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Characterisation of linear bounded operators on co-orbit spaces using tensor products of localised frames

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Contents

1 Abstract

This doctoral dissertation is made up of three articles of colleagues of mine and myself and deals with a few closely related problems in harmonic analysis and operator theory. In the first part of this dissertation we establish several properties of the tensor product of self-localised frames for Hilbert spaces and the co-orbit spaces associated with this tensor product. Valuable in their own right, these properties allow us to prove the kernel theorems for the bounded linear operators mapping the smallest and largest of the co-orbit spaces associated with self-localised frames onto one another and, in doing so, to characterise the corresponding operator spaces. In the second part of this dissertation we continue the study of properties of the tensor product of self-localised frames and show that the tensor product of two self-localised frames is a self-localised frame too. This allows one to apply the whole range of tools of co-orbit theory to tensor product frames to study corresponding operator spaces. In the third and last part of this dissertation we explore the relation between localised and structured frames and show that most prominent structured frames such as Gabor frames, wavelets, frames for α-modulation spaces, ridgelets, curvelets, shearlets and wave atoms for the corresponding decomposition spaces of complex functions of two real variables can easily be made to be $\mathcal{A}$-self-localised, $\mathcal{A}$ being a Jaffard's algebra.

2 Kurzfassung

Diese Doktorarbeit setzt sich aus drei Artikeln zusammen und befaßt sich mit einigen eng verbundenen Fragen in der harmonischen Analyse und der Operatortheorie. Im ersten Teil dieser Doktorarbeit stellen wir mehrere Eigenschaften des Tensorprodukts selbstlokalisierter Rahmen für Hilberträume und der mit solchen Rahmen verbundenen Koorbiträumen fest. Mithilfe dieser Eigenschaften beweisen wir die Kernsätze für die beschränkten linearen Operatoren, die den kleinsten und den größten der mit einem selbstlokalisierten Rahmen verbundenen Koorbiträume aufeinander abbilden, und charakterisieren dadurch die dementsprechenden Operatorenräume. Im zweiten Teil dieser Doktorarbeit setzen wir unsere Untersuchung von Eigenschaften des Tensorprodukts selbstlokalisierter Rahmen für Hilberträume fort und zeigen, daß das Tensorprodukt zweier selbstlokalisierter Rahmen auch einen selbslokalisierten Rahmen bildet. Als wichtige Folgerung können alle Werkzeuge der Koorbittheorie angewandt werden, um dementsprechende Operatorenräume zu untersuchen. Im dritten und letzten Teil dieser Doktorarbeit untersuchen wir das Verhältnis zwischen strukturierten und selbstlokalisierten Rahmen und zeigen, daß die bekanntesten strukturierten Rahmen wie zum Beispiel Gaborrahmen, Wavelets, Rahmen für α-Modulationsräume, Ridgelets, Curvelets, Shearlets und Wellenatome für dementsprechende Dekompositionsräume komplexwertiger Funktionen zweier Veränderlicher selbstlokalisiert werden lassen können.

3 Introduction

In this doctoral dissertation we deal with a few closely related problems in harmonic analysis and operator theory. Indeed, we study properties – such as self-localisation – of frames for topological vector spaces, of function spaces whose elements are characterised by the sequences of their analysis coefficients with respect to those frames being in specific weighted Lebesgue spaces and of bounded linear operators acting on those function spaces.

Kernel theorems provide very convenient bijective relations between linear bounded operators on infinite-dimensional topological vector spaces and functions or distributions, known as the kernels of those operators, and express the result of the action of the operators as integrals in a way very similar to the matrix multiplication whereby the action of finite matrices, seen as linear operators on finite-dimensional vector spaces, is expressed. The original kernel theorem for the linear bounded operators mapping the Schwartz space onto the space of tempered distributions was formulated and proved by Schwartz in 1952. In 1980, Feichtinger proved a kernel theorem for the bounded linear operators mapping the modulation space $M^1(\mathbb{R}^n)$ – now also known as the Feichtinger algebra – onto its topological dual, the modulation space $M^\infty(\mathbb{R}^n)$ [16], the modulation spaces being the spaces of choice in may areas of harmonic analysis [30]. In 2019, Balazs et al. adopted a conceptually more abstract approach and proved a kernel theorem for bounded linear operators acting on co-orbit spaces associated with integrable representations of locally compact groups [3]. In the first part of this dissertation we take a step further in this direction and show that the group structure and the corresponding representation theory are an unnecessary, albeit very elegant, setup for proving kernel theorems. Indeed, we formulate and prove the kernel theorems for the bounded linear operators mapping the smallest of the co-orbit spaces associated with a self-localised frame and the largest of those co-orbit spaces onto one another; and in doing so characterise the corresponding operator spaces. Moreover, using the Schur test, we describe bounded linear operators mapping the co-orbit space $H^1_{w_1}(\Psi_1)$ onto the co-orbit space $H^p_{w_2}(\Psi_2)$ as $p \in (1, \infty)$ – the space intermediate between the spaces $H^1_{w_2}(\Psi_2)$ and $H^\infty_{w_2}(\Psi_2)$ – and those mapping the co-orbit space $H^p_{w_1}(\Psi_1)$ onto the co-orbit space $H^\infty_{w_2}(\Psi_2)$. We conclude this part of the dissertation by considering bounded linear operators on shift-invariant spaces, an example of the problems that can be easily dealt with using our kernel theorems, but to which the above-mentioned kernel theorems for co-orbit spaces associated with integrable group representations are not applicable.

The notion of self-localisation of a frame for a Hilbert space was introduced by Gröchenig in 2004 to measure the quality of the frame and defined as its Gram matrix belonging to an inverse-closed involutive Banach matrix algebra [32]. Since then self-localised frames have been used mainly for decomposing and synthesising functions. In recent years the use of tensor product frames for representing operators proved rather successful in some numerical calculations. Hardly any theoretical explanation of this success has been provided, though. We try to supply such an explanation. It is common knowledge that the set of elementary tensor products of the elements of frames for two Hilbert spaces constitutes a frame for the tensor product of those Hilbert spaces [1]. The question that has remained open, though, is whether the tensor product of two self-localised frames is, one way or another, self-localised too. In the second part of this dissertation we show that, under the conditions that are met by all known concrete examples of self-localised frames for Hilbert spaces [36, 34, 4, 27, 42, 25], the tensor product of any such two frames is indeed a self-localised frame. To do so we first design an involutive Banach algebra of tensors of rank four made from two solid involutive Banach matrix algebras and prove that this algebra is inverse-closed. We then show that the Gram tensor made from the tensor products of the elements of self-localised frames for two Hilbert spaces belongs to the algebra we designed. As a byproduct we obtain self-localised frames that may include operators of rank higher than one for the Hilbert-Schmidt operators.

In the third and last part of this dissertation we explore the relation between localied and structured frames [9, 39, 44, 5], two concepts that have long been studied and used in parallel [26] – with the notable exception of Gabor frames, and do so on examples of frames for spaces of complex functions or distributions of two real variables. Over the last seventy years or so, a number of structured frames that allow one to decompose and synthesise functions of two variables of various classes have been designed. The most prominent examples of such frames are Gabor frames [30], wavelets [13], frames for α-modulation spaces [28], ridgelets [10], curvelets [11], shearlets [38] and wave atoms [14]. We design a set of what we call wave packet systems – of which all above-mentioned frames are particular examples – and prove that the wave packet systems are self-localised frames for a great variety of quasi-Banach spaces of complex functions or distributions of two real variables and therefore so are all of the above-mentioned examples of frames. Unlike the above-mentioned examples of structured frames, the wave packet system can be made to assume a wide range of types and degrees of time-frequency self-localisation by the suitable choice of values of the parameters of the system. This, we believe, makes the wave packet systems not only suitable for decomposing and synthesising functions of a wide range of quasi-Banach function spaces in an efficient and effective way, but also for their use for representing Fourier integral operators on those spaces by sparse and well structured matrices using the Galerkin method, the fundamental reason for this being the fact that the Fourier integral operators are known to transform wave packets into each other [12]. This, in its turn, should allow one to design efficient computer programs for solving corresponding operator equations on two-dimensional manifolds using finite element methods.

This doctoral dissertation is made up of three articles [7, 8, 6]. The core concepts and facts used throughout the dissertation are provided in Section 4. The main results of articles [7], [8] and [6] are recapitulated and briefly discussed in Sections 5.1, 5.2 and 5.3 respectively.

4 Background information and notation

In this section we provide the core concepts and facts that we shall use throughout this dissertation. The definitions of a few notions more specific to particular parts of this thesis will be supplied as they are needed.

For any positive numbers a and b, $a \lesssim b$ will indicate that there is such a constant c that $a \leqslant c\, b$ and $a \asymp b$ that $a \lesssim b$ and $b \lesssim a$.

4.1 Self-localised frames for Hilberts spaces and co-orbit spaces

Both one of the central subjects of this whole study and the main tool we used in it are so-called *frames* for vector spaces. The concept of a frame generalises that of a basis. One of the major advantages of frames over bases is that a frame can often be designed in a way that allows for a much sparser decompositions of the elements of a given vector space than a basis would do. The only possible inconvenience of using frames, rather than orthonormal bases, is that such decompositions will not necessarily be unique. Here is the definition of the frame for a separable Hilbert space [15].

Definition 4.1 *A countable set $\Psi := \{\psi_i\}_{i\in I}$ of elements of a separable Hilbert space H is called a* frame *for that space if there exist such positive numbers A_Ψ and B_Ψ – called* lower *and* upper frame bounds *of the frame – that*

$$A_\Psi \, \|f\|_H^2 \leqslant \sum_{i\in I} |\langle f, \psi_i\rangle_H|^2 \leqslant B_\Psi \, \|f\|_H^2 \tag{1}$$

for any element $f \in H$.

Here are some of the major implications of this definition. Firstly, the *analysis operator*

$$C_\Psi : H \to \ell^2(I), \quad f \mapsto \{\langle f, \psi_i\rangle_H\}_{i\in I}, \tag{2}$$

where the sequence $\{\langle f, \psi_i\rangle_H\}_{i\in I}$ is called the *generalised wavelet transform* of the element f, is bounded and injective and has a closed range. Secondly, the adjoint D_Ψ of the analysis operator C_Ψ, that is, the *synthesis operator*

$$D_\Psi : \ell^2(I) \to H, \quad \{c_i\}_{i\in I} \mapsto \sum_{i\in I} c_i \psi_i \tag{3}$$

is bounded and surjective. Thirdly, the *Gram operator*

$$\begin{aligned} G_\Psi : \ell^2(I) \to \ell^2(I), \quad \{c_i\}_{i\in I} &\mapsto \sum_{i'\in I} \langle \psi_{i'}, \psi_i\rangle_H \, c_{i'} \\ &= \sum_{i'\in I} G_{i,\,i'} \, c_{i'} = G_\Psi c = C_\Psi D_\Psi c \end{aligned} \tag{4}$$

with the *Gram matrix*

$$G_\Psi := (G_{i,\,i'})_{(i,\,i')\in I^2} := \big(\langle \psi_{i'}, \psi_i\rangle_H\big)_{(i,\,i')\in I^2} \tag{5}$$

is bounded. Fourthly, there exists at least one other frame $\Psi^{\mathbf{d}} := \{\psi_i^d\}_{i\in I}$ for the space H – called a *dual frame* for the frame Ψ – that, together with Ψ, satisfies

$$f = \sum_{i\in I} \langle f, \psi_i\rangle_H \, \psi_i^d = D_{\Psi^{\mathbf{d}}} C_\Psi f = \sum_{i\in I} \langle f, \psi_i^d\rangle_H \, \psi_i = D_\Psi C_{\Psi^{\mathbf{d}}} f \tag{6}$$

for any element $f \in H$. Finally, the *frame operator*

$$S_\Psi : H \to H, \quad f \mapsto \sum_{i\in I} \langle f, \psi_i\rangle_H \, \psi_i,$$

is bounded, self-adjoint and invertible for any frame Ψ and the sequence $\widetilde{\Psi} = \{\widetilde{\psi_i}\}_{i\in I} = \{S_\Psi^{-1}\psi_i\}_{i\in I}$ is a dual frame for the frame Ψ – known as the *canonical dual frame*.

There is also the notion of a frame for and an atomic decomposition of Banach spaces, introduced by Gröchenig [29]. These two notions coincide if the Banach space is a Hilbert space.

Definition 4.2 *Let X be a separable Banach space and X_d a Banach sequence space. A countable set $\Psi := \{\psi_i\}_{i\in I}$ of elements of the topological dual X^* of the space X is called an* X_d-frame *for the space X if, for any element $f \in X$, there are such positive numbers A_Ψ and B_Ψ that*

$$A_\Psi \left\|f\right\|_X \leqslant \left\|\{\langle f, \psi_i\rangle_{X,\, X^*}\}_{i\in I}\right\|_{X_d} \leqslant B_\Psi \left\|f\right\|_X . \tag{7}$$

Definition 4.3 *Let X be a separable Banach space and X_d a Banach space of sequences. A sequence $\Psi \times \Phi := \{\psi_i,\, \phi_i\}_{i\in I}$ in the space $X^* \times X$ is called an* atomic decomposition *of the space X with respect to the* coefficient space X_d *if the analysis operator*

$$C_\Psi : X \to X_d,\ f \mapsto \{\langle f, \psi_i\rangle_{X,\, X^*}\}_{i\in I} \tag{8}$$

is bounded and, for any $f \in X$,

$$f = \sum_{i\in I} \langle f, \psi_i\rangle_{X,\, X^*}\, \phi_i \tag{9}$$

and

$$\|f\|_X \asymp \|\{\langle f, \psi_i\rangle_{X,\, X^*}\}_{i\in I}\|_{X_d} . \tag{10}$$

Despite an apparent similarity of Definitions 4.1 and 4.2, there is a very important difference between frames for Hilbert spaces and those for Banach spaces, namely there being an X_d-frame for a Banach space in itself does not necessarily allow one to express any element of the space in a way similar to that described by (6). Therefore, additional assumptions about frames are needed to ensure the possibility of such an expression, the corresponding frames being called *Banach frames* [32]. Formally, a *X_d-frame* for the Banach space X is called a *Banach X_d-frame* if there is such a bounded operator D_Ψ – called the *synthesis operator* – that, for any $f \in X$,

$$D_\Psi : X_d \to X,\ \{\langle f, \psi_i\rangle_{X,\, X^*}\}_{i\in I} \mapsto f . \tag{11}$$

One way of designing a Banach frame for a Banach space X and indeed its topological dual X^* is to consider the so-called *Gelfand triple* $(X,\, H,\, X^*)$ where H stands for a Hilbert space – that is, such a triple $(X,\, H,\, X^*)$ that $X \subseteq H \subseteq X^*$ where the Banach space X is densely embedded in the Hilbert H and the Hilbert space H is weak-* embedded in the Banach space X^* – and make sure that the frame for the Hilbert space H is *self-localised*, as then the norm equivalence expressed by (7) itself implies the existence of a bounded synthesis operator. Here is the original concept of an *intrensically* or *self-localised frame* for a Hilbert space [31] that we believe has, among other things, the advantage of being beautifully intuitive.

Definition 4.4 *For a metric ρ on the space $\mathbb{R}^d$, a countable subset I of the space $\mathbb{R}^d$ is called* separate *if there is such a positive number C that*

$$\inf_{i,\, i' \in I;\, i \neq i'} \rho(i, i') = C \tag{12}$$

for any i and $i' \in I$. A frame $\Psi := \{\psi_i\}_{i \in I}$ for a separable Hilbert space H is said to be intrensically *or* self-localised *if, for an integer n larger than the dimension d of the space $\mathbb{R}^d$,*

$$|(G_\Psi)_{i,i'}| = |\langle \psi_i, \psi_{i'} \rangle_H| \lesssim (1 + \rho(i, i'))^{-n} \tag{13}$$

for any i and $i' \in I$.

The matrices whose elements satisfy (13) belong to the so-called *Jaffard algebra*, an example of what is known as spectral matrix algebras. This allowed Gröchenig to generalise the notion of self-localisation [32]. Here is its definition, together with an ancillary notion of the *spectral matrix algebra.*

Definition 4.5 *An involutive Banach algebra $\mathcal{A}$ of infinite matrices with indices in a countable set I^2 and a norm $\|\cdot\|_{\mathcal{A}}$ is called a* spectral matrix algebra *if*

(i) any matrix $A \in \mathcal{A}$ defines a bounded linear operator on the space $\ell^2(I)$; and

(ii) the algebra $\mathcal{A}$ is inverse-closed.

It is rather common to include yet another – desirable, but often not strictly necessary – condition on the spectral matrix algebra, namely its having to be solid. Here is the definition of this notion.

Definition 4.6 *If, for any matrices $A := (A_{i,j})_{(i,j) \in I^2}$ and $B := (B_{i,j})_{(i,j) \in I^2} \in \mathcal{A}$ with complex elements, of which the former belongs to a spectral matrix algebra $\mathcal{A}$, the fact that $|B_{i,j}| \leqslant |A_{i,j}|$ for any $(i, j) \in I^2$ implies that the matrix B also belongs to the algebra $\mathcal{A}$ and $\|B\|_{\mathcal{A}} \leqslant \|A\|_{\mathcal{A}}$, the spectral matrix algebra $\mathcal{A}$ is called* solid.

Examples of solid spectral matrix algebras include, among others, Jaffard's [36], Schur's [33] and Sjöstrand's [42] algebras, Jaffard's algebra, indeed, satisfying the criteria of Definition 4.4.

Definition 4.7 *Let I be a countable index set. The set $\Psi = \{\psi_i\}_{i \in I}$ of elements of a Hilbert space H is said to be $\mathcal{A}$*-self-localised*, or simply* self-localised*, if its Gram matrix $G_\Psi := (\langle \psi_i, \psi_{i'} \rangle_H)_{(i,\, i') \in I^2}$ belongs to a spectral matrix algebra $\mathcal{A}$.*

An $\mathcal{A}$-self-localised frame Ψ for a Hilbert space H allows one to generate a whole range of Banach spaces known as *co-orbit spaces* [21, 22], which are the Banach spaces of choice in numerous areas of harmonic analysis. Traditionally, co-orbit spaces were studied for families generated by integrable group representations. In this thesis we shall follow the approach introduced by Fornasier and Gröchenig [25] and consider co-orbit spaces that are generated by localised frames. Here is how such co-orbit spaces are defined.

Definition 4.8 *Let $\mathcal{A}$ be a spectral algebra of infinite matrices whose elements have indices in a countable set I^2. A sequence of positive real numbers $w := \{w_i\}_{i\in I} \subset \mathbb{R}_{>0}$ is called an* $\mathcal{A}$-admissible *weight if any matrix $A \in \mathcal{A}$ defines a bounded operator on the sequence space $\ell^p_w(I)$ for any $p \in [1, \infty]$.*

Definition 4.9 *Let $\mathcal{A}$ be a spectral matrix algebra, $\Psi := \{\psi_i\}_{i\in I}$ an $\mathcal{A}$-localised frame for a Hilbert space H, $\widetilde{\Psi} := \{\widetilde{\psi}_i\}_{i\in I}$ its canonical dual frame, $w := \{w_i\}_{i\in I}$ an $\mathcal{A}$-admissible weight and*

$$H_{00}(\Psi) := \left\{ \sum_{i\in I} c_i\, \psi_i : \ \{c_i\}_{i\in I} \in c_{00} \right\}, \tag{14}$$

where c_{00} stands for the space of sequences of complex numbers with only finitely many non-zero terms. The co-orbit space $H^p_w(\Psi)$ is the completion of the set $H_{00}(\Psi)$ with respect to the norm $\|f\|_{H^p_w(\Psi)} := \left\|C_{\widetilde{\Psi}} f\right\|_{\ell^p_w(I)}$ as $p \in [1, \infty)$ – where $\ell^p_w(I)$ stands for the w-weighted Lebesgue space on the set I – or its completion in the $\sigma(H, H_{00}(\Psi))$-topology as $p = \infty$.

The exact definition of the co-orbit space as $p = \infty$ is rather intricate and we shall provide it in full as we introduce co-orbit spaces associated with so-called tensor product frames later on, cf Definitions 4.15 and 4.16.

One of the main features of those co-orbit spaces is that the self-localised frame for the Hilbert space that was used to generate the range of co-orbit spaces is also a Banach frame for any of the co-orbit spaces and so it allows one to decompose and re-synthesise any element of any co-orbit space [25].

4.2 Tensor product of Banach spaces

The next three definitions clarify the notion of the tensor product of elements of Hilbert spaces, Hilbert spaces themselves and Banach spaces [40], which are central to the first two parts of this dissertation.

Definition 4.10 *The elementary tensor $f_1 \otimes f_2$ of an element f_1 of a Hilbert space H_1 and an element f_2 of a Hilbert space H_2 is defined as the rank one operator mapping the Hilbert space H_1 onto the Hilbert space H_2 according to*

$$(f_1 \otimes f_2)(f) := \langle f, f_1\rangle_{H_1} f_2 \tag{15}$$

for any $f \in H_1$.

Definition 4.11 *The tensor product $H_1 \otimes H_2$ of Hilbert spaces H_1 and H_2 is the completion of the linear span of the elementary tensors with respect to the metric induced by the scalar product defined by*

$$\langle f_1 \otimes f_2, g_1 \otimes g_2\rangle_{H_1\otimes H_2} := \overline{\langle f_1, g_1\rangle}_{H_1} \langle f_2, g_2\rangle_{H_2} \tag{16}$$

for any f_1 and $g_1 \in H_1$ and any f_2 and $g_2 \in H_2$.

Definition 4.12 *The* projective tensor product $X_1 \widehat{\otimes}_\pi X_2$ *of two Banach spaces* X_1 *and* X_2 *is the completion of the set*

$$\left\{ \sum_{i \in I} x_{1,\,i} \otimes x_{2,\,i} :\, x_{1,\,i} \in X_1,\, x_{2,\,i} \in X_2,\, \sum_{i \in I} \|x_{1,\,i}\|_{X_1} \|x_{2,\,i}\|_{X_2} < \infty \right\} \tag{17}$$

with respect to the norm

$$\|u\|_{X_1 \widehat{\otimes}_\pi X_2} := \inf \left(\sum_{i \in I} \|x_{1,\,i}\|_{X_1} \|x_{2,\,i}\|_{X_2} \right), \tag{18}$$

the infimum being taken over all representations $\sum_{i \in I} x_{1,\,i} \otimes x_{2,\,i}$ *of* $u \in X_1 \widehat{\otimes}_\pi X_2$.

The tensor product $\Psi_1 \otimes \Psi_2 := \{\psi_{1,\,i} \otimes \psi_{2,\,j}\}_{(i,\,j) \in I \times J}$ of a frame $\Psi_1 := \{\psi_{1,\,i}\}_{i \in I}$ for a Hilbert space H_1 and a frame $\Psi_2 := \{\psi_{2,\,j}\}_{j \in J}$ for a Hilbert space H_2 is known to be a frame for the tensor product $H_1 \otimes H_2$, its canonical dual $\widetilde{\Psi_1 \otimes \Psi_2}$ being $\widetilde{\Psi_1} \otimes \widetilde{\Psi_2}$ [1]. Here are the definitions of the analysis and synthesis operators and co-orbit spaces associated with the tensor product frame $\Psi_1 \otimes \Psi_2$.

Definition 4.13 *Let* $\Psi_1 := \{\psi_{1,\,i}\}_{i \in I}$ *be a frame for a Hilbert space* H_1*,* $\Psi_2 := \{\psi_{2,\,j}\}_{j \in J}$ *a frame for a Hilbert space* H_2*. Furthermore, let* $B\left(H^{p_1}_{w_1}(\Psi_1),\, H^{p_2}_{w_2}(\Psi_2)\right)$ *denote the space of bounded linear operators mapping the co-orbit space* $H^{p_1}_{w_1}(\Psi_1)$ *onto the co-orbit space* $H^{p_2}_{w_2}(\Psi_2)$*. The map*

$$C_{\Psi_1 \otimes \Psi_2} : B\left(H^{p_1}_{w_1}(\Psi_1),\, H^{p_2}_{w_2}(\Psi_2)\right) \to \ell^{2,\,2}(I \times J) \tag{19}$$

defined by

$$\begin{aligned} C_{\Psi_1 \otimes \Psi_2} O &:= \left\{ \langle O\,,\, \psi_{2,\,j} \otimes \bar{\psi}_{1,\,i} \rangle_{B\left(H^{p_1}_{w_1}(\Psi_1),\, H^{p_2}_{w_2}(\Psi_2)\right),\, B'\left(H^{p_1}_{w_1}(\Psi_1),\, H^{p_2}_{w_2}(\Psi_2)\right)} \right\}_{(i,\,j) \in I \times J} \\ &= \left\{ \langle O\psi_{1,\,i}\,,\, \psi_{2,\,j} \rangle_{H^{p_2}_{w_2}(\Psi_2),\, H^{\frac{1}{1-1/p_2}}_{1/w_2}(\Psi_2)} \right\}_{(i,\,j) \in I \times J} \end{aligned} \tag{20}$$

for any operator $O \in B\left(H^{p_1}_{w_1}(\Psi_1),\, H^{p_2}_{w_2}(\Psi_2)\right)$ *will be called the analysis operator. The operator*

$$D_{\Psi_1 \otimes \Psi_2} : \ell^{2,\,2}(I \times J) \to B\left(H^{p_1}_{w_1}(\Psi_1),\, H^{p_2}_{w_2}(\Psi_2)\right) \tag{21}$$

defined by

$$\begin{aligned} \left(D_{\Psi_1 \otimes \Psi_2} M\right)(f) &:= \left(\sum_{(i,\,j) \in I \times J} M_{i,\,j}\, \psi_{2,\,i} \otimes \bar{\psi}_{1,\,j} \right)(f) \\ &:= \sum_{i \in I} \left(\sum_{j \in I} M_{i,\,j} \langle f\,,\, \psi_{1,\,j} \rangle_{H^{p_1}_{w_1}(\Psi_1),\, H^{\frac{1}{1-1/p_1}}_{1/w_1}(\Psi_1)} \right) \psi_{2,\,i} \end{aligned} \tag{22}$$

for any matrix $M \in \ell^{2,\,2}(I \times J)$ *and any* $f \in H^{p_1}_{w_1}(\Psi_1)$ *will be called the synthesis operator.*

Definition 4.14 *Let p and $q \in [1, \infty)$,*

$$\ell^{p,\,q}_w(I \times J) := \left\{ c : I \times J \to \mathbb{C} : \ \sum_{j \in J} \Big(\sum_{i \in I} |c_{i,\,j}|^p w^p_{i,\,j} \Big)^{q/p} < \infty \right\}, \tag{23}$$

$$\mathfrak{l}^{p,\,q}_w(I \times J) := \left\{ c : I \times J \to \mathbb{C} : \quad \sum_{i \in I} \Big(\sum_{j \in J} |c_{i,\,j}|^p \, w^p_{i,\,j} \Big)^{q/p} < \infty \right\} \tag{24}$$

and

$$H_{00}(\Psi_1 \otimes \Psi_2) = \left\{ \sum_{(i,\,j) \in I \times J} c_{i,\,j}\, \psi_{1,\,i} \otimes \psi_{2,\,j} : \ \{c_{i,\,j}\}_{(i,\,j) \in I \times J} \in c_{00} \right\}. \tag{25}$$

The co-orbit spaces $H^{p,\,q}_w(\Psi_1 \otimes \Psi_2)$ and $\mathfrak{H}^{p,\,q}_w(\Psi_1 \otimes \Psi_2)$ are the completion of the set $H_{00}(\Psi_1 \otimes \Psi_2)$ with respect to the norm $\|F\|_{H^{p,\,q}_w(\Psi_1 \otimes \Psi_2)} := \|C_{\widetilde{\Psi_1 \otimes \Psi_2}} F\|_{\ell^{p,\,q}_w(I \times J)}$ and $\|F\|_{\mathfrak{H}^{p,\,q}_w(\Psi_1 \otimes \Psi_2)} := \|C_{\widetilde{\Psi_1 \otimes \Psi_2}} F\|_{\mathfrak{l}^{p,\,q}_w(I \times J)}$ respectively.

Definition 4.15 *Two weakly convergent sequences $\{F_n\}_{n \in \mathbb{N}}$ and $\{F'_n\}_{n \in \mathbb{N}}$ in the space $H_1 \otimes H_2$ are* equivalent *if*

$$\lim_{n \to \infty} \big\langle F_n - F'_n, \widetilde{\psi}_{1,\,i} \otimes \widetilde{\psi}_{2,\,j} \big\rangle_{H_1 \otimes H_2} = 0 \tag{26}$$

for any $(i,\,j) \in I \times J$.

Definition 4.16 *Let $p \in [1, \infty)$. The co-orbit space $H^{r,\,s}_{w_1 \otimes w_2}(\Psi_1 \otimes \Psi_2)$ and $\mathfrak{H}^{r,\,s}_{w_1 \otimes w_2}(\Psi_1 \otimes \Psi_2)$ where $(r,\,s) \in \{(p, \infty), (\infty, p), (\infty, \infty)\}$ is the space of equivalence classes $F = \big[\{F_n\}_{n \in \mathbb{N}}\big]$ where $\{F_n\}_{n \in \mathbb{N}}$ is such a sequence in the space $H_1 \otimes H_2$ that*

$$\lim_{n \to \infty} \big\langle F_n, \widetilde{\psi}_{1,\,i} \otimes \widetilde{\psi}_{2,\,j} \big\rangle_{H_1 \otimes H_2} =: k_{i,\,j} \tag{27}$$

exists for any $(i,\,j) \in I \times J$ and, respectively,

$$\sup_n \big\| C_{\widetilde{\Psi_1 \otimes \Psi_2}} F_n \big\|_{\ell^{r,\,s}_{w_1 \otimes w_2}(I \times J)} < \infty$$

or

$$\sup_n \big\| C_{\widetilde{\Psi_1 \otimes \Psi_2}} F_n \big\|_{\mathfrak{l}^{r,\,s}_{w_1 \otimes w_2}(I \times J)} < \infty.$$

Furthermore

$$\big(C_{\widetilde{\Psi_1 \otimes \Psi_2}} F \big)_{i,\,j} := k_{i,\,j} \tag{28}$$

for any $(i,\,j) \in I \times J$,

$$\|F\|_{H^{r,\,s}_{w_1 \otimes w_2}(\Psi_1 \otimes \Psi_2)} := \big\| C_{\widetilde{\Psi_1 \otimes \Psi_2}} F \big\|_{\ell^{r,\,s}_{w_1 \otimes w_2}(I \times J)} \tag{29}$$

and

$$\|F\|_{\mathfrak{H}^{r,\,s}_{w_1 \otimes w_2}(\Psi_1 \otimes \Psi_2)} := \big\| C_{\widetilde{\Psi_1 \otimes \Psi_2}} F \big\|_{\mathfrak{l}^{r,\,s}_{w_1 \otimes w_2}(I \times J)}. \tag{30}$$

Finally we provide the definition of the Galerkin matrix of a linear operator.

Definition 4.17 *Let $\widetilde{\Psi}_1 := \{\widetilde{\psi}_{1,\,i}\}_{i \in I}$ be a dual of a frame $\Psi_1 := \{\psi_{1,\,i}\}_{i \in I}$ for a Hilbert space H_1, $\widetilde{\Psi}_2 := \{\widetilde{\psi}_{2,\,j}\}_{j \in J}$ a dual of a frame $\Psi_2 := \{\psi_{2,\,j}\}_{j \in J}$ for a Hilbert space H_2 and O a linear operator mapping H_1 on H_2. The Galerkin matrix k of the operator O is defined by*

$$k := (k_{i,\,j})_{(i,\,j) \in I \times J} := \Big(\big\langle O \widetilde{\psi}_{1,\,i}, \widetilde{\psi}_{2,\,j} \big\rangle_{H_2} \Big)_{(i,\,j) \in I \times J} = C_{\widetilde{\Psi}_1 \otimes \widetilde{\Psi}_2} O\,. \tag{31}$$

4.3 Structured frames and decomposition spaces

In this section we briefly discuss the notion of the structured coverings of the space $\mathbb{R}^2$ used to define the so-called modulation spaces of functions of two real variables and structured frames for those spaces, the latter being the frames whose elements have the supports of their Fourier transforms confined to the elements of the coverings.

The decomposition spaces – introduced by Feichtinger and Gröbner [18, 19] – are made of three basic building blocks, namely a structured or an almost-structured covering $\mathcal{Q}$ of a Lebesgue-measurable subset of the frequency plane, a regular partition of unity on the plane $\mathbb{R}^2$ and a $\mathcal{Q}$-moderate weight. Here are the definitions of these notions.

Definition 4.18 *The family $\mathcal{Q} = \{Q_i\}_{i\in I}$, where Q_i stand for subsets of $\mathbb{R}^2$, is called an almost-structured covering of a Lebesgue-measurable subset O of $\mathbb{R}^2$ if*

1. *the number of elements in the set $\{i' \in I : Q_{i'} \cap Q_i \neq \emptyset\}$ is uniformly bounded for any $i \in I$;*

2. *there exists such a family of invertible affine-linear maps $\{\mathbb{R}^2 \to \mathbb{R}^2,\, \xi \mapsto T_i\xi + b_i\}_{i\in I}$, where T_i and b_i stand for two-by-two and one-by-two matrices respectively, and such finite families $\{Q'_n\}_{n=1}^N$ and $\{P_n\}_{n=1}^N$ of non-empty open and bounded subsets Q'_i and P_i of $\mathbb{R}^2$ that*

   (a) $\overline{P_n} \subset Q'_n$ for $1 \leqslant n \leqslant N$;

   (b) for each $i \in I$ there exists such an $n_i \in \{1, ..., N\}$ that $Q_i = \big(T_i\, Q'_{n_i} + b_i\big) \cap O$;

   (c) there exists such a constant $C_1 > 0$ that $\|T_i^{-1}T_{i'}\| \leqslant C_1$ for all such i and $i' \in I$ that $Q_i \cap Q_{i'} \neq \emptyset$; and

   (d) $\bigcup_{i\in I}(T_iP_i + b_i) = O$.

The almost-structured covering is called simply *structured* if $N = 1$. Condition 1. ensures that the elements of the covering spread more or less uniformly over a neighbourhood of the subset O of $\mathbb{R}^2$. And this is precisely the feature of the almost-structured covering that will allow us later on to make structured frames corresponding to almost-structured coverings self-localised. Point (b) of Condition 2. guarantees that the elements of the covering are made from only a finite number of bounded subsets Q'_{n_i}'s of $\mathbb{R}^2$. Point (c) of Condition 2. ensures that those elements of the covering that overlap one another are similar in size and orientation.

Definition 4.19 *For any almost-structured covering $\mathcal{Q} = \{Q_i\}_{i\in I}$ of a Lebesgue-measurable subset O of the space $\mathbb{R}^2$ and any family of invertible affine-linear maps associated with it $\{\mathbb{R}^2 \to \mathbb{R}^2,\, \xi \mapsto T_i\xi + b_i\}_{i\in I}$, the family of functions $\Phi = \{\phi_i\}_{i\in I}$ is called a regular partition of unity on $O \subseteq \mathbb{R}^2$ subordinate to $\mathcal{Q}$, if*

1. *$\phi_i \in C_c^\infty(\mathbb{R}^2,\, [0,\, 1])$ with $\operatorname{supp}\phi_i \subset Q_i$ for all $i \in I$;*

2. *$\sum_{i\in I}\phi_i \equiv 1$ on $O \subseteq \mathbb{R}^2$; and*

3. $\sup_{i\in I} \| \partial^\alpha \phi_i^\natural \|_{L^\infty} < \infty$ *for any* $\alpha \in \mathbb{N}_0^2$ *where* $\phi^\natural : \mathbb{R}^2 \to [0,\, 1]$, $\xi \mapsto \phi_i\left((T_i\, \xi + b_i) \cap O\right)$.

Conditions 1. and 2. ensure that the elements of a regular partition of unity are smooth, or regular, enough, on the one hand, and that they have their supports confined to elements of the covering $\mathcal{Q}$, on the other hand. This is of paramount importance in order for the decomposition space, defined later on by the norm (32) in Definition 4.21, to have the desired smoothness and an atomic decomposition, introduced in Definition 4.3. According to Theorem 2.8 in [43] and Proposition 1 in [9], there exists a regular partition of unity subordinate to any, respectively, structured or almost-structured covering $\mathcal{Q}$.

Definition 4.20 *A sequence* $w = \{w_i\}_{i\in I}$ *of positive numbers is called a weight. The weight is called* $\mathcal{Q}$*-moderate where* $\mathcal{Q} = \{Q_i\}_{i\in I}$ *stands for an almost-structured covering of a Lebesgue-measurable subset* O *of* $\mathbb{R}^2$*, if there exists such a positive number* C *that* $w_i \leqslant C \cdot w_{i'}$ *for all such* i *and* $i' \in I$ *that* $Q_i \cap Q_{i'} \neq \emptyset$.

The weight determines the importance that will be assigned to every component in the norm (32) of an element of a decomposition space, introduced below, in Definition 4.21, depending on the component's location in the frequency plane. Using a $\mathcal{Q}$-*moderate* weight, where $\mathcal{Q}$ stands for an almost-structured covering, will guarantee that those components that are close to one another in the frequency plane are assigned comparable amounts of importance.

Definition 4.21 *Let* $\mathcal{Q} = (Q_i)_{i\in I_0}$ *be an almost-structured covering of a Lebesgue-measurable subset* O *of* $\mathbb{R}^2$, $\Phi = (\phi_i)_{i\in I_0}$ *be a regular partition of unity on* $O \subseteq \mathbb{R}^2$ *subordinate to* $\mathcal{Q}$, $w = (w_i)_{i\in I_0}$ *be a* $\mathcal{Q}$*-moderate weight,* $p, q \in (0, \infty]$ *and*

$$\|g\|_{\mathcal{D}(\mathcal{Q},\, L^p,\, \ell^q_w)} := \left\| \left(w_i \cdot \|\mathcal{F}^{-1}(\varphi_i \cdot \widehat{g})\|_{L^p} \right)_{i\in I_0} \right\|_{\ell^q} \in [0, \infty] \tag{32}$$

where $\mathcal{F}^{-1}$ *and* $\widehat{g}$ *stand for the inverse Fourier transformation and the Fourier transform of* g *respectively. Then*

$$\mathcal{D}(\mathcal{Q},\, L^p,\, \ell^q_w) := \left\{ g \in Z^* : \|g\|_{\mathcal{D}(\mathcal{Q},\, L^p,\, \ell^q_w)} < \infty \right\} ,$$

where Z^* *stands for the topological dual of* $Z := \mathcal{F}\left(C_c^\infty(\mathbb{R}^2)\right) \subset \mathcal{S}(\mathbb{R}^2)$*, is called a decomposition space.*

Us drawing elements of the modulation space $\mathcal{D}(\mathcal{Q},\, L^p,\, \ell^q_w)$ from the pool $Z^*(\mathbb{R}^2)$ of distributions larger than the space $\mathcal{S}(\mathbb{R}^2)$ of tempered distributions – as was originally done by Feichtinger [18] – ensures that the modulation space $\mathcal{D}(\mathcal{Q},\, L^p,\, \ell^q_w)$ is complete with respect to the quasi-norm defined by (32) and independent of the choice of a particular regular partition of unity – as long as it is subordinate to the same almost-structured covering $\mathcal{Q}$ – not only as its parameter $p \in [1,\, \infty]$ but also as $p \in (0,\, 1)$. Thus the decomposition space is a quasi-Banach space. In Definition 4.21, the functions $\{\mathcal{F}^{-1}(\varphi_i \cdot \widehat{g})\}_{i\in I_0}$ play the role of components of the function g localised in the frequency domain. Therefore, for (32) to be finite, and so for the function g to belong to $\mathcal{D}(\mathcal{Q}, L^p, \ell^q_w)$, these components must be

p-integrable and their contributions $\{\|\mathcal{F}^{-1}(\varphi_i \cdot \widehat{g})\|_{L^p}\}_{i \in I_0}$ to the norm $\| \,.\, \|_{\mathcal{D}(\mathcal{Q}, L^p, \ell^q_w)}$ weighted with w_i must be q-summable. The parameter s determines what weight w_i^s will be assigned to the contribution of the i-th component $\mathcal{F}^{-1}(\varphi_i \cdot \widehat{g})$ of g to the norm (32). Therefore varying the value of s amounts to making the space $\mathcal{D}(\mathcal{Q},\, L^p,\, \ell^q_w)$ more or, for that matter, less regular, since the index i determines, among other things, the absolute value of the frequency at which the i-th component is localised.

5 Main results of this study and their brief discussion

5.1 Kernel theorems for operators on co-orbit spaces

We first establish several properties of the tensor product of self-localised frames for Hilbert spaces and the co-orbit spaces associated with this tensor product. These properties allow us, among other things, to formulate the kernel theorems for the bounded linear operators mapping the smallest and largest of the co-orbit spaces associated with self-localised frames onto one another and characterises the corresponding operator spaces.

First of all we verify that the functions defined by (29) and (30) are indeed norms and establish a few interesting facts about tensor products of localised frames. Some parts of the statement of the following theorem are rather similar to those of [2, 25] for non-tensorial frames. Their proof is quite different, though, as the fact that frames Ψ_1 and Ψ_2 are self-localised does not necessarily implies that so is the frame $\Psi_1 \otimes \Psi_2$.

