DISSERTATION | DOCTORAL THESIS

Titel | Title

Operational reconstruction of quantum statistics

verfasst von | submitted by

Nicolás Medina Sánchez

angestrebter akademischer Grad | in partial fulfilment of the requirements for the degree of

Doktor der Naturwissenschaften (Dr.rer.nat.)

Wien | Vienna, 2026

Studienkennzahl lt. Studienblatt | Degree programme code as it appears on the student record sheet:	UA 796 605 411
Dissertationsgebiet lt. Studienblatt | Field of study as it appears on the student record sheet:	Physik
Betreut von | Supervisor:	Assoz. Prof. Mag. Dr. Borivoje Dakic

Acknowledgements

You see again how far away,
every thing is from every other thing.
– Louise Glück, Averno.

We reach toward others with hopes that they might reflect something back to us, a gesture toward connection that reminds us we are not entirely confined to our own thoughts and movements. We look for resonance, for an echo, for evidence that we are someone in relation to someone else, a point of reference from one moment to the next.

Sometimes we seek something in others and find nothing, because they are also reaching outward. There is often a distance that cannot be fully bridged. Connecting with others is work, a practiced and uncertain art. Yet at times there is solace, understanding, or a small consolation that arrives without warning

And we remember that we are not so alone, not so sad, that the world is not impenetrable. You remember that Anamaria exists somewhere in São Paulo, that Pipe and Claudita exist, that Mauricio exists, that Julián exists, that Wilmar and his baby exist, that Jorge and his two babies exist; that far away in Shanghai Cristian exists, and far off in Australia Diego and another baby exist. That Padilla and Dieguito exist and have always existed. That Laura and Laura exist in some bleak corner of Germany, changing the world as best they can with whatever is at hand. That my dear “Patito” exists with a baby on the way who will surely be as tender as she is.

And even in the worst moments many people tried to meet me halfway, and Sara existed when the world was falling apart, and Poli existed, and Charlie existed, and Anna existed. And Sebastian existed, and Jimena and Bruno, who listened and spoke in words I could understand, existed. And Robin and Caroline that gave me their company in unexpected moments and Niyusha with her impossible sensitivity towards the world, and the lovebirds María and José, and Ida with her constant joy. The streets existed and the silences existed, and everything was grey and bitter, yet sometimes there was sweetness. A handful of times, existing here and there was not a permanent challenge of proving yourself.

Sometimes there was love, and for a moment it seemed that we were not so lost, that affection might save us. Thea, with her infinite charm, appeared at a crucial moment and saved my heart when I thought it was lost. And the friends who found me again in Colombia, when returning there felt like reaching the surface when you are drowning: Daniela, with her unintended, humorous existence, drifting in and out; Julieth, now surely freezing somewhere in Canada, still dreaming of Husserl; and Carito, who feels etched into the soul, and Linda, whose present and future remain a mystery.

I think every day of Santiago, of his memory, of the fear of death, of how easy it is to lose sight of oneself. And yet instability sometimes brings people closer; it creates a kind of recognition that steadier states do not. I must also mention Julian and Tristan, who always arrived with new ideas and, more importantly, with the enthusiasm for this project that I often lacked.

I have learned more from life than from physics, and the thesis ought to speak about that, though that is not how things work. I owe everything to Gerardo, María del Carmen, and Daniel, because they are all I have. I thank Časlav Brukner for the films we have shared, and Borivoje Dakić, my supervisor, for believing in me and in the possibility of this thesis.

Abstract

This thesis develops a completely novel operational perspective on the behaviour of quantum particles. The approach follows the operational reconstruction of quantum theory, in which empirical scenarios are modelled as preparations, transformations, and measurements. So far in practice, we observe two families of quantum particles up to statistical behaviour: bosons and fermions. The route by which these classes of particles crystallised in the literature predates quantum mechanics from its very beginning, and can be traced to problems of particle indistinguishability already present in classical statistical mechanics. Yet the physical principles that single out only these two types of particles in the universe remain opaque. The aim of this thesis is to explain why.

Chapter 1 reviews how indistinguishability and identity have been treated across a century of work, covering the symmetrisation postulate, algebraic and topological formulations, and space–time based arguments in quantum field theory. **Chapter 2** introduces a complete characterisation of the particle statistics compatible with universal operational scenarios. These scenarios impose symmetry constraints on the accessible quantum state spaces, derived from how interferometers act on single particles and how multi-particle detection is performed. Under full knowledge of internal degrees of freedom, only bosons and fermions remain consistent. Relaxing this condition, however, reveals new families of quantum particles, described by what we call **transtatistics**, that are fully compatible with quantum mechanics yet have not previously been discussed in the literature. **Chapter 3** reframes these results in the context of the transition from first to second quantization, and examines whether such alternative statistics introduced in chapter 2 can be generated in that way. We construct many-particle physical states as equivalence classes of states of identical but distinguishable constituents and obtain quadratic algebras that reproduce transtatistical state spaces providing also new tools for many-body systems and, ultimately, field theory.

This project formed the main part of my doctorate under the supervision of Borivoje Dakić.

Zusammenfassung

Diese Dissertation entwickelt eine neuartige operative Perspektive auf das Verhalten quantenmechanischer Teilchen. Der Ansatz folgt der operativen Rekonstruktion der Quantentheorie, in der empirische Szenarien durch Präparationen, Transformationen und Messungen modelliert werden. In der Praxis unterscheiden wir bislang zwei Familien quantenmechanischer Teilchen nach ihrem statistischen Verhalten: Bosonen und Fermionen. Der Weg, auf dem sich diese beiden Klassen in der Literatur herausgebildet haben, geht der modernen Quantentheorie historisch voraus und lässt sich auf Probleme der Ununterscheidbarkeit von Teilchen zurückführen, die bereits in der klassischen statistischen Mechanik auftreten. Die physikalischen Prinzipien jedoch, die allein diese beiden Typen von Teilchen im Universum auszeichnen sollen, bleiben unklar. Das Ziel dieser Dissertation ist es, zu klären, warum das so ist.

Kapitel 1 bietet einen Überblick darüber, wie Ununterscheidbarkeit und Identität im Laufe eines Jahrhunderts behandelt wurden, einschließlich der Symmetrisationspostulate sowie algebraischer, topologischer und raumzeitlicher Argumente in der Quantenfeldtheorie. **Kapitel 2** führt eine vollständige Charakterisierung derjenigen Teilchenstatistiken ein, die mit universellen operativen Szenarien vereinbar sind. Diese Szenarien erzwingen Symmetrieeinschränkungen auf den zugänglichen Quantenzustandsräumen, die sich aus der Wirkungsweise von Interferometern auf Einzelteilchen und der Detektion mehrerer Teilchen ergeben. Unter vollständiger Kenntnis der inneren Freiheitsgrade bleiben nur Bosonen und Fermionen konsistent. Wenn man diese Bedingung lockert, treten jedoch neue Klassen von quantenmechanischen Teilchen zutage, die wir **Transtatistiken** nennen. Sie sind vollständig mit der Quantentheorie vereinbar, wurden jedoch bislang in der Literatur nicht diskutiert. **Kapitel 3** ordnet diese Ergebnisse dem Übergang von der ersten zur zweiten Quantisierung zu und untersucht, ob die in Kapitel 2 eingeführten alternativen Statistiken auf diesem Wege erzeugt werden können. Wir konstruieren Vielteilchenzustände als Äquivalenzklassen von Zuständen identischer, aber unterscheidbarer Konstituenten und erhalten quadratische Algebren, die transtatistische Zustandsräume reproduzieren und zugleich neue Werkzeuge für Vielteilchensysteme und letztlich auch für die Feldtheorie bereitstellen.

Dieses Projekt bildete den zentralen Teil meiner Promotion unter der Betreuung von Borivoje Dakić.

Contents

1 Bosons vs. Fermions

In this section, we review the literature on quantum statistics and the issues surrounding the boson–fermion duality, tracing the discussion from its initial formulation to contemporary developments across physical, mathematical and philosophical contexts. We begin with the early combinatorial approaches that first exposed the problem of defining indistinguishable particles. We then examine the algebraic frameworks that enabled systematic treatments of many-particle quantum systems on Fock spaces. The discussion subsequently turns to the coupling of quantum mechanics with configuration space, revisiting the geometric and topological approaches that advanced more operational perspectives on the problem. At this stage, we consider questions of identity and the ontological status of many-particle systems. We conclude with remarks on the programme of operational reconstructions of quantum mechanics, its relevance to the problem, and an outline of how the thesis aims to address these issues.

1.1 The Pre–Quantum Origin of Quantum Statistics

1.1.1 The Gibbs Paradox and the Birth of Indistinguishability

In the beginning, all particles were distinguishable. Consider an ideal gas in a vessel divided into two equal halves by a partition. Each half contains the same gas at the same temperature, pressure and density. Removing the partition leaves all macroscopic variables unchanged. Classical, label-based counting of microstates, however, predicts an entropy increase corresponding to the mixing of the gases initially confined to each half,

$$\Delta S_{\text{mix}} = 2Nk_B \ln 2.$$

In the present case, the system has not changed thermodynamically, yet the mere removal of the barrier appears to increase the entropy. The reason is that once the partition is removed, classical counting includes microstates in which particles have exchanged sides. Gibbs addressed this by dividing the partition function $\mathcal{Z}$ by $N!$, thereby removing permutations that amount only to relabelling identical particles [Gib02]. Since exchanging identical particles does not produce a genuinely new physical state, this correction eliminates the overcounting and restores the expected extensivity of the entropy.

Although initially introduced as a formal device, Ehrenfest and Trkal later clarified that the transformation $\mathcal{Z} \to \mathcal{Z}/N!$ reflects physical indistinguishability rather than a mathematical convenience [ET20]. The issue can also be expressed in operational terms. If no observable distinguishes the two samples, the entropy of mixing vanishes. If a distinguishing observable exists, the entropy of mixing is nonzero, and the total entropy changes accordingly [vK84, Jay92, Swe06].

1.1.2 Bose-Einstein

Bose rederived Planck's law by counting photon occupation numbers directly [Bos24]. Einstein then extended Bose's method to material particles with fixed particle number [Ein24, Ein25], obtaining the distribution

$$\bar{n}_i = \frac{1}{e^{(\varepsilon_i-\mu)/kT}-1},$$

and predicting the phenomenon of Bose–Einstein condensation. At this stage, the indistinguishability of quanta entered as a combinatorial prescription for counting microstates rather than as a theorem derived from quantum mechanics.

1.1.3 Fermi-Dirac

Fermi [Fer26] incorporated the exclusion principle into the statistical description of an ideal electron gas by assuming that no more than a single electron may occupy a given quantum state. Independently, Dirac [**?**] formulated the same statistics in terms of energy levels and occupancy numbers $n_i \in \{0,1\}$,

$$\bar{n}_i = \frac{1}{e^{(\varepsilon_i-\mu)/kT}+1}.$$

The Fermi–Dirac distribution therefore emerged as a thermodynamic expression of the Pauli exclusion principle.

1.2 The Symmetrization Postulate

Hence the wave function of identical particles must either be unchanged when any pair of particles are interchanged (and hence when the particles are permuted in any manner), or change sign when any pair are interchanged. In the first we speak of a symmetrical wave function, and in the second case of an antisymmetrical one. - Landau & Lifshitz [LL77, p. 210]

The origin of the symmetrization postulate arises only with the development of quantum mechanics, once physical states begin to be represented by wavefunctions. Its first appearance occurs in Heisenberg's study of the helium atom, where he observed that the empirical spectrum could be reproduced only by constructing two-electron wavefunctions that remain invariant under particle exchange, thereby identifying permutation symmetry as a physical requirement rather than a combinatorial convention [Hei26b]. Dirac subsequently formulated this insight in general terms: for indistinguishable particles, the many-particle state must be represented by a wavefunction that transforms under any permutation of particles by a global phase [Dir26]. Moreover, physical consistency restricts this phase to the values $+1$ or -1. Allowing arbitrary phases would lead to superselection violations and dynamical inconsistencies, since repeated exchanges of identical particles must reproduce the identity operation on physical states and the composition of permutations must be represented consistently. Only the fully symmetric and fully antisymmetric representations satisfy these requirements. Finally, von Neumann provided the first rigorous formulation by introducing the symmetrised and antisymmetrised subspaces of the many-particle

Hilbert space as the physical state spaces for systems of identical particles [vN27]. It is worth noting that the postulate, as stated, is formulated in terms of identity rather than indistinguishability. We will clarify this distinction later.

1.3 Algebraic Foundations of Quantum Statistics

In this section we review the algebraic approach to many-body quantum theory as it was originally formulated. It begins with the introduction of second quantisation by Dirac, followed by its appearance within quantum field theory, linking the formalism to the structure of space–time and culminating in the celebrated spin–statistics theorem. We then turn to Segal's approach based on tensor algebras, and conclude with the so-called parastatistics, introduced through an algebraic generalisation of the commutation rules.

1.3.1 The Creation-Annihilation algebras

The theoretical distinction between bosons and fermions was first analysed by Dirac when he introduced the method of second quantisation [Dir26, Dir27]. Assuming indistinguishability in the sense that many-particle states are invariant under particle permutations, Dirac represents physical states by occupation numbers $\{n_i\}_{i\in I}$, where I indexes the single-particle modes associated with wavefunctions $\{\phi_i\}_{i\in I}$. He introduces creation and annihilation operators $a_i^\dagger, a_i$ that change the particle number in each mode, defined by

$$a_i^\dagger\,|\ldots,n_i,\ldots\rangle = \sqrt{n_i+1}\,|\ldots,n_i+1,\ldots\rangle\,, \qquad a_i\,|\ldots,n_i,\ldots\rangle = \sqrt{n_i}\,|\ldots,n_i-1,\ldots\rangle\,.$$

Consistency with the probabilistic interpretation of quantum mechanics fixes the square-root factors and implies the commutation relations

$$[a_i,a_j^\dagger] = \delta_{ij}, \qquad [a_i,a_j] = 0,$$

for *Bose–Einstein* statistics, where there is no restriction on occupation numbers. Alternatively we have,

$$\{a_i,a_j^\dagger\} = \delta_{ij}, \qquad \{a_i,a_j\} = 0,$$

where the occupation numbers are restricted to $n_i \in \{0,1\}$, which defines *Fermi–Dirac* statistics. Dirac also noted that other kinds of symmetry for the quantum states of several particles are mathematically possible while not applying to any of the observed systems known in that moment.

Other more complicated kinds of symmetry are possible mathematically, but do not apply to any known particles. [Dir58]

Within this construction the canonical commutation relations (CCR) for bosons and the canonical anticommutation relations (CAR) for fermions appear explicitly for the first time.

1.3.2 The Spin–Statistics Theorem

Once creation and annihilation algebras are defined and their representations fixed, one couples occupation numbers to excitations localised in spacetime. The resulting state space is a Hilbert space $\mathcal{H}$ carrying a projective unitary representation U of the proper orthochronous Poincaré group. Within this setting, the *spin–statistics theorem* provides the key restriction. Under the assumptions of relativistic covariance, positive energy, locality (microcausality), and the existence of a cyclic invariant vacuum [Pau40], integer-spin fields admit only bosonic commutation relations, whereas half-integer fields admit only fermionic anticommutation relations:

$$[a(f), a^\dagger(g)] = \langle f, g \rangle, \qquad \{a(f), a^\dagger(g)\} = \langle f, g \rangle.$$

The symmetry of relativistic spacetime thus fixes the admissible statistics encoded in the operator algebra.

This conclusion did not remain tied to Pauli's original wave–equation analysis. Throughout the 1950s and 1960s, the theorem was reconstructed within increasingly abstract formulations of quantum field theory. Lüders and Zumino (1958) gave the first fully covariant proof in the Wightman framework [LZ58], showing that Lorentz covariance, microcausality, and positivity already enforce commutation for integer-spin fields and anticommutation for half-integer ones at spacelike separation. Burgoyne reached the same conclusion independently [Bur58]. Streater and Wightman later clarified the logical structure of the theorem, separating the assumptions tied to causality from those concerning the spectral properties of the energy–momentum operator [SW64].

Further analysis by Greenberg established the rigidity of the result: any consistent modification of the commutation rules would require relinquishing locality, Lorentz invariance, or positivity of energy [Gre64]. By the late 1960s, the theorem had achieved its modern form: in any local, relativistic quantum field theory on Minkowski spacetime, integer-spin particles must satisfy Bose statistics and half-integer-spin particles must satisfy Fermi statistics. This structural fact paved the way for the algebraic reconstructions of the 1970s, where the same conclusion re-emerged from the properties of observable algebras in the Doplicher–Haag–Roberts framework.

1.3.3 Segal's tensor-algebra quantization

The transition from first to second quantisation can be formulated rigorously, as shown by Segal. He proposed a strategy based on the tensor algebra generated by the single-particle Hilbert space $\mathcal{H}$ [Seg58, Seg67], defined as

$$T(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}.$$

Segal demonstrated that, under natural assumptions, namely, non-negativity of the particle-number operator spectrum, continuity of the creation and annihilation operators, and covariance under quantum operations, only two subspaces of the tensor algebra satisfy these conditions: the symmetric tensor space (bosons) and the antisymmetric tensor space (fermions), given by

$$\mathcal{F}_{\mathrm{B}}(\mathcal{H}) = \bigoplus_n \mathrm{Sym}^n \mathcal{H}, \qquad \mathcal{F}_{\mathrm{F}}(\mathcal{H}) = \bigoplus_n \wedge^n \mathcal{H}.$$

Segal thereby proved that the Fock representations of the CCR and CAR algebras can be constructed from basic postulates concerning how information is extracted from the states of many identical but distinguishable particles, that is, systems with isomorphic single-particle Hilbert spaces but allowing asymmetries in their states to characterise individual particles. We shall examine this approach in greater detail in Chapter 3.

1.3.4 Beyond Bosons & Fermions

It is widely believed that the symmetrization postulate is necessary for treating identical particles in a consistent way. It is also widely believed that it is firmly supported by experiment. In fact, the arguments usually given to insert the symmetrization postulate in quantum mechanics are of an ad hoc nature, and do not proceed unavoidably from first principles – Messiah & Greenberg [MG64a]

Green observed that the standard commutation relations can, in principle, be derived from more general trilinear relations that remain consistent with the principles of quantum mechanics. These new commutation rules define what are now known as *parabosons* and *parafermions* [**?**]. Messiah and Greenberg later examined in detail the physical content and empirical status of the *Symmetrization Postulate* [MG64b]. They demonstrated that parastatistics are consistent with the CCR/CAR algebras, and that the additional symmetries introduced by the trilinear relations correspond to more general symmetrisations, namely, to higher-dimensional representations of the symmetric group acting on the state space beyond the trivial (bosonic) and alternating (fermionic) ones. Since the irreducible representations of the symmetric group are characterised by Young diagrams, allowing such trilinear relations permits tensor symmetries associated with diagrams extending beyond the two-box horizontal and vertical shapes corresponding to bosons and fermions, respectively. For parastatistics of order p, the parameter p controls the maximal horizontal or vertical length of the allowed Young diagrams for parabosonic and parafermionic systems.

The explicit trilinear relations introduced by Green are

$$\begin{aligned}
[[a_i, a_j^\dagger]_\pm, a_k^\dagger]_- &= 2\,\delta_{jk}\, a_i^\dagger, \\
[[a_i, a_j^\dagger]_\pm, a_k]_- &= -2\,\delta_{ik}\, a_j, \\
[[a_i, a_j]_\pm, a_k]_- &= 0, \qquad [[a_i^\dagger, a_j^\dagger]_\pm, a_k^\dagger]_- = 0,
\end{aligned}$$

where the upper (lower) sign corresponds to the parabosonic (parafermionic) case. The integer p does not appear explicitly in these trilinear relations; rather, it characterises the *order* of the parastatistics through Green's ansatz. In this construction, one introduces an internal index

$\alpha = 1,\ldots,p$ and defines

$$a_i = \sum_{\alpha=1}^{p} b_i^{(\alpha)} \quad \text{(parabosons)}, \qquad a_i = \sum_{\alpha=1}^{p} f_i^{(\alpha)} \quad \text{(parafermions)}.$$

For the parabosonic case, operators with the same internal index obey ordinary bosonic commutation relations,

$$[b_i^{(\alpha)}, b_j^{(\alpha)\dagger}] = \delta_{ij}, \qquad [b_i^{(\alpha)}, b_j^{(\alpha)}] = 0,$$

while those with different indices anticommute:

$$\{b_i^{(\alpha)}, b_j^{(\beta)\dagger}\} = 0 = \{b_i^{(\alpha)}, b_j^{(\beta)}\} \qquad (\alpha \neq \beta).$$

For the parafermionic case, the situation is reversed: for a fixed internal index the operators satisfy fermionic anticommutation relations,

$$\{f_i^{(\alpha)}, f_j^{(\alpha)\dagger}\} = \delta_{ij}, \qquad \{f_i^{(\alpha)}, f_j^{(\alpha)}\} = 0,$$

whereas those with different indices commute,

$$[f_i^{(\alpha)}, f_j^{(\beta)\dagger}] = 0 = [f_i^{(\alpha)}, f_j^{(\beta)}] \qquad (\alpha \neq \beta).$$

With this ansatz, the operators a_i and $a_i^\dagger$ constructed as sums over internal components satisfy Green's trilinear relations. This embedding reveals that a parabosonic or parafermionic system of order p can be represented as an ordinary Bose or Fermi system endowed with an internal $U(p)$ symmetry. Consequently, parastatistics are empirically equivalent to ordinary statistics under suitable constraints [GM65].

1.4 Geometric and Topological Foundations of Quantum Statistics

This part adopts a geometric perspective on quantum statistics, localising particles in physical space and analysing how the topology of configuration space constrains possible exchange laws. Whereas Dirac, Pauli, and Segal formulated statistics through algebraic relations in tensor-state algebras, the geometric viewpoint relocates the problem to the topology of the reduced configuration space of identical particles.

1.4.1 The Configuration Space of Indistinguishable Particles

The pioneers in the development of quantum mechanics took this [symmetrization postulate] as an experimentally based fact. Thus, e.g. Dirac states that: "Other more complicated kinds of symmetry are possible mathematically, but do not apply to any known particles". There were subsequent attempts, continuing up to present time, to deduce the symmetrization postulate from other physical principles – Girardeau [Gir65]

Let

$$Q_N = \mathbb{R}^N \setminus \Delta, \qquad \Delta = \{x_i = x_j \text{ for some } i \neq j\},$$

be the labelled configuration space without collisions in one dimension. In higher dimensions the definition is the same, replacing $\mathbb{R}^N$ with $\mathbb{R}^{dN}$. The configuration space of *indistinguishable* particles is the quotient

$$Q_N/S_N,$$

obtained by identifying points related by permutations of labels. The topology of this quotient determines the allowed exchange behaviour.

Girardeau's analysis of impenetrable one-dimensional particles [Gir65] is the clearest setting where topology alone fixes the effective statistics. Removing the coincidence set Δ disconnects Q_N into $N!$ ordering sectors,

$$Q_N = \bigsqcup_{\pi \in S_N} \{x_{\pi(1)} < \cdots < x_{\pi(N)}\},$$

and hard-core dynamics forbids any trajectory from crossing Δ. The time evolution of the system is therefore confined to a single ordering sector, so no continuous path exists that exchanges particle positions.

Using this structure, Girardeau constructed the Bose–Fermi mapping for the Tonks–Girardeau gas. If ψ_F is a fermionic wave function on a given ordering sector, define

$$\psi_B(x_1,\dots,x_N) = \prod_{i<j} \operatorname{sgn}(x_i - x_j)\, \psi_F(x_1,\dots,x_N) = |\psi_F(x_1,\dots,x_N)|.$$

The resulting bosonic and fermionic systems share the same densities and energies. The sign structure distinguishing Bose and Fermi wave functions arises from the disconnectedness of Q_N and the enforced boundary behaviour at Δ. In this one-dimensional hard-core setting, symmetrisation is a geometric constraint rather than a postulate.

1.4.2 Classical Indistinguishability

The unconventional role of indistinguishable classical particles is best expressed by the fact that in a deterministic setting no indistinguishable particles exist, or equivalently that indistinguishable classical particles have no trajectories. – Bach [Bac73]

Bach clarified the classical analogue of the geometric picture [Bac73]. The correct phase space for N identical classical particles is the quotient Q^N/S_N, not Q^N. A permutation-invariant Hamiltonian on Q^N induces a Hamiltonian on the quotient, but the projected Hamiltonian flow on Q^N/S_N is not deterministic at points corresponding to particle coincidences or exchanges. Several distinct lifted trajectories in Q^N map to the same point in the quotient, so the flow there is not single-valued. Any attempt to ascribe definite trajectories reinstates distinguishability by reintroducing labels. This obstruction is topological and does not disappear in the limit $\hbar \to 0$.

1.4.3 In Three Dimensions There Is Only One Way

It seems to us, however, that no completely satisfactory discussion on the consequences of indistinguishability, in the context of nonrelativistic quantum mechanics, has emerged so far – Myrheim and Leinaas [LM77]

Leinaas and Myrheim [**?**] provided the first systematic classification of quantum statistics via the topology of configuration space. For N particles in $\mathbb{R}^d$ with $d \geq 2$, let

$$Q_N = (\mathbb{R}^d)^N \setminus \Delta, \qquad Q_N/S_N \text{ the indistinguishable configuration space.}$$

The fundamental group of the quotient,

$$\pi_1(Q_N/S_N),$$

governs the allowed exchange statistics.

For $d > 2$, $\pi_1(Q_N/S_N) = S_N$, so the only possible linear unitary representations yield Bose and Fermi statistics. In $d = 2$,

$$\pi_1(Q_N/S_N) = B_N,$$

the braid group,

$$B_N = \langle \sigma_1, \ldots, \sigma_{N-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| > 1 \rangle.$$

Unitary representations of B_N lead to Abelian anyons, non-Abelian anyons, and more general topological sectors. A wave function is defined on the universal cover $\widetilde{Q}_N$, and particle exchange corresponds to parallel transport governed by the chosen braid representation.

This geometric framework later underpinned Wilczek's flux–charge composite model and the effective descriptions of fractional quantum Hall states, becoming foundational for topological quantum field theories and anyonic quantum computation.

1.4.4 Relational Geometry

York extended the geometric treatment of indistinguishability by embedding permutation symmetry into a fully relational configuration space [Yor]. Let

$$Q_N^{\text{shape}} = \big(Q_N\big) / \big(\mathrm{E}(d) \times \mathbb{R}^+ \times S_N\big),$$

where $\mathrm{E}(d)$ is the Euclidean group of translations and rotations, $\mathbb{R}^+$ rescales the system, and S_N enforces permutation invariance. The resulting *shape space* encodes only relational degrees of freedom: ratios of distances, relative angles, and the intrinsic geometry of the N-point configuration.

Permutation symmetry is thus not added as an external condition on the states but incorporated directly into the structure of the reduced configuration manifold. Observables are functions on shape space, and dynamics can be formulated through a reparametrisation-invariant action on this quotient. York's construction links the Leinaas–Myrheim topological classification of statistics with relational and gauge-theoretic methods, placing indistinguishability within the same class of geometric reductions familiar from canonical general relativity.

1.5 A Comment on Algebraic QFT

The third major development reinterprets particle statistics as a consequence of locality in algebraic quantum field theory (AQFT). Segal's tensor-algebra method and the configuration-space analysis of Leinaas–Myrheim required symmetry assumptions at the outset. By contrast, the Doplicher–Haag–Roberts (DHR) analysis showed that, given only locality and covariance of the observable net in AQFT, the Bose–Fermi alternatives follow as the unique possibilities in $3+1$ dimensions. In this sense the symmetrization postulate is removed from the foundations of QFT.

1.5.1 From Representation Dependence to Observable Structure

As quantum field theory matured it became clear that field operators and their canonical commutation relations are not representation independent. Haag and Kastler's algebraic formulation [HK64] therefore replaced fields with local C^*-algebras of observables $\mathfrak{A}(O)$ assigned to spacetime regions $O \subset \mathbb{R}^{1,3}$. The axioms of isotony, locality, Poincaré covariance, and the existence of a vacuum vector define the structure of the theory without reference to any specific Hilbert-space construction. Superselection sectors then appear as inequivalent representations of the quasilocal algebra, rather than as assumptions about field statistics.

1.5.2 The DHR Framework

DHR representations of the observable algebra $\mathfrak{A}$ [DHR71, DHR74] are those that differ from the vacuum representation π_0 only inside a bounded region. A representation π is said to be *localizable* in a region O if

$$\pi|_{\mathfrak{A}(O')} \cong \pi_0|_{\mathfrak{A}(O')},$$

where O' is the spacelike complement of O. This expresses that outside O the representation π is unitarily equivalent to the vacuum. A DHR sector is the unitary equivalence class of such representations.

If π_1 and π_2 are sectors localized in spacelike separated regions, locality of the observable algebras gives rise to a unitary

$$\varepsilon(\pi_1, \pi_2) : \pi_1 \otimes \pi_2 \to \pi_2 \otimes \pi_1,$$

interchanging the order of the localizations. Since the information localized in spacelike separated regions should behave independently, one obtains the symmetry relation

$$\varepsilon(\pi_2, \pi_1) \circ \varepsilon(\pi_1, \pi_2) = \mathrm{id},$$

which simply reflects that performing the exchange twice leaves the system unchanged.

Consequently, the DHR sectors form a *symmetric monoidal category*, namely $\mathrm{Rep}_{\mathrm{DHR}}(\mathfrak{A})$ where: objects are sectors, morphisms are intertwiners between representations, and the tensor product corresponds to combining representations localized in causally independent regions. The maps $\varepsilon(\pi_1, \pi_2)$ act as the symmetry (or braiding) morphisms of this category.

