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Dynamical Properties of Minimal Cantor Systems

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Abstract

Bratteli-Vershik systems are a powerful tool to represent, investigate and analyse minimal Cantor systems. Techniques include representations of S-adic subshifts, results about eigenvalues of the Koopman operator and, using the structure of the Bratteli diagram, computing ergodic invariant measures on the state space.

The thesis contains analysis and results from three publications:

(i) Interval translation maps (ITMs) are a generalisation of interval exchange transformations (IETs). Certain families of ITMs allow an induction map on the parameter space similar to the Rauzy induction on IETs. Using eigenvalues results from Bratteli diagrams, we analyse such a family of ITMs for weak mixing and characterise linear recurrence in these systems.

(ii) Existing results about the support of ergodic invariant measures on the path space of Bratteli diagram allow us to analyse them for measure-theoretic rigidity. The family of Toeplitz systems as 'almost-periodic' symbolic systems exhibits great variety in many dynamical properties such as entropy, factors and the number of ergodic measures. We prove the existence of different rigidity behaviour in Toeplitz systems and give sufficient conditions for rigidity on Bratteli diagrams.

(iii) A speedup of a Cantor system (X, T) is a map $T^{p(x)}$ on X that speeds up the dynamics differently on different points of X. We show examples of Toeplitz systems whose speedups are not Toeplitz. Further we give sufficient conditions for the speedup of a Toeplitz system to be Toeplitz and for it to be conjugate to a constant speedup.

Additional notes in the thesis concern IETs and their representation as Bratteli-Vershik systems. We investigate the conjugacy map between the systems, the minimal and maximal paths in the diagram, the density of linearly recurrent IETs as well as the support of ergodic invariant probability measures of circle rotations.

Zusammenfassung

Bratteli-Vershiksysteme sind ein mächtiges Werkzeug um minimale Cantorsysteme zu reprä-sentieren und zu untersuchen. Die Darstellung von S-adischen Systemen als Bratteli-Diagramme sowie Resultate zur Existenz von Eigenwerten des Koopman-Operators und der Berechnung ergodischer Maße am Zustandsraum des Diagramms werden genutzt um Systeme aus der symbolischen Dynamik zu analysieren.
Diese Dissertation enthält Resultate aus drei verschiedenen Publikationen:

(i) Intervallverschiebeabbildungen (ITMs) sind eine Verallgemeinerung der Intervalltauschtransformationen (IETs). Bestimmte Familien der ITMs lassen eine Induktion, ähnlich der Rauzy-Induktion auf IETs, zu. Wir untersuchen die Bratteli-Vershik-Darstellung dieser Systeme nach schwacher Mischung und charakterisieren die Eigenschaft der linearen Rekursion in ITMs.

(ii) Wir nutzen bekannte Ergebnisse über die Struktur von Bratteli-Diagrammen und der Trägermenge der ergodischen Maße um dynamische Systeme auf ihre maßtheoretische Rigidität zu untersuchen. Die Familie der Toeplitz-Systeme als 'fast-periodische' Systeme eignet sich aufgrund der verschiedenen Ausprägungen vieler dynamischer Eigenschaften gut, um Beispiele zu konstruieren. Wir zeigen, dass nicht-rigide Toeplitz-Systeme ohne Entropie existieren und geben hinreichende Bedingungen für rigide Bratteli-Diagramme.

(iii) Eine Beschleunigung eines dynamischen Systems (X, T) ist eine Abbildung $T^{p(x)}$, die die Dynamik an verschiedenen Punkten in X verschieden stark beschleunigt. Wir beweisen, dass nicht jede Beschleunigung eines Toeplitz-System wieder ein Toeplitz-System ist. Unter hinreichenden Bedingungen bleibt die Toeplitz-Eigenschaft jedoch erhalten und das beschleunigte System ist konjugiert zu einer konstanten Beschleunigung.

Ein zusätzlicher Teil der Dissertation beschäftigt sich mit IETs und ihrer Bratteli-Vershik-Darstellung. Wir untersuchen die Konjugationsabbildung, die minimalen und maximalen Pfade im Bratteli-Diagramm, die Häufigkeit linear-rekursiver IETs sowie die ergodischen Maße in der Bratteli-Vershik-Darstellung von Kreisrotationen.

Contents

1. Introduction

Symbolic dynamics are transformations on Cantor sets. The most commonly known Cantor set is the middle-third Cantor set. It is constructed by the iterated removal of the middle third of all contained intervals, starting with the unit interval. The resulting set is totally disconnected, metrizable, compact and has no isolated paths, further the middle-third Cantor set is uncountable, lies dense in the unit interval and has Lebesgue measure zero. A general Cantor set is defined by the first four properties and is homeomorphic to the middle-third set.

In a symbolic setting every point of the state space is a (bi-)infinite sequence of symbols from a finite set, the alphabet. For example the symbolic representation of the middle-third Cantor set is the ternary expansion of every number in $[0,1]$ with digits in $\{0,2\}$. On a symbolic sequence, the dynamics are represented by the left shift, which moves the whole sequence one position to the left. If any sequence of symbols from the alphabet is allowed in the state space, it is called the full shift, for only sequences satisfying certain restrictions or constructional conditions, the system is a subshift. Examples for such systems are subshifts generated from one or more substitutions, or constructing it by encoding the orbit of a point in an interval map. Substitutions are functions assigning a finite word to each symbol in the alphabet. This allows us to generate an infinite sequence and, by taking its orbit closure under the left shift, to generate a substitution subshift. Primitive substitutions, where every symbol appears in every word of the substitution, give rise to minimal systems. A system is called minimal if every orbit is dense in the state space.

The class of minimal Cantor systems has a lot of variety when it comes to well-studied dynamical properties. It contains non-uniquely ergodic systems, systems of positive entropy, weakly mixing systems and some with pure point spectrum.

The goal of this thesis is to use different techniques from Bratteli-Vershik systems to investigate minimal Cantor systems for their dynamical properties. Specifically we are interested in the weak mixing property, measure-theoretic rigidity and the behaviour of systems under speedups. To do this, we use results about the Bratteli-Vershik representation of S-adic symbolic systems, about the existence of continuous or measure-theoretic eigenvalues and about the support of finite ergodic invariant measures on the Bratteli diagram. Bratteli diagrams were first introduced in 1972 by O. Bratteli in [Bra72] as a way to classify approximately finite-dimensional C^*-algebras. They are infinite directed graphs, where the vertex set is partitioned into levels and the edges connect only vertices of subsequent levels. A few years later A. Vershik defined adic transformations on these graphs and so generated dynamical systems, the Bratteli-Vershik systems. The Vershik map uses a partial order on the incoming edges to any vertex. Extending this to a lexicographic ordering on the infinite paths of the diagram, the Vershik map maps each infinite path to its successor. In 1992, R. H. Herman, I. F. Putnam and C. F. Skau proved in [HPS92] that every minimal Cantor system can be represented by a Bratteli-Vershik system. This result allows us to use linear algebra to prove dynamical behaviour and to constructing examples of minimal Cantor systems. The structure of the Bratteli diagram is also closely connected to tower cutting-and-stacking procedures or Kakutani-Rokhlin towers and to S-adic subshifts generated by the levels of the Bratteli-Vershik system.

One interesting family of minimal Cantor systems are intervals translations maps, a generalization of interval exchange transformations. The dynamics of interval exchange transformations is well-studied. A result by Avila and Forni in [AF07] shows that almost every non-rotational interval exchange transformation is weakly mixing. For interval translation maps less is known. The Rauzy induction as a map on the parameter space of interval exchange transformations allows certain analysis, the lack of such a function for interval translation maps makes research less unified. We study a two-parameter family of interval translation maps and show that, under certain conditions, ITMs are weakly mixing.

Another important class are Toeplitz systems. They are characterised as minimal, symbolic, almost 1-1 extensions of odometers and exhibit a variety of topological and measure-theoretical properties, see [Dow05] for a survey on Toeplitz and odometer systems by T. Downarowicz. A Toeplitz sequence can be thought of as ‘almost periodic’, as every symbol of the infinite sequence reappears periodically but not all with the same period. Some, for example regular Toeplitz systems, are measure-theoretically isomorphic to odometers. Other Toeplitz systems have countably infinitely many ergodic measures, positive entropy or irrational eigenvalues. A result by R. Gjerde and Ø. Johansen in [GJ00] shows that the Bratteli diagram representing any Toeplitz system is expansive, simple, proper and has an equal number of paths to all vertices of a level. This structure of their Bratteli-Vershik representation and their varying dynamical behaviour makes Toeplitz systems a good class the study. Our focus is on measure-theoretic rigidity, a form of recurrence of almost every point in the system at a sequence of fixed times. Examples include odometers (or adding machines) and irrational rotations. The weaker condition of partial rigidity denotes that a fixed ratio of any measurable set returns to itself at fixed times.Our results show that uniquely ergodic, zero entropy Toeplitz systems exist that are not even partially rigid and we establish conditions for rigidity on Bratteli-Vershik diagrams in a Toeplitz and non-Toeplitz context.

Additionally we investigate Toeplitz systems in the context of speedups. A speedup of a dynamical system is a map that, at certain points can take multiple steps of the original dynamics at once. The resulting orbit can look quite differently to the original. A basic example is a constant speedup by 2, so $S = T^2$ for any map T. Papers on the topological properties of speedups such as [AO24] and [AAO18] by D. Ash, N. Ormes and L. Alvin show that odometers stay odometers and substitutions become non-conjugate substitutions under bounded topological speedups. In [ADL23], D. Ash, A. Dykstra and M. LeMasurier conjectured that the same would hold for Toeplitz systems. We disprove this conjecture and show an example of a sped up Toeplitz system which is no longer Toeplitz.

The thesis is structured as follows. In Chapter 2 preliminaries such as definitions and notations for dynamical systems, substitutions and Bratteli diagrams are introduced.

Chapter 3 concerns itself with interval translation maps. In comparison to interval exchange transformations where the positions of finitely many subintervals of $[0, 1)$ are permuted, in interval translation maps the subintervals are translated. Thus the map might be neither injective nor surjective. In [BT03] a two-parameter family of ITMs $(X, T_{\alpha,\beta})$ was first studied as there an induction similar to the Rauzy induction on IETs is possible. In Section 3.2 we characterise the S-adic representation of these ITMs as the sequence of substitutions $(\chi_{k_i})_i$ and show under which conditions on the sequence of $k_i \in \mathbb{N}$ the subshift is linearly recurrent. The main results concern themselves with weak mixing. This property has multiple, equivalent definitions, the one used here is that the Koopman operator $U_T : L^2(\mu) \to L^2(\mu)$, $U_T f = f \circ T_{\alpha,\beta}$ has no eigenfunction except for constant functions. To study this we use results about the existence of eigenvalues in Bratteli diagrams, see [DFM19] and [BDM05]. In Theorem 3.3.2 we use Lyapunov exponents to show that the stable space of such ITMs is one-dimensional. Through this we get the main result Theorem 3.4.10 that if $\liminf_i k_i < \infty$ and the vector (ξ, ξ, ξ) does not belong to the stable space, then the ITM is weakly mixing. Additional results show that ITMs with preperiodic sequences (k_i) are always weakly mixing in Theorem 3.4.1 and that (ξ, ξ, ξ) belonging to the stable space is a necessary but not sufficient condition for the existence of continuous eigenvalues, see Section 3.4.4. This chapter is based on [BR23].

In Chapter 4 we investigate the notion of measure-theoretic rigidity for Toeplitz systems and other families of Bratteli diagrams. A finite measure-preserving dynamical system (X, T, μ) is called rigid if there is an increasing sequence $t_n \in \mathbb{N}$ such that $\mu(T^{-t_n}(A)\Delta A) \to 0$ as $n \to \infty$ for all measurable sets A. This notion of rigidity as a form of recurrence was first introduced for measure-preserving transformations in [FW78]. With the help of results from [Wil84] we are able to show varying rigidity behaviour in Toeplitz systems. Regular Toeplitz systems are always rigid. For irregular Toeplitz systems we construct an example of a zero entropy system which is not partially measure-theoretically rigid with respect to its uniquely invariant measure, see Corollary 4.3.7. In Example 4.4.22 we use the Gjerde-Johansen result about the Bratteli-Vershik representations of Toeplitz systems to show that irregular Toeplitz systems exist that are rigid and not isomorphic to an odometer. For Bratteli diagrams we investigate the rigidity of ergodic invariant measures by its support in the diagram. We describe sufficient conditions in Theorem 4.4.6 and Theorem 4.4.24. Furthermore we prove non-rigidity for examples of stationary diagrams in Section 4.5. This chapter is based on [BKOR25].

Chapter 5 continuous the investigation of Toeplitz systems, this time in the context of topological speedups. Given a minimal Cantor system (X, T), a *topological speedup* is a map $S(x) = T^{p(x)}(x)$ for some function $p : X \to \mathbb{N}$. Naturally, the question of how the original system and its speedup differ occurs. In [AAO18] and [ADL23], bounded speedups where S is a minimal homeomorphism are investigated and it is shown that substitution systems speed up to non-conjugate substitution systems, odometers to conjugate odometers. Our results show that the Toeplitz property is not invariant under speeding up. Example 5.3.4 gives a bounded speedup of a Toeplitz system that is not Toeplitz. On the other hand Theorem 5.5.1 shows that under certain conditions the speedup will retain the Toeplitz property. Theorem 5.5.10 gives sufficient conditions for the speedup to be Toeplitz and conjugate to a constant speedup. Additionally we analyse the example of the Toeplitz shift of length three on two letters under a constant speedup T^c for its topological factors, see Example 5.5.7. This chapter is based on [AR25].

Chapter 6 takes a closer look at interval exchange transformations (IETs) and their Bratteli-Vershik representation as described in [GJ02]. We investigate the conjugacy map between the IETs and the Bratteli-Vershik system, the structure of the Bratteli diagram, the unique minimal and maximal paths, the density of linearly recurrent IETs in the parameter space and the support of the unique invariant measure in the Bratteli diagram of circle rotations. To describe the maximal path in the Bratteli diagram an order on the outgoing edges of every vertex is defined in Section 6.2. Proposition 6.2.5 shows that the infinite maximal path (in the incoming order) is the unique path that takes only edges maximal in the outgoing order. Further we demonstrate that the Gjerde-Johansen induction on IETs (an acceleration of the Rauzy induction) can be used to construct examples of fat Cantor sets. Proposition 6.4.2 shows that for certain parameters the set of points which stay in the second interval of the circle rotation for every step of the Gjerde-Johansen induction is Cantor set with positive Lebesgue measure.

2. Preliminaries

The *dynamical system* (X, T) has the set X as its state space and the map T acting on X as its dynamics. In this thesis the space X will be a *Cantor set*, a compact metrizable space that is totally disconnected and has no isolated points. If the map T is a homeomorphism, the system is invertible. A dynamical system (X, T) is called *minimal* if for every $x \in X$ the forward orbit $orb_+(x) = \{T^n(x) : n \in \mathbb{N}\}$ is dense in X.

Symbolic Dynamics. Let $\mathcal{A}$ be a finite *alphabet* of letters. The state space will be the *set of bi-infinite sequences of letters*, $\mathcal{A}^{\mathbb{Z}}$. Taking the product topology on $\mathcal{A}^{\mathbb{Z}}$, the open sets are generated as arbitrary unions of finite intersections of *cylinder sets*

$$[e_m, e_{m+1}, \ldots, e_n] = \{x \in \mathcal{A}^{\mathbb{Z}} : x_i = e_i \text{ for all } m \leq i \leq n\}.$$

Cylinder sets and are both open and closed. The state space $\mathcal{A}^{\mathbb{Z}}$ of a finite alphabet is a Cantor set. The map is the *left shift* σ on the bi-infinite sequences. A subset $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is called a *subshift* if it is closed and $\sigma(X) = X = \sigma^{-1}(X)$. As the sequences are bi-infinite, the left shift is invertible.

There are many different ways of generating and representing subshifts, in this thesis we will mainly look at S-adic subshifts and Bratteli-Vershik systems.

Substitutions and S-adic subshifts. Let $\mathcal{A}$ be a non-empty finite set and $\mathcal{A}^+$ be the set of all finite non-empty words over the alphabet $\mathcal{A}$. A *substitution* θ is a map from $\mathcal{A}$ to $\mathcal{A}^+$ which maps a non-empty word $\theta(a) \in \mathcal{A}^+$ to every letter $a \in \mathcal{A}$. The map θ can be extended to $\mathcal{A}^+$ (and to $\mathcal{A}^{\mathbb{Z}}$) by concatenation,

$$\theta(a_1 \ldots a_n) = \theta(a_1) \ldots \theta(a_n).$$

The entries $(a_{i,j})$ of the *associated matrix* M_θ count the number of occurrences of the letter i in $\theta(j)$. The later definition of *incidence matrices* for Bratteli-Vershik systems is the transpose of the associated matrix of a substitution. A substitution θ is *primitive* if there exists $n \in \mathbb{N}$ such that for every $a \in \mathcal{A}$ every letter $b \in \mathcal{A}$ appears in $\theta^n(a)$.

Let $\rho = \lim_{n\to\infty} \theta^n(a) \in \mathcal{A}^{\mathbb{Z}}$ be a fixed point under the substitution θ and let σ be the left shift. The tuple (X_θ, σ) is a *substitutive shift*, where $X_\theta = \overline{\mathrm{orb}_\sigma(\rho)}$ is the orbit closure under σ of a fixed point ρ of the substitution θ. If the substitution θ is primitive, then the shift (X_θ, σ) is a minimal Cantor system. If there exist letters $l, r \in \mathcal{A}$ such that for every letter $a \in \mathcal{A}$ the substitution word $\theta(a)$ starts with l and ends with r, then θ is called *proper*. In this case we can generate the unique fixed point from the word rl, i.e.

$$\rho = \lim_{n\to\infty} \theta^n(rl).$$

Let $(\mathcal{A}_n)_n$ be a sequence of non-empty finite sets and $\mathcal{A}_n^+$ be the set of all finite non-empty words over the alphabet $\mathcal{A}_n$. Take a sequence $(\theta_n)_{n\geq 2}$, where $\theta_n : \mathcal{A}_n \to \mathcal{A}_{n-1}^+$ is a substitutions on $\mathcal{A}_n$ for all n. Then (X, σ) is called an *S-adic subshift* if $X \subset \mathcal{A}_1^{\mathbb{Z}}$ is a set of sequences $x = (x_i)$ such that for all $i, j \in \mathbb{Z}$ the word $x_i x_{i+1} \dots x_j$ appears in the word $\theta_2\theta_3 \dots \theta_n(a_n)$ for some $n \in \mathbb{N}$. The sequence $(\theta_n)_n$ is the *S-adic representation* of (X, σ).

Bratteli-Vershik systems. The *Bratteli diagram* $B = (V, E)$ is an infinite, directed graph such that

- the vertex set $V = \bigcup_{n\geq 0} V_n$ is a countable union of disjoint, finite and non-empty sets V_n and root $V_0 = \{v_0\}$,
- the edge set $E = \bigcup_{n\geq 1} E_n$ is a countable union of disjoint, finite and non-empty sets E_n,
- the range $r : E \to V$ maps $r(E_n) = V_n$ for all $n \in \mathbb{N}$ and $r^{-1}(v) \neq \emptyset$ for $v \in V \setminus V_0$,
- the source $s : E \to V$ maps $s(E_n) = V_{n-1}$ for all $n \in \mathbb{N}$ and $s^{-1}(v) \neq \emptyset$ for $v \in V$.

We call the subgraph of vertices V_{n-1}, V_n and edges E_n the n-th level of the diagram and represent the number of edges E_n connecting V_{n-1} to V_n in a *incidence matrix*, also called transition matrix, M_n. This matrix is of size $|V_n| \times |V_{n-1}|$ and the entry at (i, j) is the number of edges from $j \in V_{n-1}$ to $i \in V_n$. The product of incidence matrices $M_n \cdots M_{m+1}$ for $n > m$ computes the number of paths between vertices in V_m and vertices V_n. The *heights* $h_n(v)$ count the total number of paths between the root vertex v_0 and $v \in V_n$.

If there exists a constant $d \in \mathbb{N}$ the number of vertices of each level $|V_n| < d$, then the Bratteli diagram is of finite rank.

The set X_B is the **space of infinite paths** starting in v_0

$$X_B = \{(e_1, e_2, \dots) : e_i \in E_i \wedge r(e_i) = s(e_{i+1}) \ \forall i \in \mathbb{N}\}.$$

The space X_B can be topologised by using the product topology of $\prod_{n\in\mathbb{N}} E_n$ and the family of *cylinder sets*

$$[e_1, e_2, \ldots, e_n] = \{x \in X_B : x_i = e_i \text{ for all } 1 \leq i \leq n\}$$

which form open and closed neighbourhoods. A cylinder set $C_n(v)$ contains all $x \in X_B$ such that the finite path between v_0 and $v \in V_n$ is fixed. We define a metric d_B on X_B with

$$d_B(x, y) = 2^{-\min\{n\in\mathbb{N}:x_n\neq y_n\}}$$

and $d_B(x, x) = 0$ for all $x, y \in X_B$. With this metric, the space X_B is a Cantor set.

A Bratteli diagram is *simple* if there exist infinitely many paths in X_B and for every level V_m there exists a level V_n with $n > m$ such that every vertex in V_m connects to every vertex in V_n. In other words the product of incidence matrices between these levels $M_n \cdots M_{m+1}$ is positive in every entry.

To define dynamics on the Bratteli diagram a partial order on the set of all edges is introduced. Two edges can be compared if they end in the same vertex, we call this the order of incoming edges. To compare two infinite paths $x, y \in X_B$, there needs to exist a level $n \in \mathbb{N}$ such that $x_i = y_i$ for all $i \geq n$. Such $x, y \in X_B$ are called *tail-equivalent*. An infinite path that only takes edges minimal (or maximal) in the incoming order is called a minimal (or maximal) path.

As the dynamics on an ordered Bratteli diagram we define the *Vershik map* V_B. It maps the set of all maximal infinite paths to the set of minimal infinite paths and any non-maximal path to its direct successor. A Bratteli-Vershik diagram is *proper* if it is simple and has a unique maximal and a unique minimal infinite path. Under these condition the Vershik map is a homeomorphism.

The following result by Herman, Putnam, Skau from 1992 allows us to use Bratteli-Vershik systems as representations of any minimal Cantor system.

Theorem 2.0.1. *[HPS92, Theorem 4.7] Let X be a compact, totally disconnected, metrizable space and T a homeomorphism. There exists a bijective correspondence between equivalence classes of ordered Bratteli diagrams with unique maximal and minimal infinite paths and topological conjugacy classes of dynamical systems (X, T) with a unique minimal subset $Y \subset X$.*

One way to describe level n of a Bratteli-Vershik diagram is a function χ_n, the so-called *morphism read on E_n*. It maps vertices in V_{n-1} to every $v \in V_n$ such that these vertices are the sources of incoming edges to v. Let $v \in V_n$, then $\chi_n(v) = v_i v_j \ldots v_k$ with

$v_i, v_j, \dots, v_k \in V_{n-1}$. The order of vertices in the word $v_i v_j \dots v_k$, with the minimal edge incoming to $v \in V_n$ from $v_i \in V_{n-1}$ and the maximal from $v_k \in V_{n-1}$, gives the order of incoming edges.

Through the sequence of morphism reads on E_n for all $n \in \mathbb{N}$, we can construct an ordered Bratteli diagram from any sequence of substitutions, see [DL12] for the Bratteli-Vershik representation of an S-adic shift.

Proposition 2.0.2. *[DL12, Proposition 2.2] Let (X, T) be the minimal S-adic subshift defined by a sequence of substitutions $\theta_n : \mathcal{A}_n \to \mathcal{A}_{n-1}^+$, where all θ_n are proper. Suppose that for all n the substitutions θ_n extend by concatenation to a one-to-one map from X_n to X_{n-1}. Then (X, T) is conjugate to (X_B, φ_B), where B is the Bratteli diagram such that for all $n \geq 2$ the substitution read on B at level n is θ_n.*

Another way of representing a dynamical system is a cutting-and-stacking procedure. For a properly ordered Bratteli diagram B we define a sequence of *Kakutani-Rokhlin partition* partitions

$$\mathcal{P}_n = \{T^j B_n(v) : v \in V_n, 0 \leq j < h_n(v)\},$$

where the set $B_n(v)$ is the base of the tower $v \in V_n$ and contains the minimal path from v_0 to $v \in V_n$. The edges E_n of the Bratteli diagram describe the cutting-and-stacking procedure to go from towers in $\mathcal{P}_{n-1}$ to towers in $\mathcal{P}_n$.

See [Bru22, Dur10, DFM19, BK16] for more on Cantor systems, Bratteli diagrams and their dynamics.

3. Interval Translation Maps with Weakly Mixing Attractors

We study linear recurrence and weak mixing of a two-parameter family of interval translation maps $T_{\alpha,\beta}$ for the subset of parameter space where $T_{\alpha,\beta}$ has a Cantor attractor. For this class, there is a procedure similar to the Rauzy induction which acts as a dynamical system G on parameter space, which was used previously to decide whether $T_{\alpha,\beta}$ has an attracting Cantor set, and if so, whether $T_{\alpha,\beta}$ is uniquely ergodic. In this paper we use properties of G to decide whether $T_{\alpha,\beta}$ is linearly recurrent or weak mixing.

This chapter is based on the article [BR23] which was submitted in January 2025.

3.1 Introduction

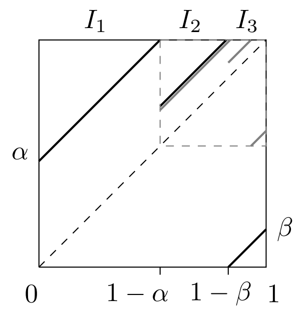
Interval translation maps (ITMs) are extensions of interval exchange transformations (IETs). Both are piecewise isometries of the unit interval $I = [0, 1)$, having a finite partition into intervals I_j of continuity. But where an IET permutes these intervals, and is hence invertible, an ITM is merely a translation on these I_js, and globally need neither be injective nor surjective.

Among the things proven for IETs is that typically they preserve only Lebesgue measure, see [Mas82, Vee82, Vee84] (although there are non-uniquely ergodic IETs, see [Kea77]), and typically, they are weakly mixing, see [AF07]. The reason we know so much about IETs is that there is a renormalization procedure, called *Rauzy* or *Rauzy-Veech induction*, that acts as a map on the parameter space, see e.g. the monograph of Viana [Via06].

Since ITMs (I, T) are in general not surjective, they don't preserve Lebesgue measure. Naturally, $T^{n+1}(I) \subset T^n(I)$ for all $n \geq 0$, but the inclusion can be strict. If we have equality for some n_0, then $\bigcap_{n\geq 0} T^n(I) = T^{n_0}(I)$ is a finite union of intervals, and restricted to this set, T is an interval exchange transformation. Such ITMs are called of *finite type*. If the inclusions $T^{n+1}(I) \subset T^n(I)$ are strict for all $n \geq 0$ (and if $\bigcap_{n\geq 0} T^n(I)$ contains no interval), then the attractor $\overline{\bigcap_{n\geq 0} T^n(I)}$ is a Cantor set. Such ITMs are called of *infinite type*. Mixtures of the two types, where $\overline{\bigcap_{n\geq 0} T^n(I)}$ is the union of intervals and a Cantor set, are possible, but not for the family $T_{\alpha,\beta}$ that we are interested in.

In general, there is no known analogue of the Rauzy induction, but there is for the family of ITMs introduced in [BT03]:

$$T_{\alpha,\beta}(x) = \begin{cases} x+\alpha, & x \in I_1 := [0, 1-\alpha), \\ x+\beta, & x \in I_2 := [1-\alpha, 1-\beta), \\ x-1+\beta, & x \in I_3 := [1-\beta, 1]. \end{cases}$$



on the parameter space $U = \{(\alpha,\beta) : 0 < \beta \leq \alpha \leq 1\}$. Viewed as circle map, $T_{\alpha,\beta}$ is a piecewise rotation on two pieces; it was studied in greater generality in [AFHS21, Bru07].

A conjecture by Boshernitzan & Kornfeld [BK95] suggests that infinite type occurs only for a set of zero Lebesgue measure in the parameter space, and for the family $T_{\alpha,\beta}$, this indeed holds, see [BT03, Theorem 6]. Determining the type for this family goes via renormalization consisting of the first return map to the interval $[1-\alpha, 1]$, see the right upper box in the above picture, and rescaling to unit size, in analogy to the Rauzy induction for IETs. This transforms $T_{\alpha,\beta}$ into a member $T_{\alpha',\beta'}$ of the same family where

$$(\alpha', \beta') = G(\alpha, \beta) = \left(\frac{\beta}{\alpha} \, , \, \frac{\beta - 1}{\alpha} + \left\lfloor \frac{1}{\alpha} \right\rfloor \right). \tag{3.1}$$

Note that $G(U) = U \cup D$ for $D = \{(\alpha,\beta) : \alpha - 1 \leq \beta \leq 0 \leq \alpha \leq 1\}$ in a countable-to-one way, see Figure 3.1 (left). Exactly the parameters in $\Omega_\infty := \{(\alpha,\beta) : G^n(\alpha,\beta) \in int(U) \text{ for all } n \geq 0\}$ have the property that $T_{\alpha,\beta}$ is of infinite type, i.e., $\overline{\bigcap_{n\geq 0} T^n_{\alpha,\beta}([0,1])}$ is a Cantor set on which $T_{\alpha,\beta}$ acts as a minimal endomorphism. See Figure 3.1 (right) for the parameter set associated to ITMs of infinity type.

Symbolically, $T_{\alpha,\beta}$ is described by an S-adic subshift (X, σ), based on a sequence of substitutions χ_{k_i}, $k_i \in \mathbb{N} = \{1, 2, 3, \dots\}$ given in (3.2) in Section 3.2 below. That is, the symbolic itinerary of 1 w.r.t. the partition $\{I_1, I_2, I_3\}$ is exactly the sequence $\rho = \lim_{i\to\infty} \chi_{k_1} \circ \cdots \chi_{k_i}(3)$, and the shift-orbit closure of $\overline{\{\sigma^n(\rho) : n \geq 0\}}$ is the S-adic subshift. The value k_i refers to the strip U_{k_i} that $G^{i-1}(\alpha,\beta)$ belongs to.

We call $T_{\alpha,\beta}$ *linearly recurrent* if the subshift (X,σ) is, i.e., there is L such that for every $x \in X$, every subword w reappears in x with gap $\leq L|w|$. Theorem 3.2.6 gives a precise condition on the sequence $(k_i)_{i\in\mathbb{N}}$ that results in linear recurrence.

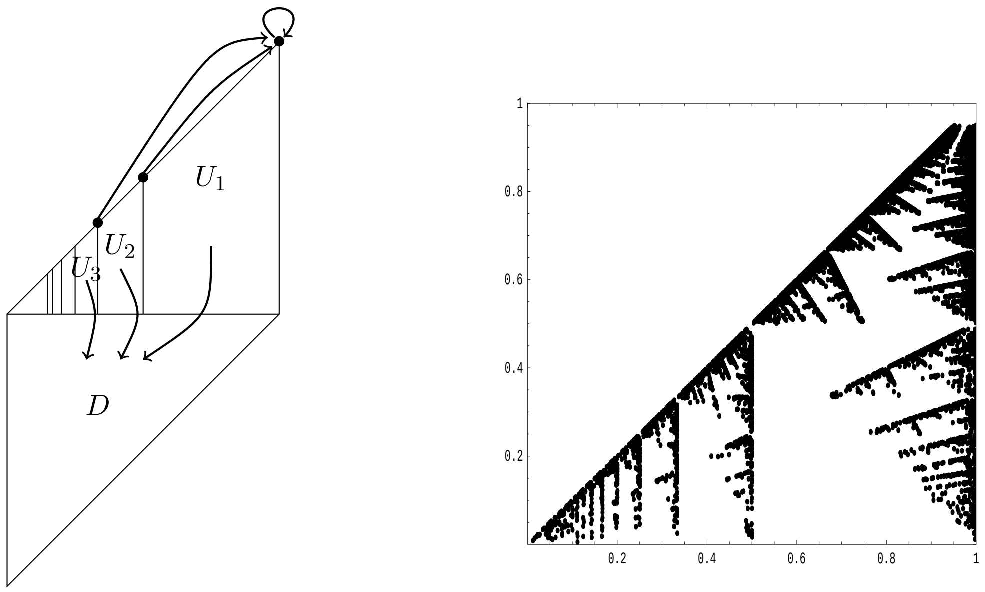


Figure 3.1: Left: U is divided into strips U_k that map to $U \cup D$.
Right: Approximation of the set Ω_∞ of type infinity parameters with $10,000$ pixels.

In Theorem 3.4.13, strengthening [BT03, Theorem 12], we show that if $\liminf_i k_i < \infty$, then $T_{\alpha,\beta}$ is uniquely ergodic, and this holds for a.e. every parameter $(\alpha, \beta) \in \Omega_\infty$ with respect to **every** G-invariant measure. The unique ergodicity of $T_{\alpha,\beta}$ fails if the sequence (k_i) increases exponentially fast, see [BT03, Theorem 11]. In this (by Theorem 3.4.13) non-generic situation, there are two ergodic measures.

We can conclude from its representation as S-adic shifts, that $T_{\alpha,\beta}$ is never strongly mixing, i.e., $\lim_{n\to\infty} \mu(A \cap T^{-n}(B)) - \mu(A)\mu(B) \not\to 0$ for any of its T-invariant measures, see e.g. [Bru22, Theorem 6.79]. Instead, the other property we investigate in this paper is weakly mixing of $T_{\alpha,\beta}$, i.e., whether the Césaro means

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \left|\mu(A \cap T^{-j}(B)) - \mu(A)\mu(B)\right| \to 0$$

. Weak mixing has many equivalent definitions, and the definition relevant for this paper is that the Koopman operator $U_T : L^2(\mu) \to L^2(\mu)$, $U_T f = f \circ T_{\alpha,\beta}$ has no eigenfunction except for $f \equiv 1$. If instead a non-trivial eigenvalue exists, we call it a *continuous eigenvalue* if the corresponding eigenfunction f is continuous, and a measurable eigenvalue if $f \in L^2(\mu)$ for the (unique) $T_{\alpha,\beta}$-invariant measure μ. General results on weak mixing for interval exchange transformations were obtained by Nogueira & Rudolph [NR97] (generic

IETs), Sinaĭ & Ulcigrai [SU05] (“periodic” IETs), Ferenczi & Zamboni [FZ11] and Avila & Forni [AF07] (Lebesgue typical IETs). Recent extensions of these results can be found in e.g. [AD16, AFS23, AL18], and for the specific family of Arnoux-Rauzy systems [AR91], conditions for the absence of eigenvalues, or the number of algebraically independent, eigenvalues, were formulated in [CFM08]. The relative simplicity of the family $\{T_{\alpha,\beta}\}$ allows us to use methods from linear algebra (rather than results of Veech and Teichmüller theory) combined with a general condition on Bratteli-Vershik (BV) systems, due to Durand, Frank & Maass [DFM19], for existence of eigenvalues of the Koopman operator. We phrase this *BV-summability condition* in (3.23) in Section 3.4.4. This condition goes back to Host [Hos86], and worked out for Bratteli-Vershik systems in more general settings in [BCBY25, BDM05, BDM10, CDHM03]. We prove that all “periodic” ITMs in our class (i.e., those for which the corresponding sequence $(k_i)_{i\in\mathbb{N}}$ is (pre-)periodic) as well as “typical” (in a sense made precise in Theorem 3.4.10) ITMs in our class are weakly mixing.

The paper is structured as follows. In Section 3.2 we characterise the linear recurrent ITMs by giving an if and only if condition (see Theorem 3.2.6) on the index sequence $(k_i)_{i\in\mathbb{N}}$ of the substitutions in the S-adic representation. When viewing the S-adic representation as sequences of toral automorphisms $(A_k)_{k\geq 1}$, one would expect that $e^{2\pi i\xi}$ is an eigenvalue the Koopman operator if and only if the vector $\vec{\xi} := (\xi,\xi,\xi) \in \mathbb{T}^3$ belongs to the stable space $W^s(\vec{0})$ of $(A_k)_{k\geq 1}$. In the linearly recurrent case, and specifically in the (pre)periodic case, this is indeed true, see Corollary 3.4.15 and Theorem 3.4.1. But beyond that, the BV-summability condition (3.23) becomes highly non-trivial. If $\vec{\xi} \notin W^s(\vec{0})$, then $e^{2\pi i\xi}$ is not a **continuous** eigenvalue (Theorem 3.4.6). This result does not explicitly depend the exact substitutions χ_{k_i} only its abelianizations. Whether $e^{2\pi i\xi}$ is not a eigenvalue at all is left open, although this can only possibly happen if $\liminf_{i\to\infty} k_i = \infty$ (Theorem 3.4.10). Further there are examples where $\vec{\xi} \in W^s(\vec{0})$ but $e^{2\pi i\xi}$ is not a continuous eigenvalue (Theorem 3.4.8).

In Section 3.3, we study the Lyapunov exponents of long concatenations $A_1\cdots A_n$ and show that $W^s(\vec{0})$ is one-dimensional. The absence of eigenvalues becomes therefore the generic situation, as studied in Section 3.4. Section 3.4.2 gives algebraic reasons why ITMs with (pre-)periodic sequences $(k_i)_{i\in\mathbb{N}}$ are weak mixing. Section 3.4.3 investigates the stable direction further, showing that it is uniquely determined by $(k_i)_{i\in\mathbb{N}}$. In Section 3.4.4, we show that $\vec{\xi} \in W^s(\vec{0})$ is a necessary, although not a sufficient, condition for the existence of continuous eigenvalues. Finally, in Section 3.4.5 we add the generic condition $\liminf_i k_i < \infty$, see Theorem 3.4.13. Under this condition the ITM is uniquely ergodic, and if $\vec{\xi} \notin W^s(\vec{0})$ for all $\xi \in (0,1)$, then $T_{\alpha,\beta}$ is weakly mixing.

Acknowledgment: We are grateful to the first referee for pointing out some unclear steps in what now became Section 3.3, as well as for drawing our attention to [Mer24], and specifically to the second referee for raising many interesting points, and correcting us in several others. We gratefully acknowledge the support of the Austrian Exchange Service (WTZ Project PL 15/2022). We also thank Reem Yassawi for the fruitful conversations.

3.2 Linear recurrence

We use symbolic dynamics w.r.t. the partition $\{I_1, I_2, I_3\} := \{[0, 1-\alpha), [1-\alpha, 1-\beta), [1-\beta, 1]\}$, with symbols $1, 2, 3$, respectively. One renormalization step is given by the substitution

$$\chi_k : \begin{cases} 1 \to 2 \\ 2 \to 31^k \\ 3 \to 31^{k-1} \end{cases} \qquad \text{for } k = \left\lfloor \frac{1}{\alpha} \right\rfloor \in \mathbb{N}. \tag{3.2}$$

The associate matrix and its inverse are

$$A_k = \begin{pmatrix} 0 & k & k-1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad \text{and} \qquad A_k^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1-k \\ -1 & 0 & k \end{pmatrix}.$$

Note that $\det(A_k) = \det(A_k^{-1}) = -1$ and the characteristic polynomial of A_k is $P_k(\lambda) = \lambda^3 - \lambda^2 - k\lambda + 1$.

By [BT03], specifically its Theorem 7, every ITM of infinite type in this family is uniquely characterised by a sequence $(k_i)_{i\in\mathbb{N}} \subset \mathbb{N}$ such that

$$k_{2i} > 1 \text{ for infinitely many } i \in \mathbb{N} \text{ and } k_{2i-1} > 1 \text{ for infinitely many } i \in \mathbb{N}. \tag{3.3}$$

Otherwise $G^n(\alpha, \beta)$ will eventually belong to the upper and right boundary alternatingly, and this behaviour belongs to the finite type case.

The itinerary of the point $1 \in [0, 1]$ is then

$$\rho = \lim_{i\to\infty} \chi_{k_1} \circ \chi_{k_2} \circ \chi_{k_3} \circ \cdots \circ \chi_{k_i}(3).$$

Let X be the closure of $\{\sigma^n(\rho)\}_{n\in\mathbb{N}}$ where σ is the left-shift. For such sequences, the attractor Ω of the ITM is a Cantor set, on which the action is isomorphic to an S-adic shift (X, σ) with associated matrices A_{k_i}. In [BT03] conditions are given under which $(\Omega, T_{\alpha,\beta})$ is uniquely ergodic, see also Theorem 3.4.13. For the first result on linear recurrence for our family of ITMs, we need certain properties for the S-adic shift.

A S-adic subshift based on substitutions $\chi_i : \mathcal{A}_i \to \mathcal{A}_{i-1}^+$ is called *primitive* if for all $m \in \mathbb{N}$ there exists $n \geq m$ such that for all $a \in \mathcal{A}_n$

$$\chi_m \circ \cdots \circ \chi_n(a) \text{ contains every } b \in \mathcal{A}_{m-1}.$$

The corresponding shift space X is the shift orbit closure of the set of accumulation points of $\{\chi_1 \circ \cdots \circ \chi_n(a) : n \in \mathbb{N}, a \in \mathcal{A}_n\}$. A subshift is *aperiodic* if there exists no $x \in X$ such that $\sigma^k(x) = x$ for some $k \in \mathbb{N}$.

A substitution is called *left-proper*, if for all letters $a \in \mathcal{A}$ the word $\chi(a)$ has the same starting letter. While the substitutions χ_{k_i} based on an ITM of infinite type are not left-proper, any telescoping $\chi_k \circ \chi_{k_{i+1}}$ is, because

$$\chi_{k_i} \circ \chi_{k_{i+1}} : \begin{cases} 1 \to 31^{k_i} \\ 2 \to 31^{k_i-1}2^{k_{i+1}} \\ 3 \to 31^{k_i-1}2^{k_{i+1}-1}. \end{cases}$$

Thus ρ, the itinerary of the point $1 \in [0,1]$, is the unique one-sided fixed point under $(\chi_{k_i})_{i \in \mathbb{N}}$.

Next we need to show that our S-adic shift (X, σ) is aperiodic. In the literature one can find some results in this direction, e.g., [BCBD$^+$21, Lemma 3.3] requires that the substitutions are unimodular, primitive and proper, where we only have left-proper. The next lemma uses a weak form of recognizability, which does hold in our case.

Definition 3.2.1. Let (X, σ) be a subshift based on the substitution χ. We call χ *combinatorially recognizable* if there is $N \in \mathbb{N}$ such that for every word w in X of length $|w| \geq N$ and every $v = v_1 v_2 v_3 \ldots v_n$ such that w appears twice in $\chi(v)$, then the first appearance of w starts at $\chi(v_1 \cdots v_i)$ if and only if the second appearance of w starts at $\chi(v_1 \cdots v_j)$ for some $0 \leq i < j < n$. (Here $i = 0$ means that $\chi(v)$ starts with w.) The S-adic subshift (X, σ) based on substitutions $(\chi_i)_{i \in \mathbb{N}}$ is recognizable if each χ_i is recognizable (not necessarily with the same N).

This definition of combinatorially recognizable is also called *recognizability in the sense of Mossé*, see [BSTY19, Definitions 2.3 and 2.4].

Lemma 3.2.2. *Let (X, σ) be an injective, combinatorially recognizable S-adic subshift based on substitutions $\chi_i : \mathcal{A}_i \to \mathcal{A}_{i-1}^+$ such that for every $n \in \mathbb{N}$ there is $m > n$ such that*

$$|\chi_n \circ \cdots \circ \chi_m(a)| > 1 \quad \textit{for every } a \in \mathcal{A}_m. \tag{3.4}$$

Then (X, σ) is aperiodic.

Proof. Assume by contradiction that (X,σ) has a periodic element $x = \sigma^k \circ \chi(y) \in X$ with period p. It follows that $x = w^\infty$ with w a word of length p. We can pick k in such a way that w^∞ starts with $\chi_1(a)$ for some $a \in \mathcal{A}_1$. As χ_{k_1} is recognizable and injective, every new occurrence of w must start with $\chi_1(a)$. In fact, there is a unique word $a \cdots b \in \mathcal{A}_1^+$ such that $\chi_1(a \cdots b) = w$ and

$$\chi_1^{-1}(x) = \chi_1^{-1}(w)^\infty = (a \cdots b)^\infty.$$

This unique $\chi_1^{-1}(x)$ is again periodic with period $p_1 \leq p$. Using (3.4) and taking preimages $\chi_m^{-1} \circ \cdots \circ \chi_1^{-1}(x)$ decreases the period: $p_m < p_1$. Since we can repeat this argument, even when the period is reduced to 1, we get a contradiction. Hence all elements of (X,σ) are aperiodic. □

Proposition 3.2.3. *The S-adic subshift (X,σ), based on substitutions $(\chi_{k_i})_{i\in\mathbb{N}}$ from an ITM of infinite type, is primitive, combinatorially recognizable and aperiodic.*

Proof. **Primitivity.** We prove that for any $i \in \mathbb{N}$ there exists n_i such that the product of matrices associate to $\chi_{k_i}, \ldots, \chi_{k_{i+n_i+1}}$ is strictly positive.

We write A_k for matrices with $k > 1$. For odd or even length blocks of substitutions with $k_i = 1$ the product of associated matrices are

$$A^{2r-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ r-1 & r & 1 \end{pmatrix}, \quad A^{2r} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r & r & 1 \end{pmatrix}.$$

Let $i \in \mathbb{N}$ be such that $k_i > 1$, as the ITM is of infinity type there exists an odd m_i such that $k_{i+m_i} > 1$.

In between k_i and k_{i+m_i} the positions of odd distance to k_i are always equal to one, the even positions may take any value $k \geq 1$, i.e.,

$$(k_i, 1, k_{i+2}, 1, \ldots, 1, k_{i+2j}, 1, \ldots, 1, k_{i+2n}, k_{i+m_i}).$$

Thus we can write the multiplication of matrices from k_i to k_{i+m_i} in the following way

$$A_{k_i} \left(\prod_{(k,r)} A^{2r-1} A_k \right) A^{2s} \; A_{k_{i+m_i}}, \tag{3.5}$$

where $s \geq 0$ and (k,r) are the values of $k_j > 1$ for $i < j < i + m_i$ and $2r - 1$ is the length of the chain of ones between the previous $k > 1$ and k_j with $r > 0$.

If the product in (3.5) is empty, it is the identity matrix. If it contains only one pair (k, r), it has the form

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ r-1 & r & 1 \end{pmatrix} \begin{pmatrix} 0 & k & k-1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & k & k-1 \\ r & (r-1)k+1 & (r-1)(k-1)+1 \end{pmatrix}.$$

Otherwise, if the product contains multiple such factors, we get

$$\prod_{(k,r)} A^{2r-1} A_k = \prod_{(k,r)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ * & * & * \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix},$$

where the $*$ represent integers greater than zero. As the identity matrix is minimal in each entry, we use it as the lowest possible bound for each entry in the matrix multiplication with $s > 0$

$$A_{k_i} \left(\prod_{(k,r)} A^{2r-1} A_k \right) A^{2s} \geq \begin{pmatrix} 0 & k_i & k_i - 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ s & s & 1 \end{pmatrix} = \begin{pmatrix} * & * & * \\ 1 & 0 & 0 \\ * & * & * \end{pmatrix}$$

In case $s = 0$, the result is A_{k_i}, which is the entry-wise lower possibility. Thus

$$\begin{aligned} A_{k_i} \left(\prod_{(k,r)} A^{2r-1} A_k \right) A^{2s} A_{k_{i+m_i}} &\geq \begin{pmatrix} 0 & k_i & k_i - 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & k_{i+m_i} & k_{i+m_i} - 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} * & * & * \\ 0 & * & * \\ * & * & * \end{pmatrix}. \end{aligned}$$

We see that the multiplication from k_i to k_{i+m_i} does not result in a full matrix. By multiplying one more step, independently of the value of k_{i+m_i+1} (it can be 1), we get

$$\begin{pmatrix} * & * & * \\ 0 & * & * \\ * & * & * \end{pmatrix} A_{k_{i+m_i+1}} = \begin{pmatrix} * & * & * \\ 0 & * & * \\ * & * & * \end{pmatrix} \begin{pmatrix} 0 & k_{i+m_i+1} & k_{i+m_i+1} - 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$$

and thus we have a full matrix by multiplying $m_i + 2$ steps together.

For an arbitrary position j with $k_j \geq 1$ there exists $n_j \geq 0$ such that there is $j \leq i \leq j + n_j$ with $k_i > 1$ and m_i odd with $k_{i+m_i} > 1$. Thus the matrix associated to the substitution $\chi_{k_j} \circ \cdots \circ \chi_{k_{i+m_i+1}}$ is strictly positive. Therefore the S-adic shift is primitive.

Combinatorial Recognizability. Let (X_i, σ) be the S-adic subshift based on $(\chi_{k_j})_{j\geq i}$, we show that substitution χ_{k_i} is combinatorially recognizable. Take $x \in X_i$, there is a unique way to decompose x into blocks $\chi_{k_i}(a)$ for $a \in \mathcal{A}$. Any 2 in x is its own block $\chi_{k_i}(1)$. Further every 3 in x starts a new block and depending on the number of 1s directly following, we can determine if the block is $\chi_{k_i}(2)$ or $\chi_{k_i}(3)$. For example

$$\begin{aligned} x &= \ldots \mid 2 \mid 3\ 1\ 1 \mid 3\ 1 \mid 2 \mid 2 \mid 3\ 1\ 1 \mid \ldots \\ &= \ldots \chi_2(1)\ \chi_2(2)\ \chi_2(3)\ \chi_2(1)\ \chi_2(1)\ \chi_2(2) \ldots \end{aligned}$$

For any word w with $|w|1$ and v such that w appears twice in $\chi_{k_i}(v)$, if w starts with letters 2 or 3 we know that there exists $i < j$ with

$$\chi_{k_i}(v) = \chi_{k_i}(v_1 \cdots v_i)w \cdots = \chi_{k_i}(v_1 \cdots v_j)w \cdots .$$

If w starts with letter 1, then there exists no such i, j.

Thus the substitution χ_{k_i} is combinatorially recognizable and therefore the S-adic subshift based by substitutions $(\chi_{k_i})_{i\in\mathbb{N}}$ is combinatorially recognizable.

Aperiodicity. By Lemma 3.2.2 there exist no periodic elements in (X, σ). □

From this proof, it is clear that a general method of telescoping the sequence A_{k_n} to obtain full matrices $\tilde{A}_i$ is the following:

$$\tilde{A}_i = \underbrace{A_1 \cdots A_1}_{r_{i,1}} \cdot A_{k_{i,1}} \cdot \underbrace{A_1 \cdots A_1}_{r_{i,2}} \cdot \cdots \cdot A_{k_{i,m}} \cdot \underbrace{A_1 \cdots A_1}_{r_{i,m+1}} \cdot A_{k_{i,m+1}} \cdot A_{k_{i,m+2}}, \tag{3.6}$$

where $k_{i,j} \geq 2$ for $1 \leq j \leq m+1$, $k_{i,m+2} \geq 1$ and $r_{i,j}$ odd for $2 \leq j \leq m$, $r_{i,1} \geq 0$ and $r_{i,m+1}$ even.

Remark 3.2.4. If the lengths of gaps between $k_i > 1$ and $k_{i+m_i} > 1$ with m_i odd, is bounded for all i, then the subshift is strongly primitive, i.e., there exists a constant $M > \max_i(m_i)$ such that the matrix associated to the substitution $\chi_{k_i} \circ \cdots \circ \chi_{k_{i+2M}}$ is strictly positive for all i.

Lemma 3.2.5. *Let (X, σ) be a primitive non-periodic S-adic shift, based on substitutions $\chi_i : \mathcal{A}_i \to \mathcal{A}_{i-1}^+$. If for every $k \in \mathbb{N}$ there are $i < j$ and $a \in \mathcal{A}_{i-1}$, $b \in \mathcal{A}_j$ such that a^k is a subword of $\chi_i \circ \cdots \circ \chi_j(b)$, then (X, σ) is not linearly recurrent.*

Proof. Let $L \in \mathbb{N}$ be arbitrary and find $i < j$ and $b \in \mathcal{A}_j$ such that $\chi_i \circ \cdots \circ \chi_j(b)$ has a^{L+2} as subword. Set $w = \chi_1 \circ \cdots \circ \chi_{i-1}(a)$; then X contains a sequence x having w^{L+2} as subword. Since x is not periodic, the word-complexity $p_x(|w|) > |w|$, and hence there is a word v of length $|v| = |w|$ that is not a subword of w^{L+2}. By primitivity, v and w^{L+2} appear infinitely often in x. But then v must have reappearances with gap $\geq L|v|$. Hence (X, σ) is not L-linearly recurrent. As L was arbitrary, the lemma follows. □

Linear recurrence is a notion very well studied by Durand and collaborators [Dur00, DDMP21]. For our purpose here, Theorem 5.4 in [BCBY25] is useful: An S-adic subshift based on recognizable substitutions is linearly recurrent if and only if we can telescope the substitutions $(\chi_{k_i})_{i\geq 1}$ into finitely many, left-proper substitutions with strictly positive transition matrices.

Theorem 3.2.6. *The subshift (X, σ) associated to an ITM $(\Omega, T_{\alpha,\beta})$ of infinite type is linearly recurrent if and only if $(k_i)_{i\in\mathbb{N}}$ is bounded and the sets $\{i : k_{2i} > 1\}$ and $\{i : k_{2i-1} > 1\}$ have bounded gaps.*

Proof. Let M and N be the maximal gap sizes of the sets $\{i : k_{2i-1} > 1\}$ and $\{i : k_{2i} > 1\}$ respectively. Every M-gap (i.e., $(k_n)_{n=i+1}^{M}$ for any $i \in \mathbb{N}$) contains at least one $k_{2i-1} > 1$ and every N-gap at least one $k_{2i} > 1$. Then there exists $\tilde{M} > \max\{M, N\}$ where for any $k_i \geq 1$ there exists $k_{i+2j-1} > 1$ and $k_{i+2l} > 1$ for $j, l < \tilde{M}$. Thus as in the proof of Proposition 3.2.3 for any $i \in \mathbb{N}$ multiplying $2\tilde{M} + 1$ matrices together from k_i to $k_{i+2\tilde{M}}$ will result in a full matrix. As there are only finitely many values $k_j \in \mathbb{N}$ can take, the set of full associated matrices is finite and the subshift (X, σ) is linearly recurrent.

On the other hand if $(k_n)_n$ unbounded we have that for any $L \in \mathbb{N}$ there exists $n \in \mathbb{N}$ with $k_n > L$ and therefore

$$\chi_{k_n}(2) = 31^{k_n}$$

contains the subword 1^{k_n}.

Similarly if the gaps of $\{i : k_i > 1\}$ are unbounded, for any $L \in \mathbb{N}$ there exists a gap of length $r_n > 2L$ and

$$\chi_1^{r_n}(2) \;=\; \begin{cases} 3^{l_n}1 & \text{for } r_n = 2l_n - 1, \\ 3^{l_n}2 & \text{for } r_n = 2l_n \end{cases}$$

contains the subword 3^{l_n}. By Lemma 3.2.5, the subshift (X, σ) is not linearly recurrent. □

3.3 Lyapunov exponents

In this section we analyse the dynamics of the matrix products $\tilde{A}_1 \cdot \tilde{A}_2 \cdots \tilde{A}_n$, $n \to \infty$, from (3.6) and show that there is one positive Lyapunov exponent (as $\liminf$), and negative

Lyapunov exponent (as lim sup), and the third Lyapunov is at least not negative (as lim inf). The usual Oseledets theory states that Lyapunov exponents exist (as limits) for typical parameters, where in our setting, typical is to be interpreted as w.r.t. a G-invariant measure on Ω_∞, in the line of [AFHS21] based on the construction of [Fou20]. We need a statement for all $(\alpha, \beta) \in \Omega_\infty$, but lim inf / lim sup-results as in Proposition 3.3.1 suffice.

Abbreviate $\widetilde{\mathbb{A}}^n = \tilde{A}_1 \cdots \tilde{A}_n$, where the $\tilde{A}_i$'s are the telescoped matrices from (3.6) in the previous section.