Theorem 5.1 *Let Ψ_1 and Φ_1 be $\mathcal{A}_1$-localised frames for a Hilbert space H_1, Ψ_2 and Φ_2 $\mathcal{A}_2$-localised frames for a Hilbert space H_2, w_1 and w_2, respectively, $\mathcal{A}_1$- and $\mathcal{A}_2$-admissible weights and $p \in [1,\, \infty]$. Then*

1. *the fact that $G_{\Psi_1,\,\Phi_1} \in \mathcal{A}_1$ and $G_{\Psi_2,\,\Phi_2} \in \mathcal{A}_2$ implies that the cross Gram matrix $G_{\Psi_1 \otimes \Psi_2,\, \Phi_1 \otimes \Phi_2}$ defines a bounded linear operator on the space $\ell^{p,\,p}_{w_1 \otimes w_2}(I \times J)$;*

2. *the fact that $G_{\Psi_1,\,\widetilde{\Phi}_1}$ and $G_{\Phi_1,\,\widetilde{\Psi}_1} \in \mathcal{A}_1$ and $G_{\Psi_1,\,\widetilde{\Phi}_1}$ and $G_{\Phi_2,\,\widetilde{\Psi}_2} \in \mathcal{A}_2$ implies that*
$$H^{p,\,p}_{w_1 \otimes w_2}(\Psi_1 \otimes \Psi_2) = H^{p,\,p}_{w_1 \otimes w_2}\big(\Phi_1 \otimes \Phi_2\big)$$
with equivalent norms and, in particular, that
$$H^{p,\,p}_{w_1 \otimes w_2}(\Psi_1 \otimes \Psi_2) = H^{p,\,p}_{w_1 \otimes w_2}\big(\widetilde{\Psi}_1 \otimes \widetilde{\Psi}_2\big);$$

3. *the synthesis operator $D_{\Psi_1 \otimes \Psi_2} : \ell^{p,\,p}_{w_1 \otimes w_2}(I \times J) \to H^{p,\,p}_{w_1 \otimes w_2}(\Psi_1 \otimes \Psi_2)$ is bounded and the expansion that the operator involves converges unconditionally strongly as $p < \infty$ or unconditionally weak-$*$ as $p = \infty$;*

4. *for any element $F \in H^{1,\,1}_{w_1 \otimes w_2}(\Psi_1 \otimes \Psi_2)$ there is such a sequence $c \in \ell^{1,\,1}_{w_1 \otimes w_2}(I \times J)$ that $F = D_{\Psi_1 \otimes \Psi_2} c$ and*
$$\|F\|_{H^{1,\,1}_{w_1 \otimes w_2}(\Psi_1 \otimes \Psi_2)} \asymp \|c\|_{\ell^{1,\,1}_{w_1 \otimes w_2}(I \times J)};$$

5. *for any* $p \in [1, \infty)$ *and the* q *that satisfies* $1/p + 1/q = 1$*,*

$$\left(H^{p,\,p}_{w_1\otimes w_2}(\Psi_1 \otimes \Psi_2)\right)^* \cong H^{q,\,q}_{1/(w_1\otimes w_2)}(\Psi_1 \otimes \Psi_2)$$

with the duality defined by

$$\langle F, G\rangle_{H^{p,\,p}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2),\, H^{q,\,q}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)} := \langle C_{\widetilde{\Psi_1\otimes\Psi_2}} F, C_{\Psi_1\otimes\Psi_2} G\rangle_{\ell^{p,\,p}_{w_1\otimes w_2}(I\times J),\, \ell^{q,\,q}_{1/(w_1\otimes w_2)}(I\times J)}\,.$$

It is unclear to us whether the statements of Theorem 5.1 apply to the co-orbit space $H^{p,\,q}_{w_1\otimes w_2}(\Psi_1 \otimes \Psi_2)$ as $p \neq q$. The main obstacle to understanding this is that one of the arguments we used to prove Statement 1. of Theorem 5.1 does not apply to the space $\ell^{p,\,q}$ as $p \neq q$. In particular, the cross Gram matrix $G_{\Psi_1\otimes\Psi_2,\,\Phi_1\otimes\Phi_2}$ is not necessarily a bounded operator on the spaces $\ell^{p,\,q}_{w_1\otimes w_2}(I \times J)$ as $p \neq q$. The proofs of Statements 2. to 5. rely crucially on the validity of Statement 1., though. That said, one of our kernel theorems will allow us later on to show that some mixed-norm co-orbit spaces are indeed independent of the specific choice of the frames used to define them, cf Corollary 5.7.

We now establish an expression for the co-orbit space $H^{\infty,\,\infty}_{1/(w_1\otimes w_2)}(\Psi_1 \otimes \Psi_2)$ and both a necessary and sufficient condition for $k \in \ell^{\infty,\,\infty}_{1/(w_1\otimes w_2)}(I \times J)$ to be the generalised wavelet transform of a vector $F \in H^{\infty,\,\infty}_{1/(w_1\otimes w_2)}(\Psi_1 \otimes \Psi_2)$ – the statement that, following the tradition established by Feichtinger and Fornasier [20, 25], may be called a *correspondence principle.*

Lemma 5.2 *Let* Ψ_1 *be an* $\mathcal{A}_1$*-localised frame for a Hilbert space* H_1*,* Ψ_2 *be an* $\mathcal{A}_2$*-localised frame for a Hilbert space* H_2 *and* w_1 *and* w_2*, respectively,* $\mathcal{A}_1$*- and* $\mathcal{A}_2$*-admissible weights. If* $F \in H^{\infty,\,\infty}_{1/(w_1\otimes w_2)}(\Psi_1 \otimes \Psi_2)$*, then*

$$F = D_{\Psi_1\otimes\Psi_2} C_{\widetilde{\Psi_1\otimes\Psi_2}} F = D_{\widetilde{\Psi_1\otimes\Psi_2}} C_{\Psi_1\otimes\Psi_2} F. \tag{33}$$

Moreover, for any $k \in \ell^{\infty,\,\infty}_{1/(w_1\otimes w_2)}(I \times J)$ *there exists such an* $F \in H^{\infty,\,\infty}_{1/(w_1\otimes w_2)}(\Psi_1 \otimes \Psi_2)$ *that*

$$k = C_{\widetilde{\Psi_1\otimes\Psi_2}} F \tag{34}$$

if and only if

$$k = G_{\widetilde{\Psi_1\otimes\Psi_2},\,\Psi_1\otimes\Psi_2}\, k = C_{\widetilde{\Psi_1\otimes\Psi_2}} D_{\Psi_1\otimes\Psi_2}\, k. \tag{35}$$

The statements also apply to $k \in \ell^{\infty,\,\infty}_{w_1\otimes w_2}(I \times J)$ *and* $F \in H^{\infty,\,\infty}_{w_1\otimes w_2}(\Psi_1 \otimes \Psi_2)$*.*

We also establish the isomorphism between the projective tensor product of two co-orbit spaces $H^1_{w_1}(\Psi_1)$ and $H^1_{w_2}(\Psi_2)$ associated with two different self-localised frames Ψ_1 and Ψ_2 and the co-orbit space $H^{1,\,1}_{w_1\otimes w_2}(\Psi_1 \otimes \Psi_2)$ associated with the tensor product $\Psi_1 \otimes \Psi_2$ of those two self-localised frames. This result of ours is summed up in the following theorem.

Theorem 5.3 *Let Ψ_1 be an $\mathcal{A}_1$-localised frame for a Hilbert space H_1, Ψ_2 be an $\mathcal{A}_2$-localised frame for a Hilbert space H_2 and w_1 and w_2, respectively, $\mathcal{A}_1$- and $\mathcal{A}_2$-admissible weights. Then*

$$H^{1,1}_{w_1\otimes w_2}\left(\Psi_1\otimes\Psi_2\right)=H^1_{w_1}\left(\Psi_1\right)\widehat{\otimes}_\pi H^1_{w_2}\left(\Psi_2\right) \tag{36}$$

with equivalent norms.

The isomorphism of the co-orbit spaces established in this theorem is very close in spirit to that of the Feichtinger algebra on a locally compact groups [17] except for lack of any group structure.

The established properties of the tensor product frames and co-orbit spaces associated with them put us in a position to prove inner and outer kernel theorems for the co-orbit spaces. Given self-localised frames Ψ_1 and Ψ_2 for Hilbert spaces H_1 and H_2 respectively, the theorems establish bijective correspondences between, on the one hand, the bounded linear operators mapping the smallest of the w_1-weighted co-orbit spaces $H^1_{w_1}(\Psi_1)$ and the largest of the $1/w_2$-weighted co-orbit spaces $H^\infty_{1/w_2}(\Psi_2)$ onto one another and, on the other hand, their kernels K. The kernel K in the inner kernel theorem belongs to the smallest of the $w_1\otimes w_2$-weighted co-orbit spaces $H^{1,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)$ while that in the outer kernel theorem belongs to the largest of the $1/(w_1\otimes w_2)$-weighted co-orbit spaces $H^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)$. This explains the names *inner* and *outer* of the theorems given to them according to the tradition established by Feichtinger and Jakobsen [23]. Here are the exact statements of our kernel theorems. Their proofs rely in the first place on representing operators by their Galerkin matrices and constructing the kernels of the operators from their matrix representations by using Lemma 5.2.

Theorem 5.4 *(Outer kernel theorem). Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be spectral matrix algebras and w_1 and w_2, respectively, $\mathcal{A}_1$- and $\mathcal{A}_2$-admissible weights. Furthermore let Ψ_1 be an $\mathcal{A}_1$-self-localised frame for a Hilbert space space H_1 and Ψ_2 an $\mathcal{A}_2$-self-localised frame for a Hilbert space H_2. Then, to any element of $K\in H^{\infty,\infty}_{1/(w_1\otimes w_2)}\left(\Psi_1\otimes\Psi_2\right)$ corresponds a unique bounded linear operator $O: H^1_{w_1}\left(\Psi_1\right)\to H^\infty_{1/w_2}\left(\Psi_2\right)$ by means of the expression*

$$\langle Of_1,f_2\rangle_{H^\infty_{1/w_2}(\Psi_2),\,H^1_{w_2}(\Psi_2)}=\langle K,f_1\otimes f_2\rangle_{H^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2),\,H^{1,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)} \tag{37}$$

that holds for any $f_1\in H^1_{w_1}\left(\Psi_1\right)$ and any $f_2\in H^1_{w_2}\left(\Psi_2\right)$. Moreover, there is the following equivalence of norms

$$\|K\|_{H^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)}\asymp\|O\|_{H^1_{w_1}(\Psi_1)\to H^\infty_{1/w_2}(\Psi_2)}\,. \tag{38}$$

Conversely, to any bounded linear operator $O: H^1_{w_1}\left(\Psi_1\right)\to H^\infty_{1/w_2}\left(\Psi_2\right)$ corresponds a unique $K\in H^{\infty,\infty}_{1/(w_1\otimes w_2)}\left(\Psi_1\otimes\Psi_2\right)$ – called the kernel *of the operator O – that satisfies the expression* (37) *for any $f_1\in H^1_{w_1}\left(\Psi_1\right)$ and any $f_2\in H^1_{w_2}\left(\Psi_2\right)$.*

Theorem 5.5 *(Inner kernel theorem). Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be spectral matrix algebras and w_1 and w_2, respectively, $\mathcal{A}_1$- and $\mathcal{A}_2$-admissible weights. Furthermore let Ψ_1 be an $\mathcal{A}_1$-self-localised* *frame for a Hilbert space space H_1 and Ψ_2 an $\mathcal{A}_2$-self-localised frame for a Hilbert space H_2. Then there is an isomorphism between the tensor product co-orbit space $H^{1,\,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)$ and the subspace of the space of bounded linear operators mapping the space $H^{\infty}_{1/w_1}(\Psi_1)$ onto the space $H^{1}_{w_2}(\Psi_2)$ for any element O of which there are such countable sequences $\{f_{1,\,i}\}_{i\in I}$ and $\{f_{2,\,i}\}_{i\in I}$ of elements of $H^1_{w_1}(\Psi_1)$ and $H^1_{w_2}(\Psi_2)$ respectively that*

$$\sum_{i\in I}\|f_{1,\,i}\|_{H^1_{w_1}(\Psi_1)}\,\|f_{2,\,i}\|_{H^1_{w_2}(\Psi_2)}<\infty \tag{39}$$

and

$$Of=\sum_{i\in I}\langle f,\,f_{1,\,i}\rangle_{H^{\infty}_{1/w_1}(\Psi_1),\,H^1_{w_1}(\Psi_1)}\,f_{2,\,i} \tag{40}$$

for any $f\in H^{\infty}_{1/w_1}(\Psi_1)$. More specifically, if either an operator $O: H^{\infty}_{1/w_1}(\Psi_1)\to H^1_{w_2}(\Psi_2)$ itself or its kernel $K\in H^{1,\,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)$ is known, the other is uniquely determined by means of the expression

$$\langle K,\,f_1\otimes f_2\rangle_{H^{1,\,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2),\,H^{\infty,\,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)}=\langle Of_1,\,f_2\rangle_{H^1_{w_2}(\Psi_2),\,H^{\infty}_{1/w_2}(\Psi_2)} \tag{41}$$

that holds for any $f_1\in H^{\infty}_{1/w_1}(\Psi_1)$ and any $f_2\in H^{\infty}_{1/w_2}(\Psi_2)$.

We can also characterise bounded linear operators mapping the co-orbit space $H^1_{w_1}(\Psi_1)$ onto the co-orbit space $H^p_{1/w_2}(\Psi_2)$ as $p\in(1,\,\infty)$ – the space intermediate between the spaces $H^1_{1/w_2}(\Psi_2)$ and $H^{\infty}_{1/w_2}(\Psi_2)$ – and those mapping the co-orbit space $H^p_{w_1}(\Psi_1)$ onto the co-orbit space $H^{\infty}_{1/w_2}(\Psi_2)$. To do so we combine the characterisation of the operators by matrices in a way similar to that used in [2, Proposition 6] with the relation that we established between Galerkin matrices representing operators and their kernels.

Theorem 5.6 *Let $p\in[1,\,\infty]$, q satisfy $\frac{1}{p}+\frac{1}{q}=1$. Furthermore let Ψ_1 be an $\mathcal{A}_1$-localised frame for a Hilbert space H_1, Ψ_2 be an $\mathcal{A}_2$-localised frame for a Hilbert space H_2 and w_1 and w_2, respectively, $\mathcal{A}_1$- and $\mathcal{A}_2$-admissible weights. Then*

1. *a linear operator $O: H^1_{w_1}(\Psi_1)\to H^p_{1/w_2}(\Psi_2)$ is bounded if and only if its kernel K belongs to the space $\mathfrak{H}^{p,\,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)$ or, alternatively, if and only if its Galerkin matrix k belongs to the space $\mathfrak{l}^{p,\,\infty}_{1/(w_1\otimes w_2)}(I\times J)$; and in that case*

$$\|O\|_{H^1_{w_1}(\Psi_1)\to H^p_{1/w_2}(\Psi_2)}\asymp\|K\|_{\mathfrak{H}^{p,\,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)}=\|k\|_{\mathfrak{l}^{p,\,\infty}_{1/(w_1\otimes w_2)}(I\times J)}; \tag{42}$$

2. *a linear operator $O: H^p_{w_1}(\Psi_1)\to H^{\infty}_{1/w_2}(\Psi_2)$ is bounded if and only if its kernel K belongs to the space $H^{q,\,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)$ or, alternatively, if and only if its Galerkin matrix k belongs to the space $\ell^{p,\,\infty}_{1/(w_1\otimes w_2)}(I\times J)$; and in that case*

$$\|O\|_{H^p_{w_1}(\Psi_1)\to H^{\infty}_{1/w_2}(\Psi_2)}\asymp\|K\|_{H^{q,\,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)}=\|k\|_{\ell^{p,\,\infty}_{1/(w_1\otimes w_2)}(I\times J)}. \tag{43}$$

This theorem allows us, in its turn, to prove that the spaces defined in Theorem 5.6 do not depend on the choice of specific localised frames in the tensor product as long as those frames are $\mathcal{A}_1$- and $\mathcal{A}_2$-localised.

Corollary 5.7 *Let Ψ_1 be an $\mathcal{A}_1$-localised frame for a Hilbert space H_1, Ψ_2 be an $\mathcal{A}_2$-localised frame for a Hilbert space H_2, w_1 and w_2, respectively, $\mathcal{A}_1$- and $\mathcal{A}_2$-admissible weights and $p \in [1, \infty]$. Then the fact that $G_{\Phi_1, \widetilde{\Psi}_1}$ and $G_{\Psi_1, \widetilde{\Phi}_1} \in \mathcal{A}_1$ and $G_{\Phi_2, \widetilde{\Psi}_2}$ and $G_{\Psi_2, \widetilde{\Phi}_2} \in \mathcal{A}_2$ implies that*

$$H^{p,\infty}_{w_1 \otimes w_2}(\Psi_1 \otimes \Psi_2) = H^{p,\infty}_{w_1 \otimes w_2}(\Phi_1 \otimes \Phi_2)$$

and

$$\mathfrak{H}^{p,\infty}_{w_1 \otimes w_2}(\Psi_1 \otimes \Psi_2) = \mathfrak{H}^{p,\infty}_{w_1 \otimes w_2}(\Phi_1 \otimes \Phi_2)$$

with their norms being equivalent.

Finally, we obtain a sufficient condition on the kernel to ensure that the corresponding operator belongs to the Schatten-p class as $p \in [1, 2]$.

Proposition 5.8 *Let Ψ_1 be an $\mathcal{A}_1$-localised frame for a Hilbert space H_1, Ψ_2 an $\mathcal{A}_2$-localised frame for a Hilbert space H_2 and $p \in [1, 2]$. If the kernel K of a bounded linear operator $O : H_1 \to H_2$ belongs to the space $\mathfrak{H}^{2,p}(\Psi_1 \otimes \Psi_2)$, then the operator O belongs to the Schatten-p class $\mathcal{S}_p(H_1, H_2)$ of operators mapping a Hilbert space H_1 onto a Hilbert space H_2. In particular, if $K \in H^{p,p}(\Psi_1 \otimes \Psi_2)$, then $O \in \mathcal{S}_p(H_1, H_2)$.*

5.2 Self-localised frames for tensor product spaces

We continue the study of properties of the tensor product of self-localised frames for Hilbert spaces that we began with Theorem 5.1 of the previous section and discover, we believe, the most striking of them all. Indeed we demonstrate that the tensor product of two self-localised frames is a self-localised frame too. This fact is important as it allows one to apply the whole range tools of co-orbit theory to tensor product frames to study corresponding operator spaces.

There are several quite ingenious methods developed to prove the inverse-closedness of Jaffard's [36], Schur's [33] and Sjöstrand's [42] algebras. However, those techniques cannot, in general, be applied to tensor products of algebras of different types or to algebras with anisotropic weights. To illustrate this, let us consider two frames $\Psi_1 = \{\psi_{1,k}\}_{k\in\mathbb{Z}}$ and $\Psi_2 = \{\psi_{2,k}\}_{k\in\mathbb{Z}}$ whose Gram matrices belong to the Jaffard class for two distinct weights, namely

$$\sup_{(k,l)\in\mathbb{Z}^2} \left|\langle \psi_{1,k}, \psi_{1,l}\rangle\right| \nu_{s_1}(k-l) < \infty, \quad \text{and} \quad \sup_{(m,n)\in\mathbb{Z}^2} \left|\langle \psi_{2,m}, \psi_{2,n}\rangle\right| \nu_{s_2}(m-n) < \infty$$

where $\nu_s(z) = (1+|z|)^s$ and both s_1 and s_2 are greater than one and differ from one another. The Gram tensor of rank four of the tensor product frame $\Psi_1 \otimes \Psi_2$ will then belong to the so-called anisotropic Jaffard class, that is,

$$\sup_{(k,l,m,n)\in\mathbb{Z}^4} \left|\langle \psi_{1,k} \otimes \psi_{2,m}, \psi_{1,l} \otimes \psi_{2,n}\rangle\right| \nu_{s_1}(k-l)\nu_{s_2}(m-n) < \infty.$$

Despite very promissing partial results by Gröchenig and Klotz [35], it has remained unclear whether this anisotropic Jaffard class in general is inverse-closed or not. One could try to circumvent this obstacle and consider the higher dimensional Jaffard class with isotropic weight $\nu_s\big((k,\,n)-(l,\,m)\big)$. This class is inverse-closed only as $s>2$, which assumes a degree of localisation that the frames Ψ_1 and Ψ_2 might not satisfy, though. In contrast to this example, our approach allows us to deal with tensor products of two algebras of the same class with distinct weights and that of two algebras of distinct classes.

This is how we achieve this. First of all we define what we call the *Gram tensor* for tensor product frames. Please note the way the elements of the tensor are indexed.

Definition 5.9 *Let $\{\Omega_{i,\,k}\}_{(i,\,k)\in I_1\times I_2}$ be a subset of the tensor product $H_1\otimes H_2$ of Hilbert spaces H_1 and H_2. The* Gram tensor of rank four G_Ω *is defined by*

$$G_\Omega := \big(\langle \Omega_{i,\,k},\Omega_{j,\,l}\rangle_{H_1\otimes H_2}\big)_{(i,\,k,\,l,\,j)\in I_1\times I_2^2\times I_1}. \tag{44}$$

If the set $\{\Omega_{i,\,k}\}_{(i,\,k)\in I_1\times I_2}$ consists only of elementary tensor products of elements of the subset Ψ_1 of the space H_1 and those of the subset Ψ_2 of the space H_2, the Gram tensor reduces to the Kronecker product of the Gram matrix of the set Ψ_1 and that of the set Ψ_2, namely

$$\begin{aligned} G_{\Psi_1\otimes\Psi_2} &= \big(\langle \psi_{1,\,j},\psi_{1,\,i}\rangle_{H_1}\,\langle \psi_{2,\,l},\psi_{2,\,k}\rangle_{H_2}\big)_{(i,\,k,\,l,\,j)\in I_1\times I_2^2\times I_1}\\ &= (G_{\Psi_1})_{(i,\,j)\in I_1^2}\,(G_{\Psi_2})_{(k,\,l)\in I_2^2}. \end{aligned} \tag{45}$$

Essential work to be done to obtain the main result of this section is to design a reasonable inverse-closed involutive algebra of tensor of rank four into which the Gram tensor G_Ω would fit. Let $\mathcal{T}$ denote the set of algebraic rank-four tensors $I_1\times I_2^2\times I_1\to\mathbb{C}$ [24]. Clearly $(\mathcal{T},\,\cdot\,,+)$ – that is, the set $\mathcal{T}$, together with pointwise addition and scalar multiplication – constitutes a vector space over the field $\mathbb{C}$. In order to define a subspace of the space $(\mathcal{T},\,\cdot\,,+)$ and make it into an involutive Banach algebra, we need to introduce a norm, a multiplication and an involution, for such tensors.

Definition 5.10 *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be involutive Banach matrix algebras with norms $\|\cdot\|_{\mathcal{A}_1}$ and $\|\cdot\|_{\mathcal{A}_2}$ whose elements are indexed by I_1^2 and I_2^2 respectively. For any tensor of rank four $A\in\mathcal{T}$,*

$$\|A\|_{\widetilde{\mathcal{A}}_1} := \left\|\left(\Big\|A\big|_{\{i\}\times I_2^2\times\{j\}}\Big\|_{\mathcal{B}(\ell^2(I_2))}\right)_{(i,\,j)\in I_1^2}\right\|_{\mathcal{A}_1} \tag{46}$$

where $A\big|_{\{i\}\times I_2^2\times\{j\}}$ stands for the restriction of A, viewed as a map $I_1\times I_2^2\times I_1\to\mathbb{C}$, to the subset $\{i\}\times I_2^2\times\{j\}$ and

$$\|A\|_{\widetilde{\mathcal{A}}_2} := \left\|\left(\Big\|A\big|_{I_1\times\{(k,\,l)\}\times I_1}\Big\|_{\mathcal{B}(\ell^2(I_1))}\right)_{(k,\,l)\in I_2^2}\right\|_{\mathcal{A}_2} \tag{47}$$

where $A\big|_{I_1\times\{(k,\,l)\}\times I_1}$ stands for the restriction of A to the subset $I_1\times\{(k,\,l)\}\times I_1$. Furthermore,

$$\widetilde{\mathcal{A}}_1 := \big\{A\in\mathcal{T}:\ \|A\|_{\widetilde{\mathcal{A}}_1}<\infty\big\},$$

$$\widetilde{\mathcal{A}}_2 := \left\{ A \in \mathcal{T} : \ \|A\|_{\widetilde{\mathcal{A}}_2} < \infty \right\}$$

and $\mathcal{A} := \mathcal{A}_1 \cap \mathcal{A}_2$ where $\mathcal{A}$ is equipped with the norm

$$\|A\|_{\mathcal{A}} := \max \left\{ \|A\|_{\widetilde{\mathcal{A}}_1}, \|A\|_{\widetilde{\mathcal{A}}_2} \right\}. \tag{48}$$

Our, perhaps somewhat unusual, way of indexing the elements of tensors of rank four is, we believe, fully justified by the fact that the *double contracted tensor multiplication* [41], which we shall formally define in a moment, functions very much like an ordinary matrix multiplication [41].

Definition 5.11 *For any two tensors of rank four $A := (A_{i,k,l,j})_{(i,k,l,j) \in I_1 \times I_2^2 \times I_1}$ and $B := (B_{i,k,l,j})_{(i,k,l,j) \in I_1 \times I_2^2 \times I_1}$, their* doubly contracted tensor product $A : B$ *will be understood as the tensor of rank four defined by*

$$\begin{aligned} &\Big((A : B)_{i,k,l,j} \Big)_{(i,k,l,j) \in I_1 \times I_2^2 \times I_1} \\ &\qquad := \left(\sum_{n \in I_2} \sum_{m \in I_1} A_{i,k,n,m} \, B_{m,n,l,j} \right)_{(i,k,l,j) \in I_1 \times I_2^2 \times I_1} . \end{aligned} \tag{49}$$

We also define the adjoint of a tensor of rank four in a way similar to that of a matrix, where the order of the indices is reversed and the elements of the matrix are complex conjugated.

Definition 5.12 *The adjoint A^* of a tensor of rank four $A \in \mathcal{T}$ will be understood as the tensor of rank four defined by*

$$\Big((A^*)_{i,k,l,j} \Big)_{(i,k,l,j) \in I_1 \times I_2^2 \times I_1} := \left(\overline{A_{j,l,k,i}} \right)_{(i,k,l,j) \in I_1 \times I_2^2 \times I_1}, \tag{50}$$

where the bar over $A_{j,l,k,i}$ stands for its complex conjugation.

These definitions allow us to obtain the desired inverse-closed involutive Banach algebra.

Theorem 5.13 *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two solid spectral algebras such that $\widetilde{\mathcal{A}}_1$ and $\widetilde{\mathcal{A}}_2$ are inverse-closed. Then*

$$(\mathcal{A}, \cdot, +, :, *, \|\cdot\|_{\mathcal{A}}) \tag{51}$$

is an inverse-closed involutive Banach algebra.

The assumption that the algebras $\widetilde{\mathcal{A}}_1$ and $\widetilde{\mathcal{A}}_2$ are inverse-closed seems quite sensible to us, as all the common inverse-closed algebras satisfy this condition. This follows from the numerous existing results of [36, 34, 4, 27, 42].

And here is the main result of this section.

Proposition 5.14 *Let Ψ_1 be an $\mathcal{A}_1$-self-localised frame for H_1 and Ψ_2 an $\mathcal{A}_2$-self-localised frame for H_2 where $\mathcal{A}_1$ and $\mathcal{A}_2$ are two such spectral matrix algebras that $\widetilde{\mathcal{A}}_1$ and $\widetilde{\mathcal{A}}_2$ are inverse-closed. Then the tensor product frame $\Psi_1 \otimes \Psi_2$ is an $\mathcal{A}$-self-localised frame for $H_1 \otimes H_2$.*

We would like to conclude this section with two applications of our main result. For any $I \subset \mathbb{R}^d$, any $s \in (d, \infty)$ and $\vartheta_s(i) := (1 + |i|)^s$ for any $i \in I$, the Jaffard algebra of complex-valued infinite matrices $I^2 \to \mathbb{C}$ is defined by

$$\mathcal{J}_s(I) := \left\{ A : \sup_{(i,j) \in I^2} |A_{i,j}| \vartheta_s(i - j) < \infty \right\}.$$

For any $I \subset \mathbb{R}^d$, any $p \in [1, \infty)$ and any positive function ϑ_δ on I, the ϑ_δ-weighted Schur algebra of complex-valued infinite matrices $I^2 \to \mathbb{C}$ is defined by

$$\mathcal{S}^p_\delta(I) := \left\{ A : \|A\|_{\mathcal{S}^p_\delta(I)} < \infty \right\}$$

where

$$\|A\|_{\mathcal{S}^p_\delta(I)} := \max \left\{ \sup_{i \in I} \left(\sum_{j \in I} |A_{i,j}|^p \vartheta_\delta(i-j)^p \right)^{\frac{1}{p}}, \quad \sup_{j \in I} \left(\sum_{i \in I} |A_{i,j}|^p \vartheta_\delta(i-j)^p \right)^{\frac{1}{p}} \right\}.$$

Finally, for any positive function ϑ on $\mathbb{Z}^d$, the Sjöstrand algebra of complex-valued infinite matrices $\mathbb{Z}^{2d} \to \mathbb{C}$ is defined by

$$\mathcal{C}_\vartheta := \left\{ A : \sum_{i \in \mathbb{Z}^d} \sup_{j \in \mathbb{Z}^d} |A_{j, j-i}| \vartheta(i) < \infty \right\}.$$

We consider operator-valued versions of the above-mentioned algebras. For a solid matrix algebra $\mathcal{A}$ and a Hilbert space H, we define

$$\mathcal{OP}(\mathcal{A}) := \left\{ (A_{i,j})_{(i,j) \in I^2} : \ A_{i,j} \in \mathcal{B}(H) \text{ and } (\|A_{i,j}\|_{\mathcal{B}(H)})_{(i,j) \in I^2} \in \mathcal{A} \right\}.$$

If $\mathcal{A}$ is, say, Jaffard's algebra, then this definition reads

$$\mathcal{OP}\big(\mathcal{J}_s(I)\big) := \left\{ A_{i,j} \in \mathcal{B}(H) : \sup_{(i,j) \in I^2} \|A_{i,j}\|_{\mathcal{B}(H)} \vartheta_s(i-j) < \infty \right\}.$$

It was shown in [37, Theorems 3.10, 4.8 and 6.3] that the operator-valued versions of Jaffard's, weighted Schur's and Sjöstrand's algebras are indeed inverse-closed. We make use of these results and show the following.

Corollary 5.15 *Let I_1 and I_2 be relatively separate discrete subsets of $\mathbb{R}^d$, $\mathcal{A}_1$ and $\mathcal{A}_2$ any two algebras from the set of algebras $\{\mathcal{J}_s(I_n), \mathcal{S}^1_\delta(I_n), \mathcal{C}_\nu\}$, $s > d$, $\delta \in (0,1]$ and ν a submultiplicative and symmetric weight that satisfies the condition $\lim_{n\to\infty} \nu(nz)^{1/n} = 1$ for any $z \in \mathbb{R}^d$. Then $\mathcal{A}$ is an inverse-closed involutive Banach algebra.*

Finally, the following result on self-localised frames for the tensor product $H_1 \otimes H_2$ of two Hilbert spaces is an almost immediate consequence of Definition 5.10 being applied to the tensor product of two self-localised frames.

Proposition 5.16 *Let Ψ_1 be an $\mathcal{A}_1$-self-localised frame for H_1 and Ψ_2 an $\mathcal{A}_2$-self-localised frame for H_2 where $\mathcal{A}_1$ and $\mathcal{A}_2$ are two spectral matrix algebras such that $\widetilde{\mathcal{A}}_1$ and $\widetilde{\mathcal{A}}_2$ are inverse-closed. Then the tensor product frame $\Psi_1 \otimes \Psi_2$ is an $\mathcal{A}$-self-localised frame for $H_1 \otimes H_2$.*

5.3 Self-localised frames for the wave packet spaces

The advantage of localised frames lies in their abstraction, making them more generally applicable in pure mathematical studies. Structured frames, on the other hand, have analytical, albeit sometimes rather cumbersome, expressions, making them more suitable for numerical calculations. We try to relate these two apparently different types of frames by close examination of the properties of the notion of the almost-structured covering that underlies that of the structured frame. More specifically, we demonstrate that most prominent structured frames such as Gabor frames [30], wavelets [13], frames for α-modulation spaces with $\alpha \in [0,\,1]$ [28], ridgelets [10], curvelets [11], shearlets [38] and wave atoms [14] for the corresponding decomposition spaces [28] of complex functions of two real variables can be made to be $\mathcal{A}$-self-localised where $\mathcal{A}$ stands for a Jaffard's algebra. To do so we design what we call the *wave packet system* and use it to introduce a family of quasi-Banach spaces – which we call *wave packet spaces* – that encompasses all the above-mentioned decomposition spaces. Then we establish the conditions under which the wave packet system constitutes a Banach frame for the wave packet spaces, the above-mentioned structured frames being specific example of such a wave packet system. Indeed, as the pair of parameters $(\alpha,\,\beta)$ of the wave packet system equals $(0,\,0)$, $(\frac{1}{2},\,\frac{1}{2})$, $(1,\,0)$, $(1,\,\frac{1}{2})$ and $(1,\,1)$, the wave packet system becomes a Gabor, wave atom, ridgelet, curvelet and wavelet frame respectively. As $\alpha = \beta \in (0,\,1)$, the wave packet system constitutes a frame for the corresponding α-modulation space. Finally, we show that the Gram matrix of the wave packet system belongs to a Jaffard's algebra. In what follows we formulate, alas, quite a few rather intricate definitions and try to clarify them in order to put ourselves in a position to state the main result of this section.

The wave packet spaces are quasi-Banach spaces defined by the decomposition method – introduced by Feichtinger and Gröbner [18, 19] – and, as such, are made of three basic building blocks, namely an almost-structured covering $\mathcal{Q}^{(\alpha,\,\beta)}$ of a Lebesgue-measurable subset O of the frequency plane $\mathbb{R}^2$, a regular partition of unity on the subset O and a $\mathcal{Q}^{(\alpha,\,\beta)}$-moderate weight. Here are their definitions.

Definition 5.17 *First, for any $\epsilon \in (0, 1/32)$,*

$$Q := (-\epsilon, 1+\epsilon) \times (-1-\epsilon, 1+\epsilon)\,; \tag{52}$$

second, for any $\alpha \in [0,\,1]$, any $\beta \in [0,\,\alpha]$, $N := 10$ and any $j^{\max} \in \mathbb{N}$,

$$I_0 := \{(0,0,0)\} \cup \{(j,m,l) \in \mathbb{N}\times\mathbb{N}_0\times\mathbb{N}_0 : j \leqslant j^{\max},\ m \leqslant m_j^{\max} \text{ and } l \leqslant l_j^{\max}\} \tag{53}$$

where

$$l_j^{\max} := \lceil N \cdot 2^{(1-\beta)j} \rceil - 1 \tag{54}$$

and

$$m_j^{\max} := \begin{cases} \lceil 2^{(1-\alpha)j-1} \rceil - 1 & \text{if } \theta_{j,l} = \theta_{j+1,l'} \text{ for an } \mathbb{N}_0 \ni l' \leqslant l_{j+1}^{\max}\ , \text{ or if } j = j^{\max} \\ \lceil 2^{(1-\alpha)j-1} \rceil & \text{if } \theta_{j,l} \neq \theta_{j+1,l'} \text{ for any } \mathbb{N}_0 \ni l' \leqslant l_{j+1}^{\max} \end{cases} \tag{55}$$

where

$$\theta_{j,l} := 2\pi \cdot \frac{2^{(\beta-1)j}}{N} \cdot l\ \ ; \tag{56}$$

finally, for $Q_0 := Q_{0,0,0} := B_4(0)$*, where* $B_4(0)$ *stands for the Euclidean ball of radius four with its centre in the origin, and*

$$Q_i := Q_{j,m,l} := (R_{j,l}(A_{j,m}Q + B_{j,m})) \cap (B_{R_{max}}(0)) \quad , \tag{57}$$

where

$$A_{j,m} = \begin{pmatrix} 2^{\alpha j} & 0 \\ 0 & 2^{(\beta-1)j}\left(2^{j-1} + (m+1)2^{\alpha j}\right) \end{pmatrix} \quad , \quad B_{j,m} = \begin{pmatrix} 2^{j-1} + m \cdot 2^{\alpha j} \\ 0 \end{pmatrix} \tag{58}$$

and

$$R_{j,l} := \begin{pmatrix} \cos\theta_{j,l} & -\sin\theta_{j,l} \\ \sin\theta_{j,l} & \cos\theta_{j,l} \end{pmatrix} \tag{59}$$

for any $i = (j,m,l) \in I_0 \setminus \{(0,0,0)\}$*, the family* $\mathcal{Q}^{(\alpha,\beta)} := \{Q_i\}_{i\in I_0}$ *is called a* wave packet covering *of* $B_{R_{max}}(0) \subseteq \mathbb{R}^2$ *where* $R_{max} := 2^{j^{\max}}$*.*

According to (57), every element of the covering $\mathcal{Q}^{(\alpha,\beta)}$ except for the disc Q_0 is made from the rectangle Q of the frequency plane $\mathbb{R}^2$ by scaling, shifting and rotating it around the origin of the plane $\mathbb{R}^2$. Figure 1 shows a chart of the generic element $Q_i := Q_{j,m,l}$ of the wave packet covering with $j \neq 0$. The design of the covering, among other things, ensures that all the elements of the covering are unique. It should be mentioned that a wave packet covering of any subset O of the plane $\mathbb{R}^2$ can be built in the exactly the same way as that of the disc $B_{R_{max}}(0)$ by replacing the disc $B_{R_{max}}(0)$ by the set O in Definition 5.17.

Here we recollect what we already mentioned in Section 4.3, namely that there is a regular partition of unity subordinate to any almost-structured covering.

Definition 5.18 *Let* $0 \leqslant \beta \leqslant \alpha \leqslant 1$*,* $s \in \mathbb{R}$*,* I_0 *be as defined by* (53) *and*

$$w_i^s := \begin{cases} 2^{js} & \text{if } i = (j,m,l) \in I_0 \setminus \{(0,0,0)\} \\ 1 & \text{if } i = 0 = (0,0,0) \quad . \end{cases} \tag{60}$$

Then $w^s = (w_i^s)_{i\in I_0}$ *is called a wave packet weight.*

These definitions allow us to introduce the wave packet spaces.

Definition 5.19 *For any* $\alpha \in [0,\,1]$*, any* $\beta \in [0,\,\alpha]$*, any* $s \in \mathbb{R}$*, any* p *and* $q \in (0,\infty]$*, the* wave packet space *is defined as the decomposition space by*

$$\mathcal{W}_s^{p,q}(\alpha,\beta) := \mathcal{D}(\mathcal{Q}^{\alpha,\beta}, L^p, \ell^q_{w^s})\,.$$

Thus the wave packet space is the decomposition space corresponding to the covering $\mathcal{Q}^{\alpha,\beta}$.

Here is the definition of what we call *wave packet system* that constitutes both a Banach frame for and a set of atoms of the wave packet spaces $\mathcal{W}_s^{p,q}(\alpha,\,\beta)$.

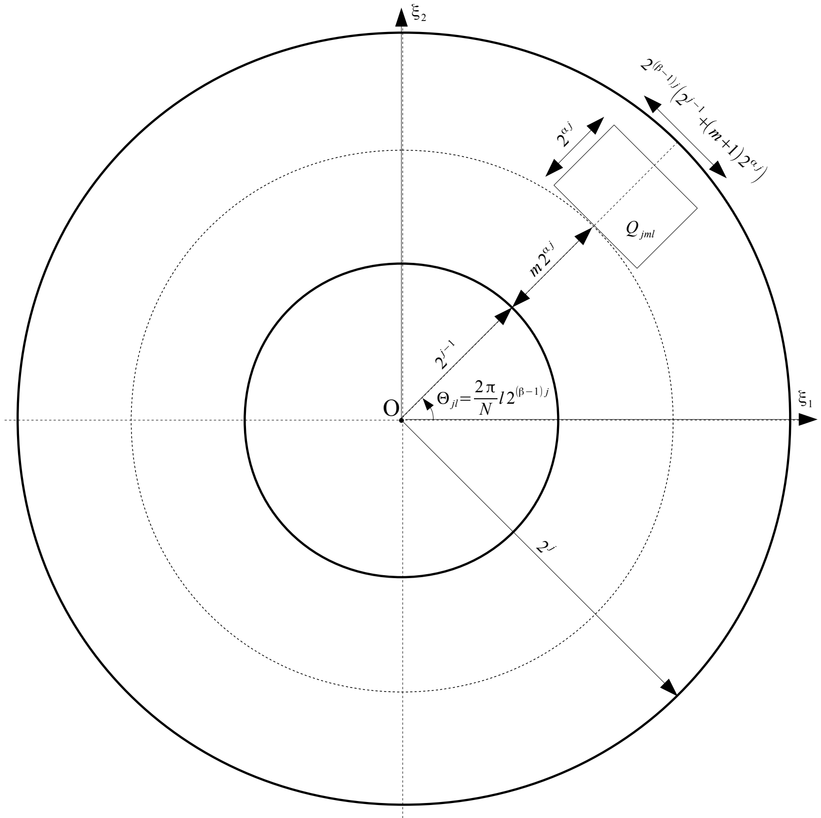

Figure 1: Chart of a generic element $Q_{j,m,l}$ of the wave packet covering introduced in Definition 5.17. The distance between the rectangle $Q_{j,m,l}$ and the origin O of the frequency plane equals $2^{j-1}+m2^{\alpha j}$. The length of $Q_{j,m,l}$, in the radial direction, equals $2^{\alpha j}$ while its width, in the angular direction, roughly approximates $2^{\beta j}$. Its axis of symmetry that intersects the origin O deviates from the ξ_1-axis by the angle $\Theta_{j,l}=\frac{2\pi}{N}\cdot l\cdot 2^{(\beta-1)j}$. For a given j, all rectangles $Q_{j,m,l}$ are contained in the dyadic ring $\{\xi\in\mathbb{R}^2 : |\xi|\asymp 2^j\}$. The indices $m_j^{\max}$ and $l_j^{\max}$ roughly approximate $2^{(1-\alpha)j}$ and $2^{(1-\beta)j}$ respectively. The bold continuous circular lines indicate boundaries between adjacent dyadic rings.

