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Dott. mag. Dott. Eric Stenhede

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Morning, meaning before noon.

Andras Bonk

Abstract

Two particularly effective and complementary ways to present contact 3-manifolds are (i) *contact surgery diagrams*, i.e., contact surgery on Legendrian links in (S^3, ξ_{st}) [DG01; DG04], and (ii) *open book decompositions*, via the Giroux correspondence [Gir02]. Motivated by the role of Kirby calculus for framed links in 3-manifold topology, the main goal is to develop a corresponding calculus for contact surgery diagrams and the tools needed to prove such a result.

The first main contribution is an explicit *algorithm for Legendrian realization*: given a Legendrian graph $G \subset (\mathbb{R}^3, \xi_{st})$ (via its front projection) and a homologically nontrivial curve on a ribbon surface of G , the algorithm outputs the front projection of a Legendrian knot that is the Legendrian realization of that curve. A key application is an algorithmic conversion from *abstract open books to Legendrian surgery diagrams*: from an abstract open book (Σ, φ) whose monodromy is presented as a product of Dehn twists along homologically nontrivial simple closed curves, one obtains a contact surgery link $L = L_1 \cup L_2$ representing the same contact manifold. The contact surgery link L_1 encodes the factorization of φ while L_2 consists of $+1$ -labelled Legendrian unknots with $tb = -1$. This is an improvement of an analogous algorithm in [Avd23].

Along the way, a *uniqueness statement* is established: Legendrian realizations of the same curve on a ribbon surface are Legendrian isotopic, and likewise for Legendrian knots on pages of open books representing the same isotopy class, strengthening a result in [LOSS09].

The second main contribution is a *contact version of Kirby's theorem*: any two contact surgery links in (S^3, ξ_{st}) presenting contactomorphic contact 3-manifolds are related by a finite sequence of moves: Legendrian isotopies, insertion/deletion of cancelling pairs, contact handleslides, and (new) lantern and chain moves—obtained by refining Avdek's ribbon-move framework [Avd23].

Zusammenfassung

Zwei besonders effektive und komplementäre Möglichkeiten, Kontakt-3-Mannigfaltigkeiten zu präsentieren, sind (i) *Kontaktchirurgie-Diagramme*, d. h. Kontaktchirurgie an Legendrianlinks in (S^3, ξ_{st}) [DG01; DG04], und (ii) *Offenbuchzerlegungen*, vermittelt durch die Giroux-Korrespondenz [Gir02]. Motiviert durch die Rolle des Kirby-Kalküls für gerahmte Links in der 3-Mannigfaltigkeiten-Topologie besteht das Hauptziel darin, ein entsprechendes Kalkül für Kontaktchirurgie-Diagramme sowie die zum Beweis eines solchen Resultats notwendigen Werkzeuge zu entwickeln.

Der erste Hauptbeitrag ist ein expliziter *Algorithmus zur Legendrian-Realisierung*: Gegeben ein Legendrian-Graph $G \subset (\mathbb{R}^3, \xi_{st})$ (über seine Frontprojektion) und eine homologisch nichttriviale Kurve auf einer Bandfläche (ribbon surface) von G , liefert der Algorithmus die Frontprojektion eines Legendrian-Knotens, der diese Kurve Legendrian realisiert. Eine zentrale Anwendung ist eine algorithmische Umwandlung von *abstrakten Offenbüchern in Legendrian-Chirurgiediagramme*: Aus einem abstrakten Offenbuch (Σ, φ) , dessen Monodromie als Produkt von Dehn-Twists entlang homologisch nichttrivialer einfacher geschlossener Kurven dargestellt ist, erhält man einen Kontaktchirurgie-Link $L = L_1 \cup L_2$, der dieselbe Kontaktmannigfaltigkeit repräsentiert. Der Kontaktchirurgie-Link L_1 kodiert die Faktorisierung von φ , während L_2 aus mit $+1$ markierten Legendrian-Unknoten mit $tb = -1$ besteht. Dies ist eine Verbesserung eines analogen Algorithmus in [Avd23].

Nebenbei wird eine *Eindeutigkeitsaussage* etabliert: Legendrian-Realisierungen derselben Kurve auf einer Bandfläche sind Legendrian-isotop; entsprechend gilt dies für Legendrian-Knoten auf Seiten von Offenbüchern, die dieselbe Isotopieklasse repräsentieren, wodurch ein Resultat aus [LOSS09] verstärkt wird.

Der zweite Hauptbeitrag ist eine *Kontakt-Version des Satzes von Kirby*: Je zwei Kontaktchirurgie-Links in (S^3, ξ_{st}) , die kontaktomorphe Kontakt-3-Mannigfaltigkeiten darstellen, sind durch eine endliche Folge von Zügen miteinander verknüpft: Legendrian-Isotopien, Einfügen/Entfernen kürzender Paare, Kontakt-Handleslides sowie (neue) Laternen- und Kettenzüge—gewonnen durch eine Verfeinerung von Avdeks Ribbon-Move-Formalismus [Avd23].

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Introduction

Contact geometry will absorb all geometry, all physics, all mathematics, everything.¹

Misha Gromov

It is not clear where to begin when trying to explain why contact geometry is important, as there are many reasons why it is regarded as a fundamental area of modern geometry. An external justification is that contact manifolds arise naturally as boundaries of symplectic manifolds, and therefore also appear in classical mechanics and in several complex variables [BW58; EG91; CM74; Arn89; Gei01]. Therefore, understanding contact manifolds allows us to control symplectic cut-and-paste operations and related symplectic invariants. A more intrinsic motivation comes from a result of É. Cartan, which shows that there exist only four classes of topologically stable distributions: line fields, even contact structures, contact structures, and Engel structures [Car99].² Of these, even contact structures are, in a way, less interesting because they are governed by algebraic-topological data [McD87], line fields have been deeply studied and belong to the realm of dynamical systems, and not much is known about Engel structures (and they only exist in dimension four). As for contact structures, they exist in every odd dimension and exhibit rich global geometry. In particular, the dichotomy between *overtwisted* and *tight* contact structures, introduced by Eliashberg, has shaped the development of the field: overtwisted contact structures are classified up to isotopy by their homotopy class [Eli89; BEM15], whereas tight contact structures display rigidity and admit subtle invariants [Ben83; Hon00; OS05]. The book [Gei08] provides a broad overview of this landscape.

Our interest in contact topology comes from the fact that 3-dimensional contact topology is tightly connected to low-dimensional topology, and in particular to the study of 3-manifolds. Work of Gromov, Lutz, and Martinet shows that every homotopy class of cooriented plane fields can be realized by a contact structure [Gro69; Mar71; Lut71]. A central theme of modern 3-dimensional contact topology is therefore to understand how much finer the contact category is than the underlying topological category.

¹This was a purposely provocative sentence said during a (very interesting!) panel discussion in 1999. https://youtu.be/Fme_r-nE4CI?si=rEpwTVCaZC13ey92&t=420

²We learned about this in the survey [Pre16], where the result is attributed to Cartan and the reference [VG87] is cited.

A recurring motif in low-dimensional topology is the search for efficient and flexible ways to present objects. Two complementary languages have proved particularly effective for studying contact manifolds:

- *contact surgery diagrams*, in which one presents a contact manifold as the result of contact surgery on a Legendrian link in (S^3, ξ_{st}) , and
- *open book decompositions*, in which a contact manifold is presented by an open book, thanks to the Giroux correspondence [Gir02].

In this thesis we study the interplay between these two presentations. The main goal is to provide an analogue of Kirby’s theorem [Kir78] for contact surgery diagrams, and to develop the tools needed to prove such a result.

The first tool is an explicit algorithm to Legendrian realize homologically nontrivial curves on ribbon surfaces of Legendrian graphs in $(\mathbb{R}^3, \xi_{\text{st}})$, together with a resulting algorithm that produces contact surgery diagrams from open books (Chapter 3). Building on this, we show that any two contact surgery links in (S^3, ξ_{st}) giving rise to the same contact manifold are related by a finite sequence of moves: Legendrian isotopies, insertion and deletion of cancelling pairs, contact handleslides, and two new types of moves that we call *lantern moves* and *chain moves*. This is joint work with Marc Kegel, Vera Vértési, and Daniele Zuddas, and forms the core of Chapter 4.

Along the way, we complete an argument from [LOSS09] and obtain a corresponding uniqueness statement. Namely, we show that any two Legendrian realizations of the same curve on a ribbon surface are Legendrian isotopic, and likewise that any two Legendrian knots lying on pages of open books compatible with the same contact structure and representing the same isotopy class on the page are Legendrian isotopic.

In the remainder of this introduction we first informally survey the relevant background in contact topology and surgery descriptions of 3-manifolds. We then state the main results of the thesis and explain how they fit into the existing literature. We conclude with a brief outline of the three main chapters.

Background

The material presented here is just the one needed to formulate and appreciate our results. Because of this, this section is by no means a comprehensive survey of 3-dimensional contact topology and surgery descriptions of 3-manifolds, as many important topics have been omitted.

3-dimensional contact topology. Let M be an oriented 3-manifold. A *positive contact form* on M is a 1-form α such that $\alpha \wedge d\alpha > 0$, and a *contact structure* is a 2-plane field $\xi \subset TM$ given as the kernel of such a form. Classical examples include the *standard contact structures*

$$\alpha_{\text{st}} = dz + x dy \quad \text{on } \mathbb{R}^3, \quad \xi_{\text{st}} = \ker \alpha_{\text{st}},$$

and

$$\xi_{\text{st}} = \ker(x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2)|_{S^3} \subset TS^3 \subset T\mathbb{R}^4.$$

Contact structures on 3-manifolds have been abundantly studied over the last decades. Early work of Darboux and then Gray shows that every contact manifold is locally isomorphic to $(\mathbb{R}^3, \xi_{\text{st}})$ and that on a closed manifold there are no nontrivial deformations

of contact structures. These phenomena are now known as Darboux’s theorem and Gray stability [Dar82; Gra59]. These results already hinted at the fact that contact geometry has no local invariants and that the interesting phenomena are global in nature. Shortly afterward, Martinet, followed by Lutz, proved that every closed oriented 3-manifold admits a contact structure in every homotopy class of cooriented 2-plane fields [Mar71; Lut71]. Together with Gromov’s h -principle techniques for open manifolds [Gro69], this established the abundance of contact structures and shifted the focus towards their classification.

When trying to understand geometric structures on manifolds, it is usually particularly useful to study submanifolds which behave well with respect to the geometric structure considered. Within a contact 3-manifold (M, ξ) , *Legendrian knots*, i.e. knots that are everywhere tangent to ξ , provide an important class of submanifolds that capture fine contact-topological information. They will be omnipresent in this thesis. Legendrian knot theory in $(\mathbb{R}^3, \xi_{\text{st}})$ was developed in parallel with the general theory of contact structures, with contributions by many mathematicians [Ben83; EF09; Che02; EH01; FI04; NRS⁺20]. In the standard contact structure $(\mathbb{R}^3, \xi_{\text{st}})$ one can work by representing Legendrian knots by their *front projection*, that is, the projection onto the (y, z) -plane: Legendrian knots are precisely those immersed curves with only semicubical cusps and no vertical tangencies. In this picture, every time we see a crossing between two Legendrian arcs in the front projection, the Legendrian condition implies that the arc with lower slope lies above the one with higher slope.

Of course, also considering surfaces in contact 3-manifolds is relevant. The modern theory of convex surfaces in contact 3-manifolds is due to Giroux and has had a profound impact on the field [Gir91]. Giroux’s key insight was to consider surfaces that are transverse to a vector field that preserves the contact structure: such surfaces are called *convex*. On a convex surface Σ , the local behavior of the characteristic foliation is encoded by the *dividing set*, a multi-curve $\Gamma_\Sigma \subset \Sigma$ which is uniquely determined up to isotopy. Giroux’s flexibility theorem (Theorem 2.8) shows that, fixing the dividing set, the characteristic foliation can be altered arbitrarily in a large class, by perturbing the surface through convex surfaces. This provides a powerful cut-and-paste technology in contact topology and has been a major tool in the quest of classifying tight contact structures.

One of the main consequences of this theory is the Legendrian realization principle (Theorem 2.9) [Kan98; Hon00; Etn04a]. In one of its common formulations, it states that if a multicurve or graph on a convex surface Σ is in suitable position with respect to the dividing set (typically “nonisolating”), then it can be realized as a Legendrian curve or graph after an arbitrarily small perturbation of Σ through convex surfaces, while keeping the dividing set fixed. In many situations of interest, the dividing set has a particularly simple form and the nonisolating condition reduces to a homological condition. In this thesis, we exploit this in the context of ribbon surfaces and pages of open books, where homologically nontrivial curves are precisely those eligible for Legendrian realization.

Surgery presentations and Kirby calculus. Dehn surgery on a knot K in a 3-manifold M can be described as follows. Consider a closed tubular neighborhood $\mathcal{N}(K)$, and the exterior $X(K) := (M \setminus \mathring{\mathcal{N}}(K))$ of K . We can see M as $X(K) \cup_{\text{Id}} \mathcal{N}(K)$, where this notation means that we are gluing $\mathcal{N}(K)$ to $X(K)$ by the identity map on their boundaries. We can create a potentially different manifold by considering $X(K) \cup_f \mathcal{N}(K)$,

where f is a diffeomorphism from $\partial\mathcal{N}(K)$ to $\partial X(K)$. This operation is called Dehn surgery on K , and up to diffeomorphism it depends only on the isotopy class in $\partial X(K)$ of the image under f of the meridian $\mathfrak{a}$ of $\partial\mathcal{N}(K)$. Therefore, to specify a Dehn surgery on a knot K it is enough to specify an isotopy class of a curve on $\partial\mathcal{N}(K)$ that we want to bound a disk after the surgery. Such a curve is called a *framing* if it intersects $\mathfrak{a}$ once. In S^3 the possible Dehn surgeries are canonically in bijection with $\mathbb{Q}$, with surgeries corresponding to framed knots being the integers.

The description of 3-manifolds by Dehn surgery on links in S^3 goes back to the independent work of Lickorish and Wallace in the 1960s: every closed, oriented 3-manifold can be obtained by integral Dehn surgery on a framed link in S^3 [Lic62; Wal60].

This immediately raises the question of when two surgery presentations define the same 3-manifold. Kirby's 1978 theorem provides a complete answer: two framed links in S^3 yield orientation-preserving diffeomorphic 3-manifolds if and only if they are related by a finite sequence of isotopies, handle slides, and blow ups/downs, i.e. the addition or removal of ± 1 -framed unknots [Kir78]. The resulting calculus on framed links, now known as *Kirby calculus*, has become a foundational tool in low-dimensional topology. See the beautiful book [GS99].

This theory allows for a very practical way of representing 3-manifolds. Every 3-manifold can be represented by a link diagram, in which every component has a rational number attached to it. In this thesis we will refer to these objects as *surgery diagrams* (which live in $\mathbb{R}^2$) in contrast to *surgery links* (which live in $\mathbb{R}^3 \subset S^3$).

Contact surgery. The contact analogue of surgery on links begins with Weinstein's introduction of contact (± 1) -surgery along Legendrian knots, viewed as attaching Weinstein 2-handles to symplectic fillings [Wei91].

In dimension three, this area was developed systematically by Ding and Geiges at the beginning of the 2000s. They showed that any closed, connected contact 3-manifold (M, ξ) can be obtained by contact (± 1) -surgery on a Legendrian link in (S^3, ξ_{st}) [DG04; DGS04]. Contact surgery will be defined properly in Section 2.2.8.

Since then, contact surgery diagrams have become a standard tool for constructing and analyzing contact 3-manifolds. They have been used to construct and analyse symplectic fillings [Eli90; Wei91; Gom98; GS99; OS04], classify tight contact structures [Hon00; LS04; LS07b; LS07a; Wan15], study Legendrian knots [GO15], analyze the Reeb dynamics [BEE12; Avd23], and in general are just useful for describing interesting contact manifolds. See the beautiful book [OS04] for an overview on this topic.

Open books and the Giroux correspondence. An *open book* on a closed 3-manifold M is a pair (B, π) where:

- $B \subset M$ is a link, called the *binding*;
- $\pi: M \setminus B \rightarrow S^1$ is a fibration which coincides with the angular coordinate on a neighborhood $B \times D^2$ of $B = B \times \{0\}$. The closure of a fiber is called a *page*.

An *abstract open book* is a pair (Σ, ϕ) where:

- Σ is a compact, oriented surface with nonempty boundary;
- $\phi \in \text{Diff}^+(\Sigma, \partial\Sigma)$ is a diffeomorphism that is the identity near $\partial\Sigma$.

In Section 2.3 it will be explained how these two notions are tightly related.

Open books have a long history in 3-manifold topology, going back to Alexander’s work, who proved that every 3-manifold admits such a structure [Ale23]. For our purposes, the interesting part of the story begins when Thurston and Winkelnkemper observed in the 1970s that any abstract open book (Σ, ϕ) carries a natural contact structure: one builds an explicit contact form on the mapping torus of ϕ and matches it with a standard model near the binding [TW75]. This provides a flexible way to construct contact manifolds from mapping class group data.

The converse direction, constructing an open book from a given contact structure, is part of one of the biggest and most fascinating results in the field. Giroux’s remarkable correspondence establishes a precise bridge between contact structures and open book decompositions: on a closed 3-manifold, there is a one-to-one correspondence between open books up to positive stabilizations and contact structures up to isotopy [Gir02]. In more detail, one can first define what it means for an open book to support a contact structure. The result of Thurston and Winkelnkemper then shows that every open book supports a contact structure. The insight of Giroux was to notice that this contact structure is unique up to isotopy and that every contact structure is supported by an open book. One then has that two open books support the same contact structure if and only if they are related by a sequence of positive stabilizations, destabilizations and isotopies.

This correspondence has become one of the central tools in 3-dimensional contact topology. It underlies the construction and refinement of Heegaard Floer invariants [OS05; HKM09], and has powerful consequences for filling and Reeb dynamics: for instance, planar open books impose strong restrictions on symplectic fillings and are closely tied to the Weinstein conjecture [Etn04b; CH13]. Moreover, this bijection yields mapping-class-group characterizations of tightness [HKM07], and plays a crucial role in relating Heegaard Floer and embedded contact homology [CGH11]. There are now several expositions [Etn06; OS04] and alternative proofs of the correspondence and its refinements [BHH24; LV24a; LV24b].

Ribbon surfaces and contact cell decompositions. An important bridge between surgery diagrams and open books is provided by ribbon surfaces of Legendrian graphs. Given a Legendrian graph G in a contact 3-manifold (M, ξ) , a *ribbon surface* of G is a compact surface $\Sigma \subset M$ containing G , obtained by “slightly thickening” G in the direction of the contact structure. A ribbon surface Σ is convex with dividing set equal to its boundary. Ribbon surfaces appeared first, without name, in Giroux’s work on the correspondence between contact structures and open books. For us they are interesting because they were used explicitly by Avdek in his study of contact surgery diagrams [Avd13].

A *contact cell decomposition*, introduced by Giroux [Gir02], is a CW-decomposition of a contact manifold (M, ξ) with Legendrian 1-skeleton and convex 2-skeleton, satisfying additional tightness conditions on 3-cells. From such a decomposition, one can build an open book supporting ξ . A page of this open book can be seen to be the ribbon surface of the Legendrian 1-skeleton. This is the reason why it is interesting to consider ribbon surfaces: because they are easy to build (they are encoded by Legendrian graphs)

and, under some conditions, they can be pages of open books. These conditions are, in particular, very controllable in (S^3, ξ_{st}) .

Main results

Legendrian realization on ribbon surfaces. The first main contribution of this thesis is an algorithm to Legendrian realize a homologically nontrivial curve on a ribbon surface of a Legendrian graph. As explained above, the Legendrian realization principle applied to ribbon surfaces ensures that a homologically nontrivial curve on a ribbon can be made Legendrian after a suitable perturbation of the surface, but the proof is highly non-constructive, as it relies on Morse-theoretic techniques. In Chapter 3 we remedy this in the setting of ribbons of Legendrian graphs in $(\mathbb{R}^3, \xi_{\text{st}})$, where the resulting Legendrian knot is shown explicitly in the front projection, that is, in the (y, z) -plane.

Informally, the algorithm can be stated as follows. For a precise statement, see Section 3.3.

Theorem 1.1 (An algorithm to Legendrian realize curves on ribbon surfaces, Theorem 3.12). *There is an algorithm with the following input and output:*

Input:

- a Legendrian graph $G \subset (\mathbb{R}^3, \xi_{\text{st}})$, given by its front projection;
- a homologically nontrivial curve $\mathfrak{a}$ on a ribbon Σ of G .

Output: the front projection of a Legendrian knot L which is the Legendrian realization of $\mathfrak{a}$.

Given the front projection of a ribbon surface in $(\mathbb{R}^3, \xi_{\text{st}})$, it is important to notice that there is no obvious way to Legendrian realize a curve on it. Indeed, many of the crossings of the boundary link of a ribbon surface violate the Legendrian condition, and so, in general, we cannot make a curve on a ribbon surface Legendrian by simply introducing cusps when needed. Consult Figure 1.1 for an example. The complete explanation of this example can be found in Section 3.5.

Very briefly, the algorithm in Theorem 1.1 works as follows. Starting from a Legendrian graph $G \subset (\mathbb{R}^3, \xi_{\text{st}})$, we use a handle decomposition of the ribbon surface R (with 0- and 1-handles induced by the vertices and edges of G) to put the given homologically nontrivial curve $\mathfrak{a}$ in a standard position. We then decompose $\mathfrak{a}$ into finitely many segments, each lying in a single handle, and for every segment in a 1-handle we assign a combinatorial quantity, the *relative gain*. Next, to each segment of $\mathfrak{a}$ we assign a Legendrian segment whose slope is dictated by the relative gain. By modifying $\mathfrak{a}$ within its isotopy class on the ribbon, we arrange that these relative gains are well understood and globally balance, so that the segments can be lifted to Legendrian arcs whose z -coordinates glue together consistently and close up to a Legendrian knot. The proof of the validity of the algorithm consists in explaining why these Legendrian segments are actually the Legendrian realization of the original segments.

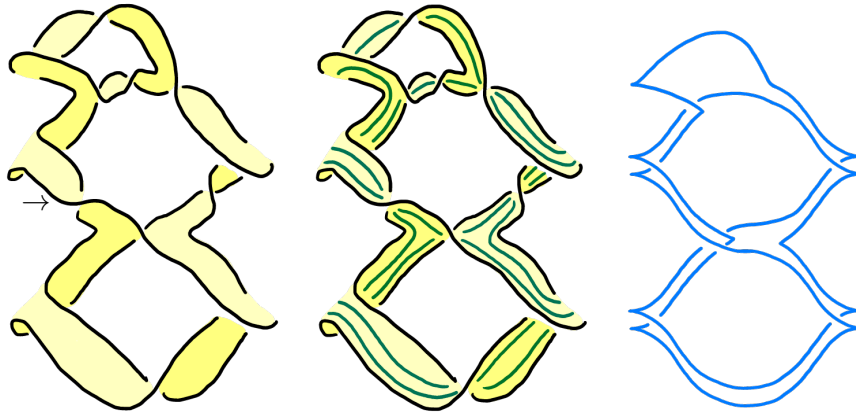


Figure 1.1. Left: the front projection of a ribbon surface Σ . The crossing indicated by an arrow is one of the crossings that makes a naive Legendrian realization impossible. Middle: a homologically nontrivial curve α on Σ . Right: the Legendrian realization of α obtained with Theorem 1.1.

An algorithm to create contact surgery diagrams. Theorem 1.1 is then used to construct an explicit algorithm for producing contact surgery diagrams from abstract open books.

Corollary 1.2. *Given an abstract open book decomposition (Σ, ϕ) whose monodromy ϕ is given explicitly as a composition of Dehn twists along homologically nontrivial simple closed curves, there exists an algorithm that produces a contact surgery diagram for the contact manifold supported by the open book (Σ, ϕ) .*

This algorithm produces a contact surgery link $\mathbf{L} = \mathbf{L}_1 \cup \mathbf{L}_2$ for which the components of $\mathbf{L}_1$ are in bijection with the Dehn twists in the factorization of ϕ , and all the components of $\mathbf{L}_2$ are $+1$ -framed Legendrian unknots with Thurston–Bennequin number $tb = -1$.

This algorithm is basically a refinement of an algorithm by Avdek in [Avd13], where the result is proven for Σ with connected boundary and ϕ given as a product of Dehn twists along the Lickorish generators.

Several related algorithms already exist in the literature, both creating contact surgery diagrams from open books and vice versa [AO01; Sti05; Ozb06; Avd13]. None of them, however, suited our purposes, hence the need to develop a new one.

Very roughly, the proof of Corollary 1.2 proceeds in three steps. Starting from an abstract open book (Σ, ϕ) with monodromy written as a product of Dehn twists along homologically nontrivial simple closed curves, we first realize Σ as the ribbon R_G of a Legendrian graph $G \subset (S^3, \xi_{st})$. This is done by concatenating explicit building blocks whose ribbons have prescribed genus and number of boundary components, producing a contact cell decomposition of (S^3, ξ_{st}) whose 1-skeleton is G . The associated ribbon R_G is then a page of an open book (B, π) supporting (S^3, ξ_{st}) , with monodromy ψ given explicitly as a product of right-handed Dehn twists along curves coming from the building blocks.

Next, we use an orientation-preserving diffeomorphism $f: \Sigma \rightarrow R_G$ to transport the factorization of ϕ to a factorization of a mapping class $\phi' = f\phi f^{-1} \in \text{MCG}(R_G, \partial R_G)$. Since the curves in this factorization are homologically nontrivial on R_G , they are eligible for Legendrian realization. Applying the algorithm of Theorem 1.1 (i.e. Theorem 3.12) to

each of these curves, and similarly to the curves appearing in the factorization of ψ , we obtain Legendrian knots lying in distinct pages of the open book (B, π) .

Finally, we assign contact surgery coefficients to these Legendrian knots so that the components corresponding to ψ contribute ψ^{-1} and those corresponding to ϕ' contribute ϕ in the mapping class group. Proposition 4.22 identifies the effect of contact ± 1 -surgery on such knots with composition by the corresponding Dehn twists on the page. With the chosen coefficients, contact surgery along the resulting Legendrian link $\mathbf{L} = \mathbf{L}_1 \cup \mathbf{L}_2$ transforms the open book (B, π) for (S^3, ξ_{st}) into an open book with page R_G and monodromy ϕ' . Since (R_G, ϕ') is equivalent to (Σ, ϕ) via f , the contact manifold obtained by surgery on $\mathbf{L}$ is exactly the one supported by (Σ, ϕ) . The construction depends only on the explicit Dehn twist factorization of ϕ , which makes the whole procedure algorithmic.

Uniqueness of the Legendrian realization. Along the way, we complete an argument from [LOSS09] and obtain a corresponding uniqueness statement. Namely, we show that any two Legendrian realizations of the same curve on a ribbon surface are Legendrian isotopic, and likewise that any two Legendrian knots lying on pages of open books compatible with the same contact structure and representing the same isotopy class on the page are Legendrian isotopic. Precise formulations are given in Theorem 3.6 and Theorem 3.10.

All the necessary ingredients already appear in [Gir02; Etn06; LOSS09]; what is missing is a brief analysis of how two isotopies of a contact manifold that induce the same path of contact structures can differ. This analysis forms the core of the second part of the proof of Theorem 3.6.

A contact version of the Kirby theorem. The second main contribution of the thesis is to establish a contact analogue of Kirby's theorem for contact surgery diagrams in (S^3, ξ_{st}) . Starting from the result of Ding and Geiges that any closed contact 3-manifold arises from contact ± 1 -surgery on a Legendrian link in (S^3, ξ_{st}) , it is natural to ask when two such surgery diagrams present the same contact manifold, and whether there is a list of moves that generates all possible relations. This problem was addressed by Avdek, who introduced the notion of ribbon moves: two contact surgery links differ by a ribbon move if both are compatible with a ribbon surface R and they define the same element in the mapping class group of R .¹ He showed that any two contact surgery links of the same contact manifold are related by a finite sequence of Legendrian isotopies and ribbon moves [Avd13]. However, while this remains the first result on the topic, the notion of a ribbon move is arguably too general to be fully satisfactory. First, these moves do not come in a finite family. Furthermore, these moves are not always presented as diagrammatic moves in front projections, and thus it is particularly difficult to prove that two contact surgery presentations differ by a ribbon move.

In Chapter 4, in joint work with Marc Kegel, Vera Vértési, and Daniele Zuddas, we refine Avdek's result and obtain the following theorem.

¹These notions are made precise in Chapter 4.

Theorem 1.3 (A contact version of the Kirby theorem). *Let $\mathbf{L}$ and $\mathbf{L}'$ be two contact surgery links in (S^3, ξ_{st}) that describe contactomorphic contact 3-manifolds. Then $\mathbf{L}$ and $\mathbf{L}'$ differ by a sequence of Legendrian isotopies, addition or deletion of cancelling pairs, contact handleslides, standard lantern moves, and standard chain moves.*

The moves can be seen in Figures 1.2–1.5. All components are decorated by a $\pm$ coefficient indicating that we want to perform ± 1 contact surgery.



Figure 1.2. A cancelling pair.

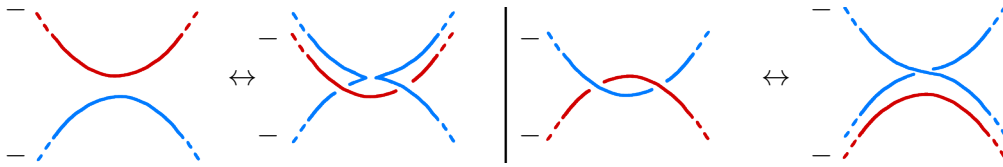


Figure 1.3. Two contact handleslides.

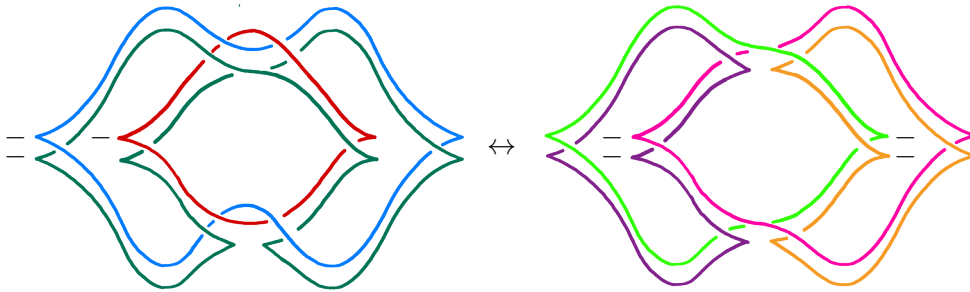


Figure 1.4. The standard lantern move.

Of course, this theorem can be stated for contact surgery diagrams by replacing “Legendrian isotopies” by “Legendrian Reidemeister moves”.

It is worth stressing that, unlike in the purely topological Kirby calculus, there is no obvious notion of blow up/down for contact surgery diagrams: contact ± 1 -surgery along a Legendrian unknot never gives (S^3, ξ_{st}) . For this reason, it might be hopeless to aim for a contact version of the Kirby theorem whose statement is as short as the topological one.

Many “contact Kirby moves” have appeared before in the literature. Cancelling pairs are a contact-geometric version of adding and removing a framed knot with a 0-framed meridian and go back to Ding–Geiges and Ding–Geiges–Stipsicz [DG04; DGS04]. Contact handleslides were also introduced and used by Ding–Geiges [DG09]. The standard chain move already appears in the work of Avdek [Avd13]. On the other hand, the standard chain move and the standard lantern move are particular cases of a new collection of moves that we introduce in Chapter 4, called *chain moves* and *lantern moves*. We also

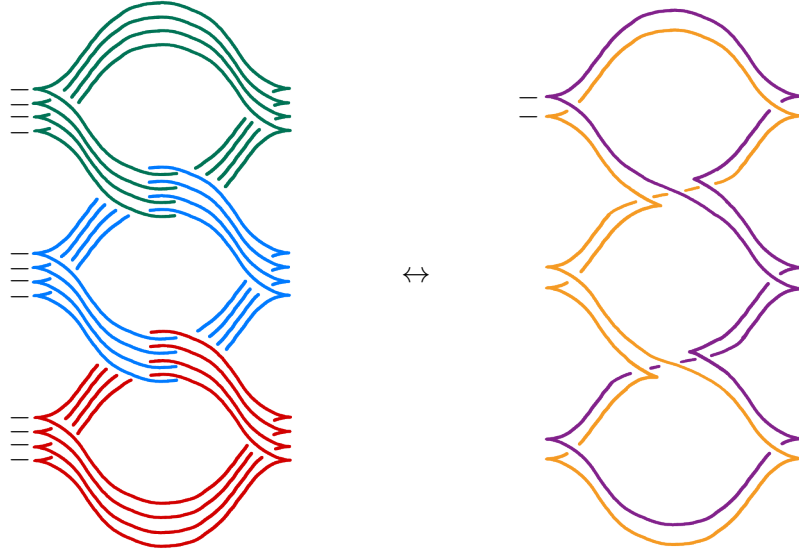


Figure 1.5. The standard chain move.

state a contact version of the Kirby theorem (Theorem 4.36) that includes also these moves, which is of course weaker because of this, but more useful for applications, as having more moves makes it easier to work with them.

The proof of Theorem 1.3 proceeds as follows. Starting from two contact surgery links in (S^3, ξ_{st}) that give the same contact 3-manifold, Avdek shows that there is a sequence of ribbon moves and Legendrian isotopies relating them; we can assume that they differ by a single ribbon move, and so, in a way that will be explained in Chapter 4, these links correspond to two factorizations of the same mapping class $\tau_L = \tau_{L'}$ in the mapping class group of the ribbon R .

After some manipulations of R , we can use a presentation of the mapping class group by Gervais (Theorem 2.6) to relate these two factorizations by a finite sequence of relations producing always homologically nontrivial curves. The Legendrian realization principle (Theorem 2.9) and Proposition 4.22 ensure that we can find a sequence of contact surgery links

$$\mathbf{L} = \mathbf{L}^0, \mathbf{L}^1, \dots, \mathbf{L}^k = \mathbf{L}'$$

such that two consecutive links $\mathbf{L}^i, \mathbf{L}^{i+1}$ produce factorizations that differ by a single relation.

By previous work understanding examples of contact surgery links differing by a ribbon move corresponding to a single relation and by a uniqueness result of Legendrian links embedded in open books adapted from [LOSS09], we conclude that, up to Legendrian isotopy, each $\mathbf{L}^i$ differs from $\mathbf{L}^{i+1}$ by either an addition or deletion of a cancelling pair, a contact handleslide, a lantern move, or a chain move.

Organization of the thesis

The thesis is divided into three chapters, in addition to this introduction.

Chapter 2. This chapter collects the necessary background and fixes notation. We recall the basic theory of mapping class groups of surfaces with boundary and state the presentation of the mapping class group by Gervais that will be used for the proof of Theorem 4.1. We then review the essentials of 3-dimensional contact topology: contact structures, Legendrian and transverse knots, front and Lagrangian projections, convex surfaces, and the Legendrian realization principle. Finally, we introduce open book decompositions and abstract open books, and we state the Giroux correspondence between contact structures and open books, both in its embedded and abstract formulations.

Chapter 3. This chapter contains the first main part of the work. We introduce ribbon surfaces of Legendrian graphs, first in a general contact 3-manifold and then specifically in $(\mathbb{R}^3, \xi_{\text{st}})$, and describe their characteristic foliations and handle decompositions. We then state and prove the algorithm to Legendrian realize a curve on a ribbon surface (Theorem 3.12), and we give an example of how to apply it. The final section applies Theorem 3.12 to obtain an algorithm that produces contact surgery diagrams out of abstract open book decompositions whose monodromy is given in terms of Dehn twists along homologically nontrivial curves (Corollary 3.25).