Proposition 3.3.1. *For every sequence $(k_i)_{i\in\mathbb{N}}$ satisfying* (3.3)*, the sequences of eigenvalues $(\lambda_{n,i})_{n\geq 1}$, $i = 1, 2, 3$, of $\widetilde{\mathbb{A}}^n$ satisfy*

$$\limsup_{n\to\infty} \frac{1}{n} \log \lambda_{n,3} < 0 \leq \liminf_{n\to\infty} \frac{1}{n} \log |\lambda_{n,2}| \leq \limsup_{n\to\infty} \frac{1}{n} \log |\lambda_{n,2}| \leq \liminf_{n\to\infty} \frac{1}{n} \log \lambda_{n,1}.$$

Proof. Let $\lambda_{n,i}$, $i = 1, 2, 3$, indicate the eigenvalues of $\widetilde{\mathbb{A}}^n$. Their product is ± 1, because $\det(A_{k_i}) = -1$ for all i. Each $\tilde{A}_j$ is strictly positive and maps the positive octant $Q^+ = \{x \in \mathbb{R}^3 : x_i \geq 0\}$ strictly into itself (except for the origin). As in (the proof of) the Perron-Frobenius Theorem (see [Kat95, Theorem 1.9.11] or [Bru22, Theorem 5.58]), this implies that $\liminf_n \frac{1}{n} \log \lambda_{n,1} > 0$ and the corresponding unstable eigenspace E_1 goes through the interior of Q^+.

For row vectors $x \in Q^- = \{x \in \mathbb{R}^3 : x_1, x_2 \geq 0 \geq x_3\}$ the product $xA_{k_i}^{-1}$ is again in Q^- for every i. Thus the octant $Q^- = \{x \in \mathbb{R}^3 : x_1, x_2 \geq 0 \geq x_3\}$ is preserved by each inverse matrix $A_{k_i}^{-1}$. A change of coordinates changes Q^- into Q^+. Indeed, take

$$U := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad B_{k_i} = UA_{k_i}^{-1}U^{-1} = \begin{pmatrix} 0 & 1 & k_i - 1 \\ 1 & 0 & 0 \\ 0 & 1 & k_i \end{pmatrix}, \tag{3.7}$$

and Q^+ is preserved under the multiplication with B_{k_i} (*i.e.,* for every column vector $x \in Q^+$ it follows that $B_{k_i}x \in Q^+$ for all i). Let $\widetilde{\mathbb{B}}^n = \tilde{B}_n \cdots \tilde{B}_2 \cdot \tilde{B}_1$, where the $\tilde{B}_i$'s are blocks of matrices B_{k_j} telescoped in exactly the same way (but in the other direction) as the A_{k_i}'s.

Then the Perron-Frobenius Theorem applies and we obtain $\liminf_n \frac{1}{n} \log(\lambda_{n,3})^{-1} > 0$ (because the eigenvalues of $\widetilde{\mathbb{A}}^n$ and $\widetilde{\mathbb{B}}^n$ are each other reciprocals). This gives a negative $\limsup_n \frac{1}{n} \log \lambda_{n,3} < 0$.

We will continue the proof using the original (*i.e.,* non-telescoped) matrices A_k and B_k. The corresponding transition graphs of the 3×3 matrices $A_k = (a_{i,j})_{i,j}$ and $B_k = (b_{i,j})_{i,j}$ is constructed by giving each edge between the three difference states the weight of its matrix entry $a_{i,j}$ or $b_{i,j}$. To follow paths in the graph each edge is labeled by a letter

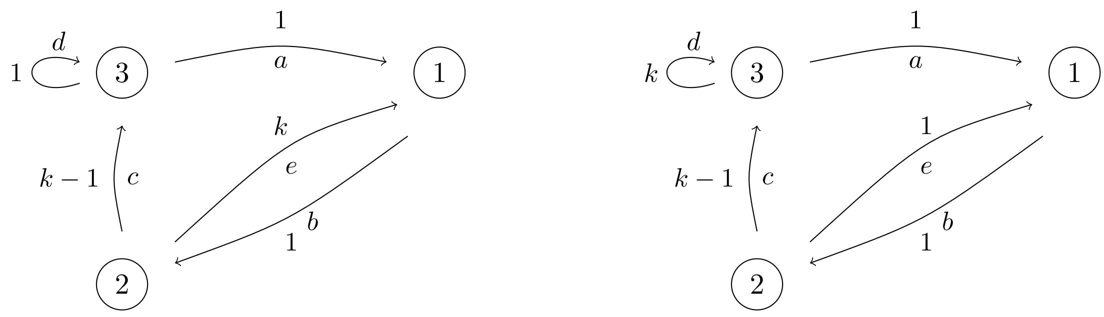


Figure 3.2: Transition graphs of A_k and B_k. The numbers at the edge-labels stand for weights (= multiplicities) of the edges.

in $\{a, b, c, d, e\}$. We can compute the weight of each finite path by multiplying its edge weights.

For our choice of B_k, the transition graphs of A_k and B_k in Figure 3.2 are the same except that the multiplicities of the 12-entry of 33-entry are swapped. (In particular, $A_k = B_k$ if $k = 1$.) The fact that B_k's have a loop $3 \to 3$ with multiplicity k (which has only multiplicity 1 in A_k) is the reason why $B_{k_n} \cdots B_{k_1}$ dominates $A_{k_1} \cdots A_{k_n}$ for sufficiently large n. Note that the matrix multiplication goes in opposite direction. This can be remedied by taking the transpose, amounting to a reversal of arrows in the transition graph, which has no influence on the number of paths.

The u, v-entries of the matrices $A_{k_1} \cdots A_{k_n}$ (or $B_{k_n} \cdots B_{k_1}$) represent the number of n-paths from u to v. Note, that in every iterate (i.e., multiplication with A_{k_i} or $B_{k_{N+1-i}}$, with N the total number of matrices involved in $\widetilde{\mathbb{B}}^n$) we use a different matrix, but their structure is the same, so that the corresponding transition graphs are the same, except for the labelling, see Figure 3.2. Let $x_1 \dots x_n$ be the edge-coding of an n-path. Then we call the product of the labels of the edges x_i the *weight* $w(x_1 \dots x_n)$. It represents the total number of paths corresponding to the single edge-code $x_1 \cdots x_n$.

We first look at the loops from vertex 1 back to 1, that is, edge-coded, from edge b to a. These are represented by words $bx_2 \dots x_{n-1}a$ where $x_j \neq a$.

Let $w_A(bx_2 \dots x_{n-1}a)$ and $w_B(bx_2 \dots x_{n-1}a)$ denote the weights of these paths in the transition graphs of A_{k_i} and $B_{k_{N+1-i}}$. If none of the symbols $x_j = a$, then there are only $\lfloor (n-1)/2 \rfloor$ possibilities, characterised by the symbol c at its unique even position. We call the collection of these words a *cluster of loops.*

We claim that for every $r \geq 1$, the sum of the weights of the $2r+1$-cluster satisfies

$$\sum_{j=0}^{r-1} w_A((be)^j bcd^{2(r-j-1)}a) = k_2 k_4 \cdots k_{2r} - 1 \leq \sum_{j=0}^{r-1} w_B((be)^j bcd^{2(r-j-1)}a), \tag{3.8}$$

and the inequality is strict if $k_{2j}, k_{2j'+1} > 1$ for some $1 \le j \le j' < r$.

Clusters of even length $2r+2$ are obtained from odd length clusters by adding an extra d at position $2r+1$. Therefore (3.8) implies that

$$\sum_{j=0}^{r-1} w_A((be)^j bcd^{2(r-j)-1}a) = k_2k_4\cdots k_{2r} - 1 \le (k_2k_4\cdots k_{2r}-1)k_{2r+1}$$
$$\le \sum_{j=0}^{r-1} w_B((be)^j bcd^{2(r-j)-1}a),$$

with strict inequality if $k_{2r-1} > 1$ or under the same condition as for (3.8).

Each edge b has weight $w_A(b) = w_B(b) = 1$; the edges c in position $2j$ have weight $w_A(c) = w_B(c) = k_{2j} - 1$ and the remaining edges in position i have weight $w_A(e) = w_B(d) = k_i$, $w_A(d) = w_B(e) = 1$.

We prove the equality regarding w_A in (3.8) by induction from 1 up to r. For $r = 1$ we have $w_A(bca) = k_2 - 1$. If the statement holds for $r-1$, then

$$\begin{aligned}
\sum_{j=0}^{r-1} w_A((be)^j bcd^{2(r-j-1)}a) &= \sum_{j=0}^{r-2} w_A((be)^j bcd^{2(r-j-1)}a) + k_2k_4\dots(k_{2r}-1) \\
&= k_2k_4\cdots k_{2r-2} - 1 + k_2k_4\dots(k_{2r}-1) \\
&= k_2k_4\cdots k_{2r} - 1.
\end{aligned}$$

The inequality regarding w_B in (3.8) is also proved by induction, now from r down to 1, and assuming that all odd-indexed $k_{2j+1} = 1$. That is, we show that

$$\sum_{j=m}^{r-1} w_B((be)^j bcd^{2(r-j-1)}a) = k_{2m+2}\cdots k_{2r} - 1$$

for all $r \in \mathbb{N}$ and $0 \le m < r$. This is done by induction too, but now working downwards, first taking the For $m = r-1$, we get $w_B((be)^{r-1}bca) = k_{2r} - 1$. If the statement holds for m, then

$$\begin{aligned}
\sum_{j=m-1}^{r-1} w_B((be)^j bcd^{2(r-j-1)}a) &= \sum_{j=m}^{r-1} w_B((be)^j bcd^{2(r-j-1)}a) + k_{2m}k_{2m+2}\cdots(k_{2r}-1) \\
&= k_{2m+2}\cdots k_{2r-2} - 1 + k_{2m}\cdots(k_{2r}-1) \\
&= k_{2m}\cdots k_{2r} - 1.
\end{aligned}$$

This concludes the induction. Since each instance $w_B(d) > 1$ at an odd position (provided $w_B(c) > 1$ at the previous occurrence of symbol c) increases the terms in the sum of w_B-weights by a factor $\ge \frac{3}{2}$, the inequality follows. This proves (3.8).

For paths starting in vertices 2 and 3, the proof that the B-weight of clusters dominates the A-weight are similar.

Now decompose an n-path from u to $v \in \{1,2,3\}$ into loops, with a potential remaining piece that is shorter than a full loop[1]. Group together all n-paths with the same loop structure (i.e., the same endpoints of their loops) as they fall into the same cluster pattern. The above estimates show that the combined B-weight of the concatenation of clusters dominates the combined A-weight of the concatenation. Summing over all cluster patterns, we arrive at $(\widetilde{\mathbb{A}}^n)_{u,v} \geq (\widetilde{\mathbb{B}}^n)_{u,v}$ for all $u, v \in \{1,2,3\}$ and for sufficiently large n. Thus the leading eigenvalue of $\widetilde{\mathbb{B}}^n$ is larger than the leading eigenvalue of $\widetilde{\mathbb{A}}^n$. Since $\widetilde{\mathbb{A}}^n$ and $\widetilde{\mathbb{B}}^n$ have reciprocals eigenvalues, $\liminf_{n\to\infty} \frac{1}{n} \log |\lambda_{n,2}| \geq 0$. □

The next theorem establishes the Lyapunov exponents and invariant directions of $(\widetilde{\mathbb{A}}^n)_{n\geq 1}$.

Theorem 3.3.2. *Let $\lambda_{n,i}$, $i = 1,2,3$, be the eigenvalues of $\widetilde{\mathbb{A}}^n$. There exist a constant $C \geq 1$ and vectors $\vec{v}_i \in \S^2$, $i = 1,2,3$, such that*

$$\frac{1}{C}\lambda_{n,i} \leq \|\vec{v}_i \widetilde{\mathbb{A}}^n\| \leq C\lambda_{n,i},$$

for $i = 1,2,3$ and for all $n \geq 1$.

The importance of this result is that the stable space is one-dimensional; this is used in Section 3.4 to verify the BV-summability condition (3.23), which decides whether the ITM is weak mixing or not. If $\vec{u} := \vec{\xi} A_{k_1} \cdots A_{k_r} \bmod \mathbb{Z}^3 \in span(\vec{v}_2 A_{k_1} \cdots A_{k_r})$ for some $r \in \mathbb{N}$, then $\vec{\xi}$ doesn't contract. However, the arguments in Section 3.4 to get the estimates of (3.23) are all necessary and the same, regardless of whether $\vec{u} A_{k_{r+1}} \cdots A_{k_n}$ increases, increases exponentially or just stays bounded. Because the matrices A_{k_i} have determinant -1, they can be viewed as automorphisms $M_k : \mathbb{T}^3 \to \mathbb{T}^3$ of the 3-torus $\mathbb{T}^3$, given by $M_k \vec{x} = \vec{x} A_k \bmod 1$. Condition (3.23) can then be interpreted as that the line in $\mathbb{T}^3$ spanned by $(1,1,1)$ intersects the stable direction of $(0,0,0)$ for the infinite system $(M_{k_i} \circ \cdots \circ M_{k_2} \circ M_{k_1})_{i\geq 1}$ in $\mathbb{T}^3$. This explains the need to compute the dimension of the stable direction of this system in the first place.

Proof. From the proof of Proposition 3.3.1 we have

$$\limsup_{n\to\infty} \frac{1}{n} \log \lambda_{n,3} < 0 \leq \liminf_{n\to\infty} \frac{1}{n} \log |\lambda_{n,2}| \leq \limsup_{n\to\infty} \frac{1}{n} \log |\lambda_{n,2}| \leq \liminf_{n\to\infty} \frac{1}{n} \log \lambda_{n,1},$$

and corresponding eigenvectors $\vec{v}_{n,1} \in Q^+ \cap \S^2$, $\vec{v}_{n,3} \in Q^- \cap \S^2$ and $\vec{v}_{n,2} \cap \S^2$ has a uniformly positive angle from the other eigenvectors.

[1]For these short pieces, similar cluster inequalities hold. The worst case is $w_A(be) + w_A(bc) = 2k-1 > w_B(be) = w_B(bc) = k$ if k is the multiplicity of the e at the second position, but this single factor $k/(2k-1)$ is negligible for large n.

Next define

$$Q_n^+ = \{\vec{v} \in \S^2 : \vec{v}(\widetilde{\mathbb{A}}^n)^{-1} \in Q^+\} \quad \text{and} \quad Q_n^- = \{\vec{v} \in \S^2 : \vec{v}(\widetilde{\mathbb{A}}^n) \in Q^-\}.$$

Since Q^+ is forward invariant and Q^- is backward invariant (see the proof of Proposition 3.3.1), these are both nested sequences of non-empty (because $\vec{v}_{n,1} \in Q_n^+$ and $\vec{v}_{n,3} \in Q_n^-$), compact sets, so we can choose $\vec{v}_1 \in \bigcap Q_n^+$ and $\vec{v}_3 \in \bigcap Q_n^-$. Recall that $\widetilde{\mathbb{A}}^n = \tilde{A}_1 \cdots \tilde{A}_n$. Since $\tilde{A}_i$ uniformly contracts Q^+ (in Hilbert semi-metric, see [Bru22, Section 8.6]),

$$\frac{\vec{v}_1\widetilde{\mathbb{A}}^n}{\|\vec{v}_1\widetilde{\mathbb{A}}^n\|} - \frac{\vec{v}_{n,1}\widetilde{\mathbb{A}}^n}{\|\vec{v}_{n,1}\widetilde{\mathbb{A}}^n\|} \to 0 \quad \text{exponentially as } n \to \infty,$$

so there is $C_1 \geq 1$ such that

$$\frac{1}{C_1}\lambda_{n,1} \leq \|\vec{v}_1\widetilde{\mathbb{A}}^n\| \leq C_1\lambda_{n,1} \qquad \text{for all } n \geq 0. \tag{3.9}$$

The same argument shows that there is $C_3 \geq 1$ such that

$$\frac{1}{C_3}\lambda_{n,3} \leq \|\vec{v}_3\widetilde{\mathbb{A}}^n\| \leq C_3\lambda_{n,3} \qquad \text{for all } n \geq 0. \tag{3.10}$$

Next define

$$P_n^+ = \{\vec{v} \in \S^2 : v(\widetilde{\mathbb{A}}^n) \notin int(Q^+)\} \quad \text{and} \quad P_n^- = \{\vec{v} \in \S^2 : v(\widetilde{\mathbb{A}}^n)^{-1} \notin int(Q^-)\}.$$

Then $P_n^+ \cap P_n^-$ forms a nested sequences of non-empty (because $\vec{v}_{n,2} \in P_n^+ \cap P_n^-$) compact sets, so we can find $\vec{v}_2 \in \bigcap_n (P_n^+ \cap P_n^-)$. Since $\widetilde{\mathbb{A}}^1 = \tilde{A}_1$ is a strictly positive matrix, there is a neighborhood V^+ of $Q^+ \cap \S^2$ such that $\vec{v}\widetilde{\mathbb{A}}^1 \in Q^+$ for each $\vec{v} \in V^+$. Therefore $P^+ \cap V^+ = \emptyset$ for $P^+ := \bigcap_n P_n^+$. The same argument gives a neighborhood V^- of $Q^- \cap \S^2$ such that $P^- \cap V^- = \emptyset$ for $P^- := \bigcap_n P_n^-$. This shows that $\vec{v}_2\widetilde{\mathbb{A}}^n \in P^- \cap P^+$ has uniformly positive angles with $\vec{v}_3\widetilde{\mathbb{A}}^n$ and $\vec{v}_1\widetilde{\mathbb{A}}^n$, so there is a constant $C' \geq 1$ such that

$$|\lambda_{n,1}\lambda_{n,2}\lambda_{n,3}|\frac{4\pi}{3} = \mathrm{Vol}(\widetilde{\mathbb{A}}^n(B_1(\vec{0})) \leq C'\|\vec{v}_1\widetilde{\mathbb{A}}^n\|\|\vec{v}_2\widetilde{\mathbb{A}}^n\|\|\vec{v}_3\widetilde{\mathbb{A}}^n\| \leq C'C_1\lambda_{n,1}C_3\lambda_{n,3}\|\vec{v}_{n,2}\widetilde{\mathbb{A}}^n\|.$$

A similar argument gives a lower bound, so there is $C_2 \geq 1$ such that

$$\frac{1}{C_2}|\lambda_{n,2}| \leq \|\vec{v}_2\widetilde{\mathbb{A}}^n\| \leq C_2|\lambda_{n,2}| \qquad \text{for all } n \geq 0. \tag{3.11}$$

Combining (3.9), (3.11) and (3.10) finishes the proof the theorem for $C = \max\{C_1, C_2, C_3\}$.

□

For further investigation into the number of concatenations of clusters, we use a lemma in terms of weighted trees.

A tree is a graph without loops; we assume it is directed, with a single vertex, called the *root* v_0, with only outgoing edges. Consider a tree with root v_0 and denote the vertex and edge sets V and E respectively[2], let $V_n = \{v \in V : v \text{ is } n \text{ edges away from } v_0\}$. Clearly $V_0 = \{v_0\}$ and $V = \bigsqcup_{n \geq 0} V_n$.

We assign to each $e \in E$ a weight $w(e) \geq 1$. Extend the weight function to V by setting $w(v_0) = 1$ and $w(v) = \prod\{w(e) : e \text{ lies between } v_0 \text{ and } v\}$ for $v \in V_n$, $n \geq 1$.

Lemma 3.3.3. *Assume that in the tree above, each vertex $v \in V_{n-1}$ has $1 + b_n$ edges to V_n and one of these edges has weight $1 + \eta_n$ and the others have weight 1. Then, since $V_1(v) = 1 + b_n$ for each $v \in V_{n-1}$,*

$$\sum_{v \in V_n} w(v) = \#V_n \prod_{r=1}^{n} \left(1 + \frac{\eta_r}{1 + b_r}\right). \tag{3.12}$$

Proof. The proof of the claim goes by induction on n. For $n = 1$, this is trivially true. Assume now that (3.12) holds for $n - 1$. For $v \in V_{n-1}$, write $V_1(v)$ for the set of vertices $v' \in V_n$ such that there is an edge between v and v'. Then

$$\begin{aligned}
\sum_{v \in V_n} w(v) &= \sum_{v \in V_{n-1}} \sum_{v' \in V_1(v)} w(v') = \sum_{v \in V_{n-1}} w(v) \left(\eta + \sum_{v' \in V_1(v)} 1\right) \\
&= \sum_{v \in V_{n-1}} w(v) \sum_{v' \in V_1(v)} \left(1 + \frac{\eta}{\#V_1(v)}\right) = \left(1 + \frac{\eta_n}{1 + b_n}\right)(1 + b_n) \sum_{v \in V_{n-1}} w(v) \\
&= (1 + b_n)\#V_{n-1} \prod_{r=1}^{n} \left(1 + \frac{\eta_r}{1 + b_r}\right) = \#V_n \prod_{r=1}^{n} \left(1 + \frac{\eta_r}{1 + b_r}\right).
\end{aligned}$$

The completes the induction and the proof. □

Assume that $\widetilde{\mathbb{A}}^n$ comprises of s matrices A_{k_j}, $1 \leq j \leq s$. We call j a *marked position* if $k_j \geq 2$. Hence, each telescoped matrix $\tilde{A}_i$ contains at least two and $\widetilde{\mathbb{A}}^n$ contains at least $2n$ marked positions.

In (3.8) we computed the total weight of *clusters* of loops with the same start and end point. Each such cluster has a unique symbol m which must be at a marked position, and it must take an even position in the loop. If there were an earlier marked position at an even place in the loop, there is another loop in the same cluster. Hence, to identify a cluster, we need the *earliest* midpoint m (which we call the *mark* of the cluster) and an integer b such that $m - 2b + 1$ is the start of the cluster, and there are no marked position

[2]Despite the choice of notation, this should not be thought of as a Bratteli diagram as in Section 3.4.4 as it has no vertex with multiple incoming edges.

at an even place between $m-2b+1$ and m. The end of the cluster is one before the start of the next cluster, and hence determined by the next cluster.

A *cluster pattern* P is the concatenation of clusters that fits within the integer interval $[1,s]$, so its first cluster starts at 1 and its last cluster ends at s. According to (3.8), a cluster c has weight $w_B(c) \geq (\frac{3}{2})^t w_A(c)$ if it contains t marked positions m' after its mark m such that $m'-m$ is odd. If $t=0$, then $w_B(c) = w_A(c)$. Let $w_B(P)$ (resp. $A(P)$) stand for the B-weight (resp. A-weight) of a cluster pattern, i.e., the product of the B-weights (resp. $A(P)$-weights) of its clusters.

Since $\sum_P w_B(P)/\sum_P w_A(P)$, where the sum runs over all cluster patterns, is an indication for the second eigenvalue $\lambda_{n,2}$, we need to identify all cluster patterns P and compare $w_B(P)$ and $w_A(P)$. To do that, we order the cluster patterns in a tree-structure and finally apply Lemma 3.3.3.

Among the marked positions, we choose a sequence as follows: j_0 is the first **even** marked position, and if j_{r-1} is chosen, then $j_r > j_{r-1}$ is the first marked position such that $j_r - j_{r-1}$ is **odd**. Let R be maximal such that $j_R < s$. We build a tree, thinking of its vertex sets V_r, $r = 0,\dots,R$, as being associated to the marked position j_r. $V_0 = \{v_0\}$ is the root. For $r \geq 1$, set $b_r = \lfloor (j_r - j_{r-1})/2 \rfloor$, and to each $v \in V_{r-1}$, attach $1+b_r$ edges to vertices in V_r, namely

- one representing cluster patterns having no cluster that ends between j_{r-1} and j_r; this edge gets weight $1+\eta_r \geq \frac{3}{2}$;
- one for each $1 \leq b \leq b_r$, representing cluster patterns having a cluster that ends at position $j_r - 2b$; such an edge gets weight 1.

Then every cluster pattern P is associated to a single R-path in the tree (i.e., a single $v \in V_R$), namely according whether P contains a cluster ending between j_r and j_{r-1}, and where that endpoint is. Moreover, $w_B(P) \geq w(v) w_A(P)$ where the weight $w(v)$ of $v \in V_R$ is the product of the weights of the edge between v_0 and v. If all these edge-weights are 1, then we have equality $w_B(P) = w_A(P)$.

Example 3.3.4. *This example shows that it is possible that the second Lyapunov exponent is zero, and a fortiori, there is no expansion in the direction $\vec{v}_2$.*

Take the sequence $k_j = 2$ for $j = 2^n - ((n+1) \bmod 2)$, that is, the marked positions are $j = 2,3,8,15,32,63,\dots$ with $k_j = 2$, and $k_j = 1$ otherwise. Since these marked positions are alternatively even and odd, $j_r = 2,3,8,15,32,63,\dots$, so $j_r = 2^{r+1}$ or $2^{r+1}-1$ for $r \geq 0$. Also $\widetilde{\mathbb{A}}^n$ comprises 4^n matrices A_k and $2n$ marked positions. The corresponding *numbers* $b_r = 0, 2, 3, 8, 15, 32, \dots$, *so* $b_r = 2^{r-1} - (r \bmod 2)$ *for* $1 \le r \le R = 2n-1$. *From* (3.12) *with* $1 + \eta_r \equiv 2$, *we derive*

$$\sum_P w_B(P) \le \sum_P w_A(P) \prod_{r=1}^{R} \left(1 + \frac{1}{1+b_r}\right) \le 2e \sum_P w_A(P)$$

for the sums of weights of cluster patterns. Therefore, the sequence of second eigenvalues $(\lambda_{n,2})_{n\ge 1}$ *is bounded.*

Proposition 3.3.5. *Every linearly recurrent ITM satisfying* (3.3) *has two strictly positive (as* lim inf*) and one strictly negative (as* lim sup*) Lyapunov exponent.*

Proof. From Proposition 3.3.1 we have already that $\limsup_n \frac{1}{n} \log \lambda_{n,3} < 0 < \liminf_n \frac{1}{n} \log \lambda_{n,1}$. The second eigenvalue $\lambda_{n,2}$ is comparable to the quotient of the B-weight of all n-paths and the A-weight of all n-paths. For this, we construct the tree as described above. The characterisation of linear recurrence in Theorem 3.2.6 implies now, that there is $K \in \mathbb{N}$ such that $1 + b_r \le K$ for all $1 \le r \le R$ and also that $R \ge n/K$. Hence, (3.12) of Lemma 3.3.3 for $\eta_r = \frac{1}{2}$ gives

$$\sum_P w_B(P) \ge \left(1 + \frac{1}{2K}\right)^R \sum_P w_A(P) \ge \left(1 + \frac{1}{2K}\right)^{n/K} \sum_P w_A(P).$$

This shows that $\liminf_n \frac{1}{n} \log \lambda_{n,2} \ge \frac{1}{K} \log(1 + \frac{1}{2K}) > 0$. Theorem 3.3.2 then gives the two positive and one negative Lyapunov exponents. □

3.4 Weak mixing

3.4.1 Host's eigenvalue condition

Host [Hos86] formulated a condition for primitive recognizable, substitution shifts, to have an eigenvalue: $e^{2\pi i \xi}$ is an eigenvalue the Koopman operator for some $\xi \in (0,1)$ if and only if there is a coboundary h such that

$$\vec{\xi} A^n \to h(\xi) \qquad \vec{\xi} = (\xi, \xi, \xi). \tag{3.13}$$

Also, every eigenfunction is continuous.

The condition is of the same gist as the Veech criterion [Vee84] for non-weak-mixing of IETs, although there A can be seen as the matrix representing the action of Rauzy induction on the homology $H^1(\mathbb{Z})$ of the translation surface obtain as suspension flow over the IET.

Later works, see e.g. [CDHM03, DFM19], showed that in the context of linearly recurrent S-adic shifts (i.e., subshifts based on a sequence of finitely many primitive substitutions rather than a single one), a necessary and sufficient condition for the existence of continuous eigenvalues is

$$\sum_{n=1}^{\infty} \|\vec{\xi} A_1 \cdots A_n\| < \infty, \qquad \vec{\xi} = (\xi, \xi, \xi), \tag{3.14}$$

where $\|x\|$ is the distance of a vector to the nearest integer lattice point and A is the associated matrix of the substitution.

For more general S-adic subshifts, more complicated conditions are needed, and we come back to them in Sections 3.4.4 and 3.4.5 where we distinguish between continuous and measurable eigenvalues.

3.4.2 Weak mixing for (pre-)periodic ITMs

Theorem 3.4.1. *For every (pre-)periodic sequence $(k_i)_{i\in\mathbb{N}}$ satisfying (3.3), the corresponding system $(\Omega, T_{\alpha,\beta})$ is weakly mixing.*

Proof. Suppose that the sequence $(k_i)_{i\in\mathbb{N}}$ has period n and also satisfies (3.3), then the corresponding S-adic shift is linearly recurrent (and uniquely ergodic). If $A^n = A_{k_1} \cdots A_{k_n}$, then its eigenvalues are $\tilde{\lambda}_i$ where $\tilde{\lambda}_3 < 1 < |\tilde{\lambda}_2| < \tilde{\lambda}_1$. That is, there is only a one-dimensional stable subspace E_3 spanned by a vector $(u_1, u_2, -1) \in Q^-$. Since $\det(A^n) = \pm 1$, its characteristic polynomial can be written as

$$P_{A^n}(\lambda) = a_n\lambda^3 + b_n\lambda^2 + c_n\lambda \pm 1 = \lambda(a_n\lambda^2 + b_n\lambda + c_n) \pm 1$$

for some $a_n, b_n, c_n \in \mathbb{Z}$. If $\lambda \in \mathbb{Z} \setminus \{0\}$ is a zero of this polynomial, then also $0 \equiv P_{A^n}(\lambda) \mod \lambda \equiv \pm 1 \mod \lambda$, and this cannot be true for any integer different from ± 1. But we know already from (the techniques of) Proposition 3.3.1, that the eigenvalues satisfy $0 < \lambda_3 < 1 < |\lambda_2| < \lambda_1$. Therefore P_{A^n} is irreducible over $\mathbb{Z}$ and the $\tilde{\lambda}_i$ are cubic numbers.

As (k_i) is periodic, telescoping gives a stationary sequence, so condition (3.14) decides on the eigenvalues of the the Koopman operator. To prove that there is a non-trivial eigenvalue, we need to show that $\vec{\xi}$ lies in some integer translation of E_3. That is

$$(\xi, \xi, \xi) + s(u_1, u_2, -1) = (p, q, r) \tag{3.15}$$

for some integers p, q, r and reals $u_1, u_2 > 0$. Eliminating s and ξ from this system, we obtain

$$(q - r)u_1 = (p - r)u_2 + (p - q). \tag{3.16}$$

For the matrix $A^n = (a_{ij})_{i,j=1}^3$, the third eigenvalue equation $\tilde{\lambda}_3(u_1, u_2, -1) = (u_1, u_2, -1)A$ gives

$$\begin{aligned} \tilde{\lambda}_3 u_1 &= a_{11}u_1 + a_{21}u_2 - a_{31}, \\ \tilde{\lambda}_3 u_2 &= a_{12}u_1 + a_{22}u_2 - a_{32}. \end{aligned}$$

Multiplying these equations and inserting (3.16) to eliminate u_1, we obtain

$$\begin{aligned} \tilde{\lambda}_3((p-r)u_2 + (p-q)) &= a_{11}((p-r)u_2 + (p-q)) + a_{21}(q-r)u_2 - (q-r)a_{31}, \\ \tilde{\lambda}_3(q-r)u_2 &= a_{12}((p-r)u_2 + (p-q)) + a_{22}(q-r)u_2 - (q-r)a_{32}. \end{aligned}$$

Making u_2 subject of the second equation (so that the RHS is a fractilinear expression in $\tilde{\lambda}_3$) and then inserting it in the first equation gives an equation of two fractilinear expressions in $\tilde{\lambda}_3$. Therefore $\tilde{\lambda}_3$ is the solution of a quadratic equation contradicting that $\tilde{\lambda}_3$ is a cubic number.

For the preperiodic case, with preperiod m, we need to replace (3.15) by

$$(\xi, \xi, \xi) \cdot A_{k_1} \cdots A_{k_m} + s(u_1, u_2, -1) = (p, q, r).$$

This gives a more cumbersome version of (3.16) but the argument is essentially the same. $\square$

Remark 3.4.2. Very recently[3], Mercat [Mer24] proved a general criterion for weak-mixing S-adic shifts, which is directly computable if the involved sequence of substitutions is preperiodic. Mercat's criterion confirms our Theorem 3.4.1, see [Mer24], Example 7.9.

3.4.3 The stable direction.

Let $(\Omega, T_{\alpha,\beta})$ be an infinite type ITM based on a sequence $(k_n)_{n\in\mathbb{N}}$ satisfying (3.3). We call $W^s(\vec{0}) := \bigcap_n A_{k_1}^{-1} \circ \cdots \circ A_{k_n}^{-1}(Q^-) = \mathrm{span}(\vec{v}_3)$ the *stable space* of the sequence $(A_{k_i})_{i\in\mathbb{N}}$. In order to find $W^s(\vec{0})$, we iterate the matrices B_{k_i} from (3.7). This gives

$$(u, v, w) = \lim_{n\to\infty} \frac{\mathbb{B}^n(\vec{a})}{\|\mathbb{B}^n(\vec{a})\|_1} \quad \text{for } \mathbb{B}^n = B_{k_n} \circ \cdots \circ B_{k_1} \text{ and all } a \in Q^+.$$

Note that we don't need unique ergodicity for this statement; $\bigcap A_1 \cdots A_n(Q^+)$ need not be a single half-line, but $\bigcap B_1 \cdots B_n(Q^+)$ always is.

Apart from the assumption of unique ergodicity, the choice of the vector $\vec{a} \in Q^+$ can be arbitrary. Then $(v, u, -w) = (u, v, w)U$ (with U from (3.7)) is the stable direction of $A_{k_1} \cdot A_{k_2} \cdots$, normalised so that $u + v + w = 1$.

[3]Mercat's preprint was uploaded on the arXiv four months after we submitted our paper.

Lemma 3.4.3. *The direction (u, v, w) is uniquely determined by the sequence $(k_i)_{i\in\mathbb{N}}$, provided it satisfies* (3.3)*.*

Thus, even if $T_{\alpha,\beta}$ fails to be uniquely ergodic and there is no unique unstable direction in the first octant, the stable direction in Q^- is always well-defined.

Proof. Since $(k_i)_{i\in\mathbb{N}}$ satisfies (3.3), there are infinitely many i and integers $r \geq 0$ such that $k_i \geq 2$, $k_{i-1} = \cdots = k_{i-2r} = 1$ and $k_{i-2r-1} \geq 2$. Abbreviate $a = k_i \geq 2$, $b = k_{i-2r-1} \geq 2$ and $c = k_{i-2r-2} \geq 1$. The telescoped block from k_i to k_{i+2r+2} gives

$$\begin{aligned}\tilde{B} &= B_a \cdot B_1^{2r} \cdot B_b \cdot B_c \\ &= \begin{pmatrix} (a-1)(r+1) & (a-1)b(r+1)+1 & c((a-1)b(r+1)+1) - (r(a-1)+1) \\ 1 & b-1 & c(b-1) \\ a(r+1) & ab(r+1)+1 & c(ab(r+1)+1) - (ra+1) \end{pmatrix},\end{aligned}$$

which is a strictly positive matrix. Therefore it represents a strict contraction in the Hilbert metric on the first octant, see e.g. [Bru22, Section 8.6]. The contraction factor is bounded by $\tanh(\frac{1}{2}\log(\rho))$, where[4]

$$\rho = \max_{1\leq j,j'\leq 3} \sqrt{\frac{\max\{\tilde{B}_{j,k}/\tilde{B}_{j',k} : 1 \leq k \leq 3\}}{\min\{\tilde{B}_{j,k}/\tilde{B}_{j',k} : 1 \leq k \leq 3\}}}. \tag{3.17}$$

The shape of $\tilde{B}$, where the rows are almost multiples of each other, yields that ρ is bounded, and hence the contraction factor smaller than 1, uniformly in a, b, c. As there are infinitely many such telescoped blocks $\tilde{B}$, the infinite matrix product contracts the positive octant into a single half-line ℓ. Thus $U\ell$ for the matrix U from (3.7) represents the unique stable direction $W^s(\vec{0})$. □

To find an eigenvalue, we have to solve

$$(\xi, \xi, \xi) = (p, q, r) + s(v, u, u + v - 1) \text{ for some } p, q, r \in \mathbb{Z}. \tag{3.18}$$

Solving for ξ and s gives arcs $\ell_{p,q,r} = \{(u, v) \in \Delta : (1 - u)(r - q) = (1 - v)(r - p)\}$ in the simplex $\Delta = \{(u, v) : 0 \leq u \leq 1 - v\}$, or equivalently

$$\ell_{p,q,r} = \{(u, v) \in \Delta : u(q - r) = v(p - r) + q - p\}. \tag{3.19}$$

Expressed in terms of p, q, r, ξ, we find

$$u = \frac{\xi - q}{\xi + r - p - q}, \qquad v = \frac{\xi - p}{\xi + r - p - q}, \tag{3.20}$$

[4]We phrase this different from formula (8.29) in [Bru22] because we are using left-multiplication with row vectors instead of right multiplication with column vectors as in [Bru22].

so that $\xi = \frac{r-p}{1-u} + p + q - r = \frac{r-q}{1-v} + p + q - r \in \mathbb{Q}$ if and only if both $u, v \in \mathbb{Q}$. We define $\ell_{p,q,r}(\xi)$ as those points $(u,v) \in \Delta$ such that (3.20) holds. Let $H_k(u,v)$ indicate the first two coordinates of $(u, v, 1-(u+v))B_k$ normalised to unit length. Then $H_k : \Delta \to \Delta_k := H_k(\Delta)$ is a map from the unit simplex Δ, and it has the formula

$$H_k : (u,v) = \frac{1}{D_k}(v \ , \ 1-v) \quad \text{for} \quad D_k = k(1-v) + 1 - u, \tag{3.21}$$

$$H_k^{-1} : (x,y) \mapsto \left(\frac{x + (k+1)y - 1}{x+y} \ , \ \frac{x}{x+y} \right) \quad \text{for} \quad (x,y) \in \Delta_k. \tag{3.22}$$

The derivative and its determinant are

$$DH_k(u,v) = \frac{1}{D_k^2} \begin{pmatrix} v & k+1-u \\ 1-v & u-1 \end{pmatrix}, \qquad \det DH_k(u,v) = -\frac{1}{D_k}.$$

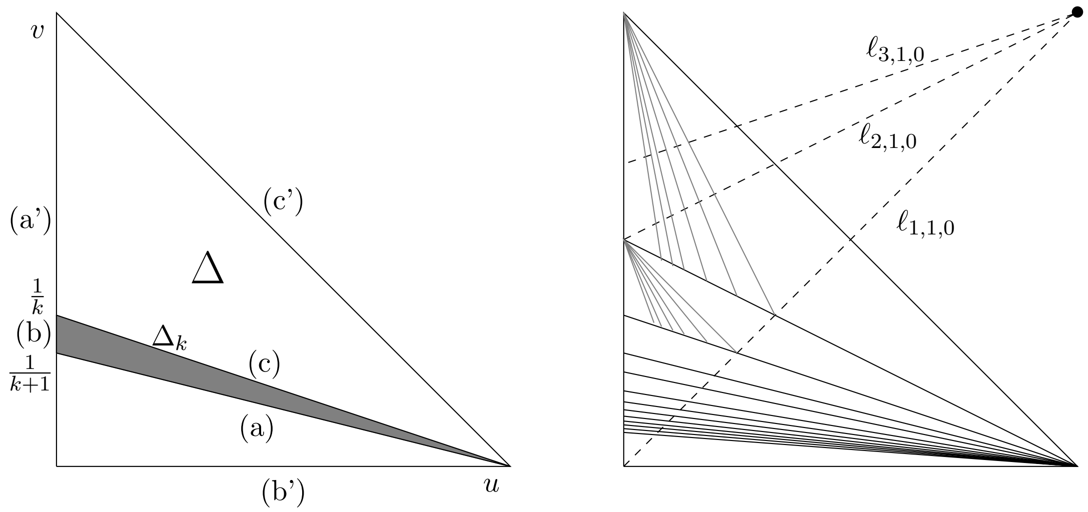


Figure 3.3: The simplex Δ and image $\Delta_k = H_k(\Delta)$ (left) and further images (right).

To find the set Δ_k, we compute the images of the three boundary lines of Δ:

$$\begin{aligned}
(a') \to (a) \qquad & H_k(\{0\} \times [0,1]) = \left\{ \frac{1}{k(1-v)+1}(v, 1-v) : v \in [0,1] \right\}, \\
(b') \to (b) \qquad & H_k([0,1] \times \{0\}) = \left\{ \frac{1}{k+1-u}(0,1) : u \in [0,1] \right\}, \\
(c') \to (c) \qquad & H_k(\{(1-v,v) : v \in [0,1]\}) = \left\{ \frac{1}{k(1-v)+v}(v, 1-v) : v \in [0,1] \right\},
\end{aligned}$$

see Figure 3.3, left. Comparing (a) and (c) we can see that the triangles Δ_k and Δ_{k+1} are adjacent, and Δ_1 is adjacent to the upper boundary of Δ. That is, the Δ_k's have disjoint interiors and $\Delta = \overline{\bigcup_{k\geq 1} \Delta_k}$. Furthermore, unless $k = k' = 1$, $H_k \circ H_{k'}$ maps $\partial\Delta$ into the interior of Δ, and hence, for every $(u, v) \in \Delta$, there are no common boundaries of Δ_k's that are the image of any point $H_k \circ H_{k'} \circ H_{k''}\dots$. It follows that for each $(u, v) \in \Delta$, there is at most one sequence $(k_i)_{i\in\mathbb{N}}$ such that $\lim_{n\to\infty} H_{k_1} \circ H_{k_2} \circ \cdots \circ H_{k_n}(\vec{a}) = (u, v)$.

The boundary points of the Δ_k cannot be reached as limit from an interior point $(u, v) \in \Delta^\circ$, and under later iterates of the H_k's, the iterated boundary points $E := \bigcup_{n\in\mathbb{N}} \bigcup_{k_1,\dots,k_n} H_{k_1} \circ \cdots \circ H_{k_n}(\partial\Delta)$ cannot be equal to any point in $\bigcap_n \bigcup_{k_1,\dots,k_n} H_{k_1} \circ \cdots \circ H_{k_n}(\Delta^\circ)$.

Remark 3.4.4. The points $(x, y) \in \Delta$ which give rise to parameters $(k_n)_{n\in\mathbb{N}}$ such that the subshift associated is linearly recurrent, are bounded away from $\partial\Delta$ for all preimages $H_{k_n}^{-1} \circ \cdots \circ H_{k_1}^{-1}(x, y)$. For $(k_n)_{n\in\mathbb{N}}$ unbounded, a subsequence of preimages goes to the edges (a'), (b'); in case of unbounded blocks of 1's in $(k_n)_{n\in\mathbb{N}}$, a subsequence of preimages goes to (c'). Conjecturally, this set has full Hausdorff dimension, but for each fixed bound C, the set of parameters for which $\limsup_n k_n \leq C$ and the lengths of blocks of 1's is eventually bounded by C has Hausdorff dimension strictly less than 1. Since the maps H_k are non-conformal, the current techniques of fractal geometry seem insufficient to prove this.

The maps H_k send points with rational coordinates to points with rational coordinates, and hence the lines between such points (thus with rational slope) to lines of the same properties.

Proposition 3.4.5. *The sides of every triangle $H_{k_1} \circ \cdots \circ H_{k_n}(\partial\Delta)$ have negative slopes for all $n \geq 1$ and $k_1, \dots, k_n \in \mathbb{N}$.*

This proposition shows that the lines $\ell_{p,q,r}$, which all have positive slopes, intersect the boundaries of these triangles $H_{k_1} \circ \cdots \circ H_{k_n}(\partial\Delta)$ transversally, so there cannot be an overlap of positive measure.

Proof. With some stretch of terminology, the statement holds for $n = 0$, so for Δ itself: here the "bottom" side $\partial_b\Delta$ has slope 0 and the left side $\partial_l\Delta$ needs to be interpreted as with slope $-\infty$. Only the "right" side $\partial_r\Delta$ has a proper negative slope -1.

Now $\bigcup_{k\in\mathbb{N}} \partial\Delta_k$ consists of $\partial_b\Delta$ and a fan F of straight lines stretching from $(1, 0)$ to $\partial_l\Delta$, see Figure 3.3, right. The slope of these lines are therefore all negative.

The maps H_k reverse orientation: they are reflections in lines of positive slope combined with a (non-linear) contraction. Since the vertex $V_k := (0, \frac{1}{k}) = H_k(1, 0)$ of Δ_k is the intersection of $\partial_l\Delta$ and $\partial_r\Delta_k$, $H_k(F)$ is a fan of lines stretching from V_k to the opposite boundary $\partial_b \Delta_k$. These lines have slopes between the slopes of $\partial_l \Delta_k$ and $\partial_r \Delta_k$, so they are negative. The proof follows from repeating this argument inductively. $\square$

3.4.4 Continuous eigenvalues for the general case

To look for eigenvalues in the non-periodic case, we use the representation of the S-adic subshift as a Bratteli-Vershik system. For these systems conditions about the existence of continuous and measurable eigenvalues are known. We specifically use [DFM19].

Let $(E_n, V_n, \succ)$ be an ordered Bratteli diagram with path space X, Vershik map τ and associated matrices A_i. Every set of edges E_i of level i in the BV-diagram is described by the substitutions χ_i. Let $\mathfrak{s}(e)$ denote the source vertex of edge e and $\mathfrak{t}(e)$ the target vertex of edge e. (See [DFM19] or [Bru22, Chapter 5.4] for the precise definitions.) Theorem 2 in [DFM19] states the following BV-summability condition for **positive** transition matrices M_j, $j \geq 1$: $e^{2\pi i \xi}$ is a continuous eigenvalue if and only if

$$\sum_{n=1}^{\infty} \max_{x \in X} \|\langle s_n(x), \vec{\xi} M_1 \cdots M_n \rangle\| < \infty, \qquad \vec{\xi} = (\xi, \xi, \xi), \tag{3.23}$$

where $(s_n(x))_v = \#\{e \in E_{n+1} \ : \ e \succ x_{n+1}, \mathfrak{s}(e) = v\}$. Thus in the ordered Bratteli-Vershik diagram (with $\succ$ indicating the order of the incoming edges) the vector $s_n(x)$ counts the number of incoming edges that are higher in the order than edge x_{n+1} in the path x. This definition of s_n is phrased in terms of ordered Bratteli diagrams. In terms of S-adic shifts (X, σ) based on substitutions $\chi_n : \mathcal{A}_n \to \mathcal{A}_{n-1}$, every sequence $x \in X$ can be written as

$$x = \lim_{n \to \infty} \sigma^{j_1} \circ \chi_1 \circ \sigma^{j_2} \circ \chi_2 \circ \cdots \circ \sigma^{j_n} \circ \chi_n(a_n)$$

for some integer sequence (j_n) and symbols $a_n \in \mathcal{A}_n$ such that a_n is the first letter of $\sigma^{j_{n+1}} \circ \chi_{k_{n+1}}(a_{n+1})$. Then for every $x \in X$ we can think of the entries of the vector $s_n(x)$ as

$$(s_n(x))_b = |\sigma^{j_{n+1}} \circ \chi_{n+1}(a_{n+1})|_b, \qquad b \in \mathcal{A}_n,$$

where $|w|_b$ indicates the number of symbols b in the word w. This is equivalent to the definition in the Bratteli-Vershik context, as the substitution $\chi_{n+1}(a_{n+1})$ describes the edges incoming to $a_{n+1} \in \mathcal{A}_{n+1}$ from vertices in $\mathcal{A}_n$ and their order. Thus $|\sigma^{j_{n+1}} \circ \chi_{n+1}(a_{n+1})|_b$ counts the number of occurrences of b in the later parts of the word $\chi_{n+1}(a_{n+1})$.

We will use condition (3.23) with χ_{k_i} and A_{k_i} (or in fact, the telescoped version $\tilde{A}_i$ from (3.6)) replacing χ_i and A_i. The paper [DFM19] is formulated for invertible S-adic shifts, but the Bratteli-Vershik system having a unique minimal path (which is true in our case because $\chi_{k_i} \circ \chi_{k_{i+1}}$ is left-proper) is sufficient.

One would expect (in accordance with the Veech criterion [Vee84] and results from IETs [AF07]) the Koopman operator to have a non-trivial eigenvalue if and only if $(u, v) \in \ell_{p,q,r}$ for some $p, q, r \in \mathbb{Z}$ (see (3.18)). If otherwise no non-trivial eigenvalue exists, the ITM would be weak mixing. However, because we have to deal with summability condition (3.23), especially the factor $s_n(x)$, the story is not as clear-cut as that. In one direction, it is as expected:

Theorem 3.4.6. *If $\vec{\xi}$ does not belong to the stable space $W^s(\vec{0}) \bmod 1$ for every $\xi \in (0, 1)$, then the corresponding ITM of infinite type has no continuous eigenvalue.*

Proof. Let $\vec{\xi}_0 = \vec{\xi} \bmod 1$ and $\vec{\xi}_{n+1} = \vec{\xi}_n \tilde{A}_{n+1} \bmod 1 \in (-\frac{1}{2}, \frac{1}{2}]^3$. Assume by contradiction that condition (3.23) holds, i.e., $\max_{x \in X} |||\langle \tilde{s}_n(x), \vec{\xi}_n \rangle|||$ is summable. Take $n_0 \in \mathbb{N}$ so that $\max_{x \in X} |||\langle \tilde{s}_n(x), \vec{\xi}_n \rangle||| < 1/4$ for all $n \geq n_0$.

The vectors $\tilde{s}_n(x)$ have non-negative entries, but are smaller than the v-th column of $\tilde{A}_{n+1}$ if the path $x \in X$ is such that $\mathfrak{t}(x_{n+1}) = v$. Using the language of BV-diagrams, if x_{n+1} is the smallest incoming edge to v, then $\tilde{s}_n(x)$ has the largest possible value, say $s_n^+(x)$, which is the v-th column of $\tilde{A}_{n+1}$ with one of the three entries decreased by 1. Taking paths $x \in X$ using successive incoming edges x_{n+1} on level $n+1$, the vector $\tilde{s}_n(x)$ decreases each time by 1 in one of its entries, depending on the source vertex $\mathfrak{s}(x_{n+1})$.

Now if $\vec{\xi}_n \in B_\varepsilon(\vec{0})$, but $\langle \tilde{s}_n^+(x), \vec{\xi}_n \rangle$ is close to a non-zero integer, then choosing paths x with x_{n+1} equal to successive incoming edges to v, $\langle \tilde{s}_n(x), \vec{\xi}_n \rangle$ decreases every step by an amount $< \varepsilon$. At some step, $|||\langle \tilde{s}_n(x), \vec{\xi}_n \rangle||| > \frac{1}{3}$, contradicting that $\max_{x \in X} |||\langle \tilde{s}_n(x), \vec{\xi}_n \rangle||| < 1/4$.

This implies that $\vec{\xi}_n \cdot \tilde{A}_{n+1} \cdots \tilde{A}_{n+m} \to \vec{0}$ as $m \to \infty$, contrary to the assumption that $\vec{\xi} \notin W^s \bmod 1$, and the proof is complete. $\square$

The next proposition together with (3.20) shows that there cannot be an eigenvalue $e^{2\pi i \xi}$ for rational ξ, because $\frac{1}{\xi + r - p - q}(\xi - q, \xi - p) \in \Delta$ is contained in $H_{k_1} \circ \cdots \circ H_{k_n}(\partial \Delta)$ for n sufficiently large, and that means there is no sequence $(k_i)_{i \in \mathbb{N}}$ satisfying (3.3) such that $\ell_{p,q,r}(\xi) = \lim_{n \to \infty} H_{k_1} \circ \cdots \circ H_{k_n}(\Delta^\circ)$.

Proposition 3.4.7. *Every rational point in Δ belongs to $\bigcup_{n \geq 0} \bigcup_{k_1, \dots, k_n \in \mathbb{N}} H_{k_1} \circ \cdots \circ H_{k_n}(\partial \Delta)$.*

Proof. Suppose $(x, y) \in \Delta^\circ$ has rational coordinates. We can give them the same denominator, i.e., we write $(x, y) = (\frac{p}{q}, \frac{p'}{q})$ with $p + p' < q$ because $x + y < 1$. Take $k_1 \in \mathbb{N}$ such that $(x, y) \in H_{k_1}(\Delta)$. If $(x, y) \in H_{k_1}(\partial \Delta)$ we are done, so assume that $(x, y) \in H_{k_1}(\Delta^\circ)$.

Then using (3.22) we get

$$H_{k_1}^{-1}(x,y) = \left(\frac{x+(k_1+1)y-1}{x+y} \ , \ \frac{x}{x+y} \right) = \left(\frac{p+(k_1+1)p'-q}{p+p'} \ , \ \frac{p}{p+p'} \right),$$

which are fractions of common denominator $p+p' < q$, independently of k_1. Continuing this way, we find $n < q$ such that $H_{k_n}^{-1} \circ \cdots \circ H_{k_1}^{-1}(x,y)$ lies on the line $\{x+y=1\} \subset \partial\Delta$. $\square$

Theorem 3.4.8. *There exist parameters (α,β) such that $T_{\alpha,\beta}$ is of infinite type and $\vec{\xi}$ does belong to the stable space $W^s(\vec{0}) \bmod 1$ for $\xi \in (0,1)$, but $e^{2\pi i\xi}$ is not a continuous eigenvalue of the Koopman operator.*

Proof. From (3.19) it follows that a parameter $(u,v) \in \Delta$ is such that $\vec{\xi}$ is in the stable subspace $W^s(\vec{0})$ only if it belongs to $\ell_{p,q,r}$ for some $p,q,r \in \mathbb{Z}$. The lines $\ell_{p,q,r}$ all pass through $(1,1)$ and have positive slope if they indeed intersect Δ. Therefore if $\ell_{p,q,r}$ intersections some subtriangle $H_{k_1} \circ \cdots \circ H_{k_n}(\Delta)$, it goes through its upper side.

Assume now that $\tilde{A}_1 \cdots \tilde{A}_i$ is a block of telescoped matrices, each of the form (3.6). Then $\varepsilon_i := \|\vec{\xi} A_{k_1} \cdots A_{k_n}\|$ is exponentially small in i (or even smaller). For the block $\tilde{A}_{i+1}$ as in (3.6), we choose $m=1$, and let $r_{i+1,1}$ be some large integer and $k_{i+1,1} = k_{i+1,2} = 2$, then $\tilde{A}_{i+1} = A_1^r \cdot A_{k_{i+1,1}} \cdot A_{k_{i+1,2}} \cdot A_{k_{i+1,3}}$ is the next telescoped matrix, and the corresponding value $\tilde{s}_i \approx r$. Choosing r sufficiently large, we can assure that $\frac{1}{i} \leq |\langle \tilde{s}_i, \vec{\xi} \tilde{A}_1 \cdots \tilde{A}_i \rangle| \leq \frac{1}{2}$. Then the BV-summability condition (3.23) for a continuous eigenvalue fails. $\square$

Theorem 3.4.9. *The set of parameters $(\alpha,\beta) \in \Omega_\infty$ such that $T_{\alpha,\beta}$ does not have an eigenvalue $e^{2\pi i\xi}$ with $\xi \in (0,1)$ contains a dense G_δ-subset of Ω_∞.*

Hence, for a dense G_δ-subset of parameters $(\alpha,\beta) \in \Omega_\infty$, the map $G_{\alpha,\beta}$ is weak mixing.