Definition 5.20 *First, let $\alpha\in[0,1]$, $\beta\in[0,\alpha]$, $N:=10$, $s_0>0$, $s\in[-s_0,s_0]$, $p_0\in(0,1]$, $p\in[p_0,\infty]$, $q_0\in(0,1]$, $q\in[q_0,\infty]$, $\delta\in(0,\delta_0]$ where $\delta_0=\delta_0(\alpha,\beta,s_0,p_0,q_0,\gamma,\psi)$. Second, let γ and $\psi\in C^1(\mathbb{R}^2)$ be such that*

$$\sup_{t\in\mathbb{R}^2}(1+|t|)^{\kappa_0}\cdot|\gamma(t)|<\infty \quad \textit{and} \quad \sup_{t\in\mathbb{R}^2}(1+|t|)^{\kappa_0}\cdot|\psi(t)|<\infty \tag{61}$$

where $\kappa_0:=10+2\cdot p_0^{-1}$ and $\partial^a\gamma$ and $\partial^a\psi\in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$ for any such $a\in\mathbb{N}_0^2$ that $|a|\leqslant 1$

Third, let $\hat\gamma$ and $\hat\psi\in C^\infty(\mathbb{R}^2)$ be such that $\hat\gamma(\xi)\neq 0$ for any $\xi:=(\xi_1,\xi_2)\in\overline{B}_4(0)$, $\hat\psi(\xi)\neq 0$

for any $\xi \in [-\epsilon, 1+\epsilon] \times [-1-\epsilon, 1+\epsilon]$,

$$|\partial^a \widehat{\partial^b \gamma}\,(\xi)| \lesssim (1+|\xi|)^{-\kappa} \cdot (1+|\xi_1|)^{-\kappa_1} \cdot (1+|\xi_2|)^{-\kappa_2} \tag{62}$$

and

$$|\partial^a \widehat{\partial^b \psi}\,(\xi)| \lesssim (1+|\xi|)^{-\kappa} \cdot (1+|\xi_1|)^{-\kappa_1} \cdot (1+|\xi_2|)^{-\kappa_2} \tag{63}$$

for any such $a \in \mathbb{N}_0^2$ *that* $|a| \leqslant 12$*, any such* $b \in \mathbb{N}_0^2$ *that* $|b| \leqslant 1$*, any* $\kappa \geqslant 10$*, any* $\kappa_1 \geqslant 2$ *and any* $\kappa_2 \geqslant 10$*. Fourth, let, for any* $j^{\max} \in \mathbb{N}$,

$$I := \{(0,0,0,k) : k \in \mathbb{Z}^2\} \cup \{(j,m,l,k) \in \mathbb{N}\times\mathbb{N}_0\times\mathbb{N}_0\times\mathbb{Z}^2 : j \leqslant j^{\max},\ m \leqslant m_j^{\max} \text{ and } l \leqslant l_j^{\max}\}\ , \tag{64}$$

where $m_j^{\max}$ *and* $l_j^{\max}$ *defined by* (55) *and* (54) *respectively, and*

$$\psi_i(t) := \begin{cases} \gamma(t - \delta \cdot k) & \text{if } (j,m,l) = (0,0,0) \\ |\det A_{jm}|^{1/2} \cdot e^{-2\pi i \langle R_{jl} B_{jm}, t\rangle} \cdot \left(\psi \ \circ A_{jm} \circ R_{jl}^{-1}\right)(t - \delta \cdot R_{jl} A_j^{-1} k) & \text{if } (j,m,l) \neq (0,0,0) \end{cases}\ , \tag{65}$$

where A_{jm}*,* B_{jm} *and* R_{jl} *defined by* (58) *and* (59)*. Then the set* $W(\alpha,\, \beta) := \{\psi_i\}_{i\in I}$ *is called a* wave packet system*.*

The coefficient space that is associated with the wave packet space $\mathcal{W}_s^{p,\,q}(\alpha,\beta)$ through its frame, the wave packet system $W(\alpha,\, \beta)$, is given by the following definition.

Definition 5.21 *Let* $\alpha \in [0,\, 1]$*,* $\beta \in [0,\, \alpha]$*,* $s \in \mathbb{R}$*,* p *and* $q \in (0,\infty]$*, and* I_0 *be as defined in* (53)*. The set of sequences of complex numbers*

$$\mathcal{C}_s^{p,\,q}(\alpha,\beta) := \left\{ c = \{c_i^{(k)}\}_{i\in I_0,\, k\in\mathbb{Z}^2} \in \mathbb{C}^{I_0\times\mathbb{Z}^2}\ :\ \|c\|_{\mathcal{C}_s^{p,\,q}} < \infty \right\}\ ,$$

where

$$\|c\|_{\mathcal{C}_s^{p,\,q}} := \left\| \left(w_i^{s+(\alpha+\beta)\cdot(\frac{1}{2}-\frac{1}{p})} \cdot \left\| (c_k^{(i)})_{k\in\mathbb{Z}^2} \right\|_{\ell^p} \right)_{i\in I_0} \right\|_{\ell^q} \in [0,\infty]$$

is called a coefficient space *associated with the wave packet space* $\mathcal{W}_s^{p\,q}(\alpha,\beta)$*.*

The Fourier transform of ψ_i, as defined by (65), is given by

$$\hat{\psi}_i(\xi) := \begin{cases} e^{-2\pi i\langle \delta\cdot k, \xi\rangle} \cdot \hat{\gamma}(\xi) & \text{if } (j,m,l) = (0,0,0) \\ |\det A_{jm}|^{1/2} \cdot e^{-2\pi i \langle \delta \cdot R_{jl} A_j^{-1} k, \xi\rangle} & \\ \qquad\qquad \cdot \left(\hat{\psi} \circ A_{jm}^{-1} \circ R_{jl}^{-1}\right)(\xi - R_{jl} A_j B_{jm}) & \text{if } (j,m,l) \neq (0,0,0) \end{cases} \tag{66}$$

for any $\xi \in \mathbb{R}^2$. The functions γ and ψ play the role of prototypes of the wave packets ψ_i with $i \in I$. Specifically all such wave packets ψ_i with an i that $(j,m,l) = (0,0,0)$ are generated from γ only by shifting it in the time plane, while the wave packets ψ_i with such an i that $(j,m,l) \neq (0,0,0)$ are generated from ψ by scaling, modulating, rotating and shifting it in the time plane. The index $j \in \mathbb{N}$ identifies the measurements $2^{-\alpha j} \times 2^{-\beta j}$ of the essential support of the wave packet $\psi_i(t)$ – compared to those of its prototype – and,

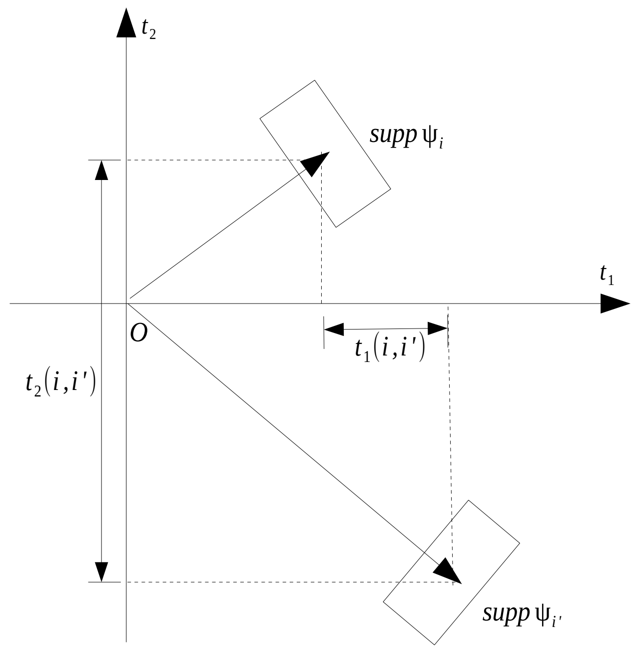

Figure 2: Chart of the supports of two wave packets ψ_i and ψ_i', together with the lengths $t_1(i,\, i')$ and $t_2(i,\, i')$ of the orthogonal projections of the difference $t_i - t_{i'}$ between the radii vectors t_i and $t_{i'}$ from the origin of the time plane to the maxima of the wave packets ψ_i and $\psi_{i'}$ on the horizontal and vertical axes of the Cartesian coordinate system of the time plane.

indeed, the measurements $2^{\alpha j} \times 2^{\beta j}$ of the essential support of its Fourier transform $\hat{\psi}_i$ – compared to those of the Fourier transform of its prototype (see Figure 1). The indices j, m and l determine the position ξ_i of the maximum of the Fourier transform $\hat{\psi}_i$ of the wave packet ψ_i in the frequency plane, namely

$$\xi_i := \xi_{jml} := \begin{cases} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{if } (j,m,l) = (0,0,0) \\ R_{jl} B_{jm} = R_{jl} \begin{pmatrix} 2^{j-1} + m \cdot 2^{\alpha j} \\ 0 \end{pmatrix} & \text{if } (j,m,l) \neq (0,0,0) \quad . \end{cases} \tag{67}$$

The indices j and m in particular determine the distance r_i of the maximum of the of Fourier transform $\hat{\psi}_i(\xi)$ of the wave packet $\psi_i(t)$ from the origin of the frequency plane, namely

$$r_i := |\xi_i| := |\xi_{jml}| := \begin{cases} 0 & \text{if } (j,m,l) = (0,0,0) \\ 2^{j-1} + m \cdot 2^{\alpha j} & \text{if } (j,m,l) \neq (0,0,0) \quad , \end{cases} \tag{68}$$

while the indices j and l determine the angle θ_i at which the Fourier transform $\hat{\psi}_i(\xi)$ of the

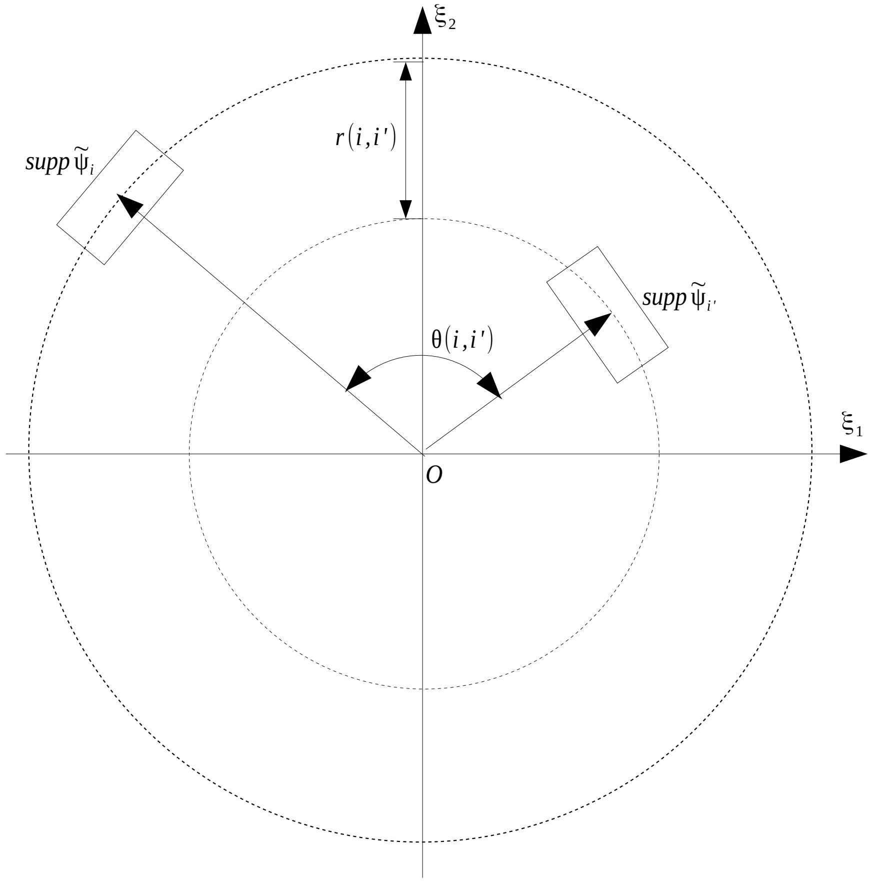

Figure 3: Chart of the supports of the Fourier transforms $\hat{\psi}_i$ and $\hat{\psi}_{i'}$ of two wave packets ψ_i and ψ'_i, together with the absolute values of the differences $r(i,i')$ and $\theta(i,i')$ between the polar coordinates of the maxima of the Fourier transforms $\hat{\psi}_i$ and $\hat{\psi}_{i'}$ of the wave packets ψ_i and $\psi_{i'}$ on the frequency plane

wave packet $\psi_i(t)$ is inclined to the ξ_1-axis in the frequency plane, namely

$$\theta_i := \begin{cases} 0 & \text{if } (j,m,l) = (0,0,0) \\ \theta_{j,l} & \text{if } (j,m,l) \neq (0,0,0) \end{cases} \tag{69}$$

with $\theta_{j,l}$ defined by (56). In other words r_i and θ_i are polar coordinates of the maximum of the Fourier transform of the wave packet $\hat{\psi}_i$ in the frequency plane. The indexes k, j and l determine the position of the maximum of the wave packet $\psi_i(t)$ in the time plane, namely

$$t_i := t_k^{(jl)} := \begin{cases} \delta \cdot k = \delta \cdot \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} & \text{if } (j,m,l) = (0,0,0) \\ \delta \cdot R_{jl} A_{jm}^{-1} k = \delta \cdot R_{jl} \begin{pmatrix} k_1 \cdot 2^{-\alpha j} \\ k_2 \cdot 2^{-\beta j} \end{pmatrix} & \text{if } (j,m,l) \neq (0,0,0) \end{cases} . \tag{70}$$

In order to formulate the main theorem of this part of the dissertation we have to be in a position to measure the distance between elements of the index set I_0, defined by (53), for the wave packet spaces $\mathcal{W}_s^{p,q}(\alpha,\beta)$. Here is the definition of the distance.

Definition 5.22 *Let $i = (j, m, l, k)$ and $i' = (j', m', l', k') \in I$ where I defined by* (64)*, $r(i, i') := |r_i - r_{i'}| \in \mathbb{R}_+$ where r_i and $r_{i'}$ defined by* (68)*, $\theta(i, i') := |\theta_i - \theta_{i'}| \in [0, \pi]$ where θ_i and $\theta_{i'}$ defined by* (69)*, $t_1(i, i') := |(t_i - t_{i'})_1| \in \mathbb{R}_+$ and $t_2(i, i') := |(t_i - t_{i'})_2| \in \mathbb{R}_+$ where t_i and $t_{i'}$ defined by* (70)*. Then*

$$\rho(i, i') := r(i, i') + \theta(i, i') + t_1(i, i') + t_2(i, i') \tag{71}$$

will be called a distance *between the indices i and i' of the wave packets ψ_i and $\psi_{i'}$.*

Thus the distance $\rho(i, i')$ between the indices i and i' and, indeed between the wave packets ψ_i and $\psi_{i'}$ indexed by them themselves, is defined as the sum of four terms, viz. the lengths $|(t_i - t_{i'})_1|$ and $|(t_i - t_{i'})_2|$ of the orthogonal projections of the difference $t_i - t_{i'}$ between the radii vectors t_i and $t_{i'}$ from the origin of the time plane to the maxima of the wave packets ψ_i and $\psi_{i'}$ on the horizontal and vertical axes of the Cartesian coordinate system of the time plane (see Figure 2.) and the absolute values of the differences $r(i, i')$ and $\theta(i, i')$ between the polar coordinates of the maxima of the Fourier transforms $\hat{\psi}_i$ and $\hat{\psi}_{i'}$ of the wave packets ψ_i and $\psi_{i'}$ on the frequency plane (see Figure 3.). A special mention should be made of the possibility to re-balance the individual contributions of each of the terms r, θ, t_1 and t_2 to the distance ρ by multiplying those terms by appropriate positive numbers without changing in any meaningful way the facts about the distance and its use in this study.

We now establish the following lemmata.

Lemma 5.23 *The the metric space I_0 defined by* (64) *with any $j^{\max} \in \mathbb{N}$ and the distance ρ defined by* (71) *is separate, that is, there is such a positive number C that*

$$\inf_{i,\, i' \in I_0,\, i \neq i'} \rho(i, i') = C > 0 \ \ . \tag{72}$$

Lemma 5.24 *For any $\alpha \in [0,\, 1]$, any $\beta \in [0,\, \alpha]$ and any $s \in \mathbb{R}$,*

$$\mathcal{W}_s^{2,\,2}(\alpha, \beta)(\mathbb{R}^2) = H^s(\mathbb{R}^2) \tag{73}$$

where $H^s(\mathbb{R}^2)$ stands for the Sobolev space

$$H^s(\mathbb{R}^2) := \left\{ f \in \mathbf{S}'(\mathbb{R}^2) : (1 + |\xi|^2)^{s/2}\, \widehat{f} \in L^2(\mathbb{R}^2) \right\} . \tag{74}$$

That means, among other things, that $\mathcal{W}_s^{2,\,2}(\alpha, \beta)(\mathbb{R}^2)$ is a Hilbert space.

Lemma 5.25 *Let p_1, q_1, p_2 and $q_2 \in (0,\, \infty]$. For any $s \in \mathbb{R}$,*

$$\mathcal{W}_{s_1}^{p_1,\, q_1}(\alpha, \beta)(\mathbb{R}^2) \hookrightarrow \mathcal{W}_{s_2}^{p_2,\, q_2}(\alpha, \beta)(\mathbb{R}^2)\, , \tag{75}$$

where the symbol $\hookrightarrow$ stands for a continuous embedding, if and only if $p_1 \leqslant p_2$ and either $q_1 \leqslant q_2$ and

$$s_1 \geqslant s_2 + (p_1^{-1} - p_2^{-1})(\alpha + \beta) \tag{76}$$

or $q_1 > q_2$ and

$$s_1 > s_2 + (p_1^{-1} - p_2^{-1})(\alpha + \beta) + (2 - \alpha - \beta)(q_2^{-1} - q_1^{-1})\, . \tag{77}$$

We are now in a position to formulate the main theorem of this section.

Theorem 5.26 *Let κ_0 and κ introduced in Definition 5.20 not be smaller than 24. Then, for any $s \in \mathbb{R}$, the wave packet system $W(\alpha, \beta)$ is a $\mathcal{A}$-self-localied frame for the wave packet space $\mathcal{W}_s^{2,2}(\alpha, \beta)$, $\mathcal{A}$ standing for a Jaffard's algebra. More specifically, for any two elements ψ_i and $\psi_{i'}$ of the wave packet system $W(\alpha, \beta)$ and any $n > 5$,*

$$|\langle \psi_i | \psi_{i'} \rangle| \lesssim (1 + \rho(i, i'))^{-n} \ . \tag{78}$$

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7 Publications that make up this dissertation

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7.1 Publication 1

J. Math. Anal. Appl. 551 (2025) 129678

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Regular Articles

Kernel theorems for operators on co-orbit spaces associated with localised frames

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A B S T R A C T

Kernel theorems provide a convenient representation of bounded linear operators. For the operator acting on a concrete function space, this means that its action on any element of the space can be expressed as a generalised integral operator, in a way reminiscent of the matrix representation of linear operators acting on finite dimensional vector spaces. We prove kernel theorems for bounded linear operators acting on co-orbit spaces associated with localised frames. Our two main results characterise the spaces of operators whose generalised integral kernels belong to the co-orbit spaces of test functions and distributions associated with the tensor product of the localised frames respectively. Moreover, using a version of Schur's test, we establish a characterisation of the bounded linear operators between some specific co-orbit spaces and kernels in mixed-norm co-orbit spaces.

1. Introduction

This paper is concerned with co-orbit spaces, which were originally introduced in the late 1980's by Feichtinger and Gröchenig [22–24], and the bounded linear operators that act on these spaces. Co-orbit spaces, such as modulation [20] and Besov spaces [7], are defined as Banach spaces of functions or distributions whose membership is determined by integrability conditions imposed on their generalised wavelet transform. Co-orbit theory provides a convenient way of analysing and synthesising spaces of functions or distributions. The original definition of co-orbit spaces relied on an integrable representation of a locally compact group on a Hilbert space. In the early 2000's, Gröchenig and Fornasier showed how localised frames can be used to

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introduce a novel class of co-orbit spaces and guarantee their stable decomposition [26,32,33]. In this paper, we prove kernel theorems for bounded linear operators acting on co-orbit spaces associated with localised frames.

The main purpose of any kernel theorem is to express a given operator as a generalised integral operator. The original kernel theorem, proved by Laurent Schwartz in 1952, states that for any continuous linear operator O mapping the Schwartz space $S(\mathbb{R}^n)$ on the space of tempered distributions $S'(\mathbb{R}^n)$ there is a unique tempered distribution $K \in S'(\mathbb{R}^{2n})$, called the *kernel*, that satisfies

$$\langle Of_1, f_2\rangle_{S'(\mathbb{R}^n),S(\mathbb{R}^n)} = \langle K, f_1 \otimes f_2\rangle_{S'(\mathbb{R}^{2n}),S(\mathbb{R}^{2n})}, \tag{1.1}$$

for any $f_1, f_2 \in S(\mathbb{R}^n)$. Furthermore, if K happens to be a locally integrable function, then

$$\langle Of_1, f_2\rangle_{S'(\mathbb{R}^n),S(\mathbb{R}^n)} = \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} K(x,y)f_1(y)\overline{f_2(x)}dydx, \tag{1.2}$$

and so the operator O is indeed expressed as an integral operator. Other classical kernel theorems were established, for example, for the continuous operators mapping $\mathcal{D}(\mathbb{R}^d)$ on $\mathcal{D}'(\mathbb{R}^d)$ [35, Theorem 5.2], and for operators mapping Gelfand-Shilov spaces on their distribution spaces [30].

In 1980, Feichtinger proved a kernel theorem for bounded linear operators mapping the modulation spaces $M^1(\mathbb{R}^n)$ to $M^\infty(\mathbb{R}^n)$ [19], the spaces of choice in time-frequency analysis [20]. Using a version of Schur's test, Cordero and Nicola characterised the kernel of operators mapping $M^1(\mathbb{R}^n)$ on $M^p(\mathbb{R}^n)$ as well as those mapping $M^p(\mathbb{R}^n)$ on $M^\infty(\mathbb{R}^n)$ [17]. Finally, Balazs, Gröchenig and Speckbacher took a more abstract approach to prove their kernel theorem [6] for operators acting on co-orbit spaces associated with integrable representations of locally compact groups. Indeed, they considered irreducible unitary integrable representations $\pi_n : G_n \to \mathcal{H}_n$, $n = 1, 2$, of locally compact groups G_n that generate two scales of co-orbit spaces and showed a one-to-one correspondence between bounded linear operators mapping $\mathrm{Co}L^1(G_1)$ to $\mathrm{Co}L^\infty(G_2)$ and their kernel belonging to the co-orbit space $\mathrm{Co}L^\infty(G_1 \times G_2)$ generated by the tensor product representation $\pi_1 \otimes \pi_2 : G_1 \times G_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2$.

In this paper, we go even further and prove kernel theorems for bounded linear operators acting on co-orbit spaces generated by localised frames [26,27] and, by doing so, show that the group structure and the corresponding representation theory are an unnecessary, albeit very elegant setup for proving the kernel theorem. Here is the essence of our main results. See Sections 2 and 3 for the definitions of the involved co-orbit spaces and Section 4 for detailed statements and proofs of the subsequent results.

Theorem *(Outer kernel theorem). Let Ψ_n be $\mathcal{A}_n$-localised frames and w_n an admissible weight for $n = 1, 2$. A linear operator O from $\mathcal{H}^1_{w_1}(\Psi_1)$ on $\mathcal{H}^\infty_{1/w_2}(\Psi_2)$ is bounded if and only if there exists a unique kernel $K \in \mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1 \otimes \Psi_2)$ such that*

$$\langle Of_1, f_2\rangle_{\mathcal{H}^\infty_{1/w_2}(\Psi_2),\mathcal{H}^1_{w_2}(\Psi_2)} = \langle K, f_1 \otimes f_2\rangle_{\mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2),\mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)}, \tag{1.3}$$

for all $f_1 \in \mathcal{H}^1_{w_1}(\Psi_1)$ and $f_2 \in \mathcal{H}^1_{w_2}(\Psi_2)$.

Theorem *(Inner Kernel Theorem). Let $\Psi_n \subset \mathcal{H}_n$ be an $\mathcal{A}_n$-localised frame, and w_n an $\mathcal{A}_n$-admissible weight for $n = 1, 2$. There is an isomorphism between kernels in $\mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1 \otimes \Psi_2)$ and the space*

$$\mathcal{B} := \Big\{ O \in B\big(\mathcal{H}^\infty_{1/w_1}(\Psi_1), \mathcal{H}^1_{w_2}(\Psi_2)\big) : \textit{ there exist } f_r \in H^1_{w_1}(\Psi_1), g_r \in H^1_{w_2}(\Psi_2) \textit{ s.t.}$$
$$O = \sum_{r\in R} \langle\,\cdot\,, f_r\rangle_{H^\infty_{1/w_1}(\Psi_1),H^1_{w_1}(\Psi_1)}\, g_r \textit{ and } \sum_{r\in R} \|f_r\|_{H^1_{w_1}(\Psi_1)}\, \|g_r\|_{H^1_{w_1}(\Psi_1)} < \infty \Big\}.$$

The terminology of outer and inner kernel theorems [25] originates in the fact that the corresponding kernels belong, respectively, to the outer- and innermost spaces on the scale of co-orbit spaces, as can be seen in the following chain of topological inclusions

$$\mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)\subset\mathcal{H}^{p,p}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)\subset\mathcal{H}^{p,p}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)\subset\mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2),$$

for weights $w_1, w_2 \geqslant 1$, and $p \in (1, \infty)$.

Following the ideas developed in [17], we apply a version of Schur's test to characterise the kernels of the bounded operators mapping $\mathcal{H}^1_{w_1}(\Psi_1)$ to $\mathcal{H}^p_{w_2}(\Psi_2)$ and those mapping $\mathcal{H}^p_{w_1}(\Psi_1)$ to $\mathcal{H}^\infty_{w_2}(\Psi_2)$, see Theorem 4.4.

Let us shortly motivate to study kernel theorems within this setup. There is every reason to use specifically localised frames here. Firstly, localised frames allow for essentially sparse atomic decompositions of the elements of their associated co-orbit spaces. Secondly, the Gram matrices of localised frames, by definition, are essentially sparsely populated. Therefore, using localised frames, together with Galerkin's method [3,29], can be expected to produce sparse matrices that represent the bounded linear operators that act on co-orbit spaces [28]. This in itself can, as highlighted in [3,5], be very useful for solving corresponding operator equations numerically. Moreover, as we shall show in this paper, the kernel of any bounded linear operator mapping one co-orbit space on the other can be synthesised from the matrix that represents the operator, therefore leading to a sparse decomposition of the kernel.

This paper is organised as follows. In Section 2 we provide the necessary frame theoretical background, namely the basics of the theory of localised frames. In Section 3 we show that co-orbit spaces generated by tensor products of localised frames share some properties with those generated by localised frames. In Section 4 we state and prove our main results, namely we prove the outer and inner kernel theorems for co-orbit spaces. Moreover, we prove statements about projective tensor products of two co-orbit spaces, boundedness of linear operators between certain intermediate co-orbit spaces and provide easily verifiable conditions under which a linear operator belongs to the Schatten-p classes.

Note that in [11] the same authors stated the outer kernel theorem in a simplified form and gave a sketch of its proof.

2. Background information and notation

Throughout this paper we will write $A \lesssim B$ if there exists $C \geqslant 1$ such that $A \leqslant CB$ and $A \asymp B$ if $C^{-1}B \leqslant A \leqslant CB$. Moreover, $\mathcal{H}, \mathcal{H}_1, \mathcal{H}_2$ will always denote separable Hilbert spaces. We shall use brackets $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ for the scalar product in the Hilbert space and $\langle\cdot,\cdot\rangle_{X,X^*}$ for the scalar product of the element of the Banach space X and that of its dual X^*. Furthermore, we denote the space of bounded operators between normed spaces X, Y by $B(X, Y)$ and write $\|\cdot\|_{X\to Y}$ for the operator norm.

2.1. Tensor products and Hilbert-Schmidt operators

Tensor products and simple tensors are the essential objects to understand kernel theorems for operators. See [41] for an overview of the subject. In this paper, we use the following convention to define the *simple tensor* of $f_1 \in \mathcal{H}_1$ and $f_2 \in \mathcal{H}_2$: $f_1 \otimes f_2$ will be understood as the rank one operator mapping $\mathcal{H}_1$ on $\mathcal{H}_2$ according to

$$(f_1 \otimes f_2)(f) := \langle f, f_1\rangle_{\mathcal{H}_1} f_2. \tag{2.1}$$

Note that this tensor is homogeneous in the following sense: $\alpha(f_1 \otimes f_2) = (\overline{\alpha} f_1) \otimes f_2 = f_1 \otimes (\alpha f_2)$. The tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ is then defined as the completion of the linear span of all simple tensors with respect to the metric induced by the inner product

$$\langle f_1 \otimes f_2, g_1 \otimes g_2 \rangle_{\mathcal{H}_1 \otimes \mathcal{H}_2} = \overline{\langle f_1, g_1 \rangle_{\mathcal{H}_1}} \langle f_2, g_2 \rangle_{\mathcal{H}_2}. \tag{2.2}$$

We note here that the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ can be identified with the space of *Hilbert-Schmidt operators* $\mathcal{HS}(\mathcal{H}_1, \mathcal{H}_2)$, which is defined as the space of bounded linear operators mapping $\mathcal{H}_1$ on $\mathcal{H}_2$ and equipped with the inner product

$$\langle O_1, O_2 \rangle_{\mathcal{HS}(\mathcal{H}_1,\mathcal{H}_2)} := \sum_{i \in \mathbb{N}} \langle O_1 e_i, O_2 e_i \rangle_{\mathcal{H}_2}$$

where $\{e_i\}_{i\in\mathbb{N}}$ is any orthonormal basis of the Hilbert space $\mathcal{H}_1$. We also note that this identification constitutes a non-trivial kernel theorem [16]. If $O \in \mathcal{HS}(\mathcal{H}_1, \mathcal{H}_2)$, $f_1 \in \mathcal{H}_1$ and $f_2 \in \mathcal{H}_2$, then

$$\langle O, f_1 \otimes f_2 \rangle_{\mathcal{HS}(\mathcal{H}_1,\mathcal{H}_2)} = \langle O f_1, f_2 \rangle_{\mathcal{H}_2}.$$

2.2. Frame theory

Throughout this work we use *frames* to decompose or synthesise functions and operators. See [14,15] for an overview of frame theory. Let us recollect the definition of a frame originally given in [18].

Definition 2.1. A countable set $\Psi := \{\psi_i\}_{i\in I} \subset \mathcal{H}$ is called a *frame* for $\mathcal{H}$ if there are positive numbers A_Ψ and B_Ψ, called *lower* and *upper frame bounds*, such that, for any $f \in \mathcal{H}$

$$A_\Psi \|f\|_{\mathcal{H}}^2 \leqslant \sum_{i\in I} |\langle f, \psi_i \rangle_{\mathcal{H}}|^2 \leqslant B_\Psi \|f\|_{\mathcal{H}}^2. \tag{2.3}$$

Let us list a few implications of this definition that we shall make use of later. The *analysis operator*

$$C_\Psi : \mathcal{H} \to \ell^2(I), \quad f \mapsto \{\langle f, \psi_i \rangle_{\mathcal{H}}\}_{i\in I},$$

is bounded and injective and has a closed range. The *synthesis operator*

$$D_\Psi : \ell^2(I) \to \mathcal{H}, \quad \{c_i\}_{i\in I} \mapsto \sum_{i\in I} c_i \psi_i,$$

is bounded and surjective. The *Gram operator*

$$G_\Psi : \ell^2(I) \to \ell^2(I), \quad \{c_i\}_{i\in I} \mapsto \sum_{i'\in I} \langle \psi_{i'}, \psi_i \rangle_{\mathcal{H}}\, c_{i'} = \sum_{i'\in I} G_{i,i'}\, c_{i'} = C_\Psi D_\Psi c, \tag{2.4}$$

with the *Gram matrix* given by

$$G_\Psi := (G_{i,i'})_{(i,i')\in I^2} := \big(\langle \psi_{i'}, \psi_i \rangle_{\mathcal{H}}\big)_{(i,i')\in I^2}, \tag{2.5}$$

is bounded. Occasionally, we will also need the definition of the *cross Gram matrix*

$$G_{\Psi,\,\widetilde{\Psi}} = \big(\langle \psi_{i'},\, \widetilde{\psi}_i \rangle\big)_{(i,\,i')\in I^2}. \tag{2.6}$$

There exists at least one other frame $\Psi^{\mathbf{d}} := \{\psi_i^d\}_{i\in I}$ for $\mathcal{H}$, called a *dual frame*, that, together with Ψ, satisfies

$$f = \sum_{i\in I} \langle f, \psi_i \rangle_{\mathcal{H}}\, \psi_i^d = D_{\Psi^{\mathbf{d}}} C_\Psi f = \sum_{i\in I} \langle f, \psi_i^d \rangle_{\mathcal{H}}\, \psi_i = D_\Psi C_{\Psi^{\mathbf{d}}} f. \tag{2.7}$$

Finally the *frame operator*

$$S_\Psi : \mathcal{H} \to \mathcal{H}, \quad f \mapsto \sum_{i\in I} \langle f, \psi_i \rangle_{\mathcal{H}}\, \psi_i,$$

is bounded, self-adjoint and invertible for any frame Ψ and the sequence $\widetilde{\Psi} = \{\widetilde{\psi}_i\}_{i\in I} = \{S_\Psi^{-1}\psi_i\}_{i\in I}$, known as the *canonical dual frame*, is a dual frame for Ψ.

2.3. Localised frames and their co-orbit spaces

One of the major advantages of frames over bases is that the former allow much more freedom in their construction and so can often be designed in a way that allows sparse decompositions of the functions of a given class. Any element of a Hilbert space can be expressed as a linear combination of the elements of a frame, see (2.7). To guarantee this reconstruction property in a Banach space, a more refined notion of frames has to be used, namely that of a *Banach frame* [31]. In order to construct a Banach frame that achieves a sparse decomposition of a concrete Banach space, say, a specific decomposition space [21], the theory of structured Banach frames [8,39,40,44] can be employed and indeed proved very useful [9,10,12]. The Banach frames of choice for more abstract co-orbit spaces are the so-called *localised frames*.

Let us start with an intuitive notion of localisation [32] that should serve as an inspiration for the general concept of $\mathcal{A}$-localisation. For a given a metric ρ on the indexing set I that satisfies

$$\inf_{i,\, i' \in I;\, i\neq i'} \rho(i, i') = C > 0,$$

the frame is called *localised* if the absolute values of the elements of the Gram matrix G_Ψ decay polynomially as they deviate from the main diagonal of the matrix [32], i.e., if

$$|(G_\Psi)_{i,i'}| = |\langle \psi_i, \psi_{i'} \rangle_{\mathcal{H}}| \lesssim (1 + \rho(i, i'))^{-n}, \quad i, i' \in I, \tag{2.8}$$

for some $n > n_0$. The matrices whose elements satisfy (2.8) belong to a spectral matrix algebra. This allows the generalization of the notion of localisation [33].

Definition 2.2. An involutive Banach algebra $\mathcal{A}$ of infinite matrices with norm $\|\cdot\|_{\mathcal{A}}$ is called a *spectral matrix algebra* if

(i) every $A \in \mathcal{A}$ defines a bounded operator on $\ell^2(I)$, i.e. $\mathcal{A} \subset B(\ell^2(I))$,
(ii) $\mathcal{A}$ is inverse-closed in $B(\ell^2(I))$, i.e. if $A \in \mathcal{A}$ is invertible on $B(\ell^2(I))$, then $A^{-1} \in \mathcal{A}$, and
(iii) $\mathcal{A}$ is *solid*, i.e. if $A \in \mathcal{A}$ and $|b_{i,j}| \leqslant |a_{i,j}|$ for any $i, j \in I$, then $B \in \mathcal{A}$ and $\|B\|_{\mathcal{A}} \leqslant \|A\|_{\mathcal{A}}$.

Examples of solid spectral matrix algebras include, among others, the Jaffard class [36] (which corresponds to (2.8) if ρ is the standard Euclidean metric), the Schur class [34] and the Sjöstrand class [42].

Definition 2.3. Let I be a countable index set. A frame $\Psi = \{\psi_i\}_{i\in I} \subset \mathcal{H}$ is said to be $\mathcal{A}$-*intrinsically localised*, or $\mathcal{A}$-*self-localised*, or simply $\mathcal{A}$-*localised*, if its Gram matrix $G_\Psi := (\langle \psi_i, \psi_{i'} \rangle_{\mathcal{H}})_{(i,i')\in I^2}$ belongs to a spectral matrix algebra $\mathcal{A}$.

Having a localised frame Ψ for a Hilbert space $\mathcal{H}$ allows to build a full range of so-called *co-orbit spaces*. To do so, we consider a weight $w = \{w_i\}_{i\in I},\ w_i > 0$, and define the weighted $\ell^p_w(I)$-spaces as the space of sequences on I for which $\|c\|_{\ell^p_w(I)} = \|c \cdot w\|_{\ell^p(I)} < \infty,\ 1 \leqslant p \leq \infty$.

Definition 2.4. Let $\mathcal{A}$ be a spectral matrix algebra indexed by $I \times I$. A weight sequence $w := \{w_i\}_{i\in I} \subset \mathbb{R}^+$ is called $\mathcal{A}$-*admissible* if every $A \in \mathcal{A}$ defines a bounded operator on $\ell^p_w(I)$ for every $1 \leqslant p \leqslant \infty$.

Definition 2.5. Let $\mathcal{A}$ be a spectral matrix algebra, $\Psi := \{\psi_i\}_{i\in I} \subset \mathcal{H}$ be an $\mathcal{A}$-localised frame, $\widetilde{\Psi} := \{\widetilde{\psi}_i\}_{i\in I} \subset \mathcal{H}$ its canonical dual frame, $w := \{w_i\}_{i\in I}$ an $\mathcal{A}$-admissible weight and

$$\mathcal{H}_{00}(\Psi) = \left\{ \sum_{i\in I} c_i\, \psi_i : \ \{c_i\}_{i\in I} \in c_{00} \right\}, \tag{2.9}$$

where c_{00} denotes the space of complex sequences with finitely many non-zero terms. For $1 \leqslant p < \infty$, the *co-orbit space* $\mathcal{H}^p_w(\Psi)$ is defined as the norm completion of $\mathcal{H}_{00}(\Psi)$ with respect to the norm $\|f\|_{\mathcal{H}^p_w(\Psi)} := \left\|C_{\widetilde{\Psi}} f\right\|_{\ell^p_w}$, while for $p = \infty$, $\mathcal{H}^\infty_w(\Psi)$ is defined as the completion of $\mathcal{H}_{00}(\Psi)$ with respect to the $\sigma(\mathcal{H}, \mathcal{H}_{00}(\Psi))$-topology. If $w \equiv 1$ we write $\mathcal{H}^p(\Psi) = \mathcal{H}^p_1(\Psi)$.

Remark 2.6. More specifically, $\mathcal{H}^\infty_w(\Psi)$ is defined as a set of equivalence classes of sequences from $\mathcal{H}$, see [5, Definition 3]. We do not provide any details for the moment, but shall do so when we consider co-orbit spaces generated by tensor products of localised frames in Section 3.

Also note that, while in [5,26] one may use any dual frame $\Psi^{\mathbf{d}}$ to define the co-orbit norms, we restrict ourselves to the canonical dual for the sake of simplicity.

In the following proposition, we summarise the main properties of co-orbit spaces that we shall need. See [5,26] for the corresponding original statements.

Proposition 2.7. *Let Ψ and Φ be two $\mathcal{A}$-localised frames for a Hilbert space $\mathcal{H}$, w be an $\mathcal{A}$-admissible weight, and $1 \leqslant p \leqslant \infty$. Then the following statements hold:*

(i) $D_\Psi : \ell^p_w(I) \to \mathcal{H}^p_w(\Psi)$ *is continuous;*
(ii) $\mathcal{H} = \mathcal{H}^2_1(\Psi)$*;*
(iii) duality: for $1 \leqslant p < \infty$*, and* q *satisfying* $1/p + 1/q = 1$*, one has* $(\mathcal{H}^p_w(\Psi))^* = \mathcal{H}^q_{1/w}(\Psi)$*, where the duality is defined by*

$$\langle f, g\rangle_{\mathcal{H}^p_w(\Psi), \mathcal{H}^q_{1/w}(\Psi)} = \langle C_{\widetilde{\Psi}} f, C_\Psi g\rangle_{\ell^p_w(I), \ell^q_{1/w}(I)};$$

(iv) inclusions: $\mathcal{H}^{p_1}_{w_1}(\Psi) \subset \mathcal{H}^{p_2}_{w_2}(\Psi)$ *for* $w_1 \geqslant w_2$*, and* $1 \leqslant p_1 < p_2 \leqslant \infty$*;*
(v) $G_{\widetilde{\Psi}} \in \mathcal{A}$ *and* $G_{\widetilde{\Psi},\Psi} \in \mathcal{A}$*, i.e., the canonical dual frame* $\widetilde{\Psi}$ *is also* $\mathcal{A}$*-localised;*
(vi) independence of Ψ*: if* $G_{\Psi,\widetilde{\Phi}} \in \mathcal{A}$ *and* $G_{\widetilde{\Psi},\Phi} \in \mathcal{A}$*, then* $\mathcal{H}^p_w(\Psi) = \mathcal{H}^p_w(\Phi)$ *with equivalent norms; in particular,* $\mathcal{H}^p_w(\Psi) = \mathcal{H}^p_w\big(\widetilde{\Psi}\big)$*.*

The following lemma sets an upper bound of the $\mathcal{H}^1_w(\Psi)$-norm of the elements of Ψ and $\widetilde{\Psi}$.

