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Definable Witnesses

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Contents

Abstract

The topic of this doctoral work is at the intersection of descriptive set theory, set theory of the reals, and forcing. We study methods of obtaining models of ZFC plus the negation of the continuum hypothesis in which there exist various combinatorially significant subsets of the real line that are definable by a projective formula of provably minimal complexity, i.e. is optimal. We present a general framework for obtaining the existence of coanalytic combinatorial sets of reals given the existence of a Σ^1_2 such witness. This is done by collecting a series of these reduction theorems appearing sporadically throughout the literature, extracting from their proofs an overarching pattern, and presenting them in a uniform fashion. These theorems improve previous constructions which required the assumption $V = L$. The general strategy is applied to provide a new reduction theorem for the case of Hausdorff gaps. We then study forcing notions relevant to the theory of cardinal characteristics; first we show a proper forcing given by Shelah in 1984 and its countable support iterations preserve tight mad families, providing an alternative proof of the consistency of $\mathfrak{b} = \mathfrak{a} < \mathfrak{s}$. This is applied to show that $\mathfrak{a} < \mathfrak{s}$ is consistent with a Δ^1_3 wellordering of the reals and a coanalytic tight mad family witnessing $\mathfrak{a} = \aleph_1$, responding to questions of Fischer and Friedman. We extend the theory of projective witnesses for values of cardinal characteristics by considering the spectrum of $\mathfrak{a}$, the set of possible cardinalities of a mad family. By showing a forcing notion of Friedman and Zdomskyy also preserves tight mad families in a strong sense, we obtain the consistency of $\mathfrak{a} = \aleph_1 < \mathfrak{c} = \aleph_2$, and for every κ in the spectrum of $\mathfrak{a}$ there exists a projective tight mad family of size κ with definition of optimal complexity. In the end we show the compatibility of our results and construct a model witnessing all of the conclusions simultaneously.

Zusammenfassung

Das Thema dieser Doktorarbeit liegt an der Schnittstelle zwischen der deskriptiven Mengenlehre, der Mengenlehre der reellen Zahlen und dem Forcing. Wir untersuchen Methoden zur Konstruktion von Modellen von ZFC plus der Negation der Kontinuumshypothese, in denen verschiedene kombinatorisch bedeutende Teilmengen der reellen Zahlen existieren, die durch eine projektive Formel von nachweislich minimaler Komplexität, d. h. optimal, definierbar sind. Wir präsentieren einen allgemeinen Rahmen, um die Existenz von koanalytischen kombinatorischen Mengen von reellen Zahlen zu erhalten, wenn ein solcher Zeuge Σ^1_2 existiert. Dies geschieht durch das Sammeln einer Reihe dieser Reduktionstheoreme, die sporadisch in der Literatur auftauchen, das Extrahieren eines übergreifenden Musters aus ihren Beweisen und deren einheitliche Darstellung. Diese Theoreme verbessern frühere Konstruktionen, die die Annahme $V = L$ erforderten. Die allgemeine Strategie wird angewendet, um ein neues Reduktionstheorem für den Fall von Hausdorff-Gaps zu liefern. Wir untersuchen dann Forcing-Begriffe, die für die Theorie der Kardinalcharakteristiken relevant sind. Zunächst zeigen wir, dass ein 1984 von Shelah angegebenes geeignetes Forcing und seine abzählbaren Iterationen enge MAD-Familien bewahren, was einen alternativen Beweis für die Konsistenz von $\mathfrak{b} = \mathfrak{a} < \mathfrak{s}$ liefert. Dies wird angewendet, um zu zeigen, dass $\mathfrak{a} < \mathfrak{s}$ mit einer Δ^1_3-Wohlordnung der reellen Zahlen und einer koanalytischen engen MAD-Familie, die $\mathfrak{a} = \aleph_1$ bezeugt, konsistent ist, und damit auf Fragen von Fischer und Friedman antwortet. Wir erweitern die Theorie der projektiven Zeugen für Werte von Kardinalcharakteristika, indem wir das Spektrum von $\mathfrak{a}$ betrachten, die Menge der möglichen Kardinalitäten einer MAD-Familie. Indem wir zeigen, dass ein Forcing-Begriff von Friedman und Zdomskyy auch enge MAD-Familien in einem starken Sinne bewahrt, erhalten wir die Konsistenz von $\mathfrak{a} = \aleph_1 < \mathfrak{c} = \aleph_2$, und für jedes κ im Spektrum von $\mathfrak{a}$ gibt es eine projektive enge MAD-Familie der Größe κ mit einer Definition optimaler Komplexität. Am Ende zeigen wir die Kompatibilität unserer Ergebnisse und konstruieren ein Modell, das alle Schlussfolgerungen gleichzeitig belegt.

Dedication

To Bonnie.

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Chapter 1

Introduction

Descriptive set theory studies the properties of the definable subsets of the real line, while set theory of the reals studies the combinatorial properties of subsets of the real line. At the intersection of these two we find the study of the definability of combinatorial sets of reals in various models of set theory, where we take the notion of definability to mean admitting a level in the projective hierarchy. More recently do we find the study of projective sets in forcing extensions in which the continuum hypothesis fails, though this line of inquiry traces back to investigations into the reals of the constructible universe L in the decades after its definition by Gödel in 1938, with notable advances in the 1980's. It is with the modern developments in the theory of iterated forcing that we are able to ask, and have the techniques to answer, more nuanced questions about the potential cardinalities of the combinatorially distinguishing structures of the continuum, such as those giving rise to cardinal characterstics. We are interested particularly in when the witness of these special subsets of the real line has a projective definition of minimal descriptive complexity, i.e., is optimal. In this doctoral work we study methods of extending a theory of optimal definable witnesses to models in which the continuum hypothesis fails, and moreover, in which there is a nontrivial theory of the cardinal characteristics of the continuum.

To explain how this is done and motivate our work, we begin by giving the context in which our results appear.

1.1 Historical context and relevant results

Another moment in the extension of the concept of number in the history of mathematics was when Georg Cantor's work in analysis in the late 19th century necessitated his introduction of the transfinite ordinals and a well-defined, systematic study of infinite sets and their sizes. The methods developed by Cantor and number theorist Richard Dedekind in the newborn theory of sets, though robust, were insufficient for answering a seemingly simple question, "How many points are there on a straight line in Euclidean space? In other terms, the question is: How many different sets of integers do there exist?" (Gödel, 1947 [Gö47]).

Cantor's Continuum Hypothesis CH, later to become the first of David Hilbert's list of problems in the 1900 International Congress of Mathemeticians in Paris, is the assertion that the cardinality of the continuum $|\mathbb{R}| = \mathfrak{c}$ is $\aleph_1$, the first infinite cardinal strictly greater than the size $\aleph_0 = |\mathbb{N}|$ of countable sets; equivalently, there is no uncountable subset of reals with cardinality strictly less than $\mathfrak{c}$. Cantor would not survive to see the resolution of CH, though he and Ivar Bendixson were able to establish it for the closed subsets of the real line ([Ben83], [Can83], [Can84]).

Cantor's theory was brought to the analysts in France in the late 19th century by by Emile Borel [Bor98], who defined the class of Borel sets: the smallest class of subsets of reals containing the closed sets and closed under the operations of complementation and countable unions. These would later be shown to verify the Continuum Hypothesis. Henri Baire [Bai99] and René Lebesgue [Leb02] also introduced an axiomatization of a theory of nicely behaved sets; each of these classes subsumed the class of Borel sets. Later joined by mathematicians such as Nikolai Lusin, Mikhail Souslin, Wacław Sierpiński, and Petr Novikov, these authors would set the foundation for the subject of descriptive set theory.

Correcting a famous mistake of Lebesgue [Leb05], Souslin [Sou17] showed that projections of Borel sets need not be Borel themselves; the class of sets obtained in this way were defined to be the class of analytic (Σ^1_1) sets. This class gives rise in turn to a proper hierarchy of sets called the projective hierarchy, which is climbed through processes of complementation and projection ([Sie25], [Lus25a]). For much of the early 20th century, work in the newly born field of descriptive set theory was motivated by establishing that the regularity properties enjoyed by the Borel sets also held for the higher projective pointclasses; these are properties such as being Lebesgue measurable, having the property of Baire, or satisfying the perfect set property. A subset of the real line is *perfect* if and only if it is a nonempty closed set without isolated points; the perfect set property is the property which states that any subset of reals is either countable or contains a perfect subset, and must therefore have cardinality of the continuum, $2^{\aleph_0}$. Perfect sets were the intial tool of Cantor and Bendixson for establishing CH for the closed sets, as well as in Alexandrov's 1916 proof of the perfect set property for Borel sets. It could be shown that the analytic sets had the above three regularity properties, and by virtue of satisfying the perfect set property, there can be no analytic counterexample to CH. The Axiom of Choice, or equivalently a wellordering of the continuum, on the other hand allows one to obtain examples that these properties need not hold for all sets in general. However, very little progress in the study of the projective sets was being made, and a famous quote of Lusin in 1925 [Lus25b] expressed the preemptive fear that perhaps "one could never know" if the desirable properties of the Borel and analytic sets extended further upwards in the projective hierarchy.

The word preemptive above is appropriate, as it predates the work of Kurt Gödel in Vienna in the 1930's. Shortly after showing that any recursively enumerable theory advanced enough for developing basic Peano Arithmetic is necessarily incomplete—meaning there are mathematical statements whose truth cannot be decided on the basis of that theory alone—Gödel [Göd39] established that if the axioms of Zermelo-Fraenkel set theory, ZF, are consistent (have a model), then so are the axioms of ZF together with Axiom of Choice and the Generalized Continuum Hypothesis, GCH. This is done by defining the constructible universe L which satisfies this latter theory.

Though the construction of L also established that consistently, the classical theorems attesting to the desirable regularity properties of the Borel and analytic sets already admit counterexamples at the lowest possible level in the projective hierarchy, Δ^1_2: in L there exists Δ^1_2 examples of a subset of the real line which is not Lebesgue measurable and of a set without the property of Baire, and there exists an uncountable coanalytic (Π^1_1) set without the perfect set property. This is because of the how Axiom of Choice is satisfied in L. Specifically, the set of constructible reals and their corresponding wellordering are defined by Σ_1-formulas of set theory and are thus Σ^1_2-definable sets; since the wellordering is total, it is in fact Δ^1_2-definable. An iterative construction along the wellorder yields those Σ^1_2-definable witnesses obtained by the Axiom of Choice, and in particular Σ^1_2 examples of certain subsets of the real line with nontrivial, distinguishing combinatorial properties—such as are the objects of interest in set theory of the reals and the study of cardinal characteristics. These strengthenings of existence results to yield optimally definable witnesses "place a premium on models of set theory for which there are Δ^1_2 wellorderings of [the real line]" (Mansfield and Weitkamp, [MW85, p. 31]).

Indeed we find the use of the axiom $V = L$ in set theory of the reals so to strengthen theorems asserting the existence of various combinatorial sets of interest, namely by showing the witness of such a set can be taken to be projective. This is of particular interest when the object is normally obtained by the Axiom of Choice, hence by nonconstructible means. An early example is given by Erdős, Kunen, and Mauldin, who construct a coanalytic, uncountable subset of the real line containing no perfect subset and exhibiting nontrivial additive properties with respect to the Lebesgue measure [EKM81, Theorem 13]. Their crucial insight towards a construction of a coanalytic witness was the reduction of complexity of the rather easily obtained Δ^1_2-definable such set. For this, a robust coding technique was introduced, which was streamlined a few years later by Arnold Miller [Mil89]. Assuming $V = L$, he produced coanalytic witnesses of various other maximal combinatorial sets such as Hamel bases, maximal almost disjoint families, and maximal independent families; previous theorems asserted the non-existence of analytic examples of such sets, so these results indeed yield witnesses of optimal complexity. Vidnyánszky [Vid14] has now formulated the technique into a black-box condition, systemetizing Miller's method and yielding shorter, simpler coanalytic existence proofs in L.

Reducing the complexity of a Σ^1_2 set amounts to removing an initial existential quantification over the reals; the success in doing in the above cases hinges upon deep, structural properties about the class Σ^1_2. Indeed, the class of Σ^1_2 sets admits what may be called desirable properties: classical theorems of Donald Martin, Richard Mansfield, and Robert Solovay established that

that every Σ^1_2 wellordering of a set of reals is of size at most $\aleph_1$, and any Σ^1_2 set of cardinality greater than $\aleph_1$ has a perfect set of nonconstructible elements. Therefore, the existence of a Σ^1_2 total wellordering of the reals implies all reals are constructible; in particular CH holds.

Paul Cohen [Coh63] introduced the method of forcing in 1963 and established the consistency of ZFC with the negation of CH. His construction of a model in which the cardinality of the real line is strictly greater than $\aleph_1$ identified the first among many independence results in set theory, and the technique of forcing soon became the standard tool for revealing further properties of the continuum that are left undecided by the usual axioms of set theory. Among some of these assertions we find those concerning the regularity properties of the higher levels of the projective hierarchy. For example, Solovay in 1970 [Sol70] showed it is consistent with ZFC that every set of reals is Lebesgue measurable, has the property of Baire, and has the perfect set property—in stark contrast with the situation in Gödel's L.

In 1977 Leo Harrington [Har77] used the method of forcing to construct a model of ZFC in which various properties of Σ^1_2 do not hold for sets already at the next level in the projective hierarchy. Specifically, he shows that it is consistent with ZFC that the continuum is arbitrarily large, there exists a Π^1_2 wellordering of a set of reals of cardinality $\mathfrak{c}$, and there exists a $\boldsymbol{\Delta}^1_3$ total wellorder of the reals. His proof relies upon powerful methods originating in the work of Solovay, and Ronald Jensen, and Johnsbråten ([SJ70], [JJ74]); these techniques were used, for instance, in constructing nonconstructible sets having the best possible projective definition.

Forcing has undoubtedly become a central tool in the more recently developed field of cardinal invariants of the continuum. A *cardinal characteristic* or *cardinal invariant of the continuum* is the minimal size of a set of reals exhibiting certain distinguishing combinatorial properties, often objets previously studied in fields such a topology, measure theory, or algebra. Of particular interest in our current work will be the property of being maximal almost disjoint:

Definition 1.1.1. A family $\mathcal{A} \subseteq [\omega]^\omega$ is called *almost disjoint (ad)* if for all $x, y \in \mathcal{A}$, $x \cap y$ is finite. An infinite ad family is *maximal almost disjoint (mad)* if $\mathcal{A}$ is not a proper subset of any other almost disjoint family; equivalently,

$\mathcal{A}$ is mad if and only if for every $y \in [\omega]^\omega$ there exists $x \in \mathcal{A}$ such that $x \cap y$ is infinite.

Mad families have been of interest for their applications in general topology and functional analysis, and define the cardinal characteristic

$$\mathfrak{a} = \min\{|\mathcal{A}| \mid \mathcal{A} \subseteq [\omega]^\omega,\ \mathcal{A} \text{ is mad}\}.$$

Further examples of combinatorial families giving rise to combinatorial invariants include a non-Lebesgue measurable set (the cardinal $\mathrm{non}(\mathcal{N})$), or a family of functions unbounded with respect to the order $\leq^*$ (the cardinal $\mathfrak{b}$), where for functions f, g on the natural numbers, $f \leq^* g$ if and only if $f(n) \leq g(n)$ for all but finitely many natural numbers n. A comprehensive exposition of the rich theory of cardinal characteristics can be found in [Bla10]. The majority of the cardinal invariants lie in the interval $[\aleph_1, 2^{\aleph_0}]$, so in models of CH, the theory of these numbers becomes trivial. While some inequalities between these numbers can be derived from ZFC, forcing allows for a systematic study of the boundary of what ZFC decides about the cardinalities of the special subsets of the real line and how they relate to one another, in some cases revealing further independencies in set theory.

One such instance of this is in regards to the cardinal $\mathfrak{b}$ mentioned above, and the cardinal $\mathfrak{s}$, defined as the minimal size of a collection $\mathcal{S}$ of infinite subsets of natural numbers which is *splitting*: for any infinite set $b \subseteq \mathbb{N}$ there is $a \in \mathcal{S}$ such that b meets both a and the complement of a in an infinite set. It was known by 1980 that there is a forcing extension in which the inequality $\mathfrak{s} < \mathfrak{b}$ held, and the construction of a model in which $\mathfrak{b} < \mathfrak{s}$ required the invention of a new forcing notion by Saharon Shelah in 1984 [She84]. This shows the relation between the cardinals $\mathfrak{b}$ and $\mathfrak{s}$ is independent of ZFC, and furthemore illustrates how unanswered questions about cardinal characteristic inequalities can lead to advancements in forcing techniques, as Shelah's forcing now indicates just one instance of a broad class of forcing notions which add new reals known as creature forcings [RS99].

1.2 Guiding questions and motivation

In summary, the Δ^1_2 wellordering of the continuum given by the axiom $V = L$ allows for a theory of nicely definable witnesses, and much investigation has been done on how to obtain witnesses with a projective definition strictly less complex than that of the wellorder. However the fact $\mathfrak{c} = \aleph_1$ under this axiom leaves no possibility for a nontrivial theory of cardinal characteristics. The method of forcing lends itself both to the theory of projective wellorders and to the theory of cardinal characteristics; for the latter it gives a way of obtaining nontrivial structure of the cardinal invariants, and for the former can be used to yield a $\boldsymbol{\Delta}^1_3$ wellorder of the reals of length greater than $\aleph_2$. In Harrington's 1977 construction, however, Martin's Axiom held, and so for a while, what had not yet been addressed was the question of a projective wellordering of the reals in models with a nontrivial theory of cardinal characteristics.

The first progress in this direction was made in 2010 by Vera Fischer and Sy David Friedman [FF10]. Improving upon Harrington's result, they showed that consistently there is a Δ^1_3 wellorder of the reals, and various cardinal characteristic constellations can be realized. This then opens the question as to the projective definability of the sets witnessing these cardinal inequalities, and namely if such witnesses can be found with a complexity strictly less than that of the wellorder.

The work in this dissertation will be guided by the following questions.

Question 1.2.1. *How can we obtain models of $\neg\mathsf{CH}$ while still obtaining $\aleph_1$-sized coanalytic combinatorial sets as is done under $V = L$?*

Question 1.2.2. *What cardinal characteristic inequalities are compatible with such $\aleph_1$-sized definable witnesses?*

Question 1.2.3. *How can we obtain definable witnesses of size strictly greater than $\aleph_1$?*

Our first part, Chapter 2, will focus on the first item of this list.

1.2.1 Coanalytic witnesses obtained from Σ^1_2 sets

While Adrian Mathias [Mat77] showed that no analytic set can be a mad family, Miller [Mil89] proved that assuming $V = L$ there exists a coanalytic mad family, and hence this is optimal. In 2014, Asger Törnquist [T09] gave a different theorem asserting the existence of a coanalytic mad family, however requiring only the assumption of ZF and the existence of a Σ^1_2 mad family. Theorems of a similar nature then began reappearing in the literature for the cases of other combinatorially significant families known to have no analytic instantiations. The major import of these results is that they may be combined with methods of preservation and the study of indestructibility, central aspects of the modern theory of iterated forcing, and thereby offer a way of extending the coanalytic existence results under $V = L$ to models with nonconstructible reals.

These proofs resembling Törnquist's are found sporadically in the literature and their structure was never made explicit, though it is evident there is a general schema and underlying structure common to each. In Chapter 2 we collect these similar examples and present them in a uniform manner, thereby extracting a general framework which is applied to give the following:

Theorem 1 (Theorem 2.4.2)**.** *If there exists a Hausdorff gap (A, B) such that both A, B are Σ^1_2-definable, then there exists a Hausdorff gap (A', B') with A', B' being Π^1_1-definable.*

A *gap* is a pair (A, B) where A, B are infinite subsets of natural numbers such that: every $a \in A$ is almost disjoint from every $b \in B$; A, B are wellordered by the relation $\subseteq^*$ of set inclusion modulo a finite set; and moreover (A, B) cannot be separated, meaning there does not exist any infinite subset $c \subseteq \mathbb{N}$ with the property that c is almost disjoint from every member of A and $c \subseteq^* b$ for every b in B. Hausdorff gaps are a special type of gap with an even stronger combinatorial property, the first examples being constructed by Hausdorff in 1909 [Hau09] and 1936 [Hau36]. Gaps played a role in Hausdorff's work towards a theory of linearly ordered sets, and their study dates back to work of Du Bois-Reymond and Hadamard.

The definability here is optimal, as Todorčević [Tod96, Theorem 1] proved in 1996 that there do not exist gaps (A, B) whenever A or B is analytic.

1.2.2 Definable wellorders, cardinal characteristics, and the mad spectrum

Chapter 3 proceeds by treating Question 1.2.2 and Question 1.2.3 simultaneously, by looking at methods of constructing models of $\mathfrak{c} \geq \aleph_2$ while preserving a Π^1_1 witness in the ground model via forcings which add generic combinatorial objects of size continuum. In particular we focus on the specific case of projective mad families. It was known that models with large continuum and a Δ^1_3 wellorder were compatible with $\mathfrak{a} = \aleph_1 < \mathfrak{c} = \aleph_2$, $\mathfrak{b} = \mathfrak{c}$, and $\mathfrak{b} < \mathfrak{a} = \mathfrak{s} = \mathfrak{c}$, and in particular, the first two inequalities can have the value of $\mathfrak{a}$ being witnessed by a projective mad family of optimal complexity ([BFB22] and [BK13]). Recently, [FFST25] has given the consistency of $\aleph_1 < \mathfrak{a} < \mathfrak{c} < \aleph_\omega$, with an optimal projective witness for $\mathfrak{a}$. Thus far, the attention has been on finding definable witnesses for the value $\mathfrak{a}$, the minimal size of an infinite mad family.

We extend these previous results by turning our attention more generally to the set

$$\operatorname{spec}(\mathfrak{a}) = \{|\mathcal{A}| \mid \mathcal{A} \subseteq [\omega]^\omega, \mathcal{A} \text{ is mad}\},$$

i.e. the set of possible cardinalities of mad families. The study of this set was initiated by Stephen Hechler in 1972 [Hec72] and followed up in the work of Blass [Bla93] and Shelah and Spinas [SS15]. By now there is significant understanding in methods for controlling the realization of the set $\operatorname{spec}(\mathfrak{a})$ in forcing extensions, and the results mentioned above show that likewise there exist ways of controlling the projective complexity of mad families in such extensions. It is natural then, the question of how to simultaneously control $\operatorname{spec}(\mathfrak{a})$ and the projective complexity of witnesses to every cardinal belonging to the spectrum. We will show that there exists a cardinal preserving forcing iteration yielding a model with continuum of size $\aleph_2$ and $\operatorname{spec}(\mathfrak{a}) = \{\aleph_1, \aleph_2\}$, such that each inclusion of a cardinal $\kappa \in \operatorname{spec}(\mathfrak{a})$ is witnessed by a tight mad family of size κ with an optimal projective definition; in fact, we will show this can be done in a model with a Δ^1_3 wellorder of the reals, in which the inequality $\mathfrak{b} < \mathfrak{s}$ is obtained. This obtains the following new consistency result, our cumulative theorem of the third chapter:

Theorem 2 (Theorem 3.5.2)**.** *It is consistent that $\aleph_1 = \mathfrak{a} < \mathfrak{s} = \aleph_2$, there* *exists a Δ^1_3 wellorder of the reals, as well as tight mad families of cardinality $\aleph_1$ and $\aleph_2$, which are respectively Π^1_1 and Π^1_2 definable.*

The proof of the Theorem 2 will be divided into three steps, each yielding a result of its own independent interest.

The first intermediary result towards our final model is to reveal a stronger combinatorial property of the aforementioned, well-known forcing defined by Shelah in 1984 [She84], used to show the consistency of $\mathfrak{b} < \mathfrak{s}$. In the original paper of Shelah, the proper forcing $\mathbb{Q}$ introduced is shown to have the property of being *almost ω^ω-bounding*, ensuring that countable support iterations of $\mathbb{Q}$ preserve the unboundedness of the ground model set of reals, thereby witnessing $\mathfrak{b} = \aleph_1$ in a countable support iteration of $\mathbb{Q}$ over a model of CH. With the goal of preserving a coanalytic mad family in a ground model satisfying $V = L$, we show that Shelah's forcing in fact satisfies a preservation property introduced in 2020 by Guzman, Hrušák, and Tellez [GHT20] known as *strong preservation of tightness*; it is a property of proper forcing notions ensuring the preservation of tight mad families under countable support iterations. A tight mad family is an almost disjoint family with a combinatorial property stronger than maximality; they were initially studied by Malykhin in 1989 [Mal89], and these strongly mad families have been shown to be rather impervious with respect to forcings which add new reals—that is, they are known to be indestructible by a large class of forcings.

Proposition 1 (Proposition 3.2.28)**.** *Shelah's forcing notion $\mathbb{Q}$ strongly preserves the tightness of ground model tight mad families.*

Since $\mathfrak{b} \leq \mathfrak{a}$ is a theorem of ZFC, the fact $\mathfrak{b} = \aleph_1$ holds in a countable support iteration of $\mathbb{Q}$ is now a consequence of the ZFC inequality and our preservation result. Thus we obtain an alternative proof of Shelah's theorem [She84, Theorem 3.1] on the consistency of $\mathfrak{b} = \aleph_1 < \mathfrak{s} = \aleph_2$; see the proof of Theorem 3.2.32.

We proceed by introducing the construction of the Δ^1_3 wellorder in a model of $\mathfrak{c} = \aleph_2$ due to Fischer and Friedman [FF10], which gave the first instance of a model with a projective wellorder and a nontrivial theory of cardinal invariants. In 2022, Bergfalk, Fischer, and Switzer [BFB22] showed this forcing satisfies strong preservation of tightness, given the assumption that the mad family to be preserved has a projective definition and provably consists of constructible reals. With their result and an appropriate minor modification to this construction, we obtain the consistency of another cardinal characteristic constellation in a model with a Δ^1_3 wellorder, as well as definable witness to the value of $\mathfrak{a}$:

Theorem 3 (Theorem 3.3.35)**.** *It is consistent with $\mathfrak{a} = \aleph_1 < \mathfrak{s} = \mathfrak{c} = \aleph_2$ that there exists a Δ^1_3 wellorder of the reals, and $\mathfrak{a} = \aleph_1$ is witnessed by a coanalytic tight mad family.*

Our third and final intermediary step is to obtain the existence of a tight mad family of size $\mathfrak{c} = \aleph_2$. Towards this end we consider a forcing notion given by Friedman and Zdomskyy in 2010 [FZ10], used to establish the consistency of $\mathfrak{b} = \mathfrak{c} = \aleph_2$ with a Π^1_2 tight mad family witnessing $\mathfrak{a} = \aleph_2$; the definability here is optimal by a result of Raghavan [Rag09] and a classical theorem of Mansfield and Solovay. We isolate the iterand responsible for adding the definable tight mad family and show that this forcing and its iterations strongly preserve tight mad families, and as an immediate corollary we get that this forcing notion does not add dominating reals. This forcing relies on techniques of almost-disjoint coding, a robust method developed by Solovay and Jensen [SJ70] and an integral feature of Harrington's construction. With this we have the following:

Theorem 4 (Theorem 3.4.20)**.** *It is consistent with $\mathfrak{a} = \aleph_1 < \mathfrak{c} = \aleph_2$ that there exists a Π^1_1 tight mad family of size $\aleph_1$, and a Π^1_2 tight mad family of size $\aleph_2$.*

In the end, we show that all of the above results are compatible by weaving together the various forcing notions considered, yielding a model whose existence proves Theorem 2.

The thesis is structured as follows. The remaining section of Chapter 1 gives preliminaries. Chapter 2 surveys the results on reducing the complexity of Σ^1_2 combinatorial sets and proves Theorem 1; the final section of this chapter includes open questions. In Chapter 3, Section 3.2, we define Shelah's creature forcing $\mathbb{Q}$ and show the preservation result Proposition 1. Section 3.3 of Chapter 3 contains the construction of a Δ^1_3 wellorder and the proof of Theorem 3. The following Section 3.4 defines the Friedman-Zdomskyy forcing and proves the main preservation result, giving the proof of Theorem 4. Chapter 3, Section 3.5 gives the proof of Theorem 2. The final section contains concluding remarks and open questions.

1.3 Preliminaries

Our notation is fairly standard. We use ω to denote the set $\mathbb{N} = \{0, 1, 2, \dots\}$ of natural numbers and the set of real numbers is denoted by $\mathbb{R}$. For a set X, we let $\mathcal{P}(X)$ denote the set of subsets of X and $[X]^\omega$, $[X]^{<\omega}$ the infinite and the finite subsets of X, respectively. We denote by X^ω by the set of infinite sequences of elements of X and $X^{<\omega}$ the set of finite sequences. For x an element of X^ω or $X^{<\omega}$, we use $x(n)$ to denote the nth entry in x, so $x = (x(0), x(1), \dots)$. For a function $f\colon X \to Y$ we use $f[A]$ to denote the pointwise image of $A \subseteq X$. The relation $\subseteq^*$ on $\mathcal{P}(\omega)$ is defined as $x \subseteq^* y$ if and only if $x \setminus y$ is finite.

A *tree* on a set X is a subset $T \subseteq X^{<\omega}$ such that T is partially ordered by some relation $\prec$, and for every $t \in T$, the set of predecessors of t, $\{s \in T \mid s \prec t\}$ is wellordered. A *branch* through T is an element $x \in X^\omega$ such that for all $n \in \omega$, $x \upharpoonright n = (x(0), x(1), \dots, x(n-1)) \in T$; we let $[T]$ denote the set of branches through T.

Descriptive Set Theory

Descriptive set theory studies definable subsets of the real line. One reasonable way of delineating the potentially vague term "definable" is to use the notion of topological simplicity.

Viewing the real line as a topological space with the topology generated by the open intervals, $\mathbb{R}$ is *separable* (admits a countable dense set, namely the set of rational numbers) and *completely metrizable* (the standard Euclidean metric on $\mathbb{R}$ is complete). A *Polish space* is a topological space which is separable and admits a complete metric compatible with the topology. Some common examples aside from $\mathbb{R}$ are the Baire space ω^ω, the Cantor space 2^ω, the Hilbert cube $[0,1]^\omega$, and the set of infinite subsets of natural numbers $[\omega]^\omega$. The Baire space is universal for Polish spaces in the sense that any nonempty Polish space is the continuous image of ω^ω; the Cantor space is universal for compact metric spaces in that any nonempty compact metrizable space is the continuous image of 2^ω. Polish spaces are closed under countable Cartesian products, and elements of Polish spaces are referred to as *reals*.

Fix a Polish space X. The *Borel σ-algebra* $\mathcal{B}(X)$ is the σ-algebra generated by the open subsets of X, this being the closure of the collection of open sets under complements and countable unions. We stratify this class into a hierarchy of Borel pointclasses by letting $\boldsymbol{\Sigma}^0_1$ be the open subsets of X, $\boldsymbol{\Pi}^0_1$ the closed subsets of X, and for $n \geq 1$,

$$\begin{aligned}
\boldsymbol{\Sigma}^0_{n+1} =& \{A \subseteq X \mid A = \textstyle\bigcup_{n\in\omega} A_n,\ A_n \in \boldsymbol{\Pi}^0_n\},\\
\boldsymbol{\Pi}^0_n =& \{A \subseteq X \mid \omega \setminus A \in \boldsymbol{\Sigma}^0_n\},\\
\boldsymbol{\Delta}^0_n =& \boldsymbol{\Pi}^0_n \cap \boldsymbol{\Sigma}^0_n.
\end{aligned}$$

The pointclass of Borel sets is denoted by $\boldsymbol{\Delta}^1_1$. Borel sets are Lebesgue measurable, have the property of Baire, and satisfy the perfect set property.

The class $\boldsymbol{\Sigma}^1_1$ of *analytic sets* are subsets $A \subseteq X$ such that there exists a Polish space Y and a continuous function $f \colon Y \to X$ such that $f[Y] = X$. A theorem of Lusin and Souslin shows that all Borel sets are analytic, however the inclusion is strict by the existence of 2^ω-universal sets for the class $\boldsymbol{\Sigma}^1_1$. $\boldsymbol{\Pi}^1_1$ denotes the complements of analytic sets, the *coanalytic sets*.

For the results mentioned above and a more in depth treatment of the theory of Borel and projective sets, see [Kec95].

$$\underbrace{\begin{array}{ccccccccccccc}
 & & \boldsymbol{\Sigma}^0_1 & & & & \boldsymbol{\Sigma}^0_2 & & & & \boldsymbol{\Sigma}^0_3 & & \\
 & \diagup & & \diagdown & & \diagup & & \diagdown & & \diagup & & \diagdown & \\
\boldsymbol{\Delta}^0_1 & & & & \boldsymbol{\Delta}^0_2 & & & & \boldsymbol{\Delta}^0_3 & & & & \cdots \\
 & \diagdown & & \diagup & & \diagdown & & \diagup & & \diagdown & & \diagup & \\
 & & \boldsymbol{\Pi}^0_1 & & & & \boldsymbol{\Pi}^0_2 & & & & \boldsymbol{\Pi}^0_3 & &
\end{array}}_{\text{Borel hierarchy}}
\quad
\underbrace{\begin{array}{ccccccccccccc}
 & & \boldsymbol{\Sigma}^1_1 & & & & \boldsymbol{\Sigma}^1_2 & & & & \boldsymbol{\Sigma}^1_3 & & \\
 & \diagup & & \diagdown & & \diagup & & \diagdown & & \diagup & & \diagdown & \\
\boldsymbol{\Delta}^1_1 & & & & \boldsymbol{\Delta}^1_2 & & & & \boldsymbol{\Delta}^1_3 & & & & \cdots \\
 & \diagdown & & \diagup & & \diagdown & & \diagup & & \diagdown & & \diagup & \\
 & & \boldsymbol{\Pi}^1_1 & & & & \boldsymbol{\Pi}^1_2 & & & & \boldsymbol{\Pi}^1_3 & &
\end{array}}_{\text{projective hierarchy}}$$

Another specification of the notion "definable set" we can take is that a definable set A is such that membership in A is defined by a first order formula $\varphi(x)$ in the language $\mathcal{L} = \{\in\}$ of set theory. That is, a set $x \in A$ if and only if $\varphi(x)$ holds. This allows for a stratification of definable subsets into complexity classes corresponding to syntactical complexity of the defining formulas. The more alternations of the quantifiers $\forall$ and $\exists$ appearing in a formula, the more complex it is deemed to be. The Lévy hierarchy in this way classifies all formulas written in the language of set theory. Restricting the quantification to the natural numbers and Polish spaces gives rise to the Borel and projective hierarchies; the Borel sets are defined by a formula whose quantifiers range only over ω, and alternations of quantification over reals indicates an increase in the level of the projective hierarchy.

Effective descriptive set theory is branch of descriptive set theory based on the notion of a recursive function in computability theory, and provides a tool for proving results in the classical case. The work in this area was initiated largely by Stephen Kleene in the 1930's and 1940's; for more on the history of this subject, see [Mos80] and references therein.

We let Σ^1_1 denote the pointclass of (effectively) analytic subsets; these are $A \subseteq \omega^\omega$ defined by a formula which is Σ^1_1. Equivalently, A is analytic if and only if there is a closed set $C \subseteq \omega^\omega \times \omega^\omega$ such that

$$A = \{x \in \omega^\omega \mid \exists y \in \omega^\omega (x, y) \in C\}.$$

Analytic sets admit a *tree representation* in the following sense. A classical fact in descriptive set theory is that the closed subsets are in one-to-one correspondence with the set of branches of trees. Therefore, we have A is Σ^1_1 if and only if there is a tree T on $\omega \times \omega$, consisting of (s, t) such that s, t are of the same length, and such that $x \in A$ if and only if $T_x = \{t \in \omega^{<\omega} \mid (x \restriction \mathrm{lh}(t), t) \in T\}$ is ill-founded, meaning there exists a branch through T_x. The class Σ^1_1 is closed under countable union, countable intersection, and projections.

The class Π^1_1 denotes the class of (effectively) coanalytic sets, these being the complements of analytic sets. From the tree representation of analytic sets we obtain the equivalent characterization of the coanalytic sets: a set $B \subseteq \omega^\omega$ is Π^1_1 if and only if there is a tree T on $\omega \times \omega$ as above, such that $x \in B$ if and only if T_x is well-founded, meaning $[T_x] = \emptyset$. The Π^1_1 sets are closed under countable union, countable intersection, and quantification over ω^ω. The class of effectively Borel sets is $\Delta^1_1 = \Sigma^1_1 \cap \Pi^1_1$; these are the same, by a theorem of Kleene, as the class of hyperarithmetic sets.

Moving up in the projective hierarchy is by means of taking projections and complements, analogous to how the Borel hierarchy is climbed through the operations of countable union and complementation. The class of Σ^1_2 sets consists of projections of Π^1_1 sets, so for $A \subseteq \omega^\omega$, $A \in \Sigma^1_2$ if and only if there exists a Π^1_1 set $F \subseteq \omega^\omega \times \omega^\omega$ such that $x \in A$ if and only if there exists y

such that $(x,y) \in F$. More generally, $A \in \Sigma^1_{n+1}$ if it is the projection of some $F \subseteq \omega^\omega \times \omega^\omega$ with $F \in \Pi^1_n$, and $F \in \Pi^1_n$ if it is the complement of a Σ^1_n set.

The topological notion of definability, giving rise to the boldface pointclasses, is connected to the recursion-theoretic notion of definability generating the lightface pointclasses, in the following way. For Γ a pointclass as above and $x \in \omega^\omega$, we denote by $\Gamma(x)$ the relativized pointclass; these are the sets $A \subseteq \omega^\omega$ such that there is some $B \in \omega^\omega \times \omega^\omega$, $B \in \Gamma$, such that $y \in A$ if and only if $(x,y) \in B$. While we will restrict our attention to the lightface pointclasses, the constructions apply to the Borel and projective sets, or the boldface pointclasses $\boldsymbol{\Delta}^1_1$, $\boldsymbol{\Sigma}^1_n$. Indeed, if $\boldsymbol{\Gamma} \in \{\boldsymbol{\Sigma}^0_n, \boldsymbol{\Pi}^0_n, \boldsymbol{\Sigma}^1_n, \boldsymbol{\Pi}^1_n\}$ and Γ is the corresponding lightface pointclass, we have $A \in \boldsymbol{\Gamma}$ if and only if there is some $x \in \omega^\omega$ such that $A \in \Gamma(x)$. When $x \in \Delta^1_1(y)$, we say x is hyperarithmetic in y, or constructible from y.

A function $f\colon X \to Y$ is Borel if its graph, $\mathrm{graph}(f)$, is Borel as a subset of $X \times Y$. For any Polish spaces X, Y of the same cardinality, there exists a Borel isomorphism $f\colon X \to Y$, meaning f is a Borel bijection and f^{-1} is Borel; so, $A \in \mathcal{B}(X)$ if and only if $f[A] \in \mathcal{B}(Y)$ (see [Mos80, Theorem 3I.4]). From this it follows that such X and Y have the same Borel structure. By the universality of ω^ω for Polish spaces, we therefore lose no generality when considering the Borel and projective hierarchies on ω^ω.

The coanalytic sets enjoy a certain definable choice principle. A *uniformization* of a set $B \subseteq X \times Y$ is a subset $B' \subseteq B$ such that for each $x \in X$, there is a unique $y \in Y$ with $x, y \in B'$, i.e., B' is the graph of a function. The Axiom of Choice implies the existence of uniformizations; the question is, when the uniformization can be assumed to be nicely definable. A pointclass Γ is said to have the *uniformization* property if for every $B \subseteq X \times Y$ with $B \in \Gamma$, B admits a uniformization which is also a member of Γ.

Theorem 1.3.1 (Novikov-Kondô 1938, Addison 1950's)**.** *The class Π^1_1 and the class $\boldsymbol{\Pi}^1_1$ have the uniformization property.*

A proof in the effective case can be found in [Mos80, Theorem 4E.4]; see [Kec95, Theorem 36.14] for the corresponding boldface result.

Another deep result in descriptive set theory is the following; essentially it says that the class of Π^1_1 sets is closed under existential quantification when the quantification is over the class Δ^1_1. For its proof, see [Mil95, Corollary 29.3] or [MW85, Corollary 4.19].

Theorem 1.3.2 (Spector–Gandy)**.** *Let $F \subseteq X \times Y$ be a Π^1_1 set. Then, if $A \subseteq X$ defined by*

$$x \in A \Leftrightarrow \exists y \in \Delta^1_1(x)\ ((x, y) \in F),$$

then A is also Π^1_1.

Constructibility, Absoluteness

Gödel's constructible universe is the proper class defined recursively on the ordinals as follows, using only the axioms of ZF. We let $L_0 = \emptyset$, and for an ordinal α, $L_{\alpha+1}$ is the set of $A \subseteq L_\alpha$ such that A is definable by a formula (in the language of set theory) using parameters from L_α.

$$L = \bigcup_{\alpha \in \mathrm{Ord}} L_\alpha,$$

where Ord denotes the class of ordinals. L satisfies all the axioms of ZFC; in particular, L is a model of the Axiom of Choice, since L admits a global wellorder: one defines a wellordering on each $L_{\alpha+1}$ using the Gödel numbering of formulas with parameters from L_α (see [Kun13, Section II.6] or [Mil95, Theorem 18.1]). The wellorder on L is denoted $\leq_L$. The Axiom of Constructibility is the statement that the universe of all sets V is equal to L and is denoted $V = L$; we say a real r is *constructible* if $r \in L$. $L[A]$ denotes the analogous construction of L where we begin with $L_0[A] = A$.

Given V_0, V_1 transitive models of a sufficient fragment of the ZF axioms and a formula φ in the language of set theory, we say φ is *absolute* between V_0 and V_1 when $V_0 \models \varphi$ if and only if $V_1 \models \varphi$. A formula φ is Σ_1 if it is of the form $\exists\psi$, where ψ contains only bounded quantifiers. The Σ_1 formulas are upwards absolute, meaning if $V_0 \subseteq V_1$ are models of set theory and φ is a Σ_1 formula, then $V_0 \models \varphi$ implies $V_1 \models \varphi$. A formula φ is Π_1 if it is the negation of a Σ_1 formula; such formulas are downwards absolute, meaning that the previous implication is reversed.

The theory of the class Σ^1_2, and hence of the Borel and analytic sets, is in a sense already decided in models of $V = L$, in that it cannot be changed by forcing extensions; this is by *Shoenfield Absoluteness*. Shown by Shoenfield in 1961, this is the statement that for every real r and all inner models $\mathcal{M}, \mathcal{N}$ of the theory ZF plus the axiom of Dependent Choices (DC) which contain r, any $\Sigma^1_2(r)$ formula is satisfied in $\mathcal{M}$ if and only if it is satisfied in $\mathcal{N}$; in other words, Σ^1_2 formulas are absolute between inner models of $\mathsf{ZF} + \mathsf{DC}$.[1] In particular this implies that if $\mathcal{M}$ is a model of $V = L$ and $\mathcal{M}$ satisfies some Σ^1_2 formula φ, then any forcing extension of $\mathcal{M}$ satisfies the same formula φ. More on Shoenfield Absoluteness and its proof can be found, for example, in [BF70], [Jec97, Theorem 98], or [MW85, Lemma 3.17].

Jech remarks that implicit in Shoenfield's proof was a tree representation for Σ^1_2 sets (see [Jec97, Section 40]). Namely, if A is $\Sigma^1_2(r)$ for some real r, then there exists a tree T on $\omega \times \omega_1$ such that $x \in A$ if and only if there exists $\alpha < \omega_1$ such that T_x is ill-founded; moreover $T \in L[r]$. The fundamental facts about the coanalytic sets listed in the previous section also suggest that the Σ^1_2 sets inherit many of the desirable properties of the Borel and analytic sets. The following, for instance, gives a sort of perfect set property for sets in this class.

Theorem 1.3.3 (Mansfield, Solovay; see [Jec97, Theorem 99])**.** *Suppose A is a Σ^1_2 set of reals such that there exists $a \in A$ with $a \notin L$. Then A contains a perfect set of nonconstructible reals.*

In fact, the existence of combinatorially signficant Σ^1_2 sets can have strong implications on the real line, and in particular its cardinality.

Theorem 1.3.4 (Mansfield, 1975; see [Jec03, Theorem 25.39])**.** *If $\leq$ is a Σ^1_2 wellordering of all the reals, then every real is constructible.*

Forcing

We will assume the reader has familiarity with the theory of forcing, especially for the later Chapter 3, though Chapter 2 can be read without advanced technical knowledge. In particular we assume familiarity with the techniques of finite support and countable support iterations; standard references are

[1]An *inner model* is a transitive model containing the class Ord of ordinals and satsifying the theory ZF , or of ZF plus some if not all of the Axiom of Choice. The constructible universe L is the minimal inner model.

[Kun13], [Jec03], or [Hal25]. Our forcing notation largely follows that of [Kun13]. In particular we adopt the convention of forcing downward, meaning that when a condition q is an extension of or is stronger than a condition p, we write $q \leq p$. We denote the ground model by V, and when G is a $\mathbb{P}$-generic filter over V and $\dot{x}$ is a $\mathbb{P}$-name, $\dot{x}^G$ denotes the evaluation of the name $\dot{x}$ in $V[G]$. Names for ground model objects $x \in V$ are denoted by the check notation $\check{x}$. For a formula φ in the forcing language, we write $\Vdash_{\mathbb{P}}$ to mean the maximal element of $\mathbb{P}$, $1_{\mathbb{P}}$, forces φ.

Chapter 2

Coanalytic witnesses

2.1 Introduction

The Borel and analytic sets constitute a class of sets which not only realize Cantor's goal of proving the Continuum Hypothesis, but also satisfy the notions of measurability defined by Lebesgue and Baire; for this reason the Borel and analytic sets are considered well-behaved or nicely definable. The Axiom of Choice (AC) shows that these regularity properties holding for the Borel and analytic sets do not hold in general; for instance, AC is used to construct a non-Lebesgue measurable set.

