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On the Cox rings of nef complete intersections in projective space

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Abstract

We introduce a new cohomological method to compute new global sections of a complete intersection in a projective variety. This is more direct than previous methods and yields equations defining these new sections in the Cox ring. As examples, we show that general codimension $c \geqslant 1$ complete intersections in $\mathbb{P}^1 \times \mathbb{P}^n$, subject to some numerical conditions on n and the degrees of the defining equations, are Mori Dream Spaces, computing the Cox ring in full. We do the same for codimension c complete intersections in $\mathbb{P}^c \times \mathbb{P}^n$. We display families of Calabi-Yau varieties that further support Morrison's cone conjecture. We provide also new examples of general type Mori Dream Spaces. In a separate example, we show how our method may be used to translate between two models of a degree one del Pezzo surface admitting a conic bundle structure – namely one model as a hypersurface in a rational scroll and its second anticanonical model in projective space.

Zusammenfassung

Wir führen eine neue kohomologische Methode zur Berechnung neuer globaler Sektionen eines vollständigen Schnitts in einer projektiven Varietät ein. Diese Methode ist direkter als bisherige Methoden und liefert Gleichungen, die diese Sektionen im Cox-Ring definieren. Als Beispiel zeigen wir, indem wir den Cox-Ring berechnen, dass allgemeine vollständige Schnitte der Kodimension $c \geqslant 1$ in $\mathbb{P}^1 \times \mathbb{P}^n$ unter gewissen numerischen Bedingungen an n und die Grade der definierenden Gleichungen Mori-Dream-Spaces sind. Dasselbe gilt für vollständige Schnitte der Kodimension c in $\mathbb{P}^c \times \mathbb{P}^n$. Wir konstruieren Familien von Calabi–Yau-Varietäten, die Morrisons Kegel-Vermutung stützen, und geben neue Beispiele von Varietäten allgemeinen Typs, die Mori-Dream-Spaces sind. In einem weiteren Beispiel zeigen wir, wie unsere Methode eine Übersetzung zwischen zwei Modellen einer del-Pezzo-Fläche vom Grad 1 mit konischer Bündelstruktur ermöglicht – nämlich ihrem Modell als Hyperfläche in einem rationalen Scroll und ihrem antikanonischen Modell im projektiven Raum.

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The most sincere recognition must first be given to my Mum, Jane. In being homeschooled from 12-15 years old, Mum's role would extend from the familial to the academic. With little resources, she took on the role of teacher with no formal teaching qualifications or higher education. Before this I had no plans to go to university, and this journey culminates now in this academic work. It wouldn't feel correct to not give her story some light. Thus the achievement of this PhD is at least to be shared with Mum and my brother Peter, and I am proud of the three of us for where we have reached today as a collective.

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As a final acknowledgement, I also mention my hometown Grimsby, which I am sure I will always hold dearly in my identity. It is not the most fashionable place (quite unjustly), and may not be the most opportunity-rich, but it will always be home and I hope to represent it in a positive light wherever I am.

Contents

1 Introduction

Throughout this thesis we work with varieties over $\mathbb{C}$, although many arguments generalise to other fields, particularly those algebraically closed fields of characteristic 0. Our general setting will be that of a smooth complete intersection Z embedded in a smooth projective ambient space X, whose Cox ring is finitely generated – that is to say that X is a Mori Dream Space. Each section of the Cox ring of X restricts to a section on Z. There may, however, be Cox ring sections on Z that are not the restriction of any section on X.

Cox rings were first introduced for toric varieties in [12] as a generalisation of the homogeneous coordinate ring of projective space. They appeared earlier as *universal torsors* in number theory, for example as seen in [10]. Our general reference for Cox rings is [2]. The application we shall focus most on appears in birational geometry; the notion of Mori Dream Spaces is introduced in [21]. One can apply the minimal model program (see [22]) to Mori Dream Spaces. Being a Mori Dream Space is equivalent to having finitely generated Cox ring under certain usual assumptions on the projective variety (for example $\mathbb{Q}$-factoriality). Having a presentation of the Cox ring of a smooth projective variety X thus allows one to describe its birational geometry combinatorially.

The principal achievement of this thesis is to introduce a new method, using Čech cohomology, to compute spaces of sections on Z that are not restrictions of those on X. In some examples we also use this to compute the Cox rings of the given Z. We see that elements of higher degree sheaf cohomology groups on X allow for sections on complete intersection subvarieties in X. Whether such an element of a cohomology group on X can provide us with a new section on Z, canonical up to the restriction of a section on X, is determined by linear algebra conditions on the induced maps of cohomology given by the defining equations of Z.

With this method, we strengthen a result of Ottem in [28], who computes the Cox rings of general degree (d, e) hypersurfaces Z in $\mathbb{P}^1 \times \mathbb{P}^n$ with $d, e \geqslant 1$ and $n \geqslant 3$. Not only can we give the exact conditions for which the Cox ring here takes the general presentation as in [28, Theorem 1.1], we can generalise to the case of $Z \subset \mathbb{P}^1 \times \mathbb{P}^n$ where Z is a complete intersection of dimension at least 3 under certain open conditions and bounds on the sum of first components of the degrees of the defining equations. Allowing for degenerate cases, where some polynomial coefficients of the defining equations are zero, we compute the Cox rings of a class of general type varieties, containing examples of arbitrary codimension and arbitrarily positive canonical class, showing that these are Mori Dream Spaces.

We also apply our method to codimension $c \geqslant 2$ complete intersections $Z \subset \mathbb{P}^c \times \mathbb{P}^n$, where $n > c$. We are able to find new sections on Z, and in the simplest two cases – that where each defining equation is linear in the coordinates on $\mathbb{P}^c$ and that where all but one defining equation is linear in the coordinates of $\mathbb{P}^c$ and the remaining is quadratic – we can compute the Cox rings in the general setting, stating exactly when this general picture occurs. In particular, we show that these varieties are Mori Dream Spaces. The latter case includes the Calabi-Yau setting.

It is already known that Fano varieties have finitely generated Cox ring [6]. Calabi-Yau varieties are not necessarily Mori Dream Spaces, as shown in an example of Kawamata in [24]. Morrison's cone conjecture states that the moveable cone of a Calabi-Yau variety will decompose into rational polyhedral cones, each giving a minimal model, and that the group of birational automorphisms acts on the moveable cone with rational polyhedral fundamental domain [27]. Our results for Calabi-Yau complete intersections in products of projective spaces support this conjecture.

Calabi-Yau varieties are of interest to physicists active in string theory. In string theory, six further spatial dimensions are required in addition to our usual three (with a further time dimension giving a full ten), and Calabi-Yau threefolds over $\mathbb{C}$ are a candidate for providing these. We refer to [20] for the mathematics and physics of this correspondence. As explained in [7, 8], one wishes in particular to compute the vector bundle cohomology of Calabi-Yau threefolds. Our results and examples provide answers for the zeroth degree line bundle cohomologies on classes of Calabi-Yau varieties, including some of those considered in the afforementioned articles.

We are also able to provide new families of general type varieties that are Mori Dream Spaces, computing their Cox rings in full.

Our method is particularly suited to understanding where Cox ring behaviour, including size of the effective and moveable cones, changes in degenerate cases. We are able to compute Cox rings of a degenerate case in the examples of $Z \subset \mathbb{P}^1 \times \mathbb{P}^n$ and $Z \subset \mathbb{P}^c \times \mathbb{P}^n$ above. The structure of the Cox ring is closely linked to the structure of syzygy modules defined by the polynomial coefficients of the defining equations viewed in the coordinates of $\mathbb{P}^c$. This partially answers questions posed in [7, 8] on how degenerations of the polynomial coefficients lead to changes in the effective and moveable cones.

Structure of thesis and originality

In Chapter 2, we state our main results.

In Chapter 3, we give the necessary introduction and definitions of Cox rings and Mori Dream Spaces. Then we introduce the method allowing us to find new Cox ring sections of a subvariety Z embedded in a variety X, whose Cox ring we understand, under certain conditions. Furthermore, these conditions ensure that we may obtain all Cox ring sections of Z up to some linear algebra calculations; these are in practice difficult to solve. We also explain how our method continues to give sections and information under weaker assumptions. The work developing this method is my own, though we rely on standard theory of Cox rings.

Chapter 4 treats the case of complete intersections Z of dimension at least 3 in $\mathbb{P}^1 \times \mathbb{P}^n$. For hypersurfaces, the material here is found in [29], published on the Arxiv in collaboration with Ito and Szendrői. All the work of Chapter 4 not featured in this paper, including the interpretation of the sections $w_{m,k}$ in Section 4.4.1, is my own.

The method utilised to find the results of Chapter 4 reduces the question of finite generation to a linear algebra problem, solving the kernel of matrices that are constant along diagonals, so-called Toeplitz matrices in the literature. Chapter 5 is dedicated to this linear algebra problem, solving independently the statement used in the proofs of Chapter 4. The work here is also my own, and generalises over integral rings, thus warranting its own Chapter. We see how algebraic geometry and Cox ring computations may allow us to solve systems of polynomial equations.

We split the work on codimension $c \geqslant 2$ complete intersections $Z \subset \mathbb{P}^c \times \mathbb{P}^n$ into two Chapters, both of which are entirely my work. For $c = 2$, much of the method can still be computed by hand, and we can provide detail in each step. This allows us not only to give Cox ring presentations for the two simplest cases of Z, but also allows us to compute Cox rings of a degenerate example in the second case. This makes up Chapter 6. In Chapter 7, we take inspiration from our findings in the $c = 2$ case explored in Chapter 6 to give analogous results for arbitrary $c \geqslant 2$.

Chapter 8 shows an example where our method provides information where some of the assumptions don't hold. We can find sections of a degree one del Pezzo surface Z embedded in a rational scroll $\mathbb{F}(0, 1, 1)$, which allows us to translate this model into its anticanonical embedding in weighted projective space. Section 8.1 is my own work, whereas Section 8.2 is joint work with Szendrői.

We conclude in Chapter 9 with remarks and comments how one may progress. This includes generalising the results of Chapters 6–7 to higher degrees, as well as exploring subvarieties Z in $(\mathbb{P}^c)^k \times \mathbb{P}^n$. In particular, we can show the latter generally gives non-Mori Dream Spaces even in our degree 2 setting - the $c = 1$ case was already proven in [34], following the work of [24]. Using the method of this thesis, we give a conjecture for the Cox ring of these varieties, even though they are not finitely generated. Once more, the work of this Chapter is my own.

2 Statement of main results

Here we lay out the main results of the thesis.

2.1 On Cox rings of nef complete intersections in $\mathbb{P}^1 \times \mathbb{P}^n$

The following results will be the main results in Chapter 4.

Fix $c \geqslant 1$ and $n \geqslant c+2$ and fix $X = \mathbb{P}^1 \times \mathbb{P}^n$ over $\mathbb{C}$. Let x_0, x_1, respectively $y_0, \dots, y_n$, be homogeneous coordinates on $\mathbb{P}^1$, respectively $\mathbb{P}^n$. If π_1, π_2 are the projection maps to the respective factors $\mathbb{P}^1, \mathbb{P}^n$, then

$$\operatorname{Pic} X \cong \operatorname{Pic} \mathbb{P}^1 \oplus \operatorname{Pic} \mathbb{P}^n \cong \mathbb{Z}^2,$$

generated by the pullbacks $\pi_1^* \mathcal{O}_{\mathbb{P}^1}(1)$ and $\pi_2^* \mathcal{O}_{\mathbb{P}^n}(1)$ respectively. The Cox ring of X is

$$\mathcal{R}(X) = \mathbb{C}[x_0, x_1, y_0, \dots, y_n],$$

with the $\mathbb{Z}^2$-grading given by $\deg(x_i) = (1,0)$ and $\deg(y_j) = (0,1)$. The ring

$$R = \mathcal{R}(\mathbb{P}^n) = \mathbb{C}[y_0, \dots, y_n]$$

is naturally a $\mathbb{Z}^2$-graded subring of $\mathcal{R}(X)$, giving $\mathcal{R}(X)$ the structure of a $\mathbb{Z}^2$-graded R-module.

We define our primary varieties of interest.

Definition 2.1. Fix $d \geqslant 0$ and $e \geqslant 1$. Let V be an irreducible and normal hypersurface

$$V = \left\{ g_0 x_0^d + g_1 x_0^{d-1} x_1 + \ldots + g_d x_1^d = 0 \right\} \subset \mathbb{P}^1 \times \mathbb{P}^n \tag{1}$$

defined by a set of homogeneous degree e elements $g_0, \dots, g_d \in R$.

(i) An *index sequence of length* τ, where $0 \leqslant \tau \leqslant d$, is a sequence of indices $0 = i_0 < \cdots < i_\tau = d$. By convention, $\tau = 0$ if and only if $d = 0$. We say V admits the index sequence $\{i_0, \dots, i_\tau\}$ of length τ if $g_i \neq 0$ if and only if $i \in \{i_0, \dots, i_\tau\}$; we call the set of indices $\{i_0, \dots, i_\tau\}$ the *index sequence* of V.

(ii) We call V *regular with index sequence* $\{i_0, \dots, i_\tau\}$, if it is non-singular, admits the index sequence $\{i_0, \dots, i_\tau\}$, and the homogeneous polynomials $g_{i_0}, \dots, g_{i_\tau}$ form a regular sequence in R.

Definition 2.2. For given $c \geqslant 1$ and $n \geqslant c+2$, the variety $Z \subset X = \mathbb{P}^1 \times \mathbb{P}^n$ will be said to have form $(*)$ with respect to hypersurfaces $V_1, \dots, V_c$ if $Z = \bigcap_{m=1}^c V_m$, and the following conditions are met.

(i) For each $m = 1, \dots, c$, the hypersurface $V_m \subset X$ is irreducible and normal, defined by an equation

$$r_m = g_{m,0} x_0^{d_m} + g_{m,1} x_0^{d_m - 1} x_1 + \ldots + g_{m,d} x_1^{d_m},$$

with $d_m \geqslant 0$, $e_m \geqslant 1$, and V_m admits an index sequence $\{i_{m,0}, \dots, i_{m,\tau_m}\}$ of length τ_m, with $0 \leqslant \tau_m \leqslant d_m$ as in Definition 2.1.

(ii) The sequence $r_1, \dots, r_c$ is regular in $\mathcal{R}(X)$ and the intermediary complete intersections

$$Z_m = \bigcap_{m'=1}^{m} V_{m'}$$

are non-singular for $m = 1, \ldots, c$.

(iii) The sequence

$$g_{1,0}, \ldots, g_{1,d_1}, \ldots\ldots, g_{c,0}, \ldots, g_{c,d_c}$$

is regular in R. In particular, we have the numerical condition

$$c + \sum_{m=1}^{c} d_m \leqslant n + 1. \tag{2}$$

We note that, so long as the numerical condition (2) holds, the conditions in parts (ii) and (iii) are open with respect to the data of part (i) in Definition 2.2.

Main Theorem 2.3. *Let $c \geqslant 1$ and $n \geqslant c + 2$ and set $X = \mathbb{P}^1 \times \mathbb{P}^n$. Suppose $Z \subset X$ satisfies conditions (i) and (ii) from Definition 2.2 with respect to hypersurfaces $V_1, \ldots, V_c$.*

(i) Suppose that Z has form $(*)$ with respect to $V_1, \ldots, V_c$. or equivalently, that condition (iii) in Definition 2.2 is also satisfied. Then there is an isomorphism of $\mathbb{Z}^2$-graded algebras

$$\mathcal{R}(Z) \cong \mathcal{R}(X)[w_{1,1}, \ldots, w_{1,\tau_1}, \ldots\ldots, w_{c,\tau_c}]/I, \tag{3}$$

where $\deg t = \deg_{\mathcal{R}(X)} t$ for $t \in \mathcal{R}(X)$ and $\deg(w_{m,k}) = (i_{m,k-1} - i_{m,k}, e_m)$ for each $1 \leqslant m \leqslant c$, $1 \leqslant k \leqslant \tau_m$, and I is the homogeneous ideal generated by the equations

$$x_1^{i_{m,k+1}-i_{m,k}} w_{m,k+1} + g_{i_{m,k}} - x_0^{i_{m,k}-i_{m,k-1}} w_{m,k}, \quad 1 \leqslant m \leqslant c, 1 \leqslant k \leqslant \tau_m, \tag{4}$$

with variables of undefined index set to zero. In particular, Z is a Mori Dream Space.

(ii) Conversely, suppose that there exists an isomorphism of the form (3), with degrees and ideal I as given. Then the sequence

$$g_{1,0}, \ldots, g_{1,d_1}, \ldots\ldots, g_{c,0}, \ldots, g_{c,d_c}$$

is regular in R and thus Z has the form $(*)$ with respect to $V_1, \ldots, V_c$.

The method used to prove Main Theorem 2.3 gives information in more general settings. We give one such setting in the following remark, postponing further comments to the end of this Chapter, for they apply also to the results in Section 2.3

Remark 2.4. The only properties we use of $\mathbb{P}^n$ to prove Theorem 2.3 is that it has dimension at least $c + 2$ and that its cohomology on all line bundles vanishes in degrees i with $1 \leqslant i \leqslant c$. In particular, we may replace $\mathbb{P}^n$ with a product W of say $s \geqslant 2$ projective spaces, each having dimension at least $c + 1$, so long as $\dim W \geqslant c + 2$. Instead of a $\mathbb{Z}^2$-grading we have a $\mathbb{Z} \bigoplus \mathbb{Z}^s$ grading, and the degrees e_m of Definition 2.1 are taken in $\mathbb{Z}^s$ and assumed positive in each component.

In Sections 4.3 and 4.4 we apply Theorem 2.3 to a variety of examples of general interest. We give just one result here for $c = 1$, applying to families of Calabi-Yau hypersurfaces in $\mathbb{P}^1 \times \mathbb{P}^3$.

Theorem 2.5. *Choose three general polynomials $g_0, g_1, g_2 \in \mathbb{C}[y_0, y_1, y_2, y_3]$ of degree $e = 4$, forming a regular sequence. Consider the family of varieties $q\colon \mathcal{Z} \to \mathbb{A}^1_t$ defined by*

$$\mathcal{Z} = \left\{g_0x_0^2 + tg_1x_0x_1 + g_2x_1^2 = 0\right\} \subset \mathbb{P}^1 \times \mathbb{P}^3 \times \mathbb{A}^1_t.$$

For $t \neq 0$, the $\mathbb{Z}^2$-graded Cox ring $\mathcal{R}(Z_t)$ of the Calabi–Yau hypersurface $Z_t \subset \mathbb{P}^1 \times \mathbb{P}^3$ can be presented as

$$\mathcal{R}(Z_t) \cong \mathbb{C}[x_0, x_1, y_0, y_1, y_2, y_3, z_1, z_2]/\langle x_1z_1 + g_0, x_1z_2 + g_1 - x_0z_1, g_2 - x_0z_2\rangle,$$

with variables of bidegrees $(1,0)$, $(0,1)$ and $(-1,4)$ respectively. For $t = 0$, we have

$$\mathcal{R}(Z_0) \cong \mathbb{C}[x_0, x_1, y_0, y_1, y_2, y_3, w]/\langle x_1^2w + g_0, g_2 - x_0^2w\rangle,$$

with variables of bidegrees $(1,0)$, $(0,1)$ and $(-2,4)$ respectively. In particular, every member of the family is a Mori dream space, with a complete intersection Cox ring, but the effective cone and Cox ring jump discontinuously in the family.

2.2 On kernels of banded Toeplitz matrices with polynomial entries

The following results are the main results in Chapter 5.

Definition 2.6. Let S be a ring and $a \geqslant 1$, $d \geqslant 0$. The banded Toeplitz matrix of degree a associated to a sequence $g_0, \dots, g_d \in S$ is the $(a-1) \times (a+d-1)$ matrix $(\varepsilon_{s,t})$ where $\varepsilon_{s,t} = g_i$ if $t - s = i$ and $\varepsilon_{s,t}$ is zero otherwise. By convention, the degree 1 matrix is the zero matrix. If

$$r = g_0x_0^d + \cdots + g_dx_1^d \in S[x_0, x_1]$$

is a general homogeneous degree d polynomial in x_0, x_1 with coefficients in S, then we define, for $a \geqslant 1$, the matrix $A_{a,r}$ to be the degree a banded Toeplitz matrix associated to the sequence $g_0, \dots, g_d \in S$:

$$A_{a,r} = \begin{pmatrix} g_0 & g_1 & g_2 & \cdots & g_d & 0 & 0 & \cdots\cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & g_0 & g_1 & \cdots & g_{d-1} & g_d & 0 & \cdots\cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ & \vdots & & \ddots & & \vdots & & \ddots & \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots\cdots & 0 & g_0 & g_1 & \cdots & g_d & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots\cdots & 0 & 0 & g_0 & \cdots & g_{d-1} & g_d \end{pmatrix}.$$

We will be interested in the case that $S = \mathbb{C}[y_0, \dots, y_n]$, so that an irreducible and normal subvariety $V \subset \mathbb{P}^1 \times \mathbb{P}^n$ is defined by an equation $r \in S[x_0, x_1]$. To prove Main Theorem 2.3 for hypersurfaces, we compute the kernel of $A_{a,r}$ in each degree $a \geqslant 1$. For codimension c complete intersections, we need a slightly more general matrix in each degree.

Definition 2.7. Fix $c \geqslant 1$, $n \geqslant c+2$ and $S = \mathbb{C}[y_0, \dots, y_n]$. Suppose for each $m = 1, \dots, c$, we have an equation

$$r_m = g_{m,0}x_0^{d_m} + \cdots + g_{m,c}x_1^{d_m} \in S[x_0, x_1],$$

where $d_m \geqslant 0$ and $g_{m,0}, \dots, g_{m,c}$ is a sequence of homogeneous polynomials of degree $e_m \geqslant 1$ in S. Define to this data the matrix

$$A_a = (A_{a,r_1}|\cdots|A_{a,r_c})$$

for each $a \geqslant 1$, which may be viewed as a map of free S-modules

$$S^{a+d_1-1} \times \cdots \times S^{a+d_c-1} \to S^{a-1}.$$

Define the trivial spaces $\eta_a \subset \ker A_a$ to be generated by columns of the matrices

$$A^*_{a,(m,m')} = (0|\cdots|0|A^t_{m',a+d_m}|0|\cdots|0|-A^t_{m,a+d_{m'}}|0|\cdots|0)^t$$

for each $1 \leqslant m < m' \leqslant c$. Then set $K_a = (\ker A_a)/\eta_a$ and $K = \bigoplus_{a\geqslant 1} K_a$, which may be viewed as a graded S-module.

Theorem 2.8. *Suppose that we have the data as in Definition 2.7.*

(i) There are natural truncation operators $\tilde{x}_0, \tilde{x}_1 : K_{a+1} \to K_a$ for each $a \geqslant 1$. Furthermore, let U' be the free graded S-algebra

$$U' = S[z_{1,1}, \ldots z_{1,d_c}, \ldots\ldots, z_{c,1}, \ldots, z_{c,d_c}],$$

with each $z_{m,k}$ in degree one. There is a graded homomorphism of S-modules $\psi' : U' \to K$.

(ii) Suppose that for each $m = 1, \ldots, c$, the equation r_m corresponds to a hypersurface (not necessarily irreducible and normal) $V \subset \mathbb{P}^1 \times \mathbb{P}^n$ admitting an index sequence of length τ_m as in Definition 2.1 and use the same notation. If U is the free graded S-algebra

$$U = S[w_{1,1}, \ldots, w_{1,\tau_1}, \ldots\ldots, w_{c,1}, \ldots, w_{c,\tau_c}],$$

with each $w_{m,k}$ in degree $i_{m,k} - i_{m,k-1}$, then there is a map $\psi : U \to K$ of graded S-modules. If $\tau_m = d_m$ for each $m = 1, \ldots, c$, we have $U = U'$ and $\psi = \psi'$.

(iii) Suppose in addition to the assumptions of part (ii), that the sequence

$$g_{1,0}, \ldots, g_{1,\tau_1}, \ldots\ldots, g_{c,1}, \ldots, g_{c,\tau_c}$$

is regular in R. Then all elements of K may be realised by applying the truncation operators $\tilde{x}_0, \tilde{x}_1$ to an element in the image of ψ. If also $\tau_m = d_m$ for each $m = 1, \ldots, c$, then ψ is already surjective, giving K the structure of a finitely generated S-algebra. In this case, the kernel of ψ is generated in degrees one and two.

As in the previous Section, the above Definitions are tailored to our applications and we add the following remarks.

Remark 2.9. In the data in Definition 2.7, we may allow S to be any finitely generated algebra graded by a free abelian group; if S is the Cox ring of a Mori Dream space W and for each $m = 1, \ldots, c$ the polynomials $g_{m,0}, \ldots, g_{m,c}$ are sections of an ample line bundle on W, then there is an analogous version of Theorem 2.8 in this setting. This has applications in the case of $Z \subset \mathbb{P}^1 \times W$ as remarked in Remark 2.20.

Remark 2.10. Since we apply this algebraic result to varieties in complex projective space, we have defined S to be a $\mathbb{C}$-algebra. All of the constructions work over any field and even any integral ring. The only statement from Theorem 2.8 that we can't deduce over a general integral ring is the final statement of part (iii) concerning the kernel of ψ.

2.3 On Cox rings of codimension c nef complete intersections in $\mathbb{P}^c \times \mathbb{P}^n$

The following results will be proven in Chapters 6–7.

Fix $c \geqslant 2$ and $n > c$ and fix $X = \mathbb{P}^c \times \mathbb{P}^n$ over $\mathbb{C}$. Let $x_0, \ldots, x_c$, respectively $y_0, \ldots, y_n$, be homogeneous coordinates on $\mathbb{P}^1$, respectively $\mathbb{P}^n$. If π_1, π_2 are the projection maps to the respective factors $\mathbb{P}^c, \mathbb{P}^n$, then

$$\operatorname{Pic} X \cong \operatorname{Pic} \mathbb{P}^c \oplus \operatorname{Pic} \mathbb{P}^n \cong \mathbb{Z}^2,$$

generated by the pullbacks $\pi_1^* \mathcal{O}_{\mathbb{P}^c}(1)$ and $\pi_2^* \mathcal{O}_{\mathbb{P}^n}(1)$ respectively. The Cox ring of X is

$$\mathcal{R}(X) = \mathbb{C}[x_0, \ldots, x_c, y_0, \ldots, y_n],$$

with the $\mathbb{Z}^2$-grading given by $\deg(x_i) = (1, 0)$ and $\deg(y_j) = (0, 1)$. The ring

$$R = \mathcal{R}(\mathbb{P}^n) = \mathbb{C}[y_0, \ldots, y_n]$$

is naturally a $\mathbb{Z}^2$-graded subring of $\mathcal{R}(X)$, giving $\mathcal{R}(X)$ the structure of a $\mathbb{Z}^2$-graded R-module.

We define a form our subvarieties of interest will take.

Definition 2.11. For each $m = 1, \ldots, c$, let V_m be an irreducible and normal hypersurface defined by a degree (d_m, e_m) equation $r_m \in \mathcal{R}(X)$ for some $d_m, e_m \geqslant 1$. The variety $Z \subset X = \mathbb{P}^c \times \mathbb{P}^n$ will be said to have form (†) with respect to the hypersurfaces $V_1, \ldots, V_c$ if the following hold.

(i) We have $Z = \bigcap_{m=1}^c V_m$.

(ii) The sequence $r_1, \ldots, r_c$ is regular in $\mathcal{R}(X)$ and the intermediary complete intersections

$$Z_m = \bigcap_{m'=1}^{m} V_{m'}$$

are non-singular for $m = 1, \ldots, c$.

2.3.1 The case $(d_1, d_2, \ldots, d_c) = (1, 1, \ldots, 1)$

Suppose that $c \geqslant 1, n \geqslant 2$ and $Z \subset \mathbb{P}^c \times \mathbb{P}^n$ has form (†) with respect to hypersurfaces $V_1, \ldots, V_c$ of degree $[V_m] = (1, e_m) \in \operatorname{Pic} X$ for each $m = 1, \ldots, c$ and some $e_m \geqslant 1$. There are then degree e_m polynomials $f_{m,0}, \ldots, f_{m,c}$ such that V_m is defined by

$$r_m = f_{m,0} x_0 + \ldots f_{m,c} x_c$$

for each $m = 1, \ldots, c$.

Definition 2.12. Define to the data above the $c \times (c+1)$ matrix

$$M = \begin{pmatrix} f_{1,0} & \cdots & f_{1,c} \\ \vdots & \ddots & \vdots \\ f_{c,0} & \cdots & f_{c,c} \end{pmatrix}$$

with entries in R. One notes that the (m, i) entry of this matrix is $\frac{\partial r_m}{\partial x_i}$. For each $i = 0, \ldots, c$, let ν_i be the maximal minor of M obtained ignoring the i^{th} column, which is a degree $e' = \sum_{m=1}^{c} e_m$ element of R.

Theorem 2.13. *Suppose that $c \geqslant 1$, $n \geqslant 2$ and $Z \subset X = \mathbb{P}^c \times \mathbb{P}^n$ has form (†) with respect to hypersurfaces $V_1, \ldots, V_c$, where $V_m = (1, e_m)$ for each $m = 1, \ldots, c$ and some $e_m \geqslant 1$. Define the matrix M and polynomials $\nu_i \in R$ of degree e' as in Definition 2.12. Then Z is isomorphic to the blow-up of $\mathbb{P}^n$ with centre the codimension 2 subvariety defined by the maximal minors ν_i of M. There is a section $v \in \mathcal{R}(Z)$ in degree $(-1, e')$, which is not the restriction of a global section on X satisfying the equations*

$$vx_i - (-1)^i \nu_i = 0, \quad 0 \leqslant i \leqslant c,$$

whose vanishing locus corresponds to the exceptional divisor. The Cox ring of Z admits the presentation

$$\mathcal{R}(Z) \cong \mathbb{C}[x_0, \ldots, x_c, y_0, \ldots, y_n, v]/\langle vx_0 - \nu_0, \ldots, vx_c - (-1)^c \nu_c \rangle.$$

In particular, Z is a Mori Dream Space.

We note that in Theorem 2.13, the description of Z as a blow-up does not give us information as to whether it is a Mori Dream Space. See [17] for an example of a blow-up of a Mori Dream Space in a codimension two locus that is itself not a Mori Dream Space.

2.3.2 The case $(d_1, d_2, \ldots, d_c) = (2, 1, \ldots, 1)$

We now consider a class of varieties that include the Calabi-Yau case. Suppose that $c \geqslant 1$, $n \geqslant 2$ and $Z \subset \mathbb{P}^c \times \mathbb{P}^n$ has form (†) with respect to hypersurfaces $V_1, \ldots, V_c$, where $[V_1] = (2, e_1)$ and $[V_m] = (1, e_m)$ in $\operatorname{Pic} X$ for each $m = 2, \ldots, c$ and some $e_1, \ldots, e_c \geqslant 1$. There are degree e_1 polynomials $f_{ij} \in R$ for each $0 \leqslant i \leqslant j \leqslant c$ and for each $m = 2, \ldots, c$ degree e_m polynomials $g_{m,0}, \ldots, g_{m,c}$, such that V_1 is defined by

$$r_1 = \sum_{0 \leqslant i \leqslant j \leqslant c} f_{ij} x_i x_j$$

and V_m is defined by

$$r_m = g_{m,0} x_0 + \ldots g_{m,c} x_c$$

for $m = 2, \ldots, c$.

Definition 2.14. To this data we define the $(c-1) \times (c+1)$ matrix

$$M = \begin{pmatrix} g_{2,0} & \cdots & g_{2,c} \\ \vdots & \ddots & \vdots \\ g_{c,0} & \cdots & g_{c,c} \end{pmatrix}$$

with entries in R, with (m,i) entry equal to $\frac{\partial r_m}{\partial x_i}$. For each $0 \leqslant i < j \leqslant c$ define μ_{ij} to be the maximal minor of M obtained ignoring the i^{th} and j^{th} columns. We define also $\mu_{ij} = -\mu_{ji}$ for $i > j$. Define further the augmented $c \times (c+1)$ matrix

$$M' = \begin{pmatrix} g'_{1,0} & \cdots & g'_{1,c} \\ g_{2,0} & \cdots & g_{2,c} \\ \vdots & \ddots & \vdots \\ g_{c,0} & \cdots & g_{c,c} \end{pmatrix},$$

where $g'_{1,i} = \frac{1}{2}\frac{\partial r_1}{\partial x_i}$ for each $i = 0, \dots, c$. For each $i = 0, \dots, c$, let ν_i be the maximal minor of M' obtained ignoring the i^{th} column, which has degree $e' - e_1$ in R, where $e' = \sum_{m=1}^{c} e_m$.

Lemma 2.15. *For each $0 \leqslant i < j \leqslant c$, we may multiply r_1 by μ_{ij} and use the equations $r_2, \dots, r_c$ to eliminate all x_k with $k \neq i, j$ to obtain a quadratic equation in $R[x_i, x_j]$ which vanishes on Z. More precisely, we obtain degree $2e' - e_1$ polynomials F_i for each $i = 0, \dots, c$ and G_{ij} for each $0 \leqslant i < j \leqslant c$ so that*

$$F_j x_i^2 + G_{ij} x_i x_j + F_i x_j^2 = 0$$

on Z for all $0 \leqslant i < j \leqslant c$. Furthermore, the subvariety $Y \subset \mathbb{P}^n$ defined by the F_i, G_{ij} has codimension at most 3.

Proposition 2.16. *Suppose that $c \geqslant 1$, $n \geqslant 2$ and $Z \subset \mathbb{P}^c \times \mathbb{P}^n$ has form (†) with respect to hypersurfaces $V_1, \dots, V_c$, where $[V_1] = (2, e_1)$ and $[V_m] = (1, e_m)$ in $\operatorname{Pic} X$ for each $m = 2, \dots, c$ and some $e_1, \dots, e_c \geqslant 1$. With notation as in Definition 2.14 and Lemma 2.15, there are sections $u \in \mathcal{R}(Z)$ in degree $(0, e')$, sections $z_0, \dots, z_c \in \mathcal{R}(Z)$ in degree $(-1, 2e' - e_1)$, and a degree $2e'$ polynomial ω in R satisfying on Z the equations*

$$\begin{gathered} x_i z_i - F_0, \quad i = 0, \dots, c \\ x_i z_j - (-1)^{j-i}\mu_{ij} u + \tfrac{1}{2} G_{01}, \quad 0 \leqslant i, j \leqslant c, \quad i \neq j \\ x_i u - (-1)^i \nu_i, \quad z_i u + (-1)^i \nu_i, \quad i = 0, \dots, c, \\ u^2 - \omega, \end{gathered} \tag{5}$$

Main Theorem 2.17. *Suppose $Z \subset \mathbb{P}^c \times \mathbb{P}^n$ is as in Proposition 2.16, with all notation as in Definition 2.14, Lemma 2.15.*

(i) If $Y \subset \mathbb{P}^n$ has codimension 3, then there is a presentation of the Cox ring $\mathcal{R}(Z)$ of Z given by

$$\mathcal{R}(Z) \cong \mathbb{C}[x_0, \dots, x_c, y_0, \dots, y_n, z_0, \dots, z_c, u]/I, \tag{6}$$

where I is the homogeneous ideal generated by the $r_m(x_i, y_j), r_m(z_i, y_j)$ and the equations (5). In particular, Z is a Mori Dream Space. It admits a unique flop to a second minimal model, itself isomorphic to Z, interchanging the two points of a general fibre over $\mathbb{P}^n$.

(ii) Conversely, suppose that there exists a presentation of the form (6). Then $Y \subset \mathbb{P}^n$ has codimension 3.

Remark 2.18. Main Theorem 2.17 may not hold if we allow $e_m = 0$ for some $m = 0, \dots, c$. For example, if $c = 2$, $X = \mathbb{P}^2 \times \mathbb{P}^n$ and $[V_1] = (3, 0)$, then Z is a hypersurface of $C \times \mathbb{P}^n$, where C is an elliptic curve, and the Picard group of Z is uncountable. If, however, $c > 3$, then we may allow some e_m to be equal zero, as long as the Lefschetz hyperplane theorem guarantees isomorphisms of Picard groups between the intermediary inclusions Z_m. If $(d_c, e_c) = (0, 1)$, then we may assume that $r_c = x_c$ and Z can be viewed as a codimension $c-1$ complete intersection in $\mathbb{P}^{c-1} \times \mathbb{P}^n$. We will use this observation in a trick to confirm the equations in x_i, z_k, u given in (5).

2.4 Comments

In more general settings than those assumed in Subsections 2.1, 2.3, we may still utilise our method to find information on sections (and thus, for example, the birational geometry of Z).

Remark 2.19. The non-singularity conditions in Definition 2.2(ii) and Definition 2.11 may be weakened without changing the statements in Main Theorems 2.3, 2.17. In particular, it suffices that, for each $m = 1, \ldots, c$, the intermediary intersection Z_m is $\mathbb{Q}$-factorial, and that the inclusions $Z_{m-1} \hookrightarrow Z_m$ induce isomorphisms of Picard groups, where $Z_0 = X$ by convention.

Remark 2.20. Suppose $X = \mathbb{P}^c \times T$, where $c = 1$ in the setting of Section 2.1 and $c \geqslant 2$ in the setting of Section 2.3, where T is an arbitrary Mori Dream Space. We can consider irreducible and normal subvarieties $Z \subset X$ of the forms analogous to $(*)$ and $(\dagger)$ accordingly. Whilst we may not have the isomorphisms of $\operatorname{Pic} X$-graded algebras as given in the respective Main Theorems, we do obtain the equivalent sections in the same degrees, satisfying the same equations in the Cox ring. In each case, we find a graded subalgebra of the Cox ring. We can apply this for example with $X = (\mathbb{P}^c)^k \times \mathbb{P}^n$ for arbitrary $k \geqslant 1$ and $n > c$, where we get a collection of sections as in the Main Theorem statements for each copy of $\mathbb{P}^c$ in the product.

3 A method to find Cox ring sections on subvarieties

We develop the method used to established those results given in Chapter 2. A reference for the standard theory on Cox rings is given by [2].

3.1 Cox rings

3.1.1 Basic definitions

Let X be a smooth projective variety over $\mathbb{C}$ with free abelian Picard group $\operatorname{Pic} X \cong \mathbb{Z}^\rho$. Fix a subgroup H of the free abelian group of Weil divisors on X mapping isomorphically to $\operatorname{Pic} X$. For each $V \in H$, let $D = [V]$ denote its class in $\operatorname{Pic} X$ and define the vector space

$$\mathcal{R}(V)_D = \mathcal{R}(X)_{[V]} = H^0(X, \mathcal{O}_X(V)).$$

Definition 3.1. The Cox ring of X is the $\operatorname{Pic} X$-graded $\mathbb{C}$-algebra

$$\mathcal{R}(X) = \bigoplus_{V \in H} \mathcal{R}(X)_{[V]},$$

where multiplication is given by the canonical maps

$$H^0(X, \mathcal{O}_X(V)) \bigotimes H^0(X, \mathcal{O}_X(V')) \to H^0(X, \mathcal{O}_X(V + V')).$$

Different choices of H yield isomorphic Cox rings. It is thus standard to fix a H and write, slightly loosely,

$$\mathcal{R}(X) = \bigoplus_{D \in \operatorname{Pic} X} \mathcal{R}(X)_D = \bigoplus_{D \in \operatorname{Pic} X} H^0(X, \mathcal{O}_X(D)).$$

We follow this convention throughout. For X smooth and projective, the Cox ring is integral and normal.

Suppose that $Z \to X$ is a morphism of smooth projective varieties. There is an induced morphism of Picard groups $\operatorname{Pic} X \to \operatorname{Pic} Z$ and an induced morphism $\mathcal{R}(X) \to \mathcal{R}(Z)$ of Cox rings. In the case that $\iota : Z \hookrightarrow X$ is an inclusion, the induced map ι^* on Cox rings is simply restriction of sections. These maps are first studied in [1, 15], finding conditions under which ι^* is surjective for a hypersurface $Z \subset X$ defined by a homogeneous equation $r \in \mathcal{R}(X)$.

In this thesis, we consider complete intersections $Z \subset X$, where X is a product of projective spaces. We compute the Cox rings of various such Z for which the inclusion does not induce a surjection of Cox rings. We follow from and generalise greatly the results of Ottem in [28].

3.1.2 Finite generation and Mori Dream Spaces

A key application of Cox rings lies in birational geometry.

Definition 3.2. We say that a $\mathbb{Q}$-factorial projective variety X over $\mathbb{C}$ is a Mori Dream Space if $\mathcal{R}(X)$ is a finitely generated $\mathbb{C}$-algebra.

The birational geometry of a Mori Dream Space can be understood combinatorically via its Cox ring. This correspondence is given by the following theorem, which is a main result in [21].

Theorem 3.3. *Suppose that a smooth projective variety X is a Mori Dream Space. Then there are finitely many minimal models of X. The moveable cone of X decomposes into a finite set of Mori chambers, one of which is the ample cone of X. Each Mori chamber gives a $\mathbb{Q}$-factorial variety X^+, birational to X, whose ample cone is the closure of its chamber. The birational map $X \dashrightarrow X^+$ corresponds to that given by a variation of GIT on the action of the torus $\operatorname{Spec}\operatorname{Pic} X$ on $\bar{X} = \operatorname{Spec}\mathcal{R}(X)$.*

3.2 Cohomology and Cox rings of a hypersurface

We can now introduce the new method of this thesis, starting with hypersurfaces $Z \subset X$.

3.2.1 Basics

Let X be a smooth projective variety with free abelian Picard group $\operatorname{Pic} X$. Let $\iota : Z \hookrightarrow X$ be a smooth hypersurface with class $E = [Z] \in \operatorname{Pic} X$ inducing an isomorphism of Picard groups $\operatorname{Pic} X \cong \operatorname{Pic} Z$. We denote by Pic the identification of $\operatorname{Pic} X$ and $\operatorname{Pic} Z$ via this isomorphism.

The Cox rings $\mathcal{R}(X)$ and $\mathcal{R}(Z)$ are both Pic-graded algebras

$$\mathcal{R}(X) = \bigoplus_{D \in \operatorname{Pic}} \mathcal{R}(X)_D, \quad \mathcal{R}(Z) = \bigoplus_{D \in \operatorname{Pic}} \mathcal{R}(Z)_D,$$

connected by the graded homomorphism $\iota^* : \mathcal{R}(X) \to \mathcal{R}(Z)$. The interesting case for us is when the inclusion $\iota^*\mathcal{R}(X) \hookrightarrow \mathcal{R}(Z)$ is not surjective.

The hypersurface Z is given by a section $r \in \mathcal{R}(X)_E$. For $D \in \operatorname{Pic}$, consider the standard exact sequence

$$0 \longrightarrow \mathcal{O}_X(D-E) \xrightarrow{\cdot r} \mathcal{O}_X(D) \xrightarrow{\iota^*} \mathcal{O}_Z(D) \longrightarrow 0. \tag{7}$$

The associated long exact sequence yields the short exact sequence of vector spaces

$$0 \longrightarrow \iota^*\mathcal{R}(X)_D \longrightarrow \mathcal{R}(Z)_D \xrightarrow{\delta} N_D \longrightarrow 0, \tag{8}$$

where $\iota^*\mathcal{R}(X)_D \cong \mathcal{R}(X)_D / r \cdot \mathcal{R}(X)_{D-E}$, and $N_D = \ker r_{D*}$ is the kernel of the induced map on first sheaf cohomology

$$r_{D*} : H^1(X, \mathcal{O}_X(D-E)) \longrightarrow H^1(X, \mathcal{O}_X(D)). \tag{9}$$

It will sometimes be natural to put all these maps together, to get the map

$$r_* : \bigoplus_{D \in \operatorname{Pic}} H^1(X, \mathcal{O}_X(D-E)) \longrightarrow \bigoplus_{D \in \operatorname{Pic}} H^1(X, \mathcal{O}_X(D)).$$

We have $\iota^*\mathcal{R}(X) = \mathcal{R}(Z)$ if and only if the map r_* is injective.

3.2.2 Čech cohomology considerations

Fix a totally ordered affine cover $\mathcal{U} = \{U_i\}_{i \in I}$ of X. From (7), we get the short exact sequence of complexes:

$$\begin{CD}
@. 0 @. 0 \\
@. @VVV @VVV \\
0 @>>> \prod_i \mathcal{O}_X(D-E)(U_i) @>>> \prod_{i<j} \mathcal{O}_X(D-E)(U_i \cap U_j) @>>> \dots \\
@. @VV{\cdot r}V @VV{\cdot r}V \\
0 @>>> \prod_i \mathcal{O}_X(D)(U_i) @>{d_0}>> \prod_{i<j} \mathcal{O}_X(D)(U_i \cap U_j) @>>> \dots \\
@. @VV{\iota^*}V @VV{\iota^*}V \\
0 @>>> \prod_i \mathcal{O}_Z(D)(U_i) @>>> \prod_{i<j} \mathcal{O}_Z(D)(U_i \cap U_j) @>>> \dots \\
@. @VVV @VVV \\
@. 0 @. 0
\end{CD} \tag{10}$$

Lemma 3.4. *Consider Čech 0-cochains $(v_i) \in \prod_i \mathcal{O}_X(D)(U_i)$ on X. Call a 0-cochain (v_i) compatible (with respect to Z) if for each $i < j$ there exists $q_{i,j} \in \mathcal{O}_X(D-E)(U_i \cap U_j)$ such that $v_i|_{U_i \cap U_j} - v_j|_{U_i \cap U_j} = q_{i,j} r$. Call compatible 0-cochains $(v_i), (v_i')$ equivalent, if for each i, there exists $q_i \in \mathcal{O}_X(D-E)(U_i)$ such that $v_i' - v_i = q_i r$.*

(i) Elements $v \in \mathcal{R}(Z)_D$ are naturally in bijection with equivalence classes of compatible 0-cochains

$$(v_i) \in \prod_i \mathcal{O}_X(D)(U_i).$$

Under this equivalence, the restriction of a section $g \in \mathcal{R}(X)_D$ corresponds to the Čech 0-cocycle $(g|_{U_i}) \in \prod_i \mathcal{O}_X(D)(U_i)$.

