MASTERARBEIT | MASTER'S THESIS

Titel | Title

On the Construction and Properties of Lévy White Noise as a Generalized Random Field

verfasst von | submitted by

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angestrebter akademischer Grad | in partial fulfilment of the requirements for the degree of

Master of Science (MSc)

Wien | Vienna, 2026

Studienkennzahl lt. Studienblatt | Degree programme code as it appears on the student record sheet:	UA 066 821
Studienrichtung lt. Studienblatt | Degree programme as it appears on the student record sheet:	Masterstudium Mathematik
Betreut von | Supervisor:	Assoz. Prof. Jose Luis Romero Privatdoz. PhD

Abstract (German)

Diese Arbeit etabliert einen präzisen mathematischen Rahmen für Lévy-weißes Rauschen, das als verallgemeinertes Zufallsfeld mit Werten im Raum der temperierten Distributionen $\mathcal{S}'$ behandelt wird. Wir rezensieren die Ergebnisse von Biermé, Durieu und Wang [6], liefern eine detailliertere Darstellung und korrigieren kleinere Ungenauigkeiten in der bestehenden Literatur. Durch die Verwendung einer Multi-Sequenz-Darstellung des Schwartz-Raums $\mathcal{S}$ auf Basis von Hermite-Funktionen schlägt diese Arbeit eine Brücke zwischen der klassischen Wahrscheinlichkeitstheorie und der Untersuchung von Maßen auf unendlichdimensionalen Räumen. Diese Methodik liefert einen in sich geschlossenen Beweis des Satzes von Minlos-Bochner; orientiert an der Arbeit von Unser und Fageot [11, 13] nutzen wir diesen Satz als zentrales Existenzresultat, um Lévy-weißes Rauschen auf $\mathcal{S}'$ mittels unendlich teilbarer Verteilungen zu definieren. Des Weiteren bietet die Arbeit eine vereinheitlichte Darstellung, indem sie den konstruktiven Ansatz — die Definition von Lévy-weißem Rauschen als Distributionsableitung eines Lévy-Prozesses — mit dem formalen funktionalanalytischen Rahmen verknüpft. Letztlich ermöglicht diese Formulierung eine rein distributionelle Behandlung stochastischer Differentialgleichungen (SDEs) und liefert eine in der Operatortheorie verwurzelte Lösungstheorie.

Abstract

This thesis establishes a rigorous mathematical framework for Lévy white noise, treated as a generalized random field with values in the space of tempered distributions $\mathcal{S}'$. We review the results of Biermé, Durieu, and Wang [6], provide a more detailed exposition and address minor inaccuracies in the existing literature. By utilizing a multi-sequence representation of the Schwartz space $\mathcal{S}$ based on Hermite functions, this work bridges the gap between classical probability theory and the study of measures on infinite-dimensional spaces. This methodology yields a self-contained proof of the Minlos-Bochner theorem; guided by the work of Unser and Fageot [11, 13], we utilize this theorem as the primary existence result for defining Lévy white noise on $\mathcal{S}'$ via infinitely divisible distributions. Furthermore, the thesis provides a unified presentation by connecting the constructive approach—defining Lévy white noise as the distributional derivative of a Lévy process—with the formal functional framework. Ultimately, this formulation enables a purely distributional treatment of stochastic differential equations (SDEs), providing a solution theory rooted in operator theory.

Contents

1 Introduction

In many areas of applied mathematics such as signal processing, mathematical finance and other areas where random fluctuation occurs, the concept of white noise is central to model random background noises ([21], [22]). Informally, it is regarded as a centered Gaussian Process, which is independent at different times and has infinite variance. That is, the density over all frequencies in the Fourier domain, the so called spectral density, is constant. This gives it the name "white", resembling white light where all colors are present in the spectral decomposition. We can also think of it as the derivative of Brownian motion ([22],[26]). However, since this is not well-defined in the classical sense, we have the need for an expanded framework of so called Generalized Random Fields for a mathematically rigorous treatment. It originates from the idea of a generalized stochastic process, where instead of a one dimensional time variable, the index set is some infinite dimensional function space ([24]). We can think of these functions as "observation windows" by which we can observe the modeled object through some noisy measurement. They are also of use in quantum field theory ([9], [29]) and stochastic partial differential equations ([24]). This approach was first taken by Hida in the 70s to formally introduce white noise as a generalized function on an infinite dimensional function space ([20]) and and to provide the framework for white noise analysis ([25]). Motivated by this, the first part of this thesis is concerned with the formal introduction of generalized random fields. We will view them as a generalization of random variables which take values in the space of tempered distributions. However, in general, infinite-dimensional measures on topological vector spaces are very difficult objects to handle ([44], [28]). So to deal with this difficulty we use a multi-sequence representation of $\mathcal{S}$ and $\mathcal{S}'$ as sequence spaces which allows us to transfer classical results in $\mathbb{R}^d$ to infinite-dimensional distributions. In [6] this approach is taken by Biermé, Durieu and Wang to give a self contained proof for two central results characterizing the distributions of generalized random fields. A Bochner-type theorem (Minlos-Bochner theorem) characterizing the distribution of generalized random fields by their characteristic functional and a Lévy-type continuity theorem concerning the notion of convergence. We will revisit their work, but also provide more details and address gaps in the paper where we think things are left unclear.

The formal construction of white noise in [20] is based on the Minlos-Bochner theorem via the characteristic function of the Gaussian distribution. Nonetheless, many phenomena require modeling jumps and heavy tails, which leads us to the more general class of Lévy processes. Therefore, guided by the works of [11] and [42], we will apply the Minlos-Bochner theorem not only to Gaussian distributions, but to any infinitely divisible distribution. In this way we will be able to introduce the more general concept of Lévy white noise. Thus, we can identify Gaussian white noise as a special case of Lévy white noise. Furthermore, we are able to connect the two important approaches to white noise. We show that under a simplified integrability condition the definition of Lévy white noise via Minlos-Bochner theorem coincides with the constructive approach of Lévy white noise as the distributional derivative of a Lévy process. Additionally, this framework allows for an easy distributional formulation of generalized stochastic differential equations driven by Lévy white noise. These have proven to be a useful tool to model sparse stochastic signals ([41]). We will work out a basic existence result for solutions and a characterization of their distributions.

The thesis is organized as follows. In section 1-4 we will introduce the Multi-sequence representation of the Schwartz space and its dual. This allows us in section 5 to define generalized random fields and present the two main results, Minlos-Bochner theorem and Lévy's continuity theorem. In section 6 we will review some useful results of Lévy process and infinitely divisible distributions which we need to introduce Lévy white noise. Lastly, in section 7, we will set up Lévy white noise on the space of tempered distributions via the Lévy-Schwartz condition and give a thorough analysis of its properties and applications in generalized SDEs.

2 The Hermite Functions

The first part of this thesis concerns the formal introduction of generalized random fields. That is, random variables which take values in the space of tempered distributions. However, to be able to define distributions on the Schwartz space or its dual, we first have to introduce some measurability structure. And we have to choose this structure in a way that it aligns with the topological properties of these spaces. The key will be to represent Schwartz functions as multi-dimensional sequences. For this we will decompose the Schwartz space $\mathcal{S}$ into a sequence of decreasing Hilbert spaces by using an L^2-representation based on Hermite functions. The Hermite functions were first studied in the early 19th century and then properly introduced by Charles Hermite in 1864 ([19]). They are relevant in many fields of mathematics such as numerical analysis and mathematical physics (see [30],[47]). In the first section we will introduce them and show that they form an orthonormal basis of L^2. We use [47], [24] and [39] as our main references.

The family of Hermite functions $\{h_n\}_{n\in\mathbb{N}_0}$ on $\mathbb{R}$ is defined as

$$h_n(x) := (-1)^n (2^n n! \sqrt{\pi})^{-\frac{1}{2}} e^{\frac{x^2}{2}} \left(\frac{d}{dx}\right)^n e^{-x^2}, \quad n \in \mathbb{N}_0, x \in \mathbb{R}, \tag{1}$$

and we set $h_n := 0$ for $n < 0$.
Additionally, we define the Hermite polynomials as

$$H_n(x) := (-1)^n e^{x^2} \left(\frac{d}{dx}\right)^n e^{-x^2}, \quad n \in \mathbb{N}_0, x \in \mathbb{R}.$$

and again $H_n := 0$ for $n < 0$. Thus, we have $h_n(x) = (2^n n! \sqrt{\pi})^{-\frac{1}{2}} H_n(x) e^{-\frac{x^2}{2}}$.
The following properties of the Hermite polynomials will be useful to us.

Lemma 2.1. *For the Hermite polynomials H_n the following holds:*

(i) $H_n(x) = 2xH_{n-1}(x) - H'_{n-1}(x)$, *for* $n \geq 1$

(ii) $H''_n(x) - 2xH'_n(x) + 2nH_n(x) = 0$

(iii) $H'_n(x) = 2nH_{n-1}(x)$.

Proof. (i) By the definition of H_n we get

$$\begin{aligned} H_n(x) &= (-1)^n e^{x^2} \frac{d}{dx} \left(\frac{d}{dx}\right)^{n-1} e^{-x^2} = -e^{x^2} \frac{d}{dx} \left(e^{-x^2} H_{n-1}(x)\right) \\ &= 2xH_{n-1}(x) - H'_{n-1}(x). \end{aligned}$$

(ii) Equality trivially holds for $n = 0$ since $H_0 = 1$. Now suppose that it holds for $n-1$, by (i) when then get

$$\begin{aligned} H_n'(x) &= 2H_{n-1}(x) + 2xH_{n-1}'(x) - H_{n-1}''(x) \\ &= 2H_{n-1}(x) + 2xH_{n-1}'(x) - 2xH_{n-1}'(x) + 2(n-1)H_{n-1}(x) \\ &= 2nH_{n-1}(x). \end{aligned}$$

Thus, $H_n''(x) = 2nH_{n-1}'(x)$ and therefore

$$\begin{aligned} &H_n''(x) - 2xH_n'(x) + 2nH_n(x) \\ &\quad = 2nH_{n-1}'(x) - 4xnH_{n-1}(x) + 4nxH_{n-1}(x) - 2nH_{n-1}'(x) = 0. \end{aligned}$$

(iii) If we replace n by $n-1$ in (ii) and use (i), we get

$$H_n'(x) = 2xH_{n-1}'(x) + 2H_{n-1}(x) - H_{n-1}''(x) = 2nH_{n-1}(x).$$

$\square$

By (i) we inductively see that $H_n(x)$ is a polynomial with highest order term $(2x)^n$.
With this at hand, we can now prove the following Theorem in one dimension.

Theorem 2.1. *The family of Hermite functions $\{h_n\}_{n\in\mathbb{N}_0}$ is a countable orthonormal basis of $L^2(\mathbb{R})$.*

Proof. First, we show that $\{h_n\}_{n\in\mathbb{N}_0}$ are orthonormal and then that they indeed form a complete system. Let w.o.l.g. $n \geq m$, then

$$\begin{aligned} \langle h_n, h_m\rangle_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} h_n(x)h_m(x)\,dx \\ &= (-1)^n\left(2^{n+m}n!m!\right)^{-1/2}\int_{\mathbb{R}}\left(\left(\frac{d}{dx}\right)^n e^{-x^2}\right)(-1)^m e^{x^2}\left(\frac{d}{dx}\right)^m e^{-x^2}\,dx \\ &= (-1)^n\left(2^{n+m}n!m!\right)^{-1/2}\int_{\mathbb{R}}\left(\left(\frac{d}{dx}\right)^n e^{-x^2}\right)H_m(x)\,dx \\ &= \left(2^{n+m}n!m!\right)^{-1/2}\int_{\mathbb{R}} e^{-x^2}\left(\frac{d}{dx}\right)^n H_m(x)\,dx. \end{aligned}$$

Where for the last equality we used integration by parts. Any boundary terms vanish since $\left(\frac{d}{dx}\right)^k e^{-x^2} = p_k(x)e^{-x^2}$ for some polynomial p_k and the

exponential decay of e^{-x^2}. Note that H_m is a polynomial of degree m, therefore, for $n > m$ we have $\langle h_n, h_m\rangle_{L^2(\mathbb{R})} = 0$.
If $n = m$, then

$$\begin{aligned}\langle h_n, h_n\rangle_{L^2(\mathbb{R})} &= \frac{1}{2^n n!\sqrt{\pi}} \int\limits_{\mathbb{R}} \left(\left(\frac{d}{dx}\right)^n H_n(x)\right) e^{-x^2}\, dx \\ &= \frac{1}{2^n n!\sqrt{\pi}} \int\limits_{\mathbb{R}} 2^n n! e^{-x^2}\, dx = 1.\end{aligned}$$

To show that $\{h_n\}_{n\in\mathbb{N}_0}$ forms a basis, we show that if for $f \in L^2(\mathbb{R})$, $\langle f, h_n\rangle = 0\ \forall n \in \mathbb{N}_0$, then $f = 0$.
To do so we first claim $span\{H_n\}_{n\in\mathbb{N}_0} = span\{x^n, n = 0, 1, ...\}$:
Obviously, $span\{H_n\}_{n\in\mathbb{N}_0} \subset span\{x^n, n = 0, 1, ...\}$. For the other inclusion we argue inductively. For $n = 0$ both sets agree and for $n \geq 0$ we can write

$$x^{n+1} = \frac{1}{2^{n+1}} H_{n+1}(x) - P(x),$$

where P is some polynomial of degree n and thus, the claim follows from induction.
So now it suffices to prove

$$\int\limits_{\mathbb{R}} f(x)x^n e^{-x^2/2}\, dx = 0 \quad \forall n \in \mathbb{N}_0 \implies f = 0.$$

Let us assume $\int\limits_{\mathbb{R}} f(x)x^n e^{-x^2/2}\, dx = 0 \quad \forall n \in \mathbb{N}_0$ and let $F(x) := f(x)e^{-x^2/2}$.
By Cauchy-Schwarz $F \in L^1(\mathbb{R})$ and hence $\mathcal{F}(F)$ is well defined. Here $\mathcal{F}$ denotes the Fourier transform. We can compute

$$\begin{aligned}\mathcal{F}(F)(\omega) &= \int\limits_{\mathbb{R}} e^{-i\omega x} f(x) e^{-x^2/2}\, dx \\ &= \int\limits_{\mathbb{R}} \sum_{n=0}^{\infty} \frac{(-i\omega x)^n}{n!} f(x) e^{-x^2/2}\, dx \\ &= \sum_{n=0}^{\infty} \frac{(-i\omega)^n}{n!} \int\limits_{\mathbb{R}} x^n f(x) e^{-x^2/2}\, dx = 0.\end{aligned}$$

In the last equality we used the Dominated Convergence Theorem to exchange sum and integral. We have

$$\left|\sum_{n=0}^{\infty} \frac{(-i\omega x)^n}{n!} f(x) e^{-x^2/2}\right| \leq e^{|\omega x|} e^{-x^2/2} f(x).$$

and the right hand side is integrable again by Cauchy Schwarz.
By injectivity of the Fourier transform it follows that $F = 0$ a.e. and therefore $f = 0$ almost everywhere.

$\square$

We now aim to extend this result to $\mathbb{R}^d$ for $d \geq 1$. For this we define the Hermite functions $\{h_n\}_{n\in\mathbb{N}_0^d}$ by

$$h_n(x) := \prod_{i=1}^{d} h_{n_i}(x_i), \quad n = (n_1, ..., n_d) \in \mathbb{N}_0^d, x = (x_1, ..., x_d) \in \mathbb{R}^d. \tag{2}$$

Note that $h_n \in \mathcal{S}(\mathbb{R}^d)$, $\forall n \in \mathbb{N}_0^d$. Analogously to Theorem 2.1 we have the following.

Theorem 2.2. *The family of Hermite functions $\{h_n\}_{n\in\mathbb{N}_0^d}$ on $\mathbb{R}^d$ is an orthonormal basis of $L^2(\mathbb{R}^d)$.*

Proof. The orthonormality follows directly from the one dimensional case. To show completeness we again assume $\langle f, h_n\rangle_{L^2(\mathbb{R}^d)} = 0, \quad \forall n \in \mathbb{N}_0^d$ and $f \in L^2(\mathbb{R}^d)$. Hence,

$$\int\limits_{\mathbb{R}} \left(\int\limits_{\mathbb{R}^d} f(x_1, ..., x_d) \prod_{i=1}^{d-1} h_{n_i}(x_i)\, dx_1 ... dx_{d-1} \right) h_n(x_d)\, dx_d = 0$$

$$\forall n \in \mathbb{N}_0, (n_1, ..., n_{d-1}) \in \mathbb{N}_0^{d-1}.$$

Since $\{h_n\}_{n\in\mathbb{N}_0}$ is a basis of $L^2(\mathbb{R})$ we obtain

$$\int\limits_{\mathbb{R}^d} f(x_1, ..., x_d) \prod_{i=1}^{d-1} h_{n_i}(x_i)\, dx_1 ... dx_{d-1} = 0$$

for all $(n_1, ..., n_{d-1}) \in \mathbb{N}_0^{d-1}$ and almost every $x_d \in \mathbb{R}$. Iterating this argument shows $f(x_1, ..., x_d) = 0$ for almost every $(x_1, ..., x_d) \in \mathbb{R}^d$.

$\square$

This allows us to write any function $f \in L^2(\mathbb{R}^d)$ as

$$f = \sum_{n\in\mathbb{N}_0^d} \langle f, h_n\rangle h_n,$$

where we abbreviate the $L^2(\mathbb{R}^d)$ inner product $\langle\cdot,\cdot\rangle_{L^2(\mathbb{R})}$ just by $\langle\cdot,\cdot\rangle$. We want to use this representation in the particular case of Schwartz functions, i.e. smooth functions of rapid decay. Therefore, as in[6], we introduce now an additional structure of increasing norms on $L^2(\mathbb{R}^d)$. For $p \in \mathbb{N}_0$ and $f \in L^2(\mathbb{R})$, let

$$\|f\|_p^2 := \sum_{n \in \mathbb{N}_0^d} (1+n)^{2p} \langle f, h_n \rangle^2, \tag{3}$$

where we use the notation

$$(1+n)^q = \prod_{i=1}^{d} (1+n_i)^q, \quad \text{for } q \in \mathbb{Z}, n = (n_1, ..., n_d) \in \mathbb{N}_0^d.$$

Accordingly, for every $p \in \mathbb{N}_0$ we define the space of finite $\|\cdot\|_p$-norm

$$\mathcal{S}_p := \mathcal{S}_p(\mathbb{R}^d) := \{f \in L^2(\mathbb{R}^d) |\, \|f\|_p < \infty\}. \tag{4}$$

Since $\|\cdot\|_p$ is the norm corresponding to the inner product

$$\langle f, g \rangle_p = \sum_{n \in \mathbb{N}_0^d} (1+n)^{2p} \langle f, h_n \rangle \langle g, h_n \rangle,$$

every $\mathcal{S}_p$ is a separable Hilbert space with countable orthonormal basis $\{h_n^{(p)} := (n+1)^{-p} h_n \,|n \in \mathbb{N}_0^d\}$. We can see that this defines a sequence of decreasing Hilbert spaces such that $\mathcal{S}_0 = L^2(\mathbb{R}^d)$ and for each $p \in \mathbb{N}_0$, $\mathcal{S}_{p+1} \subset \mathcal{S}_p$ with $\|\cdot\|_p \leq \|\cdot\|_{p+1}$. So we have

$$\bigcap_{p \geq 0} \mathcal{S}_p \subset ... \subset \mathcal{S}_1 \subset \mathcal{S}_0 = L^2.$$

Furthermore, we can equip $\bigcap_{p\geq 0} \mathcal{S}_p$ with the topology induced by the family of semi-norms $\{\|\cdot\|_p \mid p \geq 0\}$.

3 Schwartz Space $\mathcal{S}$ and the Space of Tempered Distributions $\mathcal{S}'$

The Schwartz space $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$ is the linear vector space consisting of infinitely differentiable functions that are rapidly decreasing. That is, for all $\alpha, \beta \in \mathbb{N}_0^d$,

$$\|f\|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^d} \left| x^\alpha \partial^\beta f(x) \right| < \infty.$$

Here $\partial^\beta f(x) = \frac{\partial^{\beta_1}...\partial^{\beta_d}}{\partial x_1^{\beta_1}...\partial x_d^{\beta_d}} f(x)$ denotes the partial derivative of order β and $x^\alpha = x_1^{\alpha_1}...x_d^{\alpha_d}$. $\mathcal{S}$ is naturally equipped with the topology induced by the family of semi-norms $\{\|\cdot\|_{\alpha,\beta} \,|\, \alpha, \beta \in \mathbb{N}_0^d\}$. This family is total, i.e., $\|f\|_{\alpha,\beta} = 0$ for all $\alpha, \beta \in \mathbb{N}_0^d$ if and only if $f = 0$ and hence, $\mathcal{S}$ is a locally convex space. Furthermore, its topology is generated by countably many semi-norms and so it is metrizable ([7, Proposition 2.1]). We can even show that it is complete.

Proposition 3.1. *The Schwartz space $\mathcal{S}$ is as a complete metric space.*

Proof. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{S}$, i.e.

$$\forall \alpha, \beta \in \mathbb{N}_0^d: \quad \|\varphi_n - \varphi_m\|_{\alpha,\beta} \to 0 \quad \text{as } n, m \to \infty.$$

Since the space of continuous bounded functions on $\mathbb{R}^d$, $C_b(\mathbb{R}^d)$ is complete with respect to the supremum norm, for every $\alpha, \beta \in \mathbb{N}_0^d$ there exists $\varphi_{\alpha,\beta} \in C_b(\mathbb{R}^d)$ s.t.

$$\sup_{x\in\mathbb{R}^d} \left| x^\alpha \partial^\beta \varphi_n(x) - \varphi_{\alpha,\beta}(x) \right| \to 0 \quad (n \to \infty).$$

Let $\varphi := \varphi_{0,0}$ then $\varphi_n \to \varphi$ uniformly and hence also point-wise. To conclude the proof we have to show that $\varphi \in C^\infty(\mathbb{R}^d)$ and $\varphi_{\alpha,\beta}(x) = x^\alpha \partial^\beta \varphi(x)$.
Let e_i be the i-th unit vector in $\mathbb{R}^d$, then by the Fundamental Theorem of Calculus we have

$$\varphi_n(x + te_i) - \varphi_n(x) = \int\limits_o^t \partial^{e_i} \varphi_n(x + se_i)\, ds \quad \forall n \in \mathbb{N}.$$

By uniform convergence if we take $n \to \infty$ we obtain

$$\varphi(x + te_i) - \varphi(x) = \int\limits_o^t \varphi_{0,e_i}(x + se_i)\, ds.$$

Hence, again by the Fundamental Theorem of Calculus, $\varphi \in C^1(\mathbb{R}^d)$ and $\partial^{e_i}\varphi = \varphi_{0,e_i}$. By induction over $|\beta|$ we then get $\varphi \in C^\infty(\mathbb{R}^d)$ and $\partial^\beta \varphi = \varphi_{0,\beta}$, $\forall \beta \in \mathbb{N}_0^d$.
To finish let $\epsilon > 0$ and $N > 0$ s.t. $\forall n, m \geq N$:

$$\sup_{x\in\mathbb{R}^d} \left| x^\alpha \left(\partial^\beta \varphi_n(x) - \partial^\beta \varphi_m(x) \right) \right| < \epsilon.$$

Thus, point-wise it holds

$$\forall n, m \geq N, x \in \mathbb{R}^d: \quad \left| x^\alpha \left(\partial^\beta \varphi_n(x) - \partial^\beta \varphi_m(x) \right) \right| < \epsilon.$$

Here we can let $m \to \infty$ and therefore have $\forall n \geq N, x \in \mathbb{R}^d$:

$$\left| x^\alpha \left(\partial^\beta \varphi_n(x) - \partial^\beta \varphi(x) \right) \right| < \epsilon.$$

Since this holds for every $x \in \mathbb{R}^d$ we have for $n \geq N$:

$$\|\varphi_n - \varphi\|_{\alpha,\beta} < \epsilon.$$

Therefore, $\varphi \in \mathcal{S}$ and $\varphi_n \to \varphi$ in $\mathcal{S}$.

□

3.1 $\mathcal{S}$ as a Sequence of decreasing Hilbert Spaces

Schwartz functions are defined by two main properties, smoothness and rapid decay. This makes the $\|\cdot\|_{\alpha,\beta}$-norms difficult to work with, since it involves any arbitrary derivative. For our purpose it is convenient to characterize the Schwartz space $\mathcal{S}$ in a different way. For this we will use the structure of decreasing Hilbert spaces we identified on L^2 via the Hermite basis. In the following part we will show that

$$\mathcal{S} = \bigcap_{p \geq 0} \mathcal{S}_p,$$

and that both topologies induced by the family of semi-norms $\{\|\cdot\|_{\alpha,\beta} \,|\, \alpha, \beta \in \mathbb{N}_0^d\}$ and $\{\|\cdot\|_p \,|\, p \in \mathbb{N}_0\}$ coincide. The following Lemmas show that we can bound both families of semi-norms against each other. Our approach is along the lines of [24, Section 1.3].

Lemma 3.1. *Let $p \in \mathbb{N}_0$. Then $\exists\, \gamma \in \mathbb{N}_0^d$, $C > 0$ s.t.*

$$\|f\|_p \leq C \max_{0 \leq |\alpha|, |\beta| \leq |\gamma| + d} \|f\|_{\alpha,\beta}, \quad \forall f \in \mathcal{S}.$$

Proof. We first show the statement for $d = 1$ in detail. Since the proof for arbitrary $d \geq 1$ is not more complicated but simply heavy in notation, we will present it afterwards without much explanation.

$d = 1$:

Remember that $h_n(x) = (2^n n! \sqrt{\pi})^{-\frac{1}{2}} H_n(x) e^{-\frac{x^2}{2}}$ and by Lemma 2.1 (iii) $H_n'(x) = 2n H_{n-1}(x)$. We use this to compute

$$\begin{aligned}
\langle f, h_n \rangle &= (2^n n! \sqrt{\pi})^{-1/2} \int_{\mathbb{R}} f(x) e^{-x^2/2} H_n(x)\, dx \\
&= (2^n n! \sqrt{\pi})^{-1/2} \frac{1}{2(n+1)} \int_{\mathbb{R}} f(x) e^{-x^2/2} H_{n+1}'(x)\, dx \\
&= (2^n n! \sqrt{\pi})^{-1/2} \frac{-1}{2(n+1)} \int_{\mathbb{R}} \left(f(x) e^{-x^2/2} \right)' H_{n+1}(x)\, dx.
\end{aligned}$$

For the last equality we used integration by parts. Any boundary terms vanish by the exponential decay of $e^{-x^2/2}$ and as $f \in \mathcal{S}$, fH_{n+1} is bounded. Repeating this k-times for any $k \in \mathbb{N}$ we obtain

$$\langle f, h_n \rangle = (2^n n! \sqrt{\pi})^{-1/2} \frac{(-1)^k}{2^k (n+1)...(n+k)} \int_{\mathbb{R}} \left(f(x) e^{-x^2/2} \right)^{(k)} H_{n+k}(x)\, dx.$$

Now, by the Leibniz rule

$$\left(f(x) e^{-x^2/2} \right)^{(k)} = \sum_{j=0}^{\infty} \binom{k}{j} f^{(k-j)}(x) \left(e^{-x^2/2} \right)^{(j)}.$$

Clearly, we can write $\left(e^{-x^2/2} \right)^{(j)} = p_j(x) e^{-x^2/2}$, where p_j is a polynomial of order j inductively defined by $p_0 = 1$ and $p_j(x) = p'_{j-1}(x) - x p_{j-1}(x)$. Now, for $j \geq 0$ let

$$D_j := \sup_{x \in \mathbb{R}} |p_j(x)| \frac{1 + |x|}{1 + |x|^{j+1}} < \infty.$$

Then

$$\begin{aligned}
\left| \left(f(x) e^{-x^2/2} \right)^{(k)} \right| &\leq \sum_{j=0}^{k} \binom{k}{j} D_j \frac{1 + |x|^{j+1}}{1 + |x|} \, |f^{k-j}(\, x)| e^{-x^2/2} \\
&\leq \sum_{j=0}^{k} \binom{k}{j} D_j \frac{e^{-x^2/2}}{1 + |x|} \left(\|f\|_{0,k-j} + \|f\|_{j+1,k-j} \right) \\
&\leq 2 \sum_{j=0}^{k} \binom{k}{j} D_j \frac{e^{-x^2/2}}{1 + |x|} \max_{0 \leq \alpha, \beta \leq k+1} \|f\|_{\alpha,\beta}.
\end{aligned}$$

With this at hand we can estimate

$$\begin{aligned}
|\langle f, h_n \rangle|^2 &\leq (2^n n! \sqrt{\pi})^{-1} \left(\frac{1}{2^k (n+1)...(n+k)} \right)^2 4 \left(\sum_{j=0}^{k} \binom{k}{j} D_j \right)^2 \\
&\qquad \max_{0 \leq \alpha, \beta \leq k+1} \|f\|_{\alpha,\beta}^2 \left(\int_{\mathbb{R}} \frac{e^{-x^2/2}}{1 + |x|} H_{n+k}(x)\, dx \right)^2 \\
&\leq 4 \left(\sum_{j=0}^{k} \binom{k}{j} D_j \right)^2 \max_{0 \leq \alpha, \beta \leq k+1} \|f\|_{\alpha,\beta}^2 \int_{\mathbb{R}} (1 + |x|)^{-2}\, dx \\
&\qquad (2^n n! \sqrt{\pi})^{-1} \left(\frac{1}{2^k (n+1)...(n+k)} \right)^2 \int_{\mathbb{R}} \left(e^{-x^2/2} H_{n+k}(x) \right)^2 dx.
\end{aligned}$$

We can identify $\left(e^{-x^2/2}H_{n+k}(x)\right)^2 = h_{n+k}^2(x)(2^{n+k}(n+k)!\sqrt{\pi})$. Further, we set $\tilde{C} := 4\left(\sum\limits_{j=0}^{k}\binom{k}{j}D_j\right)^2 \int_{\mathbb{R}}(1+|x|)^{-2}\,dx$. Since $\langle h_{n+k}, h_{n+k}\rangle = 1$, we have

$$\begin{aligned}
|\langle f, h_n\rangle|^2 &\leq \tilde{C} \max_{0\leq\alpha,\beta\leq k+1} \|f\|_{\alpha,\beta}^2\, 2^k \frac{(n+k)!}{n!}\left(\frac{1}{2^k(n+1)...(n+k)}\right)^2 \\
&= \frac{\tilde{C}}{2^k} \max_{0\leq\alpha,\beta\leq k+1} \|f\|_{\alpha,\beta}^2\, \frac{1}{(n+1)...(n+k)} \\
&\leq \frac{\tilde{C}}{2^k} \max_{0\leq\alpha,\beta\leq k+1} \|f\|_{\alpha,\beta}^2\, (n+1)^{-k}.
\end{aligned}$$

