MASTERARBEIT | MASTER'S THESIS

Titel | Title

Entanglement properties of a relativistic massive spin-1 particle

verfasst von | submitted by

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angestrebter akademischer Grad | in partial fulfilment of the requirements for the degree of

Master of Education (MEd)

Wien | Vienna, 2025

Studienkennzahl lt. Studienblatt | Degree programme code as it appears on the student record sheet:	UA 199 520 523 02
Studienrichtung lt. Studienblatt | Degree programme as it appears on the student record sheet:	Masterstudium Lehramt Sek (AB) Unterrichtsfach Mathematik Unterrichtsfach Physik
Betreut von | Supervisor:	Mag. Dr. Beatrix Hiesmayr Privatdoz.

Zusammenfassung

In dieser Arbeit wird die Abhängigkeit der sogenannten Spin-Impuls-Verschränkung eines massiven Spin-1-Teilchens vom Bewegungszustand eines Beobachters in einem relativistischen Kontext analysiert. Alle Eigenschaften, die benötigt werden, um ein freies quantenmechanisches Teilchen zu beschreiben, werden aus dem Relativitätsprinzip abgeleitet, nahezu ohne zusätzliche Annahmen. Zwei hintereinander ausgeführte Lorentz-Boosts ergeben einen Boost sowie eine Drehung. Die dadurch resultierende Differenz des Ausmaßes der Verschränkung in unterschiedlichen Bezugssystemen ist auf die Drehung zurückzuführen. Diese Änderung der Verschränkung hängt von den relativen Geschwindigkeiten zwischen Teilchen und Beobachter, den Drehwinkeln, die die Rotation des Teilchenzustandes beschreiben, den Zustandsparametern, welche die Superposition charakterisieren, und dem Winkel zwischen der Spin-Quantisierungsachse und dem Geschwindigkeitsvektor des Beobachters ab. Letzterer hat den größten Einfluss auf die Form des Graphen, der die Unterschiede der Verschränkung zwischen ruhendem und bewegtem Bezugssystem darstellt. Ein Vergleich mit der Situation im Falle eines Spin-1/2-Teilchens wird angestellt, indem einer der Zustandsparameter gleich null gesetzt wird. Ein Mathematica-Notebook, entwickelt zur Durchführung der Berechnungen, wird präsentiert.

Abstract

In this work the dependence of the spin-momentum entanglement of a massive spin-1 particle on the state of motion of an observer is analysed in a relativistic framework. All properties needed to describe a free quantum-mechanical particle are derived from the principle of relativity, almost without additional assumptions. Two consecutive Lorentz boosts can be written as one boost and a rotation. It is the rotation that causes the variation of the entanglement between two different frames of reference. The change of entanglement is affected by the relative velocities between particle and observer, the angles which specify the rotations of the particle's state vectors, the preparation parameters of the state that characterise the superposition and the angle between the spin quantisation axis and the velocity of the observer. The latter has the greatest influence on the shape of the graph of the difference in the entanglement between rest and boosted frame. A comparison with the case of a spin-1/2 particle is performed by setting one of the preparation parameters equal to zero. A Mathematica notebook developed for carrying out the calculations is presented.

Contents

1 Introduction

Energy consists of quanta that are recognised as particles and described by quantum mechanics. The quantum numbers (i.e. properties) of these particles can be undetermined before they are measured. This hypothesis has been confirmed in a variety of experiments which are based on the entanglement of certain features of quantum particles [1, 2]. Entanglement means a quantum phenomenon where specific properties of particles remain interlinked. When a property of one of the entangled particles is measured, it instantaneously affects the other, no matter the distance between them.

While quantum entanglements have been extensively analysed within the framework of quantum mechanics for low-energy systems [3], open questions still remain in the relativistic context. This particularly concerns particles whose velocities approach the speed of light. The analysis of quantum mechanical entanglement in moving reference frames is an area that requires further exploration [4, 5]. These considerations could one day become practically relevant, especially in areas such as satellite-based quantum communication, where the principles of quantum mechanics could revolutionire secure information transfer.

This thesis aims to analyse the entanglement of the momentum and the spin (intrinsic angular momentum) of a relativistic massive particle, whose spin can take three discrete values. Another purpose of this work is to complement and further develop insights gained from the study of single spin-1/2 particles [6, 7]. The starting point for this investigation is an analysis of fundamental symmetries [8], especially Lorentz invariance, that is a foundation of both special relativity and quantum field theory. Lorentz invariance implies that the laws of physics are the same for all observers, regardless of their constant velocity relative to one another. From this principle, the transformation behaviour of the spin, when changing reference frames, can be derived. This allows for determining how the spin-momentum entanglement changes under these transformations. Finally, the size of this change is compared for varying preparation parameters of the spin state and different arrangements of the movement directions of particle and observer, which can offer insights into the behaviour of quantum systems at relativistic speeds.

In summary, this research contributes to a deeper theoretical understanding of relativistic quantum mechanics. It thoroughly investigates the mathematical foundations and physical implications of spin-momentum entanglement in a relativistic scenario, thereby addressing some intriguing questions concerning the amount of entanglement, when it is regarded by a rapidly moving observer.

2 Representations of Poincare Symmetry

In this section, we analyse the physical structures that are allowed with as few assumptions as possible. This will finally lead to the concept of fields which describe the evolution of a system.

2.1 The principle of relativity

Albert Einstein taught us that, when making physical considerations, one should always think of concrete things. Let us regard several laboratories which are moving with different constant velocities through space.

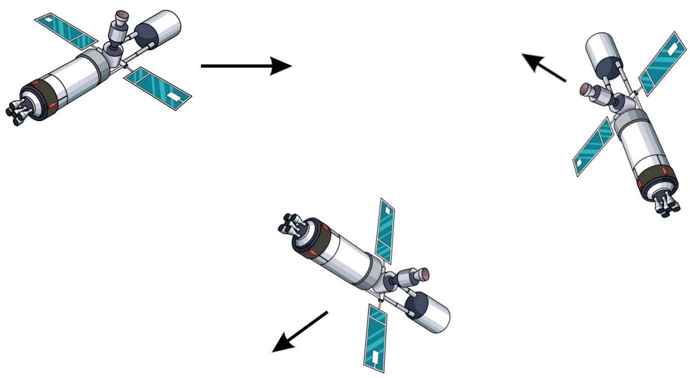

Figure 1: Laboratories in universe

The results of physical experiments performed in the different labs, that are translated, rotated and moving relative to each other, must be the same as no reason for divergent outcomes (e.g. an aether) has been found. This fact is called the principle of relativity [8].

Every pointlike event can be described by one time and three space coordinates that are collected in a four-dimensional vector $x^\mu = (t, x, y, z)$, using units described in Sec. 2.3. Each lab uses its own clock and a differing coordinate system leading to unequal sets of coordinates. Under the assumption, that finite coordinates are always mapped to finite ones, a transformation to the coordinates $\bar{x}^\mu$ used by another lab has to be of the form

$$\bar{x}^\mu = \Lambda^\mu_\nu x^\nu + k^\mu, \tag{1}$$

where Λ^μ_ν is a 4x4 matrix. The invariance of physical laws under coordinate transformations is called Poincare symmetry. Such transformations are classified into rotations, boosts and translations as explained in the next three subsections.

2.2 Rotations

A rotation is a transformation according to Eq. (1) that only affects spatial coordinates (for their indices Latin letters will be used, which run from 1 to 3). Such a transformation can be denoted in the following form [8]:

$$\Lambda^0_0 = 1,\ \Lambda^0_i = \Lambda^i_0 = 0,\ \Lambda^i_j = R_{ij}. \tag{2}$$

with the 3-dimensional rotation matrix R.
An infinitesimal rotation around the unit vector[1] $\mathbf{n}$ by an angle $d\alpha = \alpha/N$ (for large N) is given by making use of the vector product $\mathbf{n}\times\mathbf{x}$ (see Fig. 2). The components of the infinitesimal rotated vector $\bar{\mathbf{x}} = R(\mathbf{d}\boldsymbol{\alpha})\,\mathbf{x} = d\alpha\ \mathbf{n}\times\mathbf{x}$ are given by

$$R_{ij}(\mathbf{d}\boldsymbol{\alpha})\,x_j = x_i + d\alpha\,\epsilon_{ikj}\,n_k\,x_j. \tag{3}$$

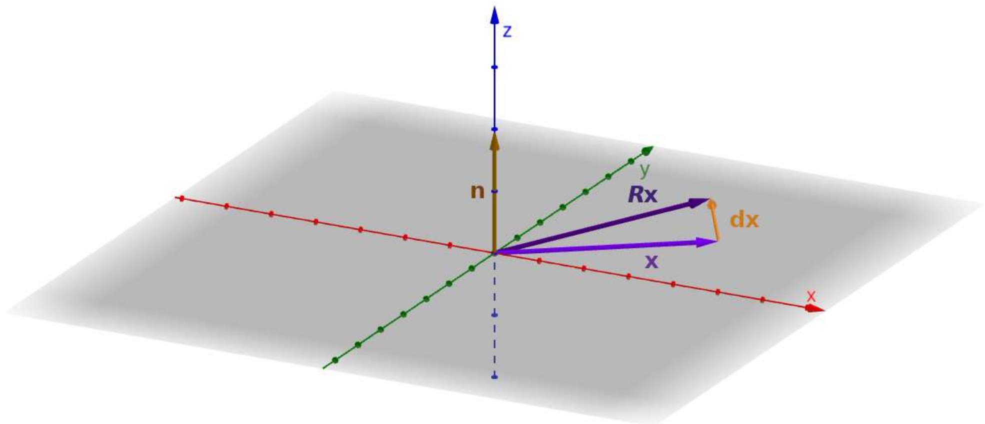

Figure 2: Infinitesimal rotation

The infinitesimal rotation can also be expressed using generators G_1, G_2, and G_3, one for each direction in space:

$$R(\mathbf{d}\boldsymbol{\alpha}) = \mathbb{1} + d\alpha\ \mathbf{G}\cdot\mathbf{n}. \tag{4}$$

A comparison with Eq. (3) leads to

$$(G_k)_{ij} = -\,\epsilon_{kij}. \tag{5}$$

In matrix form:

$$G_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},\ G_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},\ G_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6}$$

[1]Bold letters represent 3-dimensional vectors.

A finite rotation is generated by a large number of consecutive infinitesimal rotations:

$$R(\boldsymbol{\alpha}) = \lim_{N\to\infty} \left[R\left(\frac{\boldsymbol{\alpha}}{N}\right)\right]^N = \lim_{N\to\infty} \left[\mathbb{1} + \frac{\alpha}{N}\, \mathbf{n}\cdot\mathbf{G}\right]^N = \mathrm{e}^{\boldsymbol{\alpha}\cdot\mathbf{G}}, \qquad (7)$$

where $\boldsymbol{\alpha} = \alpha\,\mathbf{n} = (\alpha_1, \alpha_2, \alpha_3)$ is the rotation vector that points into the direction of the rotation axis and whose norm is the rotation angle.

The matrices G_i are obviously antisymmetric and traceless. So it follows that the rotation matrices $R(\alpha)$ are orthogonal and have a determinant equal to 1:

$$\mathbf{G} + \mathbf{G}^{\mathrm{T}} = 0 \quad \Rightarrow \quad R(\boldsymbol{\alpha})R^{\mathrm{T}}(\boldsymbol{\alpha}) = e^{\boldsymbol{\alpha}\,\mathbf{G}} e^{\boldsymbol{\alpha}\,\mathbf{G}^{\mathrm{T}}} = e^{\alpha(\mathbf{G}+\mathbf{G}^{\mathrm{T}})} = \mathbb{1}\,, \qquad (8)$$

$$\mathrm{Sp}\, G_i = 0 \quad \Rightarrow \quad \det R(\boldsymbol{\alpha}) = \mathrm{e}^{\mathrm{Sp}(\boldsymbol{G}\cdot\boldsymbol{\alpha})} = 1. \qquad (9)$$

The product of two orthogonal matrices with determinant 1 is again a matrix with the same properties, i.e. the rotations form a group that is called special orthogonal group in three dimensions, abbreviated $SO(3)$. To calculate the product of two rotation matrices $R(\boldsymbol{\alpha})R(\boldsymbol{\beta}) = e^{\boldsymbol{\alpha}\,\mathbf{G}} e^{\boldsymbol{\beta}\,\mathbf{G}}$ with the help of the Baker-Campbell-Hausdorff identity [9]

$$e^X \circ e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X,Y]]-\frac{1}{12}[Y,[X,Y]]+\dots} \qquad (10)$$

one needs the commutators of the generators of the rotation group. It is easy to verify the following relation for the generators of Eq. (6):

$$[G_i, G_j] = \epsilon_{ijk}\, G_k. \qquad (11)$$

In general, the commutator relation of the generators specifies most of the properties of the corresponding group [10] because it governs the group operation given by Eq. (10).

The rotation group is an example of a symmetry group as a rotation leaves the physical laws unchanged because of the principle of relativity. We want to find the most general mathematical structures which are allowed by Poincare symmetry. So one has to look for an extension of the group $SO(3)$ by using complex numbers (the biggest algebraically closed number system). On the contrary to the real case, it is possible to find complex 2×2 matrices that obey the commutator relation (11):

$$G_1^{\mathbb{C}} = \frac{1}{2\mathrm{i}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad G_2^{\mathbb{C}} = \frac{1}{2\mathrm{i}}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad G_3^{\mathbb{C}} = \frac{1}{2\mathrm{i}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (12)$$

They act on complex vectors and generate the group of the special unitary matrices, the $SU(2)$. ‘Special’ means the determinant of the matrix is 1. The rotation vector $\boldsymbol{\alpha}$ is kept real in order to work with the same symmetry.

There is a two-to-one homomorphism of $SU(2)$ onto $SO(3)$ which can be recognized by taking into account a rotation by an angle α around the z-axes:

$$R\big((0,0,\alpha)\big) = \exp\left(\alpha \cdot G_3\right) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{13}$$
$$R^{\mathbb{C}}((0,0,\alpha)) = \exp\left(\alpha \cdot G_3^{\mathbb{C}}\right) = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\,\alpha/2} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\,\alpha/2} \end{pmatrix}.$$

It holds $R((0,0,\alpha{+}2\pi)) = R((0,0,\alpha))$, but $R^{\mathbb{C}}((0,0,\alpha{+}2\pi)) \neq R^{\mathbb{C}}((0,0,\alpha))$. R has a period of 2π and $R^{\mathbb{C}}$ a period of 4π. The standard representation of $SU(2)$ has twice as many elements as $SO(3)$. That means, $SU(2)$ is in fact an extension of $SO(3)$, a so called *overlay group*. We redefine the generators as $\tilde{J}_i = \mathrm{i}\, G_i^{\mathbb{C}}$ in order to get Hermitian matrices. This leads to

$$[\tilde{J}_i, \tilde{J}_j] = \mathrm{i}\, \epsilon_{ijk}\, \tilde{J}_k \quad \text{and} \quad R^{\mathbb{C}}(\boldsymbol{\alpha}) = \exp\left(-\mathrm{i}\, \boldsymbol{\alpha} \cdot \tilde{\mathbf{J}}\right). \tag{14}$$

The complex rotation group $SU(2)$ has representations for each dimension n obtained by use of appropriate generators fulfilling Eq. (14). The case $n = 1$ is trivial:

$$\tilde{J}_i = 0. \tag{15}$$

The 2-dimensional generators we already know:

$$\tilde{J}_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{J}_2 = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \tilde{J}_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{16}$$

For $n = 3$ we have [11]:

$$\tilde{J}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \tilde{J}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \quad \tilde{J}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{17}$$

To find three linear independent traceless Hermitian matrices that fulfill relation (14) for $n = 4$ and so on is left as an exercise for the interested reader. The matrices above in general act respectively on n-dimensional vectors ϕ_i, whose physical meaning will be clarified later.