The constructible sets, these being the members of Gödel's universe L, also realize Cantor's program, as the Generalized Continuum Hypothesis (GCH) holds in L. However L also satisfies the Axiom of Choice, since there exists a global wellordering $\leq_L$ of the entire class L. In fact in Gödel's original paper [Göd39], it was stated (without proof) that in his model L there exist Δ^1_2 instances of a non-Lebesgue measurable set, a set without the property of Baire, and a set without the perfect set property. The class of Lebesgue measurable sets and sets without the property of Baire are closed under complementation, giving that Δ^1_2 is the minimal possible complexity of such examples, however this is not the case for the perfect set property. Nonetheless, the example of this last witness in L can be taken to be coanalytic, and the ability to reduce the complexity from Σ^1_2 to the optimal Π^1_1 follows from the uniformization property for Π^1_1 sets (Theorem 1.3.1); for a proof of this, see [Jec03, Corollary 25.37].

This phenomenon happens more generally with respect to those classes of sets with a particular, nontrivial combinatorial property, for which there can exist no analytic examples of such a set. Usually this is the case for those combinatorial families whose existence follows from an application of the Axiom of Choice, and thus have an inherently non-constructive nature, making them recalcitrant to being nicely definable. In this chapter we overview a certain type of theorem asserting the existence of coanalytic examples of such families.

When an set is obtained by the Axiom of Choice or equivalently a wellordering of the continuum, a recursive construction along the Σ^1_2-good[1] wellorder of the reals in L (i.e., at each step in the recursion one chooses the $\leq_L$ least object satisfying the desired conditions) will produce a projectively definable family; the set thus obtained inherits the definability of the wellorder. To reduce this complexity to the level Π^1_1 is a nontrivial task and, given the strict lower bound of Σ^1_1, yields a family witnessing the first projective level at which the theorems asserting non-analyticity fail, or in a more positive sense, a witness to the existence of a combinatorially interesting collection of provably optimal complexity. By now, there exists fairly substantial knowledge of how to perform these reductions under the assumption $V = L$, and it is natural to ask if this assumption can be weakened, or, which specific consequences of $V = L$ are needed for such a reduction. The main result of this chapter is the following.

Theorem (Theorem 2.4.2)**.** *If there exists a Hausdorff gap (A, B), where A, B are Σ^1_2-definable subsets of $[\omega]^\omega$, then there exists a Hausdorff gap (A', B') such that A', B' are Π^1_1-definable subsets of $[\omega \times \omega]^\omega$.*

Erdős, Kunen, and Mauldin [EKM81] introduced a robust method for this complexity reduction of Σ^1_2 sets in L that are usually obtained by a recursion along a wellordering of the continuum. The method was streamlined by Miller [Mil89] and applied to other distinguishing combinatorial structures. The success of this method is attested to in the literature, and for example has

[1]A Σ^1_2 wellorder $\leq$ is *good* the class Σ^1_2 is closed under quantification bounded by $\leq$ (see, for example, [Mos80, Chapter 5]).

produced (Π^1_1, Π^1_1)-Hausdorff gaps, towers, and ultrafilter bases; see [FKK16, Section 4], [FS22a], and [Sch22], respectively.

The deeper set theoretic reasons for the success of this technique were extracted and compiled into an efficient, formal theorem by Vidnyánszky [Vid14], allowing for shorter proofs of the above Π^1_1 existence results via a sort of black-box condition.

In particular, the above techniques have been applied to those combinatorial structures important for the study of cardinal characteristics. Adrian Mathias was one of the first authors to combine descriptive set theory with the study of combinatorial sets of reals by studying the definability of mad families. He showed in his thesis that no mad family can be analytic [Mat77]; one of the results of Miller [Mil89, Theorem 8.23] gives the existence of a projectively definable mad family under $V = L$, with the projective definition of the family being of the lowest possible complexity not contradicting Mathias's theorem.

But all of these complexity reductions mentioned above required the axiom $V = L$ and thus were limited to models of CH, in which the theory of cardinal characteristics is trivialized. In 2014, Törnquist [T09] provided a concise construction of a Π^1_1 mad family from one which is Σ^1_2, showing that the existence of a Σ^1_2 example is equivalent to the existence of a Π^1_1 example of the same character. Similar such constructions have then been done for the other combinatorial sets under consideration, including maximal independent families and maximal families of orthogonal measures. While such results give shorter and simpler proofs of coanalytic existence theorems following from $V = L$, the principal application of these types of theorems is to obtain optimally definable combinatorial sets in models of $V \neq L$, or more specifically of $\neg\mathsf{CH}$, such as is the context of the study of cardinal characteristics of the continuum.

To show the existence of these coanalytic, maximal combinatorial sets is independent of the cardinality of the continuum, the main obstacle to overcome is that in the introduction of new reals, one breaks the maximality of the ground model combinatorial set; the Π^1_1 definability of the set holds in any forcing extension, by Shoenfield absoluteness. We illustrate this through an example.

Suppose $\mathcal{A}$ is a Π^1_1-definable subset of $[\omega]^\omega$. Suppose also $\mathcal{A}$ is mad, where

$\varphi(\mathcal{A})$ is the formula

$$(\mathcal{A} \text{ is almost disjoint}) \wedge (\forall y \in \omega^\omega \exists x \in \mathcal{A}(|x \cap y| = \aleph_0)).$$

Let $\varphi_1(\mathcal{A})$ be the formula expressing "$\mathcal{A}$ is almost disjoint", and let $\varphi_2(\mathcal{A})$ express that $\mathcal{A}$ is maximal with this property, i.e. so that $\varphi(\mathcal{A})$ above is the conjunction $\varphi_1(\mathcal{A}) \wedge \varphi_2(\mathcal{A})$. By unravelling the statements of $\varphi_1(\mathcal{A})$ and $\varphi_2(\mathcal{A})$ we can compute their descriptive complexity:

$$\varphi_1(\mathcal{A}) \sim \underbrace{\forall x, y \in \mathcal{A}}_{\forall \Pi^1_1} \underbrace{[x = y \vee \exists n \in \omega(x \cap y \subseteq n)]}_{\Sigma^0_1}$$
$$\varphi_2(\mathcal{A}) \sim \underbrace{[\forall y \in [\omega]^\omega}_{\forall} \underbrace{\exists x \in \mathcal{A}}_{\exists \Pi^1_1} \underbrace{(\forall m \in \omega \exists n > m(n \in x \cap y))]}_{\Pi^0_2}$$

Observe $\varphi_2(\mathcal{A})$ is a formula of complexity $\forall\exists\Pi^1_1$, equivalently Π^1_3, and thus is not under the scope of Shoenfield absoluteness. Therefore it is this latter clause in the definition of mad family that will not necessarily persist in forcing extensions. This is the case when some forcing notion $\mathbb{P}$ adds a generic real witnessing non-maximality of $(\mathcal{A})^V$ in $V^{\mathbb{P}}$; such $\mathbb{P}$ is said to *destroy* the mad family $\mathcal{A}$. In the case $(\mathcal{A})^V$ retains its maximality in $V^{\mathbb{P}}$, we say $\mathbb{P}$ *preserves* $\mathcal{A}$, or $\mathcal{A}$ is $\mathbb{P}$-*indestructible*.

An important area in the theory of iterated forcing is the study of indestructibility and preservation. Currently there exist many known techniques for constructing $\mathbb{P}$-indestructible families for given $\mathbb{P}$, and these often are in the form of inductive constructions along a wellordering of the reals and therefore, under $V = L$, can be used to produce Σ^1_2 maximal combinatorial sets which remain both maximal and Σ^1_2-definable in a $\mathbb{P}$-generic extension. Applying the reduction theorems presented in the current chapter gives then a Π^1_1 combinatorial set of the same type.

The purpose of this chapter is to collect such results into a single place so to reveal their similarities and differences, as well as to present the advantages of various coding techniques. We believe this also offers an interesting comparison of the combinatorics particular to each of the families, and a resource to future authors. Indeed, the new Theorem 2.4.2 comes as a consequence of the study of these proofs.

A strategy for deriving coanalytic sets from Σ^1_2 sets

For any uncountable Polish space X, suppose $A \subseteq X$ is a Σ^1_2 set satisfying both some local combinatorial property (such as, all of its elements are pairwise almost disjoint subsets of ω, or A is linearly ordered by some partial order $\prec$ on X) as well as some global combinatorial property (such as, A is maximal with respect to inclusion among other almost disjoint families, or A cannot be propertly extended to another set linearly ordered by $\prec$). Denote by $\varphi_1(A)$ the local property and $\varphi_2(A)$ the global property, as we had done previously when discussing the preservation of a coanalytic mad family.

The fact that A is Σ^1_2 means there exists a Polish space Y and a Π^1_1 set $F \subseteq X \times Y$ such that

$$A = \{x \in X \mid \exists y \ (x, y) \in F\}.$$

Using Kôndo-Addison uniformization (Theorem 1.3.1), we may take F to be the graph of the function; in this way, the real y acts as a unique witness to the membership $x \in A$. The strategy is now to define a continuous function $g\colon X \times Y \to Z$, where Z is some appropriately chosen Polish space, so that $g[F]$ is a set of reals satisfying the same local and global combinatorial properties holding for the original A, i.e. both $\varphi_1(g[F])$ and $\varphi_2(g[F])$ hold. To ensure $\varphi_1(g[F])$, the image $g(x, y)$ should be a real very similar to x. To ensure $\varphi_2(g[F])$, we should be able to translate or project sets in Z to elements of X; thus, a potential "intruder" (an element not belonging to $g[F]$) translates to a potential intruder for A, and the maximality of A is then invoked.

The set $g[F]$ has the following definition:

$$z \in g[F] \Leftrightarrow \underbrace{\exists x, y}_{\exists}[\underbrace{(x, y) \in F}_{\Pi^1_1} \wedge \underbrace{g(x, y) = z}_{\Delta^1_1}]$$

Observe that the only obstruction to the above being a Π^1_1 definition is the initial existential quantification. To overcome this we aim to bound this existential quantification to the set $\Delta^1_1(z)$, and then apply the theorem of Spector-Gandy (Theorem 1.3.2). The function g is therefore defined so that there is a recursive procedure of recovering the sets x, y, given the set $z = g(x, y)$. Since $g[F]$ must also satisfy the same combinatorics as A, recovering $x \in \Delta^1_1(z)$ is fairly straightforward, as z should be a real very similar to x, though now g has put x into a form such that x also codes that y such that $(x, y) \in F$. Then given x, recovered from z, and the fact $y \in \Delta^1_1(x)$, we get $x, y \in \Delta^1_1(z)$ as desired.

Choosing the right function g as well as an appropriate codomain Z relies on the particular combinatorics of the desired family to be constructed. We hope that the presentation of the known strategies for the more standard combinatorial families offers authors a resource for adapting this structure of proof to other particular cases.

Each of the following sections treats a specific case of a combinatorial set of reals, and the sections are structured as follows. We first give the definition of the combinatorial family in question, if applicable we mention connections to cardinal invariants of the continuum, and then we overview the known definability results from the literature. This will often reference *determinacy axioms*; these are statements asserting that some player has a winning strategy in a two-player game played on ω, and have as consequences the imposition of stronger regularity properties for sets of reals than those decided by ZF. More on determinacy can be found in, for example, [Kec95]; we will not be using determinacy in this work other than as illustrative of definable witnesses in other extensions of ZF. We then give a proof that the existence of a Σ^1_2 witness of such a family implies the existence of a Π^1_1 same such family. In all but the last case considered, that of Hausdorff gaps, the proofs are credited to their original authors.

In the following proofs, when we appeal to the definition of Σ^1_2 as it appears above, we will take Y to be one of the sets ω^ω, $[\omega]^\omega$, or 2^ω. Note that there is no loss of generality in doing this, as 2^ω and $[\omega]^\omega$ are easily seen to be Δ^1_1-isomorphic, and ω^ω is Δ^1_1-isomorphic with the dense Π^0_2 set $C := \{x \in 2^\omega \mid x(n) = 1 \text{ for infinitely many } n \in \omega\}$; see [Kec95, 3.12]. As C is a Π^0_2 subset of a Polish space, C is also an uncountable Polish space. Then if A is Σ^1_2 and $F \subseteq X \times \omega^\omega$ is Π^1_1 such that $A = \{x \in X \mid \exists y\ (x, y) \in F\}$, letting $\iota \colon \omega^\omega \to C$ be a Δ^1_1 isomorphism, we have $F' = \{(x, z) \in X \times C \mid (x, \iota^{-1}(z)) \in F\}$ is a Π^1_1 subset of $X \times C$ such that $x \in A$ if and only if there exists $y \in C$ with $(x, y) \in F'$. Therefore when we take our parameter set to be 2^ω or $[\omega]^\omega$ we will use this convention.

2.2 Almost disjointedness

For convenience we recall the definition of mad family.

Definition 2.2.1. A family $\mathcal{A} \subseteq [\omega]^\omega$ is called *almost disjoint (ad)* if for all $x, y \in \mathcal{A}$, $x \cap y$ is finite. An ad family is *maximal almost disjoint (mad)* if $\mathcal{A}$ is not a proper subset of any other almost disjoint family; equivalently, $\mathcal{A}$ is mad if and only if for every $y \in [\omega]^\omega$ there exists $x \in \mathcal{A}$ such that $x \cap y$ is infinite.

We will always refer to mad families which are infinite, since any partition of ω into finite sets is a mad family. The cardinal $\mathfrak{a}$ is defined as:

$$\mathfrak{a} = \min\{|\mathcal{A}| \mid \mathcal{A} \subseteq [\omega]^\omega \text{ is mad}\}.$$

A diagonalization argument shows that $\mathfrak{a} > \aleph_0$. It it is also not too difficult to see that in ZFC there exists a mad family of size $\mathfrak{c}$. Though it is consistent with ZFC that the cardinals $\mathfrak{c}$ and $\mathfrak{a}$ are equal, it is also consistent with ZFC that $\mathfrak{a} < \mathfrak{c}$. An early proof of this consistency was given by Kunen, who showed that assuming CH, one can construct a mad family which is indestructible by any length finite support iteration of Cohen forcing [Kun80, Theorem 2.3]. Thus in an extension over a model of CH by Cohen forcing adding κ-many new reals (for $\kappa \geq \aleph_2$ a regular cardinal), the inequality $\mathfrak{a} = \aleph_1 < \mathfrak{c}$ holds, and $\mathfrak{a} = \aleph_1$ is witnessed by the ground model mad family.

Mad families became an object subject to descriptive set theoretic inquiry beginning with the work of Mathias in 1969 [Mat77], where he showed that no analytic almost disjoint family is maximal. A different proof of this fact is given by Törnquist [Tö18, Theorem 2.1], which relies on the fact that if $\mathcal{A}$ is an infinite analytic almost disjoint family, one can use its tree representation $\mathcal{A} = p([T])$ to produce a countable sequence $\{B_n \mid n \in \omega\}$ of subsets $B_n \subseteq \omega$ such that

- for all $x \in \mathcal{A}$, there exists $n < \omega$ such that $x \subseteq^* \bigcup_{i \leq n} B_i$, yet
- for all $n \in \omega$, the set $\omega \setminus \bigcup_{i \leq n} B_i$ is infinite.

In other words, $\{B_n \mid n \in \omega\}$ generates a proper ideal $\mathcal{I}$ such that $\mathcal{A} \subseteq \mathcal{I}$. The existence of such a collection allows one to construct $y \in [\omega]^\omega$ witnessing non-maximality of $\mathcal{A}$.

In that same paper Törnquist answered a question of Mathias by showing there are no mad families in Solovay's model, this being the canonical model of ZF + Axiom of Dependent Choice (DC) in which every set is Lebesgue measurable, has the property of Baire, and has the perfect set property (see [Sol70]). Moreover each of the following implies the nonexistence of mad families:

- $\mathsf{ZF} + \mathsf{DC}$ + "All sets have the Ramsey property" [ST19];
- $V = L(\mathbb{R})$ + Axiom of Determinacy (AD) [BHST22];
- $\mathsf{ZF} + \mathsf{AD}^+$ (Woodin's technical strengthening of the Axiom of Determinacy) [NN18].

Another result in [NN18] is that under ZF+DC+ Projective Determinacy (PD), there are no projectively definable mad families. From these results it is clear there is an antagonism between determinacy axioms and definable mad families; however it seems to remain open if AD implies there are no mad families. Regarding extensions of ZFC by forcing axioms, we find the more recent work of [FSW21]; the authors show that under the Bounded Proper Forcing Axiom (BPFA) and a particular anti-large cardinal assumption, there exists a Π^1_2-definable mad family. Törnquist [Tö18, Corollary 2.8] proves that Martin's Axiom at $\aleph_1$ ($\mathsf{MA}_{\aleph_1}$) implies there exist no Σ^1_2 mad families.

The axiom of constructibility $V = L$ implies the existence of a projectively definable mad family in the best way possible, as Miller's construction of a Π^1_1 mad family in L shows. Törnquist's Theorem 2.2.2 below weakens the hypothesis of Miller's theorem and is therefore able to be applied in models of $\neg\mathsf{CH}$. The proof is of a more purely combinatorial nature and is the starting point for proofs of this kind.

Theorem 2.2.2 ([T09])**.** *If there exists a Σ^1_2 mad family then there exists a Π^1_1 mad family.*

Proof. Let A be a Σ^1_2 mad family, and let $F \subseteq [\omega]^\omega \times [\omega]^\omega$ be Π^1_1 such that

$$x \in A \Leftrightarrow \exists y\ ((x,y) \in F).$$

By the Kondô-Addison Theorem 1.3.1, we may assume F is the graph of a function. For $x \in [\omega]^\omega$, let $e_x \colon \omega \to x$ be the increasing enumeration of elements of x.

Define functions $g_i \colon [\omega]^\omega \times [\omega]^\omega \to \mathcal{P}(3 \times \omega)$ for $i < 2$, where

$$g_0(x,y) = (\{0\} \times x) \cup (\{2\} \times \{e_x(n) \mid n \in y\}),$$

and

$$g_1(x,y) = (\{1\} \times x) \cup (\{2\} \times \{e_x(n) \mid n \notin y\}).$$

Clearly g_0, g_1 are continuous functions. Define $A' := g_0[F] \cup g_1[F]$; we first show that A' is a mad family of infinite subsets of $3 \times \omega$. That A' is almost disjoint follows immediately from the fact that A is almost disjoint. To show maximality of A', let $z \subseteq 3 \times \omega$ be an arbitrary infinite set which is not a member of A'. Denoting by $p_i(z)$ the projection of z onto the ith coordinate, there must be some some $i < 3$ such that $p_i(z) \in [\omega]^\omega$. Then the maximality of A gives some $x \in A$ such that $p_i(z) \cap x$ is infinite, and so let $y \in [\omega]^\omega$ be such that $(x,y) \in F$. If $i \in \{0,1\}$ then $g_i(x,y) \in A'$ and $\{(i,n) \mid n \in p_i(z) \cap x\} \subseteq g_i(x,y) \cap z$, so since the former is infinite, so is the latter. In the case $p_2(z)$ is infinite, since

$$p_2(g_0(x,y) \cup g_1(x,y)) = e_x[\omega] = x,$$

it must be that one of $z \cap g_0(x,y)$ or $z \cap g_1(x,y)$ is infinite.

Next, we turn to the definability of A'. We have that

$$z \in A' \Leftrightarrow \exists x, y[(x,y) \in F \wedge (g_0(x,y) = z \vee g_1(x,y) = z)].$$

We can bound the existential quantification to the set $\Delta^1_1(z)$ in the following way. Given $z \in A'$, one can recursively recover the x, y witnessing membership of z in the set $g_0[F] \cup g_1[F]$: first one recovers x as $x = p_i(z)$ for $i = 0$ or $i = 1$, and then to x apply the inverse $e_x^{-1} \colon x \to \omega$ to the set $p_2(z)$ to recover y. Since the projection functions and the function e_x^{-1} are continuous, this is a recursive procedure. Therefore,

$$z \in A' \Leftrightarrow \exists x, y \in \Delta^1_1(z)[(x,y) \in F \wedge (g_0(x,y) = z \vee g_1(x,y) = z)],$$

so A' is Π^1_1, by the Spector-Gandy theorem (Theorem 1.3.2). $\square$

The following illustrates how the previous theorem can be applied in models of $\neg\mathsf{CH}$.

Corollary 2.2.3. *It is consistent that $\mathfrak{c} \geq \aleph_2$ and there exists a coanalytic mad family $\mathcal{A}$ of size $\aleph_1$.*

Proof. Beginning in a model of $V = L$, construct a Σ^1_2-definable mad family $\mathcal{A}$ which is indestructible with respect to Cohen forcing.[2] Let $\kappa \geq \omega_2$ be a regular cardinal and let $\mathbb{P}$ be a finite support iteration of Cohen forcing of length κ, and let G be $\mathbb{P}$-generic over V. Then the cardinals of $V[G]$ are the same as those of V since $\mathbb{P}$ is ccc, and in $V[G]$, $\mathcal{A}$ is still a maximal almost disjoint family. By Shoenfield absoluteness, $\mathcal{A}$ is defined by the same Σ^1_2-formula, and therefore $V[G] \models \exists\mathcal{A}(\mathcal{A}$ is mad and $\mathcal{A}$ is Σ^1_2-definable). As $V[G]$ is a model of ZF and thus also of the above theorem, there exists a Π^1_1 mad family in $V[G]$. Since by standard arguments, $(\mathfrak{c} = \kappa)^{V[G]}$, $V[G]$ witnesses the desired consistency, as $(|\mathcal{A}| = \aleph_1)^{V[G]}$. $\square$

In other words, we have shown that then inequality $\mathfrak{a} = \aleph_1 < \mathfrak{c}$ can be realized with a coanalytic witness to the value of $\mathfrak{a}$. Further applications of Törnquist's reduction theorem include:

Theorem 2.2.4 (Brendle, Khosmkii [BK13])**.** *It is consistent that $\mathfrak{c} \geq \aleph_2$ and there exists a coanalytic mad family of size $\mathfrak{c}$.*

The above construction is very specific to mad families and to a specific property of Hechler forcing. The existence of coanalytic witnesses of size $\mathfrak{c} \geq \aleph_2$ for the other combinatorial families considered in subsequent sections is still open.

2.3 Other combinatorial families

2.3.1 Independence

A family $A \subseteq [\omega]^\omega$ is *independent* if for all $k, \ell < \omega$ and distinct $x_0, \ldots, x_k$, $y_0, \ldots, y_\ell \in A$, the set $\bigcap_{i \leq k} x_i \setminus \bigcup_{j \leq \ell} y_j$ is infinite. Such a family is *maximal*

[2] For example, this can be done by performing a recursive construction along the wellordering $\leq_L$ of the constructible reals of the mad family constructed in Proposition 3.2.9 of Chapter 3)

(mif) if it is maximal with respect to inclusion. Equivalently, for all $z \subseteq \omega$, there are $x_0, \ldots, x_k, y_0, \ldots, y_\ell \in A$ such that $z \cap \bigcap_{i \leq k} x_i \setminus \bigcup_{j \leq \ell} y_j$ is finite, or $(\bigcap_{i \leq k} x_i \setminus \bigcup_{j \leq \ell} y_j) \setminus z$ is finite.

It can be checked that if A is a projectively definable mif at level Π^1_n, then the formula expressing maximality of A is of complexity is Π^1_{n+2}.

Maximal independent families give rise to the cardinal

$$\mathfrak{i} = \min\{|\mathcal{A}| \mid \mathcal{A} \subseteq [\omega]^\omega \text{ is maximal independent}\}.$$

Diagonalization arguments show $\mathfrak{i} > \aleph_0$, and an early result of Fitchenholz and Kantorovich [FK34] shows there exists a maximal independent family of size $\mathfrak{c}$.

Miller [Mil89, Theorem 10.28] shows that there are no analytic maximal independent families, and in fact the argument shows that maximal independent families cannot have the Baire property (see also [BFK19, Corollary 3.8]). Therefore there are no mifs in a model of AD, and no projective mifs in Solovay's model or in a model of $\mathsf{ZF}+\ \mathsf{PD}$. In fact [BFK19, Theorem 3.1] shows that in a Cohen-generic extension, there is no projective mif, either.

Miller [Mil89, Theorem 10.31] shows if $V = L$ then there exists a Π^1_1 maximal independent family. Extracting implicit arguments from a construction of Shelah [She92], Brendle, Fischer, and Khomskii [BFK19] show that a Π^1_1 mif is consistent with $\mathfrak{c} = \aleph_2$; this followed from a recursive construction of a Σ^1_2 Sacks-indestructible mif in a model of $V = L$, performing an ω_2-length countable support iteration of Sacks forcing over this model, and then appealing to the following modification of Törnquist's proof:

Theorem 2.3.1 ([BFK19, Theorem 2.1])**.** *If there exists a Σ^1_2 maximal independent family then there exists a Π^1_1 maximal independent family.*

Proof. Let A be a Σ^1_2 maximal independent family, and let $F \subseteq ([\omega]^\omega)^2$ be a Π^1_1 set such that

$$x \in A \Leftrightarrow \exists y \in [\omega]^\omega((x, y) \in F).$$

Again by Π^1_1 uniformization we can assume F is the graph of a function.

Considering the space $\omega \cup 2^{<\omega}$ as a disjoint union, define $g\colon [\omega]^\omega \times [\omega]^\omega \to \mathcal{P}(\omega \cup 2^{<\omega})$ as follows:

$$g(x, y) = x \cup \{\chi_y \restriction n \mid n < \omega\},$$

where $\chi_y \colon \omega \to 2$ denotes the characteristic function of y. Clearly g is a continuous function.

Let $A' := g[F]$, and we claim this is a maximal independent family of subsets of $\omega \cup 2^{<\omega}$. To see A' is independent, let $x_0, \dots, x_n$ and $w_0, \dots, w_k$ be distinct elements of A'. Let $a_i = x_i \cap \omega$ and $b_j = w_j \cap \omega$; then a_i, b_j are distinct elements of A for all $i \leq n$ and $j \leq k$, using the fact that F is a function. Moreover,

$$\bigcap_{i \leq n} a_i \setminus \bigcup_{j \leq k} b_j \subseteq \bigcap_{i \leq n} x_i \setminus \bigcup_{j \leq k} w_j,$$

so since the former is infinite, so is the latter.

To see that A' is maximal, let z be a subset of $\omega \cup 2^{<\omega}$. Then as $z' = z \cap \omega$ is a subset of ω, maximality of A gives $a_0, \dots, a_k, b_0, \dots, b_\ell \in A$ such that $z \cap (\bigcap_{i \leq k} a_i \setminus \bigcup_{j \leq \ell} b_j)$ is finite, or $(\bigcap_{i \leq k} a_i \setminus \bigcup_{j \leq \ell} b_j) \setminus z$ is finite. Suppose without loss of generality it is the former, and let $y_i, t_j \in [\omega]^\omega$ be the unique sets such that $(a_i, y_i), (b_j, t_j) \in F$ for $i \leq k$ and $j \leq \ell$. If

$$z \cap \bigcap_{i \leq k} g(a_i, y_i) \setminus \bigcup_{j \leq \ell} g(b_j, t_j)$$

is not finite, it must be because

$$z \cap \bigcap \{\chi_{y_i} \restriction n\} \setminus \bigcup \{\chi_{t_j} \restriction n\}$$

is infinite, in which case we let a, b be two distinct members of A, also distinct from each a_i, b_j, and let y, t be so that $(a, y), (b, t) \in F$. If $y = t$, then $(g(a, y) \setminus g(b, t)) \cap 2^{<\omega} = \emptyset$, and then

$$z \cap (\bigcap_{i \leq k} g(a_i, y_i) \cap g(a, y)) \setminus (\bigcup_{j \leq \ell} g(b_j, t_j) \cup g(b, t))$$

is finite. In the case $y \neq t$, then $\{\chi_y \restriction n \mid n \in \omega\} \cap \{\chi_t \restriction n \mid n \in \omega\}$ is finite, and therefore the following set is finite:

$$z \cap (\bigcap_{i \leq k} g(a_i, y_i) \cap g(a, y) \cap g(b, t)) \setminus \bigcup_{j \leq \ell} g(b_j, t_j).$$

It remains to show that A' is Π^1_1-definable. We have that for all $z \subseteq \omega \cup 2^{<\omega}$,

$$z \in A' \Leftrightarrow \exists x, y((x, y) \in F \wedge g(x, y) = z),$$

and as before we want to bound the existential quantification to the set $\Delta^1_1(z)$. Indeed, given $z \in g[F]$, first recover x by considering $z \cap \omega$. Next, since the y such that $(x, y) \in F$ is unique, the characteristic function for y must be the unique branch through the tree $T = z \cap 2^{<\omega}$, and the property of being a unique branch can be defined in a Δ^1_1 way as follows:

1. $\forall n(\chi_y \restriction n \cap z \neq \emptyset)$;
2. $\forall n \forall s \in z \cap 2^n(s \neq \chi_y \restriction n \Rightarrow \exists m > n(\forall t \in z \cap 2^m \neg(t \sqsupseteq s)))$.

Therefore

$$z \in A' \Leftrightarrow \exists x, y \in \Delta^1_1(z)[(x, y) \in F \text{ and } g(x, y) = z],$$

which again by the Spector-Gandy theorem shows that A' is Π^1_1. □

Corollary 2.3.2 ([BFK19, Theorem 4.1])**.** *It is consistent that $\mathfrak{c} \geq \aleph_2$ and there exists a coanalytic maximal independent family of size $\aleph_1$.*

2.3.2 Towers

A collection $A \subseteq [\omega]^\omega$ is said to be a *tower* if it is wellordered by the relation $\supseteq^*$, where $y \supseteq^* x$ iff $x \setminus y$ is finite. If A is only linearly ordered by $\supseteq^*$ then we say A is a *linear tower.* A (linear) tower A is *maximal* if it cannot be properly extended by $\supseteq^*$, i.e. there is no $z \in [\omega]^\omega \setminus A$ such that for all $x \in A$, $x \supseteq^* z$. A set z exhibiting this property is called a *pseudointersection* of A.

Maximal towers give rise to the following cardinal characteristic

$$\mathfrak{t} = \min\{|A| \mid A \subseteq [\omega]^\omega \text{ is a maximal tower}\}$$

It follows from a diagonalization argument that no countable tower can be maximal so $\mathfrak{t} \geq \aleph_1$. Fischer and Schilhan [FS22a] showed that no tower can be analytic; this follows from the previous fact and the fact that no tower can contain a perfect subset. Indeed, if $P \subseteq A$ is a nonempty perfect set, then P^2 is a perfect Polish space and $\supseteq^* \cap P^2$ is a wellordering of P, so $\supseteq^* \cap P^2$ cannot have the Baire Property (see [Kec95, Theorem 8.48]). But $\supseteq^*$ is a Borel subset of P^2 and thus has the Baire property, a contradiction. Since all Σ^1_1 sets are either countable or contain a perfect subset, it follows that no maximal tower is analytic. [FS22a, Theorem 2.5] also establishes the stronger fact, that no maximal linear tower can be analytic; as in the above sections, the proof relies on the tree representation for analytic sets to construct a contradiction of maximality.

Moreover, the Mansfield-Solovay theorem (Theorem 1.3.3) states that any $\Sigma^1_2(x)$ set is either countable or contains a perfect set of reals constructible from x; as a consequence, any $\Sigma^1_2(x)$ tower must be a subset of $L[x]$. The existence of a $\Sigma^1_2(x)$ maximal (linear) tower implies $\omega_1 = \omega_1^{L[x]}$ (see [FS22a, Corollary 2.4] and [FS22a, Theorem 5.3]). More generally,

- Under $\mathsf{ZF} + \mathsf{PD}$, there are no projectively definable towers;
- there are no maximal linear towers in Solovay's model [FS22a, Theorem 6.1].

However, $V = L$ implies the existence of a Π^1_1 maximal tower in Gödel's universe L; this is shown in [FS22a, Theorem 3.2] using the method of Miller. Another way to derive this theorem is by using the fact that any Π^1_1 maximal linear tower contains a Π^1_1 maximal tower [FS22a, Theorem 7.1], constructing a Σ^1_2 maximal linear tower in L, and then applying the following theorem:

Theorem 2.3.3 ([FS22a, Theorem 7.2])**.** *If there exists a Σ^1_2 maximal linear tower, then there exists a Π^1_1 maximal linear tower.*

Proof. Let A be a Σ^1_2 maximal linear tower, and let $F \subseteq [\omega]^\omega \times 2^\omega$ be Π^1_1 such that

$$x \in A \Leftrightarrow \exists y \in 2^\omega (x, y) \in F.$$

Again, we suppose F is the graph of a function. Define $g\colon [\omega]^\omega \times 2^\omega \to [\omega \times \omega]^\omega$ by letting

$$g(x, y) = (\{0\} \times x) \cup (\bigcup_{n\in\omega} \{n + 1\} \times (x \setminus x(n + y(n)))).$$

Clearly g is continuous.

Let $A' := g[F]$. We claim this is a maximal linear tower in the space $[\omega \times \omega]^\omega$. To see A' is linearly ordered by $\supseteq^*$, let a, b be distinct elements of A', and let (x, y) and (z, t) be elements of F such that $g(x, y) = a$ and $g(z, t) = b$. Let a_n, b_n denote the nth column of a, b respectively, so note $a_0 = x$ and $b_0 = y$. As x, z are distinct members of A, without loss of generality suppose $x \subseteq^* z$ and $z \not\subseteq^* x$. Then $x \setminus z$ is finite by the former and $z \setminus x$ is infinite by the latter, so there is n sufficiently large so that for all $m \geq n$, $x(m) > z(m)$ and $x(m) \in z$. Then $(0, z(m)) <_{lex} (0, x(m))$ and $(0, x(m)) \in (\{0\} \times z)$ for all $m \geq n$. For the same n as above and all $m \geq n$,

$$x \setminus x(m + y(m)) \subseteq x \setminus x(m) \subseteq z \setminus z(m + t(m)).$$

In other words, for all $m \geq n$ we have $a_{m+1} \subseteq b_{m+1}$. For $k \leq n$, $a_k \subseteq^* b_k$, as $a_k \setminus b_k \subseteq x \setminus z$. Altogether we have that $a \subseteq^* b$.

To see A' is maximal let b be an infinite subset of $\omega \times \omega$ with $b \notin A'$, and suppose towards contradiction that b is a pseudointersection of A'. If there exists $n \in \omega$ such that b_n is infinite, then $b_n \subseteq^* x$ for all $x \in A$, contradicting maximality of A. If no such n exists, let $c := \{\min b_n \mid n \in \omega \wedge b_n \neq \emptyset\}$. If c is finite then so is b, so c must be infinite. We will show that c is a pseudointersection of A. Since $b \subseteq^* a$ for all $a \in A'$, there is n sufficiently large so that for all $m \geq n$, if $b_m \neq \emptyset$, then $\min b_m \in a_m$. Letting $x, y \in F$ be such that $g(x, y) = a$, we have that for all $m \geq n$, $\min b_m \in x$, implying $c \subseteq^* x$, contradicting the maximality of A.

It remains to show the Π^1_1-definability of A'. Given such a z, since $z_0 = x$ we have $x \in \Delta^1_1(z)$. Define $y \in 2^\omega$ as follows.

$$y(n) = \begin{cases} 0 & \text{if } \min z_{n+1} = x(n+1); \\ 1 & \text{if } \min z_{n+1} = x(n+2). \end{cases}$$

Therefore $y \in \Delta^1_1(x, z) = \Delta^1_1(z)$ and

$$z \in A' \Leftrightarrow \exists (x, y) \in \Delta^1_1(z)[(x, y) \in F \wedge g(x, y) = z],$$

showing that A' is Π^1_1.

$\square$

2.3.3 Maximal eventually different families

For functions $f, g \in \omega^\omega$, we say f and g are *eventually different* if there exists $n < \omega$ such that for all $m \geq n$, $f(m) = g(m)$. A family $A \subseteq \omega^\omega$ is *eventually different* if it consists of pairwise eventually different functions. Such a family A is *maximal (med)* if for any $g \in \omega^\omega$, there is $f \in A$ such that the set $\{n \in \omega \mid f(n) = g(n)\}$ is infinite.

Maximal eventually different families give rise to the cardinal invariant

$$\mathfrak{a}_e = \min\{|A| \mid A \subseteq \omega^\omega \text{ is med}\}.$$

Unlike other families considered thus far, there exist Borel and thus analytic med families. The Borel construction was given by Horowitz and Shelah [HS24], and recently Schrittesser [Sch17] has shown the existence of a closed (Π^1_0) such family. However these Borel med families must always be of size $\mathfrak{c}$, since any Borel set is either countable or has size $\mathfrak{c}$. A standard diagonalization argument shows $\aleph_1 \leq \mathfrak{a}_e$. Therefore it is nontrivial to show that the existence of a coanalytic maximal eventually different family of size $\aleph_1$ is consistent with $\aleph_2 \leq \mathfrak{c}$; this is achieved by Fischer and Schrittesser in 2018 [FS21a].

A strengthening of the notion of maximality for a combinatorial set of reals will typically yield families which are indestructible by some forcing notion which adds new reals, and these stronger combinatorial objects cannot be analytic; see, for instance, the notion of a tight eventually different family introduced by Fischer and Switzer [FS21b] and the arguments therein. In the context of almost disjoint families we find the notion of ω-*mad* families introduced by Malykhin already in 1989 [Mal89]. Kurilić [Kur01] studied topological properties of such objects and showed they have a close connection with Cohen-indestructible mad families. Under the name of *tight mad families*, the notion reappeared in work of Hrušák and Ferreira [HGF03], and Guzman, Hrušák, Tellez [GHT20]; these objects will become particularly important for Chapter 3. In 2008, Kastermans, Steprāns, and Zhang [KSZ08] introduce the notion of *strongly* maximal eventually different families, a strengthening of maximality for eventually different families parallel to how ω-mad strengthens the notion of maximality for almost disjoint families.

To define "strongly med", first say that $h \in \omega^\omega$ is *finitely covered* by an eventually different family $A \subseteq \omega^\omega$ if there is a nonempty finite $C \subseteq A$ such that for all but finitely many $n \in \omega$, $h(n) = f(n)$ for some $h \in C$; in other words, $\operatorname{graph}(f) \subseteq^* \bigcup_{h \in C} \operatorname{graph}(h)$. The *ideal generated by A* is the collection

$$\mathcal{I}(A) := \{h \in \omega^\omega \mid h \text{ is finitely covered by } A\}.$$

Let $\mathcal{I}(\mathcal{A})^+$ denote the set $\omega^\omega \setminus \mathcal{I}(\mathcal{A})$.

Definition 2.3.4. An eventually different family $A \subseteq \omega^\omega$ is said to be *strongly maximal (strongly med)* if for any countable collection $H \subseteq \mathcal{I}(A)^+$, there exists a single $f \in A$ such that for all $h \in H$, $f(n) = h(n)$ for all but finitely many $n \in \omega$.

In [Rag09], strongly med families are shown to be $\mathbb{P}$-indestructible, when $\mathbb{P}$ is Sacks, Miller, or more generally any forcing which does not make the ground model reals meager. He moreover shows that the notions of med and strongly med can be separated, and gives a hierarchy of combinatorial strengthenings of maximality for eventually different families, as well as conditions under which the hierarchy is proper.

A proof that strongly med families cannot be analytic can be found in [KSZ08, Theorem 2.1]. The strategy to prove this fact is to use the tree representation of A so to find $H \subseteq \mathcal{I}(A)^+$ witnessing A cannot be strongly maximal. Furthermore, the Cohen-indestructibility of these strongly maximal families can be used to show that such families cannot contain perfect sets [Rag09, Corollary 37, Corollary 38]; see also [FS21b, Section 3]. It follows by the Mansfield-Solovay theorem that any $\Sigma^1_2(x)$ strongly maximal eventually different family is a subset of $L[x]$.

Miller's method has been used to obtain a Π^1_1 strongly med family under $V = L$; see [KSZ08, Theorem 3.1]. The theorem was extended to models of $\neg\mathsf{CH}$ by Fischer and Schrittesser [FS21a], specifically by constructing a Σ^1_2 Sacks-indestructible med family in L and appealing to the following.

Theorem 2.3.5 ([FS21a, Theorem 8])**.** *If there exists a Σ^1_2 maximal eventually different family which is $\mathbb{P}$-indestructible for some forcing notion $\mathbb{P}$, then there exists a Π^1_1 maximal eventually different family which is also $\mathbb{P}$-indestructible.*

Proof. Let $A \subseteq \omega^\omega$ be a Σ^1_2 maximal eventually different family, and let $F \subseteq \omega^\omega \times \omega^\omega$ be Π^1_1 such that

$$f \in A \Leftrightarrow \exists y \ (f, y) \in F.$$

Again, we assume F is the graph of a function. First, fix a recursive bijection

$$c\colon \bigcup_{n<\omega} \omega^n \times \omega^n \to \omega.$$

For $i < 2$, let $\pi_i\colon \omega \to \bigcup_{n<\omega} \omega^n \times \omega^n$ be the coordinate maps, i.e. $\pi_i(c(s_0, s_1)) = s_i$ for all $s_0, s_1 \in \bigcup_{n<\omega} \omega^n \times \omega^n$. Next define a continuous function $g\colon \omega^\omega \times \omega^\omega \to \omega^\omega$ by letting

$$g(f, y)(n) = \begin{cases} f(\frac{n}{2}) & \text{if } n \text{ even;} \\ c(f \restriction n, y \restriction n) & \text{otherwise.} \end{cases}$$

Let $A' := g[F]$. We have that A' is an eventually different family, since for n even this follows precisely by the fact A is eventually different, and for n odd we use both the eventual difference of A and the fact c is a bijection. To see A' is maximal, let $h \in \omega^\omega$ be arbitrary. Then consider $h' \in \omega^\omega$, where $h'(n) := h(2n)$. By maximality of A, there is $f \in A$ such that $\{n \in \omega \mid f(n) = h'(n)\}$ is infinite, and therefore so is the set $\{n \in \omega \mid g(f, y)(2n) = h(2n)\}$. Therefore A' is indeed a med family of functions.

We will show A' has the following Π^1_1 definition.

$$f \in A' \Leftrightarrow \exists (h, y) \in \Delta^1_1(f)[(h, y) \in F \text{ and } f = g(h, y)].$$

For $i < 2$ and $n \in \omega$, let $p_i\colon \omega^n \times \omega^n \to \omega^n$ be the projection onto the ith coordinate. Given any $f \in \omega^\omega$, define $h \in \omega^\omega$ to be $h(n) = f(2n)$, and if it is the case that for all $i < 2$ and $n, m \in \omega$ with $n < m$, we have $p_i(c^{-1}(f(2n+1)) \subseteq p_i(c^{-1}(f(2m+1))$, then let $y \in \omega^\omega$ be defined as $y = \bigcup_{n\in\omega} p_1(c^{-1}(f(2n+1)))$. Then h, y are constructible from f, "$((h, y), f) \in \mathrm{graph}(g)$" is Δ^1_1, and "$(h, y) \in F$" is Π^1_1.

Lastly we verify that A' is $\mathbb{P}$-indestructible, assuming that A is. Let G be a $\mathbb{P}$-generic filter over V, and denote by F^* the set of $(f, y) \in V[G]$ such that $V[G] \models \varphi(f, y)$, where φ is the Π^1_1 formula defining the functional relation F. By Shoenfield absoluteness we have that the Σ^1_2 set

$$A^* = \{f \in V[G] \mid \exists! y (f, y) \in F^*\}$$

is an eventually different family containing the original family A. The assumption that A remains maximal in $V[G]$ implies $A = A^*$, and so also

$F = F^*$. Moreover, by absoluteness, F remains a Π^1_1 functional relation in $L[G]$, and therefore

$$V[G] \models f \in A' \Leftrightarrow \exists h, y \in \Delta^1_1(f)[(h,y) \in F \text{ and } g(h,y) = f].$$

So the above argument showing the maximality of A' above can be run in $V[G]$. $\square$

2.3.4 Maximal orthogonal families

Let X be a Polish space, let $\mathcal{B}(X)$ be its associated Borel σ-algebra, and let $P(X)$ be the set of Borel probability measures on X, that is, the set of functions $\mu\colon \mathcal{B}(X) \to [0,1]$ such that $\mu(\emptyset) = 0$ and $\mu(\bigcup_{n\in\mathbb{N}} B_n) = \sum_{n\in\mathbb{N}} \mu(B_n)$ for pairwise disjoint collections $B_n \subseteq \mathcal{B}(X)$. For $\mu, \nu \in P(X)$, we say μ is *absolutely continuous with respect to* ν, denoted $\nu \ll \mu$, if $\nu(B) = 0$ implies $\mu(B) = 0$, and are *equivalent* if $\nu \ll \mu$ and $\mu \ll \nu$. We say μ, ν are *orthogonal*, written $\mu \perp \nu$, if there is $B \in \mathcal{B}(X)$ such that $\nu(B) = \mu(X \setminus B) = 0$. Equivalently, $\mu \perp \nu$ if and only if there does not exist $\eta \in P(X)$ such that $\eta \ll \mu$ and $\eta \ll \nu$. We also denote by $P_c(X)$ the set of $\mu \in P(X)$ which are non-atomic, meaning $\mu(\{x\}) = 0$ for all $x \in X$, and by $P_d(X)$ the set of Dirac point measures. Note that $P_c(X)$ and $P_d(X)$ are mutually orthogonal in the sense that for all $\mu \in P_c(X)$ and all $\delta \in P_d(X)$, $\mu \perp \delta$.

A family $A \subseteq P(X)$ is an *orthogonal family* if for all $\mu, \nu \in A$, $\mu \perp \nu$. An orthogonal family A is *maximal (mof)* if it is maximal with respect to inclusion, or equivalently, for all $\nu \in P(X) \setminus A$, there is $\mu \in A$ and $\eta \in P(X)$ such that $\eta \ll \mu$ and $\eta \ll \nu$.

There is a way to consider $P(X)$ a recursively represented Polish space, allowing one to ask about the projective definability of $A \subseteq P(X)$. Press and Rataj [PR85] originally showed in 1985 that if A is an analytic family of orthogonal measures, then it cannot be maximal; Kechris and Sofrondis gave another proof of this fact in 2000 [KS01], using Hjorth's theory of turbulence. A more recent proof is given in 2024, by Mejak [Mej24].