Now choose k s.t. $2p-k<-1$ and set $C^2 := \frac{\tilde{C}}{2^k}\sum\limits_{n\geq 0}(n+1)^{2p-k} < \infty$. Then we arrive at

$$\begin{aligned}
\|f\|_p^2 = \sum_{n\geq 0}(n+1)^{2p}\langle f, h_n\rangle^2 &\leq \frac{\tilde{C}}{2^k} \max_{0\leq\alpha,\beta\leq k+1} \|f\|_{\alpha,\beta}^2 \sum_{n\geq 0}(n+1)^{2p-k} \\
&\leq C^2 \max_{0\leq\alpha,\beta\leq k+1} \|f\|_{\alpha,\beta}^2.
\end{aligned}$$

$d \geq 1$:
For $n, \gamma \in \mathbb{N}_0^d$ we use the notation $(1+n)^\gamma = \prod\limits_{i=1}^{d}(1+n_i)^{\gamma_i}$ and $H_n(x) = \prod\limits_{i=1}^{d} H_{n_i}(x_i)$ for $x \in \mathbb{R}^d$. Analogously to $d=1$ we compute

$$\begin{aligned}
\langle f, h_n\rangle &= \prod_{i=1}^{d}(2^{n_i}n_i!\sqrt{\pi})^{-1/2}\int\limits_{\mathbb{R}^d} f(x)\prod_{i=1}^{d} e^{-x_i^2/2}H_{n_i}(x_i)\,dx \\
&= \prod_{i=1}^{d}(2^{n_i}n_i!\sqrt{\pi})^{-1/2} \\
&\quad \int\limits_{\mathbb{R}} ... \int\limits_{\mathbb{R}} \prod_{i=2}^{d} e^{-x_i^2/2}H_{n_i}(x_i)\int\limits_{\mathbb{R}} f(x)e^{-x_1^2/2}H_{n_1}(x_1)\,dx_1...dx_d
\end{aligned}$$

$$= \prod_{i=1}^{d} (2^{n_i} n_i! \sqrt{\pi})^{-1/2}$$

$$\int\limits_{\mathbb{R}} \dots \int\limits_{\mathbb{R}} \prod_{i=2}^{d} e^{-x_i^2/2} H_{n_i}(x_i) \int\limits_{\mathbb{R}} f(x) e^{-x_1^2/2} \frac{1}{2(n_1+1)} H'_{n_1+1}(x_1)\, dx_1 \dots dx_d$$

$$= \prod_{i=1}^{d} (2^{n_i} n_i! \sqrt{\pi})^{-1/2}$$

$$(-1) \int\limits_{\mathbb{R}} \dots \int\limits_{\mathbb{R}} \prod_{i=2}^{d} e^{-x_i^2/2} H_{n_i}(x_i)$$

$$\int\limits_{\mathbb{R}} \partial_{x_1} \left(f(x) e^{-x_1^2/2} \right) \frac{1}{2(n_1+1)} H_{n_1+1}(x_1)\, dx_1 \dots dx_d.$$

For $\gamma \in \mathbb{N}_0^d$ we repeat this argument to obtain

$$\begin{aligned} \langle f, h_n \rangle = \prod_{i=1}^{d} (2^{n_i} n_i! \sqrt{\pi})^{-1/2} (-1)^{|\gamma|} 2^{-|\gamma|} \prod_{i=1}^{d} \frac{1}{(n_i+1)\dots(n_i+\gamma_i)} \\ \int\limits_{\mathbb{R}^d} \partial^\gamma \left(f(x) e^{-|x|^2/2} \right) H_{n+\gamma}(x)\, dx. \end{aligned}$$

By Leibniz rule,

$$\partial^\gamma \left(f(x) e^{-|x|^2/2} \right) = \sum_{\beta \le \gamma} \binom{\gamma}{\beta} \partial^{\gamma-\beta} f(x) \partial^\beta e^{-|x|^2/2},$$

and

$$\partial^\beta e^{-|x|^2/2} = \prod_{i=1}^{d} p_{\beta_i}(x_i) e^{-|x|^2/2} =: P_\beta(x) e^{-|x|^2/2}.$$

Note that P_β is a polynomial of order $|\beta|$. Let

$$D_\beta := \sup_{x \in \mathbb{R}^d} |P_\beta(x)| \frac{(1+|x|)^d}{1+|x^{\beta+\mathbf{1}}|} < \infty, \quad \forall \beta \in \mathbb{N}_0^d,$$

where $\mathbf{1} = (1, ..., 1) \in \mathbb{N}_0^d$. Thus,

$$\begin{aligned}
\left|\partial^\gamma \left(f(x)e^{-|x|^2/2}\right)\right| &\leq \sum_{\beta\leq\gamma} \binom{\gamma}{\beta} D_\beta \frac{1+\left|x^{\beta+\mathbf{1}}\right|}{(1+|x|)^d} \left|\partial^{\gamma-\beta} f(x)\right| e^{-|x|^2/2} \\
&\leq \frac{e^{-|x|^2/2}}{(1+|x|)^d} \sum_{\beta\leq\gamma} \binom{\gamma}{\beta} D_\beta \left(\|f\|_{0,\gamma-\beta} + \|f\|_{\beta+\mathbf{1},\gamma-\beta}\right) \\
&\leq 2\frac{e^{-|x|^2/2}}{(1+|x|)^d} \sum_{\beta\leq\gamma} \binom{\gamma}{\beta} D_\beta \max_{0\leq|\alpha|,|\beta|\leq|\gamma|+d} \|f\|_{\alpha,\beta}.
\end{aligned}$$

Let $\tilde{C} := \left(2\sum\limits_{\beta\leq\gamma} \binom{\gamma}{\beta} D_\beta\right)^2 \int\limits_{\mathbb{R}^d} (1+|x|)^{-2d}\, dx$, then

$$\begin{aligned}
|\langle f, h_n\rangle|^2 \leq & \prod_{i=1}^d (2^{n_i} n_i! \sqrt{\pi})^{-1} 2^{-2|\gamma|} \left(\prod_{i=1}^d \frac{1}{(n_i+1)...(n_i+\gamma_i)}\right)^2 \\
& \left(2\sum_{\beta\leq\gamma} \binom{\gamma}{\beta} D_\beta\right)^2 \max_{0\leq|\alpha|,|\beta|\leq|\gamma|+d} \|f\|_{\alpha,\beta}^2 \\
& \left(\int\limits_{\mathbb{R}^d} \frac{e^{-|x|^2/2}}{(1+|x|)^d} H_{n+\gamma}(x)\, dx\right)^2 \\
\leq & \tilde{C} \max_{0\leq|\alpha|,|\beta|\leq|\gamma|+d} \|f\|_{\alpha,\beta}^2 \prod_{i=1}^d (2^{n_i} n_i! \sqrt{\pi})^{-1} 2^{-2|\gamma|} \\
& \left(\prod_{i=1}^d \frac{1}{(n_i+1)...(n_i+\gamma_i)}\right)^2 \int\limits_{\mathbb{R}^d} \left(e^{-|x|^2/2} H_{n+\gamma}(x)\right)^2 dx.
\end{aligned}$$

We have $\left(e^{-|x|^2/2} H_{n+\gamma}(x)\right)^2 = h_{n+\gamma}^2(x) \prod\limits_{i=1}^d (2^{n_i+\gamma_i}(n_i+\gamma_i)!\sqrt{\pi})$ and therefore

$$\begin{aligned}
|\langle f, h_n\rangle|^2 &\leq \tilde{C} \max_{0\leq|\alpha|,|\beta|\leq|\gamma|+d} \|f\|_{\alpha,\beta}^2\, 2^{-|\gamma|} \prod_{i=1}^d \frac{1}{(n_i+1)...(n_i+\gamma_i)} \\
&\leq \tilde{C} 2^{-|\gamma|} \max_{0\leq|\alpha|,|\beta|\leq|\gamma|+d} \|f\|_{\alpha,\beta}^2 (n+1)^{-\gamma}.
\end{aligned}$$

If we choose γ s.t. $2p - \gamma_i < -1 \quad \forall i = 1, ..., d$ and set

$C^2 := \tilde{C} 2^{-|\gamma|} \sum_{n\in\mathbb{N}_0^d} \prod_{i=1}^{d} (1+n_i)^{2p-\gamma_i} < \infty$ then,

$$\|f\|_p^2 = \sum_{n\in\mathbb{N}_0^d} (1+n)^{2p} \langle f, h_n \rangle^2 \le C^2 \max_{0\le|\alpha|,|\beta|\le|\gamma|+d} \|f\|_{\alpha,\beta}^2.$$

$\square$

To estimate the $\|\cdot\|_{\alpha,\beta}$ against the $\|\cdot\|_p$ norm we first need some further properties of the hermite functions.

Lemma 3.2. *Let h_n be a hermite function with $n \ge 0$, then*

(i) $h_n'(x) = \sqrt{\frac{n}{2}} h_{n-1}(x) - \sqrt{\frac{n+1}{2}} h_{n+1}(x)$

(ii) $x h_n(x) = \sqrt{\frac{n}{2}} h_{n-1}(x) + \sqrt{\frac{n+1}{2}} h_{n+1}(x)$.

Proof. Using (i) and (iii) of Lemma 2.1 we get

$$\begin{aligned} h_n'(x) &= (2^n n! \sqrt{\pi})^{-1/2} e^{-x^2/2} \left(H_n'(x) - x H_n(x) \right) \\ &= (2^n n! \sqrt{\pi})^{-1/2} e^{-x^2/2} \left(H_n'(x) - \frac{1}{2} \left(H_{n+1}(x) + H_n'(x) \right) \right) \\ &= (2^n n! \sqrt{\pi})^{-1/2} e^{-x^2/2} \frac{1}{2} \left(2n H_{n-1}(x) - H_{n+1}(x) \right) \\ &= \sqrt{\frac{n}{2}} h_{n-1}(x) - \sqrt{\frac{n+1}{2}} h_{n+1}(x), \end{aligned}$$

and

$$\begin{aligned} x h_n(x) &= (2^n n! \sqrt{\pi})^{-1/2} e^{-x^2/2} x H_n(x) \\ &= (2^n n! \sqrt{\pi})^{-1/2} e^{-x^2/2} \frac{1}{2} \left(H_{n+1}(x) + H_n'(x) \right) \\ &= \sqrt{\frac{n}{2}} h_{n-1}(x) + \sqrt{\frac{n+1}{2}} h_{n+1}(x). \end{aligned}$$

$\square$

Lemma 3.3. *Let $\alpha, \beta \in \mathbb{N}_0^d$. Then there exists $M > 0$, $p \in \mathbb{N}_0$ s.t.*

$$\|f\|_{\alpha,\beta} \le M \|f\|_p, \quad \forall f \in \mathcal{S}.$$

Proof. To show this estimate we first claim that it is enough to prove the inequality

$$\|(\cdot)^\alpha \partial^\beta f(\cdot)\|_{L^2(\mathbb{R}^d)} \leq M\,\|f\|_p, \quad \text{for some } M > 0, p \in \mathbb{N}_0. \tag{5}$$

To see this, let $k > d/4$. Then for $f \in \mathcal{S}$, by using the Fourier transform and Cauchy-Schwarz inequality we can estimate

$$\begin{aligned}
\|f\|_\infty &\leq (2\pi)^{-d/2} \int\limits_{\mathbb{R}^d} |\hat{f}(p)|\, dp \\
&= (2\pi)^{-d/2} \int\limits_{\mathbb{R}^d} (1+|p|^2)^k |\hat{f}(p)| (1+|p|^2)^{-k}\, dp \\
&\leq (2\pi)^{-d/2} \left(\int\limits_{\mathbb{R}^d} (1+|p|^2)^{-2k}\, dp \right)^{1/2} \left(\int\limits_{\mathbb{R}^d} (1+|p|^2)^{2k} \|\hat{f}(p)|^2\, dp \right)^{1/2}.
\end{aligned}$$

Now let $K := (2\pi)^{-d/2} \left(\int\limits_{\mathbb{R}^d} (1+|p|^2)^{-2k}\, dp \right)^{1/2}$ and $\tilde{C} > 0$ s.t. $(1+|p|^2)^{2k} \leq \tilde{C}^2(1+|p|^{4k})$. Then

$$\begin{aligned}
\|f\|_\infty &\leq K\tilde{C} \left(\int\limits_{\mathbb{R}^d} (1+|p|^{4k}) \|\hat{f}(p)|^2\, dp \right)^{1/2} \\
&\leq K\tilde{C} \left(\|f\|_{L^2} + \|\,|\cdot|^{2k} \hat{f}(\cdot)\|_{L^2} \right) = K\tilde{C} \left(\|f\|_{L^2} + \|\Delta^k f\|_{L^2} \right).
\end{aligned}$$

For the last equality we used Parseval's Identity and the fact that

$$|\cdot|^{2k} \hat{f}(\cdot) = (-1)^k \mathcal{F}(\Delta^k f).$$

If we exchange $f(x)$ with $x^\alpha \partial^\beta f(x)$ for $\alpha, \beta \in \mathbb{N}_0^d$, we obtain

$$\|f\|_{\alpha,\beta} \leq K\tilde{C} \left(\|(\cdot)^\alpha \partial^\beta f(\cdot)\|_{L^2} + \|\Delta^k \left((\cdot)^\alpha \partial^\beta f(\cdot) \right) \|_{L^2} \right).$$

Since $\Delta^k \left((x)^\alpha \partial^\beta f(x) \right)$ is a finite linear combination of terms $x^{\alpha'} \partial^{\beta'} f(x)$ for some $\alpha', \beta' \in \mathbb{N}_0^d$ and since $\|\cdot\|_{p'} \leq \|\cdot\|_p$ for $p' \geq p$ the claim follows.
To show (5) we proceed as in the proof to Lemma 3.1. First, we give a detailed proof for $d = 1$. Since the case $d \geq 1$ works analogously to $d = 1$ we present the full proof thereafter without much detail.
$d = 1$:

Since the hermite functions $\{h_n\}_{n\in\mathbb{N}_0}$ are an orthonormal basis of $L^2(\mathbb{R}^d)$ we can expand $x^\alpha\partial^\beta f(x)$ with respect to this basis. For this we have to compute the coefficients :

$$\langle (\cdot)^\alpha\partial^\beta f(\cdot), h_n\rangle = \int_{\mathbb{R}} x^\alpha\partial^\beta f(x)h_n(x)\,dx = (-1)^\beta \int_{\mathbb{R}} f(x)\partial^\beta\left(x^\alpha h_n(x)\right)\,dx.$$

Here we once again used integration by parts where the boundary terms vanish by the same argument as in the proof of Lemma 3.1. By the previous Lemma 3.2 we can express

$$\partial^\beta\left(x^\alpha h_n(x)\right) = \sum_{j=-\alpha-\beta}^{\alpha+\beta} C_{j,n,\alpha,\beta}\, h_{n+j}(x),$$

for some $C_{j,n,\alpha,\beta}$ with

$$|C_{j,n,\alpha,\beta}| \leq \sqrt{\frac{n+1}{2}\cdots\frac{n+\alpha+\beta}{2}} \leq \left(\frac{n+\alpha+\beta}{2}\right)^{\frac{\alpha+\beta}{2}}.$$

Remember that $h_n = 0$ for $n < 0$. Therefore we get

$$\begin{aligned} x^\alpha\partial^\beta f(x) &= \sum_{n\geq 0}(-1)^\beta\left(\sum_{j=-\alpha-\beta}^{\alpha+\beta} C_{j,n,\alpha,\beta}\langle f, h_{n+j}\rangle\right)h_n(x) \\ &= \sum_{j=-\alpha-\beta}^{\alpha+\beta}(-1)^\beta\sum_{n\geq 0} C_{j,n,\alpha,\beta}\langle f, h_{n+j}\rangle h_n(x). \end{aligned}$$

By orthogonality of the h_n we then have for $j \in \{-\alpha-\beta, ..., \alpha+\beta\}$

$$\begin{aligned} \|\sum_{n\geq 0} C_{j,n,\alpha,\beta}\langle f, h_{n+j}\rangle h_n\|_{L^2}^2 &= \sum_{n\geq 0} C_{j,n,\alpha,\beta}^2|\langle f, h_{n+j}\rangle|^2 \\ &\leq \sum_{n\geq 0} 2^{-(\alpha+\beta)}(n+\alpha+\beta)^{\alpha+\beta}|\langle f, h_{n+j}\rangle|^2 \\ &= 2^{-(\alpha+\beta)}\sum_{n\geq j}(n-j+\alpha+\beta)^{\alpha+\beta}|\langle f, h_n\rangle|^2 \\ &\leq 2^{-(\alpha+\beta)}\sum_{n\geq 0}(n+2(\alpha+\beta))^{\alpha+\beta}|\langle f, h_n\rangle|^2. \end{aligned}$$

Now choose $K_{\alpha,\beta} > 0$ such that $(n+2(\alpha+\beta))^{\alpha+\beta} \leq K_{\alpha,\beta}^2(n+1)^{2(\alpha+\beta)}$, then

$$\begin{aligned} \left\|\sum_{n\geq 0} C_{j,n,\alpha,\beta}\langle f, h_{n+j}\rangle h_n\right\|_{L^2}^2 &\leq K_{\alpha,\beta}^2\, 2^{-(\alpha+\beta)}\sum_{n\geq 0}(n+1)^{2(\alpha+\beta)}|\langle f, h_n\rangle|^2 \\ &= K_{\alpha,\beta}^2\, 2^{-(\alpha+\beta)}\|f\|_{\alpha+\beta}^2. \end{aligned}$$

Thus, we can estimate

$$\begin{aligned}\|(\cdot)^\alpha \partial^\beta f(\cdot)\|_{L^2} &\le \sum_{j=-\alpha-\beta}^{\alpha+\beta} \Big\| \sum_{n\ge 0} C_{j,n,\alpha,\beta}\langle f, h_{n+j}\rangle h_n(x)\Big\|_{L^2} \\ &\le 2^{1-\frac{\alpha+\beta}{2}}(\alpha+\beta)K_{\alpha,\beta}\|f\|_{\alpha+\beta}.\end{aligned}$$

$d \ge 1$:
For $\alpha, \beta \in \mathbb{N}_0^d$, by integration by parts we have

$$\langle (\cdot)^\alpha \partial^\beta f(\cdot), h_n\rangle = \langle f, (-1)^{|\beta|}\partial^\beta\left((\cdot)^\alpha h_n(\cdot)\right)\rangle.$$

As for $d = 1$

$$\begin{aligned}\partial^\beta\left(x^\alpha h_n(x)\right) &= \prod_{i=1}^d \partial^{\beta_i} x_i^{\alpha_i} h_{n_i}(x_i) \\ &= \prod_{i=1}^d \left(\sum_{j=-\alpha_i-\beta_i}^{\alpha_i+\beta_i} C_{j,n_i,\alpha_i,\beta_i}\, h_{n_i+j}(x_i)\right) \\ &= \sum_{j_1=-\alpha_1-\beta_1}^{\alpha_1+\beta_1} \cdots \sum_{j_d=-\alpha_d-\beta_d}^{\alpha_d+\beta_d} \prod_{i=1}^d C_{j_i,n_i,\alpha_i,\beta_i}\, h_{n_i+j}(x_i) \\ &= \sum_{-\alpha-\beta\le\gamma\le\alpha+\beta} C_{\gamma,n,\alpha,\beta}\, h_{n+\gamma}(x),\end{aligned}$$

where $C_{\gamma,n,\alpha,\beta} = \prod_{i=1}^d C_{j_i,n_i,\alpha_i,\beta_i}$ with $|C_{\gamma,n,\alpha,\beta}| \le \prod_{i=1}^d \left(\frac{n_i+\alpha_i+\beta_i}{2}\right)^{(\alpha_i+\beta_i)/2}$. Hence,

$$\begin{aligned}x^\alpha \partial^\beta f(x) &= \sum_{n\in\mathbb{N}_0^d} (-1)^{|\beta|} \sum_{-\alpha-\beta\le\gamma\le\alpha+\beta} C_{\gamma,n,\alpha,\beta}\langle f, h_{n+\gamma}\rangle h_n \\ &= (-1)^{|\beta|} \sum_{-\alpha-\beta\le\gamma\le\alpha+\beta} \sum_{n\in\mathbb{N}_0^d} C_{\gamma,n,\alpha,\beta}\langle f, h_{n+\gamma}\rangle h_n.\end{aligned}$$

Furthermore we can compute

$$\begin{aligned}\Big\| \sum_{n\in\mathbb{N}_0^d} C_{\gamma,n,\alpha,\beta}\langle f, h_{n+\gamma}\rangle h_n\Big\|_{L^2}^2 &= \sum_{n\in\mathbb{N}_0^d} C_{\gamma,n,\alpha,\beta}^2 |\langle f, h_{n+\gamma}\rangle|^2 \\ &\le 2^{-|\alpha+\beta|} \sum_{n\in\mathbb{N}_0^d} \prod_{i=1}^d (n_i+\alpha_i+\beta_i)^{\alpha_i+\beta_i}\, |\langle f, h_{n+\gamma}\rangle|^2 \\ &= 2^{-|\alpha+\beta|} \sum_{n\in\mathbb{N}_0^d} \prod_{i=1}^d (n_i-\gamma_i+\alpha_i+\beta_i)^{\alpha_i+\beta_i}\, |\langle f, h_n\rangle|^2.\end{aligned}$$

For $-\alpha-\beta \le \gamma \le \alpha+\beta$ we have

$$(n_i - \gamma_i + \alpha_i + \beta_i)^{\alpha_i+\beta_i} \le (n_i + 2(\alpha_i+\beta_i))^{\alpha_i+\beta_i} \le (n_i + |\alpha+\beta|)^{|\alpha+\beta|}.$$

If we choose $K_{\alpha,\beta} > 0$ s.t.

$$\prod_{i=1}^{d} (n_i + |\alpha+\beta|)^{|\alpha+\beta|} \le K_{\alpha,\beta}^2 \prod_{i=1}^{d} (n_i+1)^{2|\alpha+\beta|},$$

then we can estimate

$$\begin{aligned}\Big\| \sum_{n\in\mathbb{N}_0^d} C_{\gamma,n,\alpha,\beta} \langle f, h_{n+\gamma}\rangle h_n \Big\|_{L^2}^2 &\le K_{\alpha,\beta} 2^{-|\alpha+\beta|} \sum_{n\in\mathbb{N}_0^d} (n+1)^{2|\alpha+\beta|} |\langle f, h_n\rangle|^2 \\ &= K_{\alpha,\beta} 2^{-|\alpha+\beta|} \|f\|_{|\alpha+\beta|}^2\end{aligned}$$

and hence we have

$$\begin{aligned}\|(\cdot)^\alpha \partial^\beta f(\cdot)\|_{L^2} &\le \sum_{-\alpha-\beta\le\gamma\le\alpha+\beta} \Big\| \sum_{n\in\mathbb{N}_0^d} C_{\gamma,n,\alpha,\beta} \langle f, h_{n+\gamma}\rangle h_n \Big\|_{L^2} \\ &\le C_{\alpha,\beta} \|f\|_{|\alpha+\beta|},\end{aligned}$$

for some $C_{\alpha,\beta} > 0$ depending on α, β.

□

With these two estimates we can now finally prove

Theorem 3.1. *Let $\mathcal{S}_p$ be as defined in* (4)*, then*

$$\mathcal{S} = \bigcap_{p\ge 0} \mathcal{S}_p.$$

Furthermore, on $\mathcal{S}$ the topologies generated by the family of semi-norms $\{\|\cdot\|_{\alpha,\beta} \,|\, \alpha,\beta \in \mathbb{N}_0^d\}$ and $\{\|\cdot\|_p \,|\, p \in \mathbb{N}_0\}$ coincide.

Proof. Let $f \in \mathcal{S}$, then by Lemma 3.1, for all $p \ge 0$ there is $\gamma \in \mathbb{N}_0^d$, $C > 0$ such that

$$\|f\|_p \le C \max_{0\le|\alpha|,|\beta|\le|\gamma|+d} \|f\|_{\alpha,\beta}.$$

Hence, $\mathcal{S} \subset \bigcap_{p\ge 0} \mathcal{S}_p$.
For the converse let $f \in \bigcap\limits_{p\ge 0} \mathcal{S}_p \subset L^2$ and let $f^{(N)} := \sum\limits_{n\in\Gamma_N} \langle f, h_n\rangle h_n$ where $\Gamma_N = \{0, ...N\}^d$. Note that $f^{(N)} \in \mathcal{S}$. Then $f^{(N)} \to f$ with respect to any

$\|\cdot\|_p$ and thus $\{f^{(N)}\}_N$ is Cauchy in $\bigcap_{p\geq 0}\mathcal{S}_p$. Now by Lemma 3.3 it is also Cauchy in $\mathcal{S}$. By Proposition 3.1, $\mathcal{S}$ is complete and hence there is $\tilde{f}\in\mathcal{S}$, such that $f^{(N)}\to\tilde{f}$ in $\mathcal{S}$. Again by Lemma 3.1 $f^{(N)}\to\tilde{f}$ with respect to any $\|\cdot\|_p$ and therefore $f=\tilde{f}\in\mathcal{S}$. The equivalence of both topologies follows immediately by Lemma 3.1 and Lemma 3.3.

□

This characterization replaces the properties smoothness and rapid decay of Schwartz function with the finiteness of all $\|\cdot\|_p$-semi-norms. In this way the factor p characterizes smoothness, it forces the coefficients with respect to the Hermite expansion of higher order to decay rapidly. It has the big advantage of a simpler topological structure. As the semi-norms $\|\cdot\|_p$ are decreasing, it makes it easier to characterize neighborhood systems. A neighborhood basis of any $f\in\mathcal{S}$ can now be given by

$$B_p(f,\epsilon):=\{g\in\mathcal{S}\,|\,\|f-g\|_p<r\},\quad p\geq 0,\epsilon>0,$$

instead of the intersection of finitely many sets of this form.

3.2 Tempered Distributions

Before we use Theorem 3.1 to characterize the dual space of $\mathcal{S}$, we briefly want to introduce some relevant distribution theory, since this will be useful for some parts of this thesis. For this we use [38] and [40] as a reference and refer to those books for details.

The space of test functions $\mathcal{D}:=\mathcal{D}(\mathbb{R}^d):=\mathcal{C}_c^\infty(\mathbb{R}^d)$ is defined as the space of infinitely differentiable functions with compact support. Clearly $\mathcal{D}\subset\mathcal{S}$ and it can furthermore be shown that it is dense. It is equipped with the following topology. A sequence $(\varphi_n)\subset\mathcal{D}$ converges to $\varphi\in\mathcal{D}$ iff the supports of the φ_n are contained in a common compact set and if $\partial^\alpha\varphi_n\to\partial^\alpha\varphi$ uniformly as $n\to\infty$ for all multi-indices $\alpha\in\mathbb{N}_0^d$.
Now, the space of distributions $\mathcal{D}'$ is defined as the dual space of the test functions, that is the set of all complex valued linear and continuous functionals $F:\mathcal{D}\to\mathbb{C}$. For $\varphi\in\mathcal{D}$ and $F\in\mathcal{D}'$ we use the notation $\langle F,\varphi\rangle=F(\varphi)$. A distribution is sometimes also called a generalized function by the following observation. If $f\in L^1_{\text{loc}}$, then it can be viewed as a distribution by the following identification,

$$\langle f,\varphi\rangle:=\int_{\mathbb{R}^d} f(x)\varphi(x)\,dx,\quad\forall\varphi\in\mathcal{D}.$$

Thus, in this case it coincides with the usual L^2-pairing. If two locally integrable functions determine the same distributions we say they are equal in the sense of distributions. By integrating, we can also identify Borel measures which are finite on compact sets of $\mathbb{R}^d$ as distributions. The most important example is the Dirac delta distribution defined by $\delta(\varphi) = \varphi(0)$ for all $\varphi \in \mathcal{D}$. A key property of distributions is that we can differentiate them arbitrarily many times. For $\alpha \in \mathbb{N}_0^d$ and $F \in \mathcal{D}$ the derivative $\partial^\alpha F$ is the distribution defined by

$$\langle \partial^\alpha F, \varphi \rangle = (-1)^{|\alpha|} \langle F, \partial^\alpha \varphi \rangle, \quad \forall \varphi \in \mathcal{D}.$$

This generalizes the classical derivative in the sense that if F is given by a (weakly) differentiable function, then by integration by parts, $\partial^\alpha F$ coincides with the (weak) derivative. By the same duality principle we can make sense of LF for any linear and continuous operator $L : \mathcal{D} \to \mathcal{D}$. In particular we have $\langle LF, \varphi \rangle = \langle F, L^* \varphi \rangle$ for all $\varphi \in \mathcal{D}$. An important example is that of a translation, we write $\langle F(\cdot + \tau), \varphi \rangle = \langle F, \varphi(\cdot - \tau) \rangle$ for all $\varphi \in \mathcal{D}$ and $\tau \in \mathbb{R}^d$.

As $\mathcal{D} \subset \mathcal{S}$, we can identify the dual of $\mathcal{S}$ as a subspace of distributions $\mathcal{S}' \subset \mathcal{D}'$, in the sense that every element of $\mathcal{S}'$ restricted to $\mathcal{D}$ is a linear and continuous functional. These are referred to as tempered distributions. To determine if a distribution given by some locally integrable function is tempered, we have the following useful condition. If $f \in L^1_{loc}$ and for some $N \geq 0$ we have

$$\int_{|x|<R} |f(x)| \, dx = O(R^N), \quad \text{as } N \to \infty, \tag{6}$$

then the corresponding distribution is tempered. Concerning the topological structure on $\mathcal{S}'$ it is natural to consider two different topologies. Firstly the *strong topology* τ_s generated by the family of semi-norms

$$q_B(F) = \sup_{f \in B} |\langle F, f \rangle|, \quad B \subset \mathcal{S} \text{ bounded}.$$

A subset B of a topological vector space is called bounded if for all neighborhoods V of 0, there exists $\lambda > 0$ such that $\lambda B \subset V$. Secondly the *weak-* topology* $\sigma(\mathcal{S}', \mathcal{S})$ inherited by the dual structure. It is induced by the family of semi-norms

$$p_\varphi(F) = |\langle F, \varphi \rangle|, \quad \varphi \in \mathcal{S}.$$

Note that in [6] this topology is referred to as weak topology. Since every singleton is bounded it is clear that the strong topology is finer than the

weak-* one.
Now let us get back to our characterization of $\mathcal{S} = \bigcap_{p\geq 0} \mathcal{S}_p$ as the intersection of decreasing spaces $\mathcal{S}_p$. If we denote the dual of $\mathcal{S}_p$ by $\mathcal{S}_p'$ then we can decompose the space of tempered distributions accordingly into

$$\mathcal{S}' = \bigcup_{p\geq 0} \mathcal{S}_p'.$$

As each $\mathcal{S}_p$ is a Hilbert space, we can consider the dual norm given by

$$\|F\|_p' := \sup_{\|f\|_p\leq 1} |\langle F, f\rangle|, \quad \forall F \in \mathcal{S}_p'.$$

4 Multi-sequence Representation of $\mathcal{S}$ and $\mathcal{S}'$

In the previous chapters we have set up the characterization of $\mathcal{S}$ and $\mathcal{S}'$ by the sequences of spaces $\mathcal{S}_p$ and their duals. In this chapter we will see why this will be so powerful to us. First we set up an equivalent structure in the multi-dimensional sequence space $\mathbb{R}^{\mathbb{N}_0^d}$, which will allow us to view $\mathcal{S}$ and $\mathcal{S}'$ as subspaces of $\mathbb{R}^{\mathbb{N}_0^d}$. For one, this gives us a simpler and concrete description of these spaces and more importantly this enables us to transfer results from $\mathbb{R}^d$ to this infinite dimensional setting. This will be the key to prove Minlos-Bochner Theorem later. The multi-sequence representation was introduced by Reed and Simon [33] and as in [6], we will follow this approach closely.

