n+1} \Vdash_{\gamma_{n+1}} (\tilde{p}_{n+1} \restriction \gamma_{n+1} \in \dot{G}_{\gamma_{n+1}})$. So we have obtained

$q_{n+1}, \tilde{p}_{n+1}, \dot{m}_{n+1}$.
Note: If we were being extremely rigorous, we would have to take our names to be translates of $\tilde{p}_{n+1}$ and $\dot{m}_{n+1}$ via the complete embedding $e_{\gamma_{n+1},\gamma_n}$, so that we actually obtain $\mathbb{P}_{\gamma_{n+1}}$-names as in the inductive hypothesis, instead of just $\mathbb{P}_{\gamma_n}$-names.

Claim. $q_{n+1},\ \tilde{p}_{n+1},\ m_{n+1}$ satisfy the required properties.

Proof. This is proved exactly like in the proof of Lemma 2.3.5(CASE (2)), using the fact that $q_{n+1} \leq_{\gamma_{n+1}} e_{\gamma_{n+1},\gamma_n}(q_n)$ and then applying Lemma 2.3.6. $\square$

Thus, we are done with the proof of Lemma 3.2.6 $\square$

Remark. Since we proved Lemma 2.3.5 in full detail, explicitly writing down complete embeddings and translation of names via these complete embeddings when going from the forcing relation of $\mathbb{P}_\alpha$ to the forcing relation of $\mathbb{P}_{\alpha'}$ for $\alpha < \alpha'$, we have not made this explicit in the above proof, but is to be understood.

Corollary 3.2.7. Let $\mathcal{A}$ be a tight almost disjoint family and let $\langle \mathbb{P}_\alpha, \dot{Q}_\beta \mid \alpha \leq \gamma \wedge b < \gamma \rangle$ be a countable support iteration of proper forcings that satisfy $\forall \beta < \gamma (1_\beta \Vdash_\beta (\dot{Q}_\beta$ strongly preserves the tightness of $\mathcal{A}))$, then $\mathbb{P}_\gamma$ strongly preserves the tightness of $\mathcal{A}$.

Proof. let $M \prec H(\lambda)$ (λ sufficiently large) be countable with $\mathbb{P}_\gamma, \mathcal{A} \in M$ and let $p \in M \cap \mathbb{P}_\gamma$ and let $B \in I(\mathcal{A})$ satisfy $\forall x \in M \cap I(\mathcal{A})^+(|x \cap B| = \aleph_0)$. We need to find an $(M, \mathbb{P}_\gamma, \mathcal{A}, B)$-generic condition $q \leq_\gamma p$. Towards finding such a q, take $\gamma_0 = 0 \in M$ and $\gamma = \gamma$ and $\tilde{p}_0 = \check{p}$ in Lemma 3.2.6, to obtain an $(M, \mathbb{P}_\gamma, \mathcal{A}, B)$-generic $q \in \mathbb{P}_\gamma$ satisfying $q \Vdash_\gamma \check{p} \in \dot{G}_\gamma$. So that the seperability of $\mathbb{P}_\gamma$ implies that $q \leq_\gamma p$. Thus, we are done. $\square$

4 Tight Eventually Different Families

In this chapter, we introduce the notion of eventually different families and also introduce the notion of tightness for these families and prove an important preservation theorem similar to Chapter 3 (see 4.2.6). This chapter is based mainly on [4]

4.1 Tight eventually different families and their consistency

Definition 4.1.1.

1. We define for a natural number $n \in \omega$, the set $\omega^n = \{s \mid s : n \longrightarrow \omega\}$.
2. $\omega^{<\omega} = \bigcup_{n\in\omega} \omega^n$.
3. $\omega^\omega = \{f \mid f : \omega \longrightarrow \omega\}$.

Definition 4.1.2. $T \subseteq \omega^{<\omega}$ is called a tree if T is closed under initial segments. More precisely,

$$\forall t \forall s (t \in T \wedge s \subseteq t \implies s \in T).$$

Definition 4.1.3. We say a tree $T \subseteq \omega^{<\omega}$ is a pruned tree if

$$\forall t \in T \exists s \in T (t \subsetneq s).$$

Note: From now on, when we say "T is a tree" we actually mean "T is a pruned tree", i.e. from here on, we only consider pruned trees.

Definition 4.1.4. We say a real $g \in \omega^\omega$ is a branch of a tree T if we have $\forall n \in \omega (g \restriction n \in T)$. We define $[T] = \{g \in \omega^\omega \mid g \text{ is a branch of } T\}$.

Note: If T is a pruned tree, then every node $t \in T$ sits in a branch of T, more precisely, $\forall t \in T (\exists g \in [T](t \subseteq g))$.

Definition 4.1.5. For a tree T and a node $t \in T$, we define the tree T_t as

$$T_t = \{s \in T \mid s \subseteq t \ \vee t \subseteq s\}.$$

Remark.

1. If T is a pruned tree then so is T_t for each $t \in T$.
2. Sometimes we use the notation $(T)_t$, instead of T_t, i.e. both notations are used, even though they denote the same object.

Definition 4.1.6. For $f, g \in \omega^\omega$, we say f and g are eventually different (denoted by $f \neq^* g$), if there is $\exists n \in \omega \forall m \geq n(f(m) \neq g(m))$.

Definition 4.1.7. We say a family of reals $\mathcal{E} \subseteq \omega^\omega$ is an eventually different family if $\forall f, g \in \mathcal{E}(f \neq g \implies f \neq^* g)$. We say such a family $\mathcal{E}$ is maximal if $\forall f \in \omega^\omega \exists g \in \mathcal{E}(|\{n \in \omega \mid f(n) = g(n)\}| = \aleph_0)$ (i.e. as in the case of maximality of almost disjoint families, $\mathcal{E}$ cannot be extended further without breaking eventual difference).

As in the case of MAD families, we have a cardinal characteristic associated with eventually different families defined as

$$\mathfrak{a}_e = min\{|\mathcal{E}| \mid \mathcal{E} \text{ is a maximal eventually different family}\}$$

Definition 4.1.8. We say a real $g \in \omega^\omega$ densely diagonalizes a tree $T \subseteq \omega^{<\omega}$ if the following holds:
$\forall t \in T \exists t' \in T \exists k \in \omega(t' \supsetneq t \wedge k \in dom(t') \setminus dom(t)(g(k) = t'(k)))$.

Definition 4.1.9. (The tree ideal and positive trees generated by an eventually different family):
Let $\mathcal{E} \subseteq \omega^\omega$ be an eventually different family, we then define the tree ideal generated by $\mathcal{E}$ as

$$I_T(\mathcal{E}) = \{T \subseteq \omega^{<\omega} \mid T \text{ is a pruned tree and } \exists t \in T \exists X \in [\mathcal{E}]^{<\omega}(\bigcup T_t \subseteq^* \bigcup X\}.$$

The positive sets of $\mathcal{E}$ is the set,

$$I_T(\mathcal{E})^+ = \{T \subseteq \omega^{<\omega} \mid T \text{ is a pruned tree and } T \notin I_T(\mathcal{E})\}.$$

Note: The tree ideal is not an ideal in the usual sense of the word ideal, but we used the term “ideal” to stress that the role of the tree ideal is similar to the role of the ideal $I(\mathcal{A})$ in the case of an almost disjoint family $\mathcal{A}$.

Definition 4.1.10. We say an eventually different family $\mathcal{E} \subseteq \omega^\omega$ is maximal if it is maximal with respect to inclusion. More precisely, if

$$\forall g \in \omega^\omega (\mathcal{E} \cup \{g\} \text{ is not eventually different})$$

Definition 4.1.11. We say a pruned tree $T \subseteq \omega^{<\omega}$ is " densely diagonalized by a real $g \in \omega^\omega$ " if

$$\forall t \in T \exists t' \in T \exists k \in \omega (t' \supsetneq t \wedge k \in dom(t') \setminus dom(t) \wedge t'(k) = g(k)),$$

i.e. every node of T sits in a branch of T, that is infinitely often equal to the real g.

Definition 4.1.12. We say an eventually different family $\mathcal{E}$ is tight if

$$\forall J \in [I_T(\mathcal{E})^+]^{\leq\omega} \exists g \in \mathcal{E} \ (\forall T \in J (g \text{ densely diagonalizes } T))$$

Lemma 4.1.13. Let $\mathcal{E}$ be an eventually different family, then we have:
$\mathcal{E}$ is tight $\implies$ $\mathcal{E}$ is maximal

Proof. Assume the hypothesis that $\mathcal{E}$ is tight. Now let $g \in \omega^\omega$ be arbitrary, we show that $\mathcal{E} \cup \{g\}$ is NOT an eventually different family. So firstly, suppose $\mathcal{E} \cup \{g\}$ is an eventually different family (towards a contradiction) and now define the tree $T_g = \{g \upharpoonright n \mid n \in \omega\}$ and claim that $T_g \in I_T(\mathcal{E})^+$. To see why, if this were not the case then let $E = \{f_1, ..., f_n\} \in [\mathcal{E}]^{<\omega}$ be such that $g \subseteq^* f_1 \cup f_2 \cup .... \cup f_n$ (Note that it is easily seen that the tree T_g has the property that for each $t \in T_g$, $\bigcup (T_g)_t = \bigcup T_g = g$), then choose an $n_0 \in \omega$ such that

$$\forall n \geq n_0 \exists i_n \in \{1, 2, ..., n\} (\langle n, g(n) \rangle \in f_{i_n}).$$

Now one can easily choose a strictly increasing sequence of natural numbers $(n_k)_{k\in\omega}$ such that $n_k \geq n_0$ for each $k \in \omega$ and $\{i_{n_k} \mid k \in \omega\} = \{r\}$ and then we have for all $k \in \omega$, $\langle n_k, g(n_k) \rangle \in f_{i_{n_k}} = f_r$. So that one obtains that $\forall k \in \omega (f_r(n_k) = g(n_k))$, which contradicts $\mathcal{E} \cup \{g\}$ is an eventually different family. Thus we conclude that $g \in I_T(\mathcal{E})^+$. Now, by tightness, choose $h \in \mathcal{E}$ such that h densely diagonalizes T_g. But then this just means that $\exists^\infty n \in \omega (h(n) = g(n))$, yet again arriving at a contradiction. Thus, we conclude that $\mathcal{E}$ is in fact maximal. $\square$

Our next goal is to answer the consistency of the existence of tight eventually different families. Towards this, we introduce a σ - centered forcing notion for eventually different families and using this forcing notion, we show that under $MA(\sigma$- centered) every eventually different family of cardinality strictly smaller than the continuum can be extended to a tight family eventually different family (for a definition of σ-centered and $MA(\sigma$- centered) see [10]).

Definition 4.1.14. We define $Fn(\omega,\omega) = \{p \mid \exists A \in [\omega]^{<\omega}(p : A \longrightarrow \omega)\}$ = the set of all finite partial functions from ω into ω.

Definition 4.1.15. For any eventually eventually different family $\mathcal{E} \subseteq \omega^\omega$ we define the poset $\mathbb{P}(\mathcal{E}) = \{\langle s, E\rangle \mid s \in Fn(\omega,\omega) \wedge E \in [\mathcal{E}]^{<\omega}\}$ and we define the ordering on $\mathbb{P}(\mathcal{E})$ as follows:

$$\langle s, E\rangle \leq \langle s', E'\rangle \iff s \supseteq s' \wedge E \supseteq E' \wedge \forall k \in dom(s) \setminus dom(s') \forall f \in E'(s(k) \neq f(k))$$

Note: It is easily seen that the above set, along with the ordering, is indeed a forcing notion.