For each irreducible sector π the operator $\varepsilon(\pi, \pi)$ becomes a multiplication by a scalar, and in four spacetime dimensions there are only two possibilities $\omega_\pi = \pm 1$, that is, either the bosonic or fermionic cases. In $2+1$ dimensions on the other hand, the category becomes non-trivially braided, hence allowing more general things like anyonic statistics.

1.5.3 Relation to Earlier Approaches

The DHR result unifies earlier algebraic and geometric insights. Pauli derived Bose–Fermi alternatives from commutation relations of local fields; Segal obtained CCR/CAR from conditions on representations; Leinaas–Myrheim derived statistics from the topology of configuration space. DHR supersedes these by working directly with the observable net. The statistics parameter is the monoidal symmetry of $\mathrm{Rep}_{\mathrm{DHR}}(\mathfrak{A})$, not an imposed operator sign. Locality of observables is therefore the structural origin of particle statistics in relativistic quantum theory.

1.6 Of Identical and Indistinguishable Particles

The terms *identity* and *indistinguishability* are often treated interchangeably in physics, usually as a preamble to the statistics problem. However, these concepts refer to different questions. Indistinguishability concerns the invariance of observable predictions under permutations of particle labels; identity concerns the deeper issue of whether quantum particles possess any primitive individuality at all. The ontology of multi-particle states is therefore not a straightforward extension of classical intuitions about objects with trajectories and persistent labels. As numerous authors have emphasised (Mirman; French and Krause; Saunders; Caulton and Butterfield), the classical picture obscures rather than illuminates the conceptual foundations of quantum statistics.

1.6.1 Empirical Identity

The concept of identical particles and the related requirements on the symmetry character of the many-particle wave function [...] tend to be used without any analysis of their experimental meaning [...] and are often stated so as to be experimentally (and even semantically) meaningless. – Mirman [Mir73]

Mirman argued that the standard textbook treatment introduces “identical particles” by fiat: one simply asserts that the wave function must be symmetric or antisymmetric. Yet the invariance of $\psi(x_1, \ldots, x_N)$ under permutations is not itself an observable. The empirical content lies instead in transition amplitudes, spectra, and interference effects. Mirman therefore insisted that the concept of identical particles must be reconstructed from operational considerations: what do experiments require us to identify as “the same kind of particle”? His critique showed that formal permutation symmetry does not, by itself, explain statistics; it must be tied to measurable consequences.

1.6.2 Emergent Individuality

Dieks and Lubberdink [DL10] provided a systematic treatment of the *index problem*: the labels $1,\dots,N$ in

$$\psi(x_1,\dots,x_N)$$

do not correspond to physical identifiers. They merely parameterise points of the configuration space Q_N. Upon symmetrisation, these indices cease to mark persistent objects; what persists are *correlations* encoded in the wave function. Dieks and Lubberdink argued that individuality is not primitive but *emergent*: in decohered regimes or effectively separable wave packets, one may track quasi-classical "particles", but these are not part of the exact ontology of the theory. The fundamental level is one of holistic relational structure, not object-based metaphysics.

1.6.3 Non-individuality

French and Krause [FK06] developed the most explicit metaphysical account of quantum particles as *non-individuals*. Standard set theory presupposes that objects have identity conditions independently of their properties. Quantum particles violate this assumption: permuting labels creates no new physical state, and no intrinsic haecceity[1] distinguishes one electron from another. To model this, French and Krause introduced *quasi-set theory*, where objects may lack identity in the classical sense. The goal is not to modify physics but to provide a coherent metaphysical framework in which quantum indistinguishability is not forced into classical individuation.

1.6.4 Relational Identity

Saunders [Sau16] challenged the idea that quantum particles must be completely non-individual. He argued that the symmetric or antisymmetric states of identical particles often instantiate *weak discernibility*: particles are distinct in virtue of irreflexive relations (for example, occupying antisymmetric spatial correlations), even though they lack intrinsic identities. On this view, individuality is relational and structural rather than haecceitistic. The key point is that discernibility does not require classical identity; it can arise from the structure of the multi-particle state.

1.6.5 Structural identity

Caulton and Butterfield [CB12] reinforced the structuralist interpretation by examining symmetries and parastatistics. Their central insight is that the formal apparatus of permutation invariance underdetermines the ontology: many different metaphysical interpretations (non-individuality, weak discernibility, structural identity) are compatible with the same physical facts. They argued that the correct moral is structuralist: multi-particle quantum states represent *patterns of relations* rather than collections of objects with primitive identities. Moreover, their analysis supports the view that parastatistics is not metaphysically exotic but a conventional mathematical extension of symmetric and antisymmetric subspaces.

[1] From the Latin *haecceitas* ("this-ness"), it refers to the property that makes an object this particular individual rather than another, even if it were to share all its qualitative features with something else. In this sense, the properties of objects are divided into qualitative and non-qualitative ones, the latter corresponding to their bare identity.

1.6.6 The conventionality of parastatistics

The status of parastatistics further clarifies the distinction between identity and indistinguishability. From a purely group-theoretic standpoint, Green's parafermions and parabosons arise by allowing higher-dimensional irreducible representations of S_N. Nothing in the formalism of indistinguishability prescribes that only one-dimensional representations should be realised. The modern consensus is that parastatistics is *conventional*: it can always be re-expressed as ordinary Bose/Fermi statistics combined with an internal symmetry (e.g. a hidden $SU(n)$ index). Thus the exclusion of paraparticles is not a fact about the identity of particles but a dynamical restriction on admissible field content. This reinforces the point that indistinguishability does not determine ontology. It restricts the symmetry of states, not the metaphysical status of the entities described.

1.7 Operational Reconstructions of Quantum Statistics

The final stage of the literature review traces the emergence of the *operational* approach, where physical theories are reconstructed from consistency conditions on experimental procedures rather than postulated formal axioms. This movement reframed the question of particle statistics once more: from algebraic or ontological assumptions to constraints on information processing, compositional structure, and probability assignment. It provides the conceptual bridge to the operational classification of quantum statistics developed in the following chapters.

1.7.1 Origins of the operational paradigm

By the end of the twentieth century, the interpretation problem had already become a central issue in quantum mechanics. The lack of conceptual clarity in the mathematical postulates of the theory made it difficult to determine what quantum mechanics was about or what physical laws of nature it referred to. The new century began with an alternative approach: understanding quantum mechanics through its observable effects and investigating which principles and which experimentally motivated constraints are sufficient to recover, or rather *reconstruct*, the mathematical formalism of the theory. In this spirit, the aim became to identify families of physically motivated principles, grounded in concrete phenomenology, that would single out the formalism characteristic of quantum theory and its statistical consequences.

Hence, rather than taking the Hilbert-space formalism as primitive, these reconstructions aim to recover it from structural conditions that constrain either the composition of physical systems, how states are represented, what are valid physical transformations, etc.

Hardy formulated one of the first systematic reconstructions of quantum theory from five axioms concerning the continuity of transformations, the simplicity of state spaces, and the composition of subsystems [Har01]. His framework used the dimension of the convex set of states as a key variable, showing that only real, complex, and quaternionic quantum theories satisfy the axioms, with the complex case singled out by continuity. This work established the modern template for operational reconstructions: identify a minimal set of physically intelligible postulates, then prove that the resulting probabilistic theory must coincide with standard quantum mechanics.

Chiribella, D'Ariano, and Perinotti (CDP) later developed a complete reconstruction based on the operational-probabilistic framework of generalised probabilistic theories (GPTs) [CDP11a]. Their axioms yielded the full Hilbert-space formalism including density matrices, unitary dynamics, and the tensor-product rule. Purification in particular encodes the reversibility of information at the composite-system level and has become a central concept linking operational and thermodynamic formulations.

Hardy subsequently proposed a second-generation set of probabilistic-structure axioms [Har11], incorporating continuous reversibility and informational locality, recovering complex Hilbert spaces within the GPT framework. Parallel categorical reconstructions, such as Coecke's categorical quantum mechanics [Coe10], expressed the operational logic of quantum processes in terms of symmetric monoidal categories, providing a diagrammatic calculus where composition and tensor product are primitive structural operations rather than added axioms.

Other reconstructions explored probabilistic and epistemic interpretations. While Hardy's and CDP's frameworks treat the theory as an objective description of operational statistics, the Quantum Bayesian (QBist) programme led by Fuchs and Schack reinterprets quantum states as expressions of coherent Bayesian belief constrained by the Born rule [FS13]. Here the Hilbert-space formalism codifies rational consistency among an agent's probabilistic expectations about measurement outcomes, rather than intrinsic properties of a system.

Several complementary efforts followed. Dakić and Brukner formulated a reconstruction of quantum theory based on limited information capacity, local tomography, and simple compositional principles [DB11]. Their work clarified how these operational constraints generate the structure of quantum state spaces, and it set the direction for later developments. Masanes and Müller subsequently provided a natural continuation of this program, deriving quantum theory from closely related information-theoretic axioms within a broader operational framework [MM11a]. In parallel, Barnum and Wilce advanced an alternative characterisation grounded in spectrality and self-duality of convex state spaces [BW14].

1.7.2 Neori and Goyal: an operational derivation of the symmetrization postulate

Neori and Goyal [NG12] consider an experiment involving two particles assumed to be indistinguishable and two possible paths. They denote by α_{12} the amplitude for the labelled process in which "particle 1 follows path 1 and particle 2 follows path 2," and by α_{21} the amplitude for the exchanged process in which "particle 1 follows path 2 and particle 2 follows path 1." These processes are labelled only because we artificially assume the existence of a particle 1 and a particle 2. At the physical level the particles are indistinguishable, and the experiment cannot discriminate between these two labelled descriptions; they correspond to the same physical event observed in the laboratory.

Since empirically there is no distinction between the two labelled processes, we require a rule that specifies, given the amplitudes α_{12} and α_{21}, what the total physical amplitude A of the indistinguishable process should be. This is expressed as

$$A = H(\alpha_{12}, \alpha_{21}),$$

where H is, *a priori*, an unknown function that maps the pair of complex amplitudes to the single complex amplitude associated with the physical process.

The function H must satisfy certain natural conditions. First, exchanging the particle labels does not change the physical experiment, so the two labelled processes represent the same physical situation. This implies that H is a *symmetric* function:

$$H(\alpha_{12}, \alpha_{21}) = H(\alpha_{21}, \alpha_{12}).$$

In addition, H must be compatible with the standard rules that quantum mechanics imposes on amplitudes:

1. **Linearity**: mutually exclusive alternatives have amplitudes that add;
2. **Product rule**: consecutive processes have amplitudes that multiply;
3. **Associativity of composition**: the amplitude of a composite process is independent of how its constituent processes are grouped;
4. **Normalization for single-particle processes**: when there is only one possible path, H must reduce to one of the input amplitudes.

Imposing these conditions on H yields the functional equation

$$H(\alpha_{12}, \alpha_{21}) = \alpha_{12} \pm \alpha_{21}.$$

The two possible signs correspond precisely to the standard exchange rules: the positive branch yields the symmetric (bosonic) composition of amplitudes, and the negative branch the antisymmetric (fermionic) one. Thus the boson–fermion dichotomy arises as a consequence of how amplitudes for empirically indistinguishable processes must combine, rather than as an independent postulate.

1.8 Ouverture

The developments reviewed in this chapter show that quantum statistics can be understood, at the formal level, through two broad perspectives. The algebraic perspective explains statistical behaviour by analysing the structure of operator algebras, symmetrised tensor products, and the representation theory of permutation groups. The geometric perspective explains the same behaviour through the topology of configuration space and the constraints imposed by particle exchanges. Both views were early attempts to formalise what became known as the symmetrization postulate. They expressed the empirical fact that nature exhibits bosons and fermions, and they did so by incorporating permutation symmetry directly into the mathematical framework.

However, the symmetrization postulate did not arise from a fundamental physical principle. It was introduced because it matched the available experimental results. This generated a long-standing conceptual concern. If symmetrization is simply a historical convenience, then it becomes important to ask whether the dichotomy between bosons and fermions is unique, what assumptions support it, and whether other consistent statistical patterns are possible. Work on parafermions, parabosons, braid statistics in two dimensions, and generalized symmetry sectors indicated that the formalism does not force a twofold classification. Instead, the restriction to Bose and Fermi behaviour reflects additional assumptions that must be identified and justified.

This observation became tied to a deeper foundational question: what exactly does it mean to say that particles are identical or indistinguishable. Philosophical analyses of quantum theory have stressed that these two notions must be separated. Identity refers to individuality. It is the question of whether the theory attributes any intrinsic property that can distinguish one particle from another. Indistinguishability is operational. It concerns the set of measurements that an agent can perform and the equivalence of predictions under permutations of labels.

Quantum mechanics breaks the classical association between these notions. In classical theory, particles possess trajectories and can be followed in time. Identity and distinguishability can be treated as equivalent. In quantum theory, the labels that appear in multi-particle wave functions do not correspond to intrinsic markers. They are merely formal indices used to parametrize points in configuration space. After imposing permutation symmetry, these labels no longer refer to individual systems. What remains is the structure of correlations within the state.

These analyses reinforce the idea that the symmetrization postulate reflects constraints on relational structure rather than an assumption about intrinsic identity. They also suggest that a more basic account of statistics should start from the operational features of experiments. Modern reconstructions of quantum theory adopt this standpoint and describe physical behaviour in terms of processes, transformations, and composition rules. In this setting indistinguishability can be expressed directly as a condition on the way experimental operations compose. The question then becomes: which statistical behaviours are compatible with these operational constraints.

If one widens enough the operational assumptions, further possibilities beyond bosons and fermions appear. In particular, one can impose consistency conditions not only on the composition of processes but also on the factorization properties of the partition function. One can also include local phase transformations that act separately on particle sectors. These additions alter the operational constraints and reveal that other statistical patterns are viable.

The present thesis develops this generalised operational approach. The starting point is an explicitly experimental scenario. A global unitary transformation describes an interferometer that mixes spatial or internal modes. Local phases describe particle counting or local adjustments in each mode. These two elements form the basic operational ingredients of any interference experiment involving identical particles. Requiring these ingredients to compose consistently constrains the possible statistical behaviours.

Our analysis shows that there is a family of operationally admissible statistics that extend beyond the usual Bose and Fermi cases. This family will be called transtatistics. Bosons and fermions arise as the special cases that satisfy the strongest consistency constraints, but other coherent behaviours satisfy weaker constraints and remain mathematically consistent.

The thesis also develops a rigorous mathematical framework for discussing commutation relations and multi-particle structures beyond the familiar categories of bosons, fermions, and parafields. This framework provides new tools for analysing how symmetries act on operational scenarios and how they constrain possible statistical laws. It allows us to study relations of commutation that have not appeared in the previous literature and to classify a broader range of admissible behaviours.

In summary, this work provides both a conceptual and a mathematical contribution to the foundations of quantum statistics. Conceptually, it interprets indistinguishability as an operational principle and shows that the usual statistics arise from the compatibility of interference and local counting. Mathematically, it classifies all statistics compatible with these principles and identifies the precise conditions that isolate the standard cases. The result is a coherent framework for the study of quantum statistics that does not rely on the symmetrization postulate but derives the relevant structure from operational reasoning.

2 Reconstruction of Quantum Statistics

Identical quantum particles exhibit only two types of statistics: bosonic and fermionic. Theoretically, this restriction is commonly established through the symmetrization postulate or (anti)commutation constraints imposed on the algebra of creation and annihilation operators. The physical motivation for these axioms remains poorly understood, leading to various generalizations by modifying the mathematical formalism in somewhat arbitrary ways. In this work, we take an opposing route and classify quantum particle statistics based on operationally well-motivated assumptions. Specifically, we consider that a) the standard (complex) unitary dynamics defines the set of single-particle transformations, and b) phase transformations act locally in the space of multi-particle systems. We develop a complete characterization, which includes bosons and fermions as basic statistics with minimal symmetry. Interestingly, we have discovered whole families of novel statistics (dubbed transtatistics) accompanied by hidden symmetries, generic degeneracy of ground states, and spontaneous symmetry breaking– effects that are (typically) absent in ordinary statistics.

2.1 Introduction

The concept of identical particles was introduced by Gibbs in 1902 [Gib10] as an alternative to solve the problem related to the extensitivity of entropy, the so-called *Gibbs paradox*. According to Gibbs, a system consists of identical particles if its physical magnitudes are invariant under any permutation of its elements. Bose has put forward this idea in quantum mechanics in his derivation of Planck's law of blackbody radiation [Bos24]. This was further developed by Dirac [DF26] and Heisenberg [Hei26a], who formulated the well-known *symmetrization postulate*: physical states must be symmetric in such a way that the exchange of particles does not give any observable effect. Put in the standard language of wavefunctions, if the state of, say, two particles is given by $\psi(x_1,x_2)$, then

$$\psi(x_2,x_1) = e^{i\varphi}\psi(x_1,x_2). \tag{2.1}$$

Applying the particle swap twice trivially reveals $e^{i\varphi} = \pm 1$. This is the origin of two types of particle statistics: *bosons* (symmetric) and *fermions* (antisymmetric).

Another approach to explain the origin of quantum statistics is the *topological* argument [LM77, LD71]. Namely, the exchange symmetry is directly related to the continuous movement of particles in a physical (configuration) space, which implies that only bosonic and fermionic phases are allowed, given that the number of spatial dimensions is three or greater. In lower dimensions, one gets fractional phases and anyonic statistics [Wil82].

The third common way of addressing the question of particle statistics is to take the *algebraic* *(field)* approach [?], i.e., by postulating the set of canonical relations

$$[a_i, a_j^\dagger]_\pm = \delta_{ij}\mathbb{1}, \tag{2.2}$$

where $\pm$ stands for (anti)commutator of operators (for fermions and bosons, respectively). Starting with an assumption of a unique vacuum state, one can build the multi-particle state space (Fock space) for two types of particle statistics.

While these approaches agree at the level of ordinary statistics (bosons and fermions), all of them have been criticized for their *ad hoc* nature [MG64a, ?, vE19]. This leaves the door open for various generalizations, many of which resort to somewhat arbitrary assumptions added to the quantum formalism. Earliest work along these lines dates back to Gentile and his attempt to interpolate between two statistics [Gj40], and since then, we have seen dozens of generalized and exotic statistics, such as parastatistics [Gre53], quons and intermediate statistics [Gre91a, Gre99, LN10, Fiv90], infinite statistics [Gre90, Med97], generalizations of fractal and topology-dependent statistics [CND96, Pol99, CB09, Sur04], ewkons [HS16] modifications of statistics due to quantum gravity [Swa08, BBCeSTS01, BWC06], non-commutative geometry [AB09] and others [Mas09, Tri09, Bag11, NG09].

2.1.1 Operational approach and particle statistics

So far, exotic statistics have never been observed in nature. This situation can be interpreted at least in two ways: we need more sophisticated and precise experiments, or (some) generalizations are in collision with basic laws of physics (believed to hold universally). An excellent example of the latter point is a question of the parity superselection rule (PSR) for fermions derived from the impossibility of discriminating a 2π-rotation from the identity in three-dimensional space [Wig95]. One may wonder how to apply this reasoning in a more abstract scenario, such as fermions occupying some discrete degrees of freedom (e.g., energy) where no notion of rotation (*a priori*) exists. An elegant study was provided in a recent work [Joh16] based on techniques from quantum information, showing that a PSR violation would allow for superluminal communication. Thus, the parity superselection rule can be derived from a more basic law, i.e., the *no signaling* principle [Ghi13]. Such an approach to physical theories (from physical laws to mathematical formalism) resembles Einstein's original presentation of special relativity. In that case, a concise set of physical postulates, namely the covariance of physical laws and the constancy of the speed of light in all frames of reference, paved the way for the formalism of Lorentz transformations. In the realm of quantum foundations, the application of this methodology was particularly successful. With the pioneering work of Hardy [Har01], the field of *operational reconstructions* of quantum theory [Har01, DB11, CDP11b, MM11b, DB16, HW17] was established where one recovers the abstract machinery of Hilbert spaces starting from a set of information-theoretic axioms. Considering the significance of identical particles in quantum information processing (such as in linear optical quantum computing [KLM01]), it becomes evident that utilizing this operational approach holds significant potential to derive particle statistics based on physically grounded assumptions. Rather than exploring possible modifications of the existing formalism, a more constructive approach may begin by defining a typical quantum experiment and addressing straightforward physical questions. For example, how do we define particle (in)distinguishability from an experimental standpoint? Is it possible to establish a clear operational differentiation between various types of identical particles, and if so, how do we characterize the corresponding mathematical formalism? Our work can be understood as an attempt to answer these questions. Along these lines, promising research studies appeared in the context of the symmetrization postulate [Goy19], anyonic statistics [Neo16], quantum field theory [DP14, EMMA22] and identical particles in the framework of generalized probabilistic theories [DMPT14, DGT+13].

2.1.2 Reconstruction, mathematical foundations and summary of the results

Following the instrumentalist approach of Hardy [Har01], we study identical quantum particles in an operationally well-defined setup composed of laboratory primitives, such as preparations, transformations, and measurements (see Fig. 2.1). Our starting point is a single quantum particle which we assume is an ordinary quantum particle described by standard formalism and unitary dynamics. This appears rather natural, as a single quantum particle is insensitive to statistics. We introduce a typical apparatus for a single-particle transformation described by a unitary channel on d-modes ($d \times d$ unitary matrix) and a set of d detectors at the output. For such a fixed circuit, we investigate the scenario with multiple identical particles at the input and analyze the probability of detecting them after the transformation. Detectors can register only particle numbers but cannot distinguish them; thus, indistinguishability is built in from the beginning. As we shall see, the Fock-space structure will naturally arise as an ambient space for multi-particle states. Two central mathematical ingredients will thus figure prominently in our reconstruction of particle statistics:

1) unitary group $U(d)$ describing single-particle transformations, and

2) the Fock space structure encompassing multi-particle states.

Paired with the locality assumption (i.e., phase transformations acting locally in Fock space), these two elements will determine how particles are organized in multiparticle states. Mathematically, the problem concerns the classification of representations of the $U(d)$ group in Fock space subjected to locality constraint. We found a one-to-one correspondence to the well-studied mathematical problem of characterizing completely-positive sequences [BD02, ASW52, Dav00, BG15]. This, in turn, provided us with a *complete categorization of particle statistics* based on integral polynomials. To be more precise, a list of integers

$$[q_0, q_1, \dots]_\pm, \tag{2.3}$$

defines a type of particle statistics, provided that $Q_\pm(x) = \sum_s (\mp 1)^s q_s x^s$ are polynomials with all negative $(+)$ or positive $(-)$ roots. We coin the term *transtatistics* for this generalized statistics. This is a natural generalization of ordinary statistics into two types: fermionic-like $[\dots]_-$ (*transfermions*) and bosonic-like $[\dots]_+$ (*transbosons*), and to the best of our knowledge, was not presented in the literature. Ordinary statistics is the simplest possibility (degree-one) $[1,1]_\pm$ with multiparticle Fock state being completely specified by irreducible representations (IR) of $U(d)$. On the other hand, general transtatistics requires additional quantum numbers to identify states of indistinguishable particles; thus, *hidden symmetries* [McI59] and *new degrees of freedom*

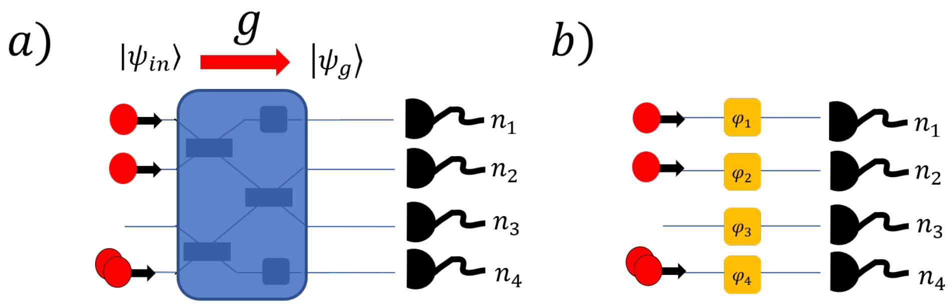

Figure 2.1: **Operational setup**. a) Quantum circuit represented by $U(d)$-transformation for a single particle (example of $d = 4$ is shown). For many particles injected into the setup, detectors can register their number only. This incorporates the notion of indistinguishability. b) Disconnected (independent) phase gates acting on particles locally, in individual modes.

emerge exclusively from these types of particles. We discuss further physical consequences by analyzing the thermodynamics of non-interacting gases. In doing so, we find an interesting inclusive degeneracy of ground-states followed by *spontaneous symmetry breaking* [Pei91], which (usually) does not exist in ordinary statistics.

Symmetry is central to our reconstructions. In particular, the $U(d)$ symmetry of single-particle transformations is uniquely related to ordinary statistics and transtatistics. This also brings the main difference to other generalized statistics, which rely on different symmetries. Apart from the foundational relevance, our findings apply to quantum information and quantum many-body physics. Concretely speaking, transtatistics brings novel theoretical models for non-interacting identical particles. The latter is relevant for studying strongly-correlated quantum systems (see [CPGSV21] and references therein), many of which are reducible to non-interacting models of indistinguishable particles [SML64]. Therefore, one may find new integrable models among strongly interacting quantum systems reducible to our non-interacting model. On the quantum information side, quantum statistics is essential in complexity theory and intermediate quantum computing models, such as in boson sampling [AA11]. In this respect, our classification is relevant as it may lead to the discovery of new intermediate computational models. These points are only summarized here and will be discussed in more detail in the last section of the manuscript.

2.2 Operational setup for indistinguishable particles

The operational framework for indistinguishable particles is illustrated in Fig. 2.1a). The apparatus consists of d modes into which particles can be injected, followed by a transformation g and a set of d detectors that register particles after the transformation. The transformation g is fixed and independent of particle number at the input and particle-statistics type. One can think of this transformation as a quantum circuit composed of elementary gates, such as beam-splitters and phase shifters used in quantum linear optics to produce a general unitary transformation $g \in U(d)$ on d modes [PCL+12], where $U(d)$ is the set of $d \times d$ unitary matrices. As long as just one particle is injected into the setup, e.g., in mode i, the jth detectors will register the particle with the probability $p_j = |g_{ji}|^2$ with g_{ji} being the matrix element of g. In other words, g represents standard complex unitary dynamics of a single quantum particle with d levels (modes). The critical question to be answered is what will happen if more than one particle is injected into such apparatus? To formalize the situation, there are three points to be addressed in the first place:

i) We shall determine the ambient Hilbert space describing the multi-particle system,

ii) we have to find the corresponding representation of transformations (as defined by the $U(d)$ group) in such a space, and

iii) finally, determine the Born rule to calculate probabilities of detection events.

To identify the Hilbert space of many particles, we use the fact that particles are indistinguishable, i.e., detectors can register only particle numbers (how many particles land in a particular detector without distinguishing them). Thus the overall measurement outcome is described by a set of numbers $(n_1, n_2, \dots, n_d)$, with $n_k = 0, 1, 2, \dots$. This outcome fully specifies the physical configuration; thus, we associate to it the measurement vector $|n_1, \dots n_d\rangle$ such that the Born rule gives detection probabilities

$$p_{n_1,\dots,n_d} = |\langle n_1, \dots, n_d|\psi_g|n_1, \dots, n_d|\psi_g\rangle|^2, \tag{2.4}$$

where $|\psi_g\rangle$ is the state of the system after the transformation g. From here, we directly see that $|\psi_g\rangle \in \mathcal{F}_d$ resides in a Fock space defined as a span over number states, i.e.,

$$\mathcal{F}_d = \text{span}\{|n_1, n_2, ..., n_d\rangle \mid n_k = 0, 1, 2, \dots p\}. \tag{2.5}$$

Here *span* denotes the complex linear span (hull) of basis vectors. Since outcomes $(n_1, n_2, \dots, n_d)$ are perfectly distinguishable, vectors $|n_1, n_2, ..., n_d\rangle$ form an orthonormal set. We introduced the possibility of there being a maximal occupation number $p \in \mathbb{N}$, which is the *generalized Pauli exclusion principle*. As we shall see, $p = 1$ will correspond to fermionic statistics, while bosons are associated with the case $p = +\infty$. At this stage, p is characteristic of statistics and is kept as an integer parameter (possibly infinite). Note that the Fock space in (2.5) shall not be *a priori* identified with the standard (textbook) Fock space constructed as a direct sum of particle sectors. Our Fock space is an ambient Hilbert space for multi-particle states naturally emerging from operational considerations and the measurement postulate defined in (2.4). Note also that the Fock space in (2.5) is of the tensor product form, i.e., $\mathcal{F}_d = \mathcal{F}_1^{\otimes d}$.

Now, we shall find an appropriate unitary representation of $g \in U(d)$ in the ambient space $\mathcal{F}_d$, i.e. $\Delta_d : U(d) \mapsto GL(\mathcal{F})$ such that

$$|\psi_g\rangle = \Delta_d(g)\,|\psi_{in}\rangle\,, \tag{2.6}$$

with $\Delta_d(g)$ being unitary representation and $|\psi_{in}\rangle \in \mathcal{F}_d$ is some input state to the circuit in Fig. 2.1a). For example, $|\psi_{in}\rangle = |1, 1, 0, \dots, 0\rangle$ represents the input state of two particles injected in mode 1 and 2. In general, $|\psi_{in}\rangle$ may involve the superposition of number states. Representation Δ_d is reducible in general, and the *group character* completely determines its decomposition into irreducible (IR) sectors [FH04], that is, a function defined over the elements of the group

$$\chi_d(g) = \mathrm{Tr}(\Delta_d(g)), \quad \forall g \in U(d). \tag{2.7}$$

As we shall see, the irreducible decomposition of Fock space (2.5) will be in one-to-one correspondence to the type of particle statistics. So, the group character will be our main object of interest.