Unlike the families considered in previous sections, however, Σ^1_2 maximal sets of orthogonal measures are never indestructible with respect to Cohen or random forcing [FT10, Proposition 4.2]. This uses that Cohen forcing implies all Δ^1_2 sets have the property of Baire. Moreover any mof is of size $\mathfrak{c}$; this is clear for $A \subseteq P_d(X)$, and an application of Mycielski's theorem in [FT10, Proposition 4.1] shows that this holds for $A \subseteq P_c(X)$. [Mej24] shows that PD precludes the existence of projectively definable mofs, while AD implies there are no such families. However, coanalytic mofs exist under $V = L$ by Fischer and Törnquist [FT10, Theorem 3.1]; implicit in their construction is the result below.

Theorem 2.3.6 ([FT10])**.** *If there exists a Σ^1_2 maximal set of orthogonal measures, then there exists a Π^1_1 maximal set of orthogonal measures.*

Proof. Let $A \subseteq P(2^\omega)$ be a Σ^1_2 maximal family of orthogonal measures. Since any $\mu \in A$ is a finite measure, μ can have at most countably many atoms $M \subseteq 2^\omega$, and we may define an atomless measure $\mu' \ll \mu$ by letting $\mu'(B) = \max(\mu(B \setminus M), 0)$. Therefore we may assume without loss of generality that $A \subseteq P_c(2^\omega)$. Take $F \subseteq P_c(2^\omega) \times 2^\omega$ to be the Π^1_1 graph of a function such that

$$\mu \in A \Leftrightarrow \exists y\ (\mu, y) \in F.$$

The idea is to define a continuous function

$$g\colon P_c(2^\omega) \times 2^\omega \to P_c(2^\omega)$$

so that $g(\mu, y)$ will be a measure equivalent to μ, which additionally codes the real y, implying $\mu, y \in \Delta^1_1(g(\mu, y))$. Instead of working with the space $P(2^\omega)$ we consider the set

$$p(2^\omega) := \{f\colon 2^{<\omega} \to [0,1] \mid f(\emptyset) = 1 \wedge \forall s(f(s) = f(s^\frown 0) + f(s^\frown 1))\}.$$

By [Kec95, 17.7], there is a unique probability measure μ on 2^ω with $f(s) = \mu(N_s)$, and conversely every Borel probability measure on 2^ω arises in this way.[3] For $f \in p(2^\omega)$, denote by μ_f the corresponding measure; we think of $p(2^\omega)$ as the codes for elements of $P(2^\omega)$. Let $p_c(2^\omega)$ be the set of those $f \in p(2^\omega)$ such that μ_f is nonatomic. That $p(2^\omega)$ can be equipped with a Polish topology and moreover is recursively presented is detailed in [FT10, Section

[3] Here, $N_s \subseteq 2^\omega$ denotes the basic open set for the Polish topology on 2^ω, that is, $N_s = \{x \in 2^\omega \mid s \subseteq x\}$.

2].[4] Moreover, [FT10, Lemma 2.2] shows $\ll$, $\perp$, and measure equivalence are Π^0_2-definable relations with this representation.

We will define a continuous function g from $p(2^\omega) \times 2^\omega$ to $p(2^\omega)$. For a measure $\mu \in P(2^\omega)$ and $s \in 2^{<\omega}$, let $t(s,\mu)$ denote the lexicographically minimal $t \in 2^{<\omega}$ extending s and such that $\mu(N_{s^\frown 0}) > 0$ and $\mu(N_{s^\frown 1}) > 0$, when such t exists. Otherwise we let $t(s,\mu) = \emptyset$. By induction on $n \in \omega$ we define a sequence $(t^f_n)_{n\in\omega} \subseteq 2^{<\omega}$. Let $t^f_0 := \emptyset$, and supposing t^f_n defined, let $t^f_{n+1} := t(t^f_n{}^\frown 0, \mu_f)$. Note that when μ_f is nonatomic, $\mathrm{length}(t^f_n) < \mathrm{length}(t^f_m)$ for all $n < m$.

Next define inductively $g(f,y) \in p(2^\omega)$ on the length of $s \in 2^{<\omega}$. For $s \in 2^n$, if $s = t^f_k$ for some k, let

$$g(f,y)(s^\frown 0) = \begin{cases} \frac{2}{3} g(f,y)(s) & \text{if } y(k) = 1; \\ \frac{1}{3} g(f,y)(s) & \text{if } y(k) = 0. \end{cases}$$

$$g(f,y)(s^\frown 1) = \begin{cases} \frac{1}{3} g(f,y)(s) & \text{if } y(k) = 1; \\ \frac{2}{3} g(f,y)(s) & \text{if } y(k) = 0. \end{cases}$$

In all other cases, let

$$g(f,y)(s^\frown i) = \begin{cases} 0 & \text{if } f(s) = 0; \\ \frac{g(f,y)(s)}{f(s)} f(s^\frown i) & \text{otherwise.} \end{cases}$$

Note that g is continuous. We will show that the Borel probability measure $\mu_{g(f,y)}$ is equivalent to μ_f. First, define $\Theta\colon 2^{<\omega} \to [0,1]$ by letting

$$\Theta(s) = \frac{g(f,y)(s)}{f(s)}$$

when $f(s) \neq 0$, and let $\Theta(s) = 0$ otherwise. Let $\{s_i \mid i \in \omega\}$ enumerate the set

$$\{s \in 2^{<\omega} \mid \forall n(s \neq t^f_n \text{ and } \exists m(s \restriction (\mathrm{lh}(s) - 1) \sqsubseteq t^f_m)\}.$$

[4] The reason for working rather with the space $p(2^\omega)$ is because any $f \in p(2^\omega)$ can be found at some countable level L_α of L, while this is not necessarily the case for μ_f. This is for the purposes of the authors; see [FT10, Remark 2.1].

Then for any Borel set $B \subseteq 2^\omega$,

$$\mu_{g(f,y)}(B) = \sum_{i=0}^{\infty} \Theta(s_i)\mu_f(B \cap N_{s_i}).$$

Therefore $\mu_f(B) = 0$ if and only if $f(s_i) = 0$, if and only if $\Theta(s_i) = 0$, if and only if $\mu_{g(f,y)}(B) = 0$. Since μ_f and $\mu_{g(f,y)}$ have the same null sets, they are measure equivalent.

Fix a recursive pairing function

$$\langle \cdot, \cdot \rangle \colon [0,1]^{(2^{<\omega})} \times 2^\omega \to 2^\omega,$$

and for $i < 2$ let $z \mapsto (z)_i$ be the maps such that $z = \langle (z)_0, (z)_1 \rangle$ for all $z \in 2^\omega$.

We define $A' \subseteq P_c(2^\omega)$ to be the set

$$A' = \{\mu_{g(f,\langle f,y \rangle)} \mid (\mu_f, y) \in F\},$$

and as $\mu_{g(f,\langle f,y \rangle)}$ and μ_f are measure equivalent, A' is clearly a maximal family of atomless Borel probability measures on 2^ω. By adding to A' the Dirac point mass measures (a closed subset of $P(2^\omega)$), we obtain a maximal orthogonal family in $P(2^\omega)$.

Moreover, we claim that A' has the following Π^1_1 definition:

$$\mu \in A' \Leftrightarrow \exists (f,y) \in \Delta^1_1(\mu)[(\mu_f, y) \in F \text{ and } \mu_{g(f,\langle f,y \rangle)} = \mu].$$

Indeed, given $\mu \in P_c(2^\omega)$, there is a recursive relation $R \subseteq p_c(2^\omega) \times 2^\omega$ expressing that the $g \in p_c(2^\omega)$ such that $\mu = \mu_g$ codes a real $z \in 2^\omega$. Specifically, let

$$\begin{aligned} R(g,z) \Leftrightarrow & \forall n \in \omega [z(n) = 0 \Leftrightarrow (g(t^g_n{}^\frown 0) = \frac{1}{3} g(t^g_n) \wedge g(t^g_n{}^\frown 1) = \frac{2}{3} g(t^g_n)) \\ & \wedge z(n) = 1 \Leftrightarrow (g(t^g_n{}^\frown 0) = \frac{2}{3} g(t^g_n) \wedge g(t^g_n{}^\frown 1) = \frac{1}{3} g(t^g_n))]. \end{aligned}$$

So if $\mu = \mu_g$, one checks if there is a $z \in 2^\omega$ such that $R(g,z)$, and in this case one recovers $f \in p_c(2^\omega)$ and $y \in 2^\omega$ such that $z = \langle f, y \rangle$. Therefore $(f,y) \in \Delta^1_1(\mu)$, finishing the proof. $\square$

A shorter proof of the above was given later by Schrittesser and Törnquist in [ST18, Lemma 4.2], and we include this below for completeness.

Proof. (of Theorem 2.3.6)

Let $A \subseteq P(2^\omega)$ be a maximal orthogonal family, and again we may suppose $A \subseteq P_c(2^\omega)$. Fix $F \subseteq P_c(2^\omega) \times 2^\omega$ such that $\mu \in A$ if and only if there is y such that $(\mu, y) \in F$. Again we assume F is the graph of a function.

For $\mu \in P_c(2^\omega)$, let y_μ be the left-most branch of $\{t \in 2^{<\omega} \mid \mu(N_t) > 0\}$. As μ is atomless we have

$$C = \{n \in \omega \mid \mu(N_{y_\mu \restriction n^\frown(0)}) > 0 \text{ and } \mu(N_{y_\mu \restriction n^\frown(1)}) > 0\}$$

is infinite. Let $C(n)$ denote the nth element of C.

Define $h\colon P_c(2^\omega) \to 2^\omega$ by letting

$$h(\mu)(i) = \begin{cases} 0 & \text{if } \mu(N_{y_\mu \restriction C(i)^\frown(0)}) \geq \mu(N_{y_\mu \restriction C(i)^\frown(1)}); \\ 1 & \text{otherwise.} \end{cases}$$

The previous proof showed that given any recursive bijection $\phi\colon 2^\omega \to P_c(2^\omega)$ there exists a continuous function $G\colon P_c(2^\omega) \times 2^\omega \to P_c(2^\omega)$ such that $G(\mu, y)$ is measure equivalent to μ and $y \in \Delta^1_1(G(\mu, y))$ for all $(\mu, y) \in P_c(2^\omega) \times 2^\omega$. Fixing such a ϕ and G, we have $h(G(\mu, y)) = y$.

Let $\langle \cdot, \cdot \rangle\colon 2^\omega \times 2^\omega \to 2^\omega$ be a recursive bijection, and for $i < 2$ and $z \in 2^\omega$ let $z \mapsto (z)_i$ be the coordinate maps, so $z = \langle (z)_0, (z)_1 \rangle$.

Let $A' = \{G(\mu, \langle \phi^{-1}(\mu), y \rangle) \mid (\mu, y) \in F\}$. Then since all measures in A' are measure equivalent to those in A, we have that A' is also a maximal orthogonal family in $P_c(2^\omega)$. Moreover, A' is Π^1_1 since for all $\mu \in P_c(2^\omega)$,

$$\begin{aligned} \mu \in A' \Leftrightarrow \forall z, \nu, y[(z = h(\mu) \wedge \nu = \phi((z)_0) \wedge y = (z)_1) \\ \Rightarrow (\mu = G(\nu, z) \wedge (\nu, y) \in F)], \end{aligned}$$

This is clearly a Π^1_1 formula. $\square$

While there is no hope of preserving a mof in an iteration of length $\geq \omega_2$, it has been of interest to ask in which models of $V \neq L$ there exists a Π^1_1, or more generally projective, mof. [FT10, Proposition 4.2] shows that if there exists a Cohen real x over L, there is no coanalytic mof in $L[x]$, and this result was shown in the case of Sacks, Miller, and Mathias forcing in [ST18]. On the other hand we remark that it is consistent that $\mathfrak{c} \geq \aleph_2$ and there exists a Π^1_2 mof; see [FFT12].

2.3.5 Ultrafilters

A collection $F \subseteq \mathcal{P}(X)$ is a *filter* on a set X if $X \in F$, $\emptyset \notin F$, and F is closed under finite intersection and superset. A filter F is an *ultrafilter* if it is maximal in the sense that for all $x \subseteq X$, either $x \in F$ or $X \setminus x \in F$. That any filter is contained in an ultrafilter is a traditional application of Zorn's lemma, though in general the statement that any filter can be extended to an ultrafilter is strictly weaker than the Axiom of Choice. If U is an ultrafilter on X, $A \subseteq \mathcal{P}(X)$ is a *base* for U if A is closed under finite intersection and for every $x \in U$, there is $y \in A$ such that $y \subseteq x$. U is *nonprincipal* if it is not generated by a single element; in the following we will only consider nonprincipal ultrafilters.

Considering ultrafilters on ω, Sierpiński showed that ultrafilters cannot be Lebesgue measurable; this follows from the fact that any filter on ω is either Lebesgue null or nonmeasurable. Under AD, there do not exist ultrafilters on ω. Since an analytic ultrafilter base would give an analytic ultrafilter, there do not exist analytic ultrafilter bases.

The next best definability of an ultrafilter base is Π^1_1; the ultrafilter thus generated will be Σ^1_2 definable and by maximality it is Δ^1_2 definable. Schilhan showed that this minimal complexity is possible, namely ,$V = L$ implies there exists a Π^1_1 base for an ultrafilter.

Of particular interest to many areas of study are ultrafilters with stronger combinatorial properties. An ultrafilter U is called a *P-point* if for any countable collection $A \subseteq U$ admits a pseudointersection y such that $y \in U$ (recall the notion of pseudointersection was introduced in Section 2.3.2). An ultrafilter U is called a *Q-point* if for any countable partition $\omega = \bigcup_{n\in\omega} P_n$ with each P_n finite, there is $x \in U$ such that for all $n \in \omega$, $x \cap P_n$ contains at most one element. U is *Ramsey* if U is both a Q-point and a P-point. Schilhan [Sch22, Theorem 1.1] shows there exists a coanalytic base for a P-point in L, and [Sch22, Theorem 1.2] establishes the analogous result for Q-points; this is in constrast with [Sch22, Theorem 1.3], which shows there cannot be a Π^1_1

base for a Ramsey ultrafilter.

For ultrafilters in general, Schilhan shows that there is equivalence between the existence of a Δ^1_{n+1} ultrafilter and the existence of a Π^1_n ultrafilter base, for all $n \in \omega$. The fact that this complexity reduction holds for higher pointclasses, as well as the format of the proof we will see below, shows that this theorem diverges from our general schema encountered thus far.

Theorem 2.3.7 ([Sch22, Theorem 1.4])**.** *If there exists a Δ^1_2 ultrafilter on ω, then there exists an ultrafilter on $\omega \times \omega$ with a Π^1_1 base.*

Proof. Let $U \subseteq \mathcal{P}(\omega)$ be a Δ^1_2 ultrafilter, and let $F \subseteq \mathcal{P}(\omega) \times 2^\omega$ be Π^1_1 such that

$$x \in U \leftrightarrow \exists w \in 2^\omega \ (x, w) \in F.$$

The *Fubini product* of U is the collection of subsets of $\omega \times \omega$ defined by

$$U \otimes U := \{y \in \mathcal{P}(\omega \times \omega) \mid \{n \in \omega \mid \{m \in \omega \mid (n, m) \in y\} \in U\} \in U\}.$$

One can check that if U is an ultrafilter on ω then $U \otimes U$ is an ultrafilter on $\omega \times \omega$, and moreover if U is Σ^1_2 definable then so is $U \otimes U$. We will construct a Π^1_1 base $A \subseteq U \otimes U$.

Fix a recursive function $r \colon \omega \times 2^\omega \to 2^\omega$ such that for any sequence $(w_n)_{n<\omega} \subseteq 2^\omega$, there exists $z \in 2^\omega$ which is not eventually constant such that $r(n, z) = w_n$. For $y \in [\omega \times \omega]^\omega$, let $y_n = \{m \in \omega \mid (n, m) \in y\}$ denote the nth vertical section of y, and let $z(y)$ denote the set of $n \in \omega$ such that $y_n \neq \emptyset$. In the case $z(y)$ is infinite, let y^n to denote the nth element of $z(y)$.

Define a function $O \colon [\omega \times \omega]^\omega \to 2^\omega$ by letting

$$O(y)(n) = \begin{cases} 0 & \text{if } |z(y)| < \omega; \\ 0 & \text{if } \min y^n \geq \min y^{n+1}; \\ 1 & \text{if } \min y^n < \min y^{n+1}. \end{cases}$$

This is clearly a recursive function. The idea is to code a set $y \in [\omega \times \omega]^\omega$ into an element of 2^ω via the sequence of integers determined by $(\min y^n)_{n\in\omega}$. We define the Π^1_1 base for $U \otimes U$ to be the subset $X \subseteq [\omega \times \omega]^\omega$, where

$$\begin{aligned} y \in X \Leftrightarrow {} & |z(y)| = \omega \wedge (z(y), r(0, O(y))) \in F \\ & \wedge \forall n \in \omega \exists s \in [\omega]^{<\omega}[(s \cup y^n, r(n+1, O(y))) \in F]. \end{aligned}$$

In other words, we let $y \in X$ if and only if: y has infinitely many nonempty sections; the real coding y, $O(y)$, codes via the function r a countable sequence $(w_n)_{n\in\omega}$ such that $(z(y), w_0) \in F$ (so $z(y) \in U$); and for some finite set $s \in [\omega]^{<\omega}$, $(s \cup y^n, w_{n+1}) \in F$, (so $s \cup y^n$ and hence $y^n \in U$). This shows that $X \subseteq U \otimes U$.

Moreover we can show that X is indeed a base for $U \otimes U$: given any $x \in U \otimes U$, let $\tilde{y} = \bigcup_{n\in\omega}\{\{n\} \times x_n \mid x_n \in U\}$, where x_n is the nth column of x. Then $z(\tilde{y}) = \{n \in \omega \mid x_n \in U\}$. As $x \in U \otimes U$, $z(\tilde{y}) \in U$, so there is $w_0 \in 2^\omega$ such that $(z(\tilde{y}), w_0) \in F$. Similarly for $n \in z(\tilde{y})$, let w_{n+1} be such that $(x_n, w_{n+1}) \in F$. Then there exists $w \in 2^\omega$ which is not eventually constant such that for each n, $r(n, w) = w_n$. Let $e_y \colon \omega \to z(\tilde{y})$ be the increasing enumeration of the vertical sections of $\tilde{y}$ which are in U. Recursively construct a sequence of integers $(m_n)_{n\in\omega}$ with $m_n \in (\tilde{y})_{e_y(n)}$ coding w in the sense that we let $m_n < m_{n+1}$ iff $w(n) = 1$. Explicitly, whenever $w(n) = 1$, since $(\tilde{y})_{e_y(n+1)} \in U$, there is some $m_{n+1} \in (\tilde{y})_{e_y(n+1)}$ with $m_{n+1} > m_n$. Whenever there is some finite block of 0s $w(n+1), \dots, w(n+k)$, there is

$$m_{n+1} = \cdots = m_{n+k} \in \bigcap\{(\tilde{y})_{e_y(i)} \mid n < i \leq n+k\},$$

as U is closed under finite intersections. Let $y = \bigcup_{n\in\omega}\{\{e_y(n)\} \times ((\tilde{y})_{e_y(n)} \setminus m_n)\}$. Then for all $n \in \omega$, $y_{e_y(n)} \subseteq x_k$ for some $k \geq n$, so therefore $y \subseteq x$. We check that y satisfies the clauses in the definition of X. First, $z(y) = z(\tilde{y}) = \{n \in \omega \mid x_n \in U\}$ is infinite; $O(y) = w$ codes via r the sequence $(w_n)_{n\in\omega}$ and $(z(y) = z(\tilde{y}), w_0) \in F$; lastly there is some finite $s \subset \omega$, namely $((\tilde{y})_{e_y(n)}) \cap m_n$, such that $(s \cup y^n, r(n+1, w) = w_n) \in F$. Therefore $y \in X$, showing X is a base for $U \otimes U$. $\square$

2.4 Hausdorff gaps

For $A, B \subseteq [\omega]^\omega$, the pair (A, B) is called a *pre-gap* (in $(\mathcal{P}(\omega), \subseteq^*)$) if both A, B are wellordered by $\subseteq^*$, and for all $x \in A$ and $y \in B$, $x \cap y$ is finite. Such a pair is a *gap* if there is no $z \in [\omega]^\omega$ such that both $x \setminus z$ and $y \cap z$ are finite for all $x \in A$ and $y \in B$. Such a z is said to *separate* or *interpolate* (A, B). If (A, B) is a gap with $|A| = \kappa$ and $|B| = \lambda$, then (A, B) is called a (κ, λ)-gap. If (A, B) is a (κ, λ)-pre-gap, and $\kappa, \lambda < \omega_1$, then they can be separated. Let

$A = \{a_\alpha \mid \alpha < \omega_1\}$ and $B = \{b_\alpha \mid \alpha < \omega_1\}$ be an (ω_1, ω_1)-gap. We say (A, B) is a *Hausdorff gap* if the following condition is satisfied:

$$\forall \alpha < \omega_1 \forall k < \omega \ \{\gamma < \alpha \mid a_\alpha \cap b_\gamma \subseteq k\} \text{ is finite.} \tag{2.1}$$

One can prove that (ω_1, ω_1)-pre-gaps satisfying the above condition are indeed gaps. In fact, Hausdorff gaps are very impervious to forcing extensions in the sense that a Hausdorff gap in the ground model will remain unseparated in any forcing extension preserving ω_1, where a forcing $\mathbb{P}$ preserves ω_1 when $\omega_1^V = \omega_1^{V[G]}$ whenever G is $\mathbb{P}$-generic over V. On the other hand, it is consistent that not every (ω_1, ω_1)-gap satisfies the Hausdorff condition; see, for example, [BNC15]. An in depth overview of gaps, in particular gaps in other partially ordered sets such as $(\omega^\omega, <^*)$, can be found in [Sch93].

We will restrict our attention to (ω_1, ω_1)-(pre-)gaps and for such pairs (A, B), we will say (A, B) is a $(\Gamma, \cdot)$ or (Γ, Γ) (pre-)gap if Γ is a pointclass and $A \in \Gamma$ or both $A, B \in \Gamma$, respectively.

A weaker assumption than being wellordered by $\subseteq^*$ is assuming A, B are *σ-directed*, meaning that for every countable $\{a_n \mid n \in \omega\} \subseteq A$, there is $a \in A$ such that for all n, $a_n \subseteq^* a$, and analogously for B. Todorčević showed that even in this case, there is no $(\Sigma^1_1, \cdot)$ gap (see [Tod96, Corollary 1]). It is interesting to note that dropping even the assumption of σ-directedness, there exist gaps (A, B) with A, B being perfect, hence closed, sets; see [Tod96, Theorem 2] or [Kho12, Section 4.6].

Beyond ZFC:

- There are no gaps in Solovay's model [Kho12, Section 4.4];
- Under $\mathsf{ZF}+(\mathsf{AD}_\mathbb{R})$ there are no gaps [Kho14].

However, it still remains open whether the assumption of the second item above can be weakened to AD. Via a Cantor-Bendixson derivative style argument, Khomskii [Kho12, Theorem 4.2.3, Corollary 4.2.4] shows that if for all $r \in \omega^\omega$, $\omega_1^{L[r]} < \omega_1$, then there are no $(\mathbf{\Pi}^1_1, \cdot)$ gaps. Nonetheless, a recursive construction in L gives a $(\mathbf{\Sigma}^1_2, \mathbf{\Sigma}^1_2)$ Hausdorff gap, which by an application of Miller's method yields a $(\mathbf{\Pi}^1_1, \mathbf{\Pi}^1_1)$ Hausdorff gap (this is made explicit in [Kho12, Lemma 4.3.5]). A consequence of constructing a $\mathbf{\Pi}^1_1$-definable Hausdorff gap in L, together with the indestructibility of the gap by any ω_1-preserving forcing notion, gives the converse of the above implication; namely, if there is r with $\omega_1^{L[r]} = \omega_1$, then there is a $(\Pi^1_1(r), \Pi^1_1(r))$ Hausdorff gap.

Below we give show that the existence of a (Σ^1_2, Σ^1_2) Hausdorff gap is equivalent with the existence of a (Π^1_1, Π^1_1) Hausdorff gap, thereby weakening the assumption of $V = L$ in the construction presented by Khomskii. This will rely on showing that the analogue of condition (2.1) in $\mathcal{P}(\omega \times \omega)$ is sufficient for an (ω_1, ω_1) pre-gap in $(\mathcal{P}(\omega \times \omega), \subseteq^*)$ to be a gap.

Lemma 2.4.1. *Suppose (A, B) is an (ω_1, ω_1)-pre-gap in the space $\mathcal{P}(\omega \times \omega)$, and that for every $\alpha < \omega_1$ and all $k \in \omega$, the set*

$$\{\gamma < \omega_1 \mid a_\alpha \cap b_\gamma \subseteq k \times k\}$$

is finite. Then there is no $c \in [\omega \times \omega]^\omega$ such that for all $a \in A$ and for all $b \in B$, both $a \setminus c$ and $c \cap b$ are finite.

Proof. Suppose not; that is, there is $c \in [\omega \times \omega]^\omega$ such that for every $\alpha < \omega_1$ there is $k_\alpha < \omega$ such that $a_\alpha \setminus c \subseteq k_\alpha \times k_\alpha$ and $c \cap b_\alpha \subseteq k_\alpha \times k_\alpha$. Then there is an uncountable $X \subseteq \omega_1$ and $k < \omega$ such that for every $\alpha \in X$, $k_\alpha = k$. As X is uncountable we can find $\alpha \in X$, $\alpha \geq \omega$, such that $X \cap \alpha$ is infinite. Then since $a_\alpha \cap b_\gamma \subseteq (a_\alpha \setminus c) \cup (b_\gamma \cap c) \subseteq k \times k$ for all $\gamma \in X \cap \alpha$, the set

$$\{\gamma \in X \cap \alpha \mid a_\alpha \cap b_\gamma \subseteq k \times k\}$$

is infinite, a contradiction. □

Theorem 2.4.2. *Suppose A, B are Σ^1_2-definable subsets such that (A, B) is a Hausdorff gap. Then there exists a Hausdorff gap (A', B') in $\mathcal{P}(\omega \times \omega)$ such that A', B' are Π^1_1.*

Proof. Let A, B be as above, and let $F, G \subseteq [\omega]^\omega \times 2^\omega$ be Π^1_1 graphs of functions such that

$$a \in A \Leftrightarrow \exists x \in 2^\omega((a, x) \in F) \quad \text{and} \quad b \in B \Leftrightarrow \exists y \in \omega^\omega((b, y) \in G).$$

Define functions $g_A, g_B \colon [\omega]^\omega \times 2^\omega \to [\omega \times \omega]^\omega$ by letting

$$g_A(a, x) = (\{0\} \times a) \cup \bigcup_{n \in \omega} (\{2n+2\} \times (a \setminus a(n + x(n)))),$$

and

$$g_B(b, y) = (\{0\} \times b) \cup \bigcup_{n \in \omega} (\{2n+1\} \times (b \setminus b(n + y(n)))).$$

Then g_A and g_B are continuous. Let $A' := g_A[F]$, and $B' := g_B[G]$. First we show A', B' are linearly ordered by the relation $\subseteq^*$ on $[\omega \times \omega]^\omega$. While the proof goes identically as in the case of towers (see Section 5, Theorem 2.3.3), we include a proof in this context for the sake of completeness.

Let $c, d \in A'$ be distinct, and let a^c, x^c and a^d, x^d be such that $c = g_A(a^c, x^c)$ and $d = g_A(a^d, x^d)$. For $n \in \omega$ denote by c_n, d_n the n-th column of c, d, respetively. So, $a_0 = a^c$ and $d_0 = a^d$.

As $c \neq d$, it follows that a^c, a^d are distinct members of A, so without loss of generality, $a^c \subseteq^* a^d$ and $\neg(a^d \subseteq^* a^d)$. Since $a^c \setminus a^d$ is finite and $a^d \setminus a^c$ is infinite, there exists $N \in \omega$ such that for every $m \geq N$, $a^c(m) > a^d(m)$ and $a^c(m) \in a^d$.

For such $m \geq N$, we have $c_0 \subseteq d_0$. Moreover,

$$c_{2m+2} = a^c \setminus a^c(m + x^c(m)) \subseteq a^c \setminus a^c(m) \subseteq a^d \setminus a^d(m + x^d(m)) = d_{2m+2},$$

whenever $m \geq N$; in particular the latter inclusion follows by noting that if $k > a^c(m)$ and $k \in a^c$ then $k \geq m \geq N$, so therefore $k \in a^d$. Thus for all but finitely many $m \in \omega$, $c_m \subseteq d_m$, and so $c \subseteq^* d$.

To see A' and B' is a pre-gap, for any $c \in A'$ and $d \in B'$, $c_0 \cap d_0$ is finite as (A, B) was a pre-gap, and for $n \geq 1$, $c_n \cap b_n = \emptyset$, as

$$A' \cap \bigcup_{n < \omega} \{2n+1\} \times \omega = \emptyset = B' \cap \bigcup_{n < \omega} \{2n+2\} \times \omega.$$

Write $A' = \{c_\alpha \mid \alpha < \omega_1\}$ and $B' = \{d_\alpha \mid \alpha < \omega_1\}$. We claim that for every $\alpha < \omega_1$ and for all $k \in \omega$, the set

$$\{\gamma < \alpha \mid a'_\alpha \cap b'_\gamma \subseteq k \times k\}$$

is finite. Suppose not, and let $\alpha < \omega_1$ and $k < \omega$ be such that there are infinitely many ordinals $\gamma < \alpha$ with $a'_\alpha \cap b'_\gamma \subseteq k \times k$. But notice if $(n, m) \in$

$a'_\alpha \cap b'_\gamma$, by the previous observation it must be $n = 0$ and $m \in a_\alpha \cap b_\gamma$, where $a_\alpha \in A$ and $b_\gamma \in B$ are such that $g_A(a_\alpha, x_\alpha) = a'_\alpha$ and $g_B(b_\gamma, y_\gamma) = b'_\gamma$, for some $x_\alpha, y_\alpha \in 2^\omega$. Therefore

$$\{\gamma < \alpha : a_\alpha \cap b_\gamma \subseteq k\}$$

is infinite, contradicting the hypothesis on (A, B). Lemma 2.4.1 then shows that (A', B') is indeed a gap.

To see that A' can be defined by a Π^1_1 formula, we show that

$$c \in A' \Leftrightarrow \exists (a, x) \in \Delta^1_1(c)[(a, x) \in F \wedge g_A(a, x) = z],$$

and similarly this holds for membership in B'. Given $c \in \mathcal{P}(\omega \times \omega)$, we check if $c \cap \bigcup_{n\in\omega}\{2n+2\} \times \omega \neq \emptyset$ and $c \cap \bigcup_{n\in\omega}\{2n+1\} \times \omega = \emptyset$; when this is the case, one uses the set c_0 to effectively define a real $x \in 2^\omega$, by letting

$$x(n) = \begin{cases} 0 & \text{if } \min c_{2n+2} = c_0(n+1); \\ 1 & \text{if } \min c_{2n+2} = c_0(n+2). \end{cases}$$

Therefore there is a pair $(c_0, x) \in \mathcal{P}(\omega) \times 2^\omega$ constructible from c. To check if $g_A(a, x) = c$ is Δ^1_1, and $(a, x) \in F$ is Π^1_1, so the result follows. The case for the definability of B' is analogous. □

Notice that the continuous function g above can be seen as a hybrid of the functions from Törnquist's theorem and the function used for construction a tower in $[\omega \times \omega]^\omega$.

The above theorem has a variety of applications due to the following, which formalizes our comment about the absoluteness of the inseperability of Hausdorff gaps.

Lemma 2.4.3. *Let $\mathbb{P}$ be any forcing notion such that for all $p \in \mathbb{P}$, $p \Vdash \omega_1 = \check{\omega}_1$, and suppose $(A, B) \in L$ be a Hausdorff gap. Then in any $\mathbb{P}$-generic extension of L, (A, B) remains a gap.*

Proof. Let (A, B) be a Hausdorff gap in V. First observe that

$$\Vdash_{\mathbb{P}} \text{“}\forall \alpha \in \omega_1 \forall k \in \omega\ \{\gamma < \alpha \mid a_\alpha \cap b_\gamma \subseteq k\} \text{ is finite”}.$$

Suppose towards contradiction there is $p \in \mathbb{P}$ and τ a $\mathbb{P}$-name such that $p \Vdash$"τ interpolates $(\check{A}, B)$", so $p \Vdash \forall\alpha < \omega_1 \exists k_\alpha \in \omega[(a_\alpha \setminus \tau) \subseteq k_\alpha \wedge (b_\alpha \cap \tau) \subseteq k_\alpha]$. Then if G is a $\mathbb{P}$-generic filter over V containing p, in $V[G]$ there is an uncountable set $I \subseteq \omega_1^{V[G]}$ and $k \in \omega$ such that $V[G] \models \forall\alpha \in I\ [(a_\alpha \setminus \tau^G) \subseteq k \wedge (b_\alpha \cap \tau^G) \subseteq k]$, so there is $\alpha \in I$ infinite with $I \cap \alpha$ infinite, therefore $L[G] \models$"$\{\gamma \in I \cap \alpha \mid a_\alpha \cap b_\gamma \subseteq k\}$ is infinite", as $a_\alpha \cap b_\gamma \subseteq (a_\alpha \setminus \tau) \cup (b_\gamma \cap \tau) \subseteq k$. This contradicts the first observation.

□

Now suppose we are in a model of $V = L$ and fix a (Σ^1_2, Σ^1_2) Hausdorff gap (A, B). Let $\mathbb{P} \in V$ be any ω_1-preserving forcing notion—for instance any forcing which is ccc or proper. Let G be $\mathbb{P}$-generic over V. Then $((A, B)$ is a (Σ^1_2, Σ^1_2) Hausdorff gap$)^{V[G]}$, and so by Theorem 2.4.2 applied in $V[G]$, there exists a (Π^1_1, Π^1_1) Hausdorff gap. In particular, in all constructions in Chapter 3, there exists a (Π^1_1, Π^1_1) Hausdorff gap.

2.5 Questions

Above we showed positive instances of when there is a method of coding two reals into one in such a way that a desired combinatorial property was preserved. Of course, it seems that coming up with an appropriate coding for preserving other properties just amounts to more clever coding, though it is interesting to ask when this provably cannot be done, as is the case for total wellorders of the reals. Indeed, the Lebesgue and Baire measurability of coanalytic sets immediately precludes any Π^1_1 wellordering of the continuum.

Question 2.5.1. *Do there exist families for which one can prove that there consistently exist Σ^1_2 but no Π^1_1 examples?*

All of the proofs presented in this paper were specific to the classes Σ^1_2 and Π^1_1, with the exception of Theorem 2.3.7. The obstruction for lifting the proofs in the other cases is the use of Π^1_1 uniformization and the Spector–Gandy theorem.

Question 2.5.2. *How might the constructions above be lifted to higher pointclasses?*

The goal of this survey has been to overview methods which perhaps will give ideas for answering the following:

Question 2.5.3. *Are there* ZFC *derivations of* Π^1_1 *examples from* Σ^1_2 *for the following:*

- *Hamel bases (see, for example, [Vid14, Corollary 4.11] for a* Π^1_1 *construction in L);*
- *Cofinitary groups (see, for example, [FST17] for* Π^1_1 *construction in L which is Cohen indestructible).*

Chapter 3

Projective sets and large continuum

3.1 Introduction

The previous chapter gave techniques for reducing the descriptive complexity of a projective witness in such a way that the nontrivial combinatorial structure of the set is unaffected by the coding. The theorems resulting from these reductions have been used in conjunction with methods of preservation in the theory of forcing so to obtain models in which $\mathfrak{c} > \aleph_1$, and there exists a projectively definable combinatorial object of size $\aleph_1$, usually providing a witness that some cardinal characteristic has value $\aleph_1$. When the definable witness of some set defining a cardinal characteristic furthermore has a definition of minimal complexity, we will call such a witness an *optimal projective witness*.

In this chapter we look at ways of constructing models of $\mathsf{ZFC} + \neg\mathsf{CH}$ in which some nontrivial cardinal characteristic constellation is realized, while both preserving an optimal projective witness of size $\aleph_1$ and asking for an optimal projective witness of size $\kappa > \aleph_1$. This requires the orchestration of preservation techniques together with:

1. Techniques for constructing generic objects via cardinal preserving iterations of forcing notions;

2. Methods of rendering a given set projectively definable in a generic extension.

Techniques pertaining to item (1) fall into the domain of set theory of the reals and the existing literature around cardinal characteristics provides ample resources; see, for example, the later sections in [Bla10], or [Hal25]. Work on item (2) can be seen to have begun with Solovay, Jensen, Johnsbråten, and Harrington in the early period of forcing; for example, see [SJ70], [Jen70], [JJ74] or [Har77]. Only fairly recently were these two techniques combined. Our cumulative construction illustrating a study of the interplay between these lines of inquiry and their synthesis will be the following:

Theorem (Theorem 3.5.2)**.** *It is consistent with $\aleph_1 = \mathfrak{a} < \mathfrak{s} = \mathfrak{c} = \aleph_2$ that there exists a Δ^1_3 wellorder of the reals, a coanalytic tight mad family of size $\aleph_1$, and a Π^1_2 tight mad family of size $\aleph_2$.*

The theorem is proved in three steps, each of which yield generic extensions of ZFC which are of interest in their own right. The first step is to obtain a model of the inequality $\mathfrak{a} < \mathfrak{s}$ with the value of $\mathfrak{a}$ being witnessed by a coanalytic mad family. Here, $\mathfrak{s}$ denotes the *splitting number*, and will be defined in Section 3.2. Our strategy is to fix a Π^1_1-definable mad family $\mathcal{A}$ in L, and use a forcing notion which not only increases the value of $\mathfrak{s}$, but also is such that it and its iterations preserve the maximality of $\mathcal{A}$. We must therefore ensure that we do not increase the values of cardinal characteristics which are provably less than or equal to $\mathfrak{a}$, such as is the case for the cardinal $\mathfrak{b}$. The first instance of a forcing achieving $\mathfrak{a} = \aleph_1 < \mathfrak{s} = \aleph_2$ was introduced by Shelah in 1984 [She84], and was originally purposed to give the consistency of $\aleph_1 = \mathfrak{b} < \mathfrak{s} = \aleph_2$, thereby establishing that no inequality between $\mathfrak{b}$ and $\mathfrak{s}$ is provable from ZFC alone. In Section 3.2 below, we show that the forcing notion $\mathbb{Q}$ of [She84] (see Definition 3.2.17) satisfies an iterable preservation property (see Definition 3.2.3) introduced in 2020 by Guzman, Hrušák, and Tellez [GHT20] called strong preservation of tightness (see Definition 3.2.3), which guarantees the preservation of mad families under countable support iterations.

Proposition (Proposition 3.2.28)**.** *Let $\mathbb{Q}$ be the forcing notion of Definition 3.2.17, and let $\mathcal{A} \in V$ be a tight mad family. Then $\mathbb{Q}$ strongly preserves the* *tightness of $\mathcal{A}$, and therefore ($\mathcal{A}$ is a tight mad family)$^{V[G]}$, whenever G is $\mathbb{Q}$-generic over V.*

A consequence of this is that if CH holds in the ground model V and $\mathcal{A} \in V$ is a tight mad family, countable support iterations of the $\mathbb{Q}$ above do not destroy the maximality of $\mathcal{A}$.

The notion $\mathbb{Q}$ is the first example of a *creature forcing*, which now refers to a broad class of forcings that add new reals (see [RS99]). It is a proper forcing notion that adds a generic real which witnesses that no ground model family can be a witness to the cardinal $\mathfrak{s}$, and moreover has the property of being what is called almost ω^ω-bounding. This latter property implies that in a countable support iteration of $\mathbb{Q}$, the ground model reals remain an unbounded family, thus providing a witness to $\mathfrak{b} = \aleph_1$, while the former fact implies in an ω_2-length countable support iteration of $\mathbb{Q}$, there can exist no splitting families of size strictly less than $\aleph_2$—i.e., in such a generic extension $\mathfrak{s} \geq \aleph_2$. Since ZFC proves $\mathfrak{b} \leq \mathfrak{a}$, and countable support iterations of $\mathbb{Q}$ preserve a witness to $\mathfrak{a} = \aleph_1$, the fact $(\mathfrak{b} = \aleph_1)^{V[G]}$ is now a consequence of the ZFC inequality and the above preservation theorem. Thus we obtain an alternative proof of the consistency of $\mathfrak{b} = \mathfrak{a} < \mathfrak{s}$.

The second step is to use this result to obtain a new theorem in the theory of projective wellorders and in particular the intersection of this theory with that of cardinal characteristics, a relatively recent line of inquiry. The question of the existence of definable wellorders is a central question in set theory and has been studied in a variety of extensions of ZFC; while Gödel proved that if ZF is consistent then so is $\mathsf{ZFC} + \mathsf{GCH}$ and there exists a Δ^1_2 wellorder of the reals, Harrington [Har77] showed that if ZFC is consistent then so is $\mathsf{ZFC} + \neg\mathsf{CH}$ and there exists a $\boldsymbol{\Delta}^1_3$ wellorder. Fischer and Friedman [FF10] then showed a Δ^1_3 wellorder is consistent with various cardinal characteristic constellations, and moreover asked about the projective complexity of witnesses to the corresponding cardinal invariants of in such models. Note that, we can think of the projective wellorder as an optimal projective witness to $\mathfrak{c} = \aleph_2$. We will show:

Theorem (Theorem 3.3.35)**.** *It is consistent with $\aleph_1 = \mathfrak{a} < \mathfrak{s} = \mathfrak{c} = \aleph_2$ that there exists a Δ^1_3-definable wellorder of the reals and a Π^1_1 tight mad family of size $\aleph_1$.*

All of the above projective definitions are provably of minimal complexity, by classicall theorems of Mansfield and Mathias. Previous work on optimal projective witnesses for values of cardinal characteristics includes, for example, [BFB22], [FFZ11], [FS21a], [FFK13], [BK13], and [FSS25].

In our third step towards the cumulative theorem we are motivated by a generalization of the theory of cardinal characteristics. Namely, analogous to how Hechler in 1972 [Hec72] gives methods of defining a cardinal preserving forcing notion yielding a generic extension in which a given cardinal κ belongs to the set

$$\mathrm{spec}(\mathfrak{a}) = \{|\mathcal{A}| \mid \mathcal{A} \subseteq [\omega]^\omega \text{ is mad}\},$$

we wish to find a model in which $|\mathrm{spec}(\mathfrak{a})| = 2$ and every inclusion $\kappa \in \mathrm{spec}(\mathfrak{a})$ is witnessed by an optimal projective witness. Towards this we consider an S-proper forcing iteration introduced by Friedman and Zdomskyy [FZ10] (see Definition 3.4.5) designed to give an optimal projective witness for $\mathfrak{a} = \aleph_2$ in a model of $\mathfrak{b} = \mathfrak{c} = \aleph_2$. We prove a similar preservation result as in the above proposition for the iterand of their construction responsible for adding the Π^1_2 tight mad family of size $\mathfrak{c}$. This gives the existence of a model of ZFC yielding the following.

Theorem (Theorem 3.4.20)**.** *It is consistent that $\mathfrak{b} = \mathfrak{a} = \aleph_1 < \mathfrak{c} = \aleph_2$, and there exists a Π^1_1-definable tight mad family of size $\aleph_1$, and a Π^1_2-definable tight mad family of size $\aleph_2$.*

By a result of Raghavan [Rag09] and the Mansfield-Solovay theorem, this is the best possible projective definability of a tight mad family of size strictly greater than $\aleph_1$.

The chapter is structured as follows. In Section 3.2 we introduce the forcing notion $\mathbb{Q}$, the notions of a tight mad family and of strong preservation of tightness, and prove Proposition 3.2.28. In Section 3.3 we define the countable support iteration of [FF10] for adding a Δ^1_3 wellorder of the reals and prove Theorem 3.3.35. The following Section 3.4 gives the definition of the Friedman-Zdomskyy forcing, shows it preserves tight mad families, and proves Theorem 3.4.20. In Section 3.5 we show how these results are weaved together to yield Theorem 3.5.2. The last section contains our concluding remarks and open questions.

3.2 A creature forcing and tight mad families

Define the relation $\leq^*$ of *eventual domination* on the collection ω^ω by letting $f \leq^* g$ if and only if there exists $n \in \omega$ such that for all $m \geq n$, $f(m) \leq g(m)$. Define the relation *splits* on the collection $[\omega]^\omega$ by letting a split b if and only if both $b \cap a$ and $b \setminus a$ are infinite. In 1984 Shelah answered a question of Nyikos by showing that it is consistent that the minimal size of a family $\mathcal{F} \subseteq \omega^\omega$ which is unbounded with respect to $\leq^*$ can be strictly less than the minimal size of a splitting family $\mathcal{S} \subseteq [\omega]^\omega$, that is, a family $\mathcal{S}$ with the property that for any infinite $b \subseteq \omega$ there is $a \in \mathcal{S}$ which splits b. We denote with $\mathfrak{b}$ and $\mathfrak{s}$ these respective cardinalities:

$$\begin{aligned}\mathfrak{b} &= \min\{|\mathcal{F}| \mid \mathcal{F} \subseteq \omega^\omega \text{ is } \leq^* \text{-unbounded}\},\\ \mathfrak{s} &= \min\{|\mathcal{S}| \mid \mathcal{S} \subseteq [\omega]^\omega \text{ is splitting}\}.\end{aligned}$$

That $\mathfrak{s} < \mathfrak{b}$ is consistent was already shown in 1980 by Balcar and Simon [BPS80, Remark 4.7], during an investigation of topological properties of the space of all ultrafilters on ω. Shelah [She84, Section 4] also shows that $\mathfrak{s} < \mathfrak{b}$ also holds in a generic extension by a finite support iteration of Hechler forcing of length κ, for $\kappa \geq \omega_2$ a regular cardinal.