There is also an infinite-dimensional representation of the rotation group. As one cannot work with an infinite number of components, one has to use a function $\phi(\mathbf{x})$ and the generators have to be differential operators. The continuous 'indices' $\mathbf{x}$ are identified with the coordinates in order to make a connection with the space of our experience. An infinitesimal rotation of $\phi(\mathbf{x})$ is given by

$$\begin{aligned} R\big(\phi(\mathbf{x}), \mathbf{d}\boldsymbol{\alpha}\big) &= \phi\left(R(\mathbf{d}\boldsymbol{\alpha})^{-1}\,\bar{\mathbf{x}}\right) = \phi\left(R(-\mathbf{d}\boldsymbol{\alpha})\,\bar{\mathbf{x}}\right) = \\ &= \phi(\bar{\mathbf{x}} - \mathbf{d}\boldsymbol{\alpha} \times \bar{\mathbf{x}}) = \phi(\bar{\mathbf{x}}) - \mathbf{d}\boldsymbol{\alpha} \times \bar{\mathbf{x}} \cdot \boldsymbol{\nabla}\phi(\bar{\mathbf{x}}) = \\ &= \phi(\bar{\mathbf{x}}) - \mathbf{d}\boldsymbol{\alpha} \cdot \bar{\mathbf{x}} \times \boldsymbol{\nabla}\phi(\bar{\mathbf{x}})\,, \end{aligned} \tag{18}$$

The rotated position vector is denoted by $\bar{\mathbf{x}}$. Therefore, the generators of the infinite-dimensional representation of the rotation group are given by

$$\tilde{J}_i^{\text{infinite}} = \mathrm{i}\;\bar{\mathbf{x}} \times \boldsymbol{\nabla}. \tag{19}$$

The infinite-dimensional representation can be combined with a finite-dimensional representation:

$$R\big(\phi_i(\mathbf{x}), \boldsymbol{\alpha}\big) = \mathrm{e}^{\mathrm{i}\,\boldsymbol{\alpha}\cdot\tilde{\mathbf{J}}}\,\mathrm{e}^{\mathrm{i}\,\boldsymbol{\alpha}\cdot\tilde{\mathbf{J}}^{\text{infinite}}}\,\phi_i(\bar{\mathbf{x}}). \tag{20}$$

The object $\phi_i(\mathbf{x})$ is called a *vector-like field.*

Every group that is generated in the way of the rotation group, according to Eq. (7), is simultaneously a differentiable manifold, because the generators form a tangent space [10]. Such groups are called *Lie groups.* In case of $SU(2)$ a group element, a unitary matrix with determinant equal to 1, can be written in the following way (2-dimensional representation):

$$R = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \quad \text{with} \quad |a|^2 + |b|^2 = 1, \tag{21}$$

where $a = w + \mathrm{i}\,x$ and $b = y + \mathrm{i}\,z$ are complex numbers. This leads to $w^2 + x^2 + y^2 + z^2 = 1$ which is the defining equation of a hypersphere that describes the manifold associated with $SU(2)$.

2.3 Boosts

It can be shown [8] that, as a consequence of the principle of relativity, the coordinate transformation in case of a uniform rectilinear motion is given by

$$\begin{aligned} &\bar{x}^0 = \gamma(v)\,(x^0 + K\,\mathbf{v}\,\mathbf{x}) \\ &\bar{\mathbf{x}} = \mathbf{x} + \frac{\gamma(v) - 1}{v^2}\,\mathbf{v}\,(\mathbf{v}\,\mathbf{x}) - \gamma(v)\,\mathbf{v}\,x^0 \\ &\text{where}\;\; \gamma(v) = \sqrt{(1 + K\,v^2)^{-1}}. \end{aligned} \tag{22}$$

Only a constant K and the sign of $\gamma(v)$ are not determined. By setting $K = 0$ and taking the positive sign for $\gamma(v)$, one gets the Galilean boost. In general one obtains from (22)

$$(\bar{x}^0)^2 + K\,\bar{\mathbf{x}}^2 = (x^0)^2 + K\,\mathbf{x}^{\mathbf{2}}. \tag{23}$$

The equation above can also be written in differential form:

$$(d\bar{x}^0)^2 + K\,(d\bar{\mathbf{x}})^2 = (dx^0)^2 + K\,(d\mathbf{x})^2 = ds^2$$
$$\Rightarrow \left(\frac{d\mathbf{x}}{dx^0}\right)^2 = \left(\left(\frac{ds}{dx^0}\right)^2 - 1\right)/K. \tag{24}$$

If $\left(\frac{ds}{dx^0}\right)^2 = 0$ in one coordinate system, then it remains zero after an arbitrary coordinate transformation of the form (22). It follows [8] that a time-dependent motion with velocity

$$\left(\frac{d\mathbf{x}}{dx^0}\right)^2 = -1/K \tag{25}$$

has the same velocity in all other coordinate systems. The constant K (not determined by Poincare symmetry) has to be measured in experiments [8]. It turned out that the value of $-1/K$ is equal to the square of the speed of light and therefore K is negative. We are using units where the speed of light is equal to 1, so that $K = -1$. The sign of $\gamma(v) = \sqrt{(1-v^2)^{-1}}$ should be positive in order to avoid a reversal of the sense of time [8]. Then Eq. (22) can be rewritten in the form

$$\bar{x}^\rho = B^\rho_\mu x^\mu \quad \text{with} \quad B = \begin{pmatrix} \gamma(v) & -\gamma(v)\mathbf{v}^T \\ -\gamma(v)\mathbf{v} & \mathbb{1} + \frac{\gamma(v)^2}{1+\gamma(v)}\mathbf{v}\mathbf{v}^T \end{pmatrix}. \tag{26}$$

As $(\bar{x}^0)^2 - (\bar{\mathbf{x}})^2 = (x^0)^2 - (\mathbf{x})^{\mathbf{2}}$, this special kind of scalar product of the coordinate vector $x^\mu = (t, x, y, z)$ with itself is invariant under boost transformations. One introduces a 4-dimensional metric

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{27}$$

so that

$$s^2 = (x^0)^2 - (\mathbf{x})^2 = x^\mu \eta_{\mu\nu} x^\nu = x^\mu x_\mu. \tag{28}$$

For the 4-dimensional pseudo-scalar product it follows in case of a boost:

$$\bar{x}^\mu \eta_{\mu\nu} \bar{x}^\nu = x^\sigma B^\mu_\sigma\, \eta_{\mu\nu}\, B^\nu_\rho x^\rho = x^\sigma \eta_{\sigma\rho} x^\rho \tag{29}$$

which leads to

$$B^\mu_\sigma\, \eta_{\mu\nu}\, B^\nu_\rho = \eta_{\sigma\rho}. \tag{30}$$

Similar to the case of the rotations, we look at an infinitesimal boost

$$B^\mu_\sigma(\epsilon) = \delta^\mu_\sigma + \epsilon K^\mu_\sigma \tag{31}$$

in order to be able to find an expression for the boost generator K. It follows form Eq. (30)

$$(\delta^\mu_\sigma + \epsilon K^\mu_\sigma)\,\eta_{\mu\nu}\,(\delta^\nu_\rho + \epsilon K^\nu_\rho) = \eta_{\sigma\rho}. \tag{32}$$

As a consequence

$$K^\mu_\sigma \eta_{\mu\rho} + K^\nu_\rho \eta_{\sigma\nu} = 0 \quad \Rightarrow \quad K^T\eta = -\eta K, \tag{33}$$

if one neglects terms of order ϵ^2. We now look at a boost along the x-axis that has to be of the form:

$$K_1 = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{34}$$

By inserting this matrix into Eq. (33) one gets $a = d = 0$ and $b = c$ equal to a constant as solution. One chooses b and c equal to $-\mathrm{i}$ in order to be able to write the finite boost transformation in the standard form $B_1(\kappa_1) = \mathrm{e}^{\mathrm{i}\kappa_1 \cdot K_1}$ with a real parameter κ_1. In an analogous way the generators for boosts along the y- and z-axis are obtained.

$$K_2 = \begin{pmatrix} 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \end{pmatrix}. \tag{35}$$

so that $B(\boldsymbol{\kappa}) = \mathrm{e}^{\mathrm{i}\boldsymbol{\kappa}\cdot\mathbf{K}}$ with three real 3 parameters $\boldsymbol{\kappa} = (\kappa_1, \kappa_2, \kappa_3)$, that are related to the three components of the velocity vector associated with the boost. In the special case of a boost in the x-direction $\cosh(\kappa_1) = \gamma(v^2)$ and $\kappa_2 = \kappa_3 = 0$.

2.4 Translations in space and time

We are speaking here of translations in time and space. In an infinite form they are written as

$$x^\mu + \epsilon^\mu = x^\mu + \epsilon^\nu \partial_\nu x^\mu \quad \text{with} \quad \partial_\nu = \partial/\partial x^\nu. \tag{36}$$

Therefore, the translation generators are given by ∂_ν and they are made Hermitian by multiplying with i. A general infinite-dimensional representation of a translation T is given by

$$T(\phi(x^\mu), k^\mu) = \mathrm{e}^{k^\nu\partial_\nu}\phi(x^\mu) = \mathrm{e}^{-\mathrm{i}k^\nu P_\nu}\phi(x^\mu) = \phi(x^\mu + k^\mu). \tag{37}$$

with the translation generators $P_\nu = \mathrm{i}\,\partial_\nu$. Like in the case of rotations the 'vectors', they act on, are fields.

2.5 The Lorentz group

The generators G_i of the $SO(3)$ representation of the rotation group from Eq. (6) are extended in the following way to get 4-dimensional matrices.

$$J_i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \mathrm{i}\,G_i & \\ 0 & & & \end{pmatrix}. \tag{38}$$

Then the corresponding 4-dimensional rotation matrices $R^\mu_\nu = \mathrm{e}^{\mathrm{i}\,\alpha_i J_i}$ fulfill the same relation (30) as the boosts B^μ_ν, that means they preserve the pseudo-scalar product. Rotations and boosts are members of the so called *Lorentz group* $O(1,3)$, whose elements Λ^μ_ν are specified by $\Lambda^\mu_\sigma\,\eta_{\mu\nu}\,\Lambda^\nu_\rho = \eta_{\sigma\rho}$. Strictly speaking, rotations and boost together form a subgroup of $O(1,3)$, the proper orthochronous Lorentz group, that does not contain time and space reversals. By using the explicit expressions for the generators of rotations and boosts, see Eqs. (6), (34), (35) and (38), the following commutator relations, that define the proper orthochronous Lorentz group, can easily be verified:

$$[J_i, J_j] = \mathrm{i}\,\epsilon_{ijk} J_k\,, \quad [K_i, K_j] = -\mathrm{i}\,\epsilon_{ijk} J_k\,, \quad [J_i, K_j] = \mathrm{i}\,\epsilon_{ijk} K_k. \tag{39}$$

In order to get well-arranged relations the following linear combinations of the generators are introduced [11]

$$N_i^\pm = \frac{1}{2}(J_i \pm \mathrm{i} K_i). \tag{40}$$

This leads to

$$[N_i^+, N_j^+] = \mathrm{i}\,\epsilon_{ijk} N_k^+\,, \quad [N_i^-, N_j^-] = \mathrm{i}\,\epsilon_{ijk} N_k^-\,, \quad [N_i^+, N_j^-] = 0. \tag{41}$$

By comparing this with the defining equation $[\tilde{J}_i, \tilde{J}_j] = \mathrm{i}\,\epsilon_{ijk}\,\tilde{J}_k$ of the group $SU(2)$, which we have already derived in subsection 2.2, one notices that the Lorentz group is composed of two separate copies of $SU(2)$.

In case of $SU(2)$ we found n-dimensional representations for the generators, Eq. (15), (16) and (17). As the Lorentz group contains two copies of $SU(2)$ it must have $n_1 \times n_2$-dimensional representations.

The 1×1-dimensional representation is just a multiplication of a scalar ϕ by 1, as $N_i^+ = N_i^- = 0$ is the only possible choice for 1-dimensional objects, that fulfill Eq. (41), i.e. all generators are zero: $e^0\,\phi\, e^0 = \phi$. This seems to be quite boring, but the most prominent physical realisation of the 1×1-dimensional representation is the Higgs field.

In order to calculate the form of the generators J_i and K_i in the 2×1-dimensional representation of the Lorentz group [11], one can make use of the fact that $N^- = 0$ (for the same reason as in case of the 1×1 representation):

$$N_i^- = \frac{1}{2}(J_i - \mathrm{i}\, K_i) = 0 \quad \rightarrow \quad J_i = \mathrm{i}\, K_i. \tag{42}$$

Since in this case the N_i^+ generators belong to a 2-dimensional representation of $SU(2)$, they must be 2×2-dimensional traceless matrices. Such matrices contain 3 independent parameters. A convenient choice for a basis are the so called Pauli matrices, already used in subsection 2.2.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{43}$$

This leads to

$$N_i^+ = \frac{\sigma_i}{2}. \tag{44}$$

The factor 1/2 is used because the commutator relation has to be fulfilled. On the other hand

$$N_i^+ = \frac{1}{2}\,(J_i + \mathrm{i}\, K_i) = \frac{1}{2}\,(\mathrm{i}\, K_i + \mathrm{i}\, K_i) = \mathrm{i}\, K_i \quad \rightarrow \quad K_i = -\frac{\mathrm{i}}{2}\,\sigma. \tag{45}$$

By use of Eq. (42):

$$J_i = \mathrm{i}\, K_i = \frac{1}{2}\,\sigma. \tag{46}$$

Therefore, in this representation a Lorentz transformation is given by

$$e^{\mathrm{i}\,\alpha\cdot\mathbf{J} + \mathrm{i}\,\kappa\cdot\mathbf{K}} = e^{\mathrm{i}\,\alpha\cdot\frac{\sigma}{2} + \kappa\cdot\frac{\sigma}{2}}. \tag{47}$$

Because it is a 2×1-dimensional representation, those matrices act on 2×1-dimensional objects which are 2-dimensional vectors χ, called *Spinors*. They are realised in nature through the fermion fields.

Finally, in case of the 2×2-dimensional representation the transformed object has to be a 2×2 tensor Φ_{ij}. The transformation matrix from Eq. (47) is applied twice, once from the left and once from the right, according to the standard form of a linear matrix transformation:

$$e^{\mathrm{i}\,\alpha\cdot\frac{\sigma}{2} + \kappa\cdot\frac{\sigma}{2}}\, \Phi \left(e^{\mathrm{i}\,\alpha\cdot\frac{\sigma}{2} + \kappa\cdot\frac{\sigma}{2}}\right)^\dagger. \tag{48}$$

As a basis for Φ (which is not traceless) we use the Pauli matrices plus the unit matrix (that is called σ_0 in this context).

$$\Phi = \phi^\mu \, \sigma_\mu = \begin{pmatrix} \phi^0 + \phi^3 & \phi^1 - \mathrm{i}\, \phi^2 \\ \phi^1 + \mathrm{i}\, \phi^2 & \phi^0 - \phi^3 \end{pmatrix} . \tag{49}$$

We want to find a Lorentz-invariant quantity that can be derived from Φ and used as a building block for a Lorentz-invariant description of nature. The determinant of Φ does not change when transformed pursuant to Eq. (48), but it can be a complex number, which would make the connection with measurable quantities complicated. A better choice is the difference of the determinants of the Hermitian part Φ^H and the anti-Hermitian part Φ^A of the matrix Φ:

$$\mathrm{Det}(\Phi^H) - \mathrm{Det}(\Phi^A) = |\Phi^0|^2 - |\Phi^1|^2 - |\Phi^2|^2 - |\Phi^3|^2, \tag{50}$$

that turns out to be completely equivalent to the pseudo-scalar product of a 4-dimensional vector ϕ^μ with itself. If two tensors Φ_a and Φ_b are connected by a Lorentz transformation according to Eq. (48)

$$\Phi_b = \Lambda(\Phi_a) = \mathrm{e}^{\mathrm{i}\, \alpha\cdot\frac{\sigma}{2} + \kappa\cdot\frac{\sigma}{2}} \, \Phi_a \left(\mathrm{e}^{\mathrm{i}\, \alpha\cdot\frac{\sigma}{2} + \kappa\cdot\frac{\sigma}{2}}\right)^\dagger , \tag{51}$$

this establishes a linear relationship between the quantities $\phi_a^0, \phi_a^1, \phi_a^2, \phi_a^3$ and $\phi_b^0, \phi_b^1, \phi_b^2, \phi_b^3$. It follows that there must be a linear transformation connecting the vectors ϕ_a^μ and ϕ_b^μ

$$\phi_b^\mu = L^\mu_\nu \, \phi_b^\nu. \tag{52}$$

As the Lorentz transformations Λ^μ_ν of the usual 4-dimensional representation are the most general homogenous linear transformations which preserve the squared norm of a 4-vector, $L^\mu_\nu = \Lambda^\mu_\nu$ holds and we have demonstrated that there exists a mapping between the 2×2-dimensional and the 4-dimensional representation. So instead of the rather complicated matrices Φ the vectors ϕ^μ can be used.

Summing up, in this subsection we have found scalars ϕ, spinors χ and vectors ϕ^μ, with the corresponding transformation properties, as the most general objects allowed by the Lorentz symmetry up to $n_1 \times n_2 = 2 \times 2$. We will not analyse higher dimensional representations.