2.2.1 Locality assumption

To evaluate character on $U(d)$ group, recall that any unitary matrix can be diagonalized, i.e., $g = StS^\dagger$, with $t = \mathrm{diag}[x_1,\dots,x_d] \in T_d = U(1)\times\cdots\times U(1)$ being an element of the maximal torus (also known as the phase group) with $x_k = e^{i\theta_k} \in U(1)$. Therefore, the character of $U(d)$ is entirely specified by the character evaluated on T_d, that is, $\chi_d(StS^\dagger) = \mathrm{Tr}\Delta_d(StS^\dagger) = \mathrm{Tr}\Delta_d(t) = \chi_d(t)$ (i.e., class function), thus it effectively becomes a function of phase variables, i.e., $\chi_d(\vec{x}) = \chi_d(x_1,\dots,x_d)$.

Consider the case of a single-mode ($d = 1$) with the Fock space $\mathcal{F}_1 = \mathrm{span}\{|n\rangle \mid n = 0,1,\dots,p\}$ on which the group $U(1)$ acts with representation $\Delta_1(x)$, with $x = e^{i\theta}$. We can think of $\Delta_1(x)$ representing a simple device providing a phase shift to the state of a single particle placed in a mode. We can now consider the collection of d such devices disconnected from each other and operating independently in separate modes, as illustrated in Fig. 2.1b). These transformations form the phase group T_d acting in the entire Fock space, and given their operational independence, it appears natural to assume the following.

Assumption 1 (Locality)**.** *The action of the phase group T_d in Fock space is local, i.e.,*

$$\Delta_d(\vec{x}) = \Delta_1(x_1)\otimes\cdots\otimes\Delta_1(x_d), \tag{2.8}$$

for $\vec{x} \in T_d$.

By taking the trace of the last equation, one gets

$$\chi_d(\vec{x}) = \prod_{k=1}^{d} \chi_1(x_k), \tag{2.9}$$

with $\chi_1(x) = \mathrm{Tr}(\Delta_1(x))$ being the single-mode character. One can also go in the reversed direction, i.e., starting with the character factorization in (2.9), we may derive the tensor factorization in (2.8), which follows from general character theory [FH04].

Assumption 1 is our central assumption. We see that the single-mode character χ_1, a function of a single variable, entirely specifies the character of the whole $U(d)$. The problem then simplifies, and our goal is to determine the most general form of $\chi_1(x)$ such that $\chi_d(\vec{x})$ in (2.9) is a valid character of $U(d)$.

2.2.2 Generalized number operator and conserved quantities

What follows from Assumption 1 and factorization given in (2.9) is that the single-mode character $\chi_1(x)$ completely specifies the character of the whole $U(d)$ and consequently determines the decomposition of Fock space into IR sectors. Note that the action of the single-mode phase transformation $x = e^{i\theta} \in U(1)$ can be seen as an instance of the Hamiltonian evolution. Thus we can write $\theta = \varepsilon t/\hbar$, where ε is the single-particle energy associated with this mode. With this, the representation of the phase transformation becomes $\Delta_1(e^{i/\hbar \varepsilon t}) = e^{i/\hbar \hat{H} t}$, where $\hat{H}$ is the single-mode Hamiltonian (generator of phase). From the invariance under (2π)-rotations, i.e., $e^{i(\theta+2\pi)} = e^{i\theta}$, we conclude that all eigenvalues of $\hat{H}$ are integer multiples of ε, that is, $\hat{H} = \varepsilon \tilde{N}$ with $\tilde{N}$ being the operator with integer eigenvalues. This defines the *generalized number operator* or *excitation operator* $\tilde{N}$. Here, we will consider only the case $\tilde{N} \geq 0$; the possibility of negative eigenvalues of $\tilde{N}$, which would account for the most general generator of $U(1)$, will be discussed later in Section 2.6.2. Without loss of generality, we can assume $U(1)$ action to be number preserving, thus

$$\tilde{N} = \sum_{n=0}^{p} f_n \left|n\right\rangle \left\langle n\right|, \tag{2.10}$$

with f_n being non-negative integers. Note that $\tilde{N}$ is in general different from the standard number operator $\hat{N} = \sum_{n=0}^{p} n \left|n\right\rangle \left\langle n\right|$. The two will coincide only if $f_n = n$, and as we shall see, this happens only in the case of ordinary statistics.

Finally, we can write the single-mode character $\chi_1(e^{i\theta}) = \mathrm{Tr}(e^{i\theta\tilde{N}})$ as

$$\chi_1(x) = \sum_{s=0}^{+\infty} a_s x^s = x^{f_0} + x^{k_1} + \cdots + x^{f_p}, \tag{2.11}$$

with a_s being a non-negative integer. Mathematically speaking, the formula above is the decomposition of χ_1 into irreducible representations of $U(1)$. For fermions, we have $\chi_1^{(-)}(x) = 1 + x$, while for bosons $\chi_1^{(+)}(x) = 1 + x + x^2 + \cdots = \frac{1}{1-x}$.

For the case of d modes, the action of the phase group in (2.8) becomes

$$\Delta_d(\vec{x}) = e^{i\theta_1 \tilde{N}_1 + \cdots + i\theta_d \tilde{N}_d}, \tag{2.12}$$

where $\tilde{N}_k = \mathbb{1}^{\otimes(k-1)} \otimes \tilde{N} \otimes \mathbb{1}^{\otimes(d-k)}$ are generators of local phases. The vector $\vec{x} = \theta(1, 1, \ldots, 1)^T \in T_d$ corresponds to the scalar $d \times d$ matrix $e^{i\theta}\mathbb{1}_d$ commuting with all $U(d)$ matrices, thus the operator

$$\tilde{N} = \sum_{k=1}^{d} \tilde{N}_k, \tag{2.13}$$

is a conserved quantity (Casimir operator) and represents the total number of excitations. We can also write (2.12) as being generated by the following Hamiltonian

$$\hat{H} = \sum_{k=1}^{d} \varepsilon_k \tilde{N}_k, \tag{2.14}$$

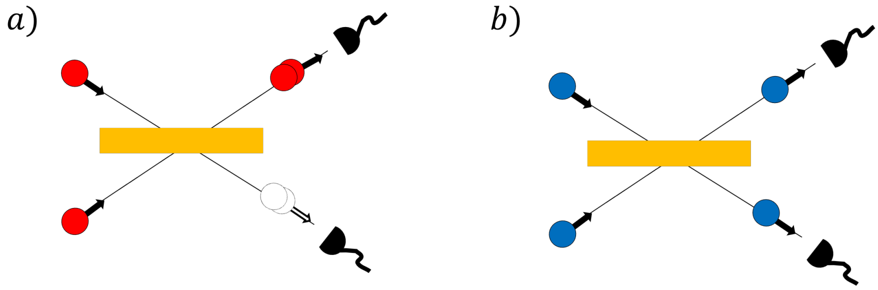

Figure 2.2: **Hong-Ou-Mandel effect** [HOM87]. a) Boson bunching, and b) fermion antibunching. See main text for details.

where $\theta_k = \varepsilon_k t/\hbar$ and ε_k is are the single-particle energies.

2.3 Particle statistics and its classification

2.3.1 On exchange symmetry

In the 1st quantization approach, particle statistics are classified via the exchange of particles and symmetrization postulate as given in equation (2.1). However, this method does not apply to the Fock-space approaches simply because there is no particle label (they are indistinguishable). A partial solution to this problem is to introduce *permutation of modes operator* [ST70]

$$\Delta_d(\sigma)\,|n_1,n_2,\ldots,n_d\rangle = \left|n_{\sigma(1)},n_{\sigma(2)},\ldots,n_{\sigma(d)}\right\rangle, \tag{2.15}$$

for some permutation $\sigma \in S_d$ of d elements. In this way, the permutation group acts in Fock space and plays the same role as the exchange of particles in the 1st-quantized picture. For ordinary statistics, we have the usual sign change, i.e., $\Delta_d(\sigma)\,|1,1,\ldots,1\rangle = (\pm)^\sigma\,|1,1,\ldots,1\rangle$, where $(..)^\sigma$ denotes the parity of permutation ($+1$ for bosons and $(-1)^\sigma$ for fermions). Nevertheless, the permutation of modes is only a discrete subgroup of the group of single-particle transformation, thus insufficient for the whole physical picture. For example, in our case, it is the subgroup of the unitary group, i.e., $S_d < U(d)$. But it can also be a subgroup of some other group, such as an orthogonal group, in which case one gets parastatistics [RS63, SdJ08]. Therefore, to fully understand how different types of particles integrate into multi-particle states in Fock space, one must study transformation properties under the action of the whole group of single-particle transformations. This work concerns $U(d)$ as our premise is that standard unitary quantum mechanics governs the physics of one particle.

2.3.2 Physical consequences

To illustrate how single-particle transformations affect the physical behavior of indistinguishable particles, take an example of two particles entering the 50/50 beamsplitter (BS) at different ports (modes), as shown in Fig. 2.2. The beam-splitter is defined via unitary matrix $u_{bs} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$.

Now, if particles are bosons, then the input state is $|1,1\rangle = a_1^\dagger a_2^\dagger |0,0\rangle$, where $a_{1(2)}^\dagger$ are bosonic ladder operators associated to two different modes (ports of BS). The output state (after BS) is given by $\frac{1}{2}(a_1^\dagger + a_2^\dagger)(a_1^\dagger - a_2^\dagger)|0,0\rangle = \frac{1}{\sqrt{2}}(|2,0\rangle - |0,2\rangle)$. We see that bosons exit the BS bunched together, and this is the well-known Hong-Ou-Mandel effect [HOM87]. In contrast, if particles were fermions, the calculation remains the same but with fermionic ladder operators $a_{1/2}^\dagger$, thus we have the output state $\frac{1}{2}(a_1^\dagger + a_2^\dagger)(a_1^\dagger - a_2^\dagger)|0,0\rangle = -a_1^\dagger a_2^\dagger|0,0\rangle = -|1,1\rangle$. This means that fermions exit the BS antibunched (in different ports). These two complementary behaviors can be deduced from the decomposition of the Fock space (2.5) into IR sectors of the $U(2)$ group and action of the u_{bs} element. In the case of two bosons, $U(2)$ reduces into three-dimensional subspace span$\{|2,0\rangle, |1,1\rangle, |2,0\rangle\}$ (bosonic IR) which encompasses the bunching effect. For two fermions, we have the one-dimensional IR spanned by $\{|1,1\rangle\}$ (fermionic IR), directly resulting in fermionic antibunching.

2.3.3 Particle statistics

As explained at the beginning of this section, the group of single-particle transformations determines the physical behavior of non-interacting indistinguishable particles, and different types of particle statistics arise due to the Fock space's $U(d)$-IR decomposition. Therefore, what we mean by classification of particle statistics is a *classification of all possible ways* the Fock space (2.5) decomposes into IR sectors, i.e.

$$\mathcal{F}_d = \bigoplus_\lambda c_\lambda \mathcal{V}_\lambda, \tag{2.16}$$

where $\mathcal{V}_\lambda$ is an $U(d)$-IR. These are indexed [FH04] by a partition (Young diagram) $\lambda = (\lambda_1, \dots \lambda_d)$ with $\lambda_1 \geq \dots \geq \lambda_d$, and $c_\lambda \in \mathbb{N}_0$ is the frequency of the IR. Now, recall that the character of a representation completely determines its decomposition into IR sectors. A well-known fact from representation theory is that IRs of $U(d)$ have *Schur polynomials* $s_\lambda(\vec{x})$ as characters (see Appendix 2.7 for definition) [1]. Thus, equation (2.16) translates to decomposition of character (2.9) into Schur-polynomials, i.e.

$$\prod_{k=1}^{d} \chi_1(x_k) = \sum_\lambda c_\lambda s_\lambda(\vec{x}), \quad c_\lambda \in \mathbb{N}_0. \tag{2.17}$$

We see that the single-mode character $\chi_1(x)$ *completely specifies particle statistics* (in the sense of definition (2.16)) and this is a direct consequence of our locality assumption 1.

To clarify the point, we provide examples of bosonic and fermionic statistics. For fermions, the maximal occupation number is $p = 1$, thus the single-mode character in (2.11) reduces to

[1] Note that this holds for polynomial representations of $U(d)$, which is our case here, as the $U(1)$ generator satisfies $\tilde{N} \geq 0$. In the most general case, irreducible representations are rational and captured by Laurent-Schur polynomials [SF99]. More details are provided in the Appendix 2.6.2.

$\chi_1^{(-)}(x) = 1+x$. For d-modes, character (2.9) can be expanded as

$$\begin{aligned}\chi_d^{(-)}(\vec{x}) &= \prod_{k=1}^{d}(1+x_k) = 1+(x_1+\cdots+x_d)+ \\ &+ (x_1x_2+\cdots+x_{d-1}x_d)+\cdots+x_1x_1\ldots x_d.\end{aligned} \tag{2.18}$$

Written in terms of Schur-polynomials, this equation reads

$$\begin{aligned}\chi_d^{(-)}(\vec{x}) &= s_{(0,0,\ldots,0)}(\vec{x})+s_{(1,0,\ldots,0)}(\vec{x})+s_{(1,1,\ldots,0)}(\vec{x})+\ldots \\ &+ s_{(1,1\ldots,1)}(\vec{x}).\end{aligned} \tag{2.19}$$

This expansion corresponds to the decomposition of the Fock space $\mathcal{F}_d = \bigoplus_{N=1}^{d} \mathcal{V}_-^{(N)}$ into fermionic irreducible subspaces $\mathcal{V}_-^{(N)}$ associated with particle sectors.

Similarly, for the case of bosons and $p=+\infty$, equation (2.11) reads $\chi_1^{(+)}(x) = 1+x+x^2+\cdots = \frac{1}{1-x}$. For d modes, (2.9) reads

$$\begin{aligned}\chi_d^{(+)}(\vec{x}) &= \prod_{k=1}^{d}\frac{1}{1-x_k} = 1+(x_1+\cdots+x_d) \\ &+ (x_1^2+x_1x_2+x_2^2+\cdots+x_{d-1}x_d+x_d^2) \\ &+ (x_1^3+x_1^2x_2+x_1x_2^2+x_2^3+\cdots+x_{d-1}x_d^2+x_d^3)\ldots\end{aligned} \tag{2.20}$$

or written in terms of bosonic Schur polynomials

$$\begin{aligned}\chi_d^{(+)}(\vec{x}) &= s_{(0,0,\ldots,0)}(\vec{x})+s_{(1,0,\ldots,0)}(\vec{x})+s_{(2,0,\ldots,0)}(\vec{x}) \\ &+ s_{(3,0,\ldots,0)}(\vec{x})+\ldots\end{aligned} \tag{2.21}$$

Again, this corresponds to the decomposition of the Fock space $\mathcal{F}_d = \bigoplus_{N=1}^{+\infty} \mathcal{V}_+^{(N)}$ into bosonic irreducible subspaces $\mathcal{V}_+^{(N)}$ associated with particle sectors.

An important remark is in order about the single-particle sector $\mathcal{F}_d^{(1)} = \mathrm{span}\{|n_1,n_2,\ldots,n_d\rangle \mid \sum_k n_k = 1\}$ which is d-dimensional. This subspace is associated with the character $s_{(1,0,\ldots,0)}(\vec{x}) = x_1+\cdots+x_d$, same for bosons and fermions, i.e. we have $\mathcal{F}_d^{(1)} = \mathcal{V}_+^{(1)} = \mathcal{V}_-^{(1)}$. This is consistent with the fact that the quantum physics of one particle is insensitive to the type of statistics. This also agrees with our operational setup in Fig. 2.1*a*), which was defined through $d\times d$ unitary matrices acting in the space of one particle. Such representation is called the *standard* or *defining* representation.

2.3.4 Partition theorem and general statistics

Generally, not all $U(1)$-characters $\chi_1(x) = \sum_{s\in\mathbb{N}_0} a_s x^s$ induce a valid $U(d)$-character in (2.9). To see this, take a simple example of $\chi_1(x) = 1+x^2$. For two-modes equation (2.9) reads $(1+x_1^2)(1+x_2^2) = s_{(0,0)}(x_1,x_2)+s_{(2,0)}(x_1,x_2)+s_{(2,2)}(x_1,x_2)-s_{(1,1)}(x_1,x_2)$ and we have a negative expansion coefficient $c_{(1,1)} < 0$, which contradicts $c_\lambda \geq 0$ in equation (2.16).

Next, suppose that the first k coefficients in the single-mode character expansion are zero. Then, we can always write $\chi_1(x) = \sum_{s=k}^{+\infty} a_s x^s = x^k \sum_{s=0}^{+\infty} a_{s+k} x^s = x^k \tilde{\chi}_1(x)$. For the general d mode character in (2.9), we will have

$$\chi_d(\vec{x}) = (x_1 \ldots x_d)^k \tilde{\chi}_d(\vec{x}). \tag{2.22}$$

The term $(x_1 \ldots x_d)^k = (\det g)^k$ equals determinant of a unitary matrix g with eigenvalues $x_1, \ldots x_d$. From here, we recognize that χ_d and $\tilde{\chi}_d$ are equivalent up-to-determinant. Therefore, without loss of generality, we will assume $a_0 > 0$.

The problem of classifying all single-mode characters that induce valid representation of $U(d)$ involves non-trivial mathematics. Luckily, we found an equivalent formulation to the well-studied combinatorial problem of characterizing completely-positive sequences [BD02, ASW52, Dav00, BG15]. Details are provided in the Appendix 2.8 together with the proof of our main theorem:

Theorem 1 (Partition)**.** For $\chi_1(x) = \sum_{s\in\mathbb{N}_0} a_s x^s$ with $a_0 > 0$, a symmetric function $\prod_{k=1}^{d} \chi_1(x_k)$ is a $U(d)$ character for all $d \in \mathbb{N}$ if and only if the generating function is of the form

$$\chi_1(x) = \frac{Q_-(x)}{Q_+(x)}, \tag{2.23}$$

where $Q_\pm(x)$ is an integral polynomial with all positive (negative) roots. Furthermore $Q_+(0) = 1$.

In other words, $Q_\pm(x) = c_\pm \prod_i (1 \mp \alpha_i x)$ are polynomials with integer coefficients, where $\alpha_1 > \alpha_2 > \cdots > 0$, $c_+ = 1$ and $c_- \in \mathbb{N}$. From here, it follows that $Q_\pm(x)$ is a polynomial with all non-zero coefficients.

Note that we are interested only in *elementary statistics*, i.e., the Fock space cannot be factorized as a tensor product $\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2$, with $\mathcal{F}_{1/2}$ being associated with different particle types. Therefore, the character of elementary statistics cannot be factorized as $\chi_1(x) = \mu_1(x)\nu_1(x)$, with $\mu_1(x)$ and $\nu_1(x)$ being of the type (2.23). Thus equation (2.23) for elementary statistics is either $\chi_1 = Q_-$ or $\chi_1 = 1/Q_+$. We conclude that statistics is of two kinds, i.e., *fermionic-like* $[\ldots]_-$ and *bosonic-like* $[\ldots]_+$ specified by

$$Q_\pm(x) = \sum_{s=0}^{\deg[Q_\pm]} (\mp 1)^s q_s x^s := [q_0, q_1, \ldots]_\pm, \;\; q_s \in \mathbb{N}, \tag{2.24}$$

with $Q_\pm(x)$ being irreducible polynomials over integers satisfying conditions in (2.23). The corresponding single-mode characters are $Q_-(x)$ and $1/Q_+(x)$, respectively. This classification naturally generalizes ordinary statistics, and we term it *transtatistics* with two possible types: *transfermions* (type $[\ldots]_-$) and *transbosons* (type $[\ldots]_+$). Here $\deg[Q_\pm]$ is the degree of $Q_\pm(x)$ to which we also refer as *order of statistics*. Order 0 is a trivial case, thus we assume $\deg[Q_\pm] \geq 1$. For $[\ldots]_-$ statistics, the generalized Pauli principle applies with $p = Q_-(1) - 1 < +\infty$ being the maximal number of particles per mode, while for $[\ldots]_+$ we have $p = +\infty$. From now on, we shall use the label $[q_0, q_1, \ldots]_\pm$ to refer to a particular type of particle statistics.

Note that one can find the eigenvalues of the excitation operator $\tilde{N}$ defined in (2.10) by solving the following equation

$$x^{f_0} + x^{k_1} + \cdots + x^{f_p} = (Q_\pm(x))^{\mp 1}\,. \tag{2.25}$$

2.4 Irreducible particle sectors: bosons and fermions

Ordinary statistics is order-one statistics of the type $[1,1]_\pm$. To answer what makes bosons and fermions special in the whole family of generalized statistics classified in (2.24), we introduce the following assumptions:

Assumption 2 (Irreducibility)**.** *All symmetries of the system of indistinguishable particles are determined by the $U(d)$ group.*

Assumption 2 essentially states that the Fock space decomposes into $U(d)$-IR sectors without multiplicity; thus, no additional symmetries (conserved quantities) are present in the system. We show now that only ordinary statistics has this property.

We start with a general single-mode character $\chi_1(x) = \sum_{s=0}^{+\infty} a_s x^s$. Character equation (2.9) for d-modes can be expanded as follows

$$\begin{aligned}\chi_d(\vec{x}) &= a_0^d + a_0^{d-1} a_1 (x_1 + \cdots + x_d) + W(\vec{x}) \\ &= a_0^d s_{(0,0,\ldots,0)}(\vec{x}) + a_0^{d-1} a_1 s_{(1,0,\ldots,0)}(\vec{x}) + W(\vec{x}),\end{aligned} \tag{2.26}$$

where $W(\vec{x})$ is the symmetric function that contains quadratic and higher-order terms in variables $\vec{x} = (x_1,\ldots,x_d)^T$. Since Schur polynomials of degree l form the basis in the space of l-degree symmetric polynomials, the constant and linear terms in the equation (2.26) are already IR-decomposed. Because assumption 2 requests no multiplicities, we have $a_0 = 0,1$ and $a_1 = 0,1$.

Now we turn to concrete cases. For transfermions, the single-mode character reads

$$\begin{aligned}Q_-(x) &= c_- \prod_i (1 + \alpha_i x) \\ &= c_- + c_- \left(\sum_i \alpha_i\right) x + c_- \left(\sum_{i<j} \alpha_i \alpha_j\right) x^2 + \ldots \\ &= a_0 + a_1 x + a_2 x^2 + \ldots\end{aligned} \tag{2.27}$$

with $\alpha_1 > \alpha_2 > \cdots > 0$ and $c_- \in \mathbb{N}$. This is consistent with the previous analysis of (2.26) only if $c_- = a_0 = 1$ and $\sum_i \alpha_i = a_1 = 1$. For the quadratic coefficient in (2.27) we have $a_2 = \sum_{i<j} \alpha_i \alpha_j = \frac{1}{2}(\sum_i \alpha_i)^2 - \frac{1}{2}\sum_i \alpha_i^2 = \frac{1}{2} - \frac{1}{2}\sum_i \alpha_i^2 \in \mathbb{N}_0$ because Q_- is an integral polynomial. This is possible only if $\alpha_1 = 1$ and $\alpha_2 = \cdots = 0$. Thus, we recover the fermionic character $\chi_1(x) = 1 + x$.

For the case of transbosons, we have the single-mode character

$$\begin{aligned}1/Q_+(x) &= 1/\prod_i (1 - \alpha_i x) \\ &= 1 + \left(\sum_i \alpha_i\right) x + \left(\sum_i \alpha_i^2 + \sum_{i<j} \alpha_i \alpha_j\right) x^2 + \ldots \\ &= a_0 + a_1 x + a_2 x^2 + \ldots\end{aligned} \tag{2.28}$$

By the same analysis as for transfermions, we conclude $\sum_i \alpha_i = 1$. For the quadratic term in (2.28), we have $a_2 = \sum_i \alpha_i^2 + \sum_{i<j} \alpha_i \alpha_j = \frac{1}{2}(\sum_i \alpha_i)^2 + \frac{1}{2}\sum_i \alpha_i^2 = \frac{1}{2} + \frac{1}{2}\sum_i \alpha_i^2 \in \mathbb{N}_0$. Again, this is satisfied only if $\alpha_1 = 1$ and $\alpha_2 = \cdots = 0$. Thus $\chi_1(x) = \frac{1}{1-x}$ and we recover the bosonic character. This concludes that only bosonic and fermionic statistics are consistent with the assumption 2.

For ordinary statistics, the excitation operator in (2.10) coincides with the standard number operator. The Casimir operator in (2.13) becomes the total number of particles which is a conserved quantity linked with N-particle sectors

$$\mathcal{F}_d^{(N)} = \mathrm{span}\{|n_1, n_2, ..., n_d\rangle \mid \sum_k n_k = N\}. \tag{2.29}$$

These are also $U(d)$-IR sectors associated with the standard bosonic (fermionic) subspaces $\mathcal{V}_{\pm}^{(N)}$.

It is worth pointing out that only in the case of ordinary statistics is the solution to the equation (2.25) for spectrum f_n of the excitation operator $\tilde{N}$ non-degenerate (in this case $f_n = n$). In all other cases, degeneracy necessarily appears. This follows from the fact that coefficients in the polynomial $Q_{\pm}(x)$ are all non-zero, and at least one of them is 2 or greater (otherwise, all coefficients are equal to 1, and we have ordinary statistics). Given this, at least one expansion coefficient on the right-hand side of (2.25) is 2 or greater. Thus at least two f_n numbers on the left-hand side of (2.25) are the same. However, it should be noted that these results are based on the assumption that $\tilde{N} \geq 0$. A generalization to the possibility of negative values of $\tilde{N}$ is discussed in Section 2.6.2.

2.5 Hidden symmetry and transtatistics

We learned from the previous analysis that multiplicities in the Fock space decomposition (2.16) will necessarily appear for all transtatistics apart from bosonic and fermionic. These multiplicities cannot be resolved without additional, so-called *hidden symmetry*, present in the system [McI59]. The latter is typically identified as a higher symmetry of the Hamiltonian required to fully resolve the degeneracy of the energy spectrum (sometimes called 'accidental' degeneracy). The classic example is the degeneration of the spectrum of the hydrogen atom not captured by the rotational symmetry (SO(3) group) of the Hamiltonian but requires a higher (hidden) symmetry for resolution, which is the SO(4) group [PJ26]. In our case, the situation is similar; the multiplicities in the Fock space decomposition are in one-to-one correspondence with the degeneration of the Hamiltonian $\hat{H} = \sum_{k=1}^{d} \varepsilon_k \tilde{N}_k$ defined in (2.14) (generator of the $U(d)$ action). The total energy is given by

$$E = \sum_{k=1}^{d} \varepsilon_k f_k, \tag{2.30}$$

with f_k being the eigenvalues of the excitation operator $\tilde{N}$ defined in (2.10). As long as this operator is non-degenerate, the energy spectrum E is well-resolved with the set of quantum numbers $(f_{k_1}, \ldots, f_{k_d})$. Nevertheless, we have seen that this happens only in the case of ordinary statistics. For all other cases, degeneracy in spectrum f_n necessarily appears, which is to be resolved by different quantum numbers unrelated to the $U(d)$ group. Without the specification of these numbers, the representation of $U(d)$ in Fock space remains unspecified, defined only up to IR-multiplicity.

We will study these effects in detail for the first non-trivial case beyond ordinary statistics, i.e., the order-one statistics $[1,q]_\pm$, with $q \in \mathbb{N}$. To make the analysis more accessible, we will separate the notation for transbosons (type $[1,\beta]_+$) and transfermions (type $[1,\alpha]_-$) with $\alpha, \beta \in \mathbb{N}$. The reason why we set the first coefficient in $[\ldots]_\pm$ to be 1 is because we will restrict our analysis only to the case of a unique vacuum state $|0\rangle^{\otimes d}$. To be more precise, we will study the cases in which the only invariant state under $U(d)$ is a vacuum state. This is possible only if the first coefficient in the single-mode character $\chi_1(x) = \sum_{s=0}^{+\infty} a_s x^s$ is set $a_0 = 1$ (see discussion around equation (2.26)).

To begin with, take an example of transfermions $[1,\alpha]_-$ with $\alpha = 2$. In this case, the singe-mode Fock space is three-dimensional (follows from $\chi_1(x) = 1 + 2x = 1 + 2e^{i\theta}$), and the maximal occupation number is $p = \alpha = 2$. For the case of two modes, the character reads $\chi_2(x_1, x_2) = 1 + 2(x_1 + x_2) + 4x_1 x_2 = s_{(0,0)}(x_1, x_2) + 2s_{(1,0)}(x_1, x_2) + 2^2 s_{(1,1)}(x_1, x_2)$. Thus, the Fock space decomposes into fermionic multiplets of the size $\alpha^N = 2^N$, for $N = 0, 1, 2$. This exponential growth of multiplicity is generic to order-one transtatistics. It is formalized in the following theorem (see Appendix 2.9 for proof)

Theorem 2. Fock-spaces for $[1,\alpha]_+$ and $[1,\beta]_-$ decompose into IR sectors as

$$\mathcal{F}_d = \bigoplus_{N=0}^{d} \alpha^N \mathcal{V}_-^{(N)}, \quad \alpha \in \mathbb{N}, \tag{2.31}$$

$$\mathcal{F}_d = \bigoplus_{N=0}^{+\infty} \beta^N \mathcal{V}_+^{(N)}, \quad \beta \in \mathbb{N}, \tag{2.32}$$

where $\mathcal{V}_-^{(N)}$ and $\mathcal{V}_+^{(N)}$ are the fermionic and bosonic IRs (N-particle sectors for ordinary statistics), respectively.

In the next section, we will build the concrete ansatz to identify auxiliary quantum numbers to resolve the degeneracy in (2.31)-(2.32). Based on this, we will construct the $U(d)$ representation in Fock space.