ZFC proves the following inequalities,

$$\begin{array}{ccccc} \mathfrak{s} & \longrightarrow & \mathfrak{d} & \longrightarrow & \mathfrak{c} \\ \uparrow & & \uparrow & & \uparrow \\ \aleph_1 & \longrightarrow & \mathfrak{b} & \longrightarrow & \mathfrak{a} \end{array}$$

where an arrow between the cardinals represents the relation "less than or equal to". The cardinal $\mathfrak{d}$, the dual of $\mathfrak{b}$, is the minimal size of a *dominating family*, that is, a family $\mathcal{F} \subseteq \omega^\omega$ such that for all $f \in \omega^\omega$ there is $g \in \mathcal{F}$ such that $f(n) \leq^* g(n)$. Hechler forcing is the standard forcing notion for increasing $\mathfrak{d}$, however it also increases the size of $\mathfrak{b}$. The cardinals $\mathfrak{a}$ and $\mathfrak{d}$ are independent.

One approach to obtaining a model in which $\mathfrak{s} = \aleph_2$ is with a countable support iteration of Mathias forcing $\mathbb{M}$, this being the partial order consisting of $(s, A) \in [\omega]^{<\omega} \times [\omega]^\omega$, with $\max s < \min A$, and the extension relation defined as $(t, B) \leq (s, A)$ if and only if t end-extends s, $B \subseteq A$, and $t \setminus s \subseteq A$. For any $X \subseteq [\omega]^\omega$, the set

$$D_X = \{(s, A) \in \mathbb{M} \mid A \subseteq X \vee A \subseteq \omega \setminus X\}$$

is dense in $\mathbb{M}$, and for this reason Mathias forcing adds a real $a \in [\omega]^\omega$ such that one of $a \cap X$ or $a \setminus X$ is finite, when X is any ground model infinite subset of ω. This implies the ground model $[\omega]^\omega$ cannot be a splitting family.

Mathias forcing cannot be used to show the consistency of $\mathfrak{b} < \mathfrak{s}$. The issue is that if G is $\mathbb{M}$-generic and $a := \bigcup\{s \mid \exists A(s, A) \in G\}$, then a is an infinite subset of ω such that the enumeration function $e_a \colon \omega \to a$ is a *dominating real over* V, meaning e_a has the property that $f \leq^* e_a$ whenever $f \in \omega^\omega \cap V$ is a ground model function, and so provides a witness that $\omega^\omega \cap V$ is no longer $\leq^*$-unbounded in the $\mathbb{M}$-generic extension. In other words, the Mathias reals grow too fast to preserve the unboundedness of $\omega^\omega \cap V$, and in an ω_2-length countable support iteration of $\mathbb{M}$ the bounding number $\mathfrak{b}$ is of size $\aleph_2 = \mathfrak{c}$.

To remediate this problem Shelah's original "creature forcing" of [She84], henceforth denoted by $\mathbb{Q}$ (Definition 3.2.17 below), slows the growth of the generic. Intuitively, this is achieved by rather than taking just $A \subseteq [\omega]^\omega$ as the second coordinate of the condition, this which guides the construction of the generic real, one considers infinite sequences of finite subsets of ω, $\langle s_i : i \in \omega\rangle$ such that $\max s_i < \min s_{i+1}$, and equips each s_i with a measure $h_i \colon \mathcal{P}(s_i) \to \omega$ (see Definition 3.2.13 of a logarithmic measure). This allows for careful selection of the integers with which one extends a given finite approximation s, so that the resulting generic real does not bound all ground model $f \in \omega^\omega$ simultaneously, and is a real not split by the ground model $[\omega]^\omega$ for similar reasons as with $\mathbb{M}$ (see Lemma 3.2.30).

A proper forcing $\mathbb{P}$ such that in any $\mathbb{P}$-generic extension, the set of ground model reals $\omega^\omega \cap V$ remains unbounded with respect to $\leq^*$, is known as *weakly ω^ω-bounding*. However this property is not preserved under countable support iterations. An example this is not the case is the product $\mathbb{P}_0 \times \mathbb{P}_1$, where $\mathbb{P}_0$ adds ω_1-many Cohen reals, and $\mathbb{P}_1$ is Hechler's forcing for adding a dominating real (over V); $\mathbb{P}_0$ is weakly ω^ω-bounding and it can be shown that $(\mathbb{P}_1 \text{ is weakly bounding})^{V^{\mathbb{P}_0}}$, though $\mathbb{P}_0 \times \mathbb{P}_1$ clearly adds a dominating real over V. To show a countable support iteration of length ω_2 of $\mathbb{Q}$ preserves unboundedness of the ground model reals it suffices to show that $\mathbb{Q}$ has the slightly stronger property of being *almost ω^ω-bounding*.

Definition 3.2.1. A forcing notion $\mathbb{P}$ is *almost ω^ω-bounding* if and only if for every $p \in \mathbb{P}$ and every $\mathbb{P}$ name $\dot{f}$ for an element of ω^ω, there exists $g \in \omega^\omega \cap V$ with the property that for every $A \in [\omega]^\omega$, there is $q_A \leq p$ such that

$$q_A \Vdash \exists^\infty n \in A(\dot{f}(n) \leq \check{g}(n)).$$

Countable support iterations of almost ω^ω-bounding forcings are weakly ω^ω-bounding; this is shown in [Abr10, Theorem 4.4], and Section 4 of [Abr10] contains more in depth discussion of these notions.

Aside from establishing the independence of $\mathfrak{s}$ and $\mathfrak{b}$, another important contribution of [She84] is responding to a question of Mathias, who asked if it is consistent that in a model of $\mathfrak{b} < \mathfrak{s}$, also $\aleph_1 < \mathfrak{a}$; Shelah gives a positive answer, but it is only after a modification of his original forcing $\mathbb{Q}$ that this can be shown. Indeed, he shows $\mathfrak{a} = \aleph_1$ in the countable support iteration of $\mathbb{Q}$ by directly constructing the specific mad family in the ground model, utilizing that CH holds for this construction, and then shows this particular mad family is indestructible under $\mathbb{Q}$ and its iterations.

In 1995 Alan Dow [Dow95] constructs a mad family that is indestructible with respect to countable support iterations of Miller forcing. Miller forcing, also known as rational perfect set forcing, was introduced by Arnold Miller in 1984 [Mil84] with the purpose of increasing the dominating number $\mathfrak{d}$ while keeping $\mathfrak{b}$ small. It is the partial order consisting of trees $T \subseteq \omega^{<\omega}$ such that for every $t \in T$ there exists $s \in T$ extending t and such that s has infinitely many immediate successors in T; the extension relation is given by $T \leq S$ if and only if T is a subtree of S. Miller forcing is proper [Dow95, Proposition 8.11], and like the creature forcing of [She84] this forcing is almost ω^ω-bounding [Dow95, Lemma 8.13]. Unlike Shelah's creature forcing however, Miller forcing does not add an unsplit real [Mil84, Proposition 3.3].

Since 1984 the theory of preservation in forcing has considerably developed; see, for example, [She17] or [Gol93]. Likewise has been developed the theory of *indestructibility* of various witnesses to cardinal characteristics, in particular the indestructibility of mad families; see [BY05], [Hru01], or [HGF03]. In 2020, Guzman, Hrušák, and Ferreira study the preservation of tight mad families, where:

Definition 3.2.2. An almost disjoint family $\mathcal{A} \subseteq [\omega]^\omega$ is *tight* if for all countable collections $\mathcal{B} \subseteq \mathcal{I}(\mathcal{A})^+$, there exists a single $a \in \mathcal{A}$ such that $a \cap b$ is infinite for each $b \in \mathcal{B}$.

Above, $\mathcal{I}(\mathcal{A})^+$ denotes the collection of sets not belonging to the *ideal generated by* $\mathcal{A}$ and the finite subsets of ω:

$$\mathcal{I}(\mathcal{A}) = \{b \subseteq \omega \mid \exists F \in [\mathcal{A}]^{<\omega} \ (b \subseteq^* \bigcup F)\}.$$

Notice that if an almost disjoint family is tight, then it is immediately maximal. This strengthening initially appeared in the work of Malykhin [Mal89] in 1989, under the name of ω-mad families, and the connection of these families to Cohen-indestructibility was studied by Kurilić in 2001 [Kur01]. In fact, the existence of tight mad families is equivalent to the existence of Cohen-indestructible mad families: on the one hand, tight mad families are Cohen-indestructible, while on the other hand if $\mathcal{A}$ is a Cohen-indestructible mad family, there exists $B \in \mathcal{I}(\mathcal{A})$ such that $\mathcal{A} \restriction B := \{A \cap B \mid A \in \mathcal{A}\}$ is a tight mad family; see [Kur01, Theorem 4]. The existence of tight mad families also follows from the assumption $\mathfrak{b} = \mathfrak{c}$ and a certain parametrized $\Diamond$-principle (see [HGF03]). Nonetheless it remains a long standing open question if ZFC proves the existence of tight mad families.

One of the important contributions of [GHT20] to the study of tight mad families was the introduction of a property of proper forcings sufficient for guaranteeing the preservation of tight mad families.

Definition 3.2.3 ([GHT20, Definition 7.1])**.** Let $\mathcal{A}$ be a tight mad family. We say a proper forcing notion $\mathbb{P}$ *strongly preserves the tightness of* $\mathcal{A}$ if for every $p \in \mathbb{P}$ and every countable elementary $\mathcal{M} \prec H_\theta$, where θ is a sufficiently large regular cardinal so that $\mathbb{P}, \mathcal{A}, p \in \mathcal{M}$, and for every $B \in \mathcal{I}(\mathcal{A})$ such that $B \cap Y$ is infinite for all $Y \in \mathcal{I}(\mathcal{A})^+ \cap \mathcal{M}$, there exists $q \leq p$ such that $q \in (\mathcal{M}, \mathbb{P})$-generic and

$$q \Vdash \text{“}\forall \dot{Z} \in (\mathcal{I}(\mathcal{A})^+ \cap \mathcal{M}[\dot{G}])(|\dot{Z} \cap B| = \omega)\text{”}.$$

Such a q is called an $(\mathcal{M}, \mathbb{P}, \mathcal{A}, B)$*-generic condition.*

Recall that an $(\mathcal{M}, \mathbb{P})$-generic condition is a condition $q \in \mathbb{P}$ such that $q \Vdash \check{D} \cap \dot{G} \cap \mathcal{M} \neq \emptyset$ whenever $D \in \mathcal{M}$ is a dense open subset of $\mathbb{P}$, and $\dot{G}$ denotes the canonical name for the $\mathbb{P}$-generic filter. One equivalent formulation of properness is that a forcing $\mathbb{P}$ is proper if and only if for all $\mathcal{M} \prec H_\theta$ with $\mathbb{P} \in \mathcal{M}$, the set of $(\mathcal{M}, \mathbb{P})$-generic conditions is dense below any $p \in \mathcal{M} \cap \mathbb{P}$.

It is easy to see that if $\mathbb{P}$ is proper and preserves the tightness of some $\mathcal{A}$, then $\mathcal{A}$ remains tight (and hence maximal) in any $\mathbb{P}$-generic extension. What is crucial for obtaining cardinal preserving generic extensions separating $\mathfrak{a} = \aleph_1$ from another cardinal characteristic is that this property is preserved under countable support iterations of $\mathbb{P}$. By first proving the following intermediary results, [GHT20] shows that this is the case.

Lemma 3.2.4 ([GHT20, Lemma 6.3]). *Let $\mathcal{A}$ be a tight mad family, let $\mathbb{P}$ be a proper forcing strongly preserving the tightness of $\mathcal{A}$, and let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a forcing notion such that $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ strongly preserves the tightness of $\check{\mathcal{A}}$". Then $\mathbb{P} * \dot{\mathbb{Q}}$ strongly preserves the tightness of $\mathcal{A}$. Furthermore if $B \in \mathcal{I}(\mathcal{A})$, and $\mathcal{M}$ is a countable elementary submodel of H_θ with $\mathcal{A}, \mathbb{P}, \dot{\mathbb{Q}} \in \mathcal{M}$, if $p \in \mathbb{P}$ is $(\mathcal{M}, \mathbb{P}, \mathcal{A}, B)$-generic and $\dot{q}$ is a $\mathbb{P}$-name for a condition in $\dot{\mathbb{Q}}$ such that $p \Vdash$ "$\dot{q}$ is $(\mathcal{M}[\dot{G}], \dot{\mathbb{Q}}, \mathcal{A}, B)$-generic", then $(p, \dot{q})$ is an $(\mathcal{M}, \mathbb{P} * \dot{\mathbb{Q}}, \mathcal{A}, B)$-generic condition.*

Proposition 3.2.5 ([GHT20, Proposition 6.4]). *Let $\gamma \leq \omega_2$ be an ordinal, let $\mathcal{A}$ be a tight mad family, and let $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \gamma, \beta < \gamma \rangle$ be a countable support iteration of proper forcings such that $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ strongly preserves the tightness of $\check{\mathcal{A}}$". Let $B \in \mathcal{I}(\mathcal{A})$, and let $\mathcal{M}$ be a countable elementary submodel of H_θ, where θ a sufficiently large regular cardinal, such that $\mathcal{A}, \mathbb{P}, \gamma \in \mathcal{M}$. Then for every $\alpha \in \mathcal{M} \cap \gamma$ and for every $p \in \mathbb{P}_\alpha$ such that p is $(\mathcal{M}, \mathbb{P}_\alpha, \mathcal{A}, B)$-generic, if $\dot{q}$ is a $\mathbb{P}_\alpha$-name such that $p \Vdash_{\mathbb{P}_\alpha}$ "$\dot{q} \in \mathbb{P}_\gamma \cap \mathcal{M}$ and $\dot{q} \restriction \alpha \in \dot{G}_\alpha$", then there exists $\overline{p} \in \mathbb{P}_\gamma$ which is $(\mathcal{M}, \mathbb{P}_\alpha, \mathcal{A}, B)$-generic, such that $\overline{p} \restriction \alpha = p$ and $\overline{p} \Vdash_{\mathbb{P}_\gamma} \dot{q} \in \dot{G}$.*

Lemma 3.2.6 ([GHT20, Corollary 6.5]). *Let $\gamma \leq \omega_2$ be an ordinal, and let $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \gamma, \beta < \gamma \rangle$ be a countable support iteration such that for all $\alpha < \gamma$, $\Vdash_{\mathbb{P}_\alpha}$ " $\dot{\mathbb{Q}}_\alpha$ strongly preserves the tightness of $\check{\mathcal{A}}$". Then $\mathbb{P}$ strongly preserves the tightness of $\mathcal{A}$.*

The proofs of the above into the standard paradigm of preservation theorems such as those found in [Abr10], the canonical example being the theorem that countable support iterations of proper forcings are proper [She82].

Definition 3.2.3 serves as a key tool in showing a forcing notion does not increase the size of $\mathfrak{a}$, and therefore also preserves the size of $\mathfrak{b}$. Some examples of forcings exhibiting this property are Miller forcing and Sacks forcing, as remarked after in [GHT20]; Miller partition forcing also is shown to satisfy strong preservation of tightness [GHT20, Proposition 6.11]. Our main result includes Shelah's creature forcing to this list, and provides an a different proof of Shelah's result that $\mathfrak{b} < \mathfrak{s}$ in generic extension via a countable support iteration of $\mathbb{Q}$.

Proposition (Proposition 3.2.28)**.** *Shelah's forcing $\mathbb{Q}$ strongly preserves the tightness of tight mad families from the ground model.*

With this we give a proof of the following:

Theorem (Theorem 3.2.32; [She84, Theorem 3.1])**.** *Let V be a model of CH and let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$ be a countable support iteration with such that for all $\alpha < \omega_2$, $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$ name for $\mathbb{Q}$. If G is $\mathbb{P}_{\omega_2}$-generic over V, then in $V[G]$ it holds that $\aleph_1 = \mathfrak{b} = \mathfrak{a} < \mathfrak{s} = \aleph_2$.*

The standard approach in the literature thus far for showing a proper forcing $\mathbb{P}$ strongly preserves tightness as in Definition 3.2.3, is to modify the construction of the $(\mathcal{M}, \mathbb{P})$-generic in the following way. We inductively construct a sequence $\langle p_n : n \in \omega \rangle \subseteq \mathcal{M} \cap \mathbb{P}$ with $p_0 = p$ and for each $n \in \omega$ $p_{n+1} \leq p_n$; in order to obtain an $(\mathcal{M}, \mathbb{P})$-generic condition, we would only demand that at each inductive stage n, $p_n \in D_n$ where D_n comes from an enumeration of the dense open subsets of $\mathbb{P}$ which are elements of $\mathcal{M}$. To furthermore obtain an $(\mathcal{M}, \mathbb{P}, \mathcal{A}, B)$-generic condition, we also require that p_{n+1} forces "$(\dot{Z}_n \cap B) \setminus n \neq \emptyset$", where $\dot{Z}_n$ comes from a fixed, repetitive enumeration of the $\mathbb{P}$-names in $\mathcal{M}$ for an elements of $\mathcal{I}(\mathcal{A})^+$, and $B \in \mathcal{I}(\mathcal{A})$ is the fixed witness that intersects each Y in the countable set $\mathcal{I}(A)^+ \cap \mathcal{M}$ in an infinite set. This is achieved by what we shall call an *outer hull argument.*

Definition 3.2.7. If $\mathbb{P}$ is a forcing notion, $p \in \mathbb{P}$ and $\dot{Z}$ is a $\mathbb{P}$-name for a subset of ω, the *outer hull of $\dot{Z}$ with respect to p* is the set

$$W_p := \{m \in \omega \mid \exists r \leq p \ (r \Vdash m \in \dot{Z})\}.$$

Fact 3.2.8. ([GHT20, Lemma 6.2]) For an almost disjoint family $\mathcal{A}$ and $\mathbb{P}$ a forcing notion, if $\dot{Z}$ is a $\mathbb{P}$-name for an element of $\mathcal{I}(\mathcal{A})^+$, then for any $p \in \mathbb{P}$, $W_p \in \mathcal{I}(\mathcal{A})^+$.

The above follows from noting that $\dot{Z}$ will be forced by p to be a subset of W_p. Thus in order to obtain a $p_{n+1} \in \mathbb{P}$ forcing that $(\dot{Z}_n \cap B) \setminus n$ is nonempty, it suffices to note that $W_{p_n} \in \mathcal{I}(\mathcal{A})^+ \cap \mathcal{M}$, so as $B \cap W_{p_n}$ is infinite, there is $m \in B$ with $m > n$ and $m \in W_{p_n}$; this latter fact yields the desired p_{n+1}.

However, in the case of the creature forcing $\mathbb{Q}$ (as well as the Friedman–Zdomskyy forcing, see Section 3.4), we must take more care into what type of extension we are taking and want to restrict the existential quantifier in the definition of the outer hull. Thus we consider a refined notion of the outer hull, and must prove that Fact 3.2.8 still holds for these refinements. These proofs can be found as Claim 3.2.29 and Lemma 3.4.12 in the current section and in Section 3.4, respectively.

Before defining $\mathbb{Q}$ and proving our preservation result, we show that in some sense, the indestructible mad families constructed by Shelah and Dow are canonical examples of tight mad families. The following construction originates in the proof of [She84, Theorem 3.1]; see also [She17, Theorem 7.1, Chapter VI].

Proposition 3.2.9. *Assume CH. Then there exists a tight mad family.*

Proof. Fix an enumeration

$$\{\langle B_n^\alpha : n \in \omega\rangle \mid \alpha < \omega_1\}$$

of sequences $\langle B_n^\alpha : n \in \omega\rangle$, with $B_n^\alpha \in [\omega]^{<\omega} \setminus \{\emptyset\}$ and $B_n^\alpha \cap B_m^\alpha = \emptyset$ for all distinct $m, n \in \omega$.

By induction on $\alpha < \omega_1$, recursively define a family $\mathcal{A} = \{A_\alpha \mid \alpha < \omega_1\}$ as follows. First, choose $\{A_n \mid n \in \omega\}$ to be any partition of ω into infinite sets.

For $\alpha \in [\omega, \omega_1)$, choose $A_\alpha \subseteq \omega$ so that A_α is almost disjoint from A_β for each $\beta < \alpha$, and for each $\beta < \alpha$:

<u>If</u> for all $k \in \omega$ and $\alpha_j < \alpha$ for $j < k$, for all $m \in \omega$ the set

$$\{n \in \omega \mid \min(B_n^\beta) > m \land \ B_n^\beta \cap \bigcup_{j<k} A_{\alpha_j} = \emptyset\}$$

is infinite,

then:

1. there exist infinitely many $n \in \omega$ such that $B_n^\beta \subseteq A_\alpha$, and

2. for all $k \in \omega$ and $\alpha_j \leq \alpha$ for $j < k$, the set

$$\{n \in \omega \mid B_n^\beta \cap (\bigcup_{j<k} A_{\alpha_j}) = \emptyset\}$$

   is infinite.

This can be done by induction, using $\mathfrak{c} = \aleph_1$. Let $\mathcal{A} = \{A_\alpha \mid \alpha < \omega_1\}$. Then clearly $\mathcal{A}$ is almost disjoint; let us see that $\mathcal{A}$ is tight.

Let $\{C_j \mid j \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$. As each C_j is an infinite subset of ω, it is a countable union of singletons, $C_j = \bigcup_{n\in\omega}\{C_j(n)\}$, where $C_j(n)$ denotes the $(n+1)$-st smallest element of C_j. Then for each $j \in \omega$ there exists $\beta_j < \omega_1$ such that $\langle\{C_j(n)\} : n \in \omega\rangle = \langle B_n^{\beta_j} : n \in \omega\rangle$. Let $\delta < \omega_1$ be such that $\beta_j < \delta$ for every $j \in \omega$, and so

$$\{\langle B_n^{\beta_j} : n \in \omega\rangle \mid j \in \omega\} \subseteq \{\langle B_n^i : n \in \omega\rangle \mid i < \delta\}.$$

Then at stage δ of the construction, for every $j \in \omega$ and $\alpha_0, \dots \alpha_{k-1} < \delta$, there are infinitely many $n \in \omega$ such that $C_j(n) > m$ and

$$C_j(n) \cap \bigcup_{\ell<k} A_{\alpha_\ell} = \emptyset,$$

so by (1) of the construction, there exist infinitely many n such that $\{C_j(n)\} = B_n^{\beta_j} \subseteq A_\delta$. Therefore for each $j \in \omega$, $C_j \cap A_\delta$ is infinite, so $\mathcal{A}$ is indeed tight. $\square$

Remark 3.2.10. Note that, if we had assumed the stronger $V = L$ in the above proof, the construction can be carried out in a Σ_2^1 way and hence the family $\mathcal{A}$ thus obtained is a Σ_2^1-definable tight mad family. Appealing to Törnquist's Theorem 2.2.2 of Chapter 2, this gives an alternative way of proving that under $V = L$ there exists a Π_1^1 definable mad family (this is Miller's result [Mil89, Theorem 8.23]).

Dow [Dow95, Proposition 8.24] shows that a generic extension over a model of CH by an ω_2-length iteration of Miller forcing, $\aleph_1 = \mathfrak{s} = \mathfrak{a} = \mathfrak{b} < \mathfrak{d} = \aleph_2$. In particular he shows that Miller forcing is almost ω^ω-bounding, giving $\mathfrak{b} = \aleph_1$; that $\mathfrak{a} = \aleph_1$ is shown by constructing a mad family in the ground model which is indestructible by countable support iterations of Miller forcing. The family constructed as well as the preservation proof are similar to those of Shelah's, but one difference between Dow's construction and that of Shelah's is that the former only requires the assumption $\mathfrak{p} = \mathfrak{c}$ instead of full CH. The cardinal invariant $\mathfrak{p}$ refers to the *pseudointersection number*; it is the minimal cardinality of a collection $\mathcal{F} \subseteq [\omega]^\omega$ with the strong finite intersection property (SFIP), meaning for any finite $F \subseteq \mathcal{F}$ the intersection $\bigcap F$ is infinite, and moreover there is no pseudointersection y of $\mathcal{F}$ (this is a $y \in [\omega]^\omega$ such that $y \subseteq^* x$ for every $x \in \mathcal{F}$). Every set of cardinality $\kappa < \mathfrak{p}$ with the SFIP has a pseudointersection; this is the key to Dow's construction.

Proposition 3.2.11 ([Dow95, Lemma 2.3])**.** *Assuming $\mathfrak{p} = \mathfrak{c}$, there exists a mad family $\mathcal{A} = \{A_\alpha \mid \alpha < \mathfrak{c}\}$ such that for any fixed enumeration*
$\{\langle B_n^\alpha : n \in \omega\rangle \mid \alpha < \mathfrak{c}\}$ of $[[\omega]^{<\omega}]^\omega$ and for any $\alpha < \mathfrak{c}$:
<u>If</u> there exists $\beta < \alpha$ such that

$$(*) : \textit{for all } I \in \mathcal{I}(\textstyle\bigcup_{\beta<\alpha} A_\beta),\ B_n^\beta \cap I = \emptyset \textit{ for infinitely many } n \in \omega,$$

<u>then</u> there are infinitely many n such that $B_n^\beta \subseteq A_\alpha$.

Proof. We define $\mathcal{A}$ inductively by first letting $\{A_n \mid n \in \omega\}$ be any partition of ω into infinite sets. Suppose A_β has been constructed for all $\beta < \alpha$, and let $\mathcal{I}_\alpha$ denote the ideal $\mathcal{I}(\bigcup_{\beta<\alpha} A_\beta)$.

We can assume without loss of generality that $(*)$ holds for all $\beta < \alpha$. Next, for each finite partial function $s\colon \omega \to \alpha$, define F_s to be the set consisting of all $x \in [\omega]^{<\omega}$ such that

- $x \cap \bigcup_{i\in\mathrm{dom}(s)} A_{s(i)} = \emptyset$, and
- $\forall i \in \mathrm{dom}(s)\ \exists m > \max\mathrm{dom}(s)\ (B_m^{s(i)} \subseteq x)$.

Let $\mathcal{F} = \{F_s \mid s\colon \omega \to \alpha, \text{ s is a finite partial function}\}$. Then $\mathcal{F} \subseteq [[\omega]^{<\omega}]^\omega$ has the SFIP. Indeed, let s, t be distinct finite partial functions from

ω to α, and define $h\colon \omega \to \alpha$ such that

$$h(i) = \begin{cases} s(i) & \text{if } i \in \mathrm{dom}(s); \\ t(j) & \text{if } j \in \mathrm{dom}(t) \text{ and } i = \max \mathrm{dom}(s) + 1 + j. \end{cases}$$

Then $F_s \cap F_t = F_h \in \mathcal{F}$.

Therefore $\mathcal{F}$ is a set with the SFIP and $|\mathcal{F}| = |\alpha^{<\omega}| < \mathfrak{c} = \mathfrak{p}$, so by definition of $\mathfrak{p}$ there exists an infinite $Y \subseteq [\omega]^{<\omega}$ such that $Y \subseteq^* F$ for each $F \in \mathcal{F}$. Let $A_\alpha := \bigcup Y$. To see A_α is as desired, let $\beta < \alpha$, and let $s\colon \omega \to \alpha$ be any finite partial function such that $s(0) = \beta$. Then

$$\begin{aligned} n \in A_\beta \cap A_\alpha &\Leftrightarrow \exists x \in Y (n \in x \cap A_\beta) \\ &\Rightarrow \exists x \in Y (n \in x \wedge x \notin F_s) \subseteq Y \setminus F_s \end{aligned}$$

and as this last set is finite, so is $A_\beta \cap A_\alpha$.

Moreover we can show that the set $\{n \in \omega \mid B^\beta_n \subseteq A_\alpha\}$ is infinite. Note that as $Y \cap F_s$ is infinite, it is in particular nonempty, and if $y \in Y \cap F_s$ then there is $m > \max \mathrm{dom}(s)$ with $B^{s(0)}_m = B^\beta_m \subseteq y \subseteq A_\alpha$. Because $\max(\mathrm{dom}(s))$ can be taken to be any $n \in \omega$, as we only require $s(0) = \beta$, the integer m above can be taken arbitrarily large. This completes the inductive construction. □

Proposition 3.2.12. *Assume $\mathfrak{p} = \mathfrak{c}$ and let $\mathcal{A} = \{A_\alpha \mid \alpha < \mathfrak{c}\}$ be the mad family constructed in Proposition 3.2.11. Then $\mathcal{A}$ is tight.*

Proof. The proof proceeds almost identically to that of Proposition 3.2.9. Let $\{C_j \mid j \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$, and for each $j \in \omega$ find $\alpha_j < \mathfrak{c}$ such that

$$\langle \{C_j(n)\} : n \in \omega \rangle = \langle B^{\alpha_j}_n : n \in \omega \rangle.$$

Then $\{\alpha_j \mid j \in \omega\}$ is bounded below $\mathfrak{c}$, so there exists $\beta < \mathfrak{c}$ such that $\alpha_j < \beta$ for all $j \in \omega$. Therefore

$$\{\langle \{C_j(n)\} : n \in \omega \rangle \mid j \in \omega\} \subseteq \{\langle B^\alpha_n : n \in \omega \rangle \mid \alpha < \beta\}.$$

For each $\alpha_j < \beta$, it cannot be that there is $I \in \mathcal{I}_\beta$ such that $B^{\alpha_j}_n = \{C_j(n)\} \cap I \neq \emptyset$ for all but finitely many $n \in \omega$, as this implies $C_j \subseteq^* I$ and so $C_j \in \mathcal{I}_\beta$.

But this in turn implies $C_j \in \mathcal{I}(\mathcal{A})$, contradicting the hypothesis on C_j. Therefore it must be that $\{n \in \omega \mid B_n^{\alpha_j} \cap I = \emptyset\}$ is infinite for each $I \in \mathcal{I}_\beta$ and each $j \in \omega$, and therefore by construction of A_β there are infinitely many n such that $B_n^{\alpha_j} = \{C_j(n)\} \subseteq A_\beta$. $\square$

The above together with the fact Miller forcing strongly preserves tightness give a different way to see that $\mathfrak{a} < \mathfrak{d}$ in the Miller model.

We proceed with the proof of Proposition 3.2.28 and Theorem 3.2.32; for this we introduce the definition of the original creature forcing, beginning with the introduction of logarithmic measures. We follow the presentation as given in Abraham [Abr10]; see also [Fis08].

Definition 3.2.13. For s a subset of ω, a *logarithmic measure on s* is a function $h\colon [s]^{<\omega} \to \omega$ such that for all $A, B \in [s]^{<\omega}$ and $\ell \in \omega$, if $h(A \cup B) \geq \ell + 1$, then either $h(A) \geq \ell$ or $h(B) \geq \ell$. A *finite logarithmic measure* is a pair (s, h) such that $s \subseteq \omega$ is finite and h is a logarithmic measure on s.

A standard induction gives the following.

Lemma 3.2.14 ([She84]; [Fis08, Lemma 2.1.3])**.** *If h is a logarithmic measure on s and $h(A_0 \cup \ldots \cup A_{n-1}) > \ell$, then there exists $j < n$ such that $h(A_j) \geq \ell - j$.*

The *level* of a finite logarithmic measure (s, h) is the value $h(s)$ and is denoted $\mathrm{level}(h)$. If $A \subset s$ such that $h(A) > 0$, then A is called *h-positive*.

Definition 3.2.15. Let $P \subseteq [\omega]^{<\omega}$ be an upwards closed collection. The logarithmic measure h on $[\omega]^{<\omega}$ *induced by P* is defined inductively on the cardinality of $s \in [\omega]^{<\omega}$ as follows:

1. $h(e) \geq 0$ for all $e \in [\omega]^{<\omega}$;

2. $h(e) > 0$ if and only if $e \in P$;

3. For all $\ell \geq 1$, $h(e) \geq \ell + 1$ if and only if $|e| > 1$ and for all $e_0, e_1 \subseteq e$ such that $e = e_0 \cup e_1$, then $h(e_0) \geq \ell$ or $h(e_1) \geq \ell$.

Then $h(e) = \ell$ if and only if $\ell \in \omega$ is maximal such that $h(e) \geq \ell$.

Note that if h is as above and e is such that $h(e) \geq \ell$, then $h(a) \geq \ell$ for all sets $a \supseteq e$. In the following we will always assume an induced logarithmic measure is *non-atomic*, meaning there are no h-positive singletons. This assumption is necessary for the proof of the next lemma, which gives a condition on P implying that the logarithmic measure it induces will take arbitrarily high values. This in turn allows for the construction of pure extensions with desired properties.

Lemma 3.2.16 ([Abr10, Lemma 4.7]). *Let $P \subseteq [\omega]^{<\omega}$ be an upwards closed collection of nonempty sets, and let h be the induced logarithmic measure. Suppose that:*

($\dagger$) *for every $n \in \omega$ and for every finite partition $\omega = A_0 \cup \ldots \cup A_{n-1}$, there exists $i < n$ such that $[A_i]^{<\omega}$ contains some $x \in P$.*

Then for every $n, k \in \omega$, and finite partition of ω into sets $A_0 \cup \ldots \cup A_{n-1}$, there exists $i < n$ and $x \subseteq A_i$ such that $h(x) \geq k$.

The above can be proved by induction on $k \in \omega$, arguing by contradiction and appealing to König's lemma; see [Fis08, Lemma 2.1.9, Lemma 2.1.10]. We now define the main forcing notion of interest.

Definition 3.2.17 ([She84, Definition 2.8], [Fis08, Definition 1.3.5]). Let $\mathbb{Q}$ be the partial order consisting of pairs $p = (u, T)$ such that $u \subseteq \omega$ is finite and T is a sequence $T = \langle t_i : i \in \omega \rangle$, where for all $i \in \omega$, t_i is a pair $t_i = (s_i, h_i)$, where h_i is a finite logarithmic measure on s_i, and such that:

1. $\max(u) < \min(s_0)$;

2. $\max(s_i) < \min(s_{i+1})$ for all $i \in \omega$;

3. $\langle h(s_i) : i \in \omega \rangle$ is unbounded and $h_i(s_i) < h_{i+1}(s_{i+1})$ for all $i \in \omega$.

For T as above, let $\mathrm{int}(t_i) = s_i$, and $\mathrm{int}(T) = \bigcup_{i \in \omega} \mathrm{int}(t_i)$. When $e \subseteq \mathrm{int}(t_i)$ is such that $h_i(e) > 0$, we say that e is *t_i-positive*. For conditions $(u_0, T_0), (u_1, T_1) \in \mathbb{Q}$, writing $T_j = \langle t_i^j : i \in \omega \rangle$, $t_i^j = (s_i^j, h_i^j)$ for $j < 2$, define $(u_1, T_1) \leq (u_0, T_0)$, if and only if:

5. u_1 end-extends u_0 and $u_1 \setminus u_0 \subseteq \mathrm{int}(T_0)$;

6. $\mathrm{int}(T_1) \subseteq \mathrm{int}(T_0)$ and there is a sequence $\langle B_i : i \in \omega\rangle$ of finite subsets of ω such that $\max(B_i) < \min(B_{i+1})$ and for each $i \in \omega$, $s_i^1 \subseteq \bigcup_{j\in B_i} s_j^0$;

7. For all $i \in \omega$ and $e \subseteq s_i^1$, if $h_i^1(e) > 0$ then there exists $j \in B_i$ such that $h_j^0(e \cap s_j^0) > 0$.

In the case $u_1 = u_0$, call (u_1, T_1) a *pure extension* of (u_0, T_0).

If $(\emptyset, T)$ is a condition in $\mathbb{Q}$, we will identify $(\emptyset, T)$ and T, and this is meant by $T \in \mathbb{Q}$. For a condition $T = \langle t_i : i \in \omega\rangle \in \mathbb{Q}$ and $k \in \omega$, let

$$i_T(k) = \min\{i \in \omega \mid k < \min(\mathrm{int}(t_i))\}$$

and write $T \setminus k = \langle t_i : i \geq i_T(k)\rangle$. Then $T \setminus k \in \mathbb{Q}$ and $T \setminus k \leq T$. Similarly if $u \subseteq \omega$ is finite, $T \setminus \max(u) \in \mathbb{Q}$. By a slight abuse of notation, we will understand by $(u, T \setminus u)$ to mean the condition $(u, T \setminus \max(u))$ in the case $\max(u) \geq \min(\mathrm{int}(t_0))$.

One way to show $\mathbb{Q}$ is proper is by showing it satisfies Axiom A (see [Abr10, Definition 2.3]); the class of Axiom A forcings was defined by Baumgartner and this is a proper sublclass of proper forcings. We will use Proposition 3.2.28 to establish properness of $\mathbb{Q}$, though that $\mathbb{Q}$ satisfies Axiom A can be established by the following.

Definition 3.2.18. For $n \in \omega$ and $(u_0, T_0), (u_1, T_1) \in \mathbb{Q}$, let $\leq_0$ be the usual partial order on $\mathbb{Q}$. Let us write $T_j = \langle t_i^j : i \in \omega\rangle$ for $j < 2$. Define

1. $(u_1, T_1) \leq_1 (u_0, T_0)$ iff $(u_1, T_1) \leq_0 (u_0, T_0)$ and $u_1 = u_0$, and
2. for $n \geq 1$ let $(u_1, T_1) \leq_{n+1} (u_0, T_0)$ iff $(u_1, T_1) \leq_1 (u_0, T_0)$ and $t_i^1 = t_i^0$ for all $i < n$.

In particular, $(u_1, T_1) \leq_1 (u_0, T_0)$ if and only if (u_1, T_1) is a pure extension of (u_0, T_0).

Definition 3.2.19. Given a sequence $\langle p_i : i \in \omega\rangle \subseteq \mathbb{Q}$, $p_i = (u, T_i)$, $T_i = \langle t_j^i : j \in \omega\rangle$ such that $p_{i+1} \leq_{i+1} p_i$ for all $i \in \omega$, define the *fusion* of $\langle p_i : i \in \omega\rangle$ to be the condition $q := (u, \langle t_j : j \in \omega\rangle)$ such that $t_j := t_j^{j+1}$ for all $j \in \omega$.

If q is the fusion of $\langle p_i : i \in \omega \rangle$, then $q \leq_{i+1} p_i$ for all $i \in \omega$. The following notion is crucial both for proving that $\mathbb{Q}$ is proper and for our preservation result.

Definition 3.2.20. For $(u, T) \in \mathbb{Q}$, with $T = \langle t_i : i \in \omega \rangle$, and D an open dense subset of $\mathbb{Q}$, we say (u, T) is *preprocessed* for D *and* $k \in \omega$ if for every $v \subseteq k$ such that v end-extends u, if $(v, \langle t_j : j \geq k \rangle)$ has a pure extension in D, then already $(v, \langle t_j : j \geq k \rangle) \in D$.

Note that if $(u, T) \in \mathbb{Q}$ is preprocessed for D and k, then any extension of (u, T) is also preprocessed for D and k. This observation and a finite induction yield the next lemma.

Lemma 3.2.21 ([Fis08, Lemma 1.3.9])**.** *For every open dense subset $D \subseteq \mathbb{Q}$, every $k \in \omega$, and every $p \in \mathbb{Q}$, there exists $q \in \mathbb{Q}$ such that $q \leq_{k+1} p$ and q is preprocessed for D and k.*

As a consequence of the existence of fusion for $\mathbb{Q}$:

Lemma 3.2.22. *For every open dense $D \subseteq \mathbb{Q}$ and every $p \in \mathbb{Q}$ there exists a pure extension $q \leq p$ such that q is preprocessed for D and every $k \in \omega$.*

Let $\dot{C}$ be a $\mathbb{Q}$-name for a subset of ω, and let $j \in \omega$. Let $\dot{C}(j)$ denote the name for the j-th smallest element of $\dot{C}$.[1] We say a condition p *decides* $\dot{C}(j)$ if there exists $\ell \in \omega$ such that $p \Vdash \dot{C}(j) = \check{\ell}$. For such $\dot{C}$ and $j \in \omega$, let

$$E_{C(j)} = \{p \in \mathbb{Q} \mid p \text{ decides } \dot{C}(j)\}.$$

Note that when $p \in \mathbb{Q}$ forces that $\dot{C}$ is infinite, the set $E_{C(j)}$ is open dense below p in $\mathbb{Q}$.

Lemma 3.2.23. *Let T be a pure condition in $\mathbb{Q}$, $\dot{C}$ a $\mathbb{Q}$-name for an infinite subset of ω, let $n, j \in \omega$, and fix $v \subseteq n$. Then there exists $R = \langle r_i : i \in \omega \rangle \in \mathbb{Q}$ such that $R \leq T$ and for all $i \in \omega$ and r_i-positive $s \subseteq \operatorname{int}(r_i)$, there exists $w \subseteq s$ such that $(v \cup w, R \setminus \max(s))$ decides $\dot{C}(j)$.*

Proof. Let $T = \langle t_i : i \in \omega \rangle$ with $t_i = (s_i, h_i)$. By Lemma 3.2.22 we can suppose that T is preprocessed for $E_{C(j)}$ and every $k \in \omega$. Let $\mathcal{P}_v(C(j))$ denote the those $x \in [\operatorname{int}(T)]^{<\omega}$ such that:

[1]So, we understand $\dot{C}(0)$ to be the minimal element of $\dot{C}$.

1. For some $k \in \omega$, $x \cap \operatorname{int}(t_k)$ is t_k-positive;

2. There exists $w \subseteq x$ such that $(v \cup w, \langle t_i : i > \max(x) \rangle)$ decides $\dot{C}(j)$.

Note that $\mathcal{P}_v(C(j))$ is upwards closed. Let $h\colon [\omega]^{<\omega} \to \omega$ be the logarithmic measure induced by $\mathcal{P}_v(C(j))$. We will show that h takes arbitrarily high values by showing property (†) of Lemma 3.2.16 holds.

Fix $M \in \omega$ and a finite partition $\omega = A_0 \cup \ldots A_{M-1}$. First,

Claim 3.2.24. *There exists $N < M$ and an extension $T' \leq T$ such that* $\operatorname{int}(T') \subseteq A_N$

Proof. Suppose towards contradiction that for all $N < M$ there is no extension $T' \leq T$ such that $\operatorname{int}(T') \subseteq A_N$. This implies that for any $T' = \langle t'_i : i \in \omega \rangle \leq T$, with $t'_i = (s'_i, h'_i)$, for all $N < M$ the sequence $\langle h'_i(s'_i \cap A_N) : i \in \omega \rangle \notin \mathbb{Q}$, and therefore it must be that $\langle h'_i(s'_i) \cap A_N) : i \in \omega \rangle$ is bounded. As $T' \leq T$ then also $\langle h_i(s_i \cap A_N) : i \in \omega \rangle$ is bounded, so let $J_N \in \omega$ be such that $h_i(s_i \cap A_N) \leq J_N$ for every $i \in \omega$. Let $J := \max_{N<M} J_N$. Since $T \in \mathbb{Q}$, by (3) of Definition 3.2.17 there exists $i \in \omega$ such that $h_i(s_i) > J + M$. But then $h_i(\bigcup_{N<M} s_i \cap A_N) \geq J + M + 1$ so by Lemma 3.2.14 there exists $N < M$ such that $h_i(s_i \cap A_N) \geq J + M - N \geq J + 1 > J$, a contradiction. $\square$

Therefore fix T' and $N < M$ as given by the above claim. Since $(v, T' \setminus v) \in \mathbb{Q}$, there exists $w \subseteq \operatorname{int}(T' \setminus v)$ and R such that $(v \cup w, R)$ is a condition in $\mathbb{Q}$ extending $(v, T' \setminus v)$ and $(v \cup w, R)$ decides the value of $\dot{C}(j)$. Since $w \subseteq \operatorname{int}(T' \setminus v)$ is finite and using the definition of the extension relation, there exists $m_0, m_1 \in \omega$ such that $w \subseteq \bigcup_{m \in [m_0, m_1]} \operatorname{int}(t'_m)$. Let

$$x := \bigcup_{m \in [m_0, m_1]} \operatorname{int}(t'_m)$$

and note $x \in [A_N]^{<\omega}$. As $T' \setminus v \in \mathbb{Q}$ we may assume there is at least one $m \in [m_0, m_1]$ such that $\operatorname{int}(t'_m)$ is t'_m-positive. Then $T' \setminus v \leq T \setminus v$ so there exists $k \in \omega$ such that $h_k(\operatorname{int}(t'_m) \cap \operatorname{int}(t_k)) = h_k(x \cap \operatorname{int}(t_k)) > 0$. Therefore $x \subseteq \operatorname{int}(T') \subseteq A_N$ satisfies (1) in the definition of $\mathcal{P}_v(C(j))$.

We can also show that (2) holds. We have that $w \subseteq x$ was such that $(v \cup w, R) \in E_{C(j)}$, and T is preprocessed for $E_{C(j)}$ and $\max w$. Since $(v \cup w, T \setminus w)$

has a pure extension into $E_{C(j)}$, we conclude that already $(v \cup w, T \setminus w) \in E_{C(j)}$. Altogether we have found $x \in \mathcal{P}_v(C(j)) \cap [A_N]^{<\omega}$, verifying (†).

We can now define $R = \langle r_n : n \in \omega \rangle$, where $r_n = (x_n, g_n)$, inductively as follows. Clearly $\mathcal{P}_v(C(j))$ is nonempty, as we just showed above, so pick $x_0 \in \mathcal{P}_v(C(j))$, and let $g_0 := h \restriction \mathcal{P}(x_0)$. Assuming $r_i = (x_i, g_i)$ defined for all $i \leq n$ so that $\max(x_i) < \min(x_{i+1})$ and $g_i(x_i) < g_{i+1}(x_{i+1})$ for $i < n$, since h takes arbitrarily high values there is $x_{n+1} \in \mathcal{P}_v(C(j))$ with $h(x_{n+1}) > h(x_n)$. We can assume $\max(x_n) < \min(x_{n+1})$, since otherwise h is bounded. Let $g_{n+1} = h \restriction \mathcal{P}(x_{n+1})$; this completes the definition of R.