2.6 The Poincare group

By combining the Lorentz and the translation group, the Poincare group is obtained. We first introduce a tensor $M_{\mu\nu}$ which contains both the rotation and the boost generator vectors (J_i and K_i) in the following way:

$$J_i = \frac{1}{2}\epsilon_{ijk} M_{jk}, \quad K_i = M_{0i}. \tag{53}$$

This procedure is similar to using a field strength tensor in electrodynamics. The tensor $M_{\mu\nu}$ is composed of a finite- and an infinite-dimensional part. The later is given in analogy to the infinite-dimensional rotation generator of Eq. (19) by

$$M_{\mu\nu}^{\text{infinite}} = \mathrm{i}\,(x_\mu \partial_\nu - x_\nu \partial_\mu). \tag{54}$$

With the help of the commutator relations (39) for J_i and K_i one gets [11]

$$[M_{\mu\nu}, M_{\rho\sigma}] = -\mathrm{i}\,(\eta_{\mu\rho} M_{\nu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\nu\sigma} M_{\mu\rho}). \tag{55}$$

This relation together with the following two relations (including the translation generator $P_\nu = \mathrm{i}\,\partial_\nu$ from Eq. (37)) specify a Poincare transformation.

$$[P_\mu, P_\nu] = 0, \quad [M_{\mu\nu}, P_\rho] = \mathrm{i}\,(\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu). \tag{56}$$

A Lorentz transformation is given by

$$\Lambda(\phi^\mu(x)) = \mathrm{e}^{\mathrm{i}\, b^{\sigma\rho} M_{\sigma\rho}} \phi^\mu(\bar{x}), \tag{57}$$

where the tensor $b^{\mu\nu}$ contains the new transformation parameters, that can be calculated from the old ones, namely $\boldsymbol{\alpha}$ and $\boldsymbol{\kappa}$. And a Poincare transformation Π looks like

$$\Pi(\phi^\mu(x)) = \mathrm{e}^{-\mathrm{i}\, k^\nu P_\nu} \Lambda(\phi^\mu(\bar{x})) = \mathrm{e}^{-\mathrm{i}\, k^\nu P_\nu} \mathrm{e}^{\mathrm{i}\, b^{\sigma\rho} M_{\sigma\rho}} \phi^\mu(\bar{x}). \tag{58}$$

As mentioned before, apart from the scalar fields and the spinors, the vector fields $\phi^\mu(x)$ are the most general objects, the transformations act on. For the rest of the work we will specify ourselves on vector fields. The eigenvectors of the generators form a basis for the vector field. For the eigenspace of the finite-dimensional generators four linear independent vectors $\epsilon_\mu^{(\lambda)}(p)$ labelled by $\lambda = 0, 1, 2, 3$ can be used as basis. The infinite-dimensional generator P^μ has the eigenfunctions $\mathrm{e}^{\mathrm{i}\, x \cdot p}$. Therefore, a vector field can be decomposed in the following way, where $a^{(\lambda)}(p)$ is the component function.

$$\phi_\mu(x) = \sum_{\lambda=0}^{3} \int d^4 p\, a^{(\lambda)}(p)\, \epsilon_\mu^{(\lambda)}(p)\, \mathrm{e}^{-\mathrm{i}\, x \cdot p}. \tag{59}$$

An example for a Poincare invariant quantity is $\phi_\mu^\dagger(x)\, \phi^\mu(x)$ as

$$\Pi(\phi_\mu(x)^\dagger\, \phi(x)^\mu)) = \phi_\mu(x)^\dagger \mathrm{e}^{-\mathrm{i}\, b^{\sigma\rho} M_{\sigma\rho}} \mathrm{e}^{\mathrm{i}\, c^\nu P_\nu} \mathrm{e}^{-\mathrm{i}\, c^\nu P_\nu} \mathrm{e}^{\mathrm{i}\, b^{\sigma\rho} M_{\sigma\rho}} \phi^\mu(x) = \phi_\mu^\dagger(x) \phi^\mu(x).$$

These pseudo-scalar products are basic building blocks for a Poincare invariant universal description of nature, as we will see in the next section.

3 Measurable quantities

Until now our formalism was quite theoretical. In this section we build a bridge to measurable quantities. But first, we focus on kinematics.

3.1 Time evolution

An arbitrary physical system is in the beginning in the state $|i\rangle$ and after a short time interval dt in the state $|f\rangle$. How these states are specified, is explained later. The operator $\hat{H}$ performs the time shift:

$$|f\rangle = \left(1 - \mathrm{i}\, dt\, \hat{H}\right) |i\rangle. \tag{60}$$

In this section we find an expression for the operator $\hat{H}$, that describes the time evolution[2]. We just know that it must be a function of $\phi^\mu(x)$ and the derivation $\frac{\partial}{\partial\phi^\mu(x)}$ because there are no other quantities. The index μ will be suppressed in the following and we work at first order in the small time interval dt.

$$\begin{aligned}
&\left(1 - \mathrm{i}\, dt\, \hat{H}\left(\tilde{\phi}(x), \frac{\partial}{\mathrm{i}\,\partial\tilde{\phi}(x)}\right)\right) |i\rangle = \\
&= \int_{-\infty}^{\infty} \mathrm{d}\phi(x) \left(1 - \mathrm{i}\, dt\, \hat{H}\left(\phi(x), -\frac{\partial}{\mathrm{i}\,\partial\phi(x)}\right)\right) \delta\left(\tilde{\phi}(x) - \phi(x)\right) |i\rangle = \\
&= \int_{-\infty}^{\infty} \mathrm{d}\phi \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \left[(1 - \mathrm{i}\, dt\, H(\phi,\omega))\, e^{\mathrm{i}\,\omega\left(\tilde{\phi}-\phi\right)}\right] |i\rangle = \\
&= \int_{-\infty}^{\infty} \mathrm{d}\phi \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \left[1 + \mathrm{i}\, dt \left(\omega\, \frac{\tilde{\phi} - \phi}{dt} - H(\phi,\omega)\right)\right] |i\rangle.
\end{aligned} \tag{61}$$

Such a functional integral can be evaluated via the so called Feynman path integral formalism [12] by splitting the range of integration into small discrete pieces and taking the continuous limit afterwards. The term path integral does not fully apply here, as not all paths, but all possible field constellations are considered, starting from the constellation of the initial state and ending with that of the final state. Terms in H, which are of higher order then two in the integration variable ω, are usually neglected, as they do not appear in concrete physical situations or their contribution is small. Therefore, the ω-integration can be done in a close form, yielding the result:

$$\lim_{N\to\infty} \prod_{n=1}^{N} \int d\phi_n\, \mathrm{e}^{\mathrm{i}\int d^4x \mathcal{L}(\phi,\partial^\mu\phi)} = \int \mathcal{D}\phi\, \mathrm{e}^{\mathrm{i}\int d^4x \mathcal{L}(\phi,\partial^\mu\phi)}, \tag{62}$$

[2]When using SI-units, the operator is divided by a constant $\hbar$, Planck's quantum of action. We are using natural units, where $\hbar = 1$ and omit this constant.

where the function $L = \int d^4x\, \mathcal{L}$ is connected to H through a Legendre transformation

$$L(\phi(x), \partial_0 \phi(x)) = \omega(x)\, \partial_0 \phi(x) - H(\phi(x), \omega(x)) \tag{63}$$

that came out automatically in the last line of Eq. (61) in a differential form.

The exponent in Eq. (62) is divided by the constant $\hbar$, which is equal to 1 in our units. If the integral $d^4x\mathcal{L}$ (called action) is much bigger than 1, the exponential function oscillates rapidly in the complex domain for any change in the action and all contributions without a stationary action cancel each other. Let us emphasise, $\delta \int d^4x\mathcal{L} = 0$ is not an assumption, it is a consequence in this case. But, if the action is of the order 1, there is no longer a deterministic evolution of the field. Then everything, that is allowed by the symmetries, occurs during the time evolution, any field constellation happens (with the same probability), as there is no reason to privilege one constellation. However, we will see that a deterministically evoluting field which is obtained by $\delta \int d^4x\mathcal{L} = 0$, is useful for many purposes.

What remains to do, is to find an expression for $\mathcal{L}$, that is called *Lagrangian.* As everything exists that is not forbidden, it must be the most general Poincare invariant expression containing $\phi^\mu(x)$ and its derivation. There are three possible terms that can contribute to $\delta \int d^4x\mathcal{L} = 0$:

$$\phi^\dagger_\mu(x)\phi(x)^\mu, \quad \partial_\mu\phi^\dagger_\nu\partial^\mu\phi^\nu(x), \quad \partial^\mu\phi^\dagger_\mu(x)\partial^\mu\phi_\nu(x). \tag{64}$$

Expressions with more than two derivations are expected to be neglectable. Every term in the Lagrangian is multiplied with a constant[3]

$$\mathcal{L}(\phi^\mu, \partial^\mu\phi_\mu) = -\frac{1}{2}(\partial_\mu\phi^\dagger_\nu\partial^\mu\phi^\nu + k_1\, \partial^\mu\phi^\dagger_\mu\partial^\nu\phi_\nu + k_2\, \phi^\dagger_\mu\phi^\mu\,). \tag{65}$$

Calculating $\delta \int d^4x\mathcal{L} = 0$ leads to

$$\partial_\nu\partial^\nu\phi^\mu + k_1\, \partial^\mu\partial^\nu\phi_\nu - k_2\, \phi^\mu = 0. \tag{66}$$

If we apply ∂_μ to this equation, we get

$$\partial_\mu\partial_\nu\partial^\nu\phi^\mu + k_1\, \partial_\mu\partial^\mu\partial^\nu\phi_\nu - k_2\, \partial_\mu\phi^\mu = 0. \tag{67}$$

To fix the constants k_i we introduce the Casimir operators of the Poincare group. These operators commute with all group generators and have a constant eigenvalue within a representation. One of them is given by $P_\nu P^\nu$ [8].

$$P_\nu P^\nu\, \phi^\mu = -\partial_\nu\partial^\nu\phi^\mu = -m^2\, \phi^\mu. \tag{68}$$

[3]The choice of -1/2 as normalisation will be explained later.

It follows that ϕ^μ can only fulfill the relations (66) and (67) if $k_1 = 0$ or -1 and $k_2 = m^2$. The second Casimir operator of the Poincare group, build from the *Pauli-Lubanski vector* (see App. A), leads to an equation which requests a relation between the four components of the vector ϕ^μ [8]. Such a relation is given by Eq. (67), if $k_1 = -1$ and $k_2 = m^2$. It is then of the form

$$\partial_\mu \phi^\mu = 0. \tag{69}$$

So the Lagrangian must look like

$$\mathcal{L}(\phi^\mu, \partial^\mu \phi_\mu) = -\frac{1}{2}(\partial_\mu \phi^\dagger_\nu \partial^\mu \phi^\nu - \partial^\mu \phi^\dagger_\mu \partial^\nu \phi_\nu + m^2\, \phi^\dagger_\mu \phi^\mu\,). \tag{70}$$

The function H, which is called Hamiltonian, can then by obtained by a Legendre transformation.

In the decomposition of ϕ^μ, given by Eq. (59), the polarisation vector $\epsilon^{(0)}_\mu(p)$ can be chosen parallel to p and the other three orthogonal to p. Then it follows from relation (69) that $a^{(0)}(p)$ must be zero and therefore the sum in the decomposition of ϕ^μ goes only from 1 to 3. Apart from that, the integration measure should be multiplied with $\delta(p^2 - m^2)$ due to Eq. (68) and a factor $\theta(p_0)$ has to be applied to get a unique representation of the Poincare group [8]:

$$\phi_\mu(x) = \sum_{\lambda=1}^{3} \int \frac{d^4p}{(2\pi)^4}\; \delta(p^2 - m^2)\, \theta(p_0)\, a^{(\lambda)}(p)\, \epsilon^{(\lambda)}_\mu(p)\, \mathrm{e}^{-\mathrm{i}\, x\cdot p}. \tag{71}$$

The following modifications can be applied to the integration measure:

$$\begin{aligned}
&dp^4\, \delta(p^2 - m^2)\, \theta(p_0) = dp^4\, \delta(p_0^2 - \mathbf{p}^2 - m^2)\, \theta(p_0) \\
&= dp^4\, \delta\left((p_0 - \sqrt{\mathbf{p}^2 + m^2})(p_0 + \sqrt{\mathbf{p}^2 + m^2})\right)\, \theta(p_0) \\
&= dp^4\, \frac{1}{2p_0}\left(\delta(p_0 - \sqrt{\mathbf{p}^2 + m^2}) + \delta(p_0 + \sqrt{\mathbf{p}^2 + m^2})\right)\, \theta(p_0) \\
&= dp^3\, \frac{dp_0}{2p_0} \delta(p_0 - \sqrt{\mathbf{p}^2 + m^2}) = \frac{dp^3}{2\sqrt{\mathbf{p}^2 + m^2}} = \frac{dp^3}{2E_p} = d^3\tilde{p}.
\end{aligned} \tag{72}$$

The decomposition of $\phi^\mu(x)$ then looks like:

$$\phi_\mu(x) = \sum_{\lambda=1}^{3} \int d^3\tilde{p}\; a^{(\lambda)}(p)\, \epsilon^{(\lambda)}_\mu(p)\, \mathrm{e}^{-\mathrm{i}\, x\cdot p}. \tag{73}$$

3.2 Noether's theorem

One of the most famous theorems of mathematical physics was published in the year 1918 [13]. Noether's theorem states that every continuous symmetry of the action of a physical system leads to a conserved quantity. Some of these quantities will be our observables. A variation of the Lagrangian is given by

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi^\eta}\delta\phi^\eta + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\eta)}\delta(\partial_\mu\phi^\eta) + \frac{\partial\mathcal{L}}{\partial x_\mu}\delta x_\mu. \tag{74}$$

This equation can be rearranged [11]:

$$\begin{aligned}\delta\mathcal{L} &= \partial_\mu\left(\frac{\mathcal{L}}{\partial(\partial_\mu\phi^\eta)}\right)\delta\phi^\eta + \left(\frac{\mathcal{L}}{\partial(\partial_\mu\phi^\eta)}\right)\delta(\partial_\mu\phi^\eta) + \frac{\partial\mathcal{L}}{\partial x_\mu}\delta x_\mu \\ &= \partial_\mu\left(\frac{\mathcal{L}}{\partial(\partial_\mu\phi^\eta)}\delta\phi^\eta\right) + \frac{\partial\mathcal{L}}{\partial x_\mu}\delta x_\mu. \end{aligned} \tag{75}$$

The variation of the field $\delta\phi^\eta$ by applying the Poincare group generators is:

$$\delta\phi^\eta = -\mathrm{i}\,P^\mu\phi^\eta(x)\; + \mathrm{i}\,b^{\mu\nu}\,M_{\mu\nu}\,\phi^\eta(x)\,\delta x_\mu. \tag{76}$$

As Poincare transformations are symmetry transformations, they do not change the action, guaranteed by $\delta\mathcal{L} = 0$.Inserting the first term of $\delta\phi^\eta$ in Eq. (75) and doing some algebraic manipulations [11] shows that the translation invariance leads to

$$\partial_0 \int d^3x\,\Theta^0_\mu = 0, \tag{77}$$

and the Lorentz invariance has the consequence

$$\partial_0\frac{1}{2}\epsilon_{ijk}\int d^3x\left(\frac{\partial\mathcal{L}}{\partial(\partial_0\phi^\eta)}\,\mathrm{i}\,b^{\mu\nu}\,M^{finite}_{jk}\phi^\eta + (\Theta_{k0}\,x_j - \Theta_{j0}\,x_k)\right) = 0, \tag{78}$$

where Θ is the energy-momentum tensor given by

$$\Theta^\nu_\mu := \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi^\eta)}\frac{\partial\phi^\eta}{\partial x^\mu} - \delta^\nu_\mu\mathcal{L}. \tag{79}$$

3.3 Observables

We now use the Lagrangian from Eq. (70) to calculate some of the conserved quantities obtained by Noether's theorem which are appropriate as measurable quantities. The Eq. (77) has four lines ($\mu = 0, 1, 2, 3$). The case $\mu = 0$ is due to time invariance and the conserved quantity is called energy:

$$\int d^3x\,\Theta^0_0 = \int d^3x\left(\frac{\partial\mathcal{L}}{\partial(\partial_0\phi^\eta)}\frac{\partial\phi^\eta}{\partial x_0} - \mathcal{L}\right). \tag{80}$$

The right side of Eq. (80) looks exactly like an integral over a Legendre transformation of $\mathcal{L}$. So it must be the Hamilton function. We have now shown that it is equal to the energy of the field, if the Lagrangian is normalised in an appropriate way as we did in Eq. (65). One can go on by inserting the decomposition (73) of ϕ^μ into Eq. (80) an gets

$$\int d^3x\, \Theta_0^0 = d^3\tilde{p}\ |a(p)|^2 \sqrt{\mathbf{p}^2 + m^2}, \tag{81}$$

which is still equal to the energy of the field. In the case $\mu = 1, 2, 3$ we obtain the momentum of the field:

$$\int d^3x\, \Theta_i^0 = \int d^3x\, \partial_0 \phi_\mu^\dagger \partial_i\, \phi^\mu = \int d^3\tilde{p}\ |a(p)|^2\, p_i. \tag{82}$$

The last two equations can be combined to

$$\int d^3x\, \Theta_\nu^0 = \int d^3x\, \partial_0 \phi_\mu^\dagger (-\mathrm{i}\, P_\nu)\, \phi^\mu = \int d^3\tilde{p}\ |a(p)|^2\, p_\nu. \tag{83}$$

If we use instead of ϕ^μ an eigenfunction of the generator P_μ, namely $\frac{1}{\sqrt{E_p}} \mathrm{e}^{-\mathrm{i}\, x \cdot p}$ (with a proper normalisation), we get p_μ as momentum vector of the field. In case of a general wave it is the weighted average given by Eq. (83). That means, the eigenvalues of the translation symmetry generator are conserved quantities (in a closed system) and therefore useful as measurable quantities. We have derived this as a consequence of the principle of relativity.

Experiments have shown that in case of a microscopic system a general wave function is changed to an eigenfunction when it comes in contact with a macroscopic environment. To explain that would go beyond the scope of this work.