We will think of $\mathbb{R}^{\mathbb{N}_0^d}$ as the space of real valued sequences indexed by the multidimensional index set $\mathbb{N}_0^d$. For two multi-sequences $a = (a_n)_{n\in\mathbb{N}_0^d}, b = (b_n)_{n\in\mathbb{N}_0^d} \in \mathbb{R}^{\mathbb{N}_0^d}$ such that $\sum_{n\in\mathbb{N}_0^d} |a_n b_n| < \infty$ we define the pairing

$$\langle a, b\rangle := \sum_{n\in\mathbb{N}_0^d} a_n b_n. \tag{7}$$

In the same way as before, we can introduce the multi-sequence analogue of the spaces $\mathcal{S}_p$. For $a = (a_n)_{n\in\mathbb{N}_0^d} \in \mathbb{R}^{\mathbb{N}_0^d}$ and $p \in \mathbb{Z}$ we define the $\|\cdot\|_p$-norm as

$$\|a\|_p^2 := \sum_{n\in\mathbb{N}_0^d} (1+n)^{2p} a_n^2$$

and we denote the set of multi-sequence with finite $\|\cdot\|_p$-norm as

$$\mathbb{S}_p := \{a \in \mathbb{R}^{\mathbb{N}_0^d} : \|a\|_p < \infty\}, \quad \text{for } p \in \mathbb{Z}.$$

Clearly, $\mathbb{S}_p$ equipped with the norm $\|\cdot\|_p$ is a Hilbert space with respect to the inner product $\langle a, b\rangle_p = \sum_{n\in\mathbb{N}_0^d}(1+n)^{2p}a_n b_n$. Again the spaces are decreasing, i.e., $\mathbb{S}_{p+1} \subset \mathbb{S}_p$ for $p \geq 0$. So it is natural to define

$$\mathbb{S} := \bigcap_{p\geq 0} \mathbb{S}_p.$$

The space $\mathbb{S}$ is then given the topology generated by the family of Hilbert norms $\{\|\cdot\|_p : p \geq 0\}$.
We can make the following easy observation. Let $a \in \mathbb{S}_p$ $p \in \mathbb{Z}$, then there exists $C_p > 0$ such that

$$|a_n| \leq C_p(1+n)^{-p}, \quad \forall n \in \mathbb{N}_0^d. \tag{8}$$

With this we can see that the multi-sequences in $\mathbb{S}$ have "rapid decay", i.e. property (8) is satisfied for every $p \geq 0$.

Lemma 4.1. *Let $a = (a_n)_{n\in\mathbb{N}_0^d} \in \mathbb{R}^{\mathbb{N}_0^d}$. Then $a \in \mathbb{S}$ if and only if for all $p \in \mathbb{N}_0$ there is some $C_p > 0$ such that for all $n \in \mathbb{N}_0^d$, $|a_n| \leq C_p(1+n)^{-p}$.*

Proof. If $a \in \mathbb{S}$, then for $p \in \mathbb{N}_0$ we have for all $n \in \mathbb{N}_0^d$

$$|a_n|(1+n)^p \leq \|a\|_p < \infty.$$

For the converse direction, let $p \in \mathbb{N}_0$ and $(a_n)_{n\in\mathbb{N}_0^d} \in \mathbb{R}^{\mathbb{N}_0^d}$ such that 8 holds for all $p' \geq 0$. Choose $p' = p+1$, then

$$\|a\|_p^2 = \sum_{n\in\mathbb{N}_0^d}(1+n)^{2p}a_n^2 \leq C_{p'}^2 \sum_{n\in\mathbb{N}_0^d}(1+n)^{-2} < \infty$$

and so $a \in \mathbb{S}$.

□

This Lemma gives us a description of $\mathbb{S}$ similar to the Schwartz space $\mathcal{S}$. It exactly contains those multi-sequences which have a faster decay than any polynomial.
To describe its dual space we proceed similar. We denote by $\mathbb{S}_p'$ the dual space of $\mathbb{S}_p$ and it is given the dual norm

$$\|F\|_p' = \sup_{\|a\|_p\leq 1} |(F, a)|, \quad \text{for } F \in \mathbb{S}_p \text{ and } \forall a \in \mathbb{S}_p.$$

Here we use the notation $(\cdot,\cdot) : \mathbb{S}_p' \times \mathbb{S}_p \to \mathbb{C}$ for the bilinear dual pairing, since we used the angled brackets $\langle\cdot,\cdot\rangle$ to define the pairing 7. However, we will see that they actually agree. The dual space of $\mathbb{S}$ is then given as

$$\mathbb{S}' = \bigcup_{p\geq 0} \mathbb{S}_p'.$$

In [6, Lemma 3.1] it is shown that we can identify the spaces $\mathbb{S}_p'$ and $\mathbb{S}_{-p}$. In particular, the map $b \mapsto \langle b,\cdot\rangle$ is an isometry from $\mathbb{S}_{-p}$ to $\mathbb{S}_p'$. By this we have

$$\mathbb{S}' = \bigcup_{p\geq 0} \mathbb{S}_{-p} \subset \mathbb{R}^{\mathbb{N}_0^d}.$$

From (8) we get that for $b \in \mathbb{S}'$ there exists $C > 0$, $N \in \mathbb{N}$ s.t.

$$|b_n| \leq C(1+n)^N, \quad \forall n \in \mathbb{N}_0^d. \tag{9}$$

Analogously to Lemma 4.1 we get the description of $\mathbb{S}'$ of those sequences of at most polynomial growth.

Lemma 4.2 ([6, Lemma 3.2])**.** *Let $b = (b_n)_{n\in\mathbb{N}_0^d} \in \mathbb{R}^{\mathbb{N}_0^d}$. Then $b \in \mathbb{S}'$ if and only if there exists $p \in \mathbb{N}$ and some $C > 0$ such that for all $n \in \mathbb{N}_0^d$, $|b_n| \leq C(1+n)^p$.*

By (8) and (9) the pairing $\langle b,a\rangle$ is always well defined for $a \in \mathbb{S}$, $b \in \mathbb{S}'$ and by the identification of $\mathbb{S}_p'$ with $\mathbb{S}_{-p}$ it coincides with the duality bracket (b,a). In a slight abuse of notation, we will from now on use the angle bracket notation $\langle\cdot,\cdot\rangle$ for the pairing $\mathcal{S}' \times \mathcal{S} \to \mathbb{C}$ as well as for the pairing $\mathbb{S}' \times \mathbb{S} \to \mathbb{C}$. The spaces $\mathcal{S}$ and $\mathbb{S}$ were constructed in analogous fashion. It is therefore reasonable to assume their equivalence. In [6, Proposition 3.3 and 3.4] Biermé, Durieu and Wang show that they are actually topological isomorphic. We state their results as

Proposition 4.1 ([6, Proposition 3.3])**.** *The map*

$$\begin{aligned}\Phi : \mathcal{S} &\longrightarrow \mathbb{S}\\ f &\longmapsto (\langle f, h_n\rangle)_{n\in\mathbb{N}_0^d}\end{aligned}$$

is a topological isomorphism, i.e. a linear isomorphism and a homeomorphism.

and for the dual spaces we have

Proposition 4.2 ([6, Proposition 3.4]). *The map*

$$\begin{aligned} \Psi : \mathcal{S}' &\longrightarrow \mathbb{S}' \\ F &\longmapsto (\langle F, h_n \rangle)_{n \in \mathbb{N}_0^d} \end{aligned}$$

is a linear isomorphism.

Next we want to introduce a strong and a weak-* topology on $\mathbb{S}'$ analogously to those on $\mathcal{S}'$ to turn this identification into a topological isomorphism. We follow the approach of [6] but provide more details to make this rigorous. It is easy to see that Ψ preserves the dual pairing.

Lemma 4.3. *Let $F \in \mathcal{S}'$, $f \in \mathcal{S}$, then*

$$\langle F, f \rangle = \langle \Psi(F), \Phi(f) \rangle = \sum_{n \in \mathbb{N}_0^d} \Psi(F)_n \Phi(f)_n$$

where Ψ and Φ denote the maps from Proposition 4.1 and 4.2.

Proof. By continuity of F and expansion of f with respect to the ONB of hermitian functions we get

$$\langle \Psi(F), \Phi(f) \rangle = \sum_{n \in \mathbb{N}_0^d} \langle f, h_n \rangle (F, h_n) = F \left(\sum_{n \in \mathbb{N}_0^d} \langle f, h_n \rangle h_n \right) = \langle F, f \rangle.$$

□

On $\mathbb{S}'$ we introduce the *strong topology* π_s generated by the family of semi-norms

$$q_B(\cdot) = \sup_{a \in B} |\langle \cdot, a \rangle|, \quad B \text{ bounded in } \mathbb{S}.$$

We already know that $\mathcal{S}$ and $\mathbb{S}$ are topologically isomorphic, in particular bounded sets in $\mathcal{S}$ correspond to bounded sets in $\mathbb{S}$. By Lemma 4.3 it is then clear that these semi-norms correspond to those which generate the strong topology τ_s on $\mathcal{S}$ and hence $\Psi : (\mathcal{S}', \tau_s) \to (\mathbb{S}', \pi_s)$ is a topological isomorphism.
Again by duality we introduce the *weak-* topology* π_w on $\mathbb{S}'$ generated by the semi-norms $|\langle \cdot, a \rangle|$, $a \in \mathbb{S}$. By Lemma 4.3 it is clear that $(\mathcal{S}', \sigma(\mathcal{S}', \mathcal{S}))$ and $(\mathbb{S}', \pi_w)$ are topologically isomorphic too. In the sequel the strong topology will always be considered on $\mathbb{S}'$ if not mentioned otherwise.

The main advantage of this multi-sequence representation is the following. Classically, the Schwartz space is described by two properties, smoothness and rapid decay. In the multi-sequence space $\mathbb{S}$ however, these two properties are now replaced by the single rapid decay property (Lemma 4.1). Furthermore we can view $\mathbb{S}$ and $\mathbb{S}'$ as subspaces of $\mathbb{R}^{\mathbb{N}_0^d}$. The objects in these spaces are much easier to handle. For example it can easily be seen that $\mathbb{S}, \mathbb{S}'$ (see [6, Proposition 3.5]) are separable. Therefore the same must be true for $\mathcal{S}$ and $\mathcal{S}'$. This idea captures the spirit to show Minlos-Bochner and Lévy's continuity theorem. We establish a result in the multi-sequence formulation and then use the topological isomorphism Φ and Ψ to transfer it to $\mathcal{S}$ and $\mathcal{S}'$.

4.1 Borel and Cylinder σ-Algebras

To define random variables with values in $\mathbb{S}'$ ($\mathcal{S}'$ respectively) we need some measurability structure on $\mathbb{S}'$ and $\mathbb{S}$. For this we follow [6, Section 3.3] and give more details to appropriate parts.
Since $\mathbb{S}$ and $\mathbb{S}'$ are subspaces of $\mathbb{R}^{\mathbb{N}_0^d}$, we start with $\mathbb{R}^{\mathbb{N}_0^d}$. The Borel σ-algebra associated to the product topology is denoted by $\mathcal{B}(\mathbb{R}^{\mathbb{N}_0^d})$. From basic measure theory ([23, Lemma 1.2]) we have

$$\mathcal{B}(\mathbb{R}^{\mathbb{N}_0^d}) = \bigotimes_{n \in \mathbb{N}_0^d} \mathcal{B}(\mathbb{R}).$$

By definition this coincides with the σ-algebra generated by the sets $\{a \in \mathbb{R}^{\mathbb{N}_0^d} : (a_i)_{i \in \Gamma} \subset B\}$ for all finite subsets Γ of $\mathbb{N}_0^d$ and all Borel sets $B \in \mathcal{B}(\mathbb{R}^{\#\Gamma})$.
On $\mathbb{S}$ we introduce the cylinder σ-algebra $\mathcal{C}(\mathbb{S})$ as the σ-algebra generated by the cylinder sets

$$\{a \in \mathbb{S} : (\langle b_1, a\rangle, ..., \langle b_m, a\rangle) \in B\}$$

for $m \geq 1$, $b_1, ..., b_m \in \mathbb{S}'$ and $B \in \mathcal{B}(\mathbb{R}^m)$. We can make the following observation

Lemma 4.4. *$\mathcal{C}(\mathbb{S})$ is the σ-algebra induced by $\mathcal{B}(\mathbb{R}^{\mathbb{N}_0^d})$ on $\mathbb{S}$, i.e.*

$$\mathcal{C}(\mathbb{S}) = \{\mathbb{S} \cap A : A \in \mathcal{B}(\mathbb{R}^{\mathbb{N}_0^d})\}.$$

Proof. For a general set X, a subset $S \subset X$ and a family of subsets $\mathcal{G}$ we have

$$\sigma(S \cap \mathcal{G}) = S \cap \sigma(\mathcal{G}).$$

Hence, if we denote the σ-algebra induced by $\mathcal{B}(\mathbb{R}^{\mathbb{N}_0^d})$ on $\mathbb{S}$ by $\mathcal{A}$, we see that $\mathcal{A}$ is the σ-algebra generated by the sets $\{a \in \mathbb{S} : (a_i)_{i\in\Gamma} \subset B\}$ for all finite subsets Γ of $\mathbb{N}_0^d$ and all Borel sets $B \in \mathcal{B}(\mathbb{R}^{\#\Gamma})$.
For $\Gamma \subset \mathbb{N}_0^d$ finite, $B \in \mathcal{B}(\mathbb{R}^{\#\Gamma})$ and $i \in \Gamma$ let $b_i = (\delta_{i,j})_{j\in\mathbb{N}_0^d}$. Then

$$\{a \in \mathbb{S} : (a_i)_{i\in\Gamma} \subset B\} = \{a \in \mathbb{S} : (\langle b_i, a\rangle)_{i\in\Gamma} \in B\} \in \mathcal{C}(\mathbb{S})$$

and therefore $\mathcal{A} \subset \mathcal{C}(\mathbb{S})$.
For the other inclusion we have to show that for $b \in \mathbb{S}'$ the function

$$\begin{aligned} \langle b, \cdot\rangle : (\mathbb{S}, \mathcal{A}) &\longrightarrow \mathbb{R} \\ a &\longmapsto \langle b, a\rangle \end{aligned}$$

is measurable. So let $m \in \mathbb{N}$ and define $b^m \in \mathbb{S}'$ by

$$b_i^m = \begin{cases} b_i, & \text{if } i \in \Gamma_m = \{0, ...m\}^d \\ 0, & \text{else} \end{cases}.$$

By (9) there is $C > 0, N \in \mathbb{N}$ s.t.

$$|b_n| \leq C(1+n)^N, \quad \forall n \in \mathbb{N}_0^d.$$

Now let $p_0 \in \mathbb{N}$ such that $\sum_{n\in\mathbb{N}^d}(1+n)^{N-p_0}$ converges. Then for $a \in \mathbb{S}$ there is $C_{p_0} > 0$ s.t.

$$|b_n a_n| \leq C C_{p_0}(1+n)^{N-p_0}, \quad \forall n \in \mathbb{N}_0^d.$$

and therefore for the sum $\sum_{n\in\mathbb{N}_0^d} a_n b_n$ is absolutely converging and by dominated convergence we get

$$\langle b, a\rangle = \sum_{n\in\mathbb{N}_0^d} a_n b_n = \lim_{m\to\infty} \sum_{n\in\mathbb{N}_0^d} a_n b_n^m = \lim_{m\to\infty} \langle b^m, a\rangle.$$

Thus, the function $\langle b, \cdot\rangle$ is the point-wise limit of $(\langle b^m, \cdot\rangle)_{m\in\mathbb{N}}$ and so it is enough to prove that $\langle b, \cdot\rangle$ is measurable, where b has only finitely many nonzero entries. But for $b = (\delta_{i,j})_{j\in\mathbb{N}_0^d}$ for some $i \in \mathbb{N}_0^d$, $\langle b, \cdot\rangle$ is measurable since

$$(\langle b, \cdot\rangle)^{-1}(B) = \{a \in \mathbb{S} : \langle b, a\rangle \in B\} = \{a \in \mathbb{S} : a_i \in B\} \in \mathcal{A}$$

for any $B \in \mathcal{B}(\mathbb{R})$. And since $\langle b, \cdot\rangle$ is linear in b and the finite sum of measurable functions is again measurable, we are done. $\square$

On $\mathbb{S}'$ the cylinder σ-algebra $\mathcal{C}(\mathbb{S}')$ is generated by the cylinder sets

$$\{b \in \mathbb{S}' : (\langle b, a_1 \rangle, ..., \langle b, a_m \rangle) \in B\}$$

for $m \geq 1$, $a_1, ..., a_m \in \mathbb{S}$ and $B \in \mathcal{B}(\mathbb{R}^m)$. Again we have the following identification.

Lemma 4.5. *$\mathcal{C}(\mathbb{S}')$ is the σ-algebra induced by $\mathcal{B}(\mathbb{R}^{\mathbb{N}_0^d})$ on $\mathbb{S}'$.*

Proof. The proof works identically to the proof of Lemma 4.4.

□

For the topologies on $\mathcal{S}$ and $\mathbb{S}$ we denote the corresponding Borel σ-algebras as $\mathcal{B}(\mathcal{S})$ and $\mathcal{B}(\mathbb{S})$. Further for the duals $\mathcal{S}'$, $\mathbb{S}'$ equipped with the strong topology τ_s, π_s respectively, the Borel σ-algebras are denoted by $\mathcal{B}(\mathcal{S}')$ and $\mathcal{B}(\mathbb{S}')$. With the topological isomorphism Φ and Ψ form Proposition 1 and 2 we have

$$\Phi^{-1}(\mathcal{B}(\mathbb{S})) = \mathcal{B}(\mathcal{S}) \text{ and } \Psi^{-1}(\mathcal{B}(\mathbb{S}')) = \mathcal{B}(\mathcal{S}').$$

Furthermore in [6, Proposition 3.7 and 3.8] it is shown, that these Borel σ-algebras corresponding to the strong topologies actually coincide with the cylinder σ-algebras, i.e.

$$\mathcal{C}(\mathbb{S}) = \mathcal{B}(\mathbb{S}) \text{ and } \mathcal{C}(\mathbb{S}') = \mathcal{B}(\mathbb{S}').$$

Moreover we can relate the cylinder σ-algebra $\mathcal{C}(\mathbb{S}')$ with the weak-* topology.

Lemma 4.6. *The cylinder σ-algebra $\mathcal{C}(\mathbb{S}')$ coincides with the Borel σ-algebra corresponding to the weak-* topology π_w on $\mathbb{S}'$.*

Proof. To see that $\mathcal{C}(\mathbb{S}')$ is contained in the Borel σ-algebra corresponding to π_w, we have to show that the functions

$$\begin{aligned} \langle \cdot, a \rangle : (\mathbb{S}', \pi_w) &\longrightarrow \mathbb{R} \\ b &\longmapsto \langle b, a \rangle \end{aligned}$$

for $a \in \mathbb{S}$ are measurable. But they are clearly continuous by definition of the weak-* topology π_w.
To show the opposite inclusion we use the fact that $\mathcal{C}(\mathbb{S}') = \mathcal{B}(\mathbb{S}')$. By definition of the topologies we have

$$\pi_w \subset \pi_s \subset \mathcal{B}(\mathbb{S}') = \mathcal{C}(\mathbb{S}').$$

Since π_w is stable under finite intersections we get ([23, Theorem 1.1]) that the Borel σ-algebra generated by π_w is contained in $\mathcal{C}(\mathbb{S}')$.

□

This shows that while on $\mathbb{S}'$ (and therefore also $\mathcal{S}'$) the weak-* and the strong topologies may be different, the corresponding Borel σ-algebras agree (see also [6, Corollary 3.9]).
The importance of this result will become clear when we define generalized random fields. The σ-algebra on $\mathbb{S}'$ should capture its topological structure, hence we are interested in the Borel σ-algebra. However, since $\mathcal{C}(\mathbb{S}) = \mathcal{B}(\mathbb{S})$ we can also describe measurable sets by duality on finite-dimensional projections.

4.2 Montel Property

In [6] it is shown that $\mathbb{S}'$ (and therefore $\mathcal{S}'$) is a Montel space, that is strongly closed and strongly bounded subsets are strongly compact. This result is important to show Lévy's continuity theorem. However, for this thesis it is not relevant. Still, we want to provide more details to [6, Proposition 6.2] and [6, Lemma 6.3]. For the later we present a slightly different and more rigorous proof. The result follows from a couple of steps.
The first step shows that $\mathcal{S}$ (and so $\mathbb{S}$) is a Montel space.

Proposition 4.3 ([6, Proposition 6.2])**.** *In $\mathcal{S}$ (and hence in $\mathbb{S}$), the bounded subsets are relatively compact.*

The main tool for this proof is Arzelà-Ascoli's Theorem:

Theorem 4.1 (Arzelà-Ascoli, [34])**.** *Let X be a compact Hausdorff space and*
$F \subset C(X)$. Then F is relatively compact if and only if F is point-wise bounded (i.e. $\forall x \in X : \sup_{f\in F} |f(x)| < \infty$) and equicontinuous (i.e. $\forall x \in X$ and $\epsilon > 0$ there exists a neighborhood U_x of x s.t. $\forall y \in U_x, f \in F : |f(x) - f(y)| < \epsilon$).

Proof of Proposition 4.3. As $\mathcal{S}$ is metrizable, it suffices to show sequential compactness. To apply Arzelà-Ascoli's theorem we need to restrict the functions to compact sets. So let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{S}$ be bounded and $(K_l)_{l\in\mathbb{N}}$ be a sequence of compact subsets of $\mathbb{R}^d$ such that $K_l \subset K_{l+1}$ and $\bigcup_{l\in\mathbb{N}} K_l = \mathbb{R}^d$. We can choose $K_l = \overline{B_l(0)}$.
Since $(f_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{S}$, it is bounded with respect to each semi-norm $\|\cdot\|_{\alpha,\beta}$, for $\alpha, \beta \in \mathbb{N}_0^d$, uniformly in n. Hence, for fixed $\alpha, \beta \in \mathbb{N}_0^d$ the functions $x \mapsto x^\alpha \partial^\beta f_n(x)$, $n \in \mathbb{N}$, are uniformly bounded and therefore point-wise too.
Next we show that $x \mapsto x^\alpha \partial^\beta f_n(x)$, $n \in \mathbb{N}$, is equicontinuous. Let $x \in \mathbb{R}^d$ and $\epsilon > 0$. We denote $h^n_{\alpha,\beta}(x) := x^\alpha \partial^\beta f_n(x)$. Then by the mean value theorem

there exists $\lambda \in [0,1]$ such that for all $y \in \mathbb{R}^d$

$$\begin{aligned}|h^n_{\alpha,\beta}(x) - h^n_{\alpha,\beta}(y)| &= |\nabla h^n_{\alpha,\beta}((1-\lambda)x + \lambda y)(x-y)| \\ &\leq |\nabla h^n_{\alpha,\beta}((1-\lambda)x + \lambda y)|\,|x-y| \\ &\leq C_{\alpha,\beta}|x-y|\end{aligned}$$

where $C_{\alpha,\beta} \geq 0$ is a constant which, by boundedness of $(f_n)_{n\in\mathbb{N}}$, can be chosen independently of n. So for $y \in \mathbb{R}^d$ such that $|x-y| < \epsilon / C_{\alpha,\beta}$ we have $|h^n_{\alpha,\beta}(x) - h^n_{\alpha,\beta}(y)| < \epsilon$.
If we restrict $(h^n_{\alpha,\beta})_{n\in\mathbb{N}}$ to K_1 we can apply Arzelà-Ascoli to extract a subsequence that converges uniformly on K_1. We can do this for every $\alpha, \beta \in \mathbb{N}_0^d$. By a diagonal extraction we can then find a subsequence $(f_{\varphi(n)})_{n\in\mathbb{N}}$ for which $(\cdot)^\alpha \partial^\beta f_{\varphi(n)}(\cdot)$ converges uniformly on K_1 for all $\alpha, \beta \in \mathbb{N}_0^d$. Repeating this procedure for K_l, $l \in \mathbb{N}$, and then once again by another diagonal extraction we find a subsequence $(f_{\psi(n)})_{n\in\mathbb{N}}$ such that $(\cdot)^\alpha \partial^\beta f_{\psi(n)}(\cdot)$ converges for all $\alpha, \beta \in \mathbb{N}_0^d$ and K_l, $l \in \mathbb{N}$. Now we claim that this subsequence converges with respect to each semi-norm $\|\cdot\|_{\alpha,\beta}$ and therefore in $\mathcal{S}$. To see this let $\alpha, \beta \in \mathbb{N}_0^d$ and $n_0 \in \mathbb{N}$, then

$$\begin{aligned}\|f_{\psi(n)} - f_{\psi(m)}\|_{\alpha,\beta} \leq &\sup_{x\in K_{n_0}} |x^\alpha \partial^\beta (f_{\psi(n)} - f_{\psi(m)})| \\ &+ \sup_{x\in\mathbb{R}^d\setminus K_{n_0}} \frac{1+|x^\alpha|}{1+|x^\alpha|} |x^\alpha \partial^\beta (f_{\psi(n)} - f_{\psi(m)})| \\ \leq &\sup_{x\in K_{n_0}} |x^\alpha \partial^\beta (f_{\psi(n)} - f_{\psi(m)})| \\ &+ \left(\sup_{x\in\mathbb{R}^d\setminus K_{n_0}} \frac{1}{1+|x^\alpha|} \right) \\ &\quad (\|f_{\psi(n)}\|_{\alpha,\beta} + \|f_{\psi(m)}\|_{\alpha,\beta} + \|f_{\psi(n)}\|_{2\alpha,\beta} + \|f_{\psi(m)}\|_{2\alpha,\beta}).\end{aligned}$$

The last term is bounded independently of n, m by the boundedness of $(f_{\psi(n)})$ in $\mathcal{S}$. We can then choose n_0 such that $\sup_{x\in\mathbb{R}^d\setminus K_{n_0}} 1/(1+|x^\alpha|)$ is arbitrary small. The first term goes to zero as $n, m \to \infty$ by the properties of the extracted subsequences $(f_{\psi(n)})$.

□

Now we turn our attention to [6, Lemma 6.3]. We will state it and then give an adjusted proof.

Lemma 4.7 ([6, Lemma 6.3]). *If $H \subset \mathbb{S}'$ is closed and bounded in the weak-* topology π_w, then it is compact with respect to π_w.*

Proof. Let us consider the map

$$\begin{aligned} \Phi : (\mathbb{S}', \pi_w) &\longrightarrow \mathbb{R}^{\mathbb{S}} \\ b &\longmapsto (\langle b, a\rangle)_{a\in\mathbb{S}} \end{aligned}$$

where we assume the product topology on $\mathbb{R}^{\mathbb{S}}$. Then Φ is clearly injective. Furthermore it is a homeomorphism onto its image since

$$b^n \to b \quad \text{in } (\mathbb{S}', \pi_w) \Leftrightarrow \langle b^n, a\rangle \to \langle b, a\rangle \quad \forall a \in \mathbb{S} \Leftrightarrow \Phi(b^n) \to \Phi(b) \quad \text{in } \mathbb{R}^{\mathbb{S}}.$$

Thus, for $H \subset \mathbb{S}'$ weak-* closed and bounded, it suffices to show that $\Phi(H)$ is compact in the product topology.
By the boundedness of H in π_w, for all $a \in \mathbb{S}$ there is $\lambda_a > 0$ such that

$$|\langle b, a\rangle| < \lambda_a \quad \forall b \in H.$$

We then have

$$\Phi(H) = \{(\langle b, a\rangle)_{a\in\mathbb{S}} \mid b \in H\} \subset \prod_{a\in\mathbb{S}} \{\langle b, a\rangle \mid b \in H\} \subset \prod_{a\in\mathbb{S}} [-\lambda_a, \lambda_a].$$

By Tychonoff's Theorem we know that $\prod_{a\in\mathbb{S}}[-\lambda_a, \lambda_a]$ is compact in $\mathbb{R}^{\mathbb{S}}$. Here we use the fact that the product topology of the subspace topology is the subspace topology of the product topology. Therefore, as a closed subset of a compact set, we conclude that $\Phi(H)$ is compact itself. $\square$

With this it is now not very difficult to show that $\mathbb{S}'$ is Montel space [6, Proposition 6.1].

5 Generalized Random Fields

The concept of general random fields was originally introduced as a way of generalizing the notion of a stochastic process. It can be seen as a collection of real random variables $X = (X(f))_{f\in\mathcal{K}}$, indexed by some topological vector space of real-valued functions $\mathcal{K}$. It is required to be linear in the sense that for all $n \geq 1, a_1, ..., a_n \in \mathbb{R}, f_1, ..., f_n \in \mathcal{K}$,

$$X\left(\sum_{i=1}^{n} a_i f_i\right) = \sum_{i=1}^{n} a_i X(f_i), \tag{10}$$

and continuous in the sense that if $f_n \to f$ in $\mathcal{K}$ then $X(f_n) \to X(f)$ in distribution. Usually, the space of the test functions $\mathcal{D}$ is used as the underlying function space (see[18]) or, in our case, the larger Schwartz space $\mathcal{S}$. The motivation behind this idea is that a generalized stochastic process does not have a point-wise representation as a random variable in time, but instead as a (noisy) measurement through some "observation window" $\varphi \in \mathcal{K}$.

5.1 Generalized Random Fields on $\mathcal{S}'$

We follow the approach of [6] to define generalized random fields. One can also view this object as a random element in the corresponding dual space. This leads to the following definition.