2.5.1 Hidden quantum numbers

We start with transfermions $[1,\alpha]_-$ for some $\alpha \geq 2$. For this case, the single-mode character reads $\chi_1(x) = 1 + \alpha x$ with $x = e^{i\theta} \in U(1)$. The single-mode Fock space is $(\alpha+1)$-dimensional $\mathcal{F}_1 = \mathrm{span}\{|n\rangle \mid n = 0, 1, 2, \ldots, p = \alpha\}$. Equation (2.25) reads

$$x^{f_0} + x^{k_1} + \cdots + x^{f_p} = 1 + \alpha x, \tag{2.33}$$

with the solution $f_0 = 0$ and $f_n = 1$ for $n = 1, \ldots, \alpha$. The generator of $U(1)$ action $\Delta_1(e^{i\theta}) = e^{i\theta\tilde{N}}$ is the excitation operator defined in (2.10) and in our case, it acts is as follows

$$\tilde{N}|n\rangle = \begin{cases} 0 & n = 0, \\ +1\,|n\rangle & n = 1, \ldots, \alpha. \end{cases} \tag{2.34}$$

Given this, one can re-interpret the single-mode states $|n\rangle$ for $n \geq 1$ as *de facto* being the single-particle excitations distinguished by some auxiliary degree of freedom with α values. Therefore, we can introduce decomposition $n = k + z$ with $k = 0, 1$ being the 'real' occupation number of the fermionic type and $z = 0, \ldots \alpha^k - 1$ as an auxiliary quantum number accounting for degeneracy. Having this, the formula (2.34) takes the standard form, i.e $\tilde{N}\,|k+z\rangle = k\,|k+z\rangle$. Now, to separate degrees of freedom captured by k and z quantum numbers, we introduce the mapping

$$L_1\,|k+z\rangle = \begin{cases} |0\rangle_F & k=0, \\ |1\rangle_F \otimes |z\rangle_A & k=1, \end{cases} \tag{2.35}$$

where $|k\rangle_F$ is the ordinary fermionic number state with $k = 0, 1$, while $|z\rangle_A$ (with $z = 0, \ldots \alpha - 1$) is a new degree of freedom emerged solely from the *statistics type*. The ansatz straightforwardly generalizes to the d-mode Fock space. We define

$$L_d\,|n_1, \ldots, n_d\rangle = \mathcal{T} L_1^{\otimes d}\,|n_1, \ldots, n_d\rangle\,, \tag{2.36}$$

where $\mathcal{T}$ is the shift operator needed to separate degrees of freedom, i.e., to shift all auxiliary states to the right. For example, $\mathcal{T}\,|k_1\rangle\,|z_1\rangle\,|k_2\rangle\,|z_2\rangle = |k_1, k_2\rangle_F \otimes |z_1, z_2\rangle_A$. To fully clarify the mapping in (2.36), let $|n_1, \ldots, n_d\rangle = |k_1 + z_1, \ldots, k_d + z_d\rangle$, where again, $k_s = 0, 1$ and $z_s = 0, \ldots, \alpha^{k_s} - 1$. We form the ordered list $(z_{s_1}, \ldots, z_{s_N})$ for which $k_{s_r} = 1$, i.e. the list of all non-zero fermionic excitations. Here $N = d - (\delta_{0,n_1} + \cdots + \delta_{0,n_d})$ is the total number of them. Then, the equation (2.36) reads

$$L_d\,|n_1, \ldots, n_d\rangle = |k_1, \ldots, k_d\rangle_F \otimes |z_{s_1}, \ldots, z_{s_N}\rangle_A \tag{2.37}$$

where $|k_1, \ldots, k_d\rangle_F$ is the ordinary N-particle fermionic state, with the auxiliary label of particles $|z_{s_1}, \ldots, z_{s_N}\rangle_A$. This brings us precisely to the decomposition in (2.31), which can also be written as

$$\mathcal{F}_d = \bigoplus_{N=0}^{d} \mathcal{V}_-^{(N)} \otimes \mathcal{H}_A^{\otimes N}, \tag{2.38}$$

where $\mathcal{H}_A = \mathrm{span}\{|z\rangle \mid z = 0, \ldots \alpha - 1\}$ is the auxiliary space. Given this factorization, it is clear that $U(d)$ acts only in the fermionic part $\mathcal{V}_-^{(N)}$, while $\mathcal{H}_A^{\otimes N}$ remains untouched. The additional $U(\alpha)$ group acting in the space $\mathcal{H}_A^{(N)}$ can be added to resolve degeneracy completely. Now, for an element $g \in U(d)$, let the standard action on the fermionic number state is $\Delta_d^{(F)}(g)\,|k_1, \ldots, k_d\rangle_F$. This induces the action $\Delta_d(g)$ in the Fock space (2.5) as

$$\Delta_d(g) = L_d^{-1}\left(\Delta_d^{(F)}(g) \otimes \mathbb{1}_A\right) L_d, \tag{2.39}$$

where L_d is the mapping given in(2.36). With this, we have defined the action of $U(d)$ in the Fock space.

In complete analogy, we provide an ansatz for transbosons of $[1,\beta]_+$ type with $\beta \geq 2$. In this case, we have the single mode character $\chi_1(x) = \frac{1}{1-\beta x}$ with $x = e^{i\theta} \in U(1)$ and the single-mode Fock space is infinte-dimensional $\mathcal{F}_1 = \mathrm{span}\{|n\rangle \mid n = 0, 1, 2, \ldots\}$. As before, we shall solve equation (2.25)

$$x^{f_0} + x^{k_1} + x^{k_2} + \cdots = \frac{1}{1-\beta x}. \tag{2.40}$$

It is convenient to write the particle number n in the form

$$n = 1 + \beta^2 + \cdots + \beta^{k-1} + z = \frac{\beta^k - 1}{\beta - 1} + z, \tag{2.41}$$

with $k = 0, 1, 2, \ldots$ and $z = 0, \ldots, \beta^k - 1$. Having this notation, the solution to (2.40) is simple, i.e., $f_{\frac{\beta^k-1}{\beta-1}+z} = k$. We have the following action of the excitation operator $\tilde{N}$

$$\tilde{N}\left|\frac{\beta^k - 1}{\beta - 1} + z\right\rangle = k\left|\frac{\beta^k - 1}{\beta - 1} + z\right\rangle. \tag{2.42}$$

Here k represents the 'new' occupation number of the bosonic type, while z is an auxiliary quantum number. Since $z = 0, \ldots, \beta^k - 1$ counts all possible states associated to k bosonic excitations, it is convenient to write z in the β-base, i.e. $z = z_{k-1}\beta^{k-1} + z_{k-2}\beta^{k-2} + \cdots + z_0\beta^0 := z_{k-1}z_{k-2}\ldots z_0$, where $z_s = 0, \ldots, \beta - 1$ are the digits. With this, we can introduce the mapping

$$L_1\left|\frac{\beta^k - 1}{\beta - 1} + z\right\rangle = \begin{cases} |0\rangle_B & k = 0, \\ |k\rangle_B \otimes |z_{k-1}z_{k-2}\ldots z_0\rangle_A & k > 0, \end{cases} \tag{2.43}$$

where $|k\rangle_B$ is the ordinary bosonic Fock (number) state with $k = 0, 1, 2, \ldots$, while $|z_{k-1}z_{k-2}\ldots z_0\rangle_A$ (with $z_s = 0, \ldots, \beta - 1$) is associated to the statistics degree of freedom. The generalization to the d-mode Fock state is as for the case of transfermions, i.e., we use the same equation (2.36). In this case, we have

$$L_d\,|n_1, \ldots, n_d\rangle = |k_1, \ldots, k_d\rangle_B \otimes |z_{s_1}, \ldots, z_{s_N}\rangle_A\,, \tag{2.44}$$

where $|k_1, \ldots, k_d\rangle_B$ is the ordinary bosonic number state with $k_s = 0, 1, 2, \ldots$, while $|z_{s_1}\ldots z_{s_N}\rangle_A$ comes from type of statistics. The action of $U(d)$ is introduced in complete analogy to the fermionic case and equation (2.39).

2.5.2 Is hidden symmetry an ordinary internal symmetry?

We may question if the hidden quantum numbers introduced in the previous section are related to some genuine degree of freedom emerging from the type of statistics. Could these numbers be associated with the standard internal degrees of freedom, such as spin? For example, degeneration in (2.30) could be potentially explained by the argument that energy is spin-independent, and then, transatistics may be just an ordinary (fermionic of bosonic) statistics where $U(d)$ affects only external degrees of freedom (such as modes represented by paths of particles in Fig. 2.1a). However, this argument cannot be well-aligned with the Fock-state decomposition in (2.31)-(2.32), even though only multiplets of ordinary statistics appear in decomposition. This is due to the dimension discrepancy between ordinary statistics and transtatistics. To see this, suppose that we deal with ordinary fermions with d real degrees of freedom (d modes on which $U(d)$ acts) and some internal degree of freedom (e.g., spin) with $z = 0, \dots, \alpha - 1$ values, which is unaffected by $U(d)$. The overall dimension of the single-particle space is αd; hence the dimension of the fermionic Fock space is $2^{\alpha d}$. This starkly contrasts the dimension α^d of the transfermionic Fock space for $[1, \alpha]_-$ type. As we shall see, this dimension discrepancy will differentiate the thermodynamics of non-interacting systems of ordinary and transtatistics. The latter will be accompanied by the effect of generic spontaneous symmetry breaking absent in ordinary statistics.

Note that analogy to, e.g., spin degree of freedom discussed here is only possible for order-one statistics. For higher-order statistics, no (obvious) similarities can be concluded. We will discuss this point later.

2.5.3 Relation to thermodynamics

To study thermodynamics, we consider the single-particle energy spectrum $\varepsilon_1, \dots, \varepsilon_d$, where ε_ks represent the energies associated with different modes. This situation is similar to the one discussed in Section 2.2.2. In that section, we examined the unitary evolution generated by the Hamiltonian given in equation (2.14). However, the system is in contact with a thermal bath in the present case. The thermodynamical quantities (e.g., for canonical ensemble) can be derived from the partition function $Z_d(\beta) = \mathrm{Tr} e^{-\beta \hat{H}}$, and its explicit form follows directly from the form of H, i.e.,

$$Z_d(\beta) = \prod_{k=1}^{d} Z_1(e^{-\beta\varepsilon_k}) = \prod_{k=1}^{d} \chi_1(e^{-\beta\varepsilon_k}) \tag{2.45}$$

with $\beta = 1/k_B T$ being the Boltzmann factor.

The physical relevance of character χ can also be understood through thermodynamics [Bal01]. This is because we can get the partition function from the character via Wick's rotation, that is, $it/\hbar\varepsilon_k \to -\beta\varepsilon_k$. In this respect, the product form of equation (2.45) arises directly from our central assumption 1, i.e., the overall partition function can be expressed as a product of individual partition functions (associated with individual modes). This aligns with the expected behavior for independent systems, such as a set of independent modes. Therefore, assumption 1 is in one-to-one correspondence with the independence in a thermodynamical sense. The formula (2.45) trivially holds for ordinary statistics (bosons and fermions) [AM76].

When the system is capable of exchanging excitations (particles) with a reservoir, we can analyze its behavior using the grand canonical partition function $\mathcal{Z}_d = \mathrm{Tr} e^{-\beta(\hat{H} - \mu\tilde{N})}$. In this expression, $\tilde{N}$ represents the excitation operator as defined in equation (2.10), and μ corresponds to the chemical potential (variable conjugated to $\tilde{N}$). The explicit form of the grand canonical partition function $\mathcal{Z}_d$ is as follows

$$\mathcal{Z}_d = \prod_{k=1}^{d} Z_1(e^{-\beta(\varepsilon_k - \mu)}) = \prod_{k=1}^{d} \chi_1(e^{-\beta(\varepsilon_k - \mu)}). \tag{2.46}$$

2.5.4 Thermodynamics of ideal gasses and spontaneous symmetry breaking

Let us examine the thermodynamical properties of a non-interacting system for general order-one statistics $[1, q]_\pm$ with $q \in \mathbb{N}$. Ordinary statistics is recovered for $q = 1$. We consider a

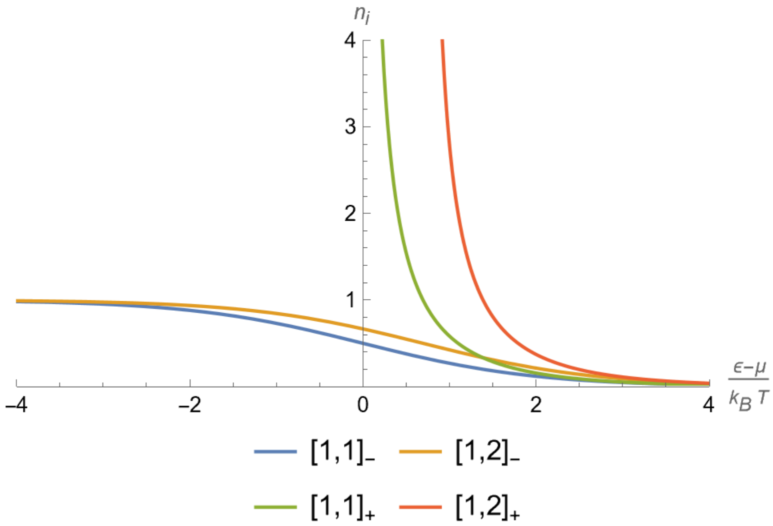

Figure 2.3: **Mean particle number** for ordinary (blue and green) and generalized (orange and red) statistics.

grand-canonical ensemble defined by a set of single-particle energies $\varepsilon_1, \dots, \varepsilon_d$ associated with different modes. The system is described by a equilibrium state $\rho = \frac{1}{\mathcal{Z}_d} e^{-\beta(\hat{H}-\mu\hat{N})}$, where $\mathcal{Z}_d$ is the grand-canonical partition function defined in (2.46). All thermodynamical quantities can be evaluated from the grand canonical potential $\Omega = -\frac{1}{\beta}\log\mathcal{Z}_d$. For example, $N = \frac{\partial\Omega}{\partial\mu}$ gives the mean particle number. For the case of transtatistics $[1,q]_\pm$, we get

$$N = \sum_k n_k = \sum_i \frac{1}{\frac{1}{q}e^{\beta(\varepsilon_k-\mu)} \mp 1}. \tag{2.47}$$

This expression reduces to the Fermi-Dirac and Bose-Einstein distributions for $q = 1$. The plots for n_i in (2.47) for various statistics are presented in Fig. 2.3. For the fermionic-type statistics, equation (2.47) reduces to the Fermi-Dirac distribution $n_k = \theta(\mu - \varepsilon_k)$ at zero temperature for all q. For the bosonic type, the mean number diverges at the values of energy $\varepsilon = \mu + \frac{1}{\beta}\log q$ when the Bose-Einstein condensation occurs. In the classical limit of $\beta(\varepsilon-\mu) \gg 1$, the formula (2.47) reduces to the standard Maxwell-Boltzmann distribution, i.e., $n_k \approx q e^{-\beta(\varepsilon_k-\mu)}$, where the factor q appears as the degeneracy factor. The same factor appears in the classical limit for standard quantum gasses with $q = 2s+1$ coming from spin s (see, for example, Chapter 8.3. in [Sal01]). This is because the energy is independent of spin, and thus, the energy spectrum degenerates.

Note that the chemical potential μ in the formulas above is temperature dependent. To be more precise, the standard approach to thermodynamics of ideal gasses is to keep total particle number N as a fixed parameter and then invert (2.47) to calculate chemical potential $\mu = \mu(N,T)$ as a function of a total number of particles and temperature [AM76]. Given this, one can introduce a simple change of variables $\mu \to \mu - \frac{1}{\beta}\log q$, and the formula (2.47) would reduce to one for ordinary statistics. This means that solution for the chemical potential for order-one transtatistics is

$$\mu_q = \mu_{q=1} - k_B T \log q, \tag{2.48}$$

where $\mu_{q=1}$ is the chemical potential of ordinary statistics. What follows is that almost all thermodynamical quantities (e.g., mean energy, heat capacity, etc.) remain the same as in the case of ordinary statistics for arbitrary q. Nevertheless, the entropy will change. To see this, note that $S = -\beta\Omega + \beta\langle E\rangle - \beta\mu N$, thus the shift of $-k_B T \log q$ in the chemical potential introduces a change in the entropy, i.e.

$$S_q = S_{q=1} + k_B N \log q. \tag{2.49}$$

The entropy of ordinary statistics $S_{q=1}$ vanishes at $T = 0$; hence, a residual entropy of $k_B N \log q$ remains at zero temperature for all $q > 1$. This is consistent with the fact that fermionic (bosonic) N-particles IRs in the Fock space decomposition (2.31)-(2.32) appear q^N times; therefore, the ground state is q^N times degenerate. This degeneration is known to result in *residual entropy at zero temperature* and is associated with *spontaneous symmetry breaking* [Pei91], here present for transtatistics. This is one of the main differences compared to ordinary quantum gasses exhibiting non-degenerate ground states.

2.6 Discussion and outlook

2.6.1 Statistics of higher order

Here we briefly analyze some of the technical and conceptual difficulties that arise when dealing with statistics of higher order. As an illustration, we take the example of statistics of order two $[1, q_1, q_2]_\pm$. A simple inspection shows that polynomial $Q_\pm(x) = 1 \mp q_1 x + q_2 x^2$ has non-negative (positive) roots for $q_1^2 > 4q_2$. To see how the Fock space decomposes in some simple cases, consider transfermions $[1, q, 1]_-$ and the corresponding two-mode character

$$\begin{aligned}
\chi_2(x_1, x_2) &= (1 + qx_1 + x_1^2)(1 + qx_2 + x_2^2) \\
&= s_{(0,0)}(x_1, x_2) + qs_{(1,0)}(x_1, x_2) + \\
&= (q^2 - 1)s_{(1,1)}(x_1, x_2) + s_{(2,0)}(x_1, x_2) + \\
&+ \; qs_{(2,1)}(x_1, x_2) + s_{(2,2)}(x_1, x_2).
\end{aligned} \tag{2.50}$$

The IR characters $s_{(2,1)}$ and $s_{(2,2)}$ that are not fermionic nor bosonic type show-up in the decomposition. This is a typical feature that appears for any higher-order statistics. In turn, finding Fock space's decomposition for general d modes, such as one provided for order-one statistics in (2.31)-(2.32), is more difficult. Next, the dimension of the single-mode Fock space is $q + 2$, and the maximal occupation number is $p = q + 1$. The solution to the single-mode character equation (2.25)

$$x^{f_0} + \cdots + x^{f_{q+1}} = 1 + qx + x^2 \tag{2.51}$$

is $f_0 = 0$ and $f_{q+1} = 2$, while $f_n = 1$ for $n = 1, \ldots, q$. Recall that these are the eigenvalues of the excitation operator $\tilde{N}$ in (2.10), and as we see, we have three distinct values $k = 0, 1, 2$. Again, we have degeneration of the spectrum, but resolving it is a more delicate issue than for the case of order-one statistics we have presented in Section 2.5. This is partially because a clear interpretation is missing. For example, we may try to label the single-mode Fock states with two quantum numbers, $k = 0, 1, 2$ (for excitations), and one auxiliary number z_k, to account for degeneracy. As before, we have $\tilde{N}\,|k+z_k\rangle = k\,|k+z_k\rangle$, with $z_k = 0$ for $k = 0, 2$, while for $k = 1$ we have $z_k = 0, \ldots, q$. This appears paradoxical because degeneracy is present for one excitation but disappears for two. From this example, we see the analysis of degeneracy and categorization of hidden quantum numbers becomes significantly more complicated due to the involvement of 'non-standard' IRs.

Let's take a look at the general case of $[1, q_1, \ldots, q_m]_\pm$, where $Q_\pm(x) = \prod_{s=1}^m (1 \pm \alpha_k x)$ with $\alpha_s > 0$ and $m = \deg[Q_\pm]$ representing the degree of statistics. Given the Cauchy identities [SF99]

$$\prod_{k=1}^{d}\prod_{s=1}^{m}(1+\alpha_s x_k) = \Sigma_{l(\lambda)\leq m} s_\lambda(x_1,...,x_d)s_\lambda(\alpha_1,...,\alpha_m), \tag{2.52}$$

$$\prod_{k=1}^{d}\prod_{s=1}^{m}\frac{1}{1-\alpha_s x_k} = \Sigma_{l(\lambda)\leq m} s_\lambda(x_1,...,x_d)s_\lambda(\alpha_1,...,\alpha_m), \tag{2.53}$$

where $l(\lambda)$ is the length of the diagram (number of rows), and $l(\lambda')$ is the conjugate partition of λ, we that the multiplicity in the Fock space decomposition (2.16) are given by Schur polynomials evaluated at the parameters α_s, i.e. $a(\lambda) = s_\lambda(\alpha_1, ..., \alpha_m)$. This multiplicity coefficient can be simplified using the so-called Kostka numbers [SF99], but we leave this for further consideration due to the involvement of non-trivial combinatorics.

On the other hand, thermodynamic considerations are simpler due to the product nature of the single-mode character. In the general case $[1, q_1, \ldots, q_m]_\pm$ the grand canonical partition function (2.46) becomes

$$\mathcal{Z}_d = \prod_{k=1}^{d}\prod_{s=1}^{m}\left(1 \mp e^{-\beta(\varepsilon_k-\mu)}\right)^{\mp 1}. \tag{2.54}$$

In complete analogy to the derivation of (2.47), we get the mean particle number as

$$n_k = \sum_{s=1}^{m}\frac{1}{\frac{1}{\alpha_s}e^{\beta(\varepsilon_k-\mu)} \mp 1}, \tag{2.55}$$

where m is the degree of statistics. This expression reduces to the Fermi-Dirac and Bose-Einstein distributions for $m = 1$ and $\alpha_1 = 1$. For transfermions, equation (2.55) reduces to $n_k = m\,\theta(\mu - \varepsilon_k)$ at zero temperature. As expected, m (degree of statistics) trasnfermions can occupy the same energy level at $T = 0$ as a consequence of the generalized Pauli exclusion principle. For N particles, transfermions will form the analogon of the Fermi sea at zero temperature. On the other hand, we have for transbosons, the mean number diverging at the values of energy $\varepsilon = \mu + \frac{1}{\beta}\log\alpha_{\max}$ when the Bose-Einstein condensation occurs. In the classical limit of $\beta(\varepsilon - \mu) \gg 1$, the formula (2.55) reduces to the standard Maxwell-Boltzmann distribution, i.e., $n_k \approx qe^{-\beta(\varepsilon_k-\mu)}$, where the factor $q = \sum_s \alpha_s(\beta_s)$ appears as the degeneracy factor.

2.6.2 'Negative occupation' numbers, exceptional statistics and extension to an infinite number of modes

So far, we have discussed only the case of non-negative eigenvalues of $U(1)$ generators, i.e., the generalized excitation operator as defined in Section (2.2.2) and eq. (2.13) satisfying $\tilde{N} \geq 0$. Here, we will discuss the extensions to 'negative occupations', allowing $\tilde{N}$ to have negative eigenvalues, which encounters the most general $U(1)$ generator. On the physical side, such a situation is particularly relevant for the theory of antiparticles introduced for ordinary statistics by Dirac [Dir30]. To do so, we shall extend the single-mode character in (2.11) to a (formal) Laurent series $\chi_1(x) = \sum_{s\in\mathbb{Z}} a_s x^s$, with a_s integer coefficients, such that the overall product $\chi_d(\vec{x}) = \prod_{k=1}^{d} \chi_1(x_k)$ is a valid $U(d)$-character. In such case, $\chi_d(\vec{x})$ decomposes over Laurent-Schur polynomials of the form $(x_1 \dots x_d)^{-k} s_\lambda(x_1,\dots,x_d)$, with $k \in \mathbb{N}_0$, encompassing the most general irreducible representations of $U(d)$ [SF99]. Details are provided in the Appendix 2.6.2, where we prove the following classification theorem:

Theorem 3. General $U(1)$-character (under conditions previously discussed) is one of the following forms

1) **transtatistical type**, i.e. either $x^k\chi_1(x)$ or $x^k\chi_1(1/x)$, where $\chi_1(x)$ is the character classified by the Theorem 1 and $k \in \mathbb{Z}$, or

2) **exceptional type**, $a\sum_{s\in\mathbb{Z}} \rho^s x^s$, with $a, \rho \in \mathbb{N}_0$.

This concludes the classification in the most general case. For the trastatistical types, the factor x^k builds to $(x_1 \dots x_d)^k = (\det(g))^k$ for the case of d modes, which corresponds to the determinant representation. Thus, representations can be constructed from representations obtained for transtatistics wired with the determinant representation (power k). Note that $\chi_1(1/x) = \chi_1(x)^*$, because $x = e^{i\varphi}$, thus this character is associated with the conjugate representation. Finally, the exceptional character is of a fundamentally different form and essentially comes from exceptional totally-positive sequences (see Appendix 2.6.2 for details). This is a very interesting case in which generalized number operator $\tilde{N}$ is unbounded both from below and above, which is not the case for transtatistical types of statistics.

Finally, we will conclude this section with a brief comment on the generalization to the infinite set of modes when $d = \infty$. This is particularly relevant for investigating the algebra of creation and annihilation operators for transtatiscics field theory. In this respect, establishing a relation to the existing results on $U(\infty)$ representations and extreme characters theory [VK82, OO97, Voi76] is very promising. In particular, striking similarities are found in the generating functions of the so-called "extreme characters" and our principal decomposition given in Theorem 1.

2.6.3 Relation to other generalized statistics

An obvious question is if and how our statistics classified in (2.24) differs from other generalized statistics presented in the literature. Of course, we are not able to exhaustively compare but rather analyze the most common cases. The first remark is that the main difference is due to the underlying symmetries. Our classification relies on the $U(d)$ group, while in most of the cases, other generalized statistics is based on a different group. Take an example of fractal statistics [Wil82, LM77] where topological defects and representation of braid groups [FRS89] play the central role. We can have, for example, the action of 2π-rotation, leaving a non-trivial phase. This contrasts the 2π-periodicity, essential to derive the integer spectrum for the excitation operator in our equation (2.10). This suggests that we speak of different kinds of particle statistics due to the involvement of different symmetry groups. On the other hand, the recent work of [BH08] suggests that fractal statistics can be phrased in terms of Jack polynomials, which generalize Schur polynomials (our primary tool to classify statistics). This relationship is worth looking into in the future.

Very similar holds for many generalized statistics related to deformed canonical commutation relations. Take an example of q-deformations (quons) with $a_i a_j^\dagger - q a_j^\dagger a_i = \delta_{ij}\mathbb{1}$ [Gre91a]. However, q-deformations introduce new symmetries even at the level of a single particle, i.e., q-deformed $U(d)$ [Jim85] group, while our statistics is directly paired to the $U(d)$ symmetry. Still, some comparison might be possible for the order-one statistics, where our ansatz of Section 2.5 provides the means to construct the algebra of creation and annihilation operators and evaluate the corresponding commutation relations.

Finally, the question is how our generalization is related to parastatistics [Gre53]. As already pointed out, the group behind the parastatistics is different [RS63, SdJ08]. This leads to the different Fock space decomposition, i.e., for parastatistics of order p, we have [HT69, SV20]

$$\mathcal{F}_{\text{parab}} = \bigoplus_{l(\lambda)\leq p} \mathcal{V}_\lambda \tag{2.56}$$

$$\mathcal{F}_{\text{paraf}} = \bigoplus_{l(\lambda')\leq p} \mathcal{V}_\lambda, \tag{2.57}$$

where the sum runs over Young diagrams λ (parabose case) or λ' (parafermi case) of the length $l(\lambda)$ (number of rows). Here λ' is the conjugated diagram of λ and $\mathcal{V}_\lambda$ is an $U(d)$-IR associated to λ. This decomposition contains no multiplicities and thus is compatible with our classification only for the case of ordinary statistics.

2.6.4 Some open questions and applications

The broad range of possibilities for generalized statistics introduced here leaves many interesting open questions and potential for applications. Firstly, an open question is what is more on the physical side (compared to new effects already discussed) that tanstatistics brings. As we already discussed, there are technical difficulties with higher-order statistics, mainly in the context of hidden quantum numbers. Nevertheless, we may study thermodynamics directly by the ansatz defined in section 2.5.4. One has to calculate partition functions given in (2.45) for more general characters. In this case, a simple shift of the chemical potential in (2.48) will not reduce thermodynamical quantities to ones given by ordinary statistics as it happens for order-one statistics. Given this, we can expect other novel physical effects to appear.

The next exciting point to analyze is the application of our method to diagonalize solid-state Hamiltonians, such as it is done for spin-chains via spin-fermion mapping (Jordan-Wigner transformation) [SML64]. For example, the transfermionic Fock space for $[1,\alpha]_+$ is isomorphic to $(\mathbb{C}^{(\alpha+1)})^{\otimes d}$ which is suitable to study higher dimensional spin chains. In complete analogy to the spin-fermion mapping, one can expect to find other integrable many-body Hamiltonians that reduce to our non-interacting model.

An interesting point to be analyzed is the question of entanglement in transtatistics. This question has raised a long-standing debate in the community regarding the case of bosons and fermions due to the apparent entanglement present in the first quantized picture, which comes solely from the (anti) symmetrization of the wave function. It is accepted nowadays that such "kinematic" entanglement is physical [BFFM20, MYF+20]. In the case of transtatistics, the starting point is different as we do not know if the first-quantized picture for such generalized statistics exists; thus, the situation is less clear. However, one can try to put transtatistics in the context of quantum-information processing and protocols designed to study the entanglement of standard indistinguishable particles (see [MYF+20] and references therein). This shall provide a more clear view of the relation between entanglement and indistinguishability in this case. Along these lines, an exciting perspective on our results comes from the quantum computational complexity of quantum statistics. Namely, it is well-known that the non-interacting bosons are computationally hard to simulate [AA11], while non-interacting fermions are not [TD02]. One can ask a similar question here, i.e., what is the computational power of the non-interacting model for transtatistics? Any answer to it is relevant and may find applications in quantum computing.