Lemma 4.1.16. Let $\mathcal{E}$ be an eventually different family, then the poset $\mathbb{P}(\mathcal{E})$ defined above is σ-centered and moreover the following sets are dense in $\mathbb{P}(\mathcal{E})$:

1. For each $n \in \omega$, the set
$$\{\langle s, E\rangle \in \mathbb{P}(\mathcal{E}) \mid n \in dom(s)\};$$
2. For each $f \in \mathcal{E}$, the set
$$\{\langle s, E\rangle \in \mathbb{P}(\mathcal{E}) \mid f \in E\};$$
3. For each $T \in I_T(\mathcal{E})^+$ and each $t \in T$, the set
$$\{\langle s, E\rangle \in \mathbb{P}(\mathcal{E}) \mid \exists t' \in T \exists k \in \omega(t' \supseteq t \wedge k \in dom(t') \setminus dom(t) \wedge s(k) = t'(k))\}.$$

Proof. For σ - centeredness, we just note that $Fn(\omega,\omega)$ is countable and that if for a fixed $s \in Fn(\omega,\omega)$, $\langle s, E_1\rangle, \langle s, E_2\rangle \in \mathbb{P}(\mathcal{E})$, then we have $\langle s, E_1 \cup E_2\rangle \in \mathbb{P}(\mathcal{E})$ and lies below both the chosen elements, showing that for any finitely many elements in the collection $C_s = \{\langle t, E\rangle \in \mathbb{P}(\mathcal{E}) \mid t = s\}$, there is a single element in the poset that lies below all of them (via an easy induction).
Now we prove the density of the sets one by one as follows:

1. Let $n \in \omega$ be arbitrary and assume that $\langle s, E\rangle \in \mathbb{P}(\mathcal{E})$ is also arbitrary and without loss of generality, let $n \notin dom(s)$. Now let $E = \{f_1, ...f_k\}$ (as E is finite) and let
$m \in \omega$ be such that $f_1(n) \neq m$, $f_2(n) \neq m$,..., $f_k(n) \neq m$ and put $s' = s \cup \{\langle n, m\rangle\}$, then $\langle s', E\rangle \in \{\langle s, E\rangle \in \mathbb{P}(\mathcal{E}) \mid n \in dom(s)\}$ and
$\langle s', E\rangle \leq \langle s, E\rangle$. Thus (1) is proved
2. Trivial

3. Firstly for notational convenience, we let for each $T \in I_T(\mathcal{E})^+$ and each $t \in T$,

$$D(T,t) = \{\langle s, E\rangle \in \mathbb{P}(\mathcal{E}) \mid \exists t' \in T \exists k \in \omega(t' \supseteq t \wedge k \in dom(t') \setminus dom(t) \wedge s(k) = t'(k))\}$$

and note that for any $t_1, t_2 \in T$ a trivial computation shows that

$$t_1 \subseteq t_2 \implies D(T, t_2) \subseteq D(T, t_1).$$

Now fix a $T \in I_T(\mathcal{E})^+$ and a $t \in T$ and let $\langle s, E\rangle \in \mathbb{P}(\mathcal{E})$ be arbitrary. Now, as T is a pruned tree and $|dom(s)|$ is finite and by the fact that $D(T, t_2) \subseteq D(T, t_1)$ whenever $t_1 \subseteq t_2$, we can, without loss of generality, assume that $dom(t) \supseteq dom(s)$.
Now we claim that

$$\exists t' \in T \exists k \in \omega(t' \supseteq t \wedge k \in dom(t') \setminus dom(t) \wedge \forall f \in E(f(k) \neq t'(k))).$$

To see why, suppose this were not true, then we would have

$$\forall t' \in T \forall k \in \omega(t' \supseteq t \wedge k \in dom(t') \setminus dom(t) \implies \exists f \in E(f(k) = t'(k))),$$

but then, this just means that

$$\forall t' \in T \forall k \in \omega(t' \supseteq t \wedge k \in dom(t') \setminus dom(t) \implies \exists f \in E(\langle k, t'(k)\rangle \in f)),$$

from which it easily follows that $\bigcup T_t \setminus t \subseteq \bigcup_{f \in E}(f) = \bigcup E$, but as t is finite this is a contradiction to the fact that $T \in I_T(\mathcal{E})^+$, so that the claim follows. Now choose such a $t' \in T$ and such a $k \in dom(t') \setminus dom(t)$ satisfying $\forall f \in E(f(k) \neq t'(k))$, as $k \notin dom(t) \supseteq dom(s)$, we have $k \notin dom(s)$, and hence $s' = s \cup \langle k, t'(k)\rangle$ is a well defined partial function. Then it is clear that $\langle s', E\rangle \leq \langle s, E\rangle$, and also, as $t'(k) = s'(k)$. We have essentially shown that $t' \in T \wedge t' \supseteq t \wedge k \in dom(t') \setminus dom(t) \wedge s'(k) = t'(k)$, which gives $\langle s', E\rangle \in D(T, t)$. Thus (3) is also proved.

□

With this lemma, we are now ready to prove the main result of this section:

Theorem 4.1.17. *Assume $MA(\sigma$ - centered), then every eventually different family of size strictly less than continuum is contained in a tight eventually different family.*

Proof. Let $\mathcal{E}_0$ be an eventually different family of size strictly less than the continuum. Now let $\{T_\alpha \mid \alpha < 2^{\aleph_0}\}$ be an enumeration of all ω-sequences of pruned subtrees of $\omega^{<\omega}$ (so that $T_\alpha(n)$ denotes the n^{th} tree in the ω-sequence T_α)

Claim. Enough to construct inductively a family $(\mathcal{E}_\alpha)_{\alpha<2^{\aleph_0}}$ satisfying the following conditions:

(a) $\forall\alpha < \beta < 2^{\aleph_0}(\mathcal{E}_\alpha \subseteq \mathcal{E}_\beta)$ and $\mathcal{E}_0 = \mathcal{E}_0$

(b) $\forall\alpha < 2^{\aleph_0}(|\mathcal{E}_\alpha| < 2^{\aleph_0} \wedge \mathcal{E}_\alpha$ is an eventually different family)

(c) $\forall\alpha < 2^{\aleph_0}(T_\alpha \in [I_T(\mathcal{E}_\alpha)^+]^{\leq\omega} \to \exists g \in \mathcal{E}_{\alpha+1}(\forall n \in \omega(g$ densely diagonalizes $T_\alpha(n))))$

Proof. To see why this is true put $\mathcal{E} = \bigcup_{\alpha<2^{\aleph_0}}(\mathcal{E}_\alpha)$, then it is easy to see that

$$I_T(\mathcal{E}) = \bigcup_{\alpha<2^{\aleph_0}}(I_T(\mathcal{E}_\alpha)),$$

taking complements this gives that $I_T(\mathcal{E})^+ = \bigcap_{\alpha<2^{\aleph_0}}(I_T(\mathcal{E}_\alpha)^+)$. So that for any $\alpha < 2^{\aleph_0}$, we get that, if $T_\alpha \in [I_T(\mathcal{E})^+]^{\leq\omega}$, then in particular $T_\alpha \in [I_T(\mathcal{E}_\alpha)^+]^{\leq\omega}$, so that by property (c), we obtain that there is a $g \in \mathcal{E}_{\alpha+1}$ such that
$\forall n \in \omega(g$ densely diagonalizes $T_\alpha(n))$. This g is also obviously in $\mathcal{E}$. This shows that $\mathcal{E}$ is a tight eventually different family.

□

Construction of such a family:
Put $\mathcal{E}_0 = \mathcal{E}_0$ and and assume we have constructed $\mathcal{E}_\alpha$ for all $\alpha < \beta$, where $\beta < 2^{\aleph_0}$. Now we construct $\mathcal{E}_\beta$ as follows:
If β is a limit ordinal just put $\mathcal{E}_\beta = \bigcup_{\alpha<\beta}(\mathcal{E}_\alpha)$.
Now assume $\beta = \beta_0 + 1$, i.e. we assume β is a successor ordinal.
If it is the case that $T_{\beta_0} \notin [I_T(\mathcal{E}_{\beta_0})^+]^{\leq\omega}$, then we consider the family of dense subsets of $\mathbb{P}(\mathcal{E}_{\beta_0})$, $\partial = \{D_n \mid n \in \omega\} \cup \{H_f \mid f \in \mathcal{E}_{\beta_0}\}$, where
$D_n = \{\langle s, E\rangle \in \mathbb{P}(\mathcal{E}_{\beta_0}) \mid n \in dom(s)\}$ and $H_f = \{\langle s, E\rangle \in \mathbb{P}(\mathcal{E}_{\beta_0}) \mid f \in E\}$. Now by inductive hypothesis we know that $|\mathcal{E}_{\beta_0}| < 2^{\aleph_0}$, so that $|\partial| < 2^{\aleph_0}$ and hence (as $\mathbb{P}(\mathcal{E}_{\beta_0})$ is σ- centered) by $MA(\sigma$- centered), we get that there is a filter $G \subseteq \mathbb{P}(\mathcal{E}_{\beta_0})$ such that $\forall D \in \partial(D \cap G \neq \emptyset)$. Let now

$$g = \bigcup\{s \mid \exists E \in [\mathcal{E}_{\beta_0}]^{<\omega}(\langle s, E\rangle \in G)\}.$$

Then an easy argument using the fact that $\forall n \in \omega(D_n \cap G \neq \emptyset)$ and the fact that any pair of elements in G is comparable in G shows that $g \in \omega^\omega$. Moreover if $f \in \mathcal{E}_{\beta_0}$, then as $H_f \cap G \neq \emptyset$, pick $\langle s, E\rangle \in H_f \cap G$, and put $n_0 = max(dom(s)) + 1$, then for any $m > n_0$, we have $g(m) = s'(m)$ for some $\langle s', E'\rangle \in G$, now choosing an $\langle s'', E''\rangle \in G$ such that $\langle s'', E''\rangle \leq \langle s', E'\rangle$, $\langle s, E\rangle$, we get $m \in dom(s'')\backslash dom(s)$ and hence $g(m) = s''(m) \neq f(m)$. Therefore we have shown that for $\forall m > n_0(f(m) \neq g(m))$ and hence $\forall f \in \mathcal{E}_{\beta_0}(f \neq^* g)$, so that if we let $\mathcal{E}_\beta = \mathcal{E}_{\beta_0+1} = \mathcal{E}_{\beta_0} \cup \{g\}$, then $\mathcal{E}_\beta$, has the required properties.

If it is the case that $T_{\beta_0} \in [I_T(\mathcal{E}_{\beta_0})^+]^{\leq\omega}$, then letting

$$D(T,t) = \{\langle s,E\rangle \in \mathbb{P}(\mathcal{E}_{\beta_0}) \mid \exists t' \in T \exists k \in \omega(t' \supseteq t \wedge k \in dom(t') \setminus dom(t) \wedge s(k) = t'(k))\}$$

(as in the proof of Lemma 4.1.16), we consider the family of dense sets (dense in $\mathbb{P}(\mathcal{E}_{\beta_0})$ of course) $\partial' = \partial \cup \{D(T_{\beta_0}(n),t) \mid n \in \omega \wedge t \in T_{\beta_0}(n)\}$, then as the latter set in the union is countable, we have yet again by $MA(\sigma$- centered), that there is a filter G that has a non empty intersection with each element of ∂'. Then, as before, G defines a real $g \in \omega^\omega$ such that $\forall f \in \mathcal{E}_{\beta_0}(f \neq^* g)$. But in this case as G intersects extra dense sets apart from the ones in ∂, we can easily show g densely diagonalizes $T_{\beta_0}(n)$ for each $n \in \omega$ as follows:
Let $n \in \omega$ be arbitrary, and let $t \in T_{\beta_0}(n)$, now as $D(T_{\beta_0}(n),t) \cap G \neq \emptyset$, choose $\langle s,E\rangle \in D(T_{\beta_0}(n),t) \cap G$, then we have by the definition of $D(T,t)$ that there is a $t' \in T_{\beta_0}(n)$ such that $t' \supseteq t$ and there is a $k \in dom(t') \setminus dom(t)$ such that $s(k) = t'(k)$, but $s(k) = g(k)$, so that what we have just shown is that

$$\forall t \in T_{\beta_0}(n) \exists t' \in T_{\beta_0}(n) \exists k \in \omega(t' \supseteq t \wedge k \in dom(t') \setminus dom(t) \wedge g(k) = t'(k)),$$

this shows that g densely diagonalizes $T_{\beta_0}(n)$ (for each $n \in \omega$). Therefore letting $\mathcal{E}_{\beta_0+1} = \mathcal{E}_\beta = \mathcal{E}_{\beta_0} \cup \{g\}$, we conclude the construction.

$\square$

4.2 Preserving tight eventually different families

In this section, we introduce the notion of "strong preservation of tightness of an eventually different family " for a forcing poset and show that this property is preserved under countable support iterations of proper forcings.