(ii) Multiplication in the algebra $\mathcal{R}(Z)$ is compatible with multiplication of cochains: if $D, D' \in \operatorname{Pic}$ and $v \in \mathcal{R}(Z)_D$, $v' \in \mathcal{R}(Z)_{D'}$ are represented by compatible 0-cochains $(v_i), (v_i')$ respectively, then $vv' \in \mathcal{R}(Z)_{D+D'}$ is represented by the compatible 0-cochain $(v_i v_i')$.

Proof. These statements follow from considering the first column of (10), recalling that the global section functor on sheaves is exact over affine schemes. Note that a 0-cochain (v_i) restricts to a 0-cocycle on Z if and only if it is compatible. □

We are interested in cases where there are equivalence classes of compatible 0-cochains (v_i) that do not come from the restriction of a section $g \in \mathcal{R}(X)$, so that (v_i) is not a 0-cocycle on X. For $D \in \operatorname{Pic}$, in order to find a complementary vector space to $\iota^*\mathcal{R}(X)_D$ in $\mathcal{R}(Z)_D$, we need to complete the first three steps below.

1. Give an explicit description of the first sheaf cohomologies of $\mathcal{O}_X(D-E)$ and $\mathcal{O}_X(D)$ and of the map r_{D*} of (9).

2. Describe the kernel $N_D = \ker r_{D*}$ as a subspace of $H^1(X, \mathcal{O}_X(D-E))$.

3. Choose a right inverse $\epsilon\colon N_D \to \mathcal{R}(Z)_D$ of δ that lifts elements of N_D to sections on Z.

4. Study the ring structure of the direct sum of the resulting spaces to reconstruct the whole structure of $\mathcal{R}(Z)$.

Steps (1) and (2) have to be completed in particular cases of interest by explicit calculation. Finding a map ϵ in step (3) comes down to untangling the connecting homomorphism δ. Given $s \in N_D$ for some $D \in \mathrm{Pic}$, let

$$(s_{i,j}) \in \prod_{i<j} \mathcal{O}_X(D-E)(U_i \cap U_j)$$

be a representative 1-cocycle. Then $(r \cdot s_{i,j})$ is cohomologically trivial, thus the image of a 0-cochain

$$(v_i) \in \prod_i \mathcal{O}_X(D)(U_i),$$

which is compatible since $v_i|_{U_i \cap U_j} - v_j|_{U_i \cap U_j} = s_{i,j}r$ for each $i < j$. This compatible 0-cochain corresponds via Lemma 3.4(i) to an element $v \in \mathcal{R}(Z)_D$ that satisfies $\delta v = s$. Note that the element v is only well-defined up to an element in $\iota^*\mathcal{R}(X)_D$, since (v_i) is well-defined only up to an element of $\ker d^0 = \mathcal{R}(X)_D$. For each s in a chosen basis of N_D, we may find a v with δv in this way and set $\epsilon s = v$. Extending linearly we obtain a well-defined lifting map

$$\epsilon\colon N_D \to \mathcal{R}(Z)_D,$$

a right inverse to δ. To complete step (4), further arguments are needed that will be based on Lemma 3.4(ii). We note that, in practice, we may only need to perform Step (2) and Step (3) for a subset of all line bundles to find generators for the full Cox ring.

The following Lemma, concerning commutativity of the map δ with maps on first cohomology induced by multiplication by a section, is used multiple times in this thesis.

Lemma 3.5. *Let g be a degree $E \in \mathrm{Pic}$ element of $\iota^*\mathcal{R}(X) \subset \mathcal{R}(Z)$, with induced maps g_* on first cohomology. Then $g_*\delta = \delta g_*$ as maps*

$$\mathcal{R}(Z) \to \bigoplus_{D \in \mathrm{Pic}} H^1(X, \mathcal{O}_X(D-E))$$

Proof. Applying δ to a cocycle on Z amounts to lifting this cocycle to X, applying the Čech differential d^0, before dividing each element by r. Each of these steps are compatible with g_*, which is component-wise multiplication by g. □

To conclude the general discussion, we make one further remark. Suppose that $v \in \mathcal{R}(Z)_D$ is represented by a compatible 0-cochain (v_i) via Lemma 3.4(i). For fixed $i \in I$, we have $v_i \in \mathcal{O}_X(U_i)(D)$, and thus $v_i = \frac{h_i}{g_i}$ for some regular functions h_i, g_i on X with g_i not vanishing on U_i. Identifying h_i and g_i with their restrictions on Z, the section $g_i v_i - h_i$ is zero on $Z \cap U_i$, which is open and dense in Z. For each i, we thus obtain an equation

$$g_i v - h_i = 0$$

in $\mathcal{R}(Z)$. This explains the relevance of localisations of $\mathcal{R}(X)$ for the problem, which is the basis of the approach taken in [19].

Having established a process for hypersurfaces, we turn to subvarieties in X of higher codimension.

3.3 The method on a complete intersection - case 1

Suppose more generally that $\iota : Z \to X$ is a smooth codimension $c \geqslant 1$ intersection of irreducible and normal hypersurfaces $V_1, \ldots, V_c \in X$, defined respectively by equations $r_m \in \mathcal{R}(X)_{E_m}$, where $E_m = [V_m]$. For each $m = 1, \ldots, c$, denote by Z_m the intermediary intersection

$$Z_m = \bigcap_{m'=1}^{m} Z_{m'}.$$

By convention set $Z_0 = X$. Suppose that for each $m = 1, \ldots, c$, the hypersurface inclusion $Z_{m-1} \hookrightarrow Z_m$ defined by r_m satisfies the conditions in Section 3.2.1, so that we may iteratively apply the procedure in Section 3.2.2. Let Pic denote the identified Picard groups of each Z_m. We will also assume that $H^i(X, \mathcal{O}_X(D)) = 0$ for each $D \in \mathrm{Pic}$ and $2 \leqslant i \leqslant c$, noting that this condition is empty in the case $c = 1$.

The induced homomorphism $\mathcal{R}(X) \to \mathcal{R}(Z)$ factors via

$$\mathcal{R}(X) = \mathcal{R}(Z_0) \to \mathcal{R}(Z_1) \to \cdots \to \mathcal{R}(Z_c) = \mathcal{R}(Z).$$

For each $m = 1, \ldots, c$, the inclusion $Z_{m-1} \hookrightarrow Z_m$ gives us short exact sequences,

$$0 \longrightarrow \iota^*\mathcal{R}(Z_{m-1})_D \longrightarrow \mathcal{R}(Z_m)_D \xrightarrow{\delta_m} N_{m,D} \longrightarrow 0, \tag{11}$$

for each $D \in \mathrm{Pic}$ as in (8), where which inclusion meant by ι is clear in context. We can work over the direct sums, and thus denote by N_m the direct sum $\bigoplus_{D \in \mathrm{Pic}} N_{m,D}$. Denote by ϵ_m the isomorphism of vector spaces

$$\epsilon_m : N_m \to \mathcal{R}(Z_m)/\iota^*\mathcal{R}(Z_{m-1})$$

induced by any right inverse of δ_m.

Lemma 3.6. *For each $1 \leqslant m \leqslant c$ we have $H^i(Z_{m-1}, \mathcal{O}_{Z_{m-1}}(D)) = 0$ for all $D \in \mathrm{Pic}$ and $2 \leqslant i \leqslant c-(m-1)$.*

Proof. The result is true for $m = 1$ by assumption. For $m > 1$ we use induction, utilising the long exact sequence of cohomology associated to the short exact sequence (7) for the inclusion $Z_{m-1} \hookrightarrow Z_m$. □

Corollary 3.7. *For each $1 \leqslant m \leqslant c-1$ and $D \in \mathrm{Pic}$, we have a natural isomorphism*

$$H^1(Z_m, \mathcal{O}_{Z_m}(D)) \cong H^1(Z_{m-1}, \mathcal{O}_{Z_{m-1}}(D))/r_{m,D*} \cdot H^1(Z_{m-1}, \mathcal{O}_{Z_{m-1}}(D - E_m))$$

Proof. This follows once more via induction in m along the long exact sequences of cohomology, using that $H^2(Z_{m-1}, \mathcal{O}_{Z_{m-1}}(D)) = 0$ for each $D \in \mathrm{Pic}$ and $1 \leqslant m \leqslant c-2$ from Lemma 3.6. □

Definition 3.8. For each $m = 1, \ldots, c$, define

$$T_m = \left(\bigoplus_{D \in \mathrm{Pic}} H^1\left(X, \mathcal{O}_X(D - E_1)\right) \right) \oplus \cdots \oplus \left(\bigoplus_{D \in \mathrm{Pic}} H^1\left(X, \mathcal{O}_X(D - E_m)\right) \right)$$

and

$$T' = \bigoplus_{D \in \mathrm{Pic}} H^1\left(X, \mathcal{O}_X(D)\right).$$

Note that $T_m = \bigoplus_{m'=1}^{m} T'(-E_{m'})$. Denote by B_m the map

$$\begin{array}{rcl} B_m : T_m & \longrightarrow & T' \\ (f_1, \dots, f_m) & \mapsto & r_{1,*}f_1 + \dots + r_{m,*}f_m, \end{array}$$

and let η_m be the subspace of of T_m generated by all elements of the form

$$(0, \dots, 0, r_{m',*}q, 0, \dots, 0, -r_{m'',*}q, 0, \dots, 0), \quad 1 \leqslant m'' < m' \leqslant m,$$

noting that $\eta_m \subset \ker B_m$.

For each $m = 1, \dots, c$, there are natural maps

$$\begin{array}{rcl} T_{m-1} & \longrightarrow & T_m \\ (f_1, \dots, f_{m-1}) & \mapsto & (f_1, \dots, f_{m-1}, 0), \end{array}$$

where $T_0 = 0$ by convention. These induce maps

$$\gamma_m : (\ker B_{m-1})/\eta_{m-1} \to (\ker B_m)/\eta_m.$$

Iterating the result of Corollary 3.7, we see that

$$N_m \cong \ker\left(r_{m,*} : T'(-E_m) \to T'\right)/(\operatorname{im} r_{1,*} + \dots + \operatorname{im} r_{m-1,*}).$$

The natural map $T_m \to T'(-E_m)$ given by projection the the final component thus induces a well defined map

$$\beta_m : (\ker B_m)/\eta_m \to N_m.$$

The following Proposition is key.

Proposition 3.9. *For each $m = 0, 1, \dots, c$, there is an isomorphism*

$$\alpha_m : (\ker B_m)/\eta_m \to \mathcal{R}(Z_m)/\iota^*\mathcal{R}(X),$$

such that the following diagram commutes for each $m = 1, \dots c$.

$$\begin{tikzcd} (\ker B_{m-1})/\eta_{m-1} \arrow[r, "r_{m,*}^{\oplus m-1}"] \arrow[d, "\alpha_{m-1}"] & (\ker B_{m-1})/\eta_{m-1} \arrow[r, "\gamma_m"] \arrow[d, "\alpha_{m-1}"] & (\ker B_m)/\eta_m \arrow[r, "\beta_m"] \arrow[d, "\alpha_m"] & \bigoplus_{b \in \mathbb{Z}} N_m \arrow[r] \arrow[d, "\epsilon_m"] & 0 \\ \mathcal{R}(Z_{m-1})/\iota^*\mathcal{R}(X) \arrow[r, "\cdot r_m"] & \mathcal{R}(Z_{m-1})/\iota^*\mathcal{R}(X) \arrow[r, "\iota^*"] & \mathcal{R}(Z_m)/\iota^*\mathcal{R}(X) \arrow[r, "\pi_m"] & \mathcal{R}(Z_m)/\iota^*\mathcal{R}(Z_{m-1}) \arrow[r] & 0 \end{tikzcd} . \tag{12}$$

Proof. We first define the maps α_m. The map α_0 is the zero isomorphism between zero vector spaces. Now let $1 \leqslant m \leqslant c$ and $f = (f_1, \dots, f_m) \in \ker B_m$. Choose for each $1 \leqslant m' \leqslant m$ a representative 1-cocycle

$$(s_{m',ij}) \in \prod_{i<j} \mathcal{O}_X(D - E_{m'})(U_i \cap U_j)$$

of $f_{m'} \in H^1(X, \mathcal{O}_X(D - E_{m'}))$. That

$$r_{1,*}f_1 + \cdots + r_{m,*}f_m = 0 \in H^1(X, \mathcal{O}_X(D))$$

is the statement that the 1-cocycle

$$(r_1 s_{1,ij} + \cdots + r_m s_{m,ij}) \in \prod_{i<j} \mathcal{O}_X(D)(U_i \cap U_j)$$

is cohomologically trivial, thus the image of a 0-cocycle $(v_i) \in \prod_i \mathcal{O}_X(D)(U_i)$ in the associated Čech complex. By construction, (v_i) restricts to a Z_m-compatible 0-cochain on Z_{m-1}, defining, via Lemma 3.4(i), a section $v \in \mathcal{R}(Z_m)$ with $\delta_m v = f_m \in N_m$. Since the choice of (v_i) is unique up to an element of $\mathcal{R}(X)$, we have a well-defined map $\overline{\alpha}_m : \ker B_m \to \mathcal{R}(Z_m)/\iota^*\mathcal{R}(X)$. If $f \in \eta_m$, then the $(s_{m',ij})$ may be chosen so that $(r_1 s_{1,ij} + \cdots + r_m s_{m,ij}) = 0$ as a cochain, thus $\eta_m \subset \ker \overline{\alpha}_m$ and we obtain the induced map α_m.

By construction, the diagrams (12) commute. The bottom row in each diagram is exact via the long exact sequence of cohomology associated with the inclusion $Z_{m-1} \hookrightarrow Z_m$. If we can show that the top rows are exact, we can conclude that each α_m is an isomorphism using the five lemma with induction.

To show that β_m is surjective, let $f_m \in N_m$, which is an element of the kernel of

$$r_{m,*} : T'(-E_m)/(\operatorname{im} r_{1,*} + \operatorname{im} r_{m-1,*}) \to T'/(\operatorname{im} r_{1,*} + \operatorname{im} r_{m-1,*}).$$

This implies the existence of $(f_1, \ldots, f_{m-1}) \in T_{m-1}$ such that $B_m(f_1, \ldots, f_m) = 0$. Then $(f_1, \ldots, f_m) \in (\ker B_m)/\eta_m$ maps to f_m under β_m.

It is immediate that $\operatorname{im} \gamma_m \subset \ker \beta_m$. Suppose for the converse that $f = (f_1, \ldots, f_m) \in \ker \beta_m$, then $\beta f = f_m \in (\operatorname{im} r_{1,*} + \cdots + \operatorname{im} r_{m-1,*})$, and thus we can find $f' \in \eta_m$ such that $f - f'$ has final component equal 0. Then $f - f'$, which is equal f in $(\ker B_m)/\eta_m$, is in the image of γ_m.

Lastly, we check $\ker \gamma_m = \operatorname{im}(r_{m,*} \oplus \cdots \oplus r_{m,*})$. Suppose $f = (f_1, \ldots, f_{m-1}) \in \ker \gamma$, so that $(f_1, \ldots, f_{m-1}, 0) \in \eta_m$. Replacing f with a suitable representative modulo η_{m-1}, we conclude that

$$(f_1, \ldots, f_{m-1}) \in \operatorname{im}(r_{m,*} \oplus \cdots \oplus r_{m,*}).$$

Conversely, suppose $f = (f_1, \ldots, f_{m-1}) \in (\ker B_{m-1})/\eta_{m-1}$. Consider

$$\gamma_m \left(r_{m,*}^{\oplus m-1}\right) f = (r_{m,*}f_1, \ldots, r_{m,*}f_{m-1}, 0) = (0, \ldots, 0, -r_{1,*}f_1 - \cdots - r_{m-1,*}f_{m-1}) = 0,$$

where the middle equality is modulo η_m and the last follows from $f \in \ker B_{m-1}$. We are done. $\square$

We are now able to give the 4-step procedure to compute Cox rings in our higher codimension setting, which we will apply to prove Theorem 2.3.

1. Give an explicit description of the first sheaf cohomology of X and define the vectors spaces T_m, T', η_m and maps B_m as in Definition 3.8.

2. Describe the space $K = (\ker B_c)/\eta_c$.

3. Choose a lift $\alpha : K \to \mathcal{R}(Z_m)$ of $\alpha_c : K \cong \mathcal{R}(Z)/\iota^*\mathcal{R}(X)$, the latter obtained as in Proposition 3.9.

4. Study the ring structure of $\mathcal{R}(Z) \cong \iota^*\mathcal{R}(X) \bigoplus K$ to reconstruct the whole structure of $\mathcal{R}(Z)$.

Remark 3.10. Our 4-step procedure generalises the one given at the end of Section 3.2.2; if $c = 1$, then we notice that $\ker B_1 = N_1$ and the maps β_1 and π_1 are the identity maps. Then α is just a left inverse of δ_1.

3.4 The method on a complete intersection - case 2

Suppose once more that $\iota : Z \to X$ is a smooth codimension $c \geqslant 1$ intersection of irreducible and normal hypersurfaces $V_1, \ldots, V_c \in X$, defined respectively by equations $r_m \in \mathcal{R}(X)_{E_m}$, where $E_m = [V_m]$. Once more denote by Z_m the intermediary intersections

$$Z_m = \bigcap_{m'=1}^{m} Z_{m'}$$

and set $Z_0 = X$. Suppose that for each $m = 1, \ldots, c$, the hypersurface inclusion $Z_{m-1} \hookrightarrow Z_m$ defined by r_m satisfies the conditions in Section 3.2.1, so that we may iteratively apply the procedure in Section 3.2.2. Let Pic denote the identified Picard groups of each Z_m. This time we assume that $H^i(X, \mathcal{O}_X(D)) = 0$ for each $D \in \mathrm{Pic}$ and $1 \leqslant i \leqslant c-1$, noting that this condition is empty in the case $c = 1$. The vanishing cohomology condition is one degree lower than in Section 3.3.

Once more the homomorphism $\mathcal{R}(X) \to \mathcal{R}(Z)$ factors through each $\mathcal{R}(Z_m)$ and we have short exact sequences as in (11) and we use the same notation for N_m and δ_m. The following are the analogues of Lemma 3.6 and Corollary 3.7.

Lemma 3.11. *For each $1 \leqslant m \leqslant c$ we have $H^i(Z_{m-1}, \mathcal{O}_{Z_{m-1}}(D)) = 0$ for all $D \in \mathrm{Pic}$ and $1 \leqslant i \leqslant c-m$.*

Proof. Exactly as in the proof of Lemma 3.6, we use induction in m along the long exact sequences associated to the inclusions $Z_{m-1} \hookrightarrow Z_m$. □

Definition 3.12. For each $m = 1, \ldots c$, let $F_m \in \mathrm{Pic}$ denote the sum $\sum_{m'=1}^{m} E_{m'}$ and define

$$T_m = \bigoplus_{D \in \mathrm{Pic}} H^c(X, \mathcal{O}_X(D - F_m)).$$

Define also

$$T'_m = \left(\bigoplus_{D \in \mathrm{Pic}} H^1(X, \mathcal{O}_X(D - F_m + E_1)) \right) \oplus \cdots \oplus \left(\bigoplus_{D \in \mathrm{Pic}} H^1(X, \mathcal{O}_X(D - F_m + E_m)) \right),$$

noting that $T'_m = \bigoplus_{m'=1}^{m} T_m(E_m)$. Denote by B_m the map

$$\begin{aligned} B_m : T_m &\longrightarrow T'_m \\ f &\mapsto (r_{1,*}f_1, \ldots, r_{m,*}f_m). \end{aligned}$$

Proposition 3.13. *For each $m = 1, \ldots, c-1$ the map*

$$\delta_1 \circ \cdots \circ \delta_m : H^{c-m}(Z_m, \mathcal{O}_X(D)) \to \ker B_m$$

is an isomorphism. We also have an isomorphism

$$\delta_1 \circ \cdots \circ \delta_{c-1} \circ \overline{\delta}_c : \mathcal{R}(Z)/\iota^*\mathcal{R}(X) \to \ker B_c,$$

which we denote by δ.

Proof. We use induction in m, beginning from $m = 0$, where we have B_0 equal the zero map by convention. Suppose $1 \leqslant m \leqslant c - 1$ and that we have established the result for $m - 1$. By Lemma 3.11 we have isomorphisms

$$\delta_m : H^{c-m}(Z_m, \mathcal{O}_X(D)) \to \ker\left(r_{m*} : H^{c-m+1}(Z_{m-1}, \mathcal{O}_X(D - E_m)) \to H^{c-m+1}(Z_{m-1}, \mathcal{O}_X(D))\right).$$

Thus $\delta_1 \circ \cdots \circ \delta_m$ is an isomorphism onto its image, which is shown to be $\ker B_{m-1}$ by induction.

Now let $s = (s_{i_0 \ldots i_{c-m}})$ be an element of $H^{c-m}(Z_m, \mathcal{O}_X(D))$. Then $\delta_m s = s' = (s'_{i_0 \ldots i_{c-m+1}})$ is an element of $\ker r_{m*}$, so that $r_{m*}s' = (r_m s'_{i_0 \ldots i_{c-m+1}})$ is cohomologically trivial. By induction, this is if and only if $\delta_1 \circ \cdots \delta_{m-1} r_{m*} s'$ is equal zero. By Lemma 3.5, $\delta_{m'}$ commutes with r_{m*}, so we have

$$r_{m*}\delta_1 \circ \cdots \delta_m s = r_{m*}\delta_1 \circ \cdots \circ \delta_{m-1} s' = \delta_1 \circ \cdots \circ \delta_{m-1} r_{m*} s' = 0,$$

so $\delta_1 \circ \cdots \circ \delta_m s \in \ker B_m$.

If $m = c$, then δ_1 is not an isomorphism, for it vanishes on $\iota^*\mathcal{R}(X)$. Factoring this out, we can extend the above proof by induction to $m = c$ and we are done. □

We once more obtain a 4-step procedure to compute the Cox ring $\mathcal{R}(Z)$.

1. Give an explicit description of the sheaf cohomology groups $H^c(X, \mathcal{O}_X(D))$ on X and define the vector spaces T_m, T'_m and maps B_m as in Definition 3.12.
2. Describe the space $K = \ker B_c$.
3. Choose a lift $\alpha : K \to \mathcal{R}(Z)$ of the isomorphism δ^{-1}, as obtained in Proposition 3.13.
4. Study the ring structure of $\mathcal{R}(Z) \cong \iota^*\mathcal{R}(X) \bigoplus K$ to reconstruct the whole structure of $\mathcal{R}(Z)$.

3.5 Comments

Remark 3.14. In Section 3.2.1, we made the assumption that the inclusion $Z \hookrightarrow X$ induces an isomorphism of Picard groups. Weakening to an injection of Picard groups, we can still apply the cohomological method. In this case we compute only sections of the Veronese subring

$$\bigoplus_{D \in \iota^* \operatorname{Pic} X} \mathcal{R}(Z)_D \subset \mathcal{R}(Z).$$

One may still recover information from such a Veronese subring. We will use this in Chapter 8.

Remark 3.15. The cohomology vanishing conditions on X in Sections 3.3 and 3.4 are strict, but do include products of projective spaces and, for example, Grassmannian varities. These conditions are only necessary to find all new sections on $\mathcal{R}(Z)$ with our method. Dropping the vanishing cohomology conditions in either case, we may still define the spaces and kernels in Definitions 3.8 and 3.12. The lifts given by α are no longer isomorphisms, but injections. The method may then be useful in more general settings, such as complete intersections in more general trivial $\mathbb{P}^c$-bundles.

4 On Cox rings of nef complete intersections in $\mathbb{P}^1 \times \mathbb{P}^n$

In this Chapter we apply the programme developed in Section 3.3 to non-singular codimension $c \geqslant 1$ complete intersections Z in $X = \mathbb{P}^1 \times \mathbb{P}^n$, where $n \geqslant c+2$. We will prove Main Theorem 2.3.

For $c = 1$, which is the hypersurface case, this has first been published in [29], Szendrői and Ito as collaborators, following on from work of Ottem.

As stated in the Structure and Originality Section of the introduction, I have generalised the result on hypersurfaces to complete intersections, so to general $c \geqslant 1$. These results are to appear in a second paper of my own, which is still in preparation.

4.1 Finding sections

4.1.1 Basics

We recall the set-up as at the beginning of Section 2.1. Fix $c \geqslant 1, n \geqslant c+2$ and let $X = \mathbb{P}^1 \times \mathbb{P}^n$, with homogeneous coordinates x_0, x_1 and $y_0, \dots, y_n$ on the respective factors. Then $\operatorname{Pic} X \cong \mathbb{Z}^2$, with generators corresponding the the pullback of hyperplane sections on each factor in X. The Cox ring is

$$\mathcal{R}(X) = \mathbb{C}[x_0, x_1, y_0, \dots, y_n]$$

with its usual $\mathbb{Z}^2$-grading and it contains $R = \mathcal{R}(\mathbb{P}^n) = \mathbb{C}[y_0, \dots, y_n]$ as a graded subring, making $\mathcal{R}(X)$ a graded R-module.

We recall the Definitions of some subvarieties of X from Section 2.1.

Definition 4.1. Fix $d \geqslant 0$ and $e \geqslant 1$. Let V be an irreducible and normal hypersurface

$$V = \left\{ g_0 x_0^d + g_1 x_0^{d-1} x_1 + \ldots + g_d x_1^d = 0 \right\} \subset \mathbb{P}^1 \times \mathbb{P}^n \tag{13}$$

defined by a set of homogeneous degree e elements $g_0, \dots, g_d \in R$.

(i) An *index sequence of length* τ, where $0 \leqslant \tau \leqslant d$, is a sequence of indices $0 = i_0 < \cdots < i_\tau = d$. By convention, $\tau = 0$ if and only if $d = 0$. We say V admits the index sequence $\{i_0, \dots, i_\tau\}$ of length τ if $g_i \neq 0$ if and only if $i \in \{i_0, \dots, i_\tau\}$; we call the set of indices $\{i_0, \dots, i_\tau\}$ the *index sequence* of V.

(ii) We call V *regular with index sequence* $\{i_0, \dots, i_\tau\}$, if it is non-singular, admits the index sequence $\{i_0, \dots, i_\tau\}$, and the homogeneous polynomials $g_{i_0}, \dots, g_{i_\tau}$ form a regular sequence in the R.

Definition 4.2. For given $c \geqslant 1$ and $n \geqslant c+2$, the variety $Z \subset X = \mathbb{P}^1 \times \mathbb{P}^n$ will be said to have form $(*)$ with respect to hypersurfaces $V_1, \dots, V_c$ if $Z = \bigcap_{m=1}^c V_m$, and the following conditions are met.

(i) For each $m = 1, \dots, c$, the hypersurface $V_m \subset X$ is irreducible and normal, defined by an equation

$$r_m = g_{m,0} x_0^{d_m} + g_{m,1} x_0^{d_m - 1} x_1 + \ldots + g_{m,d} x_1^{d_m},$$

with $d_m \geqslant 0$, $e_m \geqslant 1$, and V_m admits an index sequence $\{i_{m,0}, \dots, i_{m,\tau_m}\}$ of length τ_m, with $0 \leqslant \tau_m \leqslant d_m$ as in Definition 4.1.

(ii) The sequence $r_1, \ldots, r_c$ is regular in $\mathcal{R}(X)$ and the intermediary complete intersections

$$Z_m = \bigcap_{m'=1}^{m} V_{m'}$$

are non-singular for $m = 1, \ldots, c$.

(iii) The sequence

$$g_{1,0}, \ldots, g_{1,d_1}, \ldots \ldots, g_{c,0}, \ldots, g_{c,d_c}$$

is regular in R. In particular, we have the numerical condition

$$c + \sum_{m=1}^{c} d_m \leqslant n + 1. \tag{14}$$

By the Künneth formula, we have $H^i(X, \mathcal{O}_X(D)) = 0$ for all $2 \leqslant i \leqslant c$ and $D \in \operatorname{Pic} X$. We assume that condition (ii) of Definition 4.2 holds, then the conditions presumed in the method of Section 3.3 are met – the isomorphisms of Picard groups implied by the Lefschetz hyperplane theorem. For an ordered open cover $\mathcal{W}$ of X, let $\mathcal{U} = \{U_0, U_1\}$, where $U_i = \{x_i \neq 0\}$ for $i = 0, 1$ be the standard open cover of $\mathbb{P}^1$ and let $\mathcal{V} = \{V_0, \ldots, V_n\}$, where $V_j = \{y_j \neq 0\}$ is the standard open cover of $\mathbb{P}^n$. Then let $\mathcal{W} = \{W_{ij}\}_{i,j}$ be the open cover of X given by $W_{ij} = U_i \times V_j$ for each i, j, with lexicographic ordering. If $ij < kl$, then denote $W_{ij} \cap W_{kl}$ by $W_{ij,kl}$.

4.1.2 Describing the maps on first cohomology

We begin with Step (1) of our procedure laid out at the end of Section 3.3. Since we assume throughout that $Z \subset X$ is a subvariety satisfying condition (ii) of Definition 4.2, the inclusion induces an isomorphism of Picard groups $\operatorname{Pic} X \cong \operatorname{Pic} Z$ and we denote their common identification by $\operatorname{Pic} \cong \mathbb{Z}^2$.

Lemma 4.3. *(i) There are natural $\mathbb{Z}^2$-graded vector space isomorphisms*

$$\bigoplus_{(a,b)\in\mathbb{Z}^2} H^1(X, \mathcal{O}_X(a,b)) \cong \frac{1}{x_0 x_1} R[x_0^{-1}, x_1^{-1}] \cong S[x_0^{-1}, x_1^{-1}]/L, \tag{15}$$

where L is the vector subspace of $S[x_0^{-1}, x_1^{-1}]$ generated by all terms in which x_0 or x_1 appear with non-negative exponent.

(ii) For each $m = 1, \ldots, c$, the map

$$r_{m,*} : \bigoplus_{(a,b)\in\operatorname{Pic}} H^1(X, \mathcal{O}_X(a - d_m, b - e_m)) \longrightarrow \bigoplus_{D\in\operatorname{Pic}} H^1(X, \mathcal{O}_X(a,b))$$

can be identified with the map

$$\begin{array}{rccl} r_{m,*} : S[x_0^{-1}, x_1^{-1}]/L & \longrightarrow & S[x_0^{-1}, x_1^{-1}]/L \\ f & \mapsto & r_m \cdot f \mod L \end{array}$$

that multiplies an element $f \in S[x_0^{-1}, x_1^{-1}]/L \cong \frac{1}{x_0 x_1} R[x_0^{-1}, x_1^{-1}]$ by r_m, and ignores those terms in which x_0 or x_1 occur with non-negative exponent.

Proof. This first isomorphism in (15) is clear from the Künneth formula. This identifies an element $f \in \frac{1}{x_0x_1}R[x_0^{-1}, x_1^{-1}]_{(a,b)}$ with class of the 1-cocycle $(s_{ij,kl})$ representing an element of $H^1(X, \mathcal{O}_X(a, b))$, where $s_{ij,kl} = f$ if $i \neq k$ and $s_{ij,kl} = 0$ if $i = k$. The second isomorphism in (15) is immediate. The description in (ii) is also clear from the explicit cocycle description. $\square$

Note that the isomorphisms (15) are also isomorphisms of graded R-modules, which we will use.

Definition 4.4. For each $m = 1, \ldots, c$, we define the spaces

$$T_{m,(a,b)} \cong \bigoplus_{m'=1}^{m} H^1(X, \mathcal{O}_X(a - d_{m'}, b - e_{m'})), \quad T_m \cong \bigoplus_{(a,b)\in \mathrm{Pic}}^{m} T_{m,(a,b)}$$

and the space

$$T' = \bigoplus_{(a,b)\in \mathrm{Pic}} H^1(X, \mathcal{O}_X(a, b)).$$

Let $\eta_m \subset T_m$ be the subspace generated by elements of the form

$$(0, \ldots, 0, r_{m',*}q, 0, \ldots, 0, -r_{m'',*}q, 0, \ldots, 0), \quad 1 \leqslant m'' < m' \leqslant m,$$

and define $B_m : T_m \to T'$ to be the map

$$\begin{aligned} B_m : T_m &\longrightarrow T' \\ (f_1, \ldots, f_m) &\mapsto r_{1,*}f_1 + \cdots + r_{m,*}f_m, \end{aligned}$$

with $B_{m,(a,b)}$ denoting this map restricted to $T_{m,(a,b)}$ for each $(a, b) \in \mathrm{Pic}$. We denote by $K_{(a,b)}$ the kernel of $B_{c,(-a,b)}$, set $K_a = \bigoplus_{b\in\mathbb{Z}} K_{(a,b)}$ and $K = \ker B_c = \bigoplus_{a\in\mathbb{Z}} K_a$, noting the sign switching convention on a in the grading.

With this we have completed Step (1) of the programme.

4.1.3 Describing the kernel of B_c

We move onto Step (2). Since $H^1(X, \mathcal{O}_X(a, b))$ is only non-zero for $a < 0$, we will use the index $-a$ rather than a. For fixed $a > 0$, there are R-module isomorphisms

$$\begin{aligned} R^{a-1} &\xrightarrow{\sim} \bigoplus_{b\in\mathbb{Z}} \frac{1}{x_0x_1} R[x_0^{-1}, x_1^{-1}]_{(-a,b)} \\ (f_1, \ldots, f_{a-1}) &\mapsto f = \sum_{k=1}^{a-1} \frac{f_k}{x_0^{a-k}x_1^k}. \end{aligned} \tag{16}$$

We will use the following Lemma often.

Lemma 4.5. *Let $d \geqslant 0$, $e \geqslant 1$ and $r = g_0x_0^d + \cdots + g_dx_1^d \in \mathcal{R}(X)_{(d,e)}$. The map*

$$r_*^{(a)} : \bigoplus_{b\in\mathbb{Z}} H^1(X, \mathcal{O}_X(-a - d, b - e)) \to \bigoplus_{b\in\mathbb{Z}} H^1(X, \mathcal{O}_X(-a, b))$$

is given by the $(a-1) \times (a+d-1)$ *matrix*

$$A_{a,r} = \begin{pmatrix} g_0 & g_1 & g_2 & \cdots & g_d & 0 & 0 & \cdots\cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & g_0 & g_1 & \cdots & g_{d-1} & g_d & 0 & \cdots\cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ & \vdots & & \ddots & & \vdots & & \ddots & \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots\cdots & 0 & g_0 & g_1 & \cdots & g_d & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots\cdots & 0 & 0 & g_0 & \cdots & g_{d-1} & g_d \end{pmatrix}$$

under the identifications of Lemma 4.3 and (16).

Proof. This follows from the description of $r_*^{(a)}$ given by Lemma 4.3(ii). □

Proposition 4.6. *For $(-a, b) \in \operatorname{Pic}$, the space K_a can only be non-zero if $-a \leqslant \max_m d_m - 2$.*

(i) For $-1 \leqslant -a \leqslant \max_m d_m - 2$, we have a graded isomorphism

$$K_a \cong R^{a+d_1-1}[e_1] \times \cdots \times R^{a+d_c-1}[e_c],$$

where by convention $R^{a'} = 0$ if $a' \leqslant 0$ and $[e']$ denotes a shift in grading.

(ii) For $-a \leqslant -1$, we have a graded isomorphism

$$\bigoplus_{b \in \mathbb{Z}} K_{(-a,b)} \cong \ker A_a,$$

where A_a is the R-module homomorphism

$$A_a : R^{a+d_1-1}[e_1] \times \cdots \times R^{a+d_c-1}[e_c] \to R^{a-1}$$

given by the matrix of the same name $A_a = (A_{a,r_1}|\cdots|A_{a,r_c})$.

Proof. We note that $R^{a+d_1-1}[e_1] \times \cdots \times R^{a+d_c-1}[e_c]$ corresponds to $\bigoplus_{b\in\mathbb{Z}} T_{c,(-a,b)}$ in (15) and that by Lemma 4.5, the matrix A_a is the matrix given by the map $\bigoplus_{b\in\mathbb{Z}} B_{c,(-a,b)}$. □

4.1.4 Lifting elements of K to the Cox ring

In Step (3) we choose a map $\alpha : K \to \mathcal{R}(Z)$ inducing an isomorphism $\alpha_c : K \to \mathcal{R}(Z)/\iota^*\mathcal{R}(X)$ fitting into the setting of Proposition 3.9.

Suppose that $f = (f_1; \ldots; f_c) \in K$, homogeneous of degree $(-a, b) \in \operatorname{Pic}$. By (16), we may identify f_m with an element

$$(f_{m,1}, \ldots, f_{m,a+d_m-1}) \in R^{a+d_m-1}$$

for each $1 \leqslant m \leqslant c$, so that $(f_1; \ldots; f_c) \in \ker A_a$. Each $(f_{m,1}, \ldots, f_{m,a+d_m-1})$ is in turn identified with

$$u_m = \sum_{k=1}^{a+d_m-1} \frac{f_{m,k}}{x_0^{a+d_m-k} x_1^k} \in \frac{1}{x_0 x_1} R[x_0^{-1}, x_1^{-1}].$$

The element $B_c f$ is then identified via Lemma 4.3 with $u = r_1 u_1 + \cdots + r_c u_c$. Since $f \in \ker B_c$, this is cohomologically trivial on X, thus each term contains at least one of x_0 or x_1 with non-negative exponent.

Write

$$u = u^{(0)} + u^{(1)} + u^{(2)}$$

where each term of $u^{(0)}$, respectively $u^{(1)}$, has x_0, respectively x_1 occuring with negative exponent, and $u^{(2)}$ consists of the remaining terms of u. Note that

$$u^{(i)} \in \mathcal{O}_X(-a,b)(W_{ij}) \text{ for all } j \text{ and } i = 0,1,$$
$$u^{(2)} \in S_{(-a,b)}.$$

The 0-cochain (v_{ij}) on X given by

$$v_{ij} = \begin{cases} u^{(0)} + \frac{1}{2}u^{(2)} & \text{if } i = 0, \\ -u^{(1)} - \frac{1}{2}u^{(2)} & \text{if } i = 1, \end{cases} \tag{17}$$

restricts to a section $v \in \mathcal{R}(Z)$. This induces a map $\bar{\alpha} : K \to \mathcal{R}(Z)$. In summary we obtain the following.

Proposition 4.7. *Let $\eta = \eta_c$. The map $\bar{\alpha} : K \to \mathcal{R}(Z)$ defined above induces the isomorphism*

$$\alpha_c : K/\eta \to \mathcal{R}(Z)/\iota^*\mathcal{R}(X)$$

as in Proposition 3.9. Furthermore, α is a map of R-modules and isomorphic over degrees $(-a,b) \in \mathrm{Pic}$ with $-a \leqslant -1$.

Proof. The construction above is compared with the proof of Proposition 3.9. The final statement follows since $\mathcal{R}(X)_{(-a,b)} = 0$ if $-a \leqslant -1$. □

4.2 The ring structure of $\mathcal{R}(Z)$

4.2.1 The Cox ring in higher degrees

In this Section, we evaluate α in degrees $(-1,b)$, where $b \in \mathbb{Z}$, which will produce sections in $\mathcal{R}(Z)$ that, along with those sections restricted from X, generate the Cox ring $\mathcal{R}(Z)$ in degrees $(-a,b) \in \mathrm{Pic}$ with $-a \geqslant 1$.

Proposition 4.8. *Let $Z \subset X$ be a subvariety satisfying condition (ii) in Definition 4.2.*

(i) For each $1 \leqslant m \leqslant c$ and $1 \leqslant k \leqslant d_m$, there are sections $z_{m,k} \in \mathcal{R}(Z)$, with $\deg z_{m,k} = (-1, e_m)$ satisfying the equations

$$x_1 z_{m,k+1} + g_{m,k} - x_0 z_{m,k} = 0, \quad 1 \leqslant m \leqslant c, 0 \leqslant k \leqslant d_m, \tag{18}$$

where variables of undefined index are set equal zero.

(ii) The subalgebra $\mathcal{R}(Z)_{z_{m,k}}$ of $\mathcal{R}(Z)$ generated by the $x_i, y_j, z_{m,k}$ for the relevant i, j, m, k, contains $\mathcal{R}(Z)_{(-a,b)}$ for all $-a \geqslant -1, b \in \mathbb{Z}$

Proof. By Proposition 4.6 we have

$$K_1 \cong R^{d_1}[e_1] \times \cdots \times R^{d_c}[e_c] \tag{19}$$

For $1 \leqslant m \leqslant c$, $1 \leqslant k \leqslant d_m$, let $h_{m,k}$ be the k^{th} standard basis vector of $R^{d_m}[e_m]$ in (19). This is identified with $\frac{1}{x_0^{d_m-k+1}x_1^k} \in \frac{1}{x_0x_1}R[x_0^{-1}, x_1^{-1}]$. Setting $z_{m,k} = \alpha h_{m,k}$ gives us R-linearly independent sections in $\bigoplus_{b\in\mathbb{Z}} \mathcal{R}(Z)_{(-1,b)}$. By the definition of α, each section $z_{m,k}$ is given by the following 0-cochain $(z_{m,k,ij})$ on X.

$$z_{m,k,ij} = \begin{cases} \sum_{l-k\geqslant 0} g_{m,l}\frac{x_1^{l-k}}{x_0^{l-k+1}} & \text{if } i = 0, \\ -\sum_{l-k<0} g_{m,l}\frac{x_0^{k-l-1}}{x_1^{k-l}} & \text{if } i = 1. \end{cases} \tag{20}$$

Since for any choice of i, j, m, k we have $x_1 z_{m,k+1,ij} + g_{m,k} - x_0 z_{m,k,ij} = 0$, we conclude that

$$x_1 z_{m,k+1} + g_{m,k} - x_0 z_{m,k} = 0$$

for each $1 \leqslant m \leqslant c, 0 \leqslant k \leqslant d_m$, where undefined variables are set to be zero. This concludes part (i).

For part (ii), note that part (i) shows that $\bigoplus_{b\in\mathbb{Z}} \mathcal{R}(Z)_{(-1,b)} \subset \mathcal{R}(Z)_{z_{m,k}}$. It suffices to show that $\alpha K_a \subset \mathcal{R}(Z)_{z_{m,k}}$ for each $(-a, b) \in \mathrm{Pic}$ with $-a \geqslant 0$. By Proposition 4.6(ii), an R-module basis of K_a is given by all $h_{-a,m,k}$ with $1 \leqslant m \leqslant c, 0 \leqslant -a \leqslant d_m-2, 0 \leqslant k \leqslant a+d_m-1$, where $h_{-a,m,k}$ is the k^{th} standard basis vector of $R^{a+d_m-1}[e_m]$, which is a direct summand of K_a. This is identified with $\frac{1}{x_0^{a+d_m-k}x_1^k} \in \frac{1}{x_0x_1}R[x_0^{-1}, x_1^{-1}]$. Setting $v_{-a,m,k} = \alpha h_{-a,m,k}$, we find for example the equations

$$v_{-a,m,k} = z_{m,k}x_0^{1-a} - \frac{1}{2}\sum_{0\leqslant l-k\leqslant -a} g_{m,l}x_0^{-a-l+k}x_1^{l-k}.$$

Each $v_{-a,m,k}$ is in $\mathcal{R}(Z)_{z_{m,k}}$ and we are done. $\square$

Remark 4.9. Part (i) of Proposition 4.8 can be proven just applying the method of Section 3.3 in codimension one. The section $z_{m,k}$ is the restriction of a section given by the inclusion $V_m \hookrightarrow X$. Similarly, each $v_{-a,m,k}$ in the proof of Proposition 4.8 are restrictions of sections on V_m.

Corollary 4.10. *Let*

$$S' = \mathbb{C}[X_0, X_1, Y_0, \ldots, Y_n, Z_{1,1}, \ldots, Z_{1,d_1}, \ldots\ldots, Z_{m,d_m}]$$

be the Pic*-graded* $\mathbb{C}$*-algebra with generators having degrees* $\deg X_i = (1, 0)$, $\deg Y_j = (0, 1)$, $\deg Z_{m,k} = (-1, e_m)$. *Let* $I \lhd S'$ *be the ideal generated by the equations*

$$X_1 Z_{m,k+1} + g_{m,k}(Y_j) - X_0 Z_{m,k}, \quad 1 \leqslant m \leqslant c, 0 \leqslant k \leqslant d_m,$$

with variables of undefined index set to zero. The algebra homomorphism

$$\phi : S' \to \mathcal{R}(Z)$$

defined by $X_i \mapsto x_i$, $Y_j \mapsto y_j$, $Z_{m,k} \mapsto z_{m,k}$ *has kernel containing* I*, inducing isomorphisms* $(S'/I)_{(a,b)} \cong \mathcal{R}(Z)_{(a,b)}$ *whenever* $a \geqslant -1$.

Proof. By Proposition 4.8, the ideal I is contained in $N = \ker\phi$. To prove the corollary, it suffices to show that $(N/I)_{(a,b)} = 0$ for $a \geqslant -1$. Firstly, let $r'_m = r_m(X_i, Y_j)$, which maps via ϕ onto the defining equation

r_m of V_m, so that $r'_m \in N$. Since

$$\begin{aligned} r'_m =& X_0^{d_m}\left(X_1 Z_{m,1} + g_{m,0}(Y_j)\right) + X_0^{d_1-1}X_1\left(X_1 Z_{m,2} + g_{m,1}(Y_j) - X_0 Z_{m,1}\right) + \\ & \cdots + X_0 X_1^{d_m-1}\left(X_1 Z_{m,d_m-1} + g_{m,d-1}(Y_j) - X_0 Z_{m,d}\right) + X_1^{d_m}\left(g_{d_m}(Y_j) - X_0 Z_{m,d_m}\right), \end{aligned}$$

we have $r'_m \in I$. Let $f = f(X_i, Y_j, Z_{m,k}) \in N$ be homogeneous of degree (a,b), with $a \geqslant -1$. We prove that the image $\overline{f}$ of f in N/I is zero by considering cases in a. Note that any term in f is a monomial in the Y_j multiplied by a monomial $M = X_0^{\lambda_0}X_1^{\lambda_1}Z_{1,1}^{\mu_{1,1}}\dots Z_{m,d_m}^{\mu_{m,d_m}}$ such that $\lambda_0 + \lambda_1 - \mu_{1,1} - \cdots - \mu_{m,d_m} = a$.