Now we come to the second term of Eq. (76). If this variation of the field is applied to Noether's theorem, one gets the following conserved quantity:

$$\frac{1}{2}\, \epsilon_{ijk} \int d^3x \left((\Theta_{k0}\, x_j - \Theta_{j0}\, x^k) + \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi^\mu)}\, \mathrm{i}\, b^{\mu\nu}\, M_{jk}^{finite} \phi^\mu \right). \tag{84}$$

The first term is the angular momentum of the field. We will not analyse it in detail. The second term is a kind of intrinsic angular momentum, the spin. In subsection 2.6 we introduced the definition $J_i = \frac{1}{2}\epsilon_{ijk}\, M_{jk}^{finite}$, so we get for the spin:

$$S_i = \int d^3x\, \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi^\mu(x))} (-\mathrm{i}\, J_i^{finite})\, \phi^\mu(x) = \int d^3x\, \partial_0 \phi_\mu^\dagger(x) (-\mathrm{i} J_i^{finite})\, \phi^\mu(x). \tag{85}$$

The finite-dimensional representation of J_i, that we use, is in principle 4-dimensional, but only the 3-dimensional submatrix that generates the rotations is relevant for spin description. This submatrix has three eigenvalues and therefore three eigenvectors $\epsilon_\mu^{(\lambda)}$, which form a basis in the subspace. So every component of the spin vector can take one of three discrete values. If one considers the commutator relation of the generators J_i, which is forced by Lorentz symmetry, one notices, that only one of the generators can be diagonalised simultaneously, so that in Eq. (85) only the eigenvectors of one of the three generators J_i can be used as a replacement of ϕ_μ. The other components of the spin are then a mixture of the three possible values.

3.4 Quanta

In subsection 3.1 the time shifting operator $\hat{H}$ is introduced. It is a function of the multiplicative operator $\phi^\mu(x)$ and the differential operator $\frac{\partial}{\mathrm{i}\,\partial\phi^\mu(x)}$. This is only one of different possible representations. In an abstract form the two operators are denoted by $\hat{\phi}^\mu(x)$ and $\hat{\omega}^\mu(x)$. It is easy to show that

$$[\hat{\phi}^\mu(x), \frac{\partial}{\mathrm{i}\,\partial\phi_\nu(x)}] = \mathrm{i}\,\delta(x-y)\,g^{\mu\nu}. \qquad (86)$$

As such a commutator relation does not depend on a concrete realisation, it also holds

$$[\hat{\phi}^\mu(x), \hat{\omega}^\nu(x)] = \mathrm{i}\,\delta(x-y)\,g^{\mu\nu}. \qquad (87)$$

Also in subsection 3.1 we have shown that it is equivalent to describe time evolution by a path integral containing the Hamiltonian $H(\phi^\mu, \omega^\mu)$, that is the same function as the operator $\hat{H}(\hat{\phi}^\mu, \hat{\omega}^\mu)$, with the difference that the arguments are no operators anymore. The deterministic field is obtained by the condition $\delta(\omega_\mu \partial_0 \phi^\mu - H(\phi^\mu, \omega^\mu)) = \delta L(\phi^\mu, \omega^\mu) = 0$, that also implicates

$$\omega_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi^\mu)} = \partial_0 \phi_\mu^\dagger. \qquad (88)$$

In Eq. (71) we have specialised ourselves on positive energies

$$E_p = \sqrt{\mathbf{p}^2 + m^2} > 0\,, \qquad (89)$$

but there is also a representation of the Poincare group with $E_p < 0$ that we will now include simultaneously (with $b^{(\lambda)}(E_p, \mathbf{p}) = a^{\dagger(\lambda)}(-E_p, \mathbf{p})$):

$$\phi_\mu(x) = \sum_{\lambda=1}^{3} \int d^3\tilde{p}\ \left(a^{(\lambda)}(p)\,\epsilon_\mu^{(\lambda)}(p)\,\mathrm{e}^{-\mathrm{i}\,x\cdot p} + b^{\dagger(\lambda)}(p)\,\epsilon_\mu^{\dagger(\lambda)}(p)\,\mathrm{e}^{\mathrm{i}\,x\cdot p}\right) \qquad (90)$$

and due to Eq. (88):

$$\omega_\mu(x) = \frac{1}{\mathrm{i}} \sum_{\lambda=1}^{3} \int d^3\tilde{p}\; E_p \; \left(a^{(\lambda)}(p)\, \epsilon_\mu^{(\lambda)}(p)\, \mathrm{e}^{-\mathrm{i}\, x\cdot p} - b^{\dagger(\lambda)}(p)\, \epsilon_\mu^{\dagger(\lambda)}(p)\, \mathrm{e}^{\mathrm{i}\, x\cdot p} \right). \quad (91)$$

The operators $\hat{\phi}^\mu(x)$ and $\hat{\omega}^\mu(x)$ can be decomposed in the same way with operators $\hat{a}^{(\lambda)}(p)$ and $\hat{b}^{(\lambda)}(p)$. Then the relation $[\hat{\phi}^\mu(x), \hat{\omega}^\nu(x)] = \mathrm{i}\, \delta(x-y)\, g^{\mu\nu}$ leads to

$$[\hat{a}^{(\lambda)}(p), \hat{a}^{\dagger(\lambda')}(p')] = [\hat{b}^{(\lambda)}(p), \hat{b}^{\dagger(\lambda')}(p')] = (2\pi)^3\, 2E_p\, \delta^{(3)}(\mathbf{p} - \mathbf{p}')\, \delta^{(\lambda)(\lambda')} \quad (92)$$

and 0 for all other commutators.

The explicit form of the Hamilton function is given by $H = \int dx^3\, \Theta^0_0$, where Θ is the energy-momentum tensor of Eq. (79). By inserting the field operators and subtracting the expression of the zero point energy one gets:

$$\hat{H} = \sum_{\lambda=1}^{3} \int \frac{d\tilde{p}^3}{(2\pi)^3} E_p \left(\hat{a}^{\dagger(\lambda)}(p)\, \hat{a}^{(\lambda)}(p) + \hat{b}^{\dagger(\lambda)}(p)\, \hat{b}^{(\lambda)}(p) \right). \quad (93)$$

For a fixed λ this leads to

$$\begin{aligned}
[\hat{H}, \hat{a}^{\dagger(\lambda)}(p)] &= \int \frac{d\tilde{p}^3}{(2\pi)^3} E_p\, [\hat{a}^{\dagger(\lambda')}(p')\, \hat{a}(p')^{(\lambda')}, \hat{a}^{\dagger(\lambda)}(p)] \\
&= \int \frac{d\tilde{p}^3}{(2\pi)^3} E_p\, \hat{a}^{\dagger(\lambda')}(p')\, [\hat{a}(p')^{(\lambda')}, \hat{a}^{\dagger(\lambda)}(p)] \\
&= \int d\tilde{p}^3 2E_p^2\, \hat{a}^{\dagger(\lambda)}(p)\, \delta^3(\mathbf{p} - \mathbf{p}') = E_p\, \hat{a}^{\dagger(\lambda)}(p) \qquad (94)
\end{aligned}$$

and

$$[\hat{H}, \hat{a}^{(\lambda)}(p)] = -E_p\, a^{(\lambda)}(p). \quad (95)$$

There are analog relations for the b-operators. The acting of $\hat{H}$ on an energy eigenstate is defined by

$$\hat{H}|E\rangle = E\,|E\rangle. \quad (96)$$

Now we let $\hat{H}$ act on $\hat{a}^{\dagger(\lambda)}(p)\,|E\rangle$:

$$\begin{aligned}
\hat{H}\, \hat{a}^{\dagger(\lambda)}(p)\,|E\rangle &= \hat{a}^{\dagger(\lambda)}(p) E\,|E\rangle + [\hat{H}, \hat{a}^{\dagger(\lambda)}(p)]|E\rangle \\
&= \hat{a}^{\dagger(\lambda)}(p) E\,|E\rangle + E_p\, \hat{a}^{\dagger(\lambda)}(p)|E\rangle \\
&= (E + E_p)\hat{a}^{\dagger(\lambda)}(p)\,|E\rangle. \qquad (97)
\end{aligned}$$

What we see is that the energy of the state $\hat{a}^{\dagger(\lambda)}(p)\,|E\rangle$ is higher by E_p compared to the state $|E\rangle$. This can be interpreted in the following way:

The application of $\hat{a}^{\dagger(\lambda)}(p)$ has enlarged the energy by a certain amount E_p. Applying it twice enlarges the energy by $2E_p$ and so on. Contrariwise, the operator $\hat{a}^{(\lambda)}(p)$ decreases the energy by E_p. Therefore, the energy is not a continuous quantity, it can only vary by fixed quanta. That is the reason for the name *quantum theory*. And also for the fact that we live in a world of particles, because a particle is such an energy quantum. The 3-dimensional momentum $\mathbf{p}$ of a particle is given by the relation $E_p^2 = \mathbf{p}^2 + m^2$. The energy of a particle at rest is m, the mass of the particle.

If one makes the same considerations with the spin operator, see Eq. (85),

$$\hat{S}_i = \int d^3x \, \partial_0 \, \hat{\phi}^\dagger_\mu(x) \, \mathrm{i} \, J_i^{finite} \hat{\phi}^\mu(x) \tag{98}$$

instead of the Hamiltonian, one gets e.g. for the third component:

$$\hat{S}_3 \, \hat{a}^{\dagger(\lambda)}(p) \, |0\rangle = \sigma_\lambda \, \hat{a}^{\dagger(\lambda)}(p) \, |0\rangle \quad \text{with} \quad \sigma_\lambda \in \{1, 0, -1\}, \tag{99}$$

where $|0\rangle$ is the vacuum state. One recognises, the particle created by $\hat{a}^{\dagger(\lambda)}(p)$ is in a defined spin state with respect to the spin quantisation axis, that is in the direction of the 3-dimensional momentum of the particle due to the choice of the polarisation vectors in subsection 3.1. But any other spin quantisation axis can be established by performing a rotation.

By using the other three components of the energy-momentum tensor Θ^0_μ the Hamilton operator can be extended to a 4-momentum operator

$$\hat{P}^\mu = \sum_{\lambda=1}^{3} \int \frac{d\tilde{p}^3}{(2\pi)^3} \, p^\mu \left(\hat{a}^{\dagger(\lambda)}(p) \, \hat{a}^{(\lambda)}(p) + \hat{b}^{\dagger(\lambda)}(p) \, \hat{b}^{(\lambda)}(p) \right). \tag{100}$$

Its eigenvalues are the states created by $\hat{a}^{\dagger(\lambda)}(p)$ and $\hat{b}^{\dagger(\lambda)}(p)$. It can be shown that the particles created by $\hat{b}^{\dagger(\lambda)}(p)$ have opposite electric charge and are interpreted as antiparticles [14].

All in all, a one-particle state with mass m and spin j is specified by its momentum and the spin component measured on a certain axis. Therefore, a general state is denoted by[4]

$$|p, \sigma\rangle, \tag{101}$$

where in our case the spin component σ can have the values 1, 0 and -1, because we focus on vector bosons. They are realised in nature as W-bosons and ρ-mesons. The latter are bound states of two quarks.

[4]This answers the question, how the states $|i\rangle$ and $|f\rangle$ can be defined, that we have introduced in subsection 3.1.

4 Entanglement

When a composite physical system, as a whole, takes up a well-defined pure state without the possibility to assign a well-defined state to each of the subsystems, one speaks of *entanglement*. The correlations of entangled states differ from classical correlations.

4.1 Spin-momentum entanglement

A particle can be in a superposition of two or more states, e.g.:

$$|\psi\rangle = a_1\,|p_1,\sigma_1\rangle + a_2\,|p_2,\sigma_2\rangle. \tag{102}$$

Performing a measurement in this basis choice, one detects either $|p_1,\sigma_1\rangle$ or $|p_2,\sigma_2\rangle$. The probability to detect $|p_1,\sigma_1\rangle$ is given by $|a_1|^2$ and analog by $|a_2|^2$ for the second state. It holds $|a_1|^2 + |a_2|^2 = 1$, because the overall probability is assumed to be 1. The interesting fact is, that if the momentum is measured, the spin is automatically determined and vice versa.

Such a correlation can be described by different measures [15, 16]. We will use the von Neumann entropy, with the usual choice of the basis of the logarithm in quantum information theory, given by

$$S(\rho) = -\mathrm{Tr}(\rho\log_2\rho) = -\sum_i \lambda_i \log_2\lambda_i, \tag{103}$$

where $\rho = |\psi\rangle\langle\psi|$ is the density matrix of a pure state $|\psi\rangle$ and the λ_i are the eigenvalues of the density matrix, calculated in App. B. For a composite system of two Hilbert spaces $H_A \otimes H_B$, the reduced density matrix of the subsystem A is obtained by tracing out subsystem B, i.e. $\rho_A = \mathrm{Tr}_B\rho_{AB}$.

4.2 Dependence on the state of motion of the observer

If a moving particle is monitored by an observer, which is moving in another direction, two consecutive Lorentz transformations are needed to link the rest frame of the particle with the rest frame of the observer. By use of the Baker-Campbell-Hausdorff identity (10) one can show that two consecutive boost can be transmuted into one boost and a rotation:

$$\mathrm{e}^{\mathrm{i}\,\kappa_a\cdot\mathbf{K}}\mathrm{e}^{\mathrm{i}\,\kappa_b\cdot\mathbf{K}} = \mathrm{e}^{\mathrm{i}\,\kappa_c\cdot\mathbf{K}}\mathrm{e}^{\mathrm{i}\,\alpha\cdot\mathbf{J}}. \tag{104}$$

That means, in such a case the spin quantisation axis appears rotated from the point of view of the observer. And the one-particle state transforms in the following way [17]:

$$U(\Lambda)|p,\sigma\rangle = \sum_{\sigma'} D_{\sigma\sigma'}|\Lambda p,\sigma'\rangle, \tag{105}$$

where $D_{\sigma\sigma'}$ is the *Wigner rotation matrix*. When the Euler angles are used to describe the rotation mentioned above

$$R(\alpha,\beta,\gamma) = e^{-\mathrm{i}\alpha J_z} e^{-\mathrm{i}\beta J_y} e^{-\mathrm{i}\gamma J_z}, \tag{106}$$

the Wigner rotation matrix is of the form

$$D^j_{\sigma\sigma'}(\alpha,\beta,\gamma) \equiv \langle j\sigma'|R(\alpha,\beta,\gamma)|j\sigma\rangle = e^{-\mathrm{i}\sigma'\alpha} d^j_{\sigma'\sigma}(\beta) e^{-\mathrm{i}\sigma\gamma} \tag{107}$$

with Wigner's little d-matrix:

$$d^j_{\sigma'\sigma}(\beta) = \langle j\sigma'|e^{-\mathrm{i}\beta J_y}|j\sigma\rangle = D^j_{\sigma'\sigma}(0,\beta,0)\,. \tag{108}$$

For $j = 1$ it takes the form [18]

$$d^1(\beta) = \begin{pmatrix} \frac{1}{2}(1+\cos(\beta)) & -\frac{1}{2}\sqrt{2}\sin(\beta) & \frac{1}{2}(1-\cos(\beta)) \\ \frac{1}{2}\sqrt{2}\sin(\beta) & \cos(\beta) & -\frac{1}{2}\sqrt{2}\sin(\beta) \\ \frac{1}{2}(1-\cos(\beta)) & \frac{1}{2}\sqrt{2}\sin(\beta) & \frac{1}{2}(1+\cos(\beta)) \end{pmatrix}. \tag{109}$$

In our first investigation we analyse a configuration (Fig. 3) where the spin quantisation axis is in the z-direction and the velocity vectors $\mathbf{u_i}$ and $\mathbf{v}$ of the particle and the observer are in the xz-plane.

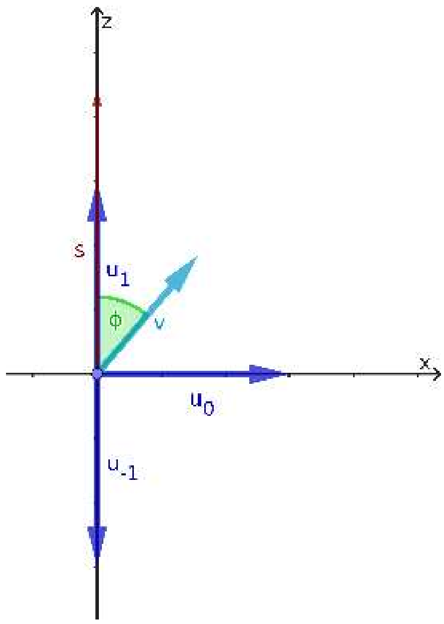

Figure 3: Configuration of the velocity vectors in the first setup

If the particle has $\sigma = 1$, it moves into the positive z-direction. In the case

$\sigma = 0$ along the x-axis, and for $\sigma = -1$ along the negative z-axis. The angle between particle's and observer's velocity vector is denoted as ϕ, e.g. ϕ_+ in case of a momentum p_+. In the rest frame, the particle's state is assumed as

$$|\psi\rangle = \cos(\eta)\,\sin(\kappa)\,|p_+, 1\rangle + \sin(\eta)\,\sin(\kappa)\,|p_0, 0\rangle + \cos(\kappa)\,|p_-, -1\rangle\,, \quad (110)$$

where only three distinct momentum states are allowed. The Lorentz transformed state in the observer's rest frame is obtained by applying Eq. (105). The dependence of the Wigner rotation angle on the angle ϕ [19] is given by

$$\beta(\phi) = 2\arctan\left(\frac{\sin(\phi)}{\cos(\phi) + \mathcal{D}(u,v)}\right), \quad \phi \in [0,\pi] \quad (111)$$

with

$$\mathcal{D}(u,v) = \sqrt{\frac{(\gamma_u+1)(\gamma_v+1)}{(\gamma_u-1)(\gamma_v-1)}}\,, \quad \gamma_x = \frac{1}{\sqrt{1-x^2}}\,, \quad (112)$$

where u and v are the velocities of the particle and the observer. By use of the von Neumann entropy the amount of entanglement is calculated. The maximal change of this amount caused by two consecutive Lorentz transformations is about 4% for $u = v = 0.7$ and about 50% if $u = v = 0.99$. For $\phi_+ = 0.3\,\pi$, $u = 0.7$ and $v = 0.9$ the dependence on η and κ of the entropy difference ΔE between the rest and the boosted frame is shown in Fig. 4.