Chapter 4. This chapter is devoted to the contact version of the Kirby theorem. After fixing conventions, we study compatibility between Legendrian links and ribbon surfaces and provide many examples. We then review Avdek’s notion of ribbon move and his result that any two surgery diagrams of the same contact manifold are related by ribbon moves and Legendrian isotopies. Using Gervais’ presentation of the mapping class group, we define elementary R -equivalences and analyze how they can be realized as diagrammatic moves on surgery diagrams, introducing cancelling pairs, contact handleslides, lantern and chain moves and their standard representatives. Finally, we assemble these ingredients to prove the contact version of the Kirby theorem and its variant, thereby obtaining an explicit list of moves that relate any two contact surgery presentations of the same contact 3-manifold.

Preliminaries

In this chapter we recall basic notions and fix notation for mapping class groups, contact structures, and open book decompositions. Most of the material is standard in low-dimensional topology, but we include it to make the thesis self-contained and to establish conventions that will be used throughout. A few results (for instance Theorem 2.6) are perhaps less familiar, but they will play a central role later on.

2.1. Mapping class group

In this section we recall the definition of the mapping class group, we show some relations, and we state a presentation that will be useful in later chapters.

2.1.1. Basic definitions and conventions. Let Σ be a smooth, compact, oriented surface with non-empty boundary. We denote by $\text{Diff}^+(\Sigma, \partial\Sigma)$ the group of orientation-preserving diffeomorphisms of Σ that restrict to the identity on a neighborhood of the boundary, and by $\text{Diff}_0^+(\Sigma, \partial\Sigma)$ the connected component containing the identity. The *mapping class group* $\text{MCG}(\Sigma, \partial\Sigma)$ ¹ is the quotient

$$\text{MCG}(\Sigma, \partial\Sigma) := \text{Diff}^+(\Sigma, \partial\Sigma) / \text{Diff}_0^+(\Sigma, \partial\Sigma).$$

This group is generated by the classes of simple diffeomorphisms called *Dehn twists*. Throughout this thesis, we use Dehn twists extensively, but refer to [FM12] for relevant definitions and results.

We denote by $\tau_{\mathfrak{a}}^{\pm}$ the mapping class represented by a positive or negative Dehn twist along a simple closed curve $\mathfrak{a}$ in Σ . We continue to refer to this element as a *Dehn twist along $\mathfrak{a}$* .

2.1.2. Some relations in the mapping class group. We now recall a few standard relations among Dehn twists; for proofs of these facts, consult [FM12]. A basic observation is that Dehn twists along isotopic simple closed curves define the same element of $\text{MCG}(\Sigma, \partial\Sigma)$. Thus the subscript in $\tau_{\mathfrak{a}}^{\pm}$ can be regarded as the isotopy class of $\mathfrak{a}$, rather than the curve itself.

¹This is often called the *relative mapping class group* to differentiate it from the group obtained when considering no restrictions with respect to the boundary.

Example 2.1 (Cancelling Dehn twists). For any simple closed curve $\mathfrak{a}$ we have

$$\tau_{\mathfrak{a}}^+ \tau_{\mathfrak{a}}^- = \tau_{\mathfrak{a}}^- \tau_{\mathfrak{a}}^+ = 1.$$

In particular, $\tau_{\mathfrak{a}}^-$ is the inverse of $\tau_{\mathfrak{a}}^+$.

Example 2.2 (Commutativity). If $\mathfrak{a}$ and $\mathfrak{b}$ are disjoint simple closed curves, then the corresponding Dehn twists commute:

$$\tau_{\mathfrak{a}}^+ \tau_{\mathfrak{b}}^+ = \tau_{\mathfrak{b}}^+ \tau_{\mathfrak{a}}^+ \quad \text{if } \mathfrak{a} \cap \mathfrak{b} = \emptyset.$$

Now let $\mathfrak{a}$ and $\mathfrak{b}$ be simple closed curves on Σ that intersect transversely. We define a new curve $\mathfrak{ab}$ by smoothing each intersection point of the graph $\mathfrak{a} \cup \mathfrak{b}$ as follows: when traversing $\mathfrak{b}$ and approaching an intersection point, we turn onto the strand of $\mathfrak{a}$ lying to the right. See Figure 2.1.

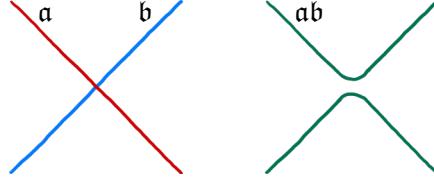


Figure 2.1. The curves $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{ab}$ near the intersection point $\mathfrak{a} \cap \mathfrak{b}$.

If $\mathfrak{a}$ and $\mathfrak{b}$ intersect transversely at a single point, then $\mathfrak{ab}$ is isotopic to $\tau_{\mathfrak{a}}^+(\mathfrak{b})$ (this fails in general when there is more than one intersection point).

Example 2.3 (Braid relations). Let $\mathfrak{a}$ and $\mathfrak{b}$ be simple closed curves intersecting transversely at a single point. Then

$$\tau_{\mathfrak{a}}^+ \tau_{\mathfrak{b}}^+ = \tau_{\mathfrak{ab}}^+ \tau_{\mathfrak{a}}^+.$$

This is a *braid relation*; see Figure 2.2.

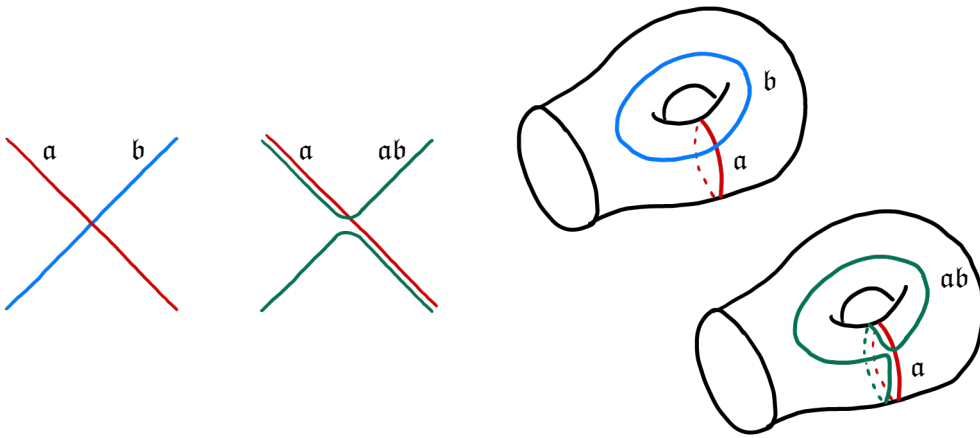


Figure 2.2. Left: the curves $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{ab}$ near their intersection point. Right: a neighborhood of $\mathfrak{a} \cup \mathfrak{b}$ with the three curves drawn on it.

Example 2.4 (Lantern relations). Let $\mathbf{a}$ and $\mathbf{b}$ be simple closed curves that intersect transversely in two points and algebraically zero times. Then

$$\tau_{\mathbf{a}}^+ \tau_{\mathbf{b}}^+ \tau_{\mathbf{ab}}^+ = \tau_{\partial_4}^+ \tau_{\partial_3}^+ \tau_{\partial_2}^+ \tau_{\partial_1}^+,$$

where $\partial_1, \partial_2, \partial_3, \partial_4$ are the boundary components of a regular neighborhood of $\mathbf{a} \cup \mathbf{b}$. This is a *lantern relation*; see Figure 2.3.

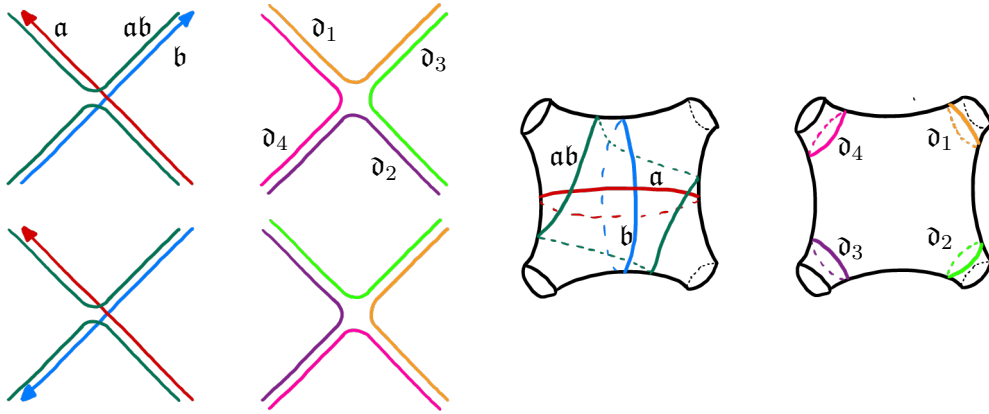


Figure 2.3. Left: the curves in a lantern relation near the two intersection points of $\mathbf{a}$ and $\mathbf{b}$ (which have opposite sign). Right: a neighborhood of $\mathbf{a} \cup \mathbf{b}$ with all the curves drawn.

Example 2.5 (Chain relations). Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be simple closed curves such that $\mathbf{a}$ and $\mathbf{b}$ intersect transversely in one point, $\mathbf{b}$ and $\mathbf{c}$ intersect transversely in one point, and $\mathbf{a} \cap \mathbf{c} = \emptyset$. Then

$$(\tau_{\mathbf{a}}^+ \tau_{\mathbf{b}}^+ \tau_{\mathbf{c}}^+)^4 = \tau_{\partial_2}^+ \tau_{\partial_1}^+,$$

where ∂_1 and ∂_2 are the boundary components of a regular neighborhood of $\mathbf{a} \cup \mathbf{b} \cup \mathbf{c}$. This is a *chain relation*; see Figure 2.4.

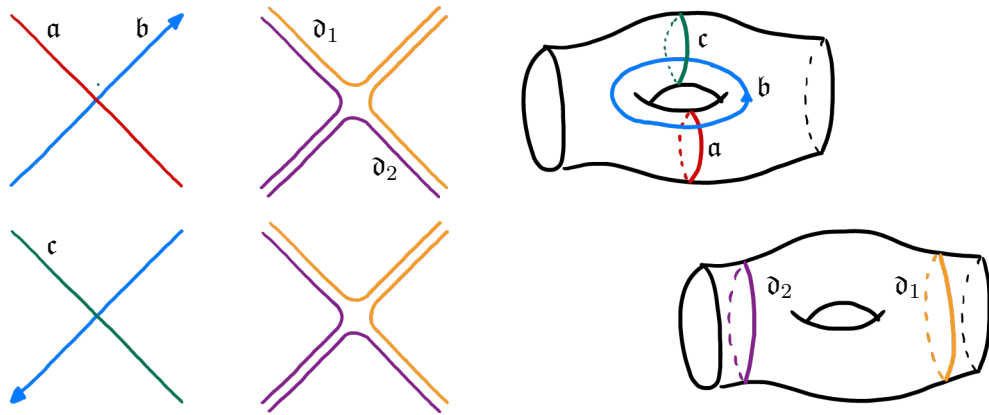


Figure 2.4. Left: the curves in a chain relation near the intersection points of $\mathbf{a}$ with $\mathbf{b}$ and of $\mathbf{b}$ with $\mathbf{c}$. Right: a neighborhood of $\mathbf{a} \cup \mathbf{b} \cup \mathbf{c}$ with the curves $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, ∂_1 , ∂_2 drawn.

2.1.3. A presentation of the mapping class group. We now state a convenient presentation of the mapping class group that will be used in Chapter 4. Recall that a simple closed curve is *nonseparating* if its complement is connected.

Theorem 2.6 ([Ger96, Theorem B]). *Let Σ be a compact, oriented surface of genus $g \geq 2$. Then the mapping class group $\text{MCG}(\Sigma, \partial\Sigma)$ admits the following presentation:*

- **Generators:** all Dehn twists $\tau_{\mathbf{a}}^{\pm}$ along nonseparating simple closed curves $\mathbf{a}$ in Σ .
- **Relations:**
 - (a) $\tau_{\mathbf{a}}^+ \tau_{\mathbf{a}}^- = \tau_{\mathbf{a}}^- \tau_{\mathbf{a}}^+ = 1$ for all $\mathbf{a}$ (cancelling Dehn twists).
 - (b) $\tau_{\mathbf{a}}^+ \tau_{\mathbf{b}}^+ = \tau_{\mathbf{b}}^+ \tau_{\mathbf{a}}^+$ whenever $\mathbf{a} \cap \mathbf{b} = \emptyset$ (commutativity relations).
 - (c) All braid relations between Dehn twists along curves intersecting in a single point.
 - (d) One lantern relation involving only Dehn twists along nonseparating curves.
 - (e) One chain relation involving only Dehn twists along nonseparating curves.

Here “one lantern/chain relation” means that we may choose any specific lantern/chain configuration of nonseparating curves and impose the corresponding relation; that single relation suffices in the presentation.

2.2. Contact structures

In this section we recall some definitions and well-known facts about 3-dimensional contact topology.

2.2.1. Basic definitions. A *contact form* on a smooth, oriented 3-manifold M is a 1-form $\alpha \in \Omega^1(M)$ such that

$$\alpha \wedge d\alpha > 0$$

with respect to the given orientation.¹ A *contact structure* ξ on M is a 2-plane field that can be written as the kernel of a contact form, i.e. $\xi = \ker \alpha$ for some α as above.²

On $\mathbb{R}^3$ we consider the *standard contact form*

$$\alpha_{\text{st}} := dz + x dy$$

and the associated *standard contact structure*

$$\xi_{\text{st}} := \ker \alpha_{\text{st}}.$$

On $S^3 \subset \mathbb{R}^4$ we take the *standard contact structure* to be

$$\xi_{\text{st}} = \ker(x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2)|_{S^3}.$$

These two contact structures are denoted by the same symbol because for any point $p \in S^3$ there is a contactomorphism

$$(S^3 \setminus \{p\}, \xi_{\text{st}}|_{S^3 \setminus \{p\}}) \cong (\mathbb{R}^3, \xi_{\text{st}})$$

(see [Gei08, Proposition 2.1.8]).

¹This is what is usually called a *positive* contact form.

²More precisely, this is a *positive and cooriented contact structure*; note that ξ is in particular a cooriented plane field.

More generally, every oriented 3-manifold admits a contact structure. This next result is due to Gromov for the open case and to Lutz and Martinet for the closed case. See also the discussion at the beginning of Chapter 4 of [Gei08].

Theorem 2.7 ([Gro69; Mar71; Lut71]). *In every homotopy class of coorientable plane fields on an oriented 3-manifold M there exists a positive contact structure. In particular, every oriented 3-manifold admits a contact structure.*

Given a contact form α , its *Reeb vector field* R_α is uniquely defined by

$$d\alpha(R_\alpha, -) \equiv 0, \quad \alpha(R_\alpha) \equiv 1.$$

For the standard form α_{st} the Reeb vector field is ∂_z . A *contact vector field* X for a contact structure ξ is a vector field whose flow preserves ξ . Every Reeb vector field is contact.

2.2.2. Legendrian and transverse. Let (M, ξ) be a contact 3-manifold. A smooth curve

$$r: I \longrightarrow M, \quad I \subset \mathbb{R} \text{ or } I = S^1,$$

is *Legendrian* if $r'(s) \in \xi_{r(s)}$ for all $s \in I$. Equivalently, if $\xi = \ker \alpha$, this means $\alpha(r') \equiv 0$. A smooth 1-dimensional submanifold $K \subset M$ (possibly with boundary and/or not properly embedded) is *Legendrian* if $T_p K \subset \xi_p$ for all $p \in K$, or equivalently if any local parametrization is a Legendrian curve.

In this thesis we will mostly consider Legendrian knots and links, usually denoted by L . A *Legendrian graph* $G \subset (M, \xi)$ is a properly embedded 1-dimensional CW-complex whose edges are Legendrian segments and such that at each vertex the edges meet with distinct oriented tangencies in ξ .

Two Legendrian graphs G_0 and G_1 in (M, ξ) are said to be *Legendrian isotopic* if there is a one-parameter family of Legendrian graphs $\{G_t\}_{t \in [0,1]}$ joining them and such that, at each vertex, the cyclic order of the incident edges in the contact plane ξ is preserved.

A knot $K \subset (M, \xi)$ is *transverse* if $T_p K$ is transverse to ξ_p for all $p \in K$. If K is oriented, we say it is a *positive transverse knot* if, at every point p , the orientation of $T_p K$ followed by the orientation of ξ_p agrees with the orientation of $T_p M$; otherwise K is a *negative transverse knot*. A *transverse link* is a link whose components are all transverse knots.

In $(\mathbb{R}^3, \xi_{\text{st}})$ there are two particularly useful ways to depict Legendrian and transverse objects: the *front projection* onto the (y, z) -plane and the *Lagrangian projection* onto the (x, y) -plane. We adopt the coordinate convention in Figure 2.5. In particular, when drawing front projections we implicitly use the coordinates in Figure 2.5 (center), and when drawing Lagrangian projections we use those in Figure 2.5 (right).

If $\star$ is a Legendrian (or transverse) object in $(\mathbb{R}^3, \xi_{\text{st}})$, we denote its front projection by $\star_F$ and its Lagrangian projection by $\star_L$.

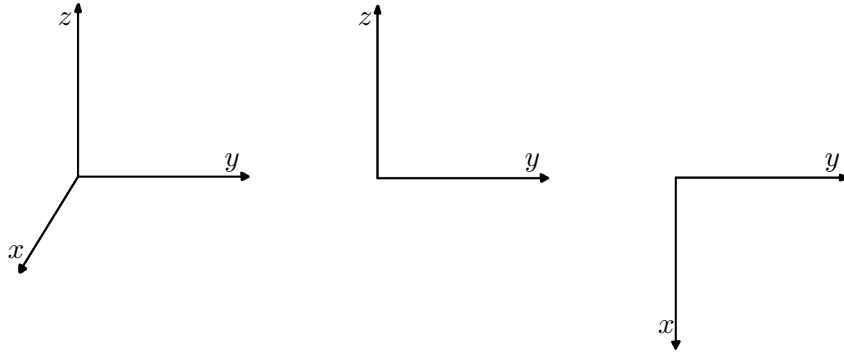


Figure 2.5. Left: coordinates on $\mathbb{R}^3$. Center: the front projection. Right: the Lagrangian projection.

2.2.3. The front projection of Legendrian graphs. Let $r = (r_x, r_y, r_z): I \rightarrow (\mathbb{R}^3, \xi_{\text{st}})$ be a Legendrian immersion, where $I \subset \mathbb{R}$ is an open subset or $I = S^1$. Its front projection is $r_F = (r_y, r_z)$. A point $r_F(s_0)$ is a *cuspid point* if $r'_y(s_0) = 0$, and a *crossing* if it is not a cusp and there are distinct $s, s' \in I$ with $r_F(s) = r_F(s')$. Any other point of the image of r_F has a neighborhood $\mathcal{U}$ in the (y, z) -plane where $(\mathcal{U}, r_F(I) \cap \mathcal{U})$ is modeled on Figure 2.6(a) (in this figure, what is important is that the segment is not vertical).

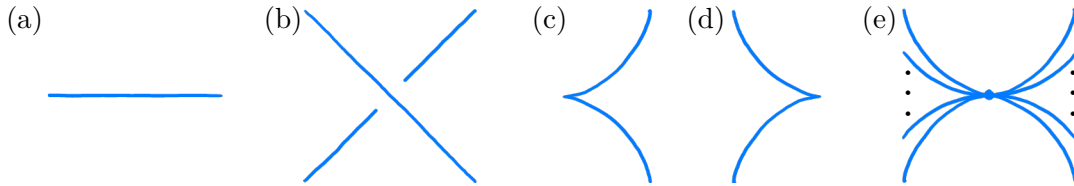


Figure 2.6. Local models for points in the front projection of a generic Legendrian graph.

If r is an embedding, we say that a crossing in the front projection is *generic* if it has a neighborhood $\mathcal{U}$ such that $(\mathcal{U}, \text{Im } r_F \cap \mathcal{U})$ looks like Figure 2.6(b): the two branches intersect transversely and neither is vertical.¹ A cusp point $r_F(s_0)$ is *generic* if s_0 is the only preimage and $r''_y(s_0) \neq 0$; in this case, if $\mathcal{U}$ is a neighborhood of $r_F(s_0)$, the couple $(\mathcal{U}, \text{Im } r_F \cap \mathcal{U})$ looks like Figure 2.6(c) or 2.6(d). A Legendrian curve r is *generic* if it is embedded and all its crossings and cusps are generic. Any Legendrian curve is Legendrian isotopic and arbitrarily C^∞ -close to a generic one (see [Gei08, Lemma 3.2.3]²).

Now let $G \subset (\mathbb{R}^3, \xi_{\text{st}})$ be a Legendrian graph. A point in the front projection G_F is a *vertex* if it is the image of a vertex of G . Its *cuspid points* and *crossings* are those points that are not vertices and arise as cusps or crossings of the interiors of the edges. A point of G_F that is neither a vertex, a crossing, nor a cusp still looks like Figure 2.6(a); a generic crossing still looks like Figure 2.6(b); and a generic cusp still looks like Figure 2.6(c) or (d).

¹In the figure, part of a segment is omitted to emphasize that the branch with smaller slope in the projection corresponds to a segment with larger x -coordinate in $\mathbb{R}^3$.

²Beware that the terminology for cusp points in that reference differs slightly from ours.

A vertex of G_F is *generic* if it is the image of a unique vertex $v \in G$ and none of the incident edges are tangent to ∂_x at v . A generic vertex has a neighborhood $\mathcal{U}$ in the (y, z) -plane where the couple $(\mathcal{U}, G_F \cap \mathcal{U})$ looks like Figure 2.6(e): there are $n \geq 0$ edges coming from the left and $m \geq 0$ from the right, and the tangency at the vertex is nonvertical.¹ A Legendrian graph is *generic* if all its vertices, crossings and cusps are generic. As before, any Legendrian graph is Legendrian isotopic and arbitrarily C^∞ -close to a generic one.

Conversely, any planar graph in the (y, z) -plane whose points have neighborhoods modeled on Figure 2.6(a)–(e) lifts uniquely to a (generic) Legendrian graph in $(\mathbb{R}^3, \xi_{\text{st}})$.²

2.2.4. Legendrian Reidemeister moves for Legendrian graphs. It is well known that the front projections of Legendrian isotopic Legendrian links in $(\mathbb{R}^3, \xi_{\text{st}})$ are related by the Reidemeister moves I, II, III in Figure 2.7.

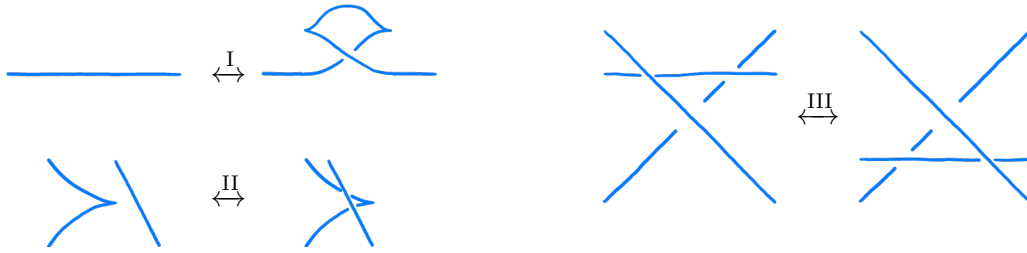


Figure 2.7. Legendrian Reidemeister moves for Legendrian links. Horizontal and vertical reflections of these moves are also allowed (with crossings adjusted appropriately).

For Legendrian graphs, generic front projections of Legendrian isotopic graphs are related by the moves I, II, III together with the additional moves II_G and R shown in Figure 2.8 (see [BI09, Proposition 4.2]).



Figure 2.8. Additional Legendrian Reidemeister moves for Legendrian graphs. Horizontal and vertical reflections of these moves are again allowed.

¹The dot at the center indicates the presence of a vertex.

²To ensure smoothness at cusps one needs a behavior close to that of a semicubical cusp; see again [Gei08, Lemma 3.2.3] or [Etn05]. We will always tacitly assume our drawings satisfy this.

2.2.5. The Lagrangian projection of Legendrian graphs. The Lagrangian projection r_L of a Legendrian immersion r is a curve immersed in the (x, y) -plane. The z -coordinate of r can be recovered from r_L and the z -coordinate of a single point $r(s_0)$ via

$$r_z(s) = r_z(s_0) - \int_{s_0}^s r_x(u) r'_y(u) du.$$

In particular, if r is a closed Legendrian curve, then

$$\oint_{r_L} x dy = 0.$$

Moreover, r is embedded if and only if every loop in r_L (except, in the closed case, the full loop r_L itself) encloses a nonzero oriented area.

Conversely, any immersed curve in the (x, y) -plane is the Lagrangian projection of a Legendrian curve in $(\mathbb{R}^3, \xi_{\text{st}})$, unique up to translation in the z -direction. A closed immersed curve r_L lifts to a Legendrian immersion of S^1 if and only if $\oint_{r_L} x dy = 0$.

Figure 2.9 shows the Lagrangian projections corresponding to the local models in Figure 2.6.

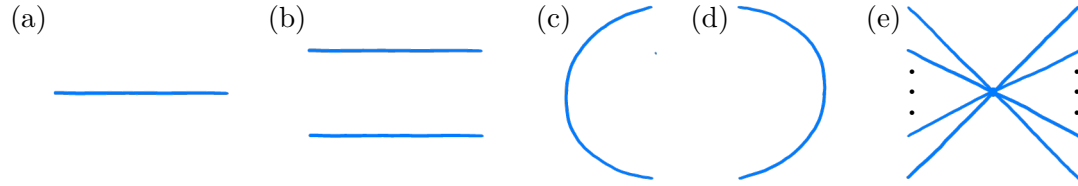


Figure 2.9. Lagrangian projections corresponding to the front projections in Figure 2.6. The y -coordinate agrees with that of the front projection, and the x -coordinate parametrizes the vertical direction (top to bottom is positive).

2.2.6. Convex surfaces. Given an oriented surface S in a contact manifold (M, ξ) , we denote by S_ξ its characteristic foliation. This is a singular foliation in S defined by the distribution in TS obtained by considering $T_p S \cap \xi_p$, at the points $p \in S$ where $T_p S \neq \xi_p$, and by considering $\{0\} \subset T_p S$ otherwise. When nonzero, the characteristic foliation is oriented by the following convention: if (v_1, v_2) is a positive basis for $T_p S$ with $v_1 \in \xi_p$, then we want v_2 to be positively transverse to ξ_p .

We say that a surface S is *convex* if there exists a contact vector field X transverse to S . Equivalently, S is convex if it admits a *vertically invariant* neighborhood: there exists a neighborhood $S \times (-\varepsilon, \varepsilon) \subset M$ of $S \cong S \times \{0\}$ in which the contact structure is invariant under translation in the second factor. In suitable coordinates the contact structure is then given as the kernel of a 1-form

$$\beta + u ds,$$

where $\beta \in \Omega^1(S)$, $u \in C^\infty(S)$, and $s \in (-\varepsilon, \varepsilon)$ is the coordinate in the second factor. The contact condition implies that $d\beta$ is a positive area form on the subset of S where $u \neq 0$.

To transition between these descriptions, note that if X makes S convex, then flowing S along X produces a vertically invariant neighborhood. Conversely, in a vertically invariant neighborhood the vector field ∂_s is a contact vector field transverse to S .

The *dividing set* Γ_S (or simply Γ) is the set of points where X lies in ξ ; in the vertically invariant model, this is precisely the zero set of u . If S is closed, Γ is a multicurve that separates S into positive and negative regions. In this thesis, however, we are primarily interested in convex surfaces with transverse boundary, for which there exists a contact vector field X that makes the dividing set Γ_X either

- Γ_S coincides with ∂S , or
- Γ_S is disjoint from ∂S (in this case, Γ_X is again a separating multicurve).

The dividing set Γ_S depends on the choice of contact vector field X . However, if S is closed and Y is another contact vector field transverse to S , then the dividing set induced by Y is isotopic to Γ_S through multicurves always transverse to the characteristic foliation S_ξ . Conversely, any multicurve obtained from Γ_S by such an isotopy arises as the dividing set for some transverse contact vector field. When S has boundary, the situation is similar but slightly more subtle.¹ In what follows we will always assume that, for surfaces with boundary, the dividing set either coincides with the boundary or is disjoint from it.

Before stating Giroux's flexibility theorem, we introduce a notion of a foliation divided by a multicurve. A singular foliation $\mathcal{F}$ on S is *divided* by a multicurve Γ if:

- Γ is transverse to $\mathcal{F}$, and
- there exists an area form ω on S and a vector field v defining $\mathcal{F}$ such that
 - $\mathcal{L}_v \omega \neq 0$ on $S \setminus \Gamma$, and
 - if $S_\pm := \{p \in S \mid \pm \operatorname{div}_\omega(v) > 0\}$, then v points out of S_+ and into S_- along Γ .

If S is convex, then its characteristic foliation S_ξ is divided by the dividing set Γ_S .

The closed case of the next result was proven by Giroux. A version for surfaces with nonempty transverse boundary and dividing set disjoint from the boundary is stated in [EV11, Theorem 2.2]. Our setting (boundary equal to the dividing set) is treated by a minor adaptation of Giroux's original proof.

Theorem 2.8 (Giroux's flexibility theorem [Gir91; EV11]). *Let S be a convex surface in a contact manifold (M, ξ) with respect to a contact vector field X . If S has boundary, assume that ∂S coincides with the dividing set Γ_S . Let $\mathcal{F}$ be a singular foliation on S divided by Γ_S , and let $\mathcal{N}(S)$ be any neighborhood of S of the form $S \times (-\varepsilon, \varepsilon)$ (where $X = \partial_z$, and z is the coordinate in the second factor) in M . Then there exists an isotopy*

$$\psi_t: S \longrightarrow \mathcal{N}(S), \quad t \in [0, 1],$$

of embeddings with the following properties:

- ψ_0 is the inclusion $S \hookrightarrow M$.

¹For example, suppose ∂S is the dividing curve for some X . Gluing two copies of a vertically invariant neighborhood $S \times (-\varepsilon, \varepsilon)$ along $\partial S \times (-\varepsilon, \varepsilon)$ yields a vertically invariant neighborhood of a closed surface $\tilde{S}$ with dividing set ∂S . Any dividing set on $\tilde{S}$ restricts to a possible dividing set on one copy of S , but this restriction need not be isotopic to ∂S relative to the boundary.

- For all $t \in [0, 1]$, the surface $\psi_t(S)$ is convex with respect to X and has dividing set $\psi_t(\Gamma_S)$.
- The characteristic foliation $(\psi_1(S))_\xi$ coincides with the $\psi_1(\mathcal{F})$.

Moreover, if $\mathcal{F}$ coincides with S_ξ on a neighborhood of Γ_S in S , then the isotopy may be chosen fixed on a small enough neighborhood of Γ_S (so, in particular, $\Gamma_{\psi_t(S)} = \Gamma_S$).

Now let S be a convex surface, and in the case $\partial S \neq \emptyset$ assume again that $\partial S = \Gamma_S$. We say that a graph $C \subset S$ is *nonisolating* if:

- C is properly embedded in S ,
- C is disjoint from ∂S , and
- C is transverse to the dividing set Γ_S (when they intersect).

In our applications the dividing set will often coincide with the boundary, so the last condition is just says that C is disjoint from Γ_S .¹

Theorem 2.9 (Legendrian realization principle [Hon00; Etn04a]). *Let S be a convex surface in a contact manifold (M, ξ) with respect to a contact vector field X . If S has boundary, assume that ∂S coincides with the dividing set Γ_S . Let $C \subset S$ be a nonisolating graph, and let $\mathcal{N}(S)$ be any neighborhood of S of the form $S \times (-\varepsilon, \varepsilon)$ (where $X = \partial_z$, and z is the coordinate in the second factor) in M . Then there exists an isotopy*

$$\psi_t: S \longrightarrow \mathcal{N}(S), \quad t \in [0, 1],$$

of embeddings such that:

- ψ_0 is the inclusion $S \hookrightarrow M$, and ψ_t is fixed on a neighborhood of Γ_S .
- For all $t \in [0, 1]$, the surface $\psi_t(S)$ is convex with respect to X and has dividing set $\Gamma_{\psi_t(S)} = \Gamma_S$.
- The graph $\psi_1(C) \subset \psi_1(S)$ is Legendrian.

Proof. The case where S is closed and C is a multicurve is proved in [Hon00, Theorem 3.7].² The case of closed S and general graphs C is discussed in [Etn04a, Theorem 2.28]. For the case where S has boundary and ∂S agrees with the dividing set Γ_S , we find (as done in [Etn04a, Theorem 2.28]) a singular foliation $\mathcal{F}$ in S divided by Γ_S containing C as a collection of leaves and singular points and matching S_ξ along a neighborhood of $\Gamma_S = \partial S$. Then we use Theorem 2.8. $\square$

2.2.7. Foundational results for contact 3-manifolds. The following collection of results constitutes the core of the foundational theory of contact topology in dimension three. Together, they show that contact structures admit no local invariants, and in many fundamental respects behave like ordinary smooth manifolds.

The statements are formulated here in the way is done in [Gei08].

¹This is the definition of nonisolating that works for us. The definition for convex surfaces with Legendrian boundary or for convex surfaces with transverse boundary and dividing set disjoint from the boundary is different.

²That reference also treats the case of convex surfaces with Legendrian boundary and a collection of properly embedded curves and arcs.

Theorem 2.10 (Darboux’s theorem [Dar82]). *Let α be a contact form on a 3-manifold M , and let $p \in M$ be any point. Then there exist local coordinates (x, y, z) on a neighborhood U of p , with $p = (0, 0, 0)$, such that*

$$\alpha|_U = dz + x dy.$$

In particular, every contact structure is locally diffeomorphic to the “standard” contact structure on $(\mathbb{R}^3, dz + x dy)$.

Theorem 2.11 (Gray’s stability theorem [Gra59]). *Let $\{\xi_t\}_{t \in [0,1]}$ be a smooth family of contact structures on a closed manifold M . Then there exists an isotopy $\{\psi_t\}_{t \in [0,1]}$ of M such that*

$$T\psi_t(\xi_0) = \xi_t \quad \text{for all } t \in [0, 1].$$

In other words, on a closed manifold, contact structures cannot be continuously deformed into nonequivalent ones: any smooth path of contact structures stays within a single equivalence class up to isotopy.

The proof is an application of the famous *Moser trick* to a path of contact forms α_t such that $\ker \alpha_t = \xi_t$. One looks for an isotopy ψ_t as the flow of a time-dependent vector field v_t on M , chosen so that $v_t \in \xi_t$ for all t . The fact that this is possible, and that one can integrate v_t to find the desired ψ_t , relies crucially on the condition that each ξ_t is a contact structure. The resulting isotopy depends uniquely on the path of contact forms α_t .

Theorem 2.12 (Neighborhood theorem for Legendrian graphs). *Let (M_i, ξ_i) , $i = 0, 1$, be contact 3-manifolds, and suppose there are diffeomorphic compact Legendrian graphs $G_i \subset (M_i, \xi_i)$. Then there exist neighborhoods of G_0 and G_1 that are contactomorphic.*

This shows that any two embeddings of the same Legendrian graph are locally the same up to contactomorphism.

Theorem 2.13 (Giroux’s surface neighborhood theorem [Gir91]). *Let S_i be closed surfaces embedded in contact 3-manifolds (M_i, ξ_i) , for $i = 0, 1$, and let*

$$\phi : S_0 \rightarrow S_1$$

be a diffeomorphism which identifies the characteristic foliation induced by ξ_0 on S_0 with that induced by ξ_1 on S_1 (with orientation). Then there is a contactomorphism of neighborhoods

$$\psi : \mathcal{N}(S_0) \rightarrow \mathcal{N}(S_1),$$

sending S_0 to S_1 , and such that on S_0 the restriction $\psi|_{S_0}$ is isotopic to ϕ via an isotopy preserving the characteristic foliation.