If $a > \max_m d_m - 2$, the generating equations of I allow us to rewrite such an M purely as a polynomial in the X_i and Y_j modulo I. Without changing the class $\overline{f}$, we replace f with a polynomial $f(X_i, Y_j)$ independent of the $Z_{m,k}$. Since $\mathcal{R}(Z)_{(a,b)} = R[x_0,x_1]/\langle r_1,\dots,r_c\rangle$ by Proposition 4.6, that $f \in N$ implies that $f(x_i,y_j) \in \langle r_1,\dots,r_c\rangle$, thus $f \in I$ and $\overline{f} = 0$.

If $-1 \leqslant a \leqslant d_c - 2$, the procedure is similar. The generating equations of I allow us to rewrite M as a $\mathbb{C}[Y_j]$-linear combination of $X_0^{a+1}Z_{1,1},\dots,X_0^{a+1}Z_{m,d_m-a-1}$, with monomials containing undefined index ignored, plus a polynomial independent of the $Z_{m,k}$ modulo I. We can thus replace f, without changing $\overline{f}$, with a polynomial of the form

$$f(X_i,Y_j,Z_{m,k}) = f_1(Y_j)X_0^{a+1}Z_{1,1} + \cdots + f_{d-a-1}(Y_j)X_0^{a+1}Z_{m,d_m-a-1} + p(X_i,Y_j).$$

Following the proof of Proposition 4.8, we have

$$\phi(f) = f_{1,1}(y_j)v_{a,1,1} + \cdots + f_{m,d_m-a-1}(y_j)v_{a,m,d_m-a-1} + q(x_i,y_j)$$

for some $q \in R[x_0,x_1]_{(a,b)} \cong \iota^*\mathcal{R}(X)_{(a,b)}$. But $q, v_{a,1,1},\dots,v_{a,m,d_m-a-1}$ are R-linearly independent, and so $\phi(f) = 0$ implies $f_{1,1} = \cdots = f_{m,d_m-a-1} = 0$, and in turn $q = p = 0$. Then $\bar{f} = 0$ as required. □

Remark 4.11. Proposition 4.6 implies that any Cox ring generators except the $x_i, y_j, z_{m,k}$ exist only in degrees $(-a,b)$ with $-a \leqslant -2$. Corollary 4.10 implies that any relations as yet unfound between generators also occur in degrees $(-a,b)$ with $-a \leqslant -2$.

4.2.2 Multiplying sections

As explained in Remark 4.11, we only need to explore $\mathcal{R}(Z)$ in degrees $(-a,b)$ with $-a \leqslant -1$. By Proposition 4.6(ii), this can be reduced to computing the kernels of the maps A_a for $a \geqslant 1$. The map A_1 is the zero map, and the corresponding sections are understood in Proposition 4.8(i). To find elements of $\ker A_a$ for $a \geqslant 2$ we use the map α, which is an isomorphism in these degrees, to interpret the maps corresponding to multiplication by the x_i and the $z_{m,k}$. More precisely, let $a \geqslant 1$ and consider the isomorphism $\alpha : K_a/\eta \cong \bigoplus_{b\in\mathbb{Z}} \mathcal{R}(Z)_{(-a,b)}$. We get maps

$$\tilde{x}_i = \alpha^{-1}(\cdot x_i)\alpha : K_{a+1}/\eta \to K_a/\eta, \quad \tilde{z}_{m,k} = \alpha^{-1}(\cdot z_{m,k})\alpha : K_a/\eta \to K_{a+1}/\eta$$

Before describing these maps explicitly, we give an explicit description of α on K_a/η for $a \geqslant 1$.

Lemma 4.12. *Suppose that*

$$f = (f_1;\cdots;f_c) = (f_{1,1},\dots,f_{1,a+d_1-1};\dots\dots;f_{m,1},\dots,f_{m,a+d_m-1}) \in K_a/\eta.$$

Then αf is given by the 0-cocycle (v_{ij}) on Z defined by

$$v_{ij} = \begin{cases} f_{1,1}g_{1,d_1}\frac{x_1^{d_1-1}}{x_0^{a+d_1-1}} + \cdots + (f_{1,1}g_{1,1} + \cdots + f_{1,d_1}g_{1,d_1})\frac{1}{x_0^a} \\ \qquad + \cdots + & \text{if } i = 0, \\ f_{c,1}g_{c,d_c}\frac{x_1^{d_c-1}}{x_0^{a+d_c-1}} + \cdots + (f_{c,1}g_{c,1} + \cdots + f_{c,d_c}g_{c,d_c})\frac{1}{x_0^a} \\ \\ -(f_{1,a}g_{1,0} + \cdots + f_{1,a+d_1-1}g_{1,d_1-1})\frac{1}{x_0^a} - \cdots - f_{1,a+d_1-1}g_{1,0}\frac{x_0^{d_1-1}}{x_1^{a+d_1-1}} \\ \qquad - \cdots - & \text{if } i = 1. \\ -(f_{c,a}g_{c,0} + \cdots + f_{c,a+d_c-1}g_{c,d_c-1})\frac{1}{x_0^a} - \cdots - f_{c,a+d_c-1}g_{c,0}\frac{x_0^{d_c-1}}{x_1^{a+d_c-1}} \end{cases} \tag{21}$$

Proof. This follows immediately from the definition of α, remembering that

$$f_{1,l}g_{1,0} + \cdots + f_{1,l+d_1}g_{1,0} + \cdots\cdots + f_{c,l}g_{c,0} + \cdots + f_{c,l+d_c}g_{1,d_c} = 0$$

for each $l = 1, \ldots, a-1$ since $f \in K_a$. $\square$

Proposition 4.13. *Let $a \geqslant 1$. The operators $\tilde{x}_0, x_1 : K_{a+1}/\eta \to K_a/\eta$ are defined as follows. Suppose that*

$$u = \left((f_{1,l})_{l=1}^{a+d_1}; \ldots; (f_{m,l})_{l=1}^{a+d_m}\right) \in K_{a+1},$$

then

$$\tilde{x}_0 u = \left((f_{1,l})_{l=1}^{a+d_1-1}; \ldots; (f_{c,l})_{l=1}^{a+d_1-1}\right), \quad \tilde{x}_1 u = \left((f_{1,l})_{l=2}^{a+d_1}; \ldots; (f_{c,l})_{l=2}^{a+d_1}\right).$$

That is, $\tilde{x}_0$, respectively $\tilde{x}_1$, is truncation of each tuple in each of the c components of u by one entry on the right, respectively left.

Proof. Using Proposition 4.12, we see that $x_0\alpha u$ and $\alpha\tilde{x}_0 u$ agree as rational sections on $W_{0j} \cap Z$, and thus must be the same section. Similarly $x_1\alpha u$ and $\alpha\tilde{x}_1 u$ agree on each W_{1j}. One may also use Lemma 3.5. $\square$

Proposition 4.14. *Let $a \geqslant 1$, $1 \leqslant m \leqslant c$, $1 \leqslant k \leqslant d_m$. The operator $\tilde{z}_{m,k} : K_a/\eta \to K_{a+1}/\eta$ is defined as follows. Suppose that*

$$u = \left((f_{1,l})_{l=1}^{a+d_1-1}; \ldots; (f_{m,l})_{l=1}^{a+d_m-1}\right) \in K_a.$$

Then $\tilde{z}_{m,k}u$ is given by

$$\left((f'_{m',l})_{l=1}^{a+d_m}\right)_{m'=1}^{c},$$

where

$$f'_{m',l} = \begin{cases} f_{m',l}g_{m,k} + \cdots + f_{m',l+d_m-k}g_{m,d_m} & \textit{if} \quad 1 \leqslant m' \leqslant c, \quad 1 \leqslant l \leqslant a + d_{m'} - d_m + k - 1, \\ \\ -f_{1,l-k}g_{1,0} - \cdots - f_{1,a+d_1-1}g_{1,d_1+a+k-l-1} - \cdots \\ \quad -f_{m,l-k}g_{m,0} - \cdots - f_{m,l-1}g_{m,k-1} - \cdots & \textit{if} \qquad m' = m, \quad a + k \leqslant l \leqslant a + d_m, \\ -f_{c,l-k}g_{c,0} - \cdots - f_{c,a+d_c-1}g_{c,d_c+a+k-l-1} \\ \\ f_{m',l}g_{m,k} + \cdots + f_{m',d_m+l-k-1}g_{m,d_m-1} & \textit{if} \quad m' \neq m, \quad a + d_{m'} - d_m + k \leqslant l \leqslant a + d_{m'}. \end{cases}$$

Proof. We fix m and use reverse induction in k. If $k = d_m$ then we notice that

$$x_0\alpha\left(\tilde{z}_{m,d_m}u\right) = \alpha\left(\tilde{x}_0\tilde{z}_{m,d_m}u\right) = \alpha(g_d u) = g_d\alpha u = x_0 z_{m,d_m}\alpha u,$$

where we have used Proposition 4.13 and the equation $g_{m,d_m} - x_0 z_{m,d_m} = 0$ in $\mathcal{R}(Z)$ from Proposition 4.8(i). Since the Cox ring is an integral domain, multiplication by x_0 is injective and the result is true for $k = d_m$.

Suppose now that $1 \leqslant k < d_m$. One checks from the definition of $\tilde{z}_{m,k}$ that for any $u' \in K_a$, we have

$$\tilde{x}_0\tilde{z}_{m,k}(u') = g_{m,k}u' + \tilde{x}_1\tilde{z}_{m,k+1}(u').$$

As such we have

$$x_0\alpha(\tilde{z}_{m,k}u) = \alpha(\tilde{x}_0\tilde{z}_{m,k}u) = \alpha((g_{m,k} + \tilde{x}_1\tilde{z}_{m,k+1})u) = (g_{m,k} + x_1 z_{m,k+1})\alpha u = x_0 z_{m,k}\alpha u,$$

where the last two equalities invoke the induction hypothesis and the equations in Proposition 4.8 respectively. Once more, as x_0 is not a zero divisor, we establish the result for k. □

Remark 4.15. Suppose we have a new generator of the Cox ring $t \in \mathcal{R}(Z)$ of degree $(-a, b) \in \mathrm{Pic}$, with $-a \leqslant -2$. Then t is identified with an element

$$\left((f_{1,l})_{l=1}^{a+d_1-1}; \cdots; (f_{c,l})_{l=1}^{a+d_c-1}\right).$$

Using Proposition 4.13 we obtain equations

$$x_0^{a-j}x_1^{j-1}t = f_{1,j}z_{1,1} + \cdots + f_{1,d_1+j}z_{1,d_1} + \cdots\cdots + f_{c,j}z_{c,1} + \cdots + f_{c,d_c+j}z_{c,d_c}$$

for each $1 \leqslant j \leqslant a$. It follows that multiplication by any section can be understood via multiplication by the sections $z_{m,k}$ and the R-module structure on $\mathcal{R}(Z)$.

4.2.3 Proof of Main Theorem 2.3

We are now able to prove Main Theorem 2.3. The following Theorem is part (i).

Theorem 4.16. *Fix $c \geqslant 1$ and $n \geqslant c + 2$. Let $X = \mathbb{P}^1 \times \mathbb{P}^n$ and suppose $Z \subset X$ has form $(*)$ with respect to irreducible hypersurfaces $V_1, \ldots V_c$, with all notation as in Definition 4.2. Then there are sections $w_{m,k} \in \mathcal{R}(Z)_{(i_{m,k-1}-i_{m,k}, e_m)}$ for each $1 \leqslant m \leqslant c$ and $0 \leqslant k \leqslant \tau_m$ satisfying the equations*

$$x_1^{i_{m,k+1}-i_{m,k}}w_{m,k+1} + g_{i_{m,k}} - x_0^{i_{m,k}-i_{m,k-1}}w_{m,k}, \tag{22}$$

where variables of undefined index are set to zero. Furthermore, these sections, along with the sections x_0, x_1, $y_0, \ldots, y_n$ restricted from X, generate the Cox ring $\mathcal{R}(Z)$. The ideal of relations among these generators is generated by the equations (22). More precisely, let

$$U = \mathbb{C}[X_0, X_1, Y_0, \ldots, Y_n, W_{1,1}, \ldots, W_{1,\tau_1}, \ldots\ldots, W_{c,\tau_c}]$$

be the free $\mathbb{Z}^2$-graded algebra with $\deg X_i = (1,0)$, $\deg Y_j = (0,1)$ and $\deg W_{m,k} = (i_{m,k-1} - i_{m,k}, e_m)$ for relevant i, j, m, k, and $J \lhd U$ the homogeneous ideal generated by the equations

$$X_1^{i_{m,k+1}-i_{m,k}} W_{m,k+1} + g_{i_{m,k}}(Y_j) - X_0^{i_{m,k}-i_{m,k-1}} W_{m,k}, \quad 1 \leqslant m \leqslant c, 0 \leqslant k \leqslant \tau_m.$$

Then there is a surjective map $\psi : U \to \mathcal{R}(Z)$, inducing an isomorphism $\mathcal{R}(Z) \cong U/J$. In particular, Z is a Mori Dream Space.

Proof. The sections

$$\begin{aligned} w_{m,k} &= \frac{g_{m,i_{m,k}} x_0^{d_m - i_{m,k}} + \cdots + g_{m,i_{m,l_m}} x_1^{d_m - i_{m,k}}}{x_0^{d_m - i_{m,k-1}}} \\ &= -\frac{g_{m,i_{m,0}} x_0^{m,i_{m,k-1}} + \cdots + g_{m,i_{m,k-1}} x_1^{i_{m,k-1}}}{x_1^{i_{m,k}}} \in \mathcal{R}(Z)_{(i_{m,k-1}-i_{m,k}, e_m)}, \end{aligned} \tag{23}$$

defined for each $1 \leqslant m \leqslant c$ and $1 \leqslant k \leqslant \tau_m$, are defined globally on Z and satisfy the equations (22). One checks that

$$\alpha^{-1} w_{m,k} = \left((f_{1,l})_{l=1}^{d_1}; \cdots ; (f_{m,l})_{l=1}^{d_m} \right) \in (\ker A_{i_{m,k}-i_{m,k-1}})/\eta,$$

where $f_{m',l} = 1$ if $(m', l) = (m, i_{m,k})$ and all other $f_{m',l}$ are equal zero. We have a well-defined map $\psi : U \to \mathcal{R}(Z)$ sending $X_i, Y_j, W_{m,k}$ to the sections $x_i, y_j, w_{m,k}$ respectively.

The sections $z_{m,k'} \in \mathcal{R}(Z)$ found in Proposition 4.8(i) are in the image of ψ; namely, for each $1 \leqslant m \leqslant c$, $1 \leqslant k \leqslant \tau_m$ and each $1 \leqslant j \leqslant i_{m,k} - i_{m,k-1}$, one computes that

$$z_{i_{m,k-1}+j} = \psi(Z_{i_{m,k-1}+j}) \text{ with } Z_{i_{m,k-1}+j} = X_0^{j-1} X_1^{i_{m,k}-i_{m,k-1}-j} W_{m,k}. \tag{24}$$

So $\operatorname{im}\psi$ contains $\mathcal{R}(Z)_{(-a,b)}$ for all $-a \leqslant -1$ by Proposition 4.8(ii). Furthermore, the equations (18) become exactly the equations (22) and any degree (a,b) monomial of U with $a \geqslant -1$ can be rewritten as a monomial in the $X_i, Y_j, Z_{m,k'}$. Thus, by Corollary 4.10, we only need to consider $\mathcal{R}(Z)$ in degrees $(-a, b)$ with $-a \leqslant -2$. Since we can apply the method developed in Section 1, surjectivity of ψ in degrees $(-a, b)$ with $-a \leqslant -1$ is implied by Proposition 4.6(ii) and Theorem 5.17, proven algebraically in Chapter 5. It remains to show that ψ is injective.

Since ψ is surjective, Z is a Mori Dream Space, and thus $\dim \mathcal{R}(Z) = \dim Z + \rho(Z) = n - c + 3$ by [21, Proposition 2.9]. If we can show that $\mathcal{R} = U/J$ is normal and integral of dimension $n - c + 3$ then we are done. To conclude the proof of the theorem, it therefore suffices to show that the algebra $\mathcal{R} = U/J$ is normal and integral of dimension $n - c + 3$. The following argument generalises a method of Ito in [29].

Localising $\mathcal{R}$ at X_0, for each $m = 1, \ldots, c$ we have the equations

$$W_{m,k} = \frac{g_{m,i_k}(Y_j)}{X_0^{i_{m,k}-i_{m,k-1}}}, \ldots, W_{m,1} = \frac{X_1^{i_{m,2}-i_{m,1}} W_{m,2} + g_{i_{m,1}}(Y_j)}{X_0^{i_{m,1}-i_{m,0}}}$$

in $\mathcal{R}_{X_0}$, implying the surjectivity of

$$\mathbb{C}[X_0, X_1, Y_0, \ldots, Y_n]_{X_0} \to \mathcal{R}_{X_0}.$$

Substituting the above equations into $X_{m,0}^{i_{m,1}-i_{m,0}} W_{m,1} + g_{i_{m,0}}(Y_j) = 0$ in $\mathcal{R}_{X_0}$ we obtain $\frac{r_m(X_i,Y_j)}{X_0^d} = 0$ and thus

$$\mathcal{R}_{X_0} = (\mathbb{C}[X_0, X_1, Y_0, \ldots, Y_n]/\langle r_1(X_i, Y_j), \ldots, r_c(X_i, Y_j)\rangle)_{X_0}. \tag{25}$$

Similarly one obtains

$$\mathcal{R}_{X_1} = (\mathbb{C}[X_0, X_1, Y_0, \ldots, Y_n]/\langle r_1(X_i, Y_j), \ldots, r_c(X_i, Y_j)\rangle)_{X_1}. \tag{26}$$

On the other hand, we see

$$\mathcal{R}/\langle X_0, X_1\rangle = \mathbb{C}[Y_0, \ldots, Y_n, W_{1,1}, \ldots\ldots, W_{c,\tau_c}]/\langle g_{1,i_0}(Y_j), \ldots\ldots, g_{c,\tau_c}(Y_j)\rangle. \tag{27}$$

Let $T_1 = \operatorname{Spec}(\mathcal{R}/\langle X_0, X_1\rangle)$ and $T_2 = \operatorname{Spec}(\mathcal{R}/\langle Y_0, \ldots, Y_n\rangle)$. Since $r_1, \ldots, r_c$ form a regular sequence in $\mathcal{R}(X)$, the rings $\mathcal{R}_{X_0}, \mathcal{R}_{X_1}$ are integral domains of dimension $n-c+3$ by (25), (26) respectively. Additionally, since

$$g_{1,i_{1,0}}, \ldots\ldots, g_{c,i_{c,\tau_c}}$$

forms a regular sequence in R, we have $\dim T_1 = n - c + 1$ by (27). One sees that $\dim(T_2 \cap \operatorname{Spec} \mathcal{R}_{X_0}) = 2$ via

$$T_2 \cap \operatorname{Spec} \mathcal{R}_{X_0} = \operatorname{Spec}(\mathbb{C}[X_0, X_1, Y_0, \ldots, Y_n]_{X_0}/\langle r_m(X_i, Y_j), Y_0, \ldots, Y_n\rangle) = \operatorname{Spec} \mathbb{C}[X_0, X_1]_{X_0}.$$

In the same way, $\dim(T_2 \cap \operatorname{Spec} \mathcal{R}_{X_1}) = 2$. Since

$$T_2 = (T_1 \cap T_2) \cup (\operatorname{Spec} \mathcal{R}_{X_0} \cap T_2) \cup (\operatorname{Spec} \mathcal{R}_{X_1} \cap T_2),$$

we conclude that $\dim T_2 \leqslant \max\{\dim T_1, 2\} = n - c + 1$. We may now consider the morphism

$$\operatorname{Spec} \mathcal{R} \backslash (T_1 \cup T_2) \to Z,$$

which is a $(\mathbb{C}^*)^2$-bundle. This implies that $\operatorname{Spec} \mathcal{R} \backslash (T_1 \cup T_2)$ is smooth and irreducible with dimension $n - c + 3$.

But $J \lhd U$ is generated by $\sum_{m=1}^{c} \tau_m + c$ elements, and thus each irreducible component of $\operatorname{Spec} \mathcal{R}$ has dimension at least $n - c + 3$. Since $\dim(T_1 \cup T_2) = n - c + 1$, it cannot be its own irreducible component, and thus $\operatorname{Spec} \mathcal{R}$ is irreducible of dimension $n - c + 3$. It follows that J is a complete intersection ideal and that $\operatorname{Spec} \mathcal{R}$ is smooth in codimension one, indeed it is smooth away from $T_1 \cup T_2$. Since J is a complete intersection ideal and $\operatorname{Spec} \mathcal{R}$ smooth in codimension one, it follows that $\mathcal{R}$ is normal and integral, and we are done. ∎

We now prove Main Theorem 2.3(ii).

Proposition 4.17. *Let $c \geqslant 1$ and $n \geqslant c+2$. Let $X = \mathbb{P}^1 \times \mathbb{P}^n$ and suppose $Z \subset X$ satisfies conditions (i), (ii) in Definition 4.2 and adopt the corresponding notation. Let*

$$U = \mathbb{C}[X_0, X_1, Y_0, \dots, Y_n, W_{1,1}, \dots\dots, W_{c,\tau_c}]$$

be the $\mathbb{Z}^2$-graded algebra with $\deg X_i = (1,0)$, $\deg Y_j = (0,1)$ and $\deg W_{m,k} = (i_{m,k-1} - i_{m,k}, e_m)$ for the relevant i, j, m, k, and $J \lhd U$ the homogeneous ideal generated by the equations

$$X_1^{i_{m,k+1}-i_{m,k}} W_{m,k+1} + g_{i_{m,k}}(Y_j) - X_0^{i_{m,k}-i_{m,k-1}} W_{m,k}, \quad 1 \leqslant m \leqslant c, 0 \leqslant k \leqslant \tau_m.$$

If there is an isomorphism $\mathcal{R}(Z) \cong \mathcal{R}$, where $\mathcal{R} = U/J$, then the sequence

$$g_{1,i_{1,0}}, \dots g_{1,i_{1,\tau_1}}, \dots\dots, g_{c,i_{c,1}}, \dots, g_{c,i_{c,o_c}}$$

is regular in $\mathbb{C}[y_0, \dots, y_n]$. In other words, Z is of the form $(*)$.

Proof. Again, it is an argument of Ito allowing us to prove this statement in its strongest form. By degree considerations, the isomorphism $\mathcal{R} \cong \mathcal{R}(Z)$ sends the $\mathbb{C}$-span of X_0, X_1 to the $\mathbb{C}$-span of x_0, x_1. By [21, Proposition 2.7], we conclude X_0, X_1 is a regular sequence in $\mathcal{R}$, and thus

$$\dim \mathcal{R}/\langle X_0, X_1\rangle = \dim \mathcal{R} - 2 = \dim \mathcal{R}(Z) - 2 = n - c + 1,$$

where again the final equality uses Z being a Mori Dream Space. Using (27) as found in the proof of the previous Theorem, we conclude that

$$\dim \mathbb{C}[Y_0, \dots, Y_n]/\langle g_{1,i_{1,0}}(Y_j), \dots\dots, g_{c,i_{c,o_c}}(Y_j)\rangle = n - \sum_{m=1}^{c} \tau_m - c + 1,$$

implying that $g_{1,i_{1,0}}, \dots\dots, g_{c,i_{c,o_c}}$ is regular in $\mathbb{C}[y_0, \dots, y_n]$, as claimed. ∎

4.3 Geometry of hypersurfaces

In this Section we apply the techniques developed to some geometric examples with $c = 1$.

4.3.1 The degeneracy locus of the second projection

Consider the projection $p_2 \colon X \to \mathbb{P}^n$ restricted to Z. This is generically a d-to-one cover, but it has a degeneracy locus

$$Y = \{g_0 = g_1 = \dots = g_d = 0\} \subset \mathbb{P}^n,$$

over which the fibres are isomorphic to $\mathbb{P}^1$. Let

$$I_Y = \langle g_0, g_1, \dots g_d\rangle \lhd R = \mathbb{C}[y_0, \dots, y_n]$$

be the corresponding ideal. The quotient R/I_Y has a finite free R-module resolution of the form

$$\ldots \longrightarrow R^\gamma \xrightarrow{C} R^\beta \xrightarrow{B} R^{d+1} \xrightarrow{A} R \longrightarrow R/I_Y \longrightarrow 0, \tag{28}$$

where we suppressed degrees, with $A = (g_0, g_1, \ldots, g_d)$, and B, C matrices of homogeneous elements of the appropriate sizes and degrees.

Of particular interest to us is the case where Z is regular with index sequence $\{i_0, \ldots, i_l\}$. It is easy to check that there are always such choices with $Z \subset X$ nonsingular. Then (28) is the sum of the standard Koszul complex for $\{g_{i_0}, \ldots, g_{i_l}\}$ and a trivial complex. The locus $Y \subset \mathbb{P}^n$ is a complete intersection of codimension $l+1$, and the inverse image $E = p_2^{-1}(Y) \subset X$ has codimension l in Z. In the extreme case, where $l = 1$ and $r = g_0 x_0^d + g_d x_1^d$, the variety E is a divisor ruled over Y.

4.3.2 Examples

Example 4.18. Let $n = 3$, $d = 2$ and choose three general polynomials $g_0, g_1, g_2 \in \mathbb{C}[y_0, y_1, y_2, y_3]$ of degree $e = 4$, forming a regular sequence. Consider the family of varieties $q\colon \mathcal{Z} \to \mathbb{A}^1_t$ defined by

$$\mathcal{Z} = \left\{ g_0 x_0^2 + t g_1 x_0 x_1 + g_2 x_1^2 = 0 \right\} \subset \mathbb{P}^1 \times \mathbb{P}^3 \times \mathbb{A}^1_t.$$

For every $t \in \mathbb{A}^1$, the hypersurface fibre $Z_t \subset \mathbb{P}^1 \times \mathbb{P}^3$ of the family $q\colon \mathcal{X} \to \mathbb{A}^1_t$ is a smooth Calabi–Yau threefold. Let $Z_t \xrightarrow{f_t} \bar{Z}_t \xrightarrow{g_t} \mathbb{P}^3$ be the Stein factorization of the second projection p_2. The map $g_t\colon \bar{Z}_t \to \mathbb{P}^3$ is a double cover in all cases, ramified over the divisor

$$D_t = \{t^2 g_1^2 - 4 g_0 g_2 = 0\} \subset \mathbb{P}^3$$

that is singular along the degeneracy locus

$$Y_t = \{g_0 = t g_1 = g_2 = 0\} \subset \mathbb{P}^3.$$

For $t \neq 0$, $Y_t \subset \mathbb{P}^3$ is a set of 64 points, and the map $f_t\colon Z_t \to \bar{Z}_t$ is a small contraction, contracting a set of 64 rational curves to nodes. For $t = 0$ on the other hand, the map $Z_0 \to \bar{Z}_0$ is a divisorial contraction, contracting a divisor $E \subset Z_0$ to a genus-33 curve $C \subset \bar{Z}_0$ isomorphic to $Y_0 \subset \mathbb{P}^3$. As already observed by [7], see in particular [11, 3.3.2-3.3.2], for numbers of global sections, the Cox ring detects this change in birational behaviour.

Theorem 4.19. *For $t \neq 0$, the $\mathbb{Z}^2$-graded Cox ring $\mathcal{R}(Z_t)$ of the Calabi–Yau hypersurface $Z_t \subset \mathbb{P}^1 \times \mathbb{P}^3$ can be presented as*

$$\mathcal{R}(Z_t) \cong k[x_0, x_1, y_0, y_1, y_2, y_3, z_1, z_2]/\langle x_1 z_1 + g_0, x_1 z_2 + g_1 - x_0 z_1, g_2 - x_0 z_2 \rangle,$$

with variables of bidegrees $(1,0)$, $(0,1)$ and $(-1,4)$ respectively. For $t = 0$, we have

$$\mathcal{R}(Z_0) \cong \mathbb{C}[x_0, x_1, y_0, y_1, y_2, y_3, w]/\langle x_1^2 w + g_0, g_2 - x_0^2 w \rangle,$$

with variables of bidegrees $(1,0)$, $(0,1)$ and $(-2,4)$ respectively. In particular, every member of the family is a Mori dream space, with a complete intersection Cox ring, but the effective cone and Cox ring jump discontin-

uously in the family.

Proof. By the assumptions, for $t \neq 0$, the hypesurface Z_t is regular with full index sequence $\{0, 1, 2\}$. On the other hand, the hypersurface Z_0 is regular, admitting index sequence $\{0, 2\}$. Hence the statements follow from Theorem 4.16. □

Antonio Laface has informed us that the Cox rings in these examples can also be computed using the method of [19].

Example 4.20. Let $n \geqslant 3$ and $d, e \geqslant 1$ with $d \leqslant n$. Choose general polynomials $g_0, \ldots, g_d \in \mathbb{C}[y_0, \ldots, y_n]$ of degree e, forming a regular sequence. Then by Theorem 2.3 we have

$$\mathcal{R}(Z) \cong \mathbb{C}[x_0, x_1, y_0, \ldots, y_n, z_1, \ldots, z_d]/\langle x_1 z_1 + g_0, x_1 z_2 + g_1 - x_0 z_1, \ldots, g_d - x_0 z_d\rangle.$$

Let S_{Z^+} be the subalgebra of $\mathcal{R}(Z)$ generated by $y_0, \ldots, y_n, z_1, \ldots, z_d$. By Proposition 5.19, proved in Chapter 5, we have an isomorphism

$$S_{Z^+} \cong \mathbb{C}[y_0, \ldots, y_n, z_1 \ldots, z_d]/J',$$

where J' is the ideal generated by the equations

$$g_k(z_{k'+1} z_{k''} - z_{k'} z_{k''+1}) - g_{k'}(z_{k+1} z_{k''} - z_k z_{k''+1}) + g_{k''}(z_{k+1} z_{k'} - z_k z_{k'+1}), \tag{29}$$

for $0 \leqslant k < k' < k'' \leqslant d$. The ring S_{Z^+}, defined by the equations (29), is the homogeneous coordinate ring of a birational model $Z^+ \subset \mathbb{P}^d_{z_k} \times \mathbb{P}^n_{y_j}$ of Z; compare [28, Section 2].

We introduce one further, singular, member of this deformation family of varieties with interesting behaviour.

Example 4.21. Choose general linear, respectively cubic polynomials $a_0, a_1, a_2 \in R_1$ and $b_0, b_1, b_2 \in R_3$. Consider the determinantal hypersurface

$$Z = \left\{ \begin{vmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ x_0^2 & x_0 x_1 & x_1^2 \end{vmatrix} = 0 \right\} = \{g_0 x_0^2 + g_1 x_0 x_1 + g_2 x_1^2 = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^3,$$

with $g_0 = a_1 b_2 - a_2 b_1, g_1 = a_2 b_0 - a_0 b_2, g_2 = a_0 b_1 - a_1 b_0$. Then the degeneracy locus is the curve

$$Y = \left\{ \operatorname{rk} \begin{pmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{pmatrix} \leqslant 1 \right\} \subset \mathbb{P}^3,$$

a smooth space curve of genus 21 and degree 13. Its ideal $I_Y \lhd R$ has a resolution (28) in Hilbert–Burch form

$$0 \longrightarrow R^2 \xrightarrow{B} R^3 \xrightarrow{A} R \longrightarrow R/I_Y \longrightarrow 0. \tag{30}$$

Here, $B = \begin{pmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{pmatrix}^t$, and $A = \bigwedge^2 B = (g_0, g_1, g_2)^t$. As a determinantal hypersurface, Z is singular along the locus given by the 2×2 minors of its defining matrix, which gives the locus

$$\operatorname{Sing} Z = \{g_0 = g_1 = g_2 = a_1^2 - a_0 a_2 = x_0 a_1 + x_1 a_0 = x_0 a_2 + x_1 a_1 = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^3,$$

a set of 26 isolated ordinary double points all lying on the ruled surface $E \subset Z$. Blowing up this ruled surface, a Weil divisor through each of the ODP's, gives a small resolution $\widetilde{Z} \to Z$, a smooth Calabi–Yau model.

Since $Z \subset X = \mathbb{P}^1 \times \mathbb{P}^3$ has isolated nodal singularities, the restriction map $\mathrm{Pic}(X) \to \mathrm{Pic}(Z)$ is still an isomorphism. However, Z is not $\mathbb{Q}$-Cartier, so the ring $\mathcal{R}(Z)$ as defined above only contains sections of Cartier divisors.

By Proposition 4.6(ii), the columns of B give us elements $f_1 \in N_{(-2,5)}$ and $f_2 \in N_{(-2,7)}$ that together generate the R-module $\bigoplus_{b\in\mathbb{Z}} N_{(-2,b)}$. Let $u = \alpha f_1 \in \mathcal{R}(Z)_{(-2,5)}$ and $w = \alpha f_2 \in \mathcal{R}(Z)_{(-2,7)}$ be the lifts of these module generators to elements of the Cox ring. By Proposition 4.13, these satisfy equations

$$x_0 u = a_1 z_1 + a_2 z_2, \quad x_1 u = a_2 z_1 + a_3 z_2,$$
$$x_0 w = b_1 z_1 + b_2 z_2, \quad x_1 w = b_2 z_1 + b_3 z_2.$$

Furthermore, by comparing $\delta(u), \delta(w), \delta(z_1^2), \delta(z_1 z_2), \delta(z_2^2)$ we have

$$z_1^2 + b_2 u - a_2 w = z_1 z_2 - b_1 u + a_1 w = z_2^2 + b_0 u - a_0 w = 0.$$

Consider the free $\mathbb{Z}^2$-graded k-algebra

$$T = \mathbb{C}[X_0, X_1, Y_0, \dots, Y_n, Z_1, Z_2, U, W]$$

with generators in degrees $(1,0), (0,1), (-1,4), (-2,5)$ and $(-2,7)$ respectively. Let $I \lhd T$ be the ideal

$$I = \langle X_1 Z_1 + g_0(Y_i), \quad X_1 Z_2 + g_1(Y_i) - X_0 Z_1, \quad g_2(Y_i) - X_0 Z_2,$$
$$X_0 U - a_1(Y_i) Z_1 - a_2(Y_i) Z_2, \quad X_1 U - a_2(Y_i) Z_1 - a_3(Y_i) Z_2,$$
$$X_0 W - b_1(Y_i) Z_1 + b_2(Y_i) Z_2, \quad X_1 W - b_2(Y_i) Z_1 - b_3(Y_i) Z_2,$$
$$Z_1^2 + b_2(Y_i) U - a_2(Y_i) W, \quad Z_1 Z_2 - b_1(Y_i) U + a_1(Y_i) W, \quad Z_2^2 + b_0(Y_i) U - a_0(Y_i) W\rangle.$$

Then our observations so far prove the existence of an algebra homomorphism

$$\phi : T/I \to \mathcal{R}(Z),$$

to the (Cartier) Cox ring $\mathcal{R}(Z)$ of Z that induces isomorphisms $(T/I)_{(a,b)} \cong \mathcal{R}(Z)_{(a,b)}$ whenever $a \geqslant -2$.

We sketch an argument provided to us by Antonio Laface that proves that for a general determinantal hypersurface Z, the map ϕ is an isomorphism. By [19, Thm.1], the Cox ring $\mathcal{R}(Z)$ is the intersection of certain localizations of quotients of $\mathcal{R}(\mathbb{P}^1 \times \mathbb{P}^3)$. This intersection can be computed using the ideas of [19, Cor.2.4], which gives the result that ϕ is surjective under certain dimension and saturation conditions. The latter can be checked for general a_i, b_j using computer algebra.

Example 4.22. Consider the hypersurface

$$Z = \{y_0 x_0^4 + (-\lambda_0 y_0 - \lambda_1 y_1 - \lambda_2 y_2 - \lambda_3 y_3) x_0^3 x_1 + y_1 x_0^2 x_1^2 + y_2 x_0 x_1^3 + y_3 x_1^4 = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^3$$

of bidegree $(d,e) = (4,1)$ with non-zero scalars $\lambda_j \in \mathbb{C}$, noting that a general degree $(4,1)$ hypersurface in $\mathbb{P}^1 \times \mathbb{P}^3$ has this form after a coordinate transformation. As the five linear forms are linearly dependent, by

Theorem 4.17, the Cox ring is not of the form given in (4.16), contrary to the statement [28, Thm.1.1(iv)]. We investigate the Cox ring of this particular family in this example.

The kernel of A_2 as defined in Proposition 4.6 contains $(\lambda_0, 1, \lambda_1, \lambda_2, \lambda_3)$, defining a section $w \in \mathcal{R}(Z)$ of degree $(-2, 1)$, which satisfies the equations

$$x_0 w = \lambda_0 z_1 + z_2 + \lambda_1 z_3 + \lambda_2 z_4, \quad x_1 w = z_1 + \lambda_1 z_2 + \lambda_2 z_3 + \lambda_3 z_4. \tag{31}$$

These equations imply that for general choices of λ_j, the subalgebra $\mathcal{R}'$ of $\mathcal{R}(Z)$ generated by x_0, x_1, w, z_3, z_4 contains z_1, z_2. Note also that the relations

$$x_1 z_1 + y_0 = 0, \quad x_1 z_3 + y_1 - x_0 z_2 = 0, \quad x_1 z_4 + y_2 - x_0 z_3 = 0, \quad y_3 - x_0 z_4 = 0$$

hold in $\mathcal{R}(Z)$. Hence the subalgebra $\mathcal{R}'$ contains y_0, y_1, y_2, y_3 as well. In fact, we can show that $\mathcal{R}(Z) = \mathcal{R}'$, which is a polynomial ring as follows.

Proposition 4.23. *For general constants $\lambda_j \in \mathbb{C}$, the $\mathbb{Z}^2$-graded Cox ring $\mathcal{R}(Z)$ of the hypersurface $Z \subset \mathbb{P}^1 \times \mathbb{P}^3$ is isomorphic to the polynomial ring $\mathbb{C}[x_0, x_1, w, z_3, z_4]$.*

Proof. Let $r \in H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(4, 1)) = H^0(\mathcal{O}_{\mathbb{P}^1}(4)) \otimes H^0(\mathcal{O}_{\mathbb{P}^3}(1))$ be the section which defines Z. Then r induces an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-4) \xrightarrow{r} H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{E} \to 0$$

on $\mathbb{P}^1$ and we have $Z = \mathbb{P}_{\mathbb{P}^1}(\mathcal{E}) \subset \mathbb{P}^1 \times \mathbb{P}^3$. We note that $\mathcal{E}$ is locally free of rank 3 since r is general. Let $\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b) \oplus \mathcal{O}_{\mathbb{P}^1}(c)$ with $a+b+c = 4$. Since $\mathcal{E}$ is a quotient of a trivial bundle, we have $a, b, c \geqslant 0$. If $a = 0$, then r must be contained in $H^0(\mathcal{O}_{\mathbb{P}^1}(4)) \otimes V'$ for $V' = \ker H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \to H^0(\mathcal{O}_{\mathbb{P}^1}(a)) = \mathbb{C}$, which contradicts the generality of r. Hence $a, b, c \geqslant 1$ and $\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$.

Since the embedding $Z = \mathbb{P}_{\mathbb{P}^1}(\mathcal{E}) \subset \mathbb{P}^1 \times \mathbb{P}^3$ over $\mathbb{P}^1$ is induced by $H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \otimes \mathcal{O}_{\mathbb{P}^1} \twoheadrightarrow \mathcal{E}$, the tautological line bundle $\mathcal{O}_{\mathcal{E}}(1)$ on $\mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$ is identified with $\mathcal{O}_Z(0, 1)$, where $\mathcal{O}_Z(a, b) := \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(a, b)|_Z$. Since $\mathcal{E}(-1) \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$, the morphism

$$\mu : Z = \mathbb{P}_{\mathbb{P}^1}(\mathcal{E}(-1)) \to P = \mathbb{P}H^0(\mathcal{O}_Z(-1, 1)) \cong \mathbb{P}^3$$

induced by $|\mathcal{O}_Z(-1, 1)|$ is the blow-up along a line l. We note that P is different from the second factor of $\mathbb{P}^1 \times \mathbb{P}^3$.

Let

$$D_i = (x_i = 0), \quad D'_k = (z_k = 0), \quad E = (w = 0)$$

be divisors on Z. Since w is a non-zero section of $H^0(\mathcal{O}_Z(-2, 1)) = H^0(\mathcal{E}(-2)) \simeq \mathbb{C}$, E is the exceptional divisor of the blow-up μ.

Since z_1, z_2, z_3, z_4 are linearly independent over R by Proposition 4.8, so are over $\mathbb{C}$. Hence z_1, z_2, z_3, z_4 form a basis of $H^0(\mathcal{O}_Z(-1, 1)) = H^0(\mathcal{E}(-1)) \simeq \mathbb{C}^4$. By (31), $x_0 w, x_1 w, z_3, z_4$ also form a basis of

$$H^0(\mathcal{O}_Z(-1, 1)) = H^0(P, \mathcal{O}(1)).$$

We regard $P = \mathbb{P}^3$ as a toric variety by this basis. Let $H_i = (x_i w = 0) \subset P$ be the torus invariant planes for $i = 0, 1$. Since $\mu^* H_i = D_i + E$, the line $l = \mu(E)$ is $H_0 \cap H_1$, which is torus invariant. Hence μ is a toric morphism and Z is a toric variety as well. Then torus invariant prime divisors of Z are D_0, D_1, E, D_3', D_4'. Since the Cox ring of a smooth projective toric variety is a polynomial ring whose variables correspond to torus invariants prime divisors, we conclude that $\mathcal{R}(Z) \simeq \mathbb{C}[x_0, x_1, w, z_3, z_4]$. □

4.4 Geometry in the general case

In this Section we explore the geometry of various complete intersections $Z \subset X$ of the form $(*)$. We focus generally on those with each constitutive hypersurface having full index sequence $\{0, 1, \dots, d_m\}$. We start by interpreting the sections $z_{m,k}$ from Proposition 4.8(i) geometrically.

4.4.1 Geometric interpretation of the sections

Recall that $Z \subset X$, of the form $(*)$, is defined by degree (d_m, e_m) equations

$$r_m = g_{m,0} x_0^{d_m} + \cdots + g_{m,d_m} x_1^m$$

for $m = 1, \dots c$. Let $1 \leqslant \sigma \leqslant c$ be such that $d_m = 0$ if and only if $m > \sigma$. Then Z is a subvariety of $\mathbb{P}^1 \times Y$ where $Y \subset \mathbb{P}^n$ is given by

$$Y = \{g_{\sigma+1,0} = \cdots = g_{c,0} = 0\} \subset \mathbb{P}^n.$$

Suppose that a point

$$(t, y) = ([t_0 : t_1], [y_0, \dots, y_n]) \in Z$$

is given. For each $m = 1, \dots, \sigma$ such that $g_{m,0}, \dots, g_{m,d_m}$ don't all vanish on y, we have that t is a root of r_m evaluated at y, which is a homogenous degree d_m polynomial in $r_m(y) = \mathbb{C}[x_0, x_1]$. For such an m, the sections $z_{m,k}$ as found in Proposition 4.8 do not all vanish. The equations satisfied by the $z_{m,k}$ imply that

$$r_m(y) = (-t_1 x_0 + t_0 x_1)(z_{m,1} x_0^{d_m - 1} + \cdots + z_{m,d_m} x_1^{d_m - 1}) \in \mathbb{C}[x_0, x_1], \tag{32}$$

where the sections $z_{m,k}$ are evaluated at (t, y). The sections $z_{m,k}$ thus define the coefficients of the polynomial r_m evaluated at y divided by the linear form determined by the chosen root t of r_m over y. If $m = 2$ then the factorisation is

$$r_m(y) = (-t_1 x_0 + t_0 x_1)(z_{m,1} x_0 + z_{m,2} x_1)$$

and $[z_{m,2} : -z_{m,1}]$ is simply the other root of $r_m(y)$. One recovers the matrices given at the start of [28]. Indeed over the locus where not all $g_{m,k}$ vanish, $[t_0 : t_1]$ is only a root of r_m if such a factorisation (32) exists, which is if and only if the kernel of

$$\begin{pmatrix} -t_1 & t_0 & 0 & \cdots & 0 \\ 0 & -t_1 & t_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -t_1 & t_0 \\ g_{m,0}(y) & g_{m,1}(y) & \cdots & g_{m,d_m-1}(y) & g_{m,d_m}(y) \end{pmatrix}^T$$

is one dimensional, which is if and only if its determinant vanishes. But the factorisation also exists if and only if the matrix

$$\begin{pmatrix} z_{m,1} & z_{m,2} & \cdots & z_{m,d_m} & 0 \\ 0 & z_{m,1} & \cdots & z_{m,d_m-1} & z_{m,d_m} \\ g_{m,0}(y) & g_{m,1}(y) & \cdots & g_{m,d_m-1}(y) & g_{m,d_m}(y) \end{pmatrix}^T \tag{33}$$

has one dimensional kernel, which is if and only if its 3×3 minors vanish.