Lemma 2.8. *Let $1 \leqslant p < \infty$, Ψ be an $\mathcal{A}$-localised frame for $\mathcal{H}$, and w an $\mathcal{A}$-admissible weight. Then*

$$\|\psi_i\|_{\mathcal{H}^p_w(\Psi)} \lesssim w_i, \quad \textit{and} \quad \big\|\widetilde{\psi}_i\big\|_{\mathcal{H}^p_w(\Psi)} \lesssim w_i, \quad i \in I. \tag{2.10}$$

Proof. Remember that the Gram matrix $G_{\widetilde{\Psi}}$ of $\widetilde{\Psi}$ as well as the cross-Gram matrix of Ψ and $\widetilde{\Psi}$ are bounded operators on $\ell^p_w(I)$ for $p \in [1, \infty]$, as Ψ and $\widetilde{\Psi}$ are $\mathcal{A}$-localised by assumption and Proposition 2.7 (v), and w is $\mathcal{A}$-admissible. Therefore,

$$\begin{aligned}\|\widetilde{\psi}_i\|^p_{\mathcal{H}^p_w(\Psi)} &= \sum_{k\in I} \left|\langle \widetilde{\psi}_i, \widetilde{\psi}_k\rangle_{\mathcal{H}}\right|^p w_k^p = \sum_{k\in I} \left|\left(G_{\widetilde{\Psi}}\right)_{k,\,i}\right|^p w_k^p \\ &= \sum_{k\in I} \left|\left(G_{\widetilde{\Psi}} e_i\right)_k\right|^p w_k^p = \left\|G_{\widetilde{\Psi}} e_i\right\|^p_{\ell^p_w(I)} \lesssim \|e_i\|^p_{\ell^p_w(I)} = w_i^p,\end{aligned}$$

where e_i stands for the i-th vector of the standard basis of $\ell^1_w(I)$. The upper bound of $\|\psi_i\|_{\mathcal{H}^p_w(\Psi)}$ follows from a similar argument, replacing $G_{\widetilde{\Psi}}$ by $G_{\widetilde{\Psi},\Psi}$. $\square$

Finally, we recollect a result from [26] concerning the representation of elements of $\mathcal{H}^1_w(\Psi)$ as a series expansion.

Proposition 2.9. *Let Ψ be an $\mathcal{A}$-localised frame for $\mathcal{H}$ and w be an $\mathcal{A}$-admissible weight. For any $f \in \mathcal{H}^1_w(\Psi)$ there is a sequence of numbers $c := \{c_i\}_{i\in I} \in \ell^1_w(I)$ such that*

$$f = \sum_{i\in I} c_i\, \psi_i,$$

where the series converges absolutely and $\|c\|_{\ell^1_w(I)} \asymp \|f\|_{\mathcal{H}^1_w(\Psi)}$.

3. Tensor products of localised frames

If we consider the tensor product $\Psi_1 \otimes \Psi_2 = \{\psi_{1,\,i} \otimes \psi_{2,\,j}\}_{(i,j)\in I\times J}$ of two frames $\Psi_1 := \{\psi_{1,\,i}\}_{i\in I} \subset \mathcal{H}_1$ and $\Psi_2 := \{\psi_{2,\,j}\}_{j\in J} \subset \mathcal{H}_2$, then it is straightforward to show [4] that $\Psi_1 \otimes \Psi_2$ forms a frame for the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ and that the canonical dual frame is given by $\widetilde{\Psi_1 \otimes \Psi_2} = \widetilde{\Psi}_1 \otimes \widetilde{\Psi}_2$. If both frames are localised with respect to a solid spectral matrix algebra, then it is possible to also define co-orbit spaces. For the sake of completeness, we provide here all necessary definitions. Let us first define the mixed-norm sequence spaces

$$\ell^{p,q}_w(I\times J) := \left\{ c : I\times J \to \mathbb{C} :\ \sum_{j\in J} \Big(\sum_{i\in I} |c_{i,j}|^p w^p_{i,j} \Big)^{q/p} < \infty \right\}$$

for $1 \leqslant p, q \leqslant \infty$, with the usual adaptations if $p = \infty$, or $q = \infty$. Since the order of summation matters if $p \neq q$, we also introduce the spaces

$$\mathfrak{l}^{p,q}_w(I\times J) := \left\{ c : I\times J \to \mathbb{C} :\quad \sum_{i\in I} \Big(\sum_{j\in J} |c_{i,j}|^p\, w^p_{i,j} \Big)^{q/p} < \infty \right\}. \tag{3.1}$$

Let

$$\mathcal{H}_{00}(\Psi_1 \otimes \Psi_2) = \left\{ \sum_{(i,j)\in I\times J} c_{i,\,j}\, \psi_{1,\,i} \otimes \psi_{2,\,j} :\ \{c_{i,\,j}\}_{(i,j)\in I\times J} \in c_{00} \right\},$$

and $p, q \in [1, \infty)$. The *co-orbit spaces* $\mathcal{H}^{p,\,q}_w(\Psi_1 \otimes \Psi_2)$ are defined as the completion of $\mathcal{H}_{00}(\Psi_1 \otimes \Psi_2)$ with respect to the norm $\|F\|_{\mathcal{H}^{p,q}_w(\Psi_1\otimes\Psi_2)} := \|C_{\widetilde{\Psi_1\otimes\Psi_2}} F\|_{\ell^{p,q}_w(I\times J)}$, and $\mathfrak{H}^{p,\,q}_w(\Psi_1 \otimes \Psi_2)$ as the completion of $\mathcal{H}_{00}(\Psi_1 \otimes \Psi_2)$ with respect to the norm $\|F\|_{\mathfrak{H}^{p,q}_w(\Psi_1\otimes\Psi_2)} := \|C_{\widetilde{\Psi_1\otimes\Psi_2}} F\|_{\mathfrak{l}^{p,q}_w(I\times J)}$. Note that $\mathcal{H}_{00}(\Psi_1 \otimes \Psi_2)$ is indeed contained in $\mathcal{H}^{p,q}_w(\Psi_1 \otimes \Psi_2)$ since by Lemma 2.8

$$\begin{aligned}\|F\|_{\mathcal{H}^{p,q}_w(\Psi_1\otimes\Psi_2)} &\leqslant \sum_{(i,j)\in I\times J} |c_{i,j}|\,\|\psi_{1,i}\otimes\psi_{2,j}\|_{\mathcal{H}^{p,q}_w(\Psi_1\otimes\Psi_2)}\\ &= \sum_{(i,j)\in I\times J} |c_{i,j}|\,\|\psi_{1,i}\|_{\mathcal{H}^{p}_{w_1}(\Psi_1)}\|\psi_{2,j}\|_{\mathcal{H}^{q}_{w_2}(\Psi_2)}\\ &\lesssim \sum_{(i,j)\in I\times J} |c_{i,j}| w_{1,i}w_{2,j} < \infty,\end{aligned}$$

as $c \in C_{00}$. Similarly, $\mathcal{H}_{00}(\Psi_1\otimes\Psi_2) \subset \mathfrak{H}^{p,q}_w(\Psi_1\otimes\Psi_2)$. If $w \equiv 1$, we simply write $\mathcal{H}^{p,q}(\Psi_1\otimes\Psi_2) = \mathcal{H}^{p,q}_1(\Psi_1\otimes\Psi_2)$, and $\mathfrak{H}^{p,q}(\Psi_1\otimes\Psi_2) = \mathfrak{H}^{p,q}_1(\Psi_1\otimes\Psi_2)$. We shall primarily consider the case where $p = q$, but $p \neq q$ will make an appearance in Theorem 4.4.

As mentioned in [26], dealing with $\mathcal{H}^\infty_w(\Psi)$ for a localised frame Ψ is technically rather tricky; a detailed exposition of the construction can be found in [5]. Here we have to grapple with this issue again as now the tensor product frame cannot be assumed to be localised. The major obstacle to simply applying established theory to the tensor product of an $\mathcal{A}_1$-localised frame and an $\mathcal{A}_2$-localised frame is the fact that the natural candidate for a tensor product algebra, the projective tensor product $\mathcal{A}_1\widehat{\otimes}_\pi\mathcal{A}_2$, is in general neither solid nor inverse-closed.

In order to properly define the co-orbit spaces where one of the indices or both of them are ∞, we need the following notion. We call two sequences $\{F_n\}_{n\in\mathbb{N}}$, $\{F'_n\}_{n\in\mathbb{N}} \subset \mathcal{H}_1\otimes\mathcal{H}_2$ *equivalent* if

$$\lim_{n\to\infty} \big\langle F_n - F'_n, \widetilde{\psi}_{1,\,i}\otimes\widetilde{\psi}_{2,\,j}\big\rangle_{\mathcal{H}_1\otimes\mathcal{H}_2} = 0,$$

for all $(i, j) \in I\times J$.

Definition 3.1. Let $1 \leqslant p < \infty$, $n = 1, 2$, and w_n be $\mathcal{A}_n$-admissible weights. We define the spaces $\mathcal{H}^{r,s}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)$ and $\mathfrak{H}^{r,s}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)$ where $(r, s) \in \{(p,\infty), (\infty, p), (\infty,\infty)\}$, as the spaces of equivalence classes $F = \big[\{F_n\}_{n\in\mathbb{N}}\big]$ with $\{F_n\}_{n\in\mathbb{N}} \subset \mathcal{H}_1\otimes\mathcal{H}_2$ such that

$$\lim_{n\to\infty} \big\langle F_n, \widetilde{\psi}_{1,\,i}\otimes\widetilde{\psi}_{2,\,j}\big\rangle_{\mathcal{H}_1\otimes\mathcal{H}_2} =: k_{i,\,j} \tag{3.2}$$

exists for all $(i, j) \in I\times J$ and

$$\sup_n \big\|C_{\widetilde{\Psi_1\otimes\Psi_2}}F_n\big\|_{\ell^{r,s}_{w_1\otimes w_2}(I\times J)} < \infty, \quad \text{and} \quad \sup_n \big\|C_{\widetilde{\Psi_1\otimes\Psi_2}}F_n\big\|_{\mathfrak{l}^{r,s}_{w_1\otimes w_2}(I\times J)} < \infty$$

respectively. Furthermore we define $\big(C_{\widetilde{\Psi_1\otimes\Psi_2}}F\big)_{i,j} := k_{i,j}$, $(i, j) \in I\times J$, and

$$\|F\|_{\mathcal{H}^{r,s}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)} := \big\|C_{\widetilde{\Psi_1\otimes\Psi_2}}F\big\|_{\ell^{r,s}_{w_1\otimes w_2}(I\times J)}, \tag{3.3}$$

as well as

$$\|F\|_{\mathfrak{H}^{r,s}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)} := \big\|C_{\widetilde{\Psi_1\otimes\Psi_2}}F\big\|_{\mathfrak{l}^{r,s}_{w_1\otimes w_2}(I\times J)}. \tag{3.4}$$

It is clear that $\|\cdot\|_{\mathcal{H}^{r,s}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)}$ is a seminorm. Moreover, $\|F\|_{\mathcal{H}^{r,s}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)} = 0$ implies that all limits in (3.2) equal zero for any sequence $\{F_n\}_{n\in\mathbb{N}}$ representing F. In other words, all such sequences $\{F_n\}_{n\in\mathbb{N}}$ are equivalent to the sequence of zeros in $\mathcal{H}_1\otimes\mathcal{H}_2$ and so $F = 0$. Consequently, $\|\cdot\|_{\mathcal{H}^{r,s}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)}$ is indeed a norm. The same reasoning shows that $\|\cdot\|_{\mathfrak{H}^{r,s}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)}$ is a norm.

The spaces $\mathcal{H}^{p,p}_{w_1,w_2}(\Psi_1\otimes\Psi_2)$ are structurally very similar to the co-orbit spaces of Section 2 and results analoguous to Propositions 2.7 and 2.9 hold. These results however are not a direct consequence of the existing theory as, e.g., Propositions 2.7 and 2.9 assume the localisation of the frame with respect to a solid spectral algebra. We conjecture that $\Psi_1 \otimes \Psi_2$ would be $\mathcal{A}_1 \widetilde{\otimes} \mathcal{A}_2$-localised (for an appropriately chosen tensor product structure satisfying the assumptions of Definition 2.2) given that Ψ_1 and Ψ_2 are $\mathcal{A}_1$- and $\mathcal{A}_2$-localised respectively. This would allow us to directly apply existing theory. This is however not a simple matter and is the subject of an ongoing project of ours. Fortunately, the assumption that both Ψ_1 and Ψ_2 are localised suffices to prove all the statements that we need here.

Proposition 3.2. *Let $\Psi_n, \Phi_n \subset \mathcal{H}_n$ be $\mathcal{A}_n$-localised frames, and w_n be $\mathcal{A}_n$-admissible weights, $n = 1, 2$, and $1 \leqslant p \leqslant \infty$. Then the following statements hold:*

(i) if $G_{\Psi_n,\Phi_n} \in \mathcal{A}_n$, $n = 1, 2$, then the cross Gram matrix $G_{\Psi_1\otimes\Psi_2,\Phi_1\otimes\Phi_2}$ defines a bounded operator on $\ell^{p,p}_{w_1\otimes w_2}(I \times J)$;

(ii) independence of $\Psi_1 \otimes \Psi_2$: if $G_{\Psi_n,\widetilde{\Phi}_n} \in \mathcal{A}_n$ and $G_{\Phi_n,\widetilde{\Psi}_n} \in \mathcal{A}_n$, $n = 1, 2$, then

$$\mathcal{H}^{p,p}_{w_1\otimes w_2}(\Psi_1 \otimes \Psi_2) = \mathcal{H}^{p,p}_{w_1\otimes w_2}\big(\Phi_1 \otimes \Phi_2\big)$$

with equivalent norms. In particular,

$$\mathcal{H}^{p,p}_{w_1\otimes w_2}(\Psi_1 \otimes \Psi_2) = \mathcal{H}^{p,p}_{w_1\otimes w_2}\big(\widetilde{\Psi}_1 \otimes \widetilde{\Psi}_2\big);$$

(iii) $D_{\Psi_1\otimes\Psi_2} : \ell^{p,p}_{w_1\otimes w_2}(I \times J) \to \mathcal{H}^{p,p}_{w_1\otimes w_2}(\Psi_1 \otimes \Psi_2)$ is bounded and the expansion converges absolutely for $p < \infty$, and weak-$*$ unconditionally for $p = \infty$;

(iv) for any $F \in \mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1 \otimes \Psi_2)$, there exists a sequence $c \in \ell^{1,1}_{w_1\otimes w_2}(I \times J)$ such that $F = D_{\Psi_1\otimes\Psi_2} c$ and $\|F\|_{\mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)} \asymp \|c\|_{\ell^{1,1}_{w_1\otimes w_2}(I\times J)}$;

(v) duality: for $1 \leqslant p < \infty$ and q satisfying $1/p + 1/q = 1$,

$$\big(\mathcal{H}^{p,p}_{w_1\otimes w_2}(\Psi_1 \otimes \Psi_2)\big)^* \cong \mathcal{H}^{q,q}_{1/(w_1\otimes w_2)}(\Psi_1 \otimes \Psi_2)$$

with the scalar product defined by

$$\langle F, G\rangle_{\mathcal{H}^{p,p}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2),\mathcal{H}^{q,q}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)} := \langle C_{\widetilde{\Psi_1\otimes\Psi_2}} F, C_{\Psi_1\otimes\Psi_2} G\rangle_{\ell^{p,p}_{w_1\otimes w_2}(I\times J),\ell^{q,q}_{1/(w_1\otimes w_2)}(I\times J)}.$$

Proof. Ad (i): As the weight functions w_n are $\mathcal{A}_n$-admissible, it follows that the cross Gram matrix G_{Ψ_n,Φ_n} defines a bounded operator on $\ell^p_{w_n}(I_n)$ [5]. Therefore, for $1 \leqslant p < \infty$, and $c \in \ell^{p,p}_{w_1\otimes w_2}(I \times J)$, we obtain from (2.2) that

$$\begin{aligned}\big\|G_{\Psi_1\otimes\Psi_2,\Phi_1\otimes\Phi_2} c\big\|^p_{\ell^{p,p}_{w_1\otimes w_2}(I\times J)} &= \sum_{j\in J}\sum_{i\in I}\Big|\sum_{(i',j')\in I\times J} \overline{(G_{\Psi_1,\Phi_1})_{i,i'}}(G_{\Psi_2,\Phi_2})_{j,j'} c_{i',j'}\Big|^p w^p_{1,i} w^p_{2,j} \\ &\lesssim \sum_{j\in J}\sum_{i\in I}\Big|\sum_{j'\in J}(G_{\Psi_2,\Phi_2})_{j,j'} c_{i,j'}\Big|^p w^p_{1,\,i} w^p_{2,\,j} \\ &\lesssim \sum_{j\in J}\sum_{i\in I}|c_{i,j}|^p w^p_{1,\,i} w^p_{2,\,j} = \|c\|^p_{\ell^{p,p}_{w_1\otimes w_2}(I\times J)}.\end{aligned}$$

A similar argument can be used to prove the statement in the case where $p = \infty$. This concludes the proof of (i).

Ad (ii): Since $F = D_{\Psi_1\otimes\Psi_2} C_{\widetilde{\Psi_1\otimes\Psi_2}} F$ for every $F \in \mathcal{H}_{00}(\Psi_1 \otimes \Psi_2)$, we see that the frame coefficients with respect to $\widetilde{\Phi}_1 \otimes \widetilde{\Phi}_2$ are given by

$$C_{\widetilde{\Phi_1\otimes\Phi_2}}F = C_{\widetilde{\Phi_1\otimes\Phi_2}}D_{\Psi_1\otimes\Psi_2}C_{\widetilde{\Psi_1\otimes\Psi_2}}F = G_{\Psi_1\otimes\Psi_2,\widetilde{\Phi_1\otimes\Phi_2}}C_{\widetilde{\Psi_1\otimes\Psi_2}}F.$$

Using (i) we then conclude that for every $F \in \mathcal{H}_{00}(\Psi_1 \otimes \Psi_2)$

$$\begin{aligned}\|F\|_{\mathcal{H}^{p,p}_{w_1\otimes w_2}(\Phi_1\otimes\Phi_2)} &= \|C_{\widetilde{\Phi_1\otimes\Phi_2}}F\|_{\ell^{p,p}_{w_1\otimes w_2}(I\times J)} \lesssim \|C_{\widetilde{\Psi_1\otimes\Psi_2}}F\|_{\ell^{p,p}_{w_1\otimes w_2}(I\times J)} \\ &= \|F\|_{\mathcal{H}^{p,p}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)}.\end{aligned}$$

Therefore, as $\mathcal{H}_{00}(\Psi_1\otimes\Psi_2)$ is dense in $\mathcal{H}^{p,p}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)$ it follows that $\mathcal{H}^{p,p}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2) \subseteq \mathcal{H}^{p,p}_{w_1\otimes w_2}(\Phi_1\otimes\Phi_2)$. Exchanging the roles of $\Psi_1 \otimes \Psi_2$ and $\Phi_1 \otimes \Phi_2$ results in $\mathcal{H}^{p,p}_{w_1\otimes w_2}(\Phi_1 \otimes \Phi_2) \subseteq \mathcal{H}^{p,p}_{w_1\otimes w_2}(\Psi_1 \otimes \Psi_2)$ and the norm equivalence.

Ad (iii): The proofs of (iii) and (iv) follow closely the arguments of [26, Proposition 2.4] and [5, Lemmas 2 and 7] respectively. First, let $c \in \ell^{p,p}_{w_1\otimes w_2}(I \times J)$ and set $F = D_{\Psi_1\otimes\Psi_2}c$. Then

$$\begin{aligned}\|F\|_{\mathcal{H}^{p,p}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)} &= \left\|C_{\widetilde{\Psi_1\otimes\Psi_2}}F\right\|_{\ell^{p,p}_{w_1\otimes w_2}(I\times J)} \\ &= \left\|G_{\widetilde{\Psi_1\otimes\Psi_2},\Psi_1\otimes\Psi_2}\, c\right\|_{\ell^{p,p}_{w_1\otimes w_2}(I\times J)} \lesssim \left\|c\right\|_{\ell^{p,p}_{w_1\otimes w_2}(I\times J)},\end{aligned} \tag{3.5}$$

where we used (i), and Proposition 2.7 (v). The absolute convergence for $p < \infty$ follows immediately from Lemma 2.8.

The proof of the unconditional weak-$*$ convergence for $p = \infty$ is postponed until the end of the proof of this proposition.

Ad (iv): Let F be an element of $\mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1 \otimes \Psi_2)$. Then F is the limit of a sequence $\{F_k\}_{k\in\mathbb{N}} \subseteq \mathcal{H}_{00}(\Psi_1 \otimes \Psi_2)$ with respect to the $\mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1 \otimes \Psi_2)$-norm. Being convergent, $\{F_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1 \otimes \Psi_2)$. From this and the first equality in (3.5) we infer that $\{C_{\widetilde{\Psi_1\otimes\Psi_2}}F_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $\ell^{1,1}_{w_1\otimes w_2}(I \times J)$ converging to $c = C_{\widetilde{\Psi_1\otimes\Psi_2}}F \in \ell^{1,1}_{w_1\otimes w_2}(I \times J)$. Let $F' := D_{\Psi_1\otimes\Psi_2}c$. From (iii) we infer that $F' \in \mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1 \otimes \Psi_2)$. Using the fact that

$$C_{\widetilde{\Psi_1\otimes\Psi_2}}H = C_{\widetilde{\Psi_1\otimes\Psi_2}}D_{\Psi_1\otimes\Psi_2}C_{\widetilde{\Psi_1\otimes\Psi_2}}H$$

for any $H \in \mathcal{H}_{00}(\Psi_1 \otimes \Psi_2)$ and the boundedness of $D_{\Psi_1\otimes\Psi_2}$ results in

$$\begin{aligned}\lim_{k\to\infty}\left\|F' - F_k\right\|_{\mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)} &= \lim_{k\to\infty}\left\|C_{\widetilde{\Psi_1\otimes\Psi_2}}(F' - F_k)\right\|_{\ell^{1,1}_{w_1\otimes w_2}(I\times J)} \\ &= \lim_{k\to\infty}\left\|C_{\widetilde{\Psi_1\otimes\Psi_2}}D_{\Psi_1\otimes\Psi_2}C_{\widetilde{\Psi_1\otimes\Psi_2}}(F - F_k)\right\|_{\ell^{1,1}_{w_1\otimes w_2}(I\times J)} \\ &\lesssim \lim_{k\to\infty}\left\|C_{\widetilde{\Psi_1\otimes\Psi_2}}(F - F_k)\right\|_{\ell^{1,1}_{w_1\otimes w_2}(I\times J)} = 0.\end{aligned}$$

Therefore $F' = F = D_{\Psi_1\otimes\Psi_2}c$ and we have proved that $D_{\Psi_1\otimes\Psi_2}$ is surjective. Finally, $D_{\Psi_1\otimes\Psi_2}$ is a bijective operator mapping $\ell^{1,1}_{w_1\otimes w_2}(I \times J)/\ker(D_{\Psi_1\otimes\Psi_2})$ on $\mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1 \otimes \Psi_2)$, which shows that there exists $c \in \ell^{1,1}_{w_1\otimes w_2}(I \times J)$ such that $D_{\Psi_1\otimes\Psi_2}c = F$ and

$$\|c\|_{\ell^{1,1}_{w_1\otimes w_2}(I\times J)} \lesssim \|F\|_{\mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)}, \tag{3.6}$$

according to the inverse mapping theorem. Combining (3.5) and (3.6) results in the norm equivalence of (iv).

Ad (v): We follow the line of arguments of [5, Proposition 2]. Let us define a sesquilinear form $\mathcal{H}^{p,p}_{w_1\otimes w_2}(\Psi_1 \otimes \Psi_2) \times \mathcal{H}^{q,q}_{1/(w_1\otimes w_2)}(\Psi_1 \otimes \Psi_2) \to \mathbb{C}$ by

$$(F, G) \mapsto \langle C_{\widetilde{\Psi_1\otimes\Psi_2}}F, C_{\Psi_1\otimes\Psi_2}G\rangle_{\ell^{p,p}_{w_1\otimes w_2}(I\times J),\ell^{q,q}_{1/(w_1\otimes w_2)}(I\times J)}.$$

Fixing $G \in \mathcal{H}^{q,q}_{1/(w_1\otimes w_2)}(\Psi_1 \otimes \Psi_2)$ defines a bounded linear functional $\mathcal{W}(G) : \mathcal{H}^{p,p}_{w_1\otimes w_2}(\Psi_1 \otimes \Psi_2) \to \mathbb{C}$ according to

$$\mathcal{W}(G)(F) = \langle C_{\widetilde{\Psi_1\otimes\Psi_2}} F, C_{\Psi_1\otimes\Psi_2} G\rangle_{\ell^{p,p}_{w_1\otimes w_2}(I\times J),\ell^{q,q}_{1/(w_1\otimes w_2)}(I\times J)}.$$

Indeed, by (ii)

$$\begin{aligned}|\mathcal{W}(G)(F)| &\leqslant \|C_{\widetilde{\Psi_1\otimes\Psi_2}} F\|_{\ell^{p,p}_{w_1\otimes w_2}(I\times J)} \|C_{\Psi_1\otimes\Psi_2} G\|_{\ell^{q,q}_{1/(w_1\otimes w_2)}(I\times J)}\\ &= \|F\|_{\mathcal{H}^{p,p}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)} \|G\|_{\mathcal{H}^{q,q}_{1/(w_1\otimes w_2)}(\widetilde{\Psi}_1\otimes\widetilde{\Psi}_2)}\\ &\lesssim \|F\|_{\mathcal{H}^{p,p}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)} \|G\|_{\mathcal{H}^{q,q}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)},\end{aligned}$$

for any $F \in \mathcal{H}^{p,p}_{w_1\otimes w_2}(\Psi_1 \otimes \Psi_2)$. This shows that $\mathcal{W} : \mathcal{H}^{q,q}_{1/(w_1\otimes w_2)}(\Psi_1 \otimes \Psi_2) \to \left(\mathcal{H}^{p,p}_{w_1\otimes w_2}(\Psi_1 \otimes \Psi_2)\right)^*$ is bounded as

$$\|\mathcal{W}(G)\|_{\left(\mathcal{H}^{p,p}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)\right)^*} \lesssim \|G\|_{\mathcal{H}^{q,q}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)}.$$

Conversely, let $H \in \left(\mathcal{H}^{p,p}_{w_1\otimes w_2}(\Psi_1 \otimes \Psi_2)\right)^*$ and $c \in \ell^{p,p}_{w_1\otimes w_2}(I \times J)$ with $p \in [1, \infty)$. Then $D_{\Psi_1\otimes\Psi_2} c \in \mathcal{H}^{p,p}_{w_1\otimes w_2}(\Psi_1 \otimes \Psi_2)$ by (iii) and

$$H\left(D_{\Psi_1\otimes\Psi_2} c\right) = \sum_{i\in I}\sum_{j\in J} c_{i,j} H(\psi_{1,\,i} \otimes \psi_{2,\,j})$$

where the series converges unconditionally for any $c \in \ell^{p,\,q}_{1/w_1\otimes w_2}(I\times J)$. Therefore $\{H(\psi_{1,\,i}\otimes\psi_{2,\,j})\}_{(i,j)\in I\times J} \in \ell^{q,\,q}_{1/(w_1\otimes w_2)}(I \times J)$ [16]. If we define the mapping $\mathcal{V} : \left(\mathcal{H}^{p,\,p}_{w_1\otimes w_2}(\Psi_1 \otimes \Psi_2)\right)^* \to \mathcal{H}^{q,q}_{1/(w_1\otimes w_2)}(\Psi_1 \otimes \Psi_2)$ by

$$\mathcal{V}(H) := \sum_{i\in I}\sum_{j\in J} \overline{H(\psi_{1,\,i} \otimes \psi_{2,\,j})}\, \widetilde{\psi}_{1,\,i} \otimes \widetilde{\psi}_{2,\,j}, \quad H \in \left(\mathcal{H}^{p,p}_{w_1\otimes w_2}(\Psi_1 \otimes \Psi_2)\right)^*,$$

then, for $F \in \mathcal{H}^{p,p}_{w_1\otimes w_2}(\Psi_1 \otimes \Psi_2)$,

$$\begin{aligned}(\mathcal{W}(\mathcal{V}(H)))(F) &= \langle C_{\widetilde{\Psi_1\otimes\Psi_2}} F, C_{\Psi_1\otimes\Psi_2}\mathcal{V}(H)\rangle_{\ell^{p,p}_{w_1\otimes w_2}(I\times J),\ell^{q,q}_{1/(w_1\otimes w_2)}(I\times J)}\\ &= \sum_{i\in I}\sum_{j\in J}\langle F, \widetilde{\psi}_{1,\,i} \otimes \widetilde{\psi}_{2,\,j}\rangle \overline{\left\langle \sum_{k\in I}\sum_{l\in J} \overline{H(\psi_{1,\,k} \otimes \psi_{2,\,l})}\, \widetilde{\psi}_{1,\,k} \otimes \widetilde{\psi}_{2,\,l}, \psi_{1,\,i} \otimes \psi_{2,\,j}\right\rangle}\\ &= \sum_{k\in I}\sum_{l\in J} H(\psi_{1,\,k} \otimes \psi_{2,\,l})\left\langle \sum_{i\in I}\sum_{j\in J}\langle F, \widetilde{\psi}_{1,\,i} \otimes \widetilde{\psi}_{2,\,j}\rangle\psi_{1,\,i} \otimes \psi_{2,\,j}, \widetilde{\psi}_{1,\,k} \otimes \widetilde{\psi}_{2,\,l}\right\rangle\\ &= \sum_{k\in I}\sum_{l\in J} H(\psi_{1,\,k} \otimes \psi_{2,\,l})\langle F, \widetilde{\psi}_{1,\,k} \otimes \widetilde{\psi}_{2,\,l}\rangle\\ &= H\left(\sum_{k\in I}\sum_{l\in J}\langle F, \widetilde{\psi}_{1,\,k} \otimes \widetilde{\psi}_{2,\,l}\rangle\psi_{1,\,k} \otimes \psi_{2,\,l}\right) = H(F),\end{aligned}$$

where the change of the order of summations, taking scalar products and application of H is justified by the unconditional convergence of the series involved, the continuity of H, and a density argument. The operator $\mathcal{W}$ is therefore surjective. Moreover, for $G \in \mathcal{H}^{q,q}_{1/(w_1\otimes w_2)}(\Psi_1 \otimes \Psi_2)$,

$$\mathcal{V}(\mathcal{W}(G)) = \sum_{i\in I}\sum_{j\in J}\overline{\mathcal{W}(G)(\psi_{1,i}\otimes\psi_{2,j})}\,\widetilde{\psi}_{1,i}\otimes\widetilde{\psi}_{2,j}$$
$$= \sum_{i\in I}\sum_{j\in J}\langle G,\psi_{1,i}\otimes\psi_{2,j}\rangle\widetilde{\psi}_{1,i}\otimes\widetilde{\psi}_{2,j} = G,$$

which shows that $\mathcal{W}$ is also injective and thus invertible.

It remains to prove the unconditional weak-$*$ convergence of $D_{\Psi_1\otimes\Psi_2}$ when $p=\infty$. Let $c\in\ell^{\infty,\infty}_{w_1\otimes w_2}(I\times J)$ and $H\in\mathcal{H}_{00}(\Psi_1\otimes\Psi_2)$. Then using (ii) and (v) results in

$$\begin{aligned}
&\left|\langle D_{\Psi_1\otimes\Psi_2}c,H\rangle_{\mathcal{H}^{\infty,\infty}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2),\mathcal{H}^{1,1}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)}\right| \\
&\qquad\leqslant \sum_{(i,j)\in I\times J}|c_{i,j}|\left|\langle C_{\widetilde{\Psi_1\otimes\Psi_2}}\psi_{1,i}\otimes\psi_{2,j},C_{\Psi_1\otimes\Psi_2}H\rangle_{\ell^{\infty,\infty}_{w_1\otimes w_2}(I\times J),\ell^{1,1}_{1/(w_1\otimes w_2)}(I\times J)}\right| \\
&\qquad= \sum_{(i,j)\in I\times J}|c_{i,j}|\left|(C_{\Psi_1\otimes\Psi_2}H)_{i,j}\right| \lesssim \|c\|_{\ell^{\infty,\infty}_{w_1\otimes w_2}(I\times J)}\|H\|_{\mathcal{H}^{1,1}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)}.
\end{aligned}$$

By density of $\mathcal{H}_{00}(\Psi_1\otimes\Psi_2)$ in $\mathcal{H}^{1,1}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)$, the chain of inequalities also holds for any $H\in\mathcal{H}^{1,1}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)$, showing that $D_{\Psi_1\otimes\Psi_2}c$ converges weak-$*$ absolutely and in particular weak-$*$ unconditionally. □

Remark 3.3. It is unclear to us whether the statements of Proposition 3.2 apply to the co-orbit spaces $\mathcal{H}^{p_1,p_2}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)$ if $p_1\neq p_2$. The main problem is that the argument used in proving (i) does not apply to mixed $\ell^{p,q}$-spaces. In particular, the cross Gram matrix $G_{\Psi_1\otimes\Psi_2,\Phi_1\otimes\Phi_2}$ is not necessarily a bounded operator on the spaces $\ell^{p_1,p_1}_{w_1\otimes w_2}(I\times J)$ when $p_1\neq p_2$. The proofs of the statements (ii)-(v) rely crucially on (i). As a consequence of one of our kernel theorems, we shall however show in Corollary 4.5 that some mixed-norm co-orbit spaces are indeed independent of the specific frames defining them as long as the individual cross Gram matrices belong to $\mathcal{A}_n$, $n=1,2$.

In the following we establish the reconstruction formula on $\mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)$ and a necessary and sufficient condition for $k\in\ell^{\infty,\infty}_{1/(w_1\otimes w_2)}(I\times J)$ to be the generalised wavelet transform of a vector $F\in\mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)$ – which, following the tradition of [22,26], can be called the 'correspondence principle'.

Lemma 3.4. *Let $\Psi_n,\Phi_n\subset\mathcal{H}_n$ be $\mathcal{A}_n$-localised frames, and w_n be $\mathcal{A}_n$-admissible weights for $n=1,2$. If $F\in\mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)$, then*

$$F = D_{\Psi_1\otimes\Psi_2}C_{\widetilde{\Psi_1\otimes\Psi_2}}F = D_{\widetilde{\Psi_1\otimes\Psi_2}}C_{\Psi_1\otimes\Psi_2}F. \tag{3.7}$$

Moreover, for any $k\in\ell^{\infty,\infty}_{1/(w_1\otimes w_2)}(I\times J)$ there exists $F\in\mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)$ such that

$$k = C_{\widetilde{\Psi_1\otimes\Psi_2}}F \tag{3.8}$$

if and only if

$$k = G_{\widetilde{\Psi_1\otimes\Psi_2},\,\Psi_1\otimes\Psi_2}\,k = C_{\widetilde{\Psi_1\otimes\Psi_2}}D_{\Psi_1\otimes\Psi_2}\,k. \tag{3.9}$$

The statements also apply to $k\in\ell^{\infty,\infty}_{w_1\otimes w_2}(I\times J)$ and $F\in\mathcal{H}^{\infty,\infty}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)$.

Proof. Let $F\in\mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)$, $H\in\mathcal{H}_{00}(\Psi_1\otimes\Psi_2)$, and let $\{I_n\}_{n\in\mathbb{N}}$ and $\{J_n\}_{n\in\mathbb{N}}$ be sequences of finite subsets of I and J respectively such that $I_n\subset I_{n+1}\subseteq I$, and $J_n\subset J_{n+1}\subseteq J$ for any $n\in\mathbb{N}$,

$\cup_{n\in\mathbb{N}} I_n = I$ and $\cup_{n\in\mathbb{N}} J_n = J$. By Proposition 3.2 (iii) $D_{\Psi_1\otimes\Psi_2} C_{\widetilde{\Psi_1\otimes\Psi_2}} F$ is well-defined and converges weak-$*$ unconditionally. Moreover,

$$\begin{aligned}
&\langle D_{\Psi_1\otimes\Psi_2} C_{\widetilde{\Psi_1\otimes\Psi_2}} F, H\rangle_{\mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2),\mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)} \\
&= \lim_{n\to\infty} \sum_{(i,j)\in I_n\times J_n} \langle F, \widetilde{\psi}_{1,\,i}\otimes\widetilde{\psi}_{2,\,j}\rangle_{\mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2),\mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)} \langle \psi_{1,\,i}\otimes\psi_{2,\,j}, H\rangle \\
&= \big\langle F, D_{\widetilde{\Psi_1\otimes\Psi_2}} C_{\Psi_1\otimes\Psi_2} H\big\rangle_{\mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2),\mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)} \\
&= \langle F, H\rangle_{\mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2),\mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)}.
\end{aligned}$$

And so $D_{\Psi_1\otimes\Psi_2} C_{\widetilde{\Psi_1\otimes\Psi_2}} F = F$. The second reconstruction formula follows from a similar argument once we recall that $\mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2) = \mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\widetilde{\Psi}_1\otimes\widetilde{\Psi}_2)$.

To prove the second statement, let $k \in \ell^{\infty,\infty}_{1/(w_1\otimes w_2)}(I\times J)$ satisfy (3.9). Then

$$F_n := \sum_{i\in I_n}\sum_{j\in J_n} k_{i,j}\, \psi_{1,\,i}\otimes\psi_{2,\,j} \in \mathcal{H}_1\otimes\mathcal{H}_2,$$

for any $n\in\mathbb{N}$ and

$$\begin{aligned}
\lim_{n\to\infty} \big\langle F_n, \widetilde{\psi}_{1,\,r}\otimes\widetilde{\psi}_{2,\,s}\big\rangle_{\mathcal{H}_1\otimes\mathcal{H}_2} &= \lim_{n\to\infty} \Big\langle \sum_{i\in I_n}\sum_{j\in J_n} k_{i,\,j}\,\psi_{1,\,i}\otimes\psi_{2,\,j}, \widetilde{\psi}_{1,\,r}\otimes\widetilde{\psi}_{2,\,s}\Big\rangle_{\mathcal{H}_1\otimes\mathcal{H}_2} \\
&= \lim_{n\to\infty} \sum_{i\in I_n}\sum_{j\in J_n} k_{i,\,j}\,\big\langle \psi_{1,\,i}\otimes\psi_{2,\,j},\, \widetilde{\psi}_{1,\,r}\otimes\widetilde{\psi}_{2,\,s}\big\rangle_{\mathcal{H}_1\otimes\mathcal{H}_2} \\
&= \Big(C_{\widetilde{\Psi_1\otimes\Psi_2}} D_{\Psi_1\otimes\Psi_2}\, k\Big)_{r,\,s} \\
&= \Big(G_{\widetilde{\Psi_1\otimes\Psi_2},\,\Psi_1\otimes\Psi_2}\, k\Big)_{r,\,s} = k_{r,\,s}.
\end{aligned}$$

From this we infer that

$$\big\langle F_n, \widetilde{\psi}_{1,\,r}\otimes\widetilde{\psi}_{2,\,s}\big\rangle_{\mathcal{H}_1\otimes\mathcal{H}_2} = \Big(G_{\widetilde{\Psi_1\otimes\Psi_2},\,\Psi_1\otimes\Psi_2}\, k\big|_{I_n\times J_n}\Big)_{r,\,s}$$

and so

$$\begin{aligned}
\sup_{r,\,s} \big|\big\langle F_n, \widetilde{\psi}_{1,\,r}\otimes\widetilde{\psi}_{2,\,s}\big\rangle_{\mathcal{H}_1\otimes\mathcal{H}_2}\big| (w_{1,\,r}\, w_{2,\,s})^{-1} &\lesssim \big\|k\big|_{I_n\times J_n}\big\|_{\ell^{\infty,\infty}_{1/(w_1\otimes w_2)}(I\times J)} \\
&\leqslant \|k\|_{\ell^{\infty,\infty}_{1/(w_1\otimes w_2)}(I\times J)}.
\end{aligned}$$

Therefore $F = [F_n] \in \mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)$ satisfies (3.8) and is, as an equivalence class, uniquely defined by k via (3.8).

Conversely, let $k \in \ell^{\infty,\infty}_{1/(w_1\otimes w_2)}(I\times J)$ and $F \in \mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)$ be such that (3.8) is satisfied. Proposition 3.2 (iii) indicates that $D_{\Psi_1\otimes\Psi_2}$ is a bounded operator mapping $\ell^{\infty,\infty}_{1/(w_1\otimes w_2)}(I\times J)$ to $\mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)$. Applying $C_{\widetilde{\Psi_1\otimes\Psi_2}} D_{\Psi_1\otimes\Psi_2}$ to both sides of (3.8) and using (3.7) results in

$$C_{\widetilde{\Psi_1\otimes\Psi_2}} D_{\Psi_1\otimes\Psi_2} k = C_{\widetilde{\Psi_1\otimes\Psi_2}} D_{\Psi_1\otimes\Psi_2} C_{\widetilde{\Psi_1\otimes\Psi_2}} F = C_{\widetilde{\Psi_1\otimes\Psi_2}} F = k.$$

Finally, if w_n is an $\mathcal{A}_n$-admissible weight then so is $1/w_n$, $n = 1, 2$. Therefore, the statement of the theorem also applies to $k \in \ell^{\infty,\infty}_{w_1\otimes w_2}(I\times J)$ and $F \in \mathcal{H}^{\infty,\infty}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)$. $\square$

4. Kernel theorems

We are now ready to state and prove our main results. To prove the outer kernel theorem, we essentially follow the proof strategy of [6], but use somewhat more involved arguments in those steps where we cannot rely on either the group structure or submultiplicativity of the weight function. The basic idea is to represent the operator by its Galerkin matrix and to derive the kernel of the operator from this representation.