Thus the germ of a contact manifold along a surface is determined (up to contactomorphism) by the characteristic foliation on the surface. This gives a powerful tool to reconstruct (or compare) contact structures from easily visualizable data on surfaces.

Turning now to submanifolds and their isotopies, one obtains the following:

Theorem 2.14 (Legendrian isotopy extension theorem). *Let $\{j_t\}_{t \in [0,1]} : L \rightarrow (M, \xi)$ be a Legendrian isotopy of a Legendrian link $L \subset M$. Then there exists a compactly supported contact isotopy $\{\psi_t\}$ of (M, ξ) such that*

$$\psi_t \circ j_0 = j_t$$

for all t .

This result says that any isotopy of a Legendrian link can be extended to a global contact isotopy: from the contact-geometric point of view, Legendrian links behave like submanifolds of smooth manifolds under isotopy.

Theorem 2.15 (Contact disk theorem). *Any two contact embeddings j_i , $i = 0, 1$, of the 3-ball B^3 into a connected contact 3-manifold (M, ξ) are contact isotopic.*

In particular, small Darboux balls are unique up to contact isotopy: hence the contact structure on a ball is canonical, and there is essentially only one “local model” for a contact 3-manifold. This complements Darboux’s theorem and ensures that local contact geometry is uniquely trivial.

Finally, one has a version of the isotopy extension theorem for embedded surfaces:

Theorem 2.16 (Isotopy extension for surfaces with fixed characteristic foliation). *Let $j_t : S \rightarrow (M, \xi)$, $t \in [0, 1]$, be an isotopy of embeddings of a closed surface S . If for all t , the embedding j_t induces the same characteristic foliation on S , then there exists a compactly supported isotopy $\psi_t : M \rightarrow M$ such that*

$$\psi_t \circ j_0 = j_t.$$

Thus, when the characteristic foliation on a surface is kept fixed, the embedded surface moves only by an ambient contact isotopy, reinforcing the idea that the induced foliation captures exactly the relevant contact-geometric data up to isotopy.

2.2.8. Contact surgery. Let L be a Legendrian knot in a contact manifold (M, ξ) . A standard neighborhood $\mathcal{N}(L)$ of L is a solid torus whose boundary $\partial\mathcal{N}(L)$ is a convex surface. The dividing set Γ on $\partial\mathcal{N}(L)$ consists of two parallel curves, and their slope defines a natural framing of L , called the *contact framing*.

Remove the interior of $\mathcal{N}(L)$ from M and glue back a solid torus H by performing $(\pm 1/n)$ -surgery, $n \in \mathbb{N}$, with respect to the contact framing. Denote the resulting manifold by M' . There is a unique way to extend $\xi|_{M \setminus \mathring{\mathcal{N}}(L)}$ to a contact structure on M' such that the restriction to H is tight.¹ This operation is called *contact $(\pm 1/n)$ -surgery* on L and is well defined up to contactomorphism, i.e. it does not depend on the choice of the standard neighborhood $\mathcal{N}(L)$.

In this thesis we will only consider contact (± 1) -surgery. Thus, for us a *contact surgery link* is a Legendrian link whose components are decorated with signs $\pm$ indicating that we perform contact (± 1) -surgery on those components. If L is a Legendrian knot, we write $L^\pm$ for L equipped with the corresponding surgery coefficient. A *contact surgery diagram* is the front projection of a contact surgery link.

We will adopt the following convention: bold symbols such as $\mathbf{L}$ denote contact surgery links, while the corresponding plain symbol L denotes the underlying Legendrian link. In other words,

$$\mathbf{L} = \bigcup_{i=1}^n L_i^{\delta_i}, \quad \delta_i \in \{+, -\}.$$

¹On $\partial H = \partial\mathcal{N}(L)$ the contact structure induces a singular foliation divided by two curves intersecting each meridian once. By a result of Honda [Hon00, Theorem 2.3], there is a unique (up to isotopy relative to the boundary) tight contact structure on H realizing this boundary condition.

2.3. Open book decompositions

Let M be a closed, oriented 3-manifold. An *open book* on M is a pair (B, π) where:

- $B \subset M$ is a link, called the *binding*;
- $\pi: M \setminus B \rightarrow S^1$ is a fibration which coincides with the angular coordinate on a neighborhood $B \times D^2$ of $B = B \times \{0\}$.

The closures of the fibers of π are called the *pages* of the open book; these are Seifert surfaces for B . The orientations of M and S^1 induce an orientation on each page and hence on the binding B . We always regard this orientation of B as part of the data of (B, π) .

Two open books (B, π) on M and (B', π') on M' are *diffeomorphic* if there exists an orientation-preserving diffeomorphism $\phi: M \rightarrow M'$ such that $\phi(B) = B'$ and, on $M \setminus B$, we have $\pi' \circ \phi = \pi$. If both open books live on the same manifold M , we say that (B, π) and (B', π') are *isotopic* if they are diffeomorphic via a diffeomorphism isotopic to the identity.

An *abstract open book* is a pair (Σ, ϕ) where:

- Σ is a compact, oriented surface with nonempty boundary;
- $\phi \in \text{Diff}^+(\Sigma, \partial\Sigma)$ is a diffeomorphism that is the identity near $\partial\Sigma$.

Two abstract open books (Σ, ϕ) and (Σ', ϕ') are *equivalent* if there exists an orientation-preserving diffeomorphism $\psi: \Sigma \rightarrow \Sigma'$ such that ϕ is isotopic (through diffeomorphisms equal to the identity near the boundary) to $\psi^{-1} \circ \phi' \circ \psi$.

The relationship between open books and abstract open books is well known; see, for example, [Etn06]. We summarize the key facts:

- Starting from an open book (B, π) on M , choosing a page Σ determines an abstract open book (Σ, ϕ) , well defined up to isotopy of ϕ through diffeomorphisms fixed near $\partial\Sigma$.
- Choosing a different page Σ' of (B, π) produces an abstract open book that is equivalent to (Σ, ϕ) .
- Conversely, given an abstract open book (Σ, ϕ) , one can construct a 3-manifold equipped with an open book; this manifold is well defined up to diffeomorphism.
- Equivalent abstract open books give rise to diffeomorphic 3-manifolds equipped with diffeomorphic open books.

In particular, abstract open books up to equivalence correspond to open books up to diffeomorphism. Note, however, that isotopy of open books on a fixed manifold is a finer relation than diffeomorphism in general.

2.4. The Giroux correspondence

We conclude this chapter by recalling the fundamental correspondence between contact structures and open books.

Let (M, ξ) be a contact 3-manifold and (B, π) an open book decomposition of M . We say that ξ is *compatible with* (B, π) (or that (B, π) *supports* ξ) if there exists a contact form α for ξ such that:

- the binding B is a positive transverse link, and
- $d\alpha$ restricts to a positive area form on the interior of every page.

Equivalently, the Reeb vector field R_α is required to be positively tangent to B and positively transverse to the interior of the pages. It is easy to see that these two formulations are equivalent: a contact form satisfies one set of conditions if and only if it satisfies the other. From this characterization it follows in particular that the interior of each page is a convex surface. It is also true that the pages themselves are convex, with dividing sets that can be assumed to coincide with the boundary.

The following result, originally formulated in [Gir02], is of immense importance in 3-dimensional contact topology. For a gentle introduction to the theorem and related definitions (e.g., positive stabilization), see [Etn06].

Theorem 2.17 (Giroux’s correspondence [Gir02; BHH24; LV24a; LV24b]). *Let M be a closed, oriented 3-manifold. There is a one-to-one correspondence between positive cooriented contact structures on M up to isotopy and open books on M up to isotopy and positive stabilization.*

In Theorem 2.17, the correspondence associates to an open book (B, π) a contact structure compatible with it as the one constructed in [TW75].

It is often convenient to phrase the correspondence in terms of abstract open books and contact manifolds up to contactomorphism:

Theorem 2.18 (Giroux correspondence, abstract version). *There is a one-to-one correspondence between contact 3-manifolds up to contactomorphism and abstract open books up to equivalence and positive stabilization.*

An algorithm to Legendrian realize a curve on a ribbon surface

Ribbon surfaces are compact surfaces obtained by thickening a Legendrian graph in the direction of the contact structure. They are naturally convex, with dividing set equal to their boundary so on these surfaces, homologically nontrivial curves are precisely those eligible for Legendrian realization. The classical Legendrian realization principle guarantees the existence of Legendrian representatives of such curves after a suitable perturbation of the surface, but it is by nature a nonconstructive result. Our goal is to replace this soft existence statement by a concrete algorithm (Theorem 3.12) that provides the front projection of such Legendrian realization when working with in the standard contact structure $(\mathbb{R}^3, \xi_{\text{st}})$.

Along the way, we complete an argument from [LOSS09] and obtain a corresponding uniqueness statement. Namely, we show that any two Legendrian realizations of the same curve on a ribbon surface are Legendrian isotopic, and likewise that any two Legendrian knots lying on pages of open books compatible with the same contact structure and representing the same isotopy class on the page are Legendrian isotopic. Precise formulations are given in Theorem 3.6 and Theorem 3.10.

At the end of this chapter, in Section 3.7, we show that Theorem 3.12 yields an explicit algorithm that, starting from an abstract open book whose monodromy is written as a product of Dehn twists along homologically nontrivial curves, produces a contact surgery diagram of the contact manifold supported by the given open book (Corollary 3.25).

3.1. Ribbon surfaces

Throughout this chapter, G denotes a Legendrian graph with no vertices of valency 1.

Definition 3.1. Let G be a Legendrian graph in a contact manifold $(, \xi)$. A *ribbon surface*¹ of G is a compact oriented surface $\Sigma \subset$ such that:

- $G \subset \Sigma$.
- There exists a contact form α for $(, \xi)$ whose Reeb vector field R_α is positively transverse to Σ .
- There exists a vector field v on Σ that directs the characteristic foliation of Σ , is positively transverse to $\partial\Sigma$, and whose time- t flow Φ_v^t satisfies

$$\bigcap_{t \in (0, \infty)} \Phi_v^{-t}(\Sigma) = G.$$

The graph G is called the *Legendrian skeleton* of Σ .

An example of a ribbon surface is shown in Figure 3.1.²

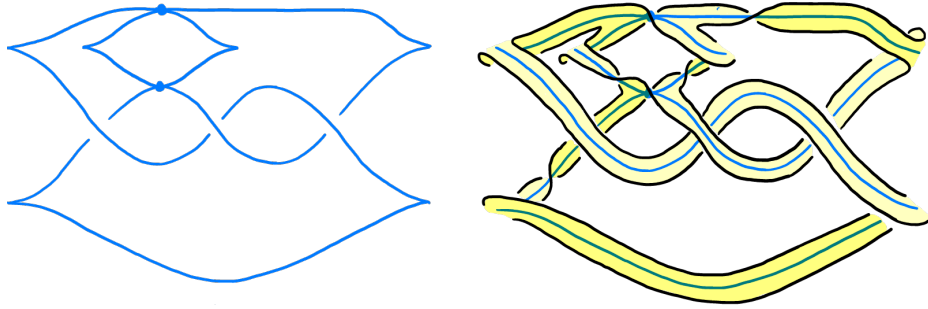


Figure 3.1. Left: a Legendrian graph in $(\mathbb{R}^3, \xi_{st})$. Right: a ribbon surface of the Legendrian graph.

Every Legendrian graph G admits a ribbon surface: one can take a sufficiently small “thickening” of G in the direction of the contact structure. Any such ribbon is automatically convex, since it is transverse to a Reeb vector field. However, we can say more about this.

Lemma 3.2. *Let Σ be a ribbon surface. Then Σ is convex with respect to a contact vector field whose dividing set Γ_Σ coincides with the boundary $\partial\Sigma$.*

Thus, from now on a ribbon surface will always be regarded as a convex surface with dividing set equal to its boundary.

¹Giroux first introduced such surfaces in [Gir02] (without giving them a name). The formulation here is taken from [Avd13, Definition 2.2].

²It will become clear later why this is a ribbon surface: it arises from an algorithm that constructs ribbon surfaces from Legendrian graphs, together with an application of the cancellation lemma.

Proof of Lemma 3.2. Use the notation of Definition 3.1. Flowing Σ along the Reeb vector field R_α produces a vertically invariant neighborhood $\Sigma \times [-\varepsilon, \varepsilon]$ in which

$$\alpha = \beta + ds,$$

with $\beta \in \Omega^1(\Sigma)$, and $s \in [-\varepsilon, \varepsilon]$. We have that

$$d\alpha|_\Sigma = d\beta =: \omega,$$

and ω is a positive area form on Σ since R_α is positively transverse to Σ . Up to rescaling the vector field v defining the characteristic foliation, we may assume that $\beta = i_v d\beta$. Then

$$\operatorname{div}_{d\beta}(v) d\beta = d(i_v d\beta) = d\beta,$$

so $\operatorname{div}_{d\beta}(v) = 1 > 0$ everywhere. Together with the fact that $\partial\Sigma$ is a positive transverse link, this implies that the characteristic foliation of Σ is divided by the multicurve $\partial\Sigma$.

The proof of [Gir91, Proposition II.2.1] shows that we can find a contact vector field transverse to Σ which induces $\partial\Sigma$ as dividing set. For completeness, we reproduce the argument here (in the form adapted from [Gei08, Theorem 4.8.5]).

Consider the 1-form $\tilde{\alpha}$ on $\Sigma \times [0, 1]$ given by

$$\tilde{\alpha} = \beta + \tilde{u} ds,$$

where $\tilde{u} \in C^\infty(\Sigma \times [-\varepsilon, \varepsilon])$ is a function to be chosen. The condition for $\tilde{\alpha}$ to be a contact form is

$$\tilde{u} \operatorname{div}_\omega(v) - v(\tilde{u}) > 0,$$

which in our case reduces to

$$\tilde{u} - v(\tilde{u}) > 0,$$

since $\operatorname{div}_\omega(v) = 1$.

Now use the flow of $-v$ to identify a collar neighborhood of $\partial\Sigma$ with $\partial\Sigma \times [0, \varepsilon]$. In these coordinates, the parameter $t \in [0, \varepsilon]$ is such that $v = -\partial_t$.

Define $\tilde{u}$ to be equal to 1 outside $\partial\Sigma \times [0, \varepsilon]$. In this region the contact condition for $\tilde{\alpha}$ is clearly satisfied.

On $\partial\Sigma \times [0, \varepsilon]$, choose $\tilde{u}$ of the form

$$\tilde{u}(t) = g(t)e^{-t}.$$

Then

$$\tilde{u} - v(\tilde{u}) = \tilde{u} + \frac{\partial \tilde{u}}{\partial t} = \frac{\partial g(t)}{\partial t} e^{-t},$$

so the contact condition is satisfied provided we choose g with $\partial g / \partial t > 0$.

We can easily find a smooth function g with this property, and we may further assume that $g = 1/e^{-t}$ near $t = \varepsilon$ (so that $\tilde{u}$ matches the value 1 outside $\partial\Sigma \times [0, \varepsilon]$) and that $g = 0$ along $\partial\Sigma \times \{0\}$. The latter condition implies that $\tilde{u} = 0$ precisely on $\partial\Sigma \times \{0\}$. Hence ∂_s is a contact vector field for the contact form $\tilde{\alpha}$, transverse to Σ and inducing $\partial\Sigma$ as dividing set.

To conclude, note that $\ker \tilde{\alpha}$ induces on Σ the same characteristic foliation as $\ker \alpha$. By Theorem 2.13, this implies that Σ is also convex with respect to $\ker \alpha$, with dividing set equal to $\partial\Sigma$ for a suitable contact vector field. $\square$

A ribbon surface Σ of a Legendrian graph G carries a natural handle decomposition (which depends on G). The 0–handles are the thickenings of pairwise disjoint regular closed neighborhoods of the vertices of G , and the 1–handles are the connected components of the closure of the complement of the union of the 0–handles. See Figure 3.2.

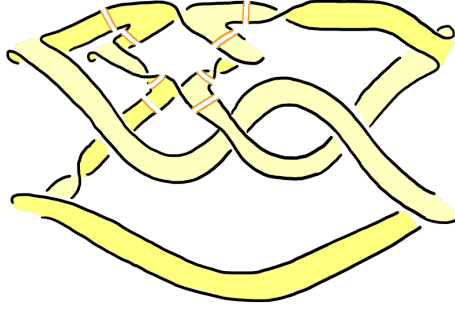


Figure 3.2. The handle decomposition of the ribbon from Figure 3.1.

A ribbon surface is called *generic*¹ if its characteristic foliation has the following form:

- every 0–handle contains exactly one singular point, which is a positive elliptic point;
- each 1–handle contains finitely many singular points, all of which are positive elliptic or positive hyperbolic.

In practice, the characteristic foliation on the handles of a generic ribbon looks as in Figure 3.3.

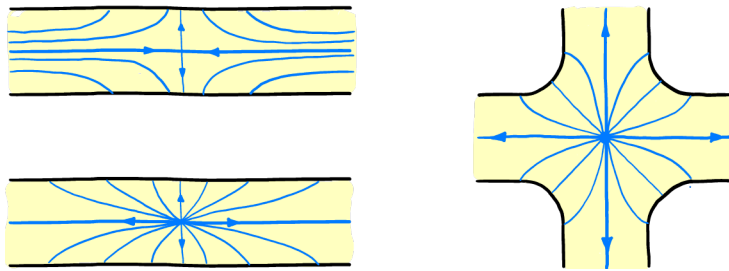


Figure 3.3. Left: the characteristic foliation on a 1–handle of a generic ribbon, obtained by concatenating pieces like those at the top-left and bottom-left. Right: the characteristic foliation on a 0–handle. Here the vertex has valency 4, but the general case is analogous.

Since this foliation is divided by $\partial\Sigma$, Giroux’s flexibility Theorem 2.8 implies that any ribbon can be perturbed to a generic one.

¹This is non-standard terminology, adopted here for convenience.

3.1.1. Ribbon surfaces in the standard contact structure. We now specialize to $(\mathbb{R}^3, \xi_{\text{st}})$. Given the front projection of a generic Legendrian graph $G \subset (\mathbb{R}^3, \xi_{\text{st}})$, there is an explicit procedure (adapted from [Avd13, Algorithm 2]) for drawing the front projection of a ribbon surface Σ for G .

For each piece of the front projection of G as in Figure 2.6, we associate a surface as indicated in the top row of Figure 3.4.

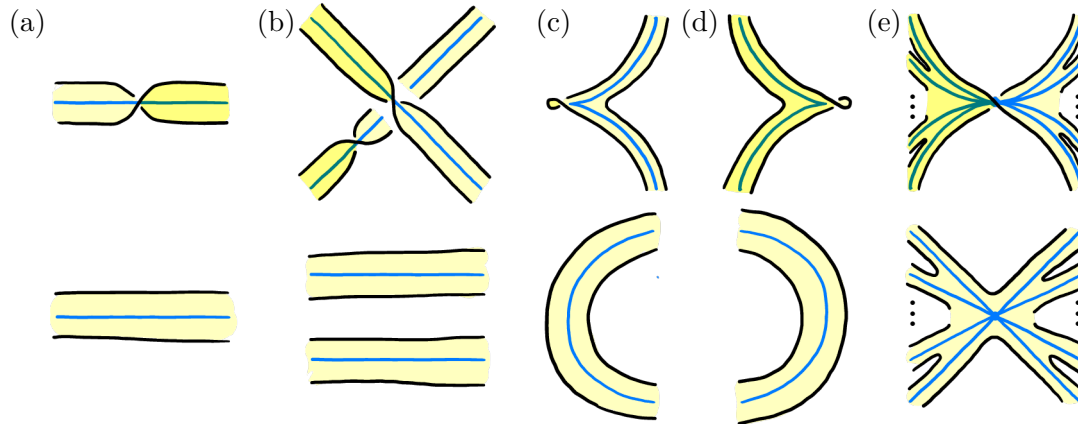


Figure 3.4. Top row: front projection of the ribbon near a non-vertical edge, a crossing, cusps, and a vertex. Bottom row: corresponding Lagrangian projections.

If the vertex in Figure 2.6(e) only has edges coming from one side, the ribbon looks locally as in Figure 3.5.

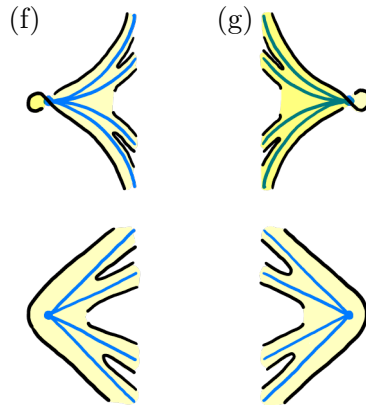


Figure 3.5. Top row: front projection of the ribbon near a one-sided vertex. Bottom row: the corresponding Lagrangian projections.

The front projection of a surface does not fully determine the surface, but it provides a good intuition for how Σ looks. One additional piece of information is that we impose the ribbon Σ to be everywhere transverse to ∂_z (a Reeb vector field), and we choose an orientation so that the lighter side is positive (that is, we orient Σ using the orientation of $\mathbb{R}^3$ and ∂_z). In particular, the Lagrangian projection of Σ is an immersion and looks like the bottom rows of Figures 3.4 and 3.5.

We can (and will) arrange that on each local piece in Figure 3.4 the characteristic foliation has exactly one critical point: a positive hyperbolic point for the piece in (a), and a positive elliptic point for the others. In the front projection, we can imagine this critical point as being located at the crossing of the ribbon surface's boundary.

Figure 3.6 shows an example of this construction applied to a Legendrian graph G .

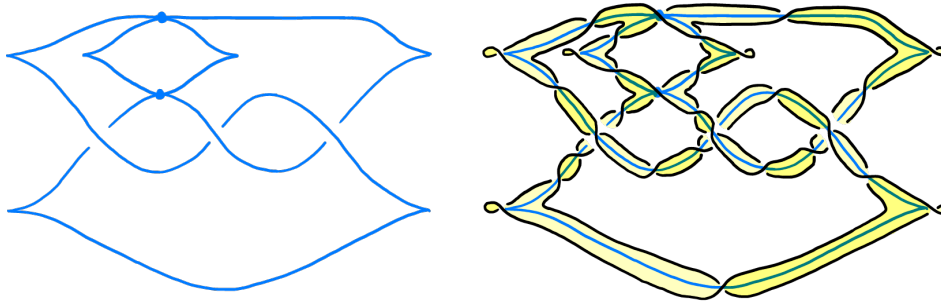


Figure 3.6. Left: a Legendrian graph. Right: a ribbon surface of the graph obtained by the local procedure above.

Note that, in particular, we assume that the surfaces created in this way are generic ribbon surfaces.

The elliptic and hyperbolic points in the 1–handles satisfy the hypotheses of the Elimination Lemma (see [Gir00, Lemma 2.15]¹), so they can be canceled in pairs by a small perturbation of Σ . The local picture is shown in Figure 3.7.



Figure 3.7. Local simplifications of the ribbon via elliptic–hyperbolic cancellation. Horizontal and vertical reflections are also allowed (with crossings and shading adjusted).

A global example is given in Figure 3.8. In each 1–handle at least one hyperbolic singularity must remain.

¹See [Mas14, Lemma 31] for an English version.

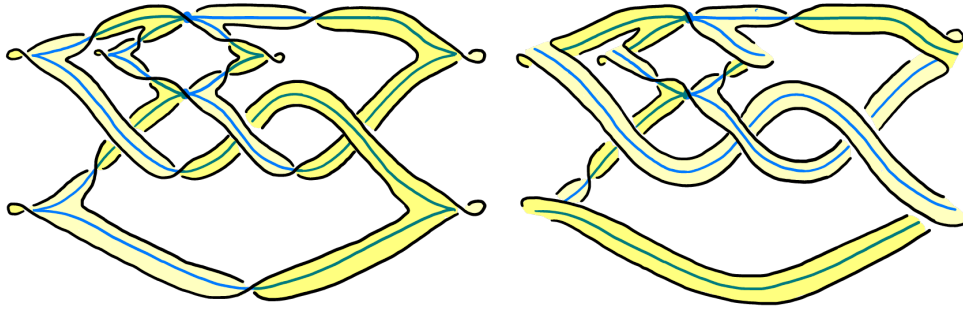


Figure 3.8. Simplifications of the ribbon in Figure 3.6. Left: after canceling elliptic points near crossings. Right: after also canceling elliptic points near cusps.

3.1.2. Legendrian realization for ribbon surfaces. The Legendrian realization principle (Theorem 2.9) gives a sufficient condition (being nonisolating) for a graph on a convex surface to be Legendrian realizable. While this condition is well known to be necessary, we provide a proof in this specific context for the sake of completeness. To this end, note that since a ribbon surface Σ is a convex surface whose dividing set coincides with its boundary, a curve α in Σ is nonisolating if and only if it is homologically nontrivial.¹

Lemma 3.3. *Let L be a Legendrian knot in a contact manifold (M, ξ) . Suppose that L is embedded in a convex surface Σ whose dividing set Γ_Σ coincides with the boundary $\partial\Sigma$. Then L is homologically nontrivial in Σ .*

Proof. Assume by contradiction that L is homologically trivial. Then L bounds a subsurface $\tilde{\Sigma} \subset \Sigma$. In a vertically invariant neighborhood $\Sigma \times (-\varepsilon, \varepsilon)$ we may write $\xi = \ker(\beta + u ds)$, with $\beta \in \Omega^1(\Sigma)$. Since L is Legendrian, we have

$$\int_L \beta = 0.$$

On the other hand, the contact condition and the assumption that $u > 0$ on $\tilde{\Sigma}$ imply that $d\beta$ is a positive area form on $\tilde{\Sigma}$, so

$$\int_{\tilde{\Sigma}} d\beta > 0.$$

This contradicts Stokes' theorem, which would give

$$0 = \int_L \beta = \int_{\tilde{\Sigma}} d\beta.$$

□

Thus homological nontriviality is a necessary condition. For our algorithm it is convenient to use the following equivalent combinatorial characterization.

Lemma 3.4. *Let Σ be a compact orientable surface with non-empty boundary, equipped with a handle decomposition. A simple closed curve $\alpha \subset \Sigma$ is homologically nontrivial if and only if there exists a 1-handle whose cocore intersects α an odd number of times.*

¹In this chapter, we prefer to phrase results using the term “homologically nontrivial” rather than “nonisolating,” as it is independent of the choice of contact vector field that makes the ribbon convex.

Proof. Consider the algebraic intersection pairing

$$\langle \cdot, \cdot \rangle: H_1(\Sigma, \partial\Sigma; \mathbb{Z}_2) \times H_1(\Sigma; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2,$$

which counts transverse intersection points modulo 2.¹

Suppose first that there is a cocore $\mathfrak{d}$ of a 1-handle intersecting $\mathfrak{a}$ transversely in an odd number of points. Then $[\mathfrak{a}] \in H_1(\Sigma; \mathbb{Z}_2)$ and $[\mathfrak{d}] \in H_1(\Sigma, \partial\Sigma; \mathbb{Z}_2)$ satisfy

$$\langle [\mathfrak{a}], [\mathfrak{d}] \rangle = 1,$$

so $[\mathfrak{a}] \neq 0$.

Conversely, assume that all cocores $\mathfrak{d}_i$ of 1-handles intersect $\mathfrak{a}$ in an even number of points. Since the pairing is nondegenerate and $[\mathfrak{a}] \neq 0$, there exists some properly embedded arc $\mathfrak{d}$ with $\langle [\mathfrak{a}], [\mathfrak{d}] \rangle = 1$. We can express $[\mathfrak{d}]$ as a $\mathbb{Z}_2$ -linear combination

$$[\mathfrak{d}] = \sum_i c_i [\mathfrak{d}_i], \quad c_i \in \mathbb{Z}_2,$$

of cocores. If every $\langle [\mathfrak{a}], [\mathfrak{d}_i] \rangle$ were zero, then

$$1 = \langle [\mathfrak{a}], [\mathfrak{d}] \rangle = \left\langle [\mathfrak{a}], \sum_i c_i [\mathfrak{d}_i] \right\rangle = \sum_i c_i \langle [\mathfrak{a}], [\mathfrak{d}_i] \rangle = 0,$$

a contradiction. Hence some cocore intersects $\mathfrak{a}$ an odd number of times. $\square$

3.2. Uniqueness of the Legendrian realization

Just for this section, we drop the assumption that we are working on $(\mathbb{R}, \xi_{\text{st}})$. This section is not technically relevant for the rest of the chapter. However, we find the results quite interesting, and we think this is the right moment to state them. These results will be used in Chapter 4.

All the necessary ingredients already appear in [Gir02; Etn06; LOSS09]; what is missing is a brief analysis of how two isotopies of a contact manifold that induce the same path of contact structures can differ. This analysis forms the core of the second part of the proof of Theorem 3.6.

3.2.1. Uniqueness of the Legendrian realization for ribbon surfaces.

Definition 3.5. Let Σ be a compact, oriented surface. Let ξ be a vertically invariant contact structure on $\Sigma \times I$, where either $I = [-\varepsilon, \varepsilon]$ or $I = \mathbb{R}$.

We call an isotopy ϕ_t of $\Sigma = \Sigma \times \{0\}$ in $\Sigma \times I$ *admissible* if $\phi_t(\Sigma)$ is always transverse to ∂_z and is the identity on a neighborhood of $\partial\Sigma$ (if $\partial\Sigma \neq \emptyset$).

Let $\mathfrak{a}$ be the isotopy class of a simple closed curve on Σ . We say that a Legendrian knot $L \subset \Sigma \times I$ is a *Legendrian realization* of $\mathfrak{a}$ if there is an admissible isotopy ϕ_t such that $L \subset \phi_1(\Sigma)$ and the isotopy class of $\phi_1^{-1}(L)$ on Σ coincides with $\mathfrak{a}$.

When we say that (L, ϕ_t) is a Legendrian realization of $\mathfrak{a}$, we are assuming we have a pair where L is a Legendrian realization of $\mathfrak{a}$ and ϕ_t is an admissible isotopy of Σ with the properties just described.

¹By Lefschetz duality this is dual to the nondegenerate pairing $H^1(\Sigma; \mathbb{Z}_2) \times H^1(\Sigma, \partial\Sigma; \mathbb{Z}_2) \rightarrow H^2(\Sigma, \partial\Sigma; \mathbb{Z}_2) \cong \mathbb{Z}_2$ induced by the cup product.

The result of the Legendrian realization principle (Theorem 2.9) is indeed a Legendrian realization of $\mathfrak{a}$, justifying the name.

Theorem 3.6. *Let Σ be a compact, oriented surface with $\partial\Sigma \neq \emptyset$. Let $\xi = \ker \alpha$ be a vertically invariant contact structure on $\Sigma \times I$, where $I = [-\varepsilon, \varepsilon]$ or $I = \mathbb{R}$, for which ∂_z is a Reeb vector field, and assume $\partial\Sigma$ (identified with $\partial(\Sigma \times \{0\})$) is a positive transverse link.*

Then any two Legendrian realizations of $\mathfrak{a}$ are Legendrian isotopic.

Remark 3.7. Let Σ be a compact, oriented surface in a contact manifold (M, ξ) . Assume that Σ has nonempty positive transverse boundary and is transverse to a Reeb vector field R_α (for example, Σ is a ribbon surface). Consider a neighborhood $\Sigma \times [-\varepsilon, \varepsilon]$ obtained by flowing Σ using R_α . Then Lemma 3.2 shows that Σ is also convex with respect to a contact vector field X that makes the dividing set Γ_Σ equal to $\partial\Sigma$.

Moreover, by carefully analyzing the proof of Lemma 3.2 we can see that we are free to choose any small enough collar neighborhood A of $\partial\Sigma$ and then construct X so that

- X is vertically invariant;
- $X_p \in T_p(\partial\Sigma \times [-\varepsilon, \varepsilon])$ for all $p \in \partial\Sigma$;
- $X = \partial_z$ on $(\Sigma \setminus A) \times \mathbb{R}$.

This in particular tells us that, given a neighborhood of Σ obtained by flowing Σ with X , it contains a neighborhood of Σ obtained by flowing Σ with R_α and vice versa. Moreover, given a Legendrian realization (L, ϕ_t) with respect to R_α , we can find such a contact vector field X so that (L, ϕ_t) is also a Legendrian realization for X .

We can probably say much more about this relation, but since we do not need it we stop here.

Before proving Theorem 3.6 we need a couple more definitions and a Lemma.

We say that an isotopy ϕ_t of $(\Sigma \times \mathbb{R}, \xi)$ is a *Moser isotopy* if it is obtained by applying the Moser trick to a family α_t of contact forms defining $\xi_t := T\phi_t^{-1}(\xi)$. More precisely, we are assuming that the Moser trick applied to α_t gives an isotopy $\tilde{\phi}_t$ such that $\xi_t = T\tilde{\phi}_t(\xi_0)$, and we are imposing that $\tilde{\phi}_t = \phi_t^{-1}$.

It is important to note that not all isotopies are Moser isotopies. For example, a contact isotopy ϕ_t of ξ will never be of this form, as in this case the path ξ_t would be constant and so $\tilde{\phi}_t = \text{Id}$.

We say that a contact form α is *compatible* with $\Sigma \times \mathbb{R}$ if

- (1) α is vertically invariant;
- (2) R_α is positively transverse to Σ ;
- (3) $\partial\Sigma$ is a positive transverse link for $\ker \alpha$.

If we are in the hypothesis of Theorem 3.6, then the contact form α (whose Reeb vector field is ∂_z) is clearly compatible with $\Sigma \times I$.

The following is an obvious adaptation of Etnyre's proof of Giroux's intuition that two contact structures compatible with the same open book are isotopic [Gir02; Etn06]. More precisely, the space C of contact forms compatible with an open book is connected and simply connected, and the same holds in our case.

Lemma 3.8. *Let Σ be an oriented, compact surface with $\partial\Sigma \neq \emptyset$. The space C of contact forms compatible with $\Sigma \times \mathbb{R}$ is connected and simply connected.*

Proof. Since we only consider z -invariant forms, any $\alpha \in C$ can be written uniquely as

$$\alpha = \lambda + h dz,$$

where λ is the pullback of a 1-form on Σ and $h : \Sigma \rightarrow \mathbb{R}$ is a function, both independent of z .

Step 1. Characterization of C .

For a contact form α , the following are clearly equivalent:

- (a) α is a positive contact form and R_α is positively transverse to Σ .
- (b) α is a positive contact form and $d\alpha|_{T(\Sigma)}$ is a positive area form on each slice.

This gives a different way of characterizing forms in C .

Step 2. The ray $\alpha + t dz$ stays inside C .

If $\alpha \in C$ and $t \geq 0$, then $\alpha_t := \alpha + t dz$ also lies in C .

Indeed, vertical invariance is clear. Since $d(\alpha_t) = d\alpha$, we still have $d\alpha_t|_{T\Sigma} = d\alpha|_{T\Sigma}$, which is a positive area form because $d\alpha$ is.

On $\partial\Sigma$, a positive tangent vector v is horizontal, so $dz(v) = 0$ and

$$\alpha_t(v) = \alpha(v) > 0.$$

Thus the boundary remains a positive transverse link.

To check the contact condition, compute

$$\alpha_t \wedge d\alpha_t = \alpha_t \wedge d\alpha = \alpha \wedge d\alpha + t dz \wedge d\alpha.$$

The first term on the right is clearly positive since α is a contact form. The second term is also nonnegative, since $d\alpha$ is a volume form on the pages, dz vanishes on the pages, and dz is positive on the oriented normal to the pages. Hence $\alpha_t \wedge d\alpha_t$ is a positive volume form for all $t \geq 0$.

So for each $\alpha \in C$, the ray $\{\alpha + t dz \mid t \geq 0\}$ lies entirely in C .

Step 3. Convex combinations after adding a large multiple of dz .

Let $\alpha_0, \alpha_1 \in C$ and write

$$\alpha_i = \lambda_i + h_i dz, \quad i = 0, 1,$$

with z -independent λ_i and h_i .