4.4.2 The space $\operatorname{Spec} \mathcal{R}(Z)$

We remain in the case that $Z \subset X$ has form $(*)$ with respect to hypersurfaces each with full index set. Once more suppose that $d_m > 0$ if and only if $m \leqslant \sigma$. For each $1 \leqslant m < m' \leqslant \sigma$ let $\operatorname{res}(r_m, r_{m'})$ be the determinant of the matrix

$$\left(\frac{A_{d_{m'}+1,r_m}}{A_{d_m+1,r_{m'}}}\right).$$

For a given $y \in \mathbb{P}^n$, there is a point $x \in \mathbb{P}^1 \times \mathbb{P}^n$ over y satisfying $r_m = r_{m'} = 0$ if and only if $\operatorname{res}(r_m, r_{m'}) = 0$.

Since Z is a Mori Dream space, we have an affine variety $\hat{Z} = \operatorname{Spec} \mathcal{R}(Z)$, which is a closed subvariety of the Toric variety

$$T = \operatorname{Spec} \mathbb{C}[x_0, x_1, y_0, \dots, y_n, z_{1,1}, \dots\dots, z_{\sigma,d_\sigma}].$$

Fix $1 \leqslant m < m' \leqslant \sigma$. Set theoretically, Z is defined by the all of the following equations:

(i) The resultant $\operatorname{res}(r_m, r_{m'})$ and the equations $g_{\sigma+1,0}, \dots, g_{c,0}$.

(ii) The 3×3 minors of the matrices (33) associated with r_m and $r_{m'}$.

(iii) The equations in Proposition 4.8.

Example 4.24. Let $n \geqslant 3$ and $d, e \geqslant 1$. Choose general polynomials $g_0, \dots, g_n \in \mathbb{C}[y_0, \dots, y_n]$ of degree e, forming a regular sequence. Then

$$\mathcal{R}(Z) \cong \mathbb{C}[x_0, x_1, y_0, \dots, y_n, z_1, \dots, z_d]/\langle x_1 z_1 + g_0, x_1 z_2 + g_1 - x_0 z_1, \dots, g_d - x_0 z_d \rangle.$$

Let S_{Z^+} be the subalgebra of $\mathcal{R}(Z)$ generated by $y_0, \dots, y_n, z_1, \dots, z_d$. By Proposition 5.19, proved in Chapter 5, we have an isomorphism

$$S_{Z^+} \cong \mathbb{C}[y_0, \dots, y_n, z_1 \dots, z_d]/J',$$

where J' is the ideal generated by the equations

$$g_k(z_{k'+1} z_{k''} - z_{k'} z_{k''+1}) - g_{k'}(z_{k+1} z_{k''} - z_k z_{k''+1}) + g_{k''}(z_{k+1} z_{k'} - z_k z_{k'+1}),$$

for $0 \leqslant k < k' < k'' \leqslant d$. The ring S_{Z^+} is the homogeneous coordinate ring of a birational model $Z^+ \subset \mathbb{P}^d_{z_k} \times \mathbb{P}^n_{y_j}$ of Z; compare [28, Section 2].

4.4.3 Examples

Example 4.25. Let $c \geqslant 2$ and $n = c+2$. Take $X' = \mathbb{P}^n$ and $X = \mathbb{P}^1 \times X'$, with Picard group $\operatorname{Pic} X \cong \mathbb{Z}^2$ and Cox ring $\mathcal{R}(X) = \mathbb{C}[x_0, x_1, y_0, \dots, y_n]$ with its usual $\mathbb{Z}^2$-grading. For $1 \leqslant m \leqslant c$, let $(d_m, e_m) \in \operatorname{Pic} X \in \mathbb{Z}^2$ with $d_m \geqslant 0, e_m \geqslant 1$. Let

$$r_m = g_{m,0} x_0^{d_m} + g_{m,1} x_0^{d_m - 1} x_1 + \dots + g_{m,d_m} x_1^{d_m} \in \mathcal{R}(X)_{(d_m, e_m)}$$

define the irreducible hypersurface $V_m \subset X$. Suppose that Z has form $(*)$ with respect to $V_1, \dots, V_c$. Suppose further that

(i) $e_1 + \dots + e_c = c+3$.

(ii) $d_1 = d_2 = 1, \quad d_m = 0$ for $m > 2$.

We can assume none of the (d_m, e_m) are equal $(0,1)$. There are such choices of (d_m, e_m) for $2 \leqslant c \leqslant 5$. The variety Z is a Calabi-Yau threefold.

The map $Z \to \mathbb{P}^n$ given by the second projection factors through

$$Y = \{g_{10} g_{21} - g_{11} g_{20} = g_{30} = \dots = g_{c0} = 0\} \subset \mathbb{P}^n.$$

Let $Z \to \overline{Z} \to Y$ be the Stein factorisation of this map. The map $Z \to \overline{Z}$ is a small contraction, collapsing the $\mathbb{P}^1$ fibres defined over those $e_1^2 e_2^2 e_3 \cdots e_c$ points in Y where $g_{10} = g_{11} = g_{20} = g_{21} = 0$. The map $\overline{Z} \to \mathbb{P}^n$ is a degree one cover, singular on the collapsed $\mathbb{P}^1$ fibres. By Theorem 4.16, the Cox ring of Z is

$$\mathcal{R}(Z) = \mathbb{C}[x_0, x_1, y_0, \dots, y_n, w_1, w_2]/(x_1 w_1 + g_{10}, g_{11} - x_0 w_1, x_1 w_2 + g_{20}, g_{21} - x_0 w_2, g_{30}, \dots, g_{c0}).$$

If $e_1 = e_2$ then the moveable and effective cones of Z coincide, generated by $(-1, e_1)$ and $(1,0)$. If $e_1 \neq e_2$ we may assume that $e_1 < e_2$. The effective cone of Z is generated by $(-1, e_1), (1,0)$ and the moveable cone by $(-1, e_2), (1,0)$. In both cases, we can consider the variety

$$Z^+ = \{g_{20} w_1 - g_{10} w_2 = g_{21} w_1 - g_{11} w_2 = g_{30} = \dots = g_{c0} = 0\} \subset \mathbb{P}^1_{w_k} \times \mathbb{P}^n_{y_j}.$$

The map $Z \to Z^+$ determined by the sections w_1, w_2 is birational, giving the second model of Z corresponding to the cone generated by the y_j and w_k. If $e_1 < e_2$ then the immoveable divisor defined by $w_1 = 0$ is the preimage of the locus $g_{10} = g_{11} = 0$ in Y.

Example 4.26. We take the same setting as in Example 4.25, except with numerical conditions

(i) $e_1 + \dots + e_c = c+3$.

(ii) $d_1 = 2, \quad d_m = 0$ for $m > 1$.

Assuming no $(d_m, e_m) = (0,1)$, there are such choices for $1 \leqslant c \leqslant 4$ defining a Calabi-Yau threefold $Z \subset \mathbb{P}^1 \times \mathbb{P}^n$.

The map $Z \to \mathbb{P}^n$ given by the second projection factors through

$$Y = \{g_{20} = \dots = g_{c0} = 0\} \subset \mathbb{P}^n.$$

First assume that V_1 has full index sequence $\{0, 1, 2\}$, so that $g_{11} \neq 0$. Let $Z \to \overline{Z} \to Y$ be the Stein factorisation of this map. The map $Z \to \overline{Z}$ is a small contraction, collapsing the $\mathbb{P}^1$ fibres defined over those $e_1^3 e_2 \cdots e_c$ points in Y where $g_{10} = g_{11} = g_{12} = 0$. The map $\overline{Z} \to \mathbb{P}^n$ is a 2-to-1 cover, ramified over the divisor

$$\{g_{11}^2 - 4g_{10}g_{12} = 0\} \subset Y$$

and singular on the points corresponding to the collapsed $\mathbb{P}^1$ fibres in Z. By Theorem 4.16, the Cox ring of Z is given by

$$\mathcal{R}(Z) = \mathbb{C}[x_0, x_1, y_0, \ldots, y_n, w_1, w_2]/(x_1w_1 + g_{10}, x_1w_2 + g_{11} - x_0w_1, g_{12} - x_0w_2, g_{30}, \ldots, g_{c0}).$$

The moveable and effective cones of Z coincide, generated by $(-1, e_1)$ and $(1, 0)$. We may consider the variety

$$Z^+ = \{g_{12}w_1^2 - g_{11}w_1w_2 + g_{10}w_1^2 = g_{20} = \cdots = g_{c0} = 0\} \subset \mathbb{P}^1_{w_k} \times \mathbb{P}^n_{y_j},$$

which is isomorphic to Z. The map $Z \to Z^+$ determined by the sections w_1, w_2 is birational but not an isomorphism, giving the second model of Z corresponding to the Mori chamber generated by the $(-1, e_1)$, $(0, 1)$.

If we allow the coefficient g_{11} to deform to zero, so that V_1 has index sequence $\{0, 2\}$, then the resulting variety $Z_0 \subset \mathbb{P}^1 \times \mathbb{P}^n$ is still a Mori Dream Space; its Cox ring is given by Theorem 4.16 as

$$\mathcal{R}(Z_0) \cong \mathbb{C}[x_0, x_1, y_0, \ldots, y_n, w]/(x_1^2w + g_{10}, g_{12} - x_0^2w, g_{20}, \cdots, g_{c0}).$$

The moveable cone of Z_0 is smaller, generated by $(0, 1), (1, 0)$. The effective cone is larger; it is generated by $(-2, e_1), (1, 0)$.

Example 4.27. Let $c \geqslant 1$ and $n \geqslant c + 2$. Take $X' = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$, where $s \geqslant 2$ and each $n_i \geqslant c + 1$ and $n_1 + \cdots n_s = n$ and $X = \mathbb{P}^1 \times X'$. For $1 \leqslant m \leqslant c$, let $(d_m, e_m) \in \operatorname{Pic} X \cong \mathbb{Z} \times \mathbb{Z}^s$, with $d_m \geqslant 0$ and e_m ample (or non-zero semi-ample, as long as we can apply Lefschetz on the subproduct of projective spaces where e_m is ample) and let

$$r_m = g_{m,0}x_0^{d_m} + g_{m,1}x_0^{d_m-1}x_1 + \cdots + g_{m,d_m}x_1^{d_m} \in \mathcal{R}(X)_{(d_m, e_m)}$$

define the irreducible hypersurface $V_m \subset X$, each with full index sequence $\{0, 1, \ldots, d_m\}$. Suppose that Z has form (*) with respect to $V_1, \ldots, V_c$. Suppose further that

(i) $3 \leqslant d_1 + \ldots d_c$.

(ii) $e_1 + \cdots + e_c - (n_1 + 1, \ldots, n_s + 1)$ is positive in each component.

This produces infinitely many classes of general type varieties Z with ample canonical bundle, each a Mori Dream Space by Theorem 4.16.

Example 4.28. Let $c = 2$ and $X = \mathbb{P}^1 \times \mathbb{P}^5$. Then $\operatorname{Pic} X \cong \mathbb{Z}^2$ and $\mathcal{R}(X) = \mathbb{C}[x_0, x_1, y_0, \ldots, y_5]$. Define irreducible hypersurfaces

$$V_m = \{r_m = g_{m0}x_0^2 + g_{m1}x_0x_1 + g_{m2}x_1^2 = 0\}$$

of degree $(2, e_m)$ admitting full index sequence $\{0, 1, 2\}$ for $m = 1, 2$. Suppose $e_2 \geqslant e_1$ and that Z has form $(*)$ with respect to V_1, V_2.

The second projection map $Z \to \mathbb{P}^5$ factors through $Y \subset \mathbb{P}^5$ defined by the equation

$$s = (g_{12}g_{20} - g_{10}g_{22})^2 - (g_{11}g_{22} - g_{12}g_{21})(g_{10}g_{21} - g_{11}g_{20}).$$

Indeed, vanishing of this equation implies that r_1, r_2 share a root. The map $Z \to Y$ is 1-to-1 except over the divisor $Y^- \subset Y$ defined by the vanishing of the 2×2 minors of

$$\begin{pmatrix} g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{pmatrix}, \tag{34}$$

where the map is 2-to-1 since here the roots of r_1, r_2 agree. By Theorem 4.16, Z has Cox ring

$$\mathcal{R}(Z) = \mathcal{R}(X)[w_{11}, w_{12}, w_{21}, w_{22}] \Big/ \left\langle \begin{pmatrix} x_1w_{11} + g_{10}, x_1w_{12} + g_{11} - x_0w_{11}, g_{12} - x_0w_{12} \\ x_1w_{21} + g_{20}, x_1w_{22} + g_{21} - x_0w_{21}, g_{22} - x_0w_{22} \end{pmatrix} \right\rangle.$$

Define, for $m = 1, 2$, the variety $Z_m^+ \subset \mathbb{P}^1_{w_{m,k}} \times \mathbb{P}^5_{y_j}$ by the equations

$$s = g_{m0}w_{m2}^2 - g_{m1}w_{m1}w_{m2} + g_{m2}w_{m1}^2 = 0,$$

and define $Z_m^+ \dashrightarrow Z$ by

$$\begin{array}{rcccl} x_0 & = & -\frac{g_{m0}}{w_{m1}} & = & \frac{-g_{m1}w_{m2}+g_{m2}w_{m1}}{w_{m2}^2} \\ x_1 & = & \frac{g_{m2}}{w_{m2}} & = & \frac{g_{m1}w_{m1}-g_{m0}w_{m2}}{w_{m1}^2} \end{array},$$

defined away from the locus where g_{m0}, g_{m1}, g_{m2} all vanish. This map has birational inverse $Z \dashrightarrow Z_m^+$ given by the sections w_{m1}, w_{m2}, defined once more away from the locus $g_{m0} = g_{m1} = g_{m2} = 0$. These birational maps can interpreted geometrically; they simply swap the roots $[x_0, x_1]$ and $[w_{m2} : -w_{m1}]$ of r_m.

The variety Z_2^+ gives the birational model of Z corresponding to the Mori chamber generated by $(-1, e_2)$, $(1, 0)$. If $e_1 = e_2$ then the birational map $Z_1^+ \dashrightarrow Z_2^+$ is the isomorphism transposing the roots of r_1 and r_2 that are not shared. If $e_1 < e_2$, then Z_1^+ gives a third minimal model corresponding to the Mori chamber generated by $(-1, e_1)$, $(-1, e_2)$; indeed we may consider the image of $Z_1^+ \subset \mathbb{P}^1_{w_{1k}} \times \mathbb{P}^n_{y_j}$ under a suitable Segre embedding.

4.5 Comments

Remark 4.29. All arguments in this Chapter work in the following more general setting. Let $c \geqslant 1$ and $X = \mathbb{P}^1 \times T$, where T is a Mori Dream Space of dimension $n \geqslant c+2$. Assume further that T has free abelian Picard group and that $H^i(T, \mathcal{O}_T(D)) = 0$ for all line bundles $D \in \operatorname{Pic} T$ and $i = 1, \ldots, c$. Definitions 4.1–4.2 can be generalised to this setting: we let e, e_m be ample line bundles in $\operatorname{Pic} T$ and the g_i are homogeneous sections of the corresponding line bundles. All proofs remain the same. In particular, the Main Theorem holds with the $\mathbb{P}^n$ factor of X replaced with various products of projective spaces or Grassmannians.

Remark 4.30. Let $Z \subset \mathbb{P}^1 \times T$ be of the form $(*)$ (with the generalised definition as explained in the previous Remark), where T is a Mori Dream Space with free abelian Picard group. If $H^i(T, \mathcal{O}_T(e))$ does not vanish for all line bundles $e \in \operatorname{Pic} T$ and $1 \leqslant i \leqslant c$, then $H^i(X, \mathcal{O}_X(d,e))$ may not vanish for all $(d,e) \in \operatorname{Pic} X$ and all $i = 2, \dots, c$. As explained in Remark 3.15, we no longer obtain the full Cox ring, rather a subalgebra. In Chapter 9 we revisit this setting for the ambient space $(\mathbb{P}^1)^k \times \mathbb{P}^n$ with $k \geqslant 2$.

5 On kernels of Toeplitz style matrices with polynomial entries

The purpose of this Chapter is to prove the remaining algebraic statement that shows the map ψ in Theorem 4.16 is surjective. This is done via a systematic solving of the kernels of the matrices in each degree a that appear.

We recall the definitions first mentioned in Section 2.2.

Definition 5.1. Let S be a ring and $a \geqslant 1, d \geqslant 0$. The banded Toeplitz matrix of degree a associated to a sequence $g_0, \ldots, g_d \in S$ is the $(a-1) \times (a+d-1)$ matrix $(\varepsilon_{s,t})$ so that $\varepsilon_{s,t} = g_i$ if $t - s = i$ and $\varepsilon_{s,t}$ is zero otherwise. By convention the degree 1 matrix is the zero matrix. If

$$r = g_0 x_0^d + \cdots + g_d x_1^d \in S[x_0, x_1]$$

is a general homogeneous degree d polynomial in x_0, x_1 with coefficients in S, then we define, for $a \geqslant 1$, the matrix $A_{a,r}$ to be the degree a banded Toeplitz matrix associated to the sequence $g_0, \ldots, g_d \in S$:

$$A_{a,r} = \begin{pmatrix} g_0 & g_1 & g_2 & \cdots & g_d & 0 & 0 & \cdots\cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & g_0 & g_1 & \cdots & g_{d-1} & g_d & 0 & \cdots\cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ & \vdots & & \ddots & & \vdots & & \ddots & \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots\cdots & 0 & g_0 & g_1 & \cdots & g_d & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots\cdots & 0 & 0 & g_0 & \cdots & g_{d-1} & g_d \end{pmatrix}.$$

We will be interested in the case that $S = \mathbb{C}[y_0, \ldots, y_n]$, the Cox ring of $\mathbb{P}^n$, and the kernels of banded Toeplitz matrices associated with an equation

$$r = g_0 x_0^d + \cdots + g_d x_1^d \in S[x_0, x_1] \cong \mathcal{R}(\mathbb{P}^1 \times \mathbb{P}^n),$$

defining an normal irreducible hypersurface in $\mathbb{P}^1 \times \mathbb{P}^n$ for some $d \geqslant 0$, where the g_i are homogeneous polynomials in S of some common degree $e \geqslant 1$. In fact, we consider a more general form of matrix.

Definition 5.2. Let $S = \mathbb{C}[y_0, \ldots, y_n]$ and $c \geqslant 1, d_1, \ldots, d_c \geqslant 0$. For each $m = 1, \ldots, c$, let

$$r_m = g_{m,0} x_0^{d_m} + \cdots + g_{m,c} x_1^{d_m} \in S[x_0, x_1],$$

where $g_{m,0}, \ldots, g_{m,c}$ are homogeneous polynomials in S of some degree $e_m \geqslant 1$. Define to this data the matrix

$$A_a = (A_{a,r_1} | \cdots | A_{a,r_c})$$

for each $a \geqslant 1$, which may be viewed as a map of free S-modules

$$S^{a+d_1-1} \times \cdots \times S^{a+d_c-1} \to S^{a-1}.$$

Define the trivial spaces $\eta_a \subset \ker A_a$ to be generated by columns of the matrices

$$A^*_{a,(m,m')} = (0|\cdots|0|A^t_{m',a+d_m}|0|\cdots|0| - A^t_{m,a+d_{m'}}|0|\cdots|0)^t$$

for each $1 \leqslant m < m' \leqslant c$. Then set $K_a = (\ker A_a)/\eta_a$ and $K = \bigoplus_{a \geqslant 1} K_a$, which may be viewed as a graded S-module.

5.1 Universal case

5.1.1 The set-up

Fix $c \geqslant 1$. For each $1 \leqslant m \leqslant c$, let $d_m \geqslant 0$ and fix an index sequence $\{i_{m,0}, \ldots, i_{m,\tau_m}\}$ of length τ_m, with $0 \leqslant \tau_m \leqslant d_m$. Recall that by our conventions, $\tau_m = 0$ if and only if $d_m = 0$. We associate further this data to an element of the set

$$\mathcal{K} = \{(c; d_1, \ldots, d_c) : c \geqslant 1, d_1, \ldots, d_c \geqslant 0\},$$

which we equip with the lexicographic ordering $\prec$.

We define the free $\mathbb{C}$-algebra

$$S = \mathbb{C}[Y_{i_{1,0}}, \ldots, Y_{i_{1,\tau_1}}, \ldots, Y_{i_{c,0}}, \ldots, Y_{i_{c,\tau_c}}].$$

For each $m = 1, \ldots, c$ and $a \geqslant 1$, define a map of free S-modules

$$A_{m,a} : S^{a+d_m-1} \to S^{a-1}$$

by the $(a-1) \times (a + d_m - 1)$ matrix with (s,t) entry equal to $Y_{i_{m,k}}$ if $t - s = i_{m,k}$ and 0 otherwise. Then define

$$A_a = (A_{1,a}| \cdots |A_{m,a}) : S^{a+d_1-1} \times \cdots \times S^{a+d_m-1} \to S^{a-1}.$$

For each $1 \leqslant m < m' \leqslant c$, let $\eta_{m,m'}$ be the submodule of $S^{a+d_1-1} \times \cdots \times S^{a+d_m-1}$ generated by the $(a + d_m + d_{m'} - 1)$ columns of the matrix

$$A^*_{a,(m,m')} = (0| \cdots |0|A^t_{m',a+d_m}|0| \cdots |0| - A^t_{m,a+d_{m'}}|0| \cdots |0)^t.$$

Let $\eta = \sum_{1 \leqslant m < m' \leqslant c} \eta_{m,m'}$ be the sum of these submodules. The calculations

$$A_{m,a}A^*_{m',a+d_{m'}} - A_{m',a}A^*_{m',a+d_m}, \quad 1 \leqslant m < m' \leqslant c,$$

imply that $\eta \subset \ker A_a$.

For each $a \geqslant 1$, we set $K_a = (\ker A_a)/\eta$ and $K = \bigoplus_{a \geqslant 1} K_a$, which we consider as a graded S-module. We note that A_1 is the zero map since $S^0 = 0$ by convention. In particular

$$K_1 \cong S^{d_1} \times \cdots \times S^{d_c}.$$

Example 5.3. Suppose that $c = 2$, $d_1 = 5$ with index sequence $\{0, 3, 5\}$, and $d_2 = 1$ with index sequence $\{0, 1\}$. Then A_3 is given by

$$\begin{pmatrix} Y_{1,0} & 0 & 0 & Y_{1,3} & 0 & Y_{1,5} & 0 & Y_{2,0} & Y_{2,1} & 0 \\ 0 & Y_{1,0} & 0 & 0 & Y_{1,3} & 0 & Y_{1,5} & 0 & Y_{2,0} & Y_{2,1} \end{pmatrix}.$$

To describe the structure of K we mostly employ algebraic methods. The following auxiliary variety will however be useful in some steps.

Definition 5.4. Associated to the data $(c; d_1, \dots, d_c) \in \mathcal{K}$ and index sequences $\{i_{m,0}, \dots, i_{m,o_c}\}$ of d_m for each $m = 1, \dots, c$ is the variety $Z' \subset \mathbb{P}^1_{x_i} \times \mathbb{P}^N_{Y_{i_{m,k}}}$ defined by the c equations

$$r_m = Y_{i_{m,0}} x_0^{d_m} + \cdots + Y_{i_{m,d_m}} x_1^{d_m}, \quad 1 \leqslant m \leqslant c.$$

Following the arguments of Chapter 4, the Cox ring of Z' contains sections $w_{m,k}$ satisfying equations as in Theorem 4.16. By the equations (24), the Cox ring $\mathcal{R}(Z')$ also contains the sections $z_{m,k'}$ as given in Proposition 4.8(i). We also have an isomorphism

$$K \cong \bigoplus_{-a \leqslant -1} \bigoplus_{b \in \mathbb{Z}} \mathcal{R}(Z')_{(-a,b)} \tag{35}$$

as graded S-modules via Lemma 4.5 and Proposition 4.6(ii), where the degree of each $Y_{i_{m,k}}$ in the Cox ring is equal to $(0, 1)$.

5.1.2 Operators on K

We define operators on the space K, which we will see correspond to multiplication by sections in the Cox ring of Z'.

Definition 5.5. Fix $a \geqslant 1$.

(i) Define the operators $\tilde{x}_0, \tilde{x}_1 : K_{a+1} \to K_a$ as follows. Suppose that

$$u = (f_1; \dots; f_c) = \left((f_{1,l})_{l=1}^{a+d_1}; \dots; (f_{c,l})_{l=1}^{a+d_c} \right) \in K_{a+1},$$

then

$$\tilde{x}_0 u = \left((f_{1,l})_{l=1}^{a+d_1-1}; \dots; (f_{c,l})_{l=1}^{a+d_1-1} \right), \quad \tilde{x}_1 u = \left((f_{1,l})_{l=2}^{a+d_1}; \dots; (f_{c,l})_{l=2}^{a+d_1} \right).$$

That is, $\tilde{x}_0$, respectively $\tilde{x}_1$, is truncation of each of $f_1, \dots, f_c$ by one entry on the right, respectively left.

(ii) For each $1 \leqslant m \leqslant c$ and $1 \leqslant k' \leqslant d_m$ we define $\tilde{z}_{m,k'} : K_a \to K_{a+1}$ as follows. Suppose that $f \in K_a$. Then $\tilde{z}_{m,k'} f$ is given by $f' \in K_{a+1}$, where

$$f'_{m',k''} = \begin{cases} f_{m',k''} Y_{m,k''} + \cdots + f_{m',k''+d_m-k'} Y_{m,d_m} & \text{if} \quad 1 \leqslant m' \leqslant c, \quad 1 \leqslant k'' \leqslant a + d_{m'} - d_m + k' - 1, \\[2ex] \begin{array}{c} -f_{1,k''-k'} Y_{1,0} - \cdots - f_{1,a+d_1-1} Y_{1,d_1+a+k'-k''-1} - \dots \\ -f_{m,k''-k'} Y_{m,0} - \cdots - f_{m,k''-1} Y_{m,k'-1} - \dots \\ -f_{c,k''-k'} Y_{c,0} - \cdots - f_{c,a+d_c-1} Y_{c,d_c+a+k'-k''-1} \end{array} & \text{if} \quad m' = m, \quad a + k' \leqslant k'' \leqslant a + d_m, \\[2ex] f_{m',k''} Y_{m,k'} + \cdots + f_{m',d_m+k''-k'-1} Y_{m,d_m-1} & \text{if} \quad m' \neq m, \quad a + d_{m'} - d_m + k' \leqslant k'' \leqslant a + d_{m'}, \end{cases}$$

where Y variables of undefined index are set to be zero.

(iii) For each $1 \leqslant m \leqslant c$ and $1 \leqslant k \leqslant \tau_m$, we define $\tilde{w}_{m,k} : K_a \to K_{a+i_{m,k}-i_{m,k-1}}$ by stipulating that

$$\tilde{x}_0^{j-1} \tilde{x}_1^{i_{m,k}-i_{m,k-1}-j} \tilde{w}_{m,k} = \tilde{z}_{i_{m,k-1}+j}$$

for each $1 \leqslant j \leqslant i_{m,k} - i_{m,k-1}$.

Proposition 5.6. *The operators $\tilde{x}_i, \tilde{z}_k, \tilde{w}_k \colon K \to K$ in Definition 5.5 are well defined.*

Proof. Consider the auxiliary variety Z' in Definition 5.4. The operators in Definition 5.5 correspond to multiplication by the corresponding sections $x_i, z_{m,k'}, w_{m,k} \in \mathcal{R}(Z')$ by Propositions 4.13–4.14 under the identification (35), and are thus well defined. □

Let

$$U = S[X_0, X_1, W_{1,1}, \dots, W_{1,\tau_1}, \dots\dots, W_{c,1}, \dots, W_{c,\tau_c}]$$

be the free graded S-algebra with $\deg(X_i) = 1$ and $\deg(W_{m,k}) = i_{m,k-1} - i_{m,k}$ for relevant i, m, k. For a general graded S-module U', we denote by $U'_{\leqslant -1}$ the graded submodule $\bigoplus_{a \leqslant -1} U'_a$. It is also useful to define the elements

$$Z_{i_{m,k-1}+j} = X_0^{j-1} X_1^{i_{m,k}-i_{m,k-1}-j} W_{m,k} \in U_{-1}, \tag{36}$$

for $1 \leqslant m \leqslant c$, $1 \leqslant k \leqslant \tau_m$ and $1 \leqslant j \leqslant i_{m,k} - i_{m,k-1}$.

Proposition 5.7. *There is a map of S-modules*

$$\psi : U_{\leqslant -1} \to K$$

defined as follows. For each $1 \leqslant m \leqslant c$ and $1 \leqslant k \leqslant \tau_m$, set $\psi(W_{m,k}) = f \in K_{i_{m,k}-i_{m,k-1}}$, where $f_{m',l} = 1$ if $(m', l) = (m, i_{m,k})$ and all other $f_{m',l}$ equal 0. The grading on U implies that any monomial M in $U_{\leqslant -1}$ divides by $W_{m,k}$ for some m, k. If $M = X_0^{\alpha_0} X_1^{\alpha_1} \prod_{(m,k)} W_{m,k}^{\beta_{m,k}}$ with say $\beta_{m',k'} > 0$, then we set

$$\psi(M) = \tilde{x}_0^{\alpha_0} \tilde{x}_1^{\alpha_1} \tilde{w}_{1,1}^{\beta_{1,1}} \dots \tilde{w}_{m',k'}^{\beta_{m',k'}-1} \dots \tilde{w}_{m,d_m}^{\beta_{m,d_m}} \psi(z_{l,k}).$$

Furthermore, the elements $\psi(Z_{m,k'}) \in K_1$, for $1 \leqslant m \leqslant c$ and $1 \leqslant k' \leqslant d_m$, give the standard basis of $K_1 \cong S^{d_1} \times \dots \times S^{d_c}$.

Proof. To argue that ψ is well defined, we need only argue the operators $\tilde{x}_i$ and $\tilde{w}_{m,k}$ all commute with each other, as well as establish that $\tilde{w}_{m,k}\psi(W_{m',k'}) = \tilde{w}_{m',k'}\psi(W_{m,k})$ for each pair $(m,k), (m',k')$.

As above, the operators correspond to multiplication of sections in the Cox ring $\mathcal{R}(Z')$ of the auxiliary hypersurface Z'. Since multiplication of these sections is commutative, all of the operators must commute with each other, and the extra property $\tilde{w}_{m,k}\psi(W_{m',k'}) = \tilde{w}_{m',k'}\psi(W_{m,k})$ for each pair $(m,k), (m',k')$ also holds.

The final statement is easily verified. □

5.2 The case $c = 1$ with full index sequence

We briefly focus on the case $c = 1$ with full index sequence. That is let $d_1 = \tau_1 = d \geqslant 0$. We may write $Y_k = Y_{1,k}$ for each $k = 0, \dots, d$ and thus

$$Z' = \{Y_0 x_0^d + \dots + Y_d x_1^d = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^d.$$

5.2.1 Commutativity using just algebra

In this case, we can show that the operators $\tilde{z}_k$ commute directly from the formula, offering an alternative proof to Proposition 5.9. Whilst this argument is lengthier, it avoids using Cox rings, and thus works over any ring.

Example 5.8. One explicitly checks the formula

$$\tilde{z}_k(z_l) = \big(Y_{k+l-1}, Y_{k+l-2}, \ldots, Y_{\max\{k,l\}}, 0, \ldots, 0, -Y_{\min\{k,l\}-1}, \ldots, -Y_{k+l-d}, -Y_{k+l-d-1}\big) \in K_2, \tag{37}$$

for each $1 \leqslant k, l \leqslant d$; variables of undefined indices are set to be zero. This formula is symmetric in (k, l), thus $\tilde{z}_k(z_l) = \tilde{z}_l(z_k) \in K_2$. We can identify this element of K_2 with the degree 2 monomial $z_k z_l$. Under this identification, these $\binom{d+1}{2}$ degree two monomials in the z_k give us a set of S-generators of K_2.

For $a > 1$, the explicit expressions for these elements are getting more complicated. For example, for $d = 3$ the entries of $\tilde{z}_1\tilde{z}_1(z_2), \tilde{z}_1\tilde{z}_2(z_3), \tilde{z}_2\tilde{z}_2(z_2)$ are given respectively by the columns of the following matrix:

$$\begin{pmatrix} Y_1Y_2 - Y_0Y_3 & Y_2Y_3 & 2Y_2Y_3 \\ -Y_0Y_2 & 0 & Y_2^2 - Y_1Y_3 \\ 0 & -Y_0Y_3 & -Y_0Y_3 - Y_1Y_2 \\ Y_0^2 & 0 & -Y_0Y_2 + Y_1^2 \\ 0 & Y_0Y_1 & 2Y_0Y_1 \end{pmatrix}$$

Proposition 5.9. *Suppose that $1 \leqslant k, l \leqslant d$ and $a \geqslant 1$. Then*

$$\tilde{z}_k\tilde{z}_l = \tilde{z}_l\tilde{z}_k.$$

Whilst elementary in formulation, proving Proposition 5.9 from the definition is not easy. We start with some preliminary lemmas.

Lemma 5.10. *If $w = (f_m)_{m=1}^{a+d-1} \in K_a$, then w is uniquely determined by any d consecutive entries. More precisely, if $w' = (f'_m)_{m=1}^{a+d-1}$ is also in K_a, and there exists m with $1 \leqslant m \leqslant a$ such that $f'_{m+n} = f_{m+n}$ for $n = 0, 1, \ldots, d-1$, then $w = w'$.*

Proof. This is immediate using the equations defined by w being in K_a. □

We will prove Proposition 5.9 by induction on d. The following notations will be helpful.

Definition 5.11. Assume $d > 1$. Let

$$\bar{A}_a : R[Y_0, \ldots, Y_{d-1}]^{a+d-2} \to R[Y_0, \ldots, Y_{d-1}]^{a-1}$$

be the R-module maps as defined above, but for the case of d variables rather than $d+1$ variables. Denote $\bar{K}_a = \ker \bar{A}_a$. Define the operators $\tilde{\zeta}_l \colon \bar{K}_a \to \bar{K}_{a+1}$ for $l = 1, \ldots, d-1$ as in Definition 5.5 with parameter $d-1$.

Lemma 5.12. *Suppose $d > 1$ and consider the element $\tilde{z}_{l_a} \cdots \tilde{z}_{l_1} z_{l_0} \in K_{a+1}$, where no l_i is equal d. Setting $Y_d = 0$ in the resulting element of S^{a+d} and discarding the final component gives us an element of $R[Y_0, \ldots, Y_{d-1}]^{a+d-1}$. This coincides with $\tilde{\zeta}_{l_a} \cdots \tilde{\zeta}_{l_1}\zeta_{l_0} \in \bar{K}_{a+1}$. Similarly, consider an element $\tilde{z}_{l_a} \cdots \tilde{z}_{l_1} z_{l_0} \in K_{a+1}$, where no l_i is equal 1. Setting $Y_0 = 0$ in the corresponding element of S^{a+d} and discarding the first component gives us an element of $R[Y_1, \ldots, Y_d]^{a+d-1}$. Reducing the index of each Y variable by one, we obtain precisely $\tilde{\zeta}_{l_a-1} \cdots \tilde{\zeta}_{l_1-1}\zeta_{l_0-1} \in \bar{K}_{a+1}$.*

Proof. These statements follow immediately from the definitions. □

Proof of Proposition 5.9. We use induction in d. If $d = 1$, then there is only one operator and the result is trivial. Let $d > 1$, then in the notation introduced in Definition 5.11, the induction hypothesis is that the corresponding result holds for the operators $\tilde{\zeta}_l$. Let $a \geqslant 1$ and define an R-module map ${}^{-}(\cdot) : K_a \to K_1$ sending an element w to ${}^{-}w$, obtained by discarding all but the first d entries. Note that for any $l = 1, \ldots, d$, the formula for the first d entries of $\tilde{z}(w)$ depend only on the entries of ${}^{-}w$ and thus

$$ {}^{-}(\tilde{z}_l w) = \tilde{z}_l({}^{-}w), \quad w \in K_a, l = 1, \ldots, d. $$

By Lemma 5.10, to prove the Proposition it suffices to show that

$$ {}^{-}\left(\tilde{z}_k {}^{-}\left(\tilde{z}_l(z_n)\right)\right) = {}^{-}\left(\tilde{z}_l {}^{-}\left(\tilde{z}_k(z_n)\right)\right) \tag{38} $$

for all $1 \leqslant k, l, n \leqslant d$. Note that ${}^{-}(\cdot) \circ z_d$ is multiplication by Y_d, thus

$$ {}^{-}\left(\tilde{z}_d {}^{-}\left(\tilde{z}_l(z_n)\right)\right) = Y_d {}^{-}\left(\tilde{z}_l(z_n)\right) = {}^{-}\left(\tilde{z}_l(Y_d z_n)\right) = {}^{-}\left(\tilde{z}_l {}^{-}\left(\tilde{z}_d(z_n)\right)\right). $$

In a similar fashion, by using the map $(\cdot)^{-} : K_a \to K_1$ taking an element to its last d entries and noticing that $(\cdot)^{-} \circ z_1$ is multiplication by $-Y_0$, we get

$$ {}^{-}\left(\tilde{z}_1 {}^{-}\left(\tilde{z}_l(z_n)\right)\right) = {}^{-}\left(\tilde{z}_l {}^{-}\left(\tilde{z}_1(z_n)\right)\right). $$

We are left only needing to show (38) is true for $2 \leqslant k, l, n \leqslant d-1$, so assume we have $2 \leqslant k, l, n \leqslant d-1$. We now use the $\tilde{\zeta}_l$ operators. By induction hypothesis the $(d+1)$-tuple

$$ \tilde{\zeta}_k \tilde{\zeta}_l \zeta_n - \tilde{\zeta}_l \tilde{\zeta}_k \zeta_n \in R[Y_0, \ldots, Y_{d-1}]^{d+1} $$

is equal to zero and by Lemma 5.12 it is obtained from

$$ \tilde{z}_k \tilde{z}_l z_n - \tilde{z}_l \tilde{z}_k z_n $$

by discarding the last entry and setting $Y_d = 0$. Thus the first $d+1$ entries of $\tilde{z}_k \tilde{z}_l z_n - \tilde{z}_l \tilde{z}_k z_n$ divide by Y_d. Similarly, we obtain

$$ \tilde{\zeta}_{k-1} \tilde{\zeta}_{l-1} \zeta_{n-1} - \tilde{\zeta}_{l-1} \tilde{\zeta}_{k-1} \zeta_{n-1} $$

is equal to zero and can be obtained from

$$ \tilde{z}_k \tilde{z}_l z_n - \tilde{z}_l \tilde{z}_k z_n $$

by discarding the first entry, setting $Y_0 = 0$, and decreasing the index of each variable by one. The last $d+1$ terms of $\tilde{z}_k \tilde{z}_l z_n - \tilde{z}_l \tilde{z}_k z_n$ are thus divisible by Y_0. We conclude that

$$ \tilde{z}_k \tilde{z}_l z_n - \tilde{z}_l \tilde{z}_k z_n = (\mu_0 Y_d, \lambda_1 Y_0 Y_d, \ldots, \lambda_d Y_0 Y_d, \mu_1 Y_0), $$

where for degree reasons the λ_j are scalars and the μ_i are linear forms in the Y_j. As this is an element of K_3, we get

$$\mu_0 = -\lambda_1 Y_1 - \cdots - \lambda_d Y_d.$$

If we can show $\mu_0 = 0$, then $\lambda_1 = \cdots = \lambda_d = 0$. Then the first d entries of $\tilde{z}_k\tilde{z}_l z_n - \tilde{z}_l\tilde{z}_k z_n$ are zero, and we would be done. Therefore we have reduced proving Proposition 5.9 to proving that the first entries of $\tilde{z}_k\tilde{z}_l z_n$ and $\tilde{z}_l\tilde{z}_k z_n$ agree for each $2 \leqslant k, l, n \leqslant d-1$. This can be shown with direct calculation, which we now perform.

Firstly, if $1 \leqslant k \leqslant l \leqslant n \leqslant d$, then using (37), the first component of $\tilde{z}_k\tilde{z}_l z_n$ is equal to

$$\sum_{j=k}^{k+l-1} Y_{k+l+n-1-j}Y_j - \sum_{j=k+n}^{d+k-1} Y_{k+l+n-1-j}Y_j.$$

Note that $k+n > k+l-1$ so there are no identical terms across these two sums. The first component of $\tilde{z}_l\tilde{z}_k z_n$ is equal to

$$\sum_{j=l}^{k+l-1} Y_{k+l+n-1-j}Y_j - \sum_{j=l+n}^{d+l-1} Y_{k+l+n-1-j}Y_j.$$

Since Y_j is not defined and thus zero for $j \geqslant d+l$ and the sums over $j = k, \ldots, l-1$ and $j = n+k, \ldots, n+l-1$ cancel, we see that the above two expressions are equal. It remains to show that $\tilde{z}_n\tilde{z}_l z_k$ has first entry also equal to this expression, which can be checked similarly. We do not need to check the remaining permutations, as we know $\tilde{z}_k z_l = \tilde{z}_l z_k$ already. □

5.3 Proof of Main Theorem 2.8

To prove Main Theorem 2.8, we first prove it in the universal case, where the non-zero polynomial coefficients $g_{i_{m,k}}$ are degree one formal variables $Y_{i_{m,k}}$.

5.3.1 Universal case

Theorem 5.13. *Let $J \lhd U$ be the homogeneous ideal generated by the equations*

$$X_1^{i_{m,k+1}-i_{m,k}} W_{m,k+1} + Y_{m,i_{m,k}} - X_0^{i_{m,k}-i_{m,k-1}} W_{m,i_{m,k}}, \quad 1 \leqslant m \leqslant c, 0 \leqslant k \leqslant \tau_m.$$

Then $\psi : U_{\leqslant -1} \to K$ is surjective with kernel $\ker\psi = J_{\leqslant -1}$. In particular there is an isomorphism of S-modules $K \cong (U/J)_{\leqslant -1}$.

We prove this by induction in the data $(c; d_1, \ldots, d_c)$ under the ordering $\prec$ on $\mathcal{K}$. The base case is with $(1;0) \in \mathcal{K}$, where the result is trivial since then $U_{\leqslant -1} = 0 = K$.

Definition 5.14. Suppose first that $d_c \geqslant 1$. Then we let

$$\bar{S} = \mathbb{C}[Y_{1,i_{1,0}}, \ldots, Y_{1,i_{1,\tau_1}}, \ldots, Y_{c,i_{c,0}}, \ldots, Y_{c,i_{c,\tau_c-1}}].$$

If instead $d_c = 0$, let

$$\bar{S} = \mathbb{C}[Y_{1,i_{1,0}}, \ldots, Y_{1,i_{1,\tau_1}}, \ldots, Y_{c-1,i_{c-1,0}}, \ldots, Y_{c-1,i_{c-1,\tau_{c-1}}}].$$

In either case let

$$\bar{A}_a : \bar{S}^{a+d_1-1} \times \dots \bar{S}^{a+d_c-2} \to \bar{S}^{a-1}.$$

be the S-module maps analogous to A_a, noting that the last factor in the domain is to be set to zero if d_c is equal zero. Let $\bar{\eta}$ be analogous to η, $\bar{K}_a = \ker \bar{A}_a/\bar{\eta}$ and $\bar{K} = \bigoplus_{a\leqslant -1} \bar{K}_a$. Define the operators $\bar{\zeta}_{m,k} \colon \bar{K}_a \to \bar{K}_{a+i_{m,k}-i_{m,k-1}}$ as in Definition 5.5. Finally define the ring $\bar{U} = S[X_0, X_1, \bar{\zeta}_{m,k}]$, the ideal $\bar{J} \lhd \bar{U}$ and the map $\bar{\psi} : \bar{U} \to \bar{K}$ analogously to U, J and ψ.

Lemma 5.15. *Suppose that $M = X_0^{\alpha_0} X_1^{\alpha_1} \prod_{(m,k)} W_{m,k}^{\beta_{m,k}}$ in $U_{\leqslant -1}$ is such that $\beta_{c,\tau_c} = 0$. Setting $Y_{c,i_{c,\tau_c}} = 0$ in $\psi(M)$ and discarding the components indexed by (c,l) with $i_{c,\tau_c-1} < l \leqslant i_{c,\tau_c}$ gives us an element of $\bar{K}$. This coincides with $\bar{\psi}(\bar{M})$, where $\bar{M} = X_0^{\alpha_0} X_1^{\alpha_1} \prod_{(m,k)} \bar{\zeta}_{m,k}^{\beta_{m,k}}$*

Proof. This follows immediately from the definitions. ☐

Lemma 5.16. *Suppose that $d_c \geqslant 1$ and*

$$f = (f_1; \dots; f_c) = \left((f_{1,l})_{l=1}^{a+d_1-1}; \cdots; (f_{c,l})_{l=1}^{a+d_c-1}\right) \in K_a$$

for $a \geqslant 1$. If $a > i_{c,\tau_c} - i_{c,\tau_c-1}$ then f is determined by $f_1, \dots, f_{c-1}$ and the first $a + i_{\tau_c-1} - 1$ entries of f_c. If $a \leqslant i_{\tau_c} - i_{\tau_c-1}$ then f is determined by $f_1, \dots, f_{c-1}$ and the first $a + i_{\tau_c-1} - 1$ entries of f_c up to free choices of the entries $f_{c,a+i_{\tau_c-1}}, \dots, f_{c,i_{\tau_c}}$. If $d_c = 0$, then an $f \in K_a$ is determined by $f_1, \dots, f_{c-1}$.

Proof. Suppose $d_c \geqslant 1$ and $f_1, \dots, f_{c-1}$ along with the first $a + i_{c,\tau_c-1} - 1$ entries of f_c of an $f \in K_a$ are known. If $a > i_{c,\tau_c} - i_{c,\tau_c-1}$, then the first $i_{c,\tau_c} - i_{c,\tau_c-1}$ rows of A_a give equations determining the remaining entries. If $a \leqslant i_{c,\tau_c} - i_{c,\tau_c-1}$, then the rows of A_a determine each of the last $a-1$ entries in f and the $a + i_{c,\tau_c-1}, \dots, i_{\tau_c}$ columns of A_a are zero columns, giving the remaining free choices. If $d_c = 0$, then each row of of A_a gives an equation determining each entry of f_c from $f_1, \dots, f_{c-1}$. ☐

Proof of Theorem 5.13. We perform induction in $\mathcal{K}$, where the result is established for $(1;0) \in \mathcal{K}$. Suppose that $(c; d_1, \dots, d_c) \succ (1;0)$ in $\mathcal{K}$ and that $f \in K_a$ is given.