Finally, on the speculative side, an interesting idea of applying generalized statistics in the context of dark-matter modeling was recently presented [HS22]. The main point is to study thermodynamics and the effects of the negative relation between pressure and energy density, emphasized in many existent dark energy candidates. Our methods provide a direct way to calculate thermodynamical properties of transtatistics and thus might be worthy of investigating relations to dark-matter models.

2.7 Schur polynomials

The linear representations of the general linear group $GL(n,\mathbb{C})$ and its maximally compact subgroup $U(n)$ are unambiguously identified by *Schur polynomials* as their characters [FH04]. These families of polynomials appear in different regions of mathematics, from pure combinatorics to algebraic geometry. This is why several standard definitions exist depending on the specific context they are discussed. Here we present them in the combinatorial definition to emphasize the combinatorial of our operational reconstruction of particle statistics. Other methods, such as the classical (determinant) definition, are standardly found in the literature [SF99].

Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be an integer partition with $\lambda_1 \geq \lambda_2 \geq \dots$, usually represented with a *Young diagram* (see Figure 2.4). The total number of boxes in a diagram is denoted by $|\lambda| = \lambda_1 + \lambda_2 + \dots$, and the partition length (the number of rows) is labeled by $l(\lambda)$.

The *semistandard* Young tableau (SSYT) of shape λ is a *filling* of the boxes in the Young diagram with positive integers such that the entries weakly increase along each row and strictly increase down each column. For given a SSYT T, we define the *type* of T, i.e., $\alpha(T) = (\alpha_1, \alpha_2, \dots)$ with $\alpha_i(T)$ being the number of repetitions of the number i in T. For example, for the SSYT given in Fig. 2.4, we have $\alpha(T) = (1,4,3,2)$. For a set of variables $x_1, x_2, \dots$ we define

$$x^T \equiv x_1^{\alpha_1(T)} x_2^{\alpha_2(T)} \dots . \tag{2.58}$$

1	2	2	2	3	4
2	3	3			
4					

Figure 2.4: Young diagram for $\lambda = (6,3,1)$. The filling of the diagram defines one SSYT.

Definition 1. Let λ be a partition. The *Schur function* $s_\lambda(x_1,\ldots,x_d)$ in $d \geq l(\lambda)$ variables associated with λ is a homogeneous symmetric polynomial of degree $|\lambda|$ defined as:

$$s_\lambda(x) = \sum_{T \in \mathrm{SSYT}_\lambda(d)} x^T, \tag{2.59}$$

where the sum runs over all semistandard Young tableaux (SSYTs) of shape λ with the filling from the set $\{1,\ldots,d\}$.

The common examples are the bosonic Schur functions (*homogeneous symmetric polynomials*)

$$s_{(n,0,\ldots,0)}(x_1,\ldots,x_d) = \sum_{k_1+\cdots+k_d=n} x_1^{k_1}\ldots x_d^{k_d}, \quad k_s = 0\ldots d, \tag{2.60}$$

associated with partitions having a single row of size n (n-particle sector), and the fermionic Schur functions (*elementary symmetric polynomials*)

$$s_{(1,\ldots,1,0,\ldots,0)}(x_1,\ldots,x_d) = \sum_{1\leq i_1<i_2\cdots<i_n\leq d} x_{i_1}x_{i_2}\ldots x_{i_n}, \tag{2.61}$$

associated with partitions having a single column of size n (n-particle sector). Notably, these bosonic and fermionic partitions are conjugate to each other. Two partitions are said to be conjugate if they can be obtained from each other by interchanging rows and columns in their Young diagrams.

2.8 Proof of main (partition) theorem

Before proceeding with the proof, we must introduce some basic definitions and important known theorems.

Definition 2. We call $f(x) = \sum_n a_n x^n$ the generating function of sequence $\{a_n\}$.

Definition 3. A matrix $(A_{ij})_{i,j\in I}$ is called Töplitz if $A_{i,j} = a_{i-j}$.

Definition 4. A sequence of real numbers $\{a_n\}_{n\in\mathbb{Z}}$ is totally positive if and only if all the minors of the Töplitz matrix $(a_{i-j})_{i,j\in\mathbb{Z}}$ are non-negative. For sequences defined only on non-negative integers $\{a_n\}_{n\in\mathbb{N}_0}$, we assume the extension $a_n = 0$ for $n < 0$. Generating function $f(x) = \sum_n a_n x^n$ is called totally positive if and only if $\{a_n\}$ is totally positive.

Proposition 1 ([BD02]). Let $f(x) = \sum_n a_n x^n$ and $D^{\lambda,\mu}_{d-1}(f) = \det\left(a_{\lambda_i-\mu_j-i+j}\right)_{1\le i,j\le d}$ with λ and μ being a pair of partitions of possibly different integers. Then every Töplitz minor of matrix $(a_{i-j})_{i,j\in\mathbb{Z}}$ is of the form $D^{\lambda,\mu}_{d-1}(f)$. Furthermore, we have

$$D^{\lambda,\mu}_{d-1}(f) = \int_{U(d)} \Phi_{n,f}(g)\overline{s_\lambda(g)}s_\mu(g)dg \tag{2.62}$$

with $\Phi_{d,f} = f(x_1)...f(x_d)$ where $x_1,...,x_d$ are the eigenvalues of g, and $\int_{U(d)}\ldots dg$ is the Haar measure integral.

Proposition 2 ([ASW52]). The sequence $\{a_n\}_{n\in\mathbb{N}_0}$ with $a_0 = 1$ is totally positive if and only if it is generated by a function $f(x)$ of the form

$$f(x) = e^{\gamma x}\frac{\Pi_i(1+\alpha_i x)}{\Pi_i(1-\beta_i x)} \tag{2.63}$$

with $\alpha,\beta,\gamma \ge 0$ and $\sum\alpha_i, \sum\beta_i$ convergent.

The last proposition is the most prominent result for characterizing totally-positive sequences. We can prove now the following simple lemma (an almost equivalent statement was presented in [BG15, Dav00]).

Lemma 1. An integral series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with $a_0 > 0$ is totally positive if and only if it is of the form

$$f(x) = \frac{g(x)}{h(x)}, \tag{2.64}$$

for some integral polynomials $g(x)$ and $h(x)$ with $h(0) = 1$, such that all complex roots of $g(x)$ are negative and all those of $h(x)$ are positive real numbers.

Proof. A sequence $\{a_n\}_{\mathbb{N}_0}$ is totally positive if and only if $\{ra_n\}_{\mathbb{N}_0}$ is totally positive for $r > 0$. We set $r = 1/a_0$ and by the Proposition 2, we get that

$$f(x) = a_0 e^{\gamma x}\frac{\Pi_i(1+\alpha_i x)}{\Pi_i(1-\beta_i x)} \tag{2.65}$$

Because $\sum_i(\alpha_i+\beta_i) < +\infty$, the function f is meromorphic in $|x| \le 1$ with a finite number of poles and zeros inside the unit disc. By the theorem of Salem [Sal45], this function is rational, i.e., of the form $g(x)/h(x)$. It is not difficult to show that for rational function $g(x)/h(x) = \sum_{n\in\mathbb{N}_0} a_n x^n$ with a_n integers, polynomials $g(x)$ and $h(x)$ are relatively prime with $h(0) = 1$ (see exercise $2(a)$ in Chapter 4 of [Sta11]). □

Having these in mind, we can write down the proof of the Partition theorem 1. For clarity, we repeat the statement.

Theorem 4 (Partition). For $\chi_1(x) = \sum_{s\in\mathbb{N}_0} a_s x^s$ with $a_0 > 0$, a symmetric function $\prod_{k=1}^{d}\chi_1(x_k)$ is a $U(d)$ character for all $d \in \mathbb{N}$ if and only if the generating function is of the form

$$\chi_1(x) = \frac{Q_-(x)}{Q_+(x)}, \tag{2.66}$$

where $Q_{\pm}(x)$ is an integral polynomial with all positive (negative) roots. Furthermore $Q_{+}(0) = 1$.

Proof. Clearly $\{a_n\}$ is integral because $\chi_1(x)$ is a character of $U(1)$. The character $\chi_d(x_1,...,x_d) = \prod_{k=1}^{d} \chi_1(x_k)$ is a class function over $U(d)$ and $x_1,...,x_d$ are variables on the maximal torus in $U(d)$, i.e., $x_k = e^{i\theta_k}$. For any symmetric function $f(x_1,...,x_d)$, the notation $f(g)$ assumes f being evaluated at eigenvalues $x_1,\ldots,x_d$ of matrix $g \in U(d)$. Now, as a consequence of the Littlewood-Richardson rule, the symmetric function $\chi_d(x_1,...,x_d)$ is Schur-positive (expands in non-negative coefficients over Schur polynomials) if and only if the product $s_\mu(x_1,...,x_d)\chi_d(x_1,..,x_d)$ is also Schur-positive. Using this, we expand $s_\mu(x_1,...,x_d)\chi(x_1,...,x_d)$ in Schur polynomials as

$$s_\mu(x_1,...,x_d)\chi_d(x_1,...,x_d) = \sum_{\lambda} c_{\mu\lambda} s_\lambda(x_1,...,x_d) \tag{2.67}$$

where the sum runs over Young diagrams λ and $c_{\mu\lambda} \geq 0$. Since Schur polynomials are orthogonal under the Haar measure

$$\langle s_\lambda, s_\mu \rangle = \int_{U(d)} \overline{s_\lambda(g)} s_\mu(g) dg = \delta_{\lambda,\mu}, \tag{2.68}$$

we get the expression

$$c_{\mu\lambda} = \int_{U(d)} \chi_d(g)\overline{s_\lambda}(g)s_\mu(g)dg \tag{2.69}$$

Using the fact that character is of factorization form $\chi_d(x_1,\ldots,x_d) = \prod_{k=1}^{d} \chi_1(x_k)$, the conditions of the Proposition 1 are met and we can rewrite the previous expression as

$$c_{\mu\lambda} = \det\Big(a_{\lambda_i-\mu_j-i+j}\Big)_{1\leq i,j\leq d} = D_{d-1}^{\lambda,\mu}(\chi_1), \tag{2.70}$$

where $\{a_n\}$ is a sequence that generates the single-mode character $\chi_1(x)$. Schur-positivity $c_{\mu\lambda} \geq 0$ thus reads

$$D_{d-1}^{\lambda,\mu}(\chi_1) = \det\Big(a_{\lambda_i-\mu_j-i+j}\Big)_{1\leq i,j\leq d} \geq 0, \tag{2.71}$$

which is by Proposition 1 equivalent to the condition that all minors of the Töplitz matrix $(a_{i-j})_{i,j\in\mathbb{Z}}$ are non-negative. This means $\{a_n\}$ is totally positive. $\{a_n\}$ is also an integral sequence with $a_0 > 0$, therefore the main result follows by Lemma 1. $\square$

2.9 Fock space decomposition for order-one statistics

Suppose two sets of variables $x_1,...,x_n$ and $y_1,...,y_m$ with $m \leq n$. The Cauchy identities [SF99] are the following

$$\prod_{i=1}^{n}\prod_{j=1}^{m}(1+x_iy_j) \quad = \quad \Sigma_{l(\lambda)\leq m} s_\lambda(x_1,...,x_n)s_\lambda(y_1,...,y_m), \tag{2.72}$$

$$\prod_{i=1}^{n}\prod_{j=1}^{m}\frac{1}{1-x_iy_j} \quad = \quad \Sigma_{l(\lambda)\leq m} s_\lambda(x_1,...,x_n)s_\lambda(y_1,...,y_m) \tag{2.73}$$

where $l(\lambda)$ is the length of the diagram (number of rows), and λ' is the conjugate partition of λ. Using these identities, the proof of Theorem 2 is as follows.

Proof. For statistics $[1,\alpha]_-$ and $[1,\beta]_+$ we have the following d-mode characters (defined in (2.9))

$$\chi_d^-(\vec{x}) = \prod_{i=1}^{d}(1+\alpha x_i), \tag{2.74}$$

$$\chi_d^+(\vec{x}) = \prod_{i=1}^{d}\frac{1}{1-\beta x_i}. \tag{2.75}$$

Now, we apply the first Cauchy identity by setting $m=1$ and $\alpha = y_1$ and we get

$$\chi_d^-(\vec{x}) = \Sigma_{l(\lambda)\leq 1} s_\lambda(\alpha) s_{\lambda'}(x_1,...,x_d) \tag{2.76}$$

$$= \Sigma_{l(\lambda)\leq 1} \alpha^{|\lambda|} s_{\lambda'}(x_1,...,x_d). \tag{2.77}$$

Similarly, we apply the second Cauchy identity by setting $m=1$ and $\beta = y_1$ and we get

$$\chi_d^+(\vec{x}) = \Sigma_{l(\lambda)\leq 1} s_\lambda(\beta) s_\lambda(x_1,...,x_d) \tag{2.78}$$

$$= \Sigma_{l(\lambda)\leq 1} \beta^{|\lambda|} s_\lambda(x_1,...,x_d). \tag{2.79}$$

□

2.10 Negative occupations and general character

We set the most general $U(1)$ character as a formal Laurent series

$$\chi_1(x) = \sum_{s\in\mathbb{Z}} a_s x^s, \tag{2.80}$$

with integral coefficients, i.e. $a_s \in \mathbb{N}_0$. For the set of d modes, the character decomposes as

$$\chi_d(x_1,\ldots,x_d) = \prod_{s=1}^{d}\chi_1(x_s) = \sum_{k,\lambda} c_\lambda^{(k)} s_\lambda^{(k)}(\vec{x}), \tag{2.81}$$

with $c_\lambda^{(k)} \in \mathbb{N}_0$ and irreducible factors being Laurent-Schur polynomials of the form

$$s_\lambda^{(k)}(x_1,\ldots,x_d) = (x_1\ldots x_d)^{-k} s_\lambda(x_1,\ldots,x_d), \quad k\in\mathbb{N}_0. \tag{2.82}$$

These are associated with the (rational) irreducible representations of $U(d)$ [SF99]. The factor $(x_1\ldots x_d)^{-k} = (\det g)^{-k}$ corresponds to the determinant representation, while $s_\lambda(x_1,\ldots,x_d)$ is the standard Schur polynomial.

Before proceeding further, a few comments are needed. Firstly, we adopt the notation of $k^d = (k,k,\ldots,k)$ being the partition of length d, and we define $\lambda + k^d = (l_1+k,\ldots,\lambda_d+k)$. Now, note that indexing of Laurent-Schur polynomials in (2.82) with a pair (k,λ) is not unique, i.e., $s^{(k)}_{\lambda}(\vec{x}) = s^{(l)}_{\mu}(\vec{x})$ holds whenever $\lambda + l^d = \mu + k^d$. This is because $s_{\lambda+k^d}(\vec{x}) = (x_1 \dots x_d)^k s_\lambda(\vec{x})$ [B+04]. In the expansion (2.81) we implicitly assume summation over different $s^{(k)}_{\lambda}$s only, although we write explicitly summation over pairs (k,λ) to simplify the notation. Next, functions in (2.82) are orthogonal under Haar measure, i.e.,

$$\langle s^{(k)}_{\lambda}, s^{(l)}_{\mu}\rangle = \int_{U(d)} (\det g)^k \overline{s_\lambda(g)} (\det g)^{-l} s_\mu(g) dg = \int_{U(d)} \overline{s_{\lambda+l^d}(g)} s_{\mu+k^d}(g) dg = \delta_{\lambda+l^d,\mu+k^d}, \quad (2.83)$$

With this, the proof of the main theorem 2.8 and equations (2.68)-(2.71) trivially modify by changing Schur polynomials s_λ with $s^{(k)}_{\lambda}$, and we easily conclude that the generating sequence of (2.80) has to be totally-positive. Edrei has provided their full classification:

Proposition 3 ([Edr53]). Let $\{a_s\}_{s\in\mathbb{Z}}$ be a totally-positive sequence and the Laurent series

$$\sum_{s\in\mathbb{Z}} a_s x^s \quad (2.84)$$

associated with it. If the sequence does not coincide with a sequence of the form

$$\{a\rho^s\}_{s\in\mathbb{Z}} \qquad (a,\rho > 0), \quad (2.85)$$

then the Laurent series (2.84) converges in some ring

$$r_1 < |x| < r_2 \qquad (0 \le r_1 < r_2), \quad (2.86)$$

and the analytic continuation of (2.84) is of the form

$$f(x) = Cx^k e^{c_1 x + c_{-1}x^{-1}} \frac{\Pi_{s\in\mathbb{N}}(1+\alpha_s x)}{\Pi_{s\in\mathbb{N}}(1-\beta_s x)} \frac{\Pi_{s\in\mathbb{N}}(1+\gamma_s x^{-1})}{\Pi_{s\in\mathbb{N}}(1-\delta_s x^{-1})}, \quad (2.87)$$

with $k \in \mathbb{Z}$, $C, c_1, c_{-1}, \alpha, \beta, \gamma, \delta \ge 0$, and $\sum_{s\in\mathbb{N}} \alpha_s + \beta_s + \gamma_s + \delta_s < +\infty$.

Now, we are ready to prove the following statement:

Proposition 4. Let $\{a_s\}_{s\in\mathbb{Z}}$ be an integral ($a_s \in \mathbb{N}_0$) totally-positive sequence which is not of the form (2.85). Then this sequence cannot extend to double infinity, i.e., there exists $k \in \mathbb{Z}$ such that $a_k = 0$ either for $s < k$ or $s > k$.

Proof. Suppose by contradiction that a_s is non-zero at both $\pm\infty$. We divide the associated Laurent series (2.84) into the positive $S_+(x) = \sum_{s=1}^{+\infty} a_s x^s$ and negative $S_-(x) = \sum_{s=-\infty}^{0} a_s x^s$ parts. Because a_s is non-zero (integer) at infinity, the series S_+ converges at best within the unit radius, i.e., $|x| < r \le 1$. Similarly, S_- converges within the radius $1/|x| < R \le 1$. These two convergence conditions are compatible only for $|x| = 1$. Since $\lim_{n\to\pm\infty} a_s x^s \ne 0$ (or does not exist) for $|x| = 1$, we conclude (2.84) converges nowhere. This contradicts Proposition 3 because of (2.86). □

From the last proposition, it follows that the general single-mode character (2.80) is either of the form $\chi_1(x) = x^k \sum_{s\ge 0} c_s x^s$ or $\chi_1(x) = x^k (\sum_{s\ge 0} c_s x^s)^*$ for some $k \in \mathbb{Z}$, where $*$ denotes complex conjugate (note that $x^* = 1/x$ because $x = e^{i\varphi}$). The factor x^k builds into $(x_1 \dots x_d)^k$ for d modes and corresponds to the determinant representations (power k). Our main theorem 1 applies directly to the factor $\sum_{s \geq 0} c_s x^s$. As follows from Proposition 3, the only exceptional case is the character generated by the sequence (2.85) with $a, \rho \in \mathbb{N}_0$.

3 Reconstruction of Quantum Fields

We investigate the structure of quantum statistics under minimal operational assumptions: that single-particle dynamics are unitary and that multi-particle states of indistinguishable particles are labeled by occupation numbers. While these assumptions are satisfied by standard bosonic and fermionic systems, we show that they also give rise to more general classes of particle statistics, for which the usual canonical (anti)commutation relations no longer hold. By modeling multi-particle state spaces as quotients of the tensor algebra over the single-particle Hilbert space, we classify the corresponding deviations from standard statistics in terms of algebraic ideals. This construction yields a natural characterization of alternative commutation rules and provides a unified framework for describing generalized statistics consistent with indistinguishability and operational accessibility.

3.1 Introduction

What is a particle? Despite how ubiquitous this question is in the foundations of physics, the answers so far remain unsatisfactory. On the one hand, one might conceive of particles simply as physical systems satisfying some criterion of irreducibility, which is, at the end of the day, entirely relative to how an agent defines irreducibility [FK06, Per93]. On the other hand, following a different ontological argument, a particle is understood as an emergent entity that arises in the interaction between a detection apparatus and a physical field, so that the particle appears as an artefact manifesting in the occurrence of a "click" or detection event [Mer85, Per93].

This becomes particularly problematic in the quantum setting: experimental evidence indicates that the statistical behaviour of collections of quantum particles can only be explained by imposing a redundancy in observable quantities under particle permutation. In other words, physics appears to be insensitive to the specific individuality of the particles. This strongly suggests that the concept of a particle, within quantum theory, may be artificial rather than fundamental.

Understanding the transition from the single-particle picture to the many-particle picture is thus of critical importance for making sense of collective or composite systems in physics. In standard presentations, this transition is governed by the symmetrisation postulate, which states that, assuming the single-particle description makes sense, one must project onto a subspace of the tensor product space that is invariant under permutations [MG64b]. Moreover, the standard formalism restricts us to either totally symmetric or totally antisymmetric states. In terms of representation theory of the symmetric group S_N (with N the number of particles), this amounts to allowing only the two one-dimensional irreducible representations [Gre53, MG64b].

This raises two conceptual difficulties. First, the usual justification for restricting attention to scalar representations of the permutation group is the idea that performing a particle permutation can produce at most an unobservable global phase. Equivalently, permutation operators must leave all observables invariant, so they lie in the commutant of the algebra of observables. If that algebra acts irreducibly, Schur's lemma forces the permutation operators to be proportional to the identity, so only one-dimensional characters are allowed. However, nothing prevents us from considering reducible representations of higher dimension. In that case, invariance of observables can be implemented nontrivially, because the commutant is no longer trivial. Traditional para-statistics rely precisely on such higher-dimensional sectors. More fundamentally, the operational indistinguishability of particles, that is, the physical loss of individuality, calls for a deeper explanation. [Gre53, FK06].

Assuming that a single physical system can be described by a Hilbert space $\mathcal{H}$, one expects that collections of such "1-particles" can be described through the composition postulate of quantum theory, namely via the tensor product. If distinct particles, that is, individuals, can be described using the same physical observables, the associated Hilbert spaces are isomorphic. This motivates the use of the *tensor algebra* over $\mathcal{H}$, defined as

$$T(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}, \tag{3.1}$$

where each term in the direct sum represents the n-particle sector [Seg56a, Seg56b]. In each sector, one can find states that retain the ability to distinguish particles, thereby preserving individuality. If this is not possible, and if physics intrinsically dictates that particles become indistinguishable when described collectively, this indicates the existence of a filtration of states in which only those invariant under permutations are physically relevant. What is the physical principle that underlies this filtration?

In this work, we introduce a general framework for understanding the transition from first to second quantisation. We describe this as a passage from the tensor algebra $T(\mathcal{H})$ to a many-particle state space which, by analogy with the standard terminology, we call the *Fock space*. Rather than relying directly on symmetrisation, we define many-particle states as equivalence classes under a chosen subspace of $T(\mathcal{H})$, i.e., we define the Fock space as a quotient

$$F = T(\mathcal{H})/\langle V \rangle,$$

where $\langle V \rangle$ is the ideal generated by a subspace $V \subset T(\mathcal{H})$. We will also write $I := \langle V \rangle$.

Building on a classical construction due to Segal [Seg56a, Seg56b], where symmetrisation arises from certain maximality conditions on the ideal used to form the quotient, we generalise the result. We show that if the resulting Fock space is required to support an operational notion of particle counting, i.e., that one can count particles occupying specific physical states (which we call modes), and that particle counts in different modes are non-interfering, and if we assume that the physics of single particles evolves unitarily (in at least a subspace of the full single-particle Hilbert space), then the resulting Fock spaces, and the corresponding observable statistics, generalise beyond the standard bosonic and fermionic cases.

Finally, we demonstrate that the spaces obtained through this construction are consistent with previously proposed *transtatistics*, and that they support modified (anti)commutation relations that extend the canonical ones used in standard many-body theory. This yields a mathematically and conceptually rich framework for interpreting physical systems beyond bosons and fermions, one that remains consistent with quantum mechanics and is grounded in operational principles. It provides a novel language for describing the emission and absorption of quantum particles, distinct from previous approaches [SD24].

3.2 From First to Second Quantization

3.2.1 Operational indistinguishability and information loss

We now proceed to develop the idea that physical states can be described as equivalence classes. At this stage, we are not assuming any invariance under the action of a specific group, such as the symmetric group on N particles. Instead, we begin from the more general intuition that, in practice, attempts to describe multi-particle systems may yield states that are physically indistinguishable, suggesting some intrinsic blindness or inability to detect a meaningful difference between them. In other words, experimental operations never access the full algebraic detail of the tensor product of single-particle states: they only register combinations of observables whose outcomes are insensitive to certain algebraic distinctions. This motivates representing physical states not by individual algebraic elements, but by equivalence classes that identify all mathematically distinct but empirically indistinguishable configurations.

3.2.2 Fock space as a quotient algebra

In this spirit, we define the Fock space F of a many-particle system as a quotient algebra of the tensor algebra of $\mathcal{H}$, namely

$$F \equiv T(\mathcal{H})/I, \tag{3.2}$$

where I denotes a two-sided ideal of the tensor algebra $T(\mathcal{H})$, and in particular $I = \langle V \rangle$ for some subspace $V \subset T(\mathcal{H})$ [Seg56a, Seg56b]. Physically, the ideal I encodes the relations that the theory declares operationally invisible: two formal tensor states are considered the same physical state whenever their difference belongs to I. Different choices of I correspond to different physical criteria for indistinguishability—for instance, commutation or anticommutation relations, symmetry constraints, or more general algebraic correlations.

This construction works as follows: given elements $x, y \in T(\mathcal{H})$, we declare x and y to be equivalent if and only if

$$x - y \in I,$$

so that the equivalence class associated with x is written as $x + I$. Operationally, this means that all experiments accessible to the observer yield identical results for x and y: the algebraic difference between them cannot be resolved by any measurement compatible with the relations defining I. In this way, x is treated as physically relevant only up to an ambiguity encoded by I, and the quotient operation removes redundant distinctions that have no empirical manifestation.

Since the canonical map

$$T(\mathcal{H}) \longrightarrow T(\mathcal{H})/I$$

is a surjective homomorphism with kernel equal to I, the passage from first to second quantization in this framework is understood as a principled reduction of information within the original tensor algebra $T(\mathcal{H})$ [Seg56a]. From a physical point of view, second quantization is therefore not an additional structure imposed on the one-particle Hilbert space, but a systematic procedure for discarding distinctions between mathematically different yet operationally indistinguishable multi-particle descriptions. The quotient construction expresses precisely this reduction: the structure of the physical state space arises from identifying all algebraic elements that cannot be distinguished by the available observables.

3.2.3 Segal's theorem

We are now in a position to define the notion of quantum particle statistics, generalizing a construction originally introduced by I. Segal [Seg56a, Seg56b] (who refers to "field statistics").

Definition 5. A *quantum particle statistics* is specified by:

1. A unitary representation Δ of a subgroup $G \subset U(\mathcal{H})$ acting on $F \cong T(\mathcal{H})/I$;
2. A set of operators $\psi(x)$ acting on F, representing the creation of a particle in the one-particle state $x \in \mathcal{H}$.

This generalizes Segal's definition by allowing representations of proper subgroups of the full unitary group $U(\mathcal{H})$, reflecting the idea that in practice one may only have operational control over a subset of all possible degrees of freedom. Thus, the group action on the system is only partial. Our definition aligns with an operational perspective, in which single-particle unitary dynamics are promoted to systems of many identical particles, consistent with interferometric reasoning: if an experimental device can apply a unitary transformation to a particle, then it can also apply the same operation to many identical ones [SD24].

The action of the creation operators is defined with respect to a distinguished vector $|0\rangle$, referred to as the vacuum, which is declared to be unique. The role of creation operators is to act on the vacuum to generate a basis for the Fock space [Seg56b]. With this, we have provided a complete description of the transition from first to second quantization, and, as a result, of how quantum particle statistics are constructed.

The unitary representation of G on F also maps self-adjoint operators to self-adjoint operators. In particular, if we consider a projection operator on a subspace of the one-particle Hilbert space $\mathcal{H}$, its image in F will correspond to a particle number operator for that subspace. This is analogous to how measuring the energy of a non-interacting many-particle system corresponds to summing the individual particle energies. We therefore define the vacuum state $|0\rangle$ as the unique state with eigenvalue zero under the action of every such projected operator in the Fock representation [Seg56b].

An important consequence of the representation $\Delta : G \to \mathrm{Aut}(F)$ is that the subspace V generating the ideal I must itself be invariant under the action of G. This follows from the fact that for any $u \in G$, the representation respects the quotient structure:

$$\Delta(u)(x+I) = \Gamma(u)x + I,$$

where Γ is the representation of G on the single-particle Hilbert space $\mathcal{H}$.

3.3 Quadratic spaces and Transtatistics

3.3.1 Characteristic ideals and standard statistics

We begin by recalling a classical theorem due to Segal that recovers the Fock space construction for bosons and fermions [Seg56a, Seg56b]. To this end, we introduce the notion of *characteristic ideals*.

Definition 6 (Characteristic and fully characteristic ideals). Let $\mathrm{Aut}_0(T(\mathcal{H}))$ be the subgroup of algebra automorphisms induced by number-preserving unitaries on $\mathcal{H}$. An ideal $I \subset T(\mathcal{H})$ is *characteristic* if $\phi(I) = I$ for all $\phi \in \mathrm{Aut}_0(T(\mathcal{H}))$. It is *fully characteristic* if $\psi(I) = I$ for all algebra endomorphisms $\psi : T(\mathcal{H}) \to T(\mathcal{H})$ that commute with $\mathrm{Aut}_0(T(\mathcal{H}))$.