Fix $R > 0$ and set

$$\beta_i := \alpha_i + R dz = \lambda_i + (h_i + R) dz, \quad i = 0, 1.$$

By Step 2, $\beta_i \in C$ for all $R \geq 0$.

For $s \in [0, 1]$, define the convex combination

$$\beta_s := (1 - s)\beta_0 + s\beta_1 = \lambda_s + (h_s + R) dz,$$

where

$$\lambda_s := (1 - s)\lambda_0 + s\lambda_1, \quad h_s := (1 - s)h_0 + sh_1.$$

We now show that for R sufficiently large, every β_s lies in C .

Boundary condition. Along $\partial\Sigma$ we have $dz(v) = 0$ for tangent vectors v , so

$$\beta_s(v) = (1-s)\alpha_0(v) + s\alpha_1(v),$$

which is positive since each $\alpha_i(v) > 0$. Thus the boundary remains a positive transverse link for all s .

Positivity of $d\beta_s$. On Σ , we have

$$d\beta_s|_{T\Sigma} = d\lambda_s = (1-s)d\lambda_0 + sd\lambda_1.$$

Each $d\lambda_i$ is a positive area form, so their convex combination $d\lambda_s$ is also a positive area form. In particular, there exists a smooth positive function f_s such that

$$d\lambda_s = f_s \omega$$

for some fixed background area form ω on Σ . By compactness of $\Sigma \times [0, 1]$, there exists $m > 0$ with $f_s \geq m$ everywhere, so

$$d\lambda_s \geq m \omega$$

pointwise for all s .

Contact condition for large R . Compute

$$d\beta_s = d\lambda_s + dh_s \wedge dz,$$

and hence

$$\beta_s \wedge d\beta_s = \lambda_s \wedge d\lambda_s + \lambda_s \wedge dh_s \wedge dz + (h_s + R) dz \wedge d\lambda_s.$$

Write this as

$$\beta_s \wedge d\beta_s = A_s + (h_s + R) B_s,$$

where $B_s := dz \wedge d\lambda_s$ is a positive volume form (since $d\lambda_s$ is a positive area form) and $A_s := \lambda_s \wedge d\lambda_s + \lambda_s \wedge dh_s \wedge dz$.

Consider the volume form $\Omega = \omega \wedge dz$ on $\Sigma \times \mathbb{R}$. By compactness of Σ and the fact that everything is vertically invariant, there exists $C > 0$ such that

$$|A_s| \leq C \Omega$$

for all s .¹ Since h_s is a convex combination of the two fixed functions h_0, h_1 , there exists $H > 0$ with $|h_s| \leq H$ for all s .

On the other hand $d\lambda_s \geq m\omega$ implies

$$B_s = dz \wedge d\lambda_s \geq m \omega \wedge dz = m \Omega.$$

Therefore

$$(h_s + R) B_s \geq (R - H) m \Omega.$$

Combining the estimates, we get

$$\beta_s \wedge d\beta_s \geq -C \Omega + (R - H) m \Omega = ((R - H)m - C) \Omega.$$

If we choose R so large that

$$(R - H)m - C > 0,$$

¹This notation is not really valid, because in general there is no total order on volume forms, but it is clear what it means in this case.

then $\beta_s \wedge d\beta_s$ is a positive volume form everywhere for all $s \in [0, 1]$. Thus each β_s is a positive contact form.

We have already seen that $d\beta_s|_{T\Sigma}$ is a positive area form, so by the characterization above the Reeb field R_{β_s} is positively transverse to Σ . Together with the boundary condition, and the fact that all these contact forms are obviously vertically invariant, this shows $\beta_s \in C$ for all $s \in [0, 1]$.

In particular, for any $\alpha_0, \alpha_1 \in C$ there exists $R \geq 0$ such that the path

$$s \mapsto (1-s)(\alpha_0 + Rdz) + s(\alpha_1 + Rdz)$$

lies entirely in C . This is exactly what we have just proved.

Step 4. Path-connectedness of C .

Let $\alpha_0, \alpha_1 \in C$. Choose $R \geq 0$ as in Step 3 and set $\beta_i = \alpha_i + Rdz$.

Consider the following concatenated path in C :

- (1) From α_0 to β_0 along the ray

$$t \mapsto \alpha_0 + tRdz, \quad t \in [0, 1],$$

which lies in C by Step 2.

- (2) From β_0 to β_1 along the convex path

$$s \mapsto \beta_s = (1-s)\beta_0 + s\beta_1, \quad s \in [0, 1],$$

which lies in C by Step 3.

- (3) From β_1 to α_1 along

$$t \mapsto \alpha_1 + (1-t)Rdz, \quad t \in [0, 1],$$

again in C by Step 2.

Thus any two points of C can be joined by a path in C , so C is path-connected.

Step 5. Simple connectedness of C .

Let $\{\alpha_\theta\}_{\theta \in S^1} \subset C$ be a continuous loop. We will show it is null-homotopic in C .

Stage 1: push the loop in the dz -direction. For each $\theta \in S^1$ and $t \in [0, 1]$ define

$$\alpha_{\theta,t} := \alpha_\theta + tRdz.$$

By Step 2, $\alpha_{\theta,t} \in C$ for all (θ, t) . For $t = 0$ we recover the original loop, and for $t = 1$ we get the loop

$$\theta \mapsto \beta_\theta := \alpha_\theta + Rdz.$$

This gives a homotopy in C from $\{\alpha_\theta\}$ to $\{\beta_\theta\}$.

Stage 2: contract the pushed-up loop by convex combinations. Fix some basepoint $\theta_0 \in S^1$ and consider

$$H(\theta, s) := (1-s)\beta_\theta + s\beta_{\theta_0}, \quad (\theta, s) \in S^1 \times [0, 1].$$

For each fixed θ , this is exactly the kind of convex path treated in Step 3, now with parameters (θ, s) .

The estimates in Step 3 depend continuously on θ and s , and S^1 is compact. Thus there exists a single $R > 0$ such that for this R all forms $H(\theta, s)$ lie in C . For $s = 0$ we

recover β_θ , and for $s = 1$ we have the constant loop $\theta \mapsto \beta_{\theta_0}$. Hence $\{\beta_\theta\}$ is null-homotopic in C .

Combining Stages 1 and 2, we obtain a homotopy in C from the original loop $\{\alpha_\theta\}$ to a constant loop. Thus every loop in C is null-homotopic and $\pi_1(C) = 0$.

Therefore C is both connected and simply connected. $\square$

A good part of the argument in the following proof comes from [LOSS09].

Proof of Theorem 3.6. We need to show that two Legendrian realizations L and L' of $\mathfrak{a}$ are Legendrian isotopic. That is, we need to find an isotopy $\psi_t : S^1 \rightarrow \Sigma \times I$, $t \in [0, 1]$, such that

- $\psi_0(S^1) = L$;
- $\psi_1(S^1) = L'$;
- $\psi_t(S^1)$ is Legendrian for all t .

To achieve this, we will proceed as follows. First, if we are working on $(\Sigma \times [-\varepsilon, \varepsilon], \xi = \ker \alpha)$, then we think of this manifold as being a subset of $\Sigma \times \mathbb{R}$, where α is extended by imposing it to be vertically invariant. Next we will find an (ambient) isotopy of $\Sigma \times \mathbb{R}$ that sends L to L' through Legendrian knots. By restricting the isotopy to L , this gives the desired Legendrian isotopy. Of course, in the case where the original manifold was $\Sigma \times [-\varepsilon, \varepsilon]$, we need to ensure that the image of L under this isotopy is always contained in $\Sigma \times [-\varepsilon, \varepsilon]$.

If ϕ_t is an admissible isotopy of Σ in $\Sigma \times \mathbb{R}$ as in Definition 3.5, we can extend it to an (ambient) isotopy, still denoted ϕ_t , of $\Sigma \times \mathbb{R}$ by simply setting

$$\phi_t(x, z) = (\phi_t(x)_\Sigma, \phi_t(x)_\mathbb{R} + z),$$

where $\phi_t(x)_\Sigma$ and $\phi_t(x)_\mathbb{R}$ are the coordinates of $\phi_t(x) \in \Sigma \times \mathbb{R}$. We will always do this and still call it ϕ_t . (We will sometimes abuse notation and infer properties of the ambient isotopy from properties of the isotopy of Σ , while calling both by the same name. It will be clear from the context which one we are referring to.)

Note that if ϕ_t is an admissible isotopy (we now think of these as ambient isotopies as explained above), and we consider the family of contact forms

$$\alpha_t := T\phi_t^{-1}(\alpha),$$

then each α_t is compatible with $\Sigma \times \mathbb{R}$. This is true because:

- condition (1) in the definition of compatibility is guaranteed by the fact that both ϕ_t and α are vertically invariant;
- condition (2) is guaranteed by the fact that $R_{\alpha_t} = T\phi_t^{-1}(\partial_z)$, and this is transverse to Σ because $\phi_t(\Sigma)$ is transverse to ∂_z ;
- condition (3) is guaranteed by the fact that each α_t agrees with α on $A \times \mathbb{R}$, where A is a neighborhood of $\partial\Sigma$ on which ϕ_t is the identity.

Conversely, given a path α_t of contact forms compatible with $\Sigma \times \mathbb{R}$, we have that:

- we can apply the Moser trick to them and obtain an isotopy ϕ_t (this is not automatic, as $\Sigma \times \mathbb{R}$ is not compact). It follows from the fact that the time-dependent vector field defining ϕ_t is vertically invariant;

- the isotopy ϕ_t is vertically invariant, i.e.

$$\phi_t(x, z) = (\phi_t(x, 0)_\Sigma, \phi_t(x)_\mathbb{R} + z);$$

- $\phi_t(\Sigma)$ is transverse to ∂_z .

These properties are all consequences of the fact that each α_t is vertically invariant.

Given all of this, it is easy to check that there is a surjection between:

- the space of paths α_t of contact forms compatible with $\Sigma \times \mathbb{R}$

onto:

- the space of vertically invariant Moser isotopies ϕ_t such that $\phi_t(\Sigma)$ is transverse to ∂_z .

It is then clear why having a control over the first space (given by Lemma 3.8) has consequences over the second space.

In the following, every time we say that an isotopy ϕ_t is a Moser isotopy, we are also implicitly assuming that the Moser trick was applied to the specific path of contact forms $\alpha_t := T\phi_t^{-1}(\alpha)$ (in principle, two different paths of contact forms for the same path of contact structures might yield different isotopies).

(a) Assume first that (L, ϕ_t) and (L', ϕ'_t) are two Legendrian realizations of $\mathfrak{a}$ such that ϕ_t and ϕ'_t are Moser isotopies. Then L and L' are Legendrian isotopic. Call α_t and α'_t the paths of contact forms associated to ϕ_t and ϕ'_t . Note that $\alpha_0 = \alpha'_0 = \alpha$.

The proof of this claim follows as in the proof of [LOSS09, Theorem 2.7]. The key ingredient is Lemma 3.8.

The steps of the proof are the following. We do not give the details as they are identical to those in the reference given. We, however, stress the key ingredients so that one can check that everything works in our setting.

(a.1) Let a be a curve whose isotopy class is $\mathfrak{a}$, and let ϕ_t and ϕ'_t be two admissible Moser isotopies such that $\phi_1(a)$ and $\phi'_1(a)$ are Legendrian.

(a.1.1) Assume that $\alpha_1 = \alpha'_1$ (so, in particular, we are assuming that $\xi_1 = \xi'_1$). Then $\phi_1(a)$ is Legendrian isotopic to $\phi'_1(a)$. This follows from the fact that C is simply connected and the fact that the contact forms in the paths α_t and α'_t are all compatible with $\Sigma \times \mathbb{R}$. More precisely, there exists a path α_t^s of paths such that

- $\alpha_t^0 = \alpha_t$,
- $\alpha_t^0 = \alpha_t$,
- $\alpha_0^s = \alpha$. and
- $\alpha_1^s = \alpha_1 = \alpha'_1$.

This gives rise to a path of Moser isotopies that achieves what we want.

(a.1.2) If $\alpha_1 \neq \alpha'_1$, the proof that $\phi_1(a)$ is Legendrian isotopic to $\phi'_1(a)$ uses the fact that there exists a path $\tilde{\alpha}_t$ of contact forms compatible with $\Sigma \times \mathbb{R}$ such that

- $\tilde{\alpha}_0 = \alpha_1$;
- $\tilde{\alpha}_1 = \alpha'_1$;
- the curve a is Legendrian for all the $\tilde{\alpha}_t$.

This is shown in [LOSS09, Lemma 2.9].

In practice, one considers the path from α to α'_1 obtained by concatenating a path from α to α_1 with $\tilde{\alpha}_t$. The Moser isotopy associated to this path first makes a into $\phi_1(a)$ and then transforms $\phi_1(a)$ by a Legendrian isotopy to a knot Legendrian isotopic to $\phi'_1(a)$ (by step (a.1.1)).

(a.2) Let a_t be a path of curves all representing the homology class determined by $\mathbf{a}$. Assume that ϕ_t and ϕ'_t are admissible Moser isotopies such that $\phi_1(a_0)$ and $\phi'_1(a_1)$ are Legendrian. Then $\phi_1(a_0)$ and $\phi'_1(a_1)$ are Legendrian isotopic.

The proof of this relies on the fact that there exists a path of contact forms $\tilde{\alpha}_t$ compatible with $\Sigma \times \mathbb{R}$ such that a_t is Legendrian for $\tilde{\alpha}_t$ [LOSS09, Lemma 2.8]. We can now consider a path α_t from α to $\alpha_1 = \tilde{\alpha}_0$ concatenated with the path $\tilde{\alpha}_t$. The Moser isotopy associated to this path first makes a_0 into a Legendrian knot L that is Legendrian isotopic to $\phi_1(a_0)$ by (a.1.1), and then Legendrian isotopes L to a Legendrian knot that is Legendrian isotopic to $\phi'_1(a_1)$ again by (a.1.1), proving the theorem in this case.

(b) Next assume that (L, ϕ_t) is any type of Legendrian realization (that is, we are not assuming that ϕ_t is a Moser isotopy). We will show that L is Legendrian isotopic to a Legendrian realization coming from a Moser isotopy, thereby reducing the general case to (a).

Consider the path of contact forms

$$\alpha_t := T\phi_t^{-1}(\alpha)$$

and the corresponding path of contact structures $\xi_t := \ker \alpha_t$ on $\Sigma \times \mathbb{R}$. Let us write $\xi_0 = \xi$.

By the Moser trick, there exists a Moser isotopy

$$\tilde{\phi}_t : \Sigma \times \mathbb{R} \rightarrow \Sigma \times \mathbb{R}, \quad \tilde{\phi}_0 = \text{id},$$

such that

$$T\tilde{\phi}_t^{-1}(\xi_0) = \xi_t \quad \text{for all } t.$$

Recall that $\tilde{\phi}_t$ exists because each α_t is vertically invariant and so the time-dependent vector field given by the Moser trick is also vertically invariant and thus can be integrated. Moreover, note that $\tilde{\phi}_t$ is admissible because the path α_t is compatible with $\Sigma \times \mathbb{R}$ and all α_t coincide with α on $A \times \mathbb{R}$, where A is a neighborhood of $\partial\Sigma$ on which ϕ_t is the identity. This last condition ensures that $\tilde{\phi}_t$ is the identity on $A \times \mathbb{R}$ for all t .

By definition of ξ_t and of $\tilde{\phi}_t$, we have

$$T\phi_t^{-1}(\xi_0) = \xi_t = T\tilde{\phi}_t^{-1}(\xi_0) \quad \text{for all } t \in [0, 1].$$

The “difference” isotopy

$$F_t := \tilde{\phi}_t \circ \phi_t^{-1}$$

is then a contact isotopy of $(\Sigma \times \mathbb{R}, \xi_0)$, because

$$TF_t(\xi_0) = T(\tilde{\phi}_t \circ \phi_t^{-1})(\xi_0) = T\tilde{\phi}_t(\xi_t) = \xi_0.$$

Hence F_t is a path of contactomorphisms of $(\Sigma \times \mathbb{R}, \xi_0)$, with $F_0 = \text{id}$.

Let $a := \phi_1^{-1}(L) \subset \Sigma$. By assumption (L, ϕ_t) is a Legendrian realization of $\mathfrak{a}$, so a is a curve on Σ representing the class $\mathfrak{a}$. Consider the knot

$$L' := \tilde{\phi}_1(a) \subset \tilde{\phi}_1(\Sigma).$$

Then:

- L' is Legendrian because $L' = F_1(L)$ and F_1 is a contactomorphism. Thus $(L', \tilde{\phi}_t)$ is a Legendrian realization of $\mathfrak{a}$ associated to the Moser isotopy $\tilde{\phi}_t$.
- Moreover, since F_t is a contact isotopy of $(\Sigma \times \mathbb{R}, \xi_0)$ with $F_0 = \text{id}$, the path $t \mapsto F_t(L)$ gives a Legendrian isotopy from L to L' .

Thus any Legendrian realization (L, ϕ_t) is Legendrian isotopic to a Legendrian realization $(L', \tilde{\phi}_t)$ whose isotopy $\tilde{\phi}_t$ is a Moser isotopy.

Finally, a careful (but not difficult) analysis of all these isotopies shows that in all cases the Legendrian isotopy between L and L' stays inside $\Sigma \times [-\varepsilon, \varepsilon]$ if we started with Legendrian realizations in this manifold.

Combined with (a), this implies that any two Legendrian realizations of $\mathfrak{a}$ lying in the same vertically invariant neighborhood $\Sigma \times \mathbb{R}$ are Legendrian isotopic. $\square$

3.2.2. Uniqueness of the Legendrian realization for open book decompositions.

Notice that the open book setting is very similar to our setting.

Definition 3.9. Let (M, ξ) be a contact manifold and let (B, π) be an open book compatible with ξ .

In this setting, we call an isotopy ϕ_t of M *admissible* if

- ϕ_t is the identity on a neighborhood of B ;
- there exists a contact form α for ξ such that
 - α is compatible with (B, π) in the sense that the Reeb vector field R_α is always positively transverse to the pages and positively tangent to B ;
 - for each $z \in S^1$, the surface $\phi_t(\pi^{-1}(z))$ is always transverse to R_α .

Let $\mathfrak{a}$ be the isotopy class of a simple closed curve on a page Σ of (B, π) . We say that a Legendrian knot $L \subset M \setminus B$ is a *Legendrian realization* of $\mathfrak{a}$ if there is an admissible isotopy ϕ_t such that $L \subset \phi_1(\Sigma)$ and the isotopy class of $\phi_1^{-1}(L)$ on Σ coincides with $\mathfrak{a}$.

Let ϕ_t be an admissible isotopy of $(M, (B, \pi), \xi)$ as in Definition 3.9, and let α be the corresponding contact form. We say that ϕ_t is a *Moser isotopy* if it is obtained by applying the Moser trick to the path of contact forms

$$\alpha_t := T\phi_t^{-1}(\alpha).$$

In this language, in [LOSS09, Theorem 2.7] it is proved that, given a homologically nontrivial isotopy class $\mathfrak{a}$ of a simple closed curve on a page Σ , we have:

- there exists a Legendrian realization of $\mathfrak{a}$ given by a Moser isotopy;
- any two Legendrian realizations of $\mathfrak{a}$ given by a Moser isotopy are Legendrian isotopic.

The proof uses the fact that the isotopies are Moser because it uses that the space of contact forms compatible with a given open book is convex [Gir02; Etn06].

However, a simple adaptation of the argument in part (b) of the proof of Theorem 3.6, together with the proof of [LOSS09, Theorem 2.7], shows the following:

Theorem 3.10 (Uniqueness of Legendrian knots on open books). *Let (B, π) be an open book supporting a contact manifold (M, ξ) . Let Σ be a page and let $\mathfrak{a}$ be the isotopy class of a homologically nontrivial simple closed curve on Σ . Then:*

- *There exists a Legendrian knot L embedded on a page of an open book (B', π') supporting ξ such that*
 - *(B, π) and (B', π') are isotopic through open books compatible with the same contact form α for ξ ;*
 - *if we denote by ϕ_t the isotopy between (B, π) and (B', π') , then $\phi_1^{-1}(L) \subset \Sigma$ defines the same isotopy class $\mathfrak{a}$.*
- *If L' is another Legendrian knot that is embedded on a page of an open book (B'', π'') such that*
 - *(B, π) and (B'', π'') are isotopic through open books compatible with the same contact form α' for ξ ;*
 - *if we denote by ϕ'_t the isotopy between (B, π) and (B'', π'') , then $\phi'_1(L') \subset \Sigma$ defines the same isotopy class $\mathfrak{a}$,**then L' is Legendrian isotopic to L .*

We need this stronger version (in fact, Theorem 3.6) of the result in [LOSS09] because in Chapter 4 we will encounter two Legendrian knots and show that they are both contained in admissible isotopies of the same ribbon surface. Theorem 3.6 then implies that these knots are Legendrian isotopic. However, since these Legendrian knots might not arise as Legendrian realizations of a curve on the ribbon (that is, by perturbing the ribbon via a Moser isotopy), the result of [LOSS09] alone does not suffice to guarantee that they are Legendrian isotopic.

3.3. Statement of the result

Before stating the main theorem of this chapter, we formalize the notion of an “abstract ribbon”, which allows us to separate the combinatorics on a surface from its concrete embedding in $(\mathbb{R}^3, \xi_{\text{st}})$.

Let Σ be a ribbon surface of a Legendrian graph G . We write $[\Sigma]$ for the isotopy class of Σ through embeddings that fix G pointwise. It is straightforward to check that if Σ and Σ' are ribbon surfaces of the same Legendrian graph G , then $[\Sigma] = [\Sigma']$. This leads to:

Definition 3.11. The *abstract ribbon* $\mathcal{S}$ of a Legendrian graph G is the class $[\Sigma]$, where Σ is any ribbon surface of G .

The abstract ribbon inherits a well-defined handle decomposition from any representative Σ , since the handle decomposition only depends on G .

We can now state the main result.

Theorem 3.12. *There exists an algorithm with the following input and output:*

Input:

- a generic Legendrian graph $G \subset (\mathbb{R}^3, \xi_{\text{st}})$, given by its front projection;
- a homologically nontrivial simple closed curve $\mathfrak{a}$ on the abstract ribbon $\mathcal{S}$ of G .

Output: the front projection of a generic Legendrian knot L which is a Legendrian realization of $\mathfrak{a}$ in the following sense. There exists a generic ribbon surface Σ of G and an isotopy $\psi_t: \Sigma \rightarrow \mathbb{R}^3$, $t \in [0, 1]$, such that:

- (1) ψ_0 is the inclusion $\Sigma \subset \mathbb{R}^3$, and ψ_t is fixed on a neighborhood of $\partial\Sigma$;
- (2) each surface $\psi_t(\Sigma)$ is transverse to ∂_z and hence convex;
- (3) for every $\varepsilon > 0$, the algorithm can be carried out so that

$$\text{Im}\psi_t \subset \Sigma \times [-\varepsilon, \varepsilon] \quad \text{for all } t \in [0, 1],$$

where the second factor is parametrized by the z -coordinate.

- (4) L is a Legendrian knot properly embedded in $\psi_1(\Sigma)$;
- (5) the curve $\psi_1(\mathfrak{a})$ is smoothly isotopic to L in the surface $\psi_1(\Sigma)$.

In the rest of the chapter we first describe the algorithm step by step, and then justify that it produces the Legendrian realization claimed in Theorem 3.12.

Remark 3.13. By Theorem 3.6, L is the unique, up to Legendrian isotopy, Legendrian realization of the isotopy class of $\mathfrak{a}$.

3.4. Description of the algorithm

We first fix notation. The data of the algorithm consists of:

- G , a generic Legendrian graph in $(\mathbb{R}^3, \xi_{\text{st}})$, described by its front projection;
- $\mathcal{S}$, the abstract ribbon of G ;
- $\mathfrak{a}$, a homologically nontrivial simple closed curve on $\mathcal{S}$.

As recalled above, the ribbon surfaces of G (and hence the abstract ribbon $\mathcal{S}$) come with a natural handle decomposition. We will denote a handle, either 0– or 1–handle, by H .

If H is a 1–handle, we call $G \cap H$ the *Legendrian core* of H and denote it by $\mathfrak{e}$.¹ If H is a 0–handle, we call $G \cap H$ the *Legendrian skeleton* of H , consisting of several Legendrian segments meeting at the unique vertex v contained in H ; we usually denote these segments by $\mathfrak{e}_1, \dots, \mathfrak{e}_n$. See Figure 3.9. Note that in this example, $\mathfrak{e}_1 \cup \mathfrak{e} \cup \mathfrak{e}_3$ forms an edge of G .

We now make two preliminary choices:

- Choose a 1–handle of $\mathcal{S}$ whose cocore intersects $\mathfrak{a}$ an odd number of times (this exists by Lemma 3.4). Denote this distinguished 1–handle by $H_\star$.
- Choose an orientation for the core $\mathfrak{e}$ of each 1–handle H .

¹Here the letter $\mathfrak{e}$ stands for “edge”, even though in general $\mathfrak{e}$ is only a subinterval of an edge of G .

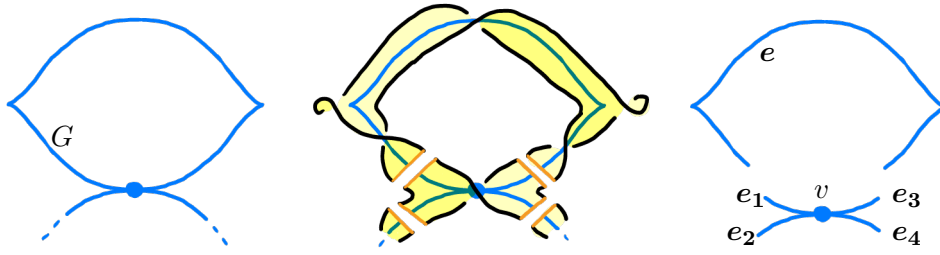


Figure 3.9. Left: a portion of a Legendrian graph G containing a vertex v and an edge. Middle: a portion of a ribbon surface subdivided into handles. Right: the Legendrian core e of the 1-handle and the Legendrian skeleton of the 0-handle.

Given such a choice, each 1-handle H can be identified with the abstract handle shown in Figure 3.10, with the blackboard orientation agreeing with the orientation of H and the core e oriented from left to right.

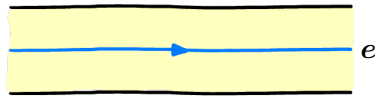


Figure 3.10. An abstract 1-handle, with oriented core e .

In all subsequent pictures of abstract 1-handles we adopt this convention. For example, if a 1-handle H looks as on the left of Figure 3.11, then the green and orange segments correspond to the green and orange segments on the right, after identifying H with the abstract handle.

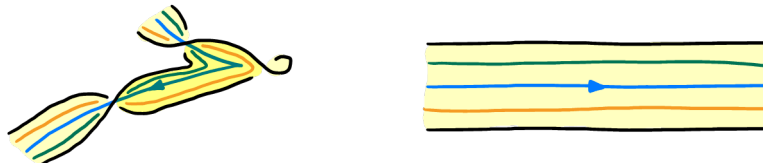


Figure 3.11. Left: a 1-handle H with a green and an orange segment and Legendrian core e . Right: the corresponding segments on the abstract handle of Figure 3.10.

Step 1: subdividing the curve. Isotope the curve $\mathfrak{a}$ (on the abstract ribbon $\mathcal{S}$) to minimize its intersections with the attaching regions of the 1-handles. After this isotopy, $\mathfrak{a}$ decomposes into a concatenation of segments, each contained in a single handle and meeting its boundary only in the attaching regions. These segments alternate between 0- and 1-handles, see Figure 3.12.

We now:

- choose an orientation of $\mathfrak{a}$;
- choose one segment of $\mathfrak{a}$ lying in a 1-handle and call it $\mathfrak{l}_1$.

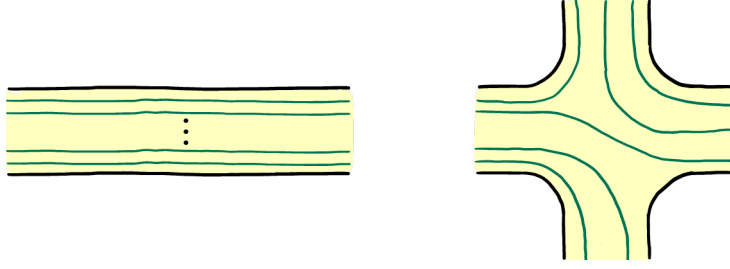


Figure 3.12. Left: a 1-handle of $\mathcal{S}$ and the segment of $\mathfrak{a}$ it contains after Step 1. Right: an example of how $\mathfrak{a}$ may look on a 0-handle after Step 1.

Let $\mathfrak{l}_2$ be the next segment encountered when traveling along $\mathfrak{a}$ in the direction of its orientation, then $\mathfrak{l}_3$, and so on, until we return to $\mathfrak{l}_1$. In total we obtain segments $\mathfrak{l}_1, \dots, \mathfrak{l}_n$ with n even. The odd-indexed segments lie in 1-handles, and the even-indexed segments lie in 0-handles. Each segment inherits an orientation from $\mathfrak{a}$.

Step 2: relative gain (displacement index) in a 1-handle. Let H be a 1-handle, and let $\mathfrak{l}_{j_1}, \dots, \mathfrak{l}_{j_k}$ be the segments of $\mathfrak{a}$ contained in H . To each segment $\mathfrak{l}_{j_i}$ we assign a rational number $\Delta(\mathfrak{l}_{j_i})$, called the *relative gain* (or *displacement index*) of $\mathfrak{l}_{j_i}$.

We first define an intermediate quantity $\tilde{\Delta}(\mathfrak{l}_{j_i})$. After identifying H with the abstract handle of Figure 3.10, we may speak of the lowest segment, the second lowest, and so on. If $\mathfrak{l}_{j_i}$ is the h -th lowest segment, set

$$a(\mathfrak{l}_{j_i}) := h - \frac{k+1}{2}.$$

Then define

$$\tilde{\Delta}(\mathfrak{l}_{j_i}) := \begin{cases} \lfloor a(\mathfrak{l}_{j_i}) \rfloor, & a(\mathfrak{l}_{j_i}) < 0, \\ 0, & a(\mathfrak{l}_{j_i}) = 0, \\ \lceil a(\mathfrak{l}_{j_i}) \rceil, & a(\mathfrak{l}_{j_i}) > 0. \end{cases}$$

When the number of segments is odd, the lowest segment has value $-d$, the middle segment has value 0, and the highest segment has value d , with all other values determined by the formula above. If there is an even number of segments, then no segment has $\tilde{\Delta} = 0$. See Figure 3.13.

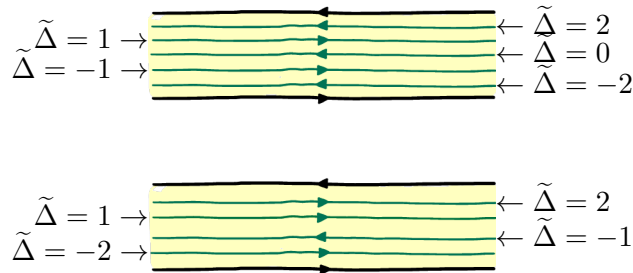


Figure 3.13. Top: a 1-handle with an odd number of segments of $\mathfrak{a}$. Bottom: a 1-handle with an even number of segments. The values $\tilde{\Delta}$ are indicated.

We then define the relative gain $\Delta(\mathfrak{l}_{j_i})$ by

$$\Delta(\mathfrak{l}_{j_i}) := \delta_{j_i} \tilde{\Delta}(\mathfrak{l}_{j_i}),$$

where

$$\delta_{j_i} := \begin{cases} 1, & \text{if the orientation of } \mathfrak{l}_{j_i} \text{ agrees with the orientation of the core } \mathbf{e} \text{ of } H, \\ -1, & \text{otherwise.} \end{cases}$$

For the segments in Figure 3.13, the corresponding values of Δ are shown in Figure 3.14 (recall that the core $\mathbf{e}$ is oriented from left to right).

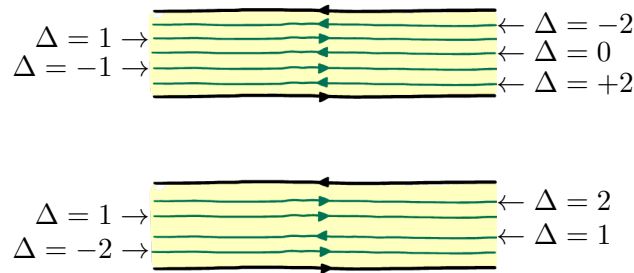


Figure 3.14. The relative gains (displacement indices) of the segments from Figure 3.13.

Remark 3.14. Intuitively, $\Delta(\mathfrak{l})$ measures how much the Legendrian realization of a segment $\mathfrak{l}$ will “gain” or “lose” height in the front projection relative to the core $\mathbf{e}$. Segments further away from the center of the handle have larger $|\Delta|$, and reversing the orientation of the segment flips the sign. The precise geometric meaning of Δ will appear in the justification of the algorithm, where it will be related to the integral $\int x dy$ along the segment.

The following discussion provides an informal explanation. Consider a 1-handle H of a ribbon surface Σ of a Legendrian graph G (see, for example, Figure 3.15).

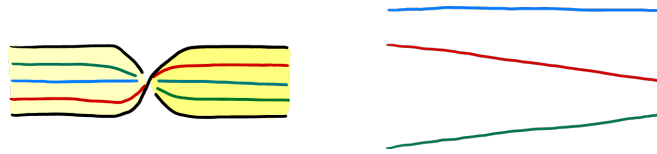


Figure 3.15. Left: The front projection of the handle H and three segments on it (one being the Legendrian core of the handle). Right: The Legendrian realization of these three segments.

Denote by $\mathbf{e}$ the Legendrian core of the handle H (the central blue segment in Figure 3.15). This is one of the inputs to our algorithm, as we know the front projection of G . Assume there are three segments of $\mathfrak{a}$ in this handle H . In Step 1, we first straighten these segments to look like the three segments (green, blue, and red) in Figure 3.15 (left). The central blue segment is the Legendrian core, the red segment is “a bit closer to us” (in the sense that it has a higher x -coordinate), and the green segment is “a bit further away from us.”

The Legendrian realization of the blue segment is a copy of itself, possibly translated in the ∂_z direction. Intuitively, the Legendrian realizations of the green and red segments will look similar to the blue segment but slightly rotated due to the contact condition (the (y, z) -slope of the contact planes decreases as x increases), and possibly translated in the ∂_z direction (see Figure 3.15, right). The relative gain Δ measures how much these Legendrian realizations lose or gain height relative to the blue segment when traversed in their oriented direction. For example, if all the segments are oriented from left to right, the red segment loses some height, while the green segment gains some height.

The amount by which we need to translate these Legendrian realizations in the ∂_z direction depends on the entire curve $\mathbf{a}$ and will be encoded by the quantity P defined later in Step 4.

Step 3: balancing the total gain. Define

$$\theta := \sum_{i=1}^n \Delta(\mathfrak{l}_i),$$

where by convention $\Delta(\mathfrak{l}_i) := 0$ if $\mathfrak{l}_i$ lies in a 0-handle. Ultimately, θ will correspond to the integral $\oint_{\mathbf{a}} x dy$, so we need to modify the Δ 's so that θ becomes zero.

We do this by adjusting the relative gains of the segments in the distinguished handle $H_\star$. Let $\mathfrak{l}_{1_1}, \dots, \mathfrak{l}_{1_k}$ be the segments of $\mathbf{a}$ that lie in $H_\star$, with k odd. Let k_+ be the number of those segments whose orientation agrees with the orientation of the core $\mathbf{e}$ of $H_\star$, and k_- the number oriented in the opposite direction. Since k is odd, $k_+ \neq k_-$.