Assume first that $d_c \geqslant 1$, so that the problem with $\bar{S}$ as defined in Definition 5.14 has data

$$(c; d_1, \dots, d_{c-1}, i_{c,\tau_c-1}) \prec (c; d_1, \dots, d_c),$$

where we may use the inductive hypothesis. Setting $Y_{c,i_{c,\tau_c}} = 0$ in each entry of f and discarding the last $i_{c,\tau_c} - i_{c,\tau_c-1}$ entries of f_c we obtain an element $\bar{f} \in \bar{K}_a$. By the induction hypothesis, we may find

$$p(X_0, X_1, \bar{\zeta}_{1,1}, \dots, \bar{\zeta}_{1,\tau_1}, \dots\dots, \bar{\zeta}_{c,\tau_c-1}) \in \bar{U}_{\leqslant -1}$$

with $\bar{\psi}(p) = \bar{f}$. By Lemma 5.15, the element

$$u = f - \psi(p(X_0, X_1, W_{1,1}, \dots, W_{1,\tau_1}, \dots\dots, W_{c,\tau_c-1})) \in K_a$$

has the form

$$(f_1' Y_{c,i_{c,\tau_c}}; \cdots; f_{c-1}' Y_{c,i_{c,\tau_c}}; f_{c,1}' Y_{c,i_{c,\tau_c}}, \dots, f_{c,a+i_{c,\tau_c-1}}' Y_{c,i_{c,\tau_c}}, F_{a+i_{c,\tau_c-1}}, \dots, F_{a+d_c-1})$$

for some $f_m' \in S^{a+d_m-1}$ and $F_k \in S$. Write $f_m' = (f_{m,l})_{l=1}^{a+d_m-1}$ as usual.

If $a \leqslant i_{c,\tau_c} - i_{c,\tau_c-1}$, consider the element

$$v = \psi(f'_{1,1}Z_{1,1} + \cdots + f'_{1,d_1}Z_{1,d_1} + \cdots\cdots + f'_{c,a+i_{c,\tau_c-1}-1}Z_{c,a+i_{c,\tau_c-1}-1}) \in K_1.$$

By considering the definition of $\tilde{w}_{c,\tau_c}$, we establish that

$$u' = u - \tilde{x}_0^{i_{c,\tau_c}-i_{c,\tau_c-1}}\tilde{w}_{c,\tau_c}(v) = \left((u'_{1,l})_{l=1}^{a+d_1-1}; \cdots ; (u'_{c,l})_{l=1}^{a+d_c-1}\right),$$

where $u'_{m,l} = 0$ if $1 \leqslant m \leqslant c-1$ and $1 \leqslant l \leqslant d_m$ or if $m = c$ and $1 \leqslant l \leqslant a + i_{c,\tau_c-1} - 1$. Furthermore, for $1 \leqslant m \leqslant c-1$ and $d_m < l \leqslant a + d_m - 1$ we have that $Y_{c,i_{c,\tau_c}}$ divides $u'_{m,l}$. Since $a \leqslant i_{c,\tau_c} - i_{c,\tau_c-1}$, we can add an element of η, thus not changing u' in K, and assume further that $u_{m,l} = 0$ for all $1 \leqslant m \leqslant c-1$ and $1 \leqslant l \leqslant a + d_m - 1$. Since u' is in the image of ψ if and only if f is, we can replace f with u'. Now $f \in K_a$ has the form

$$p = (0; \cdots ; 0; 0, \ldots, 0, f_{c,a+i_{c,\tau_c-1}}, \ldots, f_{a+d_c-1})$$

By Lemma 5.16, we have $f_{c,l} = 0$ for $i_{c,\tau_c-1} < l \leqslant a + d_c - 1$. Then we have

$$f = \psi\left((f_{c,a+i_{c,\tau_c-1}}X_1^{i_{c,\tau_c}-i_{c,\tau_c-1}-a} + \cdots + f_{c,i_{c,\tau_c}}X_0^{i_{c,\tau_c}-i_{c,\tau_c-1}-a})W_{c,\tau_c}\right),$$

in the image of ψ.

Now suppose that $a > i_{c,\tau_c} - i_{c,\tau_c-1}$. Since $Y_{c,i_{c,\tau_c}}$ is not a zero divisor in S, we have

$$f' = \left((f_{1,l})_{l=1}^{a-i_{c,\tau_c}+i_{c,\tau_c-1}+d_1-1}; \cdots ; (f_{c,l})_{l=1}^{a-i_{c,\tau_c}+i_{c,\tau_c-1}+d_c-1}\right) \in K_{a-i_{c,\tau_c}-i_{c,\tau_c-1}}.$$

By induction in a, we can find $p' \in U_{\leqslant -1}$ such that $\psi(p') = f'$. Again, checking the definition of $\tilde{w}_{c,\tau_c}$, we have

$$u' = f - \tilde{w}_{c,\tau_c}(f') = \left((u'_{1,l})_{l=1}^{a+d_1-1}; \cdots ; (u'_{c,l})_{l=1}^{a+d_c-1}\right),$$

where $u'_{m,l} = 0$ for $1 \leqslant m \leqslant c$ and $1 \leqslant l \leqslant a - i_{c,\tau_c} + i_{c,\tau_c-1} + d_m - 1$, and furthermore that $Y_{c,i_{c,\tau_c}}$ divides $u'_{m,l}$ for $1 \leqslant m \leqslant c-1$ and $l \geqslant a - i_{c,\tau_c} + i_{c,\tau_c-1} + d_m$. As before, without changing the class of u' modulo η, we can assume that $u'_{m,l} = 0$ except possibly where $m = c$ and $l \geqslant a + i_{c,\tau_c-1}$. Except these final entries are uniquely determined as in Lemma 5.16, and thus are also all zero.

It remains only to deal with the case where $c > 1$ and $d_c = 0$, where the problem for $\bar{S}$ has data $(c-1; d_1, \ldots, d_{c-1}) \in \mathcal{K}$, so that we may apply induction. Suppose that $f \in K_a$ is given. By setting $Y_{c,0} = 0$ in each entry of f and discarding f_c, we obtain $\bar{f} \in \bar{K}_a$. Using induction we find

$$p(X_0, X_1, \bar{\zeta}_{1,1}, \ldots, \bar{\zeta}_{1,o_1}, \ldots\ldots \bar{\zeta}_{c-1,\tau_{c-1}}) \in \bar{U}_{\leqslant -1}$$

with $\bar{\psi}(p) = \bar{f}$. By Lemma 5.15, we have

$$u = f - \psi(p(X_0, X_1, W_{1,1}, \ldots, W_{1,\tau_1}, \ldots\ldots W_{c-1,\tau_{c-1}})) \in K_a$$

which has the form

$$(f'_1 Y_{c,0}; \cdots ; f'_{c-1}Y_{c,0}; F_c).$$

for some $f'_m \in S^{a+d_m-1}$ and $F_c \in S^{a-1}$. Adding a suitable element of η, we can assume that $f'_m = 0$ for each

$1 \leqslant m \leqslant c-1$. That the resulting f lies in $\ker A_a$ is that $Y_{c,0} \cdot \mathrm{Id}_{a-1}(F_c) = 0$, which implies that $F_c = 0$. We are done. □

5.3.2 General case

We are now able to describe the kernels $(\ker A_a)/\eta$ defined in Proposition 4.6(ii).

Theorem 5.17. *Fix $(c; d_1, \dots, d_c) \in \mathcal{K}$ and take an index sequence $\{i_{m,0}, \dots, i_{m,\tau_m}\}$ of each d_m. Let S be a quotient of the $\mathbb{Z}$-graded ring $\mathbb{C}[y_0, \dots, y_n]$ by some homogeneous ideal, with S having projective dimension $\kappa \geqslant \sum_{m=1}^{c} \tau_m + c - 1$. Suppose $e_1, \dots, e_c \geqslant 1$ and that*

$$g_{1,i_{1,0}}, \dots, g_{1,i_{1,\tau_1}}, \dots\dots, g_{c,i_{c,\tau_c}}$$

forms a regular sequence in S, with each $g_{m,k}$ having degree e_m. Let $a \geqslant 1$ and for each $m = 1, \dots, c$, define a map of free S-modules $A_{m,a} : S^{a+d_m-1} \to S^{a-1}$ by the $(a-1) \times (a+d_m-1)$ matrix with (s,t) entry equal to $g_{m,i_{m,k}}$ if $t - s = i_{m,k}$ and 0 otherwise. Then define

$$A_a = (A_{1,a}| \cdots |A_{m,a}) : S^{a+d_1-1} \times \cdots \times S^{a+d_c-1} \to S^{a-1}.$$

For each $1 \leqslant m < m' \leqslant c$, let $\eta_{m,m'}$ be the submodule of $S^{a+d_1-1} \times \cdots \times S^{a+d_m-1}$ generated by the $(a + d_m + d_{m'} - 1)$ columns of the matrix

$$A^*_{a,(m,m')} = (0|\cdots|0|A^t_{m',a+d_m}|0|\cdots|0| - A^t_{m,a+d_{m'}}|0|\cdots|0)^t.$$

Let $\eta = \sum_{1 \leqslant m < m' \leqslant c} \eta_{m,m'}$ be the sum of these submodules. Then $\eta \subset \ker A_a$ and we may set $K_a = (\ker A_a)/\eta$ and $K = \bigoplus_{a \geqslant 1} K_a$. Let

$$U = S[X_0, X_1, W_{1,1}, \dots, W_{1,\tau_1}, \dots\dots, W_{c,\tau_c}]$$

be the free $\mathbb{Z}^2$-graded S-algebra with $\deg y_j = (0,1)$, $\deg(X_i) = (1,0)$ and $\deg(W_{m,k}) = (i_{m,k-1} - i_{m,k}, e_m)$ for relevant j, i, m, k. Then there is a surjective map

$$\psi : U_{\leqslant -1} = \bigoplus_{-a \leqslant -1} \bigoplus_{b \in \mathbb{Z}} U_{(-a,b)} \to K,$$

whose kernel contains $J_{\leqslant -1}$, where $J \lhd U$ is the homogeneous ideal generated by the equations

$$X_1^{i_{m,k+1} - i_{m,k}} W_{m,k+1} + g_{i_{m,k}} - X_0^{i_{m,k} - i_{m,k-1}} W_{m,k}, \quad 1 \leqslant m \leqslant c, 0 \leqslant k \leqslant \tau_k.$$

Proof. If $S = \mathbb{C}[Y_{1,i_{1,0}}, \dots\dots, Y_{c,i_{c,\tau_c}}]$ and $g_{m,i_{m,k}} = Y_{m,i_{m,k}}$ for each $1 \leqslant m \leqslant c$ and $0 \leqslant k \leqslant \tau_m$, then the statement is that of Theorem 5.13 with refined degrees. All of the proofs of Section 5.3.1 can be modified to the more general setting of this Theorem. The operators in Definition 5.5 are defined in the same way with the $Y_{m,i_{m,k}}$ replaced by the $g_{m,i_{m,k}}$. Proposition 5.7 holds as we symbolically only replace the $Y_{m,i_{m,k}}$ with $g_{m,i_{m,k}}$, preserving commutativity. In Definition 5.14, we set $\bar{S} = S/(g_{c,i_{c,\tau_c}})$, which has projective dimension $\kappa - 1$. Note that for this induction step the Theorem statement must allow for S to be a quotient. For Lemma 5.15, rather than setting $Y_{c,i_{c,\tau_c}} = 0$, we reduce modulo $g_{c,i_{c,\tau_c}}$. Lemma 5.16 holds with regularity of the sequence $g_{1,i_{1,0}}, \dots\dots, g_{c,i_{c,\tau_c}}$ in S.

We use induction in the triples (κ, λ, a) with lexicographic ordering, where $\lambda = (c; d_1, \ldots, d_c) \in \mathcal{K}$. If $\kappa = 0$ then $\lambda = (1; 0)$ where the result is trivially true. If $\kappa \geqslant 1$ and $\lambda = (1; 0)$, again the result is trivially true. If $\kappa \geqslant 1$, $\lambda \succ (1; 0)$ and $u \in K_a$, then reducing u modulo $g_{c, i_{c,\tau_c}}$ and discarding the last $i_{c,\tau_c} - i_{m,\tau_c - 1}$ terms of u_c puts us in a setting with parameters $(\kappa - 1, \lambda', a)$, where we may use induction, using the same method as in the proof of Theorem 5.13. This gives us the surjectivity of ψ. That $J_{\leqslant -1} \subset \ker \psi$ follows from it being true in the universal setting of Theorem 5.13. □

5.4 The negative graded algebra for $c = 1$ and $\tau = d$

Following on from Section 5.2, we assume again that $c = 1$ and that the single index sequence is full, so that $d_1 = \tau_1 = d \geqslant 0$. We write $Z_k = Z_{1,k}$ and $W_k = W_{1,k}$ for each k and note that $W_k = Z_k$ in this case. The following result is used in Example 4.20.

Proposition 5.18. *Suppose that we are in the setting of Theorem 5.13 with $c = 1$ and $d_1 = \tau_1 = d \geqslant 0$. Let $U' = S[Z_1, \ldots, Z_d]$ be the S-subalgebra of U generated by $Z_1, \ldots, Z_d$. Let $J' \lhd U'$ be the homogeneous ideal generated by the equations*

$$Y_k(Z_{k'+1}Z_{k''} - Z_{k'}Z_{k''+1}) - Y_{k'}(Z_{k+1}Z_{k''} - Z_kZ_{k''+1}) + Y_{k''}(Z_{k+1}Z_{k'} - Z_kZ_{k'+1})$$

for $0 \leqslant k < k' < k'' \leqslant d$. Then the natural map $\phi : U'_{\leqslant -1} \to (U/J)_{\leqslant -1}$ is surjective and has kernel J'.

Proof. The equations correspond exactly to the images of the standard basis vectors in the degree 3 map of the Koszul complex of the sequence $Y_0, Y_1, \ldots, Y_d$ in S, and thus are contained in J. Alternatively, that $J' \subset J$ can be checked explicitly by writing the generators of J' explicitly as S-combinations of the generators of J.

We use induction in d. If $d = 0$ the result is immediate, so suppose that $d \geqslant 1$. Let $\bar{U}'$ and $\bar{J}'$ be the respective analogues of U' and J' with parameters $d - 1$. The equations in J imply that any element in $(U/J)_{\leqslant -1}$ can be written as a polynomial in the Y_j, Z_k, thus ϕ is surjective. Notice next that $(U/J)_{-1} \cong S^d \cong (U'/J')_{-1}$. Suppose that $p(Y_j, Z_k) \in \ker \phi$ has degree $-a \leqslant -2$. Then $p(Y_j, Z_k)$ is also in $\ker \psi = J$. As in the proof of Theorem 5.13, we can assume all terms of p divide by Y_d, Z_d. The equations of J' can then be used to replace p with a polynomial dividing by Z_d, so that $p = qZ_d$. Since $\tilde{z}_d$ is an injective operator, we conclude that $q \in \ker \phi$. Using induction in the degree a, we have $q \in J'$ and thus $p \in J'$. □

Proposition 5.19. *Suppose that we are in the setting of Theorem 5.17 with $c = 1$ and $d_1 = \tau_1 = d \geqslant 0$. Let $U' = S[Z_1, \ldots, Z_d]$ be the subalgebra of U generated by $Z_1, \ldots, Z_d$. Let $J' \lhd U'$ be the homogeneous ideal generated by the equations*

$$g_k(Y_k)(Z_{k'+1}Z_{k''} - Z_{k'}Z_{k''+1}) - g_{k'}(Y_k)(Z_{k+1}Z_{k''} - Z_kZ_{k''+1}) + g_{k''}(Y_k)(Z_{k+1}Z_{k'} - Z_kZ_{k'+1})$$

for $0 \leqslant k < k' < k'' \leqslant d$. Then the natural map $\phi : U'_{\leqslant -1} \to (U/J)_{\leqslant -1}$ is surjective and has kernel J'.

Proof. The proof is identical to that of Proposition 5.19, with $\bar{U}$ being defined as an algebra over the quotient $\bar{S} = S/(g_d(Y_j))$ for the induction step in d. □

5.5 Comments

Remark 5.20. The large majority of statements of this Chapter work for fields, and even commutative rings, other than $\mathbb{C}$. Where we used the auxiliary variety Z' of Definition 5.4 to prove Proposition 5.7, we need still to justify this correspondence over rings other than $\mathbb{C}$. One can instead prove commutativity of the operators directly from the definition, as is done for the case $c = 1$ in Section 5.2.

Remark 5.21. The statement of Theorem 5.17 can be generalised to the setting where S is the quotient of the Cox ring of a Mori Dream Space T with free abelian Picard group. We require that the degrees $e_1, \ldots, e_c$ correspond to ample line bundles on T and that the $g_{m,i_0}, \ldots, g_{m,i_{\tau_m}}$ are sections of the corresponding line bundle e_m. Indeed, each proof of this Chapter can be modified to this setting. This generalisation can be made to justify the statements made in Remarks 2.20 and 3.15.

6 On Cox rings of codimension 2 complete intersections in $\mathbb{P}^2 \times \mathbb{P}^n$

In this Chapter, we apply the programme developed in Section 3.4 to codimension two complete intersections in the ambient space $X = \mathbb{P}^2 \times \mathbb{P}^n$. In the simplest two cases, which will include the Calabi-Yau setting, we are able to give a full Cox ring in the general case. We are able to do most of the steps of Section 3.4 by hand in these cases and do so. If $c > 2$, the map α of Proposition becomes too complex to work with. The $c > 2$ case is thus postponed to the next Chapter, where we can use our observations in the $c = 2$ case to derive results.

6.1 Finding sections

6.1.1 Basics

We recall the set-up as described in Section 2.3, where we take $c = 2$. Fix $n \geqslant 3$ and let $X = \mathbb{P}^2 \times \mathbb{P}^n$, homogeneous coordinates x_0, x_1, x_2 and $y_0, \ldots, y_n$ on the respective factors. The Picard group of X is isomorphic to $\mathbb{Z}^2$, generated by the pullbacks of a hyperplane section on each factor of X. Writing $R = \mathcal{R}(\mathbb{P}^n) = \mathbb{C}[y_0, \ldots, y_n]$, the Cox ring of X is

$$S = \mathcal{R}(X) = \mathcal{R}(\mathbb{P}^2) \otimes \mathcal{R}(\mathbb{P}^n) \cong R[x_0, x_1, x_2].$$

The graded subring R gives $\mathcal{R}(X)$ the structure of a graded R-module.

Definition 6.1. Let $d_1, d_2, e_1, e_2 \geqslant 1$ and suppose we have equations

$$r_1 \in \mathcal{R}(X)_{(d_1,e_1)}, \quad r_2 \in \mathcal{R}(X)_{(d_2,e_2)},$$

defining irreducible and normal hypersurfaces V_1, V_2 respectively. We say that $Z = V_1 \cap V_2$ has the form $(\dagger)$ with respect to V_1, V_2 if the sequence r_1, r_2 is regular in $\mathcal{R}(X)$ and the varieties V_1 and Z are non-singular.

All varieties $Z \subset \mathbb{P}^2 \times \mathbb{P}^n$ in this section are of the form $(\dagger)$ with respect to hypersurfaces V_1, V_2 as in Definition 6.1. For such a Z, the chain of inclusions $Z \hookrightarrow V_1 \hookrightarrow X$ induces a chain of isomorphisms $\operatorname{Pic} X \cong \operatorname{Pic} V_1 \cong \operatorname{Pic} Z$ of Picard groups by the Lefschetz hyperplane theorem, and we denote their common identification by Pic. We are in a setting to utilise the method developed in Section 3.4.

We require still an ordered affine cover of X. Let $\mathcal{W} = \{W_{ij}\}_{i,j}$ be the standard affine cover, where W_{ij} is the locus on which x_i, y_j are non-zero, and set the lexicographic ordering on $\mathcal{W}$. As usual, if $ij < kl$, denote by $W_{ij,kl}$ the intersection $W_{ij} \cap W_{kl}$.

6.1.2 Describing the maps on cohomology

We begin with Step (1) of the programme laid out in Section 3.4.

Lemma 6.2. *(i) There are natural $\mathbb{Z}^2$-graded vector space isomorphisms*

$$\bigoplus_{(a,b)\in\mathbb{Z}^2} H^2(X, \mathcal{O}_X(a,b)) \cong \frac{1}{x_0x_1x_2} R[x_0^{-1}, x_1^{-1}, x_2^{-1}] \cong S[x_0^{-1}, x_1^{-1}, x_2^{-1}]/L, \tag{39}$$

where L is the vector subspace of $S[x_0^{-1}, x_1^{-1}, x_2^{-1}]$ generated by all terms in which x_0, x_1 or x_2 appear with non-negative exponent.

(ii) For $m = 1, 2$, the map

$$r_{m,*} : \bigoplus_{(a,b)\in \mathrm{Pic}} H^2(X, \mathcal{O}_X(a - d_m, b - e_m)) \longrightarrow \bigoplus_{D\in \mathrm{Pic}} H^2(X, \mathcal{O}_X(a, b))$$

can be identified with the map

$$\begin{array}{rcl} r_{m,*} : S[x_0^{-1}, x_1^{-1}, x_2^{-1}]/L & \longrightarrow & S[x_0^{-1}, x_1^{-1}, x_2^{-1}]/L \\ f & \mapsto & r_m \cdot f \mod L \end{array}$$

that multiplies an element $f \in S[x_0^{-1}, x_1^{-1}, x_2^{-1}]/L \cong \frac{1}{x_0x_1x_2}R[x_0^{-1}, x_1^{-1}, x_2^{-1}]$ by r_m, and ignores those terms in which x_0, x_1 or x_2 occur with non-negative exponent.

Proof. This follows directly from the Künneth formula. The details are the same as in Lemma 4.3. □

We note that the isomorphisms (39) are isomorphisms of R-modules, which we will use.

Definition 6.3. Define, for each $(a, b) \in \mathbb{Z}^2$, the vector spaces

$$T_{(a,b)} = H^2\left(X, \mathcal{O}_X\left(a - d_1 - d_2, b - e_1 - e_2\right)\right), \quad T_a = \bigoplus_{b\in\mathbb{Z}} T_{(a,b)},$$

along with the spaces

$$T'_{(a,b)} = H^2\left(X, \mathcal{O}_X\left(a - d_2, b - e_2\right)\right) \bigoplus H^2\left(X, \mathcal{O}_X\left(a - d_1, b - e_1\right)\right), \quad T'_a = \bigoplus_{b\in\mathbb{Z}} T_{(a,b)}.$$

For each $a \in \mathbb{Z}$ define $B_a : T_a \to T'_a$ to be the map

$$\begin{array}{rcl} B_a : T_a & \longrightarrow & T'_a \\ h & \mapsto & (r_{1,*}h, r_{2,*}h). \end{array}$$

Where necessary we denote by $B_{(a,b)}$ the map B_a restricted to the corresponding graded pieces of T_a and T'_a. Finally, we write $K_{-a} = \ker B_a$ (note the flip of sign in a) for each $a \in \mathbb{Z}$ and $K = \bigoplus_{a\in\mathbb{Z}} K_a$.

This concludes Step (1) of our programme. By the method of Section 3.4, there is a graded isomorphism of vector spaces $\mathcal{R}(Z)/\iota^*\mathcal{R}(X) \cong K$, where the sign of the grading is flipped. To understand this isomorphism we continue with our programme

6.1.3 Describing the kernels of the maps B_a

In Step (2) we wish to describe the space K. We begin by setting up a correspondence between T_a and a free R-module for each $a \in \mathbb{Z}$.

Definition 6.4. For each $a \in \mathbb{Z}$, define the discrete 2-simplex

$$\Delta_2(a - 3) = \{(i, j, k) \in \mathbb{Z}_{\geqslant 0} : i + j + k = a - 3\} \subset \mathbb{R}^3,$$

consisting of $\binom{a-1}{2}$ points in $\mathbb{R}^3$, where we set $\binom{a-1}{2} = 0$ if $a < 3$, corresponding $\Delta_2(a - 3) = \varnothing$. For ease of notation, we write ijk for an element (i, j, k) in $\Delta_2(a - 3)$. If $ijk \in \Delta_2(a - 3)$ and $i'j'k' \in \Delta_2(d)$ then we can write $ijk + i'j'k' = (i + i', j + j', k + k') \in \Delta_2(a + d - 3)$, where $a, d \in \mathbb{Z}$.

Since $H^2(X, \mathcal{O}_X(a,b)) = 0$ for $a > -3$, we use the index $-a$ rather than a. Let $-a \leqslant -3$. If we give $R^{\binom{a-1}{2}}$ a standard basis indexed by $\Delta_2(a-3)$, there are R-module isomorphisms

$$\begin{aligned} R^{\binom{a-1}{2}} &\xrightarrow{\sim} \bigoplus_{b \in \mathbb{Z}} \frac{1}{x_0 x_1 x_2} R[x_0^{-1}, x_1^{-1}, x_2^{-1}]_{(-a,b)} \\ (h_{ijk})_{ijk \in \Delta_2(a-3)} &\mapsto h = \sum_{ijk \in \Delta_2(a-3)} \frac{h_{ijk}}{x_0^{i+1} x_1^{j+1} x_2^{k+1}}. \end{aligned} \tag{40}$$

The following Lemma describes the maps on second cohomology induced by the multiplication by a global section on X.

Lemma 6.5. *Let $d \geqslant 0$ and $e \geqslant 1$. A general polynomial $r \in \mathcal{R}(X)_{(d,e)}$ is given by a collection of degree e polynomials $(g_{ijk})_{ijk \in \Delta_2(d)}$ via*

$$(g_{ijk})_{ijk \in \Delta_2(d)} \mapsto \sum_{ijk \in \Delta_2(d)} g_{ijk} x_0^i x_1^j x_2^k.$$

If $-a \in \mathbb{Z}$, then the map

$$r_*^{(a)} : \bigoplus_{b \in \mathbb{Z}} H^2(X, \mathcal{O}_X(a-d, b-e)) \to \bigoplus_{b \in \mathbb{Z}} H^2(X, \mathcal{O}_X(a,b))$$

is identified via the isomorphisms in Lemma 6.2 and (40) with the map

$$\begin{aligned} R^{\binom{a+d-1}{2}} &\longrightarrow R^{\binom{a-1}{2}} \\ (h_{ijk})_{ijk \in \Delta_2(a+d-3)} &\mapsto (h'_{ijk})_{ijk \in \Delta_2(a-3)}, \end{aligned}$$

where for each $ijk \in \Delta_2(a-3)$, h'_{ijk} is given by

$$h'_{ijk} = \sum_{i'j'k' \in \Delta_2(d)} h_{ijk+i'j'k'} g_{i'j'k'}.$$

Proof. This follows by construction from the description of $r_*^{(a)}$ given by Lemma 6.2(ii). □

The maps in Lemma 6.5 can be visualised as in the following example.

Example 6.6. Suppose that $a = d = 2$ and that

$$r = g_{200} x_0^2 + g_{110} x_0 x_1 + g_{101} x_0 x_2 + g_{020} x_1^2 + g_{011} x_1 x_2 + g_{002} x_2^2$$

is a global section on X. The element r and a general element $(h_{ijk})_{ijk \in \Delta_2(a+d-3)} \in R^{\binom{a+d-1}{2}}$ can be represented by the respective triangles (or more formally, discrete 2-simplexes labelled by polynomials)

$$\begin{array}{ccccc} & & g_{020} & & \\ & g_{110} & & g_{011} & \\ g_{200} & & g_{101} & & g_{002} \end{array} \qquad \begin{array}{ccccccc} & & & h_{030} & & & \\ & & h_{120} & & h_{021} & & \\ & h_{210} & & h_{111} & & h_{012} & \\ h_{300} & & h_{201} & & h_{102} & & h_{003} \end{array}$$

There are three ways the triangle on the left can be translated (without rotation) over the triangle on the right, so that each $g_{i'j'k'}$ lines up with a h_{ijk}. Where we may do this we may take the scalar product of the overlapping polynomials. These give the three entries of $r_*(h_{ijk})$.

Proposition 6.7. *For each $-a \in \mathbb{Z}$, the space $K_a = \ker B_a$ may only be non-zero if $-a \leqslant -3 + d_1 + d_2$.*

(i) For $-3 + \max\{d_1, d_2\} < -a \leqslant -3 + d_1 + d_2$, we have a graded isomorphism

$$K_a \cong R^{\binom{a+d_1+d_2-1}{2}}\left[e_1 + e_2\right].$$

(ii) For $-a \leqslant -3 + \max\{d_1, d_2\}$, we have graded isomorphisms

$$K_a \cong \ker r_{1*}^{(a)} \cap \ker r_{2*}^{(a)}.$$

Proof. The first two statements follow since $T_a = 0$ if $-a > -3+d_1+d_2$ and $T'_a = 0$ for $-3+\max\{d_1, d_2\} < -a \leqslant -3 + d_1 + d_2$. The isomorphism of part (ii) is immediate from the definition of B_a. □

We note that the dimensions of the spaces T_a and T'_a increase quadratically in a, making these kernels more complex than those explored in Chapter 4. In particular, representing the maps as matrices becomes unworkable outside perhaps the first few non-trivial cases. We move onto the procedure of Step (3), which is also much more complex than the analogous Step in Chapter 4.

6.1.4 Lifting elements of K to the Cox ring

We recall the theory behind Step (3) of the programme. For a fixed $(-a, b) \in \mathbb{Z}^2$ there is a map

$$\hat{\delta} : \mathcal{R}(Z)_{(-a,b)} \to H^2(X, \mathcal{O}_X(-a - d_1 - d_2, b - e_1 - e_2)),$$

where $\hat{\delta}$ is the composition $\delta_1 \circ \delta_2$ of connecting homomorphisms δ_1, δ_2 in the long exact sequence of sheaf cohomology associated with the subschemes V_1, V_2 in X. That is, $\hat{\delta}$ is the composition

$$\mathcal{R}(Z)_{(-a,b)} \xrightarrow{\delta_2} H^1(V_1, \mathcal{O}_{V_1}(-a - d_2, b - d_2)) \xrightarrow{\delta_1} H^2(X, \mathcal{O}_X(-a - d_1 - d_2, b - e_1 - e_2)).$$

By Proposition 3.13, the map $\hat{\delta}$ induces an isomorphism

$$\delta : (\mathcal{R}(Z)/\mathcal{R}(Z))_{(-a,b)} \to \ker_{B_{(-a,b)}}.$$

As usual, we may take the direct sum over $(-a, b) \in \mathbb{Z}^2$ and we denote all these maps by δ. We wish to find a map $\alpha : K \to \mathcal{R}(Z)$ which induces an inverse isomorphism $\overline{\alpha}$ to δ.

Suppose that h is a homogeneous element of K, in the sense it is identified via (40) with a homogeneous element of $\frac{1}{x_0x_1x_2}R[x_0^{-1}, x_1^{-1}, x_2^{-1}]$ of some degree $(-a, b) \in \mathbb{Z}^2$. Formally, h is identified, via the Künneth isomorphism, with the degree $(-a, b)$ Čech 2-cochain $(t_{ij,kl,op})$ on X with

$$t_{ij,kl,op} = \begin{cases} h & \text{if } (i,k,o) = (0,1,2), \\ 0 & \text{else.} \end{cases}$$

Since $h \in K$, we know that $r_{1*}h = (r_1 \cdot t_{ij,kl,op})$ is cohomologically trivial. By Lemma 6.2, this implies each term of $r_1 \cdot h$ contains at least one of x_0, x_1, x_2 with non-negative exponent. Therefore, we may write

$$r_1 h = u^{(01)} + u^{(02)} + u^{(12)} + u^{(0)} + u^{(1)} + u^{(2)} + u^{(g)},$$

where $u^{(ij)}$ consists only of terms where x_i, x_j have negative exponent for each $0 \leqslant i < j \leqslant 2$, $u^{(i)}$ consists only of terms where x_i is the only x variable appearing with negative exponent for each $i = 0, 1, 2$ and $u^{(g)}$ is a global section on X. Consider the 1-cochain $(s_{ij,kl})$ on X given by

$$s_{ij,kl} = \begin{cases} u^{(01)} + \frac{1}{2}u^{(0)} + \frac{1}{2}u^{(2)} + \frac{1}{3}u^{(g)} & \text{if } (i,k) = (0,1), \\ -u^{(02)} - \frac{1}{2}u^{(0)} - \frac{1}{2}u^{(2)} - \frac{1}{3}u^{(g)} & \text{if } (i,k) = (0,2), \\ u^{(12)} + \frac{1}{2}u^{(1)} + \frac{1}{2}u^{(2)} + \frac{1}{3}u^{(g)} & \text{if } (i,k) = (1,2), \\ 0 & \text{else.} \end{cases} \tag{41}$$

By construction, this maps to $r_{1*}h$ under the first degree Čech complex map d^1 on X. It also restricts to a 1-cocycle on V_1. The cohomology class s of $(s_{ij,kl})$ on V_1 is the the unique element

$$s \in H^1(V_1, \mathcal{O}_{V_1}(-a - d_2, b - e_2))$$

with $\delta_1 s = h$. As shown in the proof of Proposition 3.13, that $h \in \ker r_{2*}$ implies that the 1-cocycle s on V_1 is also in the kernel of r_{2*} viewed as a map on the first cohomology on V_1. That is, the 1-coycle $(r_2 \cdot s_{ij,kl})$ on V_1 is cohomologically trivial, thus the image of a 0-cochain (v_{ij}) on V_1. Since no entry of a cochain has depended on the indices j or l, we can choose (v_{ij}) so that $v_{ij} = v_i$ is independent of the index j for $i = 0, 1, 2$. Since $\mathcal{R}(V_1) \cong \mathcal{R}(X)/(r_1)$, each v_i may be taken in $\mathcal{R}(X)[x_i^{-1}]$. We obtain equations

$$r_2 s_{ij,kl} = v_i - v_k \in \mathcal{O}_{V_1}(-a, b)(W_{ij,kl}) \cong \mathcal{R}(X)[x_i^{-1}, x_k^{-1}]/(r_1).$$

We are thus able to find $q_{ik} \in \mathcal{R}(X)[x_i^{-1}, x_k^{-1}]$ for each $0 \leqslant i < k \leqslant 2$ such that

$$v_i - v_k = r_2 s_{ij,kl} + r_1 q_{ik}, \quad 0 \leqslant i < j \leqslant 2.$$

This equation implies that (v_{ij}) is a 0-cocycle when considered on Z, which was expected by the construction. The corresponding section v, defined by v_i over the locus $x_i \neq 0$ for each $i = 0, 1, 2$, satisfies $\hat{\delta} v = h$ by construction. The choice of v is not unique; it is however unique up to an element of $\iota^* \mathcal{R}(X)$.

To define $\alpha : K \to \mathcal{R}(Z)$, we let $h_1, \ldots, h_t$ be a set of minimal (and in particular $\mathbb{C}$-linearly independent) R-module generators of K and find $v_1, \ldots, v_t$ as above so that each $\hat{\delta} v_{t'} = h_{t'}$. Set $\alpha h_{t'} = v_{t'}$ for each $t' = 1, \ldots, t$ and extend R-linearly.

In practice, we don't need to worry much about the non-uniqueness of α, since $\bigoplus_{b \in \mathbb{Z}} \iota^* \mathcal{R}(X)_{(-a,b)} = 0$ for nearly all $-a \in \mathbb{Z}$ for which $K_a \neq 0$. In our examples, we will only go through the steps of defining α in degrees $(-a, b)$ with $-a \in \{-1, 0\}$; we will show that the new sections found in these degrees are enough along with restrictions of sections on X to generate the Cox ring $\mathcal{R}(Z)$ in the cases $(d_1, d_2) = (1, 1)$ and $(d_1, d_2) = (2, 1)$ with general defining equations. Before restricting to these cases, we interpret how multiplication by the sections x_0, x_1, x_2 acts on K.

6.1.5 Multiplication by x_0, x_1, x_2

Let $-a \in \mathbb{Z}$ and consider the isomorphism $\overline{\alpha} : K \to \mathcal{R}(Z)/\iota^*\mathcal{R}(X)$ with inverse δ. For each $i = 0, 1, 2,$, consider the map

$$\tilde{x}_i = \delta(\cdot x_i)\overline{\alpha} : K \to K,$$

mapping each K_{a+1} to K_a.

Lemma 6.8. *Let $-a \in \mathbb{Z}$. The maps $\tilde{x}_0, \tilde{x}_1, \tilde{x}_2 : K_{a+1} \to K_a$ are given respectively by*

$$\tilde{x}_0 h = (h'_{ijk})_{ijk \in \Delta_2(a-3)}, \quad h'_{ijk} = h_{i+1,jk},$$

$$\tilde{x}_1 h = (h'_{ijk})_{ijk \in \Delta_2(a-3)}, \quad h'_{ijk} = h_{i,j+1,k},$$

$$\tilde{x}_2 h = (h'_{ijk})_{ijk \in \Delta_2(a-3)}, \quad h'_{ijk} = h_{ij,k+1}.$$

Proof. By Lemma 3.5, we have $\delta(\cdot r) = r_*\delta$ for any $r \in \mathcal{R}(X)$. For any $r \in \mathcal{R}(X)$ we thus have $\tilde{r} = \delta(\cdot r)\alpha = r_*\delta\alpha = r_*$. By Lemma 6.2, the maps x_{0*}, x_{1*}, x_{2*} have exactly the descriptions given in the Lemma. □

Remark 6.9. As in Example 6.6, we may view the coefficients of a $h \in K_a$ in a discrete 2-simplex labelled by the polynomial coefficients. In this pictorial representation, multiplication by x_0 corresponds to discarding the upper right edge of the triangle:

$$\tilde{x}_0 \left(\begin{array}{ccccccc} & & & h_{030} & & & \\ & & h_{120} & & h_{021} & & \\ & h_{210} & & h_{111} & & h_{012} & \\ h_{300} & & h_{201} & & h_{102} & & h_{003} \end{array}\right) = \begin{array}{ccccc} & & h_{120} & & \\ & h_{210} & & h_{111} & \\ h_{300} & & h_{201} & & h_{102} \end{array}.$$

Similarly, $\tilde{x}_1$ corresponds to discarding the bottom edge of the triangle and $\tilde{x}_2$ corresponds to discarding the upper left edge. This can be compared directly with the truncation maps of Proposition 4.13.

With this, we end our comments on the general setting, and compute Cox rings in the general cases with $(d_1, d_2) = (1, 1)$ and $(d_1, d_2) = (2, 1)$.

6.2 The case $(d_1, d_2) = (1, 1)$

Recall that $Z \subset X = \mathbb{P}^2 \times \mathbb{P}^n$ is a non-singular complete intersection of irreducible normal hypersurfaces V_1, V_2, with V_1 also non-singular, defined by equations $r_1, r_2 \in \mathcal{R}(Z)$ of degrees $(d_1, e_1), (d_2, e_2)$, where $d_1, d_2, e_1, e_2 \geqslant 1$, respectively. In this Section, we treat the case $d_1 = d_2 = 1$. There are homogeneous degree e_1, respectively degree e_2, polynomials f_1, f_2, f_3, respectively g_1, g_2, g_3, such that

$$r_1 = f_1 x_0 + f_2 x_1 + f_3 x_2, \quad r_2 = g_1 x_0 + g_2 x_1 + g_3 x_2.$$

We will see that Z is a blow-up of $\mathbb{P}^2 \times \mathbb{P}^n$ in a codimension 2 ideal. Such blow-ups need not be Mori Dream Spaces, but we will be able to compute the Cox ring of Z and show that Z is a Mori Dream Space with unique minimal model.

6.2.1 Basic geometry

Let $Z \to \mathbb{P}^n$ be projection to the second factor. This map is generically one-to-one, thus defining a birational map. The fibres are controlled by the rank of the matrix

$$M = \begin{pmatrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \end{pmatrix}.$$

Since Z has codimension 2, the matrix M has full rank over a general fibre. On the locus in $\mathbb{P}^n$ where M has full rank, the fibre is a point. On the locus in $\mathbb{P}^n$ where M has rank 1, given by the vanishing of the 2×2 minors minus the locus where all 6 coefficient polynomials vanish, the fibre is a line in $\mathbb{P}^2$. Over the locus in which $M = 0$ we get $\mathbb{P}^2$ fibres. In conclusion, we see that $Z \subset X$ is defined by the equations

$$\begin{aligned} &x_0(f_1g_3 - f_3g_1) + x_1(f_2g_3 - f_3g_2), \\ &x_0(f_1g_2 - f_2g_1) - x_2(f_2g_3 - f_3g_2), \\ &x_1(f_1g_2 - f_2g_1) + x_2(f_1g_3 - f_3g_1), \end{aligned} \tag{42}$$

which can be obtained initially by eliminating one of the x_0, x_1, x_2 from the equations r_1, r_2.

Proposition 6.10. *The variety Z is isomorphic to the blow-up of $\mathbb{P}^n$ in the codimension 2 subvariety defined by the 2×2 minors of M.*

Proof. Let $Y \subset \mathbb{P}^n$ be defined by the 2×2 minors of M. There are three defining equations with one syzygy and Y is thus a codimension 2 ideal. The equations of the blow up in $\mathbb{P}^2 \times \mathbb{P}^n$ are

$$\begin{aligned} &x_0(f_1g_3 - f_3g_1) - x_1(f_2g_3 - f_3g_2), \\ &x_0(f_1g_2 - f_2g_1) - x_2(f_2g_3 - f_3g_2), \\ &x_1(f_1g_2 - f_2g_1) - x_2(f_1g_3 - f_3g_1). \end{aligned}$$

Replacing the coordinate x_1 of $\mathbb{P}^2$ with $-x_1$, we get precisely the equations (42). $\square$

We will go through the procedure of Section 3.4, and show that $\mathcal{R}(Z)$ is generated by the restrictions of sections in $\mathcal{R}(X)$ along with one new section in degree $(-1, e_1 + e_2)$, which vanishes along the exceptional divisor of Z viewed as a blow-up via Proposition 6.10.

6.2.2 A first new section

The spaces T_{-a}, T'_{-a}, as defined in Definition 6.3, are given by

$$T_{-a} = \bigoplus_{b \in \mathbb{Z}} H^2(X, \mathcal{O}_X(-a-2, b-e_1-e_2))$$

$$T'_{-a} = \bigoplus_{b \in \mathbb{Z}} \left(H^2(X, \mathcal{O}_X(-a-1, b-e_2)) \oplus H^2(X, \mathcal{O}_X(-a-1, b-e_1)) \right),$$

for each $-a \in \mathbb{Z}$. The maps B_{-a} are given by (r_{1*}, r_{2*}) with kernel K_a. By Proposition 6.7 we have that $K_{-a} \neq 0$ if and only if $-a \leqslant -1$ and

$$K_1 \cong R[e_1 + e_2],$$

with the standard module generator corresponding to $h = \frac{1}{x_0, x_1, x_2} \in \frac{1}{x_0 x_1 x_2} R[x_0^{-1} x_1^{-1} x_2^{-1}]$ under the isomorphisms in Lemma 6.2. We apply α to h explicitly, noting that the choice is unique since $\mathcal{R}(X)_{(-1,b)} = 0$ for each $b \in \mathbb{Z}$. We follow closely the procedure given in Section 6.1.4.

If $h = \frac{1}{x_0 x_1 x_2}$ then

$$r_1 h = \frac{f_1}{x_1 x_2} + \frac{f_2}{x_0 x_2} + \frac{f_3}{x_1 x_2}.$$

The 1-cochain $(s_{ij,kl})$ on X defined by

$$s_{ij,kl} = \begin{cases} \frac{f_3}{x_0 x_1} & \text{if } (i,k) = (0,1) \\ -\frac{f_2}{x_0 x_2} & \text{if } (i,k) = (0,2) \\ \frac{f_1}{x_1 x_2} & \text{if } (i,k) = (1,2) \\ 0 & \text{else,} \end{cases}$$

maps to $r_{1*} h$ under the Čech differential, and restricts to a 1-cocycle on V_1. The 1-cochain $r_{2*}(s_{ij,kl}) = (r_2 s_{ij,kl})$, which can be viewed on X or V_1, is then given by

$$r_{2*}(s_{ij,kl})_{ij,kl} = \begin{cases} \frac{f_3 g_3 x_2}{x_0 x_1} + \frac{f_3 g_2}{x_0} + \frac{f_3 g_1}{x_1} & \text{if } (i,k) = (0,1) \\ -\frac{f_2 g_2 x_1}{x_0 x_2} - \frac{f_2 g_3}{x_0} + \frac{f_2 g_1}{x_2} & \text{if } (i,k) = (0,2) \\ \frac{f_1 g_1 x_0}{x_1 x_2} + \frac{f_1 g_3}{x_1} + \frac{f_1 g_2}{x_2} & \text{if } (i,k) = (1,2) \\ 0 & \text{else.} \end{cases}.$$

We can add the 1-cochain on X or V_1 given by

$$q'_{ij,kl} = \begin{cases} -\frac{g_3}{x_0 x_1} r_1 & \text{if } (i,k) = (0,1), \\ \frac{g_2}{x_0 x_2} r_1 & \text{if } (i,k) = (0,2), \\ -\frac{g_1}{x_1 x_2} r_1 & \text{if } (i,k) = (1,2), \\ 0 & \text{else.} \end{cases},$$

without changing the 1-cochain on V_1. Defining

$$q_{01} = -\frac{g_3}{x_0 x_1}, \quad q_{02} = \frac{g_2}{x_0 x_2}, \quad q_{12} = -\frac{g_1}{x_1 x_2},$$

we get an equality

$$(r_2 s_{ij,kl} + r_1 q_{ik})_{ij,kl} = \begin{cases} \frac{f_3g_2 - f_2g_3}{x_0} + \frac{f_3g_1 - f_1g_3}{x_1} & \text{if } (i,k) = (0,1), \\ \frac{f_3g_2 - f_2g_3}{x_0} + \frac{f_1g_2 - f_2g_1}{x_2} & \text{if } (i,k) = (0,2), \\ \frac{f_1g_3 - f_3g_1}{x_1} + \frac{f_1g_2 - f_2g_1}{x_2} & \text{if } (i,k) = (1,2), \\ 0 & \text{else,} \end{cases}.$$

of 1-cochains on X. If we define

$$v_0 = \frac{f_3g_2 - f_2g_3}{x_0}, \quad v_1 = \frac{f_1g_3 - f_3g_1}{x_1}, \quad v_2 = \frac{f_2g_1 - f_1g_2}{x_2},$$

then the equations

$$v_i - v_j = r_2 s_{ij,kl} + r_1 q_{ik}$$

are satisfied for each $0 \leqslant i < j \leqslant 2$, and we must have $\alpha h = v$, where v is given by the 0-cocycle (v_i) on Z defined by

$$v_i = \begin{cases} \frac{f_3g_2 - f_2g_3}{x_0} & \text{if } i = 0, \\ \frac{f_1g_3 - f_3g_1}{x_1} & \text{if } i = 1, \\ \frac{f_2g_1 - f_1g_2}{x_2} & \text{if } i = 2. \end{cases}$$

The section v satisfies the equations

$$vx_0 + (f_2g_3 - f_3g_2) = vx_1 - (f_1g_3 - f_3g_1) = vx_2 + (f_1g_2 - f_2g_1) = 0.$$

We note finally that the existence of this section is clear from the equations describing Z as a blow-up.