Then $R \in \mathbb{Q}$, and R extends T: as each $x_n = \mathrm{int}(r_n)$ is a finite subset of $\mathrm{int}(T)$ there are $m_0^n < m_1^n \in \omega$ with $x_n \subseteq \bigcup_{i \in [m_0^n, m_1^n]} \mathrm{int}(t_i)$, so the sequence $\langle [m_0^n, m_1^n] : n \in \omega \rangle$ witnesses (6) and (7) of Definition 3.2.17. Indeed, if $s \subseteq x_n$ is such that $g_n(s) = h(s) > 0$, by (1) of the definition of $\mathcal{P}_v(C(j))$ there is k such that $s \cap \mathrm{int}(t_k)$ is t_k-positive, and necessarily $k \in [m_0^n, m_1^n]$. Moreover, R is as desired, since if $s \subseteq x_i$ is r_i-positive, by (2) of the definition of $\mathcal{P}_v(C(j))$ there is $w \subseteq s$ such that $(v \cup w, \langle t_i : i > \max(s) \rangle)$ decides $C(j)$, but $(v \cup w, \langle r_i : i > \max(s) \rangle) \leq (v \cup w, \langle t_i : i > \max(s) \rangle)$ and so makes the same decision.

□

Remark 3.2.25. If a condition $R = \langle r_i : i \in \omega \rangle$ in $\mathbb{Q}$ has the property as in the conclusion of the above lemma, then any further extension retains this same property. Indeed, if $T = \langle t_i : i \in \omega \rangle \leq R$ and $s \subseteq \mathrm{int}(t_i)$ is t_i-positive, then there is $j \in \omega$ such that $s \cap \mathrm{int}(r_j)$ is r_j-positive. Then there is $w \subseteq s \cap \mathrm{int}(r_j)$ so that $(v \cup w, R \setminus \max(s \cap \mathrm{int}(r_j)))$ decides $C(j)$. So in particular $w \subseteq s \subseteq \mathrm{int}(t_i)$, and w is such that $(v \cup w, T \setminus \max(s))$ makes the same decision about $C(j)$, since it is an extension of the condition $(v \cup w, T \setminus \max(s \cap \mathrm{int}(r_j)))$, which is an extension of $(v \cup w, R \setminus \max(s \cap \mathrm{int}(r_j)))$.

This remark is important for establishing the next lemma:

Lemma 3.2.26. *For any $T \in \mathbb{Q}$, any $n, j \in \omega$ and $\dot{C}$ a $\mathbb{Q}$-name for an infinite subset of ω, there exists $R = \langle r_i : i \in \omega \rangle \in \mathbb{Q}$ such that $R \leq T$ and for all $v \subseteq n$, for all $i \in \omega$ and r_i-positive $s \subseteq \mathrm{int}(r_i)$, there is $w \subseteq s$ such that $(v \cup w, R \setminus s)$ decides $\dot{C}(j)$.*

Proof. Fix $T \in \mathbb{Q}$, fix $n, j \in \omega$, and let $\dot{C}$ be a $\mathbb{Q}$-name for an element of $[\omega]^\omega$. Let $\{v_k \mid k < 2^n\}$ enumerate all subsets of n, and consider v_0. By

Lemma 3.2.23 there is $T_0 = \langle t_i^0 : i \in \omega \rangle \leq T$ such that for all $i \in \omega$ and t_i^0-positive sets $s \subseteq \operatorname{int}(t_i^0)$, there is $w_0 \subseteq s$ such that $(v_0 \cup w_0, T_0 \setminus s)$ decides $\dot{C}(j)$. Next considering v_1 and T_0, by Lemma 3.2.23 and Remark 3.2.25 there is $T_1 = \langle t_i^1 : i \in \omega \rangle \leq T_0$ such that for all $\ell < 2$ and v_ℓ, for all $i \in \omega$ and t_i^1-positive set $s \subseteq \operatorname{int}(t_i^1)$, there is $w_\ell \subseteq s$ such that $(v_\ell \cup w_\ell, T_1 \setminus s)$ decides $\dot{C}(j)$. Continuing in this way, for each $k < 2^n$ we obtain $T_k = \langle t_i^k : i \in \omega \rangle \in \mathbb{Q}$ extending T and every T_ℓ for $\ell \leq k$, such that for all $\ell \leq k$, for all $i \in \omega$ and t_i^k-positive $s \subseteq \operatorname{int}(t_i^k)$, there exists $w_\ell \subseteq s$ such that $(v_\ell \cup w_\ell, T_k \setminus s)$ decides $\dot{C}(j)$. Then in particular $R := T_{2^n-1} \in \mathbb{Q}$ satisfies the conclusion of the lemma, since if v is any subset of n, there is $k < 2^n$ such that $v = v_k$, and as $R \leq T_k$, for every $i \in \omega$ and r_i-positive $s \subseteq \operatorname{int}(r_i)$ there is $w \subseteq s$ so that $(v \cup w, R \setminus s)$ decides $\dot{C}(j)$. □

Corollary 3.2.27. *For any $(u, T) \in \mathbb{Q}$, any $n, j \in \omega$, and any $\dot{C}$ a $\mathbb{Q}$-name for an infinite subset of ω, there exists $(u, R) \leq_{n+1} (u, T)$ such that for all $v \subseteq n$, for all $i \geq n$ and for every $s \subseteq \operatorname{int}(r_i)$ which is r_i-positive, there exists $w \subseteq s$ such that $(v \cup w, R \setminus s)$ decides the value of $\dot{C}(j)$.*

Proof. Fix $(u, T) \in \mathbb{Q}$ and $n, j \in \omega$, and write $T = \langle t_i : i \in \omega \rangle$. Consider $(\emptyset, \langle t_i : i \geq n \rangle) \in \mathbb{Q}$. By Lemma 3.2.26 there exists $R' = \langle r_i' : i \geq n \rangle \leq \langle t_i : i \geq n \rangle$ with the property that for any $v \subseteq n$, any $i \geq n$ and r_i'-positive $s \subseteq \operatorname{int}(r_i')$, there is $w \subseteq s$ so that $(v \cup w, R' \setminus s)$ decides $\dot{C}(j)$.

Define $R = \langle r_i : i \in \omega \rangle$ by letting $r_i = t_i$ for $i < n$, and for $i \geq n$ let $r_i = r_i'$. Then $(u, R) \leq_{n+1} (u, T)$ is as desired. □

We now come to the central result of this section.

Proposition 3.2.28. *Let $\mathcal{A}$ be a tight mad family. For every $p \in \mathbb{Q}$ and $M \prec H_\theta$ countable elementary submodel, where θ is sufficiently large, containing $\mathbb{Q}, p, \mathcal{A}$, and every $B \in \mathcal{I}(\mathcal{A})$ such that $|B \cap Y| = \aleph_0$ for all $Y \in \mathcal{I}(\mathcal{A})^+ \cap M$, and $\dot{Z}$ a $\mathbb{Q}$-name for an element of $\mathcal{I}(\mathcal{A})^+$ in M, there exists a pure extension $q \leq p$ such that q is $(M, \mathbb{Q}, \mathcal{A}, B)$-generic.*

Proof. Fix θ, M and B as above, and let (u, T_0) be a condition in $\mathbb{Q} \cap M$. Let $\{D_n \mid n \in \omega\}$ enumerate all open dense subsets of $\mathbb{Q}$ in M, and let $\{\dot{Z}_n \mid n \in \omega\}$ enumerate all $\mathbb{Q}$-names for subsets of ω in M which are forced to be in $\mathcal{I}(\mathcal{A})^+$ such that each name appears infinitely often. We inductively define a sequence $\langle q_n : n \in \omega\rangle$ of conditions in $\mathbb{Q} \cap M$, where $q_0 = (u, T_0)$, $q_n = (u, T_n)$ with $T_n = \langle t_i^n : i \in \omega\rangle$, such that the following are satisfied:

1. For all $n \in \omega$, $q_{n+1} \leq_{n+1} q_n$;
2. q_{n+1} is preprocessed for D_n and every $k \in \omega$;
3. For all $v \subseteq n$, for all $i \geq n$ and $s \subseteq \operatorname{int}(t_i^{n+1})$ which is t_i^{n+1}-positive, if v is an end-extension of u, then for some $w \subseteq s$, $((v \cup w), \langle t_j^{n+1} : j > \max(s)\rangle) \Vdash (\dot{Z}_n \cap B) \setminus n \neq \emptyset$.

Suppose q_n has been constructed thus far; we will define q_{n+1}. Consider the condition $(u, \langle t_i^n : i \geq n\rangle) \leq q_n$. By Lemma 3.2.22, there exists a pure extension $(u, \langle t_i' : i \geq n\rangle) \leq (u, \langle t_i^n : i \geq n\rangle)$ which is preprocessed for D_n and every $k \in \omega$. Let $q_n^0 = (u, \langle t_i^{n,0} : i \in \omega\rangle)$, where for $i < n$, $t_i^{n,0} = t_i^n$, and $t_i^{n,0} = t_i'$ for $i \geq n$. Then $q_n^0 \leq_{n+1} q_n$ and q_n^0 satisfies (2).

Next, consider the set

$$\begin{aligned} W_{n+1} = \{m \in \omega \mid \exists r = (u, \langle t_i' : i \in \omega\rangle) \leq_{n+1} q_n^0 \text{ satisfying:} \\ \forall v \subseteq n\ \forall i \geq n\ \forall s \subseteq \operatorname{int}(t_i') \ [s \text{ is } t_i'\text{-positive} \Rightarrow \exists w \subseteq s \\ (v \cup w, \langle t_i' : i > \max(s)\rangle) \Vdash m \in \dot{Z}_n\} \end{aligned}$$

Claim 3.2.29. *$W_{n+1} \in \mathcal{I}(\mathcal{A})^+ \cap M$.*

Proof. We have that $W_{n+1} \in M$ since it is definable from the forcing relation and from $\mathbb{Q}$, q_n^0, and $\dot{Z}_n$, which are all assumed to be elements of M.

To see $W_{n+1} \notin \mathcal{I}(\mathcal{A})$, let F be a finite subset of $\mathcal{A}$; we show $W_{n+1} \setminus \bigcup F$ is infinite. Since $\Vdash_{\mathbb{Q}} \dot{Z}_n \in \mathcal{I}(\mathcal{A})^+$, in particular q_n^0 forces that $\dot{Z}_n$ is not finitely covered by $\bigcup F$, or in other words q_n^0 forces that the set $\dot{Z}_n \setminus \bigcup F$ is infinite. Let $\dot{C}$ be a $\mathbb{Q}$-name for the set $\dot{Z}_n \setminus \bigcup F$.

Now as $q_n^0 \in \mathbb{Q}$, and $\dot{C}$ is a $\mathbb{Q}$-name for an infinite subset of ω, for all $k \in \omega$ Corollary 3.2.27 gives $q^j \leq_{n+1} q_n^0$, where $q^j = (u, R_j)$ with $R_j = \langle r_i^j : i \in \omega\rangle$, and q^j has the property that for all $v \subseteq n$, for all $i \geq n$ and r_i^j-positive $s \subseteq \operatorname{int}(r_i^j)$, there is $w \subseteq s$ such that $(v \cup w, R_j \setminus s)$ decides $\dot{C}(j)$, and $j > k$. So there exists $m_j \in \omega$ such that

$$(v \cup w, R_j \setminus s) \Vdash \dot{C}(j) = \check{m}_j.$$

Note that if $m_j = \dot{C}(j)$, then $m_j \geq j > k$. Therefore for all $k \in \omega$ there exists $m_j > k$ such that $m_j \in W_{n+1}$, witnessed by q^j as above, and moreover $m \notin \bigcup F$ since $q^j \Vdash \check{m}_j \notin \bigcup F$. Thus $W_{n+1} \setminus \bigcup F$ is infinite, and as $F \in [\mathcal{A}]^{<\omega}$ was arbitrary this proves the claim. □

By assumption on the set B, there exists $m_{n+1} \in W_{n+1} \cap B$ such that $m > n$. Let $r = (u, \langle t'_i : i \in \omega \rangle) \leq_{n+1} q_n^0$ be given by $m_{n+1} \in W_{n+1}$, and define $q_{n+1} = (u, \langle t_i^{n+1} : i \in \omega \rangle)$ such that $t_i^{n+1} = t'_i$ for all $i \in \omega$. Since $r \leq_{n+1} q_n^0 \leq_{n+1} q_n$, we have that $q_{n+1} \leq_{n+1} q_n$. This completes the inductive construction.

Let $q = (u, T)$ be the fusion of the q_n's, so $T = \langle t_i : i \in \omega \rangle$ with $t_i = t_i^{i+1}$ for all $i \in \omega$.

We show q is $(M, \mathbb{Q})$-generic by showing that for all $n \in \omega$, the set $D_n \cap M$ is predense below q.[2] Towards this end let $r = (v, R)$ be an arbitrary extension of q; as D_n is dense there exists $w \subseteq \mathrm{int}(R)$ such that $(v \cup w, R \setminus \max(w)) \in D_n$. Then as $(v \cup w, R \setminus \max(w)) \leq (u, T_{n+1} \setminus \max(w)) \leq q_{n+1}$ and q_{n+1} is preprocessed for D_n and $\max(w)$, already $r' = (v \cup w, T_{n+1} \setminus \max(w)) \in D_n$. Then $r' \in M$ and r, r' are compatible, as witnessed by the condition $(v \cup w, R \setminus \max(w))$. Therefore q is an $(M, \mathbb{Q})$-generic condition.

Next, we show $q \Vdash |\dot{Z}_n \cap B| = \omega$ for every $n \in \omega$. Fix n, and let $(v, R) \leq q$ be arbitrary; it suffices to show that for every $k \in \omega$ there exists an extension of (v, R) which forces $(\dot{Z}_n \cap B) \setminus k \neq \emptyset$. Find $i \in \omega$ such that $i > k$, $v \subseteq i$, $\dot{Z}_n = \dot{Z}_i$ and $s = \mathrm{int}(R) \cap \mathrm{int}(t_i)$ is t_i-positive. The fact that s is $t_i = t_i^{i+1}$-positive and $r \leq q \leq q_{i+1}$ implies, by item (3), that there exists $w \subseteq s$ such that

$$(v \cup w), \langle t_j^{i+1} : j > \max(s) \rangle) \Vdash m_{i+1} \in \dot{Z}_i,$$

where $m_{i+1} \in B$ and $m_{i+1} \geq i > k$.

Then $(v \cup w, R \setminus s)$ is an extension both of (v, R) and of the condition $(v \cup w, \langle t_j^{i+1} : j > \max(s) \rangle)$, so by the latter,

$$(v \cup w, R \setminus s) \Vdash m_{i+1} \in (B \cap \dot{Z}_i) \setminus k = (B \cap \dot{Z}_n) \setminus k.$$

[2]Recall a subset D of a forcing notion $\mathbb{P}$ is *predense* if and only if for every $p \in \mathbb{P}$ there exists $q \in D$ such that p, q are compatible, i.e. have a common extension. $D \subseteq \mathbb{P}$ is predense below some $p \in \mathbb{P}$ if D is predense as a subset of $\mathbb{P} \restriction p = \{q \in \mathbb{P} \mid q \leq p\}$.

This completes the proof that q is an $(M, \mathbb{Q}, \mathcal{A}, B)$-generic condition, giving that $\mathcal{A}$ remains a tight mad family in any $\mathbb{Q}$-generic extension. $\square$

In addition to the previous proposition, in order to show a countable support iteration of $\mathbb{Q}$ is proper and increases the value of $\mathfrak{s}$, we need the following.

Lemma 3.2.30. *Let G be $\mathbb{Q}$-generic over V, let $\mathcal{S} \subseteq [\omega]^\omega$ be an element of V, and let $a = \{s \subseteq \omega \mid \exists T(s,T) \in G\}$. Then for all $b \in \mathcal{S}$, (b does not split a)$^{V[G]}$.*

Proof. Fix $\mathcal{S} \subseteq [\omega]^\omega \cap V$, a $\mathbb{Q}$-generic filter G, and define a as above. To show one of $a \cap b$ or $a \setminus b$ is finite, it suffices to show the set

$$D_b = \{(s,T) \in \mathbb{Q} \mid \mathrm{int}(T) \subseteq b \vee \mathrm{int}(T) \subseteq \omega \setminus b\}$$

is dense in $\mathbb{Q}$. Indeed this is sufficient since if $(s,T) \in D_b \cap G$, then $s \subseteq a$ and any further end-extension t of s is such that $t \setminus s \subseteq \mathrm{int}(T)$, so then $a \setminus s \subseteq \mathrm{int}(T)$. As $(s,T) \in D_b$, this implies $a \setminus s \subseteq b$ or $a \setminus s \subseteq \omega \setminus b$; in the latter case $b \cap a$ is finite, and in the former $a \subseteq^* b$. So b does not split a. D_b is dense because if (s,T) is any condition, $\omega = b \cup (\omega \setminus b)$ is a finite partition of ω and therefore (see Claim 3.2.24) there exists $(s,T') \leq (s,T)$ such that $\mathrm{int}(T') \subseteq b$ or $\mathrm{int}(T') \subseteq \omega \setminus b$. $\square$

In the proof of Theorem 3.2.32 below we will appeal to the following facts about proper forcings and their iterations. For their proofs, see [Abr10, Theorem 2.7], [Abr10, Theorem 2.10], and [Abr10, Theorem 2.12], respectively.

Lemma 3.2.31. *Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \delta, \beta < \delta \rangle$ be a countable support iteration such that for all $\alpha < \delta$, $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ proper". Then:*

1. *$\mathbb{P}_\delta$ is proper.*
2. *If also CH is assumed, $\delta \leq \omega_2$, and for all $\alpha < \delta$, $\Vdash_{\mathbb{P}_\alpha}$ "$|\dot{\mathbb{Q}}_\alpha| = \omega_1$", then $\mathbb{P}_\delta$ is $\aleph_2$-cc.*
3. *If also CH is assumed, $\delta < \omega_2$, and for all $\alpha < \delta$, $\Vdash_{\mathbb{P}_\alpha}$ "$|\dot{\mathbb{Q}}_\alpha| = \omega_1$", then CH holds in $V^{\mathbb{P}_\delta}$.*

Now we are in a position to prove:

Theorem 3.2.32 ([She84, Theorem 3.1])**.** *Assume* CH*, and let* $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2\rangle$ *be a countable support iteration such that for all* $\alpha < \omega_2$*,* $\dot{\mathbb{Q}}_\alpha$ *is a* $\mathbb{P}_\alpha$*-name for the partial order* $\mathbb{Q}$ *of Definition 3.2.17. Let* G *be* $\mathbb{P}_{\omega_2}$*-generic over* V*. Then* $V[G] \models \aleph_1 = \mathfrak{a} < \mathfrak{s} = \aleph_2$*.*

Proof. Let $\mathcal{A} \in V$ be a tight mad family; such a mad family exists since CH holds in V, by Proposition 3.2.9. Let G be $\mathbb{P}_{\omega_2}$-generic over V. For all $\alpha < \omega_2$, $\mathbb{P}_\alpha$ forces that $\dot{\mathbb{Q}}_\alpha$ is a proper forcing notion which strongly preserves the tightness of $\mathcal{A}$. Then $\mathbb{P}_{\omega_2}$ is proper and moreover, using Lemma 3.2.6, we also have $\mathbb{P}_{\omega_2}$ strongly preserves the tightness and hence maximality of $\mathcal{A}$. Therefore $V[G] \models \mathfrak{a} = \aleph_1$.

Let $\mathcal{S} \subseteq [\omega]^\omega$ be family of cardinality strictly less than $\aleph_2$ with $\mathcal{S} \in V[G]$, and let $\dot{\mathcal{S}}$ be a $\mathbb{P}_{\omega_2}$-name for $\mathcal{S}$. Then there exists $\alpha < \omega_2$ such that $\mathcal{S} \in V[G_\alpha]$, where $G_\alpha = \mathbb{P}_\alpha \cap G$. This is because we can assume all names for reals are *nice*, meaning any name for a real $\dot{x}$ can be identified with countably many antichains $\{\mathcal{B}_n \mid n \in \omega\} \subseteq \mathbb{P}_{\omega_2}$, where $\mathcal{B}_n$ consists of conditions forcing $n \in \dot{x}$ or $n \notin \dot{x}$ (see also Definition 3.3.27 of the following section). As CH holds in V, the $\aleph_2$-cc-ness of $\mathbb{P}_{\omega_2}$ gives that any such $\mathcal{B}_n$ is of size at most $\aleph_1$. Then the union of the supports $\mathrm{supp}(p) = \{\alpha < \omega_2 \mid p(\alpha) \neq 1_{\mathbb{P}}\}$ of those $p \in \mathbb{P}_{\omega_2}$ such that p appears in an antichain associated to some name $\dot{x} \in \dot{\mathcal{S}}$ is bounded by some $\alpha < \omega_2$. Therefore $\dot{\mathcal{S}}$ is a $\mathbb{P}_\alpha$-name and $\dot{\mathcal{S}}^{G_\alpha} = \dot{\mathcal{S}}^G$. Then by definition, $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for $\mathbb{Q}$ so by Lemma 3.2.30, $\mathcal{S}$ is not a splitting family in $V[G_{\alpha+1}]$, as witnessed by the $\mathbb{Q}_\alpha$-generic real. As $\mathcal{S}$ was an arbitrary family of size $\kappa < \aleph_2$ this shows $(\mathfrak{s} = \aleph_2)^{V[G]}$. $\square$

3.3 Projective wellorders

Projective wellorderings of the reals are indicative of to what extent regularity properties hold for the projective classes, as the definable wellordering yields examples of nonregular sets—such as a non-Lebesgue measurable set, a set without the property of Baire, or in general any other interesting set arising from an application of the Axiom of Choice—which are of the same descriptive complexity as that of the wellorder. Accordingly, projectively definable wellorderings are typically antagonistic with large cardinal axioms and compactness principles, as these assert regularity properties and determinacy for the projective pointclasses beyond the analytic sets.

Recall that under $V = L$ there exists a Δ^1_2 wellorder of the reals [Göd39]; this complexity is optimal, since any wellordering of the reals is neither Lebesgue measurable nor has the property of Baire. Conversely, Mansfield's theorem states that if there exists a Σ^1_2 wellordering of the reals, then all reals are constructible, so in particular CH holds. Using a finite support iteration of ccc forcings, Harrington [Har77, Theorem B] showed that a $\boldsymbol{\Delta}^1_3$ wellordering is consistent with $\neg\mathsf{CH}$ and Martin's Axiom (MA). Harrington relied on the technique of *almost disjoint coding* developed by Solovay and Jensen [SJ70], as well as the work of Jensen and Johnsbråten [JJ74] on Π^1_2 singletons. Friedman and Caicedo [CF11] showed that the Bounded Proper Forcing Axiom (BFPA) and the assumption $\omega_1 = \omega_1^L$ imply the existence of a Σ^1_3 wellorder of the reals.

However in these last two constructions, the forcing axioms rendered all cardinal characteristics equivalent to $\mathfrak{c}$; the question of projective wellorderings of the reals in models with nontrivial structure of cardinal characteristics of the continuum was first addressed by Fischer and Friedman in 2010 [FF10]. Using a countable support iteration of S-proper forcing notions they showed a Δ^1_3 wellorder of the reals is compatible with $\mathfrak{c} = \aleph_2$ and each of the following inequalities: $\mathfrak{d} < \mathfrak{c}$, $\mathfrak{b} < \mathfrak{a} = \mathfrak{s}$, $\mathfrak{b} < \mathfrak{g}$. This was made possible by defining a new forcing notion, *Sacks coding* (Definition 3.3.15), which uses Sacks reals to code the wellorder, giving a way of forcing a Δ^1_3 wellorder with an ω^ω-bounding iteration.

Mad families of size continuum can be obtained given a wellordering of the continuum, and so it is natural to ask about their definability in the sense of classical descriptive set theory. We recall the exposition of this study given in Chapter 2. Mathias [Mat77] initiated this line of inquiry, when in 1969 he showed no analytic almost disjoint family can be maximal. Subsequent work revealed further the antagonism between mad families and determinacy assumptions: under ZF+DC+Projective Determinacy (PD) there are no projectively definable mad families, under AD no mad family can be an element of $L(\mathbb{R})$, there are no mad families in Solovay's model, and if $\aleph_1^{L[r]} < \aleph_1$ for some real r, then there are no $\Sigma^1_2(r)$ mad families (see, respectively, [NN18], [BHST22], and [Tö18]).

On the other hand, under $V = L$ there exists Σ^1_2 mad families, and this was significantly improved by Miller's construction of a coanalytic mad family in L [Mil89], which is achieved by robust coding techniques due to Erdős, Kunen, and Mauldin for reducing the complexity of sets obtained by the Σ^1_2 wellordering of the constructible reals. In fact it suffices to assume the existence of a Σ^1_2 mad family [T09] in order to obtain a Π^1_1 mad family. By Mathias's result, this projective complexity is optimal.

It is interesting to ask whether in ZFC alone one can construct a cardinal preserving forcing iteration yielding a generic extension with Δ^1_3 wellorder in the presence of $\neg\mathsf{CH}$, while simultaneously controlling values of cardinal characteristics as well as the definability of the witnesses to those values. In 2022 Bergfalk, Fischer, and Switzer [BFB22] established that a Δ^1_3 wellorder of the reals is consistent with $\mathfrak{a} = \mathfrak{u} < \mathfrak{i}$, $\mathfrak{a} = \mathfrak{i} < \mathfrak{u}$, and $\mathfrak{a} < \mathfrak{u} = \mathfrak{i}$, with the added feature that the witnesses to the cardinal characteristics of value $\aleph_1$ can be taken to be coanalytic. They do this by showing various preservation properties of the Sacks coding forcing, and in particular they show this coding strongly preserves tightness of tight mad families from the ground model ([BFB22, Lemma 4.3], given in Fact 3.3.18, item (5) below). In this section we use this last preservation result as well as Proposition 3.2.28 to show the following:

Theorem (Theorem 3.3.35)**.** *It is consistent with a Δ^1_3 wellorder of the reals and $\mathfrak{c} = \aleph_2$ that $\aleph_1 = \mathfrak{a} < \mathfrak{s} = \aleph_1$, and $\mathfrak{a} = \aleph_1$ is witnessed by a Π^1_1-definable tight mad family.*

The obtention of a new cardinal inequality together with a projective wellorder of the reals answers (1) of Question 7 from [FF10]. Witnessing this inequality with an optimal projective witness for $\mathfrak{a} = \aleph_1$ responds to (2) of Question 7 [FF10].

Our strategy for the above theorem will be to define a countable support iteration in a model of $V = L$, in which there exists a Π^1_1-definable tight mad family $\mathcal{A}$ by [BFB22, Lemma 4.2]. Each iterand will be a $\mathbb{P}_\alpha$-name for an S-proper forcing (see Definition 3.3.3 below) which strongly preserves the tightness of $\mathcal{A}$, and along the way we construct the wellorder $<_G = \bigcup_{\alpha<\omega_2} <_\alpha$ by defining the initial segments $<_\alpha$, where $<_\alpha$ wellorders the reals of $L^{\mathbb{P}_\alpha}$, the α-th stage of the iteration. We let $\dot{<}_\alpha$ denote a $\mathbb{P}_\alpha$-name for $<_\alpha$. This ordering $<_\alpha$ can be naturally defined using the wellordering of the nice $\mathbb{P}_\alpha$-names for the reals of $L^{\mathbb{P}_\alpha}$; using appropriate bookeeping, at stage α we add a Sacks-generic real coding a pair of reals $x, y \in V^{\mathbb{P}_\alpha}$ such that $x <_\alpha y$. The way in which the α-th generic real codes these intial stages of the iteration is done so to yield the Δ^1_3-definability of the wellorder, in the following way.

At stage α, the Sacks-generic real r_α will code a countable sequence of generic club subsets of ω_1,

$$\vec{C}_\alpha = \langle C_{\alpha+m} : m \in \Delta(x * y)\rangle$$

as well as a set $Y \subset \omega_1$, which have just been added to $L^{\mathbb{P}_\alpha}$. The set $\Delta(x * y) \subseteq \omega$ is a recursive coding of the pair (x, y), and by adding $\vec{C}_\alpha$, indicates a pattern of stationary/nonstationary in the sequence

$$\langle S_{\alpha+m} : m \in \omega\rangle \in L,$$

where $S_{\alpha+m} \subseteq \omega_1^L$ is stationary costationary in L, and $C_{\alpha+m} \cap S_{\alpha+m} = \emptyset$. Note that, if also r_α codes the pair (x, y), then

$$L[r_\alpha] \models \Delta(x * y) \subseteq \{m \in \omega \mid S_{\alpha+m} \text{ is nonstationary}\}.$$

If it can also be assured that for no $m \notin \Delta(x * y)$, the set $S_{\alpha+m}$ loses its stationarity in the final extension (an "accidental stationary kill"; this appears as Claim 3.3.33 below), the idea is that in the final generic extension $L[\langle r_\alpha : \alpha < \omega_2\rangle]$, the wellorder $<_G$ has the following definition.

$$x <_G y \Leftrightarrow \exists \alpha < \omega_2\ L[r_\alpha] \models \Delta(x * y) = \{m \in \omega \mid S_{\alpha+m} \text{ is nonstationary}\}.$$

To make the above a (lightface) projective definition of $<_G$, the set Y is added after $\vec{C}_\alpha$ in order to localize the generic nonstationarity of each $S_{\alpha+m}$ to a large class of countable transitive ZF^- models (Definition 3.3.9 below); this uses what is sometimes referred to as "David's Trick". Roughly, this allows us to bound the initial existential quantification to $\omega_2^{\mathcal{M}}$, where $\mathcal{M}$ belongs to the said class of countable transitive models, and $\mathcal{M}$ contains the real r_α. This idea will also be explained in Section 3.4, when we add a generic Π^1_2 tight mad family of size $\mathfrak{c}$.

A very important fact we will use in our complexity calculations is that the satisfaction relation $\models$ for countable models of set theory is Δ^1_1-definable; this follows since a countable model of set theory is essentially a pair (ω, E), such that E is a binary relation on ω and (ω, E) is a model of set theory. All of this information can be coded by a single real; details are given in [MW85, Example 1.20].

We next give the definitions of the three forcing notions involved in the definition of the countable support iteration giving a Δ^1_3 wellorder of large continuum in [FF10] and then outline this construction, with the appropriate modification allowing for Theorem 3.3.35.

Club shooting

Baumgartner, Harrington, and Kleinberg [BHK76] introduced a cardinal preserving forcing notion which, given a stationary costationary $S \subseteq \omega_1$, adds a closed unbounded $C \subseteq \omega_1$ such that $C \cap S = \emptyset$. The forcing is often referred to as *club shooting*.

Definition 3.3.1. Let $S \subseteq \omega_1$ be a stationary costationary set. Define $Q(S)$ to be the partial order consisting of closed, bounded subsets of $\omega_1 \setminus S$, ordered by end-extension.

Lemma 3.3.2. *The following hold:*

1. *$Q(S)$ is ω-distributive, thus does not add new reals.*[3]
2. *Let G be $Q(S)$-generic, and let $C_G = \bigcup\{d \in Q(S) \mid d \in G\}$. Then C_G is a club in $\omega_1^{V[G]}$, and witnesses $(\check{S}$ is nonstationary$)^{V[G]}$.*

Proof. See, for example, [Jec03, Chapter 25] or [Cum10, Section 6]. $\square$

By item (2) above, $Q(S)$ is not a proper forcing notion since it does not preserve the stationarity of every stationary subset of ω_1. However, it still retains many of the desirable properties of a proper forcing.

[3] A forcing notion $\mathbb{P}$ is *ω-distributive* if the intersection of countably many dense open subsets of $\mathbb{P}$ is again dense open. Equivalently, any ω-distributive forcing adds no new countable sequences of ordinals.

Definition 3.3.3. Let $S \subseteq \omega_1$ be a stationary set. A forcing notion $\mathbb{P}$ is S-*proper* if for all countable elementary submodels $M \prec H_\theta$, with θ sufficiently large, and such that $M \cap \omega_1 \in S$, for every $p \in \mathbb{P} \cap M$ there is $q \leq p$ which is $(M, \mathbb{P})$-generic.

A proof of the following can be found in [Gol98, Theorem 3.7]

Lemma 3.3.4. *Suppose $S \subseteq \omega_1$ is stationary and $\mathbb{P}$ is an S-proper forcing notion. Then $\mathbb{P}$ preserves ω_1 as well as the stationarity of any stationary subset of S.*

Lemma 3.3.5. *$Q(S)$ is $(\omega_1 \setminus S)$-proper.*

Importantly, S-properness is preserved under countable support iterations; this is proved as in the case of properness (see for example, [Abr10, Theorem 2.7]). That this proof also works to yield the second clause below was noted in [BFB22].

Lemma 3.3.6. *Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \delta, \beta < \delta \rangle$ be a countable support iteration such that for all $\alpha < \delta$, $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ is an S-proper poset". Then $\mathbb{P}_\delta$ is S-proper. Moreover, if $\mathcal{A}$ is a tight mad family in the ground model and for all $\alpha < \omega_2$, $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ strongly preserves the tightness of $\check{\mathcal{A}}$", then also $\mathbb{P}_\delta$ strongly preserves the tightness of $\mathcal{A}$.*

The next two lemmas are shown in the case of proper forcings ([Abr10, Theorem 2.10] and [Abr10, Theorem 2.12], respectively).

Lemma 3.3.7. *Assume CH, let $\delta \leq \omega_2$ be an ordinal, and let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \delta, \beta < \delta \rangle$ be a countable support iteration of S-proper forcing notions such that for all $\alpha < \delta$, $\Vdash_{\mathbb{P}_\alpha}$ "$|\dot{\mathbb{Q}}_\alpha| = \omega_1$". Then $\mathbb{P}_\delta$ is $\aleph_2$-cc.*

Lemma 3.3.8. *Assume CH, let $\delta < \omega_2$ be an ordinal, and let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \delta, \beta < \delta \rangle$ be a countable support iteration of S-proper forcing notions such that for all $\alpha < \delta$, $\Vdash_{\mathbb{P}_\alpha}$ "$|\dot{\mathbb{Q}}_\alpha| = \omega_1$". Then CH holds in $V^{\mathbb{P}_\delta}$.*

Localization

The next forcing we consider has roots in René David's work [Dav82] on absolute Π^1_2 singletons, these being nonconstructible reals which are the unique solution to a Π^1_2 formula. The forcing notion below will allow for the localization of the generic clubs to the countable instances of the following class of transitive ZF^- models, where ZF^- denotes ZF without the Powerset Axiom.

Definition 3.3.9. A transitive model $\mathcal{M}$ of ZF^- is called *suitable* if $\omega_2^{\mathcal{M}}$ exists and $\omega_2^{\mathcal{M}} = \omega_2^{L^{\mathcal{M}}}$.

Throughout the rest of this section we assume V is a generic extension of L via a cofinality preserving forcing extension.

Definition 3.3.10. For $X \subseteq \omega_1$ and a Σ_1-sentence $\varphi(\omega_1, X)$ with parameters ω_1 and X such that φ holds in all suitable models $\mathcal{M}$ with $\omega_1, X \in \mathcal{M}$, denote by $\mathcal{L}(\phi)$ the set of all functions $r\colon |r| \to 2$ where $|r| = \mathrm{dom}(r)$ is a countable limit ordinal, and such that:

1. if $\gamma < |r|$ then $\gamma \in X$ if and only if $r(2\gamma) = 1$;
2. if $\gamma \leq |r|$ and $\mathcal{M}$ is a countable suitable model such that $\gamma = \omega_1^{\mathcal{M}}$ and $r \restriction \gamma \in \mathcal{M}$, then $\mathcal{M} \models \varphi(\gamma, X \cap \gamma)$.

The extension relation is end-extension.

Each $r \in \mathcal{L}(\varphi)$ is an approximation to the characteristic function of a subset $Y \subseteq \omega_1$ such that $\mathrm{Even}(Y) = \{\gamma \mid 2\gamma \in Y\} = X$. The "odd part" of r, i.e. the values r takes on ordinals of the form $2\gamma + 1$, is used for the following lemma. See also the proof of Lemma 3.4.7.

Lemma 3.3.11 ([FF10, Lemma 1])**.** *For every $r \in \mathcal{L}(\varphi)$ and countable limit ordinal $\gamma > |r|$, there exists $r' \leq r$ such that $|r'| = \gamma$.*

Corollary 3.3.12 ([FF10, Lemma 2])**.** *If G is $\mathcal{L}(\varphi)$-generic and $\mathcal{M}$ is a countable suitable model such that $\bigcup G \restriction \omega_1^{\mathcal{M}} \in M$, then $\mathcal{M} \models \varphi(\omega_1^{\mathcal{M}}, X \cap \omega_1^{\mathcal{M}})$.*

We include a proof of the below; see also [FF10, Lemma 3].

Lemma 3.3.13. *$\mathcal{L}(\varphi)$ is proper, and moreover does not add new reals.*

Proof. We show first that $\mathcal{L}(\varphi)$ is proper. Let M be a countable elementary submodel of $L_\theta = H_\theta$ for θ a sufficiently large regular cardinal, such that M contains X and $\mathcal{L}(\varphi)$. Let $p \in M$ be a condition in $\mathcal{L}(\varphi)$ and fix an enumeration $\{D_n \mid n \in \omega\}$ of the dense subsets of $\mathcal{L}(\varphi)$ which are in M. For $i = M \cap \omega_1$, let $\langle i_n : n \in \omega \rangle$ be an increasing cofinal sequence in i with $i_n \in M$ for every n.

By induction we construct $\{p_n \mid n \in \omega\} \subseteq \mathcal{L}(\varphi) \cap M$ such that $p_0 \leq p$ and for all $n \in \omega$:

1. $p_{n+1} \leq p_n$;
2. $|p_n| \geq i_n$;
3. $p_n \in D_n \cap M$.

Suppose p_n has been constructed. Since $\{r \in \mathcal{L}(\varphi) \mid |r| \geq i_{n+1}\}$ is dense and an element of M, there is $r \leq p_n$ with $r \in M$ and $|r| \geq i_{n+1}$. As D_{n+1} is dense, $D_{n+1} \in M$ and $M \prec L_\theta$, there is $p_{n+1} \in M$ such that $p_{n+1} \leq r$ and $p_{n+1} \in D_{n+1}$. Then p_{n+1} satisfies (1)-(3) above.

Let $q := \bigcup_{n \in \omega} p_n$. We first check that q is a condition in $\mathcal{L}(\varphi)$. That q is a function with $|q| = \bigcup_{n \in \omega} |p_n| = i$ follows from the fact q is a union of cohering functions. Next, let $\mathcal{N}$ be a countable suitable model such that $\gamma = \omega_1^{\mathcal{N}}$ and $q \restriction \gamma \in \mathcal{N}$, where γ is such that $\gamma \leq i$. If $\gamma < i$, then $\gamma \leq |p_n|$ for some n, and therefore (2) of Definition 3.3.10 is verified. In the case $\gamma = i$, since $\mathcal{L}(\varphi) \in M$ and M is an elementary submodel of L_θ containing X and ω_1 as elements, $M \models$" $\varphi(\omega_1, X)$ holds in all suitable models containing ω_1, X as elements."

Let $\overline{M}$ be the transitive collapse of M. Then $\overline{M} \models$ "for all suitable models $\mathcal{N}'$ containing $\omega_1^{\overline{M}}, X \cap \omega_1^{\overline{M}}$ as elements, $\mathcal{N}' \models \varphi(\omega_1^{\overline{M}}, X \cap \omega_1^{\overline{M}})$." In particular in $\overline{M}$ it holds:

"the $\leq_L$ -least suitable model containing $\omega_1^{\overline{M}}, X \cap \omega_1^{\overline{M}}$ as elements satisfies $\varphi(\omega_1^{\overline{M}}, X \cap \omega_1^{\overline{M}})$"

Then $\overline{M}$ is a transitive model, and the above is a Σ_1-sentence, hence it is upwards absolute with respect to transitive ZF^- models, so it holds in

L_θ. Then as φ is Σ_1 and hence upwards absolute, in particular in L_θ it holds that any suitable model containing $\omega_1^{\overline{M}}, X \cap \omega_1^{\overline{M}}$ as elements satisfies $\varphi(\omega_1^{\overline{M}}, X \cap \omega_1^{\overline{M}})$. Note that $\omega_1^{\overline{M}} = i = \omega_1^{\mathcal{N}}$ and $\omega_1^{\mathcal{N}} \in \mathcal{N}$, and as $q \restriction \omega_1^{\mathcal{N}} \in \mathcal{N}$ also $X \cap \omega_1^{\mathcal{N}} \in \mathcal{N}$. Therefore by the above we have $\mathcal{N} \models \varphi(\omega_1^{\mathcal{N}}, X \cap \omega_1^{\mathcal{N}})$.

Next we show $\mathcal{L}(\varphi)$ does not add new reals. Let $\dot{f}$ be a $\mathcal{L}(\varphi)$-name for a real; it suffices to show that for every $p \in \mathcal{L}(\varphi)$, there exists a function g in the ground model and an extension $q \leq p$ such that $q \Vdash \check{g} = \dot{f}$. Let M be a countable elementary submodel of L_θ, where θ is regular and sufficiently large, and M contains $\mathcal{L}(\varphi), X$, and $\dot{f}$ as elements. Let $p \in M \cap \mathcal{L}(\varphi)$, let $i = M \cap \omega_1$ and fix $\langle i_n : n \in \omega \rangle \subseteq M$ an increasing cofinal sequence in i. Construct a sequence $\langle p_n : n \in \omega \rangle \subseteq \mathcal{L}(\varphi) \cap M$ such that $p_0 \leq p$ and for all $n \in \omega$:

1. $p_{n+1} \leq p_n$;
2. $|p_n| \geq i_n$;
3. $p_n \Vdash \dot{f}(n) = \check{m}_n$ for some $m_n \in \omega$.

Let $q := \bigcup_{n\in\omega} p_n$. Then $q\colon i \to 2$ is a condition in $\mathcal{L}(\varphi)$; this can be checked identically as above. Define $g\colon \omega \to \omega$ by letting $g(n) = m_n$. Since $q \leq p_n$ for every n, we have $q \Vdash \dot{f} = \check{g}$. $\square$

Sacks Coding

Sacks coding, or coding with perfect trees, was first defined by Fischer and Friedman [FF10]. We say a tree $T \subseteq 2^{<\omega}$ is *perfect* if for all $s \in T$ there exists an extension t of s such that $t \in T$ and $t^\frown(i) \in T$ for each $i < 2$. *Sacks forcing* is the partial order consisting of perfect trees $T \subseteq 2^{<\omega}$, and a condition S extends a condition T if S is a subtree of T. Sacks forcing is proper, ω^ω-bounding[4], adds a generic real which is the unique branch through each T in the generic filter, and is often viewed as a very minimally destructive way of forcing $\neg\mathsf{CH}$ in that it does not affect the values of other

[4]A proper forcing notion $\mathbb{P}$ is said to be ω^ω*-bounding* if the ground model reals remain a dominating family in the extension, in other words, for any $f \in \omega^\omega$ in the $\mathbb{P}$-generic extension, there exists $g \in V \cap \omega^\omega$ such that $f \leq^* g$. Such forcings preserve the ground model ω^ω as a dominating family.

cardinal characteristics. The Sacks coding forcing defined below will inherit these properties, and in particular can be seen as a minimally destructive way of forcing a Δ^1_3 wellorder in the presence of large continuum.

Throughout this section we assume $V = L[Y]$, where $Y \subseteq \omega_1$ is generic over L for a cofinality preserving forcing notion.

Definition 3.3.14. Fix $n^* \in \omega$. By induction on $i < \omega_1$, define a sequence $\overline{\mu} = \langle \mu_i : i \in \omega_1 \rangle$ such that μ_i is an ordinal and $(|\mu_i| = \aleph_0)^L$, for each $i < \omega_1$. Let $\mu_0 = \emptyset$, and supposing $\langle \mu_j : j < i \rangle$ has been defined, let μ_i be the least ordinal $\mu > \sup_{j<i} \mu_j$ such that :

1. $L_\mu[Y \cap i] \prec_{\Sigma^1_{n^*}} L_{\omega_1}[Y \cap i]$;
2. $L_\mu[Y \cap i] \models \mathsf{ZF}^-$;
3. $L_\mu[Y \cap i] \models$ "ω is the largest cardinal".

Let $\mathcal{B}_i := L_{\mu_i}[Y \cap i]$. For $r \in 2^\omega$, we say that r *codes* Y *below* i if for all $j < i$,

$$j \in Y \Leftrightarrow \underbrace{L_{\mu_j}[Y \cap j][r]}_{=\mathcal{B}_j[r]} \models \mathsf{ZF}^-.$$

For a tree $T \subseteq 2^{<\omega}$, let $|T|$ denote the least $i < \omega_1$ such that $T \in \mathcal{B}_i$.

Definition 3.3.15. ([FF10, Definition 2]) *Sacks coding* is the partial order $C(Y)$ consisting of perfect trees $T \subseteq 2^{<\omega}$ such that r codes Y below $|T|$ whenever r is a branch through T. For $T_0, T_1 \in C(Y)$, let $T_1 \leq T_0$ if and only if T_1 is a subtree of T_0.

Remark 3.3.16. The hidden parameter n^* above is needed for the preservation results pertaining to definable combinatorial objects, e.g. item (5) of Fact 3.3.18 below. Specifically, n^* is chosen to an upper bound on the complexity of the formula expressing all relevant combinatorial properties which we want to reflect down to the models $\mathcal{B}_i$. In the present case of preserving a Π^1_1 tight mad family, $n^* = 5$ is sufficient. See also [BFB22, Remark 1].

We fix some arboreal terminology. For any tree $T \subseteq 2^{<\omega}$ and $t \in T$, define t to be *splitting* if $t^\frown(i) \in T$ for each $i < 2$. Let $\mathsf{Split}_n(T)$ be the set of $t \in T$ such that t is splitting and there are exactly n many proper predecessors

$s \sqsubseteq t$ such that s is splitting. Note the stem of T is the unique element of $\mathrm{Split}_0(T)$. Define a decreasing sequence $(\leq_n)_{n\in\omega}$ of partial orders on $C(Y)$ by letting let $T_1 \leq_n T_0$ if and only if $T_1 \leq T_0$ and $\mathrm{Split}_n(T_0) = \mathrm{Split}_n(T_1)$ (and therefore also $\mathrm{Split}_n(T_0) = \mathrm{Split}_n(T_1)$ for all $m \leq n$).