We redefine the relative gain of the segments in $H_\star$ by

$$(3.4.1) \quad \begin{cases} \Delta_{\text{old}}(\mathfrak{l}_{1_*}) := \Delta(\mathfrak{l}_{1_*}), \\ \Delta(\mathfrak{l}_{1_*}) = \Delta_{\text{old}}(\mathfrak{l}_{1_*}) - \frac{\theta}{k_+ - k_-}, & \text{if } \mathfrak{l}_{1_*} \text{ is oriented as } \mathbf{e}, \\ \Delta(\mathfrak{l}_{1_*}) = \Delta_{\text{old}}(\mathfrak{l}_{1_*}) + \frac{\theta}{k_+ - k_-}, & \text{otherwise.} \end{cases}$$

Remark 3.15. Let $\{H_j\}_j$ be the collection of 1-handles, and assume $H_1 = H_\star$. For each j let k_j be the number of segments of $\mathbf{a}$ in H_j , and denote these segments by $\mathfrak{l}_{j_1}, \dots, \mathfrak{l}_{j_{k_j}}$. Then, after the redefinition (3.4.1), we have

$$(3.4.2) \quad \begin{aligned} \sum_{i=1}^n \Delta(\mathfrak{l}_i) &= \sum_{i=0}^{\frac{n}{2}-1} \Delta(\mathfrak{l}_{1+2i}) = \sum_j \sum_{i=1}^{k_j} \Delta(\mathfrak{l}_{j_i}) \\ &= \sum_{i=1}^{k_1} \Delta(\mathfrak{l}_{1_i}) + \sum_{j \neq 1} \sum_{i=1}^{k_j} \Delta(\mathfrak{l}_{j_i}) \\ &= \left(-k_+ \frac{\theta}{k_+ - k_-} + k_- \frac{\theta}{k_+ - k_-} \right) + \sum_{i=1}^{k_1} \Delta_{\text{old}}(\mathfrak{l}_{1_i}) + \sum_{j \neq 1} \sum_{i=1}^{k_j} \Delta(\mathfrak{l}_{j_i}) \\ &= -\theta + \sum_{i=1}^{k_1} \Delta_{\text{old}}(\mathfrak{l}_{1_i}) + \sum_{j \neq 1} \sum_{i=1}^{k_j} \Delta(\mathfrak{l}_{j_i}) \\ &= -\theta + \theta = 0, \end{aligned}$$

where in the last step we used the original definition of θ . Thus after this adjustment we have forced the total sum of the relative gains to vanish.

Step 4: prominence. We now introduce a discrete “height bookkeeping” function, the *prominence*, defined at the endpoints of the segments. Informally, P measures the relative height of the Legendrian realization of $\mathfrak{a}$ compared to the Legendrian graph G .

If $\mathfrak{l}_i$ is a segment, we denote its starting point by $\partial^-\mathfrak{l}_i$ and its endpoint by $\partial^+\mathfrak{l}_i$. Define P inductively by

$$\begin{cases} P(\partial^-\mathfrak{l}_1) = 0, \\ P(\partial^+\mathfrak{l}_i) = P(\partial^-\mathfrak{l}_i) + \Delta(\mathfrak{l}_i), & \text{if } i \text{ is odd (segment in a 1-handle),} \\ P(\partial^+\mathfrak{l}_i) = P(\partial^-\mathfrak{l}_i), & \text{if } i \text{ is even (segment in a 0-handle).} \end{cases}$$

Since $\partial^+\mathfrak{l}_n = \partial^-\mathfrak{l}_1$, the prominence is well-defined precisely because $\sum_i \Delta(\mathfrak{l}_i) = 0$ by (3.4.2).

For segments in 0-handles we will simply speak of “the prominence of the segment”, since the two endpoints have the same value of P .

Step 5: Legendrian realization in a 1-handle. We now explain how to Legendrian realize the segments of $\mathfrak{a}$ that lie in a fixed 1-handle H . We will use the front projection of the Legendrian core e as a template.

Remark 3.16. In this step and the next, for readability we will not distinguish notationally between subsets of $\mathbb{R}^3$ and their front projections.

Let $\mathfrak{l}_{j_1}, \dots, \mathfrak{l}_{j_k}$ be the segments of $\mathfrak{a}$ in H , and let e be the Legendrian core of H (see Figure 3.16).

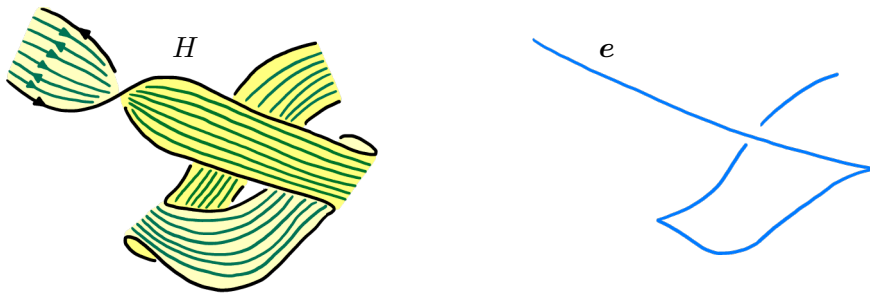


Figure 3.16. Left: a 1-handle H with six segments of $\mathfrak{a}$. Right: the Legendrian core $e = G \cap H$.

Remove from e a small open subsegment disjoint from cusps and crossings. Denote the two resulting components by e^+ and e^- , see Figure 3.17 (left).¹

Create k vertical translates of $e^+ \sqcup e^-$ in the front projection (Figure 3.17, right). These copies of $e^+ \sqcup e^-$ will form part of the Legendrian realization of the segments

¹Choosing which component is called e^+ and which e^- is arbitrary; the final result does not depend on this choice.



Figure 3.17. Left: the two components e^+ and e^- of e after removing a small open interval. Right: k vertical translates of $e^+ \sqcup e^-$.

$l_{j_1}, \dots, l_{j_k}$. What remains is to connect the copies of e^+ to the copies of e^- with a Legendrian braid.

For each segment l_{j_*} , denote by $p_*^\pm$ the endpoint of l_{j_*} lying on the same side as $e^\pm$, i.e. the endpoint in the component of the attaching region containing an endpoint of $e^\pm$ (see Figure 3.18).

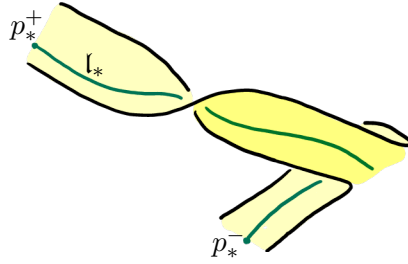


Figure 3.18. A segment l_* in H with endpoints $p_*^\pm$ on the sides of $e^\pm$.

Label the translates of $e^\pm$ by $e^\pm(1), \dots, e^\pm(k)$ as follows: the copy labeled $e^\pm(l)$ lies higher in z than the copy $e^\pm(m)$ if

- $P(p_l^\pm) > P(p_m^\pm)$, or
- $P(p_l^\pm) = P(p_m^\pm)$ but $P(p_l^\mp) > P(p_m^\mp)$.

For each $i = 1, \dots, k$, connect $e^+(i)$ to $e^-(i)$ by a straight segment in the front projection. The resulting Legendrian arc is declared to be the Legendrian realization of l_{j_i} .

Remark 3.17. Informally, the Legendrian realization of a segment l_{j_*} looks like a copy of the core e , starting at relative height $P(\partial^- l_{j_*})$ and ending at relative height $P(\partial^+ l_{j_*})$, with a jump of size $\Delta(l_{j_*})$ along the way. The combinatorics of the braid record how these jumps for different segments interleave.

Figure 3.19 shows a concrete choice of prominences for the segments in Figure 3.16, and Figure 3.20 and 3.19 show the corresponding braid.

Note that the data in Figure 3.16 is not sufficient to determine the Legendrian braid we should place in Figure 3.17, right. Indeed, only from Figure 3.16, we can at most

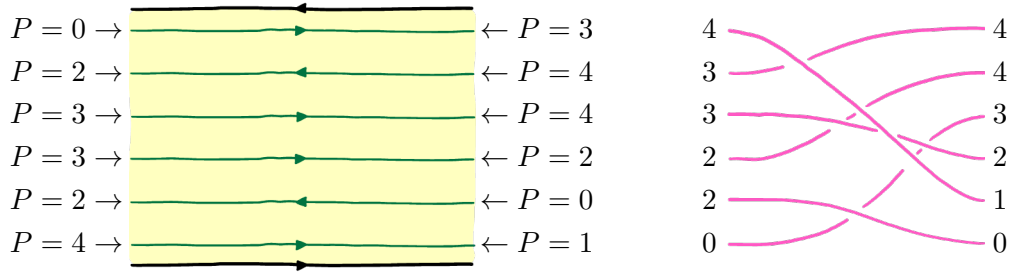


Figure 3.19. Left: prominences at the endpoints of the segments in Figure 3.16. Right: schematic picture of the braid, with strands labeled by the prominence on each side.

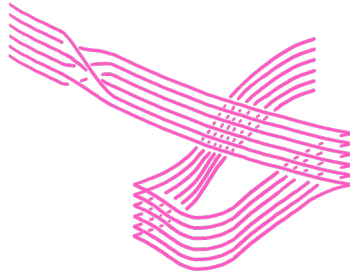


Figure 3.20. Legendrian realization of the segments of Figure 3.16, using the prominence data of Figure 3.19.

compute the displacement index Δ of all the segments. However, to determine the prominence, we need to know how the rest of the curve $\mathbf{a}$ looks outside the 1-handle we are considering. For this reason, we must specify the prominence of at least one endpoint of each segment.

Step 6: Legendrian realization in a 0-handle. Let H be a 0-handle, and let $\mathfrak{l}_{j_1}, \dots, \mathfrak{l}_{j_k}$ be the segments of $\mathbf{a}$ contained in H . Let $e_1, \dots, e_n$ be the Legendrian segments in the Legendrian skeleton of H , meeting at the vertex v (Figure 3.21).

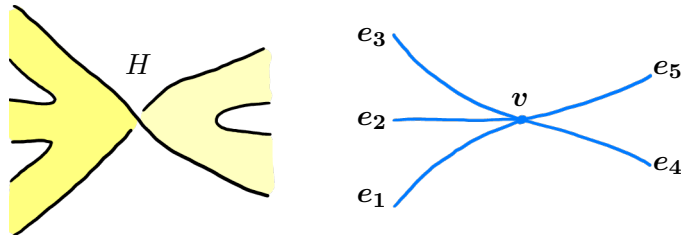


Figure 3.21. A 0-handle H and its Legendrian skeleton.

The Legendrian realization of a segment $\mathfrak{l}_*$ in H is obtained by concatenating two of the Legendrian segments $e_1, \dots, e_n$, possibly translated in the ∂_z direction and slightly perturbed. More precisely, if $\mathfrak{l}_*$ is the segment of $\mathbf{a}$ that joins a 1-handle H_1 to a 1-handle H_2 , then its Legendrian realization joins the corresponding pieces of the edges of G passing through H_1 and H_2 .

The relative position of the Legendrian realizations is determined by the prominence. Since both endpoints of a segment l_* in a 0-handle have the same prominence, we write simply $P(l_*)$ for that common value.

We place a segment l_{j_l} above a segment l_{j_m} if $P(l_{j_l}) > P(l_{j_m})$. For example, if

$$P(l_{j_4}) < P(l_{j_2}) < P(l_{j_3}) < P(l_{j_1}) < P(l_{j_5}),$$

then the Legendrian realizations are stacked vertically as in Figure 3.22.

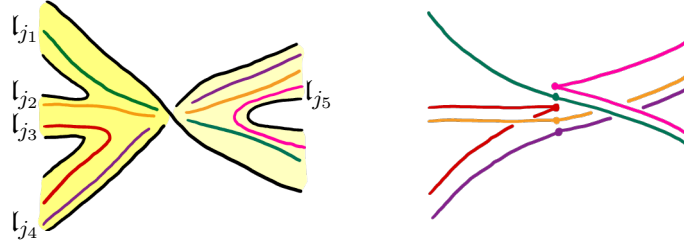


Figure 3.22. Left: a 0-handle H with several segments of $\mathbf{a}$. Right: their Legendrian realizations when all prominences are distinct. The dots indicate the relative vertical position.

In the case where $P(l_{j_l}) = P(l_{j_m})$, we again draw the Legendrian realization of these segments as the concatenation of two of the Legendrian segments of the Legendrian skeleton of H , but we perturb them slightly to avoid overlaps. The required perturbation visually matches the relative position of the segments l_{j_l} and l_{j_m} in H . See Figure 3.23 for an example.

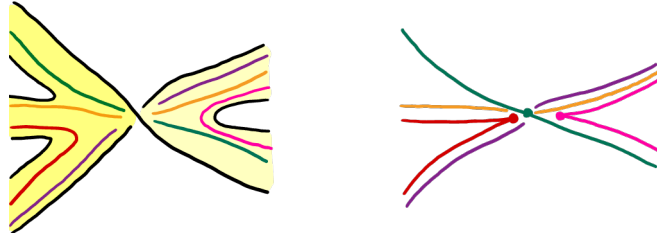


Figure 3.23. Left: a 0-handle with several segments of $\mathbf{a}$ of equal prominence. Right: their Legendrian realizations after a small perturbation.

Step 7: connecting the pieces. Finally, we connect the Legendrian realizations of consecutive segments. For each i , we join the endpoint of the Legendrian realization of l_i to the starting point of the Legendrian realization of l_{i+1} by a straight Legendrian segment.

3.5. An example

In this section we illustrate how the algorithm is applied in practice. Consider the Legendrian graph G in Figure 3.24.

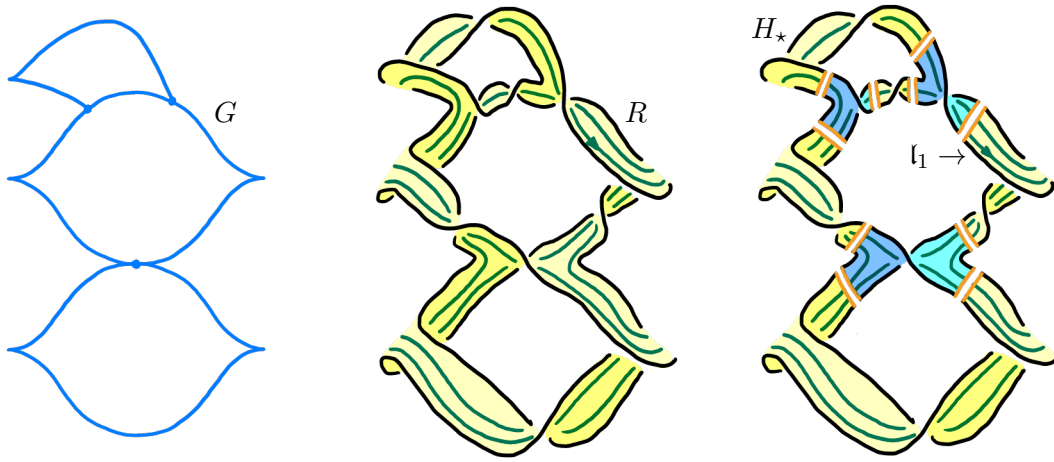


Figure 3.24. Left: the Legendrian graph G . Middle: a ribbon surface R of G and an oriented, homologically nontrivial simple closed curve α on R (in green). Right: the handle decomposition of R and the curve α subdivided into segments as in Step 1.

In the middle of Figure 3.24 we see a ribbon surface R of G with an oriented, homologically nontrivial simple closed curve α (drawn in green). On the right of Figure 3.24 we see the handle decomposition of R (three 0-handles in blue, and six 1-handles in yellow), together with the curve α subdivided into segments as prescribed after Step 1. In particular, Figure 3.24 also indicates which segment is l_1 and which 1-handle is the distinguished handle $H_\star$.

If we compute the relative gain of the segments contained in the 1-handles, we find that all such segments have the same strictly negative relative gain, except for:

- the segment in $H_\star$, whose relative gain must later be adjusted; and
- the segment in the unique 1-handle that contains only one segment and is not $H_\star$ (it is l_7 and its relative gain is 0).

More concretely, set the prominence of $\partial^- l_1$ to be zero, that is $P(\partial^- l_1) = 0$. Then

$$P(\partial^+ l_1) = -\varepsilon, \quad P(l_2) = -\varepsilon, \quad P(\partial^- l_3) = -\varepsilon, \quad P(\partial^+ l_3) = -2\varepsilon,$$

and so on, until we reach l_7 , the segment that is alone in the 1-handle which is not $H_\star$. At that point

$$P(l_6) = P(\partial^- l_7) = P(\partial^+ l_7) = P(l_8) = -3\varepsilon.$$

From there on, segments in the 1-handles continue to have relative gain $-\varepsilon$ until we reach the second-to-last segment, l_{15} , which lies in $H_\star$. We have $P(\partial^- l_{15}) = -5\varepsilon$, and since at the end we must return to prominence 0, we set

$$\Delta(l_{15}) = 5\varepsilon,$$

so that $P(\partial^+ l_{15}) = 0$. Consequently $P(l_{16}) = 0$, which agrees with $P(\partial^- l_1)$, as required.

For the discussion below, it is helpful to look at Figures 3.25, 3.26, 3.27, and 3.28 while reading. In these figures we see, successively, the Legendrian realizations of l_1 , then of l_1 and l_2 , and so on, until we obtain the Legendrian realization of the entire curve α .

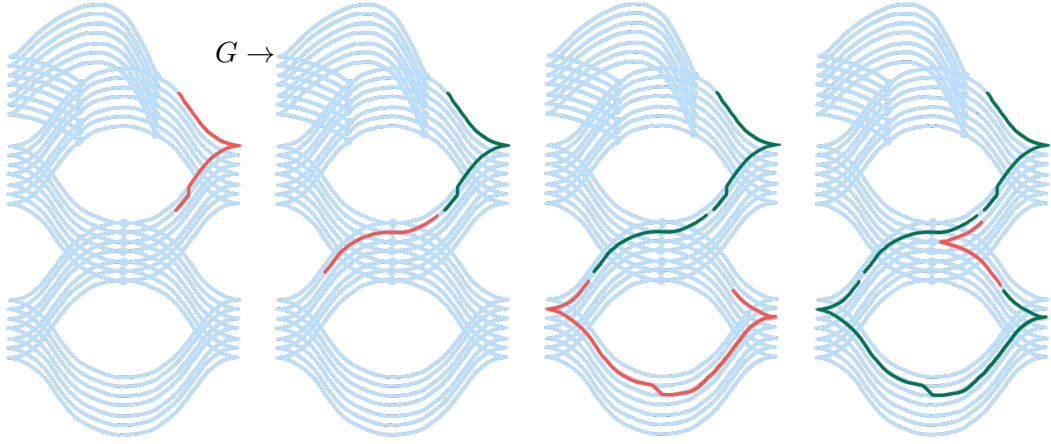


Figure 3.25. From left to right: the Legendrian realizations of the segments l_1 through l_4 .

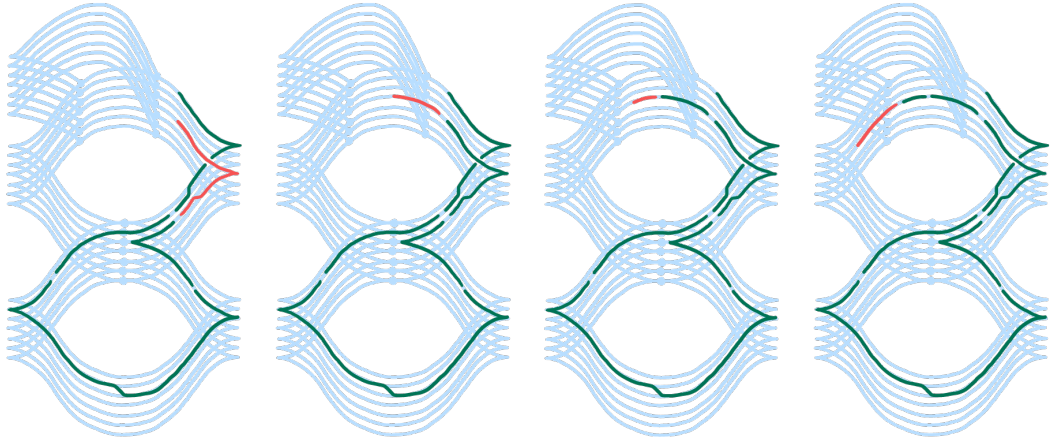


Figure 3.26. From left to right: the Legendrian realizations of the segments l_5 through l_8 .

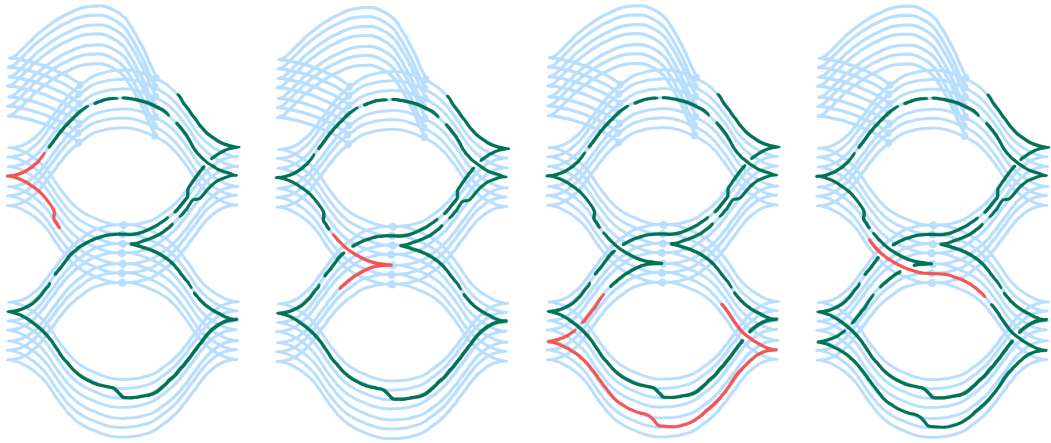


Figure 3.27. From left to right: the Legendrian realizations of the segments l_9 through l_{12} .

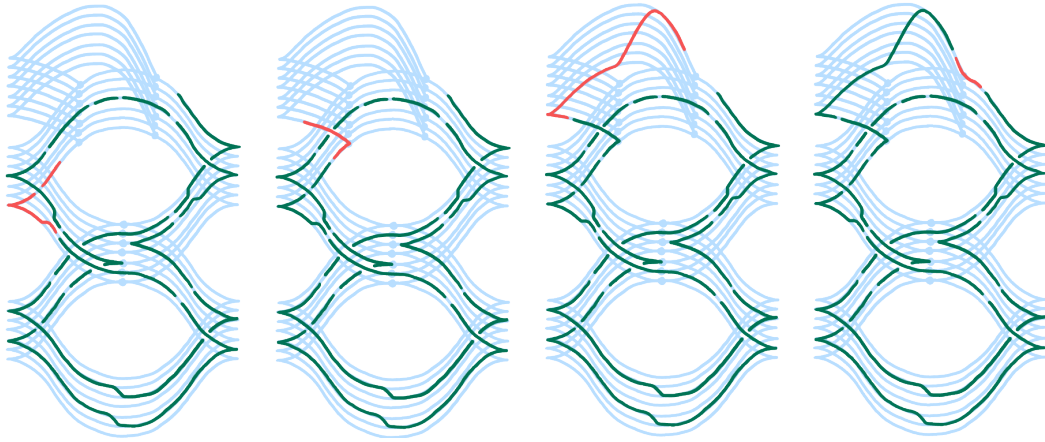


Figure 3.28. From left to right: the Legendrian realizations of the segments l_{13} through l_{16} .

To draw the Legendrian knot explicitly, we can apply the algorithm in a slightly different order from the one described in the general construction. Instead of first Legendrian realizing all segments in the 1–handles and then all segments in the 0–handles, we Legendrian realize the segments in the order in which they appear along α : first l_1 , then l_2 , and so on. This works in the present example because the prominence is monotone up to the second–to–last segment. In general one can still proceed in this way, but one may run out of space in the front projection if a new segment needs to be drawn between two previously drawn segments.

Denote by $G(s)$ the graph G translated vertically by an amount s . We draw

$$G(0), G(-\varepsilon), G(-2\varepsilon), \dots, G(-5\varepsilon).$$

These are the light blue graphs in Figures 3.25–3.28.

Consider first the Legendrian realization of l_1 . Initially it coincides with a portion of an edge of $G(0)$, and then it “jumps” by an amount $-\varepsilon$ so that it becomes a portion of an edge of $G(-\varepsilon)$; see the first picture in Figure 3.25. The Legendrian realization of l_2 stays at prominence $-\varepsilon$. The Legendrian realization of l_3 then jumps by $-\varepsilon$ to land on $G(-2\varepsilon)$, and so on, until the Legendrian realization of the segment l_{15} jumps from $G(-5\varepsilon)$ back to $G(0)$; see the second–to–last picture in Figure 3.28.

In Figure 3.29 we see the final Legendrian knot produced by the algorithm.

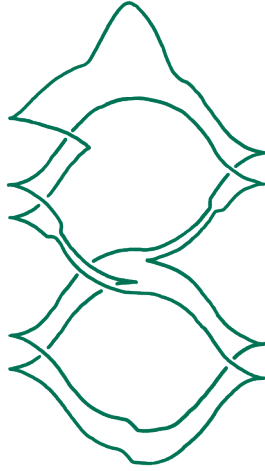


Figure 3.29. The Legendrian realization of the curve $\mathbf{a}$ from Figure 3.24 obtained by applying the algorithm.

3.6. Justification of the algorithm

The algorithm clearly produces a knot L (and not a link): we connect the Legendrian realization of a segment $\mathfrak{l}_i$ to the Legendrian realization of the consecutive segment $\mathfrak{l}_{i+1}$ of $\mathbf{a}$, and $\mathbf{a}$ is a connected closed curve. Moreover, L is a generic Legendrian knot, as all the front projections we have produced are generic and never violate the Legendrian condition.

We now make precise what remains to be shown and encapsulate it in a single statement.

Proposition 3.18 (Validity of the algorithm). *Let $G \subset (\mathbb{R}^3, \xi_{\text{st}})$ be a generic Legendrian graph, let $\mathcal{S}$ be the abstract ribbon surface of G , and let $\mathbf{a}$ be a homologically nontrivial simple closed curve on $\mathcal{S}$. Let L be the Legendrian knot produced by the algorithm of Theorem 3.12. Then the following hold:*

- *There exists a generic ribbon surface Σ of G that can be perturbed relative to the boundary, through surfaces transverse to ∂_z , so as to contain L .*
- *The isotopy class of L in the perturbation of Σ agrees with the isotopy class of $\mathbf{a}$.*
- *The perturbation of Σ can be chosen to be arbitrarily C^0 -small if the algorithm is applied carefully.*

The rest of this section is devoted to the proof of Proposition 3.18. We split the argument into several lemmas corresponding to the subsequent subsections.

3.6.1. Choosing the ribbon surface.

Lemma 3.19. *There exists a generic ribbon surface Σ of G with the following properties:*

- *In each 1-handle there is exactly one hyperbolic singularity of the characteristic foliation.*
- *This hyperbolic singularity is located precisely in the region where the braid of Step 5 is placed (compare Figure 3.16 and Figure 3.20).*

Proof. This can be achieved by first applying the algorithm that produces a ribbon surface from a Legendrian graph, then applying the simplifications of Figure 3.7, and finally sliding the remaining hyperbolic points along the core of each 1-handle until each lies precisely in the region where the braid of Step 5 is placed. $\square$

Example 3.20. The ribbon in Figure 3.16 is compatible with the braid in Figure 3.20. By contrast, the ribbons in Figure 3.30 are not: the first has too many singularities in the 1-handle, and in the other two the remaining hyperbolic point sits in the wrong place relative to the braid.

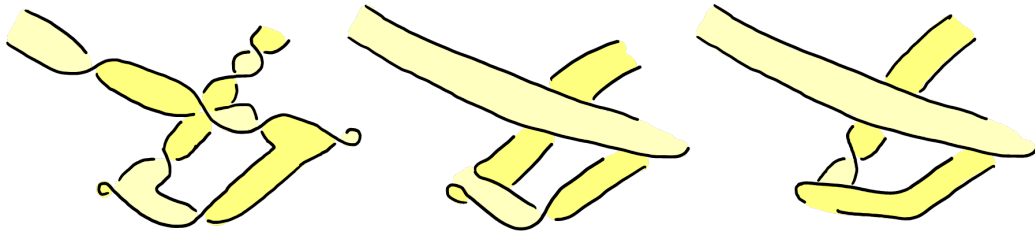


Figure 3.30. Ribbon surfaces that are not adapted to the braid in Figure 3.20: too many singularities (left) or hyperbolic point in the wrong region (middle and right).

The choice of where to place the braid in Step 5 determines a preferred representative Σ of the abstract ribbon, but Theorem 3.12 imposes no restriction on Σ beyond genericity, so this choice causes no issues.

3.6.2. Legendrian realization of individual segments. Let $\mathfrak{l}_i: [0, 1] \rightarrow (\mathbb{R}^3, \xi_{\text{st}})$ be one of the segments of $\mathfrak{a}$, parametrized compatibly with its orientation. Denote its components by

$$\mathfrak{l}_i(s) = ((\mathfrak{l}_i)_x(s), (\mathfrak{l}_i)_y(s), (\mathfrak{l}_i)_z(s)).$$

We Legendrian realize $\mathfrak{l}_i$ by keeping its Lagrangian projection (x, y) and adjusting its z -coordinate. Given a number $z_i \in \mathbb{R}$, define

$$\overline{(\mathfrak{l}_i)_z}(s) := z_i - \int_0^s (\mathfrak{l}_i)_x(u) (\mathfrak{l}_i)'_y(u) du,$$

and set

$$\overline{\mathfrak{l}_i}(s) := ((\mathfrak{l}_i)_x(s), (\mathfrak{l}_i)_y(s), \overline{(\mathfrak{l}_i)_z}(s)).$$

By construction, $\overline{\mathfrak{l}_i}$ is a Legendrian curve with $\overline{(\mathfrak{l}_i)_z}(0) = z_i$.

Remark 3.21. At this stage, the value of z_i is arbitrary for each segment, so the family $\{\overline{\mathfrak{l}_i}\}_i$ does not yet patch into a closed Legendrian curve. We are currently only interested in the local shape of the front projection of $\overline{\mathfrak{l}_i}$ inside a handle; the global choice of the z_i will be fixed later when we glue the segments together.

Next, we describe how these segments $\overline{\mathfrak{l}_i}$ look and why they match (individually) the segments produced by the algorithm.

3.6.3. 1–handles.

Lemma 3.22. *Let H be a 1–handle of the ribbon surface Σ , and let $\mathfrak{l}_i$ be the segments of $\mathfrak{a}$ contained in H . After isotoping $\mathfrak{a}$ inside H , we can assume the Legendrian realizations $\overline{\mathfrak{l}_i}$ satisfy the following:*

(1) For $H \neq H_\star$, the function

$$h_i(s) := \overline{(\mathfrak{l}_i)_z}(s) - (\mathfrak{l}_i)_z(s)$$

is constant on $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$, and monotone on $[\frac{1}{3}, \frac{2}{3}]$:

- if $\mathfrak{l}_i$ is the central (already Legendrian) segment, then h_i is constant everywhere;
- otherwise, h_i is strictly decreasing on $[\frac{1}{3}, \frac{2}{3}]$ if $\mathfrak{l}_i$ is positively transverse, and strictly increasing if it is negatively transverse;

(2) The “jump”

$$D_i := h_i(1) - h_i(0) = h_i\left(\frac{2}{3}\right) - h_i\left(\frac{1}{3}\right)$$

can be arranged, after a suitable isotopy of $\mathfrak{a}$ inside H , to satisfy

$$D_i = \varepsilon_1 \Delta(\mathfrak{l}_i),$$

for some arbitrarily small $\varepsilon_1 > 0$.

(3) For the distinguished handle $H_\star$, the same relation $D_i = \varepsilon_1 \Delta(\mathfrak{l}_i)$ holds after the additional adjustment of the $\Delta(\mathfrak{l}_i)$ prescribed in (3.4.1) and the small isotopy described in Figure 3.32.

Proof. Let H be a 1–handle distinct from $H_\star$. Isotope the segments of $\mathfrak{a}$ inside H so that:

- outside a small neighborhood of the unstable separatrix of the unique hyperbolic singularity of the characteristic foliation, each segment is tangent to the leaves of the foliation;
- inside that neighborhood, the segments become short arcs transverse to the foliation, arranged symmetrically around the hyperbolic point as in Figure 3.31(right).

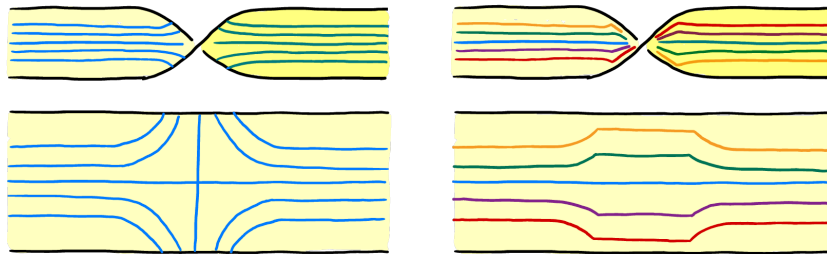


Figure 3.31. Left: a 1–handle and its characteristic foliation, with a single hyperbolic point. Right: the segments of $\mathfrak{a}$ arranged along the leaves, except near the unstable separatrix. Top pictures: front projections; bottom pictures: schematic versions.

This clearly implies (1).

By pushing the segments of $\mathfrak{a}$ closer to the central one we can make all $|D_i|$ as small as desired. Moreover, segments that are further away from the center have larger $|D_i|$. In particular, for any small $\varepsilon_1 > 0$ we can arrange, after a suitable isotopy of $\mathfrak{a}$ inside H , that

$$D_i = \begin{cases} -|\tilde{\Delta}(\mathfrak{l}_i)| \varepsilon_1, & \text{if } \mathfrak{l}_i \text{ is positively transverse,} \\ +|\tilde{\Delta}(\mathfrak{l}_i)| \varepsilon_1, & \text{if } \mathfrak{l}_i \text{ is negatively transverse.} \end{cases}$$

It follows that

$$(3.6.1) \quad D_i = \varepsilon_1 \Delta(\mathfrak{l}_i),$$

because:

- a positively transverse segment $\mathfrak{l}_i$ with $\tilde{\Delta}(\mathfrak{l}_i)$ negative is oriented as $\mathbf{e}$;
- a positively transverse segment $\mathfrak{l}_i$ with $\tilde{\Delta}(\mathfrak{l}_i)$ positive is not oriented as $\mathbf{e}$;
- a negatively transverse segment $\mathfrak{l}_i$ with $\tilde{\Delta}(\mathfrak{l}_i)$ negative is not oriented as $\mathbf{e}$;
- a negatively transverse segment $\mathfrak{l}_i$ with $\tilde{\Delta}(\mathfrak{l}_i)$ positive is oriented as $\mathbf{e}$.

This proves (2).

When $H = H_\star$, the values of $\Delta(\mathfrak{l}_i)$ were modified by (3.4.1). To keep (3.6.1) valid, we first arrange the segments in $H_\star$ symmetrically as in Figure 3.31, and then apply a small isotopy that pushes all of them slightly up or down as in Figure 3.32. This breaks the symmetry but allows us to realize the adjusted values of Δ as in (3.6.1).

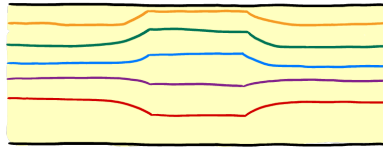


Figure 3.32. Segments of $\mathfrak{a}$ in the distinguished handle $H_\star$ after the additional isotopy used to implement the modified gains.

This proves (3). □

A good way to understand how $\bar{\mathfrak{l}}_i$ looks is as follows. Consider the ribbon surface Σ . Any surface obtained by translating Σ in the ∂_z direction is again a ribbon surface with all the properties of Σ , as ∂_z is a contact vector field. Denote by $\Sigma(q)$ the ribbon surface Σ translated vertically by an amount $q \in \mathbb{R}$.