Proof. The map $G : U \to U \cup L$ from (3.1) is piecewise continuous. Let $U_k = \{(\alpha,\beta) \in \mathbb{R}^2 : \frac{1}{k+1} < \alpha \leq \frac{1}{k}, 0 \leq \beta \leq \alpha\}$, $k \in \mathbb{N}$, be the domains on which G is continuous. The sets Ω_∞ and $\Delta_\infty = \bigcap_n \bigcup_{k_1,\dots,k_n \in \mathbb{N}^n} H_{k_1} \circ \cdots \circ H_{k_n}(\Delta^\circ)$ are homeomorphic via the coding sequences $(k_i)_{i\in\mathbb{N}}$ satisfying (3.3). Indeed, every cylinder set $[k_1,\dots,k_n]$ corresponds to open triangles $\Omega_{k_1\dots k_n}$ in parameter space and $\Delta_{k_1,\dots,k_n}$ in Δ, and these triangles form a topological basis of Ω_∞ and Δ_∞ respectively. Common boundary points of such triangles don't belong to Ω_∞ or Δ_∞, so Ω_∞ and Δ_∞ are zero-dimensional sets without isolated points, but not compact. The itinerary maps

$$\begin{cases} (\alpha,\beta) \mapsto (k_i)_{i\in\mathbb{N}}, & G^{i-1}(\alpha,\beta) \in U_{k_i}^\circ, \\ (u,v) \mapsto (k_i)_{i\in\mathbb{N}}, & H^{1-i}(u,v) \in \Delta_{k_i}^\circ, \end{cases}$$

are homeomorphisms, and the composition $h : \Delta_\infty \to \Omega_\infty$ that assigns to $(u, v) \in \Delta_\infty$ the unique parameter pair $(\alpha, \beta) \in \Omega_\infty$ so that their itineraries coincide is the required homeomorphism between Δ_∞ and Ω_∞. Since $\Delta_\infty \setminus \bigcup_{p,q,r\in\mathbb{Z}} \ell_{p,q,r}$ is clearly a dense G_δ set and no (u, v) satisfies the BV-summability condition (3.23), the h-image of this set is the required dense G_δ-subset of Ω_∞ of parameters failing (3.23). □

3.4.5 Weak mixing and measurable eigenvalues for the general case

For the Bratteli-Vershik representation $(V, E, \leq)$ of the ITM. Let $h_n \in \mathbb{Z}^3$ be the vector with entries

$$h_n(v) = \#\{\text{paths from } v_0 \text{ to } v \in V_n\} = (1, \dots, 1)M_1 \cdots M_n.$$

A necessary and sufficient condition for $e^{2\pi i\xi}$ to be a (measure-theoretic) eigenvalue is the following ([BDM05] and [Bru22, Prop 6.122]): There is a sequence of functions $\rho_n : \tilde{V}_{n+1} \to \mathbb{R}$ such that

$$g_n(x) := \Big(\tilde{S}_n(x) + \rho_n(\mathfrak{t}(x_{n+1}))\Big)\, \xi \bmod 1 \text{ converges for } \mu\text{-a.e. } x \in X_{BV} \text{ as } n \to \infty, \tag{3.24}$$

where $\tilde{S}_n(x) = \sum_{j=1}^n \langle \tilde{s}_j(x), h_j(\mathfrak{s}(x_{j+1}))\rangle$ is the minimal number $k \geq 1$ of iterates of the Vershik map such that $\tau^k(x)_{n+2} \neq x_{n+2}$. (Recall $\tilde{s}_j(x)$ from (3.23) with $\tilde{A}_k$ instead of M_k and $\mathfrak{s}(x_j)$, $\mathfrak{t}(x_j)$ as the source and target vertex of edges x_j.) Furthermore, μ is a τ-invariant and ergodic probability measure. The condition $\liminf_j k_j < \infty$ made in this section implies that there is only one such measure, see Theorem 3.4.13. The sequence of functions ρ_n does not have 'convincing interpretation', see [BDM05, Theorem 7].

Let Σ be the unit simplex in $\mathbb{R}^3$ and for $w \in \mathbb{R}^3_{\geq 0} \setminus \{0\}$, let $\pi(w) = \frac{w}{\|w\|_1}$ denote the projection of w onto Σ, that is, the frequency vector $(f_v(w))_{v=1}^3$ of symbols v in w. In terms of properties of the matrices A_{k_j}, unique ergodicity is equivalent to

$$\pi\left(\bigcap_{j>m} A_{k_{m+1}} \cdots A_{k_j}(\mathbb{R}^3_{\geq 0})\right) = \ell_m$$

is a single point in Σ, see [BT03, Lemma 17] or [Bru22, Section 6.3.3]. This enables us, for any sequence $(\varepsilon_n)_{n\in\mathbb{N}}$, to find a sequence $r_n \nearrow \infty$ and telescope the sequence of substitutions (and associated matrices) accordingly, such that for $\tilde{\chi}_{n+1} := \chi_{k_{r_{n-1}+1}} \circ \cdots \circ \chi_{k_{r_{n+1}}}$ with associated matrices $\tilde{A}_{n+1} = A_{k_{r_n+1}} \cdots A_{k_{r_{n+1}}}$ we have for all $n \in \mathbb{N}$:

$$\sup_{v\in\{1,2,3\}} \sup_{w,w'\in\{1,2,3\}} \frac{f_v(w')}{f_v(w)} \leq \sup_{v\in\{1,2,3\}} \sup_{x,y\in\mathbb{R}^3_{>0}} \frac{\pi(\tilde{A}_{n+1}x)_v}{\pi(\tilde{A}_{n+1}y)_v} \leq 1 + \varepsilon_{n+1}, \tag{3.25}$$

for the frequencies $f_v(w) := \frac{|\tilde{\chi}_{n+1}(w)|_v}{|\tilde{\chi}_{n+1}(w)|}$.

Theorem 3.4.10. *If* $\liminf_n k_n < \infty$ *and* $\vec{\xi}$ *does not belong to the stable space* $W^s(\vec{0})$ mod 1 *for* $\xi \in (0,1)$*, then the corresponding ITM is weakly mixing.*

Contrary to Theorem 3.4.6 about the absence of continuous eigenvalues, the proof below does depend explicitly on the substitutions χ_{k_i} rather than only its abelianizations. It remains an open question whether our family of ITMs contains maps with a measurable non-continuous eigenvalue.

Before proving this main result of this section, first some lemmas.

Lemma 3.4.11. *For all values of* $k_1, \dots, k_n$*, the word* $\chi_{k_1} \circ \cdots \circ \chi_{k_n}(3)$ *is a prefix of* $\chi_{k_1} \circ \cdots \circ \chi_{k_n}(2)$*. If* $k_n \geq 2$*, then* $\chi_{k_1} \circ \cdots \circ \chi_{k_n}(1)$ *is a prefix of* $\chi_{k_1} \circ \dots \chi_{k_n}(3)$*, up to its last letter which is* 1 *or* 2 *when* n *is even or odd. If* $k_n = 1$ *and* $n \geq 2$*, then* $\chi_{k_1} \circ \cdots \circ \chi_{k_n}(3)$ *is a prefix of* $\chi_{k_1} \circ \dots \chi_{k_n}(1)$*.*

Proof. Direct by induction on n. □

Lemma 3.4.12. *If* $2 \leq k_n \leq C$ *or* $k_n = 1, k_{n-1} \geq 2$*, then for every* j *such that* $A_{k_{n-j}} \cdots A_{k_n}$ *is a full matrix,*

$$|\chi_{k_{n-j}} \circ \cdots \circ \chi_{k_n}(v)|_u \leq 4C|\chi_{k_{n-j}} \circ \cdots \circ \chi_{k_n}(v')|_u$$

for all $v, v' \in V_n$ *and* $u \in \{1,2,3\}$*. In particular,* $h_n(v) \leq 4Ch_n(v')$ *for all* $v, v' \in V_n$*.*

The proof is deferred to Section 3.4.6.

Theorem 3.4.13. *If* (3.3) *holds and* $\liminf_j k_j < \infty$*, then the corresponding ITM is uniquely ergodic. The conditions hold for* ν*-a.e.* $(\alpha, \beta) \in \Omega_\infty$ *for every* G*-invariant measure* ν *supported on* Ω_∞*.*

Proof. We telescope the sequence χ_{k_j} into blocks $\chi_{k_{n_{i-1}+1}} \circ \cdots \circ \chi_{k_{n_i}}$ such that $\tilde{A}_i = A_{k_{n_{i-1}+1}} \cdots A_{k_{n_i}}$ is a full matrix and $k_{n_i} \leq C$ (and if $k_{n_i} = 1$ then $k_{n_{i-1}} \geq 2$) for all i. Using Lemma 3.4.12 we estimate

$$\rho(L, L') := \sqrt{\frac{\max\{L_u/L'_u : u = 1,2,3\}}{\min\{L_u/L'_u : u = 1,2,3\}}} \quad \text{with } L, L' \text{ columns of } \tilde{A}_i$$

for the Hilbert metric[5]. For these associated matrices, we get $\rho \leq 4C$. This gives a contraction factor $\tanh(\frac{1}{2}\log(\rho)) < 1$ for the Hilbert metric, independently of i. Hence, unique ergodicity of $T_{\alpha,\beta}$ follows.

Now if ν is any G-invariant probability measure, then there is k such that $\nu(U_k) > 0$ for the strip $U_k := \{(\alpha, \beta) \in \Omega_\infty : \frac{1}{k+1} < \alpha \leq \frac{1}{k}\}$, see Figure 3.1 (left). By Poincaré recurrence, $G^n(\alpha, \beta) \in U_k$ for infinitely many n, so $\liminf_i k_i \leq k$ for ν-a.e. (α, β). □

[5] This time we multiply by column vector on the right, so the notation is exactly as in [Bru22, Formula 8.29] and transposed compared to (3.17).

In the sequel, write $\tilde{\chi}_n = \chi_{\tilde{k}_1} \circ \cdots \circ \chi_{\tilde{k}_m}$ where $m = m(n)$. We will use Bratteli-Vershik system arguments, but still prefer to keep a metric there that agrees to the metric on the subshift. Hence if (X, τ) is the Bratteli-Vershik system containing distinct edge-labeled paths x and y, then we set $d(x,y) = 2^{-n}$ for $n = \min\{j \geq 0 : \tau^j(x)_1 \neq \tau^j(y)_1\}$.

For $x \in X_{BV}$ and $n \in \mathbb{N}$ fixed, define recursively

$$y^0 = (y^0_m)_{m \geq 1} = x \quad \text{and} \quad y^{\ell+1} = \tau^{h_n(\mathfrak{s}(y^\ell_{n+1}))}(y^\ell). \tag{3.26}$$

Lemma 3.4.14. *For every $C \geq 2$, there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$ for which $\tilde{k}_m \leq C$ and if $\tilde{k}_m = 1$, then $\tilde{k}_{m-1} \geq 2$, we have*

$$\mu(\{x \in [e] : d(x, y^\ell) < 2^{-n/16C}\}) \geq \frac{1}{16C}\mu([e]), \tag{3.27}$$

for all $\ell \geq 0$ and $e \in E_{n+1}$, where $[e] = \{x \in X_{BV} : x_{n+1} = e\}$.

Proof. The telescoping is such that $\tilde{\chi}_n(v) = \chi_{\tilde{k}_1} \circ \cdots \circ \chi_{\tilde{k}_m}(v)$ satisfies (3.25); in particular, for each $v \in V_n$, $\tilde{\chi}_n(v)$ contains every symbol in V_{n-1} multiple times, and $m = m(n)$ is so large that there are a minimal even $j' \geq 2$ and a minimal odd $j'' \geq 2$ both such that $\tilde{k}_{m-j'}, \tilde{k}_{m-j''} \geq 2$. Due to (3.3) such j' and j'' can always be found. Set $j = \max\{j', j''\}$. Then $\chi_{\tilde{k}_{m-j}} \circ \cdots \circ \chi_{\tilde{k}_m}(v)$, $v \in V_n$, have a maximal common prefix W' that contains every symbol. In the light of Lemma 3.4.11, W' equals $\chi_{\tilde{k}_{m-j}} \circ \cdots \circ \chi_{\tilde{k}_m}(1)$ minus its last letter if $\tilde{k}_m \geq 2$, and $W' = \chi_{\tilde{k}_{m-j}} \circ \cdots \circ \chi_{\tilde{k}_m}(3)$ if $\tilde{k}_{m-1} > \tilde{k}_m = 1$.

Clearly

$$W := \chi_{\tilde{k}_1} \circ \cdots \circ \chi_{\tilde{k}_{m-j-1}}(W')$$

is a common prefix of $\tilde{\chi}_n(v)$, $v \in V_n$, and it contains every symbol as well. By Lemma 3.4.11 we have

$$q := \left\lfloor \frac{1}{2}|\tilde{\chi}_1 \circ \cdots \circ \tilde{\chi}_{n-1}(W)| \right\rfloor \geq \frac{1}{16C} \max_{v \in V_n} |\tilde{\chi}_1 \circ \cdots \circ \tilde{\chi}_n(v)|.$$

Then as all paths from v_0 to $\mathfrak{t}(e)$ in the Bratteli diagram have the same mass, the union of the first q such paths in $[e]$ has mass $\geq \frac{1}{16C}\mu([e])$.

Take $v = \mathfrak{s}(e)$ and let $x_{\min}(v)$ be the minimal path from v_0 to v. For any $x \in [e] \cap \tau^j([x_{\min}(v)])$ with $0 \leq j \leq q$, if follows that $\tau^{h_n(v)}(x) \in \tau^j([x_{\min}(v')])$, where $v' \in V_n$ is the source of the successor edge to x_{n+1} (or if x_{n+1} is the maximal incoming edge, $v' \in V_n$ is the source of $\tau^{h_n(v)}(x)_{n+1}$). Because q is only half of the length of the common prefix W, $d(x, y^1) = d(x, \tau^{h_n(v)}(x)) \leq 2^{-n/16C}$. Since this is true for all $x \in \tau^j([x_{\min}(v)])$ and $0 \leq j \leq q$, and for every $\ell \geq 0$, $y^\ell \in \tau^j([x_{\min}(v'')])$ for some $v'' \in V_n$, the lemma follows. $\square$

Proof of Theorem 3.4.10. Let $C := 3 + \liminf_j k_j$. We can assume that the telescoping of the BV-diagram can be done in such a way that (3.25) holds, and the last matrix in each telescoped block either has subscript $2 \leq k \leq C$ or is equal to 1, but then the subscript of the penultimate matrix is ≥ 2. Let $\vec{\xi_0} = \vec{\xi}$ and $\vec{\zeta}_{n+1} = \vec{\xi_n} \cdot \tilde{A}_{n+1}$ and $\vec{\xi}_{n+1} = \vec{\zeta}_{n+1} - \vec{z}_{n+1} \in (-\frac{1}{2}, \frac{1}{2}]^3$, where $\vec{z}_{n+1} \in \mathbb{Z}^3$ is the closest integer vector to $\vec{\zeta}_{n+1}$. By assumption, $\vec{\xi} \notin W^s(\vec{0}) \bmod 1$, therefore there must be $\eta > 0$ such that $\|\vec{\zeta}_{n+1}\|_1 > \eta$ infinitely often.

Assume by contradiction that (3.24) holds. Let $g(x) = \lim_{n\to\infty} g_n(x)$ wherever it converges. By Lusin's Theorem, for every $\delta > 0$, there is a compact subset $X' \subset X_{BV}$ such that $\mu(X_{BV} \setminus X') < \delta$, g is uniformly continuous on X' and $\lim_{n\to\infty} g_n(x) = g(x)$ for all $x \in X'$. Pick $\delta < 1/(144C)$. Then, for $\varepsilon \in (0, \eta/(288C))$ arbitrary, there is $N \in \mathbb{N}$ such that for every $n > N$ and $x, y \in X'$ such that $d(x, y) < 2^{-N/16C}$ we have

$$|g(x) - g(y)| < \varepsilon \quad \text{and} \quad |g_n(x) - g(x)| < \varepsilon. \tag{3.28}$$

It also follows that $|g_{n+1}(x) - g_n(x)| < 2\varepsilon$ for all $n \geq N$.

Let $X_n = \{x \in \tau^j([x_{\min}(v)]) : v \in V_n, 0 \leq j < q\}$ with the notation q and $x_{\min}(v)$ from Lemma 3.4.14. In fact, X_n consists of all the paths $x \in X$ such that the finite path $(x_1, \dots, x_n)$ is among the first q paths from v_0 to $\mathfrak{s}(x_{n+1})$ and $\mu(X_n) \geq 1/(16C)$.

Recalling also the points y^ℓ from (3.26), we get from Lemma 3.4.14 that

$$d(x, y^\ell) < 2^{-n/16C} \text{ for all } x \in X_n \text{ and } \ell \geq 0. \tag{3.29}$$

Now take $x \in X_n$ and corresponding y^ℓ such that $x, y^\ell \in X'$ and $x_{n+2} = y^j_{n+2}$ for all $0 \leq j \leq \ell$. Then

$$\begin{aligned} \Big\| \sum_{j=0}^{\ell-1} \xi h_n(\mathfrak{s}(y^j_{n+1})) \Big\| &= \|\xi \cdot (\tilde{S}_n(x) - \tilde{S}_n(y^\ell))\| \\ &= \|\xi \cdot \tilde{S}_n(x) \bmod 1 - \xi \cdot \tilde{S}_n(y^\ell) \bmod 1\| = \|g_n(x) - g_n(y^\ell)\| \\ &\leq \|g_n(x) - g(x)\| + \|g(x) - g(y^\ell)\| + \|g(y^\ell) - g_n(y^\ell)\| < 3\varepsilon. \end{aligned} \tag{3.30}$$

Now take $n \geq N$ such that $\|\vec{\zeta}_{n+1}\| \geq \eta$. Notice that the components of $\vec{\xi_n}$ are $\xi_n(v) = \xi h_n(v) \bmod 1 \in (-\frac{1}{2}, \frac{1}{2}]$.

$$\begin{aligned} \vec{\zeta}_{n+1} &= \vec{\xi_n} \cdot \tilde{A}_{n+1} = (\xi_n(1), \xi_n(2), \xi_n(3)) \cdot \tilde{A}_{n+1} \\ &= \left(\sum_{j=1}^{|\tilde{\chi}_{n+1}(1)|} \xi_n(\tilde{\chi}_{n+1}(1)_j), \sum_{j=1}^{|\tilde{\chi}_{n+1}(2)|} \xi_n(\tilde{\chi}_{n+1}(2)_j), \sum_{j=1}^{|\tilde{\chi}_{n+1}(3)|} \xi_n(\tilde{\chi}_{n+1}(3)_j) \right). \end{aligned}$$

By (3.25), for each $v \in V_n$, the frequencies $f_v(w)$ differ among the $w \in V_{n+1}$ by no more than a factor $1 + \varepsilon_{n+1}$, where we choose ε_{n+1} to be the smallest positive value among the sums $\left|\sum_{v\in V_n}\xi_n(v)f_v(w)\right|$ for $w\in V_{n+1}$. Lemma 3.4.12 applied to the block of substitution that produces $\tilde{\chi}_{n+1}$ gives $|\tilde{\chi}_{n+1}(w)|\le 4C|\tilde{\chi}_{n+1}(w')|$ for all $w,w'\in V_{n+1}$.

Since $\|\vec{\zeta}_{n+1}\|_1\ge\eta$, we can pick $w\in V_{n+1}$ such that $\left|\sum_{j=1}^{|\tilde{\chi}_{n+1}(w)|}\xi_n(\tilde{\chi}_{n+1}(w)_j)\right|\ge\eta/3$. First assume that $\sum_{v\in V_n}\xi_n(v)f_v(w)>0$. Then, recalling that $\sum_{v\in V_n}|\xi_n(v)|f_v(w)\le\frac{1}{2}$, we have

$$\begin{aligned}
\frac{\eta}{24C} &\le \frac{1}{8C}\sum_{j=1}^{|\tilde{\chi}_{n+1}(w)|}\xi_n(\tilde{\chi}_{n+1}(w)_j)\le\frac{1}{4C}|\tilde{\chi}_{n+1}(w)|\cdot\frac{1}{2}\sum_{v\in V_n}\xi_n(v)f_v(w)\\
&\le \frac{1}{4C}|\tilde{\chi}_{n+1}(w)|\left(\sum_{v\in V_n}\xi_n(v)f_v(w)-\varepsilon_{n+1}\sum_{v\in V_n}|\xi_n(v)|f_v(w)\right)\\
&\le \frac{1}{4C}|\tilde{\chi}_{n+1}(w)|\left(\sum_{\xi_n(v)>0}\xi_n(v)\frac{f_v(w)}{1+\varepsilon_{n+1}}+\sum_{\xi_n(v)<0}\xi_n(v)f_v(w)(1+\varepsilon_{n+1})\right)\\
&\le |\tilde{\chi}_{n+1}(w')|\left(\sum_{\xi_n(v)>0}\xi_n(v)f_v(w')+\sum_{\xi_n(v)<0}\xi_n(v)f_v(w')\right)\\
&= |\tilde{\chi}_{n+1}(w')|\sum_{v\in V_n}\xi_n(v)f_v(w')=\sum_{j=1}^{|\tilde{\chi}_{n+1}(w')|}\xi_n(\tilde{\chi}_{n+1}(w')_j).
\end{aligned}$$

If on the other hand $\sum_{v\in V_n}\xi_n(v)f_v(w)<0$, we change signs:

$$\begin{aligned}
\frac{\eta}{24C} &\le \frac{1}{8C}\left|\sum_{j=1}^{|\tilde{\chi}_{n+1}(w)|}\xi_n(\tilde{\chi}_{n+1}(w)_j)\right|\le\left|\frac{1}{4C}|\tilde{\chi}_{n+1}(w)|\cdot\frac{1}{2}\sum_{v\in V_n}\xi_n(v)f_v(w)\right|\\
&\le -\frac{1}{4C}|\tilde{\chi}_{n+1}(w)|\left(\sum_{v\in V_n}\xi_n(v)f_v(w)-\varepsilon_{n+1}\sum_{v\in V_n}|\xi_n(v)|f_v(w)\right)\\
&\le -\frac{1}{4C}|\tilde{\chi}_{n+1}(w)|\left(\sum_{\xi_n(v)>0}\xi_n(v)f_v(w)(1+\varepsilon_{n+1})+\sum_{\xi_n(v)<0}\xi_n(v)\frac{f_v(w)}{1+\varepsilon_{n+1}}\right)\\
&\le -|\tilde{\chi}_{n+1}(w')|\left(\sum_{\xi_n(v)>0}\xi_n(v)f_v(w')+\sum_{\xi_n(v)<0}\xi_n(v)f_v(w')\right)\\
&= -|\tilde{\chi}_{n+1}(w')|\sum_{v\in V_n}\xi_n(v)f_v(w')=\left|\sum_{j=1}^{|\tilde{\chi}_{n+1}(w')|}\xi_n(\tilde{\chi}_{n+1}(w')_j)\right|.
\end{aligned}$$

Recall that the $\tilde{\chi}_{n+1}(w)$ gives the order of incoming edges to $w\in V_{n+1}$. Without loss of generality, we can assume that (3.25) applies to any subword of $\tilde{\chi}_{n+1}(w)$ of length at least $|\tilde{\chi}_{n+1}(w)|/3$. That is, if $1\le a<a+|\tilde{\chi}_{n+1}(w)|/3<b$, then

$$\frac{\eta}{72C}\le\left|\sum_{j=a}^{b-1}\xi_n(\tilde{\chi}_{n+1}(w)_j)\right|. \tag{3.31}$$

Indeed, since the frequencies of letters in all of these subwords are almost the same, these sums (consisting of $b-a$ terms of three types $\xi_n(1), \xi_n(2)$ and $\xi_n(3)$) indicate almost collinear vectors of lengths almost proportional to $|b-a|/|\tilde{\chi}_{n+1}(w)|$.

Now choose the most frequent (measure-wise) $w \in V_{n+1}$ and $v \in V_n$; that is

$$\mu(\{x \in X'_n : \mathfrak{s}(x_{n+1}) = v, \mathfrak{t}(x_{n+1}) = w\}) \geq \frac{1}{144C} - \delta > 0.$$

For a path $u := u_1 \dots u_n \in \tau^j([x_{min}(v)])$ from v_0 to $v \in V_n$ and $0 \leq j \leq q$ and for each edge $a \in E_{n+1}$ with $\mathfrak{s}(a) = v$ and $\mathfrak{t}(a) = w$, let $x^{(a)}(u)$ be a path from v_0 to $w \in V_{n+1}$ such that

(i) $x^{(a)}_{n+1}(u) = a$, and

(ii) $x^{(a)}_k(u) = u_k$ for all $1 \leq k \leq n$.

Hence for edges $a, b \in E_{n+1}$ connecting $v \in V_n$ and $w \in V_{n+1}$ such that (3.31) holds. Then

$$\mu(X' \cap [x^{(a)}(u)]) > \frac{1}{2}\mu([x^{(a)}(u)]) \quad \text{and} \quad \mu(X' \cap [x^{(b)}(u)]) > \frac{1}{2}\mu([x^{(b)}(u)]).$$

This implies that there exists $x \in [x^{(a)}(u)] \cap X'$ and $y \in [x^{(b)}(u)] \cap X'$ such that $y = y^\ell$ for some $\ell \geq 1$ in the sense of (3.26) (because $y^{(j)} \mapsto y^{(j+1)}$ is measure-preserving).

Then $\tilde{S}_{n-1}(x^{(a)}(u)) = \tilde{S}_{n-1}(x^{(b)}(u))$ and hence $g_{n-1}(x^{(a)}(u)) = g_{n-1}(x^{(b)}(u))$. Because $\mathfrak{s}(a) = \mathfrak{s}(b)$ and $b - a > |\tilde{\chi}_{n+1}(w)|/3$, it follows from (3.31) that

$$\left|g_n(x^{(b)}(u)) - g_n(x^{(a)}(u))\right| = \left|\xi \cdot \left(\tilde{S}_n(x^{(b)}(u)) - \tilde{S}_n(x^{(a)}(u))\right)\right| = \left|\sum_{j=a}^{b-1} \xi_n(\tilde{\chi}_{n+1}(w)_j)\right| \geq \frac{\eta}{72C}.$$

Therefore

$$\begin{aligned} 4\varepsilon \;\; &\geq \;\; \Big|(g_n(x^{(b)}(u)) - g_{n-1}(x^{(b)}(u))) + (g_{n-1}(x^{(b)}(u)) - g_{n-1}(x^{(a)}(u))) \\ &\qquad + (g_{n-1}(x^{(a)}(u)) - g_n(x^{(a)}(u)))\Big| \\ &= \;\; |g_n(x^{(b)}(u)) - g_n(x^{(a)}(u))| \geq \frac{\eta}{72C}, \end{aligned}$$

which contradicts the choice of ε. This finishes the proof. □

Generally, a measurable eigenvalue of a linearly recurrent Cantor system is not necessarily continuous, see [BDM05, Proposition 16]. The following corollary shows that for linearly recurrent ITMs of infinite type, measurable and continuous eigenvalues coincide.

Corollary 3.4.15. *A linearly recurrent ITM of infinite type is weakly mixing if and only if $\vec{\xi}$ does not belong to the stable space $W^s(\vec{0}) \bmod 1$ for $\xi \in (0,1)$. Furthermore, any measurable eigenvalue is continuous.*

Proof. It follows from Theorem 3.4.10 that if $\vec{\xi} \notin W^s(\vec{0})$ mod 1, then the system is weakly mixing. If on the other hand $\vec{\xi} \in W^s(\vec{0})$ mod 1, then the convergence of $\|\vec{\xi} A_{k_1} \cdots A_{k_n}\|$ to zero is exponential. By [BDM05, Theorem 1], a measurable eigenvalue $e^{2\pi i \xi}$ for linearly recurrent system exists if and only if

$$\sum_{n \geq 1} \vert\vert\vert \vec{\xi} \tilde{A}_1 \cdots \tilde{A}_n \vert\vert\vert^2 < \infty$$

and additionally the eigenvalue is continuous if and only if

$$\sum_{n \geq 1} \vert\vert\vert \vec{\xi} \tilde{A}_1 \cdots \tilde{A}_n \vert\vert\vert < \infty.$$

Thus if $\vec{\xi} \in W^s(\vec{0})$ mod 1, then the sum converges in both cases and $e^{2\pi i \xi}$ is a continuous eigenvalue of the ITM. □

3.4.6 The proof of Lemma 3.4.12

Proof of Lemma 3.4.12. The goal is to show that in the product of incidence matrices $A_{k_m} \cdots A_{k_n}$ every column is element-wise less than or equal to $4C^*$-times any other column. The proof will go though all possible states of products $A_{k_m} \cdots A_{k_n}$ before arriving at a full matrix, see the transition graph between the states in Figure 3.4. Let us write $\mathbb{A}^m = A_{k_m} \cdots A_{k_n}$ for the matrix associated to the substitution $\chi_{k_m} \circ \cdots \circ \chi_{k_n}$. Then $\mathbb{A}^n = A_{k_n}$ and the columns $\mathbb{A}^m_2 \geq \mathbb{A}^m_3$ element-wise for every $m \leq n$.

Let

$$\mathbb{D}_m := A_{k_{m-2}} \cdot A_{k_{m-1}} = \begin{pmatrix} k_{m-2} & k_{m-2} - 1 & k_{m-2} - 1 \\ 0 & k_{m-1} & k_{m-1} - 1 \\ 1 & 1 & 1 \end{pmatrix}$$

be the matrix associated to the next pair of substitutions $\chi_{k_{m-2}} \circ \chi_{k_{m-1}}$. Our proof is of algorithmic nature, illustrated by the following scheme:

The properties of the matrices in each state are:

State 1: $\mathbb{A}^m = \begin{pmatrix} 0 & k_n & k_n - 1 \\ 1 & 0 & 0 \\ r & rk_n + 1 & r(k_n - 1) + 1 \end{pmatrix}$ for some integer $r \geq 0$.

If $k_n \geq 2$, then $\mathbb{A}^n$ is in this state, with $r = 0$.

State 2:

$$\mathbb{A}^m = \begin{pmatrix} 0 & k_n & k_n - 1 \\ Q_1 & Q_2 & Q_3 \\ R_1 & R_2 & R_3 \end{pmatrix} \qquad \begin{array}{rl} \text{for integers} & 1 \leq \min_i Q_i \leq \max_j Q_j < 3C \min_i Q_i - C \\ \text{satisfying} & 1 \leq \min_i R_i \leq \max_j R_j < 3C \min_i R_i - C. \end{array}$$

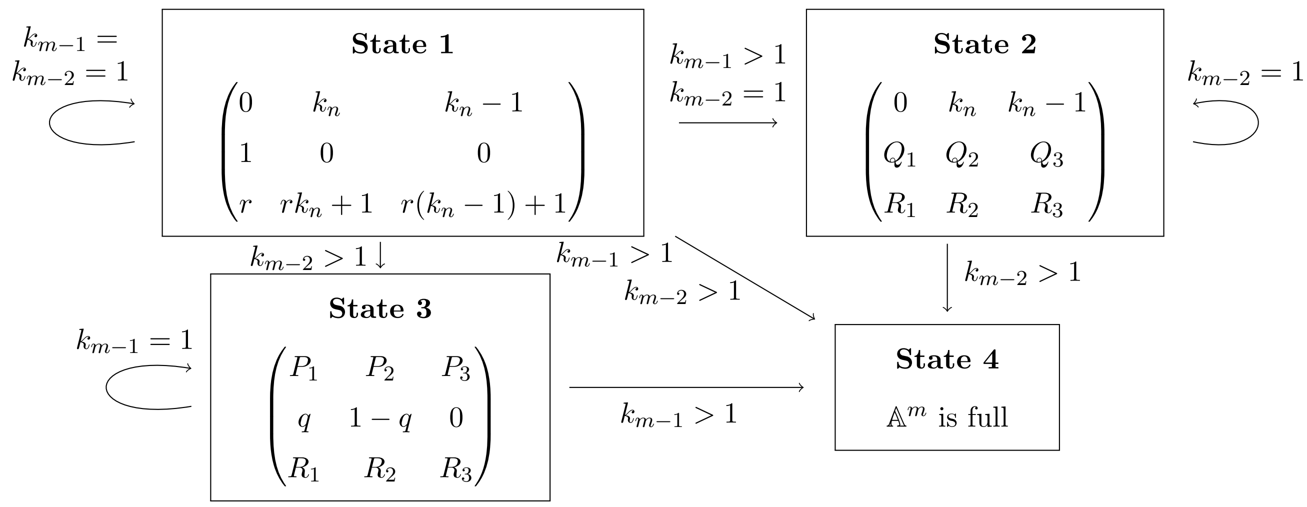


Figure 3.4: Every arrow stands for left multiplication with $\mathbb{D}_m$

State 3:

$$\mathbb{A}^m = \begin{pmatrix} P_1 & P_2 & P_3 \\ q & 1-q & 0 \\ R_1 & R_2 & R_3 \end{pmatrix} \qquad \begin{array}{rl} \text{for integers} & 1 \leq \min_i P_i \leq \max_j P_j < 3C \min_i P_i \\ \text{satisfying} & 1 \leq \min_i R_i \leq \max_j R_j < 3C \min_i R_i \\ & q \in \{0, 1\}. \end{array}$$

If $1 = k_n < k_{n-1}$, then $\mathbb{A}^n$ is in this state, with $q = 0$.

State 4: $\mathbb{A}^m$ is full and $(\max_i \mathbb{A}_i^m)_u \leq 4C(\min_j \mathbb{A}_j^m)_u$ for all $u \in \{1, 2, 3\}$. The lemma follows from these inequalities.

Every further matrix multiplication with $\mathbb{D}_m := A_{k_{m-2}} \cdot A_{k_{m-1}}$ will move the product $\mathbb{A}^m$ along the arrows in Figure 3.4 depending on the values k_{m-2}, k_{m-1} to the next state and product $\mathbb{A}^{m-2}$. Once the matrix is in State 4, it satisfies the lemma.

Now we verify the transitions between the states.

- From State 1 to State 1: $k_{m-1} = k_{m-2} = 1$. In this case

$$\mathbb{D}_m = \mathbb{J} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix},$$

  and we compute:

$$\mathbb{J}^r \cdot \mathbb{A}^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r & r & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & k_n & k_n - 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & k_n & k_n - 1 \\ 1 & 0 & 0 \\ r & rk_n + 1 & r(k_n - 1) + 1 \end{pmatrix} \tag{3.32}$$

  as required.

- From State 1 to State 2: $k_{m-1} > 1 = k_{m-2}$. From (3.32) we get

$$\begin{aligned}
\mathbb{A}^{m-2} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & k_{m-1} & k_{m-1}-1 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & k_n & k_n - 1 \\ 1 & 0 & 0 \\ r & rk_n + 1 & r(k_n-1)+1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & k_n & k_n - 1 \\ X + k_{m-1} & Xk_n + k_{m-1} - 1 & X(k_n-1) + k_{m-1} - 1 \\ r+1 & (r+1)k_n + 1 & (r+1)(k_n-1)+1 \end{pmatrix}
\end{aligned}$$

for $X = r(k_{m-1}-1) \geq 0$, and the properties of State 2 hold.

- From State 1 to State 3: $k_{m-2} > 1 = k_{m-1}$. From (3.32) we get

$$\begin{aligned}
\mathbb{A}^{m-2} &= \begin{pmatrix} k_{m-2} & k_{m-2}-1 & k_{m-2}-1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & k_n & k_n - 1 \\ 1 & 0 & 0 \\ r & rk_n + 1 & r(k_n-1)+1 \end{pmatrix} \\
&= \begin{pmatrix} Y-1 & Yk_n + k_{m-2} - 1 & Y(k_n-1) + k_{m-2} - 1 \\ 1 & 0 & 0 \\ r+1 & (r+1)k_n + 1 & (r+1)(k_n-1)+1 \end{pmatrix}
\end{aligned}$$

for $Y = r(k_{m-2}-1) + k_{m-2} \geq 2$, and the properties of State 3 hold with $q = 1$.

- From State 1 to State 4: $k_{m-2}, k_{m-1} > 1$. From (3.32) we get

$$\begin{aligned}
\mathbb{A}^{m-2} &= \begin{pmatrix} k_{m-2} & k_{m-2}-1 & k_{m-2}-1 \\ 0 & k_{m-1} & k_{m-1}-1 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & k_n & k_n - 1 \\ 1 & 0 & 0 \\ r & rk_n + 1 & r(k_n-1)+1 \end{pmatrix} \\
&= \begin{pmatrix} Y-1 & Yk_n + k_{m-2} - 1 & Y(k_n-1) + k_{m-2} - 1 \\ X + k_{m-1} & Xk_n + k_{m-1} - 1 & X(k_n-1) + k_{m-1} - 1 \\ r+1 & (r+1)k_n + 1 & (r+1)(k_n-1)+1 \end{pmatrix}
\end{aligned}$$

for $X = r(k_{m-1}-1) \geq 0$, $Y = r(k_{m-2}-1) + k_{m-2} \geq 2$, and the properties of State 4 hold.

- From State 2 to State 2: $k_{m-1} \geq 1 = k_{m-2}$. Left multiplication with

$$\mathbb{D}_m = \begin{pmatrix} 1 & 0 & 0 \\ 0 & k_{m-1} & k_{m-1}-1 \\ 1 & 1 & 1 \end{pmatrix}$$

leaves the first row of $\mathbb{A}^m$ unchanged. For the second row the inequalities for the new $\tilde{Q}_i$ follow from

$$\begin{aligned}\tilde{Q}_i + C &= k_{m-1}Q_i + (k_{m-1}-1)R_i + C \\ &< 3Ck_{m-1}Q_j + 3C(k_{m-1}-1)R_j = 3C\tilde{Q}_j\end{aligned}$$

and for the last row

$$\tilde{R}_i + C \leq Q_i + R_i + 2C < 3CQ_j + 3CR_j = 3C\tilde{R}_j.$$

So the conditions of State 2 remain valid.

- From State 3 to State 3: $k_{m-2} \geq 1 = k_{m-1}$. Left multiplication with

$$\mathbb{D}_m = \begin{pmatrix} k_{m-2} & k_{m-2}-1 & k_{m-2}-1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

keeps the conditions of State 3 valid. By $R_i + 1 \leq 3CR_j$ it holds that for the new $\tilde{P}_1$

$$\tilde{P}_1 = P_1 + R_1 + q < 3CP_i + 3CR_i \leq 3C\tilde{P}_i$$

for all i and so on for the other entries.

- From State 2 to State 4: $k_{m-2} > 1$. Left multiplication with $\mathbb{D}_m$ gives

$$\mathbb{A}^{m-2} = \mathbb{D}_m \cdot \mathbb{A}^m = \begin{pmatrix} k_{m-2} & k_{m-2}-1 & k_{m-2}-1 \\ 0 & k_{m-1} & k_{m-1}-1 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & k_n & k_n - 1 \\ Q_1 & Q_2 & Q_3 \\ R_1 & R_2 & R_3 \end{pmatrix}$$

$$= \begin{pmatrix} (k_{m-2}-1)(Q_1+R_1) & \begin{matrix} k_nk_{m-2}+ \\ (k_{m-2}-1)(Q_2+R_2) \end{matrix} & \begin{matrix} (k_n-1)k_{m-2}+ \\ (k_{m-2}-1)(Q_3+R_3) \end{matrix} \\ k_{m-1}Q_1 + (k_{m-1}-1)R_1 & k_{m-1}Q_2 + (k_{m-1}-1)R_2 & k_{m-1}Q_3 + (k_{m-1}-1)R_3 \\ Q_1 + R_1 & k_n + Q_2 + R_2 & k_n - 1 + Q_3 + R_3 \end{pmatrix}$$

keeps the inequality in the second and third row as before. For the first row

$$\begin{aligned}\tilde{P}_2 &= k_nk_{m-2} + (k_{m-2}-1)(Q_2+R_2) \\ &\leq (k_{m-2}-1)(Q_2+R_2+C) + C \\ &< 4C(k_{m-2}-1)(Q_j+R_j) \leq 4C\tilde{P}_j\end{aligned}$$

and analogously the inequalities hold for the other entries. Thus the conditions of State 4 hold.

- From State 3 to State 4: $k_{m-2} > 1$. Left multiplication with $\mathbb{D}_m$ gives

$$\mathbb{A}^{m-2} = \mathbb{D}_m \cdot \mathbb{A}^m = \begin{pmatrix} k_{m-2} & k_{m-2}-1 & k_{m-2}-1 \\ 0 & k_{m-1} & k_{m-1}-1 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} P_1 & P_2 & P_3 \\ q & 1-q & 0 \\ R_1 & R_2 & R_3 \end{pmatrix} =$$

$$= \begin{pmatrix} \begin{matrix} k_{m-2}P_1+ \\ (k_{m-2}-1)(q+R_1) \end{matrix} & \begin{matrix} k_{m-2}P_2+ \\ (k_{m-2}-1)(R_2+1-q) \end{matrix} & \begin{matrix} k_{m-2}P_3+ \\ (k_{m-2}-1)R_3 \end{matrix} \\ qk_{m-1}+(k_{m-1}-1)R_1 & (1-q)k_{m-1}+(k_{m-1}-1)R_2 & (k_{m-1}-1)R_3 \\ P_1+q+R_1 & P_2+1-q+R_2 & P_3+R_3 \end{pmatrix}$$

  keeps the required inequalities of the first and third row as before. For the second row we see that

$$\begin{aligned} \tilde{Q}_1 &= qk_{m-1} + (k_{m-1}-1)R_1 = (k_{m-1}-1)(R_1+q)+q \\ &\le 3C(k_{m-1}-1)R_j + 1 < 4C(k_{m-1}-1)R_j \le 4C\tilde{Q}_j \end{aligned}$$

  for all j and analogously for the other entries. Hence the conditions of State 4 hold.

Since for a full matrix in State 4, any further left multiplications with $\mathbb{D}_{m-2}$ etc., preserves the conditions of State 4, this proves the lemma for $k_n \ge 2$. $\square$

4. Rigidity and Toeplitz Systems

The aim of this chapter is to study measure-theoretical rigidity and partial rigidity for classes of Cantor dynamical systems including Toeplitz systems and enumeration systems. We use Bratteli diagrams to control invariant measures that are produced in our constructions. This leads to systems with desired properties. Among other things, we show that there exists a Toeplitz system with zero entropy that is not partially measure-theoretically rigid with respect to its (unique) invariant measure. We investigate enumeration systems defined by a linear recursion, prove that all such systems are partially rigid and present an example of an enumeration system which is not measure-theoretically rigid. We construct a minimal $\mathcal{S}$-adic Toeplitz subshift which has countably infinitely many ergodic invariant probability measures which are rigid for the same rigidity sequence.

This chapter is based on the article [BKOR25] which was accepted for publication in *Forum Mathematicum* in September 2025.

4.1 Introduction

This paper is devoted to the study of measure-theoretical rigidity for various classes of Cantor dynamical systems which include Toeplitz systems and enumeration systems. Rigidity is a form of recurrence in dynamical systems: a finite measure-preserving dynamical system (X, T, μ) is called rigid if there is an increasing sequence $t_n \in \mathbb{N}$ such that $\mu(T^{-t_n}(A)\Delta A) \to 0$ for all measurable sets A as $n \to \infty$. The notion of rigidity was introduced for measure preserving transformations in [FW78]. Later, a weaker property of partial rigidity was introduced in [Fri89]. The notion of rigidity in topological dynamics (i.e., topological rigidity, also called uniform rigidity) was first considered in [GM89]. An overview of the results devoted to the rigidity sequences can be found in [KL23]. Recent results on rigidity include [BdJLR14], [FK15], [Dan16], [DS17], [DMR25], [Rad24].

Toeplitz systems are minimal symbolic almost 1-1 extensions of odometers which demonstrate rich variety of topological and measure-theoretical properties (see [Dow05] for a survey). Regular Toeplitz systems are measure-theoretically isomorphic to odometers, and thus are measure-theoretically rigid with respect to their unique invariant measures. Irregular Toeplitz systems satisfying the so called Same Aperiodic Readouts (SAR) property have a measure-theoretical representation as a skew product of their maximal equicontinuous factor (which is an odometer) and a subshift (see [Dow05]). Using this representation, we construct a Toeplitz subshift of zero entropy which is not partially measure-theoretically rigid with respect to its invariant measure, see Corollary 4.3.7.

Another tool that we apply in this paper is Bratteli diagrams. These are graded graphs which are extensively used in Cantor dynamics for constructing models of homeomorphisms of Cantor spaces. Bratteli-Vershik systems proved to be very useful for classifying Cantor dynamical systems and constructing various examples of homeomorphisms (see e.g. [Dur10], [BK16], [DK18], [BK20], [DP22]). In particular, Bratteli diagrams are extremely useful in describing the simplex of invariant probability measures. In this paper, we show rigidity for the measures obtained as extensions from subdiagrams which are odometers. We apply the results to Toeplitz systems and some systems with countably infinitely many ergodic invariant probability measures. Enumeration systems are generalizations of odometers that are introduced in [BDIL00, BDL02], see also [Bru22, Section 5.3]. We show that a class of enumeration systems, coming from kneading theory, are measure-theoretically partially rigid, and that there exists an enumeration system in this class (in fact, a substitution system) that is not measure-theoretically rigid.

The outline of the paper is as follows. In Section 4.2, we give preliminaries concerning rigidity, Bratteli diagrams and Toeplitz sequences. In Section 4.3, we construct a Toeplitz subshift of zero entropy which is not partially measure-theoretically rigid with respect to any of its invariant measures. In Section 4.4, we show that a class of Bratteli-Vershik systems with the invariant probability measure which is obtained as an extension of a measure on an odometer with growing number of edges on each level is measure-theoretically rigid. Section 4.5 gives examples of non-rigid partially rigid systems. We present a class of non-simple stationary Bratteli diagrams such that the corresponding Bratteli-Vershik map is not measure-theoretically rigid but partially measure-theoretically rigid (see Definition 4.2.7) with respect to the full ergodic invariant measure. We also show that a large class of enumeration systems are partially measure-theoretically rigid, and that there exists an enumeration system which is not measure-theoretically rigid. In Section 4.6, we present a list of open problems that would be interesting to investigate further. Our main results are contained in Theorems 4.3.6, 4.3.7, 4.4.8, 4.4.6, and 4.5.5.

4.2 Preliminaries

By a Cantor dynamical system we mean a pair (X, T), where X is a Cantor set and T is a continuous surjective map. We always consider the Borel σ-algebra $\mathcal{B}$ on X and the Borel ergodic T-invariant probability measures. Throughout the paper, $\mathbb{N}, \mathbb{Z}, \mathbb{R}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ are the standard notations for the sets of numbers, and $|\cdot|$ denotes the cardinality of a set.

4.2.1 Rigidity in dynamical systems

Definition 4.2.1. A dynamical system (X, T, μ) is *measure-theoretically rigid* if there exists a sequence $t_n \to \infty$ such that

$$\mu(T^{-t_n}(A)\Delta A) \to 0 \text{ for all measurable sets } A.$$

The sequence $(t_n)_n$ is called a *rigidity sequence.*

Remark 4.2.2. This convergence is not uniform. For example, if (X, μ, T) is the dyadic odometer, then $(2^n)_{n\geq 1}$ is a rigidity sequence, but if $A_n = \{x \in X : x_{n+1} = 1\}$, then $\mu(A_n \triangle T^{2^n}(A_n)) \equiv 1$.

Remark 4.2.3. This definition of measure-theoretical rigidity is equivalent to $\mu(T^{-t_n}A \cap A) \to \mu(A)$ as $n \to \infty$ for all measurable sets A.

$$\begin{aligned}\mu(T^{-t_n}A \cap A) &= \mu(T^{-t_n}A \cup A) - \mu(T^{-t_n}A\Delta A)\\ &= \mu(T^{-t_n}A) + \mu(A) - \mu(T^{-t_n}A \cap A) - \mu(T^{-t_n}A\Delta A)\end{aligned}$$

and thus

$$\lim_{n\to\infty} 2\mu(T^{-t_n}A \cap A) = \lim_{n\to\infty} 2\mu(A) - \mu(T^{-t_n}A\Delta A) = 2\mu(A)$$

The following equivalent definition of rigidity can be found for instance in [Sil08]:

Lemma 4.2.4. *A dynamical system (X, T, μ) is measure-theoretically rigid if and only if for all measurable sets A and $\varepsilon > 0$ there exists an integer $n = n(\varepsilon) > 0$ such that*

$$\mu(T^{-n}(A)\Delta A) < \varepsilon.$$

Remark 4.2.5. Let (X, T) be a surjective dynamical system and μ be a T-invariant non-atomic probability measure on (X, T). Let F be the set of all non-invertible points of T. Assume that $\mu(F) = 0$. Then as in the case of T being a homeomorphism we can show rigidity by looking at the image of any measurable set A under T i.e.,

$$\mu(T^n(A)\Delta A) < \varepsilon.$$

Indeed, we know that

$$T^{-n}(T^n A) = \begin{cases} A \cup \bigcup_{j\le n} T^{-n+j}(x) & \text{if } T^j(x) \in T^n A \text{ for some } j \le n \text{ and } x \in F \\ A & \text{otherwise.} \end{cases}$$

Suppose $\mu(T^n A\Delta A) < \varepsilon$, then

$$\begin{aligned} \varepsilon > \mu(T^{-n}(T^n A\Delta A)) &= \mu(T^{-n}(T^n A)\Delta T^{-n}A) = \mu((A \cup \widetilde{F})\Delta T^{-n}A) \\ &= \mu(A \setminus T^{-n}A) + \mu(\widetilde{F} \setminus T^{-n}A) + \mu(T^{-n}A \setminus A) - \mu(\widetilde{F} \cap T^{-n}A \setminus A) \\ &= \mu(A\Delta T^{-n}A) \end{aligned}$$

for $\widetilde{F}$, a set of measure zero. Thus for such systems $\mu(T^{-n}A\Delta A) = \mu(T^n A\Delta A) < \varepsilon$ for all n and A, therefore (X, T) is rigid. We will see non-invertible systems in Section 4.5, where we consider enumeration systems that have one point with multiple preimages. All measure-theoretically rigid systems are invertible a.e. (see Remark 4.2.14).

Lemma 4.2.6. *If (X, d) is a compact metric space and (X, μ, T) is rigid, then for all $\varepsilon > 0$ there is an $s \in \mathbb{N}$ such that $\mu(\{x \in X : d(T^s(x), x) < \varepsilon\}) > 1 - \varepsilon$.*

Proof. Let $\varepsilon > 0$ be arbitrary. Since X is compact, there are $N \in \mathbb{N}$ and $x_i \in X$ such that $\{A_i\}_{i=1}^N := \{B(x_i; \varepsilon/2)\}_{i=1}^N$ is a cover of X. Let $s \in \mathbb{N}$ be such that $\mu(A_i \triangle T^{-s}(A_i)) < \varepsilon/N$ for all $1 \le i \le N$. Then for each $x \in A_i \cap T^{-s}(A_i)$ we have $d(x, T^s(x)) < \varepsilon$. The complement has measure $\mu(\{x \in X : d(x, T^s(x)) \ge \varepsilon\}) \le \sum_{i=1}^N \mu(A_i \triangle T^{-s}(A_i)) < N\frac{\varepsilon}{N} < \varepsilon$. □

A sufficient condition for rigidity (for ergodic measure-preserving continuous maps on first countable compact Hausdorff space) is that the ergodic measure μ has discrete spectrum. This follows by the Halmos-Von Neumann Theorem [HVN42], which says that the system is isomorphic to a minimal rotation on a compact Abelian group G with Haar measure. The conditions on X imply that G is metrizable in a way that the group rotation is an isometry, and therefore rigid, see Lemma 4.2.17

Definition 4.2.7. A dynamical system (X, T, μ) is *partially measure-theoretically rigid* if there exists a constant $\alpha > 0$ and a sequence $s_n \to \infty$ such that

$$\liminf_{n\to\infty} \mu(T^{-s_n}(A) \cap A) \ge \alpha\mu(A) \text{ for all measurable sets } A. \tag{4.1}$$

Proposition 4.2.8. *[Sil08] Let T be a finite measure-preserving transformation satisfying* (4.1) *for all sets A in a dense algebra. Then T is partially rigid along the same sequence (s_n) with the same constant α.*

Remark 4.2.9. Note that T is rigid if and only if T is partially rigid for $\alpha = 1$. Hence Proposition 4.2.8 is also true for rigid transformations.

It follows from [DMR25, Section 4] that the following proposition is true:

Proposition 4.2.10. *A finite measure-preserving transformation T is rigid if and only if there is a sequence $\alpha_n \to 1$ such that T is partially rigid with α_n for all n.*

Remark 4.2.11. In [DMR25, Remark 2.2] it is shown that if every ergodic invariant probability measure for a topological dynamical system is rigid (or partially rigid) with the same rigidity sequence $(t_n)_n$ then every invariant probability measure is rigid (or partially rigid) with the same rigidity sequence. It is also easy to see that if a topological dynamical system has countably many rigid ergodic invariant probability measures (not necessarily with the same rigidity sequence), then every invariant probability measure is partially rigid. This doesn't hold for measures for which the ergodic decomposition consists of uncountably many parts. A counterexample is the twist map $T : (x, y) \mapsto (x, x + y \pmod 1)$ defined on the cylinder $[0, 1] \times \mathbb{S}^1$ preserving Lebesgue measure, which is not partially rigid w.r.t. Lebesgue measure μ. Indeed, for every $n \geq 1$ and $\varepsilon > 0$, $\mu(\{z \in [0, 1] \times \mathbb{S}^1 : d(z, T^n(z)) < \varepsilon\}) < 3\varepsilon$. If α was the partial rigidity constant, we take $\varepsilon < \alpha/3$ and $A = (0, \varepsilon) \times \mathbb{S}^1$. Then $\mu(T^n(A) \cap A) < 3\varepsilon\mu(A) < \alpha\mu(A)$, making (4.1) impossible. This example is of course not ergodic, and hence not mixing, but it satisfies a form of mixing known as Keplerian shear, see [SB23].

Clearly, measure-theoretical rigidity, partial rigidity and the corresponding rigidity sequences are preserved under measure-theoretical conjugacy.

Lemma 4.2.12. *[Sil08] Let T be a transformation on a non-atomic probability space $(X, \mathcal{S}, \mu)$. If T is partially rigid, then T is not mixing.*

Theorem 4.2.13. *If (X, T, μ) has positive entropy, then it is not partially measure-theoretically rigid.*

Proof. Since (X, T, μ) has positive entropy, by Sinaĭ's factor theorem, it factors onto a Bernoulli shift $(S^{\mathbb{Z}}, \sigma, \nu)$ of the same entropy (see [DGS76, Theorem 12.7]). It is clear that for any α there is a cylinder set that does not satisfy condition in the definition

$$\nu(\sigma^n(A) \cap A) \geq \alpha\nu(A)$$

for any n sufficiently large. Pulling back this set for μ completes the proof. □

Remark 4.2.14. From Theorem 4.2.13 it follows that all partially measure-theoretically rigid systems have zero entropy, and by [Wal82, Corollary 4.14.3] every zero entropy probability measure preserving system on a compact metric space is invertible a.e. Thus, partially measure-theoretically rigid systems (on compact metric spaces) are invertible μ-a.e.

Remark 4.2.15. Every Cantor minimal system (X, T) has in its strong orbit equivalence class a Cantor minimal system which is non-partially measure-theoretically rigid with respect to at least one of its ergodic invariant probability measures. It follows from Theorem 4.2.13 and the result by Sugisaki [Sug07] which states that for every $\alpha \in [1, \infty)$ there is a minimal subshift of entropy $\log \alpha$ in the strong orbit equivalence class of (X, T) (see also [Dur10]).

Topological Rigidity

Definition 4.2.16. Let $T : X \to X$ be a dynamical system. We say that T is *topologically rigid* if for every $\varepsilon > 0$ there exists an $k \in \mathbb{N}$ such that $d_{\sup}(T^k, \mathrm{id}) < \varepsilon$.

In some literature topologically rigid is called uniformly rigid. Clearly, every topologically rigid system is invertible.

Lemma 4.2.17. *An isometry on a compact metric space is topologically rigid on each of its transitive components.*

Proof. Clearly isometries are invertible. Let x be arbitrary. By compactness of X, the omega-limit set $\omega(x) \neq \emptyset$. If $x \notin \omega(x)$, then $\delta := d(x, \omega(x)) > 0$. Take $n \geq 1$ such that $d(T^n(x), y) < \delta$. Then because $\omega(x)$ is backward invariant, $\delta > d(x, T^{-n}(y) \geq d(x, \omega(y))$, a contradiction. Hence, there is a sequence $n_k \to \infty$ such that $T^{n_k}(x) \to x$. Let $z \in \overline{\mathrm{orb}(x)}$, so there is $m_k \to \infty$ such that $T^{m_k}(x) \to z$. Then

$$\begin{aligned} d(T^{n_k}(z), z) &\leq d(T^{n_k}(z), T^{n_k+m_k}(x)) + d(T^{n_k+m_k}(x), T^{m_k}(x)) + d(T^{m_k}(x), z) \\ &= d(z, T^{m_k}(x)) + d(T^{n_k}(x), x) + d(T^{m_k}(x), z) \to 0 \end{aligned}$$

as $k \to \infty$. Hence $(n_k)_{k \geq 1}$ is a topological rigidity sequence for $\overline{\mathrm{orb}(x)}$. □

It follows immediately that transitive isometries (such as irrational rotations and odometers) are topologically rigid, but we cannot really weaken transitivity, because the twist map in Remark 4.2.11 is rigid on each transitive part, but Lebesgue measure, as global invariant measure, is not even partially rigid.

Recall that a measure on a topological space is called regular if every measurable set can be approximated from above by open measurable sets and from below by compact measurable sets. In particular, any Borel probability measure on a compact metric space is regular.