Definition 4.2.1. Let $\mathcal{E}$ be a tight eventually different family, let $\mathbb{P}$ be a proper forcing notion and let $\lambda > 2^{|\mathbb{P}|}$ be a sufficiently large regular cardinal. Let $M \prec H(\lambda)$ be countable with $\mathbb{P}$,$\mathcal{E} \in M$ and assume $g \in \mathcal{E}$ satisfies $\forall T \in I_T(\mathcal{E})^+ \cap M$ (g densely diagonalizes T), then we say that a condition $q \in \mathbb{P}$ is an $(M,\mathbb{P},\mathcal{E},g)$-generic condition iff

$$q \text{ is } (M,\mathbb{P})\text{-generic AND } q \Vdash_{\mathbb{P}} [\forall T \in I_T(\mathcal{E})^+ \cap M[\dot{G}](\check{g} \text{ densely diagonalizes } T)]$$

Definition 4.2.2. Let $\mathcal{E} \subseteq \omega^\omega$ be a tight eventually different family, then we say that a proper forcing poset $\mathbb{P}$ strongly preserves the tightness of $\mathcal{E}$ iff the following holds:
" for each sufficiently large regular cardinal λ (i.e. as usual $\forall\lambda > 2^{|\mathbb{P}|}$) and each countable $M \prec H(\lambda)$ with $\mathbb{P},\mathcal{E} \in M$ and each $g \in \mathcal{E}$ satisfying $\forall T \in I_T(\mathcal{E})^+ \cap M(g$ densely diagonalizes $T)$ and each $p \in M \cap \mathbb{P}$, there is a $q \leq p$ such that q is $(M,\mathbb{P},\mathcal{E},g)$-generic"

Observations

The following observations are analogous to the case of MAD families from Chapter 3, and the proofs are exactly the same:

1. for each λ sufficiently large, and each $A \in H(\lambda)$, the set

$$D(\lambda, A) = \{q \in \mathbb{P} \mid \exists M \exists g(g \in \mathcal{E} \wedge M \prec H(\lambda) \wedge |M| \leq \aleph_0 \wedge \mathcal{E}, \mathbb{P}, A \in M \wedge \\ \forall T \in M \cap I_T(\mathcal{E})^+(g \text{ densely diagonalizes } T) \wedge q \text{ is } (M, \mathbb{P}, \mathcal{E}, g)\text{-generic}\}$$

   is dense in $\mathbb{P}$, whenever $\mathbb{P}$ strongly preserves the tightness of $\mathcal{E}$.

2. If $\mathcal{E}$ is a tight eventually different family and if $\mathbb{P}$ strongly preserves the tightness of $\mathcal{E}$, then $1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\check{\mathcal{E}}$ is tight eventually different)

Moreover, we can prove the following lemmas for two-step iterations, analogous to what we did in Chapter 3:

Lemma 4.2.3. Let $\mathcal{E}$ be a tight eventually different family then:

1. Assume that $\mathbb{P}$ is a proper forcing that strongly preserves the tightness of $\mathcal{E}$ and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$ - name for a forcing poset such that

$$1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\dot{\mathbb{Q}} \text{ is a proper forcing that strongly preserves the tightness of } \mathcal{E}).$$

   Then we have that the two-step product $\mathbb{P} * \dot{\mathbb{Q}}$ strongly preserves the tightness of $\mathcal{E}$.

2. If $M \prec H(\lambda)$ (for sufficiently large λ) is countable with $\mathcal{E}$, $\mathbb{P} * \dot{\mathbb{Q}} \in M$ and $g \in \mathcal{E}$ satisfies $\forall T \in M \cap I_T(\mathcal{E})^+(g$ densely diagonalizes $T)$, then for each $p \in \mathbb{P}$ and each $\dot{q} \in V^{\mathbb{P}}$, we have,

$$\langle p, \dot{q} \rangle \text{ is } (M, \mathbb{P} * \dot{\mathbb{Q}}, \mathcal{E}, g)\text{-generic} \iff p \text{ is } (M, \mathbb{P}, \mathcal{E}, g)\text{-generic and} \\ p \Vdash_{\mathbb{P}} (\dot{q} \text{ is } (M[\dot{G}], \dot{\mathbb{Q}}, \mathcal{E}, g)\text{-generic})$$

Lemma 4.2.4. Strong tightness extension lemma (for eventually different families), for two-step products

Let $\mathcal{E}$ be a tight eventually different family and let $\mathbb{P}$ be a forcing notion that is proper and strongly preserves the tightness of $\mathcal{E}$, and assume that $\dot{\mathbb{Q}}$ is a name for a forcing notion such that

$$1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\dot{\mathbb{Q}} \text{ is proper and strongly preserves the tightness of } \mathcal{A})$$

and let $M \prec H(\lambda)$ (for sufficiently large regular λ) be countable with $\mathbb{P} * \dot{\mathbb{Q}}$, $\mathcal{E} \in M$ and let $g \in \mathcal{E}$ as usual satisfy $\forall T \in M \cap I_T(\mathcal{E})^+(g$ densely diagonalizes T). Let,

for simplicity of notation, $\mathbb{R} = \mathbb{P} * \dot{\mathbb{Q}}$.
Let $p_0 \in \mathbb{P}$ be an $(M, \mathbb{P}, \mathcal{E}, g)$-generic condition and let $\tilde{r} \in V^{\mathbb{P}}$ satisfy

$$p_0 \Vdash_{\mathbb{P}} (\tilde{r} \in \mathbb{R} \cap M \wedge \pi(\tilde{r}) \in \dot{G}_0),$$

then there is a $\mathbb{P}$ name $\tilde{p_1}$ such that the following hold:

1. $\langle p_0, \tilde{p_1} \rangle$ is $(M, \mathbb{R}, \mathcal{E}, g)$-generic and
2. $\langle p_0, \tilde{p_1} \rangle \Vdash_{\mathbb{R}} \tilde{r} \in \dot{G}$ (here as usual, $\dot{G}_0, \dot{G}$ denote the canonical names for generic filters on $\mathbb{P}$, and $\mathbb{R}$ respectively).

A NOTE ON NOTATION:

Let $F : \omega^{<\omega} \longrightarrow \omega$ denote the Gödel numbering/coding, and let $Gd : \omega \longrightarrow \omega^{<\omega}$ denote its inverse (for more on the Gödel numbering, see [2]). Then it is easily seen that Gd is an enumeration of all finite ω-sequences that satisfy the nice additional property that shorter sequences occur before longer sequences, in the enumeration. Now, if $T \subseteq \omega^{<\omega}$ is a pruned tree, we can push the enumeration forward onto T as follows:
Firstly consider the map $g_T : \omega \longrightarrow F(T)$ given by $g_T(0) = min(F(T))$ and $g_T(n+1) = min(F(T) \setminus \{g_T(0), ..., g_T(n)\})$ and then define
$Gd(T) : \omega \longrightarrow T$ by $Gd(T) = Gd \circ g_T$, then clearly $Gd(T)$ is a bijective enumeration of the tree T that satisfies the property that shorter sequences occur first in the enumeration. Moreover as g_T is definable from T and F via recursion, it is easily seen that $T \longrightarrow Gd(T)$ is a definable map (under ZFC) and hence via existential completeness of the forcing relation, given a name $\dot{T}$ for a pruned tree, and a natural number n, one has a name $\tau(\dot{T}, n) = Gd(\dot{T})(n)$ that satisfies the property that

$$\text{for each generic filter } H \ ([Gd(\dot{T})(n)]_H = (\tau(\dot{T}, n))_H = Gd(\dot{T}_H)(n)).$$

We use this name in the proof of the next lemma, which enables us to talk about nodes of trees in the generic extensions in a uniform fashion.

Lemma 4.2.5. Strong tightness extension lemma (MED version) Let $\mathcal{E}$ be a tight eventually different family, and let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \gamma, \beta < \gamma \rangle$ be a countable support iteration of proper forcings such that

$$\forall \beta < \gamma \ (1_{\mathbb{P}_\beta} \Vdash_\beta (\dot{\mathbb{Q}}_\beta \text{ strongly preserves the tightness of } \mathcal{E})),$$

let $g \in \mathcal{E}$ and let $M \prec H(\lambda)$ (λ large enough and regular) be countable with $\mathcal{E}, \mathbb{P}_\gamma \in M$ and assume that $\forall T \in M \cap I_T(\mathcal{E})^+ (g$ densely diagonalizes $T)$, then we have that:
For each $\gamma_0 \in \gamma \cap M$ and each $\mathbb{P}_{\gamma_0}$- name $\tilde{p}_0 \in V^{\mathbb{P}_{\gamma_0}}$ and each $(M, \mathbb{P}_{\gamma_0}, \mathcal{E}, g)$-generic condition $q_0 \in \mathbb{P}_{\gamma_0}$, such that $q_0 \Vdash_{\gamma_0} (\tilde{p}_0 \in \mathbb{P}_\gamma \cap M \wedge \tilde{p}_0 \restriction \gamma_0 \in \dot{G}_{\gamma_0})$ there is an $(M, \mathbb{P}_\gamma, \mathcal{E}, g)$-generic condition $q \in \mathbb{P}_\gamma$, satisfying $q \restriction \gamma_0 = q_0$ and $q \Vdash_\gamma \tilde{p}_0 \in \dot{G}_\gamma$.

Proof. Let M be as in the hypothesis and as in the previous preservation theorems we assume without loss of generality, that γ is a limit ordinal and choose a strictly increasing sequence $(\gamma_n)_{n\in\omega}$ that is cofinal in $\gamma\cap M$ and fix an enumeration $(D_n)_{n\in\omega}$ of all open dense subsets of $\mathbb{P}_\gamma$ that lie in M. Moreover we fix an enumeration $(\dot{T}_n)_{n\in\omega}$ of all $\mathbb{P}_\gamma$-names for elements of $I_T(\mathcal{E})^+$ that lie in M and finally fix a bijection $\psi : \omega \longrightarrow \omega \times \omega$ (all of the above are in V).

Claim. Enough to construct a quadruple of sequences $(q_n)_{n\in\omega}$, $(\tilde{p}_n)_{n\in\omega}$, $(\dot{t}_n)_{n\in\omega}$ and $(\dot{k}_n)_{n\in\omega}$ such that the following hold for each $n \in \omega$:

1. $q_0 = q_0$ and $\tilde{p}_0 = \tilde{p}_0$
2. q_n is an $(M, \mathbb{P}_{\gamma_n}, \mathcal{E}, g)$-generic condition
3. $q_{n+1} \restriction \gamma_n = q_n$
4. $\tilde{p}_n \in V^{\mathbb{P}_{\gamma_n}}$ and satisfies $q_n \Vdash_{\gamma_n} (\tilde{p}_n \in \mathbb{P}_\gamma \cap M \wedge \tilde{p}_n \restriction \gamma_n \in \dot{G}_{\gamma_n})$
5. $q_{n+1} \Vdash_{\gamma_{n+1}} (\tilde{p}_{n+1} \leq_\gamma \tilde{p}_n \wedge \tilde{p}_{n+1} \in D_n)$
6. $\dot{k}_n, \dot{t}_n \in V^{\mathbb{P}_{\gamma_n}}$ and they satisfy

$$q_n \Vdash_{\gamma_n} [\dot{t}_n \in (\omega^{<\omega})\check{} \wedge \dot{k}_n \in \check{\omega} \wedge \tilde{p}_n \Vdash_\gamma (\dot{t}_n \in \dot{T}_{\psi_0(n)} \wedge \dot{t}_n \supsetneq Gd(\dot{T}_{\psi_0(n)})(\psi_1(n)) \\ \wedge \dot{k}_n \in dom(\dot{t}_n) \setminus dom(Gd(\dot{T}_{\psi_0(n)})(\psi_1(n))) \wedge \check{g}(\dot{k}_n) = \dot{t}_n(\dot{k}_n))]$$

Proof. Assume we have such a quadruple of sequences satisfying (1)-(6), and put $q = \bigcup_{n\in\omega}(q_n)$. Then, as in the previous preservation theorems, conditions (1)-(5) give that q is $(M, \mathbb{P}_\gamma)$-generic. So we only need to show that

$$q \Vdash_\gamma \forall T \in I_T(\mathcal{E})^+ \cap M[\dot{G}_\gamma](\text{g densely diagonalizes } T).$$