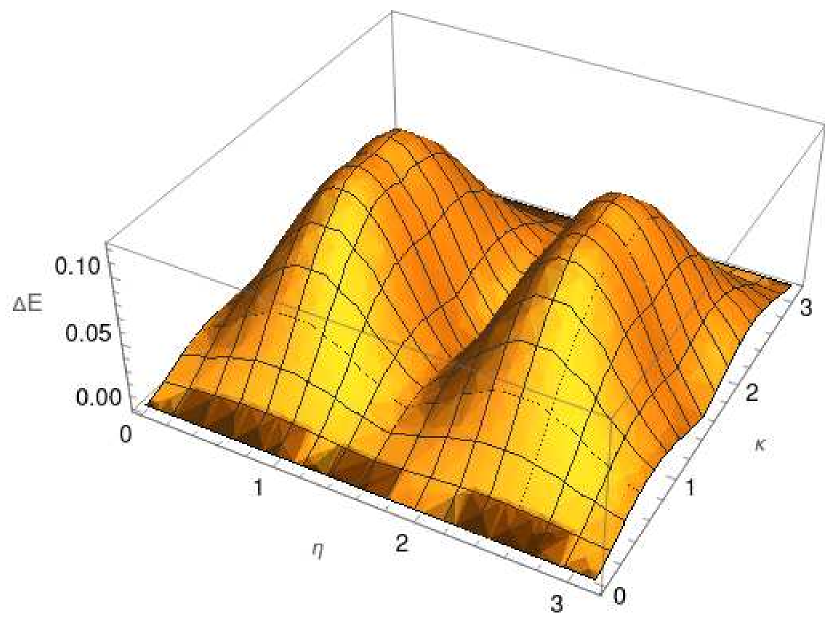

Figure 4: Entropy difference $\Delta E = E_{\text{rest}} - E_{\text{boosted}}$ for $\phi_+ = 0.3\,\pi$, $u = 0.7$ and $v = 0.9$

In Fig. 4 it can be seen that the change of the amount of the entropy is maximal for $\eta = \pi/4$ and $\eta = 3\pi/4$, due to the specific structure of the expression for ΔE.

If $\phi_+ = 0.8\,\pi$, $u = 0.9$ and $v = 0.9$ the change of the amount of the entropy is presented in Fig. 5.

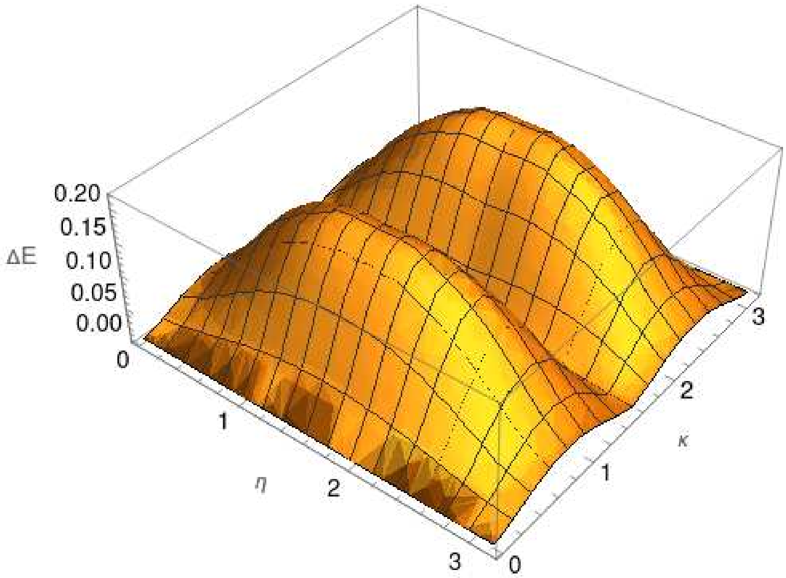

Figure 5: Entropy difference ΔE for the values $\phi_+ = 0.8\,\pi$, $u = 0.9$ and $v = 0.9$

In this case the change of the entropy is maximal for $\kappa = \pi/4$ and $\kappa = 3\pi/4$. It is twice as high compared to the situation with $u = 0.7$. The entropy in the moving frame depends strongly on the angle ϕ_+ as can be seen by comparing Fig. 4 and Fig. 5. The regions with a huge entropy difference vary significantly.

The period of the function that describes the entropy in the moving frame is π for both preparation parameters η and κ as it contains only sine and cosine functions of these variables that are at least raised to the power of 2. For the same reason this function is symmetric over the interval $[0, \pi]$ in both dimensions.

For the configuration we are discussing in this subsection, the amount of the entropy in the moving frame is always less than in the rest frame. In most cases it is maximal in the regions of the η-κ-plane where also the amount of entanglement in the rest frame is high or maximal.

4.3 Comparison with the case of a spin-1/2 particle

The single particle entanglement associated with the spin and momentum degrees of freedom in case of massive spin-$\frac{1}{2}$ particles has been analysed in [6]. In order to find out how the value of the spin of a particle influences the entanglement, we choose the following initial spin-momentum state that is similar to the one in the spin-$\frac{1}{2}$ case (for equal helicity):

$$|\psi_1\rangle_\Sigma = \cos(\eta)\,|p_+, 1\rangle + \sin(\eta)\,|p_-, -1\rangle. \tag{113}$$

This state is obtained from Eq. (110) by first setting η equal to zero and then doing the variable change $\kappa \to -\eta + \pi/2$. In such a configuration, as shown in Fig. (6), the particle is moving along the positive z-axis if $\sigma = 1$, and along the negative z-axis if $\sigma = -1$. In the rest frame of reference the z-component of the spin cannot have the value 0.

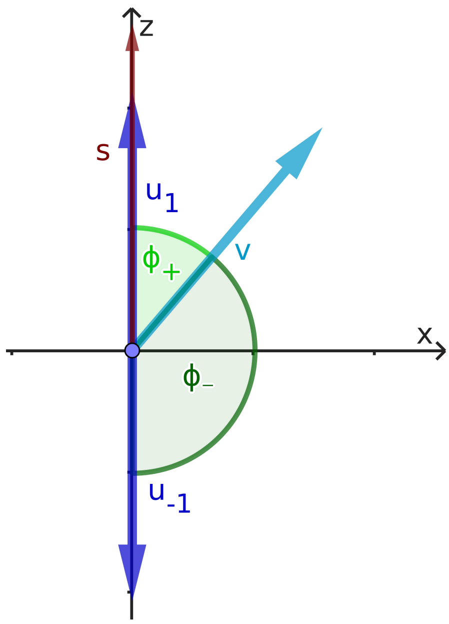

Figure 6: Configuration for comparison with spin-$\frac{1}{2}$ case

As all momenta lie in the xz-plane, the Wigner rotation of the spin states associated with a Lorentz transformation to a frame that is moving with velocity $\mathbf{v}$ is around the negative or positive y-axis (depending on the moving direction of the particle) and the corresponding rotation matrix is given by Eq. (109) with a negative or positive angle. We use a convention in which the Wigner rotation angle in case of a upward moving particle is denoted by δ_+ and for a downward moving particle by δ_-.

This leads to the expression for the state in the moving frame:

$$\begin{aligned}
|\psi_1\rangle_{\Sigma'} &= \frac{1}{2}\left(1+\cos(\delta_+)\right)\cos(\eta))\,|p'_+,1\rangle - \frac{\sin(\delta_+)\cos(\eta)}{\sqrt{2}}\,|p'_+,0\rangle \\
&+\frac{1}{2}\left(1-\cos(\delta_+)\right)\cos(\eta)\,|p'_+,-1\rangle + \frac{1}{2}\left(1-\cos(\delta_-)\right)\sin(\eta)\,|p'_-,1\rangle \\
&-\frac{\sin(\delta_-)\sin(\eta)}{\sqrt{2}}\,|p'_-,0\rangle + \frac{1}{2}\left(1+\cos(\delta_-)\right)\sin(\eta)\,|p'_-,-1\rangle. \qquad (114)
\end{aligned}$$

In [6] the following formula for calculating the Wigner angle is presented:

$$\cos\delta + 1 = \frac{(1+\gamma_u+\gamma_v+\gamma_u\gamma_v(1+uv\cos\phi))^2}{(\gamma_u+1)(\gamma_v+1)(\gamma_u\gamma_v(1+uv\cos\phi)+1)}, \quad \gamma_x = \frac{1}{\sqrt{1-x^2}}. \qquad (115)$$

A comparison with the formula from [8]

$$\cos\delta + 1 = \frac{(1+\gamma_u+\gamma_v+\gamma_u\gamma_v(1+\mathbf{u}\cdot\mathbf{v}))^2}{(\gamma_u+1)(\gamma_v+1)(\gamma_u\gamma_v(\mathbf{u}\cdot\mathbf{v})+1)} \qquad (116)$$

ensures that ϕ is the angle between $\mathbf{u}$ and $\mathbf{v}$. As shown in Fig. (6), the angle between the velocity vector of the upward moving particle and $\mathbf{v}$ is denoted by ϕ_+. On the other hand, ϕ_- is the one between the downward moving particle's velocity vector and $\mathbf{v}$. In [6] the relation

$$|\delta(\phi_+)| \stackrel{?}{=} |\delta(\phi_-)| \qquad (117)$$

is implicitly used in order to get the following rotation matrices for a spin-$\frac{1}{2}$ particle

$$U_\pm \stackrel{?}{=} \begin{pmatrix} \cos\frac{\delta}{2} & \pm\sin\frac{\delta}{2} \\ \mp\sin\frac{\delta}{2} & \cos\frac{\delta}{2} \end{pmatrix}. \qquad (118)$$

But relation (117) only holds true for $\phi_+ = n\,\pi/2$, where n is an integer. In general it is not valid. This becomes clear by the following consideration: If one has $-\mathbf{u}$ instead of $\mathbf{u}$ and fixed $|\mathbf{u}|$, $|\mathbf{v}|$ in Eq. (116), only the sign of the scalar product $\mathbf{u}\cdot\mathbf{v}$ changes, leading to a different value of δ. We give a numerical example for the case $\phi_+ = \pi/4$, $|\mathbf{u}| = 0.9$, $|\mathbf{v}| = 0.9$:

$$\begin{aligned}
|\delta(\phi_+, |\mathbf{u}|, |\mathbf{v}|)| &= |\delta(\pi/4, 0.9, 0, 9)| = 0.428144\,, \\
|\delta(\phi_-, |\mathbf{u}|, |\mathbf{v}|)| &= |\delta(\pi-\pi/4, 0.9, 0, 9)| = 0.734412\,. \qquad (119)
\end{aligned}$$

As can be seen in Fig. (6), the absolute value of ϕ_- has to be $\pi-\phi_+$. We now know that two different Wigner rotation angles have to be used, one for an upward moving particle and another one for a downward moving particle.

For the configuration we use in this subsection the amount of entanglement has to be invariant under an interchange of $\delta_+ = \delta(\phi_+)$ and $\delta_- = \delta(\phi_-)$. We give the following proof for this statement:
If the state is given by $a_1 \left|p_+, 1\right\rangle + a_2 \left|p_-, -1\right\rangle$ in the rest frame, it is in the moving frame of the form

$$\begin{aligned} \left|\psi_1\right\rangle_{\Sigma'} &= a \left|p'_+, 1\right\rangle + b \left|p'_+, 0\right\rangle + c \left|p'_+, -1\right\rangle \\ &\quad + d \left|p'_-, 1\right\rangle + e \left|p'_-, 0\right\rangle + f \left|p'_-, -1\right\rangle \end{aligned} \tag{120}$$

with

$$\begin{aligned} &a = a_1 R_{11}\left(\delta_+\right), \quad b = -a_1 R_{12}\left(\delta_+\right), \quad c = a_1 R_{13}\left(\delta_+\right), \\ &d = a_2 R_{13}\left(\delta_-\right), \quad e = -a_2 R_{12}\left(\delta_-\right), \quad f = a_2 R_{11}\left(\delta_-\right). \end{aligned} \tag{121}$$

where R_{ij} is an element of the rotation matrix, see Eq. (109). The reduced density matrix, when tracing out the spin subsystem, is then given by

$$\rho\left(\delta_+, \delta_-\right) = \begin{pmatrix} aa^* + bb^* + cc^* & 0 & ad^* + be^* + cf^* \\ 0 & 0 & 0 \\ da^* + eb^* + fc^* & 0 & dd^* + ee^* + ff^* \end{pmatrix}. \tag{122}$$

By looking at Eq. (121), it is easy to verify that $ad^* + be^* + cf^*$ is invariant under an interchange of δ_+ and δ_-. The vector (a, b, c) is the component vector of the p_+-state after rotation. As the Wigner rotation is a unitary transformation, it does not change the scalar product $aa^* + bb^* + cc^*$, that thus has to have to same value before and after rotation. Therfore it cannot depend on δ_+ or δ_-. The same applies to $dd^* + ee^* + ff^*$.
The eigenvalues of the matrix ρ are given by

$$\lambda_{1,2} = \frac{1}{2}\left(\rho_{11} + \rho_{33} \pm \sqrt{\rho_{11}^2 - 2\rho_{11}\,\rho_{33} + \rho_{33}^2 + 4\rho_{13}^2}\right), \ \lambda_3 = 0\,. \tag{123}$$

The Eqs. (122) and (123) show, that the eigenvalues are of the same form for $\rho\left(\delta_+, \delta_-\right)$ and $\rho\left(\delta_-, \delta_+\right)$. Thus the von Neumann entropy

$$-\mathrm{Tr}(\rho \log_2 \rho) = -\sum_i \lambda_i \log_2 \lambda_i\,, \tag{124}$$

which we use as entanglement measure, does not change for $\delta_+ \leftrightarrow \delta_-$ and the statement from above is proven.

It follows that the amount of the entropy E^Ψ is as well invariant under the interchange of ϕ_+ and ϕ_-. As $\phi_- = \pi - \phi_+$, the function $E^\Psi(\phi_+)$ has to be symmetric about the line $\phi_+ = \pi/2$.

A similar consideration shows that everything we have proven in the spin 1 case is also true for spin $\frac{1}{2}$ if the state of a spin-$\frac{1}{2}$ particle in the rest frame is given by

$$|\psi_{\frac{1}{2}}\rangle_{\Sigma} = \cos\eta\,|p_+,\uparrow\rangle + \sin\eta\,|p_-,\downarrow\rangle \ , \tag{125}$$

We then have the following state in the moving frame:

$$\begin{aligned} |\psi_{\frac{1}{2}}\rangle_{\Sigma'} \;&=\; \cos\eta\cos\frac{\delta_+}{2}\,|p'_+,\uparrow\rangle - \cos\eta\sin\frac{\delta_+}{2}\,|p'_+,\downarrow\rangle \\ &\quad - \sin\eta\sin\frac{\delta_-}{2}\,|p'_-,\uparrow\rangle + \sin\eta\cos\frac{\delta_-}{2}\,|p'_-,\downarrow\rangle \,, \end{aligned} \tag{126}$$

For both spin cases the explicit form of the density matrices and their eigenvalues can be calculated by use of the decompositions of the states as shown in Eq. (126) and Eq. (114). In the following tables we present a comparison of these quantities.

	Reduced density matrix
Spin $\frac{1}{2}$	$\begin{pmatrix} \cos^2(\eta) & -\frac{1}{2}\sin(2\eta)\sin\left(\frac{\delta_+ + \delta_-}{2}\right) \\ -\frac{1}{2}\sin(2\eta)\sin\left(\frac{\delta_+ + \delta_-}{2}\right) & \sin^2(\eta) \end{pmatrix}$
Spin 1	$\begin{pmatrix} \cos^2(\eta) & 0 & \frac{1}{2}\sin(2\eta)\sin^2\left(\frac{\delta_+ + \delta_-}{2}\right) \\ 0 & 0 & 0 \\ \frac{1}{2}\sin(2\eta)\sin^2\left(\frac{\delta_+ + \delta_-}{2}\right) & 0 & \sin^2(\eta) \end{pmatrix}$

	Eigenvalues
Spin $\frac{1}{2}$	$\lambda_{1,2} = \frac{1}{2}\left(1 \pm \sqrt{\cos^2(2\eta) + \sin^2(2\eta)\sin^2\left(\frac{1}{2}(\delta_+ + \delta_-)\right)}\right)$
Spin 1	$\lambda_{1,2} = \frac{1}{2}\left(1 \pm \sqrt{\cos^2(2\eta) + \sin^2(2\eta)\sin^4\left(\frac{1}{2}(\delta_+ + \delta_-)\right)}\right)$

The third eigenvalue in the spin-1 case is zero. In both cases the influence of the Wigner rotation on the reduced density matrix and their eigenvalues can be described by one parameter. But this parameter is not δ_+ as claimed in [6], it is the mean value of δ_+ and δ_-. The eigenvalues of the reduced density matrices for spin $\frac{1}{2}$ and spin 1 just differ by the exponent of $\sin\left(\frac{1}{2}(\delta_+ + \delta_-)\right)$. Some remarks on this observation and on the signs in front of of the non-diagonal elements are given in Sec. 4.4.