Proposition 4.2.18. *If T is topologically rigid, then (X, T, μ) is measure-theoretically rigid for every T-invariant measure μ.*

Proof. Since T is topologically rigid, it is invertible. Fix any Borel set A and any invariant measure μ. Let t_n be a sequence provided by topologically rigid, i.e., $\lim_{n\to\infty} d_{\sup}(f^{t_n}, \mathrm{id}) = 0$. Fix any $\varepsilon > 0$. Since μ is regular, there exists a closed set $C \subset A$ and an open set $C \subset U$ such that $\mu(A \setminus C) < \varepsilon/4$ and $\mu(U \setminus C) < \varepsilon/4$. There is $N > 0$ such that $T^{t_n}(C) \subset U$ and $T^{-t_n}(C) \subset U$ for every $n > N$. But then

$$\begin{aligned}\mu(T^{t_n}(A)\Delta A) &\leq \mu(T^{t_n}(C)\Delta C) + \mu(T^{t_n}(A \setminus C)) + \mu(A \setminus C)\\ &\leq 2\mu(U \setminus C) + 2\mu(A \setminus C) < \varepsilon\end{aligned}$$

for every $n > N$. This completes the proof. □

Definition 4.2.19. A point x of a dynamical system (X, T) is *regularly recurrent* if for every open neighborhood U of x, the set $\{n \in \mathbb{N} : T^n(x) \in U\}$ contains an infinite arithmetic progression.

The above theorem implies the following rather simple observation:

Theorem 4.2.20. *If (X, T) is a transitive and topologically rigid dynamical system and X is totally disconnected, then (X, T) is conjugate with an odometer (possibly a trivial one).*

Proof. If X has an isolated point x then it X is finite and T permutes its points, i.e., X is a single periodic orbit. Hence assume on the contrary that X does not have isolated points, hence it is a Cantor set.

Fix any clopen partition of X, say $U_1, \ldots, U_k$ and let $\varepsilon > 0$ be such that $\mathrm{dist}(U_i, U_j) > 2\varepsilon$ for all $i \neq j$. By topological rigidity, there is m such that $d(T^m(x), x) < \varepsilon$ for all $x \in X$. This in particular implies that $T^m(U_i) \subset U_i$ for any i, and therefore, for every $x \in U_i$ we have $m\mathbb{N} \subset \{n \in \mathbb{N} : T^n(x) \in U_i\}$. By the above we easily obtain that every point in x is regularly recurrent. But transitivity implies that X is an orbit closure of one of these points. The proof is completed by Theorem 4.3.1. □

Corollary 4.2.21. *If (X,T) is a topologically rigid Cantor dynamical system, then X is a disjoint union of minimal systems, each conjugate to an odometer (possibly a trivial one).*

Proof. By topological rigidity, every point $x \in X$ is recurrent. Therefore its ω-limit set defines a transitive dynamical system, so the conjugacy is obtained by Theorem 4.2.20. □

It is not hard to see that the bases of odometers in the decomposition provided Corollary 4.2.21 cannot be selected uniformly. To see this, fix a sequence of circles $S_n = \{z : |z| = r_n\}$ in the plane with diameters r_n converging to 0. Define T and $X = \sup_n C_n \cup \{0\}$ in the following way. On each S_n the map T is a rotation by angle $1/n$ and $C_n \Subset S_n$ is an arbitrarily chosen Cantor set consisting of points of period n. Then it is clear that X is a Cantor set and $T^{n!}|_{C_i} = \mathrm{id}_{C_i}$ for $i = 1, 2, \ldots, n$ showing that T is topologically rigid.

4.2.2 Rigidity and (weak) mixing

Mixing precludes partial rigidity, because by definition, for any $\alpha > 0$ and measurable set A with $\mu(A) < \alpha$, we have $\lim_{n\to\infty} \mu(A \cap T^n(A)) = \mu(A)^2 < \alpha\mu(A)$. However, rigidity is compatible with weak mixing, because then the above convergence only has to happen along sequences (n_k) of full density. The complement $\mathbb{N} \setminus \{n_k : k \in \mathbb{N}\}$ could contain the rigidity times. Rigidity together with weak mixing is the typical situation for interval exchange transformations (IETs), as shown by Ferenczi and Hubert [FH19].

It is more surprising that also topologically rigid maps can be weakly mixing. In 1977, Fathi and Herman [FH77] used fast approximation method of Anosov and Katok [AK70], to study particular diffeomorphisms of manifolds. When restricted to the torus, the residual set in the closure $\overline{\mathcal{O}(\mathbb{T}^2)}$ of maps that arise as approximations

$$\mathcal{O}(\mathbb{T}^2) = \{h \circ R_\alpha \circ h^{-1} : h \in \mathrm{Diff}^\infty(\mathbb{T}^2), \alpha \in \mathbb{T}^2\}$$

consists of weakly mixing and minimal systems, where R_α is the rotation of $\mathbb{T}^2$ by α (see [KK09] and references therein). It is also obvious that the set of topologically rigid transformations is residual in $\overline{\mathcal{O}(\mathbb{T}^2)}$. This provides examples of minimal, topologically rigid and weakly mixing diffeomorphisms of the torus. The approach from [FH77] was also motivation for Glasner and Maon [GM89] who used Baire category theorem (together with earlier results of Glasner and Weiss [GW79]) to obtain (among other examples) skew-product homeomorphism on torus of any dimension $\mathbb{T}^n$, $n \geq 2$ which is minimal, topologically rigid and weakly mixing. Such examples are not possible on the circle, however in [BCO23] the authors observed that the example of Handel [Han82] obtained by fast approximation techniques provides minimal, topologically rigid and weakly mixing homeomorphism of the pseudo-circle (a 'pathological' one-dimensional continuum). Summarizing all the above examples, we obtain the following:

Remark 4.2.22. There are examples of a connected space X of any given topological dimension $n = 1, 2, \ldots$ admitting minimal, topologically rigid and weakly mixing dynamical systems.

4.2.3 Basic definitions and facts about Bratteli diagrams

In this subsection, we present basic definitions and results about Bratteli diagrams that we will use throughout the paper. For more information on Bratteli diagrams see e.g. surveys [Dur10], [BK16],[BK20] and [Bru22, Section 5.4].

Definition 4.2.23. A *Bratteli diagram* is an infinite directed graph $B = (V, E)$ divided into disjoint union of vertex sets $V = \bigsqcup_{i \geq 0} V_i$ and edge sets $E = \bigsqcup_{i \geq 1} E_i$, where

(i) $V_0 = \{v_0\}$ is a single point;

(ii) V_i and E_i are finite sets for all i;

(iii) there exists a source map $s : E \to V$ and a target map (or range map) $t : E \to V$ such that $s(E_i) = V_{i-1}$ and $t(E_i) = V_i$ for all $i \geq 1$.

We will call the set of vertices V_i the *i-th level* of the diagram B. For readability in drawn diagrams we will show the vertices V_n in horizontal levels n and the edges E_n downward directed connecting V_{n-1} to V_n, omitting the arrows. Let

$$X_B = \{x = (x_i)_{i=1}^{\infty} : x_i \in E_i \text{ and } t(x_i) = s(x_{i+1}) \text{ for } i \geq 1\}$$

be the set of all infinite paths in B that start from v_0. The set X_B is endowed with the topology generated by cylinder sets $[\overline{e}]$, where $\overline{e} = (e_1, ..., e_n)$, $n \in \mathbb{N}$ is a finite path and $[\overline{e}] := \{x \in X_B : x_i = e_i, \ i = 1, \ldots, n\}$. With this topology, X_B is a 0-dimensional compact metric space. This topology is generated by the following metric: for $x = (x_i)$, $y = (y_i) \in X_B$, set

$$d(x, y) = \frac{1}{2^N}, \quad \text{where } N = \min\{i \in \mathbb{N} : x_i \neq y_i\}.$$

For vertices $v, w \in V$, let $E(v, w)$ denote the set of finite paths between v and w.

To define a dynamical system on the path space of a Bratteli diagram, we need to take a linear order $>$ on each set $t^{-1}(v)$ for $v \in V \setminus V_0$. This order defines a partial order on the sets of edges E_i for $i = 1, 2, \ldots$, edges e, e' are comparable if and only if $t(e) = t(e')$.

A Bratteli diagram $B = (V, E)$ together with a partial order $>$ on E is called *an ordered Bratteli diagram* $B = (V, E, >)$. We call a (finite or infinite) path $e = (e_i)$ *maximal (respectively minimal)* if every e_i has a maximal (respectively minimal) number among all elements from $t^{-1}(t(e_i))$. Denote by $X_{\max}$ and $X_{\min}$ the sets of all infinite maximal and all infinite minimal paths in X_B.

Definition 4.2.24. Let $B = (V, E, >)$ be an ordered Bratteli diagram. Given $x = (x_i)_{i=1}^{\infty} \in X_B \setminus X_{\max}$, let m be the smallest number such that x_m is not maximal. Let y_m be the successor of x_m in the set $t^{-1}(t(x_m))$. Set $\varphi_B(x) = (y_1, y_2, ..., y_{m-1}, y_m, x_{m+1}, ...)$ where $(y_0, y_1, ..., y_{m-1})$ is the unique minimal path in $E(v_0, s(y_m))$. If φ_B admits an extension to the entire path space X_B such that φ_B becomes a homeomorphism of X_B, then φ_B is called a *Vershik map*, and the system (X_B, φ_B) is called a *Bratteli-Vershik system.*

If $|X_{\min}| = |X_{\max}| = 1$, then the Vershik map can be extended to a homeomorphism of X_B by sending a unique maximal path to the unique minimal path. In the case that $|X_{\max}| > |X_{\min}| = 1$, then the Vershik map can be extended to a continuous surjection of X_B by mapping all maximal paths into the unique minimal path.

Definition 4.2.25. Let $x = (x_n)$ and $y = (y_n)$ be two paths from X_B. We say that x and y are *tail equivalent* (in symbols, $(x, y) \in \mathcal{R}$) if there exists some n such that $x_i = y_i$ for all $i \geq n$.

Let μ be a Borel non-atomic probability measure. We say that μ is $\mathcal{R}$-invariant, if $\mu([\overline{e}]) = \mu([\overline{e}'])$ for any two finite paths $\overline{e}, \overline{e}' \in E(v_0, v)$, where $v \in V_n$ is an arbitrary vertex and $n \geq 1$. Denote by $\mathcal{M}_1(\mathcal{R})$ the set of all Borel probability $\mathcal{R}$-invariant measures and by $\mathcal{M}_1(\varphi_B)$ the set of all Borel probability φ_B-invariant measures.

A homeomorphism without periodic points is called aperiodic if it has no periodic points. We say that the equivalence relation $\mathcal{R}$ is aperiodic if for every $x \in X_B$ its equivalence class is infinite.

Lemma 4.2.26. *[BKMS10] Let $B = (V, E, >)$ be an ordered Bratteli diagram which admits an aperiodic Vershik map φ_B and let the tail equivalence relation $\mathcal{R}$ be aperiodic. Then $\mathcal{M}_1(\mathcal{R}) = \mathcal{M}_1(\varphi_B)$.*

In this paper, we will consider only such Bratteli diagrams for which the tail equivalence relation and the Vershik map are aperiodic.

Let $X_n(v)$ denote the set of all paths from X_B that go through the vertex $v \in V_n$, and $h_n(v)$ denote the cardinality of the set of all finite paths between v_0 and v. We call

$X_n(v)$ the *tower* corresponding to the vertex v on level n, and $h_n(v)$ the *height* of the tower $X_n(v)$.

Definition 4.2.27. Given a Bratteli diagram B, the *n-th incidence matrix* $F_n = (f^{(n)}_{v,w})$, $n \geq 1$, is a $|V_n| \times |V_{n-1}|$ matrix such that

$$f^{(n)}_{v,w} = |\{e \in E_n : t(e) = v, s(e) = w\}| \text{ for } v \in V_n \text{ and } w \in V_{n-1}.$$

A Bratteli diagram is called *stationary* if $F_n = F$ for every $n \geq 2$. Then the matrix F is called the incidence matrix of the diagram. Unless stated otherwise, we will assume that every diagram has a *"simple hat"*, which means that there is a single edge from the vertex v_0 to each vertex $v \in V_1$.

Definition 4.2.28. Let B be a Bratteli diagram, and $n_1 = 0 < n_2 < n_3 < \ldots$ be a strictly increasing sequence of integers. The *telescoping of B with respect to* $\{n_k\}_{k=1}^{\infty}$ is the Bratteli diagram B', whose k-level vertex set V'_k is V_{n_k} and whose incidence matrices $\{F'_k\}_{k=1}^{\infty}$ are defined by

$$F'_k = F_{n_{k+1}-1} \cdots F_{n_k},$$

where $\{F_n\}_{n=1}^{\infty}$ are the incidence matrices for B.

Definition 4.2.29. We say that a Bratteli diagram $B = (V, E)$ has *finite rank* if there exists some $k \in \mathbb{N}$ such that $|V_n| \leq k$ for all $n \geq 1$. We day that a finite rank diagram B has *rank d* if d is the smallest integer such that $|V_n| = d$ for infinitely many n. A Cantor dynamical system has *topological finite rank d* if it is topologically conjugate to a Vershik map acting on a Bratteli diagram of rank d and d is the smallest such number.

Remark 4.2.30. If there is an increasing sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$ such that $|V_{n_k}| = d$ for all $k \in \mathbb{N}$ then after telescoping with respect to $\{n_k\}_{k=1}^{\infty}$ we obtain a Bratteli diagram of rank d which has exactly d vertices on each level.

A (p_n)-*odometer* is a Bratteli diagram with $V_n = \{v\}$ and p_n edges in E_n for all $n \in \mathbb{N}$. This gives a Vershik homeomorphism for any ordering of incoming edges. For more information and other definitions of odometers as adding machines or inverse limits, see [Dow05]. It was proved in [DM08], that every Cantor minimal system of finite topological rank is either an odometer or a subshift:

Theorem 4.2.31. *[DM08] Every Cantor minimal system of finite topological rank $d > 1$ is expansive.*

The following definition can be found for instance in [BKMS13]:

Definition 4.2.32. A Bratteli diagram of finite rank is of *exact finite rank* if there is a finite invariant measure μ and a constant $\delta > 0$ such that after telescoping $\mu(X_n(v)) \geq \delta$ for all $n \in \mathbb{N}$ and $v \in V_n$.

Theorem 4.2.33. *[BKMS13] Let $B = B(F_n)$ be a Bratteli diagram of finite rank. Then*

1. *if there is a constant $c > 0$ such that $\frac{m_n}{m'_n} \geq c$ for all n, where m_n and m'_n are the smallest and the largest entry of F_n respectively, then B is of exact finite rank,*
2. *if B is of exact finite rank, then B is uniquely ergodic.*

The following theorem can be found for instance in [Dan16]:

Theorem 4.2.34. *Let φ_B be a Vershik map on a Bratteli diagram of exact finite rank. Then φ_B is partially rigid with respect to the unique ergodic invariant probability measure μ on B.*

4.3 Rigidity for Toeplitz systems

A topological factor map $\pi : X \mapsto Y$ between two dynamical systems (X, T) and (Y, S) is a continuous surjective map such that $\pi \circ T = S \circ \pi$. We say it provides an *almost* 1-1 *extension* if the set of points in Y having singleton fibers is residual. For minimal systems (Y, S) it suffices to show one such singleton fiber exists.

The following is an old characterization of regularly recurrent points (e.g. see [Dow05, Theorem 5.1]):

Theorem 4.3.1. *A topological dynamical system (X, T) is a minimal almost* 1-1 *extension of an odometer (G_s, τ) if and only if it is the orbit closure of a regularly recurrent point. The set of all regularly recurrent points in (X, T) coincides with the collection of all singleton fibers, and it is a dense G_δ subset of X.*

Let ω be a regularly recurrent sequence in $\mathcal{A}^{\mathbb{Z}}$ under the left shift σ (for a finite alphabet $\mathcal{A}$). Let X be the orbit closure of ω under σ. If ω is not periodic, we call it a *Toeplitz sequence* and (X, σ) the *Toeplitz system* generated by ω.

For $p \in \mathbb{N}$ the *p-periodic part* of ω is $Per_p(\omega)$ the set of all positions $i \in \mathbb{Z}$ such that

$$\omega_i = \omega_{i+np} \text{ for all } n \in \mathbb{Z}.$$

We construct $S_p(\omega) \in (\mathcal{A} \cup \{*\})^{\mathbb{Z}}$, the *p-skeleton* of ω, by replacing all ω_i in ω by $*$ for $i \notin Per_p(\omega)$.

Then ω has *periodic structure* $p = (p_i)_i$ if p is an increasing sequence of integers with $p_0 > 1$ such that

- for every i the p_i-skeleton of ω is not periodic with any period smaller than p_i,
- $p_i \mid p_{i+1}$ for all i and
- for all $n \in \mathbb{N}$, there exists i with $n \in Per_{p_i}(\omega)$.

For more information, see [Kůr03], [Wil84].

Definition 4.3.2. Let ω be a Toeplitz sequence with periodic structure p. Set

$$q_i = |S_{p_i}(\omega)|_*$$

as the number of occurrences of $*$ in $S_{p_i}(\omega)$. The sequence ω is regular Toeplitz if

$$\delta(\omega) = \lim_{i\to\infty} \frac{q_i}{p_i} = 0.$$

Theorem 4.3.3 ([JK69]). *If ω is a regular Toeplitz sequence, then $\overline{\mathcal{O}(\omega)}$ has zero topological entropy and is strictly ergodic.*

Let D be a subset of a dynamical system (X, T, μ) such that $\mu(D) > 0$. By T_D we will denote the first return time map induced by T on D:

$$T_D(x) = T^{n(x)}(x), \qquad \text{for } n(x) = \min\{n \in \mathbb{N} : T^n(x) \in D\}.$$

The Poincaré recurrence theorem ensures that T_D is defined μ-almost everywhere on D, and it preserves conditional measure μ_D on D given by $\mu_D(B) = \mu(B)/\mu(D)$ for all Borel sets $B \subset D$. In what follows, we will associate with a Toeplitz system a special set $\underline{D}$ and return map $\sigma_{\underline{D}}$. The aim of this construction is showing that there are Toeplitz systems with zero entropy which are still not measure-theoretically rigid.

Let $\mathbf{s} = (s_m)_{m\in\mathbb{N}}$ be a sequence of positive integers such that s_m divides s_{m+1}. Denote the corresponding odometer by

$$G_{\mathbf{s}} = \varprojlim_{m} \mathbb{Z}_{s_m},$$

where the bonding maps are $f_m : \mathbb{Z}_{s_{m+1}} \to \mathbb{Z}_{s_m}$ with $f_m(x_{m+1}) = x_{m+1} \pmod{s_m}$. Denote by τ the translation by unity on $G_{\mathbf{s}}$ (i.e., addition of 1 at each coordinate). Let $(G_{\mathbf{s}}, \tau, \lambda)$ be the associated measure-theoretical dynamical system, where λ is the Haar measure on $G_{\mathbf{s}}$.

Fix a Toeplitz sequence ω and let (X_ω, σ, μ) be the associated Toeplitz system. Then (X_ω, σ) is an almost 1-1 extension of $(G_{\mathbf{s}}, \tau)$ which is its maximal equicontinuous factor (and $\mathbf{s}$ depends on X_ω, in particular on ω). Let $\pi : X_\omega \to G_\mathbf{s}$ be the associated factor map. For an odometer we have a natural map $\mathbb{Z} \to G_\mathbf{s}$, $n \mapsto \tau^n(\mathbf{1})$ where $\mathbf{1} = (1, 1, 1, \ldots)$. Then the topology on $G_\mathbf{s}$ induces by this map a natural topology on $\mathbb{Z}$ (similarly for $\mathbb{N}$). Any function $f : \mathbb{Z} \to \mathcal{A}$, where $\mathcal{A}$ is a finite set (an alphabet) with discrete topology, continuous with respect to this topology is called a semi-cocycle on $G_\mathbf{s}$. We can view $\omega : \mathbb{Z} \to \mathcal{A}, n \mapsto \omega(n)$ as a semi-cocycle (see [Dow05, Theorem 7.1]). Denote by F the graph of semi-cocycle ω in $G_\mathbf{s} \times \mathcal{A}$, where $\mathbb{Z}$ is identified with the two-sided orbit of $\mathbf{1}$ in $G_\mathbf{s}$. Denote by $D_\omega \subset G_\mathbf{s}$ the set of $\mathbf{j} \in G_\mathbf{s}$ such that the section $\{a \in \mathcal{A} : (\mathbf{j}, a) \in F\}$ is not a singleton. Note at this point that each section has always at least one point, so we can view D_ω as the set of points at which ω cannot be extended to a continuous map on $G_\mathbf{s}$. If $\lambda(D_\omega) = 0$ then (X, σ, μ) is measure-theoretically isomorphic to its maximal equicontinuous factor $(G_\mathbf{s}, \tau, \lambda)$. This is exactly the case when ω is a regular Toeplitz sequence [Mar75].

Remark 4.3.4. If a Toeplitz system (X_ω, σ) is defined by a regular Toeplitz sequence, then its unique invariant measure μ is measure-theoretically rigid (note that it is isomorphic to Haar measure of $G_\mathbf{s}$; see also comments after Lemma 4.2.6). This does not extend onto all Toeplitz systems, since there are Toeplitz systems with positive entropy, and these are not even partially measure-theoretically rigid, see Theorem 4.2.13.

In what follows, we will assume that ω is an irregular Toeplitz system, i.e., $\lambda(D_\omega) > 0$. Fix any $x \in X_\omega$ and let $\mathbf{j} = \pi(x)$. We define aperiodic part of x, denoted $\operatorname{Aper}(x) \subset \mathbb{Z}$, by putting $n \in \operatorname{Aper}(x)$ if and only if $\tau^n(\mathbf{j}) \in D_\omega$. Since ω is irregular, by the ergodic theorem there is a set $E \subset G_\mathbf{s}$ with $\lambda(E) = 1$ such that if $\mathbf{j} \in E$, then the set of $n \in \mathbb{Z}$ for which $\tau^n(\mathbf{j}) \in D_\omega$ does not have upper nor lower bound. If we write $\underline{E} = \pi^{-1}(E) \subset X_\omega$ then $\operatorname{Aper}(x)$ does not have upper nor lower bound for every $x \in \underline{E}$. For each $\mathbf{j} \in E$ we define its aperiodic readouts as

$$Y_\mathbf{j} = \{x|_{\operatorname{Aper}(x)} : \pi(x) = \mathbf{j}\}.$$

If the sets $Y_\mathbf{j}$ are the same over all $\mathbf{j} \in E$ then we say that X_ω satisfies condition (SAR) (Same Aperiodic Readouts). If we want to emphasize the unique set $Y = Y_\mathbf{j}$ then we say that (SAR) is satisfied with readouts equal to Y. The Williams' classical construction of a Toeplitz sequence satisfying (SAR) and with Y equal to an arbitrarily preset subshift is presented in [Dow05, Sec. 14].

Let (X, T) and (Y, S) be dynamical systems and let $M_T(X), M_S(Y)$ be the spaces of invariant Borel probability measures for these systems, respectively. A map $\rho : X \to Y$ is called a *Borel* conjugacy* if:

1. ρ is a Borel-measurable bijection,

2. $\rho \circ T = S \circ \rho$,

3. the map acting on measures (also denoted by ρ) defined by $\rho(\mu)(B) = \mu(\rho^{-1}(B))$ is a homeomorphism with respect to the weak* topology of measures.

The above preparation is the main step to apply the following tool:

Theorem 4.3.5. *[Dow05] Fix any subshift Y. Let ω be a Toeplitz sequence satisfying (SAR) with readouts equal to Y, let (X_ω, σ) be the associated Toeplitz system and let $(G_{\mathbf{s}}, \tau, \lambda)$ be its maximal equicontinuous factor. Denote by $D_\omega \subset G_{\mathbf{s}}$ the set of discontinuities of semi-cocycle ω. Then the Toeplitz system (X_ω, σ) is Borel* conjugate to the skew product $(G_{\mathbf{s}} \times Y, T)$, where*

$$T(\mathbf{j}, y) = \begin{cases} (\mathbf{j}+\mathbf{1}, \sigma(y)) & \text{if } \mathbf{j} \in D_\omega, \\ (\mathbf{j}+\mathbf{1}, y) & \text{if } \mathbf{j} \notin D_\omega. \end{cases} \tag{4.2}$$

Now we can prove the following theorem:

Theorem 4.3.6. *Fix any measure-theoretically strong mixing subshift (Y, σ, ν) and let $D \subset G_{\mathbf{s}}$ be a set such that $\lambda(D) > 0$, where λ is the Haar measure. Then the skew product $(G_{\mathbf{s}} \times Y, T, \lambda \times \nu)$, where*

$$T(\mathbf{j}, y) = \begin{cases} (\mathbf{j}+\mathbf{1}, \sigma(y)) & \text{if } \mathbf{j} \in D, \\ (\mathbf{j}+\mathbf{1}, y) & \text{if } \mathbf{j} \notin D \end{cases}$$

is not partially measure-theoretically rigid.

Proof. Fix any $\alpha \in (0,1]$ and fix any cylinder set $A \subset Y$ such that $\nu(A) < \alpha/4$. Fix any $\varepsilon < \nu(A)^2$ and let $N_\varepsilon > 0$ be such that $\nu(\sigma^n(A) \cap A) < \nu(A)^2 + \varepsilon$ for each $n \geq N_\varepsilon$.

For each $\mathbf{j} \in G_{\mathbf{s}}$ and $m \in \mathbb{N}$, let $r_m(\mathbf{j}) = |\{0 \leq n \leq m : \tau^n(\mathbf{j}) \in D\}|$ be the number of visits of x in D in m iterations. Denote $D_i^m = \{\mathbf{j} \in D : r_m(\mathbf{j}) = i\}$ and note that $D = \bigcup_{i=1}^m D_i^m$. Take M so large that

$$\lambda\left(\bigcup_{i=1}^{N_\varepsilon - 1} D_i^m\right) < \varepsilon\lambda(D)$$

for every $m \geq M$. Note that for each $0 \leq i \leq m$ we have

$$T^m(D_i^m \times A) = \tau^m(D_i^m) \times \sigma^i(A).$$

But then

$$\begin{aligned}\lambda \times \nu(T^m(D\times A)\cap(D\times A)) &\leq \sum_{i=1}^{m}\lambda(\tau^m(D_i^m)\cap D)\nu(\sigma^i(A)\cap A)\\ &\leq \sum_{i=1}^{N_\varepsilon-1}\lambda(D_i^m)+\sum_{i=N_\varepsilon}^{m}\lambda(D_i^m)(\nu(A)^2+\varepsilon)\\ &\leq \lambda(D)(\nu(A)^2+2\varepsilon)\leq 3\lambda(D)\nu(A)^2<\alpha(\lambda\times\nu)(D\times A).\end{aligned}$$

This implies that indeed T is not partially measure-theoretically rigid. □

In this proof, the Toeplitz shift appears as Kakutani tower over a mixing base map. Such a system would be mixing, if the return times had greatest common divisor 1, which in this case is not true. Therefore, the mixing of the Kakutani tower fails, but the mixing of the base is still powerful enough to prevent partial rigidity.

Corollary 4.3.7. *There is a uniquely ergodic Toeplitz system (X, σ) with zero entropy which is not partially measure-theoretically rigid.*

Proof. Fix any strong mixing measure μ of zero entropy. By the Jewett-Krieger theorem there exists a uniquely ergodic subshift (Y, σ) with its unique measure ν isomorphic to μ. Williams' Construction provides a Toeplitz sequence satisfying (SAR) and with aperiodic readouts equal to Y. Then by Theorem 4.3.5 we obtain a Toeplitz system (X, σ) which is Borel* conjugate to the skew product $(G_{\mathbf{s}} \times Y, T)$ given by (4.2). By Theorem 4.3.6 we obtain that $(G_{\mathbf{s}} \times Y, T, \lambda \times \nu)$ is not partially measure-theoretically rigid.

Next, let μ be any other invariant measure of T. Then induced measure $\mu_{D\times Y}$ is invariant for $T_D = \tau_D \times \sigma$. But Williams' construction ensures that τ_D is conjugate to an odometer $G_{\mathbf{s}'}$ on some scale $\mathbf{s}'$. Since (Y, σ, ν) is strongly mixing, it is disjoint from $(G_{\mathbf{s}'}, \tau, \lambda)$, and therefore $\mu_{D\times Y} = \lambda_D \times \nu$, since it is joining of λ_D and ν. But $\mu_{D\times Y} = (\lambda \times \nu)_{D\times Y}$, so we must have $\mu = \lambda \times \nu$. □

4.4 Rigidity on Bratteli diagrams

In this section we investigate how the structure of a Bratteli diagram representing a Cantor dynamical system can show rigidity. We use the diagram to control ergodic measures, specifically if the measure is a finite extension of an odometer measure. Further we use this result to study Toeplitz systems by their Bratteli-Vershik representation and show rigidity in examples with different ergodic measures.

Definition 4.4.1. An order on a Bratteli diagram is called *consecutive* if whenever we have edges e, f, g such that $e \leq f \leq g$ and $s(e) = s(g)$ then $s(f) = s(e) = s(g)$. We say that a subdiagram $\overline{B} = (\overline{W}, \overline{E})$ of $B = (V, E, >)$ is *consecutively ordered* if for any $e \leq g \in \overline{E}$ with $s(e) = s(g)$, if there exists an edge $f \in E$ with $e \leq f \leq g$ in the ordering of the full diagram B, then $s(f) = s(e) = s(g)$.

Figure 4.1 (left) illustrates the notion of a consecutive order. For more details, see e.g. [Dur10]. One of the examples of a consecutive order is a left-to-right order, when for every vertex all incoming edges are enumerated from left to right as they appear in the diagram. If B has a consecutive order, then any of its vertex subdiagrams is consecutively ordered. The stationary diagram in Figure 4.1 (right) is not consecutively ordered, but for instance the subdiagram which is a vertical odometer passing through the second vertex on each level, is consecutively ordered.

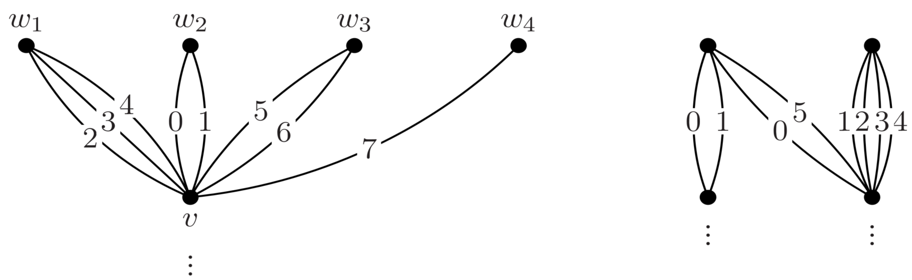


Figure 4.1: Example of a consecutive order (left) and a non-consecutively ordered diagram (right) with consecutively ordered subdiagrams

Remark 4.4.2. In general, for a Bratteli diagram of rank larger than one, the consecutive ordering is not preserved under telescoping.

A substitution $\theta : A \to (A')^+$ is called *proper* if all words $\theta(a)$ over $a \in A$ start with the same letter and end with the same letter (the starting letter and the ending letter can be different). A substitution θ is called *left proper* if only the the starting letter in each $\theta(a)$ is the same, and *right proper* if the ending letter is the same. Assume that θ is left proper and $\theta(a) = lu(a)$ for all $a \in A$. Then a substitution $\chi : A \to (A')^+$ defined by $\chi(a) = u(a)l$ is *(left) conjugate* of θ and it is right proper.

If B is an ordered Bratteli diagram, one can read a substitution $\theta_n : V_n \to V_{n-1}^+$ on B at a level $n \geq 2$ as follows. Let $v \in V_n$ and $e_1 < \ldots < e_k$ be the incoming edges to v. Then set $\theta_n(v) = w_1 \ldots w_k$, where $w_i = s(e_i)$ for $i = 1, \ldots, k$. For instance, a substitution read from Figure 4.1 is $\theta(v) = w_2 w_2 w_1 w_1 w_1 w_1 w_3 w_3 w_4$. Clearly, if diagram B is stationary then $\theta_n = \theta$ for $n \geq 2$.

In [DL12], Bratteli-Vershik representation of S-adic shifts is given. Let $(A_n)_n$ be a sequence of non-empty finite sets (alphabets) and A_n^+ be a set of all finite non-empty words over alphabet A_n. Recall that an *S-adic representation* of a subshift (X, σ) is a sequence $(\theta_n, a_n)_{n\geq 2}$, where $(\theta_n : A_n \to A_{n-1}^+)_n$ are substitutions on A_n, $a_n \in A_n$ and $X \subset A_1^{\mathbb{Z}}$ is a set of sequences $x = (x_i)$ such that all words $x_i x_{i+1} \ldots x_j$ appear in some $\theta_2\theta_3 \ldots \theta_n(a_n)$. Denote by (X_n, σ) the subshift generated by $(\theta_k, a_k)_{k\geq n}$.

Proposition 4.4.3. *[DL12] Let (X, T) be the minimal S-adic subshift defined by a sequence of substitutions $(\theta_n : A_n \to A_{n-1}^+, a_n)_n$, where all θ_n are proper. Suppose that for all n the substitutions θ_n extend by concatenation to a one-to-one map from X_n to X_{n-1}. Then (X, T) is conjugate to (X_B, φ_B), where B is the Bratteli diagram such that for all $n \geq 2$ the substitution read on B at level n is θ_n.*

Moreover, the following result holds:

Corollary 4.4.4. *[DL12] Let (X, T) be a minimal S-adic subshift defined by $(\theta_n, a_n)_{n\geq 2}$, where θ_n are left or right proper. Suppose that for all n the substitutions θ_n extend by concatenation to one-to-one maps from X_n to X_{n-1}. Then (X, T) is conjugate to (X_B, φ_B), where B is a Bratteli diagram such that for all $n \geq 2$*

(i) the substitution read on E_{2n} is left proper and equal to θ_{2n} or its conjugate;

(ii) the substitution read on E_{2n+1} is right proper and equal to θ_{2n+1} or its conjugate.

4.4.1 Measure extension from a vertex subdiagram

Let $\overline{W} = \{W_n\}_{n>0}$ be a sequence of proper, non-empty subsets $W_n \subset V_n$ and $W_0 = \{v_0\}$. The (vertex) subdiagram $\overline{B} = (\overline{W}, \overline{E})$ is a Bratteli diagram defined by the vertices $\overline{W} = \bigsqcup_{i\geq 0} W_n$ and all the edges $\overline{E}$ that have both their sources and ranges in $\overline{W}$. Consider the set $X_{\overline{B}}$ of all infinite paths whose edges belong to $\overline{B}$. Let $\widehat{X}_{\overline{B}}$ be the subset of paths in X_B that are tail equivalent to paths from $X_{\overline{B}}$. Let $\overline{\mu}$ be a probability measure on $X_{\overline{B}}$ invariant with respect to the tail equivalence relation defined on $\overline{B}$. Then $\overline{\mu}$ can be canonically extended to the measure $\widehat{\overline{\mu}}$ on the space $\widehat{X}_{\overline{B}}$ by invariance with respect to $\mathcal{R}$ (see e.g. [BKMS13], [BKK15], [ABKK17]). Specifically, for $w \in W_n$, take a finite path $\overline{e} \in \overline{E}(v_0, w)$ which lies in $\overline{B}$. For any finite path $\overline{f} \in E(v_0, w)$ from the diagram B with the same range w, we set $\widehat{\overline{\mu}}([\overline{f}]) = \overline{\mu}([\overline{e}])$. To extend $\widehat{\overline{\mu}}$ to an $\mathcal{R}$-invariant measure to the whole space X_B, set $\widehat{\overline{\mu}}(X_B \setminus \widehat{X}_{\overline{B}}) = 0$.

Set

$$\widehat{X}_{\overline{B}}^{(n)} = \{x = (x_i) \in \widehat{X}_{\overline{B}} : t(x_i) \in W_i,\ \forall i \geq n\},\ n \geq 1.$$

Then $\widehat{X}_{\overline{B}}^{(n)} \subset \widehat{X}_{\overline{B}}^{(n+1)}$ for all n and

$$\widehat{\overline{\mu}}(\widehat{X}_{\overline{B}}) = \lim_{n\to\infty} \widehat{\overline{\mu}}(\widehat{X}_{\overline{B}}^{(n)}) = 1 + \sum_{n=1}^{\infty} \widehat{\overline{\mu}}(\widehat{X}_{\overline{B}}^{(n+1)} \setminus \widehat{X}_{\overline{B}}^{(n)}). \tag{4.3}$$

Using the notation of Section 4.2.3, we can rewrite equation (4.3) as follows (see [BKK15, ABKK17]):

$$\widehat{\overline{\mu}}(\widehat{X}_{\overline{B}}) = 1 + \sum_{n=1}^{\infty} \sum_{v\in W_{n+1}} \sum_{w\in V_n\setminus W_n} f_{vw}^{(n+1)} h_n(w) \overline{\mu}(C_{n+1}(v)), \tag{4.4}$$

where $\overline{\mu}(C_{n+1}(v))$ is the $\overline{\mu}$-measure of an $n+1$-cylinder which lies inside the subdiagram $\overline{B}$ and ends in the vertex $v \in W_{n+1}$. Note that the measure extension $\widehat{\overline{\mu}}$ is finite if and only if the series in (4.3) converges. In particular, let $\overline{B}$ be an odometer, i.e., all the sets W_n are singletons. Let $W_n = \{\overline{v}\}$ (to simplify the notation in the formulas, we will use just $\overline{v}$ without the index indicating the level). It is easy to see that there is a unique invariant probability measure $\overline{\mu}$ on the odometer $\overline{B} = (\{\overline{v}\}, \overline{E})$ which is defined by its values on cylinder sets

$$\overline{\mu}(C_n(\overline{v})) = \frac{1}{\prod_{i=1}^{n} f_{\overline{v},\overline{v}}^{(i)}}. \tag{4.5}$$

Then, for the odometer, (4.4) can be rewritten in the form

$$\widehat{\overline{\mu}}(\widehat{X}_{\overline{B}}) = 1 + \sum_{n=1}^{\infty} \sum_{w\in V_n\setminus\{\overline{v}\}} \frac{f_{\overline{v},w}^{(n+1)} h_n(w)}{\prod_{i=1}^{n+1} f_{\overline{v},\overline{v}}^{(i)}}. \tag{4.6}$$

Odometers are measure-theoretically (and even topologically) rigid dynamical systems. We will see in the next subsections that the structure of a Bratteli diagram helps to preserve rigidity in the case when a measure on a Bratteli diagram is an extension of the measure defined on an odometer subdiagram according to (4.5).

In the case of stationary Bratteli diagrams the unique invariant measure can be computed directly. Let B be a stationary Bratteli diagram with the matrix A transpose to the incidence matrix F. A real number $\lambda > 1$ is called a *distinguished eigenvalue* for A if there exists a non-negative vector $\boldsymbol{x}$ such that $A\boldsymbol{x} = \lambda\boldsymbol{x}$. In [BKMS10] it was shown that all ergodic $\mathcal{R}$-invariant probability measures for X_B correspond to distinguished eigenvalues of A. Moreover, for every $n \in \mathbb{N}$ and $i \in V_n$, the measure μ corresponding to the pair $(\boldsymbol{x} = (x_i), \lambda)$ takes value

$$\mu([\overline{e}]) = \frac{x_i}{\lambda^{n-1}}$$

on a cylinder set $[\overline{e}]$ which ends at a vertex $i \in V_n$.

4.4.2 Measures supported on odometers

Proposition 4.4.5. *Let $B = (V, E)$ be a Bratteli diagram of arbitrary rank and μ be a probability invariant measure on B which is an extension of a measure on the subdiagram $\overline{B} = (\overline{W}, \overline{E})$. Then*

$$\lim_{n\to\infty} \sum_{w\in W_n} \mu(X_n(w)) = 1. \tag{4.7}$$

Proof. For every $n \in \mathbb{N}$, we have $\widehat{X}_{\overline{B}}^{(n)} \subset \bigcup_{w\in W_n} X_n(w)$. By (4.3), we obtain that

$$1 = \mu(X_B) = \mu(\widehat{X}_{\overline{B}}) = \lim_{n\to\infty} \mu\left(\widehat{X}_{\overline{B}}^{(n)}\right) \le \lim_{n\to\infty} \mu\left(\bigcup_{w\in W_n} X_n(w)\right).$$

Since the towers of level n are disjoint, we obtain (4.7). □

The next theorem is the first of two results concerning the rigidity of ergodic invariant measures supported on odometers.

Theorem 4.4.6. *Let B be an ordered Bratteli diagram with incidence matrices $F_n = (f_{v,w}^{(n)})$ and φ_B be the corresponding Vershik map. Let μ be an ergodic invariant probability measure on B. If μ is an extension of a measure on the odometer $\overline{B} = (\{\overline{v}_n\}, \overline{E})$ such that*

$$\frac{\sum_{w\in V_n\setminus\{\overline{v}_n\}} f_{\overline{v}_{n+1},w}^{(n+1)}}{f_{\overline{v}_{n+1},\overline{v}_n}^{(n+1)}} \longrightarrow 0 \quad \text{as } n\to\infty, \tag{4.8}$$

then the system (X_B, φ_B, μ) is measure-theoretically rigid with rigidity sequence $(h_n(\overline{v}_n))_n$.

Proof. Let the finite ergodic measure μ be the extension of a measure on the odometer with the vertices $\overline{v}_n$. For simplicity, we will denote by $C_n(w)$ any a cylinder set of level n ending in vertex $w \in V_n$, since all of them carry the same measure μ. Take an arbitrary cylinder set C_N ending in $v \in V_N$ and $\varepsilon > 0$. Then there exists a level $n > N$ such that

$$\sum_{w\in V_n\setminus\{\overline{v}_n\}} \mu(X_n(w)) = \sum_{w\in V_n\setminus\{\overline{v}_n\}} (F_n\cdots F_1)_{w,v_0}\mu(C_n(w)) < \varepsilon$$

and (because (4.8) implies that $f_{\overline{v}_{n+1},\overline{v}_n}^{(n+1)} \to \infty$)

$$\frac{1 + \sum_{w\in V_n\setminus\{\overline{v}_n\}} f_{\overline{v}_{n+1},w}^{(n+1)}}{f_{\overline{v}_{n+1},\overline{v}_n}^{(n+1)}} < \varepsilon.$$

Decompose C_N into cylinder sets ending in level n and observe that

$$\mu(C_N) = \sum_{w\in V_n} (F_n\cdots F_{N+1})_{w,v}\mu(C_n(w)) \le (F_n\cdots F_{N+1})_{\overline{v}_n,v}\mu(C_n(\overline{v}_n)) + \varepsilon.$$

Take one such $C_n(\overline{v}_n)$, a subcylinder of C_N ending in $\overline{v}_n \in V_n$. Under the map $T^{h_n(\overline{v}_n)}$ any path in $C_n(\overline{v}_n)$ that takes an edge in E_{n+1} which is succeeded by an edge connecting the same vertices in V_n and V_{n+1} returns to $C_n(\overline{v}_n)$. We find a lower bound by counting only edges connecting $\overline{v}_n \in V_n$ to $\overline{v}_{n+1} \in V_{n+1}$,

$$\mu(C_n(\overline{v}_n) \cap T^{h_n(\overline{v}_n)} C_n(\overline{v}_n)) \geq (f^{(n+1)}_{\overline{v}_{n+1},\overline{v}_n} - 1 - \sum_{w \in V_n \setminus \{\overline{v}_n\}} f^{(n+1)}_{\overline{v}_{n+1},w}) \mu(C_{n+1}(\overline{v}_{n+1})).$$

All paths in $C_n(\overline{v}_n)$ that use an edge in E_{n+1} that connects $\overline{v}_n \in V_n$ to $\overline{v}_{n+1} \in V_{n+1}$ and is succeeded by an edge connecting $\overline{v}_n \in V_n$ to $\overline{v}_{n+1} \in V_{n+1}$ in the incoming order to $\overline{v}_{n+1} \in V_{n+1}$ return to themselves in $C_n(\overline{v}_n)$ after $h_n(\overline{v}_n)$ steps. Any edge connecting $w \neq \overline{v}_n \in V_n$ to $\overline{v}_{n+1} \in V_{n+1}$ might interrupt such a succession and the maximal edge connecting $\overline{v}_n \in V_n$ to $\overline{v}_{n+1} \in V_{n+1}$ might also not be succeeded by such an edge, thus the lower bound for the number of edges holds. Then

$$\begin{aligned}
\mu(C_N \cap T^{h_n(\overline{v}_n)} C_N) &\geq (F_n \cdots F_{N+1})_{\overline{v}_n,v} \mu(C_n(\overline{v}_n) \cap T^{h_n(\overline{v}_n)} C_n(\overline{v}_n)) \\
&\geq (F_n \cdots F_{N+1})_{\overline{v}_n,v} f^{(n+1)}_{\overline{v}_{n+1},\overline{v}_n} \mu(C_{n+1}(\overline{v}_{n+1})) \left(1 - \frac{1 + \sum_{w \in V_n \setminus \{\overline{v}_n\}} f^{(n+1)}_{\overline{v}_{n+1},w}}{f^{(n+1)}_{\overline{v}_{n+1},\overline{v}_n}}\right) \\
&\geq (F_n \cdots F_{N+1})_{\overline{v}_n,v} f^{(n+1)}_{\overline{v}_{n+1},\overline{v}_n} \mu(C_{n+1}(\overline{v}_{n+1}))(1 - \varepsilon) \qquad (4.9) \\
&\geq (1-\varepsilon)(\mu(C_N(v)) - \sum_{w' \in V_{n+1} \setminus \{\overline{v}_{n+1}\}} (F_{n+1} \cdots F_{N+1})_{w',v} \mu(C_{n+1}(w')) \\
&\quad - \sum_{w \in V_n \setminus \{\overline{v}_n\}} (F_n \cdots F_{N+1})_{w,v} f^{(n+1)}_{\overline{v}_{n+1},w} \mu(C_{n+1}(\overline{v}_{n+1}))) \\
&\geq (1-\varepsilon)(\mu(C_N(v)) - 2\varepsilon).
\end{aligned}$$

This proves the result for cylinder sets. By using Proposition 4.2.8 for the algebra generated by cylinder sets, we get the full result. □

Remark 4.4.7. This theorem requires no information about the order of the edges, since in its proof the estimate (4.9) used the worst possible order where every edge incoming to $\overline{v}_{n+1}$, $n > 1$, from another vertex destroys the rigidity of an edge incoming to $\overline{v}_{n+1}$ from $\overline{v}_n$. If we have more information about the ordering, the extra condition of

$$\frac{\sum_{w \in V_n \setminus \{\overline{v}_n\}} f^{(n+1)}_{\overline{v}_{n+1},w}}{f^{(n+1)}_{\overline{v}_{n+1},\overline{v}_n}} \to 0$$

can be weakened or is trivially satisfied, see Corollary 4.4.15.

Theorem 4.4.8. *Let B be an ordered Bratteli diagram (of arbitrary rank) and φ_B be the corresponding Vershik map. Let μ be an ergodic invariant probability measure on*

B. Assume that μ is an extension of a measure on the consecutively ordered odometer $\overline{B} = (\{\overline{v}_n\}, \overline{E})$ such that

$$\limsup_{n\to\infty} f^{(n)}_{\overline{v}_n,\overline{v}_{n-1}} = \infty. \tag{4.10}$$

Then the system (X_B, φ_B, μ) is measure-theoretically rigid. The rigidity sequence is a subsequence of the heights $(h_n(\overline{v}_n))_n$.

Proof. Let $a_n = f^{(n)}_{\overline{v}_n,\overline{v}_{n-1}}$. Since μ is an extension of a measure on the odometer which passes through the vertex $\overline{v}_n$ on each level n of the diagram, the support of μ consists of all infinite paths that eventually passes through the vertices $\{\overline{v}_n\}$. Hence, by Proposition 4.4.5, we have

$$\lim_{n\to\infty} \mu(X_n(\overline{v}_n)) = 1 \qquad \text{and} \qquad \lim_{n\to\infty} \mu\left(\bigsqcup_{w\neq\overline{v}_n} X_n(w)\right) = 0.$$

Thus, for any cylinder set $C_N(w)$ which ends at vertex w on level N we have

$$\begin{aligned}\mu(C_N(w)) &= \lim_{n\to\infty} \mu(C_N(w) \cap X_n(\overline{v}_n) \cap X_{n+1}(\overline{v}_{n+1})) \\ &= \lim_{n\to\infty} (F_n \cdots F_{N+1})_{\overline{v}_n,w} a_{n+1} \mu(C_{n+1}(\overline{v}_{n+1})).\end{aligned}$$

Since the order on the odometer $\overline{B}$ is consecutive and $\limsup_{n\to\infty} a_n = \infty$, there exists an increasing sequence n_k such that $\lim_{k\to\infty} a_{n_k} = \infty$, and we have

$$\begin{aligned}\mu(T^{h_{n_k}(\overline{v}_{n_k})} C_N(w) \cap C_N(w)) &\geq (a_{n_k+1} - 1)(F_{n_k} \cdots F_{N+1})_{\overline{v}_{n_k},w} \mu(C_{n_k+1}(\overline{v}_{n_k+1})) \\ &= \frac{a_{n_k+1} - 1}{a_{n_k+1}} (F_{n_k} \cdots F_{N+1})_{\overline{v}_{n_k},w} a_{n_k+1} \mu(C_{n_k+1}(\overline{v}_{n_k+1})) \\ &\to \mu(C_N(w))\end{aligned}$$

as $k \to \infty$. Thus, we proved measure-theoretical rigidity for all cylinder sets and a sequence $t_k = h_{n_k}$, hence the system (X_B, φ_B, μ) is measure-theoretically rigid. □

Remark 4.4.9. In order for the measure extension of a measure defined on an odometer $\overline{B}$ to be finite, it is not necessary that (4.10) holds. For example, for reducible stationary Bratteli diagrams, the measure extension from the stationary odometer $\overline{B}$ can be finite (see [BKMS10] and Theorem 4.5.1). It is also not a sufficient condition for a measure extension from $\overline{B}$ to be finite, since the number of edges may be growing not fast enough to get the finite measure extension (see [BKMS13], [ABKK17] and Proposition 4.4.19).

Corollary 4.4.10. *Let (X_B, φ_B, μ) be as in Theorem 4.4.8. Then (X_B, φ_B, μ) has zero entropy.*

Recall that two Cantor minimal systems (X,T) and (Y,S) are called *Kakutani equivalent* if there exist clopen sets $U \subset X$ and $V \subset Y$ such that the induced systems (U,T_U) and (V,S_V) are topologically conjugate (see e.g. [Dur10]). It was proved in [HPS92] that a Bratteli-Vershik dynamical system (X_B,φ_B) associated with a simple properly ordered Bratteli diagram $(B,\geq)$ is Kakutani equivalent to a Cantor minimal system (Y,S) if and only if (Y,S) is topologically conjugate to a Bratteli-Vershik system $(X_{B'},\varphi_{B'})$, where $(B',\geq')$ and $(B,\geq)$ differ only on finite initial portions (see also [DP22]).

Corollary 4.4.11. *Let (X_B,φ_B,μ) be as in Theorem 4.4.8 and additionally let φ_B be minimal and B be properly ordered. Let (Y,T) be any Cantor minimal system which is Kakutani equivalent to (X_B,φ_B). Then there is an ergodic invariant probability measure ν for (Y,T) such that ν is an extension of a measure on an odometer and (Y,T,ν) is measure-theoretically rigid.*

Proof. Since the systems (Y,T) and (X_B,φ_B) are Kakutani equivalent, the system (Y,T) has a Bratteli-Vershik representation $(X_{B'},\varphi_{B'})$ with the diagram B' differing from B only on finitely many initial levels. As μ is the extension of a measure on an odometer in B, there is an ergodic invariant probability measure for $(X_{B'},\varphi_{B'})$ which is an extension of a measure defined on an odometer in B' which satisfies condition (4.10). Therefore, the system (Y,T) has an ergodic invariant measure ν such that the system (Y,T,ν) is rigid. $\square$

Below we give an example illustrating Corollary 4.4.11.

Example 4.4.12. *Let B be a Bratteli diagram with incidence matrices*

$$F_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad F_n = \begin{pmatrix} n^2 & 1 \\ 1 & n^2 \end{pmatrix},\ n \geq 2.$$

Endow B with the left-to-right ordering. Then B is a simple properly ordered Bratteli diagram, and the corresponding Vershik map φ_B is a minimal homeomorphism. It follows from Proposition 4.4.19, that there are two ergodic invariant probability measures μ_1 and μ_2 on B, where μ_1 is the extension of a measure on the consecutively ordered vertical odometer corresponding to the first vertex on each level, and μ_2 is the extension of a measure on the consecutively ordered vertical odometer corresponding to the second vertex on each level. Then, by Theorem 4.4.8, both systems (X_B,φ_B,μ_1) and (X_B,φ_B,μ_2) are measure-theoretically rigid. Let $m > 1$ be any natural number, and $\widetilde{B}$ be a Bratteli diagram with incidence matrices

$$F_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad F_n = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\ 2 \leq n \leq m, \quad \text{and} \quad F_n = \begin{pmatrix} n^2 & 1 \\ 1 & n^2 \end{pmatrix},\ n > m.$$

Endow $\widetilde{B}$ with the left-to-right order. Then the corresponding Vershik map $\varphi_{\widetilde{B}}$ is a minimal homeomorphism and the systems (X_B, φ_B) and $(X_{\widetilde{B}}, \varphi_{\widetilde{B}})$ are Kakutani equivalent. It follows from Corollary 4.4.11 that the system $(X_{\widetilde{B}}, \varphi_{\widetilde{B}})$ possesses two ergodic invariant probability measures $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ such that each $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ is an extension of a measure on an odometer and the systems $(X_{\widetilde{B}}, \varphi_{\widetilde{B}}, \widetilde{\mu}_1)$, $(X_{\widetilde{B}}, \varphi_{\widetilde{B}}, \widetilde{\mu}_2)$ are measure-theoretically rigid.

4.4.3 Toeplitz systems as Bratteli-Vershik systems

Below we introduce some classes of Bratteli diagrams (for more details, see e.g. [ABKK17]).

Definition 4.4.13. A non-negative integer matrix $F = (f_{ij})$ satisfies the

- *equal row sum* property (denoted $F \in ERS$ or $F \in ERS(r)$) if there is $r \in \mathbb{N}$ such that
$$\sum_j f_{ij} = r \qquad \text{for all } i.$$
- *equal column sum* property (denoted $F \in ECS$ or $F \in ECS(c)$) if there is $c \in \mathbb{N}$ such that
$$\sum_i f_{ij} = c \qquad \text{for all } j.$$

Recall that $h_n(w)$ is the number of paths between v_0 and $w \in V_n$. If $F_n \in ERS(r_n)$ for all n, then it is easy to check by induction that

$$h_n(w) = r_1 \cdots r_n$$

for every n and every $w \in V_n$. If $F_n \in ECS(c_n)$ for all n, then there is an invariant probability measure on X_B such that the measures of cylinder sets $C_n(w)$ of length n which end at a vertex $w \in V_n$ are

$$\mu(C_n(w)) = \frac{1}{c_1 \cdots c_n}.$$

In [GJ00], the ERS property for incidence matrices $(F_n)_n$ is called the *equal path number property.* The following theorem holds:

Theorem 4.4.14. *[GJ00] The family of expansive Bratteli-Vershik systems associated to simple properly ordered Bratteli diagrams with the equal path number property coincides with the family of Toeplitz systems up to conjugacy.*

For ERS systems we can use 4.4.6 to show the following.

Corollary 4.4.15. *Let B be an ordered Bratteli diagram with incidence matrices $F_n = (f^{(n)}_{v,w})$ which satisfies the $ERS(r_n)$ property. If the ergodic invariant probability measure μ is an extension of a measure on the odometer $\overline{B} = (\{\overline{v}_n\}, \overline{E})$, then the system (X_B, φ_B, μ) is measure-theoretically rigid with sequence $(h_n)_n$.*

Proof. By [ABKK17, Theorem 2.1] we know that the extension of a measure on the subdiagram $\overline{B} = (\{\overline{v}_n\}, \overline{E})$ is finite if and only if

$$\sum_{n=1}^{\infty} \sum_{w \in V_n \setminus \{\overline{v}_n\}} \frac{f^{(n+1)}_{\overline{v}_{n+1},w} h_n(\overline{v}_n)}{\prod_{i=1}^{n+1} f^{(i)}_{\overline{v}_i,\overline{v}_{i-1}}} < \infty,$$

see Subsection 4.4.1.