Definition 5.1. *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. A generalized random field on $\mathcal{S}'$ is a random variable defined on $(\Omega, \mathcal{A}, \mathbb{P})$ with values in $(\mathcal{S}', \mathcal{B}(\mathcal{S}'))$, i.e. a measurable map $X : (\Omega, \mathcal{A}) \to (\mathcal{S}', \mathcal{B}(\mathcal{S}'))$.*

Since a generalized random field X is a random element of $\mathcal{S}'$ we can form the pairing $X(f) = \langle X, f\rangle$, which is a real-valued random variable for all $f \in \mathcal{S}$. However, if we fix some $\omega \in \Omega$, the map $f \in \mathcal{S} \mapsto \langle X, f\rangle(\omega)$ is a linear and continuous functional and thus an element of $\mathcal{S}'$.
Furthermore, by duality, we can define the derivative of a generalized random field in a distributional sense. So for $\alpha \in \mathbb{N}_0^d$, the partial derivative of order α of X is defined by $\langle \partial^\alpha X, f\rangle = \langle X, (-1)^{|\alpha|}\partial^\alpha f\rangle$ for all $f \in \mathcal{S}$. The same concept obviously applies to any linear operator on $\mathcal{S}$, such as the Fourier transform or a translation. More generally, let $L : \mathcal{S} \to \mathcal{S}$ be a linear operator, then we can make sense of LX as

$$\langle LX, f\rangle = \langle X, L^* f\rangle \quad \forall f \in \mathcal{S}. \tag{11}$$

However, we will only use this concept when we define generalized stochastic differential equations and there only the case of a linear differential operator is of interest to us.
Our main interest lies in the law of generalized random fields. So the probability measure $\mathbb{P}_X = \mathbb{P} \circ X^{-1}$ on $(\mathcal{S}', \mathcal{B}(\mathcal{S}'))$ induced by X:

$$\mathbb{P}_X(B) = \mathbb{P}(\{X \in B\}) = \mathbb{P}(\{\omega \in \Omega \mid \langle X, \cdot\rangle(\omega) \in B\}), \quad \forall B \in \mathcal{B}(\mathcal{S}').$$

The central tool we use to determine its law is the so-called *characteristic functional*. We want to motivate this by giving the analog to the real-valued case. Let Y be a $\mathbb{R}^d$-valued random variable on $(\Omega, \mathcal{A}, \mathbb{P})$. It is uniquely determined by its characteristic function $\mathcal{L}_Y : \mathbb{R}^d \to \mathbb{C}$ given by

$$\mathcal{L}_Y(x) = \mathbb{E}[e^{i\langle Y,x\rangle}] = \int_\Omega e^{\mathrm{i}\langle x,y\rangle}\, d\mathbb{P}(y), \quad x \in \mathbb{R}^d.$$

It is continuous, normalized ($\mathcal{L}_Y(0) = 1$) and positive definite, i.e. for all $n \geq 1$ and all $c_1, ..., c_n \in \mathbb{C}$, $x_1, ..., x_n \in \mathbb{R}^d$, $\sum_{i,j=1}^n c_i \bar{c}_j \mathcal{L}_Y(x_i - x_j) \geq 0$. Conversely, by Bochner's theorem [36, Proposition 2.5], if $\mathcal{L} : \mathbb{R}^d \to \mathbb{C}$ is positive-definite, continuous at zero and normalized, then it is the characteristic function of some $\mathbb{R}^d$-valued random variable. The aim in the following is to translate this result to generalized random fields.

Definition 5.2. *Let X be a generalized random field on $\mathcal{S}'$ with law $\mathbb{P}_X$. Its characteristic functional $\mathcal{L}_X : \mathcal{S} \to \mathbb{C}$ is defined by*

$$\mathcal{L}_X(f) := \mathbb{E}[e^{i\langle X,f\rangle}] = \int_{\mathcal{S}'} e^{i\langle F,f\rangle}\, d\mathbb{P}_X(F), \quad f \in \mathcal{S}. \tag{12}$$

The characteristic functional of a generalized random field X is useful, because it contains the information of all finite-dimensional laws of the collection $(\langle X, f\rangle)_{f\in\mathcal{S}}$. To see this let $f_1, ..., f_n \in \mathcal{S}$, then the characteristic function of the random vector $Y = (\langle X, f_1\rangle, ..., \langle X, f_n\rangle)$ is given by

$$\begin{aligned}\mathcal{L}_Y(x) &= \mathbb{E}[e^{\mathrm{i}\langle Y,x\rangle}] = \mathbb{E}[e^{\mathrm{i}(\langle X,x_1 f_1\rangle+...+\langle X,x_n f_n\rangle)}]\\ &= \mathbb{E}[e^{\mathrm{i}(\langle X,x_1 f_1+...+x_n f_n\rangle)}] = \mathcal{L}_X(x_1 f_1 + ... + x_n f_n).\end{aligned}$$

Thus, we can also see why (12) is the natural analog to the characteristic function in the classical sense. We say a functional $\mathcal{L} : \mathcal{S} \to \mathbb{C}$ is *positive-definite* if for all $n \geq 1$, $c_1, ..., c_n \in \mathbb{C}$ and $f_1, ..., f_n \in \mathcal{S}$ we have that

$$\sum_{i,j=1}^{n} c_i \bar{c}_j \mathcal{L}(f_i - f_j) \geq 0. \tag{13}$$

In the following Proposition we summarize the main properties of the characteristic functional.

Proposition 5.1. *The characteristic functional $\mathcal{L}_X$ of a generalized random field X is normalized ($\mathcal{L}_X(0) = 1$), continuous and positive-definite.*

Proof. That $\mathcal{L}_X$ is normalized is clear by definition. To show continuity, let $f_n \to f$ in $\mathcal{S}$. Then $\langle X, f_n\rangle \to \langle X, f\rangle$ a.s. and so by a simple application of Dominated Convergence Theorem, $\mathcal{L}_X(f_n) \to \mathcal{L}_X(f)$. Lastly, let $n \geq 1, c_1, ..., c_n \in \mathbb{C}$ and $f_1, ..., f_n \in \mathcal{S}$, then

$$\begin{aligned}\sum_{i,j=1}^{n} c_i \bar{c}_j \mathcal{L}_X(f_i - f_j) &= \mathbb{E}\left[\sum_{i,j=1}^{n} c_i \bar{c}_j \exp\left(\mathrm{i}\langle X, f_i - f_j\rangle\right)\right]\\ &= \mathbb{E}\left[\left(\sum_{i=1}^{n} c_i \exp(\mathrm{i}\langle X, f_i\rangle)\right)\overline{\left(\sum_{j=1}^{n} c_j \exp(\mathrm{i}\langle X, f_j\rangle)\right)}\right]\\ &= \mathbb{E}\left[\left|\left(\sum_{i=1}^{n} c_i \exp(\mathrm{i}\langle X, f_i\rangle)\right)\right|^2\right] \geq 0.\end{aligned}$$

$\square$

We also want to talk about convergence of generalized random field or rather of their respective laws. For this, we introduce the notion of convergence in distribution analogously to the real-valued case.

Definition 5.3. *Let X, $(X_n)_{n\geq 1}$ be generalized random fields with respective laws $\mathbb{P}_X$ and $(\mathbb{P}_{X_n})_{n\geq 1}$. We say X_n converges in distribution to X with respect to the strong topology τ_s if*

$$\lim_{n\to\infty} \int\limits_{\mathcal{S}'} \varphi(F)\, d\mathbb{P}_{X_n}(F) = \int\limits_{\mathcal{S}'} \varphi(F)\, d\mathbb{P}_X(F), \quad \forall \varphi \in \mathcal{C}_b(\mathcal{S}', \tau_s),$$

where $\mathcal{C}_b(\mathcal{S}', \tau_s)$ denotes the space of continuous and bounded functions on $\mathcal{S}'$ with respect to the strong topology.

The classical Lévy's continuity theorem connects the convergence of random variables in distribution with the convergence of their respective characteristic functions. The second main objective of [6] is to formulate this result for generalized random fields.

5.2 Generalized Random Fields on $\mathbb{S}'$

The crucial part to show the Minlos-Bochner Theorem for generalized random fields in [6] is to use the multi-sequence representation of $\mathcal{S}$. This allows us to restrict the functional to finite dimensions and use the classical Bochner's Theorem. To do so, we have to extend the notion of generalized random fields to have values in $(\mathbb{S}', \mathcal{B}(\mathbb{S}'))$. However, with the topological isomorphism Ψ from Proposition 4.2 this is now easy. Recall that

$$\begin{aligned} \Psi : (\mathcal{S}', \tau_s) &\to (\mathbb{S}', \pi_s) \\ F &\mapsto ((F, h_n))_{n\in\mathbb{N}_0^d} \end{aligned}$$

is a topological isomorphism and hence $\Psi^{-1}(\mathcal{B}(\mathbb{S}')) = \mathcal{B}(\mathcal{S}')$ and $\mathcal{B}(\mathbb{S}') = \Psi(\mathcal{B}(\mathcal{S}'))$. So for a random variable X with values in $(\mathbb{S}', \mathcal{B}(\mathbb{S}'))$, $\Psi^{-1}(X)$ is a generalized random field with values in $(\mathcal{S}', \mathcal{B}(\mathcal{S}'))$ and conversely for a generalized random field with values in $(\mathcal{S}', \mathcal{B}(\mathcal{S}'))$, $X = \Psi(Y)$ is a random variable with values in $(\mathbb{S}', \mathcal{B}(\mathbb{S}')$. Thus, we also call X a *generalized random field on* $\mathbb{S}'$. By equivalence, we can translate the concepts and observations from section 5.1.

For all $\omega \in \Omega$, $X(\omega) \in \mathbb{S}'$ and for any $a \in \mathbb{S}$, $\langle X, a\rangle : \omega \to \langle X(\omega), a\rangle$ is a real-valued random variable. To study the law of a generalized random field X on $\mathbb{S}'$ we define its *characteristic functional* by

$$\mathcal{L}_X(a) := \mathbb{E}[e^{\mathrm{i}\langle X,a\rangle}] = \int\limits_{\Omega} e^{\mathrm{i}\langle X,a\rangle}\, d\mathbb{P}, \quad \forall a \in \mathbb{S}.$$

As before, it determines the finite-dimensional distributions of the collection $(\langle X, a\rangle)_{a\in\mathbb{S}}$. More generally, for a probability measure μ on $(\mathbb{R}^{\mathbb{N}_0^d}, \mathcal{B}(\mathbb{R}^{\mathbb{N}_0^d}))$ we define its characteristic functional as

$$\mathcal{L}_\mu(a) = \int\limits_{\mathbb{R}^{\mathbb{N}_0^d}} e^{\mathrm{i}\langle b,a\rangle}\, d\mu(b),$$

for all $a \in \mathbb{R}^{\mathbb{N}_0^d}$ such that $\langle b, a\rangle$ is finite for μ-almost every $b \in \mathbb{R}^{\mathbb{N}_0^d}$. In particular, it is defined for multi-sequences with only finitely many non-zero entries.

Proposition 5.2. *Let X be a generalized random field on $\mathbb{S}$ with law $\mu = \mathbb{P}\circ X^{-1}$, then we have $\mathcal{L}_X = \mathcal{L}\mu$.*

Proof. The statement seems trivial, nonetheless we have to check that the domains of definition of the two functionals coincide. To do so we observe that μ is a probability measure on $(\mathbb{R}^{\mathbb{N}_0^d}, \mathcal{B}(\mathbb{R}^{\mathbb{N}_0^d}))$ supported on $(\mathbb{S}', \mathcal{B}(\mathbb{S}'))$. This follows from Lemma 4.5 and from the identity $\mathcal{C}(\mathbb{S}') = \mathcal{B}(\mathbb{S}')$. So we need to show that if for a multi-sequence $a \in \mathbb{R}^{\mathbb{N}_0^d}$, the pairing $\langle b, a\rangle$ is finite for all $b \in \mathbb{S}'$, then $a \in \mathbb{S}$. For this, let $p \geq 0$ and define $b \in \mathbb{R}^{\mathbb{N}_0^d}$ by $b_n = \operatorname{sign}(a_n)(1+n)^p$. Obviously, we have $b \in \mathbb{S}'$ and so

$$|a_n|(1+n)^p \leq \sum_{n\in\mathbb{N}_0^d} |a_n|(1+n)^p = \sum_{n\in\mathbb{N}_0^d} a_n b_n = \langle b, a\rangle < \infty, \quad \forall n \in \mathbb{N}_0^d.$$

Thus, by Lemma 4.1 $a \in \mathbb{S}$.

□

As before we can introduce the notion of convergence by referring to the weak convergences of the corresponding measures. Let $(\mu_n)_{n\geq 1}, \mu$ be Borel probability measures on $(\mathbb{S}', \mathcal{B}(\mathbb{S}'))$. We say μ_n converges weakly to μ with respect to the strong topology π_s if

$$\lim_{n\to\infty} \int\limits_{\mathbb{S}'} \varphi(b)\, d\mu_n(b) = \int\limits_{\mathbb{S}'} \varphi(b)\, d\mu(b), \quad \forall \varphi \in \mathcal{C}_b(\mathbb{S}', \pi_s).$$

And a sequence of generalized random fields $(X_n)_{n\geq 1}$ on $\mathbb{S}'$ converges to some generalized random field X in distribution with respect to π_s, if the corresponding laws converge weakly.

5.3 Minlos-Bochner Theorem

Now we are able to present the main result of [6] which will be of central use to us. We will not restate the proof as it was given in the paper, but instead give more details to some parts and address gaps where we think this is necessary.

Theorem 5.1 (Minlos-Bochner, [6, Theorem 4.1])**.** *If a functional $\mathcal{L} : \mathbb{S} \to \mathbb{C}$ is positive-definite, continuous at 0, and $\mathcal{L}(0) = 1$, then it is the characteristic functional of a generalized random field on $\mathbb{S}'$.*

The key idea is to restrict the functional to finite dimensions and use the well known Bochner's theorem [36, Proposition 2.5]: If $f : \mathbb{R}^d \to \mathbb{C}$ is positive-definite, continuous at 0 and $f(0) = 1$, then there exists a unique probability measure μ on $\mathbb{R}^d$ with characteristic functional f, i.e.

$$f(x) = \int_{\mathbb{R}^d} e^{i\langle s,x\rangle}\, d\mu(s).$$

To then extend these measures to $\mathbb{R}^{\mathbb{N}_0^d}$ we will make use of Kolmogorov's extension theorem ([23, Theorem 5.16]) which we will introduce now. Let T be an arbitrary index set and consider a family of spaces S_t with σ-algebras $\mathcal{S}_t$ for $t \in T$. For $I \subset T$ let $S_I = \prod_{t\in I} S_t$ and $\mathcal{S}_I$ denote the product σ-algebra $\bigotimes_{t\in I} \mathcal{S}_t$. If ξ_t is a random variable with values in S_t we write $\xi_I = (\xi_t)_{t\in I}$ for the restriction of the process to I.
Moreover, let $\tilde{T}$ be the class of finite subsets of T and μ_I, $I \in \tilde{T}$ be a family of probability measures on $(S_I, \mathcal{S}_I)$. It is called *projective* if

$$\mu_J(\cdot \times S_{J\setminus I}) = \mu_I, \quad \text{for } I \subset J \text{ in } \tilde{T}.$$

Theorem 5.2 (Kolmogorov extension, [23, Theorem 5.16])**.** *For any collection of Borel spaces S_t, $t \in T$, consider a projective family of probability measures μ_I on S_I, $I \in \tilde{T}$. Then there exist some random variables ξ_t in S_t, $t \in T$, defined on a common probability space such that ξ_I has distribution μ_I for every $I \in \tilde{T}$.*

Now we are able to prove Theorem 5.1.

Proof of Theorem 5.1. Let $\mathcal{L} : \mathbb{S} \to \mathbb{C}$ be positive-definite, continuous at 0 and $\mathcal{L}(0) = 1$. For $\Gamma \subset \mathbb{N}_0^d$ finite let $\mathcal{L}_\Gamma$ be the restriction of $\mathcal{L}$ to $\mathbb{R}^\Gamma$. Here we can consider $\mathbb{R}^\Gamma \subset \mathbb{R}^{\mathbb{N}_0^d}$ by completing the multi-sequence by zeros and so clearly $\mathbb{R}^\Gamma \subset \mathbb{S}$. $\mathcal{L}_\Gamma$ is obviously positive-definite and $\mathcal{L}_\Gamma(0) = 1$. To see that it is continuous at 0 let $(a^n)_{n\in\mathbb{N}} \in \mathbb{R}^\Gamma$ with $a^n \to 0$. Then $a_i^n \to 0$ for every $i \in \Gamma$ and so for each $p \geq 0$

$$\|a^n\|_p^2 = \sum_{i\in\Gamma}(1+i)^{2p}(a_i^n)^2 \ \to 0 \quad (n \to \infty).$$

Thus, $\mathcal{L}_\Gamma(a^n) = \mathcal{L}(a^n) \to 0$.
By Bochner's theorem there exists a unique probability measure μ_Γ on the product σ-algebra of $\mathbb{R}^\Gamma$ such that for all $a \in \mathbb{R}^\Gamma$

$$\mathcal{L}(a) = \mathcal{L}_\Gamma(a) = \int\limits_{\mathbb{R}^\Gamma} e^{i\langle b,a\rangle}\, d\mu_\Gamma(b).$$

Let $\mathcal{T}$ denote the class of finite subsets of $\mathbb{R}^{\mathbb{N}_0^d}$. Now we will show that the family of probability measures μ_Γ, $\Gamma \in \mathcal{T}$ is projective. So let $\Gamma, \Gamma' \in \mathcal{T}$ with $\Gamma \subset \Gamma'$ and let $\pi_{\Gamma',\Gamma} : \mathbb{R}^{\Gamma'} \to \mathbb{R}^\Gamma$ be the canonical projection. We have to show $\mu_{\Gamma'} \circ \pi_{\Gamma',\Gamma}^{-1} = \mu_\Gamma$.
For this define $\tilde{\mu}_\Gamma := \mu_{\Gamma'} \circ \pi_{\Gamma',\Gamma}^{-1}$ and for $a \in \mathbb{R}^\Gamma$ define $a' \in \mathbb{R}^{\Gamma'}$ by

$$a_i' = \begin{cases} a_i, & \text{if } i \in \Gamma \\ 0, & \text{else} \end{cases}.$$

Then $\pi_{\Gamma',\Gamma}(a') = a$ and we have

$$\mathcal{L}_{\Gamma'}(a') = \mathcal{L}(a') = \mathcal{L}(a) = \mathcal{L}_\Gamma(a) = \int\limits_{\mathbb{R}^\Gamma} e^{i\langle b,a\rangle}\, d\mu_\Gamma(b).$$

But on the other hand,

$$\mathcal{L}_{\Gamma'}(a') = \int\limits_{\mathbb{R}^{\Gamma'}} e^{i\langle b,a'\rangle}\, d\mu_{\Gamma'}(b) = \int\limits_{\mathbb{R}^{\Gamma'}} e^{i\langle \pi_{\Gamma',\Gamma}(b),a\rangle}\, d\mu_{\Gamma'}(b) = \int\limits_{\mathbb{R}^\Gamma} e^{i\langle c,a\rangle}\, d\tilde{\mu}_\Gamma(c).$$

Hence, by uniqueness of the measure μ_Γ we obtain

$$\mu_\Gamma = \tilde{\mu}_\Gamma = \mu_{\Gamma'} \circ \pi_{\Gamma',\Gamma}^{-1}.$$

Theorem 5.2 gives us now the existence of a random variable $X : (\Omega, \mathcal{A}, \mathbb{P}) \to (\mathbb{R}^{\mathbb{N}_0^d}, \mathcal{B}(\mathbb{R}^{\mathbb{N}_0^d}))$ for some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with law $\mu = \mathbb{P} \circ X^{-1}$ such that X_Γ has distribution μ_Γ, i.e. $\mu \circ \pi_\Gamma^{-1} = \mu_\Gamma$ for all $\Gamma \in \mathcal{T}$. Here $\pi_\Gamma : \mathbb{R}^{\mathbb{N}_0^d} \to \mathbb{R}^\Gamma$ denotes the canonical projection.
In [6, proof of Theorem 4.1] it is shown that $\mu(\mathbb{S}') = 1$. Hence we can consider

X as a random variable in $(\mathbb{S}', \mathcal{C}(\mathbb{S}'))$, as by Lemma 4.5, $\mathcal{C}(\mathbb{S}')$ is the σ-algebra induced by $\mathcal{B}(\mathbb{R}^{\mathbb{N}_0^d})$ on $\mathbb{S}'$. But we have seen that $\mathcal{C}(\mathbb{S}') = \mathcal{B}(\mathbb{S}')$ and hence, X is a generalized random field on $\mathbb{S}'$.
To show that X has characteristic functional $\mathcal{L}$, let $a \in \mathbb{S}$ and $a^n = \pi_{\Gamma_n}(a) \in \mathbb{R}^{\Gamma_N}$, where $\Gamma_n = \{0, ..., n\}^d$. The functional $\mathcal{L}$ is continuous at 0 and positive definite and hence continuous ([44, Section IV.1.2, Proposition 1.1]). Then $a^n \to a$ in $\mathbb{S}$ and so by continuity of $\mathcal{L}$ and dominated convergence we get

$$\begin{aligned}
\mathcal{L}(a) &= \lim_{n\to\infty} \mathcal{L}(a^n) = \lim_{n\to\infty} \mathcal{L}_{\Gamma_n}(a^n) = \lim_{n\to\infty} \int\limits_{\mathbb{R}^{\Gamma_n}} e^{i\langle b, a^n\rangle} \, d\mu_{\Gamma_n}(b) \\
&= \lim_{n\to\infty} \int\limits_{\mathbb{R}^{\mathbb{N}_0^d}} e^{i\langle \pi_{\Gamma_n}(b), a^n\rangle} \, d\mu(b) = \lim_{n\to\infty} \int\limits_{\mathbb{R}^{\mathbb{N}_0^d}} e^{i\langle b, a^n\rangle} \, d\mu(b) \\
&= \lim_{n\to\infty} \mathcal{L}_X(a^n) = \mathcal{L}_X(a)
\end{aligned}$$

and this completes the proof.

□

With the identification of $\mathcal{S}$ and the multi-sequence space $\mathbb{S}$, we can translate this result to generalized random fields on Schwartz space.

Theorem 5.3. *Let $\mathcal{L} : \mathcal{S} \to \mathbb{C}$ be a positive-definite, continuous at 0, and $\mathcal{L}(0) = 1$, then there exists a generalized random field X on $\mathcal{S}'$ such that $\mathcal{L}_X = \mathcal{L}$.*

Theorem 5.3 allows us to compare our definition of generalized random fields with that given in the beginning of this section. If X is a generalized random field on $\mathcal{S}$, than it determines a collection of random variables $(\langle X, f\rangle)_{f\in\mathcal{S}}$ which is linear in the sense of (10) and continuous in the sense that if $f_n \to f$ in $\mathcal{S}$, then $\langle X, f_n\rangle \to \langle X, f\rangle$ a.s. and hence also in distribution. However, for a given collection of random variables $(\tilde{X}_f)_{f\in\mathcal{S}}$ which is linear and continuous in the sense above, it is not clear that it is determined by a generalized random field. Theorem 5.3 gives us a useful requirement when this is the case.

Corollary 5.1. *Let $(\tilde{X}_f)_{f\in\mathcal{S}}$ be a collection of random variables which is linear in the sense of* (10) *and continuous in the sense that if $f_n \to f$ in $\mathcal{S}$ then $\tilde{X}_{f_n} \to \tilde{X}_f$ in distribution. If the functional $f \mapsto \mathbb{E}[e^{i\tilde{X}_f}]$ is continuous at 0 on $\mathcal{S}$, then there exists a generalized random field X on $\mathcal{S}$, such that $(\langle X, f\rangle)_{f\in\mathcal{S}}$ and $(\tilde{X}_f)_{f\in\mathcal{S}}$ have the same finite dimensional distributions.*

Proof. The result follows directly from Minlos-Bochner Theorem 5.3 by observing that the functional $f \mapsto \mathbb{E}[e^{\mathrm{i}\tilde{X}_f}]$ is positive-definite by linearity of $(\tilde{X}_f)_{f\in\mathcal{S}}$.

$\square$

Note that the generalized random field X in the above corollary might not necessarily be defined on the same probability space as the collection $(\tilde{X}_f)_{f\in\mathcal{S}}$.

5.4 Lévy's Continuity Theorem

The second important result we present for generalized random fields is the generalized version of Lévy's continuity theorem. In the classical case it states that a sequence of random variables converges in distribution if and only if their characteristic functions converge point-wise (see [23, Theorem 4.3]). Biermé, Durieu and Wang show in [6] that this result translates to generalized random fields.

Theorem 5.4 ([6, Theorem 2.3])**.** *Let $(X_n)_{n\geq 1}$ be a sequence of generalized random fields on $\mathcal{S}'$. If $(\mathcal{L}_{X_n})_{n\geq 1}$ converges pointwise to a functional $\mathcal{L} : \mathcal{S} \to \mathbb{C}$ which is continuous at 0, then there exists a generalized random field X such that $\mathcal{L}_X = \mathcal{L}$ and $(X_n)_{n\geq 1}$ converges in distribution to X with respect to the strong topology.*

With this it is now easy to prove the following characterization of convergence in distributions of generalized random fields.

Corollary 5.2 ([6, Corollary 2.4])**.** *Let $(X_n)_{n\geq 1}, X$ be generalized random fields on $\mathcal{S}'$. Then the following are equivalent:*

(i) *X_n converges in distribution to X with respect to the strong topology τ_s*

(ii) *X_n converges in distribution to X with respect to the weak-* topology $\sigma(\mathcal{S}', \mathcal{S})$*

(iii) $\lim_{n\to\infty} \mathcal{L}_{X_n}(f) = \mathcal{L}_X(f)$ *for all* $f \in \mathcal{S}$

(iv) *$\langle X_n, f\rangle$ converges in distribution to $\langle X, f\rangle$ for all $f \in \mathcal{S}$*

Proof. Since $\mathcal{C}_b(\mathcal{S}', \sigma(\mathcal{S}', \mathcal{S})) \subset \mathcal{C}_b(\mathcal{S}', \tau_s)$ the implication $(i) \implies (ii)$ follows immediately. To show $(ii) \implies (iii)$ we see that the function $\varphi_f : \mathcal{S}' \to \mathbb{C}$, $\varphi_f(F) = e^{\mathrm{i}\langle F,f\rangle}$ is clearly an element of $\mathcal{C}_b(\mathcal{S}', \sigma(\mathcal{S}', \mathcal{S}))$. The implication

$(iii) \implies (i)$ is a consequence of Theorem 5.4 and the equivalence of (iii) and (iv) follows from the classical Lévy's continuity theorem and the relation

$$\mathcal{L}_{\langle X_n, f\rangle}(z) = \mathcal{L}_{X_n}(zf) \to \mathcal{L}_X(zf) = \mathcal{L}_{\langle X, f\rangle}(z)$$

for all $f \in \mathcal{S}$ and $z \in \mathbb{R}$.

□

Remark. Theorem 5.4 is shown in [6] by showing the result in $\mathbb{S}'$. It is quite detailed and we don't feel it is necessary to add anything. The key is to show that the sequence of the corresponding Borel measures on $\mathbb{S}'$ is tight, to then infer relative sequential compactness. This is done by showing that the real part of the characteristic functionals is equicontinuous.

6 Lévy processes

Before we apply the Minlos-Bochner Theorem to define, what will be called, Lévy White noise we first introduce the concept of Lévy Processes. They were introduced by Paul Lévy in the 1930's and constitute a fundamental class of stochastic processes. The content of this section is taken from [36] and [2] and we will state the relevant results without proof.
Stochastic processes are a mathematical object that model the time evolution of a random observation. Formally, a stochastic processes is a family $\{X_t : t \geq 0\}$ of random variables on $\mathbb{R}^d$ defined on a common probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We usually denote it by $(X_t)_{t\geq 0}$. For $\omega \in \Omega$ the function $t \mapsto X_t(\omega)$ is called a sample path of $(X_t)_{t\geq 0}$. The most basic processes are Brownian motion and Poisson processes. Both belong to the family of Lévy processes.

Definition 6.1. *A stochastic process $(X_t)_{t\geq 0}$ on $\mathbb{R}^d$ is called a Lévy process if:*

(i) $(X_t)_{t\geq 0}$ *has independent increments, i.e.,* $\forall n \geq 1$, $0 \leq t_0 < ... < t_n$*:* $X_{t_0}, X_{t_1} - X_{t_0}, ..., X_{t_n} - X_{t_{n-1}}$ *are independent*

(ii) $X_0 = 0$ *a.s.*

(iii) $(X_t)_{t\geq 0}$ *has stationary increments, i.e., the distribution of* $X_{s+t} - X_s$ *does not depend on* s*.*

(iv) $(X_t)_{t\geq 0}$ *is continuous in probability, i.e.,* $\forall t \geq 0$, $\epsilon > 0$ *:*

$$\lim_{s\to t} \mathbb{P}(|X_s - X_t| > \epsilon) = 0.$$

It can be shown ([36, Theorem 11.5]) that a Lévy process $(X_t)_{t\geq 0}$ has a modification $(Y_t)_{t\geq 0}$ such that it has a.s. right continuous paths for $t \geq 0$ and left limits for $t > 0$, i.e. it has a.s. cadlag paths. A modification $(Y_t)_{t\geq 0}$ of $(X_t)_{t\geq 0}$ is a stochastic process such that

$$\mathbb{P}(X_t = Y_t) = 1 \quad \forall t \geq 0.$$

6.1 Infinite Divisibility

Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ be two probability measures on $\mathbb{R}^d$, then we define its convolution as:

$$(\mu * \nu)(A) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{1}_A(x+y)\, d\mu(x) d\nu(y), \quad \forall A \in \mathcal{B}(\mathbb{R}^d).$$

It's easy to see that the convolution of two probability measures is again a probability measure. For the corresponding characteristic functions we obtain the familiar relation $(\mathcal{L}_\mu * \mathcal{L}_\nu) = \mathcal{L}_\mu \mathcal{L}_\nu$. Also, note that if X and Y are two independent random variables with law μ_X and μ_Y, then the law of $X + Y$ is given by $\mu_X * \mu_Y$.

Definition 6.2. *A probability measure $\mu \in \mathcal{P}(\mathbb{R}^d)$ on $\mathbb{R}^d$ is called infinitely divisible if $\forall n \in \mathbb{N}$, there exists a probability measure $\mu_n \in \mathcal{P}(\mathbb{R}^d)$ such that*

$$\mu = \mu_n^{*n} := \mu_n * ... * \mu_n,$$

where the convolution is taken n times.

Equivalently, $\mu \in \mathcal{P}(\mathbb{R}^d)$ is infinitely divisible if $\forall n \in \mathbb{N}$ its characteristic function $\mathcal{L}_\mu$ has a n-th root $\mathcal{L}_\mu^{1/n}$ which again is the characteristic function of a probability measure. Furthermore, we call a random variable X infinitely divisible if its law is or equivalently, if $\forall n \in \mathbb{N}$ there exists a i.i.d random variables $X_1^{(n)}, ..., X_n^{(n)}$ such that X and $X_1^{(n)} + ... + X_n^{(n)}$ are equal in distribution.
The following Theorem establishes the important connection to Lévy processes.

Theorem 6.1. ([36, Theorem 7.10]) *Let $(X_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}^d$, then*

(i) X_t is infinitely divisible for all $t \geq 0$

(ii) if $\mu \in \mathcal{P}(\mathbb{R}^d)$ is an infinitely divisible probability measure then there exists a Lévy process $(\tilde{X}_t)_{t\geq 0}$ such that $\tilde{X}_1$ has law μ

(iii) if $(X_t)_{t\geq 0}$, $(Y_t)_{t\geq 0}$ are two Lévy processes such that X_1 and Y_1 have the same law, then $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are identical in law.

This means that we have a one to one correspondence between Lévy processes and infinitely divisible distributions. We will see that this underlying distribution already contains all the information of the whole process.