6.2.3 Computing the space K

For these Z we can perform Step (2) of our procedure explicitly.

Proposition 6.11. *For each $-a \leqslant -1$, the space $K_a = \ker B_{-a}$ is isomorphic to the R-module $R[a(e_1 + e_2)]$, that is, generated by a single element in degree $(-a, a(e_1 + e_2))$. In particular, $\bigoplus_{-a \leqslant -1, b \in \mathbb{Z}} \mathcal{R}(Z) \cong \bigoplus_{-a \leqslant -1} R[a(e_1 + e_2)]$ as an R-module.*

Proof. Recalling the notation of Section 6.1.3, an element of K_a, for some $-a \leqslant -1$, is identified with a

$$h = (h_{ijk})_{ijk \in \Delta_2(a-1)} \in R^{\binom{a+1}{2}}$$

such that the equations

$$\begin{aligned} f_1 h_{i+1,jk} + f_2 h_{i,j+1,k} + f_3 h_{ij,k+1} &= 0, \\ g_1 h_{i+1,jk} + g_2 h_{i,j+1,k} + g_3 h_{ij,k+1} &= 0, \end{aligned} \tag{43}$$

are satisfied for each $ijk \in \Delta_2(a-2)$. If $a = 1$ this gives an empty condition and we recover the isomorphism $K_1 \cong R[e_1 + e_2]$ seen in the previous section. If $a \geqslant 2$ these conditions are non-empty. For a fixed $ijk \in$

$\Delta_2(a-2)$, the equations (43) imply that

$$\begin{aligned} h_{i+1,jk}(f_1g_3 - f_3g_1) - h_{i,j+1,k}(f_3g_2 - f_2g_3) &= 0, \\ h_{i+1,jk}(f_2g_1 - f_1g_2) - h_{ij,k+1}(f_3g_2 - f_2g_3) &= 0, \\ h_{i,j+1,k}(f_2g_1 - f_1g_2) - h_{ij,k+1}(f_1g_3 - f_3g_1) &= 0. \end{aligned} \tag{44}$$

$\square$

Indeed any element in the kernel of M satisfies these equations. Now the 2×2 minors share no common factor, for otherwise the section v divides by this common factor to yield a global section of degree $(0, b)$ with $b < e_1 + e_2$, contradicting our known description of K_1. The equations (44) then imply that there is a homogeneous polynomial $p \in R$ so that

$$h_{ijk} = p\,(f_3g_2 - f_2g_3)^i\,(f_1g_3 - f_3g_1)^j(f_2g_1 - f_1g_2)^k, \quad ijk \in \Delta_2(a-1).$$

In particular, K_a is a rank 1 free R-module. Its generator h has entries in degree $(a-1)(e_1+e_2)$, thus corresponds to a degree $(-a, a(e_1+e_2))$ element of K. The last statement follows since $\bigoplus_{b\in\mathbb{Z}} \mathcal{R}(Z)_{(-a,b)} \cong K_a$ for each $-a \leqslant -1$.

6.2.4 Proof of Theorem 2.13 for $c = 2$

We are now able to prove Theorem 2.13 in the case $c = 2$.

Theorem 6.12. *Suppose that $Z \subset \mathbb{P}^2 \times \mathbb{P}^n$, with $n > 2$, has form $(\dagger)$ with respect to irreducible and normal hypersurfaces*

$$V_1 = \{f_1x_0 + f_2x_1 + f_3x_2 = 0\}, \quad V_2 = \{f_1x_0 + f_2x_1 + f_3x_2 = 0\}.$$

Then the 2×2 minors of the matrix

$$M = \begin{pmatrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \end{pmatrix}$$

define a codimension 2 subvariety of $\mathbb{P}^n$ and Z is isomorphic to the blow-up of $\mathbb{P}^n$ in this locus. Furthermore, if $U = \mathbb{C}[X_0, X_1, X_2, Y_0, \dots, Y_n, V]$ is the free $\mathbb{Z}^2$-graded $\mathbb{C}$ algebra with generators having degrees $(1,0), (0,1), (-1, e_1+e_2)$ respectively, then there is a surjective map of graded $\mathbb{C}$-algebras $\psi : U \to \mathcal{R}(Z)$ sending $X_i \mapsto x_i$, $Y_j \mapsto y_j$, $V \to v$, where x_i, y_j are the restrictions of sections on X and v is the section of Z found in Section 6.2.2. The kernel of ψ is given by the ideal J generated by the equations

$$\begin{gathered} f_1(Y_j)X_0 + f_2(Y_j)X_1 + f_3(Y_j)X_3, \quad g_1(Y_j)X_0 + g_2(Y_j)X_1 + g_3(Y_j)X_3, \\ VX_0 + (f_2g_3 - f_3g_2)(Y_j), \quad VX_1 - (f_1g_3 - f_3g_1)(Y_j), \quad VX_2 + (f_1g_2 - f_2g_1)(Y_j) \end{gathered}$$

generated in degree $(0, e_1 + e_2)$.

Proof. Suppose that the minors of M generate a codimension 1 subvariety of $\mathbb{P}^n$, then they share a common factor q. This leads to the existence of a global section $\frac{v}{q}$ in a degree $(0, b)$ with $b < e_1 + e_2$, contradicting that $K_1 \cong R[e_1 + e_2]$. We conclude that such Z must be singular.

For surjectivity, we note that for each $a \geqslant 1$, the section v^a is a non-zero section in degree $(-a, a(e_1 + e_2))$. By Proposition 6.11, v^a generates $\bigoplus_{b\in\mathbb{Z}} \mathcal{R}(Z)_{(-a,b)}$ as an R-module and ψ is surjective. The equations in J certainly lie in the kernel of ψ. Suppose p is a degree $(-a, b)$ element of U/J.

If $-a \geqslant 0$, the equations in J allow us to write p as a polynomial independent of the V. Then $\psi(p) \in \iota^*\mathcal{R}(X)$ is zero if and only if it is contained in $\langle r_1, r_2\rangle$, implying that $p \in \langle r_1(X_i, Y_j), r_2(X_i, Y_j)\rangle \subset J$.

If $-a \leqslant -1$, then the equations in J allow us to write p as an element of R times V^a, and we know that v^a is R-linearly independent. There are hence no further relations and $\ker \psi = J$ as claimed, proving the Theorem. □

Remark 6.13. Since each K_a for $-a \leqslant -1$ was free of rank one, we didn't need to interpret the map

$$\tilde{v} = \delta(\cdot v)\alpha : K \to K$$

to prove the Theorem. Using Lemma 6.8 and the equations for vx_0, vx_1, vx_2 we obtain the following. If $h = (h_{ijk})_{ijk\in\Delta_2(a-1)} \in K_a$ with $-a \leqslant -1$ is so $w = \alpha h$, then as in the proof of Proposition 6.11, we can find $p \in R$ so that

$$h_{ijk} = p\,(f_3g_2 - f_2g_3)^i\,(f_1g_3 - f_3g_1)^j(f_2g_1 - f_1g_2)^k, \quad ijk \in \Delta_2(a-1).$$

Then since δvwx_0 is equal to the triangle representing vw with its upper right edge discarded, and also equal to $(f_3g_2 - f_2g_3)w$. Similarly, vw with its lower edge, respectively upper left edge, removed is equal $(f_1g_3 - f_3g_1)w$, respectively $(f_2g_1 - f_1g_2)w$. We find that $\tilde{v}h = h' = (h'_{ijk})_{ijk\in\Delta_2(a)}$ where

$$h'_{ijk} = p\,(f_3g_2 - f_2g_3)^i\,(f_1g_3 - f_3g_1)^j(f_2g_1 - f_1g_2)^k, \quad ijk \in \Delta_2(a).$$

6.3 The case $(d_1, d_2) = (2, 1)$

Once more, we recall that $Z \subset \mathbb{P}^2 \times \mathbb{P}^n$ is a non-singular complete intersection of irreducible normal hypersurfaces V_1, V_2, with V_1 non-singular, defined by equations $r_1, r_2 \in \mathcal{R}(X)$ of degrees $(d_1, e_1), (d_2, e_2)$ respectively, where $d_1, d_2, e_1, e_2 \geqslant 1$. With the case $d_1 = d_2 = 1$ considered in the previous Section, we now turn to $(d_1, d_2) = (2, 1)$. There are degree e_1 polynomials $f_1, \ldots, f_6 \in R$, respectively degree e_2 polynomials $g_1, g_2, g_3 \in R$, so that

$$r_1 = f_1x_0^2 + f_2x_0x_1 + f_3x_0x_2 + f_4x_1^2 + f_5x_1x_2 + f_6x_2^2, \quad r_2 = g_1x_0 + g_2x_1 + g_3x_2.$$

We will show that in the general case, Z is a Mori Dream Space with Cox ring of a certain form. We will also describe exactly when this general case occurs in terms of the polynomial coefficients $f_1, \ldots, g_3$ of r_1 and r_2.

6.3.1 Geometry

Let $Z \to \mathbb{P}^n$ be the projection to the second factor. The general fibre is two points (with multiplicity) in $\mathbb{P}^2$. The behaviour of remaining fibres is more complicated, we may obtain lines, degenerate conics or the entire $\mathbb{P}^2$ over loci determined by the polynomial coefficients $f_1, \ldots, g_3$.

In Chapter 4, we considered (a general) degree (d, e) hypersurface in $\mathbb{P}^1 \times \mathbb{P}^n$. In the case with $d = 2$, we constructed a birational map exchanging the two points of a general fibre. We take inspiration from this to define a birational map from our Z, a subvariety of $\mathbb{P}^2 \times \mathbb{P}^n$, to itself.

For each $i = 0, 1, 2,$ we may eliminate the variable x_i from the equations r_1, r_2. For example, multiplying r_1 by g_3^2 we may eliminate x_2:

$$0 = f_1 g_3^2 x_0^2 + f_2 g_3^2 x_0 x_1 + f_3 g_3 x_0(-g_1 x_0 - g_2 x_1) + f_4 g_3^2 x_1^2 + f_5 g_3 x_1(-g_1 x_0 - g_2 x_1) + f_6(-g_1 x_0 - g_2 x_1)^2$$

$$(f_1 g_3^2 - f_3 g_1 g_3 + f_6 g_1^2)x_0^2 + (f_2 g_3^2 - f_3 g_2 g_3 - f_5 g_1 g_3 + 2 f_6 g_1 g_2)x_0 x_1 + (f_4 g_3^2 - f_5 g_2 g_3 + f_6 g_2^2)x_1^2.$$

This equation holds for any point in Z. Similarly, we obtain

$$(f_1 g_2^2 - f_2 g_1 g_2 + f_4 g_1^2)x_0^2 + (f_2 g_2 g_3 - f_3 g_2^2 - 2 f_4 g_1 g_3 + f_5 g_1 g_2)x_0 x_2 + (f_4 g_3^2 - f_5 g_2 g_3 + f_6 g_2^2)x_2^2 = 0$$
$$(f_1 g_2^2 - f_2 g_1 g_2 + f_4 g_1^2)x_1^2 + (f_2 g_1 g_3 + f_3 g_1 g_2 - f_5 g_1^2 - 2 f_1 g_2 g_3)x_1 x_2 + (f_1 g_3^2 - f_3 g_1 g_3 + g_6 g_1^2)x_2^2 = 0$$

Each of these equations is of the form $F_j x_0^2 + G_{ij} x_0 x_1 + F_i x_j^2 = 0$ for polynomials $F_0, F_1, F_2, G_{01}, G_{02}, G_{12} \in R$ of degree $e_1 + 2e_2$. We set thus once and for all

$$\begin{aligned} F_0 &= f_4 g_3^2 - f_5 g_2 g_3 + f_6 g_2^2, \\ F_1 &= f_1 g_3^2 - f_3 g_1 g_3 + f_6 g_1^2, \\ F_2 &= f_1 g_2^2 - f_2 g_1 g_2 + f_4 g_1^2, \\ G_{01} &= f_2 g_3^2 - f_3 g_2 g_3 - f_5 g_1 g_3 + 2 f_6 g_1 g_2, \\ G_{02} &= f_2 g_2 g_3 - f_3 g_2^2 - 2 f_4 g_1 g_3 + f_5 g_1 g_2, \\ G_{12} &= f_2 g_1 g_3 + f_3 g_1 g_2 - f_5 g_1^2 - 2 f_1 g_2 g_3. \end{aligned} \tag{45}$$

These polynomials feature throughout this section and in the description of the Cox ring.

Proposition 6.14. *There are sections $z_0, z_1, z_2 \in \mathcal{R}(Z)$ of degree $(-1, e_1 + 2e_2)$ satisfying the equations*

$$\begin{aligned} x_0 z_0 &= F_0, & x_1 z_1 &= F_1, & x_2 z_2 &= F_2, \\ x_0 z_1 + x_1 z_0 &= -G_{01}, & x_0 z_2 + x_2 z_0 &= -G_{02}, & x_1 z_2 + x_2 z_1 &= -G_{12}, \end{aligned} \tag{46}$$

in $\mathcal{R}(Z)$. Moreover, there is a birational map $\phi : Z \dashrightarrow Z$ defined by

$$\phi : ([x_0 : x_1 : x_2], [y_0 : \cdots : y_n]) \mapsto ([z_0 : z_1 : z_2], [y_0 : \cdots : y_n]),$$

which is regular away from the locus over $Y \subset \mathbb{P}^n$ defined by the ideal $I_Y = \langle F_0, F_1, F_2, G_{01}, G_{02}, G_{12} \rangle$. Over the general fibre, this interchanges the two points.

Proof. We have established above that

$$F_j x_i^2 + G_{ij} x_i x_j + F_i x_j^2 = 0$$

for each $0 \leqslant i < j \leqslant 2$. The sections z_0, z_1, z_2, defined by

$$z_0 = \frac{F_0}{x_0} = -\frac{F_1x_0 + G_{01}x_1}{x_1^2} = -\frac{F_2x_0 + G_{02}x_2}{x_2^2},$$

$$z_1 = -\frac{G_{01}x_0 + F_0x_1}{x_0^2} = \frac{F_1}{x_1} = -\frac{F_2x_1 + G_{12}x_2}{x_2^2},$$

$$z_2 = -\frac{G_{02}x_0 + F_0x_2}{x_0^2} = -\frac{G_{12}x_1 + F_1x_2}{x_1^2} = \frac{F_2}{x_2},$$

are then seen to be globally defined on Z, in the degrees and satisfying the equations given in the Proposition statement. The vanishing locus of z_0, z_1, z_2 is precisely Y. If $y \in \mathbb{P}^n$ is so that all three equations $F_jx_i^2 + G_{ij}x_ix_j + F_ix_j^2$ do not vanish, then its fibre consists of two points and the equations (46) imply that $[z_i : z_j]$ is the second root of $F_jx_i^2 + G_{ij}x_ix_j + F_ix_j^2$ for each $0 \leqslant i < j \leqslant 2$. In particular, $[z_0 : z_1 : z_2]$ is the other point on Z, and ϕ is birational, interchanging the two points of a general fibre. $\square$

In the general case, which will be when the locus $Y \subset \mathbb{P}^n$ has codimension 3, we will see that Z is a Mori Dream Space. In this case, the effective cone decomposes into two full dimensional Mori chambers, whose models are each isomorphic to Z, related by the flop ϕ defined in Proposition 6.14. We can already prove this, without computing the full Cox ring. We will use our method to compute the Cox ring. As usual, our method gives us much more information on what happens in special cases. If Y has codimension less than three, then ϕ is not an isomorphism in codimension one and the Cox ring differs from the form of the general case. This will be formalised in the rest of this Section.

6.3.2 The space K_0 and its lift to $\mathcal{R}(Z)$

As in Definition 6.3 we have spaces T_{-a}, T_a' for each $-a \leqslant \mathbb{Z}$ defined by

$$T_{-a} = \bigoplus_{b \in \mathbb{Z}} H^2(X, \mathcal{O}_X(-a-3, b-e_1-e_2))$$

$$T_{-a}' = \bigoplus_{b \in \mathbb{Z}} \left(H^2(X, \mathcal{O}_X(-a-1, b-e_2)) \oplus H^2(X, \mathcal{O}_X(-a-2, b-e_1))\right),$$

and maps $B_{-a} = (r_{1*}, r_{2*}) : T_{-a} \to T_{-a}'$ with kernels K_a. By Proposition 6.7, we have $K_{-a} \neq 0$ if and only if $a \leqslant 0$ and that $K_0 \cong R[e_1 + e_2]$, with standard module generator corresponding to $h = \frac{1}{x_0x_1x_2}$. We can perform αh explicitly.

We have

$$r_1h = r_1\frac{1}{x_0x_1x_2} = \frac{f_1x_0}{x_1x_2} + \frac{f_2}{x_2} + \frac{f_3}{x_1} + \frac{f_4x_1}{x_0x_2} + \frac{f_5}{x_0} + \frac{f_6x_2}{x_0x_1}.$$

The 1-cochain $(s_{ij,kl})$ on X defined by

$$s_{ij,kl} = \begin{cases} \frac{f_6x_2}{x_0x_1} + \frac{1}{2}\frac{f_5}{x_0} + \frac{1}{2}\frac{f_3}{x_1} & \text{if } (i,k) = (0,1), \\ -\frac{f_4x_1}{x_0x_2} - \frac{1}{2}\frac{f_5}{x_0} - \frac{1}{2}\frac{f_2}{x_2} & \text{if } (i,k) = (0,2), \\ \frac{f_1x_0}{x_1x_2} + \frac{1}{2}\frac{f_3}{x_1} + \frac{1}{2}\frac{f_2}{x_2} & \text{if } (i,k) = (1,2), \\ 0 & \text{else,} \end{cases}$$

maps to $r_{1*}h$ under the Čech differential, and restricts to a 1-cocycle on V_1. The 1-cochain $r_{2*}(s_{ij,kl}) = (r_2 s_{ij,kl})$, which can be viewed on X or V_1, is then given by

$$(r_{2*}s)_{ij,kl} = \begin{cases} \frac{f_6 g_3 x_2^2}{x_0 x_1} + \frac{\frac{1}{2} f_5 g_2 x_1 + \frac{1}{2} f_5 g_3 x_2 + f_6 g_2 x_2}{x_0} + \frac{\frac{1}{2} f_3 g_1 x_0 + \frac{1}{2} f_3 g_3 x_2 + f_6 g_1 x_2}{x_1} + \frac{1}{2} f_3 g_2 + \frac{1}{2} f_5 g_1 & \text{if } (i,k) = (0,1), \\ -\frac{f_4 g_2 x_1^2}{x_0 x_2} - \frac{f_4 g_3 x_1 + \frac{1}{2} f_5 g_2 x_1 + \frac{1}{2} f_5 g_3 x_2}{x_0} - \frac{\frac{1}{2} f_2 g_1 x_0 + \frac{1}{2} f_2 g_2 x_1 + f_4 g_1 x_1}{x_2} - \frac{1}{2} f_2 g_3 - \frac{1}{2} f_5 g_1 & \text{if } (i,k) = (0,2), \\ \frac{f_1 g_1 x_0^2}{x_1 x_2} + \frac{f_1 g_3 x_0 + \frac{1}{2} f_3 g_1 x_0 + \frac{1}{2} f_3 g_3 x_2}{x_1} + \frac{f_1 g_2 x_0 + \frac{1}{2} f_2 g_1 x_0 + \frac{1}{2} f_2 g_2 x_1}{x_2} + \frac{1}{2} f_2 g_3 + \frac{1}{2} f_3 g_2 & \text{if } (i,k) = (1,2), \\ 0 & \text{else.} \end{cases}$$

Define now

$$q_{01} = -\frac{g_3}{x_0 x_1}, \quad q_{02} = \frac{g_2}{x_0 x_2}, \quad q_{12} = -\frac{g_1}{x_1 x_2}.$$

We can add to $(r_{2*}s_{ij,kl})$ the 1-cochain $r_{1*}(q_{ik,kl})$, where

$$q_{ij,kl} = \begin{cases} q_{01} & \text{if } (i,k) = (0,1), \\ q_{02} & \text{if } (i,k) = (0,2), \\ q_{12} & \text{if } (i,k) = (1,2), \\ 0 & \text{else.} \end{cases}$$

Then $r_2 s_{ij,kl} + r_1 q_{ij,kl} = u_i - u_j$ for each $0 \leqslant i < j \leqslant 2$, where

$$u_i = \begin{cases} \frac{-f_4 g_3 x_1 + \frac{1}{2} f_5 g_2 x_1 - \frac{1}{2} f_5 g_3 x_2 + f_6 g_2 x_2}{x_0} + \frac{1}{2} f_2 g_3 - \frac{1}{2} f_3 g_2 & \text{if } i = 0, \\ \frac{f_1 g_3 x_0 - \frac{1}{2} f_3 g_1 x_0 + \frac{1}{2} f_3 g_3 x_2 - f_6 g_1 x_2}{x_1} - \frac{1}{2} f_1 g_5 + \frac{1}{2} f_3 g_2 & \text{if } i = 1, \\ \frac{-f_1 g_2 x_0 + \frac{1}{2} f_2 g_1 x_0 - \frac{1}{2} f_2 g_2 x_1 + f_4 g_1 x_1}{x_2} + \frac{1}{2} f_1 g_5 - \frac{1}{2} f_2 g_3 & \text{if } i = 2. \end{cases}$$

By construction, (u_i) restricts to a 0-cocycle on Z, defining a section on $\mathcal{R}(Z)$ in degree $(0, e_1 + e_2)$, which we set to be $-u$, to make our final Cox ring compatible with the result in Chapter 7. That is, we take $u = -\alpha h$ represented by the 0-coycle $(-u_i)$ on Z defined above.

Remark 6.15. Whilst we have illustrated how to construct α in this case, we could have skipped this step using the work in Section 6.2.2. First, we rewrite the defining equation r_1 as

$$r_1 = f_1' x_0 + f_2' x_1 + f_3' x_2,$$

where

$$\begin{aligned} f_1' &= f_1 x_0 + \frac{1}{2} f_2 x_1 + \frac{1}{2} f_3 x_2, \\ f_2' &= \frac{1}{2} f_2 x_0 + f_4 x_1 + \frac{1}{2} f_5 x_2, \\ f_3' &= \frac{1}{2} f_3 x_0 + \frac{1}{2} f_5 x_1 + f_6 x_2. \end{aligned}$$

A point $(x, y) \in \mathbb{P}^2 \times \mathbb{P}^n$ satisfies $r_1 = r_2 = 0$ if and only if x lies in the kernel of

$$M' = \begin{pmatrix} f_1' & f_2' & f_3' \\ g_1 & g_2 & g_3 \end{pmatrix},$$

a matrix depending on x and y. Let ν_i, for $i = 0, 1, 2$, denote the 2×2 minor of M' obtained by deleting the i^{th} column. Now applying α to h can be performed exactly as in Section 6.2.2, with f_1, f_2, f_3 replaced by f_1', f_2', f_3'. As there, we get the section

$$\alpha h = -u = -\frac{\nu_0}{x_0} = \frac{\nu_1}{x_1} = -\frac{\nu_2}{x_2},$$

which is exactly our $-u$ obtained above.

6.3.3 The space K_1 and its lift to $\mathcal{R}(Z)$

The space T_{-1} is a free R-module of rank 3, so we equip it with a standard basis given by $\frac{1}{x_0^2 x_1 x_2}$, $\frac{1}{x_0 x_1^2 x_2}$, $\frac{1}{x_0 x_1 x_2^2}$. By Lemma 6.5, a $h = (h_1, h_2, h_3) \in T_{-1}$ is in K_1 if and only if $g_1 h_1 + g_2 h_2 + g_3 h_3 = 0$. In degrees $(-a, b)$ with $-a = -1$, the map $\alpha : K_a \to \bigoplus_{b \in \mathbb{Z}} \mathcal{R}(Z)_{(-a,b)}$ is an isomorphism, so we need not worry about choices for α. If $h \in K_1$ we can apply the procedure of Section 6.1.4 to obtain $\alpha h = z_h$, where z_h is given by the 0-cocycle $(z_{h,ij})$ on Z given by

$$z_{h,ij} = \begin{cases} \frac{-h_1 f_4 g_3 x_1 + \frac{1}{2} h_1 f_5 g_2 x_1 - \frac{1}{2} h_1 f_5 g_3 x_2 + h_1 f_6 g_2 x_2}{x_0^2} + \frac{-h_1 f_2 g_3 + h_1 f_3 g_2 - h_2 f_4 g_3 + \frac{1}{2} h_1 f_5 g_1 + h_2 f_5 g_2 + h_3 f_6 g_2}{x_0} & \text{if } i = 0, \\ \frac{h_2 f_1 g_3 x_0 - \frac{1}{2} h_2 f_3 g_1 x_0 + \frac{1}{2} h_2 f_3 g_3 x_2 - h_2 f_6 g_1 x_2}{x_1^2} + \frac{h_1 f_1 g_3 + h_2 f_2 g_3 - h_1 f_3 g_1 - \frac{1}{2} h_2 f_3 g_2 - h_2 f_5 g_1 - h_3 f_6 g_1}{x_1} & \text{if } i = 1, \\ \frac{-h_3 f_1 g_2 x_0 + \frac{1}{2} h_3 f_2 g_1 x_0 - \frac{1}{2} h_3 f_2 g_2 x_1 + h_3 f_4 g_1 x_1}{x_2^2} + \frac{-h_1 f_1 g_2 - h_2 f_2 g_2 - \frac{1}{2} h_3 f_2 g_3 - h_3 f_3 g_2 + h_2 f_4 g_1 + h_3 f_5 g_1}{x_2} & \text{if } i = 2, \end{cases}$$

By considering $x_0 z_{h,0j}$ and $h_1 u_{0j}$ for any j we spot the equation

$$x_0 z_h = -h_1 u - \frac{1}{2} h_1 f_2 g_3 + \frac{1}{2} h_1 f_3 g_2 - h_2 f_4 g_3 + \frac{1}{2} h_1 f_5 g_1 + h_2 f_5 g_2 + h_3 f_6 g_2$$

in $\mathcal{R}(Z)$. Similarly we obtain the equations

$$\begin{aligned} x_1 z_h &= -h_2 u + h_1 f_1 g_3 + \frac{1}{2} h_2 f_2 g_3 - h_1 f_3 g_1 - \frac{1}{2} h_2 f_3 g_2 - \frac{1}{2} h_2 f_5 g_1 - h_3 f_6 g_1, \\ x_2 z_h &= -h_3 u - h_1 f_1 g_2 - h_2 f_2 g_2 - \frac{1}{2} h_3 f_2 g_3 - \frac{1}{2} h_3 f_3 g_2 + h_2 f_4 g_1 + \frac{1}{2} h_3 f_5 g_1. \end{aligned}$$

It is worth noting that the way these equations are written formally in terms of the variables h_1, h_2, h_3 is not unique, since we have the relation $h_1 g_1 + h_2 g_2 + h_3 g_3 = 0$.

Consider now the Koszul generators

$$h^{(0)} = (0, -g_3, g_2), \quad h^{(1)} = (g_3, 0, -g_1), \quad h^{(2)} = (-g_2, g_1, 0).$$

We see that z_i as defined in Proposition 6.14 is equal to $\alpha h^{(i)}$ for each $i = 0, 1, 2$. We see the already known relation $g_1 z_0 + g_2 z_1 + g_3 z_2 = 0$ again.

Proposition 6.16. *The sections z_0, z_1, z_2, u, on $\mathcal{R}(Z)$ satisfy the equations*

$$\begin{aligned} x_0z_0 &= F_0, \quad x_1z_0 = g_3u - \tfrac{1}{2}G_{01}, \quad x_2z_0 = -g_2u - \tfrac{1}{2}G_{01}, \\ x_0z_1 &= -g_3u - \tfrac{1}{2}G_{01}, \quad x_1z_0 = F_1, \quad x_2z_1 = g_1u - \tfrac{1}{2}G_{12}, \\ x_0z_2 &= g_2u - \tfrac{1}{2}G_{02}, \quad x_1z_0 = -g_1u - \tfrac{1}{2}G_{12}, \quad x_2z_2 = F_2. \end{aligned}$$

These equations imply those found in Proposition 6.14. Furthermore, $u^2 = \omega$ in $\mathcal{R}(Z)$, where

$$\omega = \frac{G_{01}^2 - 4F_0F_1}{4g_3^2} = \frac{G_{02}^2 - 4F_0F_2}{4g_2^2} = \frac{G_{12}^2 - 4F_1F_2}{4g_1^2}.$$

in $R \subset \mathcal{R}(Z)$. In particular, u vanishes on those points of Z where the fibre consists of a double point.

Proof. These equations can be seen by comparing their defining 0-cocycles, which are defined explicitly above. From these equations, we can find the corresponding expressions for u^2, which must therefore be equal. For example

$$F_0F_1 = x_0z_0x_1z_1 = \left(g_3u - \frac{1}{2}G_{01}\right)\left(-g_3u - \frac{1}{2}G_{01}\right) = -g_3^2 + \frac{1}{4}G_{01}^2,$$

which rearranges to the given result. That $\omega \in R$ can be checked explicitly. From the three equal expressions for ω, we see that u^2 and thus u, vanishes when the discriminants of the quadratics $F_jx_i^2 + G_{ij}x_ix_j + F_ix_j^2$ vanish, which includes those fibres that are double points. $\square$

Corollary 6.17. *Suppose that g_1, g_2, g_3 form a regular sequence in R. Then $\bigoplus_{b \in \mathbb{Z}} \mathcal{R}(Z)_{(-1,b)}$ is isomorphic as an R-module to the quotient of $R^3[e_1 + 2e_2]$, with generators corresponding to z_0, z_1, z_2, by the submodule generated by the degree $e_1 + 3e_2$ element corresponding to $g_1z_0 + g_2z_1 + g_3z_2$.*

Proof. If g_1, g_2, g_3 is regular in R then B_{-1} is the first map of the Koszul complex for the sequence g_1, g_2, g_3 and the result follows. $\square$

Remark 6.18. There is another way one may see that u^2 is contained in R. The equations in Proposition 6.16 imply that $x_0z_1 - x_1z_0 = 2g_3u$. Let $y \in \mathbb{P}^n$ be so that $g_3(y) \neq 0$ and the fibre of Z over Y is 0-dimensional. Let x, x' be the points (possibly equal) over y. Then the sections x_0, x_1, x_2 are the coordinate functions and the sections z_0, z_1, z_2 are the coordinate functions for the other point over y. Evaluating u at both of these points we thus obtain

$$2g_3u(x,y) = (x_0z_1 - x_1z_0)(x,y) = x_0x_1' - x_1x_0' = -(x_0z_1 - x_1z_0)(x',y) = -2g_3u(x',y).$$

In particular, u takes opposite sign on the two points over a general fibre (which once again shows that u vanishes along the double point fibres). This implies that u^2 is independent of x and thus the pullback of a function on $\mathbb{P}^n$. This function is unique since Z is surjective onto $\mathbb{P}^n$.

Corollary 6.19. *Recall the degree $(1, e_1 + e_2)$ sections μ_0, μ_1, μ_2 defined in Remark 6.15. Let μ_0', μ_1', μ_2' be the same polynomials except with each x_i replaced with z_i, thus giving degree $(-1, 2e_1 + 3e_2)$ sections. Then we also have*

$$u = -\frac{\mu_0'}{z_0} = \frac{\mu_1'}{z_1} = -\frac{\mu_2'}{z_2}.$$

In particular, we obtain equations

$$uz_0 + \mu_0', \quad uz_1 - \mu_1', \quad uz_2 + \mu_2'$$

in degree $(-1, 2e_1 + 3e_2)$ on $\mathcal{R}(Z)$.

Proof. Since the sections z_0, z_1, z_2 define a rational map from Z to itself, we have

$$f_1 z_0^2 + f_2 z_0 z_1 + f_3 z_0 z_2 + f_4 z_1^2 + f_5 z_1 z_2 + f_6 z_2^2 = g_1 z_0 + g_2 z_1 + g_3 z_2 = 0,$$

and thus by symmetrical arguments as for the x_i', we conclude the existence of a section

$$u' = \frac{\mu_0'}{z_0} = -\frac{\mu_1'}{z_1} = \frac{\mu_2'}{z_2}.$$

We just need to show that $u' = -u$. To this end, let $y \in \mathbb{P}^n$ be so its fibre consists of two points x, x' in $\mathbb{P}^2$. We can evaluate u' at (x, y). By Remark 6.18, we have

$$u'(x,y) = \frac{\mu_0'(x,y)}{z_0(x,y)} = \frac{\mu_0(x',y)}{x_0(x',y)} = u(x',y) = -u(x,y).$$

Since $u' = -u$ on a dense open subset, it is true everywhere. $\square$

6.3.4 The space K_2

We have seen that K_0 is simply a free R-module of rank 1, and that K_1 can be expressed using a resolution of $R/(g_1, g_2, g_3)$. It is first with $K_2 = \ker B_{-2}$ that we have both defining equations giving conditions on coefficients. An element

$$h = (h_1, h_2, h_3, h_4, h_5, h_6) = (h_{200}, h_{110}, h_{101}, h_{020}, h_{011}, h_{001}) \in T_{-2}$$

lies in K_2 when

$$h_1 g_1 + h_2 g_2 + h_3 g_3 = 0, \qquad h_2 g_1 + h_4 g_2 + h_5 g_3 = 0, \qquad h_3 g_1 + h_5 g_2 + h_6 g_3 = 0,$$
$$h_1 f_1 + h_2 f_2 + h_3 f_3 + h_4 f_4 + h_5 f_5 + h_6 f_6 = 0,$$

are all satisfied. The matrix of $B_2 : R^6 \to R^4$ is thus given by

$$B_2 = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 \\ g_1 & g_2 & g_3 & 0 & 0 & 0 \\ 0 & g_1 & 0 & g_2 & g_3 & 0 \\ 0 & 0 & g_1 & 0 & g_2 & g_3 \end{pmatrix}.$$

Proposition 6.20. *View $B_2 : R^6 \to R^4$ by its matrix given above. Let ξ_{ij} be the 4×4 minor of B_2 obtained ignoring the i and j columns for $1 \leqslant i < j \leqslant 6$ and introduce the standard sign conventions $\xi_{ij} = -\xi_{ji}$ for $i > j$ and $\xi_{ii} = 0$ for each i. There is an R-module complex*

$$0 \longrightarrow R^4 \xrightarrow{B_2^T} R^6 \xrightarrow{P} R^6 \xrightarrow{B_2} R^4, \tag{47}$$

where P is the skew symmetric matrix $((-1)^{i+j}\xi_{ij})$. Furthermore, the columns of P, which are thus elements of K_2, map under α to the sections $z_0^2, z_0z_1, z_0z_2, z_1^2, z_1z_2, z_2^2$. The relations given by the columns of B_2^T are then those obtained from the equations $r_1(z_i) = r_2(z_i) = 0$.

Proof. The complex (47) is the Buchsbaum-Rim complex of B_2, thus a complex – see [13, §A.2.6.1] for details. We only need to show that the columns of P correspond to the degree 2 monomials in the z_i. We use the following Lemma.

Lemma 6.21. *Let $\tilde{z}_i : K \to K$ denote the maps $\delta(\cdot z_i)\alpha$ for each $i = 0, 1, 2$. Suppose $w \in \mathcal{R}(Z)$ has degree $(-1, b)$ for some $b \in \mathbb{Z}$ and*

$$\delta w = h = (h_1, h_2, h_3) = \begin{pmatrix} & h_2 & \\ h_1 & & h_3 \end{pmatrix} \in K_1.$$

For each $i = 0, 1, 2$ and $j = 1, \ldots, 6$ let h'_{ij} be so that

$$\tilde{z}_i h = \begin{pmatrix} & & h'_{i4} & & \\ & h'_{i2} & & h'_{i5} & \\ h'_{i1} & & h'_{i3} & & h'_{i6} \end{pmatrix}.$$

Then we have

$$h'_{01} = F_0h_1, \qquad h'_{02} = F_0h_2, \qquad h'_{03} = F_0h_3$$
$$h'_{04} = -F_1h_1 - G_{01}h_2, \qquad h'_{06} = -F_2h_1 - G_{02}h_3$$
$$h'_{05} = f_1g_2g_3h_1 + f_2g_2g_3h_2 - f_3g_1g_2h_1 - f_3g_2^2h_2 - f_4g_1g_3h_2 - f_6g_1g_2h_3$$

$$h'_{12} = F_1h_1, \qquad h'_{14} = F_1h_2, \qquad h'_{15} = F_1h_3$$
$$h'_{11} = -F_0h_2 - G_{01}h_1, \qquad h'_{16} = -F_2h_2 - G_{12}h_3$$
$$h'_{13} = -f_1g_2g_3h_1 + f_2g_1g_3h_1 + f_4g_1g_3h_2 - f_5g_1^2h_1 - f_5g_1g_2h_2 - f_6g_1g_2h_3,$$

$$h'_{23} = F_2h_1, \qquad h'_{25} = F_2h_2, \qquad h'_{26} = F_2h_3$$
$$h'_{21} = -F_0h_3 - G_{02}h_1, \qquad h'_{24} = -F_1h_3 - G_{12}h_2$$
$$h'_{22} = -f_1g_2g_3h_1 + f_3g_1g_2h_1 - f_4g_1g_3h_2 - f_5g_1^2h_1 - f_5g_1g_3h_3 + f_6g_1g_2h_3.$$

Proof. The equations $x_iz_i = F_i$ in $\mathcal{R}(Z)$ imply that $\tilde{x}_i\tilde{z}_ih = F_ih$. Since the maps $\tilde{x}_i$ have the description as in Lemma 6.8, we obtain the equations for

$$h'_{01}, h'_{02}, h'_{03}, h'_{12}, h'_{14}, h'_{15}, h'_{23}, h'_{25}, h'_{26}.$$

Similarly, we have $(\tilde{x}_i\tilde{z}_j + \tilde{x}_j\tilde{z}_j)h = -G_{ij}h$ for each $i \neq j$, which along with the already known h'_{ij} allow us to deduce the equations for

$$h'_{04}, h'_{06}, h'_{11}, h'_{16}, h'_{21}, h'_{24}.$$

The equations for $h'_{05}, h'_{13}, h'_{22}$ can then be deduced from the others using $\tilde{z}_i h \in K_2$ for each i. $\square$

Returning to the proof of Proposition 6.20, we can use this Lemma to compute $\delta z_i z_j$ for each $0 \leqslant i \leqslant j \leqslant 2$, and we see that these give precisely the columns of P. $\square$

Corollary 6.22. *Recall that $Y \subset \mathbb{P}^n$ is the vanishing locus of the ideal generated by $F_0, F_1, F_2, G_{01}, G_{02}, G_{12}$. If the codimension of Y is 3, then the complex (47) is exact and gives a resolution of K_2, in particular $K_2 \cong R^6[2e_1 + 4e_2]/\operatorname{im} B_2^T$. If the codimension of Y is less than 3, then the complex (47) is not exact, and K_2 is not isomorphic to $R^6[2e_1 + 4e_2]/\operatorname{im} B_2^T$.*

Proof. By [13, Theorem A.2.10], the complex (47) is exact if and only if the ideal generated by the maximal minors ξ_{ij} of B_2 has depth 3. Lemma 6.21 implies that these maximal minors vanish on Y and the statement follows. $\square$

6.3.5 Proof of Main Theorem 2.17 for $c = 2$

We conclude by proving Main Theorem 2.17 in the case $c = 2$. We recall our set-up. We define $Z \subset \mathbb{P}^2 \times \mathbb{P}^n$ to be the complete intersection of two irreducible hypersurfaces V_1, V_2 defined respectively by equations

$$r_1 = f_1x_0^2 + f_2x_0x_1 + f_3x_0x_2 + f_4x_1^2 + f_5x_1x_2 + f_6x_2^2, \quad r_2 = g_1x_0 + g_2x_1 + g_3x_2,$$

where $f_1, \ldots, f_6$, respectively g_1, g_2, g_3, are polynomials in $R = \mathbb{C}[y_0, \ldots, y_n]$ of degree $e_1 \geqslant 1$, respectively $e_2 \geqslant 1$. We assume that V_1 and Z are non-singular so that the inclusions $Z \hookrightarrow V_1 \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^n$ induce isomorphisms on Picard groups. We define the polynomials $F_0, F_1, F_2, G_{01}, G_{02}, G_{12} \in R$ as in (45). Let $Y \subset \mathbb{P}^n$ be the vanishing locus of the ideal generated by $F_0, F_1, F_2, G_{01}, G_{02}, G_{12}$. Since Y contains the locus on which $g_1 = g_2 = g_3 = 0$ we have $\operatorname{codim} Y \leqslant 3$.

Theorem 6.23. *Let $Z \subset \mathbb{P}^2 \times \mathbb{P}^n$ be as above. Let A be the free $\mathbb{Z}^2$-graded algebra*

$$A = \mathbb{C}[X_0, X_1, X_2, Y_0, \ldots, Y_n, Z_0, Z_1, Z_2, U],$$

with degrees

$$\deg X_i = (1, 0), \quad \deg Y_j = (0, 1), \quad \deg Z_i = (-1, e_1 + 2e_2), \quad \deg U = (0, e_1 + e_2),$$

for relevant i, j. There is a morphism of graded algebras $\psi : A \to \mathcal{R}(Z)$ sending $X_i \mapsto x_i$, $Y_j \mapsto y_j$, $Z_i \mapsto z_i$, $U \mapsto u$, where z_0, z_1, z_2, u are the sections defined in Proposition 6.14 and Section 6.3.2. Let $J \lhd A$ be the *homogeneous ideal generated by the equations*

$$\begin{gathered} r_1(X_i,Y_j), \quad r_2(X_i,Y_j), \quad r_1(Z_i,Y_j), \quad r_2(Z_i,Y_j), \\ X_0Z_0 - F_0(Y_j), \quad X_1Z_0 - g_3(Y_j)U + \tfrac{1}{2}G_{01}(Y_j), \quad X_2Z_0 + g_2(Y_j)U + \tfrac{1}{2}G_{02}(Y_j), \\ X_0Z_1 + g_3(Y_j)U + \tfrac{1}{2}G_{01}(Y_j), \quad X_1Z_1 - F_1(Y_j), \quad X_2Z_1 - g_1(Y_j)U + \tfrac{1}{2}G_{12}(Y_j), \\ X_0Z_2 - g_2(Y_j)U + \tfrac{1}{2}G_{02}(Y_j), \quad X_1Z_2 + g_1(Y_j)U + \tfrac{1}{2}G_{12}(Y_j), \quad X_2Z_2 - F_2(Y_j), \\ X_0U - \nu_0(X_i,Y_j), \quad X_1U + \nu_1(X_i,Y_j), \quad X_2U - \nu_2(X_i,Y_j), \\ Z_0U + \nu_0(Z_i,Y_j), \quad Z_1U\nu_1(Z_i,Y_j), \quad Z_2U - \nu_2(Z_i,Y_j), \\ U^2 + \omega(Y_j), \end{gathered}$$

where the $\nu_i(x_i,y_j)$ are defined in Remark 6.15 and $\omega(y_j)$ is defined in Proposition 6.16. Then $J \subset \ker\psi$.

(i) Suppose the space $W \subset \mathbb{P}^n$ has codimension 3. Then ψ induces an isomorphism of graded algebras $\mathcal{R}(Z) \cong A/J$. The space Z is a Mori Dream Space with two maximal Mori chambers, each giving minimal models isomorphic to Z, related by a flop which exchanges the two points of a general fibre over $\mathbb{P}^n$.

(ii) Suppose there is a presentation of graded rings of the form $\mathcal{R}(Z) \cong A/J$. Then $\operatorname{codim} W = 3$.

Proof. By Proposition 6.14, Remark 6.15, Proposition 6.16, Corollary 6.19, the map ψ is well defined and its kernel contains K.