Theorem 4.1. (Outer Kernel Theorem) *Let $\Psi_n \subset \mathcal{H}_n$ be an $\mathcal{A}_n$-localised frame, and w_n an $\mathcal{A}_n$-admissible weight for $n = 1, 2$.*

To any element of $K \in \mathcal{H}^{\infty,\infty}_{1/(w_1 \otimes w_2)}(\Psi_1 \otimes \Psi_2)$ corresponds a unique bounded linear operator $O : \mathcal{H}^1_{w_1}(\Psi_1) \to \mathcal{H}^\infty_{1/w_2}(\Psi_2)$ by means of

$$\langle Of_1, f_2\rangle_{\mathcal{H}^\infty_{1/w_2}(\Psi_2),\, \mathcal{H}^1_{w_2}(\Psi_2)} = \langle K, f_1 \otimes f_2\rangle_{\mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2),\, \mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)} \tag{4.1}$$

for any $f_1 \in \mathcal{H}^1_{w_1}(\Psi_1)$ and any $f_2 \in \mathcal{H}^1_{w_2}(\Psi_2)$. Moreover,

$$\|K\|_{\mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)} \asymp \|O\|_{\mathcal{H}^1_{w_1}(\Psi_1)\to\mathcal{H}^\infty_{1/w_2}(\Psi_2)}\,. \tag{4.2}$$

Conversely, to any bounded linear operator $O : \mathcal{H}^1_{w_1}(\Psi_1) \to \mathcal{H}^\infty_{1/w_2}(\Psi_2)$ corresponds a unique $K \in \mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1 \otimes \Psi_2)$, called the kernel *of O, that satisfies (4.1) for every $f_1 \in \mathcal{H}^1_{w_1}(\Psi_1)$ and $f_2 \in \mathcal{H}^1_{w_2}(\Psi_2)$.*

Proof. Step 1. Note that $f_1 \otimes f_2 \in \mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1 \otimes \Psi_2)$ as long as $f_1 \in \mathcal{H}^1_{w_1}(\Psi_1)$ and $f_2 \in \mathcal{H}^1_{w_2}(\Psi_2)$. So, for $K \in \mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1 \otimes \Psi_2)$,

$$\begin{aligned}
&\left|\langle K, f_1 \otimes f_2\rangle_{\mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2),\mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)}\right| \\
&\qquad = \left|\sum_{i\in I}\sum_{j\in J}\left\langle K, \widetilde{\psi}_{1,\,i} \otimes \widetilde{\psi}_{2,\,j}\right\rangle \langle \psi_{1,\,i} \otimes \psi_{2,\,j}, f_1 \otimes f_2\rangle\right| \\
&\qquad \leqslant \sum_{i\in I}\sum_{j\in J}\left|\left\langle K, \widetilde{\psi}_{1,\,i} \otimes \widetilde{\psi}_{2,\,j}\right\rangle \langle \psi_{1,\,i} \otimes \psi_{2,\,j}, f_1 \otimes f_2\rangle\right| \\
&\qquad \leqslant \left\|C_{\widetilde{\Psi_1\otimes\Psi_2}} K\right\|_{\ell^{\infty,\infty}_{1/(w_1\otimes w_2)}(I\times J)} \left\|C_{\Psi_1\otimes\Psi_2}(f_1\otimes f_2)\right\|_{\ell^{1,1}_{w_1\otimes w_2}(I\times J)} \\
&\qquad \asymp \|K\|_{\mathcal{H}^{\infty,\,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)} \|f_1\otimes f_2\|_{\mathcal{H}^{1,\,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)} \\
&\qquad < \infty,
\end{aligned}$$

where we used Proposition 3.2 (ii) for the second to last step. Now,

$$(f_1, f_2) \mapsto \langle K, f_1 \otimes f_2\rangle_{\mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2),\mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)}$$

is a sesquilinear form on $\mathcal{H}^1_{w_1}(\Psi_1) \times \mathcal{H}^1_{w_2}(\Psi_2)$. Therefore, for any fixed $f_1 \in \mathcal{H}^1_{w_1}(\Psi_1)$, the mapping $f_2 \mapsto \langle K, f_1 \otimes f_2\rangle_{\mathcal{H}^{\infty,\,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2),\mathcal{H}^{1,\,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)}$ is an antilinear functional on $\mathcal{H}^1_{w_2}(\Psi_2)$ which we shall call Of_1. The mapping $f_1 \mapsto Of_1$ is linear and so (4.1) defines a linear operator $O : \mathcal{H}^1_{w_1}(\Psi_1) \to \mathcal{H}^\infty_{1/w_2}(\Psi_2)$. This operator is bounded. Indeed,

$$\begin{aligned}
\left|\langle Of_1, f_2\rangle_{\mathcal{H}^\infty_{1/w_2}(\Psi_2),\mathcal{H}^1_{w_2}(\Psi_2)}\right| &= \left|\langle K, f_1 \otimes f_2\rangle_{\mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2),\mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)}\right| \\
&\leqslant \|K\|_{\mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)} \|f_2\|_{\mathcal{H}^1_{w_2}(\Psi_2)} \|f_1\|_{\mathcal{H}^1_{w_1}(\Psi_1)}\,,
\end{aligned}$$

and therefore

$$\|Of_1\|_{\mathcal{H}^{\infty}_{1/w_2}(\Psi_2)} \leqslant \|K\|_{\mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)} \|f_1\|_{\mathcal{H}^1_{w_1}(\Psi_1)},$$

and

$$\|O\|_{\mathcal{H}^1_{w_1}(\Psi_1)\to\mathcal{H}^{\infty}_{1/w_2}(\Psi_2)} \leqslant \|K\|_{\mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)}.$$

This proves one of the inequalities implied by (4.2) and indicates that the map $K \mapsto O$ is bounded.

Step 2. Let us now assume that the bounded linear operator $O : \mathcal{H}^1_{w_1}(\Psi_1) \to \mathcal{H}^{\infty}_{1/w_2}(\Psi_2)$ has two distinct kernels K_1 and $K_2 \in \mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)$ satisfying (4.1). This implies in particular that for every $(i,j) \in I \times J$

$$\big\langle O\widetilde{\psi}_{1,\,i}, \widetilde{\psi}_{2,\,j}\big\rangle_{\mathcal{H}^{\infty}_{1/w_2}(\Psi_2),\mathcal{H}^1_{w_2}(\Psi_2)} = \big\langle K_1, \widetilde{\psi}_{1,\,i}\otimes\widetilde{\psi}_{2,\,j}\big\rangle_{\mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2),\mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)},$$

and

$$\big\langle O\widetilde{\psi}_{1,\,i}, \widetilde{\psi}_{2,\,j}\big\rangle_{\mathcal{H}^{\infty}_{1/w_2}(\Psi_2),\mathcal{H}^1_{w_2}(\Psi_2)} = \big\langle K_2, \widetilde{\psi}_{1,\,i}\otimes\widetilde{\psi}_{2,\,j}\big\rangle_{\mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2),\mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)}.$$

Hence, for every $(i,j) \in I \times J$ it holds

$$\begin{aligned}&\big\langle K_1, \widetilde{\psi}_{1,\,i}\otimes\widetilde{\psi}_{2,\,j}\big\rangle_{\mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2),\mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)}\\ &\qquad\qquad = \big\langle K_2, \widetilde{\psi}_{1,\,i}\otimes\widetilde{\psi}_{2,\,j}\big\rangle_{\mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2),\mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)}.\end{aligned}$$

From this assumption and the fact that, according to Proposition 3.2 (iv), any $F \in \mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)$ can be expressed by an absolutely convergent series as $F = D_{\Psi_1\otimes\Psi_2}c$ we infer that

$$\begin{aligned}&\langle K_1, F\rangle_{\mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2),\mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)}\\
&\quad = \Big\langle K_1, \sum_{i\in I}\sum_{j\in J} c_{i,j}\,\psi_{1,\,i}\otimes\psi_{2,\,j}\Big\rangle_{\mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2),\mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)}\\
&\quad = \sum_{i\in I}\sum_{j\in J} c_{i,j}\,\langle K_1, \psi_{1,\,i}\otimes\psi_{2,\,j}\rangle_{\mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2),\mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)}\\
&\quad = \sum_{i\in I}\sum_{j\in J} c_{i,j}\langle K_2, \psi_{1,\,i}\otimes\psi_{2,\,j}\rangle_{\mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2),\mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)}\\
&\quad = \Big\langle K_2, \sum_{i\in I}\sum_{j\in J} c_{i,j}\,\psi_{1,\,i}\otimes\psi_{2,\,j}\Big\rangle_{\mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2),\mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)}\\
&\quad = \langle K_2, F\rangle_{\mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2),\mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)},\end{aligned}$$

and so $K_1 = K_2$ according to Proposition 3.2 (v). In other words, the map $K \mapsto O$ is injective.

Step 3. Let us now assume that the linear operator $O : \mathcal{H}^1_{w_1}(\Psi_1) \to \mathcal{H}^{\infty}_{1/w_2}(\Psi_2)$ is bounded and consider its Galerkin matrix, that is the matrix k whose (i,j)-th element is defined by

$$k_{i,j} := \big\langle O\widetilde{\psi}_{1,i}, \widetilde{\psi}_{2,j}\big\rangle_{\mathcal{H}^{\infty}_{1/w_2}(\Psi_2),\mathcal{H}^1_{w_2}(\Psi_2)}, \quad (i,j) \in I \times J. \tag{4.3}$$

From (4.3), and (2.10) we infer that

$$
\begin{aligned}
|k_{i,j}| &\leqslant \|O\|_{\mathcal{H}^1_{w_1}(\Psi_1)\to\mathcal{H}^\infty_{1/w_2}(\Psi_2)} \left\|\widetilde{\psi}_{1,i}\right\|_{\mathcal{H}^1_{w_1}(\Psi_1)} \left\|\widetilde{\psi}_{2,j}\right\|_{\mathcal{H}^1_{w_2}(\Psi_2)} \\
&\lesssim \|O\|_{\mathcal{H}^1_{w_1}(\Psi_1)\to\mathcal{H}^\infty_{1/w_2}(\Psi_2)}\, \omega_{1,i}\, \omega_{2,j}.
\end{aligned}
$$

In other words, $k \in \ell^{\infty,\infty}_{1/(w_1\otimes w_2)}(I\times J)$ as long as the operator $O : \mathcal{H}^1_{w_1}(\Psi_1) \to \mathcal{H}^\infty_{1/w_2}(\Psi_2)$ is bounded. Subsequently, we show that k satisfies (3.9), which, according to Lemma 3.4, implies that there is a unique $K \in \mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)$ that satisfies $C_{\widetilde{\Psi_1\otimes\Psi_2}}K = k$. Indeed, for any $(i,j)\in I\times J$,

$$
\begin{aligned}
&\left(C_{\widetilde{\Psi_1\otimes\Psi_2}} D_{\Psi_1\otimes\Psi_2} k\right)_{r,s} \\
&\quad= \sum_{i\in I}\sum_{j\in J} \left\langle O\widetilde{\psi}_{1,i}, \widetilde{\psi}_{2,j}\right\rangle_{\mathcal{H}^\infty_{1/w_2}(\Psi_2),\mathcal{H}^1_{w_2}(\Psi_2)} \left\langle \psi_{1,i}\otimes\psi_{2,j}, \widetilde{\psi}_{1,r}\otimes\widetilde{\psi}_{2,s}\right\rangle_{\mathcal{H}_1\otimes\mathcal{H}_2} \\
&\quad= \sum_{i\in I}\sum_{j\in J} \left\langle O\widetilde{\psi}_{1,i}, \widetilde{\psi}_{2,j}\right\rangle_{\mathcal{H}^\infty_{1/w_2}(\Psi_2),\mathcal{H}^1_{w_2}(\Psi_2)} \overline{\left\langle \psi_{1,i}, \widetilde{\psi}_{1,r}\right\rangle_{\mathcal{H}_1}} \left\langle \psi_{2,j}, \widetilde{\psi}_{2,s}\right\rangle_{\mathcal{H}_2} \\
&\quad= \sum_{i\in I} \langle O\widetilde{\psi}_{1,i}, \widetilde{\psi}_{2,s}\rangle_{\mathcal{H}^\infty_{1/w_2}(\Psi_2),\mathcal{H}^1_{w_2}(\Psi_2)} \overline{\left\langle \psi_{1,i}, \widetilde{\psi}_{1,r}\right\rangle_{\mathcal{H}_1}} \\
&\quad= \sum_{i\in I} \langle \widetilde{\psi}_{1,i}, O'\widetilde{\psi}_{2,s}\rangle_{\mathcal{H}^1_{w_1}(\Psi_1),\mathcal{H}^\infty_{1/w_1}(\Psi_1)} \left\langle \widetilde{\psi}_{1,r}, \psi_{1,i}\right\rangle_{\mathcal{H}_1} \\
&\quad= \langle \widetilde{\psi}_{1,r}, O'\widetilde{\psi}_{2,s}\rangle_{\mathcal{H}^1_{w_1}(\Psi_1),\,\mathcal{H}^\infty_{1/w_1}(\Psi_1)} \\
&\quad= \langle O\widetilde{\psi}_{1,r}, \widetilde{\psi}_{2,s}\rangle_{\mathcal{H}^\infty_{1/w_2}(\Psi_2),\mathcal{H}^1_{w_2}(\Psi_2)} = k_{r,s},
\end{aligned}
$$

where O' stands for the operator that satisfies

$$
\langle Of_1, f_2\rangle_{\mathcal{H}^\infty_{1/w_2}(\Psi_2),\mathcal{H}^1_{w_2}(\Psi_2)} = \langle f_1, O'f_2\rangle_{\mathcal{H}^1_{w_1}(\Psi_1),\mathcal{H}^\infty_{1/w_1}(\Psi_1)},
$$

for every $f_1 \in \mathcal{H}^1_{w_1}(\Psi_1)$ and $f_2 \in \mathcal{H}^1_{w_2}(\Psi_2)$. The change of the order of summation over i and j that we made here is justified by the fact that

$$
\left\{\left\langle O\widetilde{\psi}_{1,i}, \widetilde{\psi}_{2,j}\right\rangle_{\mathcal{H}^\infty_{1/w_2}(\Psi_2),\mathcal{H}^1_{w_2}(\Psi_2)} \overline{\left\langle \psi_{1,i}, \widetilde{\psi}_{1,r}\right\rangle_{\mathcal{H}_1}} \left\langle \psi_{2,j}, \widetilde{\psi}_{2,s}\right\rangle_{\mathcal{H}_2}\right\}_{(i,j)\in I^2} \in \ell^{1,1}(I\times J)
$$

for any given $(r, s) \in I\times J$, as

$$
\left\{\left\langle O\widetilde{\psi}_{1,i}, \widetilde{\psi}_{2,j}\right\rangle_{\mathcal{H}^\infty_{1/w_2}(\Psi_2),\mathcal{H}^1_{w_2}(\Psi_2)}\right\}_{(i,j)\in I\times J} = \{k_{i,j}\}_{(i,j)\in I\times J} \in \ell^{\infty,\infty}_{1/(w_1\otimes w_2)}(I\times J)
$$

and

$$
\begin{aligned}
\left\{\overline{\left\langle \psi_{1,i}, \widetilde{\psi}_{1,r}\right\rangle_{\mathcal{H}_1}} \left\langle \psi_{2,j}, \widetilde{\psi}_{2,s}\right\rangle_{\mathcal{H}_2}\right\}_{(i,j)\in I\times J} &= \left\{\overline{(C_{\Psi_1}\widetilde{\psi}_{1,r})_i}(C_{\Psi_2}\widetilde{\psi}_{2,s})_j\right\}_{(i,j)\in I\times J} \\
&\qquad\qquad \in \ell^{1,1}_{w_1\otimes w_2}(I\times J),
\end{aligned}
$$

by Proposition 2.7 (vi) and Lemma 2.8. According to what we already proved, to this unique $K \in \mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)$ corresponds a linear operator mapping $\mathcal{H}^1_{w_1}(\Psi_1)$ to $\mathcal{H}^\infty_{1/w_2}(\Psi_2)$ via (4.1). This operator is nothing but the bounded linear operator O. Indeed, let us assume that there is more than one linear operator mapping $\mathcal{H}^1_{w_1}(\Psi_1)$ to $\mathcal{H}^\infty_{1/w_2}(\Psi_2)$, say, O_1 and O_2, that correspond to the kernel $K \in \mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)$ via (4.1). So in particular for $\psi_{1,i}\otimes\psi_{2,j} \in \mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)$ we get

$$\big\langle O_1\widetilde{\psi}_{1,\,i},\widetilde{\psi}_{2,\,j}\big\rangle_{\mathcal{H}^{\infty}_{1/w_2}(\Psi_2),\mathcal{H}^{1}_{w_2}(\Psi_2)}=\big\langle K,\widetilde{\psi}_{1,\,i}\otimes\widetilde{\psi}_{2,\,j}\big\rangle_{\mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2),\mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)},$$

and

$$\big\langle O_2\widetilde{\psi}_{1,\,i},\widetilde{\psi}_{2,\,j}\big\rangle_{\mathcal{H}^{\infty}_{1/w_2}(\Psi_2),\mathcal{H}^{1}_{w_2}(\Psi_2)}=\big\langle K,\widetilde{\psi}_{1,\,i}\otimes\widetilde{\psi}_{2,\,j}\big\rangle_{\mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2),\mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)}.$$

Thus

$$\big\langle O_1\widetilde{\psi}_{1,\,i},\widetilde{\psi}_{2,\,j}\big\rangle_{\mathcal{H}^{\infty}_{1/w_2}(\Psi_2),\mathcal{H}^{1}_{w_2}(\Psi_2)}=\big\langle O_2\widetilde{\psi}_{1,\,i},\widetilde{\psi}_{2,\,j}\big\rangle_{\mathcal{H}^{\infty}_{1/w_2}(\Psi_2),\mathcal{H}^{1}_{w_2}(\Psi_2)}.$$

This equality, and the fact that, according to Proposition 2.9, any $f_1\in\mathcal{H}^1_{w_1}(\Psi_1)$ and any $f_2\in\mathcal{H}^1_{w_2}(\Psi_2)$ can be expressed by an absolutely convergent series, we infer that

$$\begin{aligned}\langle O_1 f_1, f_2\rangle_{\mathcal{H}^{\infty}_{1/w_2}(\Psi_2),\mathcal{H}^{1}_{w_2}(\Psi_2)}&=\Big\langle O_1\sum_{i\in I}c_{1,\,i}\,\widetilde{\psi}_{1,\,i},\sum_{j\in J}c_{2,\,j}\,\widetilde{\psi}_{2,\,j}\Big\rangle_{\mathcal{H}^{\infty}_{1/w_2}(\Psi_2),\mathcal{H}^{1}_{w_2}(\Psi_2)}\\&=\sum_{i\in I}\sum_{j\in J}c_{1,\,i}\,c_{2,\,j}\big\langle O_1\widetilde{\psi}_{1,\,i},\widetilde{\psi}_{2,\,j}\big\rangle_{\mathcal{H}^{\infty}_{1/w_2}(\Psi_2),\mathcal{H}^{1}_{w_2}(\Psi_2)}\\&=\sum_{i\in I}\sum_{j\in J}c_{1,\,i}\,c_{2,\,j}\,\big\langle O_2\widetilde{\psi}_{1,\,i},\widetilde{\psi}_{2,\,j}\big\rangle_{\mathcal{H}^{\infty}_{1/w_2}(\Psi_2),\mathcal{H}^{1}_{w_2}(\Psi_2)}\\&=\Big\langle O_2\sum_{i\in I}c_{1,\,i}\,\widetilde{\psi}_{1,\,i},\sum_{j\in J}c_{2,\,j}\,\widetilde{\psi}_{2,\,j}\Big\rangle_{\mathcal{H}^{\infty}_{1/w_2}(\Psi_2),\mathcal{H}^{1}_{w_2}(\Psi_2)}\\&=\langle O_2 f_1, f_2\rangle_{\mathcal{H}^{\infty}_{1/w_2}(\Psi_2),\mathcal{H}^{1}_{w_2}(\Psi_2)}.\end{aligned}$$

Thus $O_1 f_1=O_2 f_1$ for every $f_1\in\mathcal{H}^1_{w_1}(\Psi_1)$ or, in other words, $O_1=O_2$. Consequently, the mapping $O\mapsto K$ that takes the space $B\big(\mathcal{H}^1_{w_1}(\Psi_1),\mathcal{H}^{\infty}_{1/w_2}(\Psi_2)\big)$ to $\mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)$, is surjective and invertible and, as we proved before, bounded and injective. Therefore, its inverse is also bounded according to the inverse mapping theorem, which is expressed by the first inequality in (4.2). □

The so-called *tensor product property*, discovered by Feichtinger [20, Theorem 7D] (see also [37]), was generalised for co-orbit spaces associated with integrable group representations in [6]. In what follows, we shall obtain a similar result for co-orbit spaces associated with $\mathcal{A}$-localised frames. Let us recollect the definition of the *projective tensor product* of two Banach spaces B_1 and B_2, given by

$$B_1\widehat{\otimes}_\pi B_2:=\left\{F=\sum_{r\in R}f_r\otimes g_r:\quad f_r\in B_1,\,g_r\in B_2,\quad\sum_{r\in R}\|f_r\|_{B_1}\|g_r\|_{B_2}<\infty\right\}.$$

The function

$$\|F\|_{B_1\widehat{\otimes}_\pi B_2}:=\inf\left(\sum_{r\in R}\|f_r\|_{B_1}\|g_r\|_{B_2}\right),\tag{4.4}$$

the infimum being taken over all representations $\sum_{r\in R}f_r\otimes g_r$ of $F\in B_1\widehat{\otimes}_\pi B_2$, defines a norm for $B_1\widehat{\otimes}_\pi B_2$.

Theorem 4.2. *Let $\Psi_n\subset\mathcal{H}_n$ be an $\mathcal{A}_n$-localised frame, and w_n be an $\mathcal{A}_n$-admissible weight for $n=1,2$. Then*

$$\mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)=\mathcal{H}^1_{w_1}(\Psi_1)\,\widehat{\otimes}_\pi\mathcal{H}^1_{w_2}(\Psi_2)\tag{4.5}$$

with equivalent norms.

Proof. Let $F \in \mathcal{H}^{1,1}_{w_1 \otimes w_2}(\Psi_1 \otimes \Psi_2)$. Then, according to Proposition 3.2 (iv), the function F can be expressed as $F = D_{\Psi_1 \otimes \Psi_2} c$ for some $c \in \ell^{1,1}_{w_1 \otimes w_2}(I \times J)$ such that $\|c\|_{\ell^{1,1}_{w_1 \otimes w_2}(I \times J)} \asymp \|F\|_{\mathcal{H}^{1,1}_{w_1 \otimes w_2}(\Psi_1 \otimes \Psi_2)}$. Combining this with (2.10) results in

$$\begin{aligned}
\|F\|_{\mathcal{H}^1_{w_1}(\Psi_1) \widehat{\otimes}_\pi \mathcal{H}^1_{w_2}(\Psi_2)} &\leqslant \sum_{i \in I} \sum_{j \in J} |c_{i,j}| \, \|\psi_{1,\, i}\|_{\mathcal{H}^1_{w_1}(\Psi_1)} \|\psi_{2,\, j}\|_{\mathcal{H}^1_{w_2}(\Psi_2)} \\
&\lesssim \sum_{i \in I} \sum_{j \in J} |c_{i,j}| \, w_{1,\, i} \, w_{2,\, j} \\
&= \|c\|_{\ell^{1,1}_{w_1 \otimes w_2}(I \times J)} \asymp \|F\|_{\mathcal{H}^{1,1}_{w_1 \otimes w_2}(\Psi_1 \otimes \Psi_2)}.
\end{aligned}$$

Thus $\mathcal{H}^{1,1}_{w_1 \otimes w_2}(\Psi_1 \otimes \Psi_2)$ is continuously embedded in $\mathcal{H}^1_{w_1}(\Psi_1) \widehat{\otimes}_\pi \mathcal{H}^1_{w_2}(\Psi_2)$.

Let now $F \in \mathcal{H}^1_{w_1}(\Psi_1) \widehat{\otimes}_\pi \mathcal{H}^1_{w_2}(\Psi_2)$. In other words, let $F = \sum_{r \in R} f_r \otimes g_r$ where $f_r \in \mathcal{H}^1_{w_1}(\Psi_1)$ and $g_r \in \mathcal{H}^1_{w_2}(\Psi_2)$ satisfy

$$\sum_{r \in R} \|f_r\|_{\mathcal{H}^1_{w_1}(\Psi_1)} \, \|g_r\|_{\mathcal{H}^1_{w_2}(\Psi_2)} < \infty.$$

Then

$$\begin{aligned}
\|F\|_{\mathcal{H}^{1,1}_{w_1 \otimes w_2}(\Psi_1 \otimes \Psi_2)} &= \|C_{\widetilde{\Psi_1 \otimes \Psi_2}} F\|_{\ell^{1,1}_{w_1 \otimes w_2}(I \times J)} \\
&= \sum_{i \in I} \sum_{j \in J} \Big| \sum_{r \in R} \big(C_{\widetilde{\Psi_1 \otimes \Psi_2}}(f_r \otimes g_r)\big)_{i,j} \Big| \, w_{1,\, i} \, w_{2,\, j} \\
&\leqslant \sum_{r \in R} \Big(\sum_{i \in I} \big|\big(C_{\widetilde{\Psi}_1} f_r\big)_i\big| w_{1,\, i} \Big) \Big(\sum_{j \in J} \big|\big(C_{\widetilde{\Psi}_2} g_r\big)_j\big| w_{2,\, j} \Big) \\
&= \sum_{r \in R} \|C_{\widetilde{\Psi}_1} f_r\|_{\ell^1_{w_1}(I)} \, \|C_{\widetilde{\Psi}_2} g_r\|_{\ell^1_{w_2}(J)} \\
&= \sum_{r \in R} \|f_r\|_{\mathcal{H}^1_{w_1}(\Psi_1)} \, \|g_r\|_{\mathcal{H}^1_{w_2}(\Psi_2)} < \infty.
\end{aligned}$$

The change of the order of summation over i, j, and r above is justified by the absolute convergence of the series. Taking the infimum over all possible representations of F in the inequality above then shows that $\mathcal{H}^1_{w_1}(\Psi_1) \widehat{\otimes}_\pi \mathcal{H}^1_{w_2}(\Psi_2)$ is also continuously embedded in $\mathcal{H}^{1,1}_{w_1 \otimes w_2}(\Psi_1 \otimes \Psi_2)$, which concludes the proof. $\square$

The previous theorem allows us to characterise the class of operators with kernels in $\mathcal{H}^{1,1}_{w_1 \otimes w_2}(\Psi_1 \otimes \Psi_2)$.

Theorem 4.3 *(Inner Kernel Theorem). Let Ψ_n be an $\mathcal{A}_n$-localised frame for a Hilbert space $\mathcal{H}_1$, w_n an $\mathcal{A}_n$-admissible weight $n = 1, 2$. Then there is an isomorphism between $\mathcal{H}^{1,1}_{w_1 \otimes w_2}(\Psi_1 \otimes \Psi_2)$ and the space*

$$\begin{aligned}
\mathcal{B} := \Big\{ &O \in B\big(\mathcal{H}^\infty_{1/w_1}(\Psi_1), \mathcal{H}^1_{w_2}(\Psi_2)\big) : \text{ there exist } f_r \in H^1_{w_1}(\Psi_1), g_r \in H^1_{w_2}(\Psi_2) \text{ s.t.} \\
&O = \sum_{r \in R} \langle \cdot\,, f_r \rangle_{H^\infty_{1/w_1}(\Psi_1), H^1_{w_1}(\Psi_1)} \, g_r, \text{ and } \sum_{r \in R} \|f_r\|_{H^1_{w_1}(\Psi_1)} \, \|g_r\|_{H^1_{w_2}(\Psi_2)} < \infty \Big\}.
\end{aligned}$$

In particular, if any of the two,

$$K \in \mathcal{H}^{1,1}_{w_1 \otimes w_2}(\Psi_1 \otimes \Psi_2) \quad \text{or} \quad O \in \mathcal{B},$$

is given, the other is uniquely determined by means of

$$\langle K, f_1 \otimes f_2\rangle_{\mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2),\mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)} = \langle Of_1, f_2\rangle_{\mathcal{H}^1_{w_2}(\Psi_2),\mathcal{H}^\infty_{1/w_2}(\Psi_2)}$$

for any $f_1 \in \mathcal{H}^\infty_{1/w_1}(\Psi_1)$ and any $f_2 \in \mathcal{H}^\infty_{1/w_2}(\Psi_2)$.

Proof. First of all, we note that the condition

$$\sum_{r\in R} \|f_r\|_{H^1_{w_1}(\Psi_1)} \|g_r\|_{H^1_{w_2}(\Psi_2)} < \infty \tag{4.6}$$

implies that $O \in B\big(\mathcal{H}^\infty_{1/w_1}(\Psi_1), \mathcal{H}^1_{w_2}(\Psi_2)\big)$. Now, to prove the result, it suffices to establish an isomorphism between $\mathcal{B}$ and $\mathcal{H}^1_{w_1}(\Psi_1)\widehat{\otimes}_\pi\mathcal{H}^1_{w_2}(\Psi_2)$ and apply Theorem 4.2, according to which $\mathcal{H}^1_{w_1}(\Psi_1)\widehat{\otimes}_\pi\mathcal{H}^1_{w_2}(\Psi_2)$ and $\mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)$ are isomorphic. Let us define the mapping $\mathcal{I} : \mathcal{H}^1_{w_1}(\Psi_1)\widehat{\otimes}_\pi\mathcal{H}^1_{w_2}(\Psi_2) \to \mathcal{B}$ by

$$\mathcal{I}\Big(\sum_{r\in R} f_r \otimes g_r\Big) = \sum_{r\in R} \langle\,\cdot\,, f_r\rangle_{\mathcal{H}^\infty_{1/w_1}(\Psi_1),\mathcal{H}^1_{w_1}(\Psi_1)}\, g_r,$$

and first show that it is well-defined. For any two representations $\sum_{r\in R} f_r \otimes g_r$ and $\sum_{s\in S} h_s \otimes k_s$ of $H \in \mathcal{H}^1_{w_1}(\Psi_1)\widehat{\otimes}_\pi\mathcal{H}^1_{w_2}(\Psi_2)$, any $f_1 \in \mathcal{H}^\infty_{1/w_1}(\Psi_1)$ and any $f_2 \in \mathcal{H}^\infty_{1/w_2}(\Psi_2)$,

$$\begin{aligned}\Big\langle \sum_{r\in R} f_r \otimes g_r, f_1\otimes f_2\Big\rangle&_{\mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2),\mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)}\\ &= \Big\langle \sum_{s\in S} h_s \otimes k_s, f_1\otimes f_2\Big\rangle_{\mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2),\mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)}.\end{aligned}$$

Changing the order of scalar products and summation results in

$$\begin{aligned}\sum_{r\in R}\langle f_1, f_r\rangle_{\mathcal{H}^\infty_{1/w_1}(\Psi_1),\mathcal{H}^1_{w_1}(\Psi_1)}\langle g_r, f_2\rangle_{\mathcal{H}^1_{w_2}(\Psi_2),\mathcal{H}^\infty_{1/w_2}(\Psi_2)}&\\ = \sum_{s\in S}\langle f_1, h_s\rangle_{\mathcal{H}^\infty_{1/w_1}(\Psi_1),\mathcal{H}^1_{w_1}(\Psi_1)}&\langle k_s, f_2\rangle_{\mathcal{H}^1_{w_2}(\Psi_2),\mathcal{H}^\infty_{1/w_2}(\Psi_2)}\end{aligned} \tag{4.7}$$

for any $f_1 \in \mathcal{H}^\infty_{1/w_1}(\Psi_1)$ and any $f_2 \in \mathcal{H}^\infty_{1/w_2}(\Psi_2)$, from which we infer that $\mathcal{I}\big(\sum_{r\in R} f_r\otimes g_r\big) = \mathcal{I}\big(\sum_{s\in S} h_s\otimes k_s\big)$.

It is easy to see that $\mathcal{I}$ is surjective and it therefore remains to prove that is also injective. Let the elements $H_1 = \sum_{r\in R} f_r \otimes g_r$ and $H_2 = \sum_{s\in S} h_s \otimes k_s$ of $\mathcal{H}^1_{w_1}(\Psi_1)\widehat{\otimes}_\pi\mathcal{H}^1_{w_2}(\Psi_2)$ be such that $\mathcal{I}(H_1) = \mathcal{I}(H_2)$. In other words, (4.7) holds for any $f_1 \in \mathcal{H}^\infty_{1/w_1}(\Psi_1)$ and any $f_2 \in \mathcal{H}^\infty_{1/w_2}(\Psi_2)$. From this and (4.6), which holds for any element of $\mathcal{H}^1_{w_1}(\Psi_1)\widehat{\otimes}_\pi\mathcal{H}^1_{w_2}(\Psi_2)$, we deduce that

$$\begin{aligned}\langle H_1, f_1\otimes f_2\rangle&_{\mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2),\mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)}\\ &= \langle H_2, f_1\otimes f_2\rangle_{\mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2),\mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)}.\end{aligned}$$

Since $D_{\Psi_1\otimes\Psi_2} : \ell^{\infty,\infty}_{1/(w_1\otimes w_2)}(I\times J) \to \mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)$ is surjective (Lemma 3.4) and converges weak-$*$ unconditionally (Proposition 3.2 (iii)) we may consider linear combinations of $f_1\otimes f_2 = \psi_{1,i}\otimes\psi_{2,j}$ to show that $H_1 = H_2$ in $\mathcal{H}^{1,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)$. According to Theorem 4.2, this equality also holds in $\mathcal{H}^1_{w_1}(\Psi_1)\widehat{\otimes}_\pi\mathcal{H}^1_{w_2}(\Psi_2)$, which concludes the proof. $\square$

Now that we established the correspondence between operators and kernels in the extremes of the chain of co-orbit spaces, we investigate whether it is possible to characterise certain intermediate classes of operators via the properties of their kernels, in particular in terms of certain mixed-norm co-orbit spaces. The arguments crucially rely on a version of Schur's test (see, for example, [43, Propositions 5.2 and 5.4] or [38]) and the representation of an operator by its Galerkin matrix defined in (4.3) by

$$k_{i,j} := \langle O\widetilde{\psi}_{1,\,i}, \widetilde{\psi}_{2,\,j}\rangle_{\mathcal{H}^{\infty}_{1/w_2}(\Psi_2),\mathcal{H}^{1}_{w_2}(\Psi_2)}.$$

We note that the characterisation given by this matrix was established in [5, Proposition 6]. What is new here is that we relate the Galerkin matrix representing an operator to its kernel.

Theorem 4.4. *Let $1 \leqslant p, q \leqslant \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $\Psi_n \subset \mathcal{H}_n$ be an $\mathcal{A}_n$-localised frame, and w_n be an $\mathcal{A}_n$-admissible weight for $n = 1, 2$. Then*

(i) A linear operator $O : \mathcal{H}^1_{w_1}(\Psi_1) \to \mathcal{H}^p_{1/w_2}(\Psi_2)$ is bounded if and only if its kernel K is contained in $\mathfrak{H}^{p,\infty}_{1/(w_1\otimes w_2)}(\Psi_1 \otimes \Psi_2)$ or, alternatively, if and only if $k \in \mathfrak{l}^{p,\infty}_{1/(w_1\otimes w_2)}(I \times J)$, as defined by (3.1); and in that case

$$\|O\|_{\mathcal{H}^1_{w_1}(\Psi_1)\to\mathcal{H}^p_{1/w_2}(\Psi_2)} \asymp \|K\|_{\mathfrak{H}^{p,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)} = \|k\|_{\mathfrak{l}^{p,\infty}_{1/(w_1\otimes w_2)}(I\times J)}, \tag{4.8}$$

(ii) A linear operator $O : \mathcal{H}^p_{w_1}(\Psi_1) \to \mathcal{H}^\infty_{1/w_2}(\Psi_2)$ is bounded if and only if its kernel K is contained in $\mathcal{H}^{q,\infty}_{1/(w_1\otimes w_2)}(\Psi_1 \otimes \Psi_2)$ or, alternatively, if and only if $k \in \ell^{p,\infty}_{1/(w_1\otimes w_2)}(I \times J)$; and in that case

$$\|O\|_{\mathcal{H}^p_{w_1}(\Psi_1)\to\mathcal{H}^\infty_{1/w_2}(\Psi_2)} \asymp \|K\|_{\mathcal{H}^{q,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)} = \|k\|_{\ell^{p,\infty}_{1/(w_1\otimes w_2)}(I\times J)}. \tag{4.9}$$

Proof. Ad (i): The fact that $O : \mathcal{H}^1_{w_1}(\Psi_1) \to \mathcal{H}^p_{1/w_2}(\Psi_2)$ is bounded if and only if $k \in \mathfrak{l}^{p,\infty}_{1/(w_1\otimes w_2)}(I \times J)$ was established in [5, Proposition 6, eq. (24)]. To show the second equivalence, we first observe that if $O : \mathcal{H}^1_{w_1}(\Psi_1) \to \mathcal{H}^p_{1/w_2}(\Psi_2)$ is bounded, then

$$\|Of\|_{\mathcal{H}^\infty_{1/w_2}(\Psi_2)} \leqslant \|Of\|_{\mathcal{H}^p_{1/w_2}(\Psi_2)} \lesssim \|f\|_{\mathcal{H}^1_{w_1}(\Psi_1)}.$$

According to Theorem 3.4, there exists a unique kernel $K \in \mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1 \otimes \Psi_2)$ that satisfies (3.8). In particular,

$$\langle K, \widetilde{\psi}_{1,\,i} \otimes \widetilde{\psi}_{2,\,j}\rangle_{H^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2),H^{1,1}_{w_1\otimes w_2}(\Psi_1\otimes\Psi_2)} = \langle O\widetilde{\psi}_{1,\,i}, \widetilde{\psi}_{2,\,j}\rangle_{\mathcal{H}^\infty_{1/w_2}(\Psi_2),\mathcal{H}^1_{w_2}(\Psi_2)} = k_{i,j},$$

which shows that $\|K\|_{\mathfrak{H}^{p,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)} = \|k\|_{\mathfrak{l}^{p,\infty}_{1/(w_1\otimes w_2)}(I\times J)} < \infty$ if O is bounded.

If, on the other hand, $K \in \mathfrak{H}^{p,\infty}_{1/(w_1\otimes w_2)}(\Psi_1 \otimes \Psi_2) \subset \mathcal{H}^{\infty,\infty}_{1/(w_1\otimes w_2)}(\Psi_1 \otimes \Psi_2)$, then from Theorem 4.1 we infer that there exists a bounded operator $O : \mathcal{H}^1_{w_1}(\Psi_1) \to \mathcal{H}^\infty_{1/w_2}(\Psi_2)$ satisfying (3.8). Therefore, we conclude that $\|K\|_{\mathfrak{H}^{p,\infty}_{1/(w_1\otimes w_2)}(\Psi_1\otimes\Psi_2)} = \|k\|_{\mathfrak{l}^{p,\infty}_{1/(w_1\otimes w_2)}(I\times J)}$ which implies the boundedness of $O : \mathcal{H}^1_{w_1}(\Psi_1) \to \mathcal{H}^p_{1/w_2}(\Psi_2)$.

Ad (ii): The equivalence of the boundedness of the operator and the property of its Galerkin matrix is not explicitly stated in [5, Proposition 6]. This can however be done by essentially using the same arguments as those used in [5, Proposition 6] and applying [43, Proposition 2.4]. The rest of the proof is similar to the one of (i) with the only difference that here we need to make use of the inequalities

$$\|Of\|_{\mathcal{H}^\infty_{1/w_2}(\Psi_2)} \lesssim \|f\|_{\mathcal{H}^p_{w_1}(\Psi_1)} \leqslant \|f\|_{\mathcal{H}^1_{w_1}(\Psi_1)},$$

which ensures the existence of a kernel K. □

This result allows us to prove that the co-orbit spaces associated with the mixed norm spaces of Theorem 4.4 do not depend on the particular tensor product of localised frames as long as all involved cross-Gram matrices belong to $\mathcal{A}_n$.

Corollary 4.5. *Let $1 \leqslant p \leqslant \infty$, $\Psi_n, \Phi_n \subset \mathcal{H}_n$ be $\mathcal{A}_n$-localised frames, and w_n an $\mathcal{A}_n$-admissible weight $n = 1, 2$. If $G_{\Phi_n, \widetilde{\Psi}_n} \in \mathcal{A}_n$ and $G_{\Psi_n, \widetilde{\Phi}_n} \in \mathcal{A}_n$, $n = 1, 2$, then $\mathcal{H}^{p,\infty}_{w_1 \otimes w_2}(\Psi_1 \otimes \Psi_2) = \mathcal{H}^{p,\infty}_{w_1 \otimes w_2}(\Phi_1 \otimes \Phi_2)$ and $\mathfrak{H}^{p,\infty}_{w_1 \otimes w_2}(\Psi_1 \otimes \Psi_2) = \mathfrak{H}^{p,\infty}_{w_1 \otimes w_2}(\Phi_1 \otimes \Phi_2)$, with their norms being equivalent.*

Proof. We shall only prove the former identity as the latter follows from a similar argument. Let $K_\Psi \in \mathcal{H}^{p,\infty}_{w_1 \otimes w_2}(\Psi_1 \otimes \Psi_2)$ and $O : \mathcal{H}^p_{1/w_1}(\Psi_1) \to \mathcal{H}^\infty_{1/w_2}(\Psi_2)$ be the corresponding bounded operator according to Theorem 4.4 (ii). From Proposition 2.7 (ii) we know that $\mathcal{H}^p_{1/w_i}(\Psi_i) = \mathcal{H}^p_{1/w_i}(\Phi_i)$ and thus $O : \mathcal{H}^p_{1/w_1}(\Phi_1) \to \mathcal{H}^\infty_{1/w_2}(\Phi_2)$ is bounded with equivalent operator norm. Applying Theorem 4.4 (ii) again, we find an element $K_\Phi \in \mathcal{H}^{p,\infty}_{w_1 \otimes w_2}(\Phi_1 \otimes \Phi_2)$ such that

$$
\begin{aligned}
\|K_\Psi\|_{\mathcal{H}^{p,\infty}_{w_1 \otimes w_2}(\Psi_1 \otimes \Psi_2)} &\asymp \|O\|_{\mathcal{H}^p_{w_1}(\Psi_1) \to \mathcal{H}^\infty_{w_2}(\Psi_2)} \\
&\asymp \|O\|_{\mathcal{H}^p_{w_1}(\Phi_1) \to \mathcal{H}^\infty_{w_2}(\Phi_2)} \asymp \|K_\Phi\|_{\mathcal{H}^{p,\infty}_{w_1 \otimes w_2}(\Phi_1 \otimes \Phi_2)}.
\end{aligned}
$$

It remains to show that $K_\Psi = K_\Phi$. This, however, follows from the uniqueness of the kernel in Theorem 4.1 and therefore $\mathcal{H}^{\infty,\infty}_{w_1 \otimes w_2}(\Psi_1 \otimes \Psi_2) = \mathcal{H}^{\infty,\infty}_{w_1 \otimes w_2}(\Phi_1 \otimes \Phi_2)$ as long as the cross-Gram matrices of Ψ_1 and Φ_1, on the one hand, and Ψ_2 and Φ_2, on the other hand, are contained in $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively. $\square$

Finally, we give a sufficient condition on the kernel to ensure that the corresponding operator is contained in the Schatten-p class.