Remark 3.3.17. Let G be $C(Y)$-generic over $L[Y]$. Since the definition of $C(Y)$ is absolute, it still holds in $L[Y][G]$ that if T is any condition in $C(Y)$ and r is a branch through T,

$$L[Y][G] \models Y \cap |T| = \{j < |T| \mid \mathcal{B}_j[r] \models \mathsf{ZF}^-\}$$

In particular, if $r \in \bigcap G$ and $\gamma = \sup\{|T| \mid T \in G\}$, then in $L[Y][G]$,

$$Y \cap \gamma = \{j < \gamma \mid \mathcal{B}_j[r] \models \mathsf{ZF}^-\}.$$

Therefore Y is $\Delta^1_1(r)$ in $L[Y][G]$ when $\gamma = \omega_1$, as the satisfaction relation for countable transitive models of set theory is Δ^1_1.

Fact 3.3.18. The following hold.

1. ([FF10, Lemma 4]) For any $j \in \omega_1$ and any $T \in C(Y)$ with $|T| \leq j$, there exists $T' \leq T$ such that $|T'| = j$.

2. ([FF10, Lemma 7]) $C(Y)$ is proper.

3. ([FF10, Lemma 6]) Let G be $C(Y)$-generic over V and let $r := \bigcap G$. Then in $V[G]$, r codes the set Y in the sense that for all $j < \omega_1$

$$j \in Y \Leftrightarrow \mathcal{B}_j[r] \models \mathsf{ZF}^-.$$

4. ([FF10, Lemma 8]) $C(Y)$ is ω^ω-bounding.

5. ([BFB22, Lemma 4.3]) Countable support iterations of $C(Y)$ preserve the tightness of Π^1_1 tight mad families which are provably subsets of L.

Below we give the proof of items (1) and (2) of this last lemma, as they are illustrative of how the generic real is constructed and the typical strategy for proving preservation properties of the forcing.

Lemma 3.3.19. *For all $T \in C(Y)$ and every $j \geq |T|$, there is $T' \leq T$ such that $|T'| = j$.*

Proof. The proof is by induction on ordinals $j \geq |T|$. As the base case is clear, assume $j = i + 1 \geq |T|$; again by induction it suffices to suppose $|T| = i$. Suppose first $i \in Y$. Note that $T \in \mathcal{B}_i$ is a countable set partially ordered by the tree relation $\sqsubseteq$, and therefore we can view T as a forcing notion inside the model $\mathcal{B}_i$. Being countable, T is forcing isomorphic to Cohen forcing. Moreover by construction of $\vec{\mu}$, $\mathcal{B}_i \in \mathcal{B}_j$ and the latter sees $\mathcal{B}_i$ as a countable transitive model of ZF^- containing the forcing notion T. Thus all the possible $(\mathcal{B}_i, T)$-generic filters exist as elements of $\mathcal{B}_j$, by Lemma [Kun13, Lemma IV.2.3]. This allows us to define, within $\mathcal{B}_j$, a subtree $T_0 \subseteq T$ such that each branch $r \in [T_0]$ is $(\mathcal{B}_i, T)$-generic. Then T_0 is a perfect tree in $\mathcal{B}_j$ by genericity, $[T_0] \subseteq [T]$ so T_0 is a subtree of T, and $|T_0| = j$. Lastly since $r \in [T_0]$ is Cohen-generic over $\mathcal{B}_i$, we have $\mathcal{B}_i[r] \models \mathsf{ZF}^-$. Therefore $T_0 \in C(Y)$.

Next suppose $i \notin Y$. Since $\mathcal{B}_i$ is countable in $\mathcal{B}_j$, in the latter model μ_i is countable and we can pick a wellordering $E \subseteq \omega^2$ such that the order type of E is μ_i. Let $z \in 2^\omega$ be such that $z(2^m \cdot 3^n) = 0$ if and only if $\langle m, n\rangle \in E$, so z codes the countability of μ_i. Take $T_0 \subseteq T$ such that every branch in T_0 codes z; this can be done by choosing those $r \in [T]$ such that for all $n \in \omega$, $n \in z$ if and only if for $s \in \mathrm{Split}_{2n}(T)$ such that $s \sqsubseteq r$, we have $s^\frown(1) \sqsubseteq r$. Then $z \in \Delta^1_1(r)$. Then $T_0 \subseteq T$, $|T_0| = j$ as T_0 is definable from T, and $z \in \mathcal{B}_j$. We have $T_0 \in C(Y)$ since if $r \in [T_0]$ then $\mathcal{B}_i[r] = L_{\mu_i}[Y \cap i][r] \models$"$\mu_i$ is countable", and clearly $\mathcal{B}_i[r]$ cannot then be a model of ZF^-. This completes the successor case.

Let $j \geq |T|$ be a limit ordinal and let $\langle j_n : n \in \omega\rangle$ be a cofinal sequence in j, with $j_0 = |T|$. Such a sequence can be found inside $\mathcal{B}_j$ as this model sees that i is a countable limit ordinal. By induction on $n \in \omega$ we define a fusion sequence $\langle T_n : n \in \omega\rangle$ such that $T_0 = T$ and, supposing T_n defined, take $T_{n+1} \in C(Y)$, $T_{n+1} \in \mathcal{B}_{j_{n+1}}$ with $|T_{n+1}| = j_{n+1}$ and $T_{n+1} \leq_n T_n$. This can be done, using the previous case for the induction. Let $T' = \bigcap_{n<\omega} T_n$; then $T' \subseteq T$, $|T'| = j$, and $T' \in C(Y)$ since if $r \in [T']$ and $i < |T'|$, then $i < j_n = |T_n|$ for some n, and $T_n \in C(Y)$. $\square$

As a consequence, if G is $C(Y)$-generic over $L[Y]$ and r is the unique branch in $\bigcap G$, then Y is constructible from r and therefore $L[Y] \subseteq L[r]$.

Many of the preservation properties of $C(Y)$ such as listed above are dependent on the fact that $\vec{\mu}$ has nice *condensation properties*, where condensation refers to the following:

Theorem 3.3.20 (Gödel; see [Dev17, Theorem 5.2])**.** *Let X be a Σ_1-elementary submodel of L_α for some limit ordinal α. Then there exists a unique ordinal $\beta \leq \alpha$ and a unique isomorphism $\pi\colon (X, \in) \to (L_\beta, \in)$.*

What the proof of the above shows is that if $\mathcal{M}$ is a transitive model satisfying $\mathsf{ZF}^- + V = L$, then in fact $\mathcal{M} = L_\alpha$, where α is the minimal ordinal β such that $\beta \notin \mathcal{M}$. This is particularly useful when considering the transitive collapse of elementary submodels, as is the context of preservation proofs countable support iterations of proper forcings; this is demonstrated in the proof below. The properties of the Mostowski collapse of countable elementary submodels of H_θ for cardinals $\theta > \omega_1$ relevant for the proof are given in [Kun13, Lemma III.7.12].

Lemma 3.3.21 ([FF10, Lemma 7])**.** *$C(Y)$ is proper.*

Proof. Let $T \in C(Y)$, let θ be a sufficiently large cardinal and $\mathcal{M} \prec L_\theta[Y]$ a countable elementary submodel containing $C(Y), T, \vec{\mu}$, and Y as elements. Let $i = \mathcal{M} \cap \omega_1$ and let $\pi\colon \mathcal{M} \to \overline{\mathcal{M}}$ be the Mostowski collapse isomorphism; so $\overline{\mathcal{M}}$ is countable, transitive, elementarily equivalent to $\mathcal{M}$, and $\pi(\omega_1) = i = \omega_1^{\overline{\mathcal{M}}}$. By condensation, $\overline{\mathcal{M}} = L_\mu[\pi(Y)] = L_\mu[Y \cap i]$ for some $\mu < \omega_1$; note that the latter equality holds because, as Y is generic over L for some cardinal preserving forcing notion, Y is provably a subset of L, so $\pi(Y) = Y \cap \omega_1^{\overline{\mathcal{M}}}$. However, since $\overline{\mathcal{M}} \models$"$i$ is uncountable", and the property of being uncountable is expressible by a Π_1 formula and hence is downwards absolute, we must have $\mu < \mu_i$. This implies $\overline{\mathcal{M}} \in L_{\mu_i}[Y \cap i] = \mathcal{B}_i$. Moreover, using item (3) of the definition of the sequence $\vec{\mu}$, there exists a sequence $\langle i_n : n \in \omega \rangle \in \mathcal{B}_i$ which is increasing and cofinal in i, such that $i_n \in \mathcal{M}$ for all n.

We show there exists $T' \leq T$ such that T' is $(\mathcal{M}, C(Y))$-generic. Let $\{D_n \mid n \in \omega\}$ enumerate the dense subsets of $C(Y)$ in $\mathcal{M}$. By induction we define a sequence $\langle T_n : n \in \omega \rangle$ of conditions $T_n \in C(Y) \cap \mathcal{M}$ such that $T_0 \leq T$, and for all $n \in \omega$:

1. $T_{n+1} \leq_{n+1} T_n$;

2. $|T_n| \geq i_n$;

3. $T_n \Vdash \check{D}_n \cap \check{\mathcal{M}} \cap \dot{G} \neq \emptyset$.

Supposing T_m defined for all $m \leq n$, let $t \in \mathrm{Split}_{n+1}(T_n)$ and consider $T_n(t) = \{s \in T_n \mid s \sqsubseteq t \vee t \sqsubseteq s\}$. It is straightforward to check $T_n(t) \in C(Y)$. By elementarity, the previous lemma holds in $\mathcal{M}$ and thus there exists $\tilde{T}_t \leq T_n(t)$ such that $|\tilde{T}_t| \geq i_{n+1}$. This will take care of item (ii) above, and note also $\tilde{T}_t \leq_{n+1} T_n(t)$.

To address item (iii), as D_{n+1} is dense in $\mathcal{M}$ there exists $T'_t \leq \tilde{T}_t$ with $T'_t \in \mathcal{M}$ and $T'_t \in D_{n+1}$. Define $T_{n+1} = \bigcup\{T'_t \mid t \in \mathrm{Split}_{n+1}(T_n)\}$. Then T_{n+1} satisifes (iii), as well as (ii). Indeed, first note that

$$E = \{S \in C(Y) \mid \exists t \in \mathrm{Split}_{n+1}(T_{n+1})\ (S \leq T_{n+1}(t))\}$$

is dense below T_{n+1} in $C(Y)$, since if $S \leq T_{n+1}$ there is $t \in \mathrm{Split}_{n+1}(T_n) = \mathrm{Split}_{n+1}(T_{n+1})$ such that $S \leq T'_t \leq T_n(t)$. So $t \in S$ and $S(t) \leq T_{n+1}(t)$.

By density of E below T_{n+1}, if G is $C(Y)$-generic and $T_{n+1} \in G$ there is $S \in G$ such that $S \leq T_{n+1}(t) = T'(t)$ for some $t \in \mathrm{Split}_{n+1}(T_{n+1})$. Since $T'(t)$ was chosen so that $T'(t) \in D_{n+1} \cap \mathcal{M}$, as $T'(t) \geq S$ and $S \in G$, altogether we have $T'(t) \in D_{n+1} \cap \mathcal{M} \cap G$. This completes the inductive construction.

Define $T^* = \bigcap_{n \in \omega} T_n$. In order to prove this is indeed a condition in $C(Y)$ we must have that T^* is an element of $\mathcal{B}_i$, i.e. we must show that the inductive construction can be taken place within the model $\mathcal{B}_i$. Recall that the Mostowski collapse isomorphism is the identity on countable objects, such as i_n and T. Therefore we can perform the same construction with respect to $\{\pi(D_n) \mid n \in \omega\}$ and the sequence $\langle \pi(i_n) : n \in \omega\rangle = \langle i_n : n \in \omega\rangle \in \mathcal{B}_i$. Then $\{T_n \mid n \in \omega\} \in \mathcal{B}_i$ and so $T^* \in \mathcal{B}_i$; note that $|T^*| = i$. To see that T^* is a condition in $C(Y)$ let let $r \in [T^*]$, and let $j < i$. Then there is $n \in \omega$ such that $j < i_n \leq |T_n|$, so $r \in [T_n]$ and T_n is a condition. Lastly, T^* is $(\mathcal{M}, C(Y))$-generic since $T^* \leq T_n$ for all n and so $T^* \Vdash \check{D}_n \cap \check{\mathcal{M}} \cap \dot{G} \neq \emptyset$.

□

Lastly we present a novel proof of another preservation property of $C(Y)$. Recall that a tree $T \subseteq \omega_1^{<\omega_1}$ is said to be *Souslin* if T has height ω_1, but T has neither ω_1-length branches nor uncountable antichains, where an antichain of a tree T is a set $A \subseteq T$ such that for all $s, t \in A$, s and t are incomparable with respect to the tree relation $\sqsubseteq$ on T. Souslin trees can can be constructed from a $\Diamond$-sequence and thus exist in L. Moreover such objects can be used as a coding apparatus: this is one of the main tools in Jensen [Jen70] and Jensen and Johnsbråten [JJ74], in their work on Δ^1_3 nonconstructible reals. These techniques also yield the results of David [Dav82], and are more recently used by Hoffelner [Hof22] as well as Fischer, Friedman, and Zdomskii [FFZ13]. By showing that coding using $C(Y)$ preserves Souslin trees, we show these coding methods can be used independently; this provides the motivation the Proposition 3.3.23 below. The proof appeals to the following characterization, given by Miyamoto in 1993.

Lemma 3.3.22 ([Miy93])**.** *Let $\mathbb{P}$ be a proper forcing notion, let $S \in V$ be a Souslin tree, and let θ be a sufficiently large regular cardinal. Then the following are equivalent:*

1. $\Vdash_{\mathbb{P}}$ *"$\check{S}$ is Souslin;*
2. *If $\mathcal{M} \prec H_\theta$ is a countable elementary submodel containing $\mathbb{P}$ and S as elements and $p \in \mathbb{P} \cap \mathcal{M}$, there exists $q \leq p$ such that for all $t \in S_\delta$, (q, t) is $(\mathcal{M}, \mathbb{P} \times S)$-generic, where $\delta = \mathcal{M} \cap \omega_1$, and S_δ denotes all $t \in S$ such that $\{s \in S \mid s \sqsubseteq t\}$ has order type δ.*

Proposition 3.3.23. *Let $S \in V$ be a Souslin tree, and let G be $C(Y)$-generic over V. Then $(S$ is Souslin$)^{V[G]}$.*

Proof. Fix $S \in V$ a Souslin tree, let θ be sufficiently large and $\mathcal{M} \prec H_\theta$ a countable elementary submodel containing $C(Y)$ and S as elements, and let $T \in \mathcal{M} \cap C(Y)$. Let $\delta = \mathcal{M} \cap \omega_1$. We will construct $T_\infty \in C(Y)$ such that T_∞ extends T, is $(\mathcal{M}, C(Y))$-generic, and

$$T_\infty \Vdash \forall t \in S_\delta \; [t \text{ is } (\mathcal{M}[\dot{G}], \check{S})\text{-generic}],$$

where $\dot{G}$ is the canonical name for the $C(Y)$-generic filter.

Let $\pi \colon \mathcal{M} \to \overline{\mathcal{M}}$ be the Mostowski collapse isomorphism. Then $\overline{\mathcal{M}} \in \mathcal{B}_\delta$ and we can fix $\langle \delta_n : n \in \omega \rangle \in \mathcal{B}_\delta$ a cofinal sequence in δ such that $\{\delta_n \mid n \in \omega\} \subseteq \overline{\mathcal{M}}$.

Observe that if we consider the Souslin tree S as a forcing notion, then S is ω-distributive and thus preserves H_{ω_1}. Therefore whenever G_S is an S-generic branch over $\mathcal{M}$, $H_{\omega_1}^{\mathcal{M}} = H_{\omega_1}^{\mathcal{M}[G_S]}$ and this implies $C(Y)^{\mathcal{M}} = C(Y)^{\mathcal{M}[G_S]}$ and $\mathcal{M} \cap \omega_1 = \mathcal{M}[G_S] \cap \omega_1$.

Furthermore, $\overline{\mathcal{M}}[\pi(G_s)] = \overline{\mathcal{M}}[t]$ for some $t \in S_\delta$, i.e., $\pi(G_S)$ is the unique node of S at level $\delta = \omega_1^{\overline{\mathcal{M}}}$ such that $\pi(G_S) = t$, and $\overline{\mathcal{M}}$ will consider any $t \in S_\delta$ to be an uncountable branch through $S^{\overline{\mathcal{M}}}$.

Since S_δ is a countable element of $\mathcal{B}_\delta$, let $\langle t_i : i \in \omega \rangle \in \mathcal{B}_\delta$ enumerate S_δ. For each t_i we consider the model $\overline{\mathcal{M}}[t_i]$, which is isomorphic to $\mathcal{M}[G_S]$ for any G_S which is S-generic over $\mathcal{M}$ and $t_i \in G_S$. Let $\{D_k^{t_i} \mid k \in \omega\} \in \mathcal{B}_\delta$ be an enumeration of the dense open subsets of $\pi(C(Y)) = C(Y) \cap \mathcal{M} = C(Y) \cap \overline{\mathcal{M}}$ which are in $\overline{\mathcal{M}}[t_i]$, and recall $C(Y) \cap \overline{\mathcal{M}}[t_i] = C(Y) \cap \overline{\mathcal{M}}$ for all $i < \omega$.

Fix a computable bijection $\langle \cdot, \cdot \rangle \colon \omega^2 \to \omega$ and write $(n)_0, (n)_1$ for the coordinates such that $n = \langle (n)_0, (n)_1 \rangle$. Now, let $\{D_n \mid n \in \omega\} = \{D_{(n)_0}^{t_{(n)_1}} \mid n \in \omega\}$ be a re-enumeration of the dense open subsets of $C(Y) \cap \mathcal{M}$ such that $D_n = D_{(n)_0}^{t_{(n)_1}}$.

By induction on $n \in \omega$ we construct a sequence $(T_n)_{n \in \omega} \subseteq C(Y) \cap \mathcal{M}$ such that $T_0 \leq T$ and for all $n < \omega$ the following hold:

1. $T_{n+1} \leq_{n+1} T_n$;
2. $|T_n| \geq \delta_n$;
3. $T_n \Vdash D_k \cap \mathcal{M} \cap \dot{G} \neq \emptyset$, where $\dot{G}$ is the canonical name for the $C(Y)$-generic filter.

Supposing this can be done, we let $T_\infty = \bigcap_{n \in \omega} T_n \in \mathcal{B}_\delta$, and then it suffices to notice that for any $t \in S_\delta$, (t, T_∞) will be $(\mathcal{M}, S \times C(Y))$-generic: indeed, as S is ccc, *any* $t \in S$ is $(\mathcal{M}, S)$-generic, and when any such $t \in S_\delta$ is contained in an S-generic filter G_S, in $\mathcal{M}[G_S]$, T_∞ forces the $C(Y)$-generic filter to meet all dense subsets of $C(Y) \cap \mathcal{M}[G_S]$; this uses item (3) above and the fact $T_\infty \leq T_n$ for every n. Therefore for any $t \in S_\delta$, (t, T_∞) is $(\mathcal{M}, S \times C(Y))$-generic or equivalently (T_∞, t) is $(\mathcal{M}, C(Y) \times S)$-generic. By the above lemma this implies $C(Y)$ preserves that S is Souslin.

That the sequence $(T_n)_{n \in \omega}$ can be constructed within $\mathcal{B}_\delta$ follows as in the proof of properness of $C(Y)$, though now we have ensured that T_∞ is not only $(\mathcal{M}, C(Y))$-generic but also $(\mathcal{M}[G_S], C(Y)^{\mathcal{M}[G_S]})$-generic whenever G_S is an S-generic filter over $\mathcal{M}$. $\square$

This completes the discussion of Sacks coding.

3.3.1 A Δ^1_3 long wellorder

With the ingredients provided by the previous sections we proceed with the proof of Theorem 3.3.35. The definition of the forcing construction requires the establishment of some preliminaries.

Recall that $\diamondsuit$ is the assertion that there exists a sequence $\vec{A} = \langle A_\alpha : \xi < \omega_1 \rangle$ where $A_\xi \subseteq \xi$ for each $\xi < \omega_1$ and for any $X \subseteq \omega_1$, the set $\{\xi < \omega_1 \mid X \cap \xi = A_\xi\}$ is stationary. $\vec{A}$ is called a $\diamondsuit$-sequence, and such a sequence can be constructed in L so that the sequence is Σ_1-definable over L_{ω_1}; see, for example, [Dev17, Theorem 3.3].

Proposition 3.3.24. *Assume $\diamondsuit$ holds. Then there exists a sequence*

$$\vec{S} = \langle S_\alpha : \alpha < \omega_2 \rangle$$

which is Σ_1-definable over L_{ω_2} consisting of sets $S_\alpha \subseteq \omega_1$ that are stationary costationary in L, and are almost disjoint in the sense that $|S_\alpha \cap S_\beta| < \omega_1$ for all distinct $\alpha, \beta < \omega_2$. Furthermore there exists a stationary $S_{-1} \subseteq \omega_1$ such that $S_{-1} \cap S = \emptyset$ for all $S \in \vec{S}$.

Moreover, if $\mathcal{M}, \mathcal{N}$ are suitable models such that $\omega_1^{\mathcal{M}} = \omega_1^{\mathcal{N}}$, then $\vec{S}^{\mathcal{M}}$ and $\vec{S}^{\mathcal{N}}$ coincide on $\omega_2^{\mathcal{M}} \cap \omega_2^{\mathcal{N}}$. If $\mathcal{M}$ is suitable with $\omega_1^{\mathcal{M}} = \omega_1$, then $\vec{S}^{\mathcal{M}} = \vec{S} \restriction \omega_2^{\mathcal{M}}$.

Proof. Let $\vec{A} = \langle A_\alpha : \xi < \omega_1 \rangle \in L$ be a $\diamondsuit$-sequence.

For each $\alpha < \omega_2$, let $W_\alpha \subseteq \omega_1$ denote the $\leq_L$-least code for the ordinal α. Let $S_\alpha = \{\xi \in \omega_1 \mid W_\alpha \cap \xi = A_\xi \neq \emptyset\}$. Each S_α is stationary in ω_1 by the properties of $\vec{A}$, and for distinct $\alpha, \beta < \omega_2$, $S_\alpha \cap S_\beta$ is at most countable. That $\omega_1 \setminus S_\alpha$ is stationary follows from the fact $\{\xi \in \omega_1 \mid A_\xi = (\omega_1 \setminus W_\alpha) \cap \xi\}$ is stationary. Therefore $\vec{S} := \langle S_\alpha : \alpha < \omega_2 \rangle$ is as desired, and define $S_{-1} := \{\xi \in \omega_1 \mid A_\xi = \emptyset\}$. Thus S_{-1} is a stationary set disjoint from each $S \in \vec{S}$.

To see the latter statement, let $\mathcal{M}, \mathcal{N}$ be suitable models such that $\omega_1^{\mathcal{M}} = \omega_1^{\mathcal{N}}$. Since for each $\alpha < \omega_2^{\mathcal{M}} \cap \omega_2^{\mathcal{N}}$, $W_\alpha^{\mathcal{M}} = W_\alpha^{\mathcal{N}}$, we have $S_\alpha^{\mathcal{M}} = S_\alpha^{\mathcal{N}}$ for each such α. $\square$

We assume $V = L$, and therefore fix $\vec{S}$ as above. For our bookeeping function:

Lemma 3.3.25 ([FF10, Lemma 14])**.** *There exists $F\colon \omega_2 \to L_{\omega_2}$ such that for all $a \in L_{\omega_2}$, $F^{-1}(a)$ is unbounded in ω_2, and F is Σ_1-definable over L_{ω_2}. Moreover, if $\mathcal{M}$, $\mathcal{N}$ are suitable models such that $\omega_1^{\mathcal{M}} = \omega_1^{\mathcal{N}}$, then $F^{\mathcal{M}}$ and $F^{\mathcal{N}}$ coincide on $\omega_2^{\mathcal{M}} \cap \omega_2^{\mathcal{N}}$. If $\mathcal{M}$ is suitable with $\omega_1^{\mathcal{M}} = \omega_1$, then $F^{\mathcal{M}} = F \restriction \omega_2^{\mathcal{M}}$.*

Fix F as above. We will also need a tight mad family to preserve and so will use the following lemma. It is important, for the preservation by countable support iterations of $C(Y)$, that the tight mad family has a projective definition and provably consists of only constructible reals.

Lemma 3.3.26 ([BFB22, Lemma 4.2])**.** *If $V = L$ then there is a Π^1_1 tight mad family $\mathcal{A}$ such that* ZFC *proves A is a subset of L.*

By recursion on $\alpha < \omega_2$, define a countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$, where $\mathbb{P}_0$ is taken to be the trivial poset. Suppose $\mathbb{P}_\alpha$ has been defined, and let G_α be $\mathbb{P}_\alpha$-generic over L.

The wellorder $<_\alpha$ on $L[G_\alpha]$ has a natural definition using the global wellorder $<_L$ of the universe L and the collection of $\mathbb{P}_\alpha$-names for reals which we can assume to be *nice*:

Definition 3.3.27. For a forcing notion $\mathbb{P}$ and G a $\mathbb{P}$-generic filter over V, for any real $x \in V[G]$, a *nice* $\mathbb{P}$*-name for* x is a $\mathbb{P}_\alpha$-name of the form $\dot{x} = \bigcup_{n \in \omega} \{\langle \langle n, m_p^n \rangle, p \rangle \mid p \in A_n(\dot{x})\}$, where $A_n(\dot{x})$ is a maximal antichain in $\mathbb{P}_\alpha$; note that $p \Vdash \dot{x}(\check{n}) = m_p^n$.

Since every name for a real has a nice names in the above sense, this allows us to suppose that if $\alpha < \beta < \omega_2$ and $\dot{x}$ is a $\mathbb{P}_\beta$-name which is not a $\mathbb{P}_\alpha$-name, then all $\mathbb{P}_\alpha$-names precede $\dot{x}$ with respect to $\leq_L$, as it takes longer to construct $\dot{x}$. Whenever x is a real in $L[G_\alpha]$, there exists $\gamma \leq \alpha$ such that x has a nice $\mathbb{P}_\gamma$-name; let γ_x be the minimal such γ. Define σ_x^α to be the L-minimal nice $\mathbb{P}_{\gamma_x}$-name for x. In this way we can understand the reals of $L[G_\alpha]$ by considering the set

$$N = \{\sigma_x^\alpha \mid x \in L[G_\alpha] \cap \omega^\omega\}.$$

Notice that as N is a subset of L, N is canonically wellordered by $<_L$; therefore define $<_\alpha$ by letting

$$x <_\alpha y \text{ if and only if } \sigma_x^\alpha <_L \sigma_y^\alpha,$$

whenever x, y are reals of $L[G_\alpha]$. Equivalently, $x <_\alpha y$ if and only if $\gamma_x < \gamma_y$ or $\gamma_x = \gamma_y$ and $\sigma_x^\alpha <_L \sigma_y^\alpha$. Since $\sigma_x^\alpha = \sigma_x^\beta = \sigma_x^{\gamma_x}$ for any $\beta < \alpha$, $<_\beta$ is an initial segment of $<_\alpha$. Let $\dot{<}_\alpha$ denote a $\mathbb{P}_\alpha$-name for $<_\alpha$.

Define a function $*\colon \omega^\omega \times \omega^\omega \to \omega^\omega$ by letting

$$x * y = \{2n \mid n \in x\} \cup \{2n+1 \mid n \in y\}.$$

Note the pair (x, y) can be recursively recovered from $x * y$. For any real x define $\Delta(x) := x * (\omega \setminus x)$.

Lastly we require an absolute way of coding the ordinals of ω_2^L.

Definition 3.3.28. Let $\beta < \omega_2^L$ and $X \subseteq \omega_1^L$. We say X is a *sufficiently absolute code for* β if there exists a formula $\psi(x, y)$ such that for any suitable model $\mathcal{M}$ containing $X \cap \omega_1^{\mathcal{M}}$, there exists a unique $\overline{\beta} \in \omega_2^{\mathcal{M}}$ such that $\psi(\overline{\beta}, X \cap \omega_1^{\mathcal{M}})$ holds in $\mathcal{M}$, and $\overline{\beta} = \beta$ in the case $\omega_1^{\mathcal{M}} = \omega_1^L$. Moreover, if $\mathcal{M}, \mathcal{N}$ are any suitable models such that $\omega_1^{\mathcal{M}} = \omega_1^{\mathcal{N}}$ and $X \cap \omega_1^{\mathcal{M}} \in \mathcal{M} \cap \mathcal{N}$, then it is the same $\overline{\beta} \in \omega_2^{\mathcal{M}} \cap \omega_2^{\mathcal{N}}$ such that $(\psi(\overline{\beta}, X \cap \omega_1^{\mathcal{M}}))^{\mathcal{M}}$ and $(\psi(\overline{\beta}, X \cap \omega_1^{\mathcal{N}}))^{\mathcal{N}}$.

Fact 3.3.29. ([FZ10, Fact 5]) Sufficiently absolute codes exist for every $\beta < \omega_2^L$.

Proof. Let $G\colon \mathrm{Ord} \times \mathrm{Ord} \to \mathrm{Ord}$ be Gödel's pairing function, and let $\psi(x, y)$ hold if and only if x is an ordinal, $x \in \omega_2$ and y is the $\leq_L$-least subset of ω_1 such that $\langle x, \in\rangle$ is order-isomorphic to $\langle \omega_1, G^{-1}[y]\rangle$. The "moreover" of Definition 3.3.28 is satisfied because the $\leq_L$-rank of $G^{-1}[y \cap \omega_1^{\mathcal{M}}]$ is absolute for transitive models of ZF^- (see [Kun80, Lemma 4.5]). $\square$

Henceforth we fix the formula $\psi(x, y)$ above.

Working in $L[G_\alpha]$, let $\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{Q}}_\alpha^0 * \dot{\mathbb{Q}}_\alpha^1$ be a $\mathbb{P}_\alpha$-name for a two-step iteration in which $\dot{\mathbb{Q}}_\alpha^0$ is a $\mathbb{P}_\alpha$-name for the creature forcing $\mathbb{Q}$ of Definition 3.2.17, and $\dot{\mathbb{Q}}_\alpha^1$ is a $\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha^0$-name for the trivial forcing, unless the following occurs: $F(\alpha) = \{\sigma_x^\alpha, \sigma_y^\alpha\}$ for some reals $x, y \in L[G_\alpha]$ such that $x <_\alpha y$. In this case set $x_\alpha = x$ and $y_\alpha = y$, and define $\dot{\mathbb{Q}}_\alpha^1$ to be a $\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha^0$-name for a three-step iteration $\dot{\mathbb{K}}_\alpha^0 * \dot{\mathbb{K}}_\alpha^1 * \dot{\mathbb{K}}_\alpha^2$, where:

1. $\dot{\mathbb{K}}^0_\alpha$ is a $\mathbb{P}_\alpha * \dot{\mathbb{Q}}^0_\alpha$-name for the countable support iteration $\langle \mathbb{P}^0_{\alpha,\beta}, \mathbb{K}^0_{\alpha,n} : \beta \leq \omega, n \in \omega \rangle$, where $\mathbb{K}^0_{\alpha,m}$ is a $\mathbb{P}^0_{\alpha,m}$-name for $Q(S_{\alpha+m})$ for all $m \in \Delta(x_\alpha * y_\alpha)$.

2. Let R_α be $\mathbb{Q}^0_\alpha$-generic over $L[G_\alpha]$, and let H_α be $\mathbb{K}^0_\alpha$-generic over $L[G_\alpha * R_\alpha]$. In $L[G_\alpha * R_\alpha * H_\alpha]$ fix:

   - Subsets $W_\alpha, W_\eta \subseteq \omega_1$ such that W_α is a sufficiently absolute code for the ordinal α and W_η is a sufficiently absolute code for an ordinal η such that $L_\eta \models |\alpha| \leq \omega_1$;
   - A real $x_\alpha \oplus y_\alpha$ recursively coding the pair (x_α, y_α);
   - A subset $Z_\alpha \subseteq \omega_1$ coding $G_\alpha * R_\alpha * H_\alpha$.

   Fix a computable bijection $\langle \cdot, \cdot, \cdot, \cdot \rangle \colon \mathcal{P}(\omega_1) \to \mathcal{P}(\omega_1)$, and for $X \subseteq \omega_1$ and $i < 4$ write $(X)_i$ for those elements of $\mathcal{P}(\omega_1)$ such that $X = \langle (X)_i : i < 4 \rangle$. Let $X_\alpha = \langle W_\alpha, x_\alpha \oplus y_\alpha, W_\eta, Z_\alpha \rangle$.

   Let $\varphi_\alpha = \varphi_\alpha(\omega_1, X_\alpha)$ be a sentence with parameters ω_1 and X_α such that φ_α holds if and only if:

   There exists an ordinal $\overline{\alpha} \in \omega_2$ such that $(X_\alpha)_0 = \overline{\alpha}$ and there exists a pair (x, y) such that $(X_\alpha)_2 = x \oplus y$ and for all $m \in \Delta(x * y)$, $S_{\overline{\alpha}+m}$ is nonstationary.

   More formally, $\varphi_\alpha(\omega_1, X_\alpha)$ is the formula:

   $$\exists \overline{\alpha}, (x, y) \in \Delta^1_1(X_\alpha)[\overline{\alpha} \in \omega_2 \wedge \forall m \in \Delta(x * y)\ S_{\overline{\alpha}+m} \in \mathrm{NS}_{\omega_1}].$$

   where NS_{ω_1} denotes the set of nonstationary subsets of ω_1. Then φ_α is a Σ_1-sentence with parameters ω_1, X_α, and if $\mathcal{M}$ is any suitable model containing ω_1 and X_α as elements, then $\mathcal{M} \models \varphi_\alpha(\omega_1, X_\alpha)$.[5] We can therefore define $\dot{\mathbb{K}}^1_\alpha$ be a $(\mathbb{P}_\alpha * \dot{\mathbb{Q}}^0_\alpha * \mathbb{K}^0_\alpha)$-name for $\mathcal{L}(\varphi_\alpha)$.

3. Let Y_α be $\mathbb{K}^1_\alpha$-generic over $L[G_\alpha * R_\alpha * H_\alpha]$. Then as Y_α codes X_α and X_α codes $G_\alpha * R_\alpha * H_\alpha$, we have $L[G_\alpha * R_\alpha * H_\alpha * Y_\alpha] = L[Y_\alpha]$. In this model, let $\mathbb{K}^2_\alpha$ be a $(\mathbb{P}_\alpha * \dot{\mathbb{Q}}^0_\alpha * \dot{\mathbb{K}}^0_\alpha * \dot{\mathbb{K}}^1_\alpha)$-name for $C(Y_\alpha)$.

[5] This uses that $\mathcal{M}$ has the code W_η.

This completes the definition of $\mathbb{P} = \mathbb{P}_{\omega_2}$. Then $\mathbb{P}$ is a countable support iteration such that each $\mathbb{P}_\alpha$ forces that $\mathbb{Q}_\alpha$ is an S_{-1}-proper forcing notion of size $\leq \omega_1$, and by Proposition 3.2.28 and Fact 3.3.18, $\mathbb{Q}_\alpha$ is forced to strongly preserves the tightness of $\mathcal{A}$. Therefore:

Lemma 3.3.30. *$\mathbb{P}$ is S_{-1}-proper, strongly preserves the tightness of $\mathcal{A}$, and has the $\aleph_2$-cc.*

Proof. By Lemma 3.3.6, Lemma 3.2.6, Proposition 3.2.28, and Lemma 3.3.7. $\square$

Let G be $\mathbb{P}$-generic over L, and define $<_G = \bigcup <_\alpha$, where $<_\alpha = \dot{<}_\alpha^G$.

Lemma 3.3.31. *Let G be $\mathbb{P}$-generic over L, and let x, y be reals in $L[G]$. Then $x <_G y$ if and only if:*

$(*)$ *there exists a real r such that for every countable suitable $\mathcal{M}$ containing r as an element, there exists $\overline{\alpha} < \omega_2^{\mathcal{M}}$ such that for all $m \in \Delta(x * y)$, $S^{\mathcal{M}}_{\overline{\alpha}+m}$ is nonstationary in $\mathcal{M}$.*

Proof. Let x, y be reals in $L[G]$, and let $\gamma_x, \gamma_y < \omega_2$ be minimal such that x has a nice $\mathbb{P}_{\gamma_x}$-name, and y has a nice $\mathbb{P}_{\gamma_y}$-name, respectively. First suppose $x <_G y$. Then $\{\sigma_x^{\gamma_x}, \sigma_y^{\gamma_y}\} \in L_{\omega_2}$ and $F^{-1}(\{\sigma_x^{\gamma_x}, \sigma_y^{\gamma_y}\})$ is unbounded in ω_2, so there is $\alpha \geq \max(\gamma_x, \gamma_y)$ such that $F(\alpha) = \{\sigma_x^{\gamma_x}, \sigma_y^{\gamma_y}\} = \{\sigma_x^\alpha, \sigma_y^\alpha\}$. Then our definition of $\dot{\mathbb{Q}}^1_\alpha$ at this point was nontrivial, and was taken with respect to the reals $x_\alpha = x$ and $y_\alpha = y$. Let R_α be $\mathbb{Q}^0_\alpha$-generic over $L[G_\alpha]$ where $G_\alpha = \mathbb{P}_\alpha \cap G$, let H_α be $\mathbb{K}^0_\alpha$-generic over $L[G_\alpha * R_\alpha]$, let Y_α be $\mathbb{K}^1_\alpha$-generic over $L[G_\alpha * R_\alpha * H_\alpha]$, and let D_α be $\mathbb{K}^2_\alpha$-generic over $L[G_\alpha * R_\alpha * H_\alpha * Y_\alpha] = L[Y_\alpha]$.

As D_α is $C(Y_\alpha)$-generic, by Fact 3.3.18, item (3), $d_\alpha := \bigcap D_\alpha$ is a real such that for all $j \in \omega_1$, $j \in Y_\alpha$ if and only if $\mathcal{B}_j[d_\alpha] \models \mathsf{ZF}^-$. As $Y_\alpha \colon \omega_1 \to 2$ codes X_α, also X_α and hence the pair (x_α, y_α) are constructible from d_α.

Now let $\mathcal{M}$ be any countable suitable model containing d_α as an element. By the above observations and the fact $\omega_1^{\mathcal{M}} = \omega_1^{L^{\mathcal{M}}}$, the model $\mathcal{M}$ can construct $Y_\alpha \restriction \omega_1^{\mathcal{M}}$ and $X_\alpha \cap \omega_1^{\mathcal{M}}$.

Since Y_α is $\mathcal{L}(\varphi_\alpha)$ generic, by Lemma 3.3.12, $\mathcal{M} \models \varphi_\alpha(\omega_1^{\mathcal{M}}, X_\alpha \cap \omega_1^{\mathcal{M}})$, meaning in $\mathcal{M}$ it holds:

"there exists $\overline{\alpha} \in \omega_2$ such that $(X_\alpha \cap \omega_1^{\mathcal{M}})_0 = \overline{\alpha}$ and $(X_\alpha \cap \omega_1^{\mathcal{M}})_2$ is the code of a pair of reals (x', y') such that for all $m \in \Delta(x' * y')$ $(S^{\mathcal{M}}_{\overline{\alpha}+m} \in \mathrm{NS}_{\omega_1})$."

Since then $\Delta(x', y') = \Delta(x, y)$ in $\mathcal{M}$, this implies necessarily $(x', y') = (x, y)$.

To see the converse implication, fix a real r given by $(*)$. We need the following.

Claim 3.3.32. *If $\mathcal{N}$ is a suitable model of* any *cardinality with $r \in \mathcal{N}$, then there exists $\overline{\alpha} \in \omega_2^{\mathcal{N}}$ such that $S_{\overline{\alpha}+m} \in \mathrm{NS}_{\omega_1}$ for all $m \in \Delta(x * y)$.*

Proof. Suppose towards contradiction that $\mathcal{N}$ is a suitable model containing r as an element, with $|\mathcal{N}| \geq \aleph_1$, but

$$\mathcal{N} \models \forall \alpha \in \omega_2 (\Delta(x * y) \neq \{m \in \omega \mid S_{\alpha+m} \in \mathrm{NS}_{\omega_1}\}). \qquad (\otimes)$$

By the downwards Löwenheim–Skolem theorem, there exists a countable elementary submodel $\mathcal{N}_0 \prec \mathcal{N}$ with $r \in \mathcal{N}_0$. Note $\mathcal{N}_0$ is a model of ZF^-, and $\mathcal{N}_0$ models $(\otimes)$. Let $\overline{\mathcal{N}_0}$ be the transitive collapse of $\mathcal{N}_0$; as the Mostowski isomorphism is the identity on countable objects, we also have $r \in \overline{\mathcal{N}_0}$. Moreover, $\overline{\mathcal{N}_0}$ is a countable, transitive model of ZF^- plus "ω_2 exists", and $\omega_2^{\overline{\mathcal{N}_0}} = \omega_2^{\mathcal{N}} \cap \mathcal{N}_0 = (\omega_2^L)^{\mathcal{N}} \cap \mathcal{N}_0 = (\omega_2^L)^{\overline{\mathcal{N}_0}}$. Therefore $\overline{\mathcal{N}_0}$ is a countable suitable model containing r and by downwards absoluteness of the formula in $(\otimes)$, there is no $\alpha \in \omega_2^{\overline{\mathcal{N}_0}}$ such that for all $m \in \Delta(x * y)$, $S_{\alpha+m}^{\overline{\mathcal{N}_0}}$ is nonstationary in $\overline{\mathcal{N}_0}$. Then $\overline{\mathcal{N}_0}$ witnesses the failure of $(*)$, a contradiction. $\square$

Therefore we can consider the suitable model $\mathcal{M} = L_{\omega_6}[r]$. By $(*)$ there exists $\alpha \in \omega_2^{\mathcal{M}} = \omega_2$ such that for all $m \in \Delta(x * y)$, $(S_{\alpha+m}^{\mathcal{M}} \in \mathrm{NS}_{\omega_1})^{\mathcal{M}}$. As $\vec{S}$ was Σ_1-definable over L_{ω_2} and the fact $\omega_2^{\mathcal{M}} = \omega_2^L = \omega_2^{L[G]}$, we have that $S_{\alpha+m}^{\mathcal{M}} = S_{\alpha+m}$, and $S_{\alpha+m}$ is nonstationary in $\mathcal{M}$ for every $m \in \Delta(x * y)$. By upwards absoluteness, for all such m we also have $(S_{\alpha+m} \in \mathrm{NS}_{\omega_1})^{L[G]}$.

Claim 3.3.33. *In $L[G]$, suppose $\beta = \alpha + m < \omega_2$, and $m \notin \Delta(x_\alpha * y_\alpha)$. Then S_β is stationary in $L[G]$.*

Proof. By the Truth Lemma ([Kun13, Lemma IV.2.24]), there exists $p \in G$ such that $p \Vdash \beta \notin \{\alpha + m \mid m \in \Delta(x_\alpha * y_\alpha)\}$. Let $\mathbb{P} \restriction p = \{q \in \mathbb{P} \mid q \leq p\}$. Note that as G is $\mathbb{P}$-generic over L and $p \in G$, then G is also $\mathbb{P} \restriction p$-generic over L. We have that $\mathbb{P} \restriction p$ is S_β-proper, since it is a countable support iteration of the S_β-proper forcing notions. To verify this it suffices to consider the iterand $\mathbb{Q}_\alpha \restriction p(\alpha)$. But indeed, any condition in $\mathbb{Q}_\alpha \restriction p(\alpha)$ adds no new countable ordinals to $C_\beta \subseteq \omega_1 \setminus S_\beta$, and therefore $\mathbb{P} \restriction p$ is S_β-proper. $\square$

Using the claim, the fact that $S_{\alpha+m}$ is nonstationary in $L[G]$ implies $m \in \Delta(x * y)$, where $(x, y) = (x_\alpha, y_\alpha) = ((\sigma_x^\alpha)^G, (\sigma_y^\alpha)^G)$, and $x <_\alpha y$. Therefore $x <_G y$. $\square$

Lemma 3.3.34. *In $L[G]$, $<_G$ is a Δ^1_3-definable wellorder of the reals.*

Proof. By Lemma 3.3.31, we have

$$x <_G y \Leftrightarrow \Phi(x, y),$$

where $\Phi(x, y)$ is the formula

$$\underbrace{\exists r \in \mathcal{P}(\omega^\omega)}_{\exists}[\underbrace{\forall \mathcal{M} \text{ countable, suitable, } r \in \mathcal{M}}_{\forall}$$
$$\underbrace{(\exists \alpha < \omega_2^{\mathcal{M}}}_{\exists}\ (\underbrace{\mathcal{M} \models \forall m \in \Delta(x * y)\ S_{\alpha+m} \in \mathrm{NS}_{\omega_1}}_{\Delta^1_1}))]$$

Thus, $\Phi(x, y)$ is a Σ^1_3 formula. However, $<_G$ is also a total wellorder, since if x, y are any reals in $L[G]$, there is $\alpha = \max(\gamma_x, \gamma_y) < \omega_2$ such that $x = (\sigma_x^\alpha)^G, y = (\sigma_y^\alpha)^G$, where $\sigma_x^\alpha, \sigma_y^\alpha$ are $\mathbb{P}_\alpha$-names for x, y. Since $<_L$ is a total wellorder on the set of nice $\mathbb{P}_\alpha$-names for reals, either $\sigma_x^\alpha <_L \sigma_y^\alpha$ or $\sigma_y^\alpha <_L \sigma_x^\alpha$; in the case $\neg(x <_G y)$ then we must have $y <_G x$. Therefore the complement of $<_G$ is Σ^1_3-definable, giving that $<_G$ is Δ^1_3-definable. $\square$

Thus, we proved the following:

Theorem 3.3.35. *It is consistent that $\mathfrak{a} = \aleph_1 < \mathfrak{s} = \aleph_2$ that there exists a Δ^1_3 definable wellorder of the reals, and a Π^1_1 tight mad family of size $\aleph_1$.*

Proof. Suppose $V = L$, and fix a Π^1_1 definable tight mad family $\mathcal{A}$ from Lemma 3.3.26. Let $\mathbb{P}$ be the ω_2-length countable support iteration constructed in this section, and let G be $\mathbb{P}$-generic over V. First note $\mathbb{P}$ is proper by Lemma 3.3.30. Define $<_G = \bigcup_{\alpha<\omega_2} \dot{<}_\alpha^G$; by Lemma 3.3.34, $<_G$ is a Δ^1_3 wellorder of the reals. Since any set $\mathcal{S} \subseteq [\omega]^\omega \cap V[G]$ such that $|\mathcal{S}| < \omega_2$

appears at some inital stage $\alpha < \omega_2$ of the iteration and by definition of $\mathbb{P}$, $\dot{\mathbb{Q}}^0_\alpha$ is a $\mathbb{P}_\alpha$-name for the forcing $\mathbb{Q}$ of Definition 3.2.17, we have that $\mathcal{S}$ is not splitting in $V[G_{\alpha+1}]$. Therefore $(\mathfrak{s} = \aleph_2)^{V[G]}$. Notice that by Shoenfield absoluteness, $\mathcal{A}$ is defined by the same Π^1_1 formula in $V[G]$ as in V. Moreover $\mathcal{A}$ remains a tight mad family in $V[G]$ by Lemma 3.3.30, and therefore $\mathcal{A}$ provides a witness for $(\mathfrak{a} = \aleph_1)^{V[G]}$. This completes the proof. $\square$

3.4 Definable spectra

The last intermediary step towards our final model is to consider projectively definable mad families, with a definition of optimal complexity, which are of size $\kappa > \aleph_1$. We will show that we can generically add such a mad family while also preserving ground model tight mad families. First we give an account of the work on large optimal projective mad families thus far.