The Legendrian segment $\bar{\mathfrak{l}}_i|_{[0, \frac{1}{3}]}$ is embedded in the ribbon surface $\Sigma(h_i(0))$, as this segment is just a copy of $\mathfrak{l}_i|_{[0, \frac{1}{3}]}$ translated vertically by $h_i(0)$. Similarly, the Legendrian segment $\bar{\mathfrak{l}}_i|_{[\frac{2}{3}, 1]}$ is embedded in the ribbon surface $\Sigma(h_i(1))$. On the other hand, the Legendrian segment $\bar{\mathfrak{l}}_i|_{[\frac{1}{3}, \frac{2}{3}]}$ is not a simple vertical translation of $\mathfrak{l}_i|_{[\frac{1}{3}, \frac{2}{3}]}$. Instead, it is a Legendrian segment that “jumps” by an amount D_i from the ribbon surface $\Sigma(h_i(0))$ to the ribbon surface $\Sigma(h_i(1)) = \Sigma(h_i(0) + D_i)$. See Figure 3.33 for an example.

For example, if we take the segments in Figure 3.31 (right) as inputs, we can imagine that their Legendrian realizations will have “jumps” as shown in Figure 3.34, where the

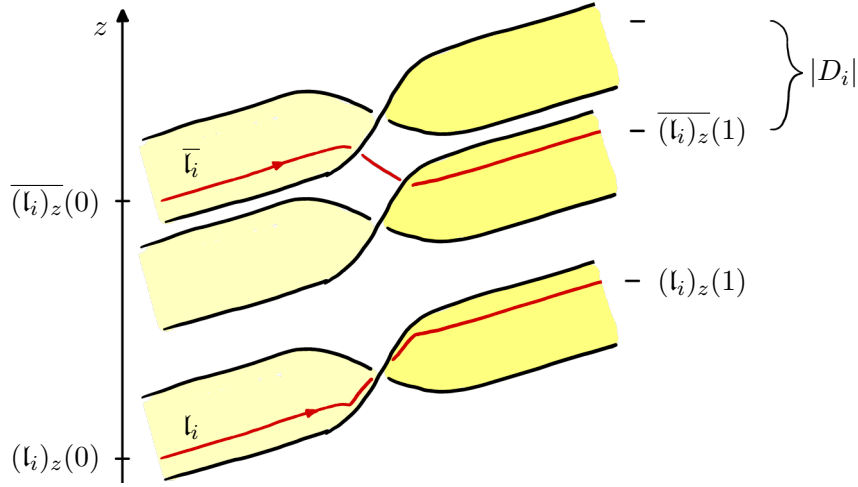


Figure 3.33. Legendrian realization $\overline{l_i}$ of a segment l_i in a 1-handle. The lower surface is Σ , the others are vertical translates.

external segments (red and orange) “jump” more than the internal segments (purple and green). The blue segment is not drawn, as its Legendrian realization will simply be a copy of itself translated vertically by a certain amount.

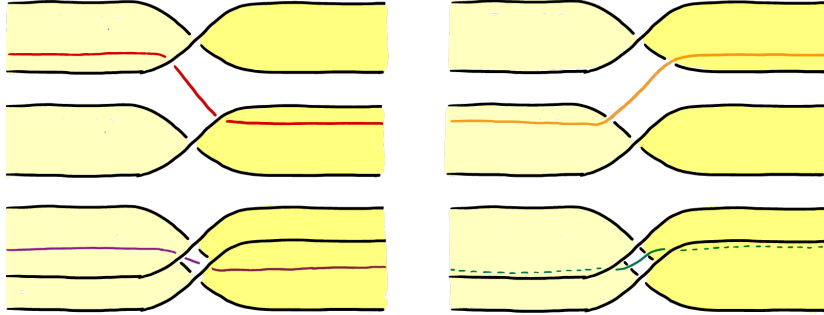


Figure 3.34. The Legendrian realization of the segments in Figure 3.31, right.

3.6.4. 0-handles.

Lemma 3.23. *Let H be a 0-handle of Σ , and let l_i be the segments of $\mathbf{a}$ contained in H . After isotoping the segments to follow the leaves of the characteristic foliation (except in a small neighborhood of the vertex $\mathbf{v}$) and modifying each segment near $\mathbf{v}$ by a small “bump” as in Figure 3.35, the associated functions*

$$h_i(s) := \overline{(l_i)_z}(s) - (l_i)_z(s)$$

satisfy

$$h_i(0) = h_i(1)$$

for all segments in 0-handles.

Proof. We isotope the segments of $\mathbf{a}$ in H so that they follow the leaves of the characteristic foliation, except in a small neighborhood of the vertex $\mathbf{v}$, where they are slightly displaced to avoid intersecting each other. For each segment $\mathfrak{l}_i$ we again consider

$$h_i(s) := \overline{(\mathfrak{l}_i)_z}(s) - (\mathfrak{l}_i)_z(s).$$

We can assume that h_i is constant on $[0, \frac{1}{3}]$ and on $[\frac{2}{3}, 1]$, since the segment is already Legendrian there. By making each segment follow the leaves of the foliation more closely, we can ensure that $|h_i(0) - h_i(1)| < \varepsilon_0$ for any prescribed $\varepsilon_0 > 0$. We then correct this discrepancy to exactly 0 by introducing a small “bump” near the vertex, as in Figure 3.35, whose Legendrian realization cancels the mismatch.

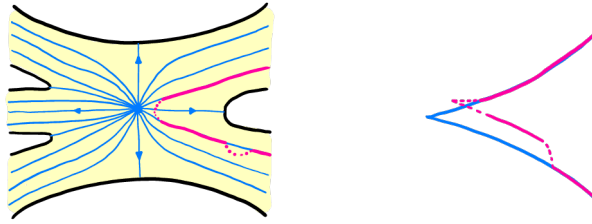


Figure 3.35. Left: a segment $\mathfrak{l}_i$ in a 0-handle, isotoped into the desired position (purple). Right: comparison between the leaves of the characteristic foliation (blue) and the Legendrian realization of $\mathfrak{l}_i$ (purple). The bump near the vertex corrects the mismatch in z at the endpoints.

After this correction, we have $h_i(0) = h_i(1)$ for all segments in 0-handles. $\square$

3.6.5. Gluing the segments together.

Lemma 3.24. *Let $\overline{\mathfrak{l}_i}$ be the Legendrian realizations of the segments of $\mathbf{a}$ constructed above. Then:*

- (1) *There exist constants z_i in the definition of $\overline{\mathfrak{l}_i}$ such that the segments $\overline{\mathfrak{l}_i}$ glue to a Legendrian knot L' .*
- (2) *The prominence P is a discrete version of the function h , in the sense that for the endpoints $\partial^\pm \mathfrak{l}_i$ of a segment $\mathfrak{l}_i$ we have*

$$h_i(0) = \varepsilon_1 P(\partial^- \mathfrak{l}_i) \quad \text{and} \quad h_i(1) = \varepsilon_1 P(\partial^+ \mathfrak{l}_i),$$

where $D_i = \varepsilon_1 \Delta(\mathfrak{l}_i)$.

- (3) *The ribbon surface Σ can be perturbed relative to $\partial\Sigma$ so that it contains L' , and this perturbation can be chosen arbitrarily C^0 -small by isotoping $\mathbf{a}$ closer to the cores of the handles.*

Proof. Set

$$z_1 := (\mathfrak{l}_1)_z(0), \quad z_{i+1} := \overline{(\mathfrak{l}_i)_z}(1) \quad \text{for } i = 1, \dots, n-1.$$

Now the segments glue together to give a Legendrian curve L' . To prove that L' closes up to a knot we need to show that $(\mathfrak{l}_n)_z(1) = (\mathfrak{l}_1)_z(0)$. This closure condition is equivalent to

$$\oint_{\mathbf{a}} x \, dy = 0,$$

and one verifies using (3.6.1) and the definition of the prominence that this holds precisely because $\sum_i \Delta(\mathfrak{l}_i) = 0$ (equation (3.4.2)). This proves (1).

Assertion (2) is an easy consequence of Lemma 3.22(2) and (3), together with the fact that, after the choice of z_i as above, $h_1(0) = 0 = P(\partial^-\mathfrak{l}_1)$.

The ribbon surface Σ can indeed be perturbed as described in Theorem 3.12, since L' only differs from $\mathfrak{a}$ in the z component, and the surface Σ is transverse to ∂_z and contains $\mathfrak{a}$ in its interior. An example of such a perturbation is shown schematically in Figure 3.36.



Figure 3.36. Left: a section of the original ribbon Σ : green points represent $\mathfrak{a}$, pink points represent L' . Right: a small perturbation of Σ that contains L' .

Because the quantities ε_1 and ε_0 in the construction of the h_i can be chosen arbitrarily small by isotoping $\mathfrak{a}$ closer to the cores of the handles, this perturbation can be made arbitrarily C^0 -small. Figure 3.37 shows schematically how bringing $\mathfrak{a}$ closer to the core reduces the vertical displacement between $\mathfrak{a}$ and L' .



Figure 3.37. Pushing $\mathfrak{a}$ closer to the core of the handle decreases the vertical distance between $\mathfrak{a}$ and its Legendrian realization L' , allowing the perturbation of Σ to be arbitrarily C^0 -small.

This proves (3). □

Proof of Proposition 3.18. The fact that the algorithm produces a generic Legendrian knot L is explained at the beginning of this section. Consider the Legendrian knot L' given by Lemma 3.24(1). Assume for a moment that $L = L'$. Then Lemma 3.24(3) shows that Σ can be perturbed relative to $\partial\Sigma$ so that it contains $L = L'$, and this perturbation can be chosen arbitrarily C^0 -small by isotoping $\mathfrak{a}$ closer to the cores of the handles. Moreover, the isotopy class of $L = L'$ in the perturbation of Σ clearly equals the isotopy class of $\mathfrak{a}$, since L' and $\mathfrak{a}$ have the same Lagrangian projection and we have perturbed Σ through surfaces transverse to ∂_z (L' is the actual Legendrian realization of $\mathfrak{a}$).

It remains to show that $L = L'$, that is, that the Legendrian segments produced by the algorithm match the Legendrian segments $\bar{\mathfrak{l}}_i$.

For segments of $\mathfrak{a}$ in a 1-handle, we have already explained how their Legendrian realizations look, up to translation in the ∂_z direction. Essentially, the Legendrian realization $\bar{\mathfrak{l}}_i$ of a segment $\mathfrak{l}_i$ initially looks like a copy of the Legendrian core $\mathfrak{e}$, translated up or down relative to the Legendrian core by an amount $h_i(0)$. When $\mathfrak{l}_i$ is transverse, it changes its relative height with respect to the Legendrian core, and then it again looks like a copy of $\mathfrak{e}$ at a relative height of $h_i(1)$.

Given a 1-handle H with k segments of $\mathfrak{a}$, Step 5 requires us to draw k different copies of $\mathfrak{e}$ minus a small interval and then merge them together using a prescribed

Legendrian braid. The connected components of the k different copies of e minus a small interval correspond to the parts of the Legendrian realization of the segments where these realizations look like copies of e . The strands of the braid that connect these components represent the Legendrian realizations of the transverse parts of the segments. This procedure produces Legendrian segments that properly represent the Legendrian realization of the k segments, as can be verified by carefully examining all possible cases.

We illustrate this with a concrete example. Assume that $\mathfrak{l}_i$ and $\mathfrak{l}_j$ are two segments of α oriented in the same direction in a 1-handle H . Their relative position in the 1-handle H might be as shown in Figure 3.38(a) or Figure 3.38(b).¹



Figure 3.38. (a) The segment that starts higher (green) is $\mathfrak{l}_i$, while the lower segment (purple) is $\mathfrak{l}_j$. (b) The positions of $\mathfrak{l}_i$ and $\mathfrak{l}_j$ are reversed.

This relative position may be determined by the prominence of their endpoints. Without loss of generality, assume $h_i(0) \geq h_j(0)$. Then we have three cases: $h_i(1) > h_j(1)$, $h_i(1) = h_j(1)$, or $h_i(1) < h_j(1)$.

In the case where $h_i(1) > h_j(1)$ and $h_i(0) > h_j(0)$, Step 5 ensures that the Legendrian realization $\bar{\mathfrak{l}}_i$ of $\mathfrak{l}_i$ is always above the Legendrian realization $\bar{\mathfrak{l}}_j$ of $\mathfrak{l}_j$. This is consistent with the definition of the function h , as this function encodes how high the Legendrian realization of a given segment is with respect to the Legendrian core of the handle. See Figure 3.39 for an example.

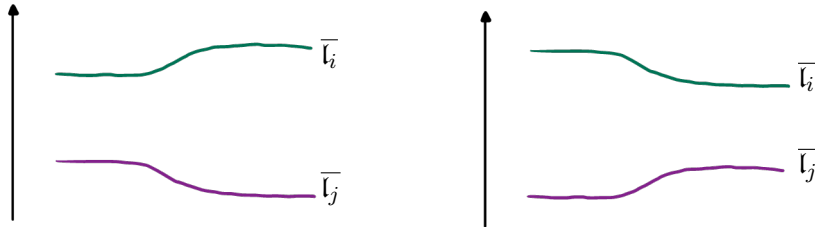


Figure 3.39. Left: the Legendrian realization of $\mathfrak{l}_i$ and $\mathfrak{l}_j$ in the situation of Figure 3.38(a). Right: the Legendrian realization of $\mathfrak{l}_i$ and $\mathfrak{l}_j$ in the situation of Figure 3.38(b).

If $h_i(1) > h_j(1)$ and $h_i(0) = h_j(0)$, then again Step 5 tells us that the Legendrian realization of $\mathfrak{l}_i$ is a Legendrian segment always above the Legendrian realization of $\mathfrak{l}_j$. The reason for this is slightly less obvious, as $h_i(0) = h_j(0)$ indicates that the Legendrian realizations of the two segments start at the same relative height. More precisely, this means that the two Legendrian realizations are initially both embedded in a copy of the ribbon surface translated vertically by an amount $h_i(0) = h_j(0)$. The fact that $h_i(1) > h_j(1)$ indicates that the Legendrian realization of the segment $\mathfrak{l}_i$ then jumps to a copy of the ribbon surface that is higher than the copy of the ribbon surface to which the

¹It is part of the example that the 1-handle looks as in these figures.

Legendrian realization of $\mathfrak{l}_j$ jumps. Because of this, it follows that the relative position of the two segments in H is as in Figure 3.38(a), and their Legendrian realizations are as in Figure 3.40(a).

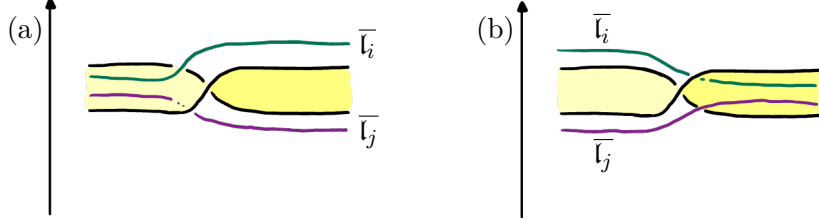


Figure 3.40. (a) The Legendrian segments $\bar{l}_i$ and $\bar{l}_j$ are initially embedded in the same ribbon surface at height $h_i(0) = h_j(0)$. The 1-handle of this ribbon surface is depicted. (b) The Legendrian segments $\bar{l}_i$ and $\bar{l}_j$ are eventually embedded in the same ribbon surface at height $h_i(1) = h_j(1)$. Again, the 1-handle of this ribbon surface is depicted.

If $h_i(1) = h_j(1)$, then we have $h_i(0) > h_j(0)$, and it follows that the relative position of the two segments in H is as in Figure 3.38(b). Therefore, the Legendrian realization of the segment $\mathfrak{l}_i$ lies entirely above the Legendrian realization of the segment $\mathfrak{l}_j$. See Figure 3.40(b).

If $h_i(1) < h_j(1)$ and $h_i(0) = h_j(0)$, then the relative position of the two segments in H is as in Figure 3.38(a). In this case, the Legendrian realization of the segment $\mathfrak{l}_j$ lies entirely above the Legendrian realization of the segment $\mathfrak{l}_i$. See Figure 3.41(a).

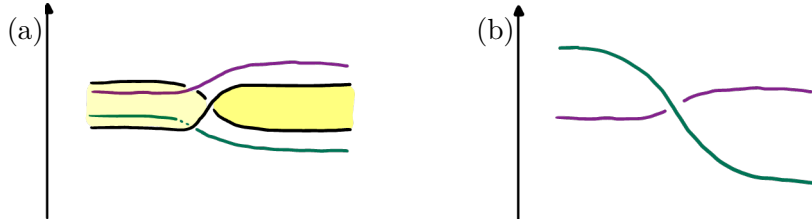


Figure 3.41. (a) The Legendrian segments $\bar{l}_i$ and $\bar{l}_j$ are initially embedded in the same ribbon surface at height $h_i(0) = h_j(0)$. The 1-handle of this ribbon surface is depicted. (b) The Legendrian segments $\bar{l}_i$ and $\bar{l}_j$ cross.

The last remaining case is $h_i(1) < h_j(1)$ and $h_i(0) > h_j(0)$. In this case, the relative position of the two segments in H is as in Figure 3.38(b), and the (front projection of the) Legendrian realizations of the two segments cross, as shown in Figure 3.41(b).

To fully justify why the braid created in Step 5 agrees with the front projection of the Legendrian realizations of the segments of $\mathfrak{a}$ in the 1-handles, we would also need to analyze all other cases,¹ but the proof in all of these cases is very similar, so we omit it.

Next, we verify that Step 6 also produces Legendrian segments that match the Legendrian realization of the segments in a 0-handle. For this, let $\mathfrak{l}_i$ and $\mathfrak{l}_j$ be two segments of $\mathfrak{a}$ in a 0-handle H , and denote by $e_1, \dots, e_n$ the segments of the Legendrian

¹That is, cases where the two segments are oriented the other way around, have opposite orientations, or when the handle looks different from the one in Figure 3.38.

skeleton of H . Step 6 claims that the Legendrian realization of a segment looks like a concatenation of two of the segments $e_1, \dots, e_n$, possibly perturbed and possibly translated vertically by a certain amount dictated by the prominence of these segments.

If $P(l_i) > P(l_j)$, then it is clear that the Legendrian realization $\bar{l}_i$ of l_i has to be higher than the Legendrian realization $\bar{l}_j$ of l_j , and therefore, independently of the precise shape of the two Legendrian segments, up to Legendrian isotopy, the relative position of these segments will be as explained in Step 6 and illustrated with an example in Figure 3.42(b) (see also Figure 3.22). If $P(l_i) = P(l_j)$, then the two Legendrian realizations $\bar{l}_i$ and $\bar{l}_j$ are both embedded in a perturbation of the same ribbon surface. Because of that, it is again clear why in Step 6 it is required that the segments $\bar{l}_i$ and $\bar{l}_j$ match the visual appearance of the segments l_i and l_j in the handle H . See Figure 3.42(c) (see also Figure 3.23) for an example showing this.

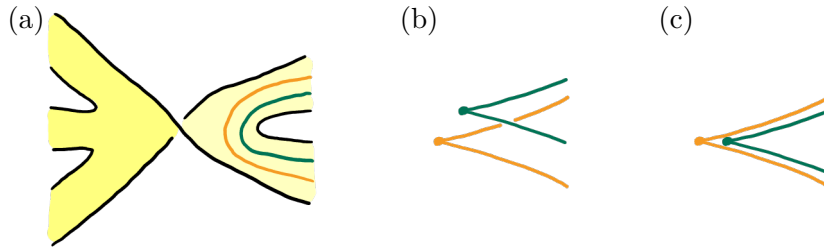


Figure 3.42. (a) Segments of $\mathbf{a}$ in a 0-handle. (b) Legendrian realizations when the prominences are distinct: higher prominence means higher z -level. (c) Legendrian realizations when the prominences agree: all lie on the same level and are separated by a small perturbation.

This proves all assertions in Proposition 3.18 and hence completes the justification of the algorithm. $\square$

3.7. Applications: from open books to contact surgery diagrams

The next result can be viewed as a refinement of Avdek's algorithm in [Avd13].

Corollary 3.25. *Given an abstract open book decomposition (Σ, ϕ) whose monodromy ϕ is given explicitly as a composition of Dehn twists along homologically nontrivial simple closed curves, there exists an algorithm that produces a contact surgery diagram for the contact manifold supported by the open book (Σ, ϕ) .*

This algorithm produces a contact surgery link $\mathbf{L} = \mathbf{L}_1 \cup \mathbf{L}_2$ for which the components of $\mathbf{L}_1$ are in bijection with the Dehn twists in the factorization of ϕ , and all the components of $\mathbf{L}_2$ are $+1$ -framed Legendrian unknots with Thurston–Bennequin number $tb = -1$.

Before delving into the proof, we first describe a suitable family of open books for (S^3, ξ_{st}) whose pages have arbitrary genus and number of boundary components. We start from two basic open books:

- one with page a genus 1 surface with connected boundary;
- one with page an annulus.

Each of these pages can be realized as the ribbon of a specific Legendrian graph in $(\mathbb{R}^3, \xi_{\text{st}})$, shown in Figure 3.43.

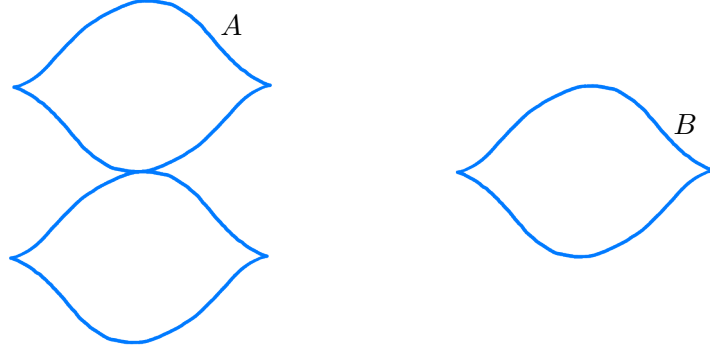


Figure 3.43. Left: the Legendrian graph A whose ribbon is a page of an open book with genus 1 and connected binding. Right: the Legendrian graph B whose ribbon is an annular page.

Let A denote the graph on the left and B the graph on the right. By assembling several copies of A and B as in Figure 3.44, we obtain a Legendrian graph whose ribbon has genus equal to the number of A -pieces and number of boundary components equal to one plus the number of B -pieces.

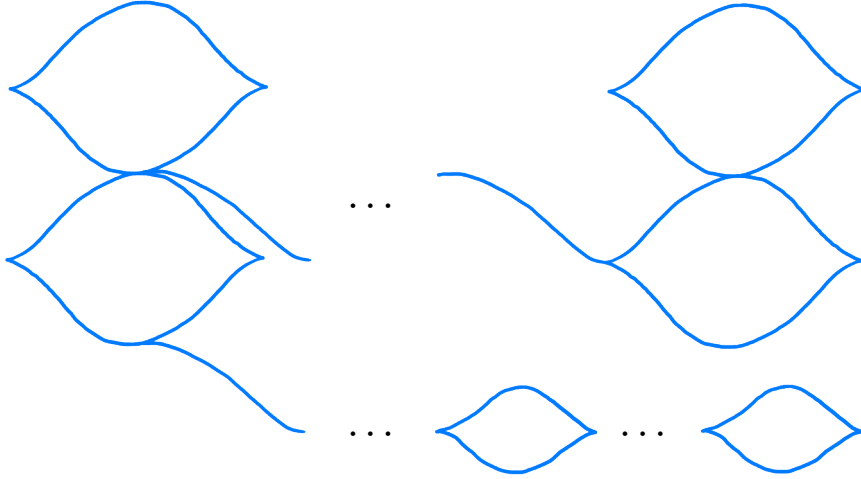


Figure 3.44. A Legendrian graph obtained by concatenating pieces of type A and B . Its ribbon has genus equal to the number of A -pieces and boundary components equal to one plus the number of B -pieces.

The ribbons of these graphs are pages of open books supporting (S^3, ξ_{st}) (because of Theorem 4.19). Thus we have explicit models of open books for (S^3, ξ_{st}) with any prescribed topology of the page.

Proof of Corollary 3.25. Let (Σ, ϕ) be an abstract open book as in the statement: Σ is a compact oriented surface with nonempty boundary, and $\phi \in \text{MCG}(\Sigma, \partial\Sigma)$ is given by

a factorization

$$\phi = \tau_{\mathfrak{a}_m}^{\delta_m} \cdots \tau_{\mathfrak{a}_1}^{\delta_1}, \quad \delta_i \in \{\pm 1\},$$

where each $\mathfrak{a}_i$ is a homologically nontrivial simple closed curve on Σ .

Step 1: Realize Σ as a ribbon in (S^3, ξ_{st}) . By the construction in Figure 3.44, we can build a Legendrian graph $G \subset (\mathbb{R}^3, \xi_{\text{st}}) \subset (S^3, \xi_{\text{st}})$ by concatenating pieces of type A and B with the following property. The ribbon R_G of this graph is a compact surface with

$$R_G \cong \Sigma$$

as oriented surfaces with boundary, and by construction it is transverse to ∂_z . Moreover, the Legendrian graph G is the 1-skeleton of a contact cell decomposition of (S^3, ξ_{st}) , so Theorem 4.19 implies that R_G is a page of an open book supporting (S^3, ξ_{st}) . We denote this open book by

$$(B, \pi) \quad \text{with corresponding abstract open book} \quad (R_G, \psi),$$

where $\psi \in \text{MCG}(R_G, \partial R_G)$ is the monodromy.

Now notice that the graph A can be seen as two Legendrian unknots touching each other in one point. Call A_1 the top one and A_2 the bottom one. Since we may have many copies of A and B in our graph G , denote them by $A^1 = A_1^1 \cup A_2^1, \dots, A^k = A_1^k \cup A_2^k$ and $B^1, \dots, B^h$. With this notation, ψ is given explicitly as a product of right-handed Dehn twists along these curves (note that each A_1^i, A_2^i and B^j can be seen as a curve on R_G). More precisely, we may write

$$\psi = \prod_{i=1}^k (\tau_{A_2^i}^+ \tau_{A_1^i}^+) \prod_{j=1}^h \tau_{B^j}^+,$$

see [Avd13].

Fix once and for all an orientation-preserving diffeomorphism

$$f: \Sigma \longrightarrow R_G.$$

Via f , we will freely identify Σ with R_G ; in particular, the abstract open book (Σ, ϕ) is equivalent to (R_G, ϕ') , where

$$\phi' := f\phi f^{-1} \in \text{MCG}(R_G, \partial R_G),$$

and the given factorization of ϕ yields a factorization

$$\phi' = \tau_{\mathfrak{a}'_m}^{\delta_m} \cdots \tau_{\mathfrak{a}'_1}^{\delta_1}, \quad \mathfrak{a}'_i := f(\mathfrak{a}_i) \subset R_G.$$

Since the $\mathfrak{a}_i$ are homologically nontrivial in Σ , the curves $\mathfrak{a}'_i$ are homologically nontrivial in R_G .

Step 2: Legendrian realization of the curves on the ribbon. The ribbon R_G is convex with dividing set equal to ∂R_G . On such a surface, a simple closed curve is nonisolating if and only if it is homologically nontrivial; in particular, each $\mathfrak{a}'_i$ is nonisolating. By the Legendrian realization principle and its explicit version given in our algorithm (Theorem 3.12), for each $i = 1, \dots, m$ we can explicitly obtain a Legendrian knot

$$L_i \subset R_G \times \{t_i\} \subset (S^3, \xi_{\text{st}})$$

isotopic to $\mathfrak{a}'_i$ in R_G . We take care that $t_i > t_j$ if $i > j$ and that $t_i < 0$ for each i .

Step 3: Constructing a surgery link that encodes the monodromy. We now build a contact surgery link

$$\mathbf{L} = \left(\bigcup_{i=1}^m L_i^{-\delta_i} \right) \cup \left(\bigcup_{i=1}^k (A_1^i \times \{2\varepsilon\})^+ \cup (A_2^i \times \{\varepsilon\})^+ \right) \cup \left(\bigcup_{j=1}^h (B^j \times \{\varepsilon\})^+ \right) \subset (S^3, \xi_{\text{st}}).$$

Thus on the components L_i we perform contact $(-\delta_i)$ -surgery, and on the others contact $+1$ -surgery. This is the desired contact surgery link.

Remark 3.26. Just for reference, the Legendrian link $(A_1^i \times \{2\varepsilon\})^+ \cup (A_2^i \times \{\varepsilon\})^+$ looks as the Legendrian link on the left of Figure 4.2 (the surgery coefficients are different though) as opposed to the Legendrian link $(A_1^i \times \{\varepsilon\})^+ \cup (A_2^i \times \{2\varepsilon\})^+$ that would look as the one of the right of Figure 4.2 (up to surgery coefficients).

We now explain why this procedure works.

Effect of surgery on the monodromy. Proposition 4.22 expresses the effect of contact ± 1 -surgery on Legendrian knots contained in distinct pages of a supporting open book in terms of Dehn twists on the page. In our setting, notice that our link $\mathbf{L}$ is compatible with R_G , and so its components are embedded in pages of the open book (B, π) (since R_G is a page of this open book). Therefore the contact manifold represented by the contact surgery link $\mathbf{L}$ is supported by the open book

$$(R_G, \psi \circ \tau_{\mathbf{L}}),$$

where $\tau_{\mathbf{L}} \in \text{MCG}(R_G, \partial R_G)$ is the mapping class obtained by composing Dehn twists along the curves represented by the components of $\mathbf{L}$, with exponents determined by the surgery coefficients. Concretely,

$$\tau_{\mathbf{L}} = \prod_{j=1}^h \tau_{B^j}^- \prod_{i=1}^k (\tau_{A_1^i}^- \tau_{A_2^i}^-) \tau_{a_m'}^{\delta_m} \cdots \tau_{a_1'}^{\delta_1}.$$

Since, by the construction of (R_G, ψ) , we know that

$$\psi = \prod_{i=1}^k (\tau_{A_2^i}^+ \tau_{A_1^i}^+) \prod_{j=1}^h \tau_{B^j}^+,$$

we obtain

$$\psi \circ \tau_{\mathbf{L}} = \psi \circ \psi^{-1} \circ \phi' = \phi'.$$

Thus the contact manifold obtained by contact surgery on the Legendrian link $\mathbf{L} \subset (S^3, \xi_{\text{st}})$ is supported by the abstract open book (R_G, ϕ') .

Identifying the resulting contact manifold. By construction, (R_G, ϕ') is equivalent to the original abstract open book (Σ, ϕ) via the diffeomorphism $f: \Sigma \rightarrow R_G$. Therefore the contact manifold supported by (R_G, ϕ') is contactomorphic to the contact manifold supported by (Σ, ϕ) .

Summarizing, starting from the explicit Dehn twist factorization of ϕ , the construction above:

- builds a Legendrian graph G whose ribbon is identified with Σ ;

- realizes the Dehn twist curves in the factorization of ϕ (and those in the factorization of ψ) as explicit Legendrian knots on different slices $R_G \times \{t\}$, using the algorithm of Theorem 3.12;
- assigns to these Legendrian knots explicit surgery coefficients ± 1 so that the resulting surgery link $\mathbf{L} \subset (S^3, \xi_{\text{st}})$ yields a contact manifold supported by (Σ, ϕ) . The sublink coming from the Legendrian realization of the curves in the factorization of ψ is used to cancel the monodromy of the open book (B, π) for the standard contact sphere, while the other components are used to produce the monodromy ϕ .

All steps are algorithmic and depend only on the data $(\Sigma, \phi = \tau_{\mathbf{a}_m}^{\delta_m} \cdots \tau_{\mathbf{a}_1}^{\delta_1})$. Hence we have produced the desired algorithm that associates to (Σ, ϕ) a contact surgery diagram for the contact manifold it supports, proving Corollary 3.25. $\square$

A contact version of the Kirby theorem

Contact surgery diagrams play for contact 3-manifolds the same role that framed link diagrams play in the topological setting. By a result of Ding and Geiges, every closed contact 3-manifold can be obtained from (S^3, ξ_{st}) by contact ± 1 -surgery along a Legendrian link, so any such manifold admits a contact surgery presentation. This naturally raises the question: given two contact surgery links in (S^3, ξ_{st}) , when do they describe the same contact manifold, and is there a manageable calculus of diagrammatic moves relating any two such presentations?

Avdek addressed this question by introducing *ribbon moves* [Avd23]. Given a contact surgery link that is compatible with a ribbon surface R , one can associate to it a factorization of a mapping class in $\text{MCG}(R, \partial R)$ as a product of Dehn twists. A ribbon move between two surgery links is, roughly speaking, a modification of the contact surgery link supported on a neighborhood of R that preserves the associated mapping class. Avdek proved that any two contact surgery links for the same contact manifold can be related by a finite sequence of ribbon moves and Legendrian isotopies. While this is a fundamental starting point, ribbon moves come in infinitely many flavors and are not always presented in a purely diagrammatic way, which makes them difficult to recognize and to use in practice.

The purpose of this chapter is to refine Avdek's result and establish a contact analogue of Kirby's theorem with an explicit list of moves. Our main result, obtained in collaboration with Marc Kegel, Vera Vértési and Daniele Zuddas, is the following.

Theorem 4.1 (A contact version of the Kirby theorem). *Let $\mathbf{L}$ and $\mathbf{L}'$ be two contact surgery links in (S^3, ξ_{st}) that describe contactomorphic contact 3-manifolds. Then $\mathbf{L}$ and $\mathbf{L}'$ differ by a sequence of Legendrian isotopies, addition or deletion of cancelling pairs, contact handleslides, standard lantern moves, and standard chain moves.*

See Section 4.6.1 for the definition of these moves

4.1. Notation and conventions

From now on we work in $(\mathbb{R}^3, \xi_{\text{st}}) \subset (S^3, \xi_{\text{st}})$ and, unless explicitly stated otherwise, all figures are drawn in the front projection. Moreover, we adopt the following conventions.

- We denote by R or by R_G a ribbon surface of a Legendrian graph G .
- Every ribbon surface we consider is assumed transverse to the Reeb vector field ∂_z .
- Whenever we write $\Sigma \times [-\varepsilon, \varepsilon]$, where Σ is a surface transverse to ∂_z , we mean the neighborhood obtained by flowing Σ along ∂_z . For a ribbon surface R , each slice $R \times \{t\}$ is again a ribbon surface.
- Let $\mathfrak{a}$ be a curve on a ribbon R , and let $R \times [-\varepsilon, \varepsilon]$ be a neighborhood of R as above. We write $\mathfrak{a} \times \{t\}$ for the corresponding curve on $R \times \{t\}$.

If $\mathfrak{a} \subset R$ is a Legendrian realizable simple closed curve, we denote by $\mathfrak{a}(t)$ the Legendrian realization of $\mathfrak{a} \times \{t\}$, embedded in a boundary–relative C^0 –small perturbation of $R \times \{t\}$ through surfaces transverse to ∂_z .

4.2. Legendrian links and ribbon surfaces

In this section, we explore some compatibility conditions between Legendrian links and ribbon surfaces.

Definition 4.2. Let $\Sigma \subset (\mathbb{R}^3, \xi_{\text{st}})$ be a compact surface with nonempty boundary that is transverse to ∂_z . A *perturbation* of Σ is a surface Σ' obtained by isotoping Σ relative to $\partial\Sigma$ through surfaces transverse to ∂_z .

Note that every ribbon surface we consider satisfies the hypotheses of Definition 4.2.

Remark 4.3. If Σ is a convex surface transverse to a contact vector field X , then in Giroux’s terminology [Gir91] an isotopy of Σ through surfaces transverse to X is called *admissible*. In our setting ∂_z is a contact vector field, so perturbations are admissible isotopies in this sense.

In Chapter 3, we introduced the term admissible isotopy for what we now call a perturbation. We have switched to “perturbation” because it shortens our sentences (e.g., “perturb R ” rather than “isotope R by an admissible isotopy”).