Proposition 4.6. *Let $\mathcal{S}_p(\mathcal{H}_1, \mathcal{H}_2)$ be the Schatten-p class of operators mapping $\mathcal{H}_1$ to $\mathcal{H}_2$, $1 \leqslant p \leqslant 2$, Ψ_n be $\mathcal{A}_n$-localised frames for $\mathcal{H}_n$ for $n = 1, 2$, and $O : \mathcal{H}_1 \to \mathcal{H}_2$ a bounded operator with a kernel K. If $K \in \mathfrak{H}^{2,p}(\Psi_1 \otimes \Psi_2)$, then $O \in \mathcal{S}_p(\mathcal{H}_1, \mathcal{H}_2)$. In particular, if $K \in \mathcal{H}^{p,p}(\Psi_1 \otimes \Psi_2)$, then $O \in \mathcal{S}_p(\mathcal{H}_1, \mathcal{H}_2)$.*

Proof. From [13, Theorem 17] we know that, if $\left\{\|Of_i\|_{\mathcal{H}_2}\right\}_{i \in I} \in \ell^p(I)$ for some frame $\{f_i\}_{i \in I} \subset \mathcal{H}_1$, then $O \in \mathcal{S}_p(\mathcal{H}_1, \mathcal{H}_2)$. Using Ψ_1 as such a frame results in

$$
\begin{aligned}
\sum_{i \in I} \|O\widetilde{\psi}_{1,i}\|^p_{\mathcal{H}_2} &\asymp \sum_{i \in I} \Big(\sum_{j \in J} \big|\langle O\widetilde{\psi}_{1,i}, \widetilde{\psi}_{2,j}\rangle_{\mathcal{H}_2}\big|^2 \Big)^{p/2} \\
&= \sum_{i \in I} \Big(\sum_{j \in J} \big|\langle K, \widetilde{\psi}_{1,i} \otimes \widetilde{\psi}_{2,j}\rangle_{\mathcal{H}_1 \otimes \mathcal{H}_2}\big|^2 \Big)^{p/2} = \|K\|_{\mathfrak{H}^{2,1}(\Psi_1 \otimes \Psi_2)}.
\end{aligned}
$$

The second statement follows immediately once we observe that $\|K\|_{\mathfrak{H}^{2,1}(\Psi_1 \otimes \Psi_2)} \leqslant \|K\|_{\mathcal{H}^{p,p}(\Psi_1 \otimes \Psi_2)}$. $\square$

5. Operators on shift-invariant spaces

We would like to conclude this paper with an example that can be dealt with using our kernel theorems, but to which kernel theorems for co-orbit spaces associated with integrable group representations [6] are not applicable. Let $T_x f(t) = f(t - x)$ denote the translation operator on $\mathbb{R}^d$. For a generator $\psi \in L^2(\mathbb{R}^d)$ we define the *shift-invariant space*

$$
V^2(\psi) = \Big\{ f \in L^2(\mathbb{R}^d) : \ f = \sum_{k \in \mathbb{Z}^d} c_k T_k \psi, \ c \in \ell^2(\mathbb{Z}^d) \Big\}.
$$

We also assume that

(i) ψ is continuous and $|\psi(x)| \lesssim (1 + |x|)^{-s}$ for $s > d$; and
(ii) $\Psi := \{T_k \psi\}_{k \in \mathbb{Z}^d}$ forms a Riesz basis for $V^2(\psi)$.

We refer to [2] for a thorough introduction to shift-invariant spaces, which includes an extensive list of references. The properties of ψ guarantee that $V^2(\psi)$ forms a reproducing kernel Hilbert space with kernel k_x^ψ. In other words

$$f(x) = \langle f, k_x^\psi \rangle, \quad f \in V^2(\psi). \tag{5.1}$$

Let $\mathcal{X} \subset \mathbb{R}^d$ be a discrete subset such that $\{k_x^\psi\}_{x\in\mathcal{X}}$ is a frame for $V^2(\psi)$. Frames of reproducing kernels are equivalently defined as *stable sets of sampling*, i.e. the inequalities

$$A\|f\|^2 \leqslant \sum_{x\in\mathcal{X}} |f(x)|^2 \leqslant B\|f\|^2, \quad f \in V^2(\psi),$$

hold for some $A, B > 0$. The problem of how densely one needs to sample has been well studied for a broad range of generators [1,2]. It was shown in [33] that $|\langle k_x^\psi, k_y^\psi\rangle| = |k_x^\psi(y)| \lesssim (1 + |x-y|)^{-s}$, i.e. the Gramian $G_{\mathcal{K}^\psi}$ is contained in the Jaffard class

$$\mathcal{J}_s(\mathcal{X}) := \{G : \mathcal{X}\times\mathcal{X} \to \mathbb{C} : \sup_{x,y\in\mathcal{X}} |G_{x,y}|(1 + |x-y|)^s < \infty\}.$$

If $\mathcal{X}$ is relatively separate and $s > d$, then $\mathcal{J}_s(\mathcal{X})$ is a spectral matrix algebra.

Let ψ_1, ψ_2 be two generators satisfying (i)-(ii), $\mathcal{X}, \mathcal{Y}$ be two stable sets of sampling on the respective shift-invariant spaces and $\mathcal{K}_1 := \{k_x^{\psi_1}\}_{x\in\mathcal{X}}$, $\mathcal{K}_2 := \{k_y^{\psi_2}\}_{y\in\mathcal{Y}}$ the corresponding frames of reproducing kernels. We consider the co-orbit spaces $\mathcal{H}^p(\mathcal{K}_i)$ and apply our results to study the boundedness of operators $O : \mathcal{H}^p(\mathcal{K}_1) \to \mathcal{H}^q(\mathcal{K}_2)$, $1 \leqslant p, q \leqslant \infty$. For example, Theorem 4.1 tells us that $O : \mathcal{H}^1(\mathcal{K}_1) \to \mathcal{H}^\infty(\mathcal{K}_2)$ is bounded if and only if its kernel is contained in $\mathcal{H}^{\infty,\infty}(\mathcal{K}_1\otimes\mathcal{K}_2)$. At this point one might ask oneself as to what one gains from this formulation. It turns out, though, that the tensor product spaces of shift-invariant spaces are nothing but shift-invariant spaces in higher dimensions. In particular, $V^2(\psi_1)\otimes V^2(\psi_2) = V^2(\psi_1\otimes\psi_2) \subset L^2(\mathbb{R}^{2d})$ and $k_x^{\psi_1} \otimes k_y^{\psi_2} = k_{(x,y)}^{\psi_1\otimes\psi_2}$. Moreover, $G_{\widetilde{\mathcal{K}_i},\Psi_i}, G_{\mathcal{K}_i,\widetilde{\Psi_i}} \in \mathcal{J}_s(\mathcal{X})$, $i = 1, 2$, by [33, Theorem 10 and Lemma 14] which, according to Proposition 3.2 (ii), implies that

$$\mathcal{H}^{p,p}(\mathcal{K}_1 \otimes \mathcal{K}_2) = \mathcal{H}^{p,p}(\Psi_1 \otimes \Psi_2).$$

Since Ψ_i are Riesz bases, the co-orbit and orbit spaces coincide. In other words $\mathcal{H}^{p,p}(\Psi_1\otimes\Psi_2) = V^p(\psi_1\otimes\psi_2)$, where

$$V^p(\psi_1 \otimes \psi_2) := \Big\{f \in L^p(\mathbb{R}^{2d}) : \ f = \sum_{k\in\mathbb{Z}^{2d}} c_k T_k(\psi_1 \otimes \psi_2), \ c \in \ell^p(\mathbb{Z}^{2d})\Big\}.$$

This implies in particular that

$$\mathcal{H}^{\infty,\infty}(\mathcal{K}_1 \otimes \mathcal{K}_2) = \mathcal{H}^{\infty,\infty}\big(\widetilde{\mathcal{K}_1} \otimes \widetilde{\mathcal{K}_2}\big) = V^\infty(\psi_1 \otimes \psi_2).$$

Finally, since the reproducing property (5.1) extends to $V^\infty(\psi_1\otimes\psi_2)$ (due to the decay condition on ψ_1, ψ_2) we deduce that

$$\|K\|_{\mathcal{H}^{\infty,\infty}(\mathcal{K}_1\otimes\mathcal{K}_2)} \asymp \|C_{\mathcal{K}_1\otimes\mathcal{K}_2} K\|_{\ell^\infty(\mathcal{X}\times\mathcal{Y})} = \sup_{(x,y)\in\mathcal{X}\times\mathcal{Y}} |K(x,y)|.$$

In other words, boundedness of the operator $O : \mathcal{H}^1(\mathcal{K}_1) \to \mathcal{H}^\infty(\mathcal{K}_2)$ boils down to pointwise boundedness of the kernel K.

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7.2 Publication 2

LOCALISED FRAMES FOR TENSOR PRODUCT SPACES

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ABSTRACT. In this paper, we investigate whether the tensor product of two frames, each individually localised with respect to a spectral matrix algebra, is also localised with respect to a suitably chosen tensor product algebra. We provide a partial answer by constructing an involutive Banach algebra of rank-four tensors that is built from two solid spectral matrix algebras. We show that this algebra is inverse-closed, given that the original algebras satisfy a specific property related to operator-valued versions of these algebras. This condition is satisfied by all commonly used solid spectral matrix algebras. We then prove that the tensor product of two self-localised frames remains self-localised with respect to our newly constructed tensor algebra. Additionally, we discuss generalisations to localised frames of Hilbert-Schmidt operators, which may not necessarily consist of rank-one operators.

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1. INTRODUCTION

The concept of localisation of frames was introduced by Gröchenig [19] to measure, in a broad sense, the quality of a frame. A frame $\Psi = \{\psi_i\}_{i\in I}$ is said to be self-localised [4, 16] if its Gram matrix $G_\Psi := (\langle\psi_{i'}, \psi_i\rangle_H)_{(i',i)\in I^2}$ belongs to an inverse-closed matrix algebra. This definition implies, among other things, that the canonical dual of a self-localised frame is also self-localised and allows to introduce a whole range of Banach spaces, the so-called co-orbit spaces, which are defined by decay conditions on the frame coefficients of its elements. It is common knowledge that the set of elementary tensor products of the elements of two frames constitutes a frame [2] – often called the tensor product frame – for the tensor product of the Hilbert spaces. The question that has remained open is whether the tensor product of two self-localised frames is, in some way, also self-localised, or, in other words, whether there is a natural way to define an inverse-closed Banach algebra of tensors of rank four that contains the Gram tensor (Definition 3.1) of a tensor product frame. Such a construction would allow for a direct application of the tools of co-orbit theory to tensor product frames [9].

In this paper, we provide an affirmative answer to this question, under what we believe to be quite reasonable additional conditions. To achieve this, we begin with two solid spectral matrix algebras and construct an algebra of rank-four tensors. We then equip this algebra with a norm derived from the norms of operator-valued variants of the original matrix algebras. We demonstrate that the resulting algebra of tensors of rank four is an inverse-closed, albeit non-solid, involutive Banach algebra provided that the operator-valued versions of the matrix algebras used in its construction are inverse-closed. Here we note that solidity of the algebra is not necessary for the intrinsically localised frames to have most of their desirable properties (see Section 4 for a discussion of that matter). It has recently been shown that the most prominent classes of inverse-closed matrix algebras indeed meet this condition [23], which makes our results widely applicable. Finally, we show that the Gram tensor of the tensor product of two self-localised frames belongs to the algebra that we constructed.

Our approach enabled us to overcome the obstacles that have prevented previous attempts to apply the methods developed for proving the inverse-closedness of the Jaffard class [22], Schur-type algebras [21, 31] and the Sjöstrand algebra [30] to tensor product algebras. More specifically, those techniques usually cannot be applied to tensor products of algebras from different classes or to algebras using anistropic weights. To illustrate this, let us, e.g., consider two frames $\Psi_1 = \{\psi_{1,k}\}_{k\in\mathbb{Z}}$ and $\Psi_2 = \{\psi_{2,k}\}_{k\in\mathbb{Z}}$ whose Gram matrices belong to the Jaffard class for two distinct weights, i.e.,

$$\sup_{(k,l)\in\mathbb{Z}^2} \big|\langle\psi_{1,k},\psi_{1,l}\rangle\big|\nu_{s_1}(k-l) < \infty, \quad \text{and} \quad \sup_{(m,n)\in\mathbb{Z}^2} \big|\langle\psi_{2,m},\psi_{2,n}\rangle\big|\nu_{s_2}(m-n) < \infty,$$

where $\nu_s(z) = (1+|z|)^s$, and s_1, s_2 are both greater than one and differ from one another. The Gram tensor of rank four of the tensor product frame $\Psi_1 \otimes \Psi_2$ will then belong to the so-called anisotropic Jaffard class, i.e.,

$$\sup_{(k,l,m,n)\in\mathbb{Z}^4} \big|\langle\psi_{1,k}\otimes\psi_{2,m},\psi_{1,l}\otimes\psi_{2,n}\rangle\big|\nu_{s_1}(k-l)\nu_{s_2}(m-n) < \infty.$$

It has remained unclear whether this anisotropic Jaffard class is inverse-closed or not. One could try to circumvent this obstacle and consider the higher dimensional Jaffard class with isotropic weight $\nu_s\big((k,n)-(l,m)\big)$. This class is inverse-closed given that $s > 2$, which assumes a degree of localisation that the frames Ψ_1 and Ψ_2 might not satisfy though. In contrast to this example, our approach allows us to deal with tensor products of two algebras of the same class with distinct weights and, of two algebras from distinct classes.

In order to make this article self-contained, we recall the definitions and established facts we drew on in this paper in Section 2 before stating and proving our new contributions in Section 3.

2. Background information and notation

2.1. **Self-localised frames for Hilbert spaces and their associated co-orbit spaces.** The concept of a *frame* generalises that of a basis of a vector space. One of the major advantages of frames over bases is that a frame can often be designed in a way that allows for a sparser decompositions of the elements of a given vector space than a basis would do.

Definition 2.1. *A countable set $\Psi := \{\psi_i\}_{i\in I}$ of elements of a separable Hilbert space H is called a* frame *if there exist positive numbers A_Ψ and B_Ψ such that*

$$A_\Psi \|f\|_H^2 \leqslant \sum_{i\in I} |\langle f,\psi_i\rangle_H|^2 \leqslant B_\Psi \|f\|_H^2 \tag{2.1}$$

for any element $f \in H$.

The inequalities in (2.1), together with the nature of Hilbert spaces, have the following implications. Firstly, the second inequality in (2.1) implies that the *analysis operator* C_Ψ

$$C_\Psi : H \to \ell^2(I), \quad f \mapsto \{\langle f, \psi_i\rangle_H\}_{i\in I},$$

is bounded. Secondly, the *synthesis operator* D_Ψ

$$D_\Psi : \ell^2(I) \to H, \quad \{c_i\}_{i\in I} \mapsto \sum_{i\in I} c_i\psi_i,$$

is bounded too. The first inequality in (2.1) implies that there exists at least one other frame $\Psi^{\mathbf{d}} := \{\psi_i^{\mathrm{d}}\}_{i\in I}$ for H, called a *dual frame*, that satisfies

$$f = \sum_{i\in I}\langle f, \psi_i\rangle_H\, \psi_i^{\mathrm{d}} = D_{\Psi^{\mathbf{d}}} C_\Psi f = \sum_{i\in I}\langle f, \psi_i^{\mathrm{d}}\rangle_H\, \psi_i = D_\Psi C_{\Psi^{\mathbf{d}}} f, \quad f \in H. \tag{2.2}$$

Fourthly, the *frame operator* S_Ψ

$$S_\Psi : H \to H, \quad f \mapsto \sum_{i\in I}\langle f, \psi_i\rangle_H\, \psi_i$$

is bounded, self-adjoint and invertible for any frame Ψ [3] and the sequence $\widetilde{\Psi} := \{S_\Psi^{-1}\psi_i\}_{i\in I}$, known as the *canonical dual frame* [10], is a dual frame for Ψ. Finally, the map G_Ψ, called the *Gram operator*,

$$G_\Psi : \ell^2(I) \to \ell^2(I), \quad \{c_i\}_{i\in I} \mapsto \sum_{i\in I}\langle \psi_{i'}, \psi_i\rangle_H\, c_i = \sum_{i\in I}(G_\Psi)_{i',\,i}\, c_i = C_\Psi D_\Psi c,$$

where

$$G_\Psi := ((G_\Psi)_{i',\,i})_{(i',\,i)\in I^2} := (\langle \psi_{i'}, \psi_i\rangle_H)_{(i',\,i)\in I^2}$$

is the so-called *Gram matrix*, is a bounded operator.

There is also the notion of a frame for Banach spaces [17].

Definition 2.2. *Let X be a separable Banach space and X_d a Banach sequence space. A countable set $\Psi := \{\psi_i\}_{i\in I}$ of elements of the topological dual X^* of the space X is called an X_d-*frame *for the space X if there are positive numbers A_Ψ and B_Ψ such that*

$$A_\Psi \|f\|_X \leqslant \left\|\{\langle f, \psi_i\rangle_{X,\,X^*}\}_{i\in I}\right\|_{X_d} \leqslant B_\Psi \|f\|_X$$

for any element $f \in X$.

Despite an apparent similarity of Definitions 2.1 and 2.2, there is a very important difference between frames for Hilbert spaces and those for Banach spaces: an X_d-frame for a Banach space in itself does not necessarily allow to express any element of the space in a way similar to that described by (2.2). Therefore, additional assumptions to ensure such a reconstruction are needed. For some Banach function spaces this problem can be solved by using so-called *structured frames* that satisfy only a few specific and easily verifiable criteria [5–8,26,27,32]. Another class of Banach spaces for which this problem can be solved are the so-called co-orbit spaces [12–14], which are the Banach spaces of choice in numerous areas of harmonic analysis. Classically, co-orbit spaces were studied for families generated by integrable group representations. Here, we shall follow the approach introduced by Fornasier and Gröchenig [16] and consider co-orbit spaces that are generated by *self-localised frames*. In essence, self-localisation of a frame Ψ amounts to the Gram matrix G_Ψ belonging to a spectral matrix algebra.

Definition 2.3. *An involutive Banach algebra $\mathcal{A}$ of infinite matrices with norm $\|\cdot\|_{\mathcal{A}}$ is called a* spectral matrix algebra *if*

(i) every $A \in \mathcal{A}$ defines a bounded operator on $\ell^2(I)$, i.e., $\mathcal{A} \subset \mathcal{B}(\ell^2(I))$;
(ii) $\mathcal{A}$ is inverse-closed, that is if $A \in \mathcal{A}$ is invertible in $\mathcal{B}(\ell^2(I))$ then $A^{-1} \in \mathcal{A}$ as well;

if, moreover,

(iii) $\mathcal{A}$ is solid, i.e., $A \in \mathcal{A}$, together with $|B_{i,j}| \leqslant |A_{i,j}|$ for any $(i, j) \in I^2$, implies that $B := (B_{i,j})_{(i,j)\in I^2} \in \mathcal{A}$ and that $\|B\|_{\mathcal{A}} \leqslant \|A\|_{\mathcal{A}}$,

then $\mathcal{A}$ is called a solid spectral matrix algebra*.*

Examples of solid spectral matrix algebras include, among others, the Jaffard class [22], Schur-type algebras [21] and the Sjöstrand algebra [30].

Definition 2.4. *Let I be a countable index set. The set $\Psi = \{\psi_i\}_{i\in I}$ of elements of H is said to be $\mathcal{A}$-*self-localised*, or simply* self-localised*, if its Gram matrix $G_\Psi := (\langle \psi_i, \psi_{i'}\rangle_H)_{(i,i')\in I^2}$ belongs to a spectral matrix algebra $\mathcal{A}$.*

An $\mathcal{A}$-self-localised frame Ψ for a Hilbert space H allows to generate a whole range of Banach spaces known as *co-orbit spaces* [12–14], which we shall introduce after the following auxiliary definition.

Definition 2.5. *Let $\mathcal{A} \subset \mathcal{B}(\ell^2(I))$ be a spectral algebra of infinite matrices. A sequence of positive real numbers $w := \{w_i\}_{i\in I} \subset \mathbb{R}_{>0}$ is called an $\mathcal{A}$-*admissible *weight if every matrix $A \in \mathcal{A}$ defines a bounded operator on the sequence space $\ell^p_w(I)$ for any $p \in [1, \infty]$.*

Definition 2.6. *Let $\mathcal{A}$ be a spectral matrix algebra, $\Psi := \{\psi_i\}_{i\in I}$ an $\mathcal{A}$-localised frame for a Hilbert space H, $\widetilde{\Psi} := \{\widetilde{\psi}_i\}_{i\in I}$ its canonical dual frame, $w := \{w_i\}_{i\in I}$ an $\mathcal{A}$-admissible weight and*

$$H_{00}(\Psi) = \left\{ \sum_{i\in I} c_i\, \psi_i : \ \{c_i\}_{i\in I} \in c_{00} \right\}, \tag{2.3}$$

where c_{00} stands for the space of sequences of complex numbers with only finitely many non-zero terms. The co-orbit space $H^p_w(\Psi)$, $1 \le p < \infty$ is the completion of $H_{00}(\Psi)$ with respect to the norm $\|f\|_{H^p_w(\Psi)} := \left\|C_{\widetilde{\Psi}} f\right\|_{\ell^p_w}$.

A proper definition of the co-orbit space $H^\infty_w(\Psi)$ is technically more involved. As our analysis focuses primarily on the properties of the localisation algebras rather than the co-orbit spaces, we will omit the details here and refer the interested reader to [4, 20] for an in depth discussion.

One of the main advantage of co-orbit space theory is that the frame for the Hilbert space that was used to generate the range of co-orbit spaces is also a frame for any of the co-orbit spaces and it allows to decompose and re-synthesise any of its elements [16].

2.2. **Tensor products of Hilbert spaces and Hilbert-Schmidt operators.** Let H_1 and H_2 be two Hilbert spaces, $f_1 \in H_1$ and $f_2 \in H_2$. The *elementary tensor* $f_1 \otimes f_2$ of f_1 and f_2 will be understood as the rank-one operator mapping H_1 to H_2 according to the expression

$$(f_1 \otimes f_2)(f) := \langle f, f_1\rangle_{H_1} f_2, \quad f \in H_1.$$

It should be pointed out that the elementary tensors are homogeneous in the following sense

$$\alpha(f_1 \otimes f_2) = (\overline{\alpha} f_1) \otimes f_2 = f_1 \otimes (\alpha f_2).$$

The tensor product $H_1 \otimes H_2$ is defined as the completion of the linear span of all elementary tensors with respect to the metric induced by the scalar product

$$\langle f_1 \otimes f_2, g_1 \otimes g_2\rangle_{H_1\otimes H_2} = \overline{\langle f_1, g_1\rangle}_{H_1} \langle f_2, g_2\rangle_{H_2}.$$

We note that the tensor product $H_1 \otimes H_2$ can be identified with the Banach algebra of *Hilbert-Schmidt operators* $HS(H_1, H_2)$, which is defined as the space of bounded linear operators mapping H_1 to H_2 equipped with the norm induced by the scalar product

$$\langle O, O'\rangle_{HS(H_1,H_2)} := \sum_{i\in\mathbb{N}} \langle Oe_i, O'e_i\rangle_{H_2},$$

where $\{e_i\}_{i\in\mathbb{N}}$ stands for an orthonormal basis of H_1.

3. Tensor products of spectral matrix algebras

The set of elementary tensor products $\Psi_1 \otimes \Psi_2 := \{\psi_{1,i} \otimes \psi_{2,j}\}_{(i,j)\in I_1\times I_2}$ of the elements of two frames $\Psi_1 := \{\psi_{1,i}\}_{i\in I_1} \subset H_1$ and $\Psi_2 := \{\psi_{2,j}\}_{j\in I_2} \subset H_2$ is known to constitute a frame for the tensor product $H_1 \otimes H_2$. Its canonical dual is given by $\widetilde{\Psi_1 \otimes \Psi_2} := \widetilde{\Psi_1} \otimes \widetilde{\Psi_2} = \{\widetilde{\psi}_{1,i} \otimes \widetilde{\psi}_{2,j}\}_{(i,j)\in I_1\times I_2}$, see [2]. In what follows we shall refer to $\Psi_1 \otimes \Psi_2$ as a *tensor product frame*. Let us now assume that Ψ_1 and Ψ_2 are self-localised according to Definition 2.4, in other words, let us assume that $G_{\Psi_1} := (\langle\psi_{1,i'}, \psi_{1,i}\rangle)_{(i,i')\in I^2} \in \mathcal{A}_1$ and $G_{\Psi_2} := (\langle\psi_{2,j'}, \psi_{2,j}\rangle)_{(j,j')\in I^2} \in \mathcal{A}_2$ where $\mathcal{A}_1$ and $\mathcal{A}_2$ are two spectral matrix algebras and show that $\Psi_1 \otimes \Psi_2$ is, in a way, self-localised too. To do so, we shall define a Gram tensor of rank four $G_{\Psi_1\otimes\Psi_2}$ and an involutive Banach algebra of tensors of rank four and prove that $G_{\Psi_1\otimes\Psi_2}$ belongs to this algebra. We first give our definition of the Gram tensor $G_{\Psi_1\otimes\Psi_2}$, which is reminiscent of the ordinary Gram matrix.

Definition 3.1. *Let* $\{\Omega_{i,k}\}_{(i,k)\in I_1\times I_2} \subset H_1 \otimes H_2$. *The* Gram tensor of rank four G_Ω *is defined by*

$$G_\Omega = \left(\langle\Omega_{i,k}, \Omega_{j,l}\rangle_{H_1\otimes H_2}\right)_{(i,k,l,j)\in I_1\times I_2^2\times I_1}. \tag{3.1}$$

If $\{\Omega_{i,k}\}_{(i,k)\in I_1\times I_2}$ consists of only elementary tensor products of two families $\Psi_1 \subset H_1$ and $\Psi_2 \subset H_2$, the Gram tensor reduces to the Kronecker product of the Gram matrices

$$\begin{aligned} G_{\Psi_1\otimes\Psi_2} &= \left(\langle\psi_{1,j}, \psi_{1,i}\rangle_{H_1} \langle\psi_{2,l}, \psi_{2,k}\rangle_{H_2}\right)_{(i,k,l,j)\in I_1\times I_2^2\times I_1} \\ &= (G_{\Psi_1})_{(i,j)\in I_1^2} (G_{\Psi_2})_{(k,l)\in I_2^2}. \end{aligned} \tag{3.2}$$

We use $\mathcal{T}$ to denote the set of rank-four tensors whose elements are indexed by $I_1 \times I_2^2 \times I_1$ [15]. Clearly $(\mathcal{T}, \cdot, +)$ – i.e., the set $\mathcal{T}$, together with pointwise addition and scalar multiplication – constitutes a vector space over the field $\mathbb{C}$. To introduce an involutive Banach algebra contained in $(\mathcal{T}, \cdot, +)$, we need to introduce a norm, a multiplication and an involution, for such tensors.

Definition 3.2. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be involutive Banach algebras of tensors of rank two with norms $\|\cdot\|_{\mathcal{A}_1}$ and $\|\cdot\|_{\mathcal{A}_2}$ whose elements are indexed by I_1^2 and I_2^2 respectively. For any tensor of rank four $A \in \mathcal{T}$ we define*

$$\|A\|_{\widetilde{\mathcal{A}}_1} := \left\| \left(\left\| A\big|_{\{i\}\times I_2^2\times\{j\}} \right\|_{\mathcal{B}(\ell^2(I_2))} \right)_{(i,j)\in I_1^2} \right\|_{\mathcal{A}_1}, \tag{3.3}$$

where $A\big|_{\{i\}\times I_2^2\times\{j\}}$ stands for the restriction of A, viewed as a map $I_1\times I_2^2\times I_1 \to \mathbb{C}$, to the subset $\{i\}\times I_2^2\times\{j\}$, and

$$\|A\|_{\widetilde{\mathcal{A}}_2} := \left\| \left(\left\| A\big|_{I_1\times\{(k,l)\}\times I_1} \right\|_{\mathcal{B}(\ell^2(I_1))} \right)_{(k,l)\in I_2^2} \right\|_{\mathcal{A}_2}, \tag{3.4}$$

where $A\big|_{I_1\times\{(k,l)\}\times I_1}$ stands for the restriction of A to the subset $I_1\times\{(k,l)\}\times I_1$. Furthermore, we set

$$\widetilde{\mathcal{A}}_1 := \left\{A\in\mathcal{T}:\ \|A\|_{\widetilde{\mathcal{A}}_1} < \infty\right\}, \quad \widetilde{\mathcal{A}}_2 := \left\{A\in\mathcal{T}:\ \|A\|_{\widetilde{\mathcal{A}}_2} < \infty\right\},$$

and $\mathcal{A} := \mathcal{A}_1 \cap \mathcal{A}_2$, where $\mathcal{A}$ is equipped with the norm

$$\|A\|_{\mathcal{A}} := \max\left\{ \|A\|_{\widetilde{\mathcal{A}}_1}, \|A\|_{\widetilde{\mathcal{A}}_2} \right\}. \tag{3.5}$$

Our, perhaps somewhat unusual, way of indexing the elements of tensors of rank four is, we believe, fully justified by the fact that the *double contracted tensor multiplication* [29], which we shall formally define in a moment, functions very much like an ordinary matrix multiplication.

Definition 3.3. *For any two tensors of rank four $A := (A_{i,k,l,j})_{(i,k,l,j)\in I_1\times I_2^2\times I_1}$ and $B := (B_{i,k,l,j})_{(i,k,l,j)\in I_1\times I_2^2\times I_1}$, their* doubly contracted tensor product $A : B$ *will be understood as the tensor of rank four defined by*

$$\begin{aligned}\Big((A:B)_{i,k,l,j}\Big)&_{(i,k,l,j)\in I_1\times I_2^2\times I_1} \\ &:= \left(\sum_{n\in I_2}\sum_{m\in I_1} A_{i,k,n,m}\, B_{m,n,l,j}\right)_{(i,k,l,j)\in I_1\times I_2^2\times I_1}.\end{aligned} \tag{3.6}$$

Subsequently, we shall prove that this product is compatible with the norm that defines $\mathcal{A}$.

Proposition 3.4. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be solid Banach algebras. Then*

$$(\mathcal{A}, \cdot, +, :, \|\cdot\|_{\mathcal{A}}) \tag{3.7}$$

is a Banach algebra.

Proof. First of all, we are going to show that the mapping $\|\cdot\|_{\mathcal{A}}$ is a norm. Indeed, the function $\|\cdot\|_{\widetilde{\mathcal{A}}_1}$ is a norm as it is the composition of the operator norm $\|\cdot\|_{\mathcal{B}(\ell^2(I_2))}$ and the norm $\|\cdot\|_{\mathcal{A}_1}$. The map $\|\cdot\|_{\widetilde{\mathcal{A}}_2}$ is a norm for a similar reason. Finally, $\|\cdot\|_{\mathcal{A}}$ is a norm as it is a maximum of the norms $\|\cdot\|_{\widetilde{\mathcal{A}}_1}$ and $\|\cdot\|_{\widetilde{\mathcal{A}}_2}$.

It remains to prove that

$$\|A : B\|_{\mathcal{A}} \leqslant \|A\|_{\mathcal{A}}\|B\|_{\mathcal{A}} \tag{3.8}$$

for any A and B in $\mathcal{A}$. According to (3.6), for any given $(i,\, j) \in I_1^2$, the norm of the restriction $(A : B)\,\big|_{\{i\}\times I_2^2\times\{j\}}$ of $A : B$ can be expressed as follows

$$(A : B)\,\big|_{\{i\}\times I_2^2\times\{j\}} = \sum_{n\in I_2}\sum_{m\in I_1} A\big|_{\{i\}\times I_2\times\{n\}\times\{m\}} B\big|_{\{m\}\times\{n\}\times I_2\times\{j\}}.$$

Using the triangle inequality and the submultiplicativity of the operator norm $\|\cdot\|_{\mathcal{B}(\ell^2(I_2))}$ results in

$$\begin{aligned}
\left\|(A : B)\,\big|_{\{i\}\times I_2^2\times\{j\}}\right\|_{\mathcal{B}(\ell^2(I_2))} &= \left\|\sum_{n\in I_2}\sum_{m\in I_1} A\big|_{\{i\}\times I_2\times\{n\}\times\{m\}} B\big|_{\{m\}\times\{n\}\times I_2\times\{j\}}\right\|_{\mathcal{B}(\ell^2(I_2))} \\
&\leqslant \sum_{m\in I_1}\left\|\sum_{n\in I_2} A\big|_{\{i\}\times I_2\times\{n\}\times\{m\}} B\big|_{\{m\}\times\{n\}\times I_2\times\{j\}}\right\|_{\mathcal{B}(\ell^2(I_2))} \\
&\leqslant \sum_{m\in I_1}\left\|A\big|_{\{i\}\times I_2^2\times\{m\}}\right\|_{\mathcal{B}(\ell^2(I_2))}\left\|B\big|_{\{m\}\times I_2^2\times\{j\}}\right\|_{\mathcal{B}(\ell^2(I_2))}.
\end{aligned} \tag{3.9}$$

Now A and $B \in \widetilde{\mathcal{A}}_1$ as both A and $B \in \mathcal{A}$ and $\mathcal{A} = \widetilde{\mathcal{A}}_1 \cap \widetilde{\mathcal{A}}_2$. This, in its turn, implies that

$$\left(\left\|A\big|_{\{i\}\times I_2^2\times\{m\}}\right\|_{\mathcal{B}(\ell^2(I_2))}\right)_{(i,\,m)\in I_1^2} \in \mathcal{A}_1,$$

and

$$\left(\left\|B\big|_{\{m\}\times I_2^2\times\{j\}}\right\|_{\mathcal{B}(\ell^2(I_2))}\right)_{(m,\,j)\in I_1^2} \in \mathcal{A}_1\,.$$

Moreover, combining (3.9) and the fact that $\mathcal{A}_1$ is a solid Banach algebra results in

$$\begin{aligned}
\|A : B\|_{\widetilde{\mathcal{A}}_1} &= \left\|\left(\left\|(A : B)\,\big|_{\{i\}\times I_2^2\times\{j\}}\right\|_{\mathcal{B}(\ell^2(I_2))}\right)_{(i,\,j)\in I_1^2}\right\|_{\mathcal{A}_1} \\
&\leqslant \left\|\left(\sum_{m\in I_1}\left\|A\big|_{\{i\}\times I_2^2\times\{m\}}\right\|_{\mathcal{B}(\ell^2(I_2))}\left\|B\big|_{\{m\}\times I_2^2\times\{j\}}\right\|_{\mathcal{B}(\ell^2(I_2))}\right)_{(i,\,j)\in I_1^2}\right\|_{\mathcal{A}_1} \\
&\leqslant \left\|\left(\left\|A\big|_{\{i\}\times I_2^2\times\{m\}}\right\|_{\mathcal{B}(\ell^2(I_2))}\right)_{(i,m)\in I_1^2}\right\|_{\mathcal{A}_1}\left\|\left(\left\|B\big|_{\{n\}\times I_2^2\times\{j\}}\right\|_{\mathcal{B}(\ell^2(I_2))}\right)_{(m,j)\in I_1^2}\right\|_{\mathcal{A}_1} \\
&= \|A\|_{\widetilde{\mathcal{A}}_1}\,\|B\|_{\widetilde{\mathcal{A}}_1}\,.
\end{aligned} \tag{3.10}$$

Using similar arguments we infer that

$$\|A : B\|_{\widetilde{\mathcal{A}}_2} \leqslant \|A\|_{\widetilde{\mathcal{A}}_2}\,\|B\|_{\widetilde{\mathcal{A}}_2}\,. \tag{3.11}$$

Combining (3.10) and (3.11) implies (3.8). Indeed

$$\begin{aligned}\|A:B\|_{\mathcal{A}} &\leqslant \max\{\|A:B\|_{\widetilde{\mathcal{A}}_1}, \|A:B\|_{\widetilde{\mathcal{A}}_2}\}\\ &\leqslant \max\{\|A\|_{\widetilde{\mathcal{A}}_1}\|B\|_{\widetilde{\mathcal{A}}_1}, \|A\|_{\widetilde{\mathcal{A}}_2}\|B\|_{\widetilde{\mathcal{A}}_2}\}\\ &\leqslant \max\{\|A\|_{\widetilde{\mathcal{A}}_1}, \|A\|_{\widetilde{\mathcal{A}}_2}\}\max\{\|B\|_{\widetilde{\mathcal{A}}_1}, \|B\|_{\widetilde{\mathcal{A}}_2}\}\\ &= \|A\|_{\mathcal{A}}\|B\|_{\mathcal{A}}.\end{aligned}$$

□

Next, we define the adjoint of a tensor of rank four.

Definition 3.5. *The adjoint A^* of a tensor of rank four $A \in \mathcal{T}$ will be understood as the tensor of rank four defined by*

$$\Big((A^*)_{i,k,l,j}\Big)_{(i,k,l,j)\in I_1\times I_2^2\times I_1} := \big(\overline{A_{j,l,k,i}}\big)_{(i,k,l,j)\in I_1\times I_2^2\times I_1}, \tag{3.12}$$

where the bar over $A_{j,l,k,i}$ stands for its complex conjugation.

This definition of the adjoint is again similar to that of a matrix, where the order of the indices is reversed and the elements of the matrix are complex conjugated.

Proposition 3.6. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be solid Banach algebras. Then*

$$(\mathcal{A},\,\cdot\,,\,+\,,\,:\,,\,*\,,\,\|\cdot\|_{\mathcal{A}}) \tag{3.13}$$

is an involutive Banach algebra.

Proof. Clearly, the adjoint of $A \in \mathcal{A}$ satisfies the following properties $(A^*)^* = A$, $(\alpha A)^* = \overline{\alpha}A^*$ and $(A+B)^* = A^* + B^*$ and therefore qualifies as an involution. Moreover, we observe that, formally,

$$\begin{aligned}(A:B)^*_{i,k,l,j} &= \sum_{m\in I_1}\sum_{n\in I_2}\overline{A_{j,l,n,m}}\,\overline{B_{m,n,k,i}} = \sum_{m\in I_1}\sum_{n\in I_2}\overline{B_{m,n,k,i}}\,\overline{A_{j,l,n,m}}\\ &= \sum_{m\in I_1}\sum_{n\in I_2}(B^*)_{i,k,n,m}\,(A^*)_{m,n,l,j} = (B^*:A^*)_{i,k,l,j}\end{aligned} \tag{3.14}$$

holds for any $(i,\,k,\,l,\,j) \in I_1 \times I_2^2 \times I_1$. The product on the right hand side is well-defined whenever $A^*, B^* \in \mathcal{A}$. To prove that $\mathcal{A}$ is an involutive Banach algebra it thus remains to show that

$$\|A^*\|_{\mathcal{A}} = \|A\|_{\mathcal{A}}$$

for every $A \in \mathcal{A}$. We note that the restriction of A^* can be understood as the adjoint of a matrix

$$(A^*)\,\big|_{\{i\}\times I_2^2\times\{j\}} = \Big(A\big|_{\{j\}\times I_2^2\times\{i\}}\Big)^*.$$

This shows that

$$\Big\|(A^*)\big|_{\{i\}\times I_2^2\times\{i\}}\Big\|_{\mathcal{B}(\ell^2(I_2))} = \Big\|A\big|_{\{j\}\times I_2^2\times\{i\}}\Big\|_{\mathcal{B}(\ell^2(I_2))},$$

and so

$$\begin{aligned}
\|A^*\|_{\widetilde{\mathcal{A}}_1} &= \left\| \left(\left\| A|_{\{j\}\times I_2^2\times\{i\}} \right\|_{\mathcal{B}(\ell^2(I_2))} \right)_{(i,j)\in I_1^2} \right\|_{\mathcal{A}_1} \\
&= \left\| \left(\left(\left\| A|_{\{i\}\times I_2^2\times\{j\}} \right\|_{\mathcal{B}(\ell^2(I_2))} \right)^* \right)_{(i,j)\in I_1^2} \right\|_{\mathcal{A}_1} \\
&= \left\| \left(\left\| A|_{\{i\}\times I_2^2\times\{j\}} \right\|_{\mathcal{B}(\ell^2(I_2))} \right)_{(i,j)\in I_1^2} \right\|_{\mathcal{A}_1} = \|A\|_{\widetilde{\mathcal{A}}_1}
\end{aligned}$$

for every $A \in \mathcal{A}$. Using similar arguments we conclude that

$$\|A^*\|_{\widetilde{\mathcal{A}}_2} = \|A\|_{\widetilde{\mathcal{A}}_2}, \quad A \in \mathcal{A},$$

which in turn implies that $\|A^*\|_{\mathcal{A}} = \|A\|_{\mathcal{A}}$, $A \in \mathcal{A}$. □

With all the preparatory work in place, our main result now follows almost immediately from previous propositions.