6.2 The Lévy–Khintchine representation

In the following we will present the Lévy–Khintchine formula. It gives us the central characterization of infinitely divisible probability measures and hence of Lévy processes through their characteristic functions.
A measure ν on $\mathbb{R}^d \setminus \{0\}$ is called a *Lévy measure* if

$$\int\limits_{\mathbb{R}^d\setminus\{0\}} \min\{1, |x|^2\}\, d\nu(x) < \infty.$$

Theorem 6.2. ([36, Theorem 8.1],[2, Theorem 1.2.14] *Let $\mu \in \mathcal{P}(\mathbb{R}^d)$ be infinitely divisible, then there exists $\gamma \in \mathbb{R}^d$, a symmetric positive-definite $d \times d$ matrix A and a Lévy measure ν such that its characteristic function admits the following representation*

$$\mathcal{L}_\mu(z) = exp\left(f(z)\right), \quad z \in \mathbb{R}^d \tag{14}$$

where $f : \mathbb{R}^d \to \mathbb{C}$ is given by

$$f(z) = i\langle\gamma, z\rangle - \frac{1}{2}\langle z, Az\rangle + \int\limits_{\mathbb{R}^d\setminus\{0\}} (e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle \mathbb{1}_{|x|\leq 1})\, d\nu(x). \tag{15}$$

This representation by γ, A and ν is unique.
Conversely, for any $\gamma \in \mathbb{R}^d$, $A \in \mathbb{R}^{d\times d}$ symmetric positive-definite and a Lévy measure ν there exists a infinitely divisible distribution on $\mathbb{R}^d$ whose characteristic function is given by (14) *and* (15).

The function f which occurs in (14) is usually referred to as *Lévy exponent* or *characteristic exponent*. The triplet (γ, A, ν) is called the *Lévy triplet* or the *characteristics* of μ.
Now let $(X_t)_{t\geq 0}$ be a Lévy process, then by Theorem 6.1 X_t is infinitely divisible for all $t \geq 0$. Hence, we can characterize X_t by its Lévy-Khintchine representation. Let μ by the law of X_1 with Lévy exponent f and Lévy

triplet (γ, A, ν). Then, by [2, Theorem 1.3.3] the characteristic function of X_t for $t \geq 0$ is given by

$$\mathcal{L}_{X_t}(z) = e^{tf(z)}, \quad \forall z \in \mathbb{R}^d.$$

Thus, X_t has Lévy exponent tf and characteristics $(t\gamma, tA, t\nu)$. Therefore, we will define the Lévy exponent and Lévy triplet of a Lévy process $(X_t)_{t\geq 0}$ as those of the random variable X_1. It becomes clear that the measure μ describes the whole distribution of the process. So in a way the study of Lévy processes reduces to the study of infinitely divisible distributions.

6.3 Examples of Lévy processes

6.3.1 Gaussian distribution and Brownian motion

Let $\gamma \in \mathbb{R}^d$ and A be a positive-definite, symmetric $d \times d$ matrix, we say $\mu \in \mathcal{P}(\mathbb{R}^d)$ is a (nondegenerate) *Gaussian distribution* with mean vector γ and covariance matrix A if it has the density

$$f_\mu(x) = \left((2\pi)^d \det(A)\right)^{-1/2} \exp\left(-\frac{1}{2}\langle x-\gamma, A^{-1}(x-\gamma)\rangle\right), \quad x \in \mathbb{R}^d$$

with respect to the Lebesgue measure. If a random variable X has distribution μ, we write $X \sim \mathrm{N}(\gamma, A)$. Its characteristic function is given by

$$\mathcal{L}_\mu(z) = \exp\left(\mathrm{i}\langle \gamma, z\rangle - \frac{1}{2}\langle z, Az\rangle\right), \quad z \in \mathbb{R}^d.$$

Thus, we can clearly see that it is infinitely divisible with Lévy triplet $(\gamma, A, 0)$. By Theorem 6.1 there exists a Lévy process $(X_t)_{t\geq 0}$ with Gaussian distribution $X_t \sim \mathrm{N}(t\gamma, tA)$.
A Brownian motion is a Lévy process $(B_t)_{t\geq 0}$ such that it has Gaussian distribution $B_t \sim \mathrm{N}(0, tI)$ and it has almost surely continuous sample paths (see Figure 1). It's characteristic function is given by $\mathcal{L}_{B_t}(z) = \exp(-1/2t|z|^2)$. It is a very important stochastic process, see for example [23] for a proof of its existence and its properties.

6.3.2 Poisson

Let $d = 1$ and $\lambda > 0$. The *Poisson distribution* μ with mean $\lambda > 0$ is given by

$$\mu(\{n\}) = \frac{\lambda^n}{n!}\mathrm{e}^{-\lambda}, \quad \forall n \in \mathbb{N}_0$$

and $\mu(B) = 0$ for every $B \in \mathcal{B}(\mathbb{R})$ that does not contain nonnegative integers. It describes the number of random events occurring with mean rate λ in a fixed time interval. Its characteristic function is given by

$$\mathcal{L}_\mu(z) = \exp(\lambda(\mathrm{e}^{\mathrm{i}z} - 1)), \quad z \in \mathbb{R}.$$

The Poisson distribution is infinitely divisible with Lévy triplet $(1, 0, \lambda\delta_1)$ where δ_1 denotes the Dirac measure at 1.
The Poisson process with rate $\lambda > 0$ is a Lévy process such that for every $t > 0$ it has Poisson distribution with mean $t\lambda$. See [36] for an explicit construction.

6.3.3 Compound Poisson

Let $\lambda > 0$ and N be a Poisson random variable with rate $\lambda > 0$. Furthermore, let $(Z_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. $\mathbb{R}^d$-valued random variables with law $\sigma \in \mathcal{P}(\mathbb{R}^d)$ and independent to N. Then the *compound Poisson random variable* X is defined by

$$X = \sum_{i=1}^{N} Z_i.$$

We can view X as a random walk with a random number of steps and a random step size determined by the underlying distribution σ. Its characteristic function is given by

$$\mathcal{L}_X(z) = \exp\left(\int_{\mathbb{R}^d} \left(\mathrm{e}^{\mathrm{i}\langle z,x\rangle} - 1\right) \lambda \, d\sigma(x)\right) = \exp(\lambda(\hat{\sigma}(z) - 1)).$$

Alternatively, we can the define the *compound Poisson distribution* directly via its characteristic function. It is infinitely divisible with Lévy triplet $\left(\int_{|x|\leq 1} x \, d\sigma(x), 0, \lambda\sigma\right)$. In dimension $d = 1$ and $\sigma = \delta_1$, we obtain the classical Poisson distribution.
In the same setting let $(N_t)_{t\geq 0}$ be a Poisson process with rate $\lambda > 0$, then the *compound Poisson process* $(Y_t)_{t\geq 0}$ is given by

$$Y_t = \sum_{i=1}^{N_t} Z_i.$$

It is a Lévy process such that for each $t > 0$ it has Compound Poisson distribution associated with λ and underlying distribution $t\sigma$.

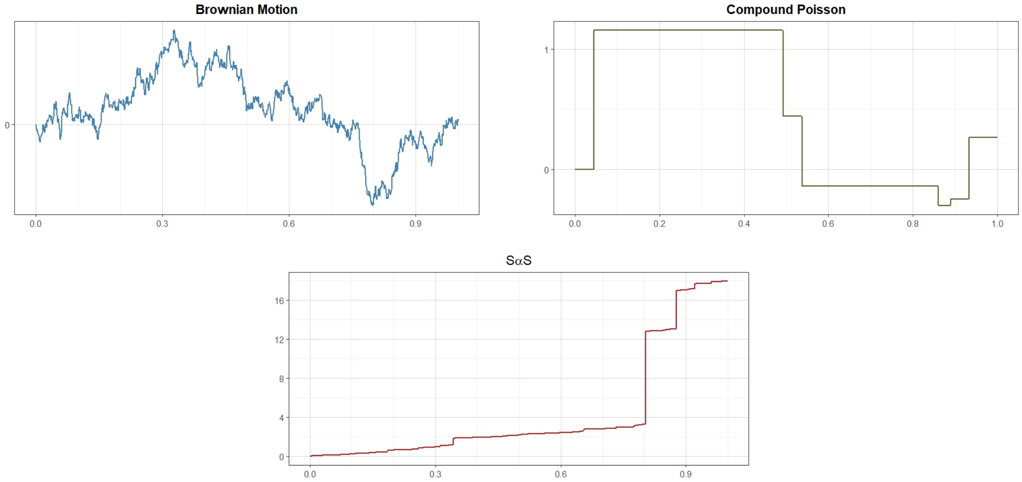


Figure 1: Simulation of the sample paths of a Brownian motion (top left), a Compound Poisson process (top right) with rate $\lambda = 5$ and underlying standard normal distribution and a $S\alpha S$ process (bottom) with $\alpha = 0.5$ and $\sigma = 1$.

6.3.4 Stable Distributions

The following part is taken out of [35, Chapter 1] and [2, Chapter 1]. For simplicity we assume $d = 1$. A random variable X on $\mathbb{R}$ is said to be *stable* if for every $A, B > 0$ there exists some $C > 0$ and $D \in \mathbb{R}$ such that

$$AX_1 + BX_2 \stackrel{d}{=} CX + D, \tag{16}$$

where X_1 and X_2 are independent copies of X. It is called *strictly stable* if (16) holds with $D = 0$. Furthermore for any stable X there is an $0 < \alpha \leq 2$ such that the constant C in (16) satisfies

$$C^\alpha = A^\alpha + B^\alpha.$$

The parameter α is referred to as the *index of stability* and X is said to be α-stable.

It is easy to see that stable random variables are infinitely divisible. For $\alpha = 2$, we obtain the Gaussian distribution from above. Of special interest are stable distributions that are also symmetric. This is because of their self-similarity. This means that a change of time scale is equivalent to a change in spatial scale (see [36, Chapter 3] for more details). It is usually denoted by $X \sim S\alpha S$ and its characteristic function is given by

$$\mathcal{L}_X(z) = \exp\left(-\sigma^\alpha |z|^\alpha\right).$$

The importance of stable distributions is in their tail behavior. For $\alpha = 2$ it is well known that the Gaussian distribution shows exponential decay. However for $\alpha < 2$ these distributions exhibit a slower polynomial decay. Because of these heavy tails they are often used to model phenomena with long range dependencies.
A Lévy process $(X_t)_{t\geq 0}$ is called stable if X_t is, for all $t \geq 0$. In the case of the $S\alpha S$ distributions its Lévy exponent is given by

$$f(z) = -\sigma^\alpha |z|^\alpha, \quad 0 < \alpha \leq 2, \sigma > 0.$$

7 Lévy White Noise

In section 5 we introduced the concept of a generalized random field as a generalization of a stochastic process. Now we want to identify an important subclass of generalized random fields by transferring the notion of a Lévy process to this more general concept. The resulting generalized random field will be called Lévy white noise.
The construction of Lévy white noise goes back to I.M. Gel'fand and N.Y. Vilenkin in [18]. It is originally defined as a random variable with values in the space of distributions $\mathcal{D}'(\mathbb{R}^d)$, where $\mathcal{D}'(\mathbb{R}^d)$ is equipped with a cylindrical σ-Algebra structure analogously to $\mathcal{S}'$. As we have seen, this approach allows us to characterize its distribution by the characteristic functional. Let $f : \mathbb{R} \to \mathbb{C}$ be continuous and consider the functional

$$\mathcal{L}(\varphi) = \exp\left(\int_{\mathbb{R}^d} f(\varphi(s))ds \right), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^d). \tag{17}$$

Gel'fand and Vilenkin give necessary and sufficient conditions on f for this to define the characteristic functional of a random variable with values in $\mathcal{D}'$.

Theorem 7.1 (Gel'fand-Vilenkin [18, Chapter III])**.** *Let f be a continuous function from $\mathbb{R}$ to $\mathbb{C}$ and $\mathcal{L}$ be the functional defined by* (17)*. Then, the following are equivalent:*

(i) $\mathcal{L}$ is a positive-definite, normalized ($\mathcal{L}(0) = 1$) and continuous functional on $\mathcal{D}$

(ii) There exists $\gamma \in \mathbb{R}$, $\sigma^2 \geq 0$ and a Lévy measure ν, such that f is a valid Lévy exponent with characteristics (γ, σ^2, ν) given by (15)*.*

If f defines a valid Lévy exponent, then the Minlos-Bochner Theorem in its full generality ([18, Chapter II, Theorem 3]) ensures the existence of a distribution on $\mathcal{D}'$ with characteristic functional (17). The corresponding random variable is referred to as *Lévy white noise*. Before we explore Lévy white noise as a generalized random field on $\mathcal{S}'$ we want to justify why the functional (17) should correspond to the generalization of a Lévy process. So far, we have only properly defined generalized random fields with values in $(\mathcal{S}', \mathcal{B}(\mathcal{S}'))$, so this should merely be seen as motivation. Let X be a $\mathcal{D}'$-valued random variable with characteristic function (17) where f is a Lévy exponent. If it can even be extended to a generalized random field on $\mathcal{S}'$ is not clear up to this point. To observe X, we have to "measure" it through some observation window $\varphi \in \mathcal{S}$. In [13] a thorough analysis of the domain of definition of Lévy white noise can be found (see [27] for a similar approach). It is shown, that it is possible to extend its domain to make sense of a random variable $\langle X, g\rangle$ where g is not necessarily an element of $\mathcal{S}$ or even $\mathcal{D}$. In particular, by approximation ([13, Proposition 3.1, Proposition 3.7]), one can identify the random variable $\langle X, \mathbb{1}_{[0,t)}\rangle$ with characteristic function

$$\mathcal{L}_{\langle X, \mathbb{1}_{[0,t)}\rangle}(z) = \mathcal{L}_X(z\mathbb{1}_{[0,t)}) = \exp(tf(z)),$$

which clearly is the characteristic function of an infinitely divisible distribution. Hence, $(\langle X, \mathbb{1}_{[0,t)}\rangle)_{t\geq 0}$ is a Lévy process as a special realization of the more general object X.

7.1 Tempered Lévy White Noise

Since $\mathcal{D} \hookrightarrow \mathcal{S}$ is dense, we can identify the space of tempered distributions $\mathcal{S}'$ as a subspace of $\mathcal{D}'$, in the sense that every tempered distribution restricted to $\mathcal{D}$ is a linear and continuous functional over the space of test functions $\mathcal{D}$. This raises the question of when a Lévy white noise can actually be extended to the space of tempered distributions. Or differently put, we can define Lévy white noise on the space of tempered distributions by (17) and use the Minlos-Bochner Theorem 5.3 to characterize it by its Lévy exponent f. For this we have to show that (17) defines a positive-definite, continuous (at 0) and normalized ($\mathcal{L}(0) = 1$) functional on $\mathcal{S}$. We see that this comes down to a more restrictive property on the Lévy measure ν.

Theorem 7.2 ([11, Theorem 3],[42, Theorem 4.8])**.** *Let f be a Lévy exponent with Lèvy triplet (γ, σ^2, ν) and $\mathcal{L} : \mathcal{S} \to \mathbb{C}$ be the functional defined by* (17). *If there exists an $\epsilon > 0$ such that*

$$\int_{|x|>1} |x|^\epsilon \, d\nu(x) < \infty, \tag{18}$$

then there exists a generalized random field $\dot{X}$ on $\mathcal{S}'$ with characteristic functional $\mathcal{L}_{\dot{X}} = \mathcal{L}$.

We call a Lévy measure $\nu \in \mathcal{P}(\mathbb{R}\setminus\{0\})$ a *Lévy-Schwartz measure* if there exists some $\epsilon > 0$ such that (18) holds.
To prove the above result we first need two technical Lemmas.

Lemma 7.1. *Let $a \in \mathbb{R}$ and $b \leq 1$,then*

(i) $|e^{ia} - 1| \leq 2^{1-b}|a|^b$

(ii) $|e^{ia} - 1 - ia| \leq |a|^2$

Proof. The Lemma follows from a simple use of Taylor's formula. For (i) we have $|e^{\mathrm{i}a} - 1| \leq 2$ so if $|a| < 2$ then by the explicit expression of the rest term in the Taylor expansion we obtain

$$|e^{\mathrm{i}a} - 1| = \left| \int_0^a e^{\mathrm{i}t}(a-t)^0 \, dt \right| \leq |a|$$

and thus $|e^{\mathrm{i}a} - 1| \leq \min\{2, |a|\} \leq 2\min\{1, |a|/2\}^b \leq 2^{1-b}|a|^b$. For (ii) by the formula for the rest term we get

$$|e^{\mathrm{i}a} - 1 - ia| \leq \left| \int_0^a \frac{e^{\mathrm{i}t}}{2!}(a-t) \, dt \right| \leq |a|^2.$$

□

Lemma 7.2. *Let f be a Lévy exponent with Lévy triplet (γ, σ^2, ν), where there exists some $\epsilon > 0$ such that (18) holds and let $\mathcal{L} : \mathcal{S} \to \mathbb{C}$ be the functional defined by (17), i.e. $\mathcal{L}(\varphi) = exp\left(\int_{\mathbb{R}^d} f(\varphi(s))\, ds\right)$ for all $\varphi \in \mathcal{S}$. Then for all $\varphi \in \mathcal{S}$ we have*

$$\begin{aligned} \|f \circ \varphi\|_{L^1} \leq & |\gamma| \|\varphi\|_{L^1} + \frac{\sigma^2}{2}\|\varphi\|_{L^2}^2 + 2^{1-\alpha}\left(\int_{|x|>1} |x|^\epsilon \, d\nu(x)\right)\|\varphi\|_{L^\alpha}^\alpha \\ & + \left(\int_{|x|\leq 1} |x|^2 \, d\nu(x)\right)\|\varphi\|_{L^2}^2. \end{aligned} \tag{19}$$

and for all $\varphi, \psi \in \mathcal{S}$

$$\begin{aligned} \|f\circ\varphi - f\circ\psi\|_{L^1} &\leq |\gamma|\|\varphi-\psi\|_{L^1} + \frac{\sigma^2}{2}\|\varphi-\psi\|_{L^2}^2 \\ &+ 2^{1-\alpha}\|\varphi-\psi\|_{L^\alpha}^\alpha \left(\int_{|x|>1} |x|^\epsilon \, d\nu(x)\right) \\ &\quad + \int_{|x|\leq 1} |x|^2 \, d\nu(x) \left(\|\varphi-\psi\|_{L^2}^2 + \|\varphi-\psi\|_{L^2}\|\psi\|_{L^2}\right), \end{aligned} \tag{20}$$

where $\alpha = min\{1, \epsilon\}$.

Proof. Let $\varphi \in \mathcal{S}$, then

$$\begin{aligned} \int_{\mathbb{R}^d} |f(\varphi(s))| \, ds &\leq |\gamma| \int_{\mathbb{R}^d} |\varphi(s)| \, ds + \frac{\sigma^2}{2}\int_{\mathbb{R}^d} |\varphi(s)|^2 \, ds \\ &\quad + \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}\setminus\{0\}} \left(e^{\mathrm{i}x\varphi(s)} - 1 - \mathrm{i}x\varphi(s)\mathbb{1}_{\{|x|\leq 1\}}\right) \, d\nu(x) ds \right| \\ &\leq |\gamma|\|\varphi\|_{L^1} + \frac{\sigma^2}{2}\|\varphi\|_{L^2}^2 + \int_{\mathbb{R}^d}\int_{|x|>1} |e^{\mathrm{i}x\varphi(s)} - 1| \, d\nu(x) ds \\ &\quad + \int_{\mathbb{R}^d}\int_{|x|\leq 1} |e^{\mathrm{i}x\varphi(s)} - 1 - \mathrm{i}x\varphi(s)| \, d\nu(x) ds. \end{aligned}$$

Let $\alpha := \min\{1, \epsilon\} \leq 1$,then by the previous Lemma we can estimate for $|x| > 1$

$$|e^{\mathrm{i}x\varphi(s)} - 1| \leq 2^{1-\alpha}|x\varphi(s)|^\alpha \leq 2^{1-\alpha}|x|^\epsilon|\varphi(s)|^\alpha$$

and for $|x| \leq 1$

$$|e^{\mathrm{i}x\varphi(s)} - 1 - \mathrm{i}x\varphi(s)| \leq |x\varphi(s)|^2.$$

Therefore,

$$\begin{aligned} \|f\circ\varphi\|_{L^1} \leq \gamma\|\varphi\|_{L^1} + \frac{\sigma^2}{2}\|\varphi\|_{L^2}^2 + 2^{1-\alpha}\left(\int_{|x|>1} |x|^\epsilon \, d\nu(x)\right)\|\varphi\|_{L^\alpha}^\alpha \\ + \left(\int_{|x|\leq 1} |x|^2 \, d\nu(x)\right)\|\varphi\|_{L^2}^2. \end{aligned}$$

To show the second inequality, let $\varphi, \psi \in \mathcal{S}$, then

$$\left| \int_{\mathbb{R}^d} f(\varphi(s)) - f(\psi(s))\, ds \right| \leq |\gamma| \|\varphi - \psi\|_{L^1} + \frac{\sigma^2}{2} \|\varphi - \psi\|_{L^2}^2$$
$$+ \int_{\mathbb{R}^d} \int_{|x|>1} |e^{\mathrm{i}x\varphi(s)} - e^{\mathrm{i}x\psi(s)}|\, d\nu(x) ds$$
$$+ \int_{\mathbb{R}^d} \int_{|x|\leq 1} |e^{\mathrm{i}x\varphi(s)} - e^{\mathrm{i}x\psi(s)} - \mathrm{i}x(\varphi(s) - \psi(s))| d\nu(x) ds.$$

Furthermore, by the previous Lemma,

$$\begin{aligned} \int_{\mathbb{R}^d} \int_{|x|>1} |e^{\mathrm{i}x\varphi(s)} - e^{\mathrm{i}x\psi(s)}|\, d\nu(x) ds &= \int_{\mathbb{R}^d} \int_{|x|>1} |e^{\mathrm{i}x(\varphi(s)-\psi(s))} - 1|\, d\nu(x) ds \\ &\leq 2^{1-\alpha} \int_{\mathbb{R}^d} \int_{|x|>1} |\varphi(s) - \psi(s)|^\alpha |x|^\alpha\, d\nu(x) ds \\ &\leq 2^{1-\alpha} \|\varphi - \psi\|_{L^\alpha}^\alpha \left(\int_{|x|>1} |x|^\epsilon\, d\nu(x) \right) \end{aligned}$$

and for $|x| \leq 1$

$$\begin{aligned} &|e^{\mathrm{i}x\varphi(s)} - e^{\mathrm{i}x\psi(s)} - \mathrm{i}x(\varphi(s) - \psi(s))| \\ &= |e^{\mathrm{i}x\psi(s)} \left(e^{\mathrm{i}x(\varphi(s)-\psi(s))} - 1 - \mathrm{i}x(\varphi(s) - \psi(s)) \right) + \mathrm{i}x(\varphi(s) - \psi(s))(e^{\mathrm{i}x\psi(s)} - 1)| \\ &\leq |e^{\mathrm{i}x(\varphi(s)-\psi(s))} - 1 - \mathrm{i}x(\varphi(s) - \psi(s))| + |x(\varphi(s) - \psi(s))(e^{\mathrm{i}x\psi(s)} - 1)| \\ &\leq |\varphi(s) - \psi(s)|^2 |x|^2 + |x(\varphi(s) - \psi(s))||x\psi(s)|, \end{aligned}$$

so

$$\begin{aligned} &\int_{\mathbb{R}^d} \int_{|x|\leq 1} |e^{\mathrm{i}x\varphi(s)} - e^{\mathrm{i}x\psi(s)} - \mathrm{i}x(\varphi(s) - \psi(s))| d\nu(x) ds \\ &\qquad \leq \int_{|x|\leq 1} |x|^2\, d\nu(x) \left(\|\varphi - \psi\|_{L^2}^2 + \|\varphi - \psi\|_{L^2} \|\psi\|_{L^2} \right). \end{aligned}$$

where for the last inequality we used Cauchy-Schwartz. In total we obtain

$$\left| \int\limits_{\mathbb{R}^d} f(\varphi(s)) - f(\psi(s))\, ds \right|$$
$$\leq |\gamma| \|\varphi - \psi\|_{L^1} + \frac{\sigma^2}{2} \|\varphi - \psi\|_{L^2}^2 + 2^{1-\alpha} \|\varphi - \psi\|_{L^\alpha}^\alpha \left(\int_{|x|>1} |x|^\epsilon \, d\nu(x) \right)$$
$$+ \int_{|x|\leq 1} |x|^2 \, d\nu(x) \left(\|\varphi - \psi\|_{L^2}^2 + \|\varphi - \psi\|_{L^2} \|\psi\|_{L^2} \right).$$

□

With this at hand we are able to prove Theorem 7.2.

Proof of Theorem 7.2. We will show that (17) defines a continuous at 0, positive definite and normalized functional on $\mathcal{S}$. The Minlos-Bochner theorem then gives us the existence of such a generalized random field $\dot{X}$. For this we will first show that (17) is actually well defined on $\mathcal{S}$ i.e., that $f \circ \varphi \in L^1(\mathbb{R}^d) \quad \forall \varphi \in \mathcal{S}$, then continuity and finally positive-definiteness. That (17) is normalized is clear.
So let f be a Lévy exponent with Lèvy triplet (γ, σ^2, ν) and $\mathcal{L}$ the functional given by (17). To see that $f \circ \varphi \in L^1$, we use (19) of Lemma 7.2 and the fact that $\mathcal{S} \subset L^p$, $\forall p > 0$.
To show continuity of $\mathcal{L}$ we proceed similar. For this let $\varphi, \psi \in \mathcal{S}$, then, since $\mathcal{S} \subset L^2$, we have $\|\psi\|_{L^2} < \infty$. Furthermore convergence in $\mathcal{S}$ implies convergence in L^p for $p > 0$. To see this let $\varphi_n \to \varphi$ $(n \to \infty)$ in $\mathcal{S}$ and $\beta \in \mathbb{N}_0^d$ s.t. $(1 + |(\cdot)^\beta|)^{-p} \in L^1$. Then

$$\begin{aligned} \|\varphi_n - \varphi\|_{L^p}^p &= \int_{\mathbb{R}^d} |\varphi_n(x) - \varphi(x)|^p \, dx \\ &= \int_{\mathbb{R}^d} \frac{\left((1 + |x^\beta|) |\varphi_n(x) - \varphi(x)| \right)^p}{(1 + |x^\beta|)^p} \, dx \\ &\leq \|(1 + |(\cdot)^\beta|)^{-p}\|_{L^1} \left(\|\varphi_n - \varphi\|_{0,0} + \|\varphi_n - \varphi\|_{\beta,0} \right) \to 0, \quad (n \to \infty). \end{aligned}$$

Thus, the continuity of $\mathcal{L}$ follows from inequality (20) of Lemma 7.2.
Whats left to show is positive definiteness of $\mathcal{L}$. This follows from Theorem 7.1 and a density argument. So let $m \geq 1$, $c_1, ..., c_m \in \mathbb{C}$ and $f_1, ..., f_m \in \mathcal{S}$. Since $\mathcal{D}$ is dense in $\mathcal{S}$, there are $(f_n^k)_{k \in \mathbb{N}} \subset \mathcal{D}$ such that $\lim\limits_{k \to \infty} f_n^k = f_n$ for all $n = 1, ..., m$. By Theorem 7.1 we know that $\mathcal{L}$ is positive definite over $\mathcal{D}$ and so using the continuity of $\mathcal{L}$ over $\mathcal{S}$ we get

$$\sum_{i,j=1}^m c_i \bar{c}_j \mathcal{L}(f_i - f_j) = \lim_{k \to \infty} \left(\sum_{i,j=1}^m c_i \bar{c}_j \mathcal{L}(f_i^k - f_j^k) \right) \geq 0.$$

$\square$

Theorem 7.2 ensures us that the following definition of *Lévy white noise* is well-defined on $\mathcal{S}'$.

Definition 7.1. *We call a generalized random field $\dot{X}$ on $\mathcal{S}'$ a Lévy white noise if its characteristic functional is given by*

$$\mathcal{L}_{\dot{X}}(\varphi) = exp\left(\int\limits_{\mathbb{R}^d} f(\varphi(s))\, ds\right), \quad \forall \varphi \in \mathcal{S},$$

where f is a Lévy exponent, i.e., of the form

$$f(z) = i\gamma z - \frac{\sigma^2}{2}z^2 + \int\limits_{\mathbb{R}\setminus\{0\}} (e^{izx} - 1 - izx\mathbb{1}_{|x|\leq 1})\, d\nu(x).$$

Here $\gamma \in \mathbb{R}, \sigma^2 \geq 0$ and ν is a Lévy-Schwartz measure.

Remark. If we are given a Lévy exponent f, then the functional (17) always defines the characteristic functional of a random variable with values in $\mathcal{D}'$. Theorem 7.2 gives us a necessary condition for this random variable to be tempered: If the corresponding Lévy measure satisfies condition (18), then it is almost surely tempered. Thus, it defines a generalized random field on $\mathcal{S}'$. It can be shown that this condition is not only necessary, but even sufficient. The functional (17) defines a generalized random field on $\mathcal{S}'$ if and only if the corresponding Lévy measure satisfies the Lévy-Schwartz condition (18) ([11, Theorem 4], [10, Theorem 2]). Furthermore, [10, Theorem 2] gives a full equivalent characterization of this condition. It is also important to note that it is nontrivial, that is, there are Lévy measures that violate the Lévy-Schwartz condition. [45] deals with a family of Lévy processes and its properties without support in $\mathcal{S}'$.

7.2 Examples of Lévy White Noise

Let us now apply Theorem 7.2 to actually construct examples of Lévy white noise on the space of tempered distributions $\mathcal{S}'$. For this we will use the examples of infinitely divisible distributions we discussed in section 5. Namely the Gaussian, the Poisson, the Compound Poisson and the $S\alpha S$ distributions discussed in section 6.3. With the Lévy–Khintchine representation at hand (see Table 1), all we have to do is check condition (18) for the corresponding Lévy measures.