Lemma 4.4. *Assume that L is a Legendrian link contained in a perturbation R' of a ribbon surface R . Let U be a neighborhood of R . Then, after a Legendrian isotopy of L , we may assume that L is embedded in a perturbation of R contained in U .*

Proof. By the Legendrian realization principle (Theorem 2.9) we can Legendrian realize a curve on R that defines the same isotopy class of L so that the result L' lies in U . By Theorem 3.6, L and L' are Legendrian isotopic. $\square$

By Lemma 4.4, whenever multiple Legendrian knots are compatible with different ribbon surfaces, we may assume that each knot lies in a small perturbation of its corresponding ribbon surface, and that these perturbations are supported in disjoint neighborhoods.

Definition 4.5. Let $L = \bigcup_{i=1}^n L_i$ be a Legendrian link and R a ribbon surface. We say that L is *compatible* with R if there exists a neighborhood $R \times [-\varepsilon, \varepsilon]$ of R , with $\varepsilon > 0$, such that each component L_i is embedded in a perturbation of $R \times \{t_i\}$ for some $t_i \in (-\varepsilon, \varepsilon)$. Moreover, we assume that the components of L are indexed so that $j > i$ implies $t_j > t_i$.

When we write $L \subset R \times [-\varepsilon, \varepsilon]$ we implicitly mean that L is compatible with R and contained in this neighborhood.

Example 4.6. Let R be a ribbon surface of a Legendrian knot L . Then L is compatible with R : in fact, L lies in the characteristic foliation of R .

Example 4.7. Consider the Legendrian graph G and its ribbon surface R from Figure 4.1.

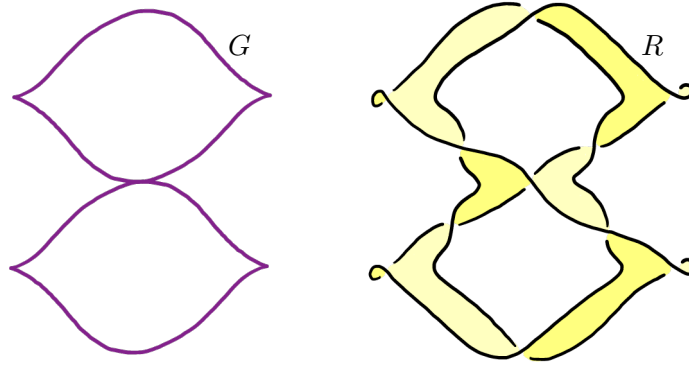


Figure 4.1. Left: a Legendrian graph G . Right: a ribbon surface R of G .

We claim that the two Legendrian links $L = L_1 \cup L_2$ and $L' = L'_1 \cup L'_2$ in Figure 4.2 are both compatible with R (individually, not simultaneously).

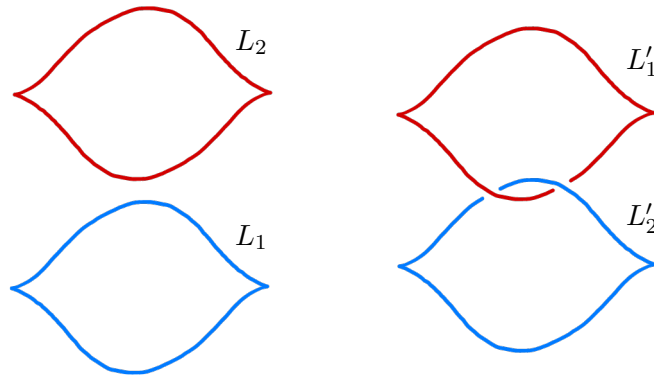


Figure 4.2. Left: a Legendrian link $L = L_1 \cup L_2$. Right: another Legendrian link $L' = L'_1 \cup L'_2$.

For L , we can embed L_1 in a perturbation of $R \times \{t_1\}$ and L_2 in a perturbation of $R \times \{t_2\}$ with $t_2 > t_1$, as illustrated in Figure 4.3.

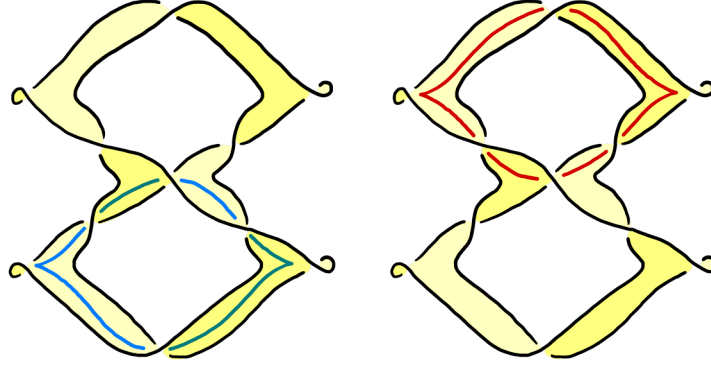


Figure 4.3. Left: L_1 embedded in $R \times \{t_1\}$. Right: L_2 embedded in $R \times \{t_2\}$ with $t_2 > t_1$.

Since $t_2 > t_1$, the surfaces $R \times \{t_2\}$ and $R \times \{t_1\}$ are disjoint and the two components of L are unlinked.

The link L' is also compatible with R . In this case L'_1 is embedded in the upper part of $R \times \{t_1\}$ and L'_2 in the upper part of $R \times \{t_2\}$ with $t_2 > t_1$, so the two components are linked.

Example 4.8. Let R be a ribbon surface and let $\mathfrak{a}$ be a homologically nontrivial simple closed curve on R . Then the Legendrian realization $\mathfrak{a}(0)$ of $\mathfrak{a}$ is compatible with R : by construction, $\mathfrak{a}(0)$ lies on a small perturbation of $R = R \times \{0\}$.

More generally, if $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ are homologically nontrivial simple closed curves on R and $t_1 < \dots < t_n$ are numbers in $(-\varepsilon, \varepsilon)$, then the Legendrian link

$$L = \mathfrak{a}_1(t_1) \cup \dots \cup \mathfrak{a}_n(t_n)$$

is compatible with R .

Before we discuss the next lemma, we need a definition. Given a Legendrian graph G whose vertices all have valency 4 and whose front projections look as on the left of Figure 4.4, a *resolution* of G is a connected component of the Legendrian graph obtained by replacing, at some of the vertices, the local picture by one of the models on the right of Figure 4.4.



Figure 4.4. Left: the local model around a generic valency 4 vertex v . Right: the two possible resolutions of v .

Lemma 4.9. Let G be a Legendrian graph whose vertices all have valency 4 and look as in Figure 4.4 (left). Assume G is embedded in a perturbation $\tilde{R}$ of a ribbon surface R . Then any resolution of G is also embedded in a (possibly different) perturbation of R .

Proof. Since $\tilde{R}$ is transverse to ∂_z , its Lagrangian projection is a local diffeomorphism. A small neighborhood of a vertex of G in $\tilde{R}$ projects as on the left of Figure 4.5.

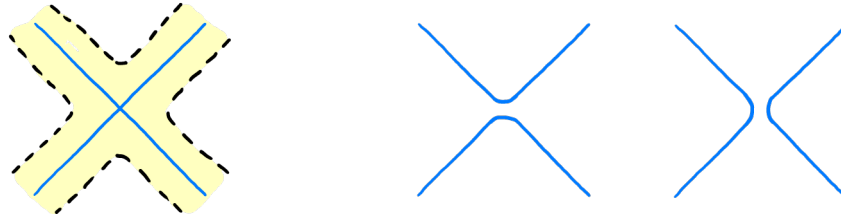


Figure 4.5. Left: Lagrangian projection of a neighborhood in $\tilde{R}$ of a vertex of G . Right: Lagrangian projections of the two possible resolutions.

On the right side of Figure 4.5 we can see the Lagrangian projection of the possible resolutions of G . Since these curves in the Lagrangian projection do not (self-)intersect (in the neighborhood considered), we see that we can further perturb the surface $\tilde{R}$ only in a neighborhood of G through surfaces transverse to ∂_z so that it contains the resolution. $\square$

Remark 4.10. Generically, a valency 4 vertex of a Legendrian graph in the front projection looks as in Figure 2.6(e), but it is not guaranteed that two edges come from the right and two from the left. If G is contained in a perturbation of a ribbon surface R , we can modify G by a Legendrian Reidemeister move R (see Section 2.2.4 and Figure 2.8) so that each valency 4 vertex looks as in Figure 4.4 (left), without losing compatibility with R .

An example is shown in Figure 4.6: on the top left we see the original front projection; the bottom left shows the corresponding Lagrangian projection in the perturbation of R . On the top right we see the front after the Reidemeister move R; the bottom right shows that this modification behaves well with respect to the Lagrangian projection and so G after this Legendrian isotopy can still be embedded in a perturbation of R .

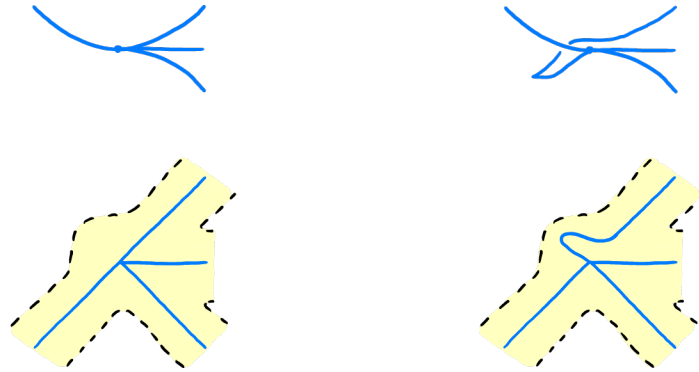


Figure 4.6. Top: front projections of a valency 4 vertex before and after a Reidemeister move R. Bottom: corresponding Lagrangian projections.

Corollary 4.11. *Any resolution of a Legendrian graph G whose vertices have valency 4 is embedded in a perturbation of any ribbon surface of G .*

Proof. By definition, G is contained in any ribbon surface R of itself. Apply Lemma 4.9. $\square$

Another useful lemma that will be used in the proof of the main theorem is the following.

Lemma 4.12. *Let L be a Legendrian link compatible with a ribbon surface R_G of a Legendrian graph G . Then L is also compatible with any ribbon surface $R_{G'}$ of any Legendrian graph G' containing G .*

Proof. A ribbon surface $R_{G'}$ can be obtained from R_G by attaching 0- and 1-handles, so R_G is a subsurface of $R_{G'}$. Any perturbation of R_G that contains L extends (by the identity on $R_{G'} \setminus R_G$) to a perturbation of $R_{G'}$ which also contains L . Thus L is compatible with $R_{G'}$. $\square$

4.3. Contact handlebodies and half open books

This section is somewhat of a digression: the material here is not strictly needed for the proofs of Theorems 4.1 and 4.36, but it provides a useful conceptual picture. It explains how ribbon surfaces arise naturally as pages of certain “half open books” on contact handlebodies. The ideas come from [Gir91; Gir02] and are at the core of many proofs in [Avd13]. In this section only we drop the standing assumption that we work in $(\mathbb{R}^3, \xi_{\text{st}})$.

Definition 4.13. Let H be a handlebody (i.e. a 3-ball with 3-dimensional 1-handles attached) in a 3-manifold M . A *half open book on H* is a pair (B, π) where:

- B is an oriented link in ∂H , and
- $\pi: H \setminus B \rightarrow [-\varepsilon, \varepsilon]$ is a fibration whose fibers are the interiors of Seifert surfaces of B .

By analogy with open books:

- B is called the *binding*,
- the closures of the fibers are called *pages*.

Example 4.14. Let (B, π) be an open book on a closed 3-manifold M , with $\pi: M \setminus B \rightarrow S^1 = \mathbb{R}/\mathbb{Z}$. For $\varepsilon < \frac{1}{2}$, the subset

$$H := \pi^{-1}([-\varepsilon, \varepsilon])$$

is a handlebody, and the restriction of (B, π) to H is a half open book.

Remark 4.15. In general, a half open book (B, π) on a handlebody $H \subset M$ does not extend to an open book of the whole manifold M . For such an extension to exist, the closure of $M \setminus H$ must also be a handlebody carrying a half open book structure with the same binding B .

Definition 4.16. Let (H, ξ) be a contact handlebody (in the sense of [Gir91]). A half open book (B, π) on H is said to *support ξ* if:

- B is positively transverse to ξ ; and
- there exists a contact form λ for ξ such that $d\lambda$ restricts to a positive area form on the interior of each page.

In this case we also say that (H, ξ) is *supported by (B, π)* .

As in the case of an open book supporting a contact structure, the pages of a half open book supporting (H, ξ) are convex surfaces whose dividing set can be taken to be the boundary of the page. Moreover, we may assume that the dividing set of ∂H is the binding B .

Remark 4.17. Any two half open books supporting a given contact handlebody are isotopic (through half open books supporting the same contact structure).

If G is a compact connected Legendrian graph in a contact manifold (M, ξ) , then a standard regular neighborhood H of G (with the restriction of ξ) is a contact handlebody.

Example 4.18. Let Σ be a ribbon surface of a Legendrian graph G in (M, ξ) (Definition 3.1). Using the flow of the Reeb vector field R_α appearing in the definition, we can construct a neighborhood U of G contactomorphic to $(\Sigma \times [-\varepsilon, \varepsilon], dz + \alpha|_{T\Sigma})$, where the z -coordinate corresponds to the flow lines of R_α ; see Figure 4.7 (left and middle).

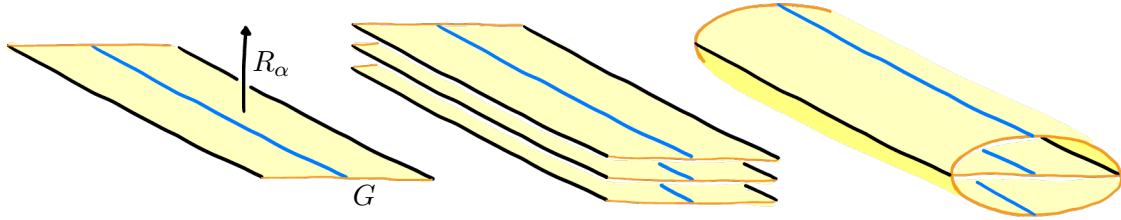


Figure 4.7. Left: a ribbon surface Σ . Middle: the product $\Sigma \times [-\varepsilon, \varepsilon]$. Right: a regular neighborhood H of G equipped with a half open book structure.

By rounding the edges of U we obtain a standard regular neighborhood H of G , which can be seen as

$$H \cong \Sigma \times [-\varepsilon, \varepsilon] / \sim,$$

where $(x, z) \sim (x, z')$ if $x \in \partial\Sigma$ (Figure 4.7, right). The projection to the second factor defines a half open book (B, π) on H whose pages are the images of $\Sigma \times \{t\}$ and whose binding is the image of $\partial\Sigma \times [-\varepsilon, \varepsilon]$. It is straightforward to check that this half open book supports the given contact structure.

Figure 4.8 shows part of this construction in (S^3, ξ_{st}) .

In particular, every ribbon surface appears as a page of a half open book on a contact handlebody.

Theorem 4.19 ([Gir02]). *A ribbon surface R_G on a contact manifold (M, ξ) is the page of an open book compatible with ξ if G is the 1-skeleton of a contact cell decomposition.*

We refer to [Etn06] for the definition of a contact cell decomposition. An important point is that, when working in $(\mathbb{R}^3, \xi_{\text{st}})$, it is particularly easy to determine when a Legendrian graph is the 1-skeleton of such a decomposition (see [Avd13]).

Theorem 4.19 is at the heart of the proof of the main result in [Avd13, Theorem 1.10], which will be our starting point.

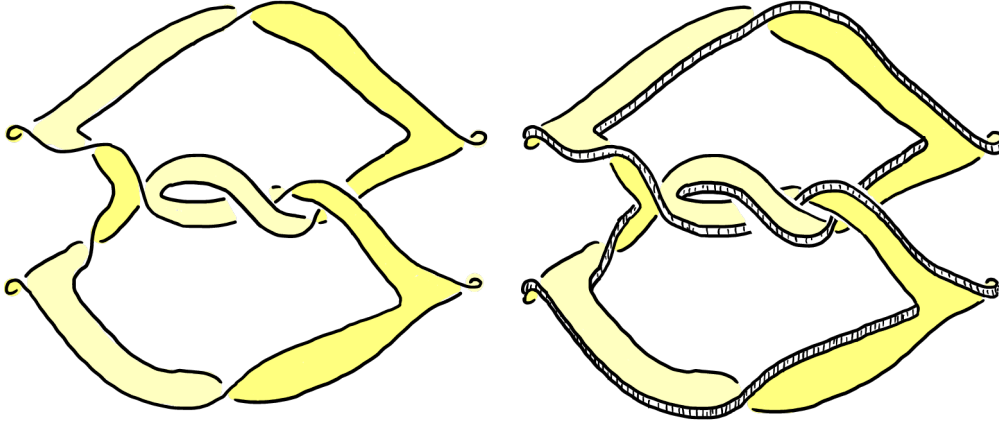


Figure 4.8. Left: a ribbon surface R . Right: the product $R \times [-\varepsilon, \varepsilon]$, viewed as a neighborhood of the Legendrian graph.

4.4. Contact surgery links and ribbon surfaces

We now recall how a contact surgery link compatible with a ribbon surface R determines a product of Dehn twists in the mapping class group $\text{MCG}(R, \partial R)$.

Before the next definition, we make a brief convention. Let L be a contact surgery link compatible with a ribbon surface R . Since each component L_i lies in a perturbation of some $R \times \{t_i\}$, we can regard L_i as a simple closed curve on R , well-defined up to isotopy: pull it back to $R \times \{t_i\}$ via the inverse of the perturbation, then project to R .

We will freely switch between these two points of view (Legendrian knot in $\mathbb{R}^3$ vs. isotopy class of a curve on R); the meaning should always be clear from the context.¹

Convention 4.20. Given a Legendrian knot L compatible with a ribbon surface R , we denote by $\bar{L}$ the simple closed curve on R defined by L up to isotopy.

Definition 4.21 ([Avd13, Definition 6.1]). Let R be a ribbon surface and

$$\mathbf{L} = \bigcup_{i=1}^n L_i^{\delta_i}, \quad \delta_i \in \{+, -\},$$

be a contact surgery link compatible with R .

The *mapping class determined by $\mathbf{L}$* is

$$\tau_{\mathbf{L}} := \tau_{L_n}^{-\delta_n} \cdots \tau_{L_1}^{-\delta_1} \in \text{MCG}(R, \partial R).$$

Note that a contact surgery link compatible with a ribbon surface determines more than just an element of the mapping class group. It provides a specific factorization in terms of Dehn twists, where each Dehn twist corresponds to a component of the contact surgery link. Since, in the following sections, the actual factorization will be important, we sometimes think of the “mapping class determined” by a contact surgery link as the data of the factorization.

The motivation for Definition 4.21 is the following well-known result.

¹Here, regarding R as an abstract surface is analogous to the familiar process of viewing a page of an open book as the surface of the corresponding abstract open book.

Proposition 4.22 ([Gay02; Etn06]). *Let*

$$\mathbf{L} = \bigcup_{i=1}^n L_i^{\delta_i}, \quad \delta_i \in \{+, -\},$$

be a contact surgery link whose components are embedded in different pages of an open book (B, π) supporting a contact structure ξ in a manifold M . Let (Σ, h) be the corresponding abstract open book, where Σ is a page chosen just before the page containing L_1 . Then the contact manifold obtained by contact surgery on $\mathbf{L}$ is contactomorphic to the contact manifold supported by the abstract open book $(\Sigma, h \circ \tau_{\mathbf{L}})$.

4.5. Ribbon moves

In this section we recall the definition of ribbon moves given in [Avd13] and we state the main result there, namely Theorem 4.29.

Definition 4.23 ([Avd13, Definition 6.2]). Let $\mathbf{L}$ and $\mathbf{L}'$ be contact surgery links in (S^3, ξ_{st}) . We say that $\mathbf{L}$ and $\mathbf{L}'$ are *R-equivalent* if:

- both links are compatible with the same ribbon surface R , and
- $\tau_{\mathbf{L}} = \tau_{\mathbf{L}'}$ in $\text{MCG}(R, \partial R)$.

Example 4.24. Consider the two contact surgery links on the left and right of Figure 4.9. On the left we have

$$\mathbf{L} := L_1^- \cup L_2^-,$$

on the right

$$\mathbf{L}' := L_1'^- \cup L_2'^-.$$

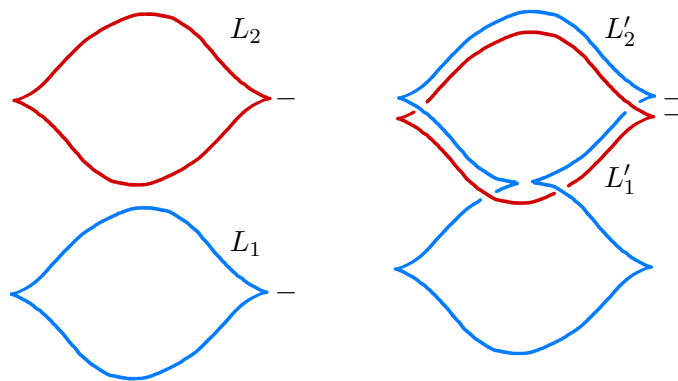


Figure 4.9. Left: the contact surgery link $\mathbf{L} = L_1^- \cup L_2^-$. Right: the contact surgery link $\mathbf{L}' = L_1'^- \cup L_2'^-$.

In Example 4.7 we already saw that $\mathbf{L}$ is compatible with the ribbon R from Figure 4.1; the same is true for $\mathbf{L}'$ (note that in this case, the component L_2' is a resolution of G and so is compatible with R because of Corollary 4.11). As the indices suggest, we have that $L_2' \subset R \times \{t_2\}$ and $L_1' \subset R \times \{t_1\}$, with $t_2 > t_1$.

We have that

$$\tau_L = \tau_{L_2}^+ \tau_{L_1}^+, \quad \tau_{L'} = \tau_{L'_2}^+ \tau_{L'_1}^+.$$

The ribbon R can be identified with the surface in Figure 2.2 so that $\overline{L_1} \mapsto \mathbf{b}$, $\overline{L_2} \mapsto \mathbf{a}$, $\overline{L'_1} \mapsto \mathbf{a}$, and $\overline{L'_2} \mapsto \mathbf{ab}$. Under this identification the two factorizations differ by a braid relation, hence

$$\tau_L = \tau_{L'} \in \text{MCG}(R, \partial R),$$

and L, L' are R -equivalent.

Lemma 4.25. *Let L and L' be two R_G -equivalent contact surgery links. Then L and L' are also $R_{G'}$ -equivalent for every Legendrian graph G' containing G .*

Proof. By Lemma 4.12, L and L' are both compatible with $R_{G'}$. Moreover, viewing R_G as a subsurface of $R_{G'}$, any diffeomorphism of R_G fixing ∂R_G extends by the identity on $R_{G'} \setminus R_G$, yielding a homomorphism

$$\text{MCG}(R_G, \partial R_G) \longrightarrow \text{MCG}(R_{G'}, \partial R_{G'}).$$

If $\tau_L = \tau_{L'}$ in $\text{MCG}(R_G, \partial R_G)$, the same holds in $\text{MCG}(R_{G'}, \partial R_{G'})$, so L and L' are $R_{G'}$ -equivalent. $\square$

Definition 4.26 ([Avd13, Definition 6.4]). Let $L \subset (S^3, \xi_{\text{st}})$ be a contact surgery link and l a contact surgery sublink. Suppose l is R -equivalent to another contact surgery link l' .

A *ribbon move* performed on l consists of replacing l by l' in L .

Example 4.27. Consider the contact surgery links $L := L_1^- \cup L_2^- \cup K^+$ and $L' := L'_1{}^- \cup L'_2{}^- \cup K^+$ in Figure 4.10.

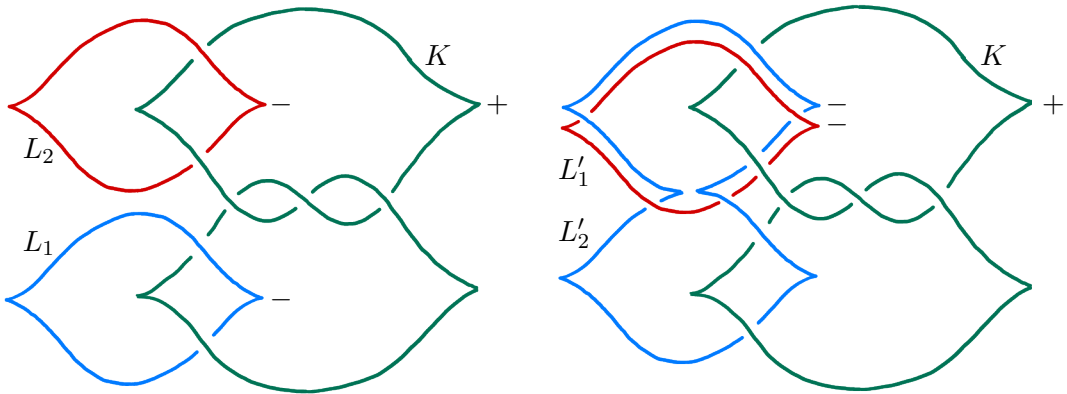


Figure 4.10. Left: the contact surgery link $L = L_1^- \cup L_2^- \cup K^+$. Right: the contact surgery link $L' = L'_1{}^- \cup L'_2{}^- \cup K^+$.

By Example 4.24, the sublinks $L_1^- \cup L_2^-$ and $L'_1{}^- \cup L'_2{}^-$ are R -equivalent, so L and L' differ by a ribbon move.

The point of introducing ribbon moves is the following result.

Proposition 4.28 ([Avd13, Proposition 6.3]). *Suppose that a contact surgery link $\mathbf{L}$ in (S^3, ξ_{st}) can be transformed into another contact surgery link $\mathbf{L}'$ by a finite sequence of ribbon moves. Then $\mathbf{L}$ and $\mathbf{L}'$ describe contactomorphic contact manifolds.*

The main result of [Avd13] is a converse to Proposition 4.28, which we will use as the starting point for our refinement.

Theorem 4.29 ([Avd13, Theorem 1.10]). *Let $\mathbf{L}$ and $\mathbf{L}'$ be contact surgery links in (S^3, ξ_{st}) that describe contactomorphic contact manifolds. Then there exists a finite sequence of ribbon moves and Legendrian isotopies transforming $\mathbf{L}$ into $\mathbf{L}'$.*

4.6. Elementary R -equivalences

We now isolate those R -equivalences whose mapping class factorizations differ by a *single* relation in the presentation of the mapping class group (Theorem 2.6). These will correspond to the local moves appearing in our contact version of the Kirby theorem.

Definition 4.30. Let $\mathbf{L}$ and $\mathbf{L}'$ be two R -equivalent contact surgery links. Assume that the factorizations of the mapping classes $\tau_{\mathbf{L}}$ and $\tau_{\mathbf{L}'}$ (as in Definition 4.21) differ by one of the following:

- insertion or deletion of a pair of cancelling Dehn twists;
- commuting two consecutive Dehn twists along disjoint curves;
- a braid relation;
- a lantern relation;
- a chain relation.

Then we say that $\mathbf{L}$ and $\mathbf{L}'$ differ by an *elementary R -equivalence*.

If two contact surgery links $\mathbf{L}$ and $\mathbf{L}'$ differ by a ribbon move performed on sublinks $\mathbf{l} \subset \mathbf{L}$ and $\mathbf{l}' \subset \mathbf{L}'$ which are elementary R -equivalent, we say that $\mathbf{L}$ and $\mathbf{L}'$ differ by an *elementary ribbon move*.

For instance, the two links in Example 4.24 differ by an elementary R -equivalence corresponding to a braid relation.

4.6.1. Examples of elementary R -equivalences. In this section we present some examples of elementary R -equivalence. These examples are very important, as they will be all the moves needed to relate any two contact surgery links of the same contact manifold.

Convention 4.31. In this section we will often draw the front projection of a 3-ball B^3 containing several Legendrian arcs, representing the intersection of a Legendrian link with the given ball. See, for example, Figure 4.13, left and right.

Our convention is the following. Whenever two or more Legendrian arcs reach the boundary of the ball (this will be indicated by dotted lines), and they do so adjacent to one another, the corresponding Legendrian arcs outside the ball are understood to remain vertical push-offs of each other in the ∂_z -direction until they encounter another ball under consideration.

4.6.1.1. *Cancellation.* Consider two Legendrian knots that are vertical push-offs of each other in the ∂_z direction, decorated with opposite surgery coefficients, as in Figure 4.11. Denote this contact surgery link by $\mathbf{L}$.



Figure 4.11. A cancelling pair: the two Legendrian knots are vertical push-offs of each other and are decorated by opposite coefficients.

It is easy to see that $\mathbf{L}$ is compatible with the Legendrian graph G in Figure 4.12 and any of its ribbon surfaces. The Legendrian graph G is just a copy of either of the two Legendrian knots in Figure 4.11.



Figure 4.12. Left: a Legendrian graph G which is a copy of either component of Figure 4.11. Right: a ribbon R of G .

The mapping class associated to $\mathbf{L}$ is the product of two cancelling Dehn twists, hence trivial in $\text{MCG}(R, \partial R)$ (see Example 2.1). The empty contact surgery diagram is also compatible with R and also determines the identity in $\text{MCG}(R, \partial R)$. Thus $\mathbf{L}$ and the empty diagram differ by an elementary R -equivalence.

A pair of Legendrian knots where one is the translation of the other in the ∂_z -direction, and with opposite surgery coefficients as in Figure 4.11 is called a *cancelling pair*.

Cancelling pairs are classical in contact surgery, going back to [DG01; DGS04]. Our interpretation via mapping class factorizations comes from Avdek [Avd13].

4.6.2. Braid relation. Let $\mathbf{L} = L_1^- \cup L_2^-$ be a contact surgery link. Up to Legendrian isotopy, we may assume that there is a ball in which $\mathbf{L}$ looks as on the left of Figure 4.13.

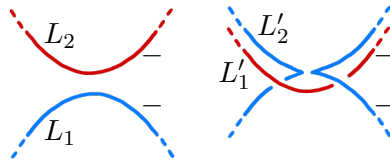


Figure 4.13. Left: the contact surgery link $\mathbf{L} = L_1^- \cup L_2^-$. Right: the contact surgery link $\mathbf{L}' = L_1'^- \cup L_2'^-$.

Let $\mathbf{L}' = L_1'^- \cup L_2'^-$ be as on the right of Figure 4.13. Both links are compatible with the ribbon R shown in Figure 4.14.

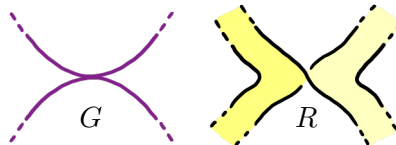


Figure 4.14. Left: the Legendrian graph G underlying the braid relation. Right: a ribbon surface R of G .

We have

$$\tau_L = \tau_{L_2}^+ \tau_{L_1}^+, \quad \tau_{L'} = \tau_{L'_2}^+ \tau_{L'_1}^+.$$

Identifying R with the surface in Figure 2.2 so that $\overline{L_2}$ and $\overline{L'_1}$ map to $\mathfrak{a}$, $\overline{L_1}$ to $\mathfrak{b}$, and $\overline{L'_2}$ to $\mathfrak{ab}$, we see that the two factorizations differ by a braid relation (see Example 2.3). Hence L and L' differ by an elementary R -equivalence.

A similar argument shows that the two contact surgery links on the left and on the right of Figure 4.15 also differ by an elementary R -equivalence.

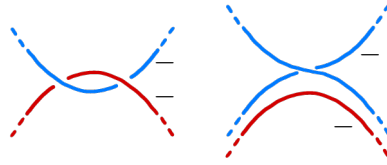


Figure 4.15. Another pair of contact surgery links differing by an elementary R -equivalence corresponding to a braid relation.

The two moves described in Example 4.6.2 are called *Legendrian handleslides*. Again, these moves are well known in the literature and were originally introduced in [DG01; DGS04]. This way of presenting them again comes from [Avd13].

4.6.3. Lantern relation. Let $L = L_1^- \cup L_2^- \cup L_3^-$ be a contact surgery link. Suppose that there are two balls in which L looks as in the left column of Figure 4.16. The arrows in the figure indicates that if we orient L_2 and L_3 as in the top ball, we want these orientations to match those drawn in the bottom ball.

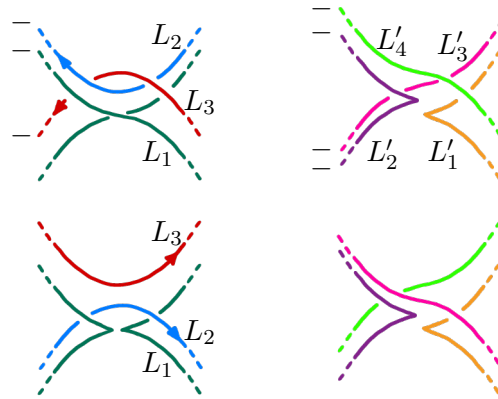


Figure 4.16. Left: the contact surgery link $L = L_1^- \cup L_2^- \cup L_3^-$. Right: the contact surgery link $L' = L'_1^- \cup L'_2^- \cup L'_3^- \cup L'_4^-$ produced by a lantern move.

Let $L' = L'_1^- \cup L'_2^- \cup L'_3^- \cup L'_4^-$ be as in the right column of Figure 4.16. As usual, both links are compatible with the ribbon surface R depicted in Figure 4.17, for instance by using Corollary 4.11.

Their associated mapping classes are

$$\tau_L = \tau_{L_3}^+ \tau_{L_2}^+ \tau_{L_1}^+, \quad \tau_{L'} = \tau_{L'_4}^+ \tau_{L'_3}^+ \tau_{L'_2}^+ \tau_{L'_1}^+.$$

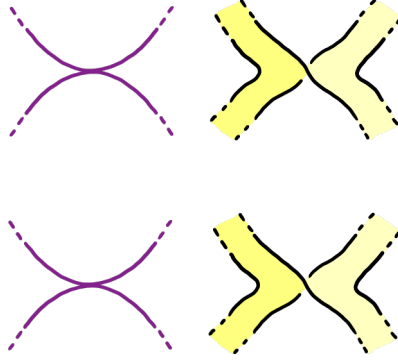


Figure 4.17. A ribbon surface R compatible with the contact surgery links appearing in the lantern move.

Identifying R with the surface in Figure 2.3 so that

$$\overline{L}_1 \mapsto \mathfrak{a}\mathfrak{b}, \quad \overline{L}_2 \mapsto \mathfrak{b}, \quad \overline{L}_3 \mapsto \mathfrak{a},$$

and

$$\overline{L}'_1, \overline{L}'_2, \overline{L}'_3, \overline{L}'_4 \mapsto \mathfrak{d}_1, \mathfrak{d}_2, \mathfrak{d}_3, \mathfrak{d}_4,$$

we see that the two factorizations differ by a lantern relation (see Example 2.4). Hence $\mathbf{L}$ and $\mathbf{L}'$ differ by an elementary R -equivalence.

With similar considerations, we can also show that the six pairs of contact surgery links shown in Figures 4.18, 4.19, and 4.20 differ by an elementary R -equivalence corresponding to a lantern relation.