Thus condition (4.8) is always satisfied as $\prod_{i=1}^{n} f^{(i)}_{\overline{v}_i,\overline{v}_{i-1}} \leq \prod_{j=1}^{n} r_j = h_n(\overline{v}_n)$ and

$$\frac{\sum_{w \in V_n \setminus \{\overline{v}_n\}} f^{(n+1)}_{\overline{v}_{n+1},w}}{f^{(n+1)}_{\overline{v}_{n+1},\overline{v}_n}} \leq \frac{\prod_{j=1}^{n} r_j \sum_{w \in V_n \setminus \{\overline{v}_n\}} f^{(n+1)}_{\overline{v}_{n+1},w}}{\prod_{i=1}^{n+1} f^{(i)}_{\overline{v}_i,\overline{v}_{i-1}}} \longrightarrow 0 \quad \text{as } n \to \infty.$$

Therefore the system is rigid by Theorem 4.4.6. □

In [Wil84] the following condition for regularity of Toeplitz systems is proven for subshifts, we have adapted it into the Bratteli-Vershik setting.

Proposition 4.4.16. *Let (X, T) be a Toeplitz system of finite rank. Let $B = (V, E)$ be a Bratteli-Vershik representation of (X, T) with substitution read $(\theta_n)_n$ where all the subsitutions are proper, primitive and of constant lengths $r_n > 2$. If*

$$\sum_{n=1}^{\infty} \frac{1}{r_n} \textit{ diverges,}$$

then the Toeplitz system is regular.

Remark 4.4.17. In [Kůr03, Section 4.6.1], the algorithm to go from a constant length, proper and primitive substitution to its Toeplitz sequence ω is explained.

Proof. We define $p_n = \prod_{i=1}^{n} r_i$ as the length of substitution words in $\theta_1 \circ \cdots \circ \theta_n$ and compute the density up to level n as

$$\delta_n = \frac{\delta_{n-1} p_{n-1}(r_n - 2)}{p_n} = \delta_{n-1} \frac{r_n - 2}{r_n}.$$

Therefore the density of unknown symbols is

$$\delta(\omega) = \lim_{n \to \infty} \prod_{i=1}^{n} \left(1 - \frac{2}{r_i}\right).$$

Thus the density converges to zero if and only if $\sum_{i=1}^{\infty} \frac{2}{r_i} = \infty$. □

In case that $\sum_{n=1}^{\infty} \frac{1}{r_n} < \infty$, there might be another Bratteli-Vershik representation such that the sum diverges. By telescoping an existing Bratteli-Vershik system we can always find a representation such that the sum converges. Thus, Proposition 4.4.16 provides only a sufficient but not necessary condition for a Toeplitz system to be regular.

Remark 4.4.18. From this follows that any Toeplitz system generated by finitely many proper substitutions (in a Bratteli-Vershik context this property is called linearly recurrent) is regular.

Proposition 4.4.19 describes ergodic invariant measures and their supports for Bratteli diagrams with 2×2 incidence matrices satisfying the ERS property (see also [FFT09]).

Proposition 4.4.19. *[ABKK17] Let B be a Bratteli diagram with 2×2 incidence matrices $F_n = (f_{v,w}^{(n)})$ satisfying ERS:*

$$F_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad F_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}, \quad \text{where } a_n + b_n = c_n + d_n = r_n \text{ for every } n \geq 2. \tag{4.11}$$

(i) Let $\overline{B} = (\{\overline{v}_n\}, \overline{E})$ be an odometer which is a vertex subdiagram of B generated by vertices $\overline{v}_n \in V_n$. The extension of the unique invariant measure $\overline{\mu}$ on the odometer $\overline{B}(\overline{v}_n)$ to the path space X_B is finite if and only if

$$\sum_{n=0}^{\infty} \frac{f_{\overline{v}_{n+1},v_n}^{(n+1)}}{r_{n+1}} < \infty, \tag{4.12}$$

where $\{v_n\} = V_n \setminus \{\overline{v}_n\}$.

(ii) There are exactly two finite ergodic invariant measures on B if and only if

$$\sum_{k=1}^{\infty} \frac{\min\{b_k, d_k\}}{r_k} < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{\min\{a_k, c_k\}}{r_k} < \infty. \tag{4.13}$$

In this case, these measures are supported on odometers that satisfy condition (4.12)*.*

(iii) If

$$\sum_{k=1}^{\infty} \frac{\min\{a_k, b_k, c_k, d_k\}}{r_k} = \infty, \tag{4.14}$$

then there is no odometer such that the unique measure μ would be the extension of a measure supported by this odometer.

Remark 4.4.20. Let B be as in (4.11). Assume that B has an ergodic invariant probability measure which is an extension of a measure on an odometer. After changing the numeration of the vertices, we may always assume that the odometer passes through the first vertex of each level. From (4.12) it follows that

$$\sum_{n=0}^{\infty} \frac{b_n}{r_n} < \infty.$$

Assume that B is simple, hence $b_n \neq 0$ for infinitely many n. Thus, we have

$$\limsup_{n\to\infty} r_n = \infty$$

and hence

$$\limsup_{n\to\infty} a_n = \infty.$$

We let B be as in (4.11) with a simple hat and investigate its rigidity behaviors. By Theorems 4.2.31, 4.4.14 and 4.4.3 if the diagram B is properly ordered then it can model either an odometer (if $a_n = b_n$ and the same order for incoming edges in both vertices) or a Toeplitz system (S-adic subshift).

Generally there are three possible situations for the ergodic measures. Either there are two ergodic measures, both finite extensions from odometers or there is a unique invariant measure which is either a finite extensions from an odometer or it is of exact finite rank, see Proposition 4.4.19. If the ergodic measures are finite extensions from odometers, Proposition 4.4.19 (4.12) shows that the conditions of Theorem 4.4.6 are satisfied and thus the measures are rigid. In the case of a unique invariant measure with exact finite rank we need further assumptions such as regularity by $\sum 1/r_n < \infty$ or $a_n = d_n$ to achieve rigidity. For example let $F_n = F$ be stationary with $a, b, c, d \in \mathbb{N}$ such that $a + b = c + d = r$, then the system is uniquely ergodic and by Proposition 4.4.16 the Toeplitz system is regular. Furthermore by Remark 4.3.4 its unique invariant measure is rigid. Another exact finite rank example that is irregular is shown later in Example 4.4.22.

In the case of a 2×2 incidence matrix additionally satisfying the ECS property we have the following result.

Theorem 4.4.21. *Let B be a non-stationary Bratteli diagram with incidence matrices*

$$F_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad F_n = \begin{pmatrix} a_n & b_n \\ b_n & a_n \end{pmatrix}, \quad \text{for} \quad n \geq 2.$$

such that $a_n + b_n \to \infty$ as $n \to \infty$. Write $r_n = c_n = a_n + b_n$ for $n \geq 2$ and $c_1 = 2$, $r_1 = 1$. Then for all $n \geq 1$ both $F_n \in ERS(r_n)$ and $F_n \in ECS(c_n)$. In particular, there exists a probability invariant measure μ (not necessarily ergodic) on B such that

$$\mu(C_n) = \frac{1}{c_1 \cdots c_n} \text{ for a cylinder set of length } n.$$

Endow B with a left-to-right ordering and let φ_B be a corresponding Vershik map. Then the system (X_B, φ_B, μ) is measure-theoretically rigid.

Proof. We take an arbitrary cylinder set C_N that ends in a vertex $w \in V_N$, then for any $n > N$ we decompose it into cylinders $C_n(v)$ of length $n > N$. There are $(F_{n-1} \cdots F_N)_{v,w}$ cylinders $C_n(v)$ ending in a vertex $v \in V_n$. Now we take all paths in $C_n(v)$ such that the edge x_{n+1} to $\tilde{v} \in V_{n+1}$ is not the last in the block of edges connecting v to $\tilde{v}$. These paths will return to the cylinder $C_n(v)$ after $h_n(v)$ steps. Thus by the special structure of F_n we get

$$\begin{aligned}
\mu(T^{h_n}(C_N) \cap C_N) &\geq (a_{n+1} + b_{n+1} - 2)((F_n \cdots F_{N+1})_{1,w} + (F_n \cdots F_{N+1})_{2,w})\mu(C_{n+1}) \\
&= (a_{n+1} + b_{n+1} - 2)\frac{(a_{N+1} + b_{N+1}) \cdots (a_n + b_n)}{2(a_2 + b_2) \cdots (a_{n+1} + b_{n+1})} \\
&= \frac{a_{n+1} + b_{n+1} - 2}{2(a_2 + b_2) \cdots (a_N + b_N)(a_{n+1} + b_{n+1})} \\
&= \frac{a_{n+1} + b_{n+1} - 2}{a_{n+1} + b_{n+1}}\mu(C_N).
\end{aligned}$$

Indeed, the equality above follows from the fact that

$$\mu(C_{n+1}) = \frac{1}{c_1 \cdots c_{n+1}} = \frac{1}{2(a_2 + b_2) \cdots (a_{n+1} + b_{n+1})}$$

and

$$\sum_{v=1}^{2} (F_n \cdots F_{N+1})_{v,w} = (a_{N+1} + b_{N+1}) \cdots (a_n + b_n).$$

Since $\frac{a_n + b_n - 2}{a_n + b_n} \to 1$ as $n \to \infty$, we obtain

$$\lim_{n \to \infty} \mu(T^{h_n}(C_N) \cap C_N) = \mu(C_N)$$

for all cylinder sets C_N. Thus, the system (X_B, φ_B, μ) is measure-theoretically rigid. □

If by Proposition 3.1 in [ABKK17] diagram B has a unique ergodic invariant probability measure, then the measure defined in the Theorem 4.4.21 is it. Thus in the case of B with exact finite rank, the system (X_B, φ_B, μ) is rigid.

Example 4.4.22 (Rigid Toeplitz system which is not measure-theoretical isomorphic to any odometer)**.** *As in Example 6 from [ADE24] we define an S-adic subshift (X, T) on the alphabet for $\mathcal{A} = \{1, 2\}$ with substitutions*

$$\theta_n : \begin{cases} 1 \mapsto (121)^{s(n)}2, \\ 2 \mapsto 1(121)^{s(n)} \end{cases}$$

and $3s(n)+1 = 5^{2n}$. *These substitutions are primitive and of constant length. By [ADE24] the generated subshift is a Toeplitz system with finite topological rank (thus zero entropy) that does not have a discrete spectrum and is not measure-theoretical isomorphic to its maximal equicontinuous factor, the odometer* $(\mathbb{Z}_5, +1)$, *or any other odometer.*

We can represent this subshift as a Bratteli-Vershik system. As the substitutions θ_n *are not proper the Bratteli diagram has different substitution reads on even and odd levels, see [DL12]*

$$\theta_{2n} : \begin{cases} 1 \mapsto (121)^{s(2n)}2, \\ 2 \mapsto 1(121)^{s(2n)} \end{cases}$$

and

$$\theta'_{2n+1} : \begin{cases} 1 \mapsto 12(121)^{s(2n+1)-1}21, \\ 2 \mapsto (121)^{s(2n+1)}1. \end{cases}$$

The incidence matrices are

$$F_n = \begin{pmatrix} 2s(n) & s(n)+1 \\ 2s(n)+1 & s(n) \end{pmatrix}$$

with ERS for all $n \in \mathbb{N}$, *thus* $h_n(1) = h_n(2) = h_n$. *The diagram is of exact finite rank and thus has a unique ergodic probability measure* μ *by Theorem 4.2.33. It follows from Theorem 4.2.34 that the system is partially rigid. To show measure-theoretical rigidity we prove that* $(3h_m)_m$ *is a rigidity sequence.*

Take any level $m \in \mathbb{N}$ *and cylinder set* $C_m(1)$, *decomposing this into sets of level* $m+1$ *gives*

$$\mu(C_m(1)) = 2s(m+1)\mu(C_{m+1}(1)) + (2s(m+1)+1)\mu(C_{m+1}(2)).$$

By the many repetitions of the letter 1 *in* $\theta_n(1)$ *(or* $\theta'_n(1)$*) in every third position, we see that all but at most 4 cylinder subsets* $C_{m+1}(1) \subseteq C_m(1)$ *return to* $C_m(1)$

$$T^{3h_m}C_{m+1}(1) \subseteq T^{3h_m}C_m(1) \cap C_m(1).$$

Similarly all but at most 2 cylinder subsets $C_{m+1}(2) \subseteq C_m(1)$ *return to* $C_m(1)$ *after* $3h_m$*-steps. For cylinder sets* $C_m(2)$ *all but at most 2 cylinder subsets* $C_{m+1}(w) \subseteq C_m(2)$ *return after* $3h_m$*-steps.*

Taking an arbitrary cylinder set $C_N(v)$ *of length* N *ending in any vertex* $v \in V_N$ *and* $\varepsilon > 0$ *we see that by Lemma 4.4.23 there exists a level* $m > N$ *such that* $10h_m\mu(C_{m+1}(w)) <$

ε for all $w \in V_{m+1}$. We decompose $C_N(v)$ into sets of length m and then

$$\begin{aligned}\mu(T^{3h_m}C_N(v)\cap C_N(v)) &\geq (F_m\cdots F_{N+1})_{1,v}\mu(T^{3h_m}C_m(1)\cap C_m(1))\\ &\quad + (F_m\cdots F_{N+1})_{2,v}\mu(T^{3h_m}C_m(2)\cap C_m(2))\\ &\geq (F_m\cdots F_{N+1})_{1,v}(\mu(C_m(1)) - 4\mu(C_{m+1}(1)) - 2\mu(C_{m+1}(2)))\\ &\quad + (F_m\cdots F_{N+1})_{2,v}(\mu(C_m(2)) - 2\mu(C_{m+1}(1)) - 2\mu(C_{m+1}(2)))\\ &\geq \mu(C_N(v)) - 10h_m\mu(C_{m+1}(w))\\ &> \mu(C_N(v)) - \varepsilon.\end{aligned}$$

Thus the system is rigid with rigidity sequence $(3h_m)_m$.

Lemma 4.4.23. *Let B be a Bratteli diagram of exact finite rank and equal row sum such that $r_n \to \infty$. Then there exists a subsequence $(n_k)_k$ such that*

$$\lim_{k\to\infty} h_{n_k-1}\mu(C_{n_k}(v)) = 0$$

for all cylinder sets $C_{n_k}(v)$ ending in vertices $v \in V_{n_k}$.

Proof. We know by the equal row sum that $h_n = h_n(v) = \prod_{i=1}^n r_i$ for all $n \in \mathbb{N}$ and $v \in V_n$. From Proposition 5.6.(2) in [BKMS13] it follows that there exists a subsequence $(n_k)_k$ such that for all $v \in V_{n_k}$ there is a constant $c_v > 0$ such that

$$\lim_{k\to\infty} c_v\mu(C_{n_k}(v))\prod_{i=1}^{n_k} r_i = 1.$$

Therefore

$$\lim_{k\to\infty} c_v h_{n_k-1}\mu(C_{n_k}(v))r_{n_k} = 1$$

and since $r_{n_k} \to \infty$, we have

$$\lim_{k\to\infty} h_{n_k-1}\mu(C_{n_k}(v)) = 0.$$

The proof is complete. □

Theorem 4.4.24. *Let (X, T) be a Toeplitz system with ergodic measure μ such that the following properties hold. Let B be a Bratteli diagram of finite rank representing (X, T) with incidence matrices $F_n = (f_{v,w}^{(n)}) \in ERS(r_n)$ and let θ_n be the substitution read for level n.*

For every level $n \in \mathbb{N}$ and $M_n \in \mathbb{N}$ let $\rho_{w,v}^{(n)}$ be the number of indices $i \in \{1, \ldots, r_n - M_n\}$ such that

$$v = \theta_n(w)_i = \theta_n(w)_{i+M_n} \text{ for } v \in V_{n-1}, w \in V_n.$$

If there exists a subsequence $(n_k)_k$ such that

$$\max_{w'\in V_{n_k+1},\ w\in V_{n_k}} h_{n_k}(f^{(n_k+1)}_{w',w} - \rho^{(n_k+1)}_{w',w})\mu(C_{n_k+1}(w')) \to 0 \text{ as } k \to \infty.$$

Then the system (X,T) is rigid with sequence $(M_{n_k}h_{n_k})_k$.

Proof. Take arbitrary cylinder set $C_N(v)$, $\varepsilon > 0$ and $m > N$ such that

$$\sum_{w\in V_m} h_m \sum_{w'\in V_{m+1}} (f^{(m+1)}_{w',w} - \rho^{(m+1)}_{w',w})\mu(C_{m+1}(w')) < \varepsilon.$$

We decompose $C_N(v)$ into subsets of length m

$$\mu(C_N(v)) = \sum_{w\in V_m} (F_m \cdots F_{N+1})_{v,w}\mu(C_m(w))$$

and then

$$\begin{aligned}\mu(C_N(v)\cap T^{M_m h_m}C_N(v)) &\geq \sum_{w\in V_m} (F_m \cdots F_{N+1})_{v,w}\mu(C_m(w)\cap T^{M_m h_m}C_m(w))\\ &\geq \sum_{w\in V_m} (F_m \cdots F_{N+1})_{v,w} \sum_{w'\in V_{m+1}} \rho^{(m+1)}_{w',w}\mu(C_{m+1}(w'))\\ &\geq \mu(C_N(v)) - \sum_{w\in V_m} h_m \sum_{w'\in V_{m+1}} (f^{(m+1)}_{w',w} - \rho^{(m+1)}_{w',w})\mu(C_{m+1}(w'))\\ &\geq \mu(C_N(v)) - \varepsilon.\end{aligned}$$

Thus the system (X,T) is rigid with sequence $(M_{n_k}h_{n_k})_k$. □

In Corollary 4.4.25 below, we use the notation from Theorem 4.4.24.

Corollary 4.4.25. *If Bratteli-Vershik representation of a Toeplitz system is of exact finite rank with $r_n \to \infty$ and*

$$\sup_{k\in\mathbb{N}} \max_{w'\in V_{n_k+1},\ w\in V_{n_k}} f^{(n_k+1)}_{w',w} - \rho^{(n_k+1)}_{w',w} < \infty,$$

then the system is rigid.

Proof. Under these assumptions and by Lemma 4.4.23 the conditions of Theorem 4.4.24 are satisfied. □

4.4.4 Toeplitz system with countably infinitely many rigid ergodic measures

As an application of Theorem 4.4.8, we get Proposition 6.8 in [DMR25] which states that for every $r \geq 1$ there is an S-adic subshift such that the number of its ergodic invariant probability measures is r and every ergodic measure is rigid for the same rigidity sequence. Moreover, the following example provides us with a minimal S-adic subshift of zero entropy which is Toeplitz and has countably infinitely many ergodic invariant probability measures which are rigid for the same rigidity sequence.

Example 4.4.26. *We present a class of Bratteli diagrams with countably infinite set of ergodic invariant probability measures. It is a slight modification of a class of diagrams presented in Subsection 6.3 of [BKK19]. To construct such diagrams, we let $V_n = \{0, 1, \ldots, n\}$ for $n = 0, 1, \ldots$, and let $\{a_n\}_{n=0}^{\infty}$ be a sequence of natural numbers such that*

$$\sum_{n=0}^{\infty} \frac{n}{a_n} < \infty. \tag{4.15}$$

To define order of all incoming edges to vertices $w \in V_n$, we use the following substitution read θ_n for all level n

$$\begin{cases} \theta_n(w) = 01\ldots(w-1)w^{a_n}(w+1)\ldots(n-2)(n-1) & \text{for all } w \in V_n \text{ with } w < n, \\ \theta_n(n) = 01\ldots(n-3)(n-2)^2(n-1)^{a_n-1} & \text{for } n \in V_n. \end{cases} \tag{4.16}$$

The incidence matrices $\widetilde{F}_n$ of B have the following form

$$\widetilde{F}_n = \begin{pmatrix} a_n & 1 & \ldots & 1 & 1 \\ 1 & a_n & \ldots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \ldots & a_n & 1 \\ 1 & 1 & \ldots & 1 & a_n \\ 1 & 1 & \ldots & 2 & a_n - 1 \end{pmatrix}$$

for $n = 1, 2, \ldots$

We observe that the Bratteli diagram defined above admits a natural order generating the Bratteli-Vershik homeomorphism. For instance, we can use the left-to-right order which is consecutive (see Definition 4.4.1). Then the minimal edge is always an edge between w and the vertex $0 \in V_n$ and the maximal edge is an edge between w and the vertex $n \in V_n$. It is easy to see that X_B has a unique infinite minimal path passing through the vertices $0 \in V_n$, $n \geq 0$ and a unique infinite maximal path passing through the vertices $n \in V_n$, $n \geq 0$. Thus, a Vershik map $\varphi_B : X_B \to X_B$ exists and it is minimal. Figure 4.2 depicts Bratteli diagram defined by matrix $\widetilde{F}_n$. It is known that all minimal Bratteli-Vershik systems with a consecutive ordering have entropy zero (see e.g. [Dur10]) hence the system that we describe in this subsection has zero entropy. By Proposition 4.4.3, the system (X_B, φ_B) is a minimal S-adic subshift defined by the substitutions read on B.

One can repeat the proof given in Proposition 6.3 of [ABKK17] to show that the diagram B has countably infinitely many ergodic invariant probability measures. These measures can be obtained as extensions of invariant measures from the odometers that pass through the sequences of vertices $\overline{w}_0 = (0,0,0,\ldots)$, $\overline{w}_1 = (0,1,1,\ldots)$, $\overline{w}_2 = (0,1,2,2,\ldots)$, $\ldots$, $\overline{w}_\infty = (0,1,2,3,\ldots)$. By Theorem 4.4.8, the systems (X_B, φ_B, μ) are measure-theoretically rigid for all ergodic invariant probability measures μ. Since each $\widetilde{F}_n$ has an equal row sum property, on each level n all towers have the same height h_n. It follows from the proof of Theorem 4.4.8 that the heights of the towers $(h_n)_n$ form a rigidity sequence for (X_B, φ_B, μ) for any ergodic invariant probability measure μ. It follows also that for any invariant probability measure ν, the system (X_B, φ_B, ν) is measure-theoretically rigid with the same rigidity sequence $(h_n)_n$.

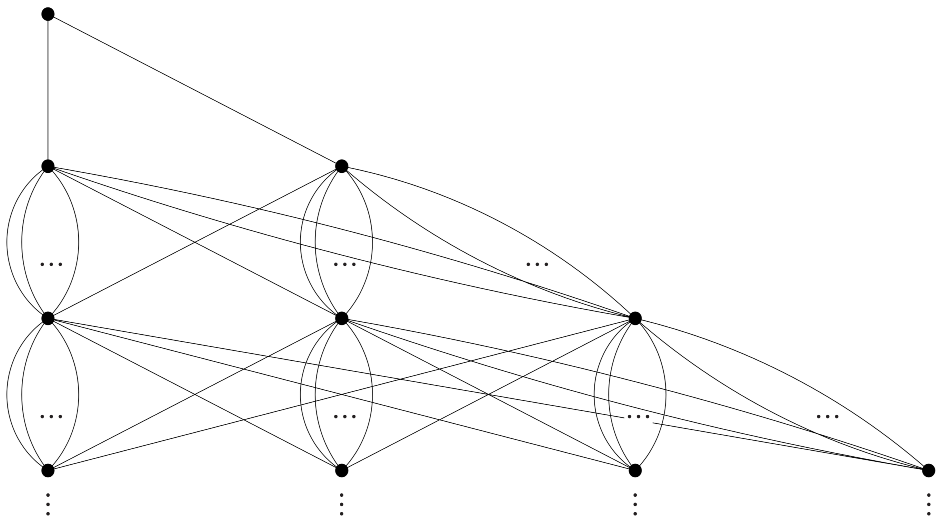


Figure 4.2: Bratteli diagrams with countably infinite set of ergodic invariant probability measures from Example 4.4.26

Lemma 4.4.27. *Example 4.4.26 is expansive and hence a Toeplitz shift.*

Proof. We observe that in the substitution read θ_n of B from V_n to V_{n-1} all maximal incoming edges have the vertex $n-1$ as its source, see (4.16).

For two edge-coded paths x and x' in the diagram, we define $H(x,x') = \min\{j \geq 1 : x_j \neq x'_j\}$. Let $\delta > 0$ be such that $d(x,x') > \delta$ if $H(x,x') \leq 2$. Take two distinct such paths x, x' and assume from now on that $k := H(x,x') \geq 3$. Then $s(x_k) = s(x'_k)$. If both x and x' represent non-maximal paths from v_0 to $t(x_k)$, and $t(x'_k)$ respectively, then $H(\varphi_B(x), \varphi_B(x')) = H(x,x')$, so we can iterate the Vershik map until one of the paths, say x, is maximal between v_0 and $t(x_k)$. Then by the previous observation $s(x_k) = k-1$.

We distinguish three cases:

1. $t(x_k) = t(x'_k)$. Since x_k is a maximal incoming edge, this can only be if $t(x_k) = k$ or $k-1$, as otherwise the definition of k is violated. In both cases $s(\varphi_B(x)_k) = 0$ and $s(\varphi_B(x')_k) = k$ or $k-1$. Hence $H(\varphi_B(x), \varphi_B(x')) < H(x, x')$.

2. $t(x_k) \neq t(x'_k)$ and x'_k is not a maximal incoming edge. Then $s(\varphi_B(x)_k) = 0$ and $s(\varphi_B(x')_k) \neq 0$ as all outgoing edges of the first vertex are either minimal (so preceded by maximal edges) or preceded by edges e_k with $s(e_k) = 0$, so we never have $s(x'_k) = k-1$, excluding this case as well. Then $H(\varphi_B(x), \varphi_B(x')) < H(x, x')$.

3. $t(x_k) \neq t(x'_k)$ and x'_k is a maximal incoming edge. Then there exists some multiple $m \in \mathbb{N}$ for the height $h = h_{k-1}(s(x_k))$ such that $s(\varphi_B^{-mh}(x)_k) \neq s(\varphi_B^{-mh}(x')_k)$, so that $H(\varphi_B^{-mh}(x), \varphi_B^{-mh}(x')) < H(x, x')$.

Hence, there is always some iterate $n \in \mathbb{Z}$ such that $H(\varphi_B^n(x), \varphi_B^n(x')) < H(x, x')$, and by induction this means that

$$\liminf_{n\in\mathbb{Z}} H(\varphi_B^n(x), \varphi_B^n(x')) \leq 2,$$

so $\limsup_{n\in\mathbb{Z}} d(\varphi_B^n(x), \varphi_B^n(x')) > \delta$ and expansivity follows. □

4.5 Examples of non-rigid partially rigid systems

In this section we give two examples of non-rigidity, both defined by stationary Bratteli diagrams. One is an invertible non-minimal system, the other based on enumeration systems is not invertible.

4.5.1 A family of non-minimal examples.

Using the methods from the previous sections, we first give a family of stationary Bratteli-Vershik systems with a non-rigid, partially rigid fully supported measure.

Theorem 4.5.1. *Let $p \geq 2$ and $q > 2$ be integers and B be the stationary Bratteli diagram with the incidence matrix*

$$F = \begin{pmatrix} 2 & 0 \\ p & q \end{pmatrix},$$

so that the incoming edges to the second vertex are ordered according to the map $\theta(2) = 1\,2^q\,1^{p-1}$.

Let φ_B be the corresponding Vershik homeomorphism. Then (X_B, φ_B) is partially rigid but not rigid w.r.t. the unique fully supported ergodic measure.

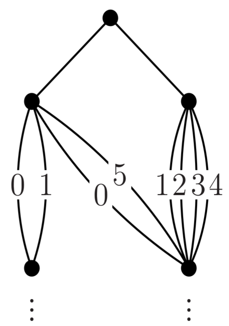


Figure 4.3: Stationary Bratteli diagram satisfying the conditions of Theorem 4.5.1

Remark 4.5.2. Figure 4.3 demonstrates the order for $p = 2$ and $q = 4$. The proof uses the mentioned order, for others such as $1^{p'} 2^q 1^{p-p'}$ for any $0 < p' < p$ it works analogously. In fact, the conclusion still holds for any stationary choice of order such that the vertical odometer corresponding to the second vertex is consecutively ordered, but for simplicity of the proof, we fixed the order as we did, with a unique minimal and a unique maximal path.

Proof. This Bratteli-Vershik systems is transitive but not minimal. The heights of the towers satisfy

$$h_n(1) = 2^{n-1}, \qquad h_n(2) = q^{n-1}\left(1 + \frac{p}{q-2}\right) - \frac{2^{n-1}p}{q-2} = qh_{n-1}(2) + ph_{n-1}(1).$$

The diagram preserves exactly two finite ergodic measures, μ_1 and μ_2 (see [BKMS10]). The measure μ_1 is supported on the odometer subdiagram $\overline{B}$ with vertex $v_n(1)$. By Theorem 4.4.6 the measure μ_1 is rigid with rigidity sequence $(h_n(1))_n$. However, this measure is not fully supported. The other measure μ_2 is fully supported, because each tail-equivalence class that eventually only goes through $v_n(2)$ is dense in B.

First, we will show that (X, φ_B, μ_2) is partially rigid. Fix $N \geq 1$ and for each $n \geq N$, let C_n be a cylinder set of length n that ends at $t(x_n) = v_n(2)$. We partition C_n into $(n+1)$-cylinders $C_{n+1}(1), \ldots, C_{n+1}(q)$ according to the edge x_{n+1}; each $C_{n+1}(a)$, $1 \leq a < q$, surely returns to C_n after $s_n = h_n(2)$ steps. Therefore

$$\liminf_{n\to\infty} \mu_2(\varphi_B^{s_n}(C_N) \cap C_N) \geq \frac{q-1}{q}\mu_2(C_N)$$

for any cylinder C_N ending in the second vertex.

In contrast, consider an N-cylinder that ends at the first vertex of the diagram. We decompose it into paths that stay at the first vertex and paths moving to the second vertex. The measure μ_2 gives zero mass to paths that stay at the first vertex for all levels n. The set of paths moving to the second vertex at some point is a countable union of subcylinders ending at the second vertex. Thus the system (X, φ_B, μ_2) is partially rigid with $\alpha \geq \frac{q-1}{q}$.

Now for the non-rigidity of μ_2, take $\varepsilon \in (0, (10q)^{-6})$ arbitrary, and take n_0 maximal such that $2^{-n_0} > 4\varepsilon$. Therefore any two distinct n_0-cylinders are at least 4ε apart. Also, take $n_1 \in \mathbb{N}$ minimal so that $h_{n_0}(2) < ph_{n_1+1}(1)$, which also means $ph_{n_1+1}(1) < 2h_{n_0}(2)$ because $h_{n_1+1}(1) = 2h_{n_1}(1)$. Without loss of generality let us assume that for all n, the vertical edges with the source $v_n(2)$ and target $v_{n+1}(2)$ are enumerated from 1 to q among all edges in $t^{-1}(v_{n+1}(2))$; cf. Remark 4.5.2. For $n \geq 1$ and $0 \leq a < p + q$, set

$$\begin{aligned} Y_{n+1}(a) &:= \{x \in X_B : t(x_{n+1}) = v_{n+1}(2) \ \text{ and } \ x_{n+1} = a\}, \\ Z_{n+1}(a) &= Y_{n+1}(q) \cap Y_{n+2}(a). \end{aligned}$$

Claim 1: The sequence $(h_n(2))_{n\geq 1}$ is not a sequence of rigidity times. More precisely, for all $n > n_1$ and $1 \leq a < q$:

$$\mu_2(\{x \in Z_{n+1}(a) : d(\varphi_B^{h_n(2)}(x), x) > 4\varepsilon\}) > \frac{1}{(q+1)^2}\mu_2(Z_{n+1}(a)).$$

Proof of Claim 1. To prove the claim, let $n > n_1$ and $C \subset Z_{n+1}(a)$ be a cylinder with edges $x_1, \dots, x_{n+2}$, $x_j \in \{1, \dots, q\}$ for $2 \leq j \leq n$, and in particular $x_{n_0+1} = q - 1$, $x_{n_1+1} = 1$, $x_{n+1} = q$ and $x_{n+2} \in \{1, \dots, q-1\}$. Since

$$\begin{aligned} p\mu_2(Y_n(0)) = p\mu_2(Y_n(q+1)) = \dots = p\mu_2(Y_n(q+p-1)) < \mu_2(Y_n(1)) = \mu_2(Y_n(2)) \\ = \ldots = \mu_2(Y_n(q)), \end{aligned}$$

the collection of such cylinders C has at least $1/(q+1)^2$ of the mass of $Z_{n+1}(a)$. Let $C' = \varphi_B^{h_{n_1}(2)}(C)$, so for every $x \in C$ and $x' = \varphi_B^{h_{n_1}(2)}(x)$ satisfies $x'_{n_1+1} = 2$, and $x'_j = x_j$ for all $j \neq n_1 + 1$, in particular $C' \subset Z_{n+1}(a)$.

It takes $h_j(2)$ iterates to change $x_{j+1} \in \{1, \dots, q-1\}$ to $x_{j+1} + 1$ and restore all the edges x_k, $k \leq j$. Similarly, it takes $h_j(2) + ph_j(1)$ iterates to change $x_{j+1} = q$ to edge 1 and restore all the edges x_k, $k \leq j$, provided that $x_{j+2} \neq q$. We compute inductively

$$\begin{aligned} h_n(2) - ph_n(1) &= qh_{n-1}(2) + ph_{n-1}(1) - ph_n(1) \\ &= (q-1)h_{n-1}(2) + h_{n-1}(2) - ph_{n-1}(1) + p(2h_{n-1}(1) - h_n(1)) \\ &= (q-1)h_{n-1}(2) + h_{n-1}(2) - ph_{n-1}(1) \\ &= (q-1)h_{n-1}(2) + \ldots + (q-1)h_1(2) + h_1(2) - ph_1(1) \\ &= (q-1)\sum_{j=1}^{n-1} h_j(2) + 1 - p \\ &= (q-1)\sum_{j=1}^{n-1} h_j(2) + p \sum_{j=1, x_{j+1}\neq 1}^{n-1} h_j(1) - d, \end{aligned}$$

where $d = p\sum_{j=1,x_{j+1}\neq 1}^{n-1} h_j(1) + p - 1$. This means that $\varphi_B^{h_n(2)}(x) = \varphi_B^{-d}(y)$, for a path y with

$$y_j = \begin{cases} x_j - 1 & \text{for } x_j > 1, \\ q & \text{for } x_j = 1 \end{cases}$$

for all $2 \leq j \leq n$, $y_{n+1} = 1$, $y_{n+2} = x_{n+2} + 1$ and $y_j = x_j$ for $j > n+2$.

Let y' be the analogue for $x' = \varphi_B^{h_{n_1}(2)}(x)$, so $\varphi_B^{h_n(2)}(x') = \varphi_B^{-d'}(y')$, but note that $d' = d + ph_{n_1+1}(1)$.

Now if $\varphi_B^{-d}(y)_j \neq x_j$ for some $j \leq n_0$, then $d(\varphi_B^{h_n(2)}(x), x) > 4\varepsilon$. So assume that $\varphi_B^{-d}(y)_j = x_j$ for all $j \leq n_0$. Then also $\varphi_B^{-d}(y')_j = x'_j$ for all $j \leq n_0$. But recall that

$$d' - d = ph_{n_1+1}(1) = h_{n_0}(2) + r \qquad \text{for some } 0 < r < h_{n_0}(2).$$

Therefore, taking $z' = \varphi_B^{-h_{n_0}(2)} \circ \varphi_B^{-d}(y')$ we find

$$\varphi_B^{h_n(2)}(x') = \varphi_B^{-r}(z') \quad \text{and} \quad z'_j = \begin{cases} x'_j & \text{for } j \leq n_0, \\ x'_{n_0+1} - 1 = q - 2 & \text{for } j = n_0 + 1. \end{cases}$$

Therefore $[\varphi_B^{-h_{n_1}(1)}(z')]_j \neq x'_j$ for at least one $j \leq n_0$. This shows that $d(\varphi_B^{h_n(2)}(x'), x') > 4\varepsilon$.

We obtain that, at least one of C and C' does not return to itself after $h_n(2)$ iterates. Recall that $x_{n_1+1} = 1$ while $x'_{n_1+1} = 2$, hence the above lack of rigidity applies to cylinders that represent at least $1/(q+1)^2$ of the mass of $Z_{n+1}(a)$, proving the claim. □

Now to prove the non-rigidity of μ_2, recall that for every $n > n_1$, any two points in the same n-cylinder are less than ε apart. Suppose by contradiction that $s \geq h_{n_1}(2)$ is a rigidity time in the sense that

$$\mu_2(U_\varepsilon(s)) > 1 - \varepsilon \quad \text{for} \quad U_\varepsilon(s) = \{x \in X_B : d(\varphi_B^s(x), x) < \varepsilon\}.$$

Let $n \geq n_1$ be maximal such that $h_n(2) \leq s$. First assume that $s < (q-1)h_n(2)$ and $1 \leq a < q$, and let $W_{n+1} := \{x \in Z_{n+1}(a) : d(\varphi_B^{h_n(2)}(x), x) > 4\varepsilon)\}$. Consider $W'_{n+1} = \varphi_B^{-s}(W_{n+1}) \subset \bigcup_{a=1}^{q-1} Y_{n+1}(a)$ and $W''_{n+1} = \varphi_B^{h_n(2)}(W'_{n+1}) \subset \bigcup_{a=2}^{q} Y_{n+1}(a)$. Note that $\mu_2(W''_{n+1}) \geq \frac{1}{(q+1)^2}\mu(Z_{n+1}(a)) > 2\varepsilon$.

If $x \in U_\varepsilon(s) \cap W'_{n+1}$, then $\varphi_B^{s-h_n(2)}(x) \in \bigcup_{a=1}^{q-1} Y_{n+1}(a)$ and

$$d(\varphi_B^{s-h_n(2)}(x), x) \leq d(\varphi_B^s(x), x) + d(\varphi_B^s(x), \varphi_B^{s-h_n(2)}(x)) < \varepsilon + \varepsilon = 2\varepsilon.$$

But then $x' := \varphi_B^{h_n(2)}(x) \in W''_{n+2}$ and $x'' := \varphi_B^{s-h_n(2)}(x')\varphi_B^s(x) \in W_{n+1}$ satisfy

$$\begin{aligned} d(\varphi_B^s(x'), x') &\geq d(\varphi_B^s(x'), \varphi_B^{s-h_n(2)}(x')) - d(\varphi_B^{s-h_n(2)}(x'), x') \\ &= d(\varphi_B^{h_n(2)}(x''), x'') - d(\varphi_B^{s-h_n(2)}(x), x) > 4\varepsilon - 2\varepsilon = 2\varepsilon. \end{aligned}$$

Hence $W''_{n+1} \cap U_\varepsilon(s) = \emptyset$, which contradicts that $\mu_2(U_\varepsilon(s)) > 1 - \varepsilon$.

Now $s < 2h_n(2)$ so that $h_{n+1}(2) - s \geq (q-1)h_n(2)$. For $n \geq 1$ and $0 \leq a < p + q$, set

$$Z'_{n+2}(a) := \{x \in X_B : t(x_{n+2}) = v_{n+2}(2),\ x_{n_0+1} = q - 1, x_{n_1+1} \in \{1, 2\},$$
$$x_{n+1} \in \{1, \dots, q-2\}, x_{n+2} = q, x_{n+3} \in \{1, \dots, q-1\}\}$$

Claim 2: Set $Q = \{x \in Z'_{n+2}(a) : d(\varphi_B^{h_{n+1}(2)}(x), x) > 4\varepsilon\}$. There is $\gamma > (q+1)^{-4}$ such that

$$\mu_2(Q) > \frac{1}{2}\,\mu_2(Z'_{n+2}(a)) > \gamma \qquad \text{for every } n > n_1.$$

The proof of Claim 2 is similar to that of Claim 1. The details are left to the reader.

Let $R = \{y \in X_B : y_j = x_j \text{ for } j \neq n+2, y_{n+2} = 1 \text{ for some } x \in Q\}$. Note that since $y_{n+2} = 1$ for every $y \in R$ we have $d(\varphi_B^{h_{n+1}(2)}(y), y) < \varepsilon$. Now fix any $x \in Q$ and associated $y \in R$ which has all coordinates, but $(n+2)$-th the same as x. Then, since $x_{n+1} < q - 1$, points $\varphi_B^s(x)$ and $\varphi_B^s(y)$ satisfy $\varphi_B^s(x)_j = \varphi_B^s(y)_j$ for $j \leq n$. Note also that $\varphi_B^{h_{n+1}(2)}(y)_j = y_j$ for all $j \leq n_0$ and $\varphi_B^{h_{n+1}(2)}(x)_j \neq x_j$ for some $j \leq n_0$. It means that there is $j \leq n_0$ such that $\varphi_B^{h_{n+1}(2)}(y)_j \neq \varphi_B^s(y)_j$ or $\varphi_B^{h_{n+1}(2)}(x)_j \neq \varphi_B^s(x)_j$. This in turn means that

$$d(\varphi_B^s(x), \varphi_B^{h_{n+1}(2)-s}(\varphi_B^s(x))) \geq \varepsilon \text{ or } d(\varphi_B^s(y), \varphi_B^{h_{n+1}(2)-s}(\varphi_B^s(y))) \geq \varepsilon.$$

As a consequence $\mu_2(U_\varepsilon(h_{n+1}(2) - s)) < 1 - \gamma$ which is a contradiction. □

Remark 4.5.3. It seems that the assumption that the Bratteli diagram in Theorem 4.5.1 is stationary is important for the result to hold. To see this, modify the stationary Bratteli diagram from Theorem 4.5.1 to the non-stationary one with the following incidence matrices:

$$F_n = \begin{pmatrix} 2 & 0 \\ p_n & q_n \end{pmatrix},$$

where $\limsup_{n\to\infty} q_n = \infty$ and the series $\sum_{n=1}^{\infty} \frac{p_n 2^{n-1}}{\prod_{k=1}^{n} q_k}$ converges. Define the order as in Theorem 4.5.1, then both ergodic invariant measures are rigid (the measure extension of a measuer on the second odometer is finite by [ABKK17, Theorem 2.1]). To prove that the measure sitting of the second vertex is rigid, use Theorem 4.4.8. Since the odometer corresponding to the second vertex is consecutively ordered and has the growing number of edges, the sequence $(h_n(2))$ is a rigidity sequence.

4.5.2 Examples from kneading theory and enumeration systems.

The non-rigid example in this subsection is minimal but not a homeomorphisms, it is one of a family of so called enumeration systems, which exhibit varying rigidity behaviors.

The following approach and notation comes from kneading theory, i.e., the symbolic dynamics of unimodal maps, see [Bru22, Section 3.6.3]. Given a *kneading map* $Q : \mathbb{N} \to \mathbb{N} \cup \{0\}$ satisfying $Q(k) < k$ for all $k \in \mathbb{N}$, we build a recursive sequence

$$S_0 = 1, \qquad S_k = S_{k-1} + S_{Q(k)}. \tag{4.17}$$

This is the sequence of *cutting times*, and it fully determines the combinatorial properties of the orbit $\mathrm{orb}(c)$ of the critical point $c = 0$ of a unimodal map $f_{\ell,a} : [0,1] \to [0,1]$, $x \mapsto a - |x|^\ell$. Here ℓ is the so-called critical order and parameter a can be chosen that $f_{\ell,a}$ has cutting times $(S_k)_{k\geq 0}$.[1] For instance, if $Q(k) \to \infty$, then $\overline{\mathrm{orb}(c)}$ is a minimal Cantor set [Bru22, Theorem 4.120], and if $\sup_k k - Q(k) < \infty$, then $f : \overline{\mathrm{orb}(c)} \to \overline{\mathrm{orb}(c)}$ is uniquely ergodic [Bru22, Corollary 6.41], and a fortiori, $\mathrm{orb}(c)$ attract Lebesgue-a.e. orbit, see [Bru98, Theorem A].

One can describe the associated dynamics by means of *enumeration scales*, introduced in [BDIL00, BDL02], see also [Bru22, Section 5.3]. It allows a more general approach to Cantor system, based on more general recursive sequences, but in this section we stick to the recursion (4.17). Every $N \in \mathbb{N}_0$ can be represented as a sequence $(x_i)_{i=0}^\infty$ of digits $x_i \in \{0,1\}$ through the greedy algorithm such that

$$\langle N \rangle = x_0 x_1 \dots \quad \text{for} \quad N = \sum_{i\geq 0} x_i S_i.$$

We define the set $X_0 = \langle \mathbb{N}_0 \rangle$ and take its closure X in the product topology on $\mathbb{N}_0^{\mathbb{N}_0}$.

As dynamics the 'addition of one' maps $T : X_0 \mapsto X_0$ by $T(\langle n \rangle) = \langle n+1 \rangle$. This can be continuously extended to the set X, provided that $Q(k) \to \infty$ (which is the assumption that we will pose from now on), and in this case $T : X \to X$ is surjective and minimal. The system (X, T) is called the *enumeration system* of the enumeration scale $(S_k)_{k\geq 0}$.

Such enumeration scales can be represented as Bratteli-Vershik systems as follows: For $i \geq 1$, the vertex set V_i consists of $1 + \#K_i$ vertices for $K_i := \{k : Q(k) < i < k\}$, indexed i and K_i, and $V_0 = \{v_0\}$. The hat is simple, i.e., E_1 n a unique edge from v_0 to each $v \in V_1$. For $i \geq 1$, E_{i+1} contains an edges (labeled 0) from $v_i \in V_i$ to $v_{i+1} \in V_{i+1}$ and from $v_i \in V_i$ to $v_k \in V_{i+1}$ if $k > i+1 > i = Q(k)$. If $Q(i+1) = i$, then there is another edge from $v_i \in V_i$ to $v_{i+1} \in V_{i+1}$, but this edge is labeled 1. The remaining edges are from $v_k \in V_i$, $k > i$, to $v_k \in V_{i+1}$ and these are all labeled 0. The labels indicate the (left-to-right) order of incoming vertices. The resulting Bratteli diagram has a simple hat and a single *spine* $v_0 \xrightarrow{0} v_1 \xrightarrow{0} v_2 \xrightarrow{0} v_3 \xrightarrow{0} \dots$ (this is the unique minimal infinite path), and for each $i \geq 2$, there is a path from v_i upward to $v_{Q(i)} \in V_{Q(i)}$ of which the lowest edge is labeled 1 and the others are labeled 0.

[1]Provided the kneading map satisfies Hofbauer's admissibility condition, see [Bru22, Formula (3.22)].

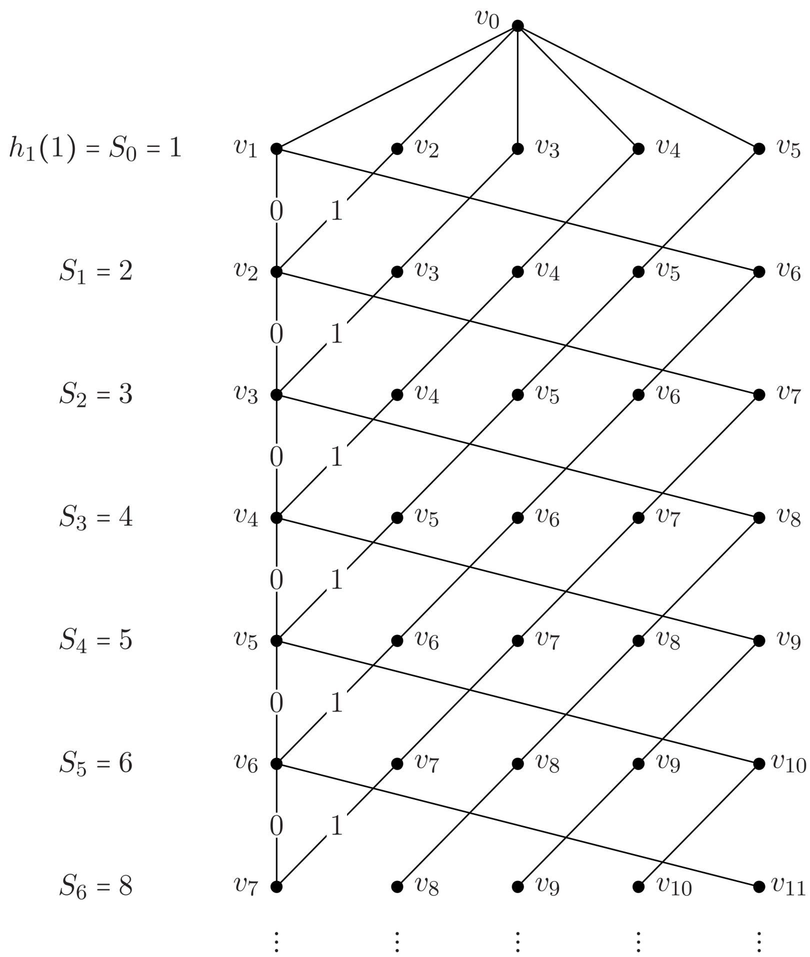


Figure 4.4: The Bratteli diagram for $S_k = S_{k-1} + S_{\max\{0,k-5\}}$

Figure 4.4 shows the Bratteli diagram for $Q(k) = \max\{k-5, 0\}$. In this (stationary) case, the cutting times are

$$(S_k)_{k\geq 0} = 1, 2, 3, 4, 5, 6, 8, 11, 15, 20, 26, 34, 42, 53, \dots$$

and the incidence matrix is

$$F = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Each infinite path has a unique labeling $(x_i)_{i\geq 1}$ where x_i is the label of the $i+1$-st edge of the path. For example, the spine represents $\langle 0 \rangle = (0, 0, 0, \dots) = x_{\min}$ and

$\langle N \rangle = \tau^N(x_{\min})$ for all $N \in \mathbb{N}$, so the Vershik map τ takes the role of the addition of one. In general, the heights are $h_i(v_i \in V_i) = S_{i-1}$ and $h_i(v_k \in V_i) = S_{Q(k)-1}$.

Proposition 4.5.4. *If* $\sup_k k - Q(k) < \infty$*, then the corresponding Cantor system is partially rigid.*

Proof. Let $d = \sup_k k - Q(k)$. Then telescoping between $2d$ levels will create a Bratteli diagram with strictly positive incidence matrices, taken from a finite number of possibilities. Therefore the BV-system has exact finite rank. Hence, the result by Danilenko [Dan16] that any exact finite rank system is partially rigid applies. □

Now we focus on the special case that $Q(k) = \max\{k - d, 0\}$ for $d = 1, 2, 3, \ldots$ If $d = 1$, then the enumeration scale is isomorphic to the dyadic odometer (and hence even topologically rigid). For $d = 2$, the enumeration system is isomorphic to both the Fibonacci substitution shift and the golden mean Sturmian shift, both well-known to be rigid. For $d = 3$ and $d = 4$, the enumeration system is isomorphic to a Pisot substitution shift with discrete spectrum, and therefore rigid as well. See [BKSP97] for the corresponding computations. In general, we expect Pisot substitution shifts to be rigid, and indeed, if the Pisot substitution conjecture holds, we obtain a discrete spectrum and hence the required isomorphism to its maximal equicontinuous factor, via the Halmos-von Neumann Theorem[2].

The interesting case comes when $d = 5$, and the enumeration system, and its characteristic equation $x^5 - x^4 - 1 = (x - e^{\pi i/3})(x - e^{5\pi i/3})(x^3 - x - 1) = 0$ are no longer Pisot.

Theorem 4.5.5. *The enumeration system* (X, T) *with* $Q(k) = \max\{0, k - 5\}$ *is not measure-theoretically rigid.*

Proof. Using techniques from Bratteli diagrams we can compute the measure of cylinder sets. We let $\xi_1, \ldots, \xi_5$ be the measures of the cylinder sets of level 1. The structure of the diagram gives that every vertex on level 1 besides the first has a unique outgoing path until it hits the spine on a later level. Therefore $\xi_i = \frac{\xi_{i-1}}{\lambda}$, where λ is the leading eigenvalue of the transition matrix F, see [BKMS10] for the construction. For cylinder sets $C_{n+1}(i)$ of lengths $n+1$ ending in $i-th$-vertex of V_{n+1} the measure is

$$\mu(C_{n+1}(1)) = \frac{\xi_1}{\lambda^n} \quad \text{and} \quad \mu(C_{n+1}(i)) = \frac{(\lambda - 1)\xi_1}{\lambda^{n+i-5}} \text{ for } i \neq 1.$$

[2] We don't know if a rigidity proof also exists without these tools (and hence without an answer to the Pisot substitution conjecture). We also don't know, whether there are any rigid non-Pisot substitution shifts, see Section 4.6.

Furthermore, the number of paths between two levels is connected to S_k. After telescoping eight levels, the incidence matrix is full and every vertex of level N connects to every vertex on level $N+8$. The number of paths between the i-th vertex in V_N and first in V_n for $n \geq N+8$ is

$$(F^{n-N})_{1,i} = S_{n-N-i-3}.$$

A simple proof by induction shows that the sequence $(S_k)_k$ satisfies the following recursive relation for sufficiently large k

$$S_{k+3} = S_{k+2} + S_{k-2} = \begin{cases} S_{k+1} + S_k + 1 & \text{if } k \equiv 0 \text{ or } 5 \bmod 6, \\ S_{k+1} + S_k - 1 & \text{if } k \equiv 2 \text{ or } 3 \bmod 6, \\ S_{k+1} + S_k & \text{if } k \equiv 1 \text{ or } 4 \bmod 6. \end{cases} \tag{4.18}$$

Define the cylinder set $A = [100000]_{0,5}$ with the digits 100000 at $x_0 \dots x_5$ and its subsets

$$\begin{aligned} B_k^0 &= [100000]_{0,5} \cap [00000000000]_{k-5,k+5}, \\ B_k^1 &= [100000]_{0,5} \cap [00000010000]_{k-5,k+5}, \\ B_k^{-1} &= [100000]_{0,5} \cap [00001000000]_{k-5,k+5}. \end{aligned}$$

These cylinders indicate paths in the Bratteli diagram that pass through the second edge in E_2 with $t(x_0) = v_2 \in V_2$ for the cylinder A), and that additionally pass through the second edge in $E_{k+1\pm1}$ with $t(x_{k\pm1}) = v_{k+1\pm1} \in V_{k+1\pm1}$ (for the cylinders $B_k^{\pm1}$), while the paths in B_k^0 go through the spine between levels $k-5$ and $k+5$. Note also that A, $T(A)$ and $T^{-1}(A)$ are pairwise disjoint, and $B_k^{\pm1} = T^{S_{k\pm1}}(B_k^0)$. The masses of these sets are

$$\mu(A) = \frac{\xi_1}{\lambda^2} \quad \text{and} \quad \mu(B_k^0) = \mu(B_k^1) = \mu(B_k^{-1}) = \mu(C_{k+3}(1))(F^{k-7})_{1,1} = \frac{\xi_1 S_{k-11}}{\lambda^{k+2}}.$$

By the Perron-Frobenius theorem, for a primitive matrix F and its leading eigenvalue λ we have

$$\lim_{n\to\infty} \frac{F^n}{\lambda^n} = \xi\eta,$$

where ξ and η are leading right and left strictly positive eigenvectors of F normalised such that $\eta\xi = 1$. Therefore, the fraction $\frac{S_n}{\lambda^n}$ converges to a positive constant and $\mu(B_k^0) = \mu(B_k^1) = \mu(B_k^{-1}) > c > 0$ for all $k \in \mathbb{N}$.

To finish the proof, let $n \geq 1$ be arbitrary and take $k = \max\{j : S_j \leq n\}$ and $n' = n - S_k$, so $n' < S_{k-4}$.

We claim that for at least one of $T^n(B_k^0) \cap A$, $T^n(B_k^1) \cap A$ and $T^n(B_k^{-1}) \cap A$ has mass $< c/2$, so n cannot be a rigidity time. This would finish the proof.

To show the claim, take $B' = B_k^0 \cap T^{-n'}(A)$, and note that $T^{n'+S_j} B' \subset A$ for all $j \geq k+2$. Indeed, since $n' < S_{k-4}$, applying $T^{n'}$ to $x \in B'$ can affect at most the first $k-1$ digits. Applying another T^{S_j} doesn't affect any digit below $k-5$, so $T^{n'+S_j}(x) \in A$.

If $\mu(B') < c/2$, then $\mu(T^{n'}(B_k^0) \setminus A) \geq c/2$ and each $x \in T^{n'}(B_k^0) \setminus A$ has 0s in positions $k-2, \dots, k+5$, but potentially a 1 in position $k-3$. Applying another S_k iterates produces a 1 in position k or $k+1$ (since $S_{k+1} = S_k + S_{k-4} = S_k + S_{k-3} - S_{k-8}$). That is, $T^n(B_k^0 \setminus B')$ is disjoint from B_k^0 but has mass $\geq c/2$. Thus the claim holds for B_k^0.

Assume therefore that $\mu(B') \geq c/2$.