Suppose that $\operatorname{codim} W = 3$. By Proposition 6.14, we have a birational map ϕ of order two from Z to itself:

$$\phi([x_0 : x_1 : x_2], [y_0 : \cdots : y_n]) = ([z_0 : z_1 : z_2], [y_0 : \cdots : y_n]).$$

Moreover ϕ is regular away from W, thus an isomorphism in codimension one. Since ϕ is also a map over $\mathbb{P}^n$, any contracted fibre is in $\mathbb{P}^2$, thus has trivial intersection with the canonical divisor K_Z, which has class $(0, e_1 + e_2 - n - 1)$ on Z. It follows that ϕ is a flop from Z to itself, inducing an isomorphism $\phi^* : \mathcal{R}(Z) \to \mathcal{R}(Z)$ of Cox rings.

Surjectivity of ψ now follows since $u, x_0, x_1, x_2, y_0, \ldots, y_n$ generate the algebra $\bigoplus_{a\geqslant 0, b\in\mathbb{Z}} \mathcal{R}(Z)_{(a,b)}$, as it is isomorphic to $\iota^*\mathcal{R}(X) \oplus K_0$ and ϕ^* interchanges $\bigoplus_{a\geqslant 0, b\in\mathbb{Z}} \mathcal{R}(Z)_{(a,b)}$ and $\bigoplus_{-a\geqslant 0, b\in\mathbb{Z}} \mathcal{R}(Z)_{(a,b+a(e_1+2e_2))}$, so the latter ring is generated by $u, z_0, z_1, z_2, y_0, \ldots, y_n$.

The equations in J ensure that any degree (a,b) element in R/J with $a \geqslant 0$ may be written as a polynomial in $u, x_0, x_1, x_2, y_0, \ldots, y_n$ and that A/J maps injectively into $\mathcal{R}(Z)$ in these degrees. Since the ideal J is symmetrical with respect to swapping X_0, X_1, X_2, U with $Z_0, Z_1, Z_2, -U$ respectively, we can use the symmetry of the Cox ring induced by ϕ^* to conclude the isomorphisms $\mathcal{R}(Z) \cong A/J$ in degrees $(-a, b)$ with $-a \leqslant 0$. This concludes part (i).

Part (ii) follows immediately from Corollary 6.22. □

6.3.6 Examples

We provide the Cox rings of some examples $Z \subset X = \mathbb{P}^2 \times \mathbb{P}^n$ of the form (†) of Definition 6.1.

The first example remarks that $\operatorname{codim} Y = 3$, where Y is as defined in Proposition 6.14, occurs in the general case for a Calabi-Yau threefold Z of form (†) in $\mathbb{P}^2 \times \mathbb{P}^3$. We can expect it to then be the general case in all choices of parameters n, e_1, e_2.

For the second example, recall that

$$r_1 = f_1x_0^2 + f_2x_0x_1 + f_3x_0x_2 + f_4x_1^2 + f_5x_1x_2 + f_6x_2^2, \quad r_2 = g_1x_0 + g_2x_1 + g_3x_2.$$

are the defining equations of Z, we compute the Cox ring for a degenerate case, where f_2, f_3, f_5, g_2 are all zero.

Example 6.24. Let $n = 3$ so $X = \mathbb{P}^2 \times \mathbb{P}^3$ and choose

$$(e_1, e_2) \in \{(1,3), (2,2), (3,1)\}.$$

In Macaulay2, one verifies that for suitably random choices of degree e_1 polynomials $f_1, \dots, f_6$ and degree e_2 polynomials g_1, g_2, g_3 in R, that $\operatorname{codim} Y = 3$. By Theorem 6.23, we conclude that a general ample complete intersection Calabi-Yau threefold is a Mori Dream Space, admitting one flop of order two to itself.

As we saw in Chapter 4, our method has particular strength in understanding degenerate cases, and how the Cox ring changes form.

Example 6.25. Assume that $f_2 = f_3 = f_5 = g_2 = 0$, so we have

$$r_1 = f_1x_0^2 + f_4x_1^2 + f_6x_2^2, \quad r_2 = g_1x_0 + g_3x_2.$$

We temporarily treat the non-zero coefficients as formal symbols. Note that $G_{ij} = 0$ for each $0 \leqslant i < j \leqslant 2$. The space K_1 is the kernel of the map of R-modules

$$R^3 \xrightarrow{\begin{pmatrix} g_1 & 0 & g_3 \end{pmatrix}} R,$$

which is a free R-module of rank 2, generated by R-linearly indepentent generators $(g_3, 0, -g_1)$ and $(0, 1, 0)$. Using the work of Section 6.3.3, we have sections $w = \alpha(0,1,0)$ and $z = \alpha(g_3, 0, -g_1)$ satisfying equations

$$\begin{aligned} x_0w &= f_4g_3, & x_1w &= u, & x_2w &= -f_4g_1, \\ x_0z &= -g_3u, & x_1z &= F_1, & x_2z &= g_1u. \end{aligned}$$

We notice that u is no longer needed as a generator.

The space K_2 is the kernel of the matrix

$$\begin{pmatrix} f_1 & 0 & 0 & f_4 & 0 & f_6 \\ g_1 & 0 & g_3 & 0 & 0 & 0 \\ 0 & g_1 & 0 & 0 & g_3 & 0 \\ 0 & 0 & g_1 & 0 & 0 & g_3 \end{pmatrix},$$

which is also a free rank 2 R-module, generated by the linearly independent elements

$$h = (0, g_3, 0, 0, -g_1, 0), \quad h' = (f_4g_3^2, 0, -f_4g_1g_3, -f_6g_1^2 - f_1g_3^2, 0, f_4g_3^2).$$

This gives us sections $v = \alpha h$ and $t = \alpha h'$. Since, using Remark 6.9,

$$\tilde{x}_1 h = \tilde{x}_1 \begin{pmatrix} & 0 & \\ g_3 & & -g_1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} & 0 & \\ g_3 & & -g_1 \end{pmatrix},$$

we have $x_1 v = z$, and z is not needed as a generator. Using the equations $x_0 w = f_4 g_3, x_2 w = -f_4 g_1$, we can calculate $t = x_1 v w$. For example

$$\tilde{x}_0 \tilde{w} \begin{pmatrix} & 0 & \\ g_3 & & -g_1 \\ 0 & 0 & 0 \end{pmatrix} = \tilde{x}_0 \tilde{w} h = f_4 g_3 h = \begin{pmatrix} & 0 & \\ f_4 g_3^2 & & -f_4 g_1 g_3 \\ 0 & 0 & 0 \end{pmatrix}$$

gives the lower left 6 entries of vw, and the lower right entries of vw are computed similarly.

Theorem 6.26. *Let*

$$A = \mathbb{C}[X_0, X_1, X_2, Y_0, \ldots, Y_n, W, V]$$

be the free $\mathbb{Z}^2$-graded algebra with degrees $\deg X_i = (1,0)$, $\deg Y_j = (0,1)$, $\deg W = (-1, e_1 + e_2)$, $\deg V = (-2, e_1 + 2e_2)$. There is a well defined map of graded algebras $\psi : U \to \mathcal{R}(Z)$ mapping $X_i \mapsto x_i$, $Y_j \mapsto y_j$, $W \mapsto w$, $V \mapsto v$. The kernel of ψ contains the ideal J generated by the equations

$$r_1(X_i, Y_j), \quad r_2(X_i, Y_j),$$
$$X_0 W - f_4(Y_j) g_3(Y_j), \quad X_1^2 W + f_1(Y_j) g_3(Y_j) X_0 - f_6(Y_j) g_1(Y_j) X_2, \quad X_2 W + f_4(Y_j) g_1(Y_j),$$
$$X_0 V - g_3(Y_j) W, \quad X_1^2 V - F_1, \quad X_2 V + g_1(Y_j) W,$$
$$W^2 - f_4(Y_j) V.$$

Suppose that g_1, g_3 are coprime and that f_4 does not divide F_1. Then ψ induces an isomorphism of graded algebras $\mathcal{R}(Z) \cong U/J$.

Proof. The previous discussion gives the existence of $\psi : A \to \mathcal{R}(Z)$ and implies $J \subset \ker \psi$.

We now suppose that g_1, g_3 is regular in R and that f_4 does not divide F_1. It suffices to show that K_a is a free R-module of rank 2 for each $a \geqslant 1$, with module generators

$$\delta\left(v^{\frac{a+1}{2}} x_1\right), \quad \delta\left(w v^{\frac{a-1}{2}}\right) \qquad \text{if } a \text{ odd}$$
$$\delta\left(w v^{\frac{a}{2}} x_1\right), \quad \delta\left(v^{\frac{a}{2}}\right) \qquad \text{if } a \text{ even.}$$

The assumptions on f_4, g_1, g_3 imply this is true for $a = 1, 2$.

If $a \geqslant 3$, then consider an element $(h_{ijk})_{ijk \in \Delta_2(a)} \in K_a$. Since g_1, g_3 is regular, the defining equations

$$g_1 h_{i+1,jk} + g_3 h_{ij,k+1} = 0$$

of K_a imply that for each $j = 0, \dots, a$ we have

$$(h_{a-j,j,0}, \dots, h_{0,j,a-j}) = h_j(g_3^{a-j}, -g_1 g_3^{a-j-1} \dots, (-1)^{a-j} g_1^{a-j}).$$

The equations

$$f_1 h_{i+2,jk} + f_4 h_{i,j+2,k} + f_6 h_{ij,k+2} = 0$$

become

$$(-1)^k g_3^i g_1^k (h_j F_1 + f_4 h_{j+2}) = 0.$$

Since f_4 does not divide F_1, it follows that h is completely determined by the entries $h_{a,0,0}$ and $h_{a-1,1,0}$, which are of the form

$$h_{a,0,0} = p_1 f_4^{\lfloor \frac{a}{2} \rfloor} g_3^a, \quad h_{a-1,1,0} = p_2 f_4^{\lfloor \frac{a-1}{2} \rfloor} g_3^{a-1}.$$

Indeed, $h_{a,0,0}$ determines all h_{ijk} with j even and $h_{a-1,1,0}$ determines all h_{ijk} with j odd. This corresponds exactly to two independent R-module generators of K_a, which must correspond to the afforementioned sections up to scalars. Indeed, they are in the same degrees and these sections are R-linearly independent. One may also verify inductively that these module generators are given by the sections as above using the known equations in $\mathcal{R}(Z)$. □

6.4 Comments

Analagously to the Remarks in Section 4.5, Theorem 6.23 generalises to codimension 2 complete intersections in ambient spaces $X \subset \mathbb{P}^2 \times T$ with T a Mori Dream Space. We spare repeating the details.

7 On Cox rings of codimension c complete intersections in $\mathbb{P}^c \times \mathbb{P}^n$

We now treat the case of a sufficiently non-singular codimension $c \geqslant 2$ complete intersections in the ambient space $X = \mathbb{P}^c \times \mathbb{P}^n$ with $n > c$. We will see that Step (2) and Step (3) become too complex to manage by hand in general. We will take inspiration from the $c = 2$ case explored in Chapter 6 to give results for the case with each defining equation being linear in the first factor, and the case with all but one equation linear and the remaining quadratic in the first factor. The results of this Chapter can be seen to generalise those of Chapter 6 (and even Chapter 4).

7.1 Finding sections

7.1.1 Basics

Fix $X = \mathbb{P}^c \times \mathbb{P}^n$ with $n > c \geqslant 1$. Take homogeneous coordinates $x_0, \ldots, x_c$ on $\mathbb{P}^c$ and $y_0, \ldots, y_n$ on $\mathbb{P}^n$. There is an isomorphism of Picard groups $\operatorname{Pic} X \cong \mathbb{Z}^2$, generated by the pullbacks of a hyperplane section of each factor defining X. Writing $R = \mathcal{R}(\mathbb{P}^n) = \mathbb{C}[y_0, \ldots, y_n]$, the Cox ring of X is

$$S = \mathcal{R}(X) = \mathcal{R}(\mathbb{P}^c) \otimes \mathcal{R}(\mathbb{P}^n) \cong R[x_0, \ldots, x_c].$$

The graded subring R gives $\mathcal{R}(X)$ the structure of a graded R-module. We recall Definition 2.11.

Definition 7.1. For each $m = 1, \ldots, c$, let V_m be an irreducible and normal hypersurface defined by a degree (d_m, e_m) equation $r_m \in \mathcal{R}(X)$ for some $d_m, e_m \geqslant 1$. The variety $Z \subset X = \mathbb{P}^c = \mathbb{P}^n$ will be said to have form $(\dagger)$ with respect to the hypersurfaces $V_1, \ldots, V_c$ if the following hold.

(i) We have $Z = \bigcap_{m=1}^{c} V_m$.

(ii) The sequence $r_1, \ldots, r_c$ is regular in $\mathcal{R}(X)$ and the intermediary complete intersections

$$Z_m = \bigcap_{m'=1}^{m} V_{m'}$$

are non-singular for $m = 1, \ldots, c$.

All varieties $Z \subset \mathbb{P}^c \times \mathbb{P}^n$ in this section are of the form $(\dagger)$ with respect to hypersurfaces $V_1, \ldots, V_c$ as in Definition 7.1. Each inclusion $Z_{m-1} \hookrightarrow Z_m$, where by convention $Z_0 = X$, induces an isomorphism of Picard groups by the Lefschetz hyperplane theorem. We denote the common identification of all Picard groups $\operatorname{Pic} Z_m$ by Pic. We are in a setting to utilise the method developed in Section 3.4.

As an ordered affine cover we take the set $\mathcal{W} = \{W_{ij}\}$, where W_{ij} is the affine open of X where x_i, y_j are non-zero, as usual with lexicographic order. For $ij < kl$ we denote by $W_{ij,kl}$ the intersection $W_{ij} \cap W_{kl}$.

7.1.2 Describing the maps on cohomology

Step (1) is as straightforward as in Chapters 4 and 6.

Lemma 7.2. *(i) There are natural $\mathbb{Z}^2$-graded vector space isomorphisms*

$$\bigoplus_{(a,b)\in\mathbb{Z}^2} H^c(X, \mathcal{O}_X(a,b)) \cong \frac{1}{x_0 \cdots x_c} R[x_0^{-1}, \ldots, x_c^{-1}] \cong S[x_0^{-1}, \ldots, x_c^{-1}]/L, \tag{48}$$

where L is the vector subspace of $S[x_0^{-1}, \dots, x_c^{-1}]$ generated by all terms in which an x_i appears with non-negative exponent.

(ii) For an $r \in \mathcal{R}(X)_{(d,e)}$, the map

$$r_* : \bigoplus_{(a,b)\in \mathrm{Pic}} H^c(X, \mathcal{O}_X(a-d, b-e)) \longrightarrow \bigoplus_{D\in \mathrm{Pic}} H^c(X, \mathcal{O}_X(a,b))$$

can be identified with the map

$$\begin{array}{rcl} r_* : S[x_0^{-1}, \dots, x_c^{-1}]/L & \longrightarrow & S[x_0^{-1}, \dots, x_c^{-1}]/L \\ f & \mapsto & r \cdot f \mod L \end{array}$$

that multiplies an element $f \in S[x_0^{-1}, \dots, x_c^{-1}]/L \cong \frac{1}{x_0 \cdots x_c} R[x_0^{-1}, \dots, x_c^{-1}]$ by r, and ignores those terms in which an x_i occurs with non-negative exponent.

Proof. The argument is the same as in Lemma 6.2, using the Künneth formula. □

Once more, we remark that the isomorphisms in Lemma 7.2 are of R-modules.

Definition 7.3. Set $d' = d_1 + \cdots + d_c$ and $e' = e_1 + \cdots + e_c$. Define, for each $(a,b) \in \mathbb{Z}^2$, the vector spaces

$$T_{(a,b)} = H^c\left(X, \mathcal{O}_X\left(a-d', b-e'\right)\right), \quad T_a = \bigoplus_{b\in\mathbb{Z}} T_{(a,b)},$$

along with the spaces

$$T'_{(a,b)} = \bigoplus_{m=1}^{c} H^c\left(X, \mathcal{O}_X\left(a-d'+d_m, b-e'+e_m\right)\right), \quad T'_a = \bigoplus_{b\in\mathbb{Z}} T_{(a,b)}.$$

For each $a \in \mathbb{Z}$ define $B_a : T_a \to T'_a$ to be the map

$$\begin{array}{rcl} B_a : T_a & \longrightarrow & T'_a \\ h & \mapsto & (r_{1,*}h, \dots r_{c,*}h). \end{array}$$

We write $K_{-a} = \ker B_a$, noting as usual the flip of sign in a, for each $a \in \mathbb{Z}$ and $K = \bigoplus_{a\in\mathbb{Z}} K_a$.

We have finished Step (1) of the programme. We know that we have isomorphisms of vector spaces $K \cong \mathcal{R}(Z)/\iota^*\mathcal{R}(Z)$, from which K inherits a grading.

7.1.3 Describing the kernels of the maps B_a

We begin with Step (2) by identifying each T_a with a free R-module.

Definition 7.4. For each $a \in \mathbb{Z}$, define the discrete c-simplex

$$\Delta_c(a-c-1) = \{(i_0, \dots, i_c) \in \mathbb{Z}_{\geqslant 0} : i_0 + \cdots + i_c = a-c-1\} \subset \mathbb{R}^{c+1},$$

consisting of $\binom{a-1}{c}$ points in $\mathbb{R}^{c+1}$, where we set $\binom{a-1}{c} = 0$ if $a < c+1$, corresponding to $\Delta_c(a-c-1) = \varnothing$ in this range. For ease of future notation, we write $i_0 \dots i_c$ for an element $(i_0, \dots, i_c)$ in $\Delta_c(a-c-1)$. If

$i_0 \dots i_c \in \Delta_c(a-c-1)$ and $i'_0, \dots, i'_c \in \Delta_c(d)$ then we can write $i_0 \dots i_c + i'_0 \dots i'_c = (i_0 + i'_0, \dots, i_c + i'_c) \in \Delta_2(a+d-c-1)$, where $a, d \in \mathbb{Z}$.

Since $H^c(X, \mathcal{O}_X(a,b)) = 0$ for $a \geqslant -c$, we use the index $-a$ rather than a. Let $-a \leqslant -c-1$. If we give $R^{\binom{a-1}{c}}$ the standard basis indexed by $\Delta_c(a-c-1)$ then there are R-module isomorphisms

$$\begin{array}{rcl} R^{\binom{a-1}{c}} & \xrightarrow{\sim} & \bigoplus_{b\in\mathbb{Z}} \frac{1}{x_0 \cdots x_2} R[x_0^{-1}, \dots, x_c^{-1}]_{(-a,b)} \\ (h_{i_0\dots i_c})_{i_0\dots i_c \in \Delta_c(a-c-1)} & \mapsto & h = \sum_{i_0\dots i_c \in \Delta_c(a-c-1)} \frac{h_{i_0\dots i_c}}{x_0^{i_0+1} \cdots x_c^{i_c+1}}. \end{array} \tag{49}$$

The following Lemma describes the maps on second cohomology induced by the multiplication by a global section described with these bases.

Lemma 7.5. *Let $d, e \geqslant 0$. A general polynomial $r \in \mathcal{R}(X)_{(d,e)}$ is given by a collection of degree e polynomials $(g_{i_0\dots i_c})_{i_0\dots i_c \in \Delta_c(d)}$ via*

$$(g_{i_0\dots i_c})_{i_0\dots i_c \in \Delta_c(d)} \mapsto \sum_{i_0\dots i_c \in \Delta_c(d)} g_{i_0\dots i_c} x_0^{i_0} \dots x_c^{i_c}.$$

If $-a \in \mathbb{Z}$, then the map

$$r_*^{(a)} : \bigoplus_{b\in\mathbb{Z}} H^c(X, \mathcal{O}_X(a-d, b-e)) \to \bigoplus_{b\in\mathbb{Z}} H^c(X, \mathcal{O}_X(a,b))$$

is identified via the isomorphisms in Lemma 7.2 and (49) with the map

$$\begin{array}{rcl} R^{\binom{a+d-1}{c}} & \longrightarrow & R^{\binom{a-1}{c}} \\ (h_{i_0\dots i_c})_{i_0\dots i_c \in \Delta_c(a+d-c-1)} & \mapsto & (h'_{i_0\dots i_c})_{i_0\dots i_c \in \Delta_c(a-c-1)}, \end{array}$$

where for each $i_0 \dots i_c \in \Delta_c(a-c-1)$, $h'_{i_0\dots i_c}$ is given by

$$h'_{i_0\dots i_c} = \sum_{i'_0\dots i'_c \in \Delta_c(d)} h_{i_0\dots i_c + i'_0 \dots i'_c} g_{i'_0\dots i'_c}.$$

Proof. This follows from the description of $r_*^{(a)}$ given by Lemma 7.2(ii). □

Similarly to in Example 6.6, the element $r \in \mathcal{R}(X)$ and $h \in T_a$ viewed as discrete c simplices labelled with their polynomial coefficients, and the map $r_*^{(a)}$ is seen as a collection of scalar products according to each way the simplex associated to r maps rigidly, without rotation, onto the simplex associated to h.

Proposition 7.6. *For each $-a \in \mathbb{Z}$, the space $K_a = \ker B_{-a}$ may only be non-zero if $-a \leqslant -c-1+d'$.*

(i) For $-c + d' - \min_m d_m \leqslant -a \leqslant -c-1+d'$, we have a graded isomorphism

$$K_a \cong R^{\binom{a+d'-1}{c}}\left[e'\right].$$

(ii) For $-a \leqslant -c-1+d'-\min_m d_m$, we have graded isomorphisms

$$K_{(-a,b)} \cong \ker r_{1*}^{(a)} \cap \cdots \cap \ker r_{c*}^{(a)}.$$

Proof. The first two statements follow since $T_a = 0$ if $-a > -c-1+d'$ and $T'_a = 0$ for $-c+d'-\min_m d_m < -a \leqslant -c-1+d'$. The isomorphism of part (ii) is immediate from the definition of B_a. ☐

Since the dimensions of the spaces T_a and T'_a increase with order c, it is not feasible to try and compute the K_a directly, outside of perhaps the simplest cases.

7.1.4 Problems defining α and multiplication by the x_i

For larger c, defining a right inverse $\alpha : K \to \mathcal{R}(Z)$, unique on those K_a with $a \geqslant 1$, explicitly is much more complex. In Section 6.1.4, we saw the situation with $c = 2$, where we must work in the polynomial rings $\mathcal{R}(X)[x_i^{-1}, x_k^{-1}]/(r_1)$. For general c we have computations in the rings $\mathcal{R}(X)[x_{i_0}^{-1}, \dots, x_{i_m}^{-1}]/(r_1, \dots, r_{c-m})$ for each $m = 1, \dots, c$ and $0 \leqslant i_0 < \cdots < i_m \leqslant c$. We will be able to derive results without completing this step explicitly. We can understand the maps on K induced by multiplication by $x_0, \dots, x_c$.

Let $-a \in \mathbb{Z}$ and consider the isomorphism $\overline{\alpha} : K \to \mathcal{R}(Z)/\iota^*\mathcal{R}(X)$ with inverse δ. For each $i = 0, \dots, c$, we have a map

$$\tilde{x}_i = \delta(\cdot x_i)\overline{\alpha} : K \to K,$$

mapping each K_{a+1} to K_a. Even though we have no explicit description for $\overline{\alpha}$, the maps $\tilde{x}_i$ have a simple description. The following Lemma generalises Lemma 6.8 to general $c \geqslant 2$.

Lemma 7.7. *Let $-a \in \mathbb{Z}$. For each $k = 0, \dots, c$, the map $\tilde{x}_k : K_{a+1} \to K_a$ is given by*

$$\tilde{x}_k h = (h'_{i_0 \dots i_c})_{i_0 \dots i_c \in \Delta_c(a+d'-c-1)}, \quad h'_{i_0 \dots i_c} = h_{i_0 \dots i_k+1 \dots i_c}$$

for each $h = (h_{i_0 \dots i_c})_{i_0 \dots i_c \in \Delta_c(a+d'-c)} \in K_{a+1}$.

Proof. As for the $c = 2$ case, this follows from Lemma 3.5 and Lemma 7.2. ☐

If we view an element $h = (h_{i_0 \dots i_c})_{i_0 \dots i_c \in \Delta_c(a+d'-c-1)}$ of K_a as the discrete c-simplex $\Delta_c(a+d'-c-1)$ with its points labelled by the polynomial entries, then the map $\tilde{x}_k$ corresponds to discarding the face of this c-simplex defined by $i_k = 0$.

We have enough tools in the general case, that we may move our focus onto the cases with $(d_1, \dots, d_c) = (1, \dots, 1)$ and $(d_1, \dots, d_c) = (2, 1, \dots, 1)$, where the geometry is accessible enough for us to derive complete results in the general case.

7.2 The case $(d_1, \dots, d_c) = (1, \dots, 1)$

Recall that $Z \subset X = \mathbb{P}^c \times \mathbb{P}^n$, with $n > c$, is of the form (†) with respect to irreducible normal hypersurfaces $V_1, \dots, V_c$ defined by equations $r_1, \dots, r_c \in \mathcal{R}(Z)$ of degrees $(d_1, e_1), \dots, (d_c, e_c)$, where $d_1, \dots, d_c, e_1, \dots, e_c \geqslant 1$, respectively. In this Section, we treat the case $d_1 = \cdots = d_c = 1$. For each $m = 1, \dots, c$, There are homogeneous degree e_m polynomials $f_{m,0}, \dots, f_{m,c} \in R = \mathbb{C}[y_0, \dots, y_n]$ such that

$$r_m = f_{m,0}x_0 + \cdots + f_{m,c}x_c.$$

We will see that Z is a blow up of $\mathbb{P}^c \times \mathbb{P}^n$ in a codimension 2 ideal. We will compute its Cox ring and show that Z is a Mori Dream Space with unique minimal model.

7.2.1 Basic geometry

Let $Z \to \mathbb{P}^n$ be projection to the second factor. This map is generally one-to-one, defining a birational map. The fibres are controlled by the rank of the matrix

$$M = \begin{pmatrix} f_{1,0} & \cdots & f_{1,c} \\ \vdots & \ddots & \vdots \\ f_{c,0} & \cdots & f_{c,c} \end{pmatrix}.$$

The fibre over a point $y \in \mathbb{P}^n$ is isomorphic to $\mathbb{P}^{c-r}$, where r is the rank of M at y. Note once more that M has full rank over the general fibre, since Z has codimension c in $\mathbb{P}^c \times \mathbb{P}^n$. For $i = 0, \ldots, c$, let $\mu_i \in R$ be the minor of M corresponding to deleting the i^{th} column, so $\deg \mu_i = e'$ in R. The following equations are then satisfied by any point on Z for each $0 \leqslant i < j \leqslant c$.

$$(-1)^i \mu_j x_i - (-1)^j \mu_i x_j = 0. \tag{50}$$

To see this, suppose $M_0 : S^{c+1} \to S^c$ is a linear map over a ring S. Then we obtain a map

$$\bigwedge^c M_0 : \bigwedge^c S^{c+1} \to \bigwedge^c S^c \cong S$$

The module $\bigwedge^c S^{c+1}$ is isomorphic to S^{c+1} with basis given by $\{e_0 \wedge \overset{\hat{e}_i}{\cdots} \wedge e_c\}_i$. Under this basis, the matrix of $\bigwedge^c M_0$ is the row vector $(\mu'_0, \ldots, \mu'_c)$, where μ'_i is the $c \times c$ minor of M_0 obtained by discarding the i^{th} row. Now $x = (x_0, \ldots, x_c) \in \ker M_0$ implies that $x \wedge e_0 \wedge \overset{\hat{e}_i, \hat{e}_j}{\cdots} \wedge e_c$ is in the kernel of $\bigwedge^{c-1} M_0$ for each $0 \leqslant i < j \leqslant c$. But for $i < j$ we have

$$x \wedge e_0 \wedge \overset{\hat{e}_i, \hat{e}_j}{\cdots} \wedge e_c = (x_i e_i + x_j e_j) \wedge e_0 \overset{\hat{e}_i, \hat{e}_j}{\cdots} \wedge e_c = (-1)^i x_i e_0 \wedge \overset{\hat{e}_j}{\cdots} \wedge e_c - (-1)^j x_j e_0 \wedge \overset{\hat{e}_i}{\cdots} \wedge e_c,$$

which maps to $(-1)^i \mu'_j x_i - (-1)^j \mu'_i x_j$ under $\bigwedge^{c-1} M_0$. Applying this to our M over the projective coordinate ring of Z we get the equations (50)

Proposition 7.8. *The variety Z is isomorphic to the blow-up of $\mathbb{P}^n$ in the codimension 2 subvariety defined by the 2×2 minors of M.*

Proof. Let $Y \subset \mathbb{P}^n$ be defined by the $c \times c$ minors $\mu_0, \ldots, \mu_c$ of M, which define a codimension 2 ideal. The equations of the blow up in $\mathbb{P}^c \times \mathbb{P}^n$ are

$$\mu_j x_i - \mu_i x_j, \quad 0 \leqslant i < j \leqslant c.$$

Replacing the coordinate x_i of $\mathbb{P}^c$ with $(-1)^i x_i$, we get precisely the equations (50). □

As for the $c = 2$ case in Section 6.2, we will see that the Cox ring of Z is generated by restrictions of sections on X and one new section in degree $(-1, e')$, which vanishes along the exceptional divisor in viewing Z as a blow-up via Proposition 7.8.

7.2.2 Proof of Theorem 2.13

Define, as in Definition 7.3, the spaces

$$T_{-a} = \bigoplus_{b \in \mathbb{Z}} H^c(X, \mathcal{O}_X(-a-c, b-e'))$$

$$T'_{-a} = \bigoplus_{m=1}^{c} \bigoplus_{b \in \mathbb{Z}} \left(H^c(X, \mathcal{O}_X(-a-c+1, b-e'+e_m)) \right),$$

for each $-a \in \mathbb{Z}$. The maps B_{-a} are given by $(r_{1*}, \ldots, r_{c*})$ with kernel K_a. We can now prove Theorem 2.13.

Theorem 7.9. *Suppose that $c \geqslant 2$, $n > c$, and $Z \subset X = \mathbb{P}^c \times \mathbb{P}^n$ is of the form (†) with respect to irreducible normal hypersurfaces V_m, with $[V_m] = (d_m, e_m) \in \operatorname{Pic} X \cong \mathbb{Z}^2$, defined by equations*

$$r_m = f_{m,0}x_0 + \cdots + f_{m,c}x_c = 0$$

for $m = 1, \ldots, c$, so that each $f_{m,l}$ is homogeneous of degree e_m. If M is the $c \times (c+1)$ matrix

$$M = \begin{pmatrix} f_{1,0} & \cdots & f_{1,c} \\ \vdots & \ddots & \vdots \\ f_{c,0} & \cdots & f_{c,c} \end{pmatrix},$$

then let μ_i be the maximal minor of M obtained by discarding the i^{th} column for $i = 0, \ldots, c$. These maximal minors define a codimension 2 subvariety $Y \subset \mathbb{P}^n$, and Z is isomorphic to the blow-up of $\mathbb{P}^n$ in Y. If

$$U = \mathbb{C}[X_0, \ldots, X_c, Y_0, \ldots, Y_n, V]$$

is the free $\mathbb{Z}^2$-graded $\mathbb{C}$-algebra with generators having degrees $(1,0)$, $(0,1)$, $(-1,e')$ respectively, then there is a surjective map of graded $\mathbb{C}$-algebras $\psi : U \to \mathcal{R}(Z)$ sending $X_i \mapsto x_i$, $Y_j \mapsto y_j$, $V \mapsto v$, where $v \in \mathcal{R}(Z)_{(-1,e')}$ is a section satisfying the equations

$$vx_i + (-1)^i \mu_i, \quad i = 0, \ldots, c.$$

The kernel of ψ is generated by the equations

$$f_{m,0}(Y_j)X_0 + \cdots + f_{m,c}(Y_j)X_c, \quad m = 1, \ldots, c,$$
$$VX_i + (-1)^i \mu_i(Y_j), \quad i = 0, \ldots, c.$$

In particular, Z is a Mori Dream space, with unique minimal model isomorphic to $\mathbb{P}^c \times \mathbb{P}^n$ given by the blow-down of the exceptional divisor on Z defined by the vanishing of the section v.

Proof. The equations (50) imply the existence of a section

$$v = -\frac{\mu_0}{x_0} = \cdots = (-1)^c \frac{\mu_c}{x_c},$$

globally defined in degree $(-1, e')$, satisfying the equations as given in the Theorem. This section is the image of V in the map ψ.

By Proposition 3.13, we have $\mathcal{R}(X) \cong \iota^*\mathcal{R}(X) \oplus K$. We have that $\mathcal{R}(X)_{(a,b)} = 0$ for $a \leqslant -1$ and by Proposition 7.6, we have $K_a = 0$ for $a \geqslant 0$ (remembering the sign flip convention on the grading of K).

It is clear that ψ is surjective in degrees (a, b) with $a \geqslant 0$. Fixing $a \geqslant 0$, the equations in J allow any degree (a, b) element p of U with $a \geqslant 0$ to be written in terms of only X_i, Y_j. Since the defining equations of Z are also contained in $\psi(J)$, we see that ψ induces isomorphisms $\mathcal{R}(Z)_{(a,b)} \cong (U/J)_{(a,b)}$ for all $a \geqslant 0, b \in \mathbb{Z}$.

We now look at the pieces graded by $(-a, b)$ with $-a \leqslant -1$. Fix $-a \leqslant -1$. The equations in J allow us to write any polynomial of a degree $(-a, b)$ in terms of just the Y_j and V. To prove the theorem, it suffices then to show that $K_a \cong R[ae']$, since then V^a maps onto the non-zero element v^a in under ψ which must correspond to a generator of K_a.

Suppose $h = (h_{i_0 \ldots i_c})_{i_0 \ldots i_c \in \Delta_c(a-1)} \in K_a$. By Lemma 7.5, for each $i_0 \ldots i_c \in \Delta_c(a-2)$, we have

$$f_{m,0} h_{i_0+1 \ldots i_c} + \cdots f_{m,c} h_{i_0 \ldots i_c+1} = 0$$

for each $m = 1, \ldots, c$. In other words, we have $(h_{i_0+1 \ldots i_c}, \ldots, h_{i_0 \ldots i_c+1}) \in \ker M$, where M is viewed as a map of free R-modules, for each $i_0 \ldots i_c \in \Delta_c(a-2)$. But then

$$(-1)^j \mu_k h_{i_0 \ldots i_j+1 \ldots i_c} - (-1)^k \mu_j h_{i_0 \ldots i_k+1 \ldots i_c}$$

for each $i_0 \ldots i_c \in \Delta_c(a-2)$. Since the μ_i share no common factor, we conclude that there is a polynomial $p \in R$ such that

$$h_{i_0 \ldots i_c} = p \mu_0^{i_0} \cdots \mu_c^{i_c}$$

for each $i_0 \ldots i_c \in \Delta_c(a-1)$. In particular, $K_a \cong R[ae']$ as a graded R-module as required. □

7.3 The case $(d_1, d_2, \ldots, d_c) = (2, 1, \ldots, 1)$

For this section, we take our general set up of $Z \subset X = \mathbb{P}^c \times \mathbb{P}^n$ of the form (†) with respect to hypersurfaces $V_1, \ldots, V_c$ of degrees $(d_1, e_1), \ldots, (d_c, e_c)$ and we assume $d_1 = 2$ and $d_2 = \cdots = d_c = 1$. Once more we set $e' = e_1 + \cdots + e_c$, and we remark that $e' = n + 1$ gives the Calabi-Yau case.

We can write

$$r_1 = \sum_{0 \leqslant i \leqslant j \leqslant c} f_{ij} x_i x_j$$

for some homogeneous degree e_1 polynomials f_{ij} in R. For each $m = 2, \ldots, c$, we can write

$$r_m = g_{m,0} x_0 + \cdots + g_{m,c} x_c$$

for some homogeneous degree e_m polynomials $g_{m,0}, \ldots, g_{m,c}$ in R.

7.3.1 Geometry and algebra

If $Z \to \mathbb{P}^n$ is the projection map, then the general fibre consists of 2 points (with multiplicity). Taking inspiration from the $c = 2$ case, explored in Section 6.3, we seek to construct a birational map from Z to itself interchanging the two points of a general fibre. Non-general fibres may be lines, smooth conic curves in $\mathbb{P}^2$, degenerate conics in $\mathbb{P}^2$ or the entire $\mathbb{P}^2$.

Define the $(c-1)\times(c+1)$ matrix

$$M=\begin{pmatrix} g_{2,0} & \cdots & g_{2,c} \\ \vdots & \ddots & \vdots \\ g_{c,0} & \cdots & g_{c,c} \end{pmatrix}$$

and let μ_{ij}, for each $0\leqslant i<j\leqslant c$, denote the $(c-1)\times(c-1)$ minor of M obtained by discarding the i and j columns of M. Each such μ_{ij} has degree $e'-e_1$ in R. If $i>j$ we set $\mu_{ij}=-\mu_{ji}$ and we also set $\mu_{ii}=0$.

If $(x,y)\in Z$, then x is in the kernel of M viewed as a map of free modules on the projective coordinate ring of Z embedded in $\mathbb{P}^c\times\mathbb{P}^n$. This implies that

$$(-1)^i\mu_{jk}x_i-(-1)^j\mu_{ik}x_j+(-1)^k\mu_{ij}x_k=0 \tag{51}$$

for each $0\leqslant i<j<k\leqslant 0$. Indeed, suppose $M_0:S^{c+1}\to S^{c-1}$ is a map of free S-modules for a ring S. Then $x\in\ker M_0$ implies that

$$x\wedge e_0\wedge \overset{\hat{e}_i,\hat{e}_j,\hat{e}_k}{\cdots}\wedge e_c\in\ker\left(\bigwedge^{c-1}M_0:\bigwedge^{c-1}S^{c+1}\to\bigwedge^{c-1}S^{c-1}\cong S\right),$$

and the image of $x\wedge e_0\wedge \overset{\hat{e}_i,\hat{e}_j,\hat{e}_k}{\cdots}\wedge e_c$ under $\bigwedge^{c-1}M_0$ is $(-1)^i\mu'_{jk}x_i-(-1)^j\mu'_{ik}x_j+(-1)^k\mu'_{ij}x_k$, where μ'_{ij} is the maximal minor of M_0 taken ignoring the i and j columns for each $0\leqslant i<j\leqslant c$.

For any $i\neq j$, we may use the equations in (51) to eliminate all variables x_k with $k\notin\{i,j\}$ from the equation $\mu_{ij}^2r_1=0$. We obtain

$$\begin{gathered} 0=\mu_{ij}^2f_{ii}x_i^2+\mu_{ij}^2f_{ij}x_ix_j+\mu_{ij}^2f_{jj}x_j^2 \\ +\mu_{ij}\textstyle\sum_{k\notin\{i,j\}}f_{ik}x_i(-(-1)^{i+k}\mu_{jk}x_i+(-1)^{j+k}\mu_{ik}x_j) \\ +\mu_{ij}\textstyle\sum_{k\notin\{i,j\}}f_{jk}(-(-1)^{i+j}\mu_{jk}x_i+(-1)^{j+k}\mu_{ik}x_j)x_j \\ +\textstyle\sum_{k,l\notin\{i,j\}}f_{kl}(-(-1)^{i+k}\mu_{jk}x_i+(-1)^{j+k}\mu_{ik}x_j)(-(-1)^{i+l}\mu_{jl}x_i+(-1)^{j+l}\mu_{il}x_j). \end{gathered}$$

Grouping the quadratic terms in x_i,x_j we obtain

$$F_{ij}x_i^2+G_{ij}x_ix_j+H_{ij}x_j^2=0,$$

where

$$F_{ij}=\mu_{ij}^2f_{ii}-\mu_{ij}\sum_{k\neq i,j}(-1)^{i+k}f_{ik}\mu_{jk}+\sum_{k,l\notin\{i,j\}}(-1)^{k+l}f_{kl}\mu_{jk}\mu_{jl},$$

$$G_{ij}=\mu_{ij}^2f_{ij}+\mu_{ij}\sum_{k\neq i,j}\left((-1)^{j+k}f_{ik}\mu_{ik}-(-1)^{i+k}f_{jk}\mu_{jk}\right)-\sum_{k,l\notin\{i,j\}}(-1)^{i+j+k+l}f_{kl}\left(\mu_{jk}\mu_{il}+\mu_{ik}\mu_{jl}\right),$$

$$H_{ij}=\mu_{ij}^2f_{jj}+\mu_{ij}\sum_{k\neq i,j}(-1)^{j+k}f_{jk}\mu_{ik}+\sum_{k,l\notin\{i,j\}}(-1)^{k+l}f_{kl}\mu_{ik}\mu_{il}.$$

Recalling our sign conventions, we notice that for each size three subset $\{i,j,k\}$ we have $F_{ij} = H_{jk} = F_j$, where

$$F_j = \sum_{k,l\neq j} (-1)^{k+l} f_{kl}\mu_{jk}\mu_{jl}.$$

We also notice that

$$G_{ij} = -\sum_{k,l}(-1)^{i+j+k+l} f_{kl}\left(\mu_{jk}\mu_{il} + \mu_{ik}\mu_{jl}\right).$$

Proposition 7.10. *There are degree $2e' - e_1$ polynomials $F_0,\dots,F_c,G_{01},\dots,G_{c-1,c}$ in R, defining a locus W of codimension at most 3 in $\mathbb{P}^n$, and sections $z_0,\dots,z_c \in \mathcal{R}(Z)$, of degree $(-1, 2e' - e_1)$, satisfying the equations*

$$\begin{aligned} x_i z_i &= F_i, \quad 0 \leqslant i \leqslant c, \\ x_i z_j + x_j z_i &= -G_{ij}, \quad 0 \leqslant i < j \leqslant c, \end{aligned} \tag{52}$$

in $\mathcal{R}(Z)$. Moreover, there is a birational map $\phi : Z \dashrightarrow Z$ defined by

$$\phi : ([x_0 : \cdots : x_c],[y_0 : \cdots : y_n]) \mapsto ([z_0 : \cdots : z_c],[y_0 : \cdots : y_n]),$$

which is regular away from W. Over the general fibre, this interchanges the two points.

Proof. The F_i, G_{ij} are as given above. The codimension of the variety $W \subset \mathbb{P}^n$ defining them is at most 3, since they vanish when all maximal minors of the $(c-1)\times(c+1)$ matrix M vanish. We have established above that

$$F_j x_i^2 + G_{ij}x_i x_j + F_i x_j^2 = 0$$

for each $0 \leqslant i < j \leqslant c$. For each $0 \leqslant i \leqslant c$, the section z_i, defined by

$$z_i = \frac{F_i}{x_i} = -\frac{F_j x_i + G_{ij}x_j}{x_j^2}, \quad j \neq i,$$

is globally defined on Z in degree $(-1, 2e' - e_1)$. These z_i satisfy the equations given in the Proposition statement. The vanishing locus of $z_0,\dots,z_c$ is precisely W. If $y \in \mathbb{P}^n$ is so that all equations $F_j x_i^2 + G_{ij}x_i x_j + F_i x_j^2$ do not vanish, then its fibre consists of two points and the equations (52) imply that $[z_i : z_j]$ is the second root of $F_j x_i^2 + G_{ij}x_i x_j + F_i x_j^2$ for each $0 \leqslant i < j \leqslant c$. In particular, $[z_0 : \cdots : z_c]$ is the other point on Z over y. Thus ϕ is birational from Z to itself, interchanging the two points of a general fibre. $\square$

We are already able to prove that Z is a Mori Dream Space if $\operatorname{codim} W = 3$, with two minimal models isomorphic to Z related by ϕ, which is in this case a flop. This exploits the induced symmetry on $\mathcal{R}(Z)$ by the flop ϕ. We wait until we have more information to give a stronger result, giving also a presentation of the Cox ring of such a Z.

7.3.2 The space K_0

The defining equation

$$r_1 = \sum_{0 \leqslant i \leqslant j \leqslant c} f_{ij}x_i x_j$$

of V_1 may also be written as

$$r_1 = g'_{1,0}x_0 + \cdots + g'_{1,c}x_c,$$

where

$$g'_{1l} = \frac{1}{2}f_{0l}x_0 + \frac{1}{2}f_{0l}x_1 + \cdots + f_{ll}x_l + \cdots + \frac{1}{2}f_{lc}x_c.$$

We may consider the $c \times (c+1)$ matrix M', with entries in the projective coordinate ring of $Z \subset X$, obtained by augmenting M with a new first row with entries $g'_{1,0}, \ldots g'_{1,c}$ as follows.

$$M' = \begin{pmatrix} g'_{1,0} & \cdots & g'_{1,c} \\ g_{2,0} & \cdots & g_{2,c} \\ \vdots & \ddots & \vdots \\ g_{c,0} & \cdots & g_{c,c} \end{pmatrix}.$$

For each $i = 0, \ldots, c$, let ν_i be the $c \times c$ minor of M' obtained by removing the i^{th} column, which are each degree $(1, e')$ sections in $\iota^*\mathcal{R}(X)$. We can assume that no ν_i is equal 0, otherwise $x_i = 0$ on Z and we can assume $Z \subset \mathbb{P}^{c-1} \times \mathbb{P}^n$. If $(x, y) \in Z$ then $x \in \ker M'(y)$, and by considering $\bigwedge^c M'(y)$, we have that

$$(-1)^i \nu_j x_i - (-1)^j \nu_i x_j = 0.$$

are satisfied. We obtain a section

$$u = \frac{\nu_0}{x_0} = \cdots = (-1)^c \frac{\nu_c}{x_c}$$

globally defined on Z in degree $(0, e')$.

Proposition 7.11. *The R-module $\bigoplus_{b \in Z} \mathcal{R}(Z)_{(0,b)}$ is isomorphic to $R \oplus R[e']$, with basis given by 1 in degree $(0,0)$ and the section u in degree $(0, e')$ defined above, which satisfies the equations*

$$ux_i - (-1)^i \nu_i = 0$$

in $\mathcal{R}(Z)$ for each $i = 0, \ldots, c$.