The reason that motivates this definition is that we aim to construct ideals that remain stable under physically meaningful transformations of the state space, ensuring in that way that no physically implementable operation can transform inaccessible information into accessible information, or vice versa. For example, under unitary transformations, the ideal must remain invariant; otherwise, one could dynamically move elements in or out of the ideal, implying that standard quantum time evolution can make visible things that before they were not, thereby undermining the physical interpretation of the quotient as a coarse-graining map. In particular, any transformation that preserves the number of particles must preserve the ideal. Full characteristicity strengthens this condition by requiring stability under all transformations that belong to the commutant of the physically meaningful transformations, though that feature is only relevant in infinite dimensions because in finite dimensions the center of the unitary group is trivial up to a scalar as a consequence of the Schur's lemma.
This leads to the Segal's characterization of bosonic and fermionic spaces:

Theorem 5 (Segal [Seg56a, Seg56b]). Let I be a characteristic ideal that is invariant under the action of the creation operators $\psi(x)$ and maximal with respect to these properties. Then the Fock space $F = T(\mathcal{H})/I$ is isomorphic to either the symmetric algebra or the exterior algebra of $\mathcal{H}$.

In other words, the bosonic and fermionic state spaces, that is, the algebras of fully symmetric or fully antisymmetric tensors, can be recovered from three conditions: characteristicity, invariance under the action of creation operators, and maximality.

3.3.2 Internal degeneracies and quantum grammar

We now introduce a generalization of this result when the physical operations only have access to a subset of the full collection of degrees of freedom. Let $\{X_{i\alpha}\}$ be a basis for the space $\mathcal{H}$, where $i \in \{1,\ldots,d\}$ indexes operationally accessible modes, and α is a degeneracy index encoding inaccessible degrees of freedom $\alpha \in \{1,...,m\}$. Without loss of generality we can assume that $X_{i\alpha} \equiv e_i \otimes k_\alpha$, hence the space of a single particle is factorizable as $\mathcal{H} = E \otimes K$ with K the space in charge of the inaccessible information.What we now do is consider a family of operators $\hat{X}_{i\alpha}^\dagger$ acting on a unique vacuum state, such that

$$\hat{X}_{i\alpha}^\dagger |0\rangle = X_{i\alpha}.$$

In this way, sequences of these operators acting on the vacuum, that is, words in the alphabet $\{X_{i\alpha}\}$, are identified with words in the abstract generators of the algebra. This identification is not free of ambiguity because the commutation relations among the operators are unknown. For this reason, for each specific state (that is, for each specific element of the algebra) we must impose a total ordering of the operators so that the correspondence is one to one. In other words, we require that

$$\hat{X}^\dagger_{1\alpha_1}\cdots\hat{X}^\dagger_{d\alpha_d}|0\rangle \longmapsto X_{1\alpha_1}\cdots X_{d\alpha_d},$$

and we also require that the images of these ordered words under the quotient map, that is, after passing to equivalence classes in complete analogy with the symmetric and antisymmetric projections for bosons and fermions, form a basis. Operationally, this means that we label the modes arbitrarily (mode 1, mode 2, and so on) in order to identify the accessible information in the system. What we are imposing is that, given this order and given the particle number measurements we perform, the ordered states we obtain form a basis of the state space. In other words, they are sufficient to describe the system completely. In this way, the identification of states with words relies on the uniqueness of the vacuum, and we require that, after the quotient projection to physical states, the images of these ordered words form a basis of the Fock space.

3.3.3 Order implies being quadratic

In terms of quotient algebras, recall that forming the quotient amounts to introducing relations among words. In other words, there is an isomorphism of graded algebras

$$T(\mathcal{H})/I \cong \mathbb{C}[\{X_{i\alpha}\}]/\mathrm{Rel}_I,$$

where $\mathbb{C}[\{X_{i\alpha}\}]$ denotes the free algebra of words in the alphabet $\{X_{i\alpha}\}$ with complex coefficients, and Rel_I denotes the relations induced by the ideal I. Since the ideal $I = \langle V\rangle$ is generated by a subspace V, we choose a basis of V written in the chosen alphabet $\{X_{i\alpha}\}$ so that

$$V = \mathrm{span}\{\, b_1(\{X_{i\alpha}\}),\ldots,b_K(\{X_{i\alpha}\})\,\},$$

and we impose the relations

$$b_1(\{X_{i\alpha}\}) = 0, \tag{3.3}$$

$$\vdots \tag{3.4}$$

$$b_K(\{X_{i\alpha}\}) = 0. \tag{3.5}$$

These relations define rewriting rules on the free algebra $\mathbb{C}[\{X_{i\alpha}\}]$. For our previous conditions to hold, that is, for the totally ordered words to form a basis after applying the rewriting rules, the rewriting rules must satisfy additional constraints. This leads to the following lemma.

Lemma 2. Let $T = \bigoplus_{n\geq 0} T_n$ be a graded free algebra on finitely many generators, let $I \triangleleft T$ be a homogeneous two sided ideal, and let $F := T/I$. Assume that for a fixed total order on the generators the classes of ordered monomials form a homogeneous basis of F. Then I is generated in degree two. Equivalently, there exists a subspace $W \subset T_2$ with $I = \langle W\rangle$.

Proof. See appendix. □

This lemma states that if we want totally ordered words to form a basis even after imposing the rewriting rules, then these rules must be quadratic. Any rewriting rule involving three or more adjacent generators must decompose into rewriting rules involving adjacent pairs. In our chosen language this means that the subspace V that generates the ideal I must satisfy $V \subset \mathcal{H} \otimes \mathcal{H}$.

3.3.4 $U(1)-$grading

We must also define a total order on the alphabet $\{X_{i\alpha}\}$, which follows the intuition described before and takes the form

$$(1,\alpha_{11}) < (1,\alpha_{12}) < \cdots < (1,\alpha_{1m}) < (2,\alpha_{21}) < \cdots < (d,\alpha_{dm}),$$

where the first index corresponds to the external mode labels and the second index labels the internal degrees of freedom. With this ordering, the projection of these words to the Fock space defines unique elements. This also allows us to define a factorizable action of the group $U(1)$, which leads to the following lemma.

Lemma 3. Suppose that $T(\mathcal{H})/I$ admits a basis of ordered monomials with respect to a total order on the generators of the form

$$(1,\alpha_{11}) < \cdots < (1,\alpha_{1m}) < (2,\alpha_{21}) < \cdots < (2,\alpha_{2m}) < \cdots < (d,\alpha_{d1}) < \cdots < (d,\alpha_{dm}).$$

Then there exists a $U(1)$-representation isomorphism

$$T(\mathcal{H})/I \;\cong\; W^{\otimes d}$$

where W is a (finite or countable) reducible representation of $U(1)$.

Proof. See appendix. □

.

3.3.5 $U(d)-$Invariance

Now, if in addition we require this generating subspace to be invariant under unitary transformations of the form $U(d)$, where d is the number of operationally accessible modes, then by extending Segal's notion of characteristic ideals and combining it with the principle of locality, which states that particle counting is mode-by-mode independent and which we interpret as a tensor-factorisation of the state space into isomorphic $U(1)$ representations (ensuring that ordered monomials form a basis after quotienting), we conclude:

Theorem 6 (Quadratic realisation of transtatistics)**.** Let $H \cong \mathbb{C}^d \otimes K$ be a finite dimensional one–particle Hilbert space, where $\mathbb{C}^d$ carries the operational $U(d)$ action on accessible modes and

K encodes internal degeneracies. Let $T(H)=\bigoplus_{n\geq 0}H^{\otimes n}$ be the tensor algebra and let $I\subset T(H)$ be a homogeneous ideal such that the Fock space

$$F \;:=\; T(H)/I$$

satisfies the following properties:

1. *PBW property.* There exists a total order on a homogeneous basis $\{X_{i\alpha}\}$ of H such that the equivalence classes of ordered monomials in the $X_{i\alpha}$ form a linear basis of F.
2. $U(d)$ *invariance.* The accessible unitary group $U(d)$ acts on H via $u\cdot(e_i\otimes f_\alpha)=(ue_i)\otimes f_\alpha$ and this action extends to $T(H)$ leaving I invariant.

Then the following holds. The single–mode generating function $G(t)$ is the character of an elementary transtatistics [SD24]. Equivalently, there exist integral polynomials $Q_\pm(t)$ with all roots real and of fixed sign,

$$Q_+(t)=\prod_i(1-\alpha_i t),\quad Q_-(t)=c_-\prod_i(1+\beta_i t),$$

for $\alpha_i,\beta_i>0$, $c_-\in\mathbb{N}$, such that

$$G(t) \;=\; Q_-(t) \quad \text{or} \quad G(t) \;=\; \frac{1}{Q_+(t)}.$$

In this case we say that F realises a transfermionic statistics of type $[q_0,q_1,\dots]_-$ or a transbosonic statistics of type $[q_0,q_1,\dots]_+$ respectively.

Proof. See appendix. □

Now, given the invariance under the $U(d)$–action we can introduce as well the following lemma,

Lemma 4. Let $\mathcal{H}\cong\mathbb{C}^d\otimes K$ be a $U(d)$–module acting only on the $\mathbb{C}^d$ factor. If $R_{\text{gen}}\subset\mathcal{H}\otimes\mathcal{H}$ is a $U(d)$–invariant quadratic subspace, then

$$R_{\text{gen}}=\text{Sym}^2(\mathbb{C}^d)\otimes W_{\text{sym}}\,\oplus\,\wedge^2(\mathbb{C}^d)\otimes W_{\text{ext}}, \tag{3.6}$$

for some subspaces $W_{\text{sym}},W_{\text{ext}}\subset K\otimes K$.

Proof. See appendix. □

The subspaces W_{sym} and W_{ext} are neither symmetric or antisymmetric in general, the suffix just reminds that they are the companion to the symmetric and antisymmetric part over the accessible modes. Thus the external behaviour of the relations is fixed (symmetric or antisymmetric), and all freedom lies in choosing the corresponding internal subspaces that scramble the degeneracy information.
More explicitly, we can calculate the exact form of the generating relations. Let

$$\mathcal{H}\cong\mathbb{C}^d\otimes K$$

that as we know is a $U(d)$ module where $U(d)$ acts only on the factor $\mathbb{C}^d$. Assume that we are given a $U(d)$ invariant quadratic subspace

$$R_{\mathrm{gen}} \subset \mathcal{H} \otimes \mathcal{H}$$

of the form as constructed previously

$$R_{\mathrm{gen}} = \mathrm{Sym}^2(\mathbb{C}^d) \otimes W_{\mathrm{sym}} \,\oplus\, \wedge^2(\mathbb{C}^d) \otimes W_{\mathrm{ext}},$$

for some subspaces $W_{\mathrm{sym}}, W_{\mathrm{ext}} \subset K \otimes K$.

Let us introduce the flip map defined as

$$\tau_d(u \otimes v) = v \otimes u.$$

and

$$P^{(d)}_{\mathrm{sym}} = \tfrac{1}{2}(\mathrm{id} + \tau_d), \qquad P^{(d)}_{\mathrm{ext}} = \tfrac{1}{2}(\mathrm{id} - \tau_d),$$

which project onto $\mathrm{Sym}^2(\mathbb{C}^d)$ and $\wedge^2(\mathbb{C}^d)$ respectively.

Additionally, choose linear projectors

$$P^{(K)}_{\mathrm{sym}} \colon K \otimes K \to W_{\mathrm{sym}}, \qquad P^{(K)}_{\mathrm{ext}} \colon K \otimes K \to W_{\mathrm{ext}},$$

and let

$$P^{(K)\perp}_{\mathrm{sym}} = \mathrm{id} - P^{(K)}_{\mathrm{sym}}, \qquad P^{(K)\perp}_{\mathrm{ext}} = \mathrm{id} - P^{(K)}_{\mathrm{ext}}$$

be the projectors onto chosen complements $W^{\perp}_{\mathrm{sym}}$ and $W^{\perp}_{\mathrm{ext}}$ in $K \otimes K$. We can write the following projectors

$$P_{\mathrm{gen}} = P^{(d)}_{\mathrm{sym}} \otimes P^{(K)}_{\mathrm{sym}} \,+\, P^{(d)}_{\mathrm{ext}} \otimes P^{(K)}_{\mathrm{ext}},$$
$$P^{\perp}_{\mathrm{gen}} = P^{(d)}_{\mathrm{sym}} \otimes P^{(K)\perp}_{\mathrm{sym}} \,+\, P^{(d)}_{\mathrm{ext}} \otimes P^{(K)\perp}_{\mathrm{ext}}.$$

realizing the direct sum decomposition of the generating subspace. In particular, P_{gen} is the projector onto R_{gen}, and $P^{\perp}_{\mathrm{gen}}$ is the projector onto a chosen complementary subspace of R_{gen} in degree two.

The results presented here are consistent with developments in algebraic combinatorics, in particular with recent work of Sam and Vandebogert, which shows that every generating function of a totally positive sequence can be realized as the Hilbert–Poincaré series of a Koszul algebra [1], and therefore of a quadratic algebra [SV24]. This connection is mathematically significant.

[1] A quadratic algebra $A = T(V)/(R)$ is Koszul if and only if for every $n \geq 3$ the subspace of relations in degree n,

$$R_n = \ker\bigl(V^{\otimes n} \to A_n\bigr),$$

satisfies

$$\ker(V^{\otimes n} \to A_n) = \bigcap_{k=0}^{n-2} \ker\Bigl(\mathrm{id}^{\otimes k} \otimes (V^{\otimes 2} \to A_2) \otimes \mathrm{id}^{\otimes n-k-2}\Bigr).$$

This condition expresses that all higher–degree relations are generated combinatorially from the quadratic ones, Koszul algebras possess several structural properties, among them that their Hilbert–Poincaré series

$$H_A(t) = \sum_{n \geq 0} \dim A_n \, t^n$$

satisfies the functional equation

$$H_A(t)\, H_{A^!}(-t) = 1,$$

where $A^!$ denotes the Koszul dual algebra [Pri70].

In our framework, these Hilbert–Poincaré series correspond precisely to the single-mode partition functions associated with each type of particle statistics. Since the algebras constructed here, which are quadratic algebras with ordered monomial bases, are Koszul [PP05], it follows that transbosons and transfermions, and in particular bosons and fermions, arise as concrete manifestations of Koszul duality.

This reveals an intrinsic symmetry between bosonic and fermionic statistics that appears naturally in our algebraic formulation of the physical state space.

3.3.6 Quadratic relations and generating ideal

Using this now we can choose bases $\{e_i\}_{i=1}^d$ of $\mathbb{C}^d$ and $\{f_\alpha\}$ of K. Set again

$$X_{i\alpha} := e_i \otimes f_\alpha \in \mathcal{H}.$$

That generate the tensor algebra $T(\mathcal{H})$. By direct application of the projector P_{gen} we get that

$$\begin{aligned} r_{ij}^{\alpha\beta} &:= P_{\text{gen}}\big((e_i \otimes e_j) \otimes (k_\alpha \otimes k_\beta)\big) \\ &= \tfrac{1}{2} \sum_{\gamma,\delta} \big(\mathcal{K}_{\text{sym}}\big)_{\alpha\beta}^{\gamma\delta} \big(X_{i\gamma} \otimes X_{j\delta} + X_{j\delta} \otimes X_{i\gamma}\big) + \tfrac{1}{2} \sum_{\gamma,\delta} \big(\mathcal{K}_{\text{ext}}\big)_{\alpha\beta}^{\gamma\delta} \big(X_{i\gamma} \otimes X_{j\delta} - X_{j\delta} \otimes X_{i\gamma}\big) \end{aligned}$$

that by definition is an element of R_{gen}.

The two sums above are respectively the symmetric and antisymmetric parts in the external indices i, j. We have then that the quadratic ideal that defines the quotient algebra

$$T(\mathcal{H}) \big/ \langle R_{\text{gen}} \rangle$$

is generated by the relations

$$r_{ij}^{\alpha\beta} = 0 \quad \text{for all } i, j \in \{1, \ldots, d\},\ k_\alpha \otimes k_\beta \in K \otimes K.$$

This relations are the ones that define the creation algebra recalling that by the unicity of vacuum we can lift the defining relations in the elements of the algebra to the action of associated creation operators acting on vacuum.

---

with no new independent relations appearing in higher degrees.[Bac82]

3.3.7 Yang-Baxter constraints

Moreover, the theory of quadratic algebras (see proof of Theorem 8.) allows us to relate our construction with the theory of quantum solutions of the Yang-Baxter equation.

Having the generating relations

$$R_{\mathrm{gen}} = \mathrm{Sym}^2(\mathbb{C}^d) \otimes W_{\mathrm{sym}} \oplus \wedge^2(\mathbb{C}^d) \otimes W_{\mathrm{ext}} \subset \mathcal{H} \otimes \mathcal{H},$$

where $\mathcal{H} \cong \mathbb{C}^d \otimes K$ and R_{gen} is $U(d)$–invariant as described earlier. Having again $P_{\mathrm{gen}} : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ as

$$P_{\mathrm{gen}} := P^{(d)}_{\mathrm{sym}} \otimes P^{(K)}_{\mathrm{sym}} + P^{(d)}_{\mathrm{ext}} \otimes P^{(K)}_{\mathrm{ext}}.$$

We can write:

Theorem 7 (Yang–Baxter conditions for the internal coefficients)**.** Assume that the ordered monomials in the generators $X_{i\alpha}$ form a PBW basis of the quotient algebra

$$T(\mathcal{H}) \big/ \langle R_{\mathrm{gen}} \rangle.$$

Then the internal projectors $P^{(K)}_{\mathrm{sym}}$ and $P^{(K)}_{\mathrm{ext}}$ each satisfy the Yang–Baxter identity on $K^{\otimes 3}$:

$$P^{(K),12}_{\mathrm{sym}} P^{(K),23}_{\mathrm{sym}} P^{(K),12}_{\mathrm{sym}} = P^{(K),23}_{\mathrm{sym}} P^{(K),12}_{\mathrm{sym}} P^{(K),23}_{\mathrm{sym}} \qquad \text{on } K^{\otimes 3}, \tag{3.7}$$

$$P^{(K),12}_{\mathrm{ext}} P^{(K),23}_{\mathrm{ext}} P^{(K),12}_{\mathrm{ext}} = P^{(K),23}_{\mathrm{ext}} P^{(K),12}_{\mathrm{ext}} P^{(K),23}_{\mathrm{ext}} \qquad \text{on } K^{\otimes 3}. \tag{3.8}$$

Proof. See appendix. □

The previous theorem expresses a key structural constraint: the internal symmetric and anti-symmetric components of the generating relations must each satisfy a Yang–Baxter equation. Equivalently, the tensors defining W_{sym} and W_{ext} must obey the corresponding cubic identities. These conditions are therefore the precise algebraic content of PBW–compatibility for the generating ideal.

3.3.8 Single-mode partition function

We know by construction that $F \cong W^{\otimes d}$, where W captures all internal combinatorics associated with the degeneracy space K. Consequently, the Hilbert–Poincaré series of F factorizes as $H_F(t) = G(t)^d$, with $G(t)$ the series of a single effective mode. This series is completely determined by the quadratic projector P_{gen} acting on $K \otimes K$. Indeed, for each $n \geq 0$ the coefficient g_n is obtained by imposing the quadratic relations on all adjacent pairs: set $V_n = K^{\otimes n}$ and define

$$P^{(n)}_k := \mathrm{id}^{\otimes(k-1)} \otimes P_{\mathrm{gen}} \otimes \mathrm{id}^{\otimes(n-k-1)} \qquad (1 \leq k \leq n-1).$$

The subspace of admissible degree-n monomials is then

$$W_n := \bigcap_{k=1}^{n-1} \ker P^{(n)}_k,$$

and the coefficient in the single–mode Hilbert–Poincaré series is

$$g_n = \dim W_n.$$

Thus the entire sequence (g_n), and hence $G(t)$, is generated quadratically from P_{gen} alone by enforcing the local vanishing of the quadratic relations on every adjacent pair of tensor factors.

3.3.9 Annihilation algebra

For the annihilation relations we proceed in a similar way. Given the quadratic projector

$$P_{\text{gen}} \colon \mathcal{H} \otimes \mathcal{H} \longrightarrow R_{\text{gen}}$$

that determines the generating relations among creation operators, the corresponding relations among destruction operators are obtained by passing to the adjoint quadratic data.

Let $\{X_{i\alpha}\}$ be the chosen homogeneous basis of $\mathcal{H}$, and let $\{X^*_{i\alpha}\}$ be the dual basis of $\mathcal{H}^*$, defined by

$$X^*_{i\alpha}(X_{j\beta}) = \delta_{ij} g_{\alpha\beta}.$$

Where we have assumed independence between the external and internal degrees of freedom and orthogonality with respect to the external ones only given that those ones are determined by the symmetric and exterior parts of the generating subspace. We will recover this discussion afterwards.

The adjoint projector

$$P^*_{\text{gen}} \colon \mathcal{H}^* \otimes \mathcal{H}^* \longrightarrow R^*_{\text{gen}}$$

is the transpose of P_{gen} under the canonical pairing

$$\langle X^*_{i\alpha} \otimes X^*_{j\beta},\, X_{k\mu} \otimes X_{\ell\nu} \rangle = \delta_{ik} g_{\alpha\mu} \delta_{j\ell} g_{\beta\nu},$$

so that $R^*_{\text{gen}} = \operatorname{im}(P^*_{\text{gen}})$ is the quadratic subspace that generates the destruction ideal.

Again we define the projectors

$$P^{(K)}_{\text{sym}} \colon K \otimes K \to W_{\text{sym}}, \qquad P^{(K)}_{\text{ext}} \colon K \otimes K \to W_{\text{ext}},$$

the adjoint quadratic relations take a form parallel to the creation relations. For each pair of external indices i, j and every $t \in K \otimes K$, define

$$\begin{aligned}
(r^{\alpha\beta}_{ij})^* &:= P^*_{\text{gen}}\big(X^*_{i\alpha} \otimes X^*_{j\beta}\big) \\
&= \tfrac{1}{2} \sum_{\gamma,\delta} \big(\mathcal{K}^*_{\text{sym}}\big)^{\gamma\delta}_{\alpha\beta} \big(X^*_{i\gamma} \otimes X^*_{j\delta} + X^*_{j\delta} \otimes X^*_{i\gamma}\big) + \tfrac{1}{2} \sum_{\gamma,\delta} \big(\mathcal{K}^*_{\text{ext}}\big)^{\gamma\delta}_{\alpha\beta} \big(X^*_{i\gamma} \otimes X^*_{j\delta} - X^*_{j\delta} \otimes X^*_{i\gamma}\big)
\end{aligned}$$

The quadratic destruction ideal is therefore generated by all tensors of the form

$$(r^{\alpha\beta}_{ij})^* = 0, \qquad i, j \in \{1, \ldots, d\},\ k^*_\alpha \otimes k^*_\beta \in K \otimes K.$$

Equivalently, the destruction algebra is the quadratic quotient

$$T(\mathcal{H}^*)\Big/\big\langle (r_{ij}^{\alpha\beta})^* \big\rangle.$$

Thus the structure of the destruction relations mirrors exactly the external–internal factorization encoded in the creation projector P_{gen}, with the order of tensor factors reversed and the internal coefficients governed by the adjoint projectors $P_{\text{sym}}^{(K)}$ and $P_{\text{ext}}^{(K)}$.

3.3.10 $(A,B)-$Bracket

Finally, we can introduce a bracket depending on the subspaces spanned by the $P^{(K)}$, acting either on creation operators or on annihilation operators, which we denote by $[\,,\,]_{A,B}$. This allows us to write in a compact way all the quadratic relations previously introduced. Explicitly we have,

Definition 7. Assuming now $A \equiv P_{\text{sym}}^{(K)}(K\otimes K)$ and $B \equiv P_{\text{ext}}^{(K)}(K\otimes K)$ for the collection of operators $\{\hat{P}_{i\alpha}, \hat{Q}_{j\beta}\}$ we write

$$[\hat{P}_{i\alpha}, \hat{Q}_{j\beta}]_{A,B} = \sum_{\eta\lambda} A_{\alpha\beta}^{\eta\lambda}\big(\hat{P}_{i\eta}\hat{Q}_{j\lambda} + \hat{Q}_{j\lambda}\hat{P}_{i\eta}\big) + \sum_{\eta\lambda} B_{\alpha\beta}^{\eta\lambda}\big(\hat{P}_{i\eta}\hat{Q}_{j\lambda} - \hat{Q}_{j\lambda}\hat{P}_{i\eta}\big),$$

.

Using this, we can give a condensed expresion then for the creation–creation and annihilation–annihilation relations introduced before simply as

$$[\hat{X}_{i\alpha}, \hat{X}_{j\beta}]_{A,B} = 0 \tag{3.9}$$

$$[\hat{X}_{i\alpha}^{\dagger}, \hat{X}_{j\beta}^{\dagger}]_{A,B} = 0 \tag{3.10}$$

We have shown that quantum statistics, such as those introduced in the previous section, can be realized via quadratically generated ideals. This construction extends Segal's original framework and offers a new algebraic perspective on the second quantization procedure.

3.4 Creation-Annihilation algebras

Having constructed the pure creation–creation algebra and the pure annihilation–annihilation algebra, the natural question is how to define the *mixed* creation–annihilation sector. For this purpose we introduce the formalism developed by Borowiec and Marcinek [BM00], which provides the algebraic machinery needed to combine both structures into a single associative algebra.

We begin by introducing a linear operator

$$C : \mathcal{H}^* \otimes \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}^*,$$

which prescribes the exchange rule between elements of the Fock algebra (generated by creation operators) and elements of the dual algebra (generated by annihilation operators). Recall that the algebras

$$A := T(\mathcal{H})/I_R, \qquad B := T(\mathcal{H}^*)/I_S,$$

were defined by the quadratic relations determined respectively by the projectors P_{gen} and P^*_{gen}. Our goal is to ensure that C induces a multiplicative map

$$\tau : B \otimes A \longrightarrow A \otimes B,$$

compatible with the quotient relations that define A and B.

The compatibility requirement is captured precisely by the following theorem of Borowiec and Marcinek.

Theorem 8 (Borowiec–Marcinek [BM00]). Let $I_R \subset \mathcal{H} \otimes \mathcal{H}$ and $I_S \subset \mathcal{H}^* \otimes \mathcal{H}^*$ be the quadratic ideals defining the algebras A and B. A homogeneous cross

$$\widetilde{\tau} : T(\mathcal{H}^*) \otimes T(\mathcal{H}) \longrightarrow T(\mathcal{H}) \otimes T(\mathcal{H}^*)$$

descends to a well–defined cross

$$\tau : B \otimes A \longrightarrow A \otimes B$$

if and only if

1. I_R is a left $\widetilde{\tau}$-ideal, and
2. I_S is a right $\widetilde{\tau}$-ideal.

Where a left $\widetilde{\tau}$-ideal means that

$$\widetilde{\tau}\big(\mathcal{H}^* \otimes I_R\big) \;\subseteq\; I_R \otimes T(\mathcal{H}^*),$$

i.e. the cross sends every generator of the ideal I_R to an element whose creation part again lies in I_R; similarly, a right $\widetilde{\tau}$-ideal means that

$$\widetilde{\tau}\big(I_S \otimes \mathcal{H}\big) \;\subseteq\; T(\mathcal{H}) \otimes I_S.$$

Equivalently, these conditions may be expressed in terms of certain quadratic identities which take the form of Yang–Baxter relations involving the operators R, S, and C.

Lemma 5 ([BM00]). Let $R : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ and $S : \mathcal{H}^* \otimes \mathcal{H}^* \to \mathcal{H}^* \otimes \mathcal{H}^*$ be the operators associated with P_{gen} and P^*_{gen}. Then the cross descends to $A \otimes B$ if and only if

$$(\mathrm{id}{\otimes}C)(C{\otimes}\mathrm{id})(\mathrm{id} - (\mathrm{id}{\otimes}R)) = (\mathrm{id} - (R{\otimes}\mathrm{id}))(\mathrm{id}{\otimes}C)(C{\otimes}\mathrm{id}), \tag{3.11}$$

$$(C{\otimes}\mathrm{id})(\mathrm{id}{\otimes}C)(\mathrm{id} - (S{\otimes}\mathrm{id})) = (\mathrm{id} - (\mathrm{id}{\otimes}S))(C{\otimes}\mathrm{id})(\mathrm{id}{\otimes}C). \tag{3.12}$$

When these conditions hold, one obtains a well–defined algebra isomorphic as vector space to $A \otimes B$ whose defining relations incorporate the exchange rule determined by C. This algebra simultaneously extends the pure creation and pure annihilation relations and remains associative.

Up to this point, the construction determines only the *quadratic* (homogeneous) part of the mixed relations. However, in the context of creation–annihilation algebras, one typically requires additional information encoding the behaviour of the operators with respect to the Fock vacuum. For operators $X_{i\alpha}$ and $X_{i\alpha}^{\dagger}$, one expects the vacuum two–point function

$$\langle 0 \mid X_{i\alpha} X_{j\beta}^{\dagger} \mid 0 \rangle$$

to be specified.

This determines a linear functional

$$G : \mathcal{H}^{*} \otimes \mathcal{H} \longrightarrow \mathbb{C}, \qquad G(B_{i\alpha}^{*} \otimes A_{j\beta}) = \langle 0 \mid X_{i\alpha} X_{j\beta}^{\dagger} \mid 0 \rangle.$$

As in the previous construction of the annihilation algebras we assume that the external degrees of freedom are orthogonal, reflecting the fact that different values of the corresponding quantum numbers label orthogonal states. For the internal degrees of freedom, however, we do not assume orthogonality *a priori*. Accordingly, we postulate that

$$\langle 0 \mid X_{i\alpha} X_{j\beta}^{\dagger} \mid 0 \rangle = g_{\alpha\beta}\, \delta_{ij},$$

where $g_{\alpha\beta}$ is a nondegenerate Hermitian form on the internal space K.