- $k \equiv 0$ or $5 \bmod 6$: Note that $T^{S_{k+1}}(B_k^0) = B_k^1$ and $T^n(B_k^1) = T^{n'+S_{k+1}+S_k}(B_k^0) \supset T^{n'+S_{k+3}-1}(B')$, which is a set of mass $\geq c/2$ contained in $T^{-1}(A)$ and hence disjoint from A. Therefore $\mu(B_k^1 \cap T^n(B_k^1)) < c/2$.

- $k \equiv 2$ or $3 \bmod 6$: Note that $T^{S_{k+1}}(B_k^0) = B_k^1$ and $T^n(B_k^1) = T^{n'+S_{k+1}+S_k}(B_k^0) \supset T^{n'+S_{k+3}+1}(B')$, which is a set of mass $\geq c/2$ contained in $T(A)$ and hence disjoint from A. Therefore $\mu(B_k^1 \cap T^n(B_k^1)) < c/2$.

- $k \equiv 4 \bmod 6$: Now $T^{S_{k-1}}(B_k^0) = B_k^{-1}$ and

  $$T^n(B_k^{-1}) = T^{n'+S_{k-1}+S_k}(B_k^0) \supset T^{n'+S_{k+2}+1}(B')$$

  , which is a set of mass $\geq c/2$ contained in $T(A)$ and hence disjoint from A. Therefore $\mu(B_k^{-1} \cap T^n(B_k^{-1})) < c/2$.

- $k \equiv 1 \bmod 6$: Now $T^{S_{k-1}}(B_k^0) = B_k^{-1}$ and

  $$T^n(B_k^{-1}) = T^{n'+S_{k-1}+S_k}(B_k^0) \supset T^{n'+S_{k+2}-1}(B')$$

  , which is a set of mass $\geq c/2$ contained in $T^{-1}(A)$ and hence disjoint from A. Therefore $\mu(B_k^{-1} \cap T^n(B_k^{-1})) < c/2$.

This finishes the proof of the claim and of the whole theorem. □

4.6 Open problems

1. Are there partially rigid but not measure-theoretically rigid Toeplitz systems? Can one use the construction from Theorem 4.3.6 with another subshift (Y, σ, ν) to get a partially rigid, but not measure-theoretically rigid Toeplitz system?

2. Suppose a minimal Cantor system (X, T, μ) is measure-theoretically rigid (or partially rigid). Does it mean that the first return time map to any clopen set of positive measure is also measure-theoretically rigid (or partially rigid)?

3. Compute the best partial rigidity constant (in [DMR25] this constant is called a "partial rigidity rate") for enumeration systems considered in Section 4.5. Determine which enumeration systems defined by linear recursion are measure-theoretically rigid.

4. If a primitive substitution shift is rigid, does it mean that its leading eigenvalue is Pisot? (Note that this is less restrictive than that the substitution itself is Pisot. For example, constant length substitutions are rigid by Proposition 4.4.16 and their leading eigenvalues are integer, hence Pisot, even though the incidence matrix may have other eigenvalues outside the unit disk.)

5. It is known that every Cantor minimal system is strongly orbit equivalent to a Cantor minimal system of entropy zero (see [BH94], [Dur10]). Is it true that every Cantor minimal system has in its strong orbit equivalence class a Cantor minimal system which is (partially) measure-theoretically rigid with respect to at least one of its ergodic invariant probability measures (see also Remark 4.2.15)? In particular, is it true that any simple Bratteli diagram can be telescoped to a diagram admitting an order such that the corresponding Vershik map is rigid with respect to at least one of its ergodic invariant probability measures?

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5. Minimal Bounded Speedups of Toeplitz Flows

We investigate minimal bounded speedups of Toeplitz flows. We demonstrate that the minimal bounded speedup of a Toeplitz flow need not be another Toeplitz flow and describe techniques for determining whether the resulting speedup is Toeplitz. We also provide sufficient conditions to guarantee that the minimal bounded speedup will be a Toeplitz flow.

This chapter is based on the article [AR25] which was submitted in September 2025.

5.1 Introduction

Given a minimal Cantor system (X, T), a *topological speedup* of (X, T) is a dynamical system (X, S) where S is a homeomorphism such that $S(x) = T^{p(x)}(x)$ for some function $p : X \to \mathbb{N}$. We call the function $p(x)$ the *jump function* for the speedup. Throughout this paper we assume that $p(x)$ is continuous (equivalently bounded by [AAO18, Proposition 2.2]) and that the resulting Cantor system (X, S) is minimal. We are interested in investigating which properties of the original Cantor system are preserved under minimal bounded speedups. In particular, this paper focuses on speedups of Toeplitz flows.

There is a rich history of studying speedups of dynamical systems, particularly in the measure-theoretical setting; see, for example, [AOW85, Nev69a, Nev69b]. More recently, there has been a focus on topological speedups. In [AO24], the authors characterize when one Cantor minimal system is the speedup of another and provide topological analogues to the measure-theoretical results of [AOW85]. In [AAO18], minimal bounded speedups of odometers and substitutions are investigated. It is shown that the minimal bounded speedup of an odometer is a conjugate odometer, whereas the minimal bounded speedup of a primitive substitution is always a different primitive substitution. In [ADL23], the authors use Bratteli diagrams to investigate minimal bounded speedups of primitive substitutions.

Because it is known that minimal bounded speedups of odometers are conjugate odometers, and Toeplitz flows are symbolic, minimal, almost one-to-one extensions of odometers, it was conjectured in [ADL23] that the minimal bounded speedup of a Toeplitz flow is a Toeplitz flow. We demonstrate that this conjecture is false. We investigate minimal bounded speedups of Toeplitz flows and provide sufficient conditions for a speedup to be Toeplitz.

In Section 5.2 we present background terminology and foundational results about substitution shifts, Toeplitz flows, minimal Cantor systems, and topological speedups of minimal Cantor systems. Section 5.3 demonstrates the existence of minimal bounded speedups of Toeplitz flows that are not themselves Toeplitz flows. In fact, we show in Theorem 5.3.5 that no bounded speedup of the Toeplitz flow (X_θ, σ) generated by the substitution $\theta(a) = ab$ and $\theta(b) = aa$ with orbit number $c = 2$ will be a Toeplitz flow.

We demonstrate in Section 5.4 that a constant speedup (X, T^c) of a Toeplitz flow (X, T) is a Toeplitz flow if and only if $\gcd(c, p_k) = 1$ for all k where (p_k) is the period structure of the original Toeplitz flow. Additionally, in this case (X, T) and (X, T^c) will have the same underlying odometer. The section concludes by presenting a method for determining whether a given speedup of a Toeplitz flow is Toeplitz. Algorithms are provided for both the case where (X, T) is generated by a constant length left proper substitution and the more general Toeplitz flow setting.

The final section of the paper investigates sufficient conditions for the minimal bounded speedup of a Toeplitz flow to be Toeplitz. Namely, if there exists some level of the Kakutani-Rokhlin partition for (X, T) such that every tower has the same orbit permutation and each orbit has the same number of levels in each tower under the jump function p, then the speedup (X, S) will be a Toeplitz flow (Theorem 5.5.1). We demonstrate that this can happen such that (X, S) is conjugate to (X, T^c) and such that (X, S) is not conjugate to (X, T^c), as (X, T^c) need not be minimal. We conjecture that our sufficient conditions in Theorem 5.5.1 are actually necessary for (X, S) to be Toeplitz because of the strong relationship between Toeplitz flows and odometers.

5.2 Preliminaries

5.2.1 Substitution Subshifts

Let $\mathcal{A}$ be a finite *alphabet*. A *substitution* over $\mathcal{A}$ is a map $\theta : \mathcal{A} \to \mathcal{A}^*$, where $\mathcal{A}^*$ denotes the set of nonempty finite words with letters from $\mathcal{A}$. We extend θ to map from $\mathcal{A}^*$ to $\mathcal{A}^*$ by concatenation such that $\theta(ab) = \theta(a)\theta(b)$; we then extend to maps from $\mathcal{A}^{\mathbb{N}}$ to

$\mathcal{A}^{\mathbb{N}}$ (similarly from $\mathcal{A}^{\mathbb{Z}}$ to $\mathcal{A}^{\mathbb{Z}}$) in the natural way. A substitution is *constant length* if $|\theta(a)| = |\theta(b)|$ for all $a, b \in \mathcal{A}$. We say a substitution is *primitive* if there exists some $n \in \mathbb{N}$ such that for every $a \in \mathcal{A}$, $\theta^n(a)$ contains every letter of $\mathcal{A}$. A substitution is *left proper* if there is some $a \in \mathcal{A}$ such that $\theta(b)$ starts with a for every $b \in \mathcal{A}$; if we also have some $c \in \mathcal{A}$ such that $\theta(b)$ ends with c for every $b \in \mathcal{A}$, then we say the substitution is *proper*.

Without loss of generality (by considering θ^n if necessary), we assume θ is primitive if and only if there is a letter $a \in \mathcal{A}$ such that $\theta(a)$ begins with a, $\lim_{n\to\infty} |\theta^n(b)| = \infty$ for all $b \in \mathcal{A}$, and $\lim_{n\to\infty} \theta^n(a)$ is a fixed point under the substitution that contains every letter in $\mathcal{A}$. We define the shift space $X_\theta \subset \mathcal{A}^{\mathbb{Z}}$ to be the set of all $x \in \mathcal{A}^{\mathbb{Z}}$ such that whenever $i, j \in \mathbb{Z}$ with $i < j$, then $x_{[i,j]}$ is a subword of $\theta^k(a)$ for some $k \in \mathbb{N}$ and $a \in \mathcal{A}$. We refer to (X_θ, σ) as the *substitution subshift* and emphasize that θ is primitive if and only if (X_θ, σ) is minimal.

5.2.2 Toeplitz Flows

A sequence $x \in \mathcal{A}^{\mathbb{N}}$ (or $\mathcal{A}^{\mathbb{Z}}$) is called a *Toeplitz sequence* if for every $i \in \mathbb{N}$ there exists a $p_i \in \mathbb{N}$ such that $x_i = x_{i+kp_i}$ for all $k \in \mathbb{N}$ (or $\mathbb{Z}$). The minimal dynamical system given by (X, σ) where $X = \overline{\{\sigma^n(x) : n \geq 0\}}$ is called a *Toeplitz flow* or *Toeplitz shift*.

Let (X, σ) be a Toeplitz flow with $X \subset \mathcal{A}^{\mathbb{Z}}$. Given $x \in X$, $p \in \mathbb{N}$, and $a \in \mathcal{A}$, we define the following sets.

- $\mathrm{Per}_p(x, a) = \{k \in \mathbb{Z} \mid x_{k'} = a \quad \forall k' \equiv k \mod p\}$.
- $\mathrm{Per}_p(x) = \bigcup_{a\in\mathcal{A}} \mathrm{Per}_p(x, a)$.
- $\mathrm{Aper}(x) = \mathbb{Z} \setminus \bigcup_{p\in\mathbb{N}} \mathrm{Per}_p(x)$.

A sequence $x \in \mathcal{A}^{\mathbb{Z}}$ is Toeplitz if and only if $\mathrm{Aper}(x) = \emptyset$. Additionally, every Toeplitz sequence x is such that for every $n \in \mathbb{N}$, there is a $p \in \mathbb{N}$ such that $\{0, 1, 2, 3, \ldots, n-1\} \subseteq \mathrm{Per}_p(x)$.

For any $p \in \mathbb{N}$ and $x \in \mathcal{A}^{\mathbb{Z}}$, the *p-skeleton* of x is the sequence obtained from x by replacing x_i by a 'hole' for all $i \notin \mathrm{Per}_p(x)$. That is, the p-skeleton of x is the part of x which is periodic with period p. We call p an *essential period* of a sequence x if the p-skeleton of x is not periodic with smaller period.

Definition 5.2.1. A *period structure* for an aperiodic Toeplitz sequence x is an increasing sequence $\{p_i\}_{i\in\mathbb{N}}$ of positive integers satisfying the following properties.

1. For all $i \in \mathbb{N}$, p_i is an essential period for x;

2. $p_i \mid p_{i+1}$;

3. $\bigcup_{i\geq 1} \mathrm{Per}_{p_i}(x) = \mathbb{Z}$.

There are many equivalent characterizations of Toeplitz sequences. We note a few of them here which will be used throughout this paper. The following theorem is stated in terms of one-sided Toeplitz sequences but can easily be generalized to two-sided sequences (see, for example, [DP22]). In fact, if $x \in \mathcal{A}^{\mathbb{Z}}$ is Toeplitz, then so is $x' = x_0x_1x_2\cdots \in \mathcal{A}^{\mathbb{N}}$. We state this one-sided version to aid in quickly identifying our KR-partitions.

Theorem 5.2.2. *[Bru22, Theorem 4.90] The one-sided sequence $x \in \mathcal{A}^{\mathbb{N}}$ is Toeplitz if and only if there is a sequence of constant length substitutions $\theta_i : \mathcal{A}_i \to \mathcal{A}_{i-1}$ on finite alphabets $\mathcal{A}_i$ with $\mathcal{A} = \mathcal{A}_0$ such that $\theta_i(a)$ starts with the same symbol for each $a \in \mathcal{A}_i$, and $x = \lim_{i\to\infty} \theta_1 \circ \theta_2 \circ \cdots \circ \theta_i(a)$, where $a \in \mathcal{A}_i$ is arbitrarily chosen.*

Observe that by using this characterization for the Toeplitz flow, we obtain a period structure (p_k) where $p_k = |\theta_1 \circ \theta_2 \circ \cdots \circ \theta_k|$ for all $k \in \mathbb{N}$. We note that by allowing $\theta_i = \theta$ for every $i \in \mathbb{N}$, we may obtain a system which is both a substitution shift and a Toeplitz flow. In this case, the underlying period structure is such that $p_k = |\theta|^k$.

Corollary 5.2.3. *If (X_θ, σ) is generated by a constant length, left proper substitution θ, then (X_θ, σ) is a Toeplitz flow.*

We now define the α-adic odometer, as there is a close relationship between odometers and Toeplitz flows.

Definition 5.2.4. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)$ be a sequence of integers with each $\alpha_i \geq 2$. Denote by X_α the set of all infinite sequences of points $\mathbf{x} = (x_1, x_2, x_3, \ldots)$ such that $0 \leq x_i \leq \alpha_i - 1$ for each $i \in \mathbb{N}$. Addition on X_α is defined by

$$(x_1, x_2, \ldots) + (y_1, y_2, \ldots) = (z_1, z_2, \ldots)$$

where $z_1 = (x_1 + y_1) \mod \alpha_1$ and for each $i \geq 2$, $z_i = x_i + y_i + r_{i-1} \mod \alpha_i$ where $r_{i-1} = 0$ if $x_{i-1} + y_{i-1} + r_{i-2} < \alpha_{i-1}$ and $r_{i-1} = 1$ otherwise (we set $r_0 = 0$). We define $T_\alpha : X_\alpha \to X_\alpha$ by

$$T_\alpha((x_1, x_2, x_3 \ldots)) = (x_1, x_2, x_3, \ldots) + (1, 0, 0, \ldots).$$

The dynamical system (X_α, T_α) is called the *α-adic odometer.*

Theorem 5.2.5. *[Dow05] Toeplitz flows can be characterized up to topological conjugacy as topological dynamical systems that are symbolic, minimal, almost one-to-one extensions of odometers. Given the period structure (p_k) of a Toeplitz point $x \in X$, the underlying odometer is (X_α, T_α) where $\alpha = \left(p_1, \dfrac{p_2}{p_1}, \dfrac{p_3}{p_2}, \ldots, \dfrac{p_i}{p_{i-1}}, \ldots\right)$.*

5.2.3 Kakutani-Rokhlin Partitions

We now introduce Kakutani-Rokhlin partitions, which will be helpful in the study of topological speedups.

Definition 5.2.6. Given a minimal Cantor system (X, T), a *Kakutani-Rokhlin partition (KR-partition)* for (X, T) is a partition

$$\mathcal{P} = \bigcup_{i=1}^{N} \{T^j(B_i) : 0 \leq j \leq h_i - 1\}$$

of X into pairwise disjoint clopen sets that cover X. We call the set $B = \bigcup_{i=1}^{N} B_i$ the *base* of the KR-partition and the integers h_i the *heights* of each *tower* $A_i = \{T^j(B_i) : 0 \leq j \leq h_i - 1\}$. Further, for all $1 \leq i \leq N$ we have $T^{h_i}(B_i) \subset B$.

Every continuous minimal Cantor system (X, T) has a sequence of KR-partitions $\mathcal{P}(k)$ satisfying the following properties [HPS92].

1. The sequence of bases is nested: $B(k+1) \subset B(k)$.

2. For all $k \in \mathbb{N}$, $\mathcal{P}(k+1)$ refines $\mathcal{P}(k)$.

3. $\bigcap_{k \in \mathbb{N}} B(k)$ is a single point $x \in X$.

4. The partitions $\mathcal{P}(k)$ form a basis for the topology of X.

5. For all $k \in \mathbb{N}$, $i \leq N(k)$ and $i' \leq N(k-1)$, there exists $0 \leq j \leq h_i^k$ such that $T^j(B_i^k) \subset B_{i'}(k-1)$.

6. $B(k) \subset B_1(k-1)$ for all $k \in \mathbb{N}$.

We now explicitly define the KR-partitions $\mathcal{P}(k)$ for Toeplitz flows and substitution systems.

Definition 5.2.7. Suppose (X, T) is a Toeplitz flow generated by the Toeplitz point $x \in X$. By Theorem 5.2.2, there is a sequence of alphabets $\{\mathcal{A}_i\}$ and a sequence of constant length substitutions $\theta_i : \mathcal{A}_i \to \mathcal{A}_{i-1}$ that generate (X, T). For every $k \in \mathbb{N}$ and $1 \leq i \leq |\mathcal{A}_k|$, let B_i^k denote the set of points $x \in X$ such that $x_{[0, |\theta_1 \circ \theta_2 \circ \cdots \circ \theta_k| - 1]} = \theta_1 \circ \theta_2 \circ \cdots \circ \theta_k(i)$ and set $h_i^k = |\theta_1 \circ \theta_2 \circ \cdots \circ \theta_k| = p_k$. The sequence of KR-partitions is defined by the sets $\mathcal{P}(k) = \bigcup_{i=1}^{|\mathcal{A}_k|} \{T^j B_i^k : 0 \leq j \leq p_k - 1\}$. For ease, we denote each tower by $A_i^k = \{T^j B_i^k : 0 \leq j \leq h_i^k - 1\}$.

Example 5.2.8. *Let $\mathcal{A} = \mathcal{A}_0 = \{0,1\}$, $\mathcal{A}_1 = \{a,b\}$, and $\mathcal{A}_i = \{c,d\}$ for all $i \geq 2$. Define $\theta_1(a) = 011$, $\theta_1(b) = 100$, $\theta_2(c) = ab$ and $\theta_2(d) = aa$, and for all $i \geq 2$, define $\theta_i(c) = ccd$, and $\theta_i(d) = cdd$. Then $\mathcal{P}(3)$ will consist of 2 towers, each of height $p_3 = 18$; the first tower will have its base floor given by the set of points $x \in X$ with $x_{[0,17]} =$ 011100011100011011 and the second tower will have its base floor given by the cylinder set $x_{[0,17]} = 011100011011011011$.*

We can similarly define the KR-partitions $\mathcal{P}(k)$ when the Toeplitz flow is generated by a substitution system. This is a special case of the Toeplitz definition where $\theta_i = \theta$ for all $i \in \mathbb{N}$.

Definition 5.2.9. Suppose that (X,T) is generated by a left proper, primitive, aperiodic substitution on an alphabet $\mathcal{A} = \{1, 2, \ldots, N\}$. For every $k \in \mathbb{N}$ and $1 \leq i \leq N$, let B_i^k denote the set of points $x \in X$ such that $x_{[0,|\theta|^k-1]} = \theta^k(i)$ and set $h_i^k = |\theta^k(i)|$. Then the sequence of KR-partitions is defined by the sets $\mathcal{P}(k) = \bigcup_{i=1}^N \{T^j B_i^k : 0 \leq j \leq h_i^k - 1\}$. For ease, we denote each individual tower by $A_i^k = \{T^j B_i^k : 0 \leq j \leq h_i^k - 1\}$

We note that intuitively, given any minimal Cantor system (X,T) we may think of the KR-partition $\mathcal{P}(k+1)$ as being a collection of towers, where each tower is comprised of a stack of towers from $\mathcal{P}(k)$ and the lowest $\mathcal{P}(k)$-tower in each $\mathcal{P}(k+1)$-tower is the same primary tower from $\mathcal{P}(k)$.

5.2.4 Speedups of Minimal Cantor Systems

Definition 5.2.10. Let (X,T) be a minimal Cantor system. A *bounded speedup* of (X,T) is a homeomorphism of the form (X,S) where $S(x) = T^{p(x)}(x)$ for some bounded function $p : X \to \mathbb{N}$. We refer to the bounded function $p(x)$ as the *jump function.*

In general, one can explore speedups where p is not assumed to be bounded; however, the function p is known to be continuous if and only if it is bounded [AAO18, Propostion 2.2]. Additionally, the resulting speedup of a minimal Cantor system need not be minimal. If all S-orbits are dense in X, then we say that (X,S) is a minimal speedup of (X,T). Throughout this paper, we will always assume that p is bounded and that (X,S) is a minimal speedup. Given a minimal bounded speedup (X,S) of (X,T), there is a constant $c \in \mathbb{N}$, called the *orbit number*, such that every T-orbit is the union of exactly c different S-orbits [AAO18, Lemma 2.3]. We now discuss some properties of jump functions as they relate to the sequence of KR-partitions for the underlying minimal Cantor system (X,T).

Remark 5.2.11. Let (X, T) be a minimal Cantor system. Suppose that $p : X \to \mathbb{N}$ defines a bounded topological speedup (X, S). Then the sequence of KR-partitions for (X, T) is such that for all k sufficiently large:

1. The height h_i^k of each tower A_i^k is greater than $\max_{x \in X} p(x)$.
2. p is constant on each level of $\mathcal{P}(k)$.
3. p is constant on each set $T^m(B(k))$ for $0 \leq m \leq \max_{x \in X} p(x)$.

By beginning our sequence of KR-partitions with k large enough, we may assume these properties are satisfied for our entire sequence of KR-partitions.

We may thus create an orbit labeling of the levels in $\mathcal{P}(k)$ that models the speedup. This process demonstrates how every T-orbit decomposes into c different S-orbits.

Definition 5.2.12. For all k large enough so that the properties in Remark 5.2.11 hold and for c equal to the orbit number of the speedup, let $\mathcal{L}_k : \mathcal{P}(k) \to \{1, 2, \ldots, c\}$ be the *orbit labeling function* defined as follows. For each $1 \leq i \leq N(k)$:

- Label the base floor B_i^k of tower A_i^k with a 1.
- Label any other floor F in A_i^k with a 1 if $F = S^j(B_i^k)$ and $\sum_{l=0}^{j-1} pS^l(x) < h_i^k$ for all $x \in B_i^k$.
- Label the lowest unlabeled floor $T^{d_2}(B_i^k)$ of A_i^k with a 2.
- Label any other floor F in A_i^k with a 2 if $F = S^j(T^{d_2}(B_i^k))$ and $\sum_{l=0}^{j-1} pS^l(x) < h_i^k$ for all $x \in T^{d_2}(B_i^k)$.
- Label the lowest unlabeled floor in A_i^k with a 3 and continue.
- Continue in this manner until all floors of A_i^k are labeled with a value from $\{1, 2, \ldots, c\}$.

Note that by assuming the properties in Remark 5.2.11 hold, we guarantee that any two floors in different columns with the same height below the $\max p(x)$ floor will have the same labeling. Suppose that F is the maximal floor in A_i^k that is labeled l; then for all $x, y \in F$, the label of the floor containing $S(x)$ is the same as the label of the floor containing $S(y)$. Hence, for k large enough so that the properties in Remark 5.2.11 hold, define $\pi_i^{(k)}(l)$ to be the label of the floor containing $S(x)$ for all x in the floor of maximal height in A_i^k labeled l.

Proposition 5.2.13. *If (X, S) is a minimal speedup of (X, T) with jump function $p : X \to \mathbb{N}$, then each $\pi_i^{(k)}$ is a permutation of the set $\{1, 2, \ldots, c\}$.*

We now list some known results about speedups of odometers and substitutions that will be helpful in our study of Toeplitz flows. Recall that every KR-partition for an odometer consists of a single tower.

Lemma 5.2.14. *[AAO18, Lemma 3.7] Suppose $(X,T) = (X_\alpha, T_\alpha)$ is an odometer and (X_α, S) is a bounded speedup of (X_α, T_α). Then (X_α, S) is minimal if and only if for all k sufficiently large, $\pi^{(k)}$ is a cyclic permutation on $\{1, 2, \ldots, c\}$, where c is the orbit number.*

The following theorem is a combination of several results from [AAO18].

Theorem 5.2.15. *[AAO18] If $(X,T) = (X_\alpha, T_\alpha)$ is an odometer with $\alpha = (\alpha_1, \alpha_2, \ldots)$, then there is a minimal bounded speedup with orbit number $c > 1$ if and only if there is some $N \in \mathbb{N}$ such that $\gcd(c, \alpha_i) = 1$ for all $i \geq N$. Further, in this case the minimal speedup will be topologically conjugate to the original odometer.*

We note that the Toeplitz flows we are interested in studying are each symbolic shift spaces. It thus follows that any minimal speedup of a Toeplitz flow will necessarily result in a system that is not conjugate to the original system.

Theorem 5.2.16. *[AAO18, Theorem 4.6] If (X,T) is a minimal subshift on an infinite set X and (X,S) is a minimal bounded speedup of (X,T), then (X,T) and (X,S) are not conjugate.*

Let θ be a constant length, left proper, primitive substitution. Because there exists some $K \geq 1$ such that $\pi_i^{(K)} = \pi_i^{(jK)}$ for all $1 \leq i \leq |\mathcal{A}|$ and all $j \in \mathbb{N}$ ([AAO18]), we may assume that K is chosen large enough to satisfy Remark 5.2.11. Additionally, because the system generated by θ is conjugate to the system generated by θ^k, without loss of generality we assume that θ satisfies all of these hypotheses with $k = 1$ and $K = 1$.

Definition 5.2.17. Let θ be a constant length, left proper, primitive substitution on the alphabet $\mathcal{A}$. Set $C = \{1, 2, \ldots, c\}$ and consider the alphabet $\mathcal{A} \times C$. Define $\tau : \mathcal{A} \times C \to (\mathcal{A} \times C)^*$ by

$$\tau(a, \ell) = (a_1, \ell_1)(a_2, \ell_2) \cdots (a_n, \ell_n)$$

where $\theta(a) = a_1 a_2 \cdots a_n$, $\ell_1 = \ell$, and $\ell_{k+1} = \pi_{i_k}(\ell_k)$ for each $k \geq 1$.

Lemma 5.2.18. *[AAO18, Lemma 4.15] Suppose θ is a (left) proper, primitive, aperiodic substitution on $\mathcal{A}$ and (X,T) is the subshift generated by θ. Suppose (X,S) is a bounded speedup of (X,T) with orbit number c. Then (X,S) is minimal if and only if τ is primitive.*

Theorem 5.2.19. *[AAO18, Theorem 4.18] Suppose (X, T) is a minimal substitution system associated with a (left) proper primitive substitution θ. If (X, S) is a minimal bounded speedup of (X, T), then (X, S) is a minimal substitution system.*

We now generalize Definition 5.2.17 and Lemma 5.2.18 from the substitution setting to the more general S-adic setting for Toeplitz flows.

Definition 5.2.20. Let $(\mathcal{A}_i)_{i\geq 0}$ be a sequence of alphabets and $(\theta_i)_{i\geq 1}$ be a sequence of constant length, left proper, primitive substitutions such that $\theta_i : \mathcal{A}_i \to \mathcal{A}_{i-1}^*$. Set $C = \{1, 2, \ldots, c\}$ and consider the sequence of alphabets $((\mathcal{A}_i \times C)_{i\geq 0})$. Define $\tau_k^l : \mathcal{A}_k \times C \to (\mathcal{A}_l \times C)^*$ by

$$\tau_k^l(a, \ell) = (a_1, \ell_1)(a_2, \ell_2) \cdots (a_n, \ell_n)$$

where $\theta_{l+1} \circ \theta_{l+2} \circ \cdots \circ \theta_k(a) = a_1 a_2 \cdots a_n$, $\ell_1 = \ell$, and $\ell_{j+1} = \pi_{i_j}(\ell_j)$ for each $j \geq 1$.

Lemma 5.2.21. *Let (X, T) be the Toeplitz flow generated by the sequences $(\mathcal{A}_i)$ and (θ_i). Suppose (X, S) is a bounded speedup of (X, T) with orbit number c. Then (X, S) is minimal if and only if for every $l \in \mathbb{N}$ there is a $k \geq l$ such that $\tau_k^l(a, j)$ contains every letter from $\mathcal{A}_l \times C$ where $a \in \mathcal{A}_k$ and $j \in C$ are arbitrarily chosen.*

Proof. Suppose for all $l \in \mathbb{N}$ there exists a $k \geq l$ such that $\tau_k^l(a, j)$ contains every letter from $\mathcal{A}_l \times C$, where $a \in \mathcal{A}_k$ and $j \in C$ are arbitrarily chosen. Let $x \in X$ and consider the decomposition of x into $\theta_1 \circ \theta_2 \circ \cdots \theta_k$-words. Within each $\theta_1 \circ \theta_2 \circ \cdots \circ \theta_k$-word is every possible $\theta_1 \circ \theta_2 \circ \cdots \theta_l$-word with every possible label. Thus, the S-orbit of x intersects all floors of $\mathcal{P}(l)$. Because this is true for all $l \in \mathbb{N}$, the S-orbit of x is dense.

Conversely, suppose that S is minimal and fix $x \in X$. There is an N such that any S-orbit block of length $N + 1$ intersects every floor of $\mathcal{P}(l)$ with all labels. Set $M = \sup_{x\in X} p(x)$ and let $k > l$ be chosen such that $|\tau_k^l(a, j)| \geq MN$ for all $(a, j) \in \mathcal{A}_k \times C$. Then we may conclude that every letter $(a', j') \in \mathcal{A}_l \times C$ appears in $\tau_k^l(a, j)$ for each $(a, j) \in \mathcal{A}_k \times C$. □

5.3 The Speedup of a Toeplitz Flow Need Not be Toeplitz

In this section we present an example of a Toeplitz flow (X, T) and a jump function $p(x) : X \to \mathbb{N}$ such that the speedup $(X, S) = (X, T^{p(x)})$ is minimal and not conjugate to a Toeplitz flow. This demonstrates that the conjecture in [ADL23] that the minimal speedup of a Toeplitz flow is Toeplitz is not true.

Example 5.3.1. *Consider the primitive substitution θ on the alphabet $\mathcal{A} = \{a, b\}$ given by $\theta(a) = ab$ and $\theta(b) = aa$. Note that the subshift (X_θ, σ) is a Toeplitz flow. Let $A = [\theta^3(a)] = \{x \mid x_{[0,7]} = abaaabab\}$ and define $p : X_\theta \to \mathbb{Z}^+$ by*

$$p(x) = \begin{cases} 3 & \textit{if } x \in A \\ 1 & \textit{if } x \in \sigma(A) \\ 2 & \textit{else.} \end{cases}$$

With this given jump function p, (X_θ, S) is a minimal bounded speedup of (X_θ, σ) where $S = \sigma^{p(x)}(x)$. In this case, the orbit number is $c = 2$.

Remark 5.3.2. For (X_θ, σ), the tower A_1^k is given by $\theta^k(a)$ and the tower A_2^k is given by $\theta^k(b)$. For all $k \geq 3$, the permutation on the first tower is given by $\pi_1^{(k)}(1) = 2$ and $\pi_1^{(k)}(2) = 1$, whereas the permutation on the second tower is given by $\pi_2^{(k)}(1) = 1$ and $\pi_2^{(k)}(2) = 2$. Because the permutation depends only on the tower and not on the level of the KR-partition, we may simply use $\pi_1(i)$ and $\pi_2(i)$ for all KR partitions $\mathcal{P}(k)$.

Also observe that the height of orbit j in tower A_i^k is given by $o_i^k(j) = 2^{k-1}$ for all $i, j \in \{1, 2\}$ and for all $k \geq 3$. In particular, the height of each orbit in each tower of $\mathcal{P}(3)$ is 4. To see this more clearly, let $A = aba$, $B = aa$, $C = ba$, $D = b$, and $E = ab$. Then tracing through the first orbit of the tower A_1^3 is equivalent to tracing through $ABCC$. Tracing through the second orbit of the tower A_1^3 is equivalent to tracing through $DBEE$. Similarly, tracing through the first orbit of the tower A_2^3 is equivalent to tracing through the $EBEB$ and tracing through the second orbit of A_2^3 is equivalent to tracing through $CBCB$.

Remark 5.3.3. In Example 5.3.1, let $ABCC$ denote the first orbit path of the first tower in $\mathcal{P}(3)$, $DBEE$ denote the second orbit path in the first tower, $EBEB$ denote the first orbit path in the second tower, and $CBCB$ denote the second orbit path in the second tower in $\mathcal{P}(3)$. Note that $|ABCC| = |DBEE| = |EBEB| = |CBCB| = 4$. Define $\mathcal{C} = \{A, B, C, D, E\}$ and $\mathcal{B} = \{w_1, w_2, w_3, w_4\}$.

Define $\psi : \mathcal{B} \to \mathcal{C}^*$ by $\psi(w_1) = ABCCCBCBDBEE$, $\psi(w_2) = ABCCDBEEEBEB$, $\psi(w_3) = ABCCCBCBDBEEEBEB$, and $\psi(w_4) = ABCCDBEE$.

Define $\phi : \mathcal{B} \to \mathcal{B}^*$ by $\phi(w_1) = w_1w_2w_3w_4$, $\phi(w_2) = w_1w_2w_4w_3$, $\phi(w_3) = w_1w_2w_3w_4w_3$, and $\phi(w_4) = w_1w_2w_4$.

Then by using the return word method of Durand [Dur98], the speedup (X_θ, S) is topologically conjugate to the shift space defined by applying ψ to the fixed point of the substitution ϕ and then taking the closure of the shift on this new point. We denote this space $(\psi(X_\phi), \sigma)$.

Theorem 5.3.4. *The speedup (X_θ, S) defined in Example 5.3.1 is not a Toeplitz flow.*

Proof. Suppose that (X_θ, S) is a Toeplitz flow. Then without loss of generality, there exists some word $y \in X_\theta$ that is Toeplitz under S. Without loss of generality, we may assume that the decomposition of y into $\mathcal{C}$-words begins with $y_{[0,8]} = ABCC$. It follows that the decomposition of y into these $\mathcal{C}$-words is a Toeplitz sequence. Hence there exists some $t \in \mathbb{N}$ such that A appears in the $\mathcal{C}$-decomposition of y in every t positions. Further, because y is necessarily minimal and the substitution ϕ is recognizable, $\psi(\phi^k(w_2))$ appears in this decomposition of y for every $k \in \mathbb{N}$. Hence, A must appear in $\psi(\phi^k(w_2))$ exactly every t positions for every $k \in \mathbb{N}$.

Observe the following $\psi(\phi)$ words (formatted with spaces to ease identifying the locations of A).

$$\psi(\phi(w_1)) = ABCCCBCBDBEE\ ABCCDBEEEBEB\ ABCCCBCBDBEEEBEB\ ABCCDBEE$$

$$\psi(\phi(w_2)) = ABCCCBCBDBEE\ ABCCDBEEEBEB\ ABCCDBEE\ ABCCCBCBDBEEEBEB$$

$$\begin{aligned}\psi(\phi(w_3)) = {} & ABCCCBCBDBEE\ ABCCDBEEEBEB\ ABCCCBCBDBEEEBEB\ ABCCDBEE \\ & ABCCCBCBDBEEEBEB\end{aligned}$$

$$\psi(\phi(w_4) = ABCCCBCBDBEE\ ABCCDBEEEBEB\ ABCCDBEE$$

By inspection, it necessarily follows that $t \geq 24$. Additionally, we can look at the $\psi(\phi^2)$ words.

$$\begin{aligned}\psi(\phi^2(w_1)) &= \psi(w_1w_2w_3w_4w_1w_2w_4w_3w_1w_2w_3w_4w_3w_1w_2w_4) \\ &= ABCCCBCBDBEE\ ABCCDBEEEBEB\ ABCCCBCBDBEEEBEB\ ABCCDBEE \\ &\quad ABCCCBCBDBEE\ ABCCDBEEEBEB\ ABCCDBEE\ ABCCCBCBDBEEEBEB \\ &\quad ABCCCBCBDBEE\ ABCCDBEEEBEB\ ABCCCBCBDBEEEBEB\ ABCCDBEE \\ &\quad ABCCCBCBDBEEEBEB\ ABCCCBCBDBEE\ ABCCDBEEEBEB\ ABCCDBEE\end{aligned}$$

$$\begin{aligned}\psi(\phi^2(w_2)) &= \psi(w_1w_2w_3w_4w_1w_2w_4w_3w_1w_2w_4w_1w_2w_3w_4w_3) \\ &= ABCCCBCBDBEE\ ABCCDBEEEBEB\ ABCCCBCBDBEEEBEB\ ABCCDBEE \\ &\quad ABCCCBCBDBEE\ ABCCDBEEEBEB\ ABCCDBEE\ ABCCCBCBDBEEEBEB \\ &\quad ABCCCBCBDBEE\ ABCCDBEEEBEB\ ABCCDBEE\ ABCCCBCBDBEE \\ &\quad ABCCDBEEEBEB\ ABCCCBCBDBEEEBEB\ ABCCDBEE\ ABCCCBCBDBEEEBEB\end{aligned}$$

$$\begin{aligned}\psi(\phi^2(w_3)) &= \psi(w_1w_2w_3w_4w_1w_2w_4w_3w_1w_2w_3w_4w_3w_1w_2w_4w_1w_2w_3w_4w_3)\\ &= ABCCCBCBDBEE\ ABCCDBEEEBEB\ ABCCCBCBDBEEEBEB\ ABCCDBEE\\ &\quad ABCCCBCBDBEE\ ABCCDBEEEBEB\ ABCCDBEE\ ABCCCBCBDBEEEBEB\\ &\quad ABCCCBCBDBEE\ ABCCDBEEEBEB\ ABCCCBCBDBEEEBEB\ ABCCDBEE\\ &\quad ABCCCBCBDBEEEBEB\ ABCCCBCBDBEE\ ABCCDBEEEBEB\ ABCCDBEE\\ &\quad ABCCCBCBDBEE\ ABCCDBEEEBEB\ ABCCCBCBDBEEEBEB\ ABCCDBEE\\ &\quad ABCCCBCBDBEEEBEB,\end{aligned}$$

$$\begin{aligned}\psi(\phi^2(w_4)) &= \psi(w_1w_2w_3w_4w_1w_2w_4w_3w_1w_2w_4)\\ &= ABCCCBCBDBEE\ ABCCDBEEEBEB\ ABCCCBCBDBEEEBEB\ ABCCDBEE\\ &\quad ABCCCBCBDBEE\ ABCCDBEEEBEB\ ABCCDBEE\ ABCCCBCBDBEEEBEB\\ &\quad ABCCCBCBDBEE\ ABCCDBEEEBEB\ ABCCDBEE,\end{aligned}$$

By applying the same logic as above, we see that necessarily $t \geq 96$. Suppose that for some $k \geq 1$, A appears every t positions in $\psi(\phi^k(w_2))$ for $t = 6 \cdot 4^k$. We now outline the following facts about $\psi \circ \phi^k$ and $\psi \circ \phi^{k+1}$.

1. Facts about the lengths of the $\psi(\phi^k(w))$ words:

   - $|\psi(\phi^k(w_1))| = 6 \cdot 4^k + 6 \cdot 4^k$.
   - $|\psi(\phi^k(w_2))| = 6 \cdot 4^k + 6 \cdot 4^k$
   - $|\psi(\phi^k(w_3))| = 6 \cdot 4^k + 6 \cdot 4^k + 4 \cdot 4^k$
   - $|\psi(\phi^k(w_4))| = 6 \cdot 4^k + 2 \cdot 4^k$

2. Facts about concatenation of words in $\psi(\phi^{k+1}(w_2))$:

   - $\psi(\phi^{k+1}(w_2)) = \psi(\phi^k(w_1)\phi^k(w_2)\phi^k(w_4)\phi^k(w_3))$
   - $|\psi(\phi^{k+1}(w_2))| = |\psi(\phi^k(w_1)\phi^k(w_2)\phi^k(w_4)\phi^k(w_3))| = 6 \cdot 4^k + 6 \cdot 4^k + 6 \cdot 4^k + 6 \cdot 4^k + 6 \cdot 4^k + 2 \cdot 4^k + 6 \cdot 4^k + 6 \cdot 4^k + 4 \cdot 4^k$
   - The alignment of $\psi(\phi^k(w_4)\phi^k(w_3))$ is such that A does not appear at all multiples of $6 \cdot 4^k$. It does hold that A appears at all multiples of $6 \cdot 4^{k+1}$ in $\psi(\phi^k(w_4)\phi^k(w_3))$. Observe that A will also appear at all multiples of $6 \cdot 4^{k+1}$ in $\psi(\phi^k(w_1w_2))$.
   - We thus conclude that A appears at all multiples of $6 \cdot 4^{k+1}$ in $\psi(\phi^{k+1}(w_2))$. Therefore, it follows that if A appears at every t positions in the $\mathcal{C}$-decomposition of y, then $t \geq 6 \cdot 4^{k+1}$.

For each $k \in \mathbb{N}$, it follows that in order for A to appear within $\psi(\phi^k(w_2))$ at every t positions, then $t \geq 6 \cdot 4^k$. Since t is assumed to be a finite number that works for every $k \in \mathbb{N}$, we obtain a contradiction. It thus follows that the decomposition of y into $\mathcal{C}$-words is not a Toeplitz sequence, and hence y will not be a Toeplitz sequence under S. We conclude that no point in X_θ will be Toeplitz under S, and therefore (X_θ, S) is not a Toeplitz flow. $\square$

We now show that no minimal speedup of the Toeplitz flow (X_θ, σ), where θ is defined as in Example 5.3.1, with orbit number $c = 2$ is a Toeplitz flow.

Theorem 5.3.5. *Any minimal speedup of the Toeplitz flow (X_θ, σ) generated by the substitution $\theta(a) = ab$ and $\theta(b) = aa$ with orbit number $c = 2$ will not be a Toeplitz flow.*

Proof. Let $p(x)$ be a jump function such that (X_θ, S) is a minimal bounded speedup of (X_θ, σ) with orbit number $c = 2$. Choose a KR-partition $\mathcal{P}(k)$ with k chosen large enough such that the properties in Remark 5.2.11 hold. Consider the permutations $\pi_a^{(k)}$ and $\pi_b^{(k)}$ induced by the speedup. Observe that if $\pi_a^{(k)} = \pi_b^{(k)} = \mathrm{id}$, then the associated speedup would not be minimal. Similarly, if $\pi_a^{(k)}(1) = \pi_b^{(k)}(1) = 2$, the associated speedup will not be minimal as $\pi_a^{(k+1)} = \pi_b^{(k+1)} = \mathrm{id}$. Thus, one of tower permutations must be the identity, and other tower permutation must be the cyclical permutation. Observe that in either case, for all $k \geq K$ (there exists $K \in \mathbb{N}$), it follows that $\pi_a^{(k)}(1) = 2$ and $\pi_b^{(k)} = \mathrm{id}$. One can check via Lemma 5.2.18 that the associated speedup will be minimal.

Let $o_a^k(i)$ denote the number of levels of the $\theta^k(a)$ tower in the KR-partition $\mathcal{P}(k)$ that are labeled with i for each $i \in \{1, 2\}$; similarly define $o_b^k(i)$. Observe that $o_a^k(1) + o_a^k(2) = |\theta^k|$ and $o_b^k(1) + o_b^k(2) = |\theta^k|$. Further note that the substitution $\phi : \mathcal{B} \to \mathcal{B}^*$ defined as in Remark 5.3.3 is the same as the substitution defined by checking returns to $(a, 1)$ in the substitution defined by τ in Definition 5.2.17. Here $\mathcal{B} = \{w_1 = (a,1)(b,2)(a,2), w_2 = (a,1)(a,2)(b,1), w_3 = (a,1)(b,2)(a,2)(b,1), w_4 = (a,1)(a,2)\}$, $\phi(w_1) = w_1w_2w_3w_4$, $\phi(w_2) = w_1w_2w_4w_3$, $\phi(w_3) = w_1w_2w_3w_4w_3$, and $\phi(w_4) = w_1w_2w_4$. Observe that when we consider the orbit heights of each orbit path, the length of w_1 is $o_a^k(1) + o_b^k(2) + o_a^k(2) = |\theta|^k + o_b^k(2)$, the length of w_2 is $o_a^k(1) + o_a^k(2) + o_b^k(1) = |\theta|^k + o_b^k(1)$, the length of w_3 is $o_a^k(1) + o_b^k(2) + o_a^k(2) + o_b^k(1) = 2|\theta|^k$, and the length of w_4 is $o_a^k(1) + o_a^k(2) = |\theta|^k$.

We now follow the same logic as in the previous theorem to demonstrate that (X_θ, S) is not Toeplitz. Suppose that (X_θ, S) is Toeplitz. Then there is some sequence $x \in X_\theta$ that is Toeplitz under S. We can decompose x uniquely into its w_i representation. As x is Toeplitz, there exists some $l \in \mathbb{N}$ such that if the decomposition of x into w_i words begins with w_1, then the decomposition of $S^{l \cdot n}(x)$ begins with w_1 for all $n \in \mathbb{N}$. By our substitution, $|\phi(w_1)| = |\phi(w_2)| = 6|\theta|^k$, $|\phi(w_3)| = 8|\theta|^k$,and $|\phi(w_4)| = 4|\theta|^k$. Hence, any return between our desired occurrences of w_1 in the w_i-decomposition of x gives $l \geq 8|\theta|^k$. By looking at $\phi^2(w_1) = w_1w_2w_3w_4w_1w_2w_4w_3w_1w_2w_3w_4w_3w_1w_2w_4$, we obtain that $l \geq 12|\theta|^k$. By looking at $\phi^3(w_1)$, we obtain that $l \geq 24|\theta|^k$. In general, we will never be able to find a single value l such that our w_i-decomposition of x returns to w_1 every l steps. Therefore (X_θ, S) is not a Toeplitz flow. $\square$

This result leads to the following natural question.

Question 5.3.6. Suppose (X_θ, σ) is an arbitrary Toeplitz flow generated by the constant length substitution θ. Is there any minimal speedup (X_θ, S) of (X_θ, σ) with orbit number $c = |\theta|$ such that (X_θ, S) is Toeplitz? What if the orbit number c is such that $\gcd(c, |\theta|) \neq 1$?

5.4 Proving that a Speedup is (Not) Toeplitz

Theorem 5.4.1. *Suppose that (X, T) is a Toeplitz flow with underlying period structure (p_k). Then (X, T^c) is a Toeplitz flow if and only if* $\gcd(c, p_k) = 1$ *for all k.*

Proof. Consider the sequence of KR-partitions $\mathcal{P}(k)$ for (X, T) as defined in Definition 5.2.7. Recall that this sequence of KR-partitions can be built from the S-adic structure for an arbitrary fixed Toeplitz sequence $x \in X$. Fix a level N such that all of the towers in the KR-partition $\mathcal{P}(k)$ have height $p_k > c$ for every $k \geq N$. Look at the permutations of the orbits on each tower. By construction, each tower has the same permutation of orbits; thus set $\pi_i^{(k)} = \pi$ for each i (here we understand that π depends on the level $\mathcal{P}(k)$, but we only work with one fixed k at a time).

First suppose that $\gcd(c, p_k) = 1$ for all k. In this case π is a cyclical permutation on c elements, so $\pi^c = \text{id}$. If one starts on the first floor of any tower (i.e., the first orbit level), after tracing through exactly c towers, they will return to the base floor of some tower. By construction, the base floor of every tower in $\mathcal{P}(k)$ is contained in the same base floor of the same primary tower in $\mathcal{P}(k)$. Hence, we have that for our fixed Toeplitz sequence x, $T^{c \cdot p_k \cdot n}(x) = (T^c)^{p_k \cdot n}(x)$ is in the same cylinder set as x for all $n \in \mathbb{N}$. Since we may follow this argument for every $k \geq N$, it follows that x is a Toeplitz point under the map T^c.

It remains to show that T^c is minimal. Fix k such that $p_k > c$. Since (X, T) is minimal, there is some $t > k$ such that the primary tower of $\mathcal{P}(t-1)$, whose base floor contains the base floor of every tower in $\mathcal{P}(t)$, is a stack that contains every tower from

$\mathcal{P}(k)$. Whenever one starts in the base floor of any tower in $\mathcal{P}(t)$, one will thus trace through every orbit of the primary tower from $\mathcal{P}(t-1)$ that contains every tower from $\mathcal{P}(k)$. That is, one will trace through every possible floor of every tower from the KR-partition $\mathcal{P}(k)$. Since k was arbitrarily chosen, we see that T^c is minimal. Hence, it follows that if $\gcd(c, p_k) = 1$, then (X, T^c) is a Toeplitz flow.

Conversely, suppose that $\gcd(c, p_k) = n > 1$ for some $k \in \mathbb{N}$. By labeling the orbits of each tower, one sees c different orbit labels. However, the permutation π on these orbit labels is such that π decomposes into n distinct cyclic permutations. As a result, the speedup (X, T^c) will not be minimal, since one cannot switch between the n distinct collections of orbit labels. Thus, if $\gcd(c, p_k) \neq 1$, it follows that (X, T^c) is not a Toeplitz flow. $\square$

Example 5.4.2. *Consider the primitive substitution θ on the alphabet $\mathcal{A} = \{a, b\}$ given by $\theta(a) = ab$ and $\theta(b) = aa$. The subshift (X_θ, T) is a Toeplitz flow with period structure $(p_k) = (2^k)$ for all $k \geq 1$. Consider the constant speedup (X_θ, T^c) where $c = 3$. By Theorem 5.4.1, (X_θ, T^3) is a Toeplitz flow. One can verify that (X_θ, T^3) is conjugate to the substitution shift generated by $\gamma : \{A, B, C, D, E\} \to \{A, B, C, D, E\}^*$ given by $\gamma(A) = ABAC$, $\gamma(B) = ABDE$, $\gamma(C) = AEDC$, $\gamma(D) = ABDC$, and $\gamma(E) = AEDE$. Observe that this substitution is once again constant length and left proper. One associated period structure for (X_θ, T^3) is $(q_k) = (4^k)$ for every $k \in \mathbb{N}$. In this case, we note that the underlying odometer for both (X_θ, T) and (X_θ, T^3) is the dyadic odometer; in fact (q_k) is a period structure for (X_θ, T), even though (p_k) is not a period structure for (X_θ, T^3).*

We want to emphasize that the previous example generated a minimal speedup which could also be represented as a left proper, constant length substitution. In general, a minimal speedup of a constant length substitution shift need not be conjugate to another constant length substitution. Every example we have constructed where the minimal speedup of a Toeplitz flow generated by a constant left proper substitution is again a Toeplitz flow has the property that the resulting speedup can be represented as a constant length and left proper substitution. This leads us to ask if this is always the case.

Question 5.4.3. If we begin with a Toeplitz flow generated from a constant length substitution θ, and the speedup (X, S) is also a Toeplitz flow, must there be a constant length representation of the substitution generating (X, S)?

Theorem 5.4.4. *Suppose (X, T) is a Toeplitz flow. If (X, T^c) is a Toeplitz flow, then (X, T^c) has the same underlying odometer as (X, T).*

Proof. Let (X,T) be a Toeplitz flow and suppose that (X,T) has period structure (p_k). Suppose further that $\gcd(c,p_k)=1$ for all $k\in\mathbb{N}$. First note that for every $k\in\mathbb{N}$ with $p_k>c$, if one starts at the base floor of any tower in the KR-partition $\mathcal{P}(k)$ and jumps c levels at a time, then one will return to a base floor of some tower after exactly p_k jumps. Look at the KR-partition $\mathcal{P}(k+1)$. Observe that there is a primary tower in $\mathcal{P}(k)$ such that the base of every tower in $\mathcal{P}(k+1)$ is contained in the base of the primary tower from $\mathcal{P}(k)$. Thus, if one starts in any base level of any tower of the KR-partition $\mathcal{P}(k+1)$, one is at the base level of the primary tower for $\mathcal{P}(k)$. Hence, one returns to the cylinder set defined by the primary tower of $\mathcal{P}(k)$ exactly every p_{k+1} jumps. Hence, there exists some $N\in\mathbb{N}$ (here $p_N>c$) such that we may choose p_{N+1} as the first essential period for our period structure.

Observe that the primary tower from $\mathcal{P}(N+1)$ contains every tower from $\mathcal{P}(N+2)$. Now the entire cylinder set associated with the base floor of the primary tower from $\mathcal{P}(N+1)$ will be entered into every p_{N+2} jumps. Further, note that we do not return to this cylinder set every p_{N+1} jumps, as our space X is not periodic. Hence p_{N+2} is an essential period.

By induction, we have that p_{N+k} is an essential period for every $k\in\mathbb{N}$, p_{N+k} divides p_{N+k+1} for every $k\in\mathbb{N}$, and because the Toeplitz point $x\in X$ generating the KR-partition for (X,T) will be Toeplitz under T^c (because every cylinder set of x will return to itself for some p_{N+k}), it follows that (p_{N+k}) is a period structure for (X,T^c). Additionally, as $(p_{N+k})\subseteq(p_k)$, it follows that (p_{N+k}) is a period structure for (X,T). Because (X,T) and (X,T^c) can be generated by the same period structure, it follows that they have the same underlying odometers. □

We now generalize the technique used in the Proof of Theorem 5.3.4 into an algorithm to determine whether the speedup of a Toeplitz flow is Toeplitz. We present the technique in two versions: the constant length substitutive Toeplitz case and the general (non-substitutive) Toeplitz case.

Algorithm to determine whether a given speedup of a substitutive Toeplitz flow is Toeplitz:

1. Given the constant length and left proper substitution θ and the jump function $p(x)$, find the tower permutations $\pi_i^{(k)}$ for the speedup. Observe that we may choose k large enough and consider the substitution θ^k so that without loss of generality, every KR-partition $\mathcal{P}(k)$ is such that the ith tower at every level has the same permutation and Remark 5.2.11 is satisfied.

2. Define the substitution $\tau : \mathcal{A} \times C \to (\mathcal{A} \times C)^*$ as in Definition 5.2.17. Note that this substitution can be rewritten as an equivalent substitution $\phi : \mathcal{B} \to \mathcal{B}^*$ by taking the fixed point of the substitution τ generated by $(a, 1)$ and using Durand's method (see [Dur98]) to create a left proper substitution by looking at returns to $(a, 1)$.

3. Define a substitution ψ that encodes the individual jumps within each tower orbit into a new alphabet $\mathcal{C}$; this substitution extends to the ϕ-words.

4. Take a bi-infinite fixed sequence x under ϕ and apply ψ. Then the shift space generated by $\psi(x)$ is conjugate to the speedup (X, S). Note that x can be found by extending the labeling rule defined in Definition 5.2.17 to the bi-infinite fixed point of θ where we assume $\ell_0 = 1$, $\ell_k = \pi_{i_k}(\ell_{k-1})$ if $k \geq 1$ and $\ell_j = \pi_{i_j}^{-1}(\ell_{j+1})$ if $j \leq -1$.

5. Observe if $\psi \circ \phi$ has a constant length and left proper representation, then immediately (X, S) is Toeplitz.

6. Suppose that some point $y \in X$ is Toeplitz under S. There is a unique decomposition of y into $\psi(\phi^k)$-words for every $k \in \mathbb{N}$. We call this decomposition y'. Since the 0th coordinate of y' appears every p positions (there exists some $p \in \mathbb{N}$, since y' is Toeplitz under the shift map), it follows that this symbol must appear every p positions in every $\psi(\phi^k)$ word for every $k \in \mathbb{N}$.

7. Thus, if one can show that no such p can possibly exist (through checking larger and larger k values), we obtain a contradiction and (X, S) is not Toeplitz.

8. If it follows that there is a $p \in \mathbb{N}$ such that the 0th coordinate of y' appears every p positions in y', then we set $p = p_1$ and we suppose there is some $p_2 \in \mathbb{N}$ (necessarily a multiple of p_1) such that the word $y'_{[0,p_1-1]}$ appears every p_2 positions in every $\psi(\phi^k)$ word for every $k \in \mathbb{N}$ and continue recursively. If we can find some infinite sequence (p_k) of integers, this will form our period structure for y' under the shift map and we will have verified that y is Toeplitz under S. If at any stage we cannot find a valid p_k, then the speedup will not be Toeplitz.