Theorem 3.7. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two solid spectral algebras such that $\widetilde{\mathcal{A}}_1$ and $\widetilde{\mathcal{A}}_2$ are inverse-closed. Then $\mathcal{A}$ is an involutive inverse-closed Banach algebra.*

Proof. The only thing that remains to be proved is that $\mathcal{A}$ is inverse-closed. Let $A \in \mathcal{A}$ be invertible in $\mathcal{B}(\ell^2(I_1 \times I_2))$. Then $A^{-1} \in \widetilde{\mathcal{A}}_1$ and $A^{-1} \in \widetilde{\mathcal{A}}_2$ as both $\widetilde{\mathcal{A}}_1$ and $\widetilde{\mathcal{A}}_2$ are inverse-closed. Therefore, $A^{-1} \in \mathcal{A}$. □

The assumption that the algebras $\widetilde{\mathcal{A}}_1$ and $\widetilde{\mathcal{A}}_2$ are inverse-closed seems quite sensible to us, as all the common inverse closed algebras satisfy this condition. This follows from the results of [23] where operator-valued matrix algebras were considered. For $I \subset \mathbb{R}^d$, $\vartheta_s(i) = (1 + |i|)^s$, and ϑ any weight function, the Jaffard algebra

$$\mathcal{J}_s(I) := \Big\{ A : \sup_{(i,j)\in I^2} |A_{i,j}| \vartheta_s(i-j) < \infty \Big\},$$

the weighted Schur-type algebras

$$\mathcal{S}^p_\delta(I) := \Big\{ A : \ \|A\|_{\mathcal{S}^p_\delta(I)} < \infty \Big\},$$

where

$$\|A\|_{\mathcal{S}^p_\delta(I)} := \max\left\{ \sup_{i\in I} \left(\sum_{j\in I} |A_{i,j}|^p \vartheta_\delta(i-j)^p \right)^{\frac{1}{p}}, \sup_{j\in I} \left(\sum_{i\in I} |A_{i,j}|^p \vartheta_\delta(i-j)^p \right)^{\frac{1}{p}} \right\},$$

and the Sjöstrand algebra

$$\mathcal{C}_\vartheta := \Big\{ A : \sum_{i\in\mathbb{Z}^d} \sup_{j\in\mathbb{Z}^d} |A_{j,j-i}| \vartheta(i) < \infty \Big\},$$

are a priori defined as complex-valued infinite matrices. In the following, we will consider operator-valued versions of the algebras defined above. For a solid matrix algebra $\mathcal{A}$ and a Hilbert space H, we define

$$\mathcal{OP}(\mathcal{A}) := \big\{ (A_{i,j})_{(i,j)\in I^2} : \ A_{i,j} \in \mathcal{B}(H) \text{ and } (\|A_{i,j}\|_{\mathcal{B}(H)})_{(i,j)\in I^2} \in \mathcal{A} \big\}.$$

If $\mathcal{A}$ is, e.g., the Jaffard class, then this definition reads

$$\mathcal{OP}\big(\mathcal{J}_s(I)\big) := \Big\{A_{i,j} \in \mathcal{B}(H) : \sup_{(i,j)\in I^2} \|A_{i,j}\|_{\mathcal{B}(H)} \vartheta_s(i-j) < \infty\Big\}.$$

It was shown in [23, Theorems 3.10, 4.8 and 6.3] that the operator-valued versions of the Jaffard class, the weighted Schur-type and the Sjöstrand algebra are indeed inverse-closed. We make use of these results to show the following.

Corollary 3.8. *Let $I_1, I_2 \subset \mathbb{R}^d$ be relatively separated discrete sets, and let $\mathcal{A}_n$ be chosen from $\{\mathcal{J}_s(I_n), \mathcal{S}^1_\delta(I_n), \mathcal{C}_\nu\}$, $n = 1, 2$, with $s > d, \delta \in (0, 1]$, and ν a sub-multiplicative and symmetric weight that satisfies $\lim_{n\to\infty} \nu(nz)^{1/n} = 1$, for every $z \in \mathbb{R}^d$. Then $\mathcal{A}$ is an inverse-closed involutive Banach algebra.*

Proof. Let $A \in \mathcal{A}$ be a rank-four tensor. For any given pair of indices $(i, i') \in I_1^2$, $A_{\{i\}\times I_2^2\times\{i'\}}$ defines a bounded operator on $\ell^2(I_2)$ and it is straightforward to show that $\mathcal{OP}(\mathcal{A}_1) = \widetilde{\mathcal{A}}_1$. Consequently, $\widetilde{\mathcal{A}}_1$ is inverse-closed according to the results of [23]. A similar argument holds for $\widetilde{\mathcal{A}}_2$. Applying Theorem 3.7 then concludes the proof. □

Remark 3.9. *It is not known to us whether the operator-valued analog of a spectral matrix algebra is necessarily inverse closed. An affirmative answer to this question would allow us to show that the tensor algebra $\mathcal{A}$ is inverse-closed as long as $\mathcal{A}_1$ and $\mathcal{A}_2$ are inverse-closed.*

4. Solidity in co-orbit theory

While solidity of a spectral matrix algebra $\mathcal{A}$ is one of the standard assumptions in co-orbit theory [4, 16], the tensor product algebra that we designed and use in this study is not solid in general. We are, however, convinced that solidity is not essential to prove the main statements of co-orbit theory. Instead, it is the other assumptions outlined in Definitions 2.3 and 2.5 that play the central roles. While a detailed discussion confirming this claim would go beyond the scope of this paper, we briefly discuss how the properties of the Gram matrix are used to derive the results of co-orbit theory. A recurring argument is to rewrite certain operations as the multiplication of the (cross-)Gram matrix by an $\ell^p_w(I)$-sequence and then use that elements of the spectral algebra define bounded operators on each $\ell^p_w(I)$, $p \in [1, \infty]$, as long as w is an $\mathcal{A}$-admissible weight (see Definition 2.5). This, for example, allows one to show boundedness of the synthesis operator or that the co-orbit spaces are independent of the particular self-localised frame that is used to define them. Inverse-closedness on the other hand is used to prove the fundamental result of co-orbit theory that the canonical dual of a self-localised frame is also self-localised.

To the best of our knowledge, there are only two occasions where solidity is indeed used in the literature on co-orbit theory, namely in [4, Theorem 3], where the argument could be readily adapted to work in the operator-valued setting that we consider in this paper, and in [16, Definition 4 & Lemma 2.2], where the somewhat different notion of ♯-self-localisation is introduced and used to describe the mutual localisation of two frames with indices in two distinct non-uniform index sets.

5. Localised frames for the tensor product $H_1 \otimes H_2$

The following result is an almost immediate consequence of Definition 3.2 being applied to the tensor product of two self-localised frames.

Proposition 5.1. *Let Ψ_1 be an $\mathcal{A}_1$-self-localised frame for H_1 and Ψ_2 an $\mathcal{A}_2$-self-localised frame for H_2 where $\mathcal{A}_1$ and $\mathcal{A}_2$ are two spectral matrix algebras such that $\widetilde{\mathcal{A}}_1$ and $\widetilde{\mathcal{A}}_2$ are inverse-closed. Then the tensor product frame $\Psi_1 \otimes \Psi_2$ is an $\mathcal{A}$-self-localised frame for $H_1 \otimes H_2$.*

Proof. According to (3.2) and (3.3),

$$\begin{aligned}\|G_{\Psi_1\otimes\Psi_2}\|_{\widetilde{\mathcal{A}}_1} &= \left\| \left(\left\| G_{\Psi_1\otimes\Psi_2}\big|_{\{i\}\times I_2^2\times\{j\}} \right\|_{\mathcal{B}(\ell^2(I_2))} \right)_{(i,j)\in I_1^2} \right\|_{\mathcal{A}_1} \\ &= \left\| \|G_{\Psi_2}\|_{\mathcal{B}(\ell^2(I_2))} \left((G_{\Psi_1})_{i,j} \right)_{(i,j)\in I_1^2} \right\|_{\mathcal{A}_1} \\ &= \|G_{\Psi_2}\|_{\mathcal{B}(\ell^2(I_2))} \|G_{\Psi_1}\|_{\mathcal{A}_1}\end{aligned}$$

and, similarly,

$$\|G_{\Psi_1\otimes\Psi_2}\|_{\widetilde{\mathcal{A}}_2} = \|G_{\Psi_1}\|_{\mathcal{B}(\ell^2(I_1))} \|G_{\Psi_2}\|_{\mathcal{A}_2}.$$

Therefore, both $\|G_{\Psi_1\otimes\Psi_2}\|_{\widetilde{\mathcal{A}}_1}$ and $\|G_{\Psi_1\otimes\Psi_2}\|_{\widetilde{\mathcal{A}}_2}$ are finite as $G_{\Psi_1} \in \mathcal{A}_1 \subset \mathcal{B}(\ell^2(I_1))$ and $G_{\Psi_2} \in \mathcal{A}_2 \subset \mathcal{B}(\ell^2(I_2))$. From this we infer that $\|G_{\Psi_1\otimes\Psi_2}\|_{\mathcal{A}} < \infty$ and so $G_{\Psi_1\otimes\Psi_2} \in \mathcal{A}$. $\square$

It should be pointed out here that self-localisation with respect to $\mathcal{A}$ need not be restricted to sets of rank-one operators. Frames of Hilbert-Schmidt operators have been recently studied, for example, in [24] in the context of quantum harmonic analysis [25]. In that study, the authors developed a theory of operator-valued modulation spaces by analysing operator families of the form $\{\pi(z_i)S\pi(w_k)^*\}_{(i,k)\in I^2}$, where $\pi(z)$ represents a time-frequency shift by $z \in \mathbb{R}^{2d}$ (see, e.g., [18]) and $S \in HS(L^2(\mathbb{R}^d), L^2(\mathbb{R}^d))$. They explored the localisation of operator-valued frames through the lens of classical co-orbit theory for integrable group representations. To derive their results they made extensive use of the connection between operator-valued frames and modulation spaces in double phase space. Our approach, on the other hand, is much more general and applicable to situations where no structure is available or where it is natural to combine different types of localisation.

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7.3 Publication 3

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On near orthogonality of the Banach frames for the wave packet spaces

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A B S T R A C T

In solving scientific, engineering or pure mathematical problems one is frequently faced with a need to approximate the function of a given class with a specified precision by the linear combination of a preferably small number of simpler functions. This can often achieved by choosing the simpler functions localised one way or another both in the time and frequency domain. Constructing a set of linearly independent functions, let alone a basis, with a given time-frequency localisation is a formidable and often unsolvable problem, though. A much better chance one stands in building a set of time-frequency localised functions that constitutes a so-called frame – a generalisation of the notion of the basis, whose elements need not be linear independent, rather than a basis.

Over the last seventy years or so, a range of frames have been designed to allow the decomposition and synthesis of functions of various classes. The most prominent examples of such systems are Gabor functions, wavelets, ridgelets, curvelets, shearlets and wave atoms. We recently introduced a family of quasi-Banach spaces – which we called *wave packet spaces* – that encompasses all those classes of functions whose elements have sparse expansions in one of the above-mentioned frames,

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supplied them with Banach frames – the kind of frames that ensure that any element of the class of functions for which a frame was designed can be decomposed and reconstructed using that frame – and provided their atomic decomposition. Herein we prove that the Banach frames for and sets of atoms of the wave packet spaces – which we call *wave packet systems* – are indeed localised in the time and frequency domain or, more specifically, that they are near orthogonal; and therefore so are all of the above-mentioned examples of frames. We shall also show that, unlike those examples, the wave packet system can be made to assume a wide range of types and degrees of time-frequency localisation by the suitable choice of values of the parameters of the system. This, we believe, makes the wave packet systems not only suitable for decomposing, synthesising or approximating functions of a wide range of quasi-Banach function spaces in an efficient and effective way, but also for their use for representing linear bounded operators on the quasi-Banach spaces by sparse and well structured matrices using the Galerkin method. This, in its turn, should allow one to design efficient computer programs for solving corresponding operator equations on two-dimensional manifolds using finite element methods.

1. Introduction

In this report we first formulated the notion of the wave packet system $W(\alpha, \beta)$, originally introduced in [9], in such a way that it now, among other things, forms automatically both a Banach frame for and a set of atoms of the corresponding wave packet space $\mathcal{W}_s^{p,q}(\alpha, \beta)$ and all elements of the system are unique. We then introduced the notion of the distance between the indices of the elements of the wave packet system and demonstrated that the index set with this distance constitutes a separate metric space. Finally we proved that the absolute value of the scalar product of two elements of the system decays polynomially as the distance between their indices increases. This, in its turn, can be interpreted as near-orthogonality or, in other words, intrinsic time-frequency localisation of the elements the wave packet system in the phase space. This, among other things, implies the intrinsic time-frequency localisation of Gabor functions, wavelets, ridgelets, curvelets, shearlets and wave atoms.

In science [50,7,8,51,52], engineering [15,48,45,1] and mathematics [47,55,29,37,49,41, 26,43,14], functions routinely need to be decomposed into or synthesised from those localised one way or another both in the time and frequency domain, the time-frequency localisation often resulting in much more economical or, as one usually says, sparser representations. For this purpose a number of systems of localised real functions of two real variables have been designed and successfully used over the last seventy years or so. The most important of them are Gabor functions [35], wavelets [21,53,28,29,6,5,10], α-modulation functions [33], ridgelets [11,31], curvelets [12], shearlets [44,42] and wave atoms [22]. Most of these systems are not bases, but rather so-called frames [23,25,34], a generalisation of the notion of the basis, whose elements need not be liner independent.

Two essential properties of each of these systems, except for the anisotropic wavelets [6,10], are determined by two parameters known as α and β. The parameter α defines how the length $(1+|\xi|)^{\alpha}$, so to speak, of the essential support of the Fourier transform of an element of the system along its radial axis of symmetry depends on the absolute value of the frequency ξ at which it is localised (see Fig. 1). For instance, the length of the essential support of the Gabor functions, whose α equals zero, is independent of the frequency, while that of wavelets, whose α equals one, is proportional to it. That being so, expanding a given function in a series of Gabor functions will result in identifying higher and lower frequencies present in the function with the same resolution, while in its expansion in a series of wavelets lower frequency will be much better resolved than higher ones. The parameter β determines the number $2^{(1-\beta)j}$ of elements of the systems whose Fourier transforms appear at a given frequency scale, defined by the index j, and differ from each other only in orientation. This, in its turn, defines directional resolution of different frequencies present in the function that can be achieved by expanding it into a series of a given type of localised functions. For instance, Gabor functions, whose β equals zero, have equally good direction resolution at all frequency scales, while wavelets, whose β equals one, do not allow one to distinguish different directions in the frequency domain at all.

The spaces of functions whose elements are characterised by their sparse decomposition into localised functions that belong to one or another of the systems mentioned above are worth studying in their own right. Not long ago we introduced a family of quasi-Banach spaces – which we called *wave packet spaces* – that encompasses all these spaces, studied their properties, equipped them with Banach frames – which we called *wave packet systems* – and provided their atomic decompositions [9]. We adopted this term from [20] as we believe that our wave packets unify in themselves the Banach frames [34] mentioned above just as the wave packets in [20,38,39,46] unified frames for separable Hilbert spaces [23]. Efficiency of the approximation of functions [32] of various classes in terms of a linear combination of elements of an appropriately chosen wave packet system and their potential usefulness for discretisation of bounded linear operators [3], in general, and Fourier integral operators [20,13], in particular, by the Galerkin method can be expected to depend crucially on how well these elements are localised, not least because the Fourier integral operators are known to essentially transform well-localised wave packets into each other [20]. Knowing, say from [46], that the time-frequency localisation of frames even for Hilbert spaces should not be taken for granted, we give herein a precise and thorough answer to the question as to whether and, if so, how well wave packet systems are localised.

This report is organised as follows. In Section 2, we define an almost-structured covering of improved design of a subset of the frequency plane and use it to define wave packet spaces. In section 3, we define wave packet systems [9] in such a way that they will now constitute at the same time both Banach frames for and sets of atoms of the wave packet spaces as long as their parameters α and β satisfy the condition $0 \leqslant \beta \leqslant \alpha \leqslant 1$; we also discuss their structure, remind the notion of intrinsic localisation of the set of functions in terms of near orthogonality of its elements and prove it for the wave packet systems. To make this report self-contained we allowed ourselves to provide the definitions of all the necessary rather specialised mathematical notions where they were first needed as well as a list of more general and widely accepted notations and notions used throughout this report in Section 4.

2. Design of wave packet spaces

The wave packet spaces are quasi-Banach spaces defined by the decomposition method [24] and, as such, are made of three basic building blocks, namely an almost-structured covering $\mathcal{Q}$ of a Lebesgue-measurable subset of the frequency plane, a regular partition of unity on the plane $\mathbb{R}^2$ and a $\mathcal{Q}$-moderate weight. Here are their definitions as well as those of all necessary axillary notions.

Definition 1. The set $\mathcal{Q} = \{Q_i\}_{i \in I}$ is called an *almost-structured covering* of a Lebesgue-measurable subset O of $\mathbb{R}^2$ if

1. the number of elements in the set $\{i' \in I : Q_{i'} \cap Q_i \neq \emptyset\}$ is uniformly bounded for any $i \in I$;
2. there exists a set $\{\mathbb{R}^2 \to \mathbb{R}^2,\ \xi \mapsto T_i\xi + b_i\}_{i \in I}$ of invertible affine-linear maps and finite sets $\{Q'_n\}_{n=1}^N$ and $\{P_n\}_{n=1}^N$ of non-empty open and bounded subsets Q'_i and P_i of $\mathbb{R}^2$ such that
   (a) $\overline{P_n} \subset Q'_n$ for $1 \leqslant n \leqslant N$;
   (b) for each $i \in I$ there exists an $n_i \in \{1, ..., N\}$ such that $Q_i = \left(T_i\, Q'_{n_i} + b_i\right) \cap O$;
   (c) there exists a constant $C_1 > 0$ such that $\|T_i^{-1} T_{i'}\| \leqslant C_1$ for all i and $i' \in I$ such that $Q_i \cap Q_{i'} \neq \emptyset$; and
   (d) $\bigcup_{i \in I} (T_i P_i + b_i) = O$.

Condition 1. ensures that the elements of the covering spread more or less uniformly over a neighbourhood of the subset O of $\mathbb{R}^2$. Point (b) of condition 2. guarantees that the elements of the covering are made from one of only a finite number of bounded subsets Q'_{n_i}'s of $\mathbb{R}^2$. Point (c) of condition 2. ensures that those elements of the covering that overlap one another are similar in size and orientation.

Definition 2. First, let $\epsilon \in (0, 1/32)$ and

$$Q := (-\epsilon, 1+\epsilon) \times (-1-\epsilon, 1+\epsilon) \quad ; \tag{1}$$

second, let $0 \leqslant \beta \leqslant \alpha \leqslant 1$, $N := 10$, $j^{\max} \in \mathbb{N}$,

$$I_0 := \{(0,0,0)\} \cup \{(j,m,l) \in \mathbb{N} \times \mathbb{N}_0 \times \mathbb{N}_0 : j \leqslant j^{\max},\ m \leqslant m_j^{\max} \text{ and } l \leqslant l_j^{\max}\} \quad , \tag{2}$$

where

$$l_j^{\max} := \lceil N \cdot 2^{(1-\beta)j} \rceil - 1 \quad , \tag{3}$$

$$m_j^{\max} := \begin{cases} \lceil 2^{(1-\alpha)j-1} \rceil - 1 & \text{if } \theta_{jl} = \theta_{j+1,l'} \text{ for an } \mathbb{N}_0 \ni l' \leqslant l_{j+1}^{\max} \text{ , or if } j = j^{\max} \\ \lceil 2^{(1-\alpha)j-1} \rceil & \text{if } \theta_{jl} \neq \theta_{j+1,l'} \text{ for any } \mathbb{N}_0 \ni l' \leqslant l_{j+1}^{\max} \end{cases} \tag{4}$$

and

$$\theta_{jl} := 2\pi \cdot \frac{2^{(\beta-1)j}}{N} \cdot l \quad ; \tag{5}$$

finally, let $Q_0 := Q_{000} := B_4(0)$, where $B_4(0)$ stands for the Euclidean ball of radius four with its centre in the origin and

$$Q_i := Q_{jml} := (R_{jl}(A_{jm}Q + B_{jm})) \cap (B_{R_{max}}(0)) \quad , \tag{6}$$

where

$$A_{jm} = \begin{pmatrix} 2^{\alpha j} & 0 \\ 0 & 2^{(\beta-1)j}\left(2^{j-1} + (m+1)2^{\alpha j}\right) \end{pmatrix} \; , \quad B_{jm} = \begin{pmatrix} 2^{j-1} + m \cdot 2^{\alpha j} \\ 0 \end{pmatrix} \tag{7}$$

and

$$R_{jl} := \begin{pmatrix} \cos\theta_{jl} & -\sin\theta_{jl} \\ \sin\theta_{jl} & \cos\theta_{jl} \end{pmatrix} \tag{8}$$

for all $i = (j, m, l) \in I_0 \setminus \{(0, 0, 0)\}$. The set $\mathcal{Q}^{(\alpha,\beta)} := \{Q_i\}_{i \in I_0}$ is called a *wave packet covering* of $B_{R_{max}}(0) \subseteq \mathbb{R}^2$ where $R_{max} := 2^{j^{\max}}$.

According to (6), every element of the covering $\mathcal{Q}^{(\alpha,\beta)}$ except for Q_0 is made from the subset Q of $\mathbb{R}^2$ by scaling, shifting and rotating it around the origin of the plane $\mathbb{R}^2$. Fig. 1 shows a chart of the generic element $Q_i := Q_{jml}$ of the wave packet covering with $j \neq 0$. The design of the covering, among other things, ensures that all the elements of the covering are unique. The proof that $\mathcal{Q}^{(\alpha,\beta)}$ is indeed a covering of $B_{R_{max}}(0) \subseteq \mathbb{R}^2$ and that it is almost-structured is akin, albeit a little more involved technically, to that of Lemmata 3.2 and 5.1 of [9] respectively. It should be mentioned that a wave packet covering of any subset O of $\mathbb{R}^2$ can be built exactly in the same way as that of $B_{R_{max}}(0)$ by replacing $B_{R_{max}}(0)$ by O in Definition 2.

Definition 3. Let $\mathcal{Q} = \{Q_i\}_{i \in I}$ and before be an almost-structured covering of a Lebesgue-measurable subset O of $\mathbb{R}^2$ and the set of invertible affine-linear maps associated with it respectively. The set of functions $\Phi = \{\phi_i\}_{i \in I}$ is called a *regular partition of unity on* $O \subseteq \mathbb{R}^2$ *subordinate to* $\mathcal{Q}$, if

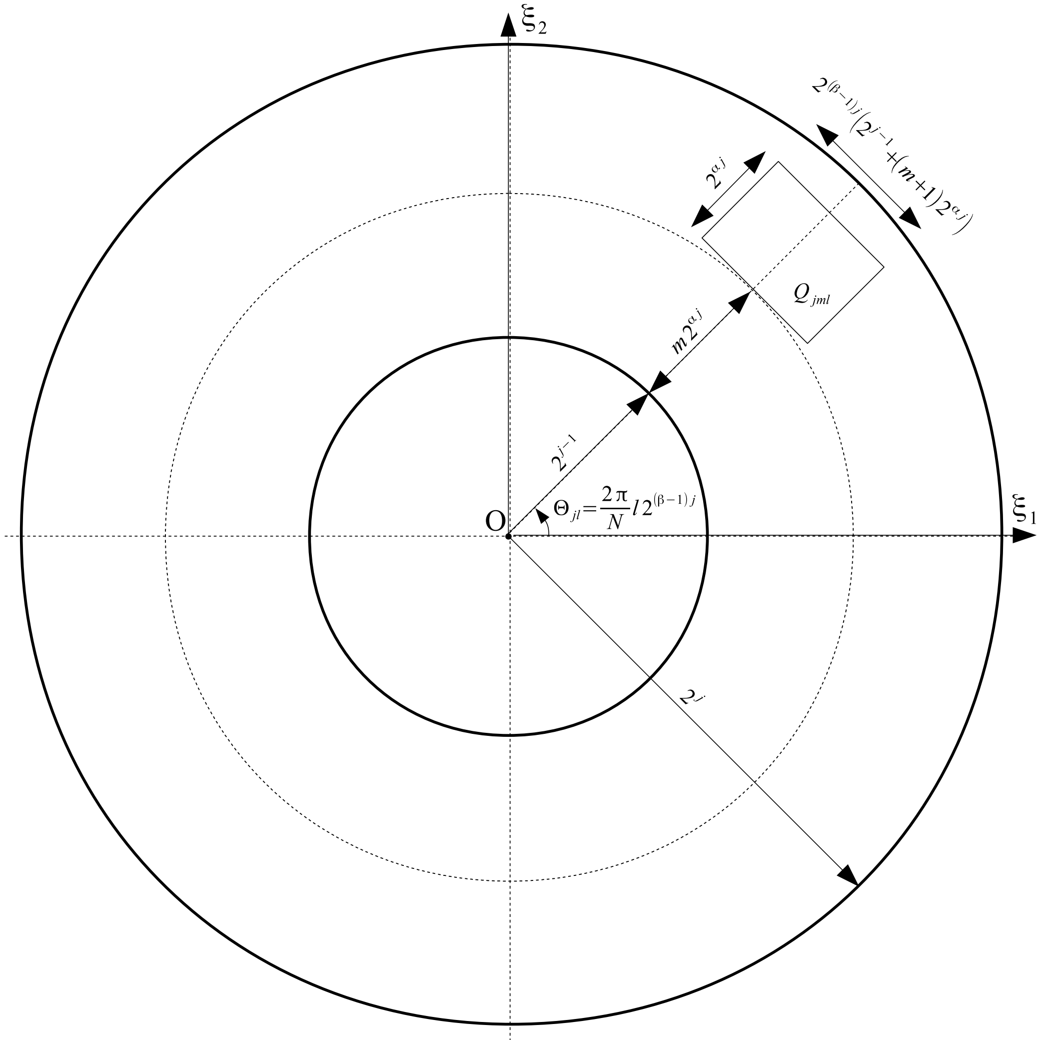

Fig. 1. Chart of a generic element Q_{jml} of the wave packet covering introduced in Definition 2. The distance between the rectangle Q_{jml} and the origin O of the frequency plane equals $2^{j-1} + m2^{\alpha j}$. The length of Q_{jml}, in the radial direction, equals $2^{\alpha j}$ while its width, in the angular direction, roughly approximates $2^{\beta j}$. Its axis of symmetry that intersects the origin O deviates from the ξ_1-axis by the angle $\Theta_{j,\ell} = \frac{2\pi}{N}\ell \cdot 2^{(\beta-1)j}$. For a given j, all rectangles $Q_{j,m,\ell}$ are contained in the dyadic ring $\{\xi \in \mathbb{R}^2 : |\xi| \asymp 2^j\}$. The indices $m_j^{\max}$ and $l_j^{\max}$ roughly approximate $2^{(1-\alpha)j}$ and $2^{(1-\beta)j}$ respectively. The bold continuous circular lines indicate boundaries between adjacent dyadic rings.

1. $\phi_i \in C_c^\infty(\mathbb{R}^2)$ with $\operatorname{supp}\phi_i \subset Q_i$ for all $i \in I$;
2. $\sum_{i\in I} \phi_i \equiv 1$ on $O \subseteq \mathbb{R}^2$; and
3. $\sup_{i\in I} \| \partial^\alpha \phi_i^\natural \|_{L^\infty} < \infty$ for any $\alpha \in \mathbb{N}_0^2$ where $\phi^\natural : \mathbb{R}^2 \to \mathbb{C}, \xi \mapsto \phi_i\left((T_i\,\xi + b_i) \cap O\right)$.

Together conditions 1. and 2. ensure that the elements of a regular partition of unity are smooth, or regular, enough, on the one hand, and that they have their supports confined to elements of the covering $\mathcal{Q}^{(\alpha,\beta)}$, on the other hand. This is of paramount importance in order for the decomposition space, defined by the norm (10) in Definition 6, to have the desired smoothness and an atomic decomposition, introduced in Definition 11. According to Theorem 2.8 in [54] inspired by Proposition 1 in [5], there exists a regular partition of unity subordinate to the wave packet covering $\mathcal{Q}^{\alpha,\beta} = \{Q_i\}_{i\in I}$ as the latter is almost-structured.

Definition 4. A sequence $w = \{w_i\}_{i\in I}$ of positive numbers is called a *weight*. The weight is called $\mathcal{Q}$-*moderate* where $\mathcal{Q} = \{Q_i\}_{i\in I}$ stands for an almost-structured covering of a Lebesgue-measurable subset O of $\mathbb{R}^2$, if there exists such a positive number C that $w_i \leqslant C \cdot w_{i'}$ for all such i and $i' \in I$ that $Q_i \cap Q_{i'} \neq \emptyset$.

The weight determines the importance that will be assigned to every component in the norm (10) of an element of a decomposition space, introduced below in Definition 6, depending on the component's location in the frequency plane. Using a $\mathcal{Q}$-*moderate* weight, where $\mathcal{Q}$ stands for an almost-structured covering, will guarantee that those components that are close to one another in the frequency plane are assigned comparable amounts of importance.

Definition 5. Let $0 \leqslant \beta \leqslant \alpha \leqslant 1$, $s \in \mathbb{R}$, I_0 be as defined by (2) and

$$w_i^s := \begin{cases} 2^{js} & \text{if } i = (j,m,l) \in I_0 \setminus \{(0,0,0)\} \\ 1 & \text{if } i = 0 = (0,0,0) \end{cases} . \tag{9}$$

Then $w^s = (w_i^s)_{i\in I_0}$ is called a *wave packet weight*.

The proof that $w^s = (w_i^s)_{i\in I_0}$ is $\mathcal{Q}^{(\alpha,\beta)}$-moderate can be found in Lemma 6.1 of [9].

Definition 6. Let $\mathcal{Q} = (Q_i)_{i\in I_0}$ be an almost-structured covering of a Lebesgue-measurable subset O of $\mathbb{R}^2$, $\Phi = (\phi_i)_{i\in I_0}$ be a regular partition of unity on $O \subseteq \mathbb{R}^2$ subordinate to $\mathcal{Q}$, $w = (w_i)_{i\in I_0}$ be a $\mathcal{Q}$-moderate weight, $p, q \in (0,\infty]$ and

$$\|g\|_{\mathcal{D}(\mathcal{Q},L^p,\ell^q_w)} := \left\| \left(w_i \cdot \|\mathcal{F}^{-1}(\varphi_i \cdot \widehat{g})\|_{L^p} \right)_{i\in I_0} \right\|_{\ell^q} \in [0,\infty] \quad . \tag{10}$$

Then

$$\mathcal{D}(\mathcal{Q},L^p,\ell^q_w) := \left\{ g \in Z^* : \|g\|_{\mathcal{D}(\mathcal{Q},L^p,\ell^q_w)} < \infty \right\} ,$$

where Z^* stands for the topological dual of $Z := \mathcal{F}\left(C_c^\infty(\mathbb{R}^2)\right) \subset \mathcal{S}(\mathbb{R}^2)$, is called a *decomposition space*.

As explained in [9], the thus defined decomposition space is a quasi-Banach space. In this definition, which differs slightly from that introduced originally [24], the functions $\{\mathcal{F}^{-1}(\varphi_i \cdot \widehat{g})\}_{i\in I_0}$ play the role of components of the function g localised in the frequency domain. Therefore, for (10) to be finite, and so for the function g to belong to $\mathcal{D}(\mathcal{Q},L^p,\ell^q_w)$, these components must be p-integrable and their contributions

$\{\|\mathcal{F}^{-1}(\varphi_i \cdot \widehat{g})\|_{L^p}\}_{i\in I_0}$ to the norm $\|\,.\,\|_{\mathcal{D}(\mathcal{Q},L^p,\ell^q_w)}$ weighted with w_i must be q-summable. With all these definitions we can now introduce the wave packet space.

Definition 7. Let $0 \leqslant \beta \leqslant \alpha \leqslant 1$, $s \in \mathbb{R}$, p and $q \in (0, \infty]$, and $\mathcal{Q}^{\alpha,\beta}$ and w_i^s be a wave packet covering and weight respectively. The *wave packet space* with parameters α, β, p, q and s is defined as the decomposition space

$$\mathcal{W}_s^{p,q}(\alpha,\beta) := \mathcal{D}(\mathcal{Q}^{\alpha,\beta}, L^p, \ell^q_{w^s})\,.$$

The wave packet space is, by construction [24], a quasi-Banach space of distributions. The parameter s determines what weight w_i^s will be assigned to the contribution of the i-th component $\mathcal{F}^{-1}(\varphi_i \cdot \widehat{g})$ of g to the norm (10). Therefore varying the value of s amounts to making $\mathcal{W}_s^{p,q}(\alpha,\beta)$ more or, for that matter, less regular, since i determines, among other things, the absolute value of the frequency at which the i-th component is localised. One of the rather desirable properties of a wave packet space $\mathcal{W}_s^{p,q}(\alpha,\beta)$ is that its elements can be expected to be approximated well by relatively short linear combinations of elements of the frame – a generalisation of the notion of an orthonormal base that we shall be discussing in the next section – associated with the covering $\mathcal{Q}^{\alpha,\beta}$. More details on the wave packet spaces, their properties and embeddings in each other or more classical function spaces such as Besov or Sobolev spaces can be found in [9].

3. Banach frames for and sets of atoms of wave packet spaces and their time-frequency localisation

To be able to decompose or synthesise the elements of $\mathcal{W}_s^{p,q}(\alpha,\beta)$ we shall use frames, or more specifically Banach frames, and sets of atoms of it. Here are the definitions of the notion of the frame for Hilbert and Banach spaces [23,34].

Definition 8. A countable set $\Psi := \{\psi_i\}_{i\in I}$ of elements of a Hilbert space H is called a *frame* for H if there are such positive numbers A_Ψ and B_Ψ, called *lower* and *upper bounds of the frame*, that

$$A_\Psi\,\|f\|_H^2 \leqslant \sum_{i\in I} |\langle f, \psi_i\rangle_H|^2 \leqslant B_\Psi\,\|f\|_H^2 \tag{11}$$

for any $f \in H$.

Definition 9. Let X be a Banach space and X_d a Banach space of sequences of scalars. A countable set $\Psi := \{\psi_i\}_{i\in I}$ of elements of the topological dual X^* of X is called an X_d-*frame* for X if there are positive numbers A_Ψ and B_Ψ such that

$$A_\Psi\,\|f\|_X \leqslant \sum_{i\in I} \|\langle f, \psi_i\rangle_{X,\,X^*}\|_{X_d} \leqslant B_\Psi\,\|f\|_X$$

for any $f \in X$.

Despite the apparent similarity between the definition of the frame for the Hilbert space and that for the Banach space, there is a subtle difference between their implications. While a frame for a Hilbert space allows one to decompose and reconstruct any element of the space. This does not necessarily apply to any frame for a Banach space. To ensure that both the decomposition and reconstruct of any element of a Banach space work, one ought to use so-called *Banach frames*. Here is its definition.

Definition 10. The frame $\Psi := \{\psi_i\}_{i\in I}$ for a Banach space X is called a *Banach frame* for X if there exists a well-defined bounded map, called an *analysis operator*, $A : X \to x, f \mapsto \{\langle\psi_i, f\rangle\}_{i\in I}$ where $x := \{\{\langle\psi_i, f\rangle\}_{i\in I} : f \in X\}$ is a solid Banach subspace of $\mathbb{C}^I$ and a bounded linear map $A_l^{-1} : x \mapsto X$ such that $A_l^{-1} \circ A = I_X$, where I_X stands for the identity operator on X.

This definition does not provide us with any instructions telling us how to design a Banach frame. One way to do so is to build a so-called intrinsically localised frame [27], whose notion is central to this work and will be specified below. In the meantime, here is the definition of a set of atoms of a Banach space.

Definition 11. A countable set $\Psi =: \{\phi_i\}_{i\in I}$ of elements of a Banach space X is called a *set of atoms* in X if there exist a well-defined bounded map, called a *synthesis operator*, $S : x \to X, \{c_i\}_{i\in I} \mapsto \sum_{i\in I} c_i\phi_i$ where the coefficient space $x := \left\{\{c_i\}_{i\in I} \subset \mathbb{C}^I : \sum_{i\in I} c_i\phi_i \in X\right\}$ associated with $\{\phi_i\}_{i\in I}$ is a solid subspace of $\mathbb{C}^I$ and a bounded linear map $S_r^{-1} : X \mapsto x$ such that $S \circ S_r^{-1} = I_X$ where I_X stands for the identity operator on X. The series expansion $g = \sum_{i\in I} c_i\phi_i$ of a given element $g \in X$ where $\Psi := \{\phi_i\}_{i\in I}$ is a set of atoms, and indeed sometimes the set of atoms itself, are called an *atomic decomposition* of g.

In Definition 9.1 of [9] we first introduced what we called a *wave packet system* and then, in Theorems 9.3 and 9.4 of the same report, established the conditions on the prototype functions of the elements of the system under which it will constitute either a Banach frame or a set of atoms for the wave packet space. By integrating these conditions with the definition, we shall now redefine the wave packet system so that it will automatically form both a Banach frame and a set of atoms of the corresponding wave packet space.

Definition 12. First, let $0 \leqslant \beta \leqslant \alpha < 1$, $s_0 \leqslant 0$, p_0 and $q_0 \in (0, 1]$. Second, let $t := (t_1, t_2) \in \mathbb{R}^2$, $\gamma(t)$ and $\psi(t) \in C^1(\mathbb{R}^2)$, $\partial^a\gamma(t)$ and $\partial^a\psi(t) \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ for any $a \in \mathbb{N}_0^2$ such that $|a| \leqslant 1$ and

$$\sup_{t\in\mathbb{R}^2} (1 + |t|)^{\kappa_0} \cdot |\gamma(t)| < \infty \quad \text{and} \quad \sup_{t\in\mathbb{R}^2} (1 + |t|)^{\kappa_0} \cdot |\psi(t)| < \infty \tag{12}$$

where $\kappa_0 := 10 + 2 \cdot p_0^{-1}$. Third, let $\xi := (\xi_1, \xi_2) \in \mathbb{R}^2$, $\hat{\gamma}(\xi)$ and $\hat{\psi}(\xi) \in C^\infty(\mathbb{R}^2)$, $\hat{\gamma}(\xi) \neq 0$ for any $\xi \in \overline{B}_4(0)$, $\hat{\psi}(\xi) \neq 0$ for any $\xi \in [-\epsilon, 1+\epsilon] \times [-1-\epsilon, 1+\epsilon]$,

$$|\partial^a \widehat{\partial^b \gamma}\,(\xi)| \lesssim (1+|\xi|)^{-\kappa} \cdot (1+|\xi_1|)^{-\kappa_1} \cdot (1+|\xi_2|)^{-\kappa_2} \tag{13}$$

and

$$|\partial^a \widehat{\partial^b \psi}\,(\xi)| \lesssim (1+|\xi|)^{-\kappa} \cdot (1+|\xi_1|)^{-\kappa_1} \cdot (1+|\xi_2|)^{-\kappa_2} \tag{14}$$

for any $a \in \mathbb{N}_0^2$ such that $|a| \leqslant 12$, any $b \in \mathbb{N}_0^2$ such that $|b| \leqslant 1$, any $\kappa \geqslant 10$, any $\kappa_1 \geqslant 2$ and any $\kappa_2 \geqslant 10$. More precise, but rather cumbersome definitions of a, b, κ, κ_1 and κ_2, which all depend on α, β, p_0 and q_0, can be found in the statements of Theorems 9.3 and 9.4 of [9]. Fourth, let $j^{\max} \in \mathbb{N}$,

$$\begin{aligned} I := \{(0,0,0,k) : k \in \mathbb{Z}^2\} \cup \{&(j,m,l,k) \in \mathbb{N} \times \mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{Z}^2 : j \leqslant j^{\max},\ m \leqslant m_j^{\max} \\ &\text{and } l \leqslant l_j^{\max}\}, \end{aligned} \tag{15}$$

where $m_j^{\max}$ and $l_j^{\max}$ defined by (4) and (3) respectively and

$$\psi_i(t) := \begin{cases} \gamma(t - \delta \cdot k) & \text{if } (j,m,l) = (0,0,0) \\ |\det A_{jm}|^{1/2} \cdot e^{-2\pi i \langle R_{jl} B_{jm}, t\rangle} & \\ \quad \cdot \left(\psi \ \circ A_{jm} \circ R_{jl}^{-1}\right)(t - \delta \cdot R_{jl} A_j^{-1} k) & \text{if } (j,m,l) \neq (0,0,0) \end{cases}, \tag{16}$$

where A_{jm}, B_{jm} and R_{jl} defined by (7) and (8). Then the set $W(\alpha,\beta) := \{\psi_i\}_{i \in I}$ is called a *wave packet system*.

Theorems 9.3 and 9.4 of [9] indicate, as a corollary, that the so defined wave packets are both Banach frames for and sets of atoms of $\mathcal{W}_s^{p\,q}(\alpha,\beta)$ with $s \in [-s_0, s_0]$, $p \in [p_0, \infty]$, $q \in [q_0, \infty]$ as long as $\delta \in (0, \delta_0]$ with $\delta_0 = \delta_0(\alpha, \beta, s_0, p_0, q_0, \gamma, \psi) > 0$ and that the coefficient space that is associated with $\mathcal{W}_s^{p\,q}(\alpha,\beta)$ through $\{\psi_i\}_{i\in I}$ is given by the following definition.

Definition 13. Let $0 \leqslant \beta \leqslant \alpha < 1$, $s \in \mathbb{R}$, p and $q \in (0, \infty]$, and I_0 be as defined in (2). The set of sequences of complex numbers

$$\mathcal{C}_s^{p\,q}(\alpha,\beta) := \left\{ c = \{c_i^{(k)}\}_{i \in I_0,\, k \in \mathbb{Z}^2} \in \mathbb{C}^{I_0 \times \mathbb{Z}^2} \,:\, \|c\|_{\mathcal{C}_s^{p\,q}} < \infty \right\} ,$$

where

$$\|c\|_{\mathcal{C}_s^{p\,q}} := \left\| \left(w_i^{s + (\alpha+\beta)\cdot(\frac{1}{2} - \frac{1}{p})} \cdot \left\| (c_k^{(i)})_{k \in \mathbb{Z}^2} \right\|_{\ell^p} \right)_{i \in I_0} \right\|_{\ell^q} \in [0, \infty]$$

is called a *coefficient space* associated with the wave packet space $\mathcal{W}_s^{p\,q}(\alpha,\beta)$.