So towards this, let G be an arbitrary $(V, \mathbb{P}_\gamma)$ generic filter which contains q as a member. Now let $T \in M[G] \cap I_T(\mathcal{E})^+$ and $t \in T$ be arbitrary. Choose $k \in \omega$ such that $T = (\dot{T}_k)_G$ and $k' \in \omega$ such that $t = Gd(T)(k') = Gd((\dot{T}_k)_G)(k')$, and choose $n \in \omega$ such that $\langle \psi_0(n), \psi_1(n)\rangle = \langle k, k'\rangle$, so that $T = (\dot{T}_k)_G = (\dot{T}_{\psi_0(n)})_G$ and $t = Gd((\dot{T}_{\psi_0(n)})_G)(\psi_1(n)) = [Gd(\dot{T}_{\psi_0(n)})(\psi_1(n))]_G$. But we know that $q \in G$, but also $q \Vdash_\gamma \tilde{p}_n \in \dot{G}_\gamma (\forall n \in \omega)$(check out the proofs of the previous preservation theorems to see why this is true). So that $p_n = (\tilde{p}_n)_G \in G$. Now as $e_{\gamma,\gamma_n}(q_n) \in G$, by (6), we then obtain that $(\dot{t}_n)_G \in (\dot{T}_{\psi_0(n)})_G$ and $(\dot{t}_n)_G \supsetneq [Gd(\dot{T}_{\psi_0(n)})(\psi_1(n))]_G$ and
$(\dot{k}_n)_G \in dom((\dot{t}_n)_G) \setminus dom([Gd(\dot{T}_{\psi_0(n)})(\psi_1(n))]_G)$ and $g((\dot{k}_n)_G) = (\dot{t}_n)_G((\dot{k}_n)_G)$, simplifying this, one obtains

$$[(\dot{t}_n)_G \in T \wedge (\dot{t}_n)_G \supsetneq t \wedge (\dot{k}_n)_G \in dom((\dot{t}_n)_G) \setminus dom(t) \wedge g((\dot{k}_n)_G) = (\dot{t}_n)_G((\dot{k}_n)_G)],$$

but as $t \in T$ was arbitrary, we conclude that g densely diagonalizes T. This shows that q is $(M, \mathbb{P}_\gamma, \mathcal{E}, g)$-generic. $\square$

Construction of such $(q_n)_{n\in\omega}$, $(\tilde{p}_n)_{n\in\omega}$, $(\dot{t}_n)_{n\in\omega}$, $(\dot{k}_n)_{n\in\omega}$**:**

This is yet again an induction, so we assume all of the above sequences have been constructed up until the nth stage, and we construct the $(n+1)^{th}$ elements of each of those sequences simultaneously as follows:
Let G_n be an arbitrary $(V, \mathbb{P}_{\gamma_n})$ generic filter with $q_n \in G_n$.
To apply existential completeness, we first show that $V[G_n]$ models the following huge statement:

$$\exists \bar{r} \exists t_{n+1} \exists k_{n+1} (k_{n+1} \in \omega \wedge t_{n+1} \in \omega^{<\omega} \wedge \bar{r} \in D_n \cap M \wedge \bar{r} \leq_\gamma (\tilde{p}_n)_{G_n} \wedge$$
$$\bar{r} \restriction \gamma_n \in G_n \wedge \bar{r} \Vdash_\gamma [t_{n+1} \in \dot{T}_{\psi_0(n+1)} \wedge t_{n+1} \supsetneq Gd(\dot{T}_{\psi_0(n+1)})(\psi_1(n+1)) \wedge$$
$$k_{n+1} \in dom(t_{n+1}) \setminus dom(Gd(\dot{T}_{\psi_0(n+1)})(\psi_1(n+1))) \wedge g(k_{n+1}) = t_{n+1}(k_{n+1})])$$

Working in $V[G_n]$:
Let $p_n = (\tilde{p}_n)_{G_n}$, so that the inductive hypothesis gives $p_n \in \mathbb{P}_\gamma \cap M \wedge p_n \restriction \gamma_n \in G_n$, now as in the proof of the two previous preservation theorems,
as $\{r \restriction \gamma_n \mid r \in D_n \wedge r \leq_\gamma p_n\}$ is in M, and is dense below $p_n \restriction \gamma_n$ and as $q_n \in G_n$ is an $(M, \mathbb{P}_{\gamma_n})$-generic condition, we obtain that
$\{r \restriction \gamma_n \mid r \in D_n \wedge r \leq_\gamma p_n\} \cap G_n \cap M \neq \emptyset$ and then using $M \prec H(\lambda)$, one can find $r \in D_n \cap M$ such that $r \leq p_n$ and $r \restriction \gamma_n \in G_n$.
Now without loss of generality, we can assume that

$$r \Vdash_\gamma (Gd(\dot{T}_{\psi_0(n+1)})(\psi_1(n+1)) = \check{s}),$$

for some $s \in \omega^{<\omega}$, in the ground model V.
This, without loss of generality assumption, can be justified as follows:
As $1 \Vdash_\gamma Gd(\dot{T}_{\psi_0(n+1)})(\psi_1(n+1)) \in (\omega^{<\omega})\check{}$, if we choose a $(V[G_n], \mathbb{P}_\gamma/G_n)$ generic filter H (so that H is also $(V, \mathbb{P}_\gamma)$-generic) such that $r \in H$, then letting $s = [Gd(\dot{T}_{\psi_0(n+1)})(\psi_1(n+1))]_H$, we obtain that $s \in \omega^{<\omega}$ and there is an $r' \leq_\gamma r$ with $r \in H$ such that $r' \Vdash_\gamma (Gd(\dot{T}_{\psi_0(n+1)})(\psi_1(n+1)) = \check{s})$ so that as $H \subseteq \mathbb{P}_\gamma/G_n$ and $r \in D_n$ and D_n is open, we get that
" $\exists r' \in D_n(r' \leq_\gamma p_n \wedge r' \restriction \gamma_n \in G_n \wedge r' \Vdash_\gamma (Gd(\dot{T}_{\psi_0(n+1)})(\psi_1(n+1)) = \check{s})$ ". But this statement in the double quotes also holds relativized to $H(\lambda)^{V[G_n]}$ and hence by the fact that $M[G_n] \prec H(\lambda)^{V[G_n]}$, this statement also holds relativized to $M[G_n]$, from which we obtain an $r' \in M[G_n] \cap D_n$ satisfying

$$r' \leq_\gamma p_n \wedge r' \restriction \gamma_n \in G_n \wedge r' \Vdash_\gamma (Gd(\dot{T}_{\psi_0(n+1)})(\psi_1(n+1)) = \check{s}),$$

but as G_n contains the $(M, \mathbb{P}_{\gamma_n})$-generic condition q_n, we get that such an r' should actually lie in M (because $M[G_n] \cap D_n \subseteq M[G_n] \cap V = M \cap V$).This justifies the without loss of generality assumption.

Now we define (similar to the proof of preservation of strong tightness for tight MAD families)

$W = \{t \in \omega^{<\omega} \mid \exists \bar{r} \leq_\gamma r(\bar{r} \restriction \gamma_n \in G_n \wedge \bar{r} \Vdash_\gamma (\check{t} \in \dot{T}_{\psi_0(n+1)} \wedge \check{t} \supsetneq \check{s}))\} \cup \{s \restriction j \mid j \leq dom(s)\}$

Claim. $W \in I_T(\mathcal{E})^+ \cap M[G_n]$

Proof. The fact that $W \in M[G_n]$ follows easily by definability considerations. Now to see that $W \in I_T(\mathcal{E})^+$, we first show that W is infact a pruned tree. Towards this, let $t \in W$ and assume that $t' \subsetneq t$. Note that, because every element in W is either extended by s or extends s, without loss of generality, we can assume that $s \subsetneq t'$. By the definition of W, we obtain an $\bar{r} \leq_\gamma r$ with $\bar{r} \restriction \gamma_n \in G_n$ and $\bar{r} \Vdash_\gamma (\check{t} \in \dot{T}_{\psi_0(n+1)} \wedge \check{t} \supsetneq \check{s})$. Then for this same $\bar{r}$, we have $\bar{r} \Vdash_\gamma (\check{t'} \in \dot{T}_{\psi_0(n+1)} \wedge \check{t'} \supsetneq \check{s})$ because $1 \Vdash_\gamma$ ($\dot{T}_{\psi_0(n+1)}$ is closed under initial segments). Thus $t' \in W$. So W is closed under initial segments. Now to see that W is pruned is easy, as, if $t \in W$ is arbitrary, then either $t \subsetneq s$ or $t \supseteq s$. In the former case, t is properly extended by s and $s \in W$. In the latter case we choose an $\bar{r}$ below r such that $\bar{r} \restriction \gamma_n \in G_n$, satisfying $\bar{r} \Vdash_\gamma (\check{t} \in \dot{T}_{\psi_0(n+1)} \wedge \check{t} \supseteq \check{s})$. Now choose a $(V[G_n], \mathbb{P}_\gamma/G_n)$ generic H with $\bar{r} \in H$ (so that this H is also $(V, \mathbb{P}_\gamma)$ generic), and as $1 \Vdash_\gamma$ ($\dot{T}_{\psi_0(n+1)}$ is pruned), let $t_1 \in (\dot{T}_{\psi_0(n+1)})_H$ be such that $t_1 \supsetneq t$ (note that this t_1 is in the ground model V), and then choose an $r_1 \in H$ satisfying $r_1 \leq_\gamma \bar{r}$ and $r_1 \Vdash_\gamma (\check{t_1} \in \dot{T}_{\psi_0(n+1)} \wedge \check{t_1} \supsetneq \check{t})$, so that we have $t_1 \in W$ and $t_1 \supsetneq t$. Thus, W is a pruned tree.
Now we show that $W \in I_t(\mathcal{E})^+$. Suppose to the contrary that $W \notin I_T(\mathcal{E})^+$, then choose $t \in W$ and $E \in [\mathcal{E}]^{<\omega}$ such that $\bigcup W_t \subseteq^* \bigcup E$. Now there are two possibilities, either $t \subseteq s$ or $t \supsetneq s$. But if it is the case that $t \subseteq s$, one can choose a $t' \in W$ such that $t' \supsetneq s$ (since we just showed that W is pruned), and then we would have $W_{t'} \subseteq W_t$, so that $\bigcup W_{t'} \subseteq \bigcup W_t \subseteq^* \bigcup E$. So without loss of generality, we can assume that $t \supsetneq s$. Now choose an $\bar{r} \leq_\gamma r$ such that $\bar{r} \restriction \gamma_n \in G_n$ and $\bar{r} \Vdash_\gamma (\check{t} \in \dot{T}_{\psi_0(n+1)} \wedge \check{t} \supsetneq \check{s})$. Now choose a $(V[G_n], \mathbb{P}_\gamma/G_n)$-generic H with $\bar{r} \in H$. We show that

$$(\bigcup((\dot{T}_{\psi_0(n+1)})_H)_t) \setminus t \subseteq \bigcup W_t,$$

by showing that

$$((\dot{T}_{\psi_0(n+1)})_H)_t \setminus \{t \restriction j \mid j \leq dom(t)\} \subseteq W_t.$$

So let $t' \in ((\dot{T}_{\psi_0(n+1)})_H)_t \setminus \{t \restriction j \mid j \leq dom(t)\}$, then obviously we have $t' \in ((\dot{T}_{\psi_0(n+1)})_H)_t$ and $t' \supseteq t$, now choose via the forcing theorem an $r' \leq_\gamma \bar{r}$ with $r' \in H$ such that $r' \Vdash_\gamma (t' \in \dot{T}_{\psi_0(n+1)} \wedge t' \supsetneq s)$ so that, yet again by the definition of W, we obtain $t' \in W$, so that $t' \in W_t$ (because $t' \supseteq t$). This shows

$$((\dot{T}_{\psi_0(n+1)})_H)_t \setminus \{t \restriction j \mid j \leq dom(t)\} \subseteq W_t,$$

so that we finally obtain

$$[\bigcup((\dot{T}_{\psi_0(n+1)})_H)_t] \setminus t \subseteq \bigcup W_t \subseteq^* \bigcup E.$$

But this contradicts $1 \Vdash_\gamma (\dot{T}_{\psi_0(n+1)} \in I_T(\mathcal{E})^+)$. Thus we conclude that $W \in I_T(\mathcal{E})^+$.
Thus, the claim is proved. $\square$