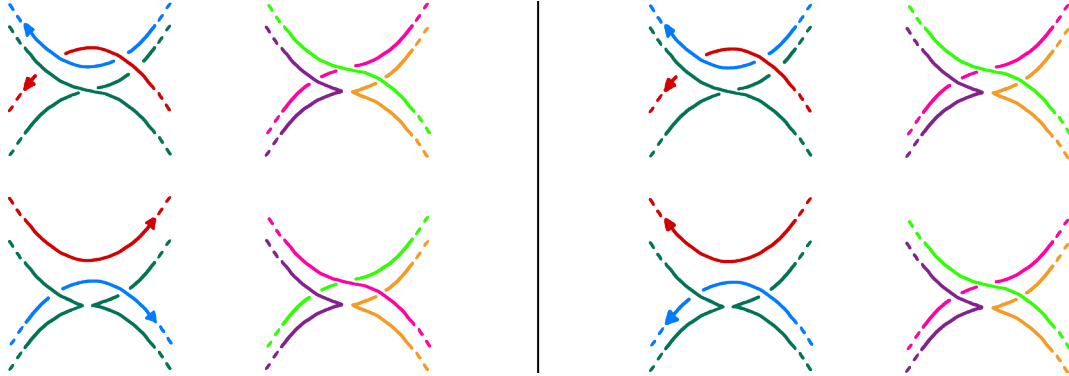


Figure 4.18. Two examples of lantern moves. All components are decorated with a $-$ sign.

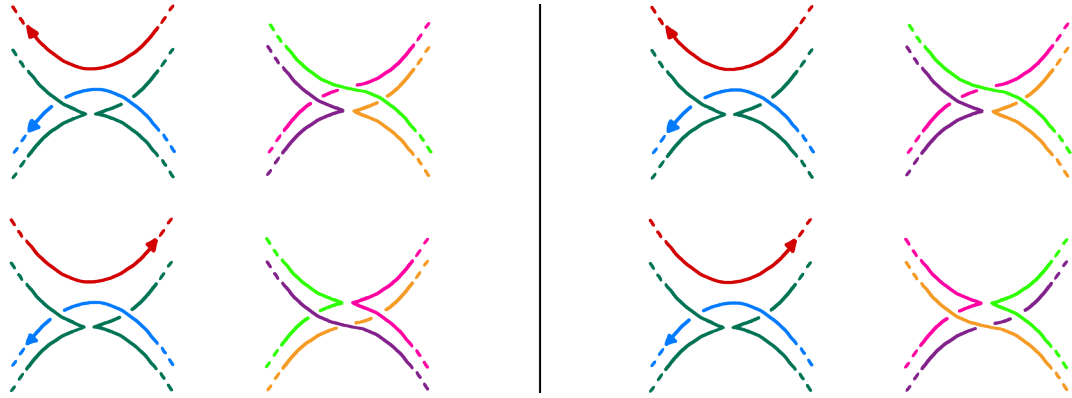


Figure 4.19. Two further examples of lantern moves. All components are decorated with a $-$ sign.

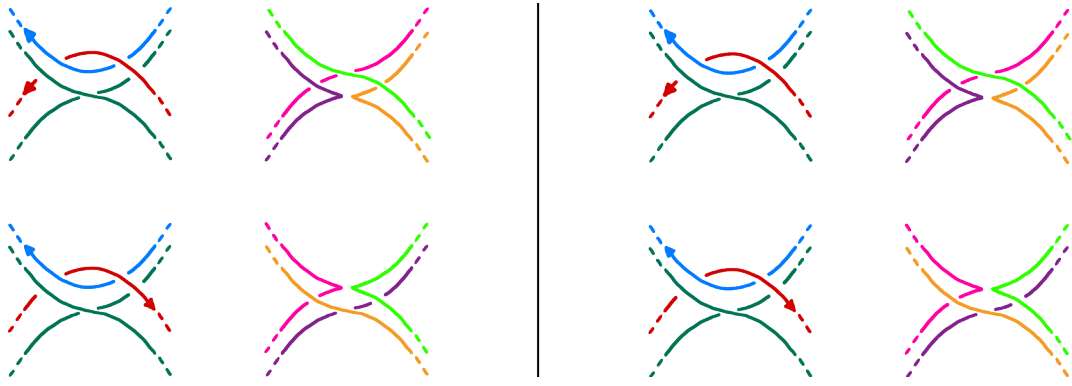


Figure 4.20. Two more examples of lantern moves arising from the lantern relation. All components are decorated with a $-$ sign.

We refer to the local moves constructed in Example 4.6.3 as *lantern moves*.

Example 4.32. From Example 4.6.3 we know that the two contact surgery links in Figure 4.21 differ by a lantern move.

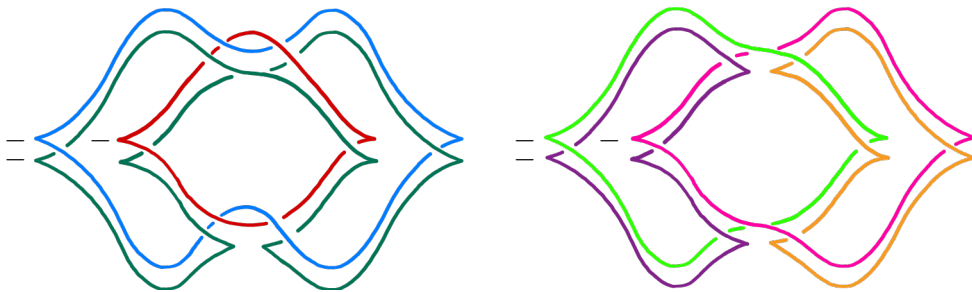


Figure 4.21. The standard lantern move.

We call the move described in Example 4.32 the *standard lantern move*.

4.6.4. Chain relation. Consider contact surgery links

$$\mathbf{L} = \bigcup_{i=1}^{12} L_i^-, \quad \mathbf{L}' = L_1'^- \cup L_2'^-$$

whose local pictures are as in Figure 4.22. Basically, as Legendrian links, L and L' differ (up to the number of strands) only inside two balls.

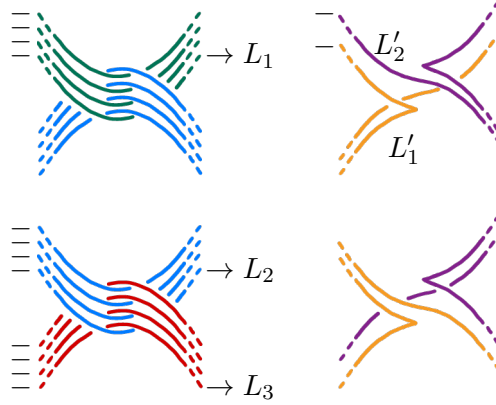


Figure 4.22. Left: the contact surgery link $\mathbf{L} = \bigcup_{i=1}^{12} L_i^-$. Right: the contact surgery link $\mathbf{L}' = L_1'^- \cup L_2'^-$.

It is straightforward to check that $\mathbf{L}$ and $\mathbf{L}'$ are compatible with the ribbon surface R in Figure 4.23. For $\mathbf{L}'$, note that L_1' and L_2' are the Legendrian realizations of the two boundary components of R given by the algorithm in Chapter 3.

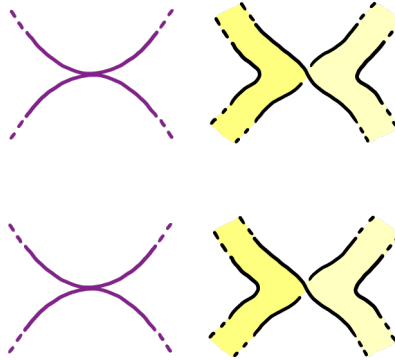


Figure 4.23. A ribbon surface R compatible with the contact surgery links appearing in the chain move.

The associated mapping classes are

$$\tau_{\mathbf{L}} = \prod_{i=1}^{12} \tau_{L_i}^+, \quad \tau_{\mathbf{L}'} = \tau_{L_2'}^+ \tau_{L_1'}^+.$$

Identifying R with the surface in Figure 2.4 so that

$$\overline{L_1}, \overline{L_4}, \overline{L_7}, \overline{L_{10}} \mapsto \mathbf{c}, \quad \overline{L_2}, \overline{L_5}, \overline{L_8}, \overline{L_{11}} \mapsto \mathbf{b}, \quad \overline{L_3}, \overline{L_6}, \overline{L_9}, \overline{L_{12}} \mapsto \mathbf{a},$$

and

$$\overline{L'_1}, \overline{L'_2} \mapsto \mathfrak{d}_1, \mathfrak{d}_2,$$

we see that the two factorizations differ by a chain relation (see Example 2.5). Hence $\mathbf{L}$ and $\mathbf{L}'$ differ by an elementary R -equivalence.

As in the lantern case, we can also show that the six pairs of contact surgery links shown in Figures 4.24, 4.25, 4.26, and 4.27 differ by an elementary R -equivalence corresponding to a chain relation.

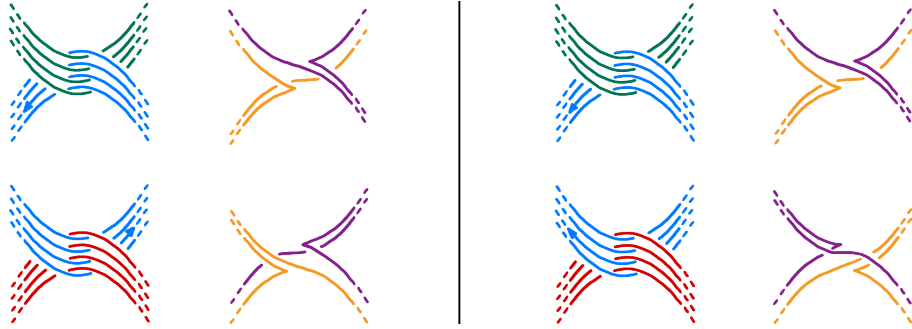


Figure 4.24. Two examples of chain moves. All components are decorated with a $-$ sign.

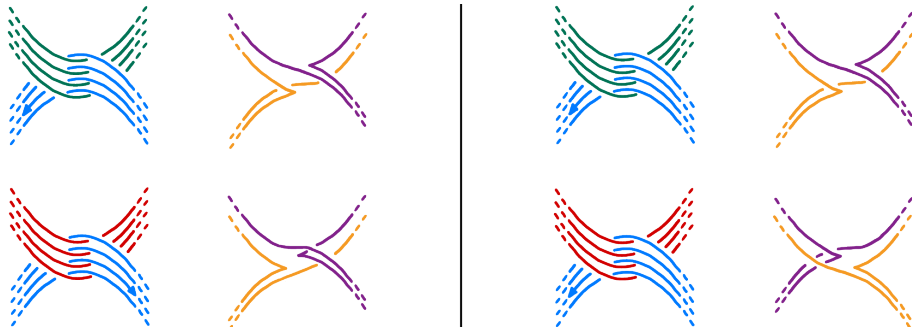


Figure 4.25. Two further examples of chain moves. All components are decorated with a $-$ sign.

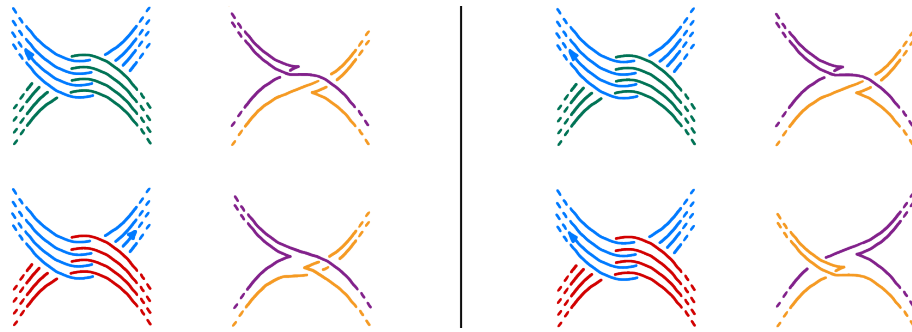


Figure 4.26. Two more examples of chain moves. All components are decorated with a $-$ sign.

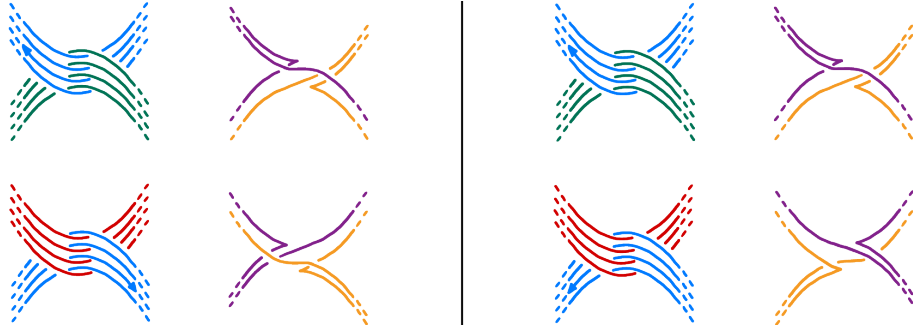


Figure 4.27. Additional examples of chain moves; again all components have coefficient $-$.

We call the R -equivalences in Example 4.6.4 *chain moves*.

Example 4.33. From Example 4.6.4 we know that the two contact surgery links in Figure 4.28 differ by a chain move.

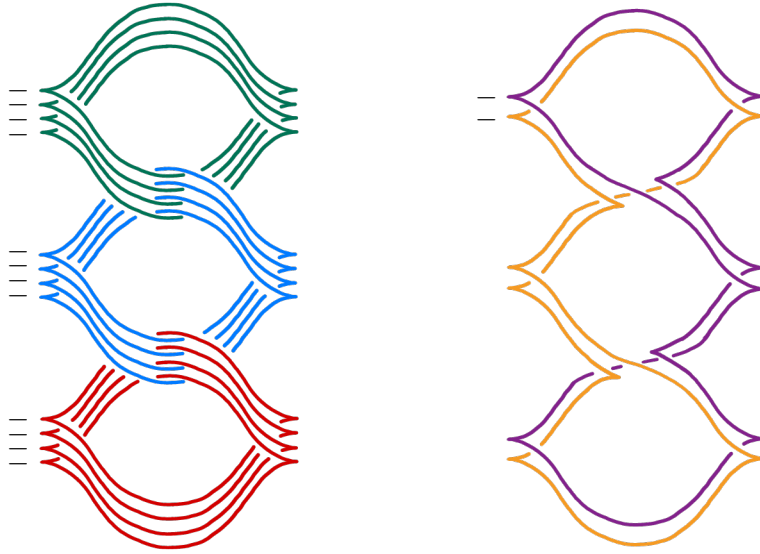


Figure 4.28. The standard chain move. All components are decorated with a $-$ sign.

We call the move described in Example 4.33 the *standard chain move*. Note that this particular case already appears in [Avd13] as an example of a ribbon move.

4.6.5. Commutativity. Let G be a Legendrian graph consisting of a chain of Legendrian knots $K_1, \dots, K_{2n+1}$: each K_j intersects K_{j-1} and K_{j+1} once, and is disjoint from all other components (with obvious exceptions at the ends of the chain). The Legendrian skeleton of the ribbon R in Example 4.6.4 is such a graph with $n = 1$.

Let R be the ribbon of G ; it is a genus n surface with two boundary components ∂_1 and ∂_2 . Let

$$L := \partial_1(-\varepsilon)^- \cup \partial_2(\varepsilon)^-, \quad L' := \partial_1(\varepsilon)^- \cup \partial_2(-\varepsilon)^-,$$

be the contact surgery links obtained by Legendrian realizing $\mathfrak{d}_1, \mathfrak{d}_2$ at different heights (using the algorithm of Chapter 3). The two contact surgery links $\mathbf{L}$ and $\mathbf{L}'$ differ by an elementary R -equivalence corresponding to exchanging two commuting Dehn twists.

We show that $\mathbf{L}$ and $\mathbf{L}'$ are in fact Legendrian isotopic. For concreteness, we assume that all intersections between K_j and K_{j+1} look as on the left of Figure 4.29: the front projection of K_{j+1} lies above that of K_j , and the local orientations are coherent as indicated. The general case is analogous (moreover, it is also true that we can Legendrian isotope G to achieve these properties).

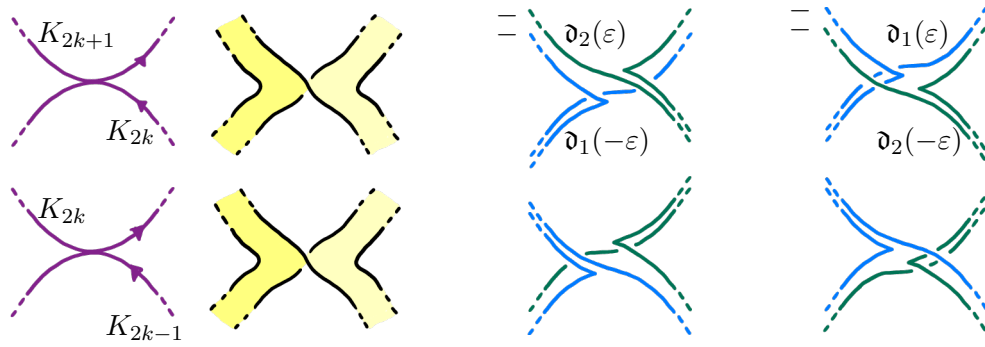


Figure 4.29. Left: local model of an intersection between successive knots in the chain. The ribbon surface R is also shown here. Right: local pictures of $\mathbf{L}$ (first column) and $\mathbf{L}'$ (second column) near a vertex; top row for intersections between K_{2k} and K_{2k+1} , bottom row for intersections between K_{2k-1} and K_{2k} for $k = 1, \dots, n$.

On the right of Figure 4.29, we can see how $\mathbf{L}$ and $\mathbf{L}'$ look in this case. We now describe a Legendrian isotopy between them.

Around the intersection of K_1 and K_2 the isotopy is as in Figure 4.30: the intersection point p between the two surgery components travels once around K_1 and reappears on the other side.

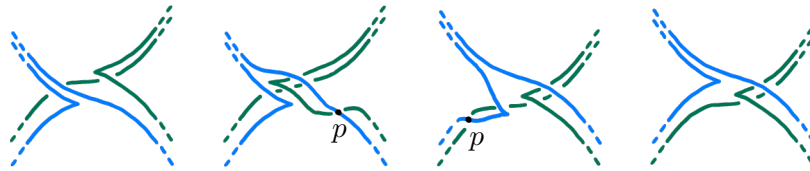


Figure 4.30. Legendrian isotopy of $\mathbf{L}$ near the intersection between K_1 and K_2 : the point p travels around K_1 .

Around the intersection of K_{2n} and K_{2n+1} the isotopy is as in Figure 4.31: the point p travels around K_{2n+1} .

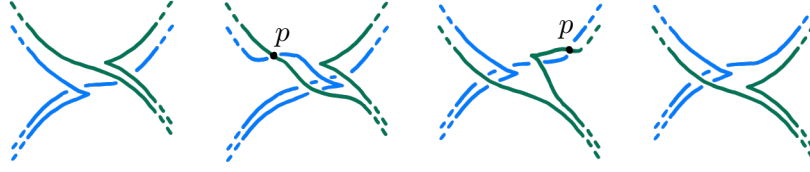


Figure 4.31. Legendrian isotopy near the intersection between K_{2n} and K_{2n+1} .

Around an intermediate pair of intersections, say between K_{2k} and K_{2k+1} and between K_{2k+1} and K_{2k+2} , the isotopy is as in Figure 4.32: the two intersection points p and q slide along K_{2k+1} from one ball to the other. This step is vacuous when $n = 1$.

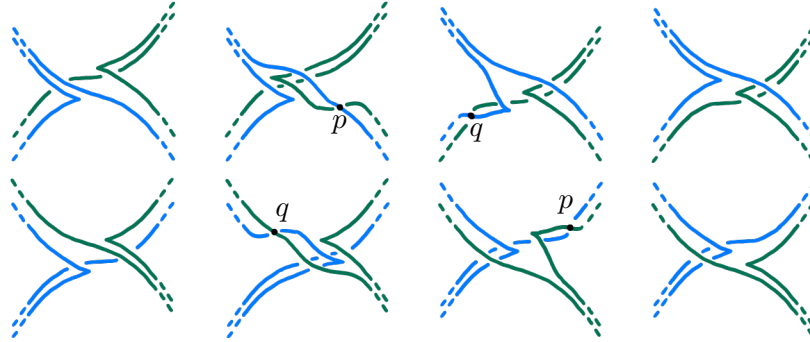


Figure 4.32. Legendrian isotopy near consecutive intersections between K_{2k}, K_{2k+1} and K_{2k+1}, K_{2k+2} : the intersection points p and q slide along K_{2k+1} .

Concatenating these local isotopies shows that L and L' are Legendrian isotopic.

4.6.6. Uniqueness of elementary R -equivalences. In the previous section, we presented examples of elementary R -equivalences. In this section, we aim to establish a kind of converse: whenever two contact surgery links differ by an elementary R -equivalence, they look, up to Legendrian isotopy, like those from the examples we discussed earlier.

We begin with a uniqueness statement for Legendrian links on a ribbon surface. This is just an obvious adaptation of Theorem 3.6.

Theorem 4.34. *Let $L = \bigcup_{i=1}^n L_i$ and $L' = \bigcup_{i=1}^n L'_i$ be Legendrian links, both compatible with the same ribbon surface R . Assume that the isotopy class of $\overline{L_i}$ equals the isotopy class of $\overline{L'_i}$ in R for each i . Then L and L' are Legendrian isotopic by an isotopy supported in any prescribed neighborhood of the form $R \times [-\varepsilon, \varepsilon]$ containing both L and L' .*

For the next proposition, it is important to note that the braid, the lantern, and the chain relations in the mapping class group described in Section 2.1.2 have an underlying graph from which one can recover all the curves involved in the relation by resolving this graph in different ways. Using the notation of Section 2.1.2, this graph is $\mathbf{a} \cup \mathbf{b}$. We refer to this graph as the *graph of the relation*.

Proposition 4.35. *Let L and L' be contact surgery links which differ by an elementary R -equivalence. Then, up to Legendrian isotopy, L and L' differ by one of the following moves:*

- *insertion or deletion of a cancelling pair;*
- *a contact handleslide;*
- *a lantern move;*
- *a chain move.*

Proof. By definition, the mapping class factorizations τ_L and $\tau_{L'}$ differ by a single relation from Theorem 2.6. Up to restricting to sublinks of L and L' by discarding components that are unchanged by the R -equivalence, we may assume that every Dehn twist appearing in the factorizations is involved in the relation.

Case 1: cancelling pair. Suppose the factorizations differ by inserting/deleting a pair of cancelling Dehn twists. Up to swapping L and L' , we can assume that $L = L_1^\pm \cup L_2^\mp$ and L' is the empty diagram.

Let $\tilde{L}_2$ be the Legendrian knot obtained from L_1 by a small vertical translation. The contact surgery link

$$\tilde{L} := L_1^\pm \cup \tilde{L}_2^\mp$$

is a cancelling pair as in Example 4.6.1.1 and is compatible with R (if L_1 lies in a perturbation of $R \times \{t_1\}$, then $\tilde{L}_2$ lies in a perturbation of $R \times \{t_1 + \varepsilon\}$). By Theorem 4.34, L is Legendrian isotopic to $\tilde{L}$, so this elementary R -equivalence is realized by a cancelling pair.

Case 2: braid, lantern, or chain relation. In these cases all curves occurring in the relation are homologically nontrivial in R (by Lemma 3.3). Since R is convex with dividing set ∂R , a homologically nontrivial curve is nonisolating, so the graph $\bar{G}$ of the relation is nonisolating.

By the Legendrian realization principle (Theorem 2.9) we can Legendrian realize $\bar{G}$ to a Legendrian graph G embedded in a perturbation of R . Using Remark 4.10, we may further assume that all vertices of G have the standard local front from Figure 4.4 (left).

Compare now with Examples 4.6.2, 4.6.3, and 4.6.4. In each of those examples, the Legendrian skeleton of the ribbon is precisely the graph of the relation, its vertices have the standard form, and every surgery component is a resolution of the Legendrian skeleton, possibly translated slightly in the ∂_z direction. Thus, for each type of relation, we can create out of G contact surgery links $\tilde{L}$ and $\tilde{L}'$ which:

- differ by the corresponding move (handleslide, lantern, or chain move),
- are compatible with R (since each component is a resolution of G , by Lemma 4.9),
- induce the same mapping class factorizations as L and L' .

More concretely, the choice of which explicit pair $(\tilde{L}, \tilde{L}')$ to use among the models in the examples is dictated by the front projection of G around vertices (orientation and relative position of the pieces of G corresponding to $\mathfrak{a}$ and $\mathfrak{b}$).

Since the factorization of τ_L corresponds to that of $\tau_{\tilde{L}}$, the components of L and $\tilde{L}$ define the same curves on R , and the same for L' and $\tilde{L}'$. By Theorem 4.34, L is Legendrian isotopic to $\tilde{L}$ and L' to $\tilde{L}'$. Therefore the elementary R -equivalence is realized, up to Legendrian isotopy, by one of our local models.

Case 3: commuting Dehn twists. Suppose $\mathbf{L} = L_1^- \cup L_2^-$ and $\mathbf{L}' = L_1'^- \cup L_2'^-$ correspond to two factorizations differing by commuting two Dehn twists along disjoint simple closed curves $\mathbf{a}$ and $\mathbf{b}$ on R . The two components L_1, L_2 (and L_1', L_2') represent $\mathbf{a}, \mathbf{b}$ respectively.

If $\mathbf{a} \cup \mathbf{b}$ is nonisolating, we can Legendrian realize it as $\mathbf{a}(0) \cup \mathbf{b}(0)$, and then consider the links

$$\tilde{\mathbf{L}} = \mathbf{a}(-\varepsilon)^- \cup \mathbf{b}(\varepsilon)^-, \quad \tilde{\mathbf{L}}' = \mathbf{a}(\varepsilon)^- \cup \mathbf{b}(-\varepsilon)^-.$$

The contact surgery link $\tilde{\mathbf{L}}$ is Legendrian isotopic to $\tilde{\mathbf{L}}'$ by translating $\mathbf{a}(-\varepsilon)$ vertically up and $\mathbf{b}(\varepsilon)$ vertically down by an amount 2ε . These two Legendrian knots do not intersect during the isotopy because we are assuming that $\mathbf{a}(0)$ and $\mathbf{b}(0)$ both are embedded simultaneously in a perturbation of R since $\mathbf{a} \cup \mathbf{b}$ is Legendrian realizable. By Theorem 4.34 we have that $\mathbf{L}$ is Legendrian isotopic to $\tilde{\mathbf{L}}$ and $\mathbf{L}'$ to $\tilde{\mathbf{L}}'$, concluding the proof in this subcase as well.

If $\mathbf{a} \cup \mathbf{b}$ is isolating, then it bounds a subsurface $\Sigma \subset R$ of genus $n \geq 1$. As in Figure 4.33, choose a chain $\tilde{G}$ of $2n + 1$ simple closed curves $\mathbf{c}_1, \dots, \mathbf{c}_{2n+1}$ in Σ such that $\mathbf{c}_j$ intersects $\mathbf{c}_{j-1}$ and $\mathbf{c}_{j+1}$ once and no other component, and such that Σ is a regular neighborhood of the union $G := \bigcup_j \mathbf{c}_j$.

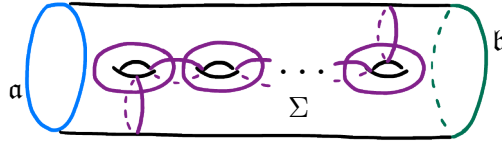


Figure 4.33. The surface Σ bounded by $\mathbf{a} \cup \mathbf{b}$ and the chain $\tilde{G}$ (in purple) whose regular neighborhood is Σ .

The graph $\tilde{G}$ is nonisolating, and so we can Legendrian realize it and get a Legendrian graph G embedded in a perturbation of R . As before, we can think of G as the graph in Example 4.6.5, where the Legendrian realization of the curve $\mathbf{c}_j$ corresponds to the Legendrian knot K_j . Reasoning as above for the other relations, we see that the two Legendrian links coming from Example 4.6.5 are compatible with R and differ by a Legendrian isotopy. Moreover, again by Theorem 4.34, these two Legendrian links are Legendrian isotopic to $\mathbf{L}$ and $\mathbf{L}'$. This completes the proof. $\square$

4.7. A contact version of the Kirby Theorem

In this section, we prove Theorems 4.1 and we state a corollary, Theorem 4.36. The key components are already in place, and we will now carefully piece them together.

Proof of Theorem 4.1. We already know that if two contact surgery links differ by Legendrian isotopies, insertion/deletion of cancelling pairs, contact handleslides, standard lantern moves, and standard chain moves, then they describe contactomorphic contact manifolds: each move is either a ribbon move (Proposition 4.28) or a Legendrian isotopy.

For the converse, let $\mathbf{L}$ and $\mathbf{L}'$ describe contactomorphic contact 3-manifolds. By Theorem 4.29, there is a sequence of ribbon moves and Legendrian isotopies taking $\mathbf{L}$ to $\mathbf{L}'$. Thus, it suffices to show the result for the case where $\mathbf{L}$ and $\mathbf{L}'$ differ by a single ribbon move. After restricting to sublinks, we can further assume that $\mathbf{L}$ and $\mathbf{L}'$ are R -equivalent.

Let

$$\tau_{\mathbf{L}} = \tau_{\overline{L}_n}^{-\delta_n} \cdots \tau_{\overline{L}_1}^{-\delta_1}, \quad \tau_{\mathbf{L}'} = \tau_{\overline{L}'_m}^{-\delta'_m} \cdots \tau_{\overline{L}'_1}^{-\delta'_1},$$

be the mapping classes in $\text{MCG}(R, \partial R)$ determined by $\mathbf{L}$ and $\mathbf{L}'$. By Lemma 3.3, every curve $\overline{L}_i$ and $\overline{L}'_j$ is homologically nontrivial in R .

We would like to use the mapping class group presentation in Theorem 2.6, which requires genus at least 2 and nonseparating curves. To arrange this, let G be the Legendrian skeleton of R and enlarge it in two steps:

- First, attach to G the purple Legendrian graph in Figure 4.34 along a Legendrian arc. The ribbon of this purple graph has genus 2 (this can be checked by constructing its ribbon explicitly), and the resulting ribbon $\hat{R}$ is the boundary connected sum of R with this genus 2 surface. Hence $\hat{R}$ has genus at least 2.
- Second, following [Avd13, Lemma 4.11], add further Legendrian segments to the enlarged graph to obtain a graph $\tilde{G}$ whose ribbon $\tilde{R}$ has connected boundary (and still genus at least 2, as attaching segments corresponds to adding 1–handles to the ribbon).

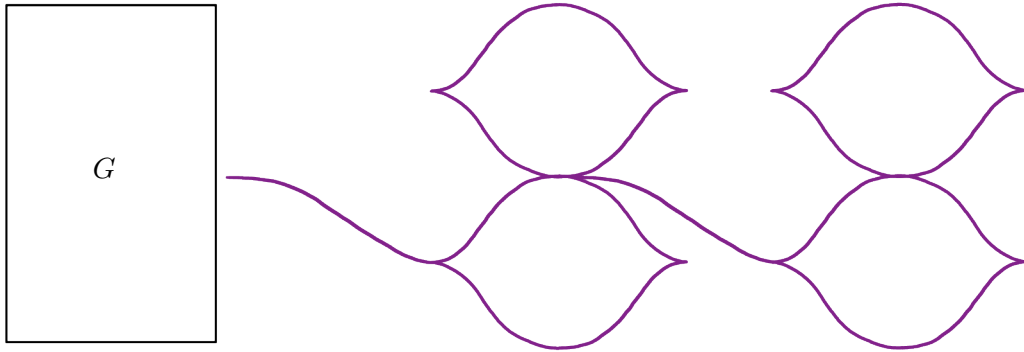


Figure 4.34. The box labelled G on the left symbolizes the Legendrian graph G . We are attaching to G (using a Legendrian arc) the purple Legendrian graph on the right of this figure.

Since $\tilde{G}$ contains G , Lemma 4.25 implies that $\mathbf{L}$ and $\mathbf{L}'$ are also $\tilde{R}$ -equivalent. We will still denote the corresponding mapping classes by $\tau_{\mathbf{L}}$ and $\tau_{\mathbf{L}'}$, now viewed in $\text{MCG}(\tilde{R}, \partial \tilde{R})$.

The advantage of passing to $\tilde{R}$ is that its genus is at least 2 and its boundary is connected. Since each curve $\overline{L}_i$ and $\overline{L}'_j$ is homologically nontrivial, it is nonseparating on $\tilde{R}$. Thus all Dehn twists in the factorizations of $\tau_{\mathbf{L}}$ and $\tau_{\mathbf{L}'}$ are along nonseparating curves, and we are in the hypotheses of Theorem 2.6.

Next, consider a sequence of relations $r_1, \dots, r_k$ from Theorem 2.6 that transforms τ_L into $\tau_{L'}$. After each relation r_i , the intermediate factorization still involves only Dehn twists along nonseparating curves.

By the Legendrian realization principle and Proposition 4.22, each intermediate factorization is realized by some contact surgery link compatible with $\tilde{R}$. Thus we obtain a sequence of contact surgery links

$$L = L^0, L^1, \dots, L^k = L'$$

such that:

- each L^i is compatible with $\tilde{R}$;
- the mapping class of L^i is the one obtained from τ_L after the relations $r_1, \dots, r_i$;
- consecutive links L^i, L^{i+1} differ by an elementary R -equivalence.

By Proposition 4.35, up to Legendrian isotopy each elementary R -equivalence is realized by inserting/deleting a cancelling pair, a contact handleslide, a lantern move, or a chain move.

Finally, to ensure that all lantern and chain moves are the *standard* ones, we modify the initial enlargement of G slightly. Along with the genus-2 piece in Figure 4.34, we also attach two Legendrian graphs G_{chain} and G_{lantern} whose ribbons support the specific chain and lantern relations underlying the standard chain and lantern moves, see Figure 4.35. In this way we can arrange that every lantern or chain relation in the sequence $r_1, \dots, r_k$ is supported on one of these subsurfaces, and therefore corresponds to the standard chain or lantern move on the surgery diagram.

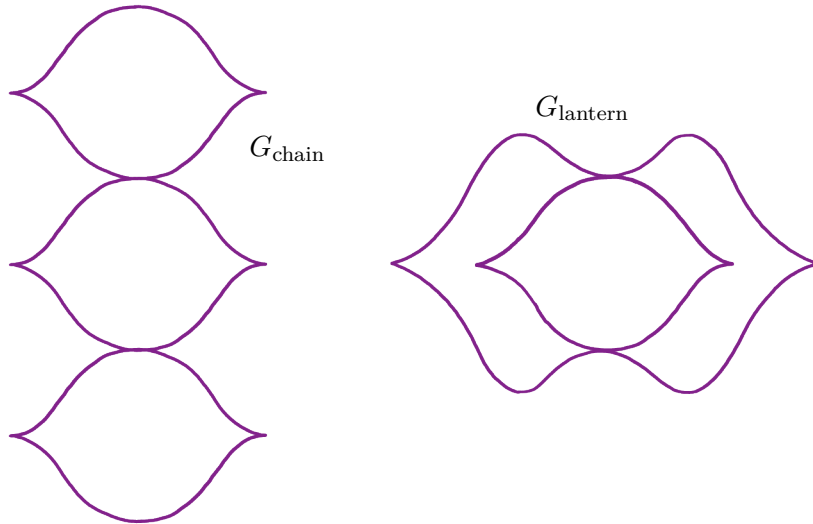


Figure 4.35. Legendrian graphs G_{chain} and G_{lantern} whose ribbons support the standard chain and lantern relations.

This concludes the proof. □

If one is not concerned with keeping track of a small generating set of moves, one can allow all lantern and chain moves, not just the standard ones. This gives the following slightly weaker but more flexible statement.

Theorem 4.36. *Let $\mathbf{L}$ and $\mathbf{L}'$ be two contact surgery links in (S^3, ξ_{st}) that describe contactomorphic contact 3-manifolds. Then $\mathbf{L}$ and $\mathbf{L}'$ differ by a sequence of Legendrian isotopies, addition or deletion of cancelling pairs, contact handleslides, lantern moves, and chain moves.*

Proof. This is an immediate corollary of Theorem 4.1. Indeed, the standard lantern move and the standard chain move are particular examples of lantern moves and chain moves, respectively. Allowing all lantern and chain moves gives a superset of the moves in Theorem 4.1, and hence suffices as well. $\square$

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