The Fourier transform of $\psi_i(t)$, as defined by (16), can be calculated to be

$$\hat{\psi}_i(\xi) := \begin{cases} e^{-2\pi i \langle \delta\cdot k, \xi\rangle} \cdot \hat{\gamma}(\xi) & \text{if } (j,m,l) = (0,0,0) \\ |\det A_{jm}|^{1/2} \cdot e^{-2\pi i \langle \delta \cdot R_{jl} A_j^{-1} k, \xi\rangle} & \\ \qquad\qquad \cdot \left(\hat{\psi} \circ A_{jm}^{-1} \circ R_{jl}^{-1}\right)(\xi - R_{jl}A_j B_{jm}) & \text{if } (j,m,l) \neq (0,0,0) \end{cases} . \tag{17}$$

The functions $\gamma(t)$ and $\psi(t)$ play the role of prototypes of the wave packets $\psi_i(t)$ with $i \in I$. Specifically all wave packets $\psi_i(t)$ with an i such that $(j,m,l) = (0,0,0)$ are generated from $\gamma(t)$ only by shifting it in the time plane, while the wave packets $\psi_i(t)$ with an i such that $(j,m,l) \neq (0,0,0)$ are generated from $\psi(t)$ by scaling, modulating, rotating and shifting it in the time plane. The index $j \in \mathbb{N}$ identifies the measurements $2^{-\alpha j} \times 2^{-\beta j}$ of the essential support of the wave packet $\psi_i(t)$ – compared with those of its prototype – and, indeed, the measurements $2^{\alpha j} \times 2^{\beta j}$ of the essential support of its Fourier transform $\hat{\psi}_i(\xi)$ – compared with those of the Fourier transform of its prototype (see Fig. 1). The indices j, m and l determine the position ξ_i of the maximum of the Fourier transform $\hat{\psi}_i(\xi)$ of the wave packet $\psi_i(t)$ in the frequency plane, namely

$$\xi_i := \xi_{jml} := \begin{cases} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{if } (j,m,l) = (0,0,0) \\ R_{jl}B_{jm} = R_{jl} \begin{pmatrix} 2^{j-1} + m \cdot 2^{\alpha j} \\ 0 \end{pmatrix} & \text{if } (j,m,l) \neq (0,0,0) \end{cases} . \tag{18}$$

The indices j and m in particular determine the distance r_i of the maximum of the of Fourier transform $\hat{\psi}_i(\xi)$ of the wave packet $\psi_i(t)$ from the origin of the frequency plane, namely

$$r_i := |\xi_i| := |\xi_{jml}| := \begin{cases} 0 & \text{if } (j,m,l) = (0,0,0) \\ 2^{j-1} + m \cdot 2^{\alpha j} & \text{if } (j,m,l) \neq (0,0,0) \end{cases} , \tag{19}$$

while the indices j and l determine the angle θ_i at which the Fourier transform $\hat{\psi}_i(\xi)$ of the wave packet $\psi_i(t)$ is inclined to the ξ_1-axis in the frequency plane, namely

$$\theta_i := \begin{cases} 0 & \text{if } (j,m,l) = (0,0,0) \\ \theta_{jl} & \text{if } (j,m,l) \neq (0,0,0) \end{cases} \tag{20}$$

with θ_{jl} defined by (5). In other words r_i and θ_i are polar coordinates of the maximum of $\hat{\psi}_i(\xi)$ in the frequency plane. The indexes k, j and l determine the position of the maximum of the wave packet $\psi_i(t)$ in the time plane, namely

$$t_i := t_k^{(jl)} := \begin{cases} \delta \cdot k = \delta \cdot \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} & \text{if } (j,m,l) = (0,0,0) \\ \delta \cdot R_{jl} A_{jm}^{-1} k = \delta \cdot R_{jl} \begin{pmatrix} k_1 \cdot 2^{-\alpha j} \\ k_2 \cdot 2^{-\beta j} \end{pmatrix} & \text{if } (j,m,l) \neq (0,0,0) \end{cases} . \tag{21}$$

Whether the set $\{\psi_i\}_{i\in I}$ of the wave packets can be orthogonalised, say, by imposing additional conditions on the prototypes γ and ϕ is the subject of a separate ongoing project. At the moment we shall show that $\langle\psi_i|\psi_{i'}\rangle$ tends to zero as i and i' become more and more different from one another. To measure the difference between i and i' we shall need the following definition.

Definition 14. Let $i = (j, m, l, k)$ and $i' = (j', m', l', k') \in I$ where I defined by (15), $r(i, i') := |r_i - r_{i'}| \in \mathbb{R}_+$ where r_i and $r_{i'}$ defined by (19), $\theta(i, i') := |\theta_i - \theta_{i'}| \in [0, \pi]$ where θ_i and $\theta_{i'}$ defined by (20), $t_1(i, i') := |(t_i - t_{i'})_1| \in \mathbb{R}_+$ and $t_2(i, i') := |(t_i - t_{i'})_2| \in \mathbb{R}_+$ where t_i and $t_{i'}$ defined by (21). Then

$$\rho(i, i') := r(i, i') + \theta(i, i') + t_1(i, i') + t_2(i, i') \tag{22}$$

will be called a *distance* between indices i and i' of the wave packets ψ_i and $\psi_{i'}$.

Thus the distance $\rho(i, i')$ between the indices i and i' and, indeed between the wave packets ψ_i and $\psi_{i'}$ indexed by them themselves, is defined as the sum of four terms, viz. the lengths $|(t_i - t_{i'})_1|$ and $|(t_i - t_{i'})_2|$ of the orthogonal projections of the difference $t_i - t_{i'}$ between the radii vectors t_i and $t_{i'}$ from the origin of the time plane to the maxima of the wave packets ψ_i and $\psi_{i'}$ on the horizontal and vertical axes of the Cartesian coordinate system of the time plane (see Fig. 2.) and the absolute values of the differences $r(i, i')$ and $\theta(i, i')$ between the polar coordinates of the maxima of the Fourier transforms $\hat{\psi}_i$ and $\hat{\psi}_{i'}$ of the wave packets ψ_i and $\psi_{i'}$ on the frequency plane (see Fig. 3.). A special mention should be made of the possibility to re-balance the individual contributions of each of the terms r, θ, t_1 and t_2 to the distance ρ by multiplying those terms by appropriate positive numbers without changing in any meaningful way the facts about the distance and its use in this work established below. The following Lemma justifies the name *distance* of $\rho(i, i')$.

Lemma 1. For any $j^{\max} \in \mathbb{N}$ and $i, i' \in I$ where I is defined by (15),

$$\rho(i, i') \geqslant 0 \quad , \tag{23}$$

$$i = i' \Leftrightarrow \rho(i, i') = 0 \quad , \tag{24}$$

$$\rho(i, i') = \rho(i', i) \tag{25}$$

and

$$\rho(i, i'') \leqslant \rho(i, i') + \rho(i', i'') \quad . \tag{26}$$

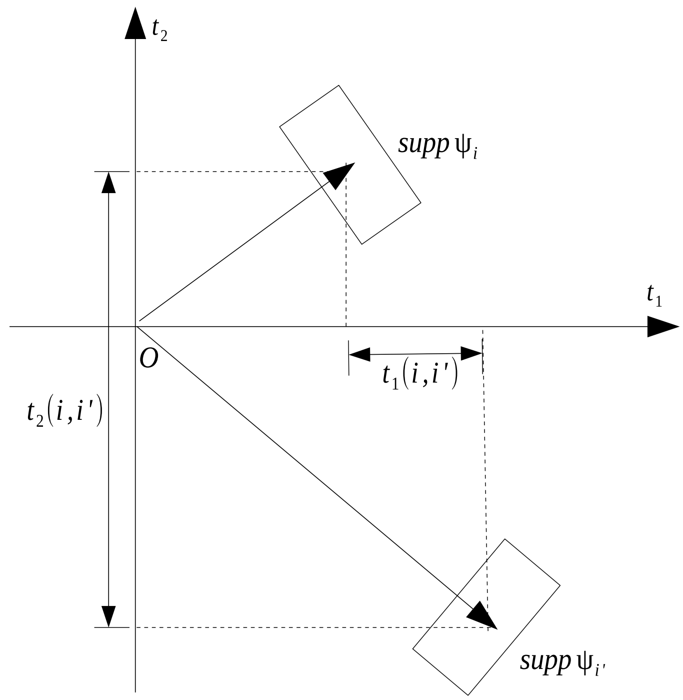

Fig. 2. Chart of the supports of two wave packets ψ_i and ψ_i', together with the lengths $t_1(i,\, i')$ and $t_2(i,\, i')$ of the orthogonal projections of the difference $t_i - t_{i'}$ between the radii vectors t_i and $t_{i'}$ from the origin of the time plane to the maxima of the wave packets ψ_i and $\psi_{i'}$ on the horizontal and vertical axes of the Cartesian coordinate system of the time plane.

Proof. First, the non-negativity of $\rho(i', i)$ expressed by (23) follows from its being the sum of four non-negative numbers.

Second, if $i = (j, m, l, k) = i' = (j', m', l', k')$, then $r(i, i')$, $\theta(i, i')$, $t_1(i, i')$ and $t_2(i, i')$ equal zero according to (19), (20) and (21) respectively. This, in its turn, implies that $\rho(i, i') = 0$. If, conversely, $\rho(i, i') = 0$, then $\theta(i, i') = 0$. These two equalities imply that $j = j'$. Indeed, if j' equalled, say, $j + 1$, then, on the one hand, $r_{i'}$ would equal or be larger than $2^{j'-1} = 2^j$, according to (19), and, on the other hand,

$$r_i \leqslant 2^{j-1} + m_j^{\max} \cdot 2^{\alpha j} = 2^{j-1} + (\lceil 2^{(1-\alpha)j-1} \rceil - 1) \cdot 2^{\alpha j} < 2^{j-1} + 2^{(1-\alpha)j-1} \cdot 2^{\alpha j} = 2^j \quad ,$$

according to (19) and (4). This, in its turn, would imply that $|r_i - r_{i'}| > 0$, which would contradict the assumption that $\rho(i, i') = 0$. If, otherwise, j' equalled or were larger than $j + 2$, then, on the one hand, $r_{i'}$ would equal or be larger than $2^{j'-1} = 2^{j+1}$, according to (19), and, on the other hand,

$$\begin{aligned} r_i &\leqslant 2^{j-1} + m_j^{\max} \cdot 2^{\alpha j} = 2^{j-1} + \lceil 2^{(1-\alpha)j-1} \rceil \cdot 2^{\alpha j} < 2^{j-1} + \left(2^{(1-\alpha)j-1} + 1\right) \cdot 2^{\alpha j} \\ &= 2^j + 2^{\alpha j} \leqslant 2^{j+1} \quad , \end{aligned}$$

according to (19) and (4). This would again imply that $|r_i - r_{i'}| > 0$, which would contradict the assumption that $\rho(i, i') = 0$. In a similar manner it can be demonstrated that the assumptions that $j' < j$ and $\rho(i, i') = 0$ are not compatible either. Furthermore,

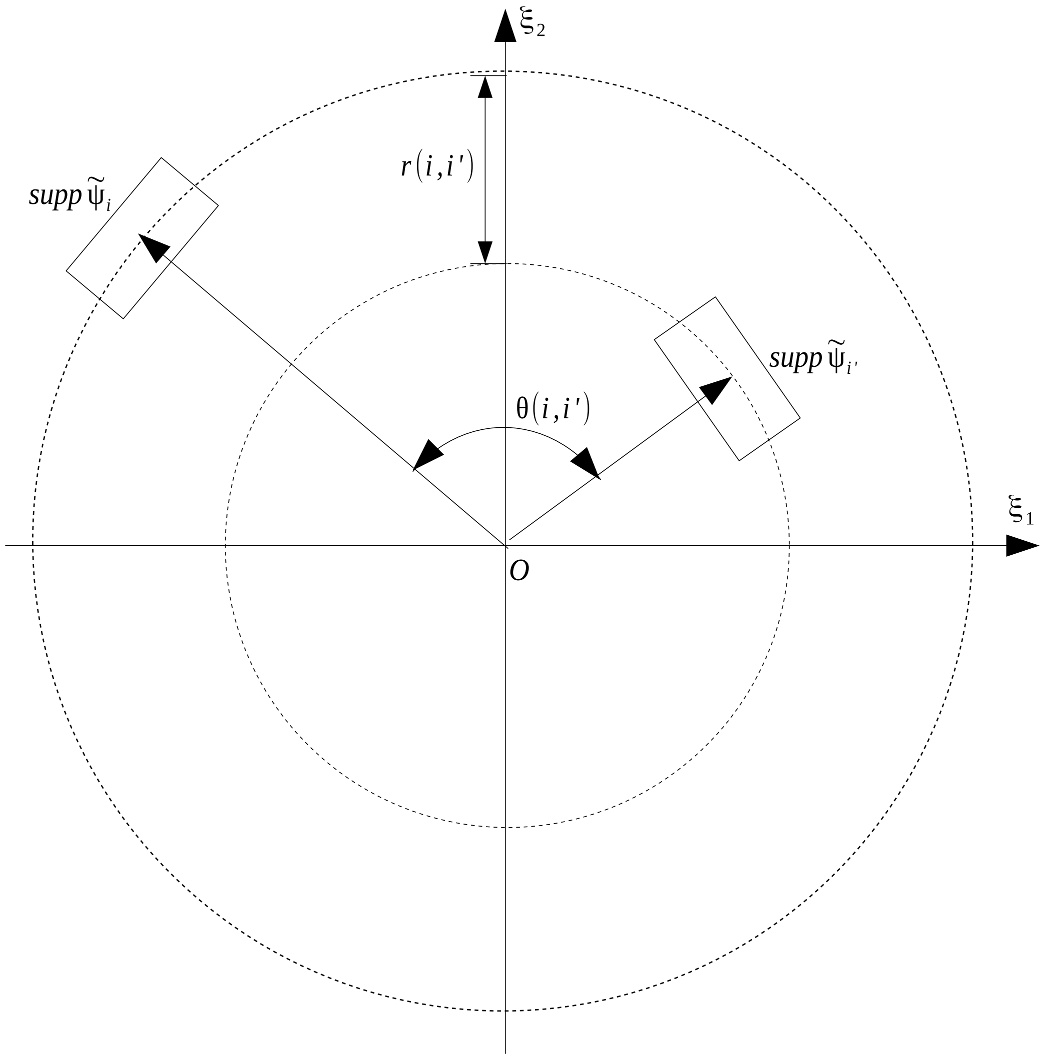

Fig. 3. Chart of the supports of the Fourier transforms $\hat{\psi}_i$ and $\hat{\psi}_{i'}$ of two wave packets ψ_i and ψ'_i, together with the absolute values of the differences $r(i,i')$ and $\theta(i,i')$ between the polar coordinates of the maxima of the Fourier transforms $\hat{\psi}_i$ and $\hat{\psi}_{i'}$ of the wave packets ψ_i and $\psi_{i'}$ on the frequency plane.

according to (20), (5) and (3), $0 \leqslant \theta_i < 2\pi$ and so $0 \leqslant \theta(i,i') < 2\pi$ too. Therefore either $l = 0 = l'$ if i is such that $j = j' = 0$ or

$$\theta(i,i') = 2\pi \cdot \frac{2^{(\beta-1)j}}{N} \cdot |l - l'| = 0$$

if it is not, which, given $2^{(\beta-1)j} > 0$ for any $j \leqslant j^{\max}$, again implies that $l = l'$. Moreover $\rho(i,i') = 0$ implies that $r(i,i') = 0$ or, according to (19), that $2^{j-1} + m \cdot 2^{\alpha j} = 2^{j'-1} + m' \cdot 2^{\alpha j'}$. This, together with $j = j'$, implies that $m = m'$. Finally, according to (21),

$$t_i - t_{i'} = \delta \cdot R_{jl} \begin{pmatrix} (k_1 - k'_1) \cdot 2^{-\alpha j} \\ (k_2 - k'_2) \cdot 2^{-\beta j} \end{pmatrix} \tag{27}$$

as $j = j'$ and $l = l'$ and so $R_{jl} = R_{j'l'}$. Therefore $\rho(i, i') = 0$ implies that

$$\begin{cases} (t_i - i_{i'})_1 = \delta \cdot \cos\theta_{jl} \cdot (k_1 - k_1') \cdot 2^{-\alpha j} - \delta \cdot \sin\theta_{jl} \cdot (k_2 - k_2') \cdot 2^{-\beta j} = 0 \\ (t_i - i_{i'})_2 = \delta \cdot \sin\theta_{jl} \cdot (k_1 - k_1') \cdot 2^{-\alpha j} + \delta \cdot \cos\theta_{jl} \cdot (k_2 - k_2') \cdot 2^{-\beta j} = 0 \end{cases} .$$

The determinant of this homogeneous system of two linear equations with unknowns $k_1 - k_2'$ and $k_1 - k_2'$ equals $\delta^2 \cdot 2^{-(\alpha+\beta)j}$, which is a positive number for any $j \leqslant j^{\max}$. Therefore the system has the trivial solution $k_1 - k_2' = 0$ and $k_1 - k_2' = 0$ only. This completes the proof of the fact that $\rho(i, i') = 0$ implies $i = i'$.

Third, the symmetry of $\rho(i, i')$ expressed by (25) results from that of $r(i, i')$, $\theta(i, i')$, $t_1(i, i')$ and $t_2(i, i')$. Fourth, the subadditivity of $\rho(i, i')$ expressed by (26) results from that of $r(i, i')$, $\theta(i, i')$, which are the absolute values of the differences between real numbers, and that of $t_1(i, i')$ and $t_2(i, i')$, which are the absolute values of the orthogonal projections of the difference between two vectors. □

The next lemma will allow us to obtain one of the major results of this work and is a corollary to Lemma 1.

Lemma 2. The metric index space I defined by (15) with any $j^{\max} \in \mathbb{N}$ and the distance ρ defined by (22) is separate, i.e.

$$\inf_{i,\, i' \in I,\, i \neq i'} \rho(i, i') = C > 0 \quad . \tag{28}$$

Proof. We note that

$$\inf_{i,\, i' \in I,\, i \neq i'} \rho(i, i') \geqslant \inf_{i,\, i' \in I,\, i \neq i'} (r(i, i') + \theta(i, i')) + \inf_{i,\, i' \in I,\, i \neq i'} (t_1(i, i') + t_2(i, i'))$$

and consider two possible situations that can occur. First let us assume that i and i' are such that at least one of the conditions $j \neq j'$, $m \neq m'$ and $l \neq l'$ is satisfied. From Lemma 1 we know that $\rho(i, i') > 0$ for any i and i' that differ one way or another from each other. In particular $r(i, i') + \theta(i, i') = \rho(i, i') > 0$ as $k_1 = k_1'$, $k_2 = k_2'$ and at least one of the conditions $j \neq j'$, $m \neq m'$ and $l \neq l'$ is satisfied. From this and the fact that $r(i, i') + \theta(i, i')$ does not depend on k_1, k_2, k_1' and k_2' we infer that $r(i, i') + \theta(i, i') > 0$ as long as at least one of the conditions $j \neq j'$, $m \neq m'$ and $l \neq l'$ is satisfied, no matter what k_1, k_2, k_1' and k_2' are. As $j^{\max}$ in a fixed whole number, so are $m^{\max}$ and $l^{\max}$. Therefore, for given k_1, k_2, k_1' and k_2', there is only a finite number of different pairs of such indices (i, i') that at least one of the conditions $j \neq j'$, $m \neq m'$ and $l \neq l'$ is satisfied and amongst those pairs there is a pair that minimises $r(i, i') + \theta(i, i')$. Let us call this positive minimum C_1, then

$$\inf_{i,\, i' \in I,\, i \neq i'} \rho(i, i') \geqslant \inf_{i,\, i' \in I,\, i \neq i'} (r(i, i') + \theta(i, i')) \geqslant C_1 > 0 \tag{29}$$

as i and i' are such that at least one of the conditions $j \neq j'$, $m \neq m'$ and $l \neq l'$ is satisfied.

Second, let us now assume that i and i' are such that $j = j'$, $m = m'$ and $l = l'$. Then $R_{jl} = R_{j'l'}$ and $t_i - t_{i'}$ is given by (27), from which we infer that

$$\begin{aligned}
t_1(i,i') + t_2(i,i') &= |(t_i - i_{i'})_1| + |(t_i - i_{i'})_2| \\
&\geqslant |\cos\theta_{jl}| \cdot |(t_i - i_{i'})_1| + |\sin\theta_{jl}| \cdot |(t_i - i_{i'})_2| \\
&= |\cos\theta_{jl} \cdot (t_i - i_{i'})_1| + |\sin\theta_{jl} \cdot (t_i - i_{i'})_2| \\
&\geqslant |\cos\theta_{jl} \cdot (t_i - i_{i'})_1 + \sin\theta_{jl} \cdot (t_i - i_{i'})_2| \\
&= \big|\delta \cdot \cos^2\theta_{jl} \cdot (k_1 - k_1') \cdot 2^{-\alpha j} - \delta \cdot \cos\theta_{jl} \cdot \sin\theta_{jl} \cdot (k_2 - k_2') \cdot 2^{-\beta j} \\
&\quad + \delta \cdot \sin^2\theta_{jl} \cdot (k_1 - k_1') \cdot 2^{-\alpha j} + \delta \cdot \sin\theta_{jl} \cdot \cos\theta_{jl} \cdot (k_2 - k_2') \cdot 2^{-\beta j}\big| \\
&= \delta \cdot |k_1 - k_1'| \cdot 2^{-\alpha j} \quad ,
\end{aligned}$$

and

$$\begin{aligned}
t_1(i,i') + t_2(i,i') &\geqslant |\sin\theta_{jl}| \cdot |(t_i - i_{i'})_1| + |\cos\theta_{jl}| \cdot |(t_i - i_{i'})_2| \\
&= |-\sin\theta_{jl} \cdot (t_i - i_{i'})_1| + |\cos\theta_{jl} \cdot (t_i - i_{i'})_2| \\
&\geqslant |-\sin\theta_{jl} \cdot (t_i - i_{i'})_1 + \cos\theta_{jl} \cdot (t_i - i_{i'})_2| \\
&= \big|-\delta \cdot \sin\theta_{jl} \cdot \cos\theta_{jl} \cdot (k_1 - k_1') \cdot 2^{-\alpha j} + \delta \cdot \sin^2\theta_{jl} \cdot (k_2 - k_2') \cdot 2^{-\beta j} \\
&\quad + \delta \cdot \cos\theta_{jl} \cdot \sin\theta_{jl} \cdot (k_1 - k_1') \cdot 2^{-\alpha j} + \delta \cdot \cos^2\theta_{jl} \cdot (k_2 - k_2') \cdot 2^{-\beta j}\big| \\
&= \delta \cdot |k_2 - k_2'| \cdot 2^{-\beta j} \quad .
\end{aligned}$$

Thus

$$\begin{aligned}
t_1(i,i') + t_2(i,i') &\geqslant 1/2 \cdot \left(\delta \cdot |k_1 - k_1'| \cdot 2^{-\alpha j} + \delta \cdot |k_2 - k_2'| \cdot 2^{-\beta j}\right) \\
&\geqslant 1/2 \cdot \delta \cdot 2^{-\alpha j} \geqslant \delta \cdot 2^{-(\alpha j^{\max}+1)} > 0
\end{aligned}$$

as long as at least one of the conditions $k_1 \neq k_1'$ and $k_2 \neq k_2'$ is satisfied. Therefore

$$\inf_{i,\, i' \in I,\, i \neq i'} \rho(i,i') \geqslant \inf_{i,\, i' \in I,\, i \neq i'} (t_1(i,i') + t_2(i,i')) > \delta \cdot 2^{-(\alpha j^{\max}+1)} > 0 \tag{30}$$

as $j = j'$, $m = m'$, $l = l'$ and at least one of the conditions $k_1 \neq k_1'$ and $k_2 \neq k_2'$ is satisfied. Combining (29) and (30) results in (28) where $C = \min(C_1,\, \delta \cdot 2^{-(\alpha j^{\max}+1)}) > 0$. $\square$

The next Lemma is an immediate corollary of the previous one and Lemma 3 in [36].

Lemma 3. Let the metric index space I be defined by (15) with any $j^{\max} \in \mathbb{N}$ and the distance ρ defined by (22). Then the distance defined by (22) satisfies the following inequality

$$\sup_{i\in I}\sum_{i'}\left(1+\rho(i,i')\right)^{-n}<\infty \tag{31}$$

for any $n>5$.

We now state and prove the main theorem of this paper.

Theorem 1. Let κ_0 and κ in Definition 10 be not smaller than 24, $\psi_i(t)$ and $\psi_{i'}(t)$ any two elements of the wave packet system $W(\alpha,\beta)$ of the wave packet space $\mathcal{W}_s^{p,q}(\alpha,\beta)$ and $\rho(i,i')$ the distance between their indexes i and $i'\in I$ where I defined by (15) with any $j^{\max}\in\mathbb{N}$. Then

$$|\langle\psi_i|\psi_{i'}\rangle|\lesssim\left(1+\rho(i,i')\right)^{-n}\quad, \tag{32}$$

where $\mathbb{N}\ni n>5$.

Proof. From (12) we infer that

$$|\gamma(t)|\lesssim\frac{1}{(1+|t|)^{\kappa_0}}\quad\text{and}\quad|\psi(t)|\lesssim\frac{1}{(1+|t|)^{\kappa_0}} \tag{33}$$

and note that the right sides of these inequalities are isotropic in the sense that they do not depend on the orientation of t in the time plane. Therefore the functions obtained from γ or ψ by rotating them around the origin of the time plane satisfy the same inequalities. We also note that $1\leqslant\det A_{jm}=2^{(\alpha+\beta)j}\leqslant 2^{(\alpha+\beta)j^{\max}}<\infty$ and that $||A_{jm}||\geqslant 1$. Furthermore from Section K.1 in [30] we know that

$$\int\limits_{\mathbb{R}^n}\frac{1}{(1+|x-a|)^m}\cdot\frac{1}{(1+|x-b|)^m}\,dx\lesssim\frac{1}{(1+|a-b|)^m} \tag{34}$$

for $m>n$. Therefore

$$\begin{aligned}|\langle\psi_i|\psi_{i'}\rangle|&\leqslant\int\limits_{\mathbb{R}^2}|\psi_i|\cdot|\psi_{i'}|\,dt\lesssim\int\limits_{\mathbb{R}^2}\frac{1}{(1+|t-t_i|)^{\kappa_0}}\cdot\frac{1}{(1+|t-t_{i'}|)^{\kappa_0}}\,dt\lesssim\frac{1}{(1+|t_i-t_{i'}|)^{\kappa_0}}\\&\leqslant\frac{1}{(1+|(t_i-t_{i'})_1|)^{\frac{\kappa_0}{2}}}\cdot\frac{1}{(1+|(t_i-t_{i'})_2|)^{\frac{\kappa_0}{2}}}\\&\leqslant\frac{1}{(1+|(t_i-t_{i'})_1|+|(t_i-t_{i'})_2|)^{\frac{\kappa_0}{2}}}=\frac{1}{(1+t_1(i,i')+t_2(i,i'))^{\frac{\kappa_0}{2}}}\quad.\end{aligned} \tag{35}$$

From (13) and (14) we infer that

$$|\hat{\gamma}(\xi)|\lesssim\frac{1}{(1+|\xi|)^{\kappa}}\quad\text{and}\quad|\hat{\psi}(\xi)|\lesssim\frac{1}{(1+|\xi|)^{\kappa}}\quad. \tag{36}$$

From this and the arguments similar to those used to obtain (35) we infer that

$$\begin{aligned}\left|\left\langle\hat{\psi}_i\middle|\hat{\psi}_{i'}\right\rangle\right| &\leqslant \int\limits_{\mathbb{R}^2}\left|\hat{\psi}_i\right|\cdot\left|\hat{\psi}_{i'}\right|\,d\xi \lesssim \int\limits_{\mathbb{R}^2}\frac{1}{\left(1+|\xi-\xi_i|\right)^{\kappa}}\cdot\frac{1}{\left(1+|\xi-\xi_{i'}|\right)^{\kappa}}\,d\xi \\ &\lesssim \frac{1}{\left(1+|\xi_i-\xi_{i'}|\right)^{\kappa}}\quad .\end{aligned}\tag{37}$$

We now note that

$$\begin{aligned}\frac{1}{1+|\xi_i-\xi_{i'}|} &= \frac{1}{1+|\xi_i-\xi_{i'}|\cdot|e^{-i\theta_{i'}}|} = \frac{1}{1+|\xi_i\cdot e^{-i\theta_{i'}}-\xi_{i'}\cdot e^{-i\theta_{i'}}|} \\ &= \frac{1}{1+|\xi_i\cdot e^{-i\theta_{i'}}-r_{i'}|} \leqslant \frac{1}{1+||\xi_i\cdot e^{-i\theta_{i'}}|-r_{i'}|} \\ &= \frac{1}{1+|r_i-r_{i'}|} = \frac{1}{1+r(i,i')}\end{aligned}\tag{38}$$

for any i and $i'\in I$. Furthermore, if i is such that $(j,m,l)\neq(0,0,0)$ and so $r_i = 2^{j-1}+m\cdot 2^{\alpha j'}\geqslant 1$, then, on the one hand,

$$\begin{aligned}\frac{1}{1+|\xi_i-\xi_{i'}|} &= \frac{1}{1+|\xi_i-\xi_{i'}|\cdot|e^{-i\theta_{i'}}|} = \frac{1}{1+|\xi_i\cdot e^{-i\theta_{i'}}-\xi_{i'}\cdot e^{-i\theta_{i'}}|} \\ &\leqslant \frac{1}{1+|(\xi_i\cdot e^{-i\theta_{i'}}-\xi_{i'}\cdot e^{-i\theta_{i'}})_2|} = \frac{1}{1+r_i\cdot|\sin(\theta_i-\theta_{i'})|} \\ &\leqslant \frac{1}{1+|\sin(\theta_i-\theta_{i'})|} = \frac{1}{1+\sin(|\theta_i-\theta_{i'}|)} \leqslant \frac{1}{1+\frac{2}{\pi}\cdot|\theta_i-\theta_{i'}|} \\ &\leqslant \frac{\pi}{2}\cdot\frac{1}{1+|\theta_i-\theta_{i'}|}\end{aligned}\tag{39}$$

as $|\theta_i-\theta_{i'}|\in[0,\frac{\pi}{2}]$; and, on the other hand,

$$\begin{aligned}\frac{1}{1+|\xi_i-\xi_{i'}|} &= \frac{1}{1+|\xi_i-\xi_{i'}|\cdot|e^{-i(\theta_{i'}+\pi/2)}|} = \frac{1}{1+\left|\xi_i\cdot e^{-i(\theta_{i'}+\pi/2)}-\xi_{i'}\cdot e^{-i(\theta_{i'}+\pi/2)}\right|} \\ &\leqslant \frac{1}{1+\left|\left(\xi_i\cdot e^{-i(\theta_{i'}+\pi/2)}-\xi_{i'}\cdot e^{-i(\theta_{i'}+\pi/2)}\right)_1\right|} \\ &= \frac{1}{1+r_i\cdot|\cos(\theta_i-\theta_{i'}-\pi/2)|} \leqslant \frac{1}{1+|\cos(\theta_i-\theta_{i'}-\pi/2)|} \\ &= \frac{1}{1+|\sin(\theta_i-\theta_{i'}-\pi)|} \leqslant \frac{1}{1+\frac{2}{\pi}\cdot|\theta_i-\theta_{i'}-\pi|} \lesssim \frac{1}{1+|\theta_i-\theta_{i'}|}\end{aligned}\tag{40}$$

as $|\theta_i-\theta_{i'}|\in[\frac{\pi}{2},\pi]$ and so $|\theta_i-\theta_{i'}-\pi|\in[0,\frac{\pi}{2}]$. Indeed, if $\theta_i-\theta_{i'}\in[\frac{\pi}{2},\pi]$, then $\theta_i-\theta_{i'}-\pi\in[0,-\frac{\pi}{2}]$ and

$$\begin{aligned}\frac{1}{1+\frac{2}{\pi}\cdot|\theta_i-\theta_{i'}-\pi|} &\leqslant \frac{1}{\left|1-\frac{2}{\pi}\cdot|\theta_i-\theta_{i'}-\pi|\right|} = \frac{1}{\left|1-\frac{2}{\pi}\cdot(\pi-|\theta_i-\theta_{i'}|)\right|} \\ &= \frac{1}{\left|1+\frac{2}{\pi}\cdot|\theta_i-\theta_{i'}|\right|} \leqslant \frac{\pi}{2}\cdot\frac{1}{1+|\theta_i-\theta_{i'}|} \quad ;\end{aligned} \tag{41}$$

and, if $\theta_i-\theta_{i'}\in[-\frac{\pi}{2},-\pi]$, then $\theta_i-\theta_{i'}-\pi\in[0,\frac{\pi}{2}]$ and

$$\begin{aligned}\frac{1}{1+\frac{2}{\pi}\cdot|\theta_i-\theta_{i'}-\pi|} &\leqslant \frac{3}{3+\frac{2}{\pi}\cdot|\theta_i-\theta_{i'}-\pi|} = \frac{3}{3+\frac{2}{\pi}\cdot(|\theta_i-\theta_{i'}|-\pi)} \\ &= \frac{3}{1+\frac{2}{\pi}\cdot|\theta_i-\theta_{i'}|} \leqslant \frac{3\pi}{2}\cdot\frac{1}{1+|\theta_i-\theta_{i'}|} \quad .\end{aligned} \tag{42}$$

From (39), (40) and (38) we infer that

$$\frac{1}{1+|\xi_i-\xi_{i'}|} \lesssim \frac{1}{1+|\theta_i-\theta_{i'}|} = \frac{1}{1+\theta(i,i')} \tag{43}$$

and

$$\frac{1}{(1+|\xi_i-\xi_{i'}|)^2} \lesssim \frac{1}{1+r(i,i')}\cdot\frac{1}{1+\theta(i,i')} \leqslant \frac{1}{1+r(i,i')+\theta(i,i')} \tag{44}$$

for any such $i\in I$ that at least one of the conditions $(j,m,l)\neq(0,0,0)$ and $(j',m',l')\neq(0,0,0)$ is satisfied.

We now consider two possible situations. First, let i and i' be such that $(j,m,l)=(0,0,0)=(j',m',l')$. Then $r_i=r_{i'}=\theta_i=\theta_{i'}=0$ and so $r(i,i')=\theta(i,i')=0$ and $\rho(i,i')=t_1(i,i')+t_2(i,i')$. Moreover $\kappa_0\geqslant 12$ according to Definition 10. Therefore (35) implies (32) for any i and i' such that $(j,m,l)=(0,0,0)=(j',m',l')$. Second, if i and i' are such that at least one of the conditions $(j,m,l)\neq(0,0,0)$ and $(j',m',l')\neq(0,0,0)$ is satisfied, then from (35), (37) and (44) and the fact that $|\langle\psi_i|\psi_{i'}\rangle|=\left|\left\langle\hat\psi_i\middle|\hat\psi_{i'}\right\rangle\right|$ we infer that

$$\begin{aligned}|\langle\psi_i|\psi_{i'}\rangle| &= \left|\,|\langle\psi_i|\psi_{i'}\rangle|^{\frac{1}{2}}\cdot\left|\left\langle\hat\psi_i\middle|\hat\psi_{i'}\right\rangle\right|^{\frac{1}{2}}\right| \\ &\lesssim \frac{1}{(1+t_1(i,i')+t_2(i,i'))^{\frac{\kappa_0}{4}}}\cdot\frac{1}{(1+r(i,i')+\theta(i,i'))^{\frac{\kappa}{4}}}\end{aligned} \quad , \tag{45}$$

which again implies (32) as long as κ_0 and κ are not smaller than 24. □

Theorem 1 indicates that the modulus of the elements of the Gram matrix $G := (\langle\psi_i|\psi_{i'}\rangle)_{(i,i')\in I\times I}$ of the wave packet system $\{\psi_i\}_{i\in I}$ decays polynomially as they deviate from the main diagonal of the matrix or, in other words, that the correlation of the wave packets ψ_i and $\psi_{i'}$ fades as their distance $\rho(i,i')$ in the phase space increases. This can be viewed as near-orthogonality, so to speak, of the wave packets as their true orthogonality would amount to their Gram matrix being diagonal. The order of the polynomial in

(32) can be increased as much as one may like by choosing larger κ and κ_1 and κ_2. This amounts to increasing the degree of orthogonality of the wave packets or that of their intrinsic time-frequency localisation, the notion that will be defined presently. To formulate the next theorem, which is a corollary of Theorem 1 and Lemma 2, we shall need the following definition [36].

Definition 15. Let $I \subset \mathbb{R}^d$ be a separate countable metric index space with a distance ρ and $n > d$. A set of functions $\{\psi_i\}_{i\in I}$ is said to be *intrinsically localised* if

$$|\langle\psi_i|\psi_{i'}\rangle| \lesssim (1+\rho(i,i'))^{-n} \tag{46}$$

for all i and $i' \in I$.

Theorem 2. Let I be the metric index space defined by (15) with an $j^{\max} \in \mathbb{N}$ and with the distance ρ defined by (22) and $W(\alpha, \beta) = \{\psi_i\}_{i\in I}$ be a wave packet system. Then the wave packet system $W(\alpha, \beta)$ is intrinsically localised.

Proof. According to (15) the set I is countable. Once equipped with the distance ρ, it becomes a separate metric space according to Lemma 2. Furthermore we know from Theorem 1 that the elements of the Gram matrix G of the wave packet system $W(\alpha, \beta) = \{\psi_i\}_{i\in I}$ satisfy (46) with $6 = n > d = 5$ as i and $i' \in I \subset \mathbb{Z}^5 \subset \mathbb{R}^5$. $\square$

Finally from Proposition 1 and 3 in [40] and Lemma 3 and Theorem 1 of this report we infer that the Gram matrix G of the wave packet system $W(\alpha, \beta) = \{\psi_i\}_{i\in I}$ with I defined by (15) will belong to a solid involutive Banach algebra closed under inversion, indeed to a so-called Jaffard algebra [40], and therefore the wave packet system will be *intrinsically localised* according to Definition 3 in [27] of this notion too.

4. Conclusions

In this report we introduced wave packet systems of improved design and proved that the elements of any of them are near orthogonal to one another. We believe that this near-orthogonality will make the wave packet systems very useful not only for the efficient approximation of functions of a wide range of classes, but also for the discretisation of Fourier integral operators.

The fundamental reason for this is that the Fourier integral operators are known to transform wave packets into each other [20]. Indeed, discretising an operator – which consists in approximating it by a finite matrix – has traditionally been done by the Galerkin method using a suitable orthonormal basis, together with the finite section method [16]. Latterly, the possibility to use frames, rather than orthonormal bases, for approximating operators on Hilbert spaces by matrices, has been demonstrated [2]. This possibility can be readily extended to operators on Banach spaces. Now the near-orthogonality of the elements of any wave packet system – which implies the Gramian made from the elements of the system being a near-diagonal matrix, together with the observation made in work [20] and mentioned above, suggests that the matrices approximating Fourier integral operators obtained by the Galerkin method using a suitable wave packet system, together with an appropriate indexing of its elements, can be expected to be a near-diagonal matrix too. Indeed, such a Galerkin matrix of a Fourier integral operator can be viewed as a matrix made of the Gram matrix of a wave packet system by shifting some of its elements to neighbouring positions. And the intrinsic time-frequency localisation of the wave packet systems with a wide range of values of the parameters α and β, namely those that satisfy $0 \leqslant \beta \leqslant \alpha \leqslant 1$, and that of the parameters κ and κ_1 and κ_2 – which we have now established – should allow one a great freedom to choose the system with both the type and degree of localisation that would ensure the maximal sparsity of the matrix representing any given Fourier integral operator. In doing so one can be expected to be in a position to further relax the restrictions on the symbol and phase of the Fourier integral operators imposed in the pioneering works [17–19] without undermining the sparsity of their representations and, as a concomitant result, to formulate propositions about the boundedness and invertibility of those operators [4,18].

Moreover, solving a matrix equation – obtained from an operator equation by the Galerkin method – is intimately related to the question of invertibility of the matrix. According to [40], the inverse of a near-diagonal matrix is near-diagonal too. This allows one to use the finite section method [16] safely to make an infinite Galerkin matrix of a Fourier integral operator obtained using a wave packet system into a finite one and proceed to solve the matrix equation numerically.

5. Notations

$\bar{A}$ denotes the *adherence* or *closure* of the set A.
$A \subset B$ indicates that the set A is a *subset* of the set B.
$\mathbb{N}$ denotes the set of all *positive integers*.
$\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.
$\mathbb{N}_0^d$ denotes the set of all *d-tuple of* numbers from $\mathbb{N}_0$.
$\mathbb{Z}^d$ denotes the set of all *d-tuple of integers*.
$\mathbb{R}_+$ denotes the set of all *non-negative real numbers*.
$\mathbb{R}^d$ denotes the set of all *d-tuple of real numbers*.
$\mathbb{C}^I$ denotes the set of all sequences of complex numbers indexed by $i \in I$.
$a \lesssim b$ indicates that there exists a constant c such that $a \leqslant c\,b$.
$a \asymp b$ indicates that $a \lesssim b$ and $b \lesssim a$.
$\lceil a \rceil$ denotes the smallest integer number greater than or equal to the real number a.
C_c^∞ denotes the set of all *infinitely differentiable functions of compact support*.

The *Fourier transform* $\hat{f}(\xi)$ of the function $f(t) \in L^1(\mathbb{R}^d)$ is defined by

$$\hat{f}(\xi) := (\mathcal{F}f)(\xi) := \int_{\mathbb{R}^d} f(t)\, e^{-2\pi i \langle t\,,\,\xi\rangle}\, dt \quad \text{for} \quad \xi \in \mathbb{R}^d \quad , \tag{47}$$

where $\langle t\,,\,\xi\rangle := \sum_{i=1}^{d} t_i \xi_i$.

The *inverse Fourier transform* $f(t) := \left(\mathcal{F}^{-1}\hat{f}\right)(t)$ of the function $\hat{f}(\xi) \in L^1(\mathbb{R}^d)$ is defined by

$$f(t) := \left(\mathcal{F}^{-1}\hat{f}\right)(t) := \int_{\mathbb{R}^d} \hat{f}(\xi)\, e^{2\pi i \langle t\,,\,\xi\rangle}\, d\xi \quad \text{for} \quad t \in \mathbb{R}^d \quad . \tag{48}$$

The *Schwartz space* $\mathcal{S}$ of functions on $\mathbb{R}^d$ is defined by

$$\mathcal{S}(\mathbb{R}^d) := \left\{ f \in C^\infty(\mathbb{R}^d) \quad : \quad ||f||_{a,b} < \infty \;\; \text{for} \;\; \forall\, a, b \in \mathbb{N}_0^d \right\} , \tag{49}$$

where

$$||f||_{a,b} := \sup_{x \in \mathbb{R}^d} |x^a (\partial^b f)(x)| \quad , \tag{50}$$

$x^a := x_1^{a_1} \cdots x_d^{a_d}$ and $\partial^b f := \frac{\partial^{b_d}}{\partial x_d^{b_d}} \cdots \frac{\partial^{b_1}}{\partial x_1^{b_1}} f$

Declaration of competing interest

There is no competing interest.

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Data availability

No data was used for the research described in the article.

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