Friedman and Zdomskyy [FZ10] established that a tight mad family with optimal projective definition is consistent with $\mathfrak{b} = \mathfrak{c} = \aleph_2$ and this was extended by Fischer, Friedman and Zdomskyy [FFZ11] to $\mathfrak{b} = \mathfrak{c} = \aleph_3$. However, a previous result of Raghavan [Rag09] shows no tight mad family can contain a perfect subset, and so by the Mansfield-Solovay theorem (Theorem 1.3.3), no Σ^1_2 tight mad family can exist in a model of $\mathfrak{b} > \aleph_1$; the optimal definition of a tight mad family of size greater than $\aleph_2$ is thus Π^1_2. Dropping the tightness requirement, Brendle and Khomskii [BK13] constructed a model in which $\mathfrak{b} = \mathfrak{c} \geq \aleph_2$, and there exists a Π^1_1 mad family; this is shown to be consistent with a Δ^1_3 wellorder of the reals in Fischer, Friedman, and Khomskii [FFK13]. The first work on projective witnesses of size κ when $\aleph_1 < \kappa < \mathfrak{c}$ is done by Fischer, Friedman, Schrittesser, and Törnquist [FFST25]; again the Mansfield-Solovay theorem, the best possible complexity of such an object is Π^1_2.

So far, the attention has been on finding definable mad families witnessing the value of $\mathfrak{a}$— the minimal element of the *mad spectrum*, this being the set

$$\mathrm{spec}(\mathfrak{a}) = \{|\mathcal{A}| \mid \mathcal{A} \subseteq [\omega]^\omega \text{ is mad}\}.$$

Hechler [Hec72] pioneered the study of the mad spectrum, and in particular he introduces techniques with which one can include a set C as a subset of the spectrum, provided C satisfies a certain list of assumptions. His work was later pursed by Blass [Bla93], and Shelah and Spinas [SS15]. Similar considerations have then since been taken with regards to other cardinal characteristics, such as $\mathfrak{a}_T$ and $\mathfrak{i}$, where $\mathfrak{a}_T$ is the minimal size of a partition of ω^ω into compact sets, and $\mathfrak{i}$ is the minimal size of a maximal independent family (See [FS25], [Bri24] for the former, and [FS19], [FS22b] for the latter). By now there is fairly substantial knowledge of the ways to control realizations of spec($\mathfrak{a}$) in various models of set theory, and likewise recent results provide methods for controlling the definability of mad families of size $\mathfrak{a}$.

One way to build upon these two lines of research is by asking that for each cardinal κ which is the possible size of a mad family, there exists a projective mad family of size κ with an optimal definition. The main result of this section is the following.

Theorem (Theorem 3.4.20)**.** *It is consistent with $\mathfrak{b} = \mathfrak{a} = \aleph_1 < \mathfrak{c} = \aleph_2$ that there exists a Π^1_1-definable tight mad family of size $\aleph_1$, and a Π^1_2-definable tight mad family of size $\aleph_2$.*

The strategy will be as follows. We begin in a model of $V = L$ and fix a Π^1_1 tight mad family $\mathcal{A}_1$. We recursively define a countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$, along the way constructing a Π^1_2 tight mad family $\mathcal{A}_2$ consisting of ω_2-many $\mathbb{Q}_\alpha$-generic reals a_α, and such that each $\mathbb{Q}_\alpha$ is forced by $\mathbb{P}_\alpha$ to be an S-proper poset strongly preserving the tightness of $\mathcal{A}_1$. The definition will be a slight modification of the iteration given in [FZ10], where the goal was to show the consistency of $\mathfrak{b} = \mathfrak{c} = \aleph_2$ with a Π^1_2-definable tight mad family. Specifically, to obtain such this latter result one defines a countable support iteration of S-proper forcings, $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$ such that in $V^{\mathbb{P}_\alpha}$, $\dot{\mathbb{Q}}_\alpha$ is a name for a forcing notion $\mathbb{Q}_\alpha$ such that:

1. When indicated by a bookeeping function, $\mathbb{Q}_\alpha$ is a certain poset $\mathbb{K}_\alpha$ (see Definition 3.4.5 below) which adds a generic real a_α as well as sequences $\vec{C}_\alpha$ and $\vec{Y}_\alpha$ so that:
   (a) $\mathcal{A}_\alpha \cup \{a_\alpha\}$ is almost disjoint, where $\mathcal{A}_\alpha$ is the union of elements of the tight mad family constructed thus far;

(b) For a $\mathbb{P}_\alpha$-generic filter G_α, $a_\alpha \cap x_i$ is infinite, where $\langle \dot{x}_i : i \in \omega \rangle$, given by a bookeeping function, is a sequence of $\mathbb{P}_\alpha$-names such that $\dot{x}_i^{G_\alpha} = x_i$ is an element of $\mathcal{I}(\mathcal{A}_\alpha)^+$ in $V[G_\alpha]$;

(c) $\vec{C}_\alpha = \langle C_{\alpha+m} : m \in \Delta(a_\alpha) \rangle$ is a countable sequence of generic club subsets of ω_1 and $\vec{Y}_\alpha = \langle Y_{\alpha+m} : m \in \Delta(a_\alpha) \rangle$ is a sequence of subsets of ω_1 localizing the addition of the generic clubs;

(d) Using almost disjoint coding, a_α codes $\vec{C}_\alpha$ and $\vec{Y}_\alpha$.

2. For cofinally many $\alpha < \omega_2$, $\mathbb{Q}_\alpha = \mathbb{D}$, Hechler's forcing for adding a dominating real over $V^{\mathbb{P}_\alpha}$.

This ensures that when G is a $\mathbb{P}_{\omega_2}$-generic filter, in $V[G]$ we have that $\mathcal{A}_2 := \{a_\alpha \mid \alpha < \omega_2\}$ is a tight almost disjoint family; this is handled by items (1)(a) and (1)(b). Item (1)(c) renders $\mathcal{A}_2$ a Π^1_2-definable subset of $[\omega]^\omega$ in $V[G]$, similarly to how the Δ^1_3 wellorder was obtained in the previous section. Item (2) guarantees that there are no mad families of size $\aleph_1$ in $V[G]$ so that $\mathcal{A}_2$ is a witness to $(\mathfrak{a} = \aleph_2)^{V[G]}$; this is because of the properties of Hechler's forcing $\mathbb{D}$ for adding a dominating real. This is the partial order $\mathbb{D}$ consisting of pairs $(s, f) \in \omega^{<\omega} \times \omega^\omega$ such that $s \subseteq f$, and the extension relation defined as $(t, g) \leq (s, f)$ if and only if $t \supseteq s$ and $g(n) \geq f(n)$ for every $n \in \omega$. This is a σ-centered forcing which adds a generic real g such that $f \leq^* g$ for each ground model function $f \in \omega^\omega \cap V$, and so finite support iterations of $\mathbb{D}$ are the canonical way of increasing the dominating number $\mathfrak{d}$. However such an iteration also increases the size of $\mathfrak{b}$ (see, for example, [Bla10, Section 11.6]) and therefore kills all mad families of size strictly less than the length of the iteration. This use of $\mathbb{D}$ was pointed out in [FZ10, Question 18], which asks about the consistency of $\mathfrak{b} < \mathfrak{a}$ with a Π^1_2 tight mad family, suggesting that it was unknown to the authors whether the iterand $\mathbb{K}_\alpha$ itself added a dominating real. The main result needed for the theorem of this section will show this is not the case.

Proposition (See Proposition 3.4.15)**.** *Let $\mathbb{K}$ be the Friedman-Zdomskyy forcing notion of Definition 3.4.5, and let $\mathcal{A}$ be a tight mad family in the ground model. Then $\mathbb{K}$ strongly preserves the tightness of $\mathcal{A}$.*

With this, Theorem 3.4.20 can be achieved with a countable support iteration much as that of [FZ10, Theorem 1] outlined above, however we can modify item (2):

2'. For cofinally many $\alpha < \omega_2$, $\mathbb{Q}_\alpha$ is some S-proper forcing notion which strongly preserves the tightness of $\mathcal{A}_1$.

This modification allows flexibility for further applications of the forcing, as in Theorem 3.5.1 of the next section.

Let us say more about how the Π^1_2-definability of $\mathcal{A}_2$ is achieved, as this differs subtly from the case of the Δ^1_3 wellorder of the previous section. Fixing $\langle S_\alpha : \alpha < \omega_2 \rangle$ and S_{-1} as in Proposition 3.3.24 in a model of $V = L$, at stage α in defining the iteration, we want a_α to uniquely determine a pattern of stationarity/nonstationarity in the sequence

$$\langle S_{\alpha+m} : m \in \omega \rangle,$$

namely by coding the sequence $\vec{C}_\alpha = \langle C_{\alpha+m} : m \in \Delta(a_\alpha) \rangle$, where $C_{\alpha+m}$ is a generic club disjoint from $S_{\alpha+m}$ for every $m \in \Delta(a_\alpha)$. This will give that in the final extension, membership in $\mathcal{A}_2$ can be defined as:

$$a \in \mathcal{A}_2 \Leftrightarrow \exists \alpha \in \omega_2 \; L[a_\alpha] \models \Delta(a_\alpha) = \{m \in \omega \mid S_{\alpha+m} \in \mathrm{NS}_{\omega_1}\}. \qquad (\star)$$

Therefore another job of the iterand $\mathbb{K}_\alpha$ is to add such generic clubs, and so conditions will consist of a *finite part*, taking care of approximations to the set a_α, and an *infinite part*, making countable approximations to a club in $\omega_1 \setminus S_{\alpha+m}$. It is important that for $m \notin \Delta(a_\alpha)$, $S_{\alpha+m}$ remains stationary, so this simultaneous construction must be carried out carefully. To make the definiton of $\mathcal{A}_2$ projective, the right hand side of $(\star)$ is localized to the class of countable suitable models, again relying on the localization techniques of René David appearing in Definition 3.3.10. For this purpose $\mathbb{K}_\alpha$ adds the sequence

$$\langle Y_{\alpha+m} : m \in \Delta(a_\alpha) \rangle$$

such that for each $Y_{\alpha+m}$, the set $\{\beta < \omega_1 \mid Y^\alpha_m(2\beta) = 1\}$ is a sufficiently absolute code for the ordinal $\alpha < \omega_2$. As a consequence, the initial existential quantification over ω_2 in $(\star)$ will range over the $\omega_2^{\mathcal{M}}$ for countable suitable

$\mathcal{M}$ containing the real a_α. Formally, in the final generic extension $V[G]$, for all $a \in V[G] \cap [\omega]^\omega$,

$$\begin{aligned} a \in \mathcal{A}_2 \Leftrightarrow & \forall \mathcal{M}(\mathcal{M} \text{ is a countable suitable model and } a \in \mathcal{M}) \\ & \exists \overline{\alpha} < \omega_2^{\mathcal{M}}\ \forall m \in \Delta(a)(\mathcal{M} \models S_{\overline{\alpha}+m} \text{ is nonstationary}). \end{aligned}$$

The right hand side of the above is in the form $\forall\exists$ and thus is a Π^1_2 formula.

For the proof of Theorem 3.4.20 we first establish preliminaries; throughout the rest of the section we work in a model of $V = L$ unless explicitly stated otherwise. Fix a coanalytic tight mad family $\mathcal{A}_1$ as in Lemma 3.3.26, and using Proposition 3.3.24 fix $\vec{S} = \langle S_\alpha : \alpha < \omega_2 \rangle \in L$ a sequence of pairwise almost disjoint stationary costationary in L subsets of ω_1, and a stationary subset $S_{-1} \subseteq \omega_1$ such that $S_{-1} \cap S_\alpha = \emptyset$ for all $S_\alpha \in \vec{S}$. Let $F\colon \mathrm{Lim}(\omega_2) \to L_{\omega_2}$ be such that $F^{-1}(x)$ is unbounded in ω_2 for all $x \in L_{\omega_2}$, where $\mathrm{Lim}(\omega_2)$ denotes the set of limit ordinals in ω_2. Recall $\vec{S}$ and F are Σ_1-definable over L_{ω_2}, and moreover that whenever $\mathcal{M}, \mathcal{N}$ are suitable models with $\omega_1^{\mathcal{M}} = \omega_1^{\mathcal{N}}$, then $\vec{S}^{\mathcal{M}}$ and $\vec{S}^{\mathcal{N}}$ agree on $\omega_2^{\mathcal{M}} \cap \omega_2^{\mathcal{N}}$.

Fix $\psi(x, y)$ given by Fact 3.3.29 and for all $\alpha < \omega_2^L$, let $X_\alpha \subseteq \omega_1^L$ denote a sufficiently absolute code for α.

Whereas the coding of the Δ^1_3 wellorder was achieved by the $C(Y)$-generic reals, the generic reals in the current context will result from almost disjoint coding, a method developed by Solovay and Jensen [SJ70], crucial for Harrington's construction [Har77], and for which we give a general definition.

Definition 3.4.1. Let $\mathcal{R}$ be an almost disjoint family in V, and let $X \in V$ be a subset of ω_1. The *almost disjoint coding of X with respect to $\mathcal{R}$* is the partial order $\mathbb{P}_{\mathcal{R}}(X)$ consisting of conditions (s, F) such that s is a finite subset of ω and F is a finite subset of $\{r_\xi \mid \xi \in X\} \subseteq \mathcal{R}$. The extension relation is defined by letting $(t, G) \leq (s, F)$ if and only if:

1. t end-extends s and $G \supseteq F$;

2. For all $\xi \in X$ and $r_\xi \in F$, $(t \setminus s) \cap r_\xi = \emptyset$.

Fact 3.4.2. The following hold.

- If $(s, F), (t, G)$ are compatible conditions in $\mathbb{P}_{\mathcal{R}}(X)$, then they admit a minimal lower bound $(s \cup t, F \cup G)$.

- $\mathbb{P}_{\mathcal{R}}(X)$ is σ-centered.[6]
- If G is $\mathbb{P}_{\mathcal{R}}(X)$-generic over M, let

  $$a := \bigcup\{s \mid \exists F(s, F) \in G\}.$$

  Then G and a are mutually definable in the generic extension; that is, $M[G] = M[a]$.

Lemma 3.4.3. *Let G be a $\mathbb{P}_{\mathcal{R}}(X)$-generic filter over V, and let a be the generic real defined by G as above. Then in $V[a]$, for all $\xi < \omega_1$:*

$$\xi \in X \Leftrightarrow a \cap r_\xi \text{ is finite.}$$

For proofs of the above, see, for example, [Har77] or [Jec97, Example IV]. A proof similar to that of the last lemma will be given in Lemma 3.4.16.

For the present coding purposes we fix an almost disjoint family

$$\mathcal{R} = \{R_{\langle\eta,\xi\rangle} \mid \eta \in \omega \cdot 2, \xi \in \omega_1\}$$

which is Σ_1-definable over L_{ω_1}, and such that for every suitable model $\mathcal{M}$, $\mathcal{R} \cap \mathcal{M} = \{R_{\langle\eta,\xi\rangle} \mid \eta \in \omega \cdot 2, \xi \in \omega_1^{\mathcal{M}}\}$. For an example of such a family see [FZ10, Proposition 3].

Recall in the previous section we had defined a function $\Delta\colon \omega^\omega \to \omega^\omega$ such that $\Delta(x)$ was a real coding both x and $\omega \setminus x$. Because of technical reasons in the coding we will do, we will modify the definition of this function, and in particular extend its definition to the finite subsets. Specifically, for $s \subseteq \omega$, finite or infinite, let

$$\Delta(s) = \{2n+1 \mid n \in s\} \cup \{2n+2 \mid n \in (\sup s \setminus s)\},$$

[6] A forcing notion $\mathbb{P}$ is said to be *σ-centered* if $\mathbb{P}$ can be written as a countable union of subsets,

$$\mathbb{P} = \bigcup_{n\in\omega} P_n,$$

such that each P_n is *centered*, meaning for any finite $F \subseteq P_n$, there exists $q \in \mathbb{P}$ such that $q \leq p$ for all $p \in F$. It is easy to see any σ-centered forcing notion is ccc. In the case of $\mathbb{P}_{\mathcal{R}}(X)$ above, we can assume the stronger notion, that the lower bound q of every $p \in F \subseteq P_n$ can be taken to be also a member of P_n.

and let $C(s) = \Delta(s) \cup (\omega \setminus \max(\Delta(s)))$. We think of $C(s)$ as the "coding area" associated with the finite subset s. Denote by $E(s), O(s)$ the sets $\{s(2n) \mid 2n < |s|\}$ and $\{s(2n+1) \mid 2n+1 < |s|\}$ respectively, where $s(\ell)$ denotes the nth element of s for every $\ell < |s|$. For a limit ordinal γ and a function $r \colon \gamma \to 2$, let $\mathrm{Even}(r) = \{\alpha < \gamma \mid r(2\alpha) = 1\}$ and $\mathrm{Odd}(r) = \{\alpha < \gamma \mid r(2\alpha+1) = 1\}$.

Lastly, for ordinals $\alpha < \beta$, let $\beta - \alpha$ denote the ordinal γ such that $\alpha + \gamma = \beta$. If B is a set of ordinals, let $B - \alpha = \{\beta - \alpha \mid \beta \in B\}$. If δ is an indecomposable ordinal, i.e. $\delta = \omega^\gamma$ for some ordinal γ, it straightforward to check $(\alpha + B) \cap \delta - \alpha = B \cap \delta$.

Recall our goal is to define a countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$, such that for all $\alpha < \omega_2$, $\dot{\mathbb{Q}}_\alpha$ is forced to be an S_{-1}-proper forcing notion which strongly preserves the tightness of $\mathcal{A}_1$, and along the way we construct a Π^1_2 tight mad family $\mathcal{A}_2 = \{a_\alpha \mid \alpha < \omega_2\}$ consisting of $\mathbb{Q}_\alpha$-generic reals a_α.

Continuing with the recursive definition, suppose $\alpha < \omega_2$ and $\mathbb{P}_\alpha$ has been defined. Let G_α be a $\mathbb{P}_\alpha$-generic filter, and let $\dot{\mathcal{A}}^2_\alpha$ be a $\mathbb{P}_\alpha$-name for the set of elements of $\mathcal{A}_2$ constructed up to stage α. The density arguments of Lemma 3.4.16 require the following inductive assumption:

$$\forall r \in \mathcal{R} \forall A' \in [\mathcal{A}^2_\alpha]^{<\omega} \; (|\mathrm{E}(r) \setminus \bigcup A'| = |\mathrm{O}(r) \setminus \bigcup A'| = \aleph_0), \tag{$*$}$$

Since $\mathcal{R}$ is an almost disjoint family, $(*)$ implies that for every $A' \in [\mathcal{A}^2_\alpha \cup \mathcal{R}]^{<\omega}$ and $r \in \mathcal{R} \setminus A'$, also $|E(r) \setminus \bigcup A'| = |O(r) \setminus \bigcup A'| = \aleph_0$, as otherwise $r \cap r'$ is infinite for some $r' \in A' \cap \mathcal{R}$.

If $\alpha < \omega_2$ is a successor ordinal, let $\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{Q}}^0_\alpha$ be a $\mathbb{P}_\alpha$-name for a proper forcing notion of cardinality $\aleph_1$ such that $\Vdash_{\mathbb{P}_\alpha}$"$\dot{\mathbb{Q}}^0_\alpha$ strongly preserves the tightness of $\mathcal{A}_1$".

For $\alpha \in \mathrm{Lim} \cap \omega_2$, unless explicitly mentioned, $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for the trivial forcing. Suppose $F(\alpha)$ is a sequence $\langle \dot{x}_i : i \in \omega \rangle$ of $\mathbb{P}_\alpha$-names such that $x_i = \dot{x}_i^{G_\alpha}$ is an infinite subset of ω and for all $i \in \omega$, $x_i \in \mathcal{I}(\mathcal{A}^2_\alpha)^+$.

Claim 3.4.4. *There exists a limit ordinal $\eta_\alpha \in \omega_1$ with the property that there are no finite subsets J, E of $\omega \cdot 2 \times (\omega_1 \setminus \eta_\alpha)$, $\mathcal{A}^2_\alpha$, respectively, and $i \in \omega$ such that $x_i \subseteq \bigcup_{\langle \eta, \xi \rangle \in J} R_{\langle \eta, \xi \rangle} \cup \bigcup E$.*

Proof. Let I denote the set of all $i \in \omega$ for which there exists $J_i \in [\omega \cdot 2 \times \omega_1]^{<\omega}$ and $E_i \in [\mathcal{A}^2_\alpha]^{<\omega}$ such that $x_i \subseteq \bigcup_{\langle \eta,\xi \rangle \in J_i} R_{\langle \eta,\xi \rangle} \cup \bigcup E_i$. Let $\eta_\alpha \in \omega_1$ be a limit ordinal such that for all $i \in I$, if $\langle \gamma, \eta \rangle \in J_i$, then $\eta < \eta_\alpha$. Such a limit ordinal exists, since for each $i \in I$ there are only countably many choices for the value of the second coordinate of J_i, as J_i is finite.

We will show that this η_α satisfies the conclusion of the claim. Fix $i \in I$, $J \in [\omega \cdot 2 \times (\omega_1 \setminus \eta_\alpha)]^{<\omega}$, and $E \in [\mathcal{A}^2_\alpha]^{<\omega}$; we show that

$$x_i \not\subseteq \bigcup_{\langle \eta,\xi \rangle \in J} R_{\langle \eta,\xi \rangle} \cup \bigcup E.$$

Since E, E_i are finite subsets of $\mathcal{A}^2_\alpha$, by hypothesis on x_i we have that $c_i := x_i \setminus \bigcup E \cup \bigcup E_i$ is infinite. But since $i \in I$, this means that $c_i \subseteq^* \bigcup_{\langle \eta,\xi \rangle \in J_i} R_{\langle \eta,\xi \rangle}$. Notice that as $\mathcal{R}$ is an almost disjoint family and $J \cap J_i = \emptyset$, we have that $\bigcup_{\langle \eta,\xi \rangle \in J_i} R_{\langle \eta,\xi \rangle} \cap \bigcup_{\langle \eta,\xi \rangle \in J} R_{\langle \eta,\xi \rangle}$ is finite. Therefore $c_i \cap \bigcup_{\langle \eta,\xi \rangle \in J} R_{\langle \eta,\xi \rangle}$ is finite. In particular $c_i \not\subseteq \bigcup_{\langle \eta,\xi \rangle \in J} R_{\langle \eta,\xi \rangle}$, and thus $x_i \not\subseteq \bigcup_{\langle \eta,\xi \rangle \in J} R_{\langle \eta,\xi \rangle} \cup \bigcup E$ as desired. $\square$

Fix η_α as in the conclusions of the previous claim. Fix also $Z_\alpha \subset \omega$ coding a surjection of ω onto η_α. The following is the original definition from [FZ10, Section 3].

Definition 3.4.5 ([FZ10])**.** The partial order $\mathbb{K}_\alpha$ consists of conditions of the form $p = \langle \langle s, s^* \rangle, \langle c_k, y_k : k \in \omega \rangle \rangle$, such that:

1. $c_k \subseteq \omega_1 \setminus \eta_\alpha$ is a closed bounded subset such that $S_{\alpha+k} \cap c_k = \emptyset$;
2. $y_k \colon |y_k| \to 2$ is a function from a countable limit ordinal $|y_k| \in \omega_1$ such that
   - $|y_k| > \eta_\alpha$, $y_k \restriction \eta_\alpha = 0$;
   - for all $\gamma < |y_k|$, $y_k(\eta_\alpha + 2\gamma) = 1$ if and only if $\gamma = \eta_\alpha$ or $\gamma > \eta_\alpha$ and $\mathrm{Even}(y_k) = \{\eta_\alpha\} \cup (\eta_\alpha + X_\alpha)$.[7]
3. $s \in [\omega]^{<\omega}$ and s^* is a finite subset of the set

   $$\{R_{\langle m,\xi \rangle} \mid m \in \Delta(s), \xi \in c_m\} \cup \{R_{\langle \omega+m,\xi \rangle} \mid m \in \Delta(s),\ y_m(\xi) = 1\} \cup \mathcal{A}^2_\alpha.$$

[7]Recall X_α denotes the sufficiently absolute code for α given by Fact 3.3.29.

Additionally, for all $n \in \omega$ such that $2n < |s \cap R_{\langle 0,0\rangle}|$, $n \in Z_\alpha$ if and only if there exists $m \in \omega$ such that $(s \cap R_{\langle 0,0\rangle})(2n) = R_{\langle 0,0\rangle}(2m)$;

4. For all $k \in C(s)$ and for all limit ordinals $\gamma \in \omega_1$ such that $\eta_\alpha < \gamma \leq |y_k|$, if γ is a limit point of c_k and $\gamma = \omega_1^{\mathcal{M}}$ for some countable suitable model $\mathcal{M}$ containing both $y_k \restriction \gamma$ and $c_k \cap \gamma$ as elements, then the following holds in $\mathcal{M}$: " $[\mathrm{Even}(y_k) - \min(\mathrm{Even}(y_k))] \cap \gamma$ is the code of some limit ordinal $\overline{\alpha} \in \omega_2$ such that $S^{\mathcal{M}}_{\overline{\alpha}+k}$ is nonstationary."

For $p = \langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle$ and $q = \langle\langle t, t^*\rangle, \langle d_k, z_k : k \in \omega\rangle\rangle$ conditions in $\mathbb{K}_\alpha$, define q to be an extension of p and write $q \leq p$ if and only if:

1. t end-extends s, $t^* \supseteq s^*$, and for all $x \in s^*$, $(t \setminus s) \cap x = \emptyset$;
2. For all $k \in C(t)$, d_k end-extends c_k and $z_k \supseteq y_k$.

For $p = \langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle \in \mathbb{K}_\alpha$, let $\mathrm{Fin}(p) = \langle s, s^*\rangle$ denote the finite part of p, and let $\mathrm{Inf}(p) = \langle c_k, y_k : k \in \omega\rangle$ denote the infinite part of p. When $p, q \in \mathbb{K}_\alpha$ and $q \leq p$, we say q is a *pure extension of* p if $\mathrm{Fin}(p) = \mathrm{Fin}(q)$.

Remark 3.4.6. The notion $\mathbb{K}_\alpha$ can be seen as a hybrid of the forcing notions of almost disjoint coding (Definition 3.4.1), club shooting (Definition 3.3.1), and localization (Definition 3.3.10). This is made more explicit in the proof of Lemma 3.4.12 below.

The following is our main result of this section; it will establish both the preservation of cardinals and of the tight mad family $\mathcal{A}_1$.

Proposition (Proposition 3.4.15)**.** *For every $p = \langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle \in \mathbb{K}_\alpha$, every countable elementary submodel $M \prec H_\theta$, for θ a sufficiently large cardinal, containing $p, \mathbb{K}_\alpha, \mathcal{A}_1$, and every $B \in \mathcal{I}(\mathcal{A}_1)$ such that $B \cap Y$ is infinite for all $Y \in \mathcal{I}(\mathcal{A}_1)^+ \cap M$, if $M \cap \omega_1 = j \notin \bigcup_{k \in C(s)} S_{\alpha+k}$, then there is an $(M, \mathbb{K}_\alpha, \mathcal{A}_1, B)$-generic condition $q \leq p$ such that* $\mathrm{Fin}(q) = \mathrm{Fin}(p)$*.*

We will need the following intermediary lemmas.

Lemma 3.4.7 ([FF10, Lemma 1])**.** *For every $p = \langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle$ in $\mathbb{K}_\alpha$ and every $\gamma \in \omega_1$, there exists a pure extension $q \leq p$ with* $\mathrm{Inf}(q) = \langle d_k, z_k : k \in \omega\rangle$*, such that $|z_k| \geq \gamma$ and* $\max(d_k) \geq \gamma$*, for every $k \in \omega$.*

Proof. Fix $p = \langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle$ in $\mathbb{K}_\alpha$ and $k \in \omega$, and suppose $\gamma \in \omega_1$ is such that $\gamma > |y_k|$ and $\gamma > \max(c_k)$.

First we extend c_k. Since $\omega_1 \setminus S_{\alpha+k}$ is stationary, and for any $\eta \in \omega_1$ the set

$$T_\eta = \{\xi \in \omega_1 \mid \xi \text{ is a limit ordinal, } \xi \geq \eta\}$$

is a club, there exists $\xi \in T_{\eta_\alpha} \cap T_\gamma \cap (\omega_1 \setminus S_{\alpha+k})$. Let $\langle \xi_n : n \in \omega\rangle$ be an increasing cofinal sequence with limit ξ such that $\xi_n \geq |y_k|$ and $\xi_n \notin S_{\alpha+k}$ for all $n \in \omega$. Define $d_k := c_k \cup \{\xi_n \mid n \in \omega\} \cup \{\xi\}$. Then d_k is a bounded subset of $\omega_1 \setminus \eta_\alpha$ with $d_k \cap S_{\alpha+k} = \emptyset$, and it is closed since any increasing sequence in $d_k \setminus c_k$ has limit $\xi \in d_k$.

Next we extend y_k. Let $W \subseteq \omega$ be any code for a bijection between ω and ξ. Define $z_k \colon \xi \to 2$ so that $z_k \restriction |y_k| = y_k$, and $\mathrm{Odd}(z_k \restriction [|y_k|, |y_k|+\omega)) = W$. For $\beta \in [|y_k| + \omega, \xi)$, let $z_k(\beta) = 0$.

Then $\langle\langle s, s^*\rangle, \langle d_k, z_k : k \in \omega\rangle\rangle$ is a condition in $\mathbb{K}_\alpha$; the only item to be checked is (4) of Definition 3.4.5, for limit ordinals η such that $|y_k| < \eta \leq |z_k|$. For such η, $\gamma \geq |y_k| + \omega$, so if $\mathcal{M}$ is a transitive model such that $\eta = \omega_1^{\mathcal{M}}$ and $z_k \restriction \eta \in \mathcal{M}$, then $\mathcal{M} \models$"$\eta = \omega_1^{\mathcal{M}}$ is countable", and hence $\mathcal{M}$ cannot be a model of ZF^-. Therefore (4) is vacuously true. $\square$

The following notion appears implicitly in [FZ10].

Definition 3.4.8. For a condition $p = \langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle$ in $\mathbb{K}_\alpha$ and open dense $D \subseteq \mathbb{K}_\alpha$, we say p is *preprocessed for D* if and only if for every extension $q = \langle\langle t, t^*\rangle, \langle d_k, z_k : k \in \omega\rangle\rangle \leq p$, if $q \in D$, then already there is some t_2^* such that $q' = \langle\langle t, t_2^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle$ is a condition in $\mathbb{K}_\alpha$ extending p, and $q' \in D$.

Lemma 3.4.9. *Suppose $p \in \mathbb{K}_\alpha$ is preprocessed for a dense open set D, and let $r \leq p$. Then r is preprocessed for D.*

Proof. Let $p = \langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle$ be preprocessed for D and let $r \leq p$, with $r = \langle\langle t, t^*\rangle, \langle d_k, z_k : k \in \omega\rangle\rangle$. If $q = \langle\langle t_1, t_1^*\rangle, \langle d'_k, z'_k : k \in \omega\rangle\rangle$ is any extension of r such that $q \in D$, then also $q \leq p$ and so by preprocessedness of p there is t_2^* such that $\langle\langle t_1, t_2^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle \in D$. Let $t_3^* = t_2^* \cup t^*$. Then $\langle\langle t_1, t_3^*\rangle, \langle d_k, z_k : k \in \omega\rangle\rangle$ is an extension of r which is also an element of D, as D is open and $\langle\langle t_1, t_3^*\rangle, \langle d_k, z_k : k \in \omega\rangle\rangle \leq \langle\langle t_1, t_1^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle \in D$.

□

The properness of Shelah's forcing $\mathbb{Q}$ required the set of pure extensions which are preprocessed for a given open dense set to be dense in $\mathbb{Q}$. This will also be the case for establishing the S_{-1}-properness of $\mathbb{K}_\alpha$ and motivates the next lemma, whose proof we give in detail for transparency.

Lemma 3.4.10 ([FZ10, Claim 9])**.** *For any $p \in \mathbb{K}_\alpha$ and open dense $D \subseteq \mathbb{K}_\alpha$, there exists a pure extension $q \leq p$ such that q is preprocessed for D.*

Proof. Let $p = \langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle \in \mathbb{K}_\alpha$ and let $D \subseteq \mathbb{K}_\alpha$ be an open dense set. Let $M \prec H_\theta$ be a countable elementary submodel, where θ is sufficiently large and M contains $p, \mathbb{K}_\alpha, X_\alpha$, and D as elements. Let $j = M \cap \omega_1$. Since $S_{\alpha+k}$ is costationary for all $k \in C(s)$, without loss of generality we may assume $j \notin \bigcup_{k \in C(s)} S_{\alpha+k}$.[8]

Let $\{\langle r_n, s_n\rangle \mid n \in \omega\}$ enumerate all pairs $\langle r, s\rangle \in (\mathbb{K}_\alpha \cap M) \times [\omega]^{<\omega}$ such that $r \leq p$ and each pair appears infinitely often. Let also $\langle j_n : n \in \omega\rangle$ be an increasing cofinal sequence in j with $\{j_n \mid n \in \omega\} \subseteq M$. By induction on $n \in \omega$, construct sequences $\langle d_k^n, z_k^n : k \in \omega\rangle \in M$ such that:

1. $d_k^0 = c_k$, $z_k^0 = y_k$ for all $k \in \omega$;
2. <u>If</u> there exists some $\tilde{r} \in \mathbb{K}_\alpha \cap M$ such that:
   (a) $\tilde{r} \leq r_n$;
   (b) $\tilde{r} \leq \langle\langle s, s^*\rangle, \langle d_k^n, z_k^n : k \in \omega\rangle\rangle$;
   (c) $\tilde{r} \in D$;
   (d) $\mathrm{Fin}(\tilde{r}) = \langle s_n, t^*\rangle$ and $\mathrm{Inf}(\tilde{r}) = \langle d_k', z_k' : k \in \omega\rangle$ for some t^* and some $\langle d_k', z_k' : k \in \omega\rangle$;

[8] Specifically, we use the fact that

$$E = \{N \cap \omega_1 \mid N \prec H_\theta \text{ is countable, } \mathbb{K}_\alpha, p, D, X_\alpha \in N\}$$

is a club in ω_1.

then let $\langle d_k^{n+1}, z_k^{n+1} : k \in \omega\rangle$ be an extension of $\mathrm{Inf}(\tilde{r})$, in the sense that d_k^{n+1} end-extends d'_k and $z_k^{n+1} \supseteq z'_k$, so that $\langle\langle s, s^*\rangle, \langle d_k^{n+1}, z_k^{n+1} : k \in \omega\rangle\rangle$ is a condition in $\mathbb{K}_\alpha$ with $\max(d_k^{n+1}) \geq j_n$ and $|z_k^{n+1}| \geq j_n$ for each $k \in C(s)$. To be precise, for all $k \in C(s_n)$, Lemma 3.4.7 gives the existence of d_k^{n+1} and z_k^{n+1} as desired, though to ensure $\langle\langle s, s^*\rangle, \langle d_k^{n+1}, z_k^{n+1} : k \in \omega\rangle\rangle$ is a condition in $\mathbb{K}_\alpha$ below $p_n := \langle\langle s, s^*\rangle, \langle d_k^n, z_k^n : k \in \omega\rangle\rangle$, we need to also take care of $k \in C(s) \setminus C(s_n)$; this is still taken care of by Lemma 3.4.7 applied to p_n.

If no such $\tilde{r}$ exists, let $\langle d_k^{n+1}, z_k^{n+1} : k \in \omega\rangle$ be any extension of $\mathrm{Inf}(\tilde{r})$ such that $\langle\langle s, s^*\rangle, \langle d_k^{n+1}, z_k^{n+1} : k \in \omega\rangle\rangle$ is a condition in $\mathbb{K}_\alpha$ with $\max(d_k^{n+1}) \geq j_n$ and $|z_k^{n+1}| \geq j_n$ for each $k \in C(s)$.

Set $d_k = \bigcup_{n\in\omega} d_k^n \cup \{j\}$ and $z_k = \bigcup_n z_k^n$ for all $k \in C(s)$, and for $k \notin C(s)$ let $d_k = z_k = \emptyset$.

We verify that $q := \langle\langle s, s^*\rangle, \langle d_k, z_k : k \in \omega\rangle\rangle$ is a condition in $\mathbb{K}_\alpha$. That clauses (1) and (2) hold in the definition of $\mathbb{K}_\alpha$ follows from the fact $j = M \cap \omega_1$ is a countable limit ordinal. As $\mathrm{Fin}(q) = \mathrm{Fin}(p)$, (3) is immediately satisfied. It suffices to check item (4) for $k \in C(s)$ and the limit ordinal $\gamma = j$. Fix $k \in C(s)$ and let $\mathcal{M}$ be a countable suitable model with $\omega_1^{\mathcal{M}} = \gamma$ and $z_k \restriction \gamma, d_k \cap \gamma \in \mathcal{M}$. Since γ is an indecomposable ordinal, $[\mathrm{Even}(z_k) - \min(\mathrm{Even}(z_k))] \cap \gamma = X_\alpha \cap \gamma \in \mathcal{M}$. Since X_α is a sufficiently absolute code for α, in $\mathcal{M}$ it holds that there exists a unique $\tilde{\alpha} \in \omega_2^{\mathcal{M}}$ such that $\mathcal{M} \models$"$X_\alpha \cap \gamma$ codes $\tilde{\alpha}$".

Let $\pi \colon M \to \overline{M}$ be the Mostowski collapse isomorphism, and so note that $\omega_1^{\overline{M}} = \pi(M \cap \omega_1) = \gamma$, i.e. $\omega_1^{\mathcal{M}} = \omega_1^{\overline{M}}$. As M was an elementary submodel of H_θ and $X_\alpha \in M$, in M the ordinal α is uniquely coded by X_α, and so in $\overline{M}$ the ordinal $\pi(\alpha) = \overline{\alpha}$ is the unique solution to $\psi(x, \pi(X_\alpha))$, that is, $\overline{\alpha}$ is uniquely coded by $\pi(X_\alpha) = X_\alpha \cap \gamma$. Again since we had chosen X_α to be a sufficiently absolute code for α and $\overline{M}, \mathcal{M}$ are suitable models with $\omega_1^{\overline{M}} = \omega_1^{\mathcal{M}}$, we have that in both $\mathcal{M}$ and $\overline{M}$, the unique ordinal coded by $X_\alpha \cap \gamma$ is $\overline{\alpha} = \tilde{\alpha}$.

In M, $(d_k \cap S_{\alpha+k}) \cap \gamma = \emptyset$, so by isomorphism, in $\overline{M}$, $\pi(d_k) = d_k \cap \gamma$ is disjoint from $\pi(S_{\alpha+k}) = S^{\overline{M}}_{\overline{\alpha}+k} = S_{\overline{\alpha}+k} \cap \gamma$. Again as $\omega_1^{\overline{M}} = \omega_1^{\mathcal{M}}$, by choice of the sequence $\vec{S}$, we have $S^{\overline{M}}_{\overline{\alpha}+k} = S^{\mathcal{M}}_{\overline{\alpha}+k}$, so also in $\mathcal{M}$ it holds that $d_k^{\mathcal{M}} = d_k \cap \gamma$ is disjoint from $S^{\mathcal{M}}_{\overline{\alpha}+k}$. Since $d_k \cap \gamma \in \mathcal{M}$ and $d_k \cap \gamma$ is a closed unbounded set in $\omega_1^{\mathcal{M}}$, $S^{\mathcal{M}}_{\overline{\alpha}+k}$ is nonstationary in $\mathcal{M}$. Therefore $q \in \mathbb{K}_\alpha$.

Let us see that $q \in \mathbb{K}_\alpha$ is as desired. Take any $r \leq q$; without loss of generality $r \in D$. Write $r = \langle\langle t, t^*\rangle, \langle d'_k, z'_k : k \in \omega\rangle\rangle$. Since both t, t^* are finite objects and for all $n \in \omega$, $\langle d^n_k, z^n_k : k \in \omega\rangle \in M$, there exists $m \in \omega$ such that $r^M = \langle\langle t, t^* \cap M\rangle, \langle d^m_k, z^m_k : k \in \omega\rangle\rangle$ is a condition in $\mathbb{K}_\alpha \cap M$, where $t^* \cap M$ is a finite subset of

$$\{R_{\langle \ell,\xi\rangle} \mid \ell \in \Delta(t), \xi \in d'_\ell \cap j\} \cup \{R_{\langle \omega+\ell,\xi\rangle} \mid \ell \in \Delta(t), y'_\ell \restriction j(\xi) = 1\} \\ \cup \{a_\beta \mid \beta < \alpha \cap M\}.$$

Note that $r \leq r^M$.

Then $r^M \leq p$: we have immediately that t is an end-extension of s since $r \leq q$, and if $R_{\langle \ell,\xi\rangle} \in s^*$, then $\ell \in \Delta(s)$ so $\ell \in \Delta(t)$. Since $\xi \in c_\ell$ and d^m_ℓ end-extends c_ℓ, also $\xi \in d^m_\ell$. A similar argument works for all $R_{\langle \omega+\ell,\xi\rangle} \in s^*$. Likewise, if $a_\beta \in s^*$, then $a_\beta \in t^*$ and $\beta < \alpha \cap M$, so $a_\beta \in t^* \cap M$ as $p \in M$. That $(t \setminus s) \cap R = \emptyset$ for all $R \in s^*$ follows from the fact $r \leq p$. Next, if $k \in C(t)$, then d'_k end-extends d^m_k which in turn end-extends c_k, since $k \in C(s)$. The argument is similar for the case of z'_k and y_k. Altogether we have indeed that $r^M \leq p$.

Therefore there exists $n \geq m$ such that $r^M = r_n$ and $t = s_n$. Note that $r \in \mathbb{K}_\alpha$ has properties (a)-(d) above: (a) and (b) follow from the fact r is a common extension of q and r^M, (c) is immediate by assumption, and (d) follows from the fact $t = s_n$. Then in H_θ it holds:

$$\exists x(x \in \mathbb{K}_\alpha \wedge x \text{ satisfies (a)-(d)}),$$

so by elementarity M satisifes the same formula, i.e., there exists some condition $\tilde{r} \in \mathbb{K}_\alpha \cap M$, $\tilde{r}$ has properties (a)-(d). In particular there are t^*_2 and $\langle \tilde{d}_k, \tilde{z}_k : k \in \omega\rangle$ in M with the properties that $t^*_2 \supseteq s^* \cap (t^* \cap M)$ and $\tilde{d}_k, \tilde{z}_k$ end-extend d^n_k, z^n_k respectively for $k \in C(s_n)$, and

$$\tilde{r} = \langle\langle s_n = t, t^*_2\rangle, \langle \tilde{d}_k, \tilde{z}_k : k \in \omega\rangle\rangle \in \mathbb{K}_\alpha \cap M.$$

Then at this stage of the inductive construction we chose d^{n+1}_k, z^{n+1}_k to be appropriate end-extensions of $\tilde{d}_k, \tilde{z}_k$ respectively for all $k \in C(s)$.

Claim 3.4.11. *$\overline{r} = \langle\langle s_n, t^*_2\rangle, \langle d_k, z_k : k \in \omega\rangle\rangle$ is a condition in $\mathbb{K}_\alpha$ extending q.*

Proof. First we verify $\langle\langle t, t_2^*\rangle, \langle d_k, z_k : k \in \omega\rangle\rangle$ is a condition in $\mathbb{K}_\alpha$. Clauses (1) and (2) follow because $q \in \mathbb{K}_\alpha$ and $\mathrm{Inf}(\overline{r}) = \mathrm{Inf}(q)$. For clause (3), clearly $t \in [\omega]^{<\omega}$, and as $\tilde{r} \in \mathbb{K}_\alpha$, t_2^* is a finite subset of

$$\{R_{\langle m,\xi\rangle} \mid m \in \Delta(t), \xi \in \tilde{d_m}\} \cup \{R_{\langle \omega+m,\xi\rangle} \mid m \in \Delta(t), \tilde{z_m}(\xi) = 1\} \cup (\mathcal{A}_\alpha^2 \cap M),$$

so as d_k is an end-extension of $\tilde{d_k}$ and $z_k \supseteq \tilde{z}_k$ for all $k \in C(s)$, this is also true for all $k \in \Delta(t)$, and therefore t_2^* is similarly a finite subset of

$$\{R_{\langle m,\xi\rangle} \mid m \in \Delta(t), \xi \in d_m\} \cup \{R_{\langle \omega+m,\xi\rangle} \mid m \in \Delta(t), z_m(\xi) = 1\} \cup \mathcal{A}_\alpha^2.$$

The fact that $q \in \mathbb{K}_\alpha$ and $k \in C(t) \subseteq C(s)$ gives that item (4) is verified for each $k \in C(t)$ and the limit ordinal $\gamma = j$. Therefore $\overline{r} \in \mathbb{K}_\alpha$.