Algorithm to determine whether a given speedup of a general Toeplitz flow is Toeplitz:

1. Given the sequence of alphabets $(\mathcal{A}_i)$ and the sequence of constant length left proper substitutions $(\theta_i : \mathcal{A}_i \to \mathcal{A}_{i-1})$ that generate the Toeplitz flow (X, T) and the jump function $p(x)$ for the speedup, find the tower permutations $\pi_i^{(k)}$ for the speedup for each KR-partition $\mathcal{P}(k)$.

2. Define a substitution $\tau_{i+1} : \mathcal{A}_{i+1} \times I \to (\mathcal{A}_i \times I)^*$ by $\tau_{i+1}(A, j) = (b_1, j_1)(b_2, j_2) \cdots (b_n, j_n)$ where $\theta_{i+1}(A) = b_1, b_2 \ldots b_n$, $j_1 = j$, and for all $i \geq 2$, $j_i = \pi^{(i)}_{b_{i-1}}(j_{i-1})$.

3. Define a substitution ψ_l that encodes the individual jumps within each tower orbit $(a, j) \in \mathcal{A}_l \times C$ into a new alphabet $\mathcal{C}$.

4. Take a bi-infinite sequence $x \in X$ and decode it into into $\mathcal{A}_l \times C$-words and apply ψ_l; denote this new sequence as x' and note that the shift space generated by x' is conjugate to the speedup (X, S). Note that if we choose $x \in X$ to be a Toeplitz sequence (under T), then the decomposition of x into $\mathcal{A}_l \times C$ words can be found by extending the labeling rule defined in Definition 5.2.20 to x where we assume $l_0 = 1$, $l_j = \pi_{i_j}(l_{j-1})$ if $j \geq 1$ and $l_j = \pi^{-1}_{i_j}(l_{j+1})$ if $j \leq -1$. Hence we may apply ψ_l to this bi-infinite sequence and generate the same shift space conjugate to (X, S).

5. Suppose that some point $y \in X$ is Toeplitz under S. There is a unique decomposition of y into $\psi_l(\mathcal{A}_l \times C)$-words for every $l \in \mathbb{N}$. We call this decomposition y' and recall that the shift map on y' acts in the same way that S acts on y. Since the 0th coordinate of y' appears every p positions (there exists $p \in \mathbb{N}$ since y' is Toeplitz under the shift map), it follows that this symbol must appear every p positions in every $\psi_l(\mathcal{A}_l \times C)$-word for every $l \in \mathbb{N}$.

6. Thus, if one can show that no such p can possibly exist (through checking larger and larger l values), we obtain a contradiction and (X, S) is not Toeplitz.

7. If it follows that there is a $p \in \mathbb{N}$ such that the 0th coordinate of y' appears every p positions in y', then we set $p = p_1$ and we suppose there is some $p_2 \in \mathbb{N}$ (necessarily a multiple of p_1) such that the word $y'_{[0,p_1-1]}$ appears every p_2 positions in every $\psi_l(\mathcal{A}_l \times C)$ word for every $l \in \mathbb{N}$ and continue recursively. If we can find some infinite sequence (p_k) of integers, this will form our period structure for y' under the shift map and we will have verified that y is Toeplitz under S. If at any stage we cannot find a valid p_k, then the speedup will not be Toeplitz.

5.5 Sufficient Conditions for Obtaining Toeplitz Flows

In this section we provide sufficient conditions to guarantee that a minimal bounded speedup of a Toeplitz flow will result in another Toeplitz flow. Recall the following definition: for each $k \in \mathbb{N}$ large enough such that Remark 5.2.11 holds and for each $j \in \{1, 2, \ldots, c\}$, we let $o_i^k(j)$ denote the *height of orbit* j *in tower* A_i^k; that is, $o_i^k(j)$ is precisely the number of floors in tower A_i^k that are labeled j.

Theorem 5.5.1. *Suppose that (X,T) is a Toeplitz flow and (X,S) is a minimal speedup of (X,T). If there exists some $k \in \mathbb{N}$ such that for the KR-partition we have that every tower has the same orbit permutation and each orbit has the same 'height' in each tower (i.e, for every $1 \leq j \leq c$, the total number of levels labeled with orbit number j in each tower is the same), then (X,S) will be a Toeplitz flow.*

Proof. First observe that if every tower has the same permutation for some fixed $k \in \mathbb{N}$, then we may denote $\pi_i^{(k)} = \pi$ for simplicity (when k is understood). Also note that if (X,S) is minimal, then π must be a cyclic permutation on $\{1, 2, \ldots, c\}$. If π were not a cyclic permutation, then there would be some subcycle of length $\ell < c$ such that $\pi^\ell(1) = \pi \circ \cdots \circ \pi(1) = 1$. In this case we would have points in X that do not enter into every orbit labeling, which contradicts the minimality of (X,S). We thus have that $\pi^c(j) = j$ for all $j \in \{1, 2, \ldots, c\}$. By the construction of the KR-partitions, the orbit labeling, and the fact that (X,T) is a Toeplitz flow, it will follow that for every $k' \geq k$, each tower in $\mathcal{P}(k')$ will have the same permutation.

Now suppose for some $k \in \mathbb{N}$ we have that for every pair of towers A_i^k and A_l^k and every orbit label $j \in \{1, 2, \ldots, c\}$ it holds that $o_i^k(j) = o_l^k(j)$. Because (X,T) is Toeplitz, $\mathcal{P}(k+1)$ has precisely $N(k+1)$ towers each formed by stacking p_{k+1}/p_k towers from level $\mathcal{P}(k)$ on top of each other (where (p_k) is the period structure for (X,T)), and each tower in $\mathcal{P}(k)$ has the same permutation and the same orbit labeling, it follows that each tower in $\mathcal{P}(k+1)$ will also have the same orbit labeling. That is, for all $k' > k$, for any two towers $A_i^{k'}$ and $A_l^{k'}$ and any orbit label $j \in \{1, 2, \ldots, c\}$ it holds that $o_i^{k'}(j) = o_l^{k'}(j)$.

Fix $k \in \mathbb{N}$ and let $x \in B(k)$, the base level of the KR-partition $\mathcal{P}(k)$. Because every tower in $\mathcal{P}(k)$ has the same cyclic permutation and the same number of floors labeled j for all $j \in \{1, 2, \ldots, c\}$, we have that $S^{o_i^k(1)+o_i^k(2)+\cdots+o_i^k(c)}(x) \in B(k)$. Let $m_k = o_i^k(1) + o_i^k(2) + \cdots + o_i^k(c)$. This implies that $S^{m_k \cdot n}(x) \in B(k)$ for all $x \in B(k)$ and all $n \in \mathbb{N}$.

Let $x = \bigcap_{k \in \mathbb{N}} B(k)$. Fix $\varepsilon > 0$; there exists some $K_\varepsilon \in \mathbb{N}$ such that for all $k \geq K_\varepsilon$ if $x, y \in B(k)$, then $d(x,y) < \varepsilon$. For each k large enough, we thus have that $d(S^{m_k \cdot n}(x), x) <$

ε. Hence x is a Toeplitz point under S. Because (X, S) is a minimal system that can be generated by a Toeplitz point, (X, S) is thus a Toeplitz flow. $\square$

Question 5.5.2. If for all $\mathcal{P}(k)$, the towers have different orbit permutations, is it possible for the minimal speedup (X, S) to be a Toeplitz flow? What if the original flow is generated by a constant length substitution?

Question 5.5.3. If for all $\mathcal{P}(k)$, the towers have different orbit heights, is it possible for the minimal speedup (X, S) to be a Toeplitz flow? What if the original flow is generated by a constant length substitution?

We now investigate co-boundaries for Toeplitz speedups. Given a topological dynamical system (X, T), the continuous function $g : X \to \mathbb{R}$ is a T-*coboundary* if there exists a continuous function $f : X \to \mathbb{R}$ such that $g = f - f(T)$. In [AAO18] it was shown that if c is the orbit number for the speedup (X, S) with jump function $p(x)$, then $p(x) - c$ is a T-coboundary. Additionally, if $p(x) - c$ is an S-coboundary and T^c is a minimal action on X, then (X, S) is conjugate to (X, T^c); we note that in general, $p(x) - c$ need not be an S-coboundary.

In the following proofs we use $p(x, n) = \sum_{i=0}^{n-1} pS^i(x)$ to denote the unique natural number satisfying $S^n(x) = T^{p(x,n)}(x)$.

Proposition 5.5.4. *Let (X, S) be the minimal bounded speedup of a minimal system (X, T) with jump function $p(x)$ and orbit number c. There exists a level $\mathcal{P}(k)$ of the Kakutani-Rokhlin partition of (X, T) such that*

$$p(x, n) = cn$$

for all $n \in \mathbb{N}$ such that n is a return time of x to its cylinder set of level k under S, if and only if we can define a function $g \in C(X, \mathbb{Z})$ such that

$$p(x) = c + g(x) - g(S(x)) \text{ for all } x \in X.$$

In other words, if and only if $p(x) - c$ is an S-coboundary.

Proof. Take k such that $p(x)$ is constant on every cylinder set in $\mathcal{P}(k)$. As S is minimal, for any KR-partition $\mathcal{P}(k)$ there exists $k' > k$ such that any T-tower of $\mathcal{P}(k')$ contains all towers of $\mathcal{P}(k)$ with every possible orbit labeling (see Lemma 5.2.21). We construct the function g along the S-orbit of a base point of the partition. For $x_0 \in B_1^{k'}$ we set $g(x_0) = 0$ and then for $x_1 = S(x_0)$ we have

$$g(x_1) = c + g(x_0) - p(x_0).$$

By induction we can define g for any x in the S-orbit of x_0,

$$g(x_n) = nc + g(x_0) - p(x_0, n) \text{ for } x_n = S^n(x_0).$$

This construction gives every cylinder set $B_i^k(j)$ in $\mathcal{P}(k)$ the g-value of x_l such that $x_l \in B_i^k(j)$; here $B_i^k(j)$ indicates the j-th floor of the A_i^k-tower in $\mathcal{P}(k)$. Thus for $p(x) - c$ to be an S-coboundary, any return to a previously visited cylinder set has to give the same value of g. Hence, for any x in any base B_i^k of the KR-partition $\mathcal{P}(k)$ and n such that $S^n(x)$ returns to that base,

$$g(S^n(x)) = nc + g(x) - p(x, n) = g(x).$$

Therefore the function g is constant on every cylinder set of $\mathcal{P}(k)$.

On the other hand, if there exists a continuous function $g : X \mapsto \mathbb{Z}$ and a level k such that

$$g(x) - g(S(x)) = p(x) - c \text{ for all } x \in X \text{ and } g(x) = g(y) \text{ for all } x, y \in B_i^k(j),$$

then for any return time n of x to $B_i^k(j)$ under S

$$\begin{aligned} 0 &= g(x) - g(S^n(x)) \\ &= \sum_{m=0}^{n-1} (g(S^m(x)) - g(S^{m+1}(x)) \\ &= \sum_{m=0}^{n-1} (p(S^m(x)) - c) = p(x, n) - nc. \end{aligned}$$

□

Example 5.5.5. *Consider the length* 3*-Toeplitz shift on two symbols* $\{a, b\}$ *generated by the following substitution*

$$\chi : \begin{cases} a \to aab \\ b \to abb \end{cases}$$

and take any jump function with orbit number $c = 2$ *such that*

- *the speedup* (X, S) *is minimal,*
- *the orbit permutations are* $\pi_a^{(k)} = \pi_b^{(k)} = (12)$ *and*
- $o_a^k(1) = o_b^k(1)$ *(and therefore also* $o_a^k(2) = o_b^k(2)$*).*

Then (X,S) is conjugate to (X,T^2) and is therefore Toeplitz.

We may choose k such that the KR-partition $\mathcal{P}(k)$ has a constant jump function on every floor and such that every tower in the KR-partition $\mathcal{P}(k+2)$ contains every tower from $\mathcal{P}(k)$. To apply Proposition 5.5.4, we need to show that

$$p(x,n) = 2n \text{ for all first return times under } S \text{ of } x \text{ to its cylinder set } B_i^k.$$

The jump function $p(x,n)$ counts the number of T-steps between $x \in B_i^k$ and its return to the same cylinder set. Then

$$p(x,n) = mh^k$$

where m is the number of T-towers traversed between those consecutive returns to B_i^k and h^k is the height of the towers in $\mathcal{P}(k)$. By the permutation $\pi^{(k)}$, every return under S to the base passes through the same number of towers with orbit label j and towers with orbit label j'; thus $m = \#(j) + \#(j') = 2\#(j)$, where $\#(j)$ denotes the number of towers traversed with orbit label j. For $x \in A_i^k$ on floor d with orbit label j, we can count the number of S-steps until its return to the same floor

$$\begin{aligned} n &= o_i^k(j) + \#(i,j')o_i^k(j') + \#(i',j')o_{i'}^k(j') + \#(i',j)o_{i'}^k(j) \\ &= (\#(i,j') + \#(i',j'))o_i^k(j') + (\#(i',j)+1)o_i^k(j) \\ &= \#(j')o_i^k(j') + \#(j)o_i^k(j) = \#(j)h^k \end{aligned}$$

where $\#(i,j)$ is the number of occurrences of towers A_i^k with orbit label j between x and its return under S. Then it follows that

$$p(x,n) = mh^k = 2\#(j)h^k = 2n.$$

We now demonstrate that the only minimal speedups of the Toeplitz flow generated by χ with orbit number $c = 2$ which are also Toeplitz are conjugate to (X,T^2) and must arise as described in the above example.

Example 5.5.6. *Consider the Toeplitz shift (X,T) on two symbols $\{a,b\}$ generated by*

$$\chi : \begin{cases} a \to aab \\ b \to abb. \end{cases}$$

Every minimal speedup with orbit number $c = 2$ that is a Toeplitz flow is conjugate to (X,T^2).

Proof. **Case I:** Let S be a minimal speedup with orbit number $c = 2$. If $\pi_a^{(k)} = \pi_b^{(k)} = (12)$ for some $k \in \mathbb{N}$, then note that $\pi_a^{(l)} = \pi_b^{(l)} = (12)$ for all $l \geq k$. Consider the substitution

$\tau : (\{a,b\} \times \{1,2\}) \to (\{a,b\} \times \{1,2\})^*$ as defined in Definition 5.2.17 that models the speedup in terms of the towers and orbit paths traced.

$$\tau(a,1) = (a,1)(a,2)(b,1)$$
$$\tau(a,2) = (a,2)(a,1)(b,2)$$
$$\tau(b,1) = (a,1)(b,2)(b,1)$$
$$\tau(b,2) = (a,2)(b,1)(b,2)$$

Then τ is a primitive substitution that is conjugate to the substitution $\phi : \{A,B,C,D\} \to \{A,B,C,D\}^*$ by

$$\begin{aligned} \phi(A) &= ABC & \quad \text{where} \quad A &= (a,1)(a,2) \\ \phi(B) &= CBC & B &= (b,1)(a,2) \\ \phi(C) &= ABD & C &= (a,1)(b,2) \\ \phi(D) &= CBD & D &= (b,1)(b,2). \end{aligned}$$

Let $o_1(a)$ denote the number of jumps made by S in $(a,1)$, $o_2(a)$ denote the number of jumps made by S in $(a,2)$, and similarly for $o_1(b)$ and $o_2(b)$. Note that $o_1(a)+o_2(a) = 3^k$ and $o_1(b) + o_2(b) = 3^k$ for some fixed $k \in \mathbb{N}$.

We denote by $|A| = o_1(a) + o_2(a) = 3^k$ the number of jumps made by S in $A = (a,1)(a,2)$. We similarly define $|B| = o_1(b) + o_2(a)$, $|C| = o_1(a) + o_2(b)$, and $|D| = 3^k$.

Observe that if (X,S) is Toeplitz, then there is some point $x \in X$ that is Toeplitz under S. Further, there is a unique decomposition of x into its ϕ-words. Using the minimality of x and the definition of Toeplitz, there exist $l, j \in \mathbb{N}$ such that $S^{ln+j}(x)$ has a ϕ-decomposition that begins with A for all $n \in \mathbb{N}$.

$$\phi^2(A) = ABCCBCABD$$
$$\phi^2(B) = ABDCBCABD$$
$$\phi^2(C) = ABCCBCCBD$$
$$\phi^2(D) = ABDCBCCBD$$

Because $\phi^2(C)$ contains only one A, it follows that $l \geq |\phi^2(C)| = 8 \cdot 3^k + o_1(a) + o_2(b)$. Similarly, we can conclude that $l \geq |\phi^2(D)| = 9 \cdot 3^k$. Further observe that $|\phi^2(A)| = 9 \cdot 3^k$ and $|\phi^2(B)| = 8 \cdot 3^k + o_1(b) + o_2(a)$. Note that for all $m \in \mathbb{N}$, $\phi^m(A)$ and $\phi^m(C)$ differ in only one position and $\phi^m(C)$ and $\phi^m(D)$ differ in only one position.

$$\phi^3(A) = \phi^2(A)\phi^2(B)\phi^2(C) = ABCCBCABD\ ABDCBCABD\ ABCCBCCBD$$
$$\phi^3(B) = \phi^2(C)\phi^2(B)\phi^2(C) = ABCCBCCBD\ ABDCBCABD\ ABCCBCCBD$$
$$\phi^3(C) = \phi^2(A)\phi^2(B)\phi^2(D) = ABCCBCABD\ ABDCBCABD\ ABDCBCCBD$$
$$\phi^3(D) = \phi^2(C)\phi^2(B)\phi^2(D) = ABCCBCCBD\ ABDCBCABD\ ABDCBCCBD$$

Thus we return to A every $l = 9 \cdot 3^k$ jumps if and only if $|\phi^2(A)| = |\phi^2(B)| = |\phi^2(C)| = |\phi^2(D)| = 9 \cdot 3^k$, which occurs if and only if $|A| = |C|$ and $o_1(a) = o_1(b)$. Otherwise, $l \geq \max\{|\phi^2(B)\phi^2(D)|, |\phi^2(C)\phi^2(B)|, |\phi^2(A)\phi^2(B)|\} = \max\{17\cdot 3^k + o_1(b) + o_2(b), 18\cdot 3^k\}$. If $l = 18 \cdot 3^k$, we again deduce $o_1(a) = o_1(b)$; otherwise

$$l \geq \max\{|\phi^3(A)|, |\phi^3(B)|, |\phi^3(C)|, |\phi^3(D)|\}.$$

Inductively, we obtain that l is a finite value if and only if $o_1(a) = o_1(b)$.

We thus conclude that if (X, S) is a Toeplitz flow with orbit number $c = 2$ and $\pi_a^{(k)} = \pi_b^{(k)} = (12)$, then (X, S) is topologically conjugate to (X, T^2).

Case II: If $c = 2$ and $\pi_a^{(k)} = \pi_b^{(k)} = \mathrm{id}$, then (X, S) is not minimal.

Case III: Suppose we have that $\pi_a^{(1)} = (12)$ and $\pi_b^{(1)} = \mathrm{id}$. Then $\pi_a^{(2l+1)} = (12)$ and $\pi_b^{(2l+1)} = \mathrm{id}$ for all $l \in \mathbb{N}$. We again consider our substitution τ as defined in Definition 5.2.17, which is given by

$$\tau(a,1) = (a,1)(a,2)(b,1)(a,1)(a,2)(b,1)(a,1)(b,2)(b,2)$$
$$\tau(a,2) = (a,2)(a,1)(b,2)(a,2)(a,1)(b,2)(a,2)(b,1)(b,1)$$
$$\tau(b,1) = (a,1)(a,2)(b,1)(a,1)(b,2)(b,2)(a,2)(b,1)(b,1)$$
$$\tau(b,2) = (a,2)(a,1)(b,2)(a,2)(b,1)(b,1)(a,1)(b,2)(b,2).$$

We obtain the conjugate substitution

$$\begin{aligned}
\phi(A) &= AABCDAE & \text{where}\quad A &= (a,1)(a,2)(b,1)\\
\phi(B) &= AABDBDBCD & B &= (a,1)(b,2)(b,2)(a,2)\\
\phi(C) &= AABDBCD & C &= (a,1)(b,2)(a,2)\\
\phi(D) &= AABDBCDAEAE & D &= (a,1)(b,2)(a,2)(b,1)(b,1)\\
\phi(E) &= AABDBDBCBAEAE & E &= (a,1)(b,2)(b,2)(a,2)(b,1)(b,1).
\end{aligned}$$

As in Case I, we get $|A| = 3^k + o_1(b)$, $|B| = 3^k + 2o_2(b)$, $|C| = 3^k + o_2(b)$, $|D| = 2\cdot 3^k + o_1(b)$, and $|E| = 3 \cdot 3^k$ for some $k \in \mathbb{N}$. Additionally, we have $|\phi(A)| = 13 \cdot 3^k + o_1(b)$, $|\phi(B)| = 17 \cdot 3^k + 2o_2(b)$, $|\phi(C)| = 13 \cdot 3^k + o_2(b)$, $|\phi(D)| = 22 \cdot 3^k + o_1(b)$, and $|\phi(E)| = 27 \cdot 3^k$.

If there is a point $x \in X$ that is Toeplitz under S, then there exist $j, l \in \mathbb{N}$ such that $S^{ln+j}(x)$ has a ϕ-decomposition that begins with AAB for all $n \in \mathbb{N}$. However, we claim that there are no values for $o_1(b)$ and $o_2(b)$ such that this is possible. Hence, it is not possible for (X, S) to be Toeplitz when $c = 2$ with $\pi_a^{(1)} = (12)$ and $\pi_b^{(1)} = \text{id}$.

Case IV: Suppose we have $\pi_a^{(1)} = \text{id}$ and $\pi_b^{(1)} = (12)$. Then $\pi_a^{(2l+1)} = \text{id}$ and $\pi_b^{(2l+1)} = (12)$ for all $l \in \mathbb{N}$. We again consider the substitution τ as in Definition 5.2.17, given by

$$\begin{aligned}
\tau(a,1) &= (a,1)(a,1)(b,1)(a,2)(a,2)(b,2)(a,1)(b,1)(b,2)\\
\tau(a,2) &= (a,2)(a,2)(b,2)(a,1)(a,1)(b,1)(a,2)(b,2)(b,1)\\
\tau(b,1) &= (a,1)(a,1)(b,1)(a,2)(b,2)(b,1)(a,2)(b,2)(b,1)\\
\tau(b,2) &= (a,2)(a,2)(b,2)(a,1)(b,1)(b,2)(a,1)(b,1)(b,2).
\end{aligned}$$

Similar to the above cases, we obtain the conjugate substitution

$$\begin{aligned}
\phi(A) &= ABC\\
\phi(B) &= ABCADAEAECC\\
\phi(C) &= ABCADCC\\
\phi(D) &= ABCADAECCADAECCADAEAECC\\
\phi(E) &= ABCADAECCADAEAEDD.
\end{aligned}$$

where

$$\begin{aligned}
A &= (a,1)\\
B &= (a,1)(b,1)(a,2)(a,2)(b,2)\\
C &= (a,1)(b,1)(b,2)\\
D &= (a,1)(b,1)(a,2)(b,2)(b,1)(a,2)(b,2)(b,1)(a,2)(a,2)(b,2)\\
E &= (a,1)(b,1)(a,2)(b,2)(b,1)(a,2)(a,2)(b,2).
\end{aligned}$$

By counting the number of jumps S makes in each letter, we see that $|A| = o_1(a)$, $|B| = 2 \cdot 3^k + o_2(a)$, $|C| = 3^k + o_1(a)$, $|D| = 4 \cdot 3^k + 3o_2(a)$, and $|E| = 3 \cdot 3^k + 2o_2(a)$ for some $k \in \mathbb{N}$. We further have $|\phi(A)| = 4 \cdot 3^k + o_1(a)$, $|\phi(B)| = 22 \cdot 3^k + o_2(a)$, $|\phi(C)| = 13 \cdot 3^k + o_1(a)$, $|\phi(D)| = 48 \cdot 3^k + 3o_2(a)$, and $|\phi(E)| = 35 \cdot 3^k + 2o_2(a)$.

If there is a point $x \in X$ that is Toeplitz under S, then there exist $j, l \in \mathbb{N}$ such that $S^{ln+j}(x)$ has a ϕ-decomposition that begins with ABC for all $n \in \mathbb{N}$. Similar to Case III, due to the formulas for the number of S-jumps in each letter, there are no possible values for $o_1(a)$ and $o_2(a)$ such that any such $l \in \mathbb{N}$ can exist. We thus conclude that it is not possible for (X, S) to be Toeplitz when $c = 2$, $\pi_a^{(1)} = \text{id}$, and $\pi_b^{(1)} = (12)$.

By considering all cases, we conclude that the only minimal speedups (X, S) of (X, T) (where (X,T) is generated by χ) with orbit number $c = 2$ that are Toeplitz occur precisely when $\pi_a = \pi_b = (12)$ and $o_1(a) = o_1(b)$. Therefore, every Toeplitz speedup of (X, T) with $c = 2$ is conjugate to (X, T^2). □

To further analyze Toeplitz flows under constant speedups, we first give an algorithm on how to find the substitution of a constant speedup and then use the previous example of the length 3-Toeplitz system to look for factors of constantly sped-up systems. Let (X, T) be a Toeplitz subshift generated by a proper, primitive, constant length substitution θ. To find the substitution of the constant speedup (X, T^c) with $\gcd(|\theta|, c) = 1$ we do the following:

- Define the alphabet $\mathcal{B}$ of the sped-up substitution as the set of all words of length c in the language of θ.

- For every such word w of length c we recode the mapping $\theta(w)$ into the new alphabet $\mathcal{B}$. This recoding will be the substitution generating the subshift (X, T^c). It (or one of its powers) will be a proper, primitive, and constant length substitution, as it inherits these properties from θ.

Example 5.5.7. *Let (X_χ, T) be a Toeplitz subshift generated by $\chi : a \mapsto aab,\ b \mapsto abb$. The substitution τ of the constant 2-speedup will be on the alphabet $\mathcal{B} = \{A, B, C, D\}$ where we sort the 2-words lexicographically. Then τ is defined in the following way.*

$$\tau(A) = \chi(aa) = aabaab = ACB$$
$$\tau(B) = \chi(ab) = aababb = ACD$$
$$\tau(C) = \chi(ba) = abbaab = BCB$$
$$\tau(D) = \chi(bb) = abbabb = BCD$$

We see that τ^2 is a proper, primitive substitution of constant length 3^2.

$$\tau^2(A) = ACB\ BCB\ ACD$$
$$\tau^2(B) = ACB\ BCB\ BCD$$
$$\tau^2(C) = ACD\ BCB\ ACD$$
$$\tau^2(D) = ACD\ BCB\ BCD$$

To find a factor map $\mathcal{F}$ from (X_τ, T^2) to (X_χ, T) we use techniques from [CQY16]. First we scale both substitutions to the same length and analyze which positions of the substitution are not coincidences. For τ there are two such positions in $w_0 w_1 \cdots w_8$: at position 2 there is a choice between B and D and at position 6 there is a choice between between A and B. For the original χ we see that the only difference is at position 4

$$\chi^2(a) = aab\ aab\ abb$$
$$\chi^2(b) = aab\ abb\ abb.$$

For a factor map $\mathcal{F}$ to exist, we need to map non-singular fibers of X_τ to all non-singular fibers of X_χ. All other non-singular fibers must collapse to singular fibers. First we shift χ^2 by 7 such that the difference is at the same position as a difference in τ^2 and compare:

$$\chi'^2(a) = b\ b\ a \quad a\ b\ a \quad a\ b\ a$$
$$\chi'^2(b) = b\ b\ a \quad a\ b\ a \quad b\ b\ a$$
$$\tau^2(A) = ACB\ BCB\ ACD$$
$$\tau^2(B) = ACB\ BCB\ BCD$$
$$\tau^2(C) = ACD\ BCB\ ACD$$
$$\tau^2(D) = ACD\ BCB\ BCD.$$

We define the map $\mathcal{F}$ such that

$$\mathcal{F}(A) = b,\ \mathcal{F}(B) = a,\ \mathcal{F}(C) = b,\ \mathcal{F}(D) = a,$$

the non-singular fiber of the pair $\{B, D\}$ collapses to a singular fiber, and the pair $\{A, B\}$ stays non-singular. The choice of which fiber collapses was arbitrary, as there exists another factor map where $\{A, B\}$ collapses by shifting χ^2 by 2. Constant speedups with an even orbit number have no non-singular fiber at the middle position, thus these cases always need to shift the substitution to get overlapping differences to the original system X_χ.

Factor maps from other constant speedups of the same Toeplitz system onto the original X_χ can be found in a similar way. We now take a few moments to comment on several examples to highlight the connection between specific constant speedups of the same Toeplitz flow.

Example 5.5.8. *Let (X_χ, T) be a Toeplitz subshift generated by $\chi : a \mapsto aab,\ b \mapsto abb$.*

1. *Letting* $c = 5$*, there are 10 words of length 5 in the language of* X_χ*. The proper, primitive substitution of the speedup* (X_χ, T) *is of length 27.*

$$
\begin{aligned}
A &\mapsto AHCEHIBGD\ BHCEHIFGD\ AHDEGIBGJ\\
B &\mapsto AHCEHIBGD\ BHCEHIFGD\ AHDEGIFGJ\\
C &\mapsto AHCEHIBGJ\ BHCEGIFGD\ AHDEGIFGJ\\
D &\mapsto AHCEHIBGJ\ BHCEGIFGD\ BHDEGIFGJ\\
E &\mapsto AHCEHIBGJ\ BHCEHIFGD\ AHDEGIBGJ\\
F &\mapsto AHCEHIBGJ\ BHCEHIFGD\ AHDEGIFGJ\\
G &\mapsto AHDEHIBGD\ BHCEGIFGD\ BHDEGIBGJ\\
H &\mapsto AHDEHIBGD\ BHCEHIFGD\ BHDEGIBGJ\\
I &\mapsto AHDEHIBGJ\ BHCEGIFGD\ AHDEGIFGJ\\
J &\mapsto AHDEHIBGJ\ BHCEGIFGD\ BHDEGIFGJ
\end{aligned}
$$

We see non-singular fibers at positions 2 *with* $\{C, D\}$*, at* 8 *with* $\{D, J\}$*, at* 13 *with* $\{G, H\}$*, at* 18 *with* $\{A, B\}$*, and at* 24 *with* $\{B, F\}$*. The original substitution of length 27 is*

$$
\begin{aligned}
a &\mapsto\ a\,a\,b\,a\,a\,b\,a\,b\,b\quad a\,a\,b\,a\,a\,b\,a\,b\,b\quad a\,a\,b\,a\,b\,b\,a\,b\,b\\
b &\mapsto\ a\,a\,b\,a\,a\,b\,a\,b\,b\quad a\,a\,b\,a\,b\,b\,a\,b\,b\quad a\,a\,b\,a\,b\,b\,a\,b\,b.
\end{aligned}
$$

To collapse all but the middle fiber we map $\{A, B, E, F, H\}$ *to a and* $\{C, D, G, I, J\}$ *to b. Hence we see that the original Toeplitz flow is a factor of the Toeplitz flow* (X_χ, T^5)*.*

2. *If we take* $c = 7$*, then there are 14 words of length 7 in the language of* X_χ*. The seven non-singular fibers are at positions 1 with* $\{L, M\}$*, at 5 with* $\{H, I\}$*, at 9 with* $\{A, B\}$*, at 13 with* $\{C, E\}$*, at 17 with* $\{K, I\}$*, at 21 with* $\{B, G\}$*, and at 25 with* $\{D, N\}$*. Thus, the factor maps* $\{A, B, C, F, G, L, M\}$ *to a and* $\{D, E, H, I, J, K, N\}$ *to b. Again, the original Toeplitz flow is a factor of the Toeplitz flow* (X_χ, T^7)*.*

3. *For the constant speedup by* $c = 4$*, we take the alphabet* $\{0, 1, \dots, 7\}$ *of size 8. We find 4 non-singular fibers at positions 1 with* $\{2, 7\}$*, at 3 with* $\{0, 3\}$*, at 5 with* $\{4, 5\}$ *and at 7 with* $\{1, 2\}$*.*

   To map this onto the constant 2*-speedup (see Example 5.5.7), we shift the* 2*-speedup* *once to have the differences at the same positions.*

$$\begin{aligned}
0 &\mapsto 0\,2\,4 \quad 0\,6\,5 \quad 3\,1\,5 \\
1 &\mapsto 0\,2\,4 \quad 3\,6\,4 \quad 3\,1\,5 \\
2 &\mapsto 0\,2\,4 \quad 3\,6\,4 \quad 3\,2\,5 \\
3 &\mapsto 0\,2\,4 \quad 3\,6\,5 \quad 3\,1\,5 \\
4 &\mapsto 0\,7\,4 \quad 0\,6\,4 \quad 3\,2\,5 \\
5 &\mapsto 0\,7\,4 \quad 0\,6\,5 \quad 3\,2\,5 \\
6 &\mapsto 0\,7\,4 \quad 3\,6\,4 \quad 3\,1\,5 \\
7 &\mapsto 0\,7\,4 \quad 3\,6\,4 \quad 3\,2\,5 \\
A &\mapsto CBB\ CBA\ CDA \\
B &\mapsto CBB\ CBB\ CDA \\
C &\mapsto CDB\ CBA\ CDA \\
D &\mapsto CDB\ CBB\ CDA
\end{aligned}$$

Thus, we see that mapping $\{5\}$ *to* A*,* $\{4,6,7\}$ *to* B*,* $\{0,3\}$ *to* C *and* $\{1,2\}$ *to* D *gives us a factor map from the constant* 4*-speedup to the constant* 2*-speedup. That is,* (X_χ, T^2) *is a factor of* (X_χ, T^4)*.*

4. *For the constant speedup by* $c = 8$ *or* $c = 10$*, a similar procedure shows that the system sped-up by any divisor of* c *is a factor. The constant* 8*-speedup has the constant* 4*-speedup as a factor and therefore also the constant* 2*-speedup and the original system as factors. The constant* 10*-speedup has the constant* 2*-, the constant* 5*-speedup, and the original system as factors.*

We believe that the relationships highlighted in the previous example hold for any constant length left proper substitution θ, any $c > 1$ with $\gcd(|\theta|, c) = 1$, and any factor l of c.

Conjecture 5.5.9. *Given any constant length, left proper substitution* θ *and* $c > 1$ *such that* $\gcd(c, |\theta|) = 1$*, if* l *is a factor of* c *with* $1 \leq l < c$*, then the Toeplitz flow* (X_θ, T^l) *is a factor of* (X_θ, T^c)*.*

We now return our attention to S-coboundaries for minimal speedups of Toeplitz flows.

Theorem 5.5.10. *Suppose that* (X, T) *is a Toeplitz flow and* (X, S) *is a minimal speedup of* (X, T)*. If there exists some* $k \in \mathbb{N}$ *such that for the KR-partition* $\mathcal{P}(k)$ *we have that* *every tower has the same cyclic orbit permutation and each orbit has the same 'height' in each tower (i.e., for every* $1 \leq j \leq c$, $o_i^k(j) = o_{i'}^k(j)$ *for all towers* i, i'*), then there exists an* S*-coboundary for* $p(x) - c$.

Proof. Let k be such that the jump function $p(x)$ is constant on every cylinder set of $\mathcal{P}(k)$. For an arbitrary $x \in B_i^k$, take a return time n such that $S^n(x) = T^{p(x,n)}(x) \in B_i^k$. Therefore we know $h^k = p_k$ divides $p(x, n)$ and by the cyclic permutation $\pi^{(k)}$, it takes a multiple of c steps to return to the same label. Thus $p(x, n) = cmh^k$ where m is the number of T-towers with label j between x and $T^{p(x,n)}(x)$. To compute the number of steps under S, we take sums over all labels j and towers i between x and $S^n(x)$, recall that $\#(i, j)$ is the number of occurrences of tower A_i^k with label j,

$$\begin{aligned} n &= \sum_j \sum_i \#(i,j) o_i^k(j) = \sum_j o_{i_o}^k(j) \sum_i \#(i,j) \\ &= \sum_j o_{i_o}^k(j) \#(j) = m \sum_j o_{i_o}^k(j) \\ &= mh^k \end{aligned}$$

with $o_i^k(j) = o_{i'}^k(j)$ for all towers i, i' and by the permutation every label has the same number of occurrences $\#(j) = \#(j')$ in the cm towers between. Then $p(x, n) = cmh^k = cn$ and by Proposition 5.5.4 there exists an S-coboundary for $p(x) - c$. □

Corollary 5.5.11. *Suppose that* (X, T) *is a Toeplitz flow and* (X, S) *is a minimal speedup of* (X, T) *such that* $\gcd(c, p_k) = 1$ *for all* k *and the assumptions of the previous theorem hold. Then* (X, S) *is conjugate to* (X, T^c).

Proof. Under the condition $\gcd(c, p_k) = 1$ for all k we know that (X, T^c) is a Toeplitz flow, see Theorem 5.4.1. As (X, S) has an S-coboundary for $p(x) - c$, Proposition 2.6 of [AAO18] proves the conjugacy. □

Conjecture 5.5.12. *If* (X, T) *is a Toeplitz flow and* (X, S) *is a minimal speedup of* (X, T) *that is also a Toeplitz flow, then* $p(x) - c$ *is an* S*-coboundary. Further, if* $\gcd(c, p_k) = 1$ *for all* k*, then* (X, S) *is topologically conjugate to* (X, T^c).

We now demonstrate that there exist minimal speedups (X, S) of Toeplitz flows (X, T) with orbit number c such that (X, S) is Toeplitz and not conjugate to (X, T^c). By construction, the speedup (X, S) will have an S-coboundary for $p(x) - c$ even though $(c, p_k) > 1$ for every k.

Example 5.5.13. *Let* $\mathcal{A} = \mathcal{A}_0 = \{0, 1\}$ *and for each* $i \in \mathbb{N}$*, define* $\mathcal{A}_i = \{a, b\}$*. Define* $\theta_1 : \mathcal{A}_1 \to \mathcal{A}$ *by* $\theta_1(a) = 1001$ *and* $\theta_1(b) = 1010$*. For all* $i \geq 2$*, define* $\theta_i(a) = aab$ *and*

$\theta_i(b) = abb$. Then the S-adic shift generated by this sequence of substitutions generates a Toeplitz flow (X, T) with period structure $(p_k) = (4 \cdot 3^k)$.

Let $A = [\theta_1(\theta_2(a))] = \{x \mid x_{[0,11]} = 1001\ 1001\ 1010\}$ and $B = [\theta_1(\theta_2(b))] = \{x \mid x_{[0,11]} = 1001\ 1010\ 1010\}$ and define $p : X \to \mathbb{Z}^+$ by

$$p(x) = \begin{cases} 2 & \text{if } x \in A \cup T^4(A) \cup T^8(A) \cup B \cup T^4(B) \cup T^8(B) \\ 2 & \text{if } x \in T(A) \cup T^5(A) \cup T^9(A) \cup T(B) \cup T^5(B) \cup T^9(B) \\ 3 & \text{if } x \in T^2(A) \cup T^6(A) \cup T^{10}(A) \cup T^2(B) \cup T^6(B) \cup T^{10}(B) \\ 1 & \text{if } x \in T^3(A) \cup T^7(A) \cup T^{11}(A) \cup T^3(B) \cup T^7(B) \cup T^{11}(B). \end{cases}$$

With this given jump function p, (X, S) is a minimal bounded speedup of (X, T) where $S(x) = T^{p(x)}(x)$. In this case, the orbit number is $c = 2$. It is easy to check that for all $k \in \mathbb{N}$, the orbit permutations are $\pi_a^{(k)} = \pi_b^{(k)} = (12)$ and the orbit heights are such that $o_a^k(1) = o_a^k(2) = o_b^k(1) = o_b^k(2)$. By Theorem 5.5.1, it follows that (X, S) is a Toeplitz flow.

We can represent (X, S) as an S-adic shift generated by the sequence of alphabets $\{\mathcal{B}_i\}_{i \geq 0}$ and substitutions $(\phi_i\}_{i \geq 1}$ given by $\mathcal{B}_0 = \{0, 1, 2, \ldots, 6\}$, $\mathcal{B}_i = \{a, b, c, d\}$ for all $i \geq 1$, $\phi_1(a) = 012304230156$, $\phi_1(b) = 015604230156$, $\phi_1(c) = 012304230456$, $\phi_1(d) = 015604230456$, and for all $i \geq 2$ define $\phi_i(a) = abccbcabd$, $\phi_i(b) = abdcbcabd$, $\phi_i(c) = abccbccbd$, and $\phi_i(d) = abdcbccbd$.

Observe that (X, S) has period structure $(q_k) = (12 \cdot 9^k)$. We conclude that (X, S) and (X, T) have the same underlying odometer. Further note that although $p(x) - c$ is an S-coboundary by Theorem 5.5.10, (X, S) is not topologically conjugate to $(X, T^c) = (X, T^2)$, since (X, T^2) is not minimal.

Recall that there exists a speedup of the odometer (X_α, T) with $\alpha = (\alpha_1, \alpha_2, \ldots)$ that has orbit number c if and only if there is some $N \in \mathbb{N}$ such that $\gcd(c, \alpha_i) = 1$ for all $i \geq N$. We now demonstrate how one can generalize Theorem 5.2.15 to the Toeplitz flow setting.

Theorem 5.5.14. *If (X, T) is a Toeplitz flow with period structure (p_k) and $c \geq 1$ is an integer such that for some $N \in \mathbb{N}$, $\gcd\left(c, \frac{p_{k+1}}{p_k}\right) = 1$ for all $k \geq N$, then there is a minimal bounded speedup (X, S) of (X, T) with orbit number c such that (X, S) is a Toeplitz flow with the same underlying odometer as (X, T). Further, (X, S) and (X, T) are not conjugate.*

Proof. Let (X, T) be a Toeplitz flow with period structure (p_k) and underlying odometer given by $\alpha = (\alpha_1, \alpha_2, \ldots)$ where $\alpha_1 = p_1$, and $\alpha_i = \frac{p_i}{p_{i-1}}$ for all $i \geq 2$. Suppose there exist

$c, N \in \mathbb{N}$ such that for all $i \geq N$, $\gcd(c, \alpha_i) = 1$. By [AAO18, Theorem 3.8], there exists some speedup of the odometer (X_α, T_α) with orbit number c; further, by [AAO18, Lemma 3.7], there exists K such that for all $k \geq K$ $\pi^{(k)}$ is a cyclic permutation on $\{1, 2, \ldots, c\}$ for (X_α, T_α). Let $M = \max(N, K)$. Define the jump function $p(x)$ on the towers of the KR-partition for (X, T) at level M so that each tower has the same orbit labellings as the single tower at level M for the KR-partition for (X_α, T_α) with orbit number c. Therefore, for (X, T) with speedup $p(x)$, every tower has the same orbit labeling permutation $\pi^{(M)}$ which is a cyclic permutation on $\{1, 2, \ldots, c\}$; additionally, the heights for each orbit agree in every tower.

We first show that $(X, T^{p(x)})$ is minimal. Note that for every level k in the sequence of KR-partitions for (X, T), every tower can be interpreted as a stack of towers from the $(k-1)$st level such that the base tower is the same tower from the $(k-1)$st level, which we will call the *primary tower at level* $k-1$. Without loss of generality, suppose that every tower at level k contains every tower from level $k-1$ (we may always choose a subsequence of our KR-partitions so this is true). Since $\pi^{(k)}$ is a permutation of $\{1, 2, \ldots, c\}$ for all $k \geq M$ and every tower has the same permutation, if one starts on the base floor of any tower at level $k \geq M$, one will traverse through every orbit labelling of every tower from level $k-1$ after traversing through c towers. As this is true for all $k \geq M$, we conclude that our speedup will be minimal.

Lastly, because every tower in the KR-partition for (X, T) with k large enough has every tower with the same orbit permutation and every orbit has the same height in each tower, by Theorem 5.5.1, we conclude that $(X, S) = (X, T^{p(x)})$ is a Toeplitz flow. By Theorem 5.2.16, we conclude that (X, S) and (X, T) are not conjugate. $\square$

Remark 5.5.15. Observe that it immediately follows from Theorem 5.5.14 that every Toeplitz flow (X, T) with underlying odometer (X_α, T_α) has a minimal speedup (X, S) which is a Toeplitz flow if there exists some prime p such that p only divides finitely many α_i. Thus, given a Toeplitz flow (X, T) with underlying odometer given by α, where $\alpha = (\alpha_1, \alpha_2, \ldots)$ does not have every natural number as a divisor, it will follow that there is some minimal speedup that generates a Toeplitz flow with the same underlying odometer. In particular, a Toeplitz flow with underlying odometer given by $\alpha = (2, 3, 5, 7, \ldots)$ will have a minimal speedup which is a Toeplitz flow for every orbit number $c \in \mathbb{N}$.

Question 5.5.16. Is it possible to speedup a Toeplitz flow in such a way to obtain a Toeplitz flow with a different underlying odometer?

We conclude with two conjectures about speedups of Toeplitz flows.

Conjecture 5.5.17. *If* $\gcd\left(c, \frac{p_k}{p_{k-1}}\right) > 1$ *for infinitely many* k*, then* (X, T) *does not have a minimal speedup with orbit number* c *that is a Toeplitz flow. Hence, if* θ *is a constant length left proper substitution, then the Toeplitz flow* (X_θ, T) *does not have a minimal speedup with orbit number* $c = |\theta|$ *that is also a Toeplitz flow.*

Conjecture 5.5.18. *Suppose* (X, T) *is a Toeplitz flow with period structure* (p_k) *and suppose* (X, S) *is a bounded speedup of* (X, T) *with orbit number* c*. Then* (X, S) *is a Toeplitz flow if and only if for all sufficiently large* k*,* $\pi_i^{(k)}$ *is the same cyclic permutation on* $\{1, 2, \ldots, c\}$ *for each tower of the KR-partition* $\mathcal{P}(k)$ *and the orbit heights are the same in each tower, (i.e.,* $o_i^k(m) = o_j^k(m)$ *for each* $m \in \{1, 2, \ldots, c\}$ *and all* i, j*).*

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6. Bratteli-Vershik Systems of Interval Exchange Transformations

The purpose of this chapter is to describe the Bratteli-Vershik diagrams and the behaviour of Bratteli-Vershik systems constructed from interval exchange transformations (IETs). We investigate the conjugacy between them, look into the general structure of such diagrams and introduce an ordering on the outgoing edges. To construct the Bratteli-Vershik diagram we will mainly use the algorithm of Gjerde-Johansen. Another approach is to use the Rauzy induction.

Interval Exchange Transformation.

Let π be an irreducible permutation of letters $\mathcal{A} = \{1, \ldots, d\}$ and the vector $\lambda = (\lambda_1, \ldots, \lambda_d)$ in the $d-1$-dimensional $\mathbb{R}$-simplex. The interval $[0,1)$ is partitioned into half-open subintervals $\Delta_1, \ldots, \Delta_d$ with corresponding lengths $\lambda_1, \ldots, \lambda_d$.

The tuple (π, λ) describes an interval exchange transformation $T = T_0$ on $[0,1)$ with

$$T(x) = x - \sum_{j<i} \lambda_j + \sum_{\pi(j)<\pi(i)} \lambda_j$$

for $x \in \Delta_i$. Let the left endpoint of the interval Δ_i be γ_i, then

$$\gamma_i = \sum_{j<i} \lambda_j.$$

To insure that the transformation is minimal, we choose (π, λ) to satisfy the *Keane condition*. This means the orbits of the endpoints are disjoint: $T^n(\gamma_i) \neq \gamma_j$ for all $n \in \mathbb{N}$ and $i \in \mathcal{A}$, $j \in \mathcal{A} \setminus \{1\}$. We call $\Lambda_d(\pi)$ the region of the simplex such that the Keane condition holds for the permutation π, see [Via06] for more about IETs.

The Rauzy induction. This induction on the parameter space $\mathcal{S}^d \times \Lambda_d(\pi)$, where $\mathcal{S}^d$ is the space of permutations of d elements, maps T to its first return map to the interval $[0,1) \setminus (\Delta_d \cap T(\Delta_{\pi^{-1}(d)}))$. It maps the IET (π, λ) to $R(\pi, \lambda) = (\pi', \lambda')$. If $\lambda_d > \lambda_{\pi^{-1}(d)}$

the induction step is of Type 0, if $\lambda_d < \lambda_{\pi^{-1}(d)}$, it is of Type 1. In each induction step the longer interval is called the winner, the shorter interval the loser. Every step of the Rauzy induction can be represented by a substitution on $\mathcal{A}$. If the IET (π, λ) is of Type 0, then

$$\chi_0 : \begin{cases} e \to ed & \text{if } e = \pi^{-1}(d) \\ j \to j & \text{else.} \end{cases} \tag{6.1}$$

The induction step changes the lengths to $\lambda'_d = \lambda_d - \lambda_e$ and $\lambda'_j = \lambda_j$ for all $j \neq d$. If the IET (π, λ) is of Type 1, then

$$\chi_1 : \begin{cases} j \to j & \text{if } 1 \leq j \leq \pi^{-1}(d) \\ e+1 \to ed & \text{if } e = \pi^{-1}(d) \\ j \to j-1 & \text{if } \pi^{-1}(d)+1 < j \leq d. \end{cases} \tag{6.2}$$

Here the length vector λ' is $\lambda'_j = \lambda_j$ for all $j < e$, $\lambda'_e = \lambda_e - \lambda_d$, $\lambda'_{e+1} = \lambda_d$ and $\lambda'_j = \lambda_{j-1}$ for all $j > e+1$. We will use the substitutions χ_0 and χ_1 of each step of the Rauzy induction to construct a Bratteli-Vershik diagram with transition matrices $M_R(n)$.

As the IET satisfies the Keane condition, the intervals $\Delta_{\pi^{-1}(d)}$ and Δ_d never have the same length. Thus the Rauzy induction is possible for infinitely many steps. We know from [Via06], that the lengths of all intervals converge to zero as the number of Rauzy induction steps grows to infinity. Further from a starting permutation π of d symbols, only a certain subset of all irreducible permutations of d symbols can be reached by any number of Rauzy steps. This subset is called the *Rauzy class* of the permutation π.

The Zorich induction is an acceleration of the Rauzy induction where all consecutive steps of Type 0 or Type 1 are taken at once.

The Gjerde-Johansen induction. The Gjerde-Johansen induction, see [GJ02], maps the IET T to its first return map to the interval $[0, 1) \setminus \Delta_d$. It is an accelerated version of the Rauzy induction. The IET after n steps of the Gjerde-Johansen induction on the starting IET T_0 defined by (π, λ), will be called T_n with the permutation $\pi^{(n)}$, length vector $\lambda^{(n)}$ and subintervals $\Delta_i^{(n)}$ for all i. As with the Rauzy induction, under the assumption that the Keane condition holds, the Gjerde-Johansen induction is possible infinitely often. Further, the resulting IET T_n still has d subintervals, $\Delta_1^{(n)}, \ldots, \Delta_d^{(n)}$.

As the permutation π is irreducible we know $\pi(d) < d$. Thus we define the non-empty set

$$J = \{j \in \{1, \ldots, d-1\} : \pi(j) > \pi(d)\} \tag{6.3}$$

as the set of letters i where the interval Δ_i is mapped to the right of $T(\Delta_d)$. Further we define the interval Δ_c such that $\gamma_d \in T^{r+1}(\Delta_c)$ where $r \in \mathbb{N}_0$ is the minimal number of steps until $\gamma_d \notin T^{r+1}(\Delta_d)$ and the set

$$J' = \{j \in J : \pi(j) > \pi(c)\} \tag{6.4}$$

as the intervals behind $T(\Delta_c)$. One step of the Gjerde-Johansen induction is represented by the following substitution

$$\chi : \begin{cases} j \to j & \text{if } 1 \leq j < c,\ j \notin J \\ j \to jd^r & \text{if } 1 \leq j < c,\ j \in J \setminus J' \\ j \to jd^{r+1} & \text{if } 1 \leq j < c,\ j \in J' \\ c \to cd^r & \\ c+1 \to cd^{r+1} & \\ j \to j-1 & \text{if } c+1 < j \leq d,\ j \notin J \\ j \to (j-1)d^r & \text{if } c+1 < j \leq d,\ j \in J \setminus J' \\ j \to (j-1)d^{r+1} & \text{if } c+1 < j \leq d,\ j \in J'. \end{cases} \tag{6.5}$$

Similarly to the Rauzy induction, we can compute the lengths of the subintervals after a Gjerde-Johansen induction step

$$\begin{aligned} &\lambda_i^{(1)} = \lambda_i && \text{for } 0 \leq i < c \\ &\lambda_c^{(1)} = r\textstyle\sum_{j\in J} \lambda_j + \sum_{j'\in J'} \lambda_{j'} + \lambda_c - \lambda_d && \\ &\lambda_{c+1}^{(1)} = \lambda_d - r\textstyle\sum_{j\in J} \lambda_j - \sum_{j'\in J'} \lambda_{j'} && \\ &\lambda_i^{(1)} = \lambda_{i-1} && \text{for } c+1 < i \leq d. \end{aligned}$$

Comparing it to the Rauzy induction, the Gjerde-Johansen induction consists of $r|J|+|J'|$ Type 0-Rauzy steps followed by one Type 1-step where the interval $T^{r+1}(\Delta_c)$ is cut in two by γ_d. In the Bratteli diagram this is represented as the Gjerde-Johansen induction being a telescoped version of the Rauzy induction. Let $M_{GJ}(n)$ be a transition matrix of the Gjerde-Johansen construction. Then for any $n \in \mathbb{N}$ there exist Rauzy steps $n_0 < n_1$ such that $M_{GJ}(n) = M_R(n_1)\cdots M_R(n_0)$ where $M_R(n_1)$ is of Rauzy Type 1. The substitution χ also gives us the ordering of the incoming edges to each vertex of the Bratteli diagram constructed by the Gjerde-Johansen induction. It is a left-to-right incoming order.

6.1 Conjugacy Map

We will define a conjugacy between an IET $T = (\pi, \lambda)$ and the Bratteli-Vershik system (X_B, V_B) defined by the (Rauzy- or) Gjerde-Johansen induction. As the state space of the Bratteli diagram is a Cantor set and its dynamics a minimal homeomorphism we have to modify the IET and its state space.

Doubling the Interval Endpoints. The IET $T = (\pi, \lambda)$ is continuous from the right on every subinterval $\Delta_i \subseteq I = [0, 1)$ but can be discontinuous from the left at points γ_{i+1}. To change this and define the IET on a Cantor set, we double the orbits of interval endpoints.

Let $D_0 = \{\gamma_1, \ldots, \gamma_d\}$ be the set of left interval endpoints, we define the set of their orbits $D = \{x \in I : \exists n \in \mathbb{Z},\ T^n(\gamma) = x \text{ for } \gamma \in D_0\}$ and double these points $D^* = \{x^- : x \in D\}$. Then

$$I^* = I \cup D^*$$

where we define $0^- = 1$ and otherwise order the points $x^- < x$ for $x \in D \setminus \{0\}$. Extending the order $\leq$ on $[0, 1)$ to I^* by $y \leq x^-$ for all $y < x$ and $x^- < y$ for all $y \geq x$, gives a total order on the set I^*. The order topology of I^* has the basis

$$\mathcal{B} = \{(a, b) : a, b \in I^*\} \cup \{[0, b) : b \in I^*\} \cup \{(a, 1] : a \in I^*\}.$$

Lemma 6.1.1. *The sets $[x, y^-]$ for $x, y \in D$ are open and closed.*

Proof. We can write the complement of $[x, y^-]$ in the two ways, once with open sets and once with closed sets

$$\begin{aligned}
[x, y^-]^c &= [0, x) \cup (y^-, 1] \\
&= [0, x^-] \cup [y, 1] \\
&= (x^-, 1]^c \cup [0, y)^c
\end{aligned}$$

as there are no points between x^- and x. Thus the complement is open and closed and therefore so is $[x, y^-]$. □

It follows from this lemma that every set $[\gamma_i, \gamma_{i+1}^-]$ representing the original subinterval Δ_i is clopen and we will from now on define the set $\Delta_i = [\gamma_i, \gamma_{i+1}^-]$.