- The Lévy exponent for the Gaussian distribution is given by

$$f(z) = \mathrm{i}\gamma z - \frac{1}{2}\sigma^2|z|^2, \quad \sigma^2 > 0, \gamma \in \mathbb{R}$$

  with corresponding Lévy triplet $(\gamma, \sigma^2, 0)$. Thus, condition (18) is trivially fulfilled for all $\epsilon > 0$. So by Theorem 7.2 there exists a Lévy white noise on $\mathcal{S}'$ whose characteristic function is determined by the Gaussian distribution, i.e.

$$\mathcal{L}(\varphi) = \exp\left(\mathrm{i}\gamma \int\limits_{\mathbb{R}^d} \varphi(s)\, ds - \frac{1}{2}\sigma^2\|\varphi\|_2^2\right), \quad \forall \varphi \in \mathcal{S}. \tag{21}$$

- For the Poisson distribution we have the Lévy triplet $(1, 0, \lambda\delta_1)$ with $\lambda > 0$. As before condition (18) trivially holds for all $\epsilon > 0$. Hence, we have a Lévy white noise with characteristic function

$$\mathcal{L}(\varphi) = \exp\left(\lambda \int\limits_{\mathbb{R}^d} (e^{i\varphi(s)} - 1)\, ds\right), \quad \forall \varphi \in \mathcal{S}.$$

- In case of the Compound Poisson distribution the Lévy measure is of the form $\lambda\sigma$ where $\lambda > 0$ and σ is a probability measure on $\mathbb{R}^d$. In this full generality the Lévy-Schwarz condition is not guaranteed. Condition (18) only holds if σ is a Lévy-Schwartz measure itself, i.e., it satisfies condition (18). If so, then the Compound Poisson distribution determines a Lévy white noise with characteristic function

$$\mathcal{L}(\varphi) = \exp\left(\lambda \int\limits_{\mathbb{R}^d} (\hat{\sigma}(\varphi(s)) - 1)\, ds\right), \quad \forall \varphi \in \mathcal{S}. \tag{22}$$

- Lastly the symmetric and stable $S\alpha S$ distributions with $0 < \alpha < 2$ are generated by the Lévy exponent $f(z) = -\sigma^\alpha |z|^\alpha$ with $\sigma > 0$ and Lévy measure

$$d\nu(x) = \frac{C_{\alpha,\sigma}}{|x|^{1+\alpha}} dx, \quad C_{\alpha,\sigma} > 0.$$

  To check condition (18) we compute

$$\int\limits_{|x|>1} |x|^\epsilon\, d\nu(x) = C_{\alpha,\sigma} \int\limits_{|x|>1} |x|^{\epsilon-(1+\alpha)}\, dx < \infty$$

which holds for $\epsilon < \alpha$. Thus, Theorem 7.2 applies and gives us the existence of a Lévy white noise with characteristic function

$$\mathcal{L}(\varphi) = \exp(-\sigma^\alpha \|\varphi\|_\alpha^\alpha), \quad \forall \varphi \in \mathcal{S}.$$

Distribution	Lévy triplet	Lévy exponent
Gaussian $\sigma^2 > 0, \gamma \in \mathbb{R}$	$(\gamma, \sigma^2, 0)$	$f(z) = \mathrm{i}\gamma z - \frac{1}{2}\sigma^2 \lvert z\rvert^2$
Poisson $\lambda > 0$	$(1, 0, \lambda\delta_1)$	$f(z) = \lambda(e^{\mathrm{i}z} - 1)$
Compound Poisson $\lambda > 0, \sigma \in \mathcal{P}(\mathbb{R}^d)$	$(\int_{\lvert x\rvert \leq 1} x d\sigma(x), 0, \lambda\sigma)$	$f(z) = \lambda(\hat{\sigma}(z) - 1)$
$S\alpha S$ $\alpha \in (0,2), \sigma > 0$	$(0, 0, \frac{C_{\alpha,\sigma}}{\lvert x\rvert^{1+\alpha}} dx)$	$f(z) = -\sigma^\alpha \lvert z\rvert^\alpha$

Table 1: Infinitely divisible distributions with Lévy triplet and exponent

7.3 Lévy White Noise as a distributional Derivative

7.3.1 Gaussian White Noise

We have seen that the distribution of generalized random fields are determined by their finite dimensional projections. In general we call a generalized random field X Gaussian if for any $\varphi_1, ..., \varphi_n$ the collection $\langle X, \varphi_1\rangle, ...,$ $\langle X, \varphi_n\rangle$ has joint Gaussian distribution. Similarly to a Gaussian Process it is determined by the mean functional $m(\varphi) = \mathbb{E}[\langle X, \varphi\rangle]$ and the covariance functional $\mathrm{Cov}(\varphi, \psi) = \mathbb{E}[(\langle X, \varphi\rangle - m(\varphi))(\langle X, \psi\rangle - m(\psi))]$. The classical Gaussian white noise, or often just called white noise, is given by mean functional $m(\varphi) = 0$ and covariance functional $\mathrm{Cov}(\varphi, \psi) = \int \varphi\psi$ ([22], [18]). We have seen above that it is a Lévy white noise with characteristic functional $\mathcal{L}(\varphi) = \exp(-1/2\|\varphi\|_2^2)$. It is a central tool to model random background noises ([26]). In the following we aim to give an explicit construction of Gaussian white noise as the distributional derivative of Brownian motion. It is a well known fact and can for example be found in [22], however we provide a rigouors treatment for our framework of generalized random fields.

Let $B = (B_t)_{t\geq 0}$ be a Brownian motion. As $t \mapsto B_t$ is almost surely continuous, we can consider $B_t(\omega) \in \mathcal{D}'(\mathbb{R}_{\geq 0})$ for almost every $\omega \in \Omega$ by the identification

$$\langle B,\varphi\rangle(\omega) := \langle B(\omega),\varphi\rangle = \int_0^\infty B_t(\omega)\varphi(t)\,dt, \quad \forall\varphi\in\mathcal{D}(\mathbb{R}_{\geq 0}). \tag{23}$$

Note that by this $\langle B,\varphi\rangle$ is only defined on some subset $A\subset\Omega$ of measure one. If we set $\langle B,\varphi\rangle$ equal to zero on $\Omega\setminus A$ for all $\varphi\in\mathcal{D}(\mathbb{R}_{\geq 0})$, then it is a well-defined random variable. Furthermore, one can show ([37]) that $B_t/t\to 0$ a.s. and thus, Brownian motion B_t satisfies condition (6) on a set of measure one and is therefore almost surely tempered. So we can define (23) also for $\varphi\in\mathcal{S}(\mathbb{R}_{\geq 0})$. As a distribution, we can form the distributional derivative of a Brownian motion as

$$\langle B',\varphi\rangle = -\langle B,\varphi'\rangle = -\int_0^\infty B_t\varphi'(t)\,dt, \quad \forall\varphi\in\mathcal{S}(\mathbb{R}_{\geq 0}). \tag{24}$$

We want to show that B' is a generalized random field in the sense of definition 5.1 and that it coincides with the Lévy white noise determined by the Gaussian distribution with $\gamma=0,\sigma^2=1$. Before we do so we want to extend this definition from $\mathbb{R}_{\geq 0}$ to the whole real line. For this let $(\tilde{B}_t)_{t\geq 0}$ be another Brownian motion independent to $(B_t)_{t\geq 0}$. We define for $\varphi\in\mathcal{S}(\mathbb{R})=\mathcal{S}$,

$$\begin{aligned}\langle B,\varphi\rangle &= \int_0^\infty B_t\varphi(t)\,dt + \int_{-\infty}^0 (-\tilde{B}_{-t})\varphi(t)\,dt\\ &= \int_0^\infty B_t\varphi(t)\,dt - \int_0^\infty \tilde{B}_t\varphi(-t)\,dt.\end{aligned}$$

And accordingly for the derivative we obtain

$$\langle B',\varphi\rangle = -\int_0^\infty B_t\varphi'(t)\,dt + \int_0^\infty \tilde{B}_t\varphi'(-t)\,dt. \tag{25}$$

Let us first show that these are generalized random fields according to our definition.

Lemma 7.3. *Brownian motion B and its derivative B' are generalized random fields on $\mathcal{S}'$.*

Proof. Since B' is defined by duality it is enough to show that $B : \omega \in \Omega \mapsto (\hat{B}(\omega))_{t\in\mathbb{R}} \in \mathcal{S}'$, where $\hat{B}_t = B_t$ for $t \geq 0$ and $\hat{B}_t = -\tilde{B}_{-t}$ for $t < 0$, is a measurable map $(\Omega, \mathcal{A}) \to (\mathcal{S}', \mathcal{B}(\mathcal{S}'))$. Let $\Psi : (\mathcal{S}', \tau_s) \to (\mathbb{S}', \pi_s)$ be the topological Isomorphism from Proposition 4.2, then $\mathcal{B}(\mathcal{S}') = \Psi^{-1}(\mathcal{B}(\mathbb{S}'))$ and also $\mathcal{B}(\mathbb{S}') = \mathcal{C}(\mathbb{S}')$. Hence, $\mathcal{B}(\mathcal{S}')$ is generated by the cylindrical sets

$$\{F \in \mathcal{S}' \mid (\langle F, \varphi_1\rangle, ..., \langle F, \varphi_n\rangle) \in B\}$$

for some $\varphi_1, ..., \varphi_n \in \mathcal{S}$ and $B \in \mathcal{B}(\mathbb{R}^n)$. So let $\varphi_1, ..., \varphi_n \in \mathcal{S}$ and $B \in \mathcal{B}(\mathbb{R}^n)$, then

$$\begin{aligned} B^{-1}(\{F \in \mathcal{S}' \mid (\langle F, \varphi_1\rangle, ..., \langle F, \varphi_n\rangle) \in B\}) \\ = \{\omega \in \Omega \mid (\langle B, \varphi_1\rangle(\omega), ..., \langle B, \varphi_n\rangle(\omega)) \in B\}. \end{aligned}$$

But $\langle B, \varphi\rangle$ are measurable for any $\varphi \in \mathcal{S}$ and hence, the above set is measurable as well. Since the cylindrical sets generate the whole σ-algebra $\mathcal{B}(\mathcal{S}')$ we can conclude the measurability of B.

□

To characterize B' we compute its characteristic functional. For this we study how B' acts on $\mathcal{S}$. We will do this in a couple of steps. To start, we need a basic property of Brownian motion. Let $(B_t)_{t\geq 0}$ be a Brownian motion, then it is a Gaussian process, i.e. for every $t_1, ..., t_n \geq 0$ the collection $B_{t_1}, ..., B_{t_n}$ has a jointly normal distribution and correlation $\mathbb{E}[B_t B_s] = \min\{t, s\}$ for all $t, s \geq 0$ ([37]).

Lemma 7.4. *Let $(B_t)_{t\geq 0}$ be a Brownian motion and $\varphi \in \mathcal{S}(\mathbb{R})$, then $\int_0^\infty B_t\varphi(t)\,dt$ is normally distributed with mean 0 and for $\psi \in \mathcal{S}(\mathbb{R})$ we have the correlation formula*

$$\mathbb{E}\left[\int_0^\infty B_t\varphi(t)\,dt \int_0^\infty B_s\psi(s)\,ds\right] = \int_0^\infty\int_0^\infty min\{t, s\}\varphi(t)\psi(s)\,dtds.$$

Proof. First, let $\varphi \in \mathcal{D}(\mathbb{R})$ and $T > 0$, such that $\mathrm{supp}(\varphi) \in [-T, T]$. By continuity of B_t and φ we can approximate the integral by a sequence of finite Riemann sums. We have

$$\int_0^\infty B_t\varphi(t)\,dt = \int_0^T B_t\varphi(t)\,dt = \lim_{n\to\infty}\sum_{i=0}^{k_n} B_{t_i^{(n)}}\varphi(t_i^{(n)})(t_{i+1}^{(n)} - t_i^{(n)}), \qquad (26)$$

where for every $n \geq 1$, $0 = t_0^{(n)} < ... < t_{k_n}^{(n)} = T$ is a partition of $[0, T]$ such that $\sup\limits_{i=0,...,k_n}(t_{i+1}^{(n)} - t_i^{(n)}) \to 0$ for $n \to \infty$. Note that the limit holds almost surely. Since $(B_t)_{t\geq 0}$ is a Gaussian process, the sums in (26) are normally distributed. As the almost sure limit of normal distributed random variables is again normal distributed, we conclude the normal distribution of $\int_0^\infty B_t\varphi(t)\,dt$ for $\varphi \in \mathcal{D}(\mathbb{R})$. To extend this to Schwartz functions, we use density of $\mathcal{D} \subset \mathcal{S}$. So let $\varphi \in \mathcal{S}(\mathbb{R})$ and $(\varphi_n)_{n\geq 1} \subset \mathcal{D}(\mathbb{R})$ such that $\varphi_n \to \varphi$ in $\mathcal{S}(\mathbb{R})$.

Since $B_t \sim N(0,t)$, it follows that $\mathbb{E}[|B_t|] = \sqrt{2t/\pi}$. Hence,

$$\int_1^\infty \mathbb{E}\left[\frac{|B_t|}{t^2}\right]\,dt = \sqrt{\frac{2}{\pi}}\int_1^\infty t^{-1/2}\,dt < \infty$$

and so by Fubini-Tonelli $\mathbb{E}[\int_1^\infty |B_t|/t^2\,dt] < \infty$ and thus $\int_1^\infty |B_t|/t^2\,dt < \infty$ almost surely. Now we can estimate

$$\begin{aligned}\left|\int_0^\infty B_t\varphi(t)\,dt - \int_0^\infty B_t\varphi_n(t)\,dt\right| &\leq \int_0^1 |B_t||\varphi(t)-\varphi_n(t)|\,dt \\ &\quad + \int_1^\infty |B_t||\varphi(t)-\varphi_n(t)|\,dt \\ &\leq \sup_{t\in[0,1]}|B_t|\|\varphi-\varphi_n\|_\infty + \int_1^\infty \frac{|B_t|}{t^2}\,dt\|\varphi-\varphi_n\|_{2,0}.\end{aligned}$$

By convergence of $\varphi_n \to \varphi$ in $\mathcal{S}$, it therefore follows that

$$\int_0^\infty B_t\varphi_n(t)\,dt \to \int_0^\infty B_t\varphi(t)\,dt \quad (n\to\infty) \text{ a.s.}$$

and we can conclude that $\int_0^\infty B_t\varphi(t)\,dt$ is normal distributed for all $\varphi \in \mathcal{S}(\mathbb{R})$. By Fubini-Tonelli we then get

$$\mathbb{E}\left[\int_0^\infty B_t\varphi(t)\,dt\right] = \int_0^\infty \mathbb{E}\left[B_t\right]\varphi(t)\,dt = 0$$

and

$$\begin{aligned}\mathbb{E}\left[\int_0^\infty B_t\varphi(t)\,dt\int_0^\infty B_s\psi(s)\,ds\right] &= \int_0^\infty\int_0^\infty \mathbb{E}\left[B_tB_s\right]\varphi(t)\psi(s)\,dtds \\ &= \int_0^\infty\int_0^\infty \min\{t,s\}\varphi(t)\psi(s)\,dtds.\end{aligned}$$

$\square$

With the help of this Lemma it is now easy to determine the characteristic functional of B'. Firstly, for $\varphi, \psi \in \mathcal{S}$ we compute

$$\mathbb{E}\left[\int_0^\infty B_t\varphi'(t)\,dt \int_0^\infty B_s\psi'(s)\,ds\right] = \int_0^\infty\int_0^\infty \min\{t,s\}\varphi'(t)\psi'(s)\,dtds$$

$$= \int_0^\infty\int_0^t s\varphi'(t)\psi'(s)\,dsdt + \int_0^\infty\int_t^\infty t\varphi'(t)\psi'(s)\,dsdt$$

$$= \int_0^\infty \varphi'(t)\left(t\psi(t) - \int_0^t \psi(s)\,ds\right)\,dt - \int_0^\infty \varphi'(t)t\psi(t)\,dt$$

$$= -\int_0^\infty\int_0^t \varphi'(t)\psi(s)\,dsdt = -\int_0^\infty \psi(s)\int_s^\infty \varphi'(t)\,dtds$$

$$= \int_0^\infty \psi(s)\varphi(s)\,ds.$$

From Lemma 7.4 it follows that $\int_0^\infty B_t\varphi'(t)\,dt \sim N(0, \int_0^\infty \varphi^2(t)\,dt)$ and by the same computation

$$\mathbb{E}\left[\int_0^\infty \tilde{B}_t\varphi'(-t)\,dt \int_0^\infty \tilde{B}_s\psi'(-s)\,ds\right] = \int_{-\infty}^0 \varphi(s)\psi(s)\,ds,$$

so in particular we have $\int_0^\infty \tilde{B}_s\varphi'(-s)\,ds \sim N(0, \int_{-\infty}^0 \varphi^2(s)\,ds)$ for all $\varphi \in \mathcal{S}$. Since $(B_t)_{t\geq 0}$ and $(\tilde{B}_s)_{s\geq 0}$ are independent we have $\mathbb{E}[B_t\tilde{B}_s] = \mathbb{E}[B_t]\mathbb{E}[\tilde{B}_S] = 0$ for all $t, s \geq 0$. Hence for all $\varphi, \psi \in \mathcal{S}$ we have by Fubini-Tonelli

$$\mathbb{E}\left[\int_0^\infty B_t\varphi'(t)\,dt \int_0^\infty \tilde{B}_s\psi'(-s)\,ds\right] = \int_0^\infty\int_0^\infty \varphi'(t)\psi'(-s)\mathbb{E}[B_t\tilde{B}_s]\,dsdt = 0$$

and so the two integrals in (25) are uncorrelated and therefore independent. Hence, we conclude that $\langle B', \varphi\rangle \sim N(0, \|\varphi\|_2^2)$ for all $\varphi \in \mathcal{S}$. We have proven the following.

Proposition 7.1. *The characteristic functional of B' as a generalized random field on $\mathcal{S}'$ is given by*

$$\mathcal{L}_{B'}(\varphi) = \mathbb{E}\left[e^{i\langle B',\varphi\rangle}\right] = exp\left(-\frac{1}{2}\|\varphi\|_2^2\right), \quad \forall\varphi \in \mathcal{S}.$$

In particular, $\langle B', \varphi\rangle \sim N(0, \|\varphi\|_2^2)$ for all $\varphi \in \mathcal{S}$.

We can see that this coincides exactly with the Lévy white noise characterized by the Gaussian distribution with $\gamma = 0, \sigma^2 = 1$ (21), which we constructed using the Minlos-Bochner theorem. However, this approach allows us to make an important observation. By the independence of the two integrals in (25) and the above computation, we get for all $\varphi, \psi \in \mathcal{S}$ the correlation formula,

$$\mathbb{E}[\langle B', \varphi\rangle\langle B', \psi\rangle] = \int_{\mathbb{R}} \varphi(t)\psi(t)\, dt.$$

So, it is clear that when φ and ψ have disjoint support, then $\langle B', \varphi\rangle$ and $\langle B', \psi\rangle$ are uncorrelated and hence independent. We can therefore view gaussian white noise as the (distributional) derivative of Brownian motion.

Remark. We can also approach equation (25) by the means of Ito-integration theory. Since we do not introduce this theory in this thesis, we just want to make this remark. The advantage this approach has, is that we can identify the action of Brownian motion on the Schwartz space (25) as integration with respect to Brownian motion. However, this extends to all L^2 functions, that is we can define $\langle B', f\rangle$ for all $f \in L^2$. In particular it is then easy to see that $\langle B', \mathbb{1}_{(0,t]}\rangle = B_t$. In this way we can recover the Brownian motion $(B_t)_{t\geq 0}$ as a special realization of the random functional B'.

7.3.2 Compound Poisson White Noise

Similarly, we want to express the Compound Poisson Lévy white noise as the distributional derivative of a Compound Poisson process $X = (X_t)_{t\geq 0}$. By definition we have

$$X_t = \sum_{i=1}^{N_t} Z_i,$$

where $(Z_i)_{i\geq 1}$ are i.i.d random variables with common law $\sigma \in \mathcal{P}(\mathbb{R})$ and N_t is a Poisson process with rate $\lambda > 0$ independent to $(Z_i)_{i\geq 1}$, i.e. N_t is a Poisson random variable with parameter λt. We can view N_t as the number of random events $(x_i)_{i\geq 1}$ which happen in the time interval $(0,t]$. The events $(x_i)_{i\geq 0}$ are independent and identically distributed and happen with rate λ, i.e. the waiting times $s_1 = x_1, s_2 = x_2 - x_1, ...$ are independent and have a exponential(λ) distribution. Furthermore, if we condition on $N_t = n$, then $0 < x_1 < ... < x_n \leq t$ are independent and uniformly distributed in $(0,t]$. Also, $(Z_i)_{i\geq 1}$ and $(x_i)_{i\geq 1}$ are independent by assumption. Thus, we can also express X_t as

$$X_t = \sum_{i\geq 1} Z_i \mathbb{1}_{(0,t]}(x_i)$$

for $t > 0$ and $X_0 = 0$. Since we don't have any regularity assumption on the sample paths of a compound Poisson process, as we have for the Brownian motion, it is not clear if wether we can make sense of X_t as a distribution. However, we know that every Lévy process has a modification which almost surely has right continuous paths for $t \geq 0$ and left limits for $t > 0$. So let us assume that this is true for $(X_t)_{t\geq 0}$. Then it is almost surely locally integrable and can therefore by considered as a distribution by

$$\langle X, \varphi \rangle := \int_0^\infty X_t \varphi(t)\, dt, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}_{\geq 0}).$$

Again, $\langle X, \varphi \rangle$ is only defined on a set of measure one, so we extend to Ω by setting $\langle X, \varphi \rangle = 0$ on the corresponding complement. The question if wether it is also tempered is a more difficult one. One easy condition is that if σ has finite first moment, then $\mathbb{E}[|X_1|] < \infty$ and then by [36, Theorem 36.5] $\lim\limits_{t\to\infty} X_t/t = \mathbb{E}[X_1]$ almost surely. In this case X_t almost surely satisfies (6) and is therefore tempered. The question in general is answered in [8, Corollary 2.6]. The compound Poisson process $(X_t)_{t\geq 0}$ is almost surely tempered if and only if X_1 has some positive absolute moment, which by [36, Theorem 25.3] is equivalent to the Lévy-Schwartz condition (18) for the Lévy measure σ of $(X_t)_{t\geq 0}$. This shows that this approach is consistent with our previous results. However, our interest does not lie in this question of regularity, but we hope to gain some useful inside by expressing the Compound Poisson Lévy white noise as a derivative of a Compound Poisson process. First, let us compute the distributional derivative of X. Let $\varphi \in \mathcal{D}(\mathbb{R}_{\geq 0})$ and $T > 0$ such that $\text{supp}(\varphi) \subset [0,T]$, then

$$\langle X', \varphi \rangle = -\int_0^\infty \sum_{i\geq 1} Z_i \mathbb{1}_{(0,t]}(x_i) \varphi'(t)\, dt.$$

Since for all $t \in [0, T]$

$$\left| \sum_{i \geq 1} Z_i \mathbb{1}_{(0,t]}(x_i) \varphi'(t) \right| \leq \left(\sum_{i \geq 1} |Z_i| \mathbb{1}_{(0,T]}(x_i) \right) \sup_{t \in [0,T]} |\varphi'(t)| < \infty \quad \text{a.s.}$$

we can exchange summation and integration and obtain

$$\begin{aligned} \langle X', \varphi \rangle &= -\sum_{i \geq 1} Z_i \int_0^\infty \mathbb{1}_{(0,t]}(x_i) \varphi'(t)\, dt = -\sum_{i \geq 1} Z_i \int_{x_i}^\infty \varphi'(t)\, dt \\ &= \sum_{i \geq 1} Z_i \varphi(x_i) = \left\langle \sum_{i \geq 1} Z_i \delta(\cdot - x_i), \varphi \right\rangle, \end{aligned}$$

where δ denotes the delta distribution defined by $\delta(\varphi) = \varphi(0)$ for all $\varphi \in \mathcal{D}(\mathbb{R}_{\geq 0})$. Hence, in the sense of distributions we have

$$X' = \sum_{i \geq 1} Z_i \delta(\cdot - x_i) \in \mathcal{D}'(\mathbb{R}_{\geq 0}),$$

We can view this as a random measure (see Figure 2) in the sense that for $A \in \mathcal{B}(\mathbb{R}_{\geq 0})$ we have

$$X'(A) := \sum_{i \geq 1} Z_i \delta(\mathbb{1}_A - x_i) = \sum_{i \geq 1} Z_i \delta_{x_i}(A).$$

The action of X' on a test-function can be understood as integration with respect to this random measure.

To extend this from $\mathbb{R}_{\geq 0}$ to $\mathbb{R}$, let $(\tilde{X}_t)_{t \geq 0}$ be a Compound Poisson process

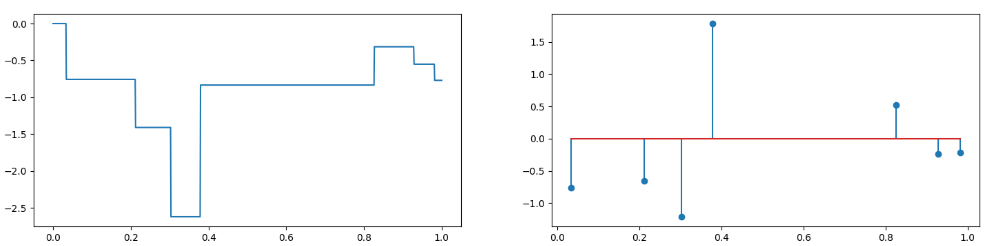


Figure 2: Sample path of a Compound Poisson Process (left) and a visualization of its derivative (right). Every jump of the process corresponds to an "impulse", i.e. a delta distribution with amplitude determined by the jump height.

identically distributed but independent to $(X_t)_{t\geq 0}$. That is $\tilde{X}_t$ is given by

$$\tilde{X}_t = \sum_{i\geq 1} \tilde{Z}_i \mathbb{1}_{(0,t]}(\tilde{x}_i),$$

where $(x_i)_{i\geq 1}$ and $(\tilde{x}_i)_{i\geq 1}$ as well as $(Z_i)_{i\geq 1}$ and $(\tilde{Z}_i)_{i\geq 1}$ are independent and identically distributed. Then we can define X almost surely as a an element of $\mathcal{D}'(\mathbb{R})$ by

$$\begin{aligned}\langle X, \varphi\rangle &= \int_0^\infty X_t \varphi(t)\, dt + \int_{-\infty}^0 (-X_{-t})\varphi(t)\, dt \\ &= \int_0^\infty X_t \varphi(t)\, dt + \int_0^\infty (-X_t)\varphi(-t)\, dt,\end{aligned}$$

for all $\varphi \in \mathcal{D}(\mathbb{R})$. Analogously to the above computation we can compute its derivative to obtain (in the sense of distributions)

$$X' = \sum_{i\geq 1} Z_i \delta(\cdot - x_i) + \sum_{i\geq 1} \tilde{Z}_i \delta(\cdot + \tilde{x}_i). \tag{27}$$

By the same proof to Lemma 7.3 we see that X and thus X' are generalized random fields. Next, we want to show that X' and the Lévy white noise generated by the underlying compound Poisson distribution coincide. To do so we compute the characteristic functional of X'.
Let $\varphi \in \mathcal{D}(\mathbb{R})$ and $T > 0$ such that $\mathrm{supp}(\varphi) \subset [-T, T]$. Furthermore, let N_φ be the number of events x_i which occur in $(0, T]$, i.e. $x_i \leq T$ for all $i = 1, ..., N_\varphi$ and $x_i > T$ for $i > N_\varphi$. It has a Poisson distribution with parameter λT. Analogously, let $\tilde{N}_\varphi$ be the number of events $\tilde{x}_1, ..., \tilde{x}_{\tilde{N}_\varphi} \in (0, T]$. By (27) we then have

$$\langle X', \varphi\rangle = \sum_{i=1}^{N_\varphi} Z_i \varphi(x_i) + \sum_{j=1}^{\tilde{N}_\varphi} \tilde{Z}_j \varphi(-\tilde{x}_j).$$

To simplify notation let us denote the first sum as $\langle X'_+, \varphi\rangle$ and the second by $\langle X'_-, \varphi\rangle$. Then, by independence we have

$$\mathbb{E}[e^{\mathrm{i}\langle X', \varphi\rangle}] = \mathbb{E}[e^{\mathrm{i}\langle X'_+, \varphi\rangle}]\mathbb{E}[e^{\mathrm{i}\langle X'_-, \varphi\rangle}].$$

Using the properties of the conditional expectation we obtain for the first term

$$\begin{aligned}\mathbb{E}\left[e^{\mathrm{i}\langle X'_+,\varphi\rangle}\right] &= \mathbb{E}\left[\mathbb{E}\left[e^{\mathrm{i}\langle X'_+,\varphi\rangle} \mid \sigma(N_\varphi)\right]\right] = \mathbb{E}\left[\mathbb{E}\left[\prod_{i=1}^{N_\varphi} e^{\mathrm{i}Z_i\varphi(x_i)} \mid \sigma(N_\varphi)\right]\right] \\ &= \mathbb{E}\left[\prod_{i=1}^{N_\varphi} \mathbb{E}\left[e^{\mathrm{i}Z_i\varphi(x_i)} \mid \sigma(N_\varphi)\right]\right] = \mathbb{E}\left[\prod_{i=1}^{N_\varphi} \mathbb{E}\left[e^{\mathrm{i}Z_1\varphi(x)}\right]\right],\end{aligned}$$

where $x \sim$ Uniform(0,T]. Let $M_{Z_1,x} := \mathbb{E}\left[e^{\mathrm{i}Z_1\varphi(x)}\right]$, then

$$\begin{aligned}\mathbb{E}\left[e^{\mathrm{i}\langle X'_+,\varphi\rangle}\right] = \mathbb{E}\left[M_{Z_1,x}^{N_\varphi}\right] &= \sum_{k\geq 0} M_{Z_1,x}^k \frac{e^{-\lambda T}(\lambda T)^k}{k!} \\ &= e^{-\lambda T}\sum_{k\geq 0}\frac{(\lambda T M_{Z_1,x})^k}{k!} \\ &= \exp\left(\lambda T(M_{Z_1,x}-1)\right).\end{aligned}$$

Furthermore,

$$\begin{aligned}M_{Z_1,x} &= \mathbb{E}\left[\mathbb{E}\left[e^{\mathrm{i}Z_1\varphi(x)} \mid \sigma(Z_1)\right]\right] = \mathbb{E}\left[\mathbb{E}\left[e^{\mathrm{i}z\varphi(x)}\right]\Big|_{z=Z_1}\right] \\ &= \mathbb{E}\left[\frac{1}{T}\int_0^T e^{\mathrm{i}Z_1\varphi(x)}\,dx\right] = \frac{1}{T}\int_0^T \mathbb{E}\left[e^{\mathrm{i}Z_1\varphi(x)}\right]\,dx \\ &= \frac{1}{T}\int_0^T\int_{\mathbb{R}} e^{\mathrm{i}z\varphi(x)}\,d\sigma(z)dx\end{aligned}$$

and so in total we have

$$\mathbb{E}\left[e^{\mathrm{i}\langle X'_+,\varphi\rangle}\right] = \exp\left(\lambda\int_0^T\int_{\mathbb{R}}(e^{\mathrm{i}z\varphi(x)}-1)\,d\sigma(z)dx\right) = \exp\left(\lambda\int_0^T(\hat{\sigma}(\varphi(x))-1)\,dx\right).$$

And by the same computation

$$\mathbb{E}\left[e^{\mathrm{i}\langle X'_-,\varphi\rangle}\right] = \exp\left(\lambda\int_{-T}^0(\hat{\sigma}(\varphi(x))-1)\,dx\right),$$

and so

$$\mathcal{L}_{X'}(\varphi) = \mathbb{E}\left[e^{\mathrm{i}\langle X',\varphi\rangle}\right] = \exp\left(\lambda \int\limits_{-T}^{T} (\hat{\sigma}(\varphi(x)) - 1)\, dx\right)$$
$$= \exp\left(\lambda \int\limits_{\mathbb{R}} (\hat{\sigma}(\varphi(x)) - 1)\, dx\right), \tag{28}$$

where for the last equality we used the fact that $\hat{\sigma}(0) = 1$, hence we can extend the integration domain from $\mathrm{supp}(\varphi)$ to the whole real line. For now this expression only holds for $\varphi \in \mathcal{D}(\mathbb{R})$. However, if we assume σ to be a Lévy-Schwartz measure, i.e. condition (18) holds, then by [8, Corollary 2.6] X' is almost surely tempered and we can extend (28) to $\mathcal{S}(\mathbb{R})$ by using Lemma 7.2 and the fact that $\mathcal{D} \subset \mathcal{S}$ is dense. We summarize our results in the following Proposition.