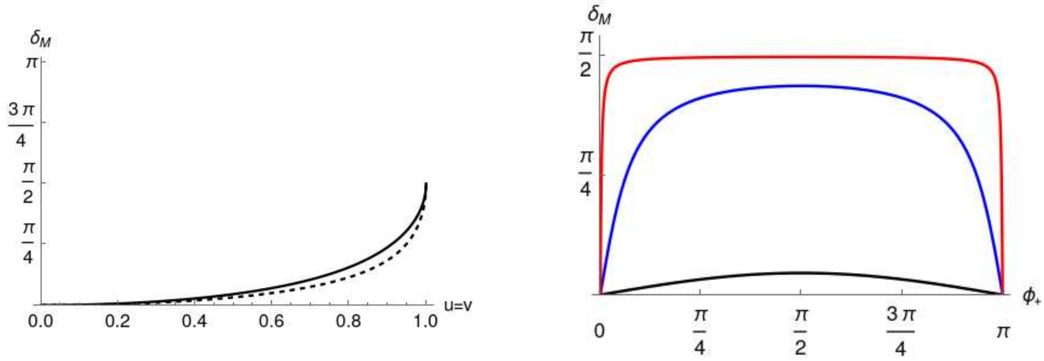

Figure 7: Left: The mean Wigner rotation angle as a function of an equal particle's and observer's velocity for $\phi_+ = \pi/2$ (black) and $\phi_+ = 3\pi/2$ (dashed). Right: The mean Wigner rotation angle as a function of ϕ_+ for $u = v = 0.5$ (black), $u = v = 0.995$ (blue) and $u = v \to 1$ (red).

We introduce the quantity $\delta_M = \frac{1}{2}(\delta_+ + \delta_-)$ and investigate its dependence on the velocity of the particle and the observer and on the angle ϕ_+, illustrated in Fig. 7. The mean Wigner angle reaches its highest possible value $\pi/2$ in the relativistic limit $u = v \to 1$. Of course, $\delta_M(\phi_+)$ is invariant under the interchange of δ_+ and δ_- and therefore also under the interchange of ϕ_+ and ϕ_-. So it has to be symmetric over the interval $[0, \pi]$. The maximum of $\delta_M(\phi_+)$ is in any case at $\phi_+ = \pi/2$.

As mentioned before, the amount of entanglement is given by the entropy $-\mathrm{Tr}(\rho \log_2 \rho) = -\sum_i \lambda_i \log_2 \lambda_i$. Using the eigenvalues shown above, one gets the following expression for the amount of the entropy in the moving frame:

$$\begin{aligned} E^{\psi}_{\Sigma'}(\delta_M) \;=\; & -\frac{1}{2}\left(1 + \sqrt{\cos^2(2\eta) + \sin^{e_j}(\delta_M)\sin^2(2\eta)}\right) \\ & \times \log_2\left(\frac{1}{2}\left(1 + \sqrt{\cos^2(2\eta) + \sin^{e_j}(\delta_M)\sin^2(2\eta)}\right)\right) \\ & -\frac{1}{2}\left(1 - \sqrt{\cos^2(2\eta) + \sin^{e_j}(\delta_M)\sin^2(2\eta)}\right) \\ & \times \log_2\left(\frac{1}{2}\left(1 - \sqrt{\cos^2(2\eta) + \sin^{e_j}(\delta_M)\sin^2(2\eta)}\right)\right). \end{aligned} \tag{127}$$

The exponent e_j is 2 in case of spin 1/2 and 4 in case of spin 1. The entanglement in the rest frame is obtained by taking the limit $\delta_M \to 0$.

The proposition 1(a) from [6] yielding that "*for any state* $|\psi_{1/2}\rangle_\Sigma$ *in the rest frame* Σ *the amount of the entropy* $E^{\psi}_{\Sigma'}$ *in the moving frame is decreasing with the Wigner rotation angle* $\mathrm{d}E^{\psi}_{\Sigma'}/\mathrm{d}\delta \leq 0$ *for* $\delta \in (0, \pi/2)$" holds true of

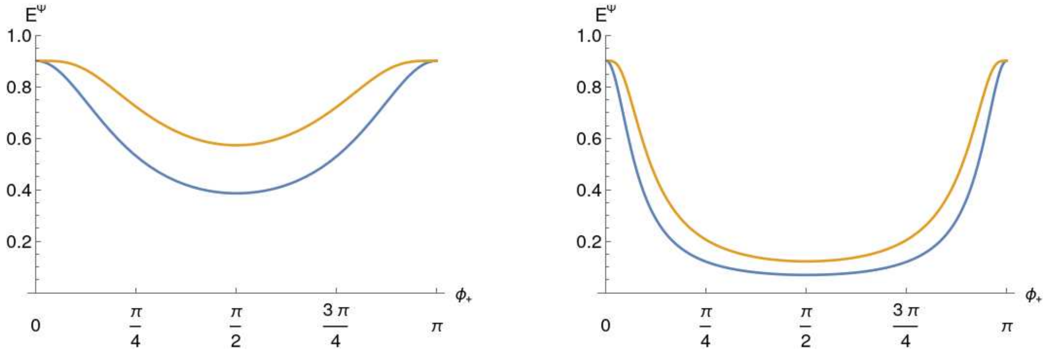

Figure 8: Left: The amount of entanglement in the moving frame as a function of the angle ϕ_+ for $\eta = 0.6$ and $u = v = 0.95$ (left), $u = v = 0.995$ (right) for spin 1/2 (blue) and spin 1 (orange).

one is using δ_M instead of δ. It is also valid in the spin 1 case. Proposition 1(b) from [6] is true as well, but δ_M never reaches higher values then $\pi/2$.

In Figure 8 the amount of the entropy in the moving frame for both spin cases is shown for $\eta = 0.6$ and different values of $u = v$. The difference of the amount of entanglement between its value in the rest and the moving frame is higher in the spin-1/2 case and always maximal for $\phi_+ = \pi/2$, that means when $\mathbf{u}$ and $\mathbf{v}$ are perpendicular. It is zero if $\mathbf{u}$ and $\mathbf{v}$ are parallel or anti-parallel. The entanglement in the moving frame is for $\phi_+ \in (0, \pi)$ in any case smaller than in the rest frame. So one can say that boosting reduces the amount of entanglement for states of equal helicity. In the limit $u, v \to 1$ there is within the open interval $(0, \pi)$ no entanglement anymore.

In Fig. 9 we present a comparison with the state $|\psi\rangle = \cos(\kappa)\, \sin(\xi)\, |p_+, 1\rangle + \sin(\kappa)\, \sin(\xi)\, |p_0, 0\rangle + \cos(\xi)\, |p_-, -1\rangle$ (with $\xi = -\eta + \pi/2$) for different κ.

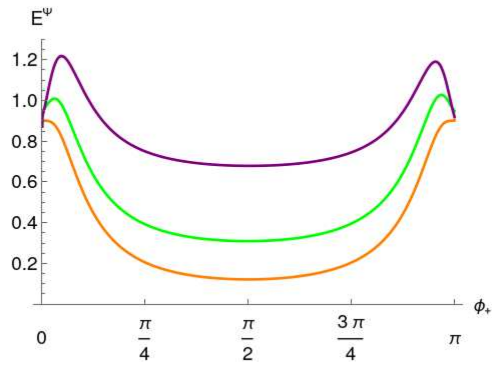

Figure 9: Entanglement in the moving frame for $\eta = 0.6$, $u = v = 0.995$ and $\kappa = 0$ (orange, as above), $\kappa = 0.3$ (green), $\kappa = 0.9$ (purple).

4.4 Results in the general case

In our setting there are three different possible directions of motion for the particle. In the rest frame, a particle with $\sigma \in \{-1, 0, 1\}$ has the velocity vector $\mathbf{u}_\sigma$. In the Mathematica notebook (presented in App. C), that is used to calculate the amount of entanglement, the three velocity vectors can be chosen arbitrary. Every vector $\mathbf{u}_\sigma$ can by expressed by its length and two angles θ_σ and φ_σ as shown in Fig. 10.

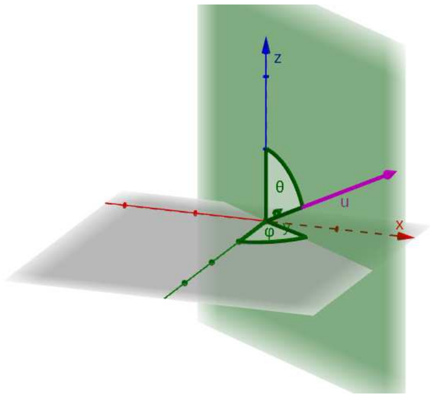

Figure 10: Velocity vector

As there exists a rotation symmetry around the z-axis the azimuth angle of the velocity vector $\mathbf{v}$ of the observer can be set to zero without loss of generality.

The Wigner rotation, in principle, can always be performed by use of a rotation matrix. But for a spin-1 particle the situation is a little bit more complicated than in the spin-1/2 case. By using the second and third generator from Eq. (6) that are

$$G_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad G_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{128}$$

one gets the standard matrices for a rotation around the y- and z-axis:

$$R_y(\beta) = \begin{pmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{pmatrix}, \quad R_z(\gamma) = \begin{pmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If we used the matrix $R_y(\beta)$ instead of Wigner's little d-matrix from Eq. (109) in Sec. 4.2 to rotate the component vector of the state, we would have obtained the same non-zero entries in the reduced density matrix as in case of a massive spin-1/2 particle. But this procedure is not correct when considering spin 1 and using an orthonormal basis, as G_3 in Eq. (128) is not diagonal. One has to use a basis in which the generator responsible for the rotation around the spin quantisation axis is diagonal. Such a basis, that is called *spherical basis*, is obtained by applying the following transition matrix to the generators.

$$T_S = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{\mathrm{i}}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{\mathrm{i}}{\sqrt{2}} & 0 \end{pmatrix} \tag{129}$$

It is easy to show that

$$T_S \cdot (\mathrm{i}\, G_3) \cdot T_S^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{130}$$

and

$$T_S \cdot R_y(\beta) \cdot T_S^{-1} = \begin{pmatrix} \frac{1}{2}(1+\cos(\beta)) & -\frac{1}{2}\sqrt{2}\sin(\beta) & \frac{1}{2}(1-\cos(\beta)) \\ \frac{1}{2}\sqrt{2}\sin(\beta) & \cos(\beta) & -\frac{1}{2}\sqrt{2}\sin(\beta) \\ \frac{1}{2}(1-\cos(\beta)) & \frac{1}{2}\sqrt{2}\sin(\beta) & \frac{1}{2}(1+\cos(\beta)) \end{pmatrix}, \tag{131}$$

which agrees with Wigner's little d-matrix for rotating around the y-axis.

If one wants to perform a general Wigner rotation on the component vector of a state around an axis that is given by a vector $\boldsymbol{\delta}$, as above, a basis-transformed rotation matrix has to be employed:

$$R_W(\boldsymbol{\delta}) = T_S \cdot R(\boldsymbol{\delta}) \cdot T_S^{-1} \tag{132}$$

The length of the rotation vector $\boldsymbol{\delta}$ is given by Eq. (116) and its direction [8] is

$$\mathbf{n}_{\boldsymbol{\delta}} = -\frac{\mathbf{u}_\sigma \times \mathbf{v}}{|\mathbf{u}_\sigma \times \mathbf{v}|}\,. \tag{133}$$

The three different angles between $\mathbf{v}$ and the three vectors $\mathbf{u}_{\boldsymbol{\sigma}}$ lead to three different Wigner rotation vectors $\boldsymbol{\delta}_{\boldsymbol{\sigma}}$ for the states in the decomposition of the particle's state in the rest frame.

$$|\psi\rangle = \cos(\eta)\,\sin(\kappa)\,|p_1, 1\rangle + \sin(\eta)\,\sin(\kappa)\,|p_0, 0\rangle + \cos(\kappa)\,|p_{-1}, -1\rangle. \tag{134}$$

We first analyse the dependence of the entanglement in the moving frame on a rotation of the velocity vector $\mathbf{u_0}$ of a particle with $\varphi_1 = 0$, $\varphi_{-1} = \pi$, $\theta_1 = 0$, $\theta_0 = \pi/2$ and $\theta_{-1} = \pi$ by an angle φ around the z-axis for $\theta_{\mathbf{v}} = 0.3\,\pi$, $|\mathbf{u}| = 0.9$, $|\mathbf{v}| = 0.9$ and three different sets of the preparation parameters η

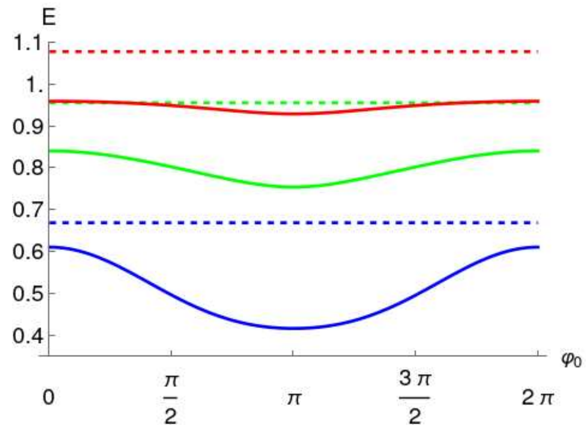

Figure 11: Dependence of the entanglement on a rotation of the velocity vector $\mathbf{u_0}$ by an angle φ_0 around the z-axis starting from the x-axis for $\eta = 1.3$, $\kappa = 0.4$ (blue), $\eta = 0.5$, $\kappa = 0.5$ (green), $\eta = 0.2$, $\kappa = 0.7$ (red) in comparison with the entropy in the rest frame (dashed lines).

and κ. It is displayed and compared with the entropy in the rest frame in Fig. 11. The entanglement is minimal for φ_0 equal to π. The graph is, of course, symmetric about the line $\varphi_0 = \pi$, due to the mirror symmetry about the plane in which the z-axis and the velocity vector $\mathbf{v}$ of the observer lie.

Next we consider the variation due to a change of the polar angle θ_0 of the velocity vector of the state where $\sigma = 0$ with $\varphi_0 = 0$, $\theta_1 = 0$, $\varphi_1 = 0$, $\theta_{-1} = \pi$, $\varphi_{-1} = 0$, $\theta_v = 0.3\,\pi$, $|\mathbf{u}| = 0.9$ and $|\mathbf{v}| = 0.9$. It is shown in Fig. 12. The dependence of the entanglement in the moving frame on the angle θ_0 is non-trivial. In the interval $(0,\, 2\pi)$ there are two local maximal and minima.

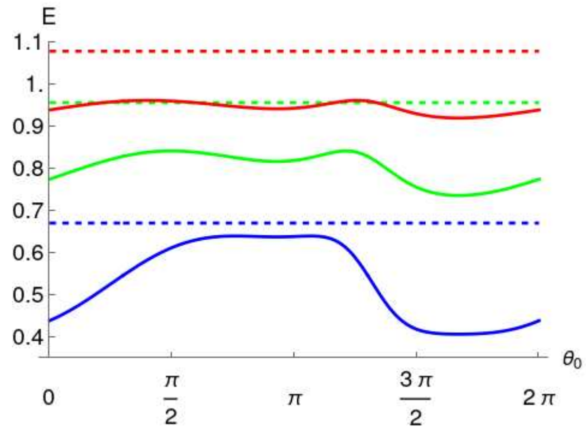

Figure 12: Dependence of the entanglement on a change of the angle θ_0 for the same values of the preparation parameters η and κ as in Fig. 11 in comparison with the entropy in the rest frame (dashed lines).

In the next setup, $\mathbf{u_1}$ is along the z-axis, $\mathbf{u_0}$ is along the x-axis and $\mathbf{u_{-1}}$ is along the y-axis. The dependence of the entropy difference between the rest and the moving frame on the parameters η and κ is shown in Fig. 13.