A key observation of Borowiec–Marcinek is that, because G takes values in the *center* of the algebra, the consistency conditions (BM1)–(BM2) remain unchanged. Thus one may incorporate G by defining the *non-homogeneous cross*

$$\tau = \widehat{\tau} + G,$$

which leads to the mixed relation in the language of the $(A,B)-$bracket

$$[X_{i\alpha}, X_{j\beta}^{\dagger}]_{A,B} = g_{\alpha\beta}\, \delta_{ij}\, \mathbf{1}.$$

Where the values of A and B depend in the Yang-Baxter compatibility given by Borowiec-Marcinek between the creation, annihilation and mixed exchange rules. Additionally, this central correction is precisely what is required to reproduce the desired vacuum expectation value and does not interfere with the Yang–Baxter consistency of the homogeneous cross.

3.4.1 Transfield representation of gl(d)

Using the above relations we can now construct a representation of the general linear algebra in terms of the creation and annihilation operators defining our transtatistical fields ("transfields"). To this end we introduce a family of operators J_{ij} defined with the help of the internal object g.

Let $g_{\alpha\beta}$ be a nondegenerate Hermitian form on the internal space K, and let $g^{\beta\alpha}$ denote the entries of the inverse matrix, so that $\sum_{\beta} g_{\alpha\beta} g^{\beta\gamma} = \delta_{\alpha}^{\gamma}$. We define

$$J_{ij} := \sum_{\alpha,\beta} g^{\beta\alpha} X_{i\alpha}^{\dagger} X_{j\beta},$$

where $i, j = 1, \ldots, d$ label the external modes and α, β label the internal degrees of freedom. If we assume that this Hermitian form is invariant under the operators R, S, and C, in the sense that the inner product of two–particle states remains unchanged under the exchanges implemented by these maps, then we may state the following lemma.

Lemma 6. The following identities hold:

(i) (Action on creators)
$$[J_{ij}, X^{\dagger}_{k\sigma}] = \delta_{jk} X^{\dagger}_{i\sigma}, \qquad 1 \leq i, j, k \leq d.$$

(ii) (Action on annihilators)
$$[J_{ij}, X_{k\sigma}] = -\delta_{ik} X_{j\sigma}, \qquad 1 \leq i, j, k \leq d.$$

(iii) (Commutators among the J)
$$[J_{ij}, J_{kl}] = \delta_{jk} J_{il} - \delta_{il} J_{kj}, \qquad 1 \leq i, j, k, l \leq d.$$

In particular, the operators J_{ij} satisfy the commutation relations of the Lie algebra $\mathfrak{gl}(d)$.

This assumption is well motivated by the fact that the operators R, S, and C encode, in a precise algebraic sense, the exchange of indistinguishable particles. Accordingly, their action on multiparticle states should not alter transition probabilities, that is, the two–particle correlations and, consequently, the correlations in systems with an arbitrary number of particles.

We thus see that the operators J_{ij} satisfy the commutation relations of the general linear algebra $\mathfrak{gl}(d)$, providing a natural way to construct representations of $\mathfrak{gl}(d)$ in terms of the transfields $X_{i\alpha}, X^{\dagger}_{i\alpha}$.

3.5 Conclusions and Outlook

We have developed an operationally motivated generalisation of the transition from first to second quantisation, without assuming symmetrisation principles as fundamental postulates. Building on the work presented in the previous chapter, we have provided an explicit algebraic formulation of families of transtatistics using quotients of tensor algebras over single-particle spaces. The assumption that an ordered basis of monomials survives the quotient then yields the operational locality condition, expressed as mode-by-mode independence of particle counting. Once this structural requirement is in place, we impose a U(d) symmetry on the accessible degrees of freedom, which fixes the external behaviour of the relations and supplies the representation-theoretic framework necessary for defining transtatistics.

From a physical perspective, this quotient-based construction offers a way of defining indistinguishability in operational terms, as the collapse of accessible information that could be used to differentiate between particles. Algebraically, the emergence of the Yang–Baxter equation as a necessary condition for statistical consistency links directly to the operational notion of locality: once states are defined solely through particle numbers, relative to additional information that is operationally inaccessible, the resulting space is quadratically generated, with the Yang–Baxter solutions ensuring invariance of the statistics under subsystem decompositions.

On the other hand, although we have defined the quadratic relations that determine the Fock space through the creation operators, the inner product on $\mathcal{H}$ allows us to introduce the annihilation operators in the standard way, namely as the adjoints of the creation operators. In the same way, we use this notion of orthogonality to formalize the creation and annihilation algebras. Moreover, this framework connects with existing work on parastatistics, which share the guiding idea of scrambling internal degrees of freedom that are operationally inaccessible, where we find the classical examples [Gre53] together with more recent developments [WH24]. In those cases the relations are introduced heuristically and by postulate, whereas here we obtain a complete characterization derived from basic operational assumptions about the quantum physics of many-particle systems.

A natural generalisation of this work is its application to quantum field theory. Since we have introduced new commutation relations, one can move from discrete degrees of freedom to measure spaces such as those modelling spacetime. This would allow the description of new types of fields compatible with transtatistics, while remaining consistent with the operational arguments presented here regarding accessible information and operational indistinguishability, but now in the continuum. Of particular interest are the consequences of imposing a causal structure on the accessible degrees of freedom, as this additional symmetry could further constrain, or even fully collapse, the Yang–Baxter symmetry of the projectors to the standard bosonic and fermionic cases. This can be viewed as an extension of the spin–statistics theorem within the present framework [Pau40, SW64, Gre91b].

3.6 Lemma 2

Proof. Given a homogeneous ideal $I \subset T(H)$ and a fixed monomial order on $T(H)$, assume that there exists an ordered monomial basis for the quotient. Without loss of generality, we may then take the set

$$\mathcal{B} := \{x_1^{n_1} \cdots x_d^{n_d} \mid n_i \in \mathbb{N}\},$$

where $\{x_1, \ldots, x_d\}$ is a basis of H, and regard its images

$$[x_1^{n_1} \cdots x_d^{n_d}] \in T(H)/I$$

as spanning the quotient. Any unordered monomial $w \in T(H)$ is projected to a linear combination of ordered monomials; equivalently, the projection induces a rewriting map

$$w \longmapsto \sum_{u \in \mathcal{B}} c_u u,$$

where the coefficients c_u are determined by the defining relations of I. In particular, the degree–two monomials carry a prescribed ordering rule, and the projection corrects any violation of this rule.

Consider now an arbitrary monomial w of degree $N > 2$. One may attempt to rewrite w into ordered form by iteratively correcting the order of all degree–two subwords. Under the assumption that the rewriting system generated by the degree–two relations is confluent, this pairwise correction uniquely determines an ordered representative of w.

Assume, for contradiction, that in addition to the degree–two relations there exists an independent higher–degree relation

$$r = \sum_i \alpha_i w_i \in I,$$

with all w_i of degree $N > 2$, which is not a consequence of the degree–two relations. Then a monomial of degree N appearing in r admits two distinct reductions:

$$w \overset{\text{pairwise}}{\longmapsto} u_{\text{pair}} \qquad \text{and} \qquad w \overset{r}{\longmapsto} u_{\text{high}},$$

where u_{pair} is the ordered monomial obtained by successive degree–two corrections, and u_{high} is the expression obtained by applying the higher–degree relation. If $u_{\text{pair}} \neq u_{\text{high}}$, then the difference provides a nontrivial linear dependence among ordered monomials:

$$u_{\text{pair}} - u_{\text{high}} = 0 \quad \text{in } T(H)/I,$$

contradicting the assumption that the ordered monomials form a basis.

Hence no additional higher–degree relations may exist independently of those generated by degree–two relations. $\square$

3.7 Lemma 3

Proof. Let $\{X_{i\alpha}\}$ be a basis of H. Assume a monomial order in which all monomials are grouped into blocks according to the index i. That is, the quotient $T(H)/I$ is spanned by elements of the form

$$[X_{1\alpha_{1,1}}\cdots X_{1\alpha_{1,m_1}}X_{2\alpha_{2,1}}\cdots X_{2\alpha_{2,m_2}}\cdots X_{d\alpha_{d,1}}\cdots X_{d\alpha_{d,m_d}}].$$

Equivalently, we may describe any ordered monomial by writing it as

$$w_1w_2\cdots w_d,$$

where each word w_i is generated by the symbols $X_{i\alpha}$, with α varying and i fixed.

Since we have assumed that the ordered monomials form a basis of the quotient, the set of all ordered monomials with fixed i spans a subspace of the quotient. Denote this subspace by

$$A_i \subset T(H)/I.$$

There is a natural map

$$A_1\otimes\cdots\otimes A_d \longrightarrow T(H)/I, \qquad w_1\otimes\cdots\otimes w_d \longmapsto [w_1\cdots w_d],$$

defined by concatenation followed by projection onto the quotient.

This map is surjective, since every ordered monomial in the quotient has the form $w_1\cdots w_d$ as described. It is also injective: the tensors $w_1\otimes\cdots\otimes w_d$ form a basis of $A_1\otimes\cdots\otimes A_d$, and the map sends this basis to a basis of the quotient. Hence we obtain a vector space isomorphism

$$T(H)/I \cong A_1\otimes\cdots\otimes A_d.$$

Moreover, by assumption the degeneracies for each mode i are identical, so all subspaces A_i are mutually isomorphic. Let W denote their common isomorphism class. Then

$$T(H)/I \cong W^{\otimes d}.$$

Finally, there is a natural $U(1)$-action on the quotient: an ordered monomial is multiplied by a phase whose argument depends on its degree. For $g\in U(1)$ we have $g\cdot w := e^{in\theta}w$, for all w of order n. In the same way, we have an action of $U(1)$ on $W^{\otimes d}$ similarly by adding up the degrees of the components on $w_1\otimes\ldots\otimes w_d$ and then multiplying by a phase accordingly. Naturally this actions commute with respect to the isomorphism hence we have that

$$T(H)/I \cong W^{\otimes d}$$

as $U(1)$-representations.

□

3.8 Theorem 6

We recall the partition theorem [SD24]

Theorem 9 (Partition)**.** Let

$$\chi_1(x) \;=\; \sum_{s\in\mathbb{N}_0} a_s x^s, \qquad a_0 > 0.$$

Then the symmetric function

$$\prod_{k=1}^{d} \chi_1(x_k)$$

is a $U(d)$–character for all $d \in \mathbb{N}$ if and only if the generating function admits a factorization

$$\chi_1(x) \;=\; \frac{Q^-(x)}{Q^+(x)}, \tag{23}$$

where $Q^{\pm}(x)$ are integral polynomials whose roots are all positive (for Q^-) or all negative (for Q^+), and satisfying the normalization

$$Q^+(0) = 1.$$

Proof. By construction: the ordered basis conditions implies the factorization property of the $U(1)$ character and additionally, the $U(d)$ invariance completes the conditions for the application of the partion theorem. The result follows. □

3.9 Lemma 4

Proof. We begin with the decomposition

$$H\otimes H \;\cong\; \mathrm{Sym}^2(\mathbb{C}^d)\otimes(K\otimes K) \;\oplus\; \wedge^2(\mathbb{C}^d)\otimes(K\otimes K).$$

Each summand is an isotypic component of the $U(d)$-representation on $H\otimes H$.

By Schur's lemma, any $U(d)$-equivariant operator

$$T : H\otimes H \longrightarrow H\otimes H$$

must act as the identity on the irreducible factor tensored with an arbitrary linear map on the multiplicity space $K\otimes K$. In particular, if $R \subset H\otimes H$ is a $U(d)$-invariant subspace, then there exists a $U(d)$-equivariant projection

$$P : H\otimes H \longrightarrow H\otimes H$$

whose image is R.

With respect to the isotypic decomposition above, the projector decomposes as

$$P = \big(\mathrm{id}_{\mathrm{Sym}^2(\mathbb{C}^d)} \otimes S_{\mathrm{sym}}\big) \;\oplus\; \big(\mathrm{id}_{\wedge^2(\mathbb{C}^d)} \otimes S_{\mathrm{ext}}\big),$$

where

$$S_{\mathrm{sym}},\, S_{\mathrm{ext}} : K \otimes K \longrightarrow K \otimes K$$

are linear projections.

Define

$$W_{\mathrm{sym}} = \mathrm{im}(S_{\mathrm{sym}}), \qquad W_{\mathrm{ext}} = \mathrm{im}(S_{\mathrm{ext}}).$$

Then the image of P, which is precisely the invariant subspace R, has the form

$$R = \mathrm{Sym}^2(\mathbb{C}^d) \otimes W_{\mathrm{sym}} \;\oplus\; \wedge^2(\mathbb{C}^d) \otimes W_{\mathrm{ext}},$$

for certain subspaces $W_{\mathrm{sym}}, W_{\mathrm{ext}} \subset K \otimes K$.

Thus every $U(d)$-invariant subspace of $H \otimes H$ is obtained by choosing arbitrary multiplicity subspaces in each isotypic component.

□

3.10 Theorem 8

This is based on [PP05].

1. PBW Bases

Let $A = T(V)/I$ be a quadratic algebra, where V has an ordered basis $\{x_1, \ldots, x_m\}$. For each multiindex $\alpha = (i_1, \ldots, i_n)$, write

$$x^\alpha = x_{i_1} \cdots x_{i_n} \in T(V),$$

and impose a lexicographic order on all multiindices.

The construction begins by identifying the degree–two monomials that survive modulo the quadratic relations. Given the order on multiindices, one considers the set

$$S^{(2)} = \{(i,j) \mid x_i x_j \text{ cannot be written modulo } I \text{ as a linear combination of } x_k x_\ell \text{ with } (k,\ell) < (i,j)\}.$$

These are the “admissible” pairs.

Higher–degree admissible multiindices are then defined inductively:

$$S^{(n)} = \{(i_1, \ldots, i_n) \mid (i_k, i_{k+1}) \in S^{(2)} \text{ for all } k\}.$$

Thus a monomial $x_{i_1} \cdots x_{i_n}$ is admissible precisely when each of its adjacent pairs is admissible in degree two.

Definition 8 (PBW Basis). The algebra $A = T(V)/I$ is said to admit a *PBW basis* if the set

$$\mathcal{B} = \{x^{\alpha} \mid \alpha \in \bigcup_{n\geq 0} S^{(n)}\}$$

projects to a vector space basis of A. In this case, the generators $x_1,\dots,x_m$ are called *PBW generators* of A.

Thus a PBW basis is a monomial basis, compatible with lexicographic order, such that all nonadmissible monomials reduce uniquely to linear combinations of admissible ones. It is the noncommutative analogue of the classical Poincaré–Birkhoff–Witt basis.

The key structural fact is that if a pair $(i,j) \notin S^{(2)}$, then

$$x_i x_j = \sum_{(k,\ell)<(i,j)} c_{i,j}^{k,\ell}\, x_k x_\ell,$$

and these quadratic reductions extend consistently to all degrees through iterated projections. A PBW basis exists precisely when these reductions are confluent.

2. PBW Theorem

Theorem 10 (PBW Theorem)**.** If the cubic monomials $x_{i_1}x_{i_2}x_{i_3}$ with $(i_1,i_2,i_3) \in S^{(3)}$ are linearly independent in degree three of $A = T(V)/I$, then the same holds in every degree. Therefore, the generators $(x_1,\dots,x_m)$ are PBW generators and $\mathcal{B}$ is a PBW basis of A.

Equivalently, the cubic independence assumption is encoded by the braid-type compatibility condition

$$\pi^{12}\pi^{23}\pi^{12} = \pi^{23}\pi^{12}\pi^{23},$$

where the maps π^{ij} apply the quadratic reduction to the i-th and j-th tensor positions of $V^{\otimes 3}$. This operator identity ensures that all overlaps of quadratic relations resolve consistently, guaranteeing global confluence of the rewriting system.

Proof. Direct application of the PBW theorem. □

3.11 Lemma 6

Proof. Recall

$$J_{ij} := \sum_{\alpha,\beta} g^{\beta\alpha} X_{i\alpha}^{\dagger} X_{j\beta},$$

and the mixed relation

$$X_{i\alpha}X_{j\beta}^{\dagger} = \sum_{\gamma,\delta} C^{\gamma\delta}{}_{\alpha\beta}\, X_{j\gamma}^{\dagger} X_{i\delta} + g_{\alpha\beta}\,\delta_{ij}\,\mathbf{1}.$$

Pure–creation and pure–annihilation exchange relations are governed by the operators S and R, respectively. By assumption, the Hermitian form g is invariant under all exchange maps C, S, and R, meaning these maps act unitarily on the Hilbert spaces defined by g and its tensor powers.

We now prove the three identities.

1. Proof of (i)

Fix i, j, k, σ. Using the definition of J_{ij},

$$[J_{ij}, X^{\dagger}_{k\sigma}] = \sum_{\alpha,\beta} g^{\beta\alpha} \left(X^{\dagger}_{i\alpha} X_{j\beta} X^{\dagger}_{k\sigma} - X^{\dagger}_{k\sigma} X^{\dagger}_{i\alpha} X_{j\beta} \right) = T_1 - T_2.$$

Term T_1. Using the mixed relation

$$X_{j\beta} X^{\dagger}_{k\sigma} = \sum_{\gamma,\delta} C^{\gamma\delta}{}_{\beta\sigma} X^{\dagger}_{k\gamma} X_{j\delta} + g_{\beta\sigma} \delta_{jk} \mathbf{1},$$

we obtain

$$T_1 = \sum_{\alpha,\beta,\gamma,\delta} g^{\beta\alpha} C^{\gamma\delta}{}_{\beta\sigma} X^{\dagger}_{i\alpha} X^{\dagger}_{k\gamma} X_{j\delta} + \delta_{jk} \sum_{\alpha,\beta} g^{\beta\alpha} g_{\beta\sigma} X^{\dagger}_{i\alpha}.$$

Since $\sum_\beta g^{\beta\alpha} g_{\beta\sigma} = \delta^{\alpha}_{\sigma}$, the inhomogeneous term reduces to

$$\delta_{jk} X^{\dagger}_{i\sigma}.$$

Term T_2. Using the pure–creation exchange relation

$$X^{\dagger}_{k\sigma} X^{\dagger}_{i\alpha} = \sum_{\gamma,\delta} S^{\gamma\delta}{}_{\sigma\alpha} X^{\dagger}_{i\gamma} X^{\dagger}_{k\delta},$$

we obtain

$$T_2 = \sum_{\alpha,\beta,\gamma,\delta} g^{\beta\alpha} S^{\gamma\delta}{}_{\sigma\alpha} X^{\dagger}_{i\gamma} X^{\dagger}_{k\delta} X_{j\beta}.$$

Cancellation of cubic terms. The Hermitian form $g_{\alpha\beta}$ on K provides an antilinear identification $K \simeq K^*$, and therefore identifies the mixed spaces $K^* \otimes K$ and $K \otimes K^*$ with $K \otimes K$. After transporting the mixed exchange map C to an operator on $K \otimes K$ via this identification, we equip $K \otimes K$ with the product inner product

$$\langle e_\alpha \otimes e_\sigma,\ e_\gamma \otimes e_\delta \rangle_g = g_{\alpha\gamma} g_{\sigma\delta}.$$

The adjointness assumption states that, with respect to this inner product, the transported mixed map C is adjoint to the pure–creation map S. Expanding this adjointness in components yields

$$\sum_{\beta} g^{\beta\alpha} C^{\gamma\delta}{}_{\beta\sigma} = \sum_{\eta} S^{\gamma\alpha}{}_{\sigma\eta}\, g_{\delta\eta}, \tag{A.1}$$

which is the identity needed to compare the cubic parts of T_1 and T_2. Using (A.1), the coefficients of each monomial $X^{\dagger}_{i\alpha} X^{\dagger}_{k\gamma} X_{j\delta}$ agree in T_1 and T_2, so all cubic terms cancel. Hence,

$$T_1 - T_2 = \delta_{jk} X^{\dagger}_{i\sigma},$$

and we obtain
$$[J_{ij}, X^\dagger_{k\sigma}] = \delta_{jk} X^\dagger_{i\sigma}.$$

2. Proof of (ii)

The computation for the annihilation operators is analogous. Using the mixed relation in the opposite order,
$$X_{k\sigma} X^\dagger_{i\alpha} = \sum_{\gamma,\delta} C^{\gamma\delta}{}_{\sigma\alpha} X^\dagger_{i\gamma} X_{k\delta} + g_{\sigma\alpha}\, \delta_{ki}\, \mathbf{1},$$
and the pure–annihilation exchange relation
$$X_{j\beta} X_{k\sigma} = \sum_{\gamma,\delta} R^{\gamma\delta}{}_{\beta\sigma} X_{k\gamma} X_{j\delta},$$
we obtain
$$[J_{ij}, X_{k\sigma}] = \sum_{\alpha,\beta,\gamma,\delta} g^{\beta\alpha} \left(R^{\gamma\delta}{}_{\beta\sigma} X^\dagger_{i\alpha} X_{k\gamma} X_{j\delta} \;-\; C^{\gamma\delta}{}_{\sigma\alpha} X^\dagger_{i\gamma} X_{k\delta} X_{j\beta} \right) \;-\; \delta_{ik} X_{j\sigma}.$$

The g-invariance of C and R, expressed by the analogue of $(\mathrm{A}.1)$ obtained after transporting C and R to $K \otimes K$, again ensures that all cubic terms cancel. Therefore,
$$[J_{ij}, X_{k\sigma}] = -\,\delta_{ik}\, X_{j\sigma}.$$

3. Proof of (iii)

Using (i) and (ii),
$$[J_{ij}, J_{kl}] = \sum_{\alpha,\beta} g^{\beta\alpha} \left([J_{ij}, X^\dagger_{k\alpha}] X_{l\beta} + X^\dagger_{k\alpha} [J_{ij}, X_{l\beta}] \right).$$
Substituting
$$[J_{ij}, X^\dagger_{k\alpha}] = \delta_{jk} X^\dagger_{i\alpha}, \qquad [J_{ij}, X_{l\beta}] = -\delta_{il} X_{j\beta},$$
gives
$$[J_{ij}, J_{kl}] = \delta_{jk} J_{il} - \delta_{il} J_{kj}.$$

Thus the operators J_{ij} satisfy the commutation relations of the Lie algebra $\mathfrak{gl}(d)$.

□

3.12 Example

We keep the internal space $K \cong \mathbb{C}^3$ with orthonormal basis $\{k_\alpha\}_{\alpha=1}^3$ and generators

$$X_{i\alpha} := e_i \otimes k_\alpha \qquad (i = 1,2)$$

where i labels the external modes. As before, we choose the unique surviving quadratic direction in $K \otimes K$ to be

$$h := \frac{1}{\sqrt{3}}\big(k_1 \otimes k_1 + k_2 \otimes k_2 + k_3 \otimes k_3\big),$$

and let $R_{\text{gen}} := h^\perp \subset K \otimes K$ generate the quadratic relations for a single mode.

For each fixed i, this implies that all off–diagonal products $X_{i\alpha}X_{i\beta}$ with $\alpha \neq \beta$ vanish and all diagonal products $X_{i\alpha}X_{i\alpha}$ are identified in the quotient. Thus the single–mode Hilbert–Poincaré series is

$$G(t) = 1 + 3t + t^2,$$

so that for $d = 2$ modes we obtain

$$H_F(t) = G(t)^2 = 1 + 6t + 11t^2 + 6t^3 + t^4.$$

In particular, the global Fock space F is supported in total degrees $0,\ldots,4$, and we have the factorization

$$F \;\cong\; W^{\otimes 2},$$

where W is the single–mode Fock space with homogeneous components $\dim W_0 = 1$, $\dim W_1 = 3$, $\dim W_2 = 1$, and $W_n = 0$ for $n \geq 3$.

Let $|0\rangle_i$ denote the vacuum of mode i and set $|0\rangle := |0\rangle_1 \otimes |0\rangle_2$. For each mode i we choose the following homogeneous basis of W:

$$W_0 = \operatorname{span}\{|0\rangle_i\}, \qquad W_1 = \operatorname{span}\{\hat{X}^\dagger_{i\alpha}\,|0\rangle_i \mid \alpha = 1,2,3\},$$

$$W_2 = \operatorname{span}\{Y_i\}, \qquad Y_i := \big(X_{i1}X_{i1} + X_{i2}X_{i2} + X_{i3}X_{i3}\big),$$

so that the vector corresponding to Y_i in the Fock representation is

$$\hat{Y}^\dagger_i\,|0\rangle_i = \big(\hat{X}^\dagger_{i1}\hat{X}^\dagger_{i1} + \hat{X}^\dagger_{i2}\hat{X}^\dagger_{i2} + \hat{X}^\dagger_{i3}\hat{X}^\dagger_{i3}\big)\,|0\rangle_i\,.$$

Using the tensor product decomposition $F \cong W_1 \otimes W_2$, the homogeneous subspaces of the full Fock space and their bases up to total degree 4 are as follows.

- **Degree** 0.

$$F_0 = \operatorname{span}\{|0\rangle\}, \qquad |0\rangle = |0\rangle_1 \otimes |0\rangle_2\,.$$

- **Degree** 1. Here $N = p + q$ with $(p,q) \in \{(1,0),(0,1)\}$, hence

$$k_1 \cong (W_1 \otimes W_0) \oplus (W_0 \otimes W_1)$$

and a basis is

$$\hat{X}^\dagger_{1\alpha}\,|0\rangle := \hat{X}^\dagger_{1\alpha}\,|0\rangle_1 \otimes |0\rangle_2\,, \qquad \hat{X}^\dagger_{2\alpha}\,|0\rangle := |0\rangle_1 \otimes \hat{X}^\dagger_{2\alpha}\,|0\rangle_2\,,$$

for $\alpha = 1,2,3$. Thus $\dim k_1 = 3+3 = 6$.

- **Degree** 2. Now $N = 2$ with $(p,q) \in \{(2,0),(1,1),(0,2)\}$, so

$$k_2 \cong (W_2 \otimes W_0)\ \oplus\ (W_1 \otimes W_1)\ \oplus\ (W_0 \otimes W_2).$$

  A convenient basis is given by:

  - one vector of type $(2,0)$:
$$\hat{Y}^\dagger_1\,|0\rangle := \hat{Y}^\dagger_1\,|0\rangle_1 \otimes |0\rangle_2\,,$$
  - nine vectors of type $(1,1)$:
$$\hat{X}^\dagger_{1\alpha}\hat{X}^\dagger_{2\beta}\,|0\rangle := \hat{X}^\dagger_{1\alpha}\,|0\rangle_1 \otimes \hat{X}^\dagger_{2\beta}\,|0\rangle_2\,, \qquad \alpha,\beta = 1,2,3,$$
  - one vector of type $(0,2)$:
$$\hat{Y}^\dagger_2\,|0\rangle := |0\rangle_1 \otimes \hat{Y}^\dagger_2\,|0\rangle_2\,.$$

  Hence $\dim k_2 = 1+9+1 = 11$, in agreement with the coefficient of t^2 in $H_F(t)$.

- **Degree** 3. Here $N = 3$ with $(p,q) \in \{(2,1),(1,2)\}$, so

$$k_3 \cong (W_2 \otimes W_1)\ \oplus\ (W_1 \otimes W_2).$$

  A basis can be chosen as

$$\begin{aligned} \hat{Y}^\dagger_1\hat{X}^\dagger_{2\beta}\,|0\rangle &:= \hat{Y}^\dagger_1\,|0\rangle_1 \otimes \hat{X}^\dagger_{2\beta}\,|0\rangle_2\,, && \beta = 1,2,3,\\ \hat{X}^\dagger_{1\alpha}\hat{Y}^\dagger_2\,|0\rangle &:= \hat{X}^\dagger_{1\alpha}\,|0\rangle_1 \otimes \hat{Y}^\dagger_2\,|0\rangle_2\,, && \alpha = 1,2,3. \end{aligned}$$

  Thus $\dim k_3 = 3+3 = 6$.

- **Degree** 4. Finally $N = 4$ forces $(p,q) = (2,2)$, so

$$F_4 \cong W_2 \otimes W_2$$

  is one–dimensional, with basis

$$\hat{Y}^\dagger_1\hat{Y}^\dagger_2\,|0\rangle := \hat{Y}^\dagger_1\,|0\rangle_1 \otimes \hat{Y}^\dagger_2\,|0\rangle_2\,.$$

With this, we can finally write the relations that generate the algebra. They take the form

$$\begin{aligned}
X_{i2}X_{i3} + X_{i1}X_{i3} + X_{i1}X_{i2} &= 0,\\
X_{i3}X_{i2} + X_{i1}X_{i3} + X_{i1}X_{i2} &= 0,\\
X_{i2}X_{i3} + X_{i3}X_{i1} + X_{i1}X_{i2} &= 0,\\
X_{i3}X_{i2} + X_{i3}X_{i1} + X_{i1}X_{i2} &= 0,\\
X_{i2}X_{i3} + X_{i1}X_{i3} + X_{i2}X_{i1} &= 0,\\
X_{i3}X_{i2} + X_{i1}X_{i3} + X_{i2}X_{i1} &= 0,\\
X_{i2}X_{i3} + X_{i3}X_{i1} + X_{i2}X_{i1} &= 0,\\
X_{i3}X_{i2} + X_{i3}X_{i1} + X_{i2}X_{i1} &= 0,
\end{aligned}$$

for $i \in \{1,2\}$, which gives a total of sixteen relations.