Now as $W \in I_T(\mathcal{E})^+ \cap M[G_n]$ and $q_n (\in G_n)$ is $(M, \mathbb{P}_{\gamma_n}, \mathcal{E}, g)$-generic, we obtain that g densely diagonalizes W, so that as $s \in W$, there is a $t_{n+1} \in W$ $(\subseteq V)$ and a $k_{n+1} \in \omega$ satisfying $t_{n+1} \supsetneq s$ and $k_{n+1} \in dom(t_{n+1}) \setminus dom(s)$ and
$g(k_{n+1}) = t_{n+1}(k_{n+1})$. But then, by definition of W, there is also an $\bar{r} \leq_\gamma r$ such that $\bar{r} \restriction \gamma_n \in G_n \wedge \bar{r} \Vdash_\gamma (\check{t}_{n+1} \in \dot{T}_{\psi_0(n+1)})$. So that for this $\bar{r}$, we have that:

$$\bar{r} \leq_\gamma r \text{ and } \bar{r} \restriction \gamma_n \in G_n \text{ and}$$

$$\bar{r} \Vdash_\gamma (\check{t}_{n+1} \in \dot{T}_{\psi_0(n+1)} \wedge \check{t}_{n+1} \supsetneq Gd(\dot{T}_{\psi_0(n+1)})(\psi_1(n+1))) \text{ and}$$

$$\bar{r} \Vdash_\gamma (\check{k}_{n+1} \in dom(\check{t}_{n+1}) \setminus dom(Gd(\dot{T}_{\psi_0(n+1)})(\psi_1(n+1))) \wedge \check{g}(\check{k}_{n+1}) = \check{t}_{n+1}(\check{k}_{n+1}))$$

Note: Here in order to conclude that $\bar{r} \Vdash_\gamma \check{t}_{n+1} \supsetneq Gd(\dot{T}_{\psi_0(n+1)})(\psi_1(n+1))$, we have used the facts that $\bar{r} \leq_\gamma r$ and $r \Vdash_\gamma \check{s} = Gd(\dot{T}_{\psi_0(n+1)})(\psi_1(n+1)))$ and $t_{n+1} \supsetneq s$.
As we have been working in $V[G_n]$ so far, what we have essentially shown is that:

$V[G_n]$ models the following huge statement:

$$\begin{gathered}\exists\bar{r}\exists t_{n+1}\exists k_{n+1}(k_{n+1} \in \omega \wedge t_{n+1} \in \omega^{<\omega} \wedge \bar{r} \in D_n \wedge \bar{r} \leq_\gamma r \leq_\gamma p_n = (\tilde{p}_n)_{G_n} \wedge \\ \bar{r} \restriction \gamma_n \in G_n \wedge \bar{r} \Vdash_\gamma [t_{n+1} \in \dot{T}_{\psi_0(n+1)} \wedge t_{n+1} \supsetneq Gd(\dot{T}_{\psi_0(n+1)})(\psi_1(n+1)) \wedge \\ k_{n+1} \in dom(t_{n+1}) \setminus dom(Gd(\dot{T}_{\psi_0(n+1)})(\psi_1(n+1))) \wedge g(k_{n+1}) = t_{n+1}(k_{n+1})]).\end{gathered}$$

Note: The forcing statements appearing in the huge statement above are all absolute for transitive models of set theory, which is why, it does not matter if they are forced with respect to V or $V[G]$.

By virtue of the largeness of λ we get that $H(\lambda)^{V[G_n]}$ models

$$\exists \bar r \exists t_{n+1} \exists k_{n+1}(k_{n+1} \in \omega \wedge t_{n+1} \in \omega^{<\omega} \wedge \bar r \in D_n \wedge \bar r \leq_\gamma (\tilde p_n)_{G_n} \wedge \\ \bar r \restriction \gamma_n \in G_n \wedge \bar r \Vdash_\gamma [t_{n+1} \in \dot T_{\psi_0(n+1)} \wedge t_{n+1} \supsetneq Gd(\dot T_{\psi_0(n+1)})(\psi_1(n+1)) \wedge \\ k_{n+1} \in dom(t_{n+1}) \setminus dom(Gd(\dot T_{\psi_0(n+1)})(\psi_1(n+1))) \wedge g(k_{n+1}) = t_{n+1}(k_{n+1})])$$

and hence it follows by $M[G_n] \prec H(\lambda)^{V[G_n]}$ that, $M[G_n]$ models the same huge statement above, so that such an $\bar r$ can be obtained in $M[G_n] \cap D_n \subseteq M$ (since G_n contains the $(M, \mathbb{P}_{\gamma_n})$-generic condition q_n).

From this we conclude that $V[G_n]$ models

$$\exists \bar r \in D_n \cap M \exists t_{n+1} \exists k_{n+1}(k_{n+1} \in \omega \wedge t_{n+1} \in \omega^{<\omega} \wedge \bar r \in D_n \cap M \wedge \bar r \leq_\gamma (\tilde p_n)_{G_n} \wedge \\ \bar r \restriction \gamma_n \in G_n \wedge \bar r \Vdash_\gamma [t_{n+1} \in \dot T_{\psi_0(n+1)} \wedge t_{n+1} \supsetneq Gd(\dot T_{\psi_0(n+1)})(\psi_1(n+1)) \wedge \\ k_{n+1} \in dom(t_{n+1}) \setminus dom(Gd(\dot T_{\psi_0(n+1)})(\psi_1(n+1))) \wedge g(k_{n+1}) = t_{n+1}(k_{n+1})])$$

But as the $(V, \mathbb{P}_{\gamma_n})$ generic filter G_n with $q_n \in G_n$ was arbitrary, we conclude that the statement

$$\exists \bar r \exists t_{n+1} \exists k_{n+1}(k_{n+1} \in \omega \wedge t_{n+1} \in \omega^{<\omega} \wedge \bar r \in D_n \cap M \wedge \bar r \leq_\gamma \tilde p_n \wedge \\ \bar r \restriction \gamma_n \in G_n \wedge \bar r \Vdash_\gamma [t_{n+1} \in \dot T_{\psi_0(n+1)} \wedge t_{n+1} \supsetneq Gd(\dot T_{\psi_0(n+1)})(\psi_1(n+1)) \wedge \\ k_{n+1} \in dom(t_{n+1}) \setminus dom(Gd(\dot T_{\psi_0(n+1)})(\psi_1(n+1))) \wedge g(k_{n+1}) = t_{n+1}(k_{n+1})])],$$

is forced by q_n (forced with respect to $\mathbb{P}_{\gamma_n}$ of course).

Now use existential completeness to obtain names $\tilde p_{n+1}$, $\dot t_{n+1}$, $\dot k_{n+1} \in V^{\mathbb{P}_{\gamma_n}}$ such that

$$q_n \Vdash_{\gamma_n} (\dot k_{n+1} \in \check\omega \wedge \dot t_{n+1} \in (\omega^{<\omega})\check{} \wedge \tilde p_{n+1} \in D_n \cap M \wedge \tilde p_{n+1} \leq_\gamma \tilde p_n \wedge \\ \tilde p_{n+1} \restriction \gamma_n \in \dot G_{\gamma_n} \wedge \tilde p_{n+1} \Vdash_\gamma [\dot t_{n+1} \in \dot T_{\psi_0(n+1)} \wedge \dot t_{n+1} \supsetneq Gd(\dot T_{\psi_0(n+1)})(\psi_1(n+1)) \wedge \\ \dot k_{n+1} \in dom(\dot t_{n+1}) \setminus dom(Gd(\dot T_{\psi_0(n+1)})(\psi_1(n+1))) \wedge g(\dot k_{n+1}) = \dot t_{n+1}(\dot k_{n+1})]).$$

Now apply the inductive hypothesis to $\gamma_n, \gamma_{n+1} < \gamma$ and the name $\tilde p_{n+1} \restriction \gamma_{n+1}$, one then obtains an $(M, \mathbb{P}_{\gamma_{n+1}}, \mathcal{E}, g)$-generic condition q_{n+1} satisfying $q_{n+1} \restriction \gamma_n = q_n$ and $q_{n+1} \Vdash_{\gamma_{n+1}} (\tilde p_{n+1} \restriction \gamma_{n+1} \in \dot G_{\gamma_{n+1}})$ and it is easily seen that this q_{n+1} also forces (with respect to $\mathbb{P}_{\gamma_{n+1}}$), the statement

$(\dot{k}_{n+1} \in \check{\omega} \wedge \dot{t}_{n+1} \in (\omega^{<\omega})\check{} \wedge \tilde{p}_{n+1} \in D_n \cap M \wedge \tilde{p}_{n+1} \leq_\gamma \tilde{p}_n \wedge \tilde{p}_{n+1} \restriction \gamma_{n+1} \in \dot{G}_{\gamma_{n+1}} \wedge$
$\tilde{p}_{n+1} \Vdash_\gamma [\dot{t}_{n+1} \in \dot{T}_{\psi_0(n+1)} \wedge \dot{t}_{n+1} \supsetneq Gd(\dot{T}_{\psi_0(n+1)})(\psi_1(n+1)) \wedge$
$\dot{k}_{n+1} \in dom(\dot{t}_{n+1}) \setminus dom(Gd(\dot{T}_{\psi_0(n+1)})(\psi_1(n+1))) \wedge g(\dot{k}_{n+1}) = \dot{t}_{n+1}(\dot{k}_{n+1})]),$

so that the inductive construction can be continued.

Note: Here, in order to be completely rigorous, we have used the fact that if $\mathbb{P}, \mathbb{Q}, \mathbb{R}$ are forcings posets and if $e : \mathbb{P} \longrightarrow \mathbb{Q}$ is a complete embedding, then for any $p \in \mathbb{P}$ and any formula $\Phi = \Phi(x_0, ..., x_{n-1})$ and any $\mathbb{P}$-names $\tau_0, ..., \tau_{n-1}, \tilde{p}_1$ we have

$$p \Vdash_{\mathbb{P}} [\tilde{p}_1 \in \mathbb{R} \wedge \tau_0, ..., \tau_{n-1} \text{ are } \mathbb{R}\text{-names} \wedge \tilde{p}_1 \Vdash_{\mathbb{R}} \Phi(\tau_0, ..., \tau_{n-1})] \implies$$
$$e(p) \Vdash_{\mathbb{Q}} [e(\tilde{p}_1) \in \mathbb{R} \wedge e(\tau_0), ..., e(\tau_{n_1}) \text{ are } \mathbb{R}\text{-names} \wedge e(\tilde{p}_1) \Vdash_{\mathbb{R}} \Phi(e(\tau_0), ..., e(\tau_{n-1}))],$$

provided the formula Φ is absolute between transitive models of set theory.
With this the construction is complete. □

Corollary 4.2.6. Let $\mathcal{E}$ be a tight eventually different family and let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \gamma \wedge \beta < \gamma \rangle$ be a countable support iteration of proper forcings such that $\forall \beta < \gamma (1_\beta \Vdash_\beta (\dot{\mathbb{Q}}_\beta$ strongly preserves the tightness of $\mathcal{E}))$, then $\mathbb{P}_\gamma$ strongly preserves the tightness of $\mathcal{E}$.

5 Miller Forcing Preserves tightness

In this chapter, we conduct a brief study of the Miller Forcing and show that the Miller Forcing is Proper, strongly preserves the tightness of every tight eventually different family and every tight maximal almost disjoint family, from the ground model(see 5.1.11, 5.2.3 and 5.2.2 respectively). This lets us keep the value of $\mathfrak{a}, \mathfrak{a}_e$ small (i.e. $= \aleph_1$), in the extension by the Miller forcing (assuming CH in the ground model). This chapter is based on [4], [8] and [9].

5.1 Miller forcing

Remark. See chapter 4 for a definition of "tree".

Definition 5.1.1. Let T be a tree and let $t \in T$, we define

$$Succ_T(t) = \{s \in T \mid \exists n \in \omega \ (s = t^\frown n)\},$$

where $t^\frown n = t \cup \{\langle dom(t), n\rangle\}$.
An element of $Succ_T(t)$ is called an "immediate successor/successor of t in T"

Definition 5.1.2. Let $T \subseteq \omega^{<\omega}$ be a tree, we say that $t \in T$ is a splitting node of T if $|Succ_T(t)| = \aleph_0$, and we denote the set of all splitting nodes of T by $split(T)$.