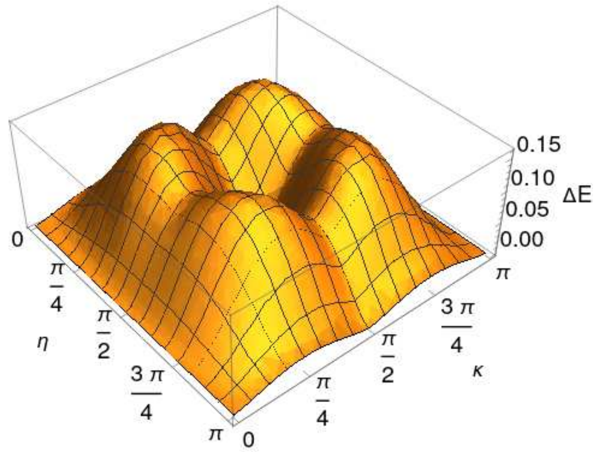

Figure 13: Entropy difference if the velocity vectors of the particle are along the coordinate axis and $\theta_v = 0.3\,\pi$, $|\mathbf{u}| = 0.9$ and $|\mathbf{v}| = 0.9$.

The maximal entropy difference 0.16 can be found at the points (0.95, 0.96), (0.95, 2.18), (2.19, 0.96) and (2.19, 2.18). If the preparation parameters are kept at the values $\eta = 1$ and $\kappa = 1$ and the velocity vectors of the particle are rotated by an angle φ around the z-axis and $\mathbf{v}$ by an angle θ_v around the y-axis, one obtains the behaviour of the entropy difference displayed in Fig. 14 with maxima of 0.46 at the points $(\theta_v,\ \varphi)$=(2.53, 1,13) and (3.75, 4.28).

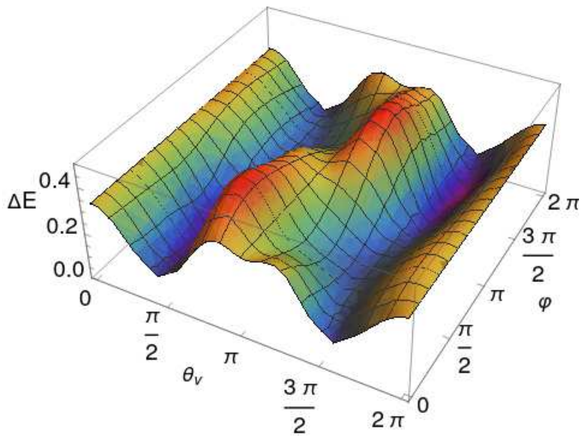

Figure 14: Entropy difference when varying θ_0 and φ.

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Finally, the three velocity vectors of the particle are rotated simultaneously by a polar angle θ and an azimuth angle φ and fixed $\theta_v = 0.3\,\pi$ We set $|\mathbf{u}| = 0.9$, $|\mathbf{v}| = 0.9$ and $(\eta, \kappa) = (1, 1)$. In Fig. 15 the resulting entropy difference is shown.

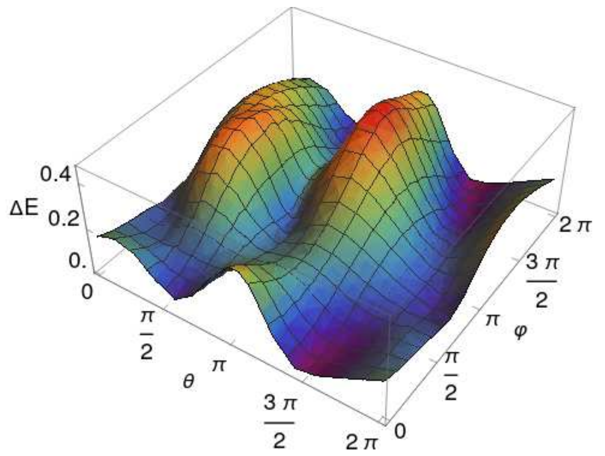

Figure 15: Entropy difference when varying the polar and azimuth angles of the velocity vectors of the particle simultaneously.

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The maximal entropy difference 0.47 appears at $(\theta, \varphi) = (3.37, 4.41)$. In this case the dependence of the entropy difference on the parameters θ and φ is rather complicated and no symmetry of the corresponding function can be recognised.

A general observation is that a change of the velocity vector $\mathbf{v}$ has a big influence on the entanglement in most cases, as it can affect all of the three angles between $\mathbf{v}$ and the vectors $\mathbf{u}_\sigma$ which determine (together with the absolute values of these vectors) the Wigner rotation angles.

When performing a Wigner rotation on the state given by Eq. (134), the entropy can never be enlarged. This is due to a feature of the von Neumann entropy which gets obvious by the following considerations: The unrotated state has a diagonal density matrix and the diagonal elements are not changed be a rotation, but the other matrix elements can become non-zero depending on the Euler angles introduced in Eq. (106), where γ can be set to zero as a rotation around the spin quantisation axis does not change the eigenvalues of the reduced density matrix:

$$\rho_{11} = \cos^2(\eta)\sin^2(\kappa),\ \rho_{22} = \sin^2(\eta)\sin^2(\kappa),\ \rho_{33} = \cos^2(\kappa)\ ,$$

$$
\begin{aligned}
\rho_{12} &= \frac{1}{\sqrt{2}} \sin(\eta)\cos(\eta)\sin^2(\kappa)\Big(\sin(\beta_+)\cos(\beta_0) \\
&- \frac{\sin(\beta_0)\left(\cos(\alpha_+ - \alpha_0 - \beta_+) + \cos(\alpha_+ - \alpha_0 + \beta_+) - 2\mathrm{i}\sin(\alpha_+ - \alpha_0)\right)}{2}\Big)\,,
\end{aligned}
$$

$$
\begin{aligned}
\rho_{13} = e^{-\mathrm{i}(\alpha_- + \alpha_+)} \cos(\eta)\sin(\kappa)\cos(\kappa) & \left(e^{\mathrm{i}\,\alpha_-} \sin\left(\frac{\beta_-}{2}\right)\cos\left(\frac{\beta_+}{2}\right) \right. \\
& \left. - e^{\mathrm{i}\,\alpha_+} \sin\left(\frac{\beta_+}{2}\right)\cos\left(\frac{\beta_-}{2}\right)\right)^2\,,
\end{aligned}
$$

$$
\begin{aligned}
\rho_{23} = \frac{1}{\sqrt{2}} \sin(\eta)\sin(\kappa)\cos(\kappa)(&-\sin(\beta_-)\cos(\beta_0) \\
&+ \sin(\beta_0)\left(\cos(\alpha_- - \alpha_0)\cos(\beta_-) - \mathrm{i}\sin(\alpha_- - \alpha_0)\right))\,.
\end{aligned}
\tag{135}
$$

The indices +, 0 and − specify the different spin states. There is the following relation between the diagonal elements d_i and the eigenvalues λ_j of ρ:

$$
d_i = \langle i|\rho|i\rangle = \sum_j |\langle i|j\rangle|^2 \lambda_j\,. \tag{136}
$$

where the set of the $|i\rangle$ is an orthonormal basis of the corresponding Hilbert space and $\{|j\rangle\}$ is an eigenbasis according to the spectral decomposition of ρ. If ρ is diagonal, the sets of the $|i\rangle$ and $|j\rangle$ coincide. In matrix form Eq. (136) can be written as:

$$
\mathbf{d} = B \cdot \boldsymbol{\lambda}, \quad 0 < B_{ij} \leq 1, \quad \sum_{i\,\mathrm{or}\,j} B_{ij} = 1\,. \tag{137}
$$

For the greatest eigenvalue λ_3 the relation $\lambda_3 \geq d_i$ is valid because assuming $\lambda_j = d_i - \Delta_j$ for every λ_j and d_i with a positive number Δ_j would lead to the contradiction

$$
d_i = \sum_j B_{ij}\lambda_j = \sum_j B_{ij} d_i - \sum_j B_{ij}\Delta_j = d_i - \sum_j B_{ij}\Delta_j\,. \tag{138}
$$

Analog, the relation $\lambda_1 \leq d_i$ can be proven for the smallest eigenvalue. As ρ is a density matrix, it holds $\sum_i d_i = \sum_i \lambda_i = 1$. For ordered diagonal elements and eigenvalues

$$
d_1 \leq d_2 \leq d_3\,, \quad \lambda_1 \leq \lambda_2 \leq \lambda_3\,, \tag{139}
$$

this also leads to the following relations:

$$\lambda_1 + \lambda_2 \leq d_1 + d_2 \quad \text{and} \quad \lambda_2 + \lambda_3 \geq d_2 + d_3 \,. \tag{140}$$

In the two-dimensional case (spin-1/2), using a state given by Eq. (125), one has

$$\lambda_1^{\rm two} \leq d_1^{\rm two} \leq d_2^{\rm two} \leq \lambda_2^{\rm two} \,. \tag{141}$$

From the fact that the curvature of the function $f(x) = -x \log_2 x$ is always negative for positive x (see Fig. 16), it follows

$$f(d_1^{\rm two} - \Delta) - f(d_1^{\rm two}) \leq f(d_2^{\rm two}) - f(d_2^{\rm two} + \Delta) \tag{142}$$

in case of positive Δ and therefore

$$f(\lambda_1^{\rm two}) - f(d_1^{\rm two}) \leq f(d_2^{\rm two}) - f(\lambda_2^{\rm two}) \tag{143}$$

or

$$f(\lambda_1^{\rm two}) + f(\lambda_2^{\rm two}) \leq f(d_1^{\rm two}) + f(d_2^{\rm two}) \,. \tag{144}$$

which demonstrates the fact that the von Neumann entropy has to be lower or equal in the moving frame of reference for a state shown in Eq. (125).

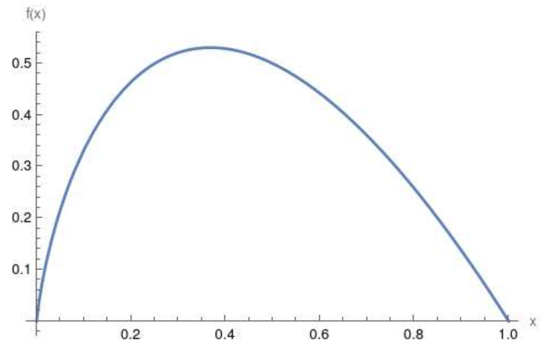

Figure 16: The function $f(x) = -x \log_2 x$.

The proof can extended to three-dimensional density matrices:

$$f(d_1 - \Delta_1) - f(d_1) + f(d_2 - \Delta_2) - f(d_2) \leq f(d_2 - \Delta_1 - \Delta_2) - f(d_2) \tag{145}$$

and

$$f(d_2 - \Delta_1 - \Delta_2) - f(d_2) \leq f(d_3) - f(d_3 + \Delta_1 + \Delta_2) \,. \tag{146}$$

This leads to

$$f(d_1 - \Delta_1) - f(d_1) + f(d_2 - \Delta_2) - f(d_2) \leq f(d_3) - f(d_3 + \Delta_1 + \Delta_2) \tag{147}$$

and the relation

$$f(\lambda_1) + f(\lambda_2) + f(\lambda_3) \leq f(d_1) + f(d_2) + f(d_3) \tag{148}$$

which had to be proven.

5 Didactic considerations

When explaining entanglement to school children, it is sensible to start with a situation where the spin states of two particles are entangled, as this is more illustrative than spin-momentum-entanglement. Of course, one should mention that spin is the quantum mechanical equivalent to the classical angular momentum and that there is an analogous conservation. One can perform

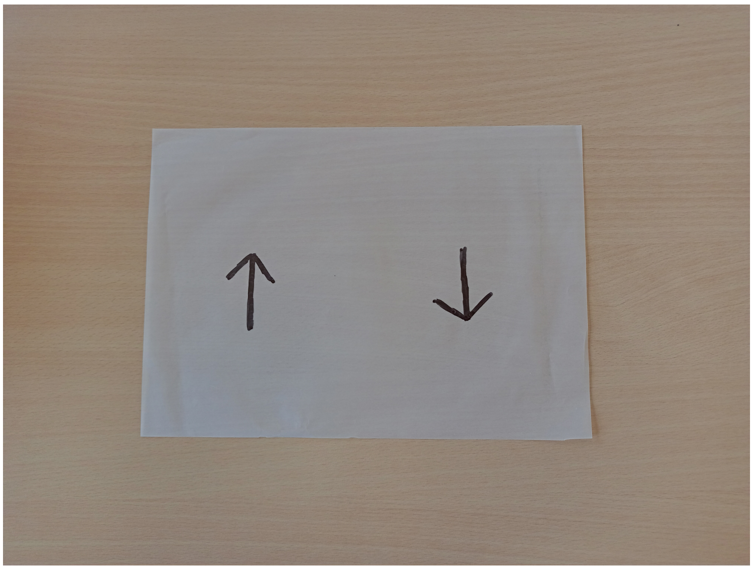

Figure 17: Transparent paper with "spins"

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the following simulation experiment: Two arrows pointing at opposite directions are drawn on a transparent paper with a marker pen. The measuring process is simulated by dropping the paper on a table. Randomly one side of the paper is facing up once it lies on the table and either the left or the right particle has spin up (see Fig. 17). But one cannot exactly determine the moment when it turns out which one of the two possibilities is realised. As long as the paper does not lie on the table, the spins are not measured, but surely they are entangled. One can argue, that a pair of completely isolated particles act as if they were in their own universe, because there is no interaction with anything in our world. Our space and time coordinates cannot be applied to them. Therefore, these particles cannot have a location or an orientation until they interact with something. The result of a location or orientation measurement has to be incidentally, otherwise it would have been fixed already before, which is impossible due to the fact that spacetime is created by the other objects and they are, by definition, of no relevance for isolated particles. Our classical picture of spacetime is caused by the fact that we are usually not dealing with isolated systems.

Concerning the Wigner rotation, it is not useful to present the corresponding mathematical formalism in a standard school lesson. But one can give the pupils a qualitative impression of the origin of a Wigner rotation. For this purpose, the transparent paper with the two spin vectors can be used again. If it moves with a velocity close to the speed of light (first Lorentz transformation) with respect to an observer, the paper appears contracted to that observer. This can be simulated by folding the paper. When the moving paper obtains an additional velocity perpendicular to the original moving direction (second Lorentz transformation), in the observer's reference frame the rear side of the paper starts the perpendicular motion earlier then the front side due to time dilation. Therefore the paper and also the spin vectors seem to be rotated from the point of view of an inertial observer.

All in all one can say, there is a common feature of education and science: One should explain the facts by use of basic principles.

6 Conclusion and outlook

Albert Einstein's principle of relativity states that the observables of physical experiments in different frames of reference must be identical. This principle arises because no evidence for divergent outcomes, such as an aether, has been found. Transformations between different reference frames take the form of a Poincare symmetry transformation, classified into rotations, boosts, and translations. A rotation transformation affects only spatial coordinates.

Entanglement arises when a composite system is in a definite overall state but its subsystems lack individual states. Measurement outcomes on one part then correlate with outcomes on the other. In spin-momentum entanglement, a single particle occupies a superposition. The degree of entanglement can be quantified by the von Neumann entropy in case of pure states as used in this work. For a bipartite system, the reduced density matrix yields the subsystem entropy. Therefore we define the amount of entanglement by the amount of entropy.

Observers moving relative to the particle require two noncollinear Lorentz boosts, which combine into one boost followed by a rotation. In such a case, the three possible spin states mix by applying the Wigner rotation matrix in the usual way that such states mix when a rotation (as a unitary transformation) is performed. This leads to a change of the amount of the entanglement.

In a specific configuration, analysed first, the spin quantisation is along the z-axis and all the momenta are placed in the xz-plane. The Wigner angles depend on the magnitudes of the velocity vectors of the particle and the angles between them and the observer ($\mathbf{u}$ and $\mathbf{v}$). We define ϕ_+ as the angle between the upward moving particle's and the observer's velocity and get the following key findings by use of η and κ as state preparation parameters that occur in the rest frame superposition:

$$|\psi\rangle = \cos\eta\, \sin\kappa, |p_1, 1\rangle + \sin\eta\, \sin\kappa, |p_0, 0\rangle + \cos\kappa\, |p_{-1}, -1\rangle$$

- Moderate boosts (u=v=0.7) alter entanglement by up to $\sim 4\%$; and ultrarelativistic boosts (u=v=0.99) by $\sim 50\%$.
- For $\phi_+ = 0.3\,\pi$, $u = 0.7$, $v = 0.9$, the entropy difference peaks at $\eta = \pi/4$ and $3\pi/4$.
- For $\phi_+ = 0.8\,\pi$, $u = v = 0.9$, the maximum shift occurs at $\kappa = \pi/4$ and $3\pi/4$, roughly doubling the effect seen at lower speeds.
- The moving-frame entropy varies periodically in η and κ with period π, and is symmetric on $[\,0, \pi]$.
- Regions of maximal change coincide with high rest-frame entanglement.

A comparison of a spin-1 superposition including only the spin up and down states with the spin-1/2 state shows that eigenvalues of the corresponding density matrices differ only by an exponent of a sine function.