Proposition 7.2. *Let $(X_t)_{t\geq 0}$ be a Compound Poisson process with underlying distribution σ and rate $\lambda > 0$. Let σ be a Lévy-Schwartz measure. Then the distributional derivative X' (27) of the process is a generalized random field on $\mathcal{S}'$ with characteristic functional*

$$\mathcal{L}_{X'}(\varphi) = exp\left(\lambda \int\limits_{\mathbb{R}} (\hat{\sigma}(\varphi(x)) - 1)\, dx\right), \quad \forall \varphi \in \mathcal{S}.$$

Again we observe that the characteristic functional of X' is the same as the one of Lévy white noise (22) that we constructed of the Compound Poisson distribution by using the Minlos-Bochner theorem. We can also make two similar remarks to those we made for the Brwonian motion. We have seen that the action of the derivative of Brownian motion on a Schwartz function gives us again a centered Gaussian random variable, where the variance depends on the "size" of the function. A similar result holds in this case too. For $\varphi \in \mathcal{S}$ we see that $\langle X', \varphi\rangle$ has a Compound Poisson distribution by computing its characteristic function:

$$\mathcal{L}_{\langle X',\varphi\rangle}(z) = \mathbb{E}\left[e^{\mathrm{i}\langle X',z\varphi\rangle}\right] = \exp\left(\lambda \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} (e^{\mathrm{i}z\varphi(s)x} - 1)\, d\nu(x) ds\right)$$
$$= \exp\left(\lambda \int\limits_{\mathbb{R}} (e^{\mathrm{i}zy} - 1)\, d\nu_{\varphi}(y)\right)$$

where the measure ν_φ is given by

$$\nu_\varphi(A) = \int\limits_{\mathrm{supp}(\varphi)} (m_{\varphi(s)})_* \nu(A)\, ds, \quad \forall A \in \mathcal{B}(\mathbb{R})$$

and $(m_{\varphi(s)})_*\nu$ denotes the push-forward of ν by the multiplication map with $\varphi(s)$, i.e. $m_{\varphi(s)} : x \mapsto \varphi(s)x$.
Furthermore, by the expression of X' as a random measure (27) we can let X' act on indicator functions. In particular, we have

$$X'((0,t]) = \langle X', \mathbb{1}_{(0,t]}\rangle = \sum_{i\geq 1} Z_i \mathbb{1}_{(0,t]}(x_i) = X_t.$$

So, as for the Brownian motion, we can realize the Compound Poisson process X_t by acting on indicator functions of the form $\mathbb{1}_{(0,t]}$.

In the example of Brownian motion and a Compound Poisson process we have seen that we can give an explicit construction of Lévy white noise by taking the distributional derivative of the Lévy process and that both approaches are equivalent. But even more, we could identify the Lévy process itself as a special realization of this more general random functional. This works not only for these two examples, but for any Lévy process.
Let $(X_t)_{t\geq 0}$ be a one dimensional Lévy process with characteristic triplet (γ, σ^2, ν). We can express the characteristic exponent as

$$\begin{aligned} f(z) &= \mathrm{i}\gamma z - \frac{1}{2}\sigma^2 z^2 + \int\limits_{\mathbb{R}\setminus\{0\}} (e^{\mathrm{i}zx} - 1 - \mathrm{i}xz\mathbb{1}_{|x|\leq 1})\, d\nu(x) \\ &= \mathrm{i}\gamma z - \frac{1}{2}\sigma^2 z^2 + \int\limits_{\mathbb{R}\setminus[-1,1]} (e^{\mathrm{i}zx} - 1)\, d\nu(x) + \int\limits_{(-1,1)\setminus\{0\}} (e^{\mathrm{i}zx} - 1 - \mathrm{i}xz)\, d\nu(x). \end{aligned}$$

This leads to the following decomposition of X_t corresponding to the four terms above:

$$X_t = \gamma t + \sigma B_t + Y_t + Z_t.$$

That is a linear drift term, a Brownian motion B_t, a Compound Poisson process Y_t with rate $\lambda_1 := \nu(\mathbb{R} \setminus [-1,1])$ and underlying distribution $\mu := 1/\lambda_1 \nu|_{\mathbb{R}\setminus[-1,1]}$ and another jump process Z_t corresponding to $\nu|_{(-1,1)\setminus\{0\}}$ (see [2, section 2.4] for a detailed description).
In the case when ν has finite first moment, i.e. $\int_{\mathbb{R}\setminus\{0\}} |x|\, d\nu(x) < \infty$, we can reduce the general case to that of the Brownian motion and the Compound Poisson process.

Theorem 7.3. *Let $(X_t)_{t\geq 0}$ be a one dimensional Lévy process with Lévy exponent f and characteristic triplet (γ, σ^2, ν) such that $\int_{\mathbb{R}\setminus\{0\}} |x|\, d\nu(x) < \infty$. Then by the above identification it defines a generalized random field on $\mathcal{S}'$ and its derivative X' has characteristic functional*

$$\mathcal{L}_{X'}(\varphi) = exp\left(\int\limits_{\mathbb{R}} f(\varphi(s))\, ds \right), \quad \forall \varphi \in \mathcal{S}.$$

In particular, X' defines a Lévy white noise.

Proof. The idea of the proof is to decompose $(X_t)_{t\geq 0}$ as above and then by linearity treat each term separately. By assumption we have $\int_{\mathbb{R}\setminus\{0\}} |x|\, d\nu(x) < \infty$ and so we can write f as

$$\begin{aligned} f(z) &= \mathrm{i}\gamma z - \frac{\sigma^2}{2} z^2 + \int\limits_{\mathbb{R}\setminus\{0\}} (e^{\mathrm{i}xz} - 1 - \mathrm{i}xz\mathbb{1}_{|x|\leq 1})\, d\nu(x) \\ &= \mathrm{i} \underbrace{\left(\gamma - \int\limits_{|x|\leq 1} x\, d\nu(x) \right)}_{=:\tilde{\gamma}} z - \frac{\sigma^2}{2} z^2 + \int\limits_{\mathbb{R}\setminus\{0\}} (e^{\mathrm{i}xz} - 1)\, d\nu(x). \end{aligned}$$

We can identify the characteristic function of a linear drift term, a Brownian motion and a Compound Poisson process. By the Lévy-Ito decomposition ([2, Theorem 2.4.16]) we can decompose X_t accordingly

$$X_t = \tilde{\gamma} t + \sigma B_t + Y_t,$$

where B_t is a Brownian motion and Y_t is a Compound Poisson process with rate $\lambda := \nu(\mathbb{R} \setminus \{0\})$ and underlying distribution $\frac{1}{\lambda}\nu$. The key of the Lévy-Ito decomposition is that B_t and Y_t can be chosen to be independent. This allows us to treat each term separately. The linear term clearly is a tempered distribution with derivative $\tilde{\gamma}$, by Proposition 7.1 the Brownian motion is a generalized random field and we already computed its derivative. Since ν has finite first moment, it is a Lévy-Schwartz measure and Proposition 7.2 applies for Y_t. In total we see that the Process $(X_t)_{t\geq 0}$ and its distributional derivative X' define generalized random fields on $\mathcal{S}'$. By independence, we can compute the characteristic functional as

$$\begin{aligned}\mathcal{L}_{X'}(\varphi) &= \mathbb{E}\left[e^{\mathrm{i}(\langle\tilde{\gamma},\varphi\rangle+\sigma\langle B',\varphi\rangle+\langle Y',\varphi\rangle)}\right] = \mathbb{E}\left[e^{\mathrm{i}\langle\tilde{\gamma},\varphi\rangle}\right]\mathbb{E}\left[e^{\mathrm{i}\sigma\langle B',\varphi\rangle}\right]\mathbb{E}\left[e^{\mathrm{i}\langle Y',\varphi\rangle}\right] \\ &= \exp\left(\mathrm{i}\int_{\mathbb{R}}\tilde{\gamma}\varphi(s)\,ds\right)\exp\left(-\frac{\sigma^2}{2}\int_{\mathbb{R}}\varphi(s)^2\,ds\right) \\ &\quad \exp\left(\lambda\int_{\mathbb{R}}\int_{\mathbb{R}\setminus\{0\}}(e^{\mathrm{i}\varphi(s)x}-1)\frac{1}{\lambda}\,d\nu(x)ds\right) \\ &= \exp\left(\int_{\mathbb{R}}f(\varphi(s))\,ds\right)\end{aligned}$$

for all $\varphi \in \mathcal{S}$. This concludes the proof.

□

In the general case where ν might not have finite first moment, the simplification to decompose the Process in these three terms might not be possible. Instead, one has to deal with a fourth term, which is another jump process, similar to that of the Compound Poisson process. However, for a proof of the statement in full generality we refer to [8]. In this work one identifies the action of the derivative of a Lévy process as integrating with respect to the process itself. This involves some technical stochastic integration theory and we just want to mention this as a remark.

7.4 Properties of Lévy White Noise

In the following, we want to work out some properties of Lévy white noise. We have seen that we could identify the classical (Gaussian) white noise as a special case of Lévy white noise. The following properties will make it clear why Lévy white noise can be considered as an appropriate generalization of white noise. Remember, we have identified white noise as a generalized random field, whose action on a Schwartz function determines a centered gaussian random variable whose variance depends on its L^2-norm. Furthermore, these random variables are independent if the L^2-inner product of the corresponding function vanishes. This happens, for example, if they have disjoint support. In [18] and [11] Lévy white noise is motivated by two important properties. It has *independent values at every point* and is *stationary.*

Definition 7.2. *Let X be a generalized random field on $\mathcal{S}$. We say X has independent values at every point if for every $\varphi, \psi \in \mathcal{S}$ such that φ and ψ have* *disjoint support, the random variables $\langle X, \varphi\rangle$ and $\langle X, \psi\rangle$ are independent. We call X stationary if the shifted version $X(\cdot - \tau)$ for $\tau \in \mathbb{R}^d$, defined by $\langle X(\cdot - \tau), \varphi\rangle = \langle X, \varphi(\cdot + \tau)\rangle$, $\varphi \in \mathcal{S}$ is equal in distribution to X.*

Since both properties concern the distribution of X, we can translate this definition again to the characteristic functional. It is clear that X has independent values at every point if and only if

$$\mathcal{L}_X(\varphi + \psi) = \mathcal{L}_X(\varphi)\mathcal{L}_X(\psi) \tag{29}$$

for all $\varphi, \psi \in \mathcal{S}$ with disjoint support. To see this, let $\varphi, \psi \in \mathcal{S}$ with disjoint support. Then $\langle X, \varphi\rangle$ and $\langle X, \psi\rangle$ are independent if and only if the random vector $(\langle X, \varphi\rangle, \langle X, \psi\rangle)$ satisfies

$$\mathcal{L}_{(\langle X,\varphi\rangle, \langle X,\psi\rangle)}(t, s) = \mathcal{L}_{\langle X,\varphi\rangle}(t)\mathcal{L}_{\langle X,\psi\rangle}(s).$$

By linearity of $\langle X, \cdot\rangle$ this is equivalent to

$$\mathcal{L}_X(t\varphi + s\psi) = \mathcal{L}_X(t\varphi)\mathcal{L}_X(s\psi)$$

and we can see that this is equivalent to (29) by choosing $t = s = 1$ and conversely considering $t\varphi(\cdot)$ and $s\psi(\cdot)$. Remember, that the interpretation here is that, if you measure X through two disjoint observation windows, you get two independent random variables. Furthermore, X is stationary if and only if

$$\mathcal{L}_X(\varphi(\cdot + \tau)) = \mathcal{L}_X(\varphi), \quad \forall \varphi \in \mathcal{S}, \tau \in \mathbb{R}^d. \tag{30}$$

With the specific form of the characteristic functional of a Lévy white noise, both properties are easy to show.

Lemma 7.5. *Let $\dot{X}$ be a Lévy white noise, then it has independent values at every point and is stationary.*

Proof. To show that $\dot{X}$ has independent values at every point, we prove (29). So let $\varphi, \psi \in \mathcal{S}$ have disjoint support and let f be the Lévy exponent of $\dot{X}$. Since $f(0) = 0$ we have

$$\int_{\mathbb{R}^d} f(\varphi(s) + \psi(s))\, ds = \int_{\mathbb{R}^d} f(\varphi(s))\, ds + \int_{\mathbb{R}^d} f(\psi(s))\, ds$$

and so $\mathcal{L}_{\dot{X}}(\varphi + \psi) = \mathcal{L}_{\dot{X}}(\varphi)\mathcal{L}_{\dot{X}}(\psi)$. That $\dot{X}$ is stationary follows from a simple change of variables in the integral.

□

These two properties justify the name "noise" in the following sense. If we would discretize a Lévy white noise by testing it against shifted and non-overlapping "observation windows", we would get a sequence of i.i.d. random variables for every instance of the discretization.

In section 5 we introduced the concept of how a linear operator acts on a generalized random field (see equation 11). Nevertheless, if some operator acts on a Lévy white noise, it is not guaranteed that the resulting generalized random field is still a Lévy white noise. However, we can identify a linear structure.

We call two generalized random fields X_1 and X_2 *independent* if their finite-dimensional distributions are, i.e. for any $n \geq 1, \varphi_1, ..., \varphi_n \in \mathcal{S}$ the collections $(\langle X_1, \varphi_i \rangle)_{i=1,...,n}$ and $(\langle X_2, \varphi_i \rangle)_{i=1,...,n}$ are independent. Since these are described by the characteristic functional, this property translates to

$$\begin{aligned} \mathcal{L}_{X_1+X_2}(\varphi) &= \mathbb{E}[e^{\mathrm{i}\langle X_1+X_2,\varphi\rangle}] = \mathbb{E}[e^{\mathrm{i}\langle X_1,\varphi\rangle}e^{\mathrm{i}\langle X_2,\varphi\rangle}] \\ &= \mathbb{E}[e^{\mathrm{i}\langle X_1,\varphi\rangle}]\mathbb{E}[e^{\mathrm{i}\langle X_2,\varphi\rangle}] = \mathcal{L}_{X_1}(\varphi)\mathcal{L}_{X_2}(\varphi), \quad \forall \varphi \in \mathcal{S}. \end{aligned} \tag{31}$$

We have the following.

Proposition 7.3. *Let $\dot{X}$ be a Lévy white noise, then*

(i) the scaled generalized random field $\dot{X}(\cdot/b)$, for $b \neq 0$, defined by $\langle \dot{X}(\cdot/b), \varphi \rangle = \langle \dot{X}, b^d \varphi(b\,\cdot) \rangle$ for all $\varphi \in \mathcal{S}$, is again a Lévy white noise,

(ii) for $a \neq 0$ the generalized random field $a\dot{X}$ is a Lévy white noise,

(iii) if $\dot{Y}$ is a Lévy white noise independent to $\dot{X}$, then $\dot{X} + \dot{Y}$ is also a Lévy white noise.

Proof. Let f be the Lévy exponent to $\dot{X}$ with Lévy triplet (γ, σ^2, ν). To show that a generalized random field is a Lévy white noise we have to show that its characteristic functional is of the form (17) with a valid Lévy exponent. Furthermore, to show that it is well defined on $\mathcal{S}$, we have to check that the Lévy-Schwartz condition (18) holds for the corresponding Lévy measure.
To show *(i)* we identify the characteristic functional of $\dot{X}(\cdot/b)$ as

$$\begin{aligned} \mathcal{L}_{\dot{X}(\cdot/b)}(\varphi) = \mathbb{E}[e^{\mathrm{i}\langle \dot{X}, b^d\varphi(b\,\cdot)\rangle}] &= \exp\left(\int_{\mathbb{R}^d} f(b^d\varphi(bs))\,ds \right) \\ &= \exp\left(\int_{\mathbb{R}^d} \frac{1}{b^d} f(b^d\varphi(x))\,dx \right), \quad \varphi \in \mathcal{S}. \end{aligned}$$

So let $\tilde{f}(z) := \frac{1}{b^d} f(b^d z)$. To check that $\tilde{f}$ admits a Lévy-Khintchine representation and that the Lévy-Schwartz condition holds, we need an explicit form of the corresponding Lévy measure. Hence, we make the following computation to determine the Lévy triplet of $\tilde{f}$.

$$\begin{aligned}
\tilde{f}(z) = \frac{1}{b^d} f(b^d z) &= \mathrm{i}\gamma z - \frac{\sigma^2}{2} b^d z^2 + \int\limits_{\mathbb{R}\setminus\{0\}} \frac{1}{b^d}(e^{\mathrm{i}xb^d z} - 1 - \mathbb{1}_{|x|\le 1}\mathrm{i}xb^d z)\, d\nu(x) \\
&= \mathrm{i}\gamma z - \frac{\sigma^2}{2} b^d z^2 + \underbrace{\int\limits_{\mathbb{R}\setminus\{0\}} \frac{1}{b^d}(e^{\mathrm{i}xb^d z} - 1 - \mathbb{1}_{|b^d x|\le 1}\mathrm{i}xb^d z)\, d\nu(x)}_{=:\alpha_b(z)} \\
&\quad + \mathrm{i}z \underbrace{\int\limits_{\mathbb{R}\setminus\{0\}} \frac{1}{b^d}(\mathbb{1}_{|xb^d|\le 1} - \mathbb{1}_{|x|\le 1}) x\, d\nu(x)}_{=:\beta_b},
\end{aligned}$$

where the last equality only holds if both terms $\alpha_b(z)$ and β_b are finite. Let $\tilde{\nu} = \frac{1}{b^d}(m_{b^d})_*\nu$, where $(m_{b^d})_*\nu$ denotes the push-forward of ν by the multiplication map $m_b : x \mapsto b^d x$. Clearly, $\tilde{\nu}$ is a Lévy-Schwartz measure too and by Lemma 7.1 we have

$$\begin{aligned}
|\alpha_b(z)| &= \left| \int\limits_{\mathbb{R}\setminus\{0\}} (e^{\mathrm{i}xz} - 1 - \mathrm{i}xz\mathbb{1}_{|x|\le 1})\, d\tilde{\nu}(x) \right| \\
&\le z^2 \int\limits_{|x|\le 1} x^2\, d\tilde{\nu}(x) + 2^{1-\min\{\epsilon,1\}} |z|^{\min\{\epsilon,1\}} \int\limits_{|x|>1} |x|^\epsilon\, d\tilde{\nu}(x) < \infty \quad \forall z \in \mathbb{R},
\end{aligned}$$

where $\tilde{\nu}$ satisfies the Lévy-Schwartz condition (18) with $\epsilon > 0$. For the second term we compute, if $|b^d| < 1$ then

$$\begin{aligned}
|\beta_b| &\le \frac{1}{|b^d|} \int\limits_{1<|x|\le \frac{1}{|b^d|}} |x|\, d\nu(x) \le \int\limits_{|x|>1} \frac{1}{|b^d|^2}\, d\nu(x) \\
&\le \frac{1}{|b^d|} \int\limits_{\mathbb{R}\setminus\{0\}} \min\{|x|^2, 1\}\, d\nu(x) < \infty.
\end{aligned}$$

If $|b^d| \ge 1$, then

$$|\beta_b| \le \frac{1}{|b^d|} \int\limits_{\frac{1}{|b^d|}<|x|\le 1} |x|\, d\nu(x) \le \frac{1}{|b^d|} \int\limits_{|x|\le 1} |x|^2 |b^d|\, d\nu(x) < \infty.$$

Let $\tilde{\gamma} = \gamma + \alpha_b$ and $\tilde{\sigma}^2 = \sigma^2 b^d$. Then $\tilde{f}$ admits a Lévy-Khintchine representation with triplet $(\tilde{\gamma}, \tilde{\sigma}^2, \tilde{\nu})$.
The proof to *(ii)* works identically to that of *(i)*.
Let $f_{\dot{Y}}$ be the Lévy exponent of $\dot{Y}$ with triplet $(\gamma_{\dot{Y}}, \sigma^2_{\dot{Y}}, \nu_{\dot{Y}})$, then by (31) we have

$$\mathcal{L}_{\dot{X}+\dot{Y}}(\varphi) = \exp\left(\int\limits_{\mathbb{R}^d} (f + f_{\dot{Y}})(\varphi(s))\, ds \right),$$

where $f + f_{\dot{Y}}$is a Lévy exponent with triplet $(\gamma + \gamma_{\dot{Y}}, \sigma^2 + \sigma^2_{\dot{Y}}, \nu + \nu_{\dot{Y}})$. The measure $\nu + \nu_{\dot{Y}}$ is clearly a Lévy measure and the Lévy-Schwartz condition holds. Thus, $\dot{X} + \dot{Y}$ is a Lévy white noise.

□

As we have seen before, the evaluation of a Lévy white noise $\dot{X}$ with characteristic exponent f at some $\varphi \in \mathcal{S}$ gives us a real-valued random variable $\langle \dot{X}, \varphi \rangle$ with characteristic function

$$\mathcal{L}_{\langle \dot{X}, \varphi \rangle}(z) = \mathbb{E}[e^{iz\langle \dot{X}, \varphi \rangle}] = \exp\left(\int\limits_{\mathbb{R}^d} f(z\varphi(s))\, ds \right).$$

This raises the question if any properties of the underlying distribution of $\dot{X}$ can be transferred to the distribution of $\langle \dot{X}, \varphi \rangle$. For the Gaussian and the Compound Poisson white noise, we have seen that the evaluation at some Schwartz function has again a Gaussian, respectively Compound Poisson distribution. Also for the $S\alpha S$ white noise, by observing its characteristic functional, it is easy to see that its action on a Schwartz function again determines a $S\alpha S$ distribution.

Theorem 7.4 (compare with [42, Theorem 9.1],[1, Theorem 3])**.** *Let $\dot{X}$ be a Lévy white noise with Lévy triplet (γ, σ^2, ν) and $\varphi \in \mathcal{S}$. Then the real-valued random variable $\langle \dot{X}, \varphi \rangle$ is infinitely divisible with Lévy triplet $(\gamma_\varphi, \sigma^2_\varphi, \nu_\varphi)$, where*

$$\begin{aligned}
\gamma_\varphi &= \gamma \int\limits_{\mathbb{R}^d} \varphi(s)\, ds + \int\limits_{supp(\varphi)} \int\limits_{\mathbb{R}\setminus\{0\}} \left(\mathbb{1}_{|x\varphi(s)|\le 1} - \mathbb{1}_{|x|\le 1} \right) x\, d\nu(x) ds, \\
\sigma^2_\varphi &= \sigma^2 \|\varphi\|_2^2 \\
\nu_\varphi &= \int\limits_{supp(\varphi)} (m_{\varphi(s)})_* \nu\, ds,
\end{aligned}$$

here $(m_{\varphi(s)})_*\nu$ denotes the push-forward of ν by the multiplication with $\varphi(s)$. In particular, ν_φ is a Lévy-Schwartz measure.

Proof. Let $\dot{X}$ be a Lévy white noise with Lévy exponent f determined by the triplet (γ, σ^2, ν) and let $\varphi \in \mathcal{S}$. As mentioned above we can identify the characteristic function of $\langle \dot{X}, \varphi\rangle$ as $\mathcal{L}_{\langle \dot{X},\varphi\rangle}(z) = e^{f_\varphi(z)}$ where f_φ is given by

$$f_\varphi(z) := \int_{\mathbb{R}^d} f(z\varphi(s))\, ds.$$

By Lemma 7.2 it is well defined. We want to show that it admits a Lévy-Khintchine representation and determine its characteristic triplet. This is pretty straightforward by using the explicit form of f, but lengthy in technicality. We have,

$$\begin{aligned} f_\varphi(z) =& \mathrm{i}\gamma \int_{\mathbb{R}^d} \varphi(s)\, ds + \frac{\sigma^2}{2}\|\varphi\|_2^2 z^2 \\ &+ \underbrace{\int_{\mathbb{R}^d}\int_{\mathbb{R}\setminus\{0\}} (e^{\mathrm{i}z\varphi(s)x} - 1 - \mathrm{i}xz\varphi(s)\mathbb{1}_{|x|\leq 1})\, d\nu(x)ds}_{=:A_\varphi(z)} \end{aligned}$$

And furthermore,

$$\begin{aligned} A_\varphi(z) =& \underbrace{\int_{\mathrm{supp}(\varphi)}\int_{\mathbb{R}\setminus\{0\}} (e^{\mathrm{i}z\varphi(s)x} - 1 - \mathrm{i}z\varphi(s)x\mathbb{1}_{|\varphi(s)x|\leq 1})\, d\nu(x)ds}_{=:a_\varphi(z)} \\ &+ \mathrm{i}z \underbrace{\int_{\mathrm{supp}(\varphi)}\int_{\mathbb{R}\setminus\{0\}} \left(\mathbb{1}_{|x\varphi(s)|\leq 1} - \mathbb{1}_{|x|\leq 1}\right) x\, d\nu(x)ds}_{=:b_\varphi}, \end{aligned}$$

where this equality holds if we can show that both terms are finite. Let ν_φ be the measure defined by

$$\nu_\varphi(A) = \int_{\mathrm{supp}(\varphi)} (m_{\varphi(s)})_*\nu(A)\, ds, \quad \forall A \in \mathcal{B}(\mathbb{R}),$$

where $(m_{\varphi(s)})_*\nu$ denotes the push-forward of ν by the multiplication with $\varphi(s)$. We will show that it is a Lévy-Schwartz measure. Let ν satisfy the

Lévy-Schwartz condition (18) with $\epsilon > 0$. We can compute

$$\int\limits_{|x|\leq 1} |x|^2 \, d\nu_\varphi(x) = \int\limits_{\operatorname{supp}(\varphi)} \int\limits_{|x|\leq 1} |x|^2 \, d\nu_{\varphi(s)}(x) \, ds = \int\limits_{\operatorname{supp}(\varphi)} \int\limits_{|x|\leq 1} |x|^2 |\varphi(s)|^2 \, d\nu(x) \, ds$$

$$\leq \int\limits_{0<|\varphi|\leq 1} |\varphi(s)|^2 \int\limits_{|x|\leq 1} |x|^2 \, d\nu(x) \, ds + \int\limits_{0<|\varphi|\leq 1} \int\limits_{1<|x|\leq \frac{1}{|\varphi(s)|}} \underbrace{|x|^2 |\varphi(s)|^2}_{\leq |x|^\epsilon |\varphi(s)|^{\min\{2,\epsilon\}}} \, d\nu(x) \, ds$$

$$+ \int\limits_{|\varphi|>1} \int\limits_{|x\varphi(s)|\leq 1} |x\varphi(s)|^2 \, d\nu(x) ds$$

$$\leq 2\|\varphi\|_2^2 \int\limits_{|x|\leq 1} |x|^2 \, d\nu(x) + \|\varphi\|_{\min\{\epsilon,2\}}^{\min\{\epsilon,2\}} \int\limits_{|x|>1} |x|^\varepsilon \, d\nu(x) < \infty$$

and

$$\int\limits_{|x|>1} |x|^\epsilon \, d\nu_\varphi(x) = \int\limits_{\operatorname{supp}(\varphi)} \int\limits_{|x\varphi(s)|>1} |x\varphi(s)|^\epsilon \, d\nu(x) ds$$

$$= \int\limits_{0<|\varphi|\leq 1} \int\limits_{|x\varphi(s)|>1} |x\varphi(s)|^\epsilon \, d\nu(x) ds + \int\limits_{1<|\varphi|} \int\limits_{|x\varphi(s)|>1} |x\varphi(s)|^\epsilon \, d\nu(x) ds$$

$$\leq \int\limits_{0<|\varphi|\leq 1} \int\limits_{|x|>1} |x\varphi(s)|^\epsilon \, d\nu(x) ds + \int\limits_{1<|\varphi|} \int\limits_{1>|x|>\frac{1}{|\varphi(s)|}} \underbrace{|x\varphi(s)|^\epsilon}_{\leq |x|^2 |\varphi(s)|^{\max\{2,\epsilon\}}} \, d\nu(x) ds$$

$$+ \int\limits_{1<|\varphi|} \int\limits_{|x|>1} |x\varphi(s)|^\epsilon \, d\nu(x) ds$$

$$\leq 2\|\varphi\|_\epsilon^\epsilon \int\limits_{|x|>1} |x|^\epsilon \, d\nu(x) + \|\varphi\|_{\max\{2,\epsilon\}}^{\max\{2,\epsilon\}} \int\limits_{|x|\leq 1} |x|^2 \, d\nu(x) < \infty.$$

Furthermore, we see that

$$a_\varphi(z) = \int\limits_{\mathbb{R}\setminus\{0\}} (e^{\mathrm{i}zx} - 1 - \mathrm{i}xz \mathbb{1}_{|x|\leq 1}) \, d\nu_\varphi(x)$$

and hence the finiteness follows from Lemma 7.1 and the fact that ν_φ is a Lévy-Schwartz measure. Since f_φ is well defined, i.e. it is finite for all $z \in \mathbb{R}^d$, the term b_φ must also be finite. Hence. f_φ admits a Lévy-Khintchine representation with triplet $(\gamma_\varphi, \sigma_\varphi^2, \nu_\varphi)$ where

$$\gamma_\varphi = \gamma \int\limits_{\mathbb{R}^d} \varphi(s) \, ds + b_\varphi$$

and $\sigma_\varphi^2 = \sigma^2 \|\varphi\|_2^2$. In particular it is infinitely divisible.

$\square$

We have actually showed more than just the infinite divisibility of $\langle \dot{X}, \varphi \rangle$, in fact, we exactly determined its distribution. This is the key for any statistical simulations. By testing a Lévy white noise against shifted "observation windows" with non-overlapping support, we get a sequence of i.i.d. random variables whose distribution is determined by Theorem 7.4 (see Figure 3 for a visualization of 2-dimensional Lévy white noise on a 100×100 gird of the square $[0,1] \times [0,1]$). Furthermore, Theorem 7.4 gives us a direct tool to transfer distributional properties of a Lévy white noise $\dot{X}$ to its observations. For example by the specific form of ν_φ it is easy to see if $\dot{X}$ has a positive absolute moment, i.e. $\int |x|^\alpha \, d\nu(x) < \infty$ for some $\alpha > 0$, then so has $\langle \dot{X}, \varphi \rangle$. We refer to [1] and [42, Chapter 9.1] for a detailed exploration of this approach.

Lastly, we want to further reinforce the connection of Lévy processes and Lévy white noise. We have seen that in a way both objects are described by the same distribution. So it is reasonable to assume that results for infinitely divisible distribution can be translated to Lévy white noise. We will show a version of the following result for infinitely divisible distributions.