Definition 5.1.3. We say a tree $T \subseteq \omega^{<\omega}$ is a superperfect tree if

$$\forall s \in T \exists t \supseteq s (t \text{ is a splitting node of } T)$$

Definition 5.1.4. We define the Miller Forcing as

$$\mathbb{M} = \{T \subseteq \omega^{<\omega} \mid T \text{ is a superperfect tree} \wedge T \neq \emptyset\},$$

with largest element $1_{\mathbb{M}} = \omega^{<\omega}$ and $\leq_{\mathbb{M}} = \subseteq$

Definition 5.1.5. Let T be a tree, we say that $t \in T$ is a stem of T if

$\exists n_1, n_2 \in \omega(n_1 \neq n_2 \wedge t^\frown n_1,\ t^\frown n_2 \in T)$ and $\forall s \subsetneq t\ (\neg \exists n_1, n_2 \in \omega(n_1 \neq n_2 \wedge s^\frown n_1, s^\frown n_2 \in T))$.

Observations

- $T \in \mathbb{M} \implies |T| = \aleph_0$
- Every superperfect tree has a unique stem, and hence it makes sense to write "$t = stem(T)$" and say "the" stem of T
- $T = T_{stem(T)}$, i.e. $\forall t \in T(t \subseteq stem(T)$ or $stem(T) \subseteq t)$
- $T, S \in \mathbb{M}$ and $stem(T) \notin S$, then S and T are incompatible.

  Proof. To see why this is true, assume the hypothesis and note that if $t \in S \cap T$, then $t \supsetneq stem(T)$ is impossible, as this would imply $stem(T) \in S$. From this, it follows that $S \cap T \subseteq \{stem(T) \restriction n \mid n \leq dom(stem(T))\}$, which shows that $S \cap T$ is finite and hence, as all superperfect trees are infinite, there cannot be a superperfect tree which is a subset of both S and T. □

Definition 5.1.6. For $T \in \mathbb{M}$, we define the n^{th} splitting level of T as:

$$split_n(T) = \{t \in split(T) \mid |\{s \in T \mid s \subseteq t \wedge s \in split(T)\}| = n + 1\}.$$

Now we prove some lemmas that will help us construct elements in $\mathbb{M}$ as nested intersections. These lemmas would be helpful in the next section as well.

Lemma 5.1.7. Let $T, S \subseteq \omega^{<\omega}$ be superperfect trees such that $T \supseteq S$, then we have that:

$$\forall r \in \omega(split_r(T) = split_r(S) \implies \forall k \leq r(split_k(T) = split_k(S))).$$

Proof. Let $k < r$ and let $t \in split_k(T)$, then choose an infinite subset of ω,say A, such that $\forall m \in A(t^\frown m \in T)$ and for each $m \in A$, choose a $t_m \in split_r(T)$ such that $t^\frown m \subseteq t_m$, then by hypothesis we have $t_m \in S$ and hence as S is closed under initial segments, we get $\forall m \in A(t^\frown m \in S)$, which implies that $t \in split(S)$. We have thus shown that $split_k(T) \subseteq split(S)$. With this, we can easily prove the lemma as follows:

Let $t \in split_k(T)$, and let $a_0 \subsetneq a_1 \subsetneq .... \subsetneq a_k = t$, be the $k + 1$ splitting notes below (or equal to) t, in T. Then, by what was just proven above, we have that $a_0, a_1, ..., a_k(= t) \in split(S)$, which shows that infact $t \in split_k(S)$. Thus $split_k(T) \subseteq split_k(S)$. Now for the other inclusion, let $t \in split_k(S)$, choose $t' \in split_r(S)$ such that $t \subsetneq t'$, but as $split_r(S) = split_r(T)$, we obtain that $t' \in split_r(T)$, showing that $t \in split_k(T)$.
Therefore, the lemma is proved.

□

Lemma 5.1.8. Let $(T_n)_{n\in\omega} \subseteq \mathbb{M}$, be a decreasing sequence of trees such that for each $k \in \omega$ the sequence $(split_k(T_n))_{n\in\omega}$ is eventually constant, then
$T_\omega = \bigcap_{n\in\omega}(T_n) \in \mathbb{M}$

Proof. For each $k \in \omega$, choose $n_k \in \omega$ such that $\forall m \geq n_k$ $(split_k(T_m) = split_k(T_{n_k}))$. Note that without loss of generality, we can assume $n_k < n_{k+1}$. Moreover, note that "for any $t \in T_\omega$, there is a $k \in \omega$ and a $t' \in split_k(T_{n_k})$ for which $t \subseteq t'$", because if this statement in the double quotes were not true then, as $t \in T_{n_k}$, choose $t'_k \subseteq t \subseteq t''_k$ such that $t'_k \in split_{m_k}(T_{n_k})$ and $t''_k \in split_{m_k+1}(T_{n_k})$. Then we would have that $m_k \geq k$. But as any node in the n^{th} splitting level of any tree has a length of at least n, we get that $|t| \geq |t'_k| \geq m_k \geq k$, i.e. we have obtained the contradiction that $\forall k \in \omega(|t| \geq k)$. Therefore, it is clear that it is enough to show that for each $k \in \omega$, $(split_k(T_\omega) = split_k(T_{n_k}))$. Because if this were true, then if $t \in T_\omega$, is arbitrary, choose by the statement in the double quotes, a $k \in \omega$ and a $t' \in split_k(T_{n_k}) = split_k(T_\omega)$, such that $t \subseteq t'$, showing that above every node of T_ω, there is a splitting node of T_ω, which means T_ω is a superperfect tree.

Claim. $\forall k \in \omega$ $(split_k(T_\omega) = split_k(T_{n_k}))$:

Proof. We prove this via induction:
For the base case we need to show that $split_0(T_{n_0}) = split_0(T_\omega)$.
Towards this, let $t \in split_0(T_{n_0}) = split_0(T_{n_1})$ (since $n_1 > n_0$). Now, let $A \in [\omega]^\omega$ be such that $\forall m \in A$ $(t^\frown m \in T_{n_1})$ and choose for each $m \in A$, a $t_m \in split_1(T_{n_1})$ such that $t_m \supsetneq t^\frown m$. So then, $t_m \in T_\omega$ and, as T_ω is closed under initial segments, we obtain that $\forall m \in A$ $(t^\frown m \in T_\omega)$, showing that $t \in split(T_\omega)$. Moreover as $T_\omega \subseteq T_{n_0}$, we obtain also that $t \in split_0(T_\omega)$. This shows that $split_0(T_{n_0}) \subseteq split_0(T_\omega)$.

Now for the other inclusion, let $t \in split_0(T_\omega)$, then obviously $t \in split(T_{n_0})$. Now if there were some $s \in split(T_{n_0})$ such that $s \subsetneq t$, choose such an s that additionally satisfies $s \in split_0(T_{n_0})$. But we already showed that $split_0(T_{n_0}) \subseteq split_0(T_\omega)$, so this would mean that $s \in split(T_\omega)$, contradicting $t \in split_0(T_\omega)$

Thus, the base case is proved. The inductive case is an exercise.

□

□

Now we prove another lemma, regarding the construction of trees in $\mathbb{M}$, which will be useful in the next subsection:

Lemma 5.1.9. Let $T \in \mathbb{M}$ and let for each $t \in split_{n+1}(T)$, $S^t \in \mathbb{M}$ be such that $S^t \subseteq T_t$, and let, $S = \bigcup_{t \in split_{n+1}(T)}(S^t)$, then we have that $S \in \mathbb{M}$ and moreover $\forall r \leq n$ $(split_r(T) = split_r(S))$. Moreover if $split_{n+1}(T) \subseteq split(S)$, then additionally $split_{n+1}(T) = split_{n+1}(S)$.

Proof. Obviously $S \in \mathbb{M}$. Now we show that $\forall r \leq n$ $(split_r(S) = split_r(T))$.
First note that for $r \leq n$ we have $split_r(T) \subseteq split(S)$. To see this, choose $t \in$

$split_r(T)$ and let $A \in [\omega]^\omega$ be such that $\forall m \in A$ $(t^\frown m \in T)$ and choose for each $m \in A$, a $t_m \in split_{n+1}(T)$ such that $t^\frown m \subseteq t_m$. Now as $t_m \in S$ (since $t_m \subseteq stem(S^{t_m}) \in S^{t_m}$ and S^{t_m} is closed under initial segments), so that
$\forall m \in A$ $(t^\frown m \in S)$, which shows that $t \in split(S)$.
Now it easily follows, as in the previous proofs (by virtue of $S \subseteq T$) that
$\forall r \leq n$ $(split_r(T) \subseteq split_r(S))$. Additionally if $split_{n+1}(T) \subseteq split(S)$, then we also obtain $split_{n+1}(T) \subseteq split_{n+1}(S)$, by the same logic.

Now for the reverse inclusion, let $s \in split_r(S)$, for $r \leq n$, now by definition of S, choose a $t \in split_{n+1}(T)$ such that $s \in S^t$, as $S^t \subseteq T_t$, we have $s \subsetneq t$ or $t \subseteq s$. The latter scenario is impossible, because if the latter scenario were true, then we would have (because of $\forall r \leq n$ $(split_r(T) \subseteq split_r(S))$) that $s \in split_{n+1}(S)$, which is obviously a contradiction. So we obtain $s \subsetneq t$, from which it readily follows that there are no extra splitting nodes (which split in T, but not in S), below s, in T, which shows that $split_r(S) \subseteq split_r(T)$. Note that, if additionally $split_{n+1}(T) \subseteq split(S)$, then we have that if $s \in split_{n+1}(S)$, then $s \supseteq t$ for some $t \in split_{n+1}(T)$, and then as $t \in split(S)$, $t \subsetneq s$, is impossible, so that $s = t$ and hence $s \in split_{n+1}(T)$. □

Now our goal is to show that $\mathbb{M}$ is proper, and towards this we describe two special antichains of $\mathbb{M}$:

Lemma 5.1.10. Let $p \in \mathbb{M}$ then we have:

1. $B(p,n) = \{(p)_t \mid t \in split_n(p)\}$ is an antichain in $\mathbb{M}$, that is maximal below p
2. for each $t \in split_n(p)$, $B'(p,n,t) = \{(p)_{t^\frown m} \mid m \in \{m \in \omega \mid t^\frown m \in p\}\}$ is an antichain in $\mathbb{M}$, that is maximal below $(p)_t$.

Proof. For (1):
Let $t_1 \neq t_2 \in split_n(p)$, then note that $(p)_{t_1} \cap (p)_{t_2}$ is finite, because if $s \in (p)_{t_1} \cap (p)_{t_2}$, then we have $s \subseteq t_1 \cap t_2$, because if $s \supseteq t_1$, then as trees are closed under initial segments, we would have, $t_1 \supseteq t_2$ or $t_2 \supseteq t_1$ ($\because$ $t_1 \in (p)_{t_2}$). But as both $t_1, t_2 \in split_n(p)$, we would then have $t_1 = t_2$. This shows that $(p)_{t_1} \cap (p)_{t_2}$ is finite, and hence we conclude that the set $B(p,n)$ is an antichain in $\mathbb{M}$. Now to show maximality below p, let $q \leq p$ be arbitrary and choose $t \in q \cap split_n(p)$. Note that $q \cap split_n(p) \neq \emptyset$, because, if $t' \in split_n(q)$, then as $q \subseteq p$, $t' \in split_{n+r}(p)$ for some $r \in \omega$, and now choose a $t \in split_n(p)$ such that $t \subsetneq t'$, then $t \in split_n(p) \cap q$. But then for such a t, we would have $(q)_t \leq (p)_t, q$, showing that $B(p,n)$ is maximal below p.

For (2):
Let $A(t) = \{m \in \omega \mid t^\frown m \in p\}$, and choose distinct elements $m_1, m_2 \in A(t)$.

Now we have as in 1, that if $s \in (p)_{t^\frown m_1} \cap (p)_{t^\frown m_2}$, then $s \subseteq t^\frown m_1 \cap t^\frown m_2$ and hence the intersection is finite. (otherwise we would get $m_1 = m_2$), showing that $(p)_{t^\frown m_1}, (p)_{t^\frown m_2}$ are incompatible.
Now for maximality below $(p)_t$, let $q \leq (p)_t$, be arbitrary, then clearly $t \in q$, also now choose an $m \in \omega$ such that $t^\frown m \in q$, then clearly $m \in A(t)$ and moreover $(q)_{t^\frown m} \leq q, (p)_{t^\frown m}$. □

Now we are ready to prove that $\mathbb{M}$ is proper.