Proof. We know that

$$\bigoplus_{b \in \mathbb{Z}} \mathcal{R}(Z)_{(0,b)} \cong \bigoplus_{b \in Z} \iota^*\mathcal{R}(X)_{(0,b)} \oplus K_0.$$

Now $\bigoplus_{b \in Z} \iota^*\mathcal{R}(X)_{(0,b)}$ is the restriction of $R = \mathcal{R}(\mathbb{P}^n)$ to Z, which is just R since the projection to $\mathbb{P}^n$ is surjective. By Proposition 7.6, we have $K_0 \cong R[e']$. Since u is a section not contained in $\iota^*\mathcal{R}(X)$, we conclude that the section u above must generate this second summand of $\bigoplus_{b \in Z} \iota^*\mathcal{R}(X)_{(0,b)}$. Clearly u satisfies the given equations. □

Remark 7.12. We can show that $\delta u \in K_0$ is given by

$$\delta u = (-1)^{\frac{c(c-1)}{2}} \frac{1}{x_0 \ldots x_c}.$$

To see this, we consider the f_{kl} and $g_{m,l}$ as formal symbols. By considering the process of applying α to $h = \frac{1}{x_0 \dots x_c}$, we see that the term

$$\frac{f_{cc} g_{2,c-1} \cdots g_{2,1}}{x_0}$$

appears with coefficient $+1$ in t_0, where (t_i) is the 0-cocycle on Z representing αh. Since the term $f_{cc} g_{2,c-1} \cdots g_{2,1}$ appears with sign $(-1)^{\frac{c(c-1)}{2}}$ in ν_0, we obtain the claim. To not have the sign $(-1)^{\frac{c(c-1)}{2}}$ present in our final Cox ring presentation, we take u to be $\alpha(-1)^{\frac{c(c-1)}{2}} h$. It is this reason that we chose $u = \alpha(-h)$ in Section 6.3.2

7.3.3 Equations in the u and z_i

Recall that in Proposition 7.10 we found sections $z_0, \dots, z_c$, in degree $(-1, 2e' - e_1)$, which generally give the coordinates of the second point of a fibre over $\mathbb{P}^n$. From Proposition 7.11 we also have a section $u \in (0, e')$. We now attempt to refine the equations in Proposition 7.10 with the section u. The equations

$$x_i z_j + x_j x_i = -G_{ij}$$

imply that, for each $0 \leqslant i < j \leqslant c$,

$$x_i z_j + \frac{1}{2} G_{ij} = -x_j z_i - \frac{1}{2} G_{ij} = v_{ij}$$

for some degree $(0, 2e' - e_1)$ section v_{ij} in $\mathcal{R}(Z)$. Then

$$-F_i F_j = -x_i z_j x_j z_i = \left(v_{ij} - \frac{1}{2} G_{ij} \right) \left(v_{ij} + \frac{1}{2} G_{ij} \right) = v_{ij}^2 - \frac{1}{4} G_{ij}^2,$$

and thus $v_{ij}^2 = \frac{1}{4} G_{ij}^2 - F_i F_j$. This is the discriminant (up to scalar) of the quadratic equation $F_j x_i^2 + G_{ij} x_i x_j + F_i x_j^2$.

Lemma 7.13. *For each $i \neq j$, we have $v_{ij} = (-1)^{j-i} \mu_{ij} u$. In particular, the equations*

$$x_i z_i - F_i = 0, \quad 0 \leqslant i \leqslant c,$$

$$x_i z_j - (-1)^{j-i} \mu_{ij} u + \frac{1}{2} G_{ij}, \quad i \neq j,$$

refine those equations in Proposition 7.10.

Proof. We first show that each v_{ij} vanishes where u vanishes. Since v_{ij}^2 is the determinant of the quadratic equation $F_j x_i^2 + G_{ij} x_i x_j + F_i x_j^2 = 0$ satisfied by any point on Z, we have that v_{ij} vanishes on double point fibres of Z over $\mathbb{P}^n$. We argue that u must vanish here also. Note that u vanishes exactly when the $c \times c$ matrix M' vanishes, and M' is the Jacobian matrix of Z with first row halved. In particular, over a fixed $y \in \mathbb{P}^n$, the maximal minors of this matrix vanish where the fibre is singular, which includes where the fibre is a double point. We conclude that the divisor defined by u is the closure of the fibres consisting of one point. Since each v_{ij} vanishes where u vanishes we have $v_{ij} \in \langle u \rangle$ for each ij. By considering degrees we have $v_{ij} = q_{ij} u$ for some degree $(0, e' - e_1)$ section q_{ij} in R.

Now we have

$$0 = \Big((-1)^i\mu_{jk}x_i - (-1)^j\mu_{ik}x_j + (-1)^k\mu_{ij}x_k\Big)\, z_k$$

$$= \left((-1)^i\mu_{jk}q_{ik} - (-1)^j\mu_{ik}q_{jk}\right) u + \left((-1)^i\frac{1}{2}G_{ik} - (-1)^j\frac{1}{2}G_{jk} + (-1)^k F_k\right),$$

which implies that $(-1)^i\mu_{jk}q_{ik} - (-1)^j\mu_{ik}q_{jk} = 0$ for each $0 \leqslant i < j < k \leqslant c$, where we have used that u and 1 are R-linearly independent. In the general case (where the $g_{m,l}$ are formal symbols), the minors μ_{ij} are irreducible and we obtain

$$q_{ij} = (-1)^{j-1}\lambda\mu_{ij}$$

for some scalar λ. Since the non-general case has the same computations, this result holds in general. We need only to show $\lambda = 1$. For $c \geqslant 2$ this can be seen from the equations in 6.16. We now suppose $c > 3$.

Once more, we only need to show that $\lambda = 1$ for one example $Z \subset \mathbb{P}^c \times \mathbb{P}^n$, since any example is obtained from the universal setting (where we treat the polynomial coefficients as formal variables) by substituting symbols. In particular, the degrees e_m are unimportant.

We suppose that $(d_c, e_c) = (0, 1)$ and that $r_c = x_c$. Throughout, we have required that $e_c \geqslant 1$, but everything works with $e_c = 0$ except possibly the isomorphism of Picard groups, which is still okay if $d_c = 1$. Then $Z \subset \mathbb{P}^{c-1}_{x_c=0} \times \mathbb{P}^n \subset \mathbb{P}^c \times \mathbb{P}^n$. If, for each $0 \leqslant i < j \leqslant c-1$, μ^-_{ij} is the $c-2 \times c-2$ minor of the matrix

$$M^- = \begin{pmatrix} g_{2,0} & \cdots & g_{2,c-1} \\ \vdots & \ddots & \vdots \\ g_{c,0} & \cdots & g_{c-1,c-1} \end{pmatrix},$$

which is obtained from M by removing the last row and the last column, then we have $\mu^-_{ij} = \mu_{ij}$. Similarly, let

$$M'^- = \begin{pmatrix} g'_{1,0} & \cdots & g'_{1,c-1} \\ g_{2,0} & \cdots & g_{2,c-1} \\ \vdots & \ddots & \vdots \\ g_{c-1,0} & \cdots & g_{c-1,c-1} \end{pmatrix}$$

be the matrix obtained by M' by removing the last row and column, then if ν^-_i, for $0 \leqslant i \leqslant c-1$, is the maximal minor of M'^- discarding the i^{th} column then $\nu^-_i = \nu_i$. Using induction in c, we now conclude that for $0 \leqslant i, j \leqslant c-1$ with $i \neq j$ we have

$$(-1)^{j-i}\lambda\mu_{ij}u = x_i z_j + \frac{1}{2}G_{ij} = (-1)^{j-i}\mu^-_{ij}u^- = (-1)^{j-i}\mu_{ij}u,$$

where u^- is the section obtained in applying the method for $Z \subset \mathbb{P}^{c-1} \times \mathbb{P}^n$, noting $u^- = u$ since each $\nu^-_i = \nu_i$. Since at least one such μ_{ij} is non-zero, we get $\lambda = 1$ as required. ☐

Corollary 7.14. *We have the equations*

$$uz_i + (-1)^i\nu'_i = 0$$

for each $i = 0, \ldots, c$, where ν'_i is obtained from ν_i by replacing each x_i with z_i. Furthermore, u satisfies the *equation $u^2 = \omega$ in $\mathcal{R}(Z)$, where*

$$\omega = \frac{G_{ij}^2 - 4F_iF_j}{4\mu_{ij}^2} \in R$$

for any choice of $0 \leqslant i < j \leqslant c$.

Proof. Since $x_iz_j - x_jz_i = 2(-1)^{j-i}\mu_{ij}u$, we conclude u takes opposite signs on the two points of a general fibre over $\mathbb{P}^n$. If (x, y) and (x', y) are the two points of a general fibre and $0 \leqslant i \leqslant c$, then

$$(uz_i + (-1)^i\nu_i')(x, y) = (-ux_i + (-1)^i\nu_i)(x', y) = 0$$

since $ux_i - (-1)^i\nu_i = 0$ on Z. It follows that $uz_i + (-1)^i\nu_i' = 0$. The equation for u^2 follows from $(x_iz_j - x_jz_i)^2 = 4\mu_{ij}^2u^2$, which follows from the previous Lemma. $\square$

7.3.4 Proof of Main Theorem 2.17

The following generalises Theorem 6.23 to all $c \geqslant 2$.

Theorem 7.15. *Let $Z \subset \mathbb{P}^c \times \mathbb{P}^n$ as above. Let A be the free $\mathbb{Z}^2$-graded algebra*

$$A = \mathbb{C}[X_0, \ldots, X_c, Y_0, \ldots, Y_n, Z_0, \ldots, Z_c, U],$$

with degrees

$$\deg X_i = (1, 0), \quad \deg Y_j = (0, 1), \quad \deg Z_i = (-1, 2e' - e_1), \quad \deg U = (0, e'),$$

for relevant i, j. There is a morphism of graded algebras $\psi : A \to \mathcal{R}(Z)$ sending $X_i \mapsto x_i$, $Y_j \mapsto y_j$, $Z_i \mapsto z_i$, $U \mapsto u$, where $z_0, \ldots, z_c, u$ are the sections defined in Proposition 7.10 and .... Let $J \lhd A$ be the homogeneous ideal generated by the equations

$$\begin{gathered}
r_m(X_i, Y_j), \quad r_m(Z_i, Y_j), \quad m = 1, \ldots c, \\
X_iZ_i - F_0(Y_j), \quad i = 0, \ldots, c \\
X_iZ_j - (-1)^{j-i}\mu_{ij}(Y_j)U + \tfrac{1}{2}G_{01}(Y_j), \quad 0 \leqslant i, j \leqslant c, \\
X_iU - (-1)^i\nu_i(X_k, Y_j), \quad Z_iU + (-1)^i\nu_i(Z_k, Y_j), \quad i = 0, \ldots, c, \\
U^2 - \omega(Y_j),
\end{gathered}$$

where the $\nu_i(x_i, y_j)$ are defined in Proposition 7.11 and $\omega(y_j)$ is defined in Corollary 7.14.Then $J \subset \ker\psi$.

- *(i) Suppose the space $W \subset \mathbb{P}^n$ has codimension 3. Then ψ induces an isomorphism of graded algebras $\mathcal{R}(Z) \cong A/J$. The space Z is a Mori Dream Space with two maximal Mori chambers, each giving minimal models isomorphic to Z, related by a flop which exchanges the two points of a general fibre over $\mathbb{P}^n$.*
- *(ii) Suppose there is a presentation of the Cox ring of Z of the form $\mathcal{R}(Z) \cong A/J$. Then $\operatorname{codim} W = 3$.*

Proof. By Proposition 7.10, Proposition 7.11, Lemma 7.13, Corollary 7.14, the map ψ is well defined and its kernel contains J.

Suppose that $\operatorname{codim} W = 3$. By Proposition 7.10, we have a birational map ϕ of order two from Z to itself:

$$\phi([x_0 : \cdots : x_c], [y_0 : \cdots : y_n]) = ([z_0 : \cdots : z_c], [y_0 : \cdots : y_n]).$$

Moreover, ϕ is regular away from W, thus an isomorphism in codimension one. Since ϕ is also a map over $\mathbb{P}^n$, any contracted fibre is in $\mathbb{P}^c$, thus has trivial intersection with the canonical divisor K_Z, which has class $(0, e' - n - 1)$. It follows that ϕ is a flop from Z to itself, inducing an isomorphism ϕ^* of Cox rings.

Surjectivity of ϕ follows since $x_0, \ldots, x_c, y_0, \ldots, y_n, u$ generated the subalgebra of $\mathcal{R}(Z)$ corresponding to degrees (a, b) with $a \geqslant 0, b \in \mathbb{Z}$. Furthermore, since $\bigoplus_{a \geqslant 0, b \in \mathbb{Z}} \mathcal{R}(Z) \cong \iota^* \mathcal{R}(X) \oplus K_0$, it is readily checked that $J_{(a,b)}$ is the kernel of ψ in degrees with $a \geqslant 0$. Now ϕ^* interchanges $\bigoplus_{a \geqslant 0, b \in \mathbb{Z}} \mathcal{R}(Z) \cong \iota^* \mathcal{R}(X) \oplus K_0$ and $\bigoplus_{a \leqslant 0, b \in \mathbb{Z}} \mathcal{R}(Z) \cong \iota^* \mathcal{R}(X) \oplus K_0$, sending the sections $x_0, \ldots, x_c, y_0, \ldots, y_c, u$ respectively to $z_0, \ldots, z_c, y_0, \ldots, y_n, -u$. The equations in J are compatible with this symmetry and it follows $A/J \to \mathcal{R}(Z)$ is an isomorphism in all degrees.

For (ii), such a presentation of the Cox ring implies that Z is a Mori Dream Space and that $\phi : Z \dashrightarrow Z$ defined by the sections z_i, y_j is a flop from Z to itself, thus a isomorphism in codimension one. But the z_i sections all vanish on W, and if the codimension of W in $\mathbb{P}^n$ is less than three, then the preimage of W in Z is a divisor on which ϕ cannot be defined. It follows that if such a presentation exists, then $\operatorname{codim} W = 3$. $\square$

7.4 Comments

Analagously to the Remarks in Section 4.5, Theorem 7.15 generalises to codimension c complete intersections in ambient spaces $X \subset \mathbb{P}^c \times T$ with T a Mori Dream Space. In Chapter 9, we apply this to codimension c complete intersections in $(\mathbb{P}^c)^k \times \mathbb{P}^n$.

8 On two hypersurface models of degree one del Pezzo surfaces with conic bundle

In this Chapter, we study two models of certain degree one del Pezzo surfaces. As in the rest of this thesis, we are working over $\mathbb{C}$, though this may be replaced with any perfect field with characteristic not equal 2.

Suppose that Z is a del Pezzo surface of degree one admitting a morphism $\pi : Z \to \mathbb{P}^1$, which equips it with the structure of a conic bundle. We will see that Z can be embedded as an ample hypersurface in the rational scroll $X = \mathbb{F}(0,1,1)$. For details on rational scrolls we refer to [30]. We will use the method of Section 3.3 with $c = 1$ – the hypersurface case – to find sections on Z that may be taken as coordinate sections for the anticanonical embedding of Z into the weighted projective space $\mathbb{P}(1,1,2,3)$. Our method gives us Cox ring equations for these new sections which we can use, with the help of Macaulay2, to translate between the defining equations of Z in X and Z in $\mathbb{P}(1,1,2,3)$. This answers a question asked twice independently in mathoverflow [31, 35].

8.1 Applying the method

8.1.1 Basics

Let X be the rational scroll $\mathbb{F}(0,1,1)$. This can be viewed as a quotient of $(\mathbb{A}^2 \setminus 0) \times (\mathbb{A}^3 \setminus 0)$ by the algebraic torus $\mathbb{G}_m^2$. Giving $\mathbb{A}^2$ coordinates t_0, t_1 and $\mathbb{A}^3$ coordinates x_0, x_1, x_2 the action is defined by

$$\lambda \cdot (t_0, t_1; x_0, x_1, x_2) = (\lambda t_0, \lambda t_1; x_0, \lambda^{-1} x_1, \lambda^{-1} x_2), \quad \lambda \in \mathbb{G}_m.$$

There is a natural map $\pi : X \to \mathbb{P}^1$ given by the coordinates t_0, t_1. The Picard group of X is isomorphic to $\mathbb{Z}^2$, with generators given by a fibre of π and the class of the coordinate section x_0. The Cox ring of X is then given by

$$\mathcal{R}(X) = \mathbb{C}[t_0, t_1, x_0, x_1, x_2],$$

with the generators having degrees

$$\deg(t_0) = \deg(t_1) = (1,0), \quad \deg(x_0) = (0,1), \quad \deg(x_1) = \deg(x_2) = (-1,1).$$

Now suppose that Z is a degree one del Pezzo surface admitting a conic bundle structure $Z \to \mathbb{P}^1$. By Theorem 5.6 in [14], we can embed Z into X as a hypersurface such that the conic bundle structure is induced by the restriction of π to Z. We denote by π also this restriction.

More precisely, Z is a degree $(1,2)$ hypersurface in X, defined by an element

$$r = A(t_0,t_1)x_0^2 + B(t_0,t_1)x_0x_1 + C(t_0,t_1)x_0x_2 + D(t_0,t_1)x_1^2 + E(t_0,t_1)x_1x_2 + F(t_0,t_1)x_2^2$$

in $\mathcal{R}(X)_{(1,2)}$, where A, B, C, D, E, F are homogeneous forms in t_0, t_1 of degrees $1, 2, 2, 3, 3, 3$ respectively. We will write

$$A(t_0,t_1) = a_0t_0 + a_1t_1, \;\; B(t_0,t_1) = b_0t_0^2 + b_1t_0t_1 + b_2t_1^2, \;\; C(t_0,t_1) = c_0t_0^2 + c_1t_0t_1 + c_2t_1^2,$$
$$D(t_0,t_1) = d_0t_0^3 + d_1t_0^2t_1 + d_2t_0t_1^2 + d_3t_1^3, \;\; E(t_0,t_1) = e_0t_0^3 + e_1t_0^2t_1 + e_2t_0t_1^2 + e_3t_1^3,$$
$$F(t_0,t_1) = f_0t_0^3 + f_1t_0^2t_1 + f_2t_0t_1^2 + f_3t_1^3.$$

The bundle structure $\pi : Z \to \mathbb{P}^1$ is given by the coordinates t_0, t_1.

On the other hand, as for all del Pezzo surfaces of degree 1, the anticanonical ring

$$\mathcal{R}(Z)^{-K_Z} = \bigoplus_{l \geqslant 0} H^0(Z, -lK_Z)$$

of Z is generated by four elements of respective degrees $1, 1, 2$ and 3, giving the anticanonical embedding

$$Z \cong \operatorname{Proj} R(Z)^{-K_Z} \hookrightarrow \mathbb{P}(1, 1, 2, 3).$$

This realises $Z \subset \mathbb{P}(1, 1, 2, 3)$ as a hypersurface of degree 6, given by a Weierstrass-like equation.

By the Lefschetz hyperplane theorem, the restriction homomorphism $\operatorname{Pic} X \to \operatorname{Pic} Z$ is injective, and we identify $\operatorname{Pic} X \cong \mathbb{Z}^2$ with its image in $\operatorname{Pic} Z$. The canonical class of X is given by $(0, -3)$ in $\mathbb{Z}^2$; by adjunction, the canonical class of Z is thus $K_Z = (1, -1)$. We conclude that x_1, x_2 restrict to linearly independent anticanonical sections of Z, giving a basis of the degree one part of the anticanonical ring $\mathcal{R}(Z)^{-K_Z}$. As mentioned above, the anticanonical ring also needs generators w, z of degrees $2, 3$. As the restriction map

$$H^0(X, \mathcal{O}_X(a, b)) \to H^0(Z, \mathcal{O}_Z(a, b))$$

is not surjective in the relevant degrees, these generators cannot be expressed as polynomials in the t_i and x_j variables.

Using the method of Section 3.2, we can find the sections w, z and give relations for them in terms of the t_i, x_j in the Cox ring of Z. For defining an ordered open cover of X, we define for each $i = 0, 1$ and $j = 0, 1, 2$ the affine open W_{ij} of X defined by $t_i \neq 0$ and $x_j \neq 0$. Then our ordered open cover is set to be $\mathcal{W} = \{W_{ij}\}_{i,j}$ with lexicographic order.

8.1.2 Maps on first cohomology

Step (1) of our procedure involves describing the first sheaf cohomology of X and describing the maps induced by multiplication by the defining equation r of Z. We can do this over all line bundles in $\operatorname{Pic} X$.

Lemma 8.1. *(i) There are natural $\mathbb{Z}^2$-graded vector space isomorphisms*

$$\bigoplus_{(a,b) \in \mathbb{Z}^2} H^1(X, \mathcal{O}_X(a, b)) \cong \frac{1}{t_0 t_1} \mathbb{C}[t_0^{-1}, t_1^{-1}, x_0, x_1, x_2] \cong \mathcal{R}(X)[t_0^{-1}, t_1^{-1}]/L, \tag{53}$$

where L is the vector subspace of $\mathcal{R}(X)[t_0^{-1}, t_1^{-1}]$ generated by all terms in which t_0 or t_1 appear with non-negative exponent.

(ii) For each $m = 1, \ldots, c$, the map

$$r_{m,*} : \bigoplus_{(a,b) \in \operatorname{Pic}} H^1(X, \mathcal{O}_X(a - 1, b - 2)) \longrightarrow \bigoplus_{D \in \operatorname{Pic}} H^1(X, \mathcal{O}_X(a, b))$$

can be identified with the map

$$\begin{array}{rcl} r_{m,*} : \mathcal{R}(X)[t_0^{-1}, t_1^{-1}]/L & \longrightarrow & \mathcal{R}(X)[t_0^{-1}, t_1^{-1}]/L \\ f & \mapsto & r \cdot f \mod L \end{array}$$

that multiplies an element $f \in \mathcal{R}(X)[t_0^{-1}, t_1^{-1}]/L \cong \frac{1}{x_0 x_1} R[t_0^{-1}, t_1^{-1}]$ by r, and ignores those terms in which t_0 or t_1 occur with non-negative exponent.

Proof. For the first part, we can consider the Čech complexes as direct sums over each monomial in the t_i and x_j. These complexes are identical to the case that $X = \mathbb{P}^1 \times \mathbb{P}^2$. The cohomology thus takes the same description; it is only the $\mathbb{Z}^2$-grading that changes. The second part then follows as usual. □

Definition 8.2. For each $(a, b) \in \mathbb{Z}^2$, define the vector spaces

$$T_{(a,b)} \cong H^1(X, \mathcal{O}_X(a-1, b-2)), \quad T'_{(a,b)} = H^1(X, \mathcal{O}_X(a, b)).$$

Let $K_{(a,b)}$ denote the kernel of $r_* : T_{(a,b)} \to T'_{(a,b)}$ and $K = \bigoplus_{(a,b) \in \mathbb{Z}^2} K_{(a,b)}$.

We have completed Step (1) of the method in Section 3.2. We will perform Step (2) only in the degrees $(2, -2)$ and $(3, -3)$, which is where we find the sections w and z on Z respectively.

8.1.3 Lifting elements of K to the Cox ring

Suppose that we have completed Step (2). In Step (3) we choose a map $\alpha : K \to \mathcal{R}(Z)$ inducing an isomorphism $\alpha_1 : K \to \mathcal{R}(Z)/\iota^*\mathcal{R}(X)$ as in the setting of Proposition 3.9.

Suppose that $f \in K$, homogeneous of degree $(-a, b)$ in $\mathbb{Z}^2$. By Lemma 8.1, we may write $u = r_* f$ as a sum

$$u = u^{(0)} + u^{(1)} + u^{(2)}$$

where each term of $u^{(0)}$, respectively $u^{(1)}$, has t_0, respectively t_1 occuring with negative exponent, and $u^{(2)}$ consists of the remaining terms of u. Note that

$$u^{(i)} \in \mathcal{O}_X(-a, b)(W_{ij}) \text{ for all } j \text{ and } i = 0, 1,$$
$$u^{(2)} \in \mathcal{R}(X)_{(-a,b)}.$$

The 0-cochain (v_{ij}) on X given by

$$v_{ij} = \begin{cases} u^{(0)} + \frac{1}{2} u^{(2)} & \text{if } i = 0, \\ -u^{(1)} - \frac{1}{2} u^{(2)} & \text{if } i = 1, \end{cases} \tag{54}$$

restricts to a section $v \in \mathcal{R}(Z)$. This induces a map $\alpha : K \to \mathcal{R}(Z)$. In summary we obtain the following.

Proposition 8.3. *The map $\alpha : K \to \mathcal{R}(Z)$ defined in the notation above by sending $f \in K$ to $v \in \mathcal{R}(Z)$ is compatible with Proposition 3.9.*

Proof. The construction of α is compared with the proof of Proposition 3.9. □

8.1.4 Finding sections w and z

We now perform Step (2) of the method of Section 3.2 in degree $(-2, 2)$, where we find a suitable section $w \in \mathcal{R}(Z)_{(-2,2)}$, and degree $(-3, 3)$, where we find a suitable section $z \in \mathcal{R}(Z)_{(-3,3)}$. By suitable we mean that the sections x_0, x_1, which are restrictions of those sections on X, along with w, z give generators of $\mathcal{R}(Z)^{-K_z}$.

By Lemma 8.1(i), an element of $T_{(-2,2)} = H^1(X, \mathcal{O}_X(-3,0))$ may be written as

$$f = \frac{\lambda_0}{t_0^2 t_1} + \frac{\lambda_1}{t_0 t_1^2}$$

for some scalars $\lambda_0, \lambda_1 \in \mathbb{C}$. Recalling the form of r given in the introduction to this Chapter, we have

$$u = r_* f = \frac{a_0\lambda_0 + a_1\lambda_1}{t_0 t_1} \in T'_{(-2,2)}.$$

Since we are assuming that r is general, we see that $f \in K$ if and only if it is given by a scalar multiple of

$$g = \frac{a_1}{t_0^2 t_1} - \frac{a_0}{t_0 t_1^2}.$$

We define the section w to be given by $\alpha(g)$. By considering the 0-cocycles defined on W_{0j} and W_{1j} for any j, we obtain the equations

$$\begin{aligned}
&2t_0^2 w - a_1 d_1 t_0^2 x_1^2 + a_0 d_2 t_0^2 x_1^2 - 2a_1 d_2 t_0 t_1 x_1^2 + 2a_0 d_3 t_0 t_1 x_1^2 \\
&\quad - 2a_1 d_3 t_1^2 x_1^2 - a_1 e_1 t_0^2 x_1 x_2 + a_0 e_2 t_0^2 x_1 x_2 - 2a_1 e_2 t_0 t_1 x_1 x_2 \\
&\quad + 2a_0 e_3 t_0 t_1 x_1 x_2 - 2a_1 e_3 t_1^2 x_1 x_2 - a_1 f_1 t_0^2 x_2^2 + a_0 f_2 t_0^2 x_2^2 - 2a_1 f_2 t_0 t_1 x_2^2 \\
&\quad + 2a_0 f_3 t_0 t_1 x_2^2 - 2a_1 f_3 t_1^2 x_2^2 - 2a_1 b_1 t_0 x_0 x_1 + 2a_0 b_2 t_0 x_0 x_1 \\
&\quad - 2a_1 b_2 t_1 x_0 x_1 - 2a_1 c_1 t_0 x_0 x_2 + 2a_0 c_2 t_0 x_0 x_2 - 2a_1 c_2 t_1 x_0 x_2 - 2a_1^2 x_0^2, \\
&2t_1^2 w - 2a_0 d_0 t_0^2 x_1^2 + 2a_1 d_0 t_0 t_1 x_1^2 - 2a_0 d_1 t_0 t_1 x_1^2 + a_1 d_1 t_1^2 x_1^2 \\
&\quad - a_0 d_2 t_1^2 x_1^2 - 2a_0 e_0 t_0^2 x_1 x_2 + 2a_1 e_0 t_0 t_1 x_1 x_2 - 2a_0 e_1 t_0 t_1 x_1 x_2 \\
&\quad + a_1 e_1 t_1^2 x_1 x_2 - a_0 e_2 t_1^2 x_1 x_2 - 2a_0 f_0 t_0^2 x_2^2 + 2a_1 f_0 t_0 t_1 x_2^2 \\
&\quad - 2a_0 f_1 t_0 t_1 x_2^2 + a_1 f_1 t_1^2 x_2^2 - a_0 f_2 t_1^2 x_2^2 - 2a_0 b_0 t_0 x_0 x_1 + 2a_1 b_0 t_1 x_0 x_1 \\
&\quad - 2a_0 b_1 t_1 x_0 x_1 - 2a_0 c_0 t_0 x_0 x_2 + 2a_1 c_0 t_1 x_0 x_2 - 2a_0 c_1 t_1 x_0 x_2 - 2a_0^2 x_0^2,
\end{aligned} \tag{55}$$

in $\mathcal{R}(Z)$.

We now wish to find $h \in K_{(-3,3)}$ so that we may define $z = \alpha h$ to be the remaining generator of the anticanonical ring of Z. An element of $T_{(-3,3)}$ is an element of $H^1(X, \mathcal{O}_X(-1,4))$, which has the form

$$f = \frac{\lambda_0}{t_0^3 t_1} x_0 + \frac{\lambda_1}{t_0^2 t_1^2} x_0 + \frac{\lambda_2}{t_0 t_1^3} x_0 + \frac{\mu_0}{t_0^2 t_1} x_1 + \frac{\mu_1}{t_0 t_1^2} x_1 + \frac{\theta_0}{t_0^2 t_1} x_2 + \frac{\theta_1}{t_0 t_1^2} x_2.$$

for some scalars $\lambda_0, \lambda_1, \lambda_2, \mu_0, \mu_1, \theta_0, \theta_1$. We know that the dimension of $\mathcal{R}(Z)_{-3K_Z}$ is equal 7, and that four of these dimensions are accounted for by the sections $x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3$, and two more are accounted for by $x_1 w$ and $x_2 w$. By Lemma 3.5, we see that $\delta x_1 w$ has only μ_0, μ_1 not equal zero when written as above, and $\delta x_2 w$ has only θ_0, θ_1 not equal zero. We conclude that any $h \in K_{(-3,3)}$ written as above with $\lambda_0, \lambda_1, \lambda_2$ not all zero can be used to define $z = \alpha h$, completing the sections $x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3, x_1 w, x_2 w$ to a basis of $\mathcal{R}(Z)^{-K_Z}$.

Up to scalar, the most symmetric choice of $h \in K_{(-3,3)}$ is given by

$$\begin{aligned} h = \frac{2a_0a_1^3}{t_0^3t_1}x_0 - \frac{2a_0^2a_1^2}{t_0^2t_1^2}x_0 + \frac{2a_0^3a_1}{t_0t_1^3}x_0 \\ -\frac{a_1^3b_0 - a_0a_1^2b_1 + a_0^2a_1b_2}{t_0^2t_1}x_1 - \frac{a_0a_1^2b_0 - a_0^2a_1b_1 + a_0^3b_2}{t_0t_1^2}x_1 \\ -\frac{a_1^3c_0 - a_0a_1^2c_1 + a_0^2a_1c_2}{t_0^2t_1}x_2 - \frac{a_0a_1^2c_0 - a_0^2a_1c_1 + a_0^3c_2}{t_0t_1^2}x_2. \end{aligned}$$

We define $z = \alpha h$ accordingly. Considering 0-cocycles, we can find polynomials $q_0(t_i, x_j), q_1(t_i, x_j) \in \mathcal{R}(X)_{(0,3)}$ such that

$$\begin{aligned} t_0^3z &= q_0(t_i, x_j) \\ t_1^3z &= q_1(t_i, x_j) \end{aligned} \tag{56}$$

in $\mathcal{R}(Z)$.

8.2 Translating between the two models of Z

8.2.1 Computing the anticanonical ring

We recall from the introduction to this Chapter, that there is a natural map of $\mathbb{Z}^2$-graded algebras

$$\psi : \mathbb{C}[t_0, t_1, x_0, x_1, w, z] \to \mathcal{R}(Z),$$

which is surjective onto the anticanonical ring $\mathcal{R}(Z)^{-K_Z}$ of Z. There are already 5 known elements of the kernel of ψ, namely the defining equation r of Z in X, the equations of (55), which we label in order h_0, h_1, and the equations of (56), which we label in order g_0, g_1.

Define the homogeneous ideal

$$J = \langle r, h_0, h_1, g_0, g_1 \rangle \lhd \mathbb{C}[t_0, t_1, x_0, x_1, x_2, w, z],$$

generated by the five aforementioned known elements of $\ker \psi$. This ideal is not prime, for in its primary decomposition will be the ideal generated by t_0, t_1, x_0. Discarding this gives us a prime ideal

$$J_0 \lhd \mathbb{C}[t_0, t_1, x_0, x_1, x_2, w, z],$$

which describes the ideal of relations between the generators defining Z inside a rank two toric variety. From J_0, we may eliminate the variables t_0, t_1 using computer algebra. This yields a principal ideal $I \lhd \mathbb{C}[x_0, x_1, x_2, w, z]$, generated by a single polynomial $p(x_1, x_2, w, z)$ of degree 6 in $\mathcal{R}(Z)^{-K_Z}$. This polynomial p is the one defining the anticanonical model of Z in the weighted projected space $\mathbb{P}(1, 1, 2, 3)$.

We are currently unable to run this process parametrically in Macaulay2, with coefficients $a_0, \ldots, f_3$ as variable inputs. However, with every concrete input vector of coefficients, primary decomposition and elimination terminate, returning the Weierstrass cubic equation p in a few seconds. For example, suppose that

A, B, C, D, E, F are given by

$$A = t_0 + t_1, \quad B = t_0^2 + t_0t_1 + t_1^2, \quad C = t_0^2 - t_0t_1 - t_1^2$$
$$D = t_0^3 + t_1^3, \quad E = t_0^3 - t_0^2t_1 - t_0t_1^2 + t_1^3, \quad F = t_0^2t_1 - t_0t_1^2.$$

Then J has two minimal ideals, one of which is (t_0, t_1, x_0). Our desired equation must lie in the second minimal ideal J_0; eliminating t_0 and t_1 gives us a principal ideal generated by the Weierstrass-like polynomial

$$\begin{aligned} p(x_1, x_2, w, z) \quad = \quad & z^2 + 4x_2wz + (4x_1^2x_2 + 6x_1x_2^2 - 2x_2^3)z \\ & -4w^3 + (3x_1^2 + 6x_1x_2 - x_2^2)w^2 \\ & +(8x_1^4 + 22x_1^3x_2 + 16x_1^2x_2^2 + 2x_1x_2^3 + 4x_2^4)w \\ & +(4x_1^6 + 16x_1^5x_2 + 23x_1^4x_2^2 + 18x_1^3x_2^3 + 12x_1^2x_2^4 + 2x_1x_2^5 + x_2^6). \end{aligned}$$

We give the full Macaulay2 code for computing p from given r in the following Section.

8.2.2 Macaulay2 Code

We give below the Macaulay2 script. The ring `Rgen` is the polynomial ring on coefficients of r and the sections t_i, x_j, w, z with their degrees. The computed elements `hgen0, hgen1, ggen0, ggen1` give the equations of (55), (56) multiplied by 2 and `Jgen` gives the ideal generated by the five known kernel elements of ψ.

We then take a specific example, giving explicit choices for the coefficients $a_0, \ldots, f_3$ in the matrix `subs`. The code below has the example

$$A = t_0 + t_1, \quad B = t_0^2 + t_0t_1 + t_1^2, \quad C = t_0^2 - t_0t_1 - t_1^2$$
$$D = t_0^3 + t_1^3, \quad E = t_0^3 - t_0^2t_1 - t_0t_1^2 + t_1^3, \quad F = t_0^2t_1 - t_0t_1^2.$$

The ring `R` is now the domain of ψ and the remaining code gives the ideal J, discards the ideal (t_0, t_1, x_0), eliminates the variables representing t_0, t_1 from the remaining primary component, and outputs the desired equation.

```
Rgen = QQ[a0,a1,b0,b1,b2,c0,c1,c2,d0,d1,d2,d3,e0,e1,e2,e3,f0,f1,f2,f3,
t0,t1,x0,x1,x2,w,z,Degrees=>{20:{0,0},2:{1,0},1:{0,1},2:{-1,1},1:{-2,2},1:{-3,3}}];
rgen = (a0*t0+a1*t1)*x0^2 + (b0*t0^2+b1*t0*t1+b2*t1^2)*x0*x1 +
  (c0*t0^2+c1*t0*t1+c2*t1^2)*x0*x2 +
  (d0*t0^3+d1*t0^2*t1+d2*t0*t1^2+d3*t1^3)*x1^2 +
  (e0*t0^3+e1*t0^2*t1+e2*t0*t1^2+e3*t1^3)*x1*x2 +
  (f0*t0^3+f1*t0^2*t1+f2*t0*t1^2+f3*t1^3)*x2^2;

rawh=(a1*t1-a0*t0)*rgen;
hpart1=sub(rawh, t0=>0);
hpart2=sub(diff(t0,rawh), t0=>0)*t0;
hpart3=sub(rawh, t1=>0);
hpart4=sub(diff(t1,rawh), t1=>0)*t1;
hpart5=rawh - hpart1 - hpart2 - hpart3 - hpart4;
```

```
hgen0 = lift(2*t0^2*w - (2*hpart1 + 2*hpart2 + hpart5)/t1^2,Rgen);
hgen1 = lift(2*t1^2*w + (2*hpart3 + 2*hpart4 + hpart5)/t0^2,Rgen);

s = 2*a0*a1^3*t1^2*x0 - 2*a0^2*a1^2*t0*t1*x0 + 2*a0^3*a1*t0^2*x0-
   (a1^3*b0 - a0*a1^2*b1 + a0^2*a1*b2)*t0*t1^2*x1 -
   (a0*a1^2*b0 - a0^2*a1*b1 + a0^3*b2)*t0^2*t1*x1 -
   (a1^3*c0 - a0*a1^2*c1 + a0^2*a1*c2)*t0*t1^2*x2 -
   (a0*a1^2*c0 - a0^2*a1*c1 + a0^3*c2)*t0^2*t1*x2;
rawg = s*rgen;
gpart1 = sub(rawg, t0=>0);
gpart2 = sub(diff(t0,rawg), t0=>0)*t0;
gpart3 = sub(diff(t0,diff(t0,rawg)), t0=>0)*t0^2/2;
gpart4 = sub(rawg, t1=>0);
gpart5 = sub(diff(t1,rawg), t1=>0)*t1;
gpart6 = sub(diff(t1,diff(t1,rawg)), t1=>0)*t1^2/2;
gpart7 = rawg - gpart1 - gpart2 - gpart3 - gpart4 - gpart5 - gpart6;
ggen0 = lift(2*t0^3*z - (2*gpart1 + 2*gpart2 + 2*gpart3 + gpart7)/t1^3,Rgen);
ggen1 = lift(2*t1^3*z + (2*gpart4 + 2*gpart5 + 2*gpart6 + gpart7)/t0^3,Rgen);

Jgen = ideal(rgen,hgen0,hgen1,ggen0,ggen1);

R=QQ[T0,T1,X0,X1,X2,W,Z,Degrees=>{2:{1,0},1:{0,1},2:{-1,1},1:{-2,2},1:{-3,3}}];
subs=matrix{{1,1,1,1,1,1,-1,-1,1,0,0,1,1,-1,-1,1,0,1,-1,0,T0,T1,X0,X1,X2,W,Z}};
f=map(R, Rgen, subs);
J=f(Jgen);
listJ=minimalPrimes J;
K=listJ#1;
I=eliminate(K, {T0,T1});
p=(gens I)_(0,0)
```

9 Final comments

In this final Chapter we give some work on the Cox rings of a general codimension c complete intersection Calabi-Yau variety Z in the ambient space $(\mathbb{P}^c)^k \times \mathbb{P}^n$, where $c \geqslant 1, n \geqslant c+1$ and $k \geqslant 2$. This uses our work in combination with [34], which originally generalised the result of [9].

We then close with ideas for further results we expect, focusing primarily on the case of extending the work in Chapters 7 beyond the two most simple cases explored.

9.1 Calabi-Yau complete intersections in $(\mathbb{P}^c)^k \times T$

Let $c \geqslant 1, k \geqslant 2$ and consider the ambient space $X = (\mathbb{P}^c)^k \times T$, where T is a product of projective spaces each having dimension at least $c+1$. Let Z be a general Calabi-Yau complete intersection. Since Z is general, the Lefschetz hyperplane theorem gives us an isomorphism of Picard groups $\operatorname{Pic} X \cong \operatorname{Pic} Z \cong \mathbb{Z}^k \oplus \operatorname{Pic} T$. Let $\mathbb{P}^c_l$ denote the l^{th} factor of X for each $l = 1, \dots, k$. Denote by T_l the subproduct

$$T_l = \left(\prod_{l' \neq l} \mathbb{P}^c \right) \times T$$

of X and let $\pi_l : X \to T_l$ be the natural projection map.

Proposition 9.1. *For each $l = 1, \dots, k$, there is an order two flop $\phi_l : Z \dashrightarrow Z$, interchanging the two points of a general fibre of π_l restricted to Z.*

Proof. We can apply the work of Chapters 4,7 accordingly to obtain sections $z_{l,0}, \dots, z_{l,c}$, for each $l = 1, \dots, c$, defining a flop ϕ_l as in the Proposition statement. □

Lemma 9.2. *The flops ϕ_l generate the birational automorphism group of Z, which is isomorphic to the free product of k copies of $\mathbb{Z}/2\mathbb{Z}$.*

Proof. This is the main result of [34]. □

Proposition 9.3. *For each $l = 1, \dots, k$, there is a section u_l, whose degree in $\mathbb{Z}^k \oplus \operatorname{Pic} T$ has 0 in each of the first k entries except the l^{th} entry.*

Proof. For each $l = 1, \dots, k$, we can view Z as a subvariety of $\mathbb{P}^c \times T_l$ and the section u_l is the section of $\mathcal{R}(Z) \backslash \iota^* \mathcal{R}(X)$ obtained as in Proposition 4.6(i) if $c = 1$ and Proposition 7.11 if $c \geqslant 2$. □

Conjecture 9.4. *The Cox ring of Z is generated by the generators of $\mathcal{R}(T)$, and the union of images of the sections*

$$x_0, \dots, x_c, u_1, \dots, u_k,$$

under the action of the birational automorphism group of Z. If $c = 1$, then the $u_1, \dots, u_k$ are redundant generators.

For work towards a proof of this conjecture, we first see that by the methods of Chapter 3, we have

$$\bigoplus_{a_1,\dots,a_k \geqslant 0} \mathcal{R}(Z)_{(a,b)} \cong \bigoplus_{a_1,\dots,a_k \geqslant 0} \iota^* \mathcal{R}(X)_{(a,b)} \oplus K_{1,0} \oplus \dots \oplus K_{k,0},$$

where $K_{l,0}$ is the kernel space K_0 from the method when viewing Z as a subvariety of $\mathbb{P}^c \times T_l$.

The birational automorphism group is generated by flops, and by considering the induced symmetry, we conclude that the conjectured generators generate all sections with image in the moving cone of Z. It requires only to show that any element of $\mathcal{R}(Z)$ defines a divisor in the interior of the moving cone of Z.

9.2 More complete intersections in $\mathbb{P}^c \times \mathbb{P}^n$

Suppose that $c \geqslant 2, n > c$ and $X = \mathbb{P}^c \times \mathbb{P}^n$. Consider a subvariety $Z \subset X$ of the form (†) with respect to hypersurfaces $V_1, \ldots, V_c$ defined by respective equations $r_1, \ldots, r_c$ of respective degrees $(d_1, e_1), \ldots, (d_c, e_c)$, with each $d_m, e_m \geqslant 1$. We have given Cox rings in the general case when $(d_1, d_2, \ldots, d_c)$ is either $(1, 1, \ldots, 1)$ or $(2, 1 \ldots, 1)$. We now speculate on the case where $d_1, \ldots, d_c$ are general. This follows the observations seen in Chapter 7.

For each choice of $0 \leqslant i < j \leqslant c$, we should be able to eliminate all variables except x_i, x_j from the equations $r_1, \ldots, r_c$, giving a degree $d' = d_1 \cdots d_c$ equation in these two variables.

In the case that $d' = 2$ seen in Section 7.3, we could do this in a way such that there existed polynomials $G_{ij} \in \mathcal{R}(\mathbb{P}^n)$ for each $0 \leqslant i \leqslant j \leqslant c$, such that each point of Z satisfied

$$G_{jj}x_i^2 + G_{ij}x_ix_j + G_{ii}x_j^2 = 0.$$

From these equations, one deduced the existence of Cox ring sections $z_0, \ldots, z_c$ on Z.

In the general case $d' > 2$ we conjecture a similar situation; there will be polynomials $G_{ij,l}$ such that

$$G_{ij,0}x_i^{d'} + \cdots + G_{ij,d'}x_j^{d'},$$

and these polynomials occur naturally in the equations in the Cox ring of Z. The polynomials $G_{ij,l}$ cut out a subvariety $Y \subset \mathbb{P}^n$, which will have an expected codimension. If Y has this expected codimension then we conjecture that Z will be a Mori Dream Space, with sections corresponding to data recording the coefficients of the above polynomials divided by the linear factor according to the values of x_i, x_j at a point of Z.

This would be analogous to the situation we observed in Chapter 4 with $c = 1$. In Chapter 4, we only have two variables x_0, x_1 and the expected codimension of Y is simply the number of polynomial coefficients of the defining equation r, namely $d + 1$. If Y has codimension $d + 1$ then the polynomial coefficients form a regular sequence and the Cox ring is finitely generated of a standard by Main Theorem 2.3. If $n > d$ then Y cannot have the expected codimension, and as is proved in [28], Z is not a Mori Dream Space.

We ask further whether this latter occurence generalises to the case of Chapter 7. Namely, if the expected codimension of Y is greater than $n + 1$, then Z will fail to be a Mori Dream Space.

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