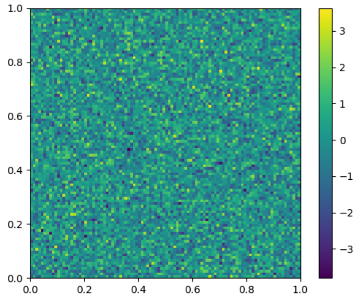
Proposition 7.4 ([36, Corollary 8.8],[2, Theorem 1.2.18])**.** *Let μ be an infinitely divisible probability measure, then there exists a series of Compound Poisson distributions which converge weakly to μ.*

We will not state the proof but rather give the main idea. We define the functions

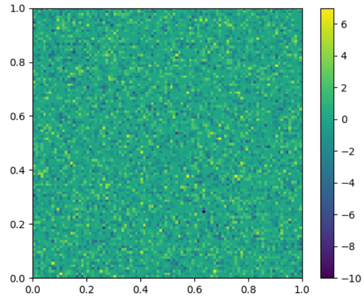
$$\mathcal{L}_n(z) := \exp(n(\hat{\mu}_n(z) - 1)) = \exp(n(\mathcal{L}_\mu^{1/n}(z) - 1)) = \exp(n(e^{1/nf(z)} - 1)),$$

where μ_n is such that $\mu = \mu_n^{*n}$ and f is the Lévy exponent of μ. It is clear that this defines the characteristic function of a Compound Poisson distribution. Now, one can show that $\mathcal{L}_n$ converges pointwise to $\mathcal{L}_\mu$ and therefore the result follows by Lévy's continuity theorem. Since Theorem 5.4 gives us a generalization of Lévy's continuity theorem for generalized random fields, we can translate this proof to the general setting. However, we can only define Lévy white noise if the Lévy measure in the characteristic triplet of the underlying distribution fulfills the Lévy-Schwartz condition. So to make this proof work we always have to make sure that we do not violate this restriction. For this we need the following Lemma.

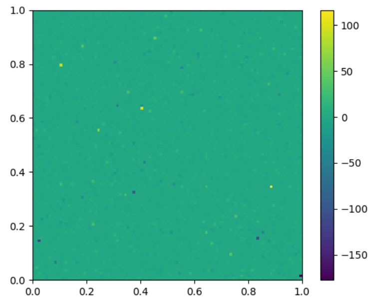
Lemma 7.6 ([36, Theorem 25.3 and Corollary 25.8])**.** *Let μ be an infinitely divisible probability measure with Lévy measure ν and let $\alpha > 0$. Then we have $\int |x|^\alpha \, d\mu(x) < \infty$ if and only if $\int_{|x|>1} |x|^\alpha \, d\nu(x) < \infty$.*



(a) Gaussian



(b) Compound Poisson



(c) $S\alpha S$

Figure 3: Simulation of 2-dimensional Lévy white noise on the square $[0,1] \times [0,1]$: Gaussian noise (3a), Compound Poisson noise (3b) with rate $\lambda = 1$ and jump height determined by a uniform distribution on $\{-2,-1,0,1,2\}$ and the heavy-tailed $S\alpha S$ noise with $\alpha = 1.5$ (3c). For more details and the code see the appendix 9.

Now we are able to prove the following result for Lévy white noises.

Theorem 7.5. *Let $\dot{X}$ be a Lévy white noise, then there exists a sequence $(\dot{X}_n)_n$ of Compound Poisson white noises which converge to $\dot{X}$ in distribution.*

Proof. Let f be the Lévy exponent of $\dot{X}$ with triplet (γ, σ^2, ν) such that the Lévy-Schwartz condition (18) holds for ν with $\epsilon > 0$. And let μ be the infinitely divisible probability measure with exponent f. Analogously to the proof in the classical case we define the functional for $\varphi \in \mathcal{S}$

$$\mathcal{L}_n(\varphi) := \exp\left(\int\limits_{\mathbb{R}^d} n(\hat{\mu}_n(\varphi(s)) - 1)\, ds\right) = \exp\left(\int\limits_{\mathbb{R}^d} n\left(\mathcal{L}_\mu^{1/n}(\varphi(z)) - 1\right)\, ds\right)$$
$$= \exp\left(\int\limits_{\mathbb{R}^d} n\left(e^{\frac{1}{n}f(\varphi(s))} - 1\right)\, ds\right),$$

where μ_n is the probability measure such that $\mu = \mu_n^{*n}$. So if μ_n is a Lévy-Schwartz measure, $\mathcal{L}_n$ is the characteristic functional of a Compound Poisson white noise $\dot{X}_n$ with rate n and underlying distribution μ_n. It is clear that μ_n is infinitely divisible with triplet $1/n(\gamma, \sigma^2, \nu)$ and so by Lemma 7.6 it follows that $\int_{|x|>1} |x|^\epsilon\, d\mu_n(x) < \infty$ iff $\int_{|x|>1} |x|^\epsilon\, d\nu(x) < \infty$ which is true by assumption. So whats left to show is that $\lim_{n\to\infty} \mathcal{L}_n(\varphi) = \mathcal{L}_X(\varphi)$ for all $\varphi \in \mathcal{S}$. The convergence in distribution of the generalized random fields follows then from the generalized Lévy's continuity Theorem 5.4. To show convergence we use the Dominated Convergence Theorem. First we note that for $\varphi \in \mathcal{S}$ by Lemma 7.1 we have

$$|f(\varphi(s))| \leq |\gamma||\varphi(s)| + \frac{\sigma^2}{2}|\varphi(s)|^2 + 2^{1-\epsilon}|\varphi(s)|^\epsilon \left(\int\limits_{|x|>1} |x|^\epsilon\, d\nu(x)\right)$$
$$+ |\varphi(s)|^2 \left(\int\limits_{|x|\leq 1} |x|^2\, d\nu(x)\right)$$
$$= a|\varphi(s)| + b|\varphi(s)|^2 + c|\varphi(s)|^\epsilon, \quad \forall s \in \mathbb{R}^d,$$

for some $a, b, c > 0$. Furthermore, since $\varphi \in \mathcal{S}$ there is some $C_{d,\varphi} > 0$ such that

$$|f(\varphi(s))| \leq \frac{C_{d,\varphi}}{(1 + |s|^{d+1})}.$$

Now we have

$$n|e^{\frac{1}{n}f(\varphi(s))} - 1| \leq \sum_{k=1}^{\infty} \frac{|f(\varphi(s))|^k}{n^{k-1}k!} \leq e^{C_{d,\varphi}} \frac{1}{1 + |x|^{d+1}}, \quad \forall s \in \mathbb{R}^d.$$

and since $n(\exp(\frac{1}{n}f(\varphi(s)) - 1) \to f(\varphi(s))$ as $n \to \infty$ we conclude that

$$\lim_{n\to\infty} \mathcal{L}_n(\varphi) = \exp\left(\lim_{n\to\infty} \int_{\mathbb{R}^d} n\left(e^{\frac{1}{n}\varphi(s)} - 1\right) ds \right)$$
$$= \exp\left(\int_{\mathbb{R}^d} f(\varphi(s))\, ds \right) = \mathcal{L}_X(\varphi). \quad \forall \varphi \in \mathcal{S}.$$

□

This result is also mentioned in [42] but without a rigorous proof.

7.5 Generalized Stochastic Differential Equations

In the classical theory for stochastic processes the concept of stochastic differential equations is introduced to model certain random behavior over time. We now want to introduce the same for generalized random fields. By their distributional character, this is relatively easy and does not involve any integration theory as it does in the classical case. We have already seen that we can define derivatives of generalized random fields and so we can consider the following *generalized stochastic (partial) differential equation*

$$Ls = \dot{X}, \tag{32}$$

where $\dot{X}$ is a given Lévy white noise and L is a linear (differential) operator. In general, the solution s is a generalized random field on $\mathcal{S}'$. These generalized SDE's are a central approach to model sparse stochastic signals ([42], [41], [43]). Here the Lévy white noise $\dot{X}$ is seen as the driving force of the signal s and the operator L is referred to as whitening operator. The aim of the theory is to characterize the distribution of the solution s.
Equation (32) is understood in a distributional sense, i.e.

$$\langle Ls, \varphi \rangle = \langle s, L^*\varphi \rangle = \langle \dot{X}, \varphi \rangle, \quad \forall \varphi \in \mathcal{S}.$$

So if we assume the suitable invertibility of L, respectively its adjoint L^*, this translates to

$$\langle s, \varphi \rangle = \langle L^{-1}\dot{X}, \varphi \rangle = \langle \dot{X}, (L^*)^{-1}\varphi \rangle, \quad \forall \varphi \in \mathcal{S}.$$

This obviously only makes sense if the action of $(L^*)^{-1}$ is well defined over $\mathcal{S}$. Since the characteristic functional of a generalized random field uniquely determines its distribution we can translate this formulation to

$$\mathcal{L}_s(\varphi) = \exp\left(\int\limits_{\mathbb{R}^d} f((L^*)^{-1}\varphi(s))\, ds\right). \tag{33}$$

However, we don't know if the right hand side of (33) actually defines a characteristic functional of a generalized random field. But since those are characterized by the Milnos-Bochner Theorem, this gives us a powerful tool to identify the existence of solutions to (33). Our computation in the proof of Theorem 7.2 already gives us the following result.

Theorem 7.6 ([11, Theorem 5])**.** *Let f be a Lévy exponent with Lèvy triplet (γ, σ^2, ν) and let ν satisfy the Lévy-Schwartz condition* (18) *for $\epsilon > 0$. Let $p_1 = \min\{1, \epsilon\}$ and $p_2 = \max\{2, \epsilon\}$ and $T : \mathcal{S} \to L^{p_1} \cap L^{p_2}$ be a linear and continuous operator. Then there exists a generalized random field on $\mathcal{S}'$ with characteristic functional*

$$\mathcal{L}(\varphi) = \exp\left(\int\limits_{\mathbb{R}^d} f(T\varphi(s))\, ds\right). \tag{34}$$

Proof. The existence is ensured by the Minlos-Bochner Theorem. For this we have to show that $\mathcal{L}$ defined by (34) is continuous at 0, positive definite and normalized. That $\mathcal{L}(0) = 1$ follows from linearity of T. To show $\mathcal{L}$ is well defined we use inequality (19) of Lemma 7.2. We obtain for $\varphi \in \mathcal{S}$

$$\begin{aligned}\|f \circ T\varphi\|_{L^1} \leq |\gamma| \|T\varphi\|_{L^1} + \frac{\sigma^2}{2} \|T\varphi\|_{L^2}^2 + 2^{1-\alpha} \left(\int_{|x|>1} |x|^\epsilon \, d\nu(x)\right) \|T\varphi\|_{L^\alpha}^\alpha \\ + \left(\int_{|x|\leq 1} |x|^2 \, d\nu(x)\right) \|T\varphi\|_{L^2}^2.\end{aligned}$$

Remember $\alpha = \min\{1, \epsilon\} = p_1$. For $\lambda \in (0, 1)$ we note that by the convexity of $x \mapsto a^x$ for $a > 0$ we have

$$\|f\|_{L^{(1-\lambda)p_1+\lambda p_2}}^{(1-\lambda)p_1+\lambda p_2} \leq (1-\lambda)\|f\|_{L^{p_1}}^{p_1} + \lambda \|f\|_{L^{p_2}}^{p_2}, \quad \forall f \in L^{p_1} \cap L^{p_2}. \tag{35}$$

And so $\|T\varphi\|_{L^1}, \|T\varphi\|_{L^\epsilon}, \|T\varphi\|_{L^2} < \infty$. Hence, $f \circ T\varphi \in L^1$ for every $\varphi \in \mathcal{S}$.

For the continuity of $\mathcal{L}$ we can use inequality (20) to obtain for $\varphi, \psi \in \mathcal{S}$

$$\left| \int_{\mathbb{R}^d} f(T\varphi(s)) - f(T\psi(s))\, ds \right|$$
$$\leq |\gamma| \|T\varphi - T\psi\|_{L^1} + \frac{\sigma^2}{2} \|T\varphi - T\psi\|_{L^2}^2 + 2^{1-\alpha} \|T\varphi - T\psi\|_{L^\alpha}^\alpha \left(\int_{|x|>1} |x|^\epsilon \, d\nu(x) \right)$$
$$+ \int_{|x|\leq 1} |x|^2 \, d\nu(x) \left(\|T\varphi - T\psi\|_{L^2}^2 + \|T\varphi - T\psi\|_{L^2} \|T\psi\|_{L^2} \right).$$

From this inequality the continuity of $\mathcal{L}$ follows now by the continuity of T and (35).
Positive definiteness follows from linearity of T and a similar density argument as before. For $\varphi \in \mathcal{S}$ we know in particular $T\varphi \in L^2$ and $\mathcal{S} \subset L^2$ is dense. Now since T is linear and by Theorem 7.2 the functional (34) is positive definite over $\mathcal{S}$, we can apply the same argument as in the proof of Theorem 7.2 by the already proven continuity.

$\square$

Let us again consider equation (32) and let assume the invertebility of L and that $(L^*)^{-1}$ fulfills the conditions of Theorem 7.6. Then the theorem not only guarantees us the existence of a solution s but also fully characterizes its distribution.
The proof is based on the two inequality (19) and (20). It can easily be seen that those can be improved. This results in a broader class of allowed operators T. In [11, Theorem 5] a more delicate approach connects the integrability of ν at zero and infinity with the range of the operator T. A similar result in a slightly more general setting is proven in [13, Proposition 5.3]. A different approach is taken in [41]. Here T is taken to be a linear and continuous map from $\mathcal{S}$ to L^p where the Lévy exponent is assumed to be p-admissible, i.e. $|f(x)| + |x|\,|f'(x)| \leq C|x|^p$ for all $x \in \mathbb{R}$ and some $1 \leq p < \infty$, $C > 0$. However, differentiability of f has to be assumed.

7.5.1 The Derivative Operator

We now want to apply Theorem 7.6 to a basic example where L is the derivative operator. For this let dimension $d = 1$ and $D : \mathcal{S} \to \mathcal{S}$, $\varphi \mapsto \varphi'$ be the derivative operator on $\mathcal{S}$. We have the following generalized SDE

$$Ds = \dot{X}, \tag{36}$$

where $\dot{X}$ is a one dimensional Lévy white noise with Lévy exponent f and the solution s is a generalized random field on $\mathcal{S}'$. By Theorem 7.3 we already know the solution of this equation. In fact, it is the corresponding Lévy process identified as a generalized random field. However, the developed theory will give us the characteristic functional of the Lévy process (as a generalized random field). We will follow [42, Section 7.4] for an investigation of this problem. To apply the theory established in Section 7.5 we have to find a suitable integral operator I. Since its adjoint I^* needs to have the right decay property we define I as

$$I(\varphi)(x) := \int_0^x \varphi(s)\, ds, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}).$$

Note that I fulfills the boundary condition $I(\varphi)(0) = 0$. We can easily see that I is a right inverse of D.

Proposition 7.5. *The adjoint of I is a continuous and linear map*

$$\begin{aligned} I^* : \mathcal{S} &\longrightarrow \bigcap_{p\geq 1} L^p \\ I^*(\varphi)(x) &= \int_{\mathbb{R}} \varphi(s)(\mathbb{1}_{[0,\infty)}(s-x) - \mathbb{1}_{(-\infty,0]}(x))\, ds \\ &= \begin{cases} \int_x^\infty \varphi(s)\, ds, & \text{if } x \geq 0 \\ -\int_{-\infty}^x \varphi(s)\, ds, & \text{if } x < 0 \end{cases}. \end{aligned}$$

Proof. To show that I and I^* are adjoint we compute for $\varphi, \psi \in \mathcal{S}$

$$\begin{aligned} \int_{\mathbb{R}} \int_0^s \varphi(x)\, dx\, \overline{\psi(s)}\, ds &= \int_{\mathbb{R}} \int_{-\infty}^s \varphi(x)\, dx\, \overline{\psi(s)} - \int_{-\infty}^0 \varphi(x)\, dx\, \overline{\psi(s)} ds \\ &= \int_{\mathbb{R}} \varphi(x) \overline{\int_{\mathbb{R}} (\mathbb{1}_{(-\infty,s]}(x) - \mathbb{1}_{(-\infty,0]}(x))\psi(s)\, ds}\, dx \\ &= \int_{\mathbb{R}} \varphi(x) \overline{\int_{\mathbb{R}} (\mathbb{1}_{[0,\infty)}(s-x) - \mathbb{1}_{(-\infty,0]}(x))\psi(s)\, ds}\, dx. \end{aligned}$$

Linearity of I^* is clear. Let us now compute the L^p norm. For this we first observe

$$H(x,s) := \mathbb{1}_{[0,\infty)}(s-x) - \mathbb{1}_{(-\infty,0]}(x) = \begin{cases} -\mathbb{1}_{(-\infty,x)}(s), & \text{if } x \leq 0 \\ \mathbb{1}_{[x,\infty)}(s), & \text{if } x > 0 \end{cases}.$$

We then obtain for $\varphi \in \mathcal{S}$

$$
\begin{aligned}
\int_{\mathbb{R}} |I^*(\varphi)(x)|\, dx &\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |H(x,s)\varphi(s)|\, ds\, dx \\
&\leq \int_{-\infty}^{0} \int_{\mathbb{R}} |\varphi(s)| \mathbb{1}_{(-\infty,x)}(s)\, ds\, dx + \int_{0}^{\infty} \int_{\mathbb{R}} |\varphi(s)| \mathbb{1}_{[x,\infty)}(s)\, ds\, dx \\
&= \int_{\mathbb{R}} |\varphi(s)| (-s\mathbb{1}_{(-\infty,0]}(s))\, ds + \int_{\mathbb{R}} |\varphi(s)| (s\mathbb{1}_{[0,\infty)}(s))\, ds \\
&= \int_{\mathbb{R}} |\varphi(s)s|\, ds < \infty.
\end{aligned}
$$

For $p \geq 1$, let $p(x) := \frac{1}{(1+x^2)C}$ with $C = \|1/(1+(\cdot)^2)\|_{L^1}$. Then by Jensen's inequality we have

$$
\begin{aligned}
\int_{\mathbb{R}} |I^*(\varphi)(x)|^p\, dx &\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |H(x,s)\varphi(s)(1+s^2)C|^p p(s)\, ds\, dx \\
&= C^{p-1} \int_{\mathbb{R}} \int_{\mathbb{R}} |H(x,s)||\varphi(s)|^p (1+s^2)^{p-1}\, ds \\
&= C^{p-1} \int_{\mathbb{R}} |\varphi(s)|^p (1+s^2)^{p-1} s\, ds < \infty.
\end{aligned}
$$

In the last equality we have used the result of the computation for $p = 1$. Both integrals are finite by the decay property of $\varphi \in \mathcal{S}$. Continuity of I^* follows from a similar argument.

□

With this result we can apply Theorem 7.6 to obtain solutions to (36). If the Lévy exponent f of $\dot{X}$ satisfies the Lévy-Schwartz condition for $\epsilon \geq 1$, Theorem 7.6 gives us the existence of a generalized random field s with characteristic functional

$$
\mathcal{L}_s(\varphi) = \exp\left(\int_{\mathbb{R}} f(I^*\varphi(s))\, ds \right). \tag{37}
$$

Since I^* is a left inverse to D^*, a solution to (36) is therefore given by $s = Iw$ since for all $\varphi \in \mathcal{S}$ we have

$$
\langle Ds, \varphi \rangle = \langle s, D^*\varphi \rangle = \langle I\dot{X}, D^*\varphi \rangle = \langle \dot{X}, I^*D^*\varphi \rangle = \langle \dot{X}, \varphi \rangle.
$$

Let us apply this to the examples of infinitely divisible distributions from before (see Table 1). Again it all comes down to checking if the Lévy-Schwartz condition (18) holds for the corresponding Lévy measures for $\epsilon \geq 1$. For the Gaussian and the Poisson distribution this is obviously true. For the Compound Poisson distribution we have to assume this on the underlying measure $\sigma \in \mathcal{P}(\mathbb{R}^d)$. Lastly for the $S\alpha S$ distribution we computed that condition (18) holds for $\epsilon < \alpha$. Thus, Theorem 7.6 is only applicable for $\alpha > 1$. In general we can observe if f is a Lévy exponent with Lévy measure ν such that ν has finite first moment, then Theorem 7.3 is applicable and we know the solution s of (36) is the corresponding Lévy process identified as a generalized random field. Furthermore, the assumptions of Theorem 7.6 are also satisfied (since I^* satisfies the Lévy-Schwartz condition with $\epsilon = 1$). In this case we can use the specific Lévy exponents in (37) to obtain the characteristic functional of the solution s. Hence, (37) gives us the characteristic function of the corresponding Lévy process identified as a generalized random field. However, the explicit form of the functionals is not very handy and hard to identify.

Remark. Nonetheless, we can make the following observation . To identify s with a stochastic process we have to "observe" s at some discrete points in time. However, we can only measure s through some observation window $\varphi \in \mathcal{S}$. An evaluation at a single point τ would correspond to applying a delta function $\delta(\cdot - \tau)$, which obviously is not an element of $\mathcal{S}$ or even a well defined function. Nevertheless, we can view $I^*\delta(\cdot - \tau)$ as a tempered distribution. By duality it is defined by

$$\begin{aligned} I^*\delta(\cdot - \tau)(\varphi) &= \delta(\cdot - \tau)(I\varphi) = I\varphi(\tau) \\ &= \int_{\mathbb{R}} \varphi(s)(\mathbb{1}_{[0,\infty)}(\tau - s) - \mathbb{1}_{(-\infty,0]}(s))\, ds. \end{aligned}$$

Thus, in the sense of distributions we have

$$I^*\delta(\cdot - \tau) = \mathbb{1}_{[0,\infty)}(\tau - \cdot) - \mathbb{1}_{(-\infty,0]}(\cdot) = \begin{cases} \mathbb{1}_{[o,\tau)}, & \text{if } \tau \geq 0 \\ -\mathbb{1}_{(\tau,0]}, & \text{if } \tau < 0 \end{cases}.$$

So it reasonable to assume that a observation of s at some positive time t is given by "observing" $\dot{X}$ through the observation window $\mathbb{1}_{[0,t)}$. Generally, this is not well defined. However we have seen that in some cases we can make sense of the expression $\langle \dot{X}, \mathbb{1}_{[0,t)} \rangle$ (in generality this is shown in [13, Proposition 3.1]) and we can recover the underlying Lévy process. This shows the consistency of this approach with the one explored in section 7.3.

8 Conclusion and Summary of own Work

In this thesis, we review and systemize the work of Biermé, Durieu, and Wang in [6]. It introduces generalized random fields as a notion of a generalized random variable with values in the space of tempered distributions to bridge the gap between classical stochastic processes and infinite-dimensional probability theory. This allows us to provide a rigorous mathematical foundation of Lévy white noise as a generalization of the classical Gaussian white noise moving beyond any heuristic description of "derivative of stochastic processes". The key to handle these infinite-dimensional distributions is to represent $\mathcal{S}$ and $\mathcal{S}'$ as multi-dimensional sequence spaces. In [6] this is the central tool to transfer classical results from finite-dimensional probability theory to infinite dimensions. We revisit this paper, provide more details and address gaps to present this theory. Firstly, in section 2, we provide a detailed proof that the Hermite functions form an orthonormal basis for $L^2(\mathbb{R}^d)$ (Theorem 2.1, Theorem 2.2). While this is a classical result, a detailed proof is often missing. Furthermore, it is the core idea behind the representation of $\mathcal{S}$ as a sequence of decreasing Hilbert spaces. In Lemma 3.1, 3.3 and Theorem 3.1 we then prove that this representation is topologically equivalent. In section 4, we introduce the sequence spaces $\mathbb{S}$ and $\mathbb{S}'$ analogously to [6], however we add some more details (Lemma 4.1, 4.3). Especially, we expand on the construction of the cylinder sigma algebras (Lemma 4.4, 4.5, 4.6) and the proof of the Montel property (Proposition 4.3, 4.7). Lemma 4.7 restates [6, Lemma 6.3], but in our opinion it is not clear why the sets H_n are closed in the proof given in [6]. Therefore, we provide a modified proof to address this issue. This cylindrical sigma-algebra structure allows us to introduce generalized random fields and describe them by their finite-dimensional projections. We again aim to give more details (Proposition 5.1, 5.2). The Minlos-Bochner theorem is shown in [6] by extending the classical Bochner's theorem to the sequence space $\mathbb{S}'$. This is based on Kolmogorov's extension theorem and we provide a more complete justification of its use (Theorem 5.1). It is the crucial existence result that allows us to define generalized random fields on $\mathcal{S}'$ by their characteristic functionals. This is the method we then use to introduce Lévy white noise via the Lévy exponent of an infinitely divisible distribution. However not all infinitely divisible distributions define a Lévy white noise on $\mathcal{S}'$. In Theorem 7.2 we carefully work out the Lévy-Schwartz condition, as it is given in [11, 42]. It separates tempered noises from non-tempered ones. An infinitely divisible distribution defines a Lévy white noise on $\mathcal{S}'$ if the corresponding Lévy measure ν satisfies $\int_{|x|>1} |x|^{\epsilon}\, d\nu(x) < \infty$ for some $\epsilon > 0$. We show this by using the idea of [11] adjusted to our framework. Although it is a non-trivial restriction, important Lévy processes such as Gaussian, Compound Poisson and symmetric α-stable processes satisfy this condition. Following this, we unify the two perspectives of looking at white noise. We show that Lévy white noise constructed via the Minlos-Bochner theorem coincides with the distributional derivative of a classical Lévy process. Firstly, we do this explicitly for the Brownian motion (Lemma 7.3, 7.4 and Proposition 7.1) and the Compound Poisson process (Proposition 7.2). Under a simplified integrability condition we then show that we can reduce the general case to those two with the help of the Lévy-Ito decomposition (Theorem 7.3). This provides a different approach without using any stochastic integration theory, as it is done in [8]. This also confirms that generalized random fields are a suitable mathematical setting to discuss the "derivative" of non-differentiable processes like the Brownian motion or Compound Poisson processes. Thereafter, we focus on properties of Lévy white noise. We work out stationary and independence results which highlight their use to model noise (Lemma 7.5, Proposition 7.3) and show how the distribution behaves under finite dimensional projections (Theorem 7.4). This is especially relevant to determine any statistical simulations. Furthermore, we show that we can approximate any Lévy white noise by a sequence of Compound Poisson white noises as a direct translation of the classical result for infinitely divisible distributions (Theorem 7.5). Lastly, we formulate the notion of a generalized SDE driven by Lévy white noise. Contrary to the classical SDE theory which is based on stochastic integration theory ([46]), the approach we present is purely distributional. This allows for a solution theory based on operator theory and the characteristic functional. Our previous results give us an immediate existence result for solutions as well as a characterization of their distributions (Theorem 7.6). By this idea, solving a generalized SDE comes down to finding a suitable inverse operator. This method has found a lot of application in the field of sparse stochastic signal modeling ([42], [14], [43]). We specifically apply this result to the derivative operator (Proposition 7.5) closing the loop between a process and its noise.

An alternative framework to generalize the notion of Lévy processes is the one of Cylindrical Lévy processes suggested in [3]. While our approach treats noise as a random element in a topological vector space ($\mathcal{S}'$), the cylindrical approach views it as a time-evolving process on a Banach space. It is closely related to the theory discussed in this thesis in the sense that the distributions of cylindrical processes are determined by finite dimensional projections by elements in the dual space. So in a similar way, this allows to transfer distributional results from finite dimensions to an infinite dimensional setting. However, the key difference is the following. In our approach we view Lévy white noise as a single random element characterized entirely by its characteristic functional. That is, we don't distinguish between space and time and we only access it by testing it against smooth observation windows. In the cylindrical approach however, time is explicit and in this way it extends the classical idea of a stochastic process. This allows for a stochastic integration theory as a tool to solve SDE's ([31]) contrary to our way of treating derivatives purely by duality. In this way they are ideal for evolution equations where time dynamics are central. Future work could rigorously bridge these two distinct mathematical constructions. We also want to mention [5] which focuses on the development of a stochastic integration theory for α-stable noises, which are important to model heavy-tailed phenomena where extreme events and sudden jumps can occur.

Finally, we want to touch on the question of regularity. In [15] the regularity of periodic Lévy white noises defined on the d-dimensional torus are studied by characterizing them in terms of Besov spaces. These are shown to be measurable subspaces of $\mathcal{S}'$. These regularity results are also applied to solutions to generalized SDE's in [16]. Analogously to classical PDE theory, it is shown that the regularity of a solution s of $Ls = \dot{X}$, where $\dot{X}$ is a Lévy white noise, increases by the order of the differential operator L. Recently, this approach was expanded to globally defined Lévy white noises by using weighted Besov spaces in [12] and refined in [4]. The core of all these works is based on a wavelet expansion to characterize local regularity. In that way the question of regularity is reduced to decay properties of the wavelet coefficients. The framework of Hermite-Besov spaces introduced in [32] (see also [17] for a recent exposition using the spectral decomposition of the Harmonic oscillator) might provide a useful tool for a similar regularity analysis. This could especially be relevant to this thesis since it might allow for a similar methodology as in [15] but in the framework of the multi-sequence spaces based on the Hermite expansion. To our knowledge, this would provide a new approach to this question of regularity and therefore might be of interest in any future extensions.

9 Appendix

Below, we want to give some details on how to simulate two-dimensional Lévy white noise on the square $[0,1]\times[0,1]$. For this we test the noise $\dot{X}$ generated by a Gaussian, a Compound Poisson and a $S\alpha S$ distribution against smooth observation windows to generate random values on a 100×100 grid on the square. In particular, we can choose a test function $\varphi \in \mathcal{D}$ with support in $[0,1/100]\times[0,1/100]$ and consider the random variable $\langle \dot{X}, \varphi\rangle$. Its distribution is determined by Theorem 7.4. Shifting φ to each cell of the grid results in a translation in $\dot{X}$ and hence, by Lemma 7.5, we get a finite number of independent and identically distributed random variables associated with each cell. By the discussion in section 7.3 it makes sense to choose φ as an approximation of the indicator function $\mathbb{1}_{[0,1/100]\times[0,1/100]}$. Note that while $\langle \dot{X}, \mathbb{1}_{[0,1/100]\times[0,1/100]}\rangle$ is not well-defined, the Lévy triplet determined in Theorem 7.4 is for any indicator function and coincides (up to a factor) with the original distribution of $\dot{X}$. Thus, for a good enough approximation φ it is reasonable to simply generate a 100×100 array with random values generated by a Gaussian, a Compound Poisson and a $S\alpha S$ distribution. All simulations were performed using Python (v3.13) with the NumPy (v2.3.5), Matplotlib (v3.10.7) and SciPy (v1.17.0) libraries. See the code below.

```python
#simulations of Levy-white noise
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import levy_stable

N = 100
rng = np.random.default_rng()

#1. Gaussian (standard normal)

noise_vals_gaussian = rng.standard_normal(size= (N,N))

plot_gauss = plt.imshow(noise_vals_gaussian, extent=[0, 1, 0, 1], origin='lower', cmap='viridis')
plt.colorbar(plot_gauss)
plt.show()

#2. Compound Poisson (\lambda = 1, random integer in {-2,-1,0,1,2})

n_jumps = rng.poisson(1, size=(N,N))
noise_vals_cp = np.zeros((N,N))
for i in range(N):
    for j in range(N):
        noise_vals_cp[i,j] = np.sum(rng.integers(-2,3,size= n_jumps[i,j]))

plot_cp = plt.imshow(noise_vals_cp, extent=[0, 1, 0, 1], origin='lower', cmap='viridis')
plt.colorbar(plot_cp)
plt.show()

#3. S\alpha S
```

```

alpha = 1.5

sas_array = levy_stable.rvs(alpha, 0.0, loc=0, scale=1.0, size=(N, N))

plot_sas = plt.imshow(sas_array, extent=[0, 1, 0, 1], origin='lower', cmap='viridis')
plt.colorbar(plot_sas)
plt.show()
```

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