Theorem 5.1.11. *The Miller forcing $\mathbb{M}$ is proper*

Proof. let λ be a large enough regular cardinal and let $M \prec H(\lambda)$ be countable with $\mathbb{M} \in M$ and let $p \in \mathbb{M} \cap M$ be arbitrary. Now enumerate all open dense subsets of $\mathbb{M}$ that lie in M, as $\{D_n \mid n \in \omega\}$.

Claim. Enough to construct a sequence $(p_n)_{n\in\omega} \subseteq \mathbb{M}$ such that the following hold for each $n \in \omega$:

1. $p_0 = p$
2. $p_{n+1} \leq p_n$ and $split_{n+1}(p_{n+1}) = split_{n+1}(p_n)$
3. $p_n \in M$
4. $\forall t \in split_{n+1}(p_{n+1}) \forall m \in \omega \ (t^\frown m \in p_{n+1} \implies (p_{n+1})_{t^\frown m} \in D_n)$

Proof. To see why this is true, assume we have such a sequence and let $q = \bigcap_{n\in\omega}(p_n)$. Then $q \in \mathbb{M}$ because of Lemma 5.1.7 and Lemma 5.1.8. It is obvious that $q \leq p_0 = p$. Moreover if G is a generic filter with $q \in G$, then for each $n \in \omega$, we have, we have $p_{n+1} \in G$, and so as $\{(p_{n+1})_t \mid t \in split_{n+1}(p_{n+1})\}$ is an antichain in $\mathbb{M}$ maximal below p_{n+1}, there is a unique $t \in split_{n+1}(p_{n+1})$ such that $(p_{n+1})_t \in G$, and now as $\{(p_{n+1})_{t^\frown m} \mid m \in \{m \in \omega \mid t^\frown m \in p_{n+1}\}\}$ is an antichain in $\mathbb{M}$, maximal below $(p_{n+1})_t (\in G)$, we get a unique m such that $t^\frown m \in p_{n+1}$ and $(p_{n+1})_{t^\frown m} \in G$ (See Lemma 5.1.10), but then as $p_{n+1}, t, m \in M$, we obtain that $(p_{n+1})_{t^\frown m} \in M$, and also by 4, we have $(p_{n+1})_{t^\frown m} \in D_n$. We have basically shown that $(p_{n+1})_{t^\frown m} \in G \cap M \cap D_n$. So that as $n \in \omega$ was arbitrary, we conclude $\forall n \in \omega \ (q \Vdash (\dot{G} \cap D_n \cap M \neq \emptyset)$. This shows that q is $(M, \mathbb{M})$-generic. Thus, the claim is proved. □

Construction of such a sequence:
Assume we have constructed $\{p_r \mid r \leq n\}$. We now construct p_{n+1} as follows:
As $p_n \in M$, it is clear that $split_{n+1}(p_n) \in M$ and $\{\langle t, C(t)\rangle \mid t \in split_{n+1}(p_n)\} \in M$, where $C(t) = \{m \in \omega \mid t^\frown m \in p_n\}$. Now for each $t \in split_{n+1}(p_n)$ and each $m \in C(t)$, as $(p_n)_{t^\frown m} \in M$ and D_n is dense in $\mathbb{M}$, choose (via $M \prec H(\lambda)$),

$r(t,m) \in M \cap D_n$ such that $r(t,m) \leq (p_n)_{t^\frown m}$. Note that this choice can be done uniformly, i.e. we can arrange for $\{r(t,m) \mid t \in split_{n+1}(p_n) \ \wedge m \in C(t)\} \in M$

(yet again because of $M \prec H(\lambda)$). Now define $p_{n+1} = \bigcup_{t \in split_{n+1}(p_n)} \bigcup_{m \in C(t)} (\ r(t,m) \ )$, then obviously $p_{n+1} \in M \cap \mathbb{M}$, $p_{n+1} \leq p_n$ and moreover, applying Lemma 5.1.9, it is easy to see that $split_{n+1}(p_{n+1}) = split_{n+1}(p_n)$. An easy computation shows that for each $t \in split_{n+1}(p_{n+1})$ and each $m \in C(t)$, $(p_{n+1})_{t^\frown m} = r(t,m) \in D_n$ (i.e. (4) holds).

This finishes the inductive construction.

$\square$

5.2 Miller forcing has the strong preservation properties

In this section, we show that the Miller forcing strongly preserves the tightness of every ground model tight MAD family and every ground model tight ED family.

Definition 5.2.1. let $p, q \in \mathbb{M}$, we then define for $n \in \omega$, $\leq_n$ as follows:
$p \leq_n q \iff p \leq q \wedge split_n(p) = split_n(q)$

Remark. By the results in the previous subsection, it is clear that if we have a sequence $(p_n)_{n \in \omega}$ such that $\forall n \in \omega \ (p_{n+1} \leq_{n+1} p_n)$, then $q = \bigcap_{n \in \omega}(p_n) \in \mathbb{M}$

Theorem 5.2.2. ***($\mathbb{M}$ strongly preserves the tightness of ground model tight MAD families)***

Let $\mathcal{A}$ be a tight MAD family (in the ground model), then the Miller forcing $\mathbb{M}$ strongly preserves the tightness of $\mathcal{A}$

Proof. Let λ be a sufficiently large regular cardinal and let $M \prec H(\lambda)$ be countable with $\mathcal{A}, \mathbb{M} \in M$ and let $B \in I(\mathcal{A})$ satisfy $\forall x \in M \cap I(\mathcal{A})^+(|x \cap B| = \aleph_0)$, and let $p \in \mathbb{M} \cap M$ be arbitrary. We need to find an $(M, \mathbb{M}, \mathcal{A}, B)$-generic condition q, which lies below p.

Let $\{D_n \mid n \in \omega\}$ be an enumeration of all open dense subsets of $\mathbb{M}$ that lie in M, and let $\{\dot{Z}_n \mid n \in \omega\}$ be an enumeration of all $\mathbb{M}$- names for elements of $I(\mathcal{A})^+$ that lie in M such that each name occurs infinitely often in the enumeration.

Claim. Enough to construct sequences $(p_n)_{n \in \omega}$, $(\dot{k}_n)_{n \in \omega}$ such that the following hold for each $n \in \omega$:

1. $p_0 = p$
2. $p_{n+1} \leq_{n+1} p_n$
3. $p_{n+1} \Vdash (\dot{k}_n \in \dot{Z}_n \cap B \setminus n)$

4. $\forall t \in split_n(p_n) \forall m \in \omega (t^\frown m \in p_n \implies (p_n)_{t^\frown m} \in D_{n-1} \cap M)$.

Remark. It is easy to see that condition 4 implies that

$$\forall n \in \omega (\forall t \in split_{n+1}(p_n) \forall m \in \omega \ (t^\frown m \in p_n \implies (p_n)_{t^\frown m} \in M \ )).$$

This is because, if $t \in split_{n+1}(p_n)$ and $m \in \omega$ are such that $t^\frown m \in p_n$, we can find $t' \in split_n(p_n)$ and an $m' \in \omega$ satisfying $t'^\frown m' \in p_n$ and $t'^\frown m' \subsetneq t^\frown m$. Now condition (4) implies that $(p_n)_{t'^\frown m'} \in M$, moreover it is easy to see that $(p_n)_{t^\frown m} = ((p_n)_{t'^\frown m'})_{t^\frown m}$. This shows that $(p_n)_{t^\frown m}$, is definable from $(p_n)_{t'^\frown m'} \in M$, $t^\frown m \in M$, and hence $(p_n)_{t^\frown m} \in M$.

Proof. To see why this claim is true, assume we have such sequences, then put $q = \bigcap_{n \in \omega}(p_n)$, then clearly $q \in \mathbb{M}$ and $q \leq p$.

Now let G be an arbitrary generic filter on $\mathbb{M}$ with $q \in G$, then $p_{n+1} \in G$. Now by Lemma 5.1.10 (part 1), we get that there is a unique $t \in split_{n+1}(p_{n+1})$ such that $(p_{n+1})_t \in G$, now by part 2 of Lemma 5.1.10, we get that there is a unique $m \in \omega$ such that $t^\frown m \in p_{n+1}$ and $(p_{n+1})_{t^\frown m} \in G$.

Note: In the above arguments, we have repeatedly used the fact that if A is an antichain maximal below some p and $p \in G$ where G is a generic filter, then $A \cap G$ is a singleton. But then, by 4, we have that $(p_{n+1})_{t^\frown m} \in D_n \cap M$. Thus we have shown that $(p_{n+1})_{t^\frown m} \in M \cap D_n \cap G$. Therefore $\forall n \in \omega (G \cap M \cap D_n \neq \emptyset)$, so that $\forall n \in \omega \ (q \Vdash (\dot{G} \cap M \cap D_n \neq \emptyset))$. This shows that q is $(\mathbb{M}, M)$-generic.

Now we show that $q \Vdash \forall x \in M[\dot{G}] \cap I(\mathcal{A})^+ \ (\ |x \cap B| = \aleph_0 \ )$:
As always let G be an arbitrary generic filter with $q \in G$, and let $x \in I(\mathcal{A})^+ \cap M[G]$. Choose $\dot{Z}_n$, such that $(\dot{Z}_n)_G = x$, and choose a strictly increasing sequence of natural numbers $(n_r)_{r \in \omega}$ such that $\forall r \in \omega \ (\dot{Z}_{n_r} = \dot{Z}_n)$. Then as each $p_{n_r+1} \in G$, we get from condition 3, that $(\dot{k}_{n_r})_G \in (\dot{Z}_{n_r})_G \cap B \setminus n_r = (\dot{Z}_n)_G \cap B \setminus n_r = x \cap B \setminus n_r$. Showing that $x \cap B$ is infinite. Thus $q \Vdash \forall x \in M[\dot{G}] \cap I(\mathcal{A})^+ \ (|x \cap B| = \aleph_0)$.

This shows that the tree q is $(M, \mathbb{M}, \mathcal{A}, B)$-generic. $\square$

Construction of such sequences:
Assume we have constructed $\{p_r \mid r \leq n\}$ (with $p_0 = p$) and $\{\dot{k}_r \mid r < n\}$, we construct p_{n+1} and $\dot{k}_n$ as follows:
For each $t \in split_{n+1}(p_n)$ $(\subseteq M)$. Let $C(t) = \{m \in \omega \mid t^\frown m \in p_n\}$ $(\subseteq M)$. Now for each $t \in split_{n+1}(p_n)$ and each $m \in C(t)$, choose $q(t,m) \in D_n \cap M$ such that $q(t,m) \leq (p_n)_{t^\frown m}$, which is possible because $M \prec H(\lambda)$, D_n is dense in $\mathbb{M}$ and because, by the remark after the claim, $(p_n)_{t^\frown m} \in M$. Now define $W_{t,m} = \{j \in \omega \mid \exists r \leq q(t,m) \ (r \Vdash \ (j \in \dot{Z}_n) \ )\}$. Then one can easily see that $W_{t,m} \in I(\mathcal{A})^+$, because, if not then $W_{t,m}$ would be almost coverable by finitely many elements from $\mathcal{A}$, but then if one chooses a generic filter G, with $q(t,m) \in G$, as

$(\dot{Z}_n)_G \in I(\mathcal{A})^+$, it would be impossible for $(\dot{Z}_n)_G$ to be almost coverable by finitely many elements from $\mathcal{A}$, but note that $(\dot{Z}_n)_G \subseteq W_{t.m}$, because if $m \in (\dot{Z}_n)_G$, then there is $r \leq q(t.m)$ such that $r \Vdash m \in \dot{Z}_n$, and hence $m \in W_{t,m}$. This gives a contradiction, showing that infact $W_{t,m} \in I(\mathcal{A})^+$. Moreover, $W_{t,m} \in M$. This shows that $W_{t.m} \in I(\mathcal{A})^+ \cap M$, so that by the assumption on B, we obtain that $W_{t,m} \cap B$ is infinite and hence choose $n_{t,m} \in W_{t,m} \cap B \setminus n$, then by definition of $W_{t,m}$ we can choose an $r(t,m) \leq q(t,m)$ such that $r(t,m) \Vdash (n_{t,m} \in \dot{Z}_n)$ and $r(t,m) \in M$. Here we have yet again used $M \prec H(\lambda)$ to obtain $r(t,m) \in M$.