4 Summary

What we have undertaken in this thesis is to establish a conceptual shift in how quantum statistics can be understood and studied, departing from the traditional approaches. These traditional perspectives were grouped into three main categories. First, there is the algebraic viewpoint, based on postulated commutation relations for creation and annihilation operators. Second, there is the geometric viewpoint, which connects quantum particles to configuration-space topology, attempting to give operational meaning to exchange symmetry and showing that quantum statistics may be interpreted as arising from the topological structure of configuration space. Third, there is a space–time or field-theoretic viewpoint, in which the coupling of fields to space–time, together with results such as the spin–statistics theorem, seems to enforce the conclusion that only bosons and fermions can exist in nature.

However, none of these perspectives derives quantum statistics from a purely quantum physical principle. They rely on additional postulates and ad hoc constructions that do not originate within quantum theory itself. The central argument advanced in this thesis is that the existence of various statistical behaviours is not a fundamental assumption but rather an operational consequence of how experiments involving many indistinguishable particles are constructed and interpreted.

The conceptual shift introduced here is to treat indistinguishability not as an ontological property of particles, but as an operational symmetry of experimental procedures. Detectors cannot label particles; interferometric or scattering devices act identically on all copies; phase-shift operations act on modes rather than on particle identities. Under these conditions, quantum statistics becomes a classification problem for the admissible representations of physically valid operations in concrete experimental scenarios.

The fundamental conclusion of this perspective is that bosons and fermions arise naturally as special cases corresponding to situations in which detectors access all relevant degrees of freedom, thereby allowing the experimentalist to determine the post-measurement state exactly. But if we allow for additional information not accessible to the measurement apparatus or experimental configuration, new statistical behaviours appear. These behave similarly to bosonic and fermionic statistics, yet differ in physically meaningful ways, including their thermodynamic and many-body properties. We have referred to these new behaviours collectively as transtatistics.

At the technical level, the approach taken in this thesis was to construct a minimal operational scenario in which individual particles evolve under a standard unitary symmetry $U(d)$. Many-particle experiments were modelled as follows: particles are injected into different modes, pass through an interferometer or interaction region, and, once the interaction has ceased, the number of particles in each mode is measured. We further introduced a notion of locality, requiring that quantum operations acting on each mode form an abelian group, namely mode-dependent phase shifts, and that these phases can be applied independently on each mode without affecting the others.

These assumptions imply that a Fock-space structure emerges naturally from representation-theoretic considerations, by demanding compatibility between the various symmetries present in the operational setting. The striking discovery is that, for such symmetries to be mutually compatible, the many-particle partition function for a single mode must be the generating function of a completely positive sequence. Completely positive sequences are objects familiar from algebraic combinatorics and representation theory, yet their connection to quantum physics had not previously been highlighted. As a result, these generating functions classify all admissible forms of transtatistics.

Once the partition function is fixed, the thermodynamics follow directly. This leads to several notable physical consequences. Transtatistical systems exhibit hidden symmetries, manifest in the appearance of multiplicities in the many-particle sectors that cannot be inferred from single-particle behaviour. They also exhibit ground-state degeneracy, even for non-interacting gases, unlike bosons and fermions, and may undergo spontaneous symmetry breaking, where such intrinsic degeneracies give rise to new mechanisms for phase transitions. Moreover, one obtains generalised occupation laws extending both Bose enhancement and the Pauli exclusion principle.

We therefore conclude that bosons and fermions constitute only a special case within a much larger family of particle types that remain fully consistent with single-particle quantum mechanics and with broadly applicable operational constraints on many-particle experiments.

On the other hand, we developed a perspective on the meaning of second quantisation that avoids the ad hoc imposition of creation and annihilation algebras in the form of CCR or CAR. Instead, second quantisation is treated as a loss-of-information map that takes, in principle, distinguishable constituents and collapses them into equivalence classes of states whose behaviour is operationally indistinguishable. In this view, second quantisation represents the transition from the distinguishable to the indistinguishable as an observational fact rather than a postulated symmetry.

More precisely, we define physical many-particle states as equivalence classes of states that contain no information that could be used to discriminate between particles. In the bosonic and fermionic cases, these equivalence classes arise because states differing only by permutations of particle labels are physically indistinguishable. Here we proceed analogously, but without assuming permutation symmetry itself. Only operational indistinguishability is imposed.

Within this framework, we show that certain families of transtatistics can be generated by imposing quadratic relations among creation and annihilation operators. Although we cannot prove in full generality that every form of transtatistics arises from such quadratic constructions, we demonstrate that consistency of the resulting many-particle states requires the introduction of Yang–Baxter-type constraints on the projectors defining the algebra. These constraints ensure associativity and consistency of exchange operations.

Thus, quadratically generated algebras can give rise to transtatistics under appropriate conditions, and we conjecture that in fact all transtatistics may be obtained in this way. This conjecture is equivalent to a form of the inverse eigenvalue problem, since the question reduces to showing that the poles of the partition function can always be realised as eigenvalues of a matrix with non-negative integer entries.

We may summarise the contributions of this thesis as follows. First, the symmetrisation postulate is not fundamental. It is a restriction that arises from the requirement of maximal compatibility among certain operations within specific observational scenarios. Second, quantum statistics are far richer than previously assumed. Transtatistics provide a family of particle types that are fully compatible with single-particle quantum mechanics and with operational invariance. Moreover, hidden symmetries turn out to be intrinsic to indistinguishability once we relax the assumption of full access to the internal degrees of freedom of a system. This establishes a fundamental link between indistinguishability, representation theory, and symmetry breaking.

Beyond this, the operational perspective serves as a conceptual framework capable of connecting algebraic, geometric, and field-theoretic approaches to identical quantum particles. In this way, we obtain a coherent structure that can support new developments across quantum information, many-body physics and condensed matter, and the foundations of quantum field theory, with potential implications for fundamental cosmological questions such as dark matter and baryon asymmetry.

Many open questions remain. One set of directions concerns applications of transtatistics to strongly correlated systems, new integrable models, and exotic phase transitions associated with emergent degeneracies. Another concerns systems in which the effective particle type changes dynamically in response to internal symmetries, a phenomenon that may lead to new classifications of universality classes. Transtatistics also offer the possibility of constructing new intermediate models of quantum computation extending boson sampling and fermion sampling. This suggests new complexity classes that are sensitive to the rational structure of the partition functions derived in this work, as well as potential implementations of hidden-mode sampling where internal degeneracies play an algorithmic role.

In a different direction, one could attempt to reformulate quantum fields using the commutation relations introduced here, examining how Doplicher–Haag–Roberts theory and spin–statistics results apply to or conflict with the algebras we derive. Reasoning of the kind developed in this thesis might serve as the basis for an operational formulation of quantum field theory, analogous to the construction presented here but adapted to the case in which modes form a continuum or couple directly to space-time structure.

Further questions concern the status of transtatistics within quantum theory itself. One may ask which quantum axioms are actually necessary for their existence, or whether other statistical behaviours could appear in generalised probabilistic theories, including mild modifications such as real or quaternionic quantum mechanics.

Finally, even if transtatistics do not occur as fundamental particle behaviours in nature, there are promising avenues for realising their operational signatures in artificial systems engineered to exhibit the indistinguishability constraints discussed here, such as photonic networks, cold atoms in synthetic dimensions, or topological phases in which particle identity becomes highly delocalised. These systems may provide favourable settings in which transtatistics could be observed or effectively simulated.

Bibliography

[AA11] Scott Aaronson and Alex Arkhipov. The computational complexity of linear optics. In *Proceedings of the forty-third annual ACM symposium on Theory of computing*, pages 333–342, 2011.

[AB09] Michele Arzano and Dario Benedetti. Rainbow statistics. *International Journal of Modern Physics A*, 24(25n26):4623–4641, 2009.

[AM76] Neil W Ashcroft and N David Mermin. *Solid state physics*. Saunders College Publishing, Philadelphia, 1976.

[ASW52] Michael Aissen, I. J. Schoenberg, and A. M. Whitney. On the generating functions of totally positive sequences i. *Journal d'Analyse Mathématique*, 2(1):93–103, Dec 1952.

[B$^+$04] Daniel Bump et al. *Lie groups*, volume 225. Springer, 2004.

[Bac73] A. Bach. Classical particle indistinguishability. *Foundations of Physics*, 3(3):365–379, 1973.

[Bac82] Jonas Backelin. On the rates of growth of the homologies of veronese subrings. In Jan-Eric Roos, editor, *Syzygies and Hilbert Functions*, volume 931 of *Lecture Notes in Mathematics*, pages 79–100. Springer, Berlin, 1982.

[Bag11] F. Bagarello. Pseudo-bosons, so far. *Reports on Mathematical Physics*, 68(2):175–210, dec 2011.

[Bal01] A. B. Balantekin. Partition functions in statistical mechanics, symmetric functions, and group representations. *Physical Review E*, 64(6), nov 2001.

[BBCeSTS01] AP Balachandran, E Batista, IP Costa e Silva, and Paulo TEOTONIO-SOBRINHO. A novel spin-statistics theorem in (2+ 1) d chern–simons gravity. *Modern Physics Letters A*, 16(20):1335–1347, 2001.

[BD02] Daniel Bump and Persi Diaconis. Toeplitz minors. *Journal of Combinatorial Theory, Series A*, 97(2):252–271, Feb 2002.

[BFFM20] Fabio Benatti, Roberto Floreanini, Fabio Franchini, and Ugo Marzolino. Entanglement in indistinguishable particle systems. *Physics Reports*, 878:1–27, 2020.

[BG15] James Borger and Darij Grinberg. Boolean witt vectors and an integral edrei–thoma theorem. *Selecta Mathematica*, 22(2):595–629, Nov 2015.

[BH08] B. Andrei Bernevig and F. D. M. Haldane. Model fractional quantum hall states and jack polynomials. *Phys. Rev. Lett.*, 100:246802, Jun 2008.

[BM00] A. Borowiec and W. Marcinek. On crossed products and wick algebras. *Journal of Mathematical Physics*, 41(11):6959–6973, 2000.

[Bos24] Bose. Plancks gesetz und lichtquantenhypothese. *Zeitschrift für Physik*, 26(1):178–181, Dec 1924.

[Bur58] N. Burgoyne. On the connection between spin and statistics. *Physical Review*, 109:471–472, 1958.

[BW14] Howard Barnum and Alexander Wilce. Local tomography and the jordan structure of quantum theory. *Foundations of Physics*, 44:192–212, 2014.

[BWC06] John C. Baez, Derek K. Wise, and Alissa S. Crans. Exotic statistics for strings in 4d bf theory, 2006.

[CB09] M. Cattani and J. M. F. Bassalo. Intermediate statistics, parastatistics, fractionary statistics and gentileonic statistics, 2009.

[CB12] Adam Caulton and Jeremy Butterfield. Symmetries and paraparticles as a motivation for structuralism. *British Journal for the Philosophy of Science*, 63(2):233–285, 2012.

[CDP11a] Giulio Chiribella, Giacomo Mauro D'Ariano, and Paolo Perinotti. Informational derivation of quantum theory. *Physical Review A*, 84:012311, 2011.

[CDP11b] Giulio Chiribella, Giacomo Mauro D'Ariano, and Paolo Perinotti. Informational derivation of quantum theory. *Physical Review A*, 84(1):012311, 2011.

[CND96] WEI CHEN, Y. JACK NG, and HENDRIK VAN DAM. A GENERALIZATION OF QUANTUM STATISTICS. *Modern Physics Letters A*, 11(10):795–803, mar 1996.

[Coe10] Bob Coecke. Quantum picturalism. In *Kindergarten Quantum Mechanics*. Oxford University Press, 2010. see also arXiv:quant-ph/0908.1787.

[CPGSV21] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product states and projected entangled pair states: Concepts, symmetries, theorems. *Rev. Mod. Phys.*, 93:045003, Dec 2021.

[Dav00] A. A. Davydov. Totally positive sequences and r-matrix quadratic algebras. *Journal of Mathematical Sciences*, 100(1):1871–1876, Jun 2000.

[DB11] Borivoje Dakić and Časlav Brukner. *Quantum Theory and Beyond: Is Entanglement Special?*, page 365–392. Cambridge University Press, 2011.

[DB16] Borivoje Dakić and Časlav Brukner. *The Classical Limit of a Physical Theory and the Dimensionality of Space*, pages 249–282. Springer Netherlands, Dordrecht, 2016.

[DF26] Paul Adrien Maurice Dirac and Ralph Howard Fowler. On the theory of quantum mechanics. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, 112(762):661–677, 1926.

[DGT$^+$13] Oscar C. O. Dahlsten, Andrew J. P. Garner, Jayne Thompson, Mile Gu, and Vlatko Vedral. Particle exchange in post-quantum theories, 2013.

[DHR71] Sergio Doplicher, Rudolf Haag, and John E. Roberts. Local observables and particle statistics i. *Communications in Mathematical Physics*, 23:199–230, 1971.

[DHR74] Sergio Doplicher, Rudolf Haag, and John E. Roberts. Local observables and particle statistics ii. *Communications in Mathematical Physics*, 35:49–85, 1974.

[Dir26] P. A. M. Dirac. Quantum mechanics and a preliminary investigation of the hydrogen atom. *Proceedings of the Royal Society A*, 110:561–579, 1926.

[Dir27] P. A. M. Dirac. The quantum theory of the emission and absorption of radiation. *Proceedings of the Royal Society of London. Series A*, 114(767):243–265, 1927.

[Dir30] Paul Adrien Maurice Dirac. A theory of electrons and protons. *Proceedings of the Royal Society of London. Series A, Containing papers of a mathematical and physical character*, 126(801):360–365, 1930.

[Dir58] Paul A. M. Dirac. *The Principles of Quantum Mechanics*. Oxford University Press, Oxford, UK, 4 edition, 1958.

[DL10] Dennis Dieks and Andrea Lubberdink. The indistinguishability of identical particles. *Foundations of Physics*, 40:1501–1524, 2010.

[DMPT14] Giacomo Mauro D'Ariano, Franco Manessi, Paolo Perinotti, and Alessandro Tosini. The feynman problem and fermionic entanglement: Fermionic theory versus qubit theory. *International Journal of Modern Physics A*, 29(17):1430025, jun 2014.

[DP14] Giacomo Mauro D'Ariano and Paolo Perinotti. Derivation of the dirac equation from principles of information processing. *Phys. Rev. A*, 90:062106, Dec 2014.

[Edr53] Albert Edrei. On the generation function of a doubly infinite, totally positive sequence. *Transactions of the American Mathematical Society*, 74(3):367–383, 1953.

[Ein24] A. Einstein. Quantentheorie des einatomigen idealen gases. erste abhandlung. *Sitzungsberichte der Preussischen Akademie der Wissenschaften*, pages 261–267, 1924.

[Ein25] A. Einstein. Quantentheorie des einatomigen idealen gases. zweite abhandlung. *Sitzungsberichte der Preussischen Akademie der Wissenschaften*, pages 3–14, 1925.

[EMMA22] Nathanaël Eon, Giuseppe Di Molfetta, Giuseppe Magnifico, and Pablo Arrighi. A relativistic discrete spacetime formulation of 3+1 qed, 2022.

[ET20] P. Ehrenfest and V. Trkal. Deduction of the dissociation-equilibrium from the theory of quanta and a calculation of the chemical constant. *Proceedings of the Royal Academy of Amsterdam*, 23:162–183, 1920.

[Fer26] E. Fermi. Zur quantelung des idealen einatomigen gases. *Zeitschrift für Physik*, 36:902–912, 1926.

[FH04] William Fulton and Joe Harris. *Representation Theory*. Springer New York, 2004.

[Fiv90] Daniel I. Fivel. Interpolation between fermi and bose statistics using generalized commutators. *Phys. Rev. Lett.*, 65:3361–3364, Dec 1990.

[FK06] Steven French and Décio Krause. *Identity in Physics: A Historical, Philosophical, and Formal Analysis*. Oxford University Press, Oxford, 2006.

[FRS89] Klaus Fredenhagen, Karl-Henning Rehren, and Bert Schroer. Superselection sectors with braid group statistics and exchange algebras: I. general theory. *Communications in Mathematical Physics*, 125:201–226, 1989.

[FS13] Christopher A. Fuchs and Rüdiger Schack. Quantum-bayesian coherence. *Reviews of Modern Physics*, 85(4):1693–1715, 2013.

[Ghi13] GianCarlo Ghirardi. Entanglement, nonlocality, superluminal signaling and cloning. *arXiv preprint arXiv:1305.2305*, 2013.

[Gib02] J. Willard Gibbs. *Elementary Principles in Statistical Mechanics*. Charles Scribner's Sons, New York, 1902.

[Gib10] Josiah Willard Gibbs. *Elementary Principles in Statistical Mechanics: Developed with Especial Reference to the Rational Foundation of Thermodynamics*. Cambridge Library Collection - Mathematics. Cambridge University Press, 2010.

[Gir65] Marvin D. Girardeau. Permutation symmetry of many-particle wave functions. *Physical Review*, 139(2B):B500–B508, 1965.

[Gj40] G. Gentile j. itosservazioni sopra le statistiche intermedie. *Il Nuovo Cimento (1924-1942)*, 17(10):493–497, Dec 1940.

[GM65] O. W. Greenberg and A. M. L. Messiah. Selection rules for parafields and the absence of paraparticles in nature. *Physical Review*, 138(4B):B1155–B1167, 1965.

[Goy19] Philip Goyal. Persistence and nonpersistence as complementary models of identical quantum particles. *New Journal of Physics*, 21(6):063031, jun 2019.

[Gre53] H. S. Green. A generalized method of field quantization. *Phys. Rev.*, 90:270–273, Apr 1953.

[Gre64] O. W. Greenberg. Spin and unitary-spin independence in a paraquark model of baryons and mesons. *Physical Review Letters*, 13:598–602, 1964.

[Gre90] O. W. Greenberg. Example of infinite statistics. *Phys. Rev. Lett.*, 64:705–708, Feb 1990.

[Gre91a] O. W. Greenberg. Particles with small violations of fermi or bose statistics. *Phys. Rev. D*, 43:4111–4120, Jun 1991.

[Gre91b] O. W. Greenberg. Particles with small violations of fermi or bose statistics. *Physical Review D*, 43(12):4111–4120, 1991.

[Gre99] O. W. Greenberg. Small violations of statistics, 1999.

[Har01] Lucien Hardy. Quantum theory from five reasonable axioms. *arXiv preprint quant-ph/0101012*, 2001.

[Har11] Lucien Hardy. Reformulating and reconstructing quantum theory. *arXiv preprint arXiv:1104.2066*, 2011.

[Hei26a] W. Heisenberg. Mehrkörperproblem und resonanz in der quantenmechanik. *Zeitschrift für Physik*, 38(6):411–426, Jun 1926.

[Hei26b] W. Heisenberg. Mehrkörperproblem und resonanz in der quantenmechanik. *Zeitschrift für Physik*, 38:411–426, 1926.

[HK64] R. Haag and D. Kastler. An algebraic approach to quantum field theory. *Journal of Mathematical Physics*, 5:848–861, 1964.

[HOM87] C. K. Hong, Z. Y. Ou, and L. Mandel. Measurement of subpicosecond time intervals between two photons by interference. *Phys. Rev. Lett.*, 59:2044–2046, Nov 1987.

[HS16] M. Hoyuelos and P. Sisterna. Quantum statistics of classical particles derived from the condition of a free diffusion coefficient. *Phys. Rev. E*, 94:062115, Dec 2016.

[HS22] Miguel Hoyuelos and Pablo Sisterna. Dark energy based on exotic statistics. 2022.

[HT69] JAMES B. HARTLE and JOHN R. TAYLOR. Quantum mechanics of paraparticles. *Phys. Rev.*, 178:2043–2051, Feb 1969.

[HW17] Philipp Andres Höhn and Christopher S. P. Wever. Quantum theory from questions. *Phys. Rev. A*, 95:012102, Jan 2017.

[Jay92] E. T. Jaynes. The gibbs paradox. In C. R. Smith, G. J. Erickson, and P. O. Neudorfer, editors, *Maximum Entropy and Bayesian Methods*, pages 1–22. Springer, Dordrecht, 1992.

[Jim85] Michio Jimbo. Aq-difference analogue of u(g) and the yang-baxter equation. *Letters in Mathematical Physics*, 10(1):63–69, Jul 1985.

[Joh16] Markus Johansson. Comment on 'reasonable fermionic quantum information theories require relativity', 2016.

[KLM01] Emanuel Knill, Raymond Laflamme, and Gerald J Milburn. A scheme for efficient quantum computation with linear optics. *Nature*, 409(6816):46–52, 2001.

[LD71] Michael G. G. Laidlaw and Cécile Morette DeWitt. Feynman functional integrals for systems of indistinguishable particles. *Phys. Rev. D*, 3:1375–1378, Mar 1971.

[LL77] L. D. Landau and E. M. Lifshitz. *Quantum Mechanics: Non-Relativistic Theory*. Pergamon Press, Oxford, 3 edition, 1977.

[LM77] J. M. Leinaas and J. Myrheim. On the theory of identical particles. *Il Nuovo Cimento B Series 11*, 37(1):1–23, Jan 1977.

[LN10] A. Lavagno and P. Narayana Swamy. Intermediate statistics as a consequence of deformed algebra. *Physica A: Statistical Mechanics and its Applications*, 389(5):993–1001, 2010.

[LZ58] G. Lüders and B. Zumino. Connection between spin and statistics. *Physical Review*, 110:1450–1453, 1958.

[Mas09] V. P. Maslov. A new distribution generalizing the bose-einstein distribution. *Theoretical and Mathematical Physics*, 159(2):684–685, May 2009.

[McI59] Harold V McIntosh. On accidental degeneracy in classical and quantum mechanics. *American Journal of Physics*, 27(9):620–625, 1959.

[Med97] M. V. Medvedev. Properties of particles obeying ambiguous statistics. *Phys. Rev. Lett.*, 78:4147–4150, Jun 1997.

[Mer85] N. David Mermin. Is the moon there when nobody looks? reality and the quantum theory. *Physics Today*, 38(4):38–47, 1985.

[MG64a] A. M. L. Messiah and O. W. Greenberg. Symmetrization postulate and its experimental foundation. *Phys. Rev.*, 136:B248–B267, Oct 1964.

[MG64b] A. M. L. Messiah and O. W. Greenberg. Symmetrization postulate and its experimental foundation. *Physical Review*, 136(1B):B248, 1964.

[Mir73] R. Mirman. Experimental meaning of the concept of identical particles. *Lettere al Nuovo Cimento*, 8(8):397–402, 1973.

[MM11a] Lluis Masanes and Markus P. Müller. A derivation of quantum theory from physical requirements. *New Journal of Physics*, 13:063001, 2011.

[MM11b] Lluí s Masanes and Markus P Müller. A derivation of quantum theory from physical requirements. *New Journal of Physics*, 13(6):063001, jun 2011.

[MYF+20] Benjamin Morris, Benjamin Yadin, Matteo Fadel, Tilman Zibold, Philipp Treutlein, and Gerardo Adesso. Entanglement between identical particles is a useful and consistent resource. *Phys. Rev. X*, 10:041012, Oct 2020.

[Neo16] Klil H. Neori. Identical particles in quantum mechanics: Operational and topological considerations, 2016.

[NG09] Robert K. Niven and Marian Grendar. Generalized classical, quantum and intermediate statistics and the pólya urn model. *Physics Letters A*, 373(6):621–626, 2009.

[NG12] R. Neori and P. Goyal. Operational derivation of feynman's rules and the symmetrization postulate. *Physical Review A*, 86:052126, 2012.

[OO97] Andrei Okounkov and Grigori Olshanski. Asymptotics of jack polynomials as the number of variables goes to infinity. *arXiv preprint q-alg/9709011*, 1997.

[Pau40] Wolfgang Pauli. The connection between spin and statistics. *Physical Review*, 58(8):716–722, 1940.

[PCL+12] Jian-Wei Pan, Zeng-Bing Chen, Chao-Yang Lu, Harald Weinfurter, Anton Zeilinger, and Marek Żukowski. Multiphoton entanglement and interferometry. *Rev. Mod. Phys.*, 84:777–838, May 2012.

[Pei91] Rudolf Peierls. Spontaneously broken symmetries. *Journal of Physics A: Mathematical and General*, 24(22):5273, 1991.

[Per93] Asher Peres. *Quantum Theory: Concepts and Methods*. Kluwer Academic Publishers, Dordrecht, 1993.

[PJ26] W Pauli Jr. Über das wasserstoffspektrum vom standpunkt der neuen quantenmechanik. *Zeitschrift für Physik A Hadrons and nuclei*, 36(5):336–363, 1926.

[Pol99] Alexios P. Polychronakos. Generalized statistics in one dimension, 1999.

[PP05] Alexander Polishchuk and Leonid Positselski. *Quadratic Algebras*, volume 37 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2005.

[Pri70] Stewart B. Priddy. Koszul resolutions. *Transactions of the American Mathematical Society*, 152:39–60, 1970.

[RS63] C Ryan and ECG169615 Sudarshan. Representations of parafermi rings. *Nuclear Physics*, 47:207–211, 1963.

[Sal45] R. Salem. Power series with integral coefficients. *Duke Mathematical Journal*, 12(1):153 – 172, 1945.

[Sal01] Silvio Salinas. *Introduction to statistical physics*. Springer New York, NY, 2001.

[Sau16] Simon Saunders. Indistinguishability. In Robert Batterman, editor, *The Oxford Handbook of Philosophy of Physics*, pages 320–356. Oxford University Press, 2016.

[SD24] Nicolás Medina Sánchez and Borivoje Dakić. Reconstruction of quantum particle statistics: Bosons, fermions, and transtatistics. *Quantum*, 8:1473, 2024.

[SdJ08] N I Stoilova and J Van der Jeugt. The parafermion fock space and explicit representations. *Journal of Physics A: Mathematical and Theoretical*, 41(7):075202, feb 2008.

[Seg56a] I. E. Segal. Tensor algebras over hilbert spaces. i. *Transactions of the American Mathematical Society*, 81(1):106–134, 1956.

[Seg56b] I. E. Segal. Tensor algebras over hilbert spaces. ii. *Annals of Mathematics*, 63(1):160–175, 1956.

[Seg58] Irving E. Segal. *Mathematical Problems of Relativistic Physics*, volume 2 of *Lectures in Applied Mathematics*. American Mathematical Society, Providence, RI, 1958.

[Seg67] Irving E. Segal. Irreducible representations of operator algebras. *Bulletin of the American Mathematical Society*, 73:127–130, 1967.

[SF99] Richard P. Stanley and Sergey Fomin. *Symmetric Functions*, volume 2 of *Cambridge Studies in Advanced Mathematics*, page 286–560. Cambridge University Press, 1999.

[SML64] Theodore D Schultz, Daniel C Mattis, and Elliott H Lieb. Two-dimensional ising model as a soluble problem of many fermions. *Reviews of Modern Physics*, 36(3):856, 1964.

[ST70] Robert H Stolt and John R Taylor. Correspondence between the first-and second-quantized theories of paraparticles. *Nuclear Physics B*, 19(1):1–19, 1970.

[Sta11] Richard P. Stanley. *Rational Generating Functions*, volume 1 of *Cambridge Studies in Advanced Mathematics*, page 464–570. Cambridge University Press, 2 edition, 2011.

[Sur04] Sumati Surya. Cyclic statistics in three dimensions. *Journal of Mathematical Physics*, 45(6):2515–2525, jun 2004.

[SV20] N.I. Stoilova and J. Van der Jeugt. Partition functions and thermodynamic properties of paraboson and parafermion systems. *Physics Letters A*, 384(21):126421, 2020.

[SV24] Steven V. Sam and Keller Vandebogert. Totally positive sequences and koszul algebras. *arXiv preprint*, 2024.

[SW64] R. F. Streater and A. S. Wightman. *PCT, Spin and Statistics, and All That*. W. A. Benjamin, New York, 1964.

[Swa08] J. Swain. EXOTIC STATISTICS FOR ORDINARY PARTICLES IN QUANTUM GRAVITY. *International Journal of Modern Physics D*, 17(13n14):2475–2484, dec 2008.

[Swe06] R. H. Swendsen. The gibbs paradox and the definition of entropy. *American Journal of Physics*, 74:187–190, 2006.

[TD02] Barbara M Terhal and David P DiVincenzo. Classical simulation of noninteracting-fermion quantum circuits. *Physical Review A*, 65(3):032325, 2002.

[Tri09] D. A. Trifonov. Pseudo-boson coherent and fock states, 2009.

[vE19] S. J. van Enk. Exchanging identical particles and topological quantum computing, 2019.

[VK82] Anatolii Moiseevich Vershik and Sergei Vasil'evich Kerov. Characters and factor representations of the infinite unitary group. In *Doklady Akademii Nauk*, volume 267, pages 272–276. Russian Academy of Sciences, 1982.

[vK84] N. G. van Kampen. The gibbs paradox. *Essays in Theoretical Physics*, pages 303–312, 1984.

[vN27] J. von Neumann. Wahrscheinlichkeitstheoretischer aufbau der quantenmechanik. *Göttinger Nachrichten*, pages 245–272, 1927.

[Voi76] Dan Voiculescu. Représentations factorielles de type ii1 de u (). *J. Math. Pures Appl*, 55(1):1–20, 1976.

[WH24] Zhiyuan Wang and Kaden R. A. Hazzard. Particle exchange statistics beyond fermions and bosons, 2024. Author-provided LaTeX version.

[Wig95] A. S. Wightman. Superselection rules; old and new. *Il Nuovo Cimento B (1971-1996)*, 110(5):751–769, May 1995.

[Wil82] Frank Wilczek. Quantum mechanics of fractional-spin particles. *Phys. Rev. Lett.*, 49:957–959, Oct 1982.

[Yor] James W. York. Permutation geometry reformulation. Unpublished manuscript, 1999. Discusses relational configuration-space approach to permutation symmetry.