In the general setup the particle can have three freely selectable velocity vectors, each parameterised by its magnitude and spherical angles (θ, φ). Exploiting rotational symmetry around the z-axis, the observer's azimuth can be set to zero without loss of generality. Starting from the rest frame superposition the entanglement in the boosted frame (where it has to be always lower or equal) was mapped under various geometric constellations:

1. *Rotation of* $\mathbf{u_0}$ *around* z: Varying the azimuth angle φ_0 with fixed $\theta_v = 0.3\,\pi$ and $|u| = |v| = 0.9$ shows a minimum entanglement at $\varphi_0 = \pi$, symmetric about this value.
2. *Change of the polar angle* θ_0 *of* $\mathbf{u_0}$: With $\varphi_0 = 0$, the entropy exhibits two local maxima and minima for $\theta_0 \in (0, 2\pi)$.
3. *Orthogonal velocity axes*: Setting $\mathbf{u_1} \parallel z$, $\mathbf{u_0} \parallel x$ and $\mathbf{u_{-1}} \parallel y$ produces an entropy difference surface in (η, κ) with a maximum ≈ 0.16.
4. *Combined rotations of particle vs. observer*: Fixing $\eta = \kappa = 1$, rotating the velocity vectors of the particle by an angle φ about the z-axis and the one of the observer by θ_v about the y-axis yields peaks of $\Delta S \approx 0.46$ at $(\theta_v, \varphi) \approx$ (2.53, 1.13) and (3.75, 4.28).

5. *Simultaneous tilt of all three* $\mathbf{u}_\sigma$: Jointly varying the polar angle θ and the azimuth φ for all three velocity vectors gives a complex entropy difference surface with a global maximum of 0.47 at (3.37, 4.41), showing no simple symmetry.

These findings illustrate how relativistic Wigner rotations can significantly and nontrivially modulate the spin–momentum entanglement. From a theorist's point of view, massive spin-1 particles offer interesting perspectives to analyse relativistic entanglement properties, despite the fact that the existing massive spin-1 particles, the weak gauge bosons and the vector mesons are difficult to manipulate and cannot be used for quantum communication due to their short lifetimes.

In this work we use the limitation that the particle's momentum is restricted to three distinct states, in order to be able to analyse a system that consists of two qutrits, focusing on particle states which are entangled in the rest frame.

Possible extensions of this work are the investigation of other states, especially states that are separable in the rest frame, or the analysis of other qutrit systems subject to transformations similar to the Wigner rotation.

Appendices

A The Pauli-Lubanski vector

There exists a vector

$$W^\mu = \frac{1}{2}\,\varepsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma} \ , \tag{149}$$

that is called Pauli-Lubanski vector. Its square $W_\mu W^\mu$ commutes with all generators of the Poincare group and is thus a Casimir operator [8]. Therfore, its eigensubspaces are invariant under a Lorentz transformation. Within a subspace, that cannot be divided into smaller subspaces, a Casimir operator has to be propotional to the unit matrix. It can be written as

$$W_\mu W^\mu = -m^2\, j(j+1)\, \mathbb{1} \tag{150}$$

where m^2 is the eigenvalue of the operator $P_\mu P^\mu$. The numbers m and j are used to classify the representations of the Poincare group. A Lorentz transformation cannot change the values of m and j. In general, the dimension of the eigensubspace (that corresponds with the number of eigenvectors) is given by $2j+1$. In case of the 2×2-dimensional representation there are a 3-dimensional and a 1-dimensional eigensubspace. We concentrate on the first one by imposing the relation $\partial_\mu \phi^\mu = 0$ on the vector field that is given by Eq. (59), as done in Sec. 3.1.

B Eigenvalues of a Hermitian 3×3 matrix

The eigenvalues λ_i of a matrix ρ are defined as

$$\rho \cdot \mathbf{v}_i = \lambda_i \, \mathbf{v}_i \,. \tag{151}$$

The calculation of the eigenvalues of a 3×3 matrix is more involved than in the 2-dimensional case [20]. As we are working with Hermitian matrices, the eigenvalues have to be real numbers. Thus, they can by calculated by use of trigonometric functions. The determinant

$$\det(\rho - \lambda \, \mathbb{1}) = 0 \,. \tag{152}$$

leads to the characteristic polynomial

$$\lambda^3 + a\,\lambda^2 + b\,\lambda + c \;=\; 0 \,. \tag{153}$$

The substitution $\lambda = z - \frac{a}{3}$ removes the quadratic term in Eq. (153).

$$z^3 + pz + q = 0 \tag{154}$$

with

$$p = b - \frac{a^2}{3} \quad \text{and} \quad q = \frac{2a^3}{27} - \frac{ab}{3} + c \tag{155}$$

$$\cos^3 \alpha \;=\; \frac{\cos 3\alpha + 3\,\cos\alpha}{4} \,. \tag{156}$$

By use of $z = r \cdot \cos\zeta$ one gets

$$\begin{aligned}
0 &= z^3 + pz + q \\
&= r^3 \cdot \cos^3\zeta + p \cdot r \cdot \cos\zeta + q \\
&= r^3 \cdot \frac{\cos(3\,\zeta) + 3\cos\zeta}{4} + p \cdot r \cdot \cos\zeta + q \\
&= \frac{r^3}{4}\cos(3\,\zeta) + \left(\frac{3}{4}r^2 + p\right) \cdot r \cdot \cos\zeta + q \\
&= \sqrt{-\frac{4}{27}\,p^3} \cdot \cos 3\zeta + q
\end{aligned} \tag{157}$$

In Eq. (157) the trigonometric relation

$$\cos^3\zeta \;=\; \frac{\cos(3\,\zeta) + 3\,\cos\zeta}{4}\,. \tag{158}$$

is used and the substitution $r = \sqrt{-\frac{4}{3}\,p}$ in order to remove the second term in the fourth line of Eq. (157). We have

$$\cos(3\,\zeta) \;=\; -\frac{q}{2}\cdot\sqrt{-\frac{27}{p^3}} \iff \zeta \;=\; \frac{1}{3}\arccos\left(-\frac{q}{2}\cdot\sqrt{-\frac{27}{p^3}}\right) + \frac{2}{3}k\pi \tag{159}$$

where k is an integer. If we insert ζ into $z = r\cdot\cos\zeta$ we obtain

$$z = -\sqrt{-\frac{4}{3}p}\cdot\cos\left(\frac{1}{3}\arccos\left(-\frac{q}{2}\cdot\sqrt{-\frac{27}{p^3}}\right) \mp \frac{k\pi}{3}\right) \tag{160}$$

by use of $\cos(\zeta \pm \frac{2k\pi}{3}) = -\cos(\zeta \mp \frac{k\pi}{3})$. For $k = -1, 0, 1$ and $\lambda = z - \frac{a}{3}$ we get the following three solutions for the eigenvalues.

$$\lambda_1 \;=\; \sqrt{-\frac{4}{3}p}\cdot\cos\left(\frac{1}{3}\arccos\left(-\frac{q}{2}\cdot\sqrt{-\frac{27}{p^3}}\right)\right) - \frac{a}{3} \tag{161}$$

$$\lambda_2 \;=\; -\sqrt{-\frac{4}{3}p}\cdot\cos\left(\frac{1}{3}\arccos\left(-\frac{q}{2}\cdot\sqrt{-\frac{27}{p^3}}\right) - \frac{\pi}{3}\right) - \frac{a}{3} \tag{162}$$

$$\lambda_3 \;=\; -\sqrt{-\frac{4}{3}p}\cdot\cos\left(\frac{1}{3}\arccos\left(-\frac{q}{2}\cdot\sqrt{-\frac{27}{p^3}}\right) + \frac{\pi}{3}\right) - \frac{a}{3} \tag{163}$$

For $|k| > 1$ the same solutions are reproduced.

C Mathematica code for the calculation of the entanglement

This notebook allows to calculate the amount of spin-momentum entanglement that a boosted observer detects in case of massive spin-1-particles.

The function MatrixCombine9 constructs a 9x9 matrix out of nine 3x3 matrices.

MatrixCombine9[mat11_, mat12_, mat13_, mat21_, mat22_, mat23_, mat31_,

mat32_, mat33_]:=Do[Clear[CombMat, CombMatPart];

```
CombMat = Array[CombMatPart, {9, 9}];
Do[Do[CombMatPart[i, j] = mat11[[i, j]], {i, 3}], {j, 3}];
Do[Do[CombMatPart[i, j + 3] = mat12[[i, j]], {i, 3}], {j, 3}];
Do[Do[CombMatPart[i, j + 6] = mat13[[i, j]], {i, 3}], {j, 3}];
Do[Do[CombMatPart[i + 3, j] = mat21[[i, j]], {i, 3}], {j, 3}];
Do[Do[CombMatPart[i + 3, j + 3] = mat22[[i, j]], {i, 3}], {j, 3}];
Do[Do[CombMatPart[i + 3, j + 6] = mat23[[i, j]], {i, 3}], {j, 3}];
Do[Do[CombMatPart[i + 6, j] = mat31[[i, j]], {i, 3}], {j, 3}];
Do[Do[CombMatPart[i + 6, j + 3] = mat32[[i, j]], {i, 3}], {j, 3}];
Do[Do[CombMatPart[i + 6, j + 6] = mat33[[i, j]], {i, 3}], {j, 3}];
Return[CombMat]]
```

The function MakeDensityMatrix9 evaluates the density matrix of a nine-dimensional quantum state:

```
MakeDensityMatrix9[statevector_, constatevector_]:=
Table[statevector[[i]] * constatevector[[j]], {i, 9}, {j, 9}]
```

TracingOut9 calculates the reduced density matrix by tracing out the spin subsystem:

```
TracingOut9[mat_]:=Table[Tr[mat[[i;;i + 2, j;;j + 2]]], {i, 1, 9, 3}, {j, 1, 9, 3}]
```

nullmat3 is a 3x3 null matrix.

```
nullmat3 = Table[0, {i, 3}, {j, 3}];
```

vecsph[lv_, θ_, ϕ_] transforms a vector that is given by spherical coordinates in cartesian coordinates.

```
vecsph[lv_, θv_, φv_]:=lv{Sin[θv] * Cos[φv], Sin[θv] * Sin[φv], Cos[θv]}
```

rotvec[a_, b_] calculates the vector product.

```
rotvec[a_, b_]:= − Cross[a, b]/Norm[Cross[a, b]]
```

γ[r_] is the Lorentz factor:

$$\gamma[\mathrm{r\_}]{:=}1/\mathrm{Sqrt}[1-r.r]$$

$$\gamma[\mathrm{a\_},\mathrm{b\_}]{:=}\gamma[a]*\gamma[b]*(1+a.b)$$

The rotation angle δ is calculated in dependence of the angle ϕ and the Lorentz factors of the moving particle and the moving observer:

$$\delta[\mathrm{u\_},\mathrm{v\_}]{:=}$$

$$\mathrm{ArcCos}[(1+\gamma[u,v]+\gamma[u]+\gamma[v])\text{^}2/((1+\gamma[u,v])(1+\gamma[u])(1+\gamma[v]))-1]$$

R[δ_, n1_, n2_, n3_] is a matrix that performs a rotation with an angle δ around an axis given by the vector (n1, n2, n3):

$$R[\delta\_,\mathrm{n1\_},\mathrm{n2\_},\mathrm{n3\_}]{:=}$$

$$\{\{\mathrm{n1}\text{^}2(1-\mathrm{Cos}[\delta])+\mathrm{Cos}[\delta],\mathrm{n1}*\mathrm{n2}(1-\mathrm{Cos}[\delta])-\mathrm{n3}*\mathrm{Sin}[\delta],$$

$$\mathrm{n1}*\mathrm{n3}*(1-\mathrm{Cos}[\delta])+\mathrm{n2}*\mathrm{Sin}[\delta]\},$$

$$\{\mathrm{n2}*\mathrm{n1}*(1-\mathrm{Cos}[\delta])+\mathrm{n3}*\mathrm{Sin}[\delta],\mathrm{n2}\text{^}2(1-\mathrm{Cos}[\delta])+\mathrm{Cos}[\delta],$$

$$\mathrm{n2}*\mathrm{n3}*(1-\mathrm{Cos}[\delta])-\mathrm{n1}*\mathrm{Sin}[\delta]\},$$

$$\{\mathrm{n3}*\mathrm{n1}*(1-\mathrm{Cos}[\delta])-\mathrm{n2}*\mathrm{Sin}[\delta],\mathrm{n2}*\mathrm{n3}*(1-\mathrm{Cos}[\delta])+\mathrm{n1}*\mathrm{Sin}[\delta],$$

$$\mathrm{n3}\text{^}2(1-\mathrm{Cos}[\delta])+\mathrm{Cos}[\delta]\}\}$$

The matrix T performs a basis change to the spherical basis :

$$T=\{\{-1/\mathrm{Sqrt}[2],i/\mathrm{Sqrt}[2],0\},\{0,0,1\},\{1/\mathrm{Sqrt}[2],i/\mathrm{Sqrt}[2],0\}\};$$

$$\mathrm{Ti}=\mathrm{Inverse}[T];$$

$$\mathrm{Rsph}[\mathrm{a\_},\mathrm{b\_}]{:=}T.(R[\delta[a,b],\mathrm{rotvec}[a,b][[1]],\mathrm{rotvec}[a,b][[2]],\mathrm{rotvec}[a,b][[3]]]).$$

$$\mathrm{Ti}$$

The vector sv1 collects the special choice of the coefficients in the superposition:

$$\mathrm{sv1}[\eta\_,\kappa\_]{:=}\{\mathrm{Cos}[\eta]\mathrm{Sin}[\kappa],0,0,0,\mathrm{Sin}[\eta]\mathrm{Sin}[\kappa],0,0,0,\mathrm{Cos}[\kappa]\}$$

The function entropy calculates the expression for the von Neumann entropy of the state vector sv1 for different choices of the preparation parameters (η , κ) and the vectors u_i and v, that are given by their lengths and their spherical coordinate angles, by solving the corresponding cubic equation (the parameter frame has to be *rest* in case of the rest frame and *moving* in case of the moving frame) :

```
entropy[η_, κ_, lu_, Θu1_, φu1_, Θu2_, φu2_, Θu3_, φu3_, lv_, Θv_, frame_]:=Do[
If[frame==moving,
rotmat9 = MatrixCombine9[Rsph[vecsph[lu, Θu1, φu1], vecsph[lv, Θv, 0]],
nullmat3, nullmat3, nullmat3,
Rsph[vecsph[lu, Θu2, φu2], vecsph[lv, Θv, 0]], nullmat3, nullmat3,
nullmat3, Rsph[vecsph[lu, Θu3, φu3], vecsph[lv, Θv, 0]]];
statev1 = rotmat9.sv1[η, κ];
ρ = Simplify[TracingOut9[MakeDensityMatrix9[statev1,
Conjugate[statev1]]]], , statev1 = sv1[η, κ];
ρ = Simplify[TracingOut9[MakeDensityMatrix9[statev1, statev1]]]];
a = −Tr[ρ];
b = −(ρ[[1, 2]]ρ[[2, 1]] − ρ[[1, 1]]ρ[[2, 2]] + ρ[[1, 3]]ρ[[3, 1]] + ρ[[2, 3]]ρ[[3, 2]]−
ρ[[1, 1]]ρ[[3, 3]] − ρ[[2, 2]]ρ[[3, 3]]);
c = −(−ρ[[1, 3]]ρ[[2, 2]]ρ[[3, 1]] + ρ[[1, 2]]ρ[[2, 3]]ρ[[3, 1]]
+ρ[[1, 3]]ρ[[2, 1]]ρ[[3, 2]] − ρ[[1, 1]]ρ[[2, 3]]ρ[[3, 2]]
−ρ[[1, 2]]ρ[[2, 1]]ρ[[3, 3]] + ρ[[1, 1]]ρ[[2, 2]]ρ[[3, 3]]);
p = b − a^2/3;
q = 2 ∗ a^3/27 − a ∗ b/3 + c;
ph = ArcCos[−q/2 ∗ Sqrt[−27/p^3]];
λ[1] = −a/3 + 2 ∗ Sqrt[−p/3] ∗ Cos[ph/3];
λ[2] = −a/3 − 2 ∗ Sqrt[−p/3] ∗ Cos[(ph − π)/3];
```

$\boldsymbol{\lambda[3] = -a/3 - 2 * \mathrm{Sqrt}[-p/3] * \mathrm{Cos}[(\mathrm{ph} + \pi)/3];}$

$\boldsymbol{\mathrm{ent} = \mathrm{Sum}[-\lambda[i] * \mathrm{Log}[2, \lambda[i]], \{i, 1, 3\}]; \mathrm{Return}[\mathrm{ent}]];}$

entropyrest is the von Neumann entropy in the rest frame of the particle.

$\boldsymbol{\mathrm{entropyrest} = \mathrm{entropy}[\eta, \kappa, 0, 0, 0, 0, 0, 0, 0, 0, 0, \mathrm{rest}];}$

entropymoving is entropy in the the moving frame for different choices of the parameters η, κ, lu, Θu1, φu1, Θu2, φu2, Θu3, φu3, lv, Θv, where lu and lv are the lengths of the velocity vectors and the different Θ and φ are the corresponding polar and azimuthal angles of the vectors:

$\boldsymbol{\mathrm{entropymoving} = \mathrm{entropy}[\eta, \kappa, 0.7, 0, 0, \pi/2, 0, \pi, 0, 0.9, \pi * 0.3, \mathrm{moving}];}$

$\boldsymbol{\mathrm{Plot3D}[\mathrm{entropyrest} - \mathrm{entropymoving}, \{\eta, 0, \pi\}, \{\kappa, 0, \pi\}]}$

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