Note:
Unlike in the proof of Theorem 5.1.11, we might NOT really have

$$\{r(t,m) \mid t \in split_{n+1}(p_n) \wedge m \in C(t)\} \in M,$$

and so it is not necessary that $p_n \in M$. This is because the formula giving the existence of the above collection has the set B as a parameter, and we might have $B \notin M$, so that one cannot directly apply $M \prec H(\lambda)$, because of the extra parameter B.

Now we put

$$p_{n+1} = \bigcup_{t \in split_{n+1}(p_n)} \Big(\bigcup_{m \in C(t)} (r(t.m)) \Big) \in \mathbb{M}$$

and put

$$\dot{k}_n = \bigcup_{t \in split_{n+1}(p_n)} \{\langle \check{n_{t,m}} \ , \ r(t.m)\rangle \mid m \in C(t)\}$$

An easy computation shows that for each $t \in split_{n+1}(p_n)$ and each $m \in C(t)$, $t^\frown m \in p_{n+1}$ and $(p_{n+1})_{t^\frown m} = r(t,m)$. Moreover taking $S^t = \bigcup_{m \in C(t)} r(t,m)$, in Lemma 5.1.9, one can easily show that $split_{n+1}(p_{n+1}) = split_{n+1}(p_n)$. Thus $p_{n+1} \leq_{n+1} p_n$. Moreover it is clear that

$$\forall t \in split_{n+1}(p_{n+1}) \forall m \in \omega (t^\frown m \in p_{n+1} \implies (p_{n+1})_{t^\frown m} = r(t,m) \in D_n \cap M).$$

Now we show that $p_{n+1} \Vdash (\dot{k}_n \in \dot{Z}_n \cap B \setminus n)$:

Let G be an arbitrary generic filter with $p_{n+1} \in G$, then applying Lemma 5.1.10 twice (as before, in this proof), let $t \in split_{n+1}(p_{n+1}) = split_{n+1}(p_n)$ and $m \in C(t)$ (t, m, unique), be such that $(p_{n+1})_{t^\frown m} = r(t,m) \in G$. Then we easily obtain that $(\dot{k}_n)_G = n_{t,m} \in (\dot{Z}_n)_G \cap B \setminus n$.
This shows that $p_{n+1} \Vdash (\dot{k}_n \in \dot{Z}_n \cap B \setminus n)$.

Thus, we are done. $\square$

Theorem 5.2.3. *Let $\mathcal{E}$ be a tight eventually different family (in the ground model), then $\mathbb{M}$ strongly preserves the tightness of $\mathcal{E}$.*

Proof. Let as usual λ be sufficiently large, and regular and let $M \prec H(\lambda)$ be countable with $\mathbb{M}, \mathcal{E} \in M$, and let $g \in \mathcal{E}$ be a real such that

$\forall T \in M \cap I_T(\mathcal{E})^+$ (g densely diagonalizes T) and let $p \in \mathbb{M} \cap M$ be arbitrary. We need to show that there is an $(M, \mathbb{M}, \mathcal{E}, g)$-generic q, which lies below p. Enumerate all $\mathbb{M}$- names for elements of $I_T(\mathcal{E})^+$, which lie in M, as $\{\dot{T}_n \mid n \in \omega\}$, and enumerate all dense open subsets of $\mathbb{M}$ which lie in M, as $\{D_n \mid n \in \omega\}$ and fix a bijection $\psi = \langle \psi_1, \psi_2 \rangle : \omega \times \omega \longrightarrow \omega$.

Claim. Enough to construct sequences $(p_n)_{n\in\omega}, (\dot{k}_n)_{n\in\omega}$ and $(\dot{t}_n)_{n\in\omega}$ such that the following hold for each $n \in \omega$:

1. $p_0 = p$
2. $p_{n+1} \leq_{n+1} p_n$
3. $\dot{t}_n, \dot{k}_n \in V^{\mathbb{M}}$ and

$$p_{n+1} \Vdash [\dot{t}_n \in \dot{T}_{\psi_0(n)} \wedge \dot{t}_n \supsetneq Gd(\dot{T}_{\psi_0(n)})(\psi_1(n)) \wedge \\ \dot{k}_n \in dom(\dot{t}_n) \setminus dom(Gd(\dot{T}_{\psi_0(n)})(\psi_1(n))) \wedge \dot{t}_n(\dot{k}_n) = g(\dot{k}_n)]$$

4. $\forall t \in split_n(p_n) \forall m \in \omega\ (t^\frown m \in p_n \implies (p_n)_{t^\frown m} \in D_{n-1} \cap M)$

Proof. To see why this is true, assume we have such sequences, and as in the previous proof, let $q = \bigcap_{n\in\omega}(p_n) \in \mathbb{M}$. Obviously $q \leq p_0 = p$.
Now if G is a generic filter with $q \in G$, $p_{n+1} \in G$ and hence, as in the previous proof (and by condition 4) we get that $(p_{n+1})_{t^\frown m} \in G \cap M \cap D_n$, for a unique $t \in split_{n+1}(p_{n+1})$ and a unique $m \in \omega$ with $t^\frown m \in p_{n+1}$. This shows that
$\forall n \in \omega\ (q \Vdash (D_n \cap M \cap \dot{G} \neq \emptyset))$ and hence q is $(M, \mathbb{M})$-generic.

Now we show that $q \Vdash \forall T \in I_T(\mathcal{E})^+ \cap M[\dot{G}]$ (g densely diagonalizes T)
Let G be an arbitrary generic filter with $q \in G$ and assume that $T \in I_T(\mathcal{E})^+ \cap M[G]$, and let $t \in T$ be arbitrary, choose $n \in \omega$ such that $T = (\dot{T}_{\psi_0(n)})_G$ and $t = Gd(T)(\psi_1(n)) = Gd((\dot{T}_{\psi_0(n)})_G)(\psi_1(n)) = [Gd(\dot{T}_{\psi_0(n)})(\psi_1(n))]_G$. Now as $p_{n+1} \in G$,

we obtain from 3, that $(\dot{t}_n)_G \in T$ and $(\dot{t}_n)_G \supsetneq t$ and $(\dot{k}_n)_G \in dom((\dot{t}_n)_G) \setminus dom(t)$ and $(\dot{t}_n)_G((\dot{k}_n)_G) = g((\dot{k}_n)_G)$. This shows that g densely diagonalizes T. Thus, we are done. $\square$

Construction of such sequences:

Assume we have constructed $\{p_r \mid r \leq n\}$ and $\{\dot{t}_r \mid r < n\}$ and $\{\dot{k}_r \mid r < n\}$. Now we construct $p_{n+1},\ \dot{t}_n,\ \dot{k}_n$ as follows:
For each $t \in split_{n+1}(p_n)$, let $C(t)$, be as in the previous proof, i.e. $C(t) = \{m \in \omega \mid t^\frown m \in p_n\}$. Now, for each $t \in split_{n+1}(p_n)$ and each $m \in C(t)$, as the inductive hypothesis gives $(p_n)_{t^\frown m} \in M$, we choose $q(t,m), s(t,m)$ such that $q(t,m) \in D_n \cap M$, $q(t.m) \leq (p_n)_{t^\frown m},\ \ s(t,m) \in \omega^{<\omega}$ and

$q(t,m) \Vdash Gd(T_{\psi_0(n)})(\psi_1(n)) = s(t,m)\check{}$. Such elements can be obtained because $M \prec H(\lambda)$, D_n is open and dense.

Now we define

$$W_{t,m} = \{u \in \omega^{<\omega} \mid u \supsetneq s(t,m) \wedge \exists r \leq q(t,m)\ (r \Vdash (\check{u} \in \dot{T}_{\psi_0(n)}))\}$$

Then one can see that $W'_{t,m} = W_{t.m} \cup \{s(t,m) \restriction j \mid j \leq dom(s(t,m))\} \in I_T(\mathcal{E})^+ \cap M$. Seeing $W'_{t,m} \in M$, is easy. To see $W'_{t,m} \in I_T(\mathcal{E})^+$, suppose not, so that there is $E \in [\mathcal{E}]^{<\omega}$ and $s \in W'_{t,m}$ such that $(W'_{t,m})_s \subseteq^* \bigcup E$, without loss of generality, let $s \supsetneq s(t,m)$, so that there is $r \leq q(t,m)$ such that $r \Vdash s \in \dot{T}_{\psi_0(n)}$, now if one chooses any generic filter G on $\mathbb{M}$ that contains r, then it is easy to see that $((\dot{T}_{\psi_0(n)})_G)_s \setminus s \restriction j \mid j \leq dom(s)\} \subseteq (W'_{t,m})_s$, and hence we obtain $((\dot{T}_{\psi_0(n)})_G)_s \subseteq^* \bigcup E$, which contradicts $1 \Vdash \dot{T}_{\psi_0(n)} \in I_T(\mathcal{E})^+$.
Therefore, by the hypothesis on g, we obtain that g densely diagonalizes each $W_{t,m}$, which means there are $r(t,m) \in \mathbb{M}$, $u(t,m) \in \omega^{<\omega}$, $k(t,m) \in \omega$ such that, $r(t,m) \leq q(t,m)$, $r(t,m) \Vdash u(t,m) \in \dot{T}_{\psi_0(n)}$, $u(t,m) \supsetneq s(t,m)$, $k(t,m) \in dom(u(t,m)) \setminus dom(s(t,m))$ and $u(t,m)(k(t,m)) = g(k(t,m))$. Moreover, because $M \prec H(\lambda)$ and D_n is open dense, we can assume $r(t,m) \in M$, $r(t,m) \in D_n$.
Now we define

$$p_{n+1} = \bigcup_{t \in split_{n+1}(p_n)} \left(\bigcup_{m \in C(t)} (r(t,m)) \right)$$

and let

$$\dot{k}_n = \bigcup_{t \in split_{n+1}(p_n)} \left(\{\langle k(t,m)\check{}, r(t,m)\rangle \mid m \in C(t)\} \right)$$

and finally let

$$\dot{t}_n = \bigcup_{t \in split_{n+1}(p_n)} \left(\{\langle u(t,m)\check{}, r(t,m)\rangle \mid m \in C(t)\} \right)$$

As in the proof of Theorem 5.2.2, we have that $split_{n+1}(p_{n+1}) = split_{n+1}(p_n)$ and for each $t \in split_{n+1}(p_{n+1})$ and each $m \in \omega$ such that $t^\frown m \in p_{n+1}$, we have $(p_{n+1})_{t^\frown m} = r(t,m)$ ($\in D_n \cap M$). From this, it immediately follows that condition (4) holds.

Now we show that condition (3) holds:

Let G be an arbitrary generic filter with $p_{n+1} \in G$, then as in the proof of Theorem 5.2.2, we get a unique $t \in split_{n+1}(p_{n+1}) = split_{n+1}(p_n)$, and a unique $m \in \omega$ with $t^\frown m \in p_{n+1}$, such that $(p_{n+1})_{t^\frown m} = r(t,m) \in G$. Then the definition of an evaluation by a filter, and the fact that we have

$r(t,m) \Vdash u(t,m) \in \dot{T}_{\psi_0(n)}$, gives that $(\dot{t}_n)_G = u(t,m) \in (\dot{T}_{\psi_0(n)})_G$ and $(\dot{k}_n)_G = k(t,m)$. So we obtain $(\dot{t}_n)_G = u(t,m) \supsetneq s(t,m) = [Gd(\dot{T}_{\psi_0(n)})(\psi_1(n))]_G$, where the last equality holds because $q(t,m) \Vdash Gd(\dot{T}_{\psi_0(n)})(\psi_1(n)) = \check{s(t,m)}$ and $q(t,m) \in G$. We also obtain, $(\dot{k}_n)_G \in dom((\dot{t}_n)_G) \setminus dom([Gd(\dot{T}_{\psi_0(n)})(\psi_1(n))]_G)$ and $(\dot{t}_n)_G((\dot{k}_n)_G) = g((\dot{k}_n)_G)$.
This shows that

$$p_{n+1} \Vdash [\dot{t}_n \in \dot{T}_{\psi_0(n)} \wedge \dot{t}_n \supsetneq Gd(\dot{T}_{\psi_0(n)})(\psi_1(n)) \wedge \\ \dot{k}_n \in dom(\dot{t}_n) \setminus dom(Gd(\dot{T}_{\psi_0(n)})(\psi_1(n))) \wedge \dot{t}_n(\dot{k}_n) = g(\dot{k}_n)],$$

i.e. 3 holds.

Hence proved.

□

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