Metric on I^*. We define a metric on I^* by giving the gap between points x^- and x a certain distance. For all $i = 2, \ldots, d$ we set the distance between these doubled points as $d^*(\gamma_i^-, \gamma_i) = g_0 = 1/(2(d-1))$. There are $d-1$ such gaps. For points $x \in D$ only certain gaps between x^- and x will have a non-zero distance after n-steps of our construction. We will call these the significant gaps of step n. Which points create such a significant gap is dependent on the number of induction steps n and the current IET $(\pi^{(n)}, \lambda^{(n)})$. If looking at the first Rauzy induction step, the significant gap is at the point where the shorter of the two intervals Δ_d and $\Delta_{\pi^{-1}(d)}$ is cut, so at either $T(\gamma_{\pi^{-1}(d)})$ or $T^{-1}(\gamma_d)$. For the faster Gjerde-Johansen induction multiple significant gaps can appear in the first induction step, the exact number depends on the number r and sets J, J'. For points $x \in D$ becoming significant at the n-th induction step we define the distances between x^- and x as g_n with $g_n > g_{n+1}$. Thus the later in the sequence of inductions a gap becomes significant, the smaller the gap between x^- and x will be. We can define the sequence $(g_n)_n$ of gap lengths the following way

$$g_{n+1} = \frac{1}{2s(n+1)} g_n$$

where $s(n+1)$ is the number of significant gaps added in the $n+1$-th induction step. This gives that $g_n > \sum_{m>n} s(m) g_m$ for all $n \in N_0$.

Example 6.1.2. *We look at an IET with $d = 3$,*

$$\pi_0 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

and $\lambda_2 > \lambda_3 > \lambda_1$, see Figure 6.1. The points γ_2 and γ_3 have significant gaps of size g_0 added before the first induction step. After one Gjerde-Johansen induction the interval Δ_2 is cut in two, and thus the cutting point $T^{-1}(\gamma_3)$ has a significant gap of size g_1. In this step the original $\Delta_3 \subset [0, 1)$ is stacked on top of the new $\Delta_3^{(1)} \subset \Delta_2$. In the second Gjerde-Johansen induction step, the interval $\Delta_1^{(1)}$ is split at the point $T^2(\gamma_3)$ and the tower on $\Delta_3^{(1)}$ is cut at $T(\gamma_2)$ and $T^2(\gamma_2)$. At these points there is a significant gap of size g_2. We see that not all (pre-)images of original endpoints under T are significant in the first two level, only a select few.

The set of points at significant gaps is still dense, as we know that for an irreducible IET for which the Keane condition holds, the Rauzy induction is possible infinitely often and the lengths of all subinterval converges to zero.

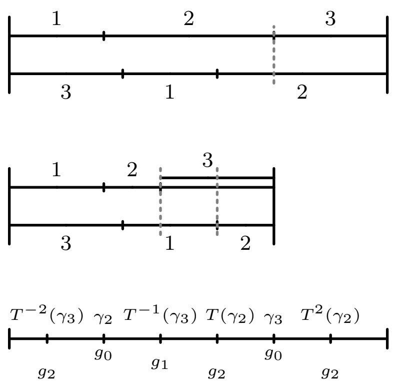


Figure 6.1: First two Gjerde-Johansen induction steps with tower cutting-and-stacking and the significant gaps on the set I^*.

For two points $x, y \in I^*$ with $x < y$, we define the following metric by summing the sizes of all gaps between x and y where $|g|$ is the length of gap g,

$$d^*(x,y) = \sum_{x<g<y} |g|.$$

And $d^*(x,y) = 0$ if and only if $x = y$ as by the denseness of the set of points at significant gaps, there is always a gap between $x \neq y$. The triangle inequality for $x < y < z$ holds as an equality

$$d^*(x,z) = d^*(x,y) + d^*(y,z).$$

The points furthest apart are 0 and $1 = 0^-$ and $d^*(0,1) < 1$.

This procedure is similar the construction of the middle-third Cantor set, at each level $l > 0$ we create a finite number gaps (for $l = 0$, $d-1$ gaps) of size g_l and the total length of all future gaps in the remaining subintervals is smaller than the gaps on either side. The position of gaps on each level is, dependent on the IET, not distributed equally. But by denseness, every subinterval gets gaps added at some level.

We observe that the metric on I^* creates gaps and stretches the remaining subintervals into certain sizes in every step. Thus if two points x, y are close in absolute value on the unit interval I, it does not follow that they are close in the metric $d*$ on I^*, as there can be a gap of fixed size in between. Although for an ε-neighbourhood of x in the absolute value $U_\varepsilon(x)$, there exists an ε-neighbourhood of x in the d^* metric $U^*_\varepsilon(x)$ in I^* such that $U_\varepsilon(x) \cap U^*_\varepsilon(x) \neq \emptyset$. On the other hand if two points $x, y \in I$ are close in the metric d^* of I^*, they must be close in absolute value. The further they are apart in absolute value the more likely it is that there are large gaps in between.

Extending the IET. We extend the IET T from $[0,1)$ to I^* by defining

$$T(x^-) = \begin{cases} \lim_{y \nearrow x} T(y) & \text{if } x \in D \text{ is a discontinuity of } T \text{ on } [0,1) \\ T(x)^- & \text{otherwise.} \end{cases}$$

Proposition 6.1.3. *(1) The space I^* is a Cantor set.*

(2) T maps D^* to D^* and is continuous on I^*.

(3) The map T on I^* is a minimal homeomorphism.

Proof. (1) We have to show that I^* is a metric, compact, totally disconnected set without isolated paths. The first follows from the definition of metric d^*. The set is compact as for any sequence (x_n) in I^* we can find a convergent subsequence with its limit in I^*. The argument, similar to the proof of the Theorem of Bolzano-Weierstraß, uses nested intervals. We start with $[\gamma_i, \gamma_{i+1}^-]$ which contains infinitely many x_n. This interval is decomposed into a finite number of closed intervals $[\gamma_i, T^{n_1}(\gamma_j)^-], [T^{n_1}(\gamma_j), T^{n_1}(\gamma_k)^-], \ldots [T^{n_1}(\gamma_l), \gamma_{i+1}^-]$ and we choose an interval which still contains infinitely many x_n. By continuing this we find a subsequence which converges in the smaller and smaller, closed subintervals of $[\gamma_i, \gamma_{i+1}^-]$.

To show total disconnectedness, take $x < y \in I^*$. If $y \in D$, we know $x \leq y^-$. Thus $x \in [0, y^-]$ which is clopen and $y \notin [0, y^-]$. If $y \notin D$, we know that D is dense and there exists some point $z \in D$ with $x < z < y$ and therefore $x \in [0, z^-]$ and $y \notin [0, z^-]$.

The space has no isolated points as by the denseness of D in $[0,1)$ we can always find points x^- or x close to any point in I^*.

(2) Taking $x^- \in D^*$, we assume $x \in D$ is a discontinuity of T on $[0,1)$, otherwise $T(x^-) = T(x)^- \in D^*$. We defined T as $T(x^-) = \lim_{y \nearrow x} T(y)$. If $T(x^-) = 1 = 0^- \in D^*$, otherwise $T(x^-)$ is the right endpoint of the set $T([z_0, x^-])$ with $z_0 \in D$. And there exists a set to the right of it which is the image of $[z_1, z_2^-]$ and $T(x^-) = T(z_1)^- \in D^*$.

The map T on I^* is continuous as the preimage of any open set is open.

(3) The same argument of continuity holds for the inverse T^{-1}. Further T is an one-to-one map on I and D^*, therefore it is bijective. To show minimality of T on I^*, we observe that for an arbitrary point $y \in I$ and $\delta > 0$, the δ-neighbourhood $U^*_\delta(y)$ contains a subinterval $I_\delta(y)$ of I with $y \in I_\delta(y)$ (which is not itself of size δ, it is contained in a δ-neighbourhood). Now we take arbitrary points $x, z \in I^*$ and $\varepsilon > 0$. In an $\varepsilon/2$-neighbourhood $U^*_{\varepsilon/2}(x)$ of x, there exists some $y_x \in I$. The same holds for z and an arbitrary $\delta > 0$, there exists a point $y_z \in I$ with $y_z \in U^*_{\delta/2}(z)$. As T satisfies the Keane

condition, it is minimal on I and there exists $n \in \mathbb{Z}$ such that $T^n(y_x) \in I_{\delta/2}(y_z) \subseteq U^*_{\delta/2}(y_z)$ and therefore

$$T^n(y_x) \in U^*_\delta(z).$$

By the continuity of T and T^{-1} in I^*, we pick δ such that

$$T^{-n}(z) \in U^*_{\varepsilon/2}(y_x),$$

thus $T^{-n}(z) \in U^*_\varepsilon(x)$ and T is a minimal homeomorphism on I^*. □

Conjugacy map. We will assume here that the Bratteli-Vershik system created by the Gjerde-Johansen induction is a minimal Cantor system, that the space of infinite paths of the Bratteli diagram X_B is a Cantor set and the Vershik map V_B is a minimal homeomorphism. We will see in Proposition 6.2.9 that these assumptions on the Bratteli diagram constructed by the Gjerde-Johansen induction are justified.

To show the conjugacy between the IET and Bratteli-Vershik system constructed by the Gjerde-Johansen induction, we define the map $g : I^* \to X_B$ which maps a point in I^* to an infinite path in X_B. We set $g(0) = x_{min}$ as the point zero stays in the first interval after any number of Gjerde-Johansen steps. Further on the orbit of zero we define $g(x_n) = g(T^n(0)) = V_B^n(g(0)) = V_B^n(x_{min})$ for $x_n = T^n(0)$.

Lemma 6.1.4. *The map g is uniformly continuous on orb(0).*

Proof. For any $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $1/2^n < \varepsilon$. Then for all points $x, y \in \text{orb}(0) \subseteq I^*$ such that $d^*(x, y) < g_n$ it follows that there are no significant gaps up to the n-th induction step between x and y. Thus both points stay in the same subinterval for at least n steps and therefore $d_B(g(x), g(y)) < 1/2^n < \varepsilon$. □

As the IET T is minimal on I^* the orbit of zero is dense and we can extend g continuously to all points of I^*. For any $x \in I^*$ there exists an sequence (x_{n_k}) where x_{n_k} is in the orbit of zero and $x_{n_k} \to x$ as $k \to \infty$. Then it follows by the uniform continuity of g on the orbit of zero and the continuity of T and V_B that

$$\begin{aligned} g(T(x)) &= g(T(\lim_{n\to\infty} x_n)) = g(\lim_{n\to\infty} T(x_n)) \\ &= \lim_{n\to\infty} g(T(x_n)) = \lim_{n\to\infty} V_B(g(x_n)) \\ &= V_B(\lim_{n\to\infty} g(x_n)) = V_B(g(\lim_{n\to\infty} x_n)) \\ &= V_B(g(x)). \end{aligned}$$

Proposition 6.1.5. *The map g is a conjugacy between I^* and X_B.*

Proof. As seen g is continuous and $g(T(x)) = V_B(g(x))$ for all $x \in I^*$. For $x \neq y \in I^*$ we know by the denseness of significant gaps that there will be a gap between x and y at some point, thus the paths $g(x)$ and $g(y)$ differ after some level and $g(x) \neq g(y)$. The map is surjective by definition of the (Rauzy or) Gjerde-Johansen construction. For the continuity of the inverse g^{-1} we take $\varepsilon > 0$. There exists $n \in \mathbb{N}$ such that $g_n < \varepsilon$. For all paths $\tilde{x}, \tilde{y} \in X_B$ such that $d^*(\tilde{x}, \tilde{y}) < 1/2^n$ the paths do not differ up to at least level n. Therefore $g^{-1}(\tilde{x})$ and $g^{-1}(\tilde{y})$ are in the same interval up to at least the n-th induction step. Thus there are no significant gaps of size g_n or larger in between and $d^*(g^{-1}(\tilde{x}), g^{-1}(\tilde{y})) < g_n$. □

Remark 6.1.6. The conjugacy g is not order-preserving. Suppose g were order-preserving and take x such that $T^n(x) < x < T^m(x)$ and $n = \min\{n_0 > m : T^{n_0}(x) < x\}$. Then as V is successor map and thus V^{-1} is also order-preserving

$$\begin{aligned} g(T^n(x)) &< g(T^m(x)) \\ V^m(g(T^{n-m}(x)) &< V^m(g(x)) \\ g(T^{n-m}(x)) &< g(x). \end{aligned}$$

This is an contradiction to g being order-preserving.

6.2 General Behaviour of IET Bratteli-Vershik diagrams

As the IET (π, λ) satisfies the Keane condition, the interval endpoints have disjoint orbits and thus the number of vertices never changes. We take $\{1, \dots, d\}$ as the vertex set on every level and generate the edges with the Gjerde-Johansen induction on the IET.

Types of Vertices and Incoming Edges. There are four distinct types of vertices and their incoming edges in a Bratteli diagram constructed by the Gjerde-Johansen induction, see Figure 6.2. The difference between the first two and last two types is that in the latter the interval Δ_i is mapped at least partly into Δ_d, so $T(\Delta_i) \cap \Delta_d \neq \emptyset$. There are two edges in the diagram which exist on every level E_n, $1 \to 1$ and $d-1 \to d$.

The number of edges incoming from vertex d depends on the minimal number of steps $r \in \mathbb{N}_0$ such that $\gamma_d \in T^{r+1}(\Delta_c)$ for $c \neq d$. In the last two types the number of edges from d is r or $r+1$, depending on whether $i \in J \setminus J'$ or J' (see definitions of sets in (6.3) and (6.4)). If Δ_d and its image $T(\Delta_d)$ have no intersection, then $r = 0$.

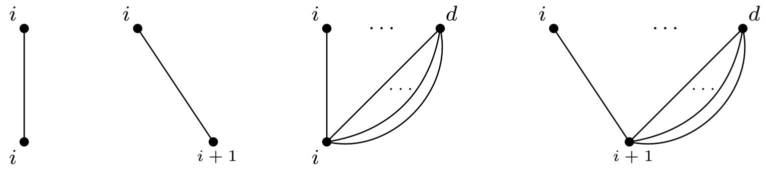


Figure 6.2: Types of vertices and their incoming edges depending on $T(\Delta_i) \cap \Delta_d$ and the position of the split interval.

The Split Interval. On every level there exists the interval Δ_c that splits in two. It is the only vertex on level n besides the vertex d that has multiple outgoing edges. Furthermore, the vertex $c+1$ on level $n+1$ has exactly $r+1>0$ incoming edges from $d \in V_n$, the vertex c on level $n+1$ has r incoming edges.

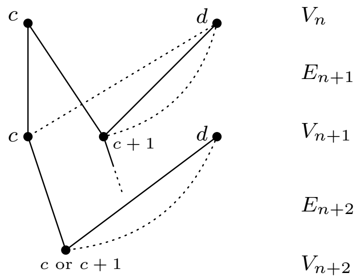


Figure 6.3: Bratteli diagram after the interval c is split, the dotted edges show the possibility of multiple edges coming from d (if $r>0$).

After the split, the vertex $c \in V_{n+1}$ represents the remaining part of the previous $\Delta_c^{(n)}$ of the IET. Then the new interval $\Delta_c^{(n+1)}$ maps to the rightmost position, $\pi^{(n)}(c) = d$. Therefore in the next step of the Gjerde-Johansen induction, it is either split up again (if the remaining $\Delta_c^{(n+1)}$ is longer than $\Delta_d^{(n+1)}$) or is a subset of $\Delta_d^{(n+1)}$. Thus there has to exist at least one edge on level $n+2$ from $d \in V_{n+1}$ to either $c \in V_{n+2}$ or $c+1 \in V_{n+2}$ (depending on the position of the split on level n+1). It follows that after the last time that the same interval c is split, there have to be multiple outgoing edges from d, at least one going to the latter part this levels split interval $c^{(n+1)}$ and at least one going to c or $c+1$ from the split before (depending of if $c < c^{(n+1)}$ or not).

There is a similar behaviour for vertices in $J \setminus J'$, see 6.4. Looking at the return map of $I^* \setminus \Delta_d$ (before renaming the intervals to be in order on the top line) we see the remaining part of Δ_c and the intervals $J \setminus J'$ are now at the end of the lower line. Their relative order to each other stays the same. Thus they are more likely to map into the new $\Delta_d^{(1)}$ than any other interval and therefore to have an incoming edge from the vertex $d \in V_{n+1}$

in the next Gjerde-Johansen induction steps. The right part of Δ_c and the intervals in J' need one step more to return to $I^* \setminus \Delta_d^{(1)}$. Thus these intervals will be mapped to the left of intervals in $J \setminus J'$. Before the next Gjerde-Johansen step, the intervals with labels $i > c$ will be renamed to $i + 1$.

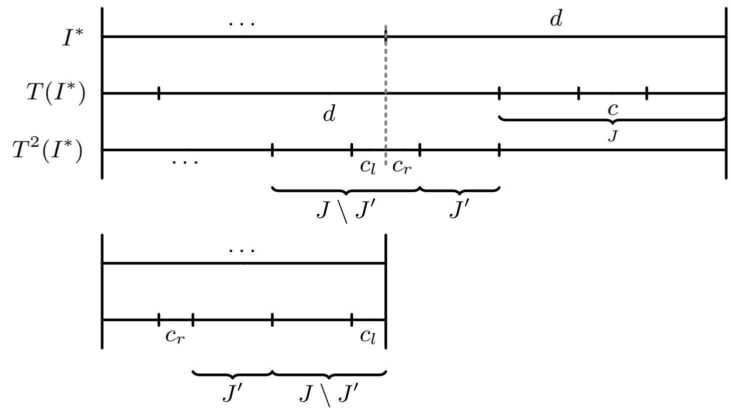


Figure 6.4: The intervals in $J \setminus (J' \cup c)$ as well as the left part of the interval Δ_c return to $I^* \setminus \Delta_d$ after two steps of the IET, the right part of Δ_c and the intervals in J' return after three steps. Inside these sets the relative order of intervals stays the same.

Simplicity of Gjerde-Johansen Bratteli-Vershik diagrams.

Definition 6.2.1. A Bratteli diagram $B = (V, E)$ is called **simple** if $|E_n| > 1$ for infinitely many levels $n \in \mathbb{N}$ and for every level m there exists $n > m$ such that for all $v \in V_m$ and for all $w \in V_n$ there exists a path $(e_{m+1}, \dots, e_n)$ with $s(e_{m+1}) = v$ and $r(e_n) = w$.

Under this condition the space X_B is a minimal Cantor set.

Proposition 6.2.2. *The Bratteli diagram constructed by the Gjerde-Johansen induction of an interval exchange transformation which satisfies the Keane condition is simple.*

We use the telescoping between the representation of the diagram by way of Gjerde-Johansen or Rauzy induction to prove simplicity. To differentiate between levels of the slower Rauzy and faster Gjerde-Johansen Bratteli diagrams, we will call the levels of the diagram from the Rauzy induction m_0, n_0 and levels of the diagram from the Gjerde-Johansen induction m, n. We know $m \le m_0, n \le n_0$.

Proof. In [Via06, Corollary 1.21] it is shown that for every $m_0 \in \mathbb{N}$ there exists an $n_0 > m_0$ such that the product of Rauzy transition matrices $P_R(m_0, n_0) = M_R(n_0) \cdots M_R(m_0 + 1)$

from the m_0-th step of the induction to the n_0-th is strictly positive. We show the same for the transition matrices of the Gjerde-Johansen induction.

For an arbitrary level $m \in \mathbb{N}$ of the Gjerde-Johansen Bratteli diagram, there exists a level $m_0 \in \mathbb{N}$ representing the same IET in the Rauzy Bratteli diagram and therefore there exists an $n_0 > m_0$ with $P_R(m_0, n_0)$ strictly positive. If the product of Rauzy matrices ends with a Type 1 matrix, we can represent $P_R(m_0, n_0)$ as the product of the corresponding Gjerde-Johansen transition matrices, $P_{GJ}(m, n) = M_{GJ}(n) \cdots M_{GJ}(m+1) = P_R(m_0, n_0)$. If the last Rauzy matrix $M_R(n_0)$ of $P_R(m_0, n_0)$ is of Type 0, we will multiply $P_R(m_0, n_0)$ by further Rauzy steps, until the last matrix $M_R(n_1)$ is of Type 1, then

$$M_R(n_1) \cdots M_R(n_0 + 1) P_R(m_0, n_0) = M_{GJ}(n) \cdots M_{GJ}(m + 1) = P_{GJ}(m, n).$$

The matrix $P_{GJ}(m, n)$ is strictly positive as the Rauzy matrices $M_R(n_i)$ are non-negative and have at least one positive entry per row. □

Infinite Minimal and Maximal Paths

The incoming edges of a Bratteli-Vershik system constructed from an IET by the Gjerde-Johansen induction are ordered left-to-right. This property does not persist under telescoping, therefore we do not telescope the diagram to the full transition matrices but stay in the Gjerde-Johansen construction.

Definition 6.2.3. A Bratteli diagram is called **properly ordered** if it is simple and has unique infinite paths x_{max} and x_{min} that only take maximal (minimal) edges in the incoming order.

The Vershik map of a properly ordered Bratteli diagram, which maps each path to its successor and x_{max} to x_{min}, is a minimal homeomorphism on X_B.

Minimal Paths. Every vertex $v \in V_n$ has one incoming edge from either $v \in V_{n-1}$ or $v - 1 \in V_{n-1}$. In the left-to-right ordering this is always the minimal incoming edge.

Proposition 6.2.4. *The path $v_0 \to 1 \to 1 \to \dots$ is the unique infinite minimal path of X_B.*

Proof. The vertex 1 on any level is connected to 1 on the next level, this connecting edge is minimal. Therefore $1 \to 1 \to \dots$ is an infinite minimal path. We take n large enough such that by the simplicity of the diagram there exists an $m < n$ with every $v \in V_n$ is connected to every vertex in V_m. The minimal path to any vertex $v \in V_n$ is found by following the minimal incoming edges up the diagram. As this edge always comes from

from v or $v-1$ in V_{n-1} after a finite number of steps it reaches the vertex 1 on some level $\tilde{m} \geq m$, from there on the minimal path follows the path $1 \to 1 \to \cdots \to 1$ to V_0. Thus $1 \to 1 \to \ldots$ is the unique infinite minimal path since n and m can be taken arbitrarily large. $\square$

The finite minimal paths to any vertex in V_n never passes through the last vertex d, as every edge outgoing from $d \in V_n$ is not minimal in the incoming order to any $v \in V_{n+1}$.

Ordering the Outgoing Edges. Finding the maximal incoming path is more complicated. We know in the IET there exists a unique point $z \in [0, 1)$ that is mapped to zero. To find the path in the Bratteli diagram that represents this point in the IET, we define an order on the outgoing edges. This order gives us a way to find the paths representing the left endpoints of the intervals in the IET, it follows the idea of a right-to-left outgoing order.

Let $T = (\pi, \lambda)$ be the IET at level $n-1$ with subintervals $\Delta_1, \ldots, \Delta_d$, the next step of the Gjerde-Johansen induction will map T to $T' = (\pi', \lambda')$ with subintervals $\Delta'_1, \ldots, \Delta'_d$. We construct the outgoing order for edges in E_n between V_{n-1} and V_n the following way:

- For $a \in V_{n-1}$, $a < d$, such that there is only one outgoing edge the order is trivial.
- For the unique $c \in V_{n-1}$, $c < d$, such that there are two outgoing edges, the outgoing order is right-to-left. The reason for this is that $T^{r+1}(\Delta_c)$ contains the left endpoint of Δ_d. The left part of Δ_c that maps outside (*i.e.*, to the left of) Δ_d corresponds to the edge $(c \to c) \in E_n$, and the right part of Δ_c that maps inside Δ_d corresponds to the edge $(c \to c+1) \in E_n$.
- For $d \in V_{n-1}$, the outgoing order is more involved. First consider the set

  $$W_{n-1} = \{a \in V_{n-1} : a < d \text{ and } T(\Delta_a) \cap \Delta_d \neq \emptyset\}.$$

  In case $r > 0$, this is the same set as J of the Gjerde-Johansen step. If $r = 0$, it is the same as $J' \cup \{c\}$ where $c \in V_{n-1}$ is the unique vertex with two outgoing edges to c and $c+1 \in V_n$.

  We put an order on W_{n-1} in which

  $$a < b \text{ if } T(\Delta_a) \text{ lies to the right of } T(\Delta_b).$$

  Let

  $$W'_n = \{a \in V_n : \exists e \in E_n, r(e) = a, s(e) \in W_{n-1}\}$$

be the set in vertices of V_n with incoming edges from vertices of W_{n-1}. To translate the order $<$ on W_{n-1} to vertices in W'_n, we set $c > c+1$ and for $a', b' \in W'_n$, setting $a' <' b'$ if $a < b$ for the unique $a, b \in W_{n-1}$ such that $(a \to a')$ and $(b \to b') \in E_n$.

Now we define $<^{out}_{n-1}$ on the set $\{e \in E_n : s(e) = d \in V_{n-1}\}$ as follows: First take the smallest incoming edges $(d \to a')$ of $a' \in W'_n$, and order them according to $<'$. Then take the second smallest incoming edges $(d \to a')$, $a' \in W'_n$, and order them according to $<'$, continuing this way until there are no edges left.

See Figures 6.5 and 6.5 for an example of the outgoing order in the IET and the Bratteli diagram on three vertices with $c = 2$ and $r = 1$. The outgoing order on the intervals can also be interpreted as the order of stacking in the cutting-and-stacking procedure of Kakutani-Rokhlin towers.

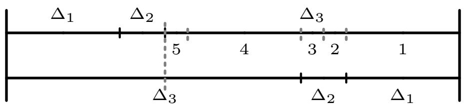


Figure 6.5: IET with cuts in Δ_3 and the value in the outgoing order of edges from vertex 3 marked.

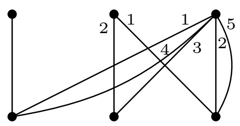


Figure 6.6: Level of the Bratteli diagram with outgoing order, the incoming order is left-to-right.

Connection between point and path. The conjugacy g maps a point x in I^* to an infinite path $\tilde{x} = (x_1, x_2, \dots)$ in the Bratteli diagram X_B. The vertices $v_n = r(x_n) \in V_n$ in the path represent the label of the interval in which the point x is in after n steps of the Gjerde-Johansen induction. Thus the cylinder set $[e_1, e_2, \dots, e_n]$ represents a subinterval in the original $\Delta_{r(e_1)}$.

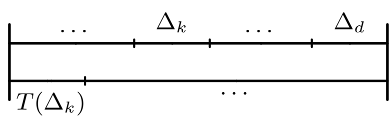
If x_{m+1} is the unique outgoing edge of v_m, it is just a renaming of the interval $\Delta^{(m)}_{v_m}$ to $\Delta^{(m+1)}_{v_{m+1}}$. If there are multiple outgoing edges, the edge x_{m+1} shows in which part of $\Delta^{(m)}_{v_m}$ the point x is. If $v_m \neq d$, the right-to-left order of outgoing edges gives us that if x_{m+1} is the minimal outgoing edge, the point x is in the right part of $\Delta^{(m)}_{v_m}$. On the other hand if x_{m+1} is the maximal outgoing edge, then the point x is in the left part of

$\Delta_{v_m}^{(m)}$. If $v_m = d$, the interval $\Delta_d^{(m)}$ is split into many parts and is distributed among the intervals mapped into $\Delta_d^{(m)}$. Similar to the case before the outgoing order shows in which subinterval of $\Delta_d^{(m)}$ the point x is.

For two paths $\tilde{x}, \tilde{y} \in [e_1, e_2, \ldots, e_n]$ and $x_{n+1} \neq y_{n+1}$ we know that the points x and y are close to each other in $\Delta_{r(e_1)}$ but after n steps the two points are separated either by $T_n(x) < \gamma_d^{(n)} \leq T_n(y)$ or (if $r(e_n) = d$) by $x < T_n(\gamma_i^{(n)}) \leq y$ for some i such that $T_n(\Delta_i^{(n)}) \subseteq \Delta_d^{(n)}$.

Paths of Maximal Outgoing Edges. Let the interval k be such that $k = \pi^{-1}(1)$, therefore the left endpoint of Δ_k is mapped to zero *i.e.,* $T(\gamma_k) = 0 = \gamma_1$. By the irreducibility of π we know that $k \neq 1$.

By the right-to-left ordering of outgoing edges of the Bratteli-Vershik diagram, the path of maximal outgoing edges x such that $r(x_1) = k$ corresponds to γ_k. In the next proposition we will show that this path of maximal outgoing edges x such that $r(x_1) = k$ is an infinite maximal path in the incoming order.



Proposition 6.2.5. *The infinite path of maximal outgoing edges $x_{(k)}$ such that $r(x_1) = k$, where $\pi(k) = 1$, is an infinite maximal path in the ordering of incoming edges* i.e., *the path $x_{(k)}$ only uses edges that are maximal in the incoming order.*

We show in Proposition 6.2.8 that the path $x_{(k)}$ is the unique maximal path *i.e.,* $x_{(k)} = x_{max}$.

Proof. If the vertex $k_n = r(x_n) \in V_n$ with $\pi^{(n)}(k_n) = 1$ has a unique incoming edge, x_n is trivially maximal. Thus we check that either k_n has no edges incoming from vertex d or that x_n is the maximal incoming edge from d. If $k_n \neq d$, it can never be the interval c that is split in the Gjerde-Johansen induction step. Comparing the interval lengths $\lambda_{k_n}^{(n)}$

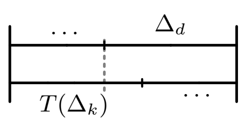


Figure 6.7: The interval k with $\pi(k) = 1$ and the interval d can only intersect if $k = d$.

and $\lambda_d^{(n)}$

$$\lambda_{k_n}^{(n)} > \sum_{i \neq d} \lambda_i^{(n)},$$

we see that Figure 6.7 is only possible if $k_n = d$. Thus the vertices $k_n \neq d$ in the path $x_{(k)}$ have only one outgoing edge until $x_{(k)}$ reaches the vertex d. Further, there are never any edges incoming from the vertex d to the vertex k_n (for this to happen $T_n(\Delta_{k_n}^{(n)}) \cap \Delta_d^{(n)} \neq \emptyset$ has to hold). Then all vertices k_n in the path $x_{(k)}$ have unique, and therefore maximal, outgoing and incoming edges until $k_n = d$. As the path $x_{(k)}$ visits the vertex d, the maximal outgoing edge goes to $c+1$ where $c \neq d$ is the interval that is split in two. This maximal outgoing edge is also the maximal incoming edge to $c+1$. Hence all edges used in the path of maximal outgoing edges are maximal incoming edges and $x_{(k)}$ is an infinite maximal path in the incoming order. □

Uniqueness of the Maximal Incoming Path. For any path $x \neq x_{(k)}$ with edges $x_n \in E_n$ and vertices $v_n = r(x_n) \in V_n$, we have to show that there exists a level n such that $v_n \neq d \in V_n$ and v_{n+1} has an incoming edge from $d \in V_n$. This edge from d is maximal in the incoming order, therefore the edge x_{n+1} taken by x is not. To show this we use the Rauzy induction to find a step that constructs such an edge.

Lemma 6.2.6. *For all paths $x \neq x_{(k)}$ there exists a Rauzy induction step $n \in \mathbb{N}$ such that $\pi^{(n)}(v_n) = d$.*

Proof. The sequence of positions $(\pi^{(n)}(v_n))_n$ is non-decreasing in n for all $v_n \in V_n$ with $\pi^{(n)}(v_n) < d$ *i.e.,* it holds for all n that

$$\pi^{(n)}(v_n) \leq \pi^{(n+1)}(v_{n+1}).$$

It strictly increases in case $\pi(v) > \pi(d)$ and the Rauzy induction is of Type 0. Then there exists some losing $j \neq v \in \mathcal{A}$ with $\pi(j) = d$ and $\lambda_j < \lambda_d$. The Rauzy step of Type 0 splits the interval $T(\Delta_d)$ in two. For the new permutation π' it holds that $\pi'(d) = \pi(d)$, $\pi'(j) = \pi(d) + 1$ and therefore $\pi'(v) = \pi(v) + 1$.

From [Via06, Proposition 1.19] we know that every interval is winner and loser of the Rauzy induction infinitely often. For k such that $\pi(k) = 1$, the maximal outgoing path $x_{(k)}$ represents the point in the IET that maps to zero, thus $\pi^{(n)}(k_n) = 1$ for all $n \in \mathbb{N}$. Therefore k_n is winner and loser only if $k_n = d$. If $k_n = d$ is the winner of a Type 0-step of the Rauzy induction, it follows that $\pi^{(n)}$ increases for all $v \in \mathcal{A}$ besides k and the losing interval. As this happens infinity often, every sequence $(v_n)_n$ will increase to $\pi^{(n)}(v_n) = d$ for $n \in \mathbb{N}$. □

Lemma 6.2.7. *For all paths $x \neq x_{(k)}$ there exists a level $n \in \mathbb{N}$ such that $v_n \neq d \in V_n$ and the following vertex $v_{n+1} \in V_{n+1}$ has an incoming edge from $d \in V_n$.*

Proof. By the previous lemma, we take $n \in \mathbb{N}$ such that $\pi^{(n)}(v_n) = d$ and define $v = v_n \in \mathcal{A}$. If $\lambda_v^{(n)} < \lambda_d^{(n)}$, we have a Type 0-Rauzy induction step and thus $v_{n+1} = v$ has an incoming edge from $d \in V_n$.

If $\lambda_v^{(n)} > \lambda_d^{(n)}$, then v has two outgoing edges. If the edge x_{n+1} of the path x goes to vertex $v_{n+1} = v + 1 \in V_{n+1}$ on the next level, we know this vertex has an incoming edge from $d \in V_n$. Otherwise the path x goes to vertex $v \in V_{n+1}$. Then in the next Rauzy step we again have $\pi^{n+1}(v) = d$. After finitely many steps of v being the winner of a Type 1-Rauzy induction step, there has to exist some level $m > n$ such that $\pi^{(m)}(v) = d$ and $\lambda_v^{(m)} < \lambda_d^{(m)}$ and thus v_{m+1} has an incoming edge from $d \in V_m$. □

Using these lemmas we can show the following proposition.

Proposition 6.2.8. *Let $x_{(k)}$ be the infinite path of maximal outgoing edges with $r(x_1) = k$ and $\pi(k) = 1$. Any path $x \neq x_{(k)}$ is not maximal in the incoming order.*

Proof. Take a level n such that the next Gjerde-Johansen step contains a Rauzy step satisfying the conditions of the previous lemma. Thus we know $T_n(\Delta_{v_n}^{(n)}) \subseteq \Delta_d^{(n)}$ and there exists an edge from $d \in V_n$ to $v_{n+1} \in V_{n+1}$. Then the edge from v_n to v_{n+1} used by x is not maximal in the incoming order to v_{n+1} and x is no infinite maximal path of the Bratteli diagram. □

Proposition 6.2.9. *The Bratteli-Vershik diagram created by the Gjerde-Johansen induction on an irreducible IET which satisfies the Keane condition, is properly ordered. Therefore the Bratteli-Vershik system is a minimal Cantor system where the Vershik map is a minimal homeomorphism.*

Proof. As shown the Bratteli diagram is simple, thus X_B is a Cantor set and it has unique infinite minimal and maximal paths with $x_{min} = (1 \to 1 \to \dots)$ and $x_{max} = x_{(k)}$, the maximal outgoing path with $r(x_1) = k$ where $k = \pi^{-1}(1)$. Therefore it is properly ordered on the Cantor set X_B and the V_B is a minimal homeomorphism mapping each path to its successor and x_{max} to x_{min}. □

Comparing the Bratteli-Vershik Representations of IETs. As mentioned before, the Bratteli-Vershik diagram of the Gjerde-Johansen induction is a telescoped version from the Rauzy induction diagram. The Type 0-blocks and the following Type 1-step of the Rauzy induction diagram are telescoped together to achieve the Bratteli-Vershik diagram of the Gjerde-Johansen induction. The incoming order is the order induced by the telescoping. We endow the Rauzy Bratteli diagram with the outgoing right-to-left order. Then telescoping to the Gjerde-Johansen Bratteli diagram maintains a right-to-left order for all but the last vertex. For the outgoing edges of the vertex d we have to compare the corresponding path in the Rauzy Bratteli diagram. The earlier the path moves away from the vertex d the lower its value in the outgoing order.

To go from a Gjerde-Johansen Bratteli diagram to a Rauzy Bratteli diagram we need to know the outgoing order of edges from vertex d. In the following example we see how otherwise the order of levels in the Rauzy Bratteli diagram is uncertain.

Example 6.2.10. *Assuming we have the level of Figure 6.8 in a Gjerde-Johansen Bratteli diagram. We observe the maximal outgoing edge of the last vertex is from 5 to 4, as it*

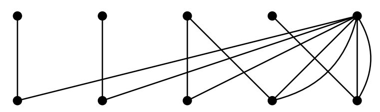


Figure 6.8: Example of a Bratteli diagram with the Gjerde-Johansen induction without orderings.

has to connect to the vertex $c+1$ on the next level. Further we know that $r = 1$ and $J = \{1,2,3,4\}$, $J' = \{4\}$. In trying to break down the Gjerde-Johansen induction into its Rauzy steps we see there are r Type 0-steps of the J block, followed by the J' block and finally one Type 1-step for the vertex $c = 3$. We can decompose the J block into first the J' block and then the vertices in $J \setminus J'$ starting with c. But without the outgoing order we can not know the order of vertices in the J' and $J \setminus J'$ blocks. Thus in the example we know in the first Type 0-step that for vertex 4, then $c = 3$ follows, but the order of steps for vertices 1 and 2 is unknown.

The dotted edges in Figure 6.9 represent an IET that maps the interval Δ_1 to the second position, the dashed edges would map Δ_1 to the third position. See Figure 6.10 for the uncertainty of positions in the IET.

Similarly if the set $J^c \setminus \{d\}$ contains more then one element, we know these intervals are mapped in front of $T(\Delta_d)$ but not their relative position to each other. The vertices in $J^c \setminus \{d\}$ all have a unique outgoing edge and its range has a unique incoming edge. Even with the outgoing order of the Gjerde-Johansen induction, these paths are not involved and we have no information on their relative order. Thus we generally do not know the exact permutation of the IET from a single level of the Gjerde-Johansen induction Bratteli diagram.

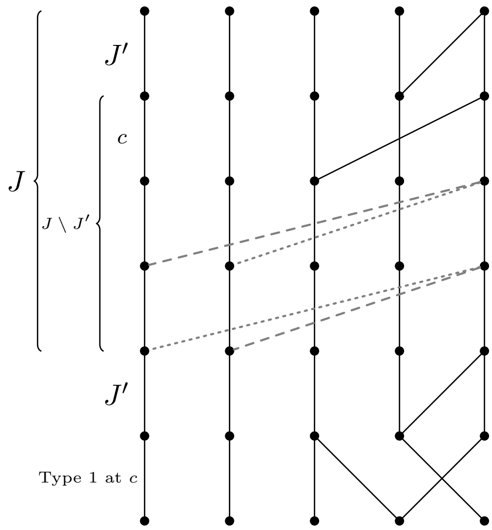


Figure 6.9: Bratteli diagram from microscoping the Gjerde-Johansen induction diagram to the Rauzy induction levels, with uncertainty in levels $J \setminus J'$.

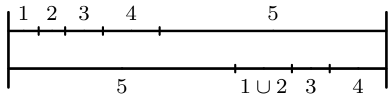


Figure 6.10: IET represented by the Bratteli diagrams with uncertain positions for $T(\Delta_1)$ and $T(\Delta_2)$

6.3 Linearly Recurrent Interval Exchange Transformations

This section investigates IETs that generated linearly recurrent subshifts and their quantity.

Definition 6.3.1. A subshift (X, σ) on letters $\mathcal{A}$ is called linearly recurrent with constant L if there exists $L > 1$ such that every word w of length $n \in \mathbb{N}$ reappears in $x \in X$ with gaps $\leq nL$.

For a minimal IET (π, λ) we defined the set

$$J = \{j \in \{1, \ldots, d-1\} : \pi(j) > \pi(d)\}.$$

The condition for linear recurrence of an IET will depend on the ratio between the lengths λ_d and λ_j for $j \in J$.

Proposition 6.3.2. *Let (π, λ) be a minimal IET. Then (π, λ) is not linearly recurrent for any constant $L < \frac{\lambda_d}{\sum_{j \in J} \lambda_j}$.*

Proof. We look at the orbit of $x \in [0,1)$ and code it by assigning each interval its letter $i \in \mathcal{A}$. For the IET to be linearly recurrent every letter has to repeat in bounded gaps smaller than a constant L in the orbit of every $x \in [0,1)$. If x gets mapped close to 1 and the intervals Δ_d and $T(\Delta_d)$ have a large intersection, x stays in the interval d for many steps. To compute this number of steps r we compare the lengths of Δ_d and intervals in J being mapped behind Δ_d (see Figure 6.11), then

$$r = \left\lfloor \frac{\lambda_d}{\sum_{j\in J} \lambda_j} \right\rfloor .$$

For any $n \leq r$ and $j \in J$, it holds that $T^n(\Delta_j) \subseteq \Delta_d$, thus the letter d gets repeated at least r times. Thus the IET can not be linearly recurrent with any constant $L \leq r$. $\square$

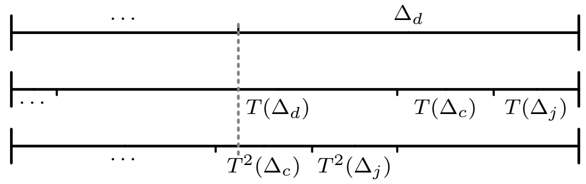


Figure 6.11: The orbit of every $x \in \Delta_j$ for $j \in J$ stays in Δ_d for at least r steps until the union of intervals in J have covered the length of Δ_d.

Remark 6.3.3. In case Δ_d and $T(\Delta_d)$ have no intersection, we know that $\sum_{j\in J} \lambda_j > \lambda_d$, hence the proposition is not applicable if $L < 1$,.

We define U_L as the set of IETs such that $L < \frac{\lambda_d}{\sum_{j\in J}\lambda_j}$, this is a subset of V_L, the set of all IETs which are not linearly recurrent with constant L.

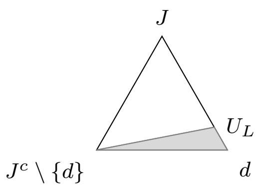


Figure 6.12: Simplified simplex with the region U_L and vertices representing the sets $J^c \setminus \{d\}$, J and d.

Theorem 6.3.4. *The set of linearly recurrent IETs has Lebesgue measure zero.*

Proof. First we show that the set of IETs, which are not linearly recurrent for a fixed constant $L > 1$ has full measure. By [Zor96] we know that the Zorich induction on the

Rauzy class $C(\pi)$ and the length vectors Λ_d admits an invariant probability measure μ which is absolutely continuous with respect to $d\pi \times Leb$ ($d\pi$ being the counting measure on the Rauzy class). Furthermore the measure μ is ergodic and its density with respect to the Lebesgue measure is a homogeneous rational function of degree $-d$ (where $\deg(f/g) = \deg f - \deg g$) which is bounded away from zero.

By the previous result

$$U_L \subseteq V_L = \{\text{IETs not linearly recurrent for constant } L\}$$

and as U_L is a slice of the simplex (to be exact of the simplex which has holes from the Keane condition on (π, λ), but has full Lebesgue measure) it follows that

$$0 < \mu(U_L) \leq \mu(V_L)$$

as the density of μ with respect to the Lebesgue is strictly positive. Then as V_L is invariant under the Zorich induction and μ is ergodic, $\mu(V_L) = 1$.

Therefore the set of IETs not linearly recurrent for constant L has full measure. This holds for any L and thus the intersection

$$\mu\left(\bigcap_{L\in\mathbb{N}} V_L\right) = 1$$

of all not linearly recurrent IETs has full μ-measure. And by the strictly positive density of μ with respect to the Lebesgue measure it follows that the set of linearly recurrent IETs has Lebesgue measure zero. □

6.4 Measures on Bratteli-Vershik diagrams of Circle Rotations

For an IET with $d = 2$ intervals which satisfy the Keane condition, the Gjerde-Johansen induction gives us the following transition matrices in its Bratteli diagram

$$F_n = \begin{pmatrix} 1 & r_n \\ 1 & r_n + 1 \end{pmatrix} \text{ with } r_n = \left\lfloor \frac{\lambda_2^{(n)}}{\lambda_1^{(n)}} \right\rfloor \tag{6.6}$$

for all n, see the Gjerde-Johansen substitutions from (6.5). The parameter r_n is the ratio of interval lengths $\lambda_1^{(n)}$ and $\lambda_2^{(n)}$ at the n-th induction step. It holds that $r_n > 0$ infinitely often. The Bratteli diagram constructed by these substitutions is primitive and properly ordered with the minimal edges of every level coming from the first vertex and the maximal edges from the second. An IET on two intervals is a circle rotation and always uniquely ergodic with the Lebesgue measure as the invariant measure. This unique invariant measure may be an expansion of different subdiagrams measures depending on r_n.

To understand more about the Bratteli-Vershik models of circle rotations we first take a look at other examples with 2×2 transition matrices.

Example 6.4.1. *We know the Bratteli diagram with transition matrices*

$$F_n = \begin{pmatrix} n^2 & 1 \\ 1 & n^2 \end{pmatrix}$$

is not uniquely ergodic, see [ABKK17, Example 3.6] for more information. The two ergodic measures are extensions of the odometer measures on the vertex subdiagrams generated by v_1 and v_2.

The Bratteli diagram with transition matrices

$$F_n = \begin{pmatrix} 1 & 1 \\ 1 & n^2 \end{pmatrix}$$

in comparison is uniquely ergodic, its invariant measure is a finite extension of the measure on vertex subdiagram generated by the second vertex v_2 of each level n. To show this using [ABKK17, Theorem 4.4] we estimate the heights of the Kakutani-Rokhlin towers. First, $h_k(v_1)$ is the height of the first tower on level k of the full diagram, $\overline{h}_k(v_1)$ is the height in the subdiagram generated by the first vertex,

$$h_k(v_1) \geq 2^k,$$
$$\overline{h}_k(v_1) = 1,$$

then

$$\frac{\overline{h}_k(v_1)}{h_k(v_1)} \leq \frac{1}{2^k} \to 0 \text{ as } k \to \infty.$$

Thus the measure extension from the subdiagram generated by v_1 is not finite. For the subdiagram generated by the vertex v_2 it follows that

$$h_k(v_2) \leq \prod_{n=2}^{k} (n^2+1),$$
$$\overline{h}_k(v_2) = \prod_{n=2}^{k} n^2$$

and thus

$$\frac{\overline{h}_k(v_2)}{h_k(v_2)} \geq \prod_{n=2}^{k} \frac{n^2}{n^2+1} \to 0 \text{ as } k \to \infty$$

as $\sum_n \frac{1}{n^2}$ converges.

To classify the different cases for circle rotation, we will first take a look at linearly recurrent Bratteli diagrams.

Linearly recurrent circle rotation. Take transition matrices

$$F_n = \begin{pmatrix} 1 & r_n \\ 1 & r_n + 1 \end{pmatrix}$$

such that the sequence $(r_n)_n$ takes only finitely many values in $\mathbb{N}$ and the distance between two consecutive occurrences of $r_n > 0$ is bounded. Then the Bratteli-Vershik system is linearly recurrent and therefore of exact finite rank. That means the only subdiagram such that the measure extension is finite is the full diagram.

General circle rotations. The following proposition shows that a finite measure supported on less than the full diagram is possible for circle rotations.

Proposition 6.4.2. *Let B be a Bratteli diagram with transition matrices F_i generated from a circle rotations as in* (6.6)*. Suppose $\overline{B}_1$ is the vertex subdiagram of B defined by vertex v_1, then $\mu(X_{\overline{B}_1}) = 0$ for the unique invariant probability measure μ on X_B. The path space of vertex subdiagram $\overline{B}_2$ of B defined by vertex v_2 has positive measure if*

$$\sum_i \frac{1}{r_i + 2} < \infty.$$

If the path spaces of both subdiagrams have zero measure, then μ is not a finite measure extension of a subdiagram.

Proof. We use [ABKK17, Theorem 4.4] and approximate the heights of towers in the diagrams B and $\overline{B}_1$,

$$h_k(v_1) \geq \prod_{i=2}^{k} (r_i + 1),$$

$$\overline{h}_k(v_1) = \prod_{i=2}^{k} 1.$$

Thus the subdiagram $\overline{B}_1$ does not support a finite measure on B as

$$\frac{\overline{h}_k(v_1)}{h_k(v_1)} \leq \prod_{i=2}^{k-1} \frac{1}{r_i + 1} \to 0 \text{ as } k \to \infty.$$

For the $\overline{B}_2$, subdiagram of B defined by vertex v_2, we have

$$h_k(v_2) \leq \prod_{i=2}^{k-1} (r_i + 2),$$

$$\tilde{h}_k(v_2) = \prod_{i=2}^{k-1} (r_i + 1)$$

and

$$\frac{\tilde{h}_k(v_2)}{h_k(v_2)} \geq \prod_{i=2}^{k-1} \frac{r_i+1}{r_i+2} = \prod_{i=2}^{k-1} \left(1 - \frac{1}{r_i+2}\right).$$

Under the assumption that $\sum_i \frac{1}{r_i+2}$ converges, the product converges to a number greater than zero. Therefore $\mu(X_{\overline{B}_2}) > 0$ for the unique ergodic probability measure. □

Circle rotation with $r_n = n$. Taking the example of a Bratteli diagram B with transition matrices

$$F_n = \begin{pmatrix} 1 & n \\ 1 & n+1 \end{pmatrix},$$

we will show that for all invariant measures, the mass of the subdiagram $\overline{B}_2$ generated by v_2 is zero. By the previous statement the same holds for $\overline{B}_1$

We can easily compute the number of paths in $\overline{B}_2$ up to level n

$$\overline{h}_n(v_2) = \prod_{k=1}^{n-1}(k+1) = n!$$

To compare this to the overall number of paths in the whole diagram B, we need to look at the following recursion

$$h_{n+1}(v_1) = h_n(v_1) + nh_n(v_2)$$
$$h_{n+1}(v_2) = h_n(v_1) + (n+1)h_n(v_2)$$

or

$$h_{n+1}(v_2) = (n+2)h_n(v_2) - h_{n-1}(v_2)$$

with initial values $h_1(v_2) = 1$ and $h_2(v_2) = 3$.

By induction we can show that

$$h_n(v_2) \geq n!\sqrt{n} \quad \text{for } n \geq 6. \tag{6.7}$$

Thus comparing the number of paths in $\overline{B}_2$ to the whole diagram B

$$\frac{\overline{h}_n(v_2)}{h_n(v_2)} \leq \frac{n!}{n!\sqrt{n}} = \frac{1}{\sqrt{n}} \longrightarrow 0 \text{ as } n \to \infty.$$

To show the induction in (6.7), we first prove that

$$h_n(v_1) \leq h_n(v_2) < (n+1)! \quad \text{for } n \geq 1.$$

We have $h_1(v_1) = h_2(v_1) = 1 < 2!$ and further

$$\begin{aligned} h_{n+1}(v_2) &= (n+2)h_n(v_2) - h_{n-1}(v_2) \\ &< (n+2)(n+1)! < (n+2)! \end{aligned}$$

With this result we can show the induction for (6.7). Firstly

$$h_6(v_2) = 2055 \geq 6!\sqrt{6} \approx 1763.63,$$

then

$$\begin{aligned} h_{n+1}(v_2) &= (n+2)h_n(v_2) - h_{n-1}(v_2) \\ &\geq (n+2)n!\sqrt{n} - n! \\ &= (n+1)!\left(\sqrt{n} + \frac{\sqrt{n}-1}{n+1}\right) \geq (n+1)!\sqrt{n+1} \end{aligned}$$

as

$$\begin{aligned} &\sqrt{n} + \frac{\sqrt{n}-1}{n+1} \geq \sqrt{n+1} \Leftrightarrow n^3 \geq 2n^2 + 15n + 16 \\ &\quad\text{which holds for all } \; n \geq \frac{1}{3}\left(2 + \sqrt[3]{359 - 12\sqrt{78}} + \sqrt[3]{359 + 12\sqrt{78}}\right) \approx 5.36. \end{aligned}$$

Thus the uniquely ergodic finite measure on B is not an expansion from any subdiagram.

Connection to the Lebesgue measure. By sketching the cutting-and-stacking procedure on the IET we can see the development of the set $\cap_n Y_n(v_2)$ of all points that stay in the second vertex for all levels $n \in \mathbb{N}$. In the case of $\sum_i \frac{1}{r_i+2}$ converging, Proposition 6.4.2

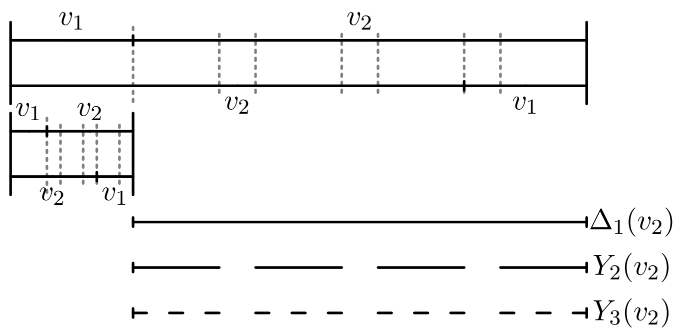


Figure 6.13: Two steps of the Gjerde-Johansen induction and the construction of $Y_n(v_2) = \{x \in X : s(x_k) = r(x_{k-1}) = v_2 \in V_k \text{ for all } k \leq n\}$

shows that the set $\cap_n Y_n(v_2)$ has positive Lebesgue measure *i.e.*, $\cap_n Y_n(v_2)$ is a fat Cantor set (also called Smith-Volterra-Cantor set), see the construction in Figure 6.13.

Sturmian subshifts. Another way of looking at circle rotations is through examining *Sturmian* (also called *rotational*) *sequences.* Their study goes back to [MH40]. Sturmians emerge in a number of settings other than rotations, for example in encoding the itinerary of an angled line intersecting vertical and horizontal grid lines. For more details and definitions about Sturmians, their word complexity $p_n(u) = n + 1$ or the frequency of letters, see [Bru22, Section 4.3]. One way of defining a Sturmian sequence is as a limit sequence generated by substitutions

$$\theta_0 : \begin{cases} 0 \mapsto 0, \\ 1 \mapsto 10 \end{cases} \qquad \theta_1 : \begin{cases} 0 \mapsto 01, \\ 1 \mapsto 1 \end{cases}$$

where both substitutions appear infinitely often. Then we can write

$$u = \lim_{n\to\infty} \theta_0^{\alpha_1}\theta_1^{\alpha_2}\theta_0^{\alpha_3}\cdots\theta_1^{\alpha_{2n}}(b)$$

for $b \in \{0, 1\}$ and $\alpha_1 \geq 0$, $\alpha_i \geq 1$ for $i \geq 2$ and the rotation angle α has the continued fraction expansion $[0; \alpha_1 + 1, \alpha_2, \alpha_3, \dots]$.

We want to make the same connection between the Gjerde-Johansen representation of a circle rotation and the continued fraction expansion $[0; a_1, a_2, a_3, \dots]$ of the length of the first interval in the IET, λ_1. Let $(n_j)_j$ be a sequence in $\mathbb{N}$ with $n_1 = 2$. For $j \geq 2$ we define n_j as the indices $i > 2$ such that $r_i > 0$. Then $a_1 = r_2 + 1$ and

$$\begin{aligned} a_{2k} &= n_{k+1} - n_k, \\ a_{2k+1} &= r_{n_{k+1}} \end{aligned}$$

for $k \in \mathbb{N}$. As mentioned previously $r_i > 0$ represents r_i Type 0-steps of the Rauzy induction followed by one Type 1-step. Successive indices i such that $r_i = 0$ describes a block of Type 1-steps in the Rauzy induction. The substitutions of a Type 0- or a Type 1-step in the Rauzy induction, are

$$\chi_0 : \begin{cases} 0 \mapsto 01, \\ 1 \mapsto 1 \end{cases} \qquad \chi_1 : \begin{cases} 0 \mapsto 0, \\ 1 \mapsto 01 \end{cases}$$

see the definition of a Rauzy step in (6.1) and (6.2). We observe that the Sturmian θ_1 substitution is the Rauzy χ_0 substitution and the Sturmian θ_0 is the once shifted substitution χ_1, see Theorem 4.4.3 for the shift. Thus, by microscoping the Gjerde-Johansen induction into its Rauzy steps, we generate the blocks

$$w = \lim_{n\to\infty} \chi_0^{a_1-1}\chi_1^{a_2}\chi_0^{a_3}\cdots\chi_1^{a_{2n}}(b)$$

for $b \in \{0, 1\}$ and λ_1 has continued fraction expansion $[0; a_1, a_2, a_3, \dots]$.

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