Lastly, we have $\overline{r} \leq q$ as this follows from the fact $\mathrm{Fin}(\overline{r}) = \mathrm{Fin}(\tilde{r})$ and $\tilde{r}$ satisfies clauses (a)-(d), so in particular $\mathrm{Fin}(\overline{r})$ extends $\mathrm{Fin}(q)$ in the almost disjoint coding. Item (2) in the definition of the extension relation is immediate. □

Moreover $\overline{r} \leq \tilde{r}$ since for all $k \in C(s)$, by construction d_k end-extends d_k^{n+1}, and similarly $z_k \supseteq z_k^{n+1} \supseteq \tilde{z}_k$, so in particular this holds for all $k \in C(s_n) \subseteq C(s)$. Therefore as D is open and $\tilde{r} \in D$ we conclude $\overline{r} \in D$. □

Recall the general fact that if $\dot{Z}$ is a $\mathbb{P}$-name for an element of $\mathcal{I}(\mathcal{A})^+$, where $\mathbb{P}$ is a forcing notion and $\mathcal{A}$ is a mad family in the ground model, for any $p \in \mathbb{P}$ the set

$$W = \{m \in \omega \mid \exists q \leq p(q \Vdash m \in \dot{Z})\}$$

is an element of $\mathcal{I}(\mathcal{A})^+$. We call W the outer hull of $\dot{Z}$ with respect to p. Since in the present context we must be careful about the integers $m \in \omega$ we add to the eventual coding area $\Delta(a_\alpha)$ of our generic real a_α, we consider a refined version of the outer hull with respect to p, namely the subset of the outer hull witnessed by only pure extensions of p. To be able to run the standard argument, however, we need that this refined outer hull still be an element of $\mathcal{I}(\mathcal{A})^+$; this requires proof.

Lemma 3.4.12. *Let $q \in \mathbb{K}_\alpha \cap M$, where $M \prec H_\theta$ is a countable elementary submodel containing $\mathbb{K}_\alpha$ and $\mathcal{A}_1$, and let $\dot{Z}$ be a $\mathbb{K}_\alpha$-name in M for an element of $\mathcal{I}(\mathcal{A}_1)^+$. Then*

$$W = \{m \in \omega \mid \exists p \leq q \ (\mathrm{Fin}(p) = \mathrm{Fin}(q) \wedge p \Vdash m \ \in \dot{Z})\}$$

is an element of $\mathcal{I}(\mathcal{A}_1)^+ \cap M$.

Proof. Fix a finite $F \subseteq \mathcal{A}_1 \cap M$. We have that

$$\Vdash_{\mathbb{K}_\alpha} \dot{Z} \setminus \bigcup F \text{ is infinite,}$$

so in particular $q \Vdash_{\mathbb{K}_\alpha} \dot{Z} \setminus \bigcup F \in [\omega]^\omega$.

Lemma 3.4.13. *For all $q \in \mathbb{K}_\alpha$ and $\dot{X}$ a $\mathbb{K}_\alpha$-name for an infinite subset of ω, there exists $p \leq q$ with* $\mathrm{Fin}(p) = \mathrm{Fin}(q)$ *and there exists $m^p \in \omega$ such that $p \Vdash m^p = \dot{X}(j)$, where $\dot{X}(j)$ denotes the j-th element of $\dot{X}$.*

Proof. First note $W \in M$ as it is definable from parameters in M. Fix q and $\dot{X}$ as above, and write $q = \langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle$. Consider the countable support product

$$\mathbb{P}_s := \prod_{k \in C(s)} Q^{\eta_\alpha}(S_{\alpha+k}) \times \mathcal{L}^{\eta_\alpha}(Y_{\alpha+k}),$$

where $Q^{\eta_\alpha}(S_{\alpha+k})$ consists of closed bounded subsets $c_k \subseteq \omega_1 \setminus \eta_\alpha$ such that $c_k \cap S_{\alpha+k} = \emptyset$ and is ordered by end extension; the partial order $\mathcal{L}^{\eta_\alpha}(Y_{\alpha+k})$ consists of functions $y_k \colon |y_k| \to 2$ with domain $|y_k|$, such that

- $|y_k| \in \omega_1 \setminus \eta_\alpha$ is a countable limit ordinal and $y_k \restriction \eta_\alpha = 0$;
- $\mathrm{Even}(y_k) = (\{\eta_\alpha\} \cup (\eta_\alpha + X_\alpha)) \cap |y_k|$;
- for all $\gamma \leq |y_k|$, if $\gamma = \omega_1^{\mathcal{M}}$ for some suitable model $\mathcal{M}$ such that $y_k \restriction \gamma \in \mathcal{M}$ and γ is a limit point of c_k, then $\mathcal{M} \models$"$\mathrm{Even}(y_k) - \min(\mathrm{Even}(y_k))$ is the code for some $\overline{\alpha} \in \omega_1$ such that $S^{\mathcal{M}}_{\overline{\alpha}+k}$ is nonstationary".

$\mathcal{L}^{\eta_\alpha}(Y_{\alpha+k})$ is ordered by end-extension. For notational simplicity we will suppress the superscript η_α in what follows.

The ordering on $\mathbb{P}_s$ is defined by $\langle d_k, z_k : k \in \omega\rangle \le \langle c_k, y_k : k \in \omega\rangle$ if and only if d_k is an end-extension of c_k and $z_k \supseteq y_k$ for all $k \in C(s)$, i.e., if and only if $(d_k, z_k) \le_{Q(S_{\alpha+k})\times\mathcal{L}(Y_{\alpha+k})} (c_k, y_k)$ for all $k \in C(s)$. Here, $\le_{Q(S_{\alpha+k})}$ denotes the ordering of the club shooting forcing of Definition 3.3.1, however with the modification that condition are closed bounded subsets containing only ordinals strictly greater than η_α. Likewise, $\le_{\mathcal{L}(Y_{\alpha+k})}$ denotes the extension relation for the localization forcing of Definition 3.3.10, where the Σ_1-formula being localized is $\varphi(\omega_1, X_\alpha)$ which asserts "Even(y_k) − min(Even(y_k)) is the code for some $\overline{\alpha} \in \omega_1$ such that $S_{\overline{\alpha}+k}$ is nonstationary".

For all $k \in C(s)$, find c'_k, y'_k such that $c'_k \le_{Q(S_{\alpha+k})} c_k$ and $y'_k \le_{\mathcal{L}(Y_{\alpha+k})} y_k$, and

$$(c_k, y_k) \Vdash_{Q(S_{\alpha+k})\times\mathcal{L}(Y_{\alpha+k})} \dot{X}(j) = \check{m}_j$$

for some $m_j \in \omega$.

Then $\langle c'_k, y'_k : k \in \omega\rangle \in \mathbb{P}_s$ is an extension of $\langle c_k, y_k : k \in \omega\rangle$ and forces $\dot{X}(j) = \check{m}_j$. Therefore $\langle \mathrm{Fin}(q), \langle c'_k, y'_k : k \in \omega\rangle\rangle \le_{\mathbb{K}_\alpha} q$ and decides $\dot{X}(j)$. □

Therefore, letting $\dot{X}$ be a $\mathbb{K}_\alpha$-name for $\dot{Z} \setminus \bigcup F$, for every $k \in \omega$ there exists $j > k$ and a pure extension $q_j \le q$ in $\mathbb{K}_\alpha$, and there exists $m_j \in \omega$ with $q \Vdash \dot{X}(j) = \check{m}_j$. Then

$$Y = \{m_j \mid j > k, q_j \le q \wedge q_j \Vdash \check{m}_j = \dot{X}(j)\}$$

is an infinite set such that for all $m_j \in Y$, $m_j \in W$ as witnessed by q_j, and moreover $m_j \notin \bigcup F$ since $q_j \Vdash m_j \in \dot{Z} \setminus \bigcup F$. This shows $W \setminus \bigcup F$ is infinite, and as F was arbitrary this implies $W \in \mathcal{I}(\mathcal{A}_1)^+$. □

Remark 3.4.14. We can make the following observations about the forcings $\mathbb{P}_s$, for $s \in [\omega]^{<\omega}$, defined in the above proof.

- If $\langle c_k, y_k : k \in \omega\rangle \Vdash_{\mathbb{P}_s} \dot{X}(j) = m_j$, then $\langle\langle \emptyset, \emptyset\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle \Vdash_{\mathbb{K}_\alpha} \dot{X}(j) = m_j$.
- $\mathbb{P}_s$ is a complete suborder of $\mathbb{P}_\emptyset$.[9]
- For all $s \in [\omega]^{<\omega}$, $\mathbb{P}_s$ does not add new reals. This follows from the fact $Q(S_{\alpha+k})$ and $\mathcal{L}(Y_{\alpha+k})$ do not add new reals.

[9]For forcing notions $\mathbb{Q}_0, \mathbb{Q}_1$, we say $\mathbb{Q}_0$ is a *complete suborder* of $\mathbb{Q}_1$ if $\mathbb{Q}_0$ is a suborder of $\mathbb{Q}_1$ and every maximal antichain of $\mathbb{Q}_0$ remains maximal as a subset of $\mathbb{Q}_1$.

The main result of this section is the following.

Proposition 3.4.15. *For every $p = \langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle \in \mathbb{K}_\alpha$, every θ sufficiently large and countable elementary submodel $M \prec H_\theta$ containing $p, \mathbb{K}_\alpha, \mathcal{A}_1$, and every $B \in \mathcal{I}(\mathcal{A}_1)$ such that $B \cap Y$ is infinite for all $Y \in \mathcal{I}(\mathcal{A}_1)^+ \cap M$, if $M \cap \omega_1 = j \notin \bigcup_{k \in C(s)} S_{\alpha+k}$, then there is an $(M, \mathbb{K}_\alpha, \mathcal{A}_1, B)$-generic condition $q \leq p$ such that* $\mathrm{Fin}(q) = \mathrm{Fin}(p)$.

Proof. Let θ be a sufficiently large regular cardinal and let $M \prec H_\theta$ be a countable elementary submodel containing $p, \mathbb{K}_\alpha, X_\alpha$ and $\mathcal{A}_1$, such that $j = M \cap \omega_1 \notin \bigcup_{k \in C(s)} S_{\alpha+k}$. Fix $B \in \mathcal{I}(\mathcal{A}_1)$ such that $B \cap Y$ is infinite for all $Y \in \mathcal{I}(\mathcal{A}_1)^+ \cap M$. Let $\{D_n \mid n \in \omega\}$ enumerate all open dense subsets of $\mathbb{K}_\alpha$ in M, and let $\{\dot{Z}_n \mid n \in \omega\}$ enumerate all $\mathbb{K}_\alpha$-names for subsets of ω in M which are forced to be in $\mathcal{I}(\mathcal{A}_1)^+$ such that each name appears infinitely often. Let $\langle j_n : n \in \omega\rangle$ be an increasing cofinal sequence of ordinals converging to j. We inductively define a descending sequence $\langle q_n : n \in \omega\rangle \subseteq M \cap \mathbb{K}_\alpha$ where $q_n = \langle\langle s, s^*\rangle, \langle d^n_k, c^n_k : k \in \omega\rangle\rangle$ and such that:

1. $d^0_k = c_k$, $z^0_k = y_k$;
2. For all $n \in \omega$ and $k \in C(s)$, d^{n+1}_k is an end-extension of d^n_k and $z^{n+1}_k \supseteq z^n_k$;
3. $\max(d^{n+1}_k), |z^{n+1}_k| \geq j_n$;
4. q_{n+1} is preprocessed for D_n;
5. $q_{n+1} \Vdash (\dot{Z}_n \cap B) \setminus n \neq \emptyset$.

Assume q_n has been constructed. First extend q_n with a pure extension q'_n such that q'_n is preprocessed for D_n. Next let

$$W_{n+1} = \{m \in \omega \mid \exists r \leq q'_n(\mathrm{Fin}(r) = \mathrm{Fin}(q'_n) \wedge\ r \Vdash m \in \dot{Z}_n)\}.$$

Since $W_{n+1} \in \mathcal{I}(\mathcal{A}_1)^+$ by Lemma 3.4.12, fix $m_n \in \omega$ such that $m_n > n$ and $m_n \in W_{n+1} \cap B$. Let $r \leq q'_n$ be given by $m_n \in W$, and let $q_{n+1} \leq r$ be a pure extension of r such that $\max(d^{n+1}_k) \geq j_n$ and $|z^{n+1}_k| \geq j_n$. Then q_{n+1} satisfies (3), and also satisfies (5), as q_{n+1} extends r and therefore forces $m_n \in W$.

Clause (4) is satisfied because $q_{n+1} \leq q'_n$ and by Lemma 3.4.9. This completes the inductive construction.

Set $d_k = \bigcup_{n\in\omega} d^n_k \cup \{j\}$ and $z_k = \bigcup_{n\in\omega} z^n_k$ for all $k \in C(s)$, and for $k \notin C(s)$ let $d_k = z_k = \emptyset$. Define $q := \langle\langle s, s^*\rangle, \langle d_k, z_k : k \in \omega\rangle\rangle$. Then q is a condition in $\mathbb{K}_\alpha$, as this can be verified as in the proof of Lemma 3.4.10. It remains to see that q is an $(M, \mathbb{K}_\alpha, \mathcal{A}_1, B)$-generic condition.

First we show q is $(M, \mathbb{K}_\alpha)$-generic by showing that for all $n \in \omega$, $D_n \cap M$ is predense below q. Fix n and let $r = \langle\langle t, t^*\rangle, \langle d'_k, z'_k : k \in \omega\rangle\rangle \leq q$, and we can assume $r \in D_n$. Then as $r \leq q_{n+1}$ and q_{n+1} is preprocessed for D_n, there is $r' = \langle\langle t, t^*_2\rangle, \langle d^{n+1}_k, z^{n+1}_k : k \in \omega\rangle \leq q_{n+1}$ for some finite $t^*_2 \in M$, such that already $r' \in D_n$. Clearly $r' \in M$. Then r and r' are compatible, as witnessed by the condition $\langle\langle t, t^* \cup t^*_2\rangle, \langle d'_k, z'_k : k \in \omega\rangle\rangle$.

Lastly, for all $n \in \omega$ we have $q \Vdash |\dot{Z}_n \cap B| = \omega$. Let $\ell \in \omega$, and take any $r \leq q$. Find $i > \ell$ such that $\dot{Z}_i = \dot{Z}_n$; then $r \leq q_{i+1}$ and so since q_{i+1} satisifes property (5), we have

$$r \Vdash \emptyset \neq (\dot{Z}_i \cap B) \setminus i = (\dot{Z}_n \cap B) \setminus i \subseteq (\dot{Z}_n \cap B) \setminus \ell.$$

As ℓ was arbitrary this shows q forces $\dot{Z}_n \cap B$ is infinite, and therefore q is an $(M, \mathbb{K}_\alpha, \mathcal{A}_1, B)$-generic condition. $\square$

Let H_α be $\mathbb{K}_\alpha$-generic over $V[G_\alpha]$, and set $Y^\alpha_k = \bigcup_{p\in H_\alpha} y_k$, $C^\alpha_k = \bigcup_{p\in H_\alpha} c_k$, $a_\alpha = \bigcup_{p\in H_\alpha} s$, and $\mathcal{A}^2_{\alpha+1} = \mathcal{A}_2 \cup \{a_\alpha\}$. The following lemma gives the consequences of forcing with $\mathbb{K}_\alpha$.

Lemma 3.4.16 ([FZ10, Claim 11])**.** *The following hold in $L[G_\alpha][H_\alpha]$.*

1. *$a_\alpha \in [\omega]^\omega$ is almost disjoint from all elements of $\mathcal{A}^2_\alpha$;*

2. *For all $i \in \omega$, $a_\alpha \cap x_i$ is infinite;*

3. *For all $m \in \Delta(a_\alpha)$, C^α_m is a club in ω_1 such that $C^\alpha_m \cap S_{\alpha+m} = \emptyset$, and for all $\xi \in \omega_1$, $\xi \in C^\alpha_m$ if and only if $a_\alpha \cap R_{\langle m,\xi\rangle}$ is finite;*

4. *For all $m \in \Delta(a_\alpha)$, $Y^\alpha_m \colon \omega_1 \to 2$ is a total function, and for all $\xi \in \omega_1$, $Y^\alpha_m(\xi) = 1$ if and only if $a_\alpha \cap R_{\langle\omega+m,\xi\rangle}$ is finite;*

5. *For all $n \in \omega$, $n \in Z_\alpha$ if and only if there exists $m \in \omega$ such that $(a_\alpha \cap R_{\langle 0,0\rangle})(2n) = R_{\langle 0,0\rangle}(2m)$;*

6. For every $r \in \mathcal{R}$ and finite $A' \subseteq \mathcal{A}^2_{\alpha+1}$, $|E(r)\setminus\bigcup A'| = |O(r)\setminus\bigcup A'| = \omega$.

Proof. For item (1), to see a_α is infinite it suffices to show that for every $n \in \omega$ the set

$$D_n = \{p \in \mathbb{K}_\alpha \mid \mathrm{Fin}(p) = \langle s, s^* \rangle,\ \exists k > n(k \in s)\}$$

is dense in $\mathbb{K}_\alpha$ for each $n \in \omega$. Let $p = \langle\langle s, s^*\rangle, \langle c_k, d_k : k \in \omega\rangle\rangle \in \mathbb{K}_\alpha$, and suppose $\max s \leq n$. Since $s^* \in [\mathcal{R} \cup \mathcal{A}^2_\alpha]^{<\omega}$, the assumption $(*)$ implies that $y := \omega \setminus (\bigcup s^* \cup R_{\langle 0,0\rangle} \cup n)$ is infinite (as it contains both $E(R), O(R)$ for any $R \in \mathcal{R} \setminus (s^* \cup \{R_{\langle 0,0\rangle}\})$). Therefore we may fix $m := \min y$ and define $p' = \langle\langle s \cup \{m\}, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle \in \mathbb{K}_\alpha$; then $p' \leq p$ and $p' \in D_n$. So D_n is dense for every n, implying a_α is infinite. Next, for any $a \in \mathcal{A}^2_\alpha$, $a_\alpha \cap a$ is finite, since first of all

$$D_a = \{p \in \mathbb{K}_\alpha \mid \mathrm{Fin}(p) = \langle s, s^* \rangle,\ a \in s^*\}$$

is dense, and if $p \in H_\alpha \cap D_a$, then $a \cap a_\alpha \subseteq a \cap s$.

For the proof of (2), fix $i, n \in \omega$, and let

$$D_{i,n} = \{p \in \mathbb{K}_\alpha \mid \mathrm{Fin}(p) = \langle s, s^* \rangle,\ \exists k > n(k \in x_i \cap s)\}.$$

We show $D_{i,n}$ is dense in $\mathbb{K}_\alpha$. Let $p = \langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle \in \mathbb{K}_\alpha$. Recall that $\eta_\alpha < \omega_1$ was a limit ordinal fixed so to ensure that $x_i \setminus \bigcup F$ is never a subset of $\bigcup_{\langle\gamma,\xi\rangle \in J} R_{\langle\gamma,\xi\rangle}$, for any $F \in [\mathcal{A}^2_\alpha]^{<\omega}$ and any $J \in [\omega \cdot 2 \times (\omega_1 \setminus \eta_\alpha)]^{<\omega}$. Since for any $\langle\gamma,\xi\rangle \in \omega \cdot 2 \times \omega_1$ with $R_{\langle\gamma,\xi\rangle} \in s^*$, items (1) and (2) of the definition of $\mathbb{K}_\alpha$ imply $\xi > \eta_\alpha$ and $\gamma > 0$, the set $y := x_i \setminus \bigcup s^*$ is infinite by the former. However by the latter we have two cases:

<u>Case I:</u> $y \setminus R_{\langle 0,0\rangle}$ is infinite. Then let

$$m := \min(y \setminus (\bigcup R_{\langle 0,0\rangle} \cup n \cup (\max s + 1))),$$

and let $p' := \langle\langle s \cup \{m\}, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle$. Then $p' \leq p$ and $p' \in D_{i,n}$.

<u>Case II:</u> $y \subseteq^* R_{\langle 0,0\rangle}$. In this case we need to take care of the additional assumption in item (3) of the definition of $\mathbb{K}_\alpha$, namely the coding of $Z_\alpha \subseteq \omega$. We may assume without loss of generality that $y \setminus R_{\langle 0,0\rangle} \subseteq n$. In the case $|s \cap R_{\langle 0,0\rangle}| = 2j$ for some $j \in \omega$, let $m := \min((y \cap R_{\langle 0,0\rangle}) \setminus (n \cup (\max s + 1)))$,

and define p' as in Case I with respect to this latter choice of integer m. Then $p' \leq p$ and $p' \in D_{i,n}$.

If $|s \cap R_{\langle 0,0 \rangle}| = 2_j - 1$ for some $j \in \omega$, we consider whether or not $j \in Z_\alpha$. If $j \in Z_\alpha$, let $m_0 := \min(E(R_{\langle 0,0 \rangle}) \setminus (n \cup (\max s + 1)))$, and note this can be done as $E(R_{\langle 0,0 \rangle}) \setminus \bigcup s^*$ is infinite by $(*)$. Let $m_1 := \min(y \setminus m_0)$, and define $p' := \langle \langle s \cup \{m_0, m_1\}, s^* \rangle, \langle c_k, y_k : k \in \omega \rangle \rangle$. Then $p' \in \mathbb{K}_\alpha$ and $p' \in D_{i,n}$

If $j \notin Z_\alpha$, take $m_0 := \min(O(R_{\langle 0,0 \rangle})) \setminus (n \cup (\max s + 1))$ and $m_1 := \min(y \setminus m_0)$, and then define p' as above; in either case $p' \leq p$ and $p' \in D_{i,n}$.

Regarding item (3), if $m \in \Delta(a_\alpha)$ then $m \in \Delta(s)$ for some $p \in H_\alpha$ with $\mathrm{Fin}(p) = \langle s, s^* \rangle$, so Lemma 3.4.7 implies C^α_m is a closed unbounded set of $\omega_1 \setminus S_{\alpha+m}$. To see a_α almost disjointly codes C^α_m using $\mathcal{R}_m = \{R_{\langle m,\xi \rangle} \mid \xi \in \omega_1\}$, first let $\xi \in \omega_1$ be an element of C^α_m. Then $\xi \in c_m$ for some $p = \langle \langle s, s^* \rangle, \langle c_k, y_k : k \in \omega \rangle \rangle \in H_\alpha$. We may assume $m \in \Delta(s)$, since otherwise as $m \in \Delta(a_\alpha) = \bigcup_{q \in H_\alpha} \{\Delta(t) \mid \mathrm{Fin}(q) = \langle t, t^* \rangle\}$, there exists $q \in H_\alpha$ with $\mathrm{Fin}(q) = \langle t, t^* \rangle$ and $m \in \Delta(t)$. Then there is $r \in H_\alpha$ which is a common extension of p, q. So without loss of generality $m \in \Delta(s)$. If $R_{\langle m,\xi \rangle} \in s^*$ then $C^\alpha_m \cap a_\alpha \subseteq s$, and if $R_{\langle m,\xi \rangle} \notin s^*$ then we use the density of the set

$$D_{m,\xi} = \{q \in \mathbb{K}_\alpha \mid \mathrm{Fin}(q) = \langle t, t^* \rangle,\ R_{\langle m,\xi \rangle} \in t^*\}$$

to find an extension $q \leq p$ with $q \in D_{m,\xi}$. Whenever such a q is also an element of H_α, then q witnesses that $C^\alpha_m \cap a_\alpha$ is finite as in the previous case.

If $\xi \notin C^\alpha_m$, let $p \in H_\alpha$ be such that $m \in \Delta(s)$, where $\mathrm{Fin}(p) = \langle s, s^* \rangle$. The set

$$D'_n = \{p \in \mathbb{K}_\alpha \mid \exists k > n\ k \in R_{\langle m,\xi \rangle} \cap s\}$$

is dense; the proof is similar to the proof of item (1), and uses the fact that $R_{\langle m,\xi \rangle} \setminus (\bigcup s^* \cup R_{\langle 0,0 \rangle})$ is infinite, by $(*)$ and the fact $R_{\langle m,\xi \rangle}$ is never an element of t^* for any $\langle \langle t, t^* \rangle, \langle d_k, z_k : k \in \omega \rangle \rangle \in H_\alpha$.

Item (4) is verified almost identically to item (3).

Next we check (5), which implies that a_α codes Z_α, using $R_{\langle 0,0 \rangle}$. Fix $n \in \omega$ such that $n \in Z_\alpha$. The set

$$D_{R,2n} = \{p \in \mathbb{K}_\alpha \mid \mathrm{Fin}(p) = \langle s, s^* \rangle,\ |s \cap R_{\langle 0,0 \rangle}| \geq 2n\}$$

is dense in $\mathbb{K}_\alpha$, using the assumption $(*)$. Indeed, if $p = \langle \langle s, s^* \rangle : \langle c_k, y_k : k \in \omega \rangle \rangle \in \mathbb{K}_\alpha$ with $|s \cap R_{\langle 0,0 \rangle}| = k < 2n$, we perform a finite induction of length

$2n-k$, picking integers m_j to be the $(k+j)$-th element of $s \cap R_{\langle 0,0\rangle}$. If $k+j$ is odd, let $m_j = \min(R_{\langle 0,0\rangle} \setminus (\bigcup s^* \cup \max s \cup \{m_\ell \mid \ell < j\}))$. If $k+j$ is even and $\frac{k+j}{2} \in Z_\alpha$, let $m_j = \min(E(R_{\langle 0,0\rangle}) \setminus (\bigcup s^* \cup \max s \cup \{m_\ell \mid \ell < j\}))$. In the case $\frac{k+j}{2} \notin Z_\alpha$, replace O with E in the previous definition of m_j. That all this is possible is by assumption $(*)$. Then it can be verified $\langle\langle s \cup \{m_j \mid j < 2n-k\}, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle$ is a condition extending p, and is an element of $D_{R,2n}$.

Lastly, to prove item (6), fix $r \in \mathcal{R}$. Since $(*)$ holds, it suffices to show that $E(r) \setminus a_\alpha$ and $O(r) \setminus a_\alpha$ are infinite. For the former, it suffices to show that the set

$$D_n^E = \{p \in \mathbb{K}_\alpha \mid \mathrm{Fin}(p) = \langle s, s^*\rangle \wedge \exists k > n(k < \max s \wedge k \in E(r) \setminus s)\}$$

is dense in $\mathbb{K}_\alpha$, and for the latter it suffices that D_n^O is dense, where D_n^O is defined analogously for $O(r)$. Letting $p = \langle\langle s, s^*\rangle, \langle c_k, d_k : k \in \omega\rangle\rangle$ be any condition in $\mathbb{K}_\alpha$, let $m_e = \min(E(r) \setminus (\bigcup s^* \cup R_{\langle 0,0\rangle} \cup (\max s + 1) \cup n))$, and let $m_o = \min(O(r) \setminus (\bigcup s^* \cup R_{\langle 0,0\rangle} \cup (\max s + 1) \cup n))$.

Define $p' = \langle\langle s \cup \{\min(\omega \setminus (m_e + m_o + 1))\}, s^*\rangle, \langle c_k, d_k : k \in \omega\rangle\rangle$. Then $p' \in D_n^E \cap D_n^O$ and extends p, as desired. □

Lemma 3.4.17 ([FZ10, Corollary 12]). *$\mathbb{K}_\alpha$ is S_{-1}-proper. Moreover, for every $p = \langle\langle s, s^*\rangle : \langle c_k, y_k : k \in \omega\rangle\rangle \in \mathbb{K}_\alpha$, the subposet $\mathbb{K}_\alpha \restriction p = \{r \in \mathbb{K}_\alpha \mid r \leq p\}$ is $(\omega_1 \setminus \bigcup_{n \in C(s)} S_{\alpha+n})$-proper.*

Proof. The first statement follows from the fact S_{-1} was a fixed stationary subset in L disjoint from each $S \in \vec{S}$. For the second statement, let $\beta = \alpha + m$ where $m \notin C(s)$, so $m < \max(\Delta(s))$ and $m \notin \Delta(s)$. Let $q = \langle\langle t, t^*\rangle, \langle d_k, z_k : k \in \omega\rangle\rangle \in \mathbb{K}_\alpha \restriction p$. Then t is an end extension of s, so $\Delta(t)$ is an end-extension of $\Delta(s)$, so in particular $m \notin C(t)$. Then define $q' = \langle\langle t, t^*\rangle, \langle d'_k, z'_k : k \in \omega\rangle\rangle$ such that $d'_k = d_k$ and $y'_k = y_k$ for all $k \in C(t)$, and let $d'_m = y'_m = \emptyset$. Then $q' \in \mathbb{K}_\alpha \restriction p$ and $q' \leq q$. Therefore it is dense in $\mathbb{K}_\alpha \restriction p$ to not be adding any ordinals to $\omega_1 \setminus S_{\alpha+m}$, and so $S_{\alpha+m}$ remains stationary in any $\mathbb{K}_\alpha \restriction p$ extension. □

This completes the definition of $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2\rangle$.

Corollary 3.4.18. *$\mathbb{P}_{\omega_2}$ is S_{-1}-proper and strongly preserves the tightness of $\mathcal{A}_1$. Moreover for all $m \in \omega \setminus \Delta(a_\alpha)$, $S_{\alpha+m}$ remains stationary in $L[G]$.*

Proof. By Proposition 3.4.15, Lemma 3.4.17 and Lemma 3.3.6. $\square$

Lemma 3.4.19 ([FZ10, Lemma 13]). *Let G be $\mathbb{P}_{\omega_2}$-generic over L, and let $\mathcal{A}_2 = \{\dot{a}_\alpha^G \mid \alpha < \omega_2\}$, where $\dot{a}_\alpha$ denotes the canonical $\mathbb{P}_{\omega_2}$-name for the $\mathbb{K}_\alpha$ generic real. Then in $L[G]$, $\mathcal{A}_2$ is definable by the following Π^1_2 formula:*

$$a \in \mathcal{A}_2 \Leftrightarrow \forall \mathcal{M}[(\mathcal{M} \text{ is a countable suitable model, } a \in \mathcal{M})$$
$$\exists \overline{\alpha} < \omega_2^{\mathcal{M}} \ (\mathcal{M} \models \forall m \in \Delta(a)(S_{\overline{\alpha}+m} \text{ is nonstationary}))]$$

Proof. Let $\varphi(a)$ denote the formula on the right-hand side of the equivalence above.

Since any countable suitable model can be recursively coded by a real and the satisfaction relation $\models$ for such models is Δ^1_1, it is easy to see $\varphi(a)$ is of the form $\forall\exists$, i.e. is a Π^1_2 formula.

Now we show $\mathcal{A}_2 = \{a \in L[G] \mid \varphi(a)\}$. Suppose first $a \in \mathcal{A}_2$. Then there exists a limit ordinal $\alpha < \omega_2$ such that $a = a_\alpha$, the generic real added by $\mathbb{K}_\alpha$. Let $\mathcal{M}$ be a countable suitable model containing a_α. By absoluteness of the family $\mathcal{R}$ and items (3),(4),(5) of Lemma 12, $\langle C^\alpha_m \cap \omega_1^{\mathcal{M}} : m \in \Delta(a_\alpha)\rangle$, $\langle Y^\alpha_m \restriction \omega_1^{\mathcal{M}} : m \in \Delta(a_\alpha)\rangle$, and Z_α are elements of $\mathcal{M}$. Note that

$$\langle \langle \emptyset, \emptyset \rangle, \langle C^\alpha_k \cap (\omega_1^{\mathcal{M}} + 1), Y^\alpha_k \restriction \omega_1^{\mathcal{M}} : k \in \omega \rangle \rangle$$

is a condition in $\mathbb{K}_\alpha$. So by (4) of Definition 3.4.5, for each $m \in \Delta(a_\alpha)$, in $\mathcal{M}$ it holds that $\mathrm{Even}(Y^\alpha_m \restriction \omega_1^{\mathcal{M}}) - \min(\mathrm{Even}(Y^\alpha_m \restriction \omega_1^{\mathcal{M}})$ is the unique code of some limit ordinal $\overline{\alpha_m}$ such that $S_{\overline{\alpha_m}+m}$ is nonstationary. By item (2) of Definition 3.4.5, $\mathrm{Even}(Y^\alpha_m \restriction \omega_1^{\mathcal{M}}) - \min(\mathrm{Even}(Y^\alpha_m \restriction \omega_1^{\mathcal{M}})) = X_\alpha$ for every $m \in \Delta(a_\alpha)$, so the unique ordinal $\overline{\alpha_m}$ is the same $\overline{\alpha} \in \omega_2^{\mathcal{M}}$ for all such m.

Converseley, suppose a is a real in $L[G]$ such that $\varphi(a)$ holds. As shown in Claim 3.3.32, $\varphi(a)$ holds in suitable models containing a of any infinite cardinality, in particular for the suitable model $\mathcal{N} = L_{\omega_{27}}[G]$. Since $\omega_2^{\mathcal{N}} = \omega_2^{L[G]} = \omega_2^L$ and $\vec{S}^{\mathcal{N}} = \vec{S}^L = \vec{S}$, if $a \in \mathcal{N}$ and there exists $\alpha < \omega_2^{\mathcal{N}}$ such that $S^{\mathcal{N}}_{\alpha+m} = S_{\alpha+m}$ is nonstationary in $\mathcal{N}$, then by upwards absoluteness of "is nonstationary", $S_{\alpha+m}$ is no longer stationary in $L[G]$. By Corollary 3.4.18, it must be that $\mathbb{K}_\alpha$ was nontrivial, and the only m for which $S_{\alpha+m}$ is nonstationary in $L[G]$ are those $m \in \Delta(a_\alpha)$, where a_α was the generic real added by $\mathbb{K}_\alpha$. Therefore $\Delta(a) = \Delta(a_\alpha)$ and so $a = a_\alpha \in \mathcal{A}_2$. $\square$

We can now prove the following.

Theorem 3.4.20. *It is consistent that* $\mathfrak{b} = \mathfrak{a} = \aleph_1 < \mathfrak{c} = \aleph_2$*, there exists a* Π^1_1*-definable tight mad family of size* $\aleph_1$*, and a* Π^1_2*-definable tight mad family of size* $\aleph_2$*.*

Proof. Assume $V = L$ and fix $\mathcal{A}_1 \in L$ a coanalytic tight mad family as in Lemma 3.3.26. Let $\mathbb{P} = \mathbb{P}_{\omega_2}$ be the countable support iteration defined above, let G be $\mathbb{P}$-generic over L, and let $\mathcal{A}^2_\alpha = \{a_\alpha \mid \alpha < \omega_2\}$, where $a_\alpha = \dot{a}^G_\alpha$ and $\dot{a}_\alpha$ is the generic real added by $\mathbb{K}_\alpha$. By Lemma 3.4.16, item (1), $\mathcal{A}_2$ is an almost disjoint family of infinite subsets of ω in L$[G]$. To see $\mathcal{A}_2$ is tight, suppose there exist $x_i \in L[G] \cap [\omega]^\omega$ such that $x_i \in \mathcal{I}(\mathcal{A}_2)^+$ for all $i \in \omega$. Let $\dot{x}_i$ denote a nice $\mathbb{P}$-name for x_i for each i. Then there exists $\alpha < \omega_2$ such that $\langle \dot{x}_i : i \in \omega \rangle \in L_{\omega_2}$ is a sequence of $\mathbb{P}_\alpha$-names. Since $F^{-1}(\langle \dot{x}_i : i \in \omega \rangle)$ is unbounded in ω_2 there exists $\beta \geq \alpha$ such that $F(\beta) = \langle \dot{x}_i : i \in \omega \rangle$ and for every $i \in \omega$, $\dot{x}^{G_\beta}_i$ is an element of $\mathcal{I}(\mathcal{A}^2_\beta)^+ \subseteq \mathcal{I}(\mathcal{A}_2)^+$, where $G_\beta = G \cap \mathbb{P}_\beta$. Then in $L[G_\beta]$, we defined $\dot{\mathbb{Q}}_\beta$ to be a $\mathbb{P}_\beta$-name for $\mathbb{K}_\beta$. Let a_β denote the $\mathbb{K}_\beta$-generic real over $L[G_\beta]$ and let H_β denote $G \cap \mathbb{K}_\beta$. By Lemma 3.4.16, item (2), $\dot{a}_\beta$ has infinite intersection with each $x_i = \dot{x}^{G_\beta}_i$ in $L[G_\beta][H_\beta] \subseteq L[G]$, so this holds in $L[G]$ as well. As $\{x_i \mid i \in \omega\} \in L[G]$ was arbitrary this shows $\mathcal{A}_2$ is a tight mad family, and moreover, $\mathcal{A}_2$ has a Π^1_2 definition in $L[G]$ by Lemma 3.4.19. Lastly, as $\mathcal{A}_1 \in L$ is a tight mad family and $\mathbb{P}$ is a proper forcing notion and strongly preserves the tightness of $\mathcal{A}_1$, in $L[G]$, $\mathcal{A}_1$ is a Π^1_1 tight mad family. Therefore $(\mathfrak{a} = |\mathcal{A}_1| = \aleph_1 < \mathfrak{c} = \aleph_2 = |\mathcal{A}_2|)^{L[G]}$. That $\mathcal{A}_1$ is still coanalytic in $L[G]$ follows from Shoenfield absoluteness, and this completes the proof.

□

3.5 Cardinal characteristic constellations and optimal projective spectra

We show the compatibility of the conclusions of Theorem 3.4 and Theorem 3.3.35 above.

Theorem 3.5.1. *It is consistent with* $\aleph_1 = \mathfrak{a} < \mathfrak{s} = \aleph_2$ *that there exists a* Π^1_1 *tight mad family of size* $\aleph_1$ *and a* Π^1_2 *tight mad family of size* $\aleph_2$*.*

Proof. The proof is almost identical to that of Theorem 3.4.20, though we modify the successor stages of the iteration. Starting in a model of $V = L$, fix a Π^1_1 tight mad family $\mathcal{A}_1$ as well as the nicely definable parameters $\vec{S}$ and $\mathcal{R}$. Let $F\colon \mathrm{Lim} \cap \omega_2 \to L_{\omega_2}$ be the bookkeeping function as in Theorem 3.4.20. Recursively we define a countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha \leq \omega_2, \beta < \omega_2\rangle$ of S_{-1}-proper forcing notions.

Let $\mathbb{P}_0$ be the trivial forcing. For successor ordinals $\alpha < \omega_2$ let $\mathbb{Q}_\alpha$ be a $\mathbb{P}_\alpha$-name for the forcing notion $\mathbb{Q}$ of Definition 3.2.17. For $\alpha \in \mathrm{Lim} \cap \omega_2$, unless the following occurs, $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$ name for the trivial forcing. Suppose that $F(\alpha)$ is the name for a sequence of reals $\langle x_i : i \in \omega\rangle$ such that no x_i is finitely covered by the almost disjoint family $\mathcal{A}^2_\alpha \in V[G_\alpha]$; in this case let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for the forcing notion $\mathbb{K}_\alpha$ of Definition 3.4.5. This completes the definition of $\mathbb{P}_{\omega_2}$.

Let G be $\mathbb{P} = \mathbb{P}_{\omega_2}$-generic over L. By Proposition 3.2.28 and Proposition 3.4.15, for every $\alpha < \omega_2$, $\mathbb{P}_\alpha$ forces that $\dot{\mathbb{Q}}_\alpha$ is a proper forcing notion which strongly preserves the tightness of $\mathcal{A}_1$. Therefore ($\mathcal{A}_1$ is a tight mad family of size $\aleph_1)^{L[G]}$ and $\aleph_1^L = \aleph_1^{L[G]}$, so $(\mathfrak{a} = \aleph_1)^{L[G]}$, as witnessed by $\mathcal{A}_1$, and by Shoenfield absoluteness $\mathcal{A}_1$ is still defined by the same Π^1_1 formula. As in the proofs of Theorem 3.2.32 and Theorem 3.4.20, it can be verified that $(\mathfrak{s} = \aleph_2)^{L[G]}$, since cofinally often, $\dot{\mathbb{Q}}_\alpha$ adds a real unsplit the ground model reals. Moreover for cofinally many $\alpha < \omega_2$, $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$ name for $\mathbb{K}_\alpha$, so by Lemma 3.4.16, denoting by a_α the $\mathbb{K}_\alpha$-generic real over $V[G_\alpha]$, where $G_\alpha = G \cap \mathbb{P}_\alpha$, $\mathcal{A}_2 = \{a_\alpha : \alpha < \omega_2\}$ is a Π^1_2 tight mad family of size $\aleph_2$ in $V[G]$.

□

The final theorem we prove synthesizes the results we have accumulated thus far. This extends the line of inquiry into the consistency of definable witnesses with a projective wellorder and nontrivial cardinal characteristic constellations, with the added nuance of spectrum considerations.

Theorem 3.5.2. *It is consistent that $\aleph_1 = \mathfrak{a} < \mathfrak{s} = \aleph_2$, there exists a Δ^1_3 wellorder of the reals, as well as tight mad families of cardinality $\aleph_1$ and $\aleph_2$, which are respectively Π^1_1 and Π^1_2 definable.*

Proof. We work in a model of $V = L$.

Lemma 3.5.3. *There exist pairwise disjoint stationary sets $T_0, T_1, T_2 \subseteq \omega_1$ such that for $i < 2$ there are sequences*

$$\vec{S}^i = \langle S^i_\alpha \subseteq T_i : \alpha < \omega_2 \rangle$$

where $S^i_\alpha \subseteq \omega_1$ is stationary costationary, and for distinct $\alpha, \beta < \omega_2$, $S^i_\alpha \cap S^i_\beta$ is bounded. Moreover, whenever $\mathcal{M}, \mathcal{N}$ are suitable models such that $\omega_1^{\mathcal{M}} = \omega_1^{\mathcal{N}}$, then $\langle (S^i_\alpha)^{\mathcal{M}} : \alpha < \omega_2^{\mathcal{M}} \cap \omega_2^{\mathcal{N}} \rangle = \langle (S^i_\alpha)^{\mathcal{N}} : \alpha < \omega_2^{\mathcal{M}} \cap \omega_2^{\mathcal{N}} \rangle$.

Proof. There exists ω_1-many pairwise disjoint stationary subsets of ω_1 by Solovay's Decomposition Theorem (see also [Kun13, Theorem III.6.10]). So in particular we can fix $T_i \subseteq \omega_1$ for $i < 3$ which are pairwise disjoint and stationary. For a stationary subset $T \subseteq \omega_1$, $\diamondsuit_T$ is the assertion that there exists $\vec{A} = \langle A_\alpha : \alpha \in T \rangle$ such that for every $X \subseteq \omega_1$, the set $\{\alpha \in T \mid T \cap \alpha = A_\alpha\}$, i.e. $\vec{A}$ is a $\diamondsuit$-sequence consisting of ordinals from T. Since $V = L$, $\diamondsuit_{T_i}$ holds for each $i < 2$ (see [Kun13, Exercise III.7.18]). Perform the same construction as in the proof of Lemma 3.3.24 to obtain $\vec{S}^i$ for $i < 2$, and it can be checked as in that proof that each $\vec{S}^i$ satisfy the conclusions above, in particular since we can take the $\diamondsuit_{T_i}$ sequences to be Σ_1-definable over L_{ω_2}, so to conclude the "moreover" clause. $\square$

Let $\mathcal{A}_1$ be a coanalytic tight mad family, and fix Σ_1-definable bookeeping function $F\colon \mathrm{Lim} \cap \omega_2 \to L_{\omega_2}$, and a Σ_1-definable almost disjoint family $\mathcal{R}$ such that $F, \mathcal{R}$ are as in the proof of Corollary 3.3.35 and Definition 3.4.5.

Define a countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$, with $\mathbb{P}_0$ being the trivial forcing. Let G_α be $\mathbb{P}_\alpha$-generic over L; the wellordering $<_\alpha$ on the reals of $L[G_\alpha]$ is defined exactly as for the proof of Theorem 3.3.35. Let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for the two-step iteration $\dot{\mathbb{Q}}^0_\alpha * \dot{\mathbb{Q}}^1_\alpha$ such that $\dot{\mathbb{Q}}^0_\alpha$ is a $\mathbb{P}_\alpha$-name for $\mathbb{Q}$ of Definition 3.2.17. Unless one of the following cases occurs, $\dot{\mathbb{Q}}^1_\alpha$ is a $\mathbb{P}_\alpha * \dot{\mathbb{Q}}^0_\alpha$-name for the trivial forcing.

<u>Case I:</u> α is a limit ordinal and $F(\alpha) = \{\dot{x}_i \mid i \in \omega\}$ is a sequence of $\mathbb{P}_\alpha * \dot{\mathbb{Q}}^0_\alpha$-names such that in $V[G_\alpha]$, $\dot{x}_i^G$ is an element of $\mathcal{I}(\mathcal{A}^2_\alpha)^+$ for each $i \in \omega$. Then let $\dot{\mathbb{Q}}^1_\alpha$ be a $\mathbb{P}_\alpha * \dot{\mathbb{Q}}^0_\alpha$ name for the forcing notion $\mathbb{K}_\alpha$ of Definition 3.4.5, with respect to the same countable limit ordinal $\eta_\alpha \in \omega_1$, and modifying item (1) by letting $c_k \subseteq \omega_1 \setminus \eta_\alpha$ be a closed bounded subset such that $S^0_{\alpha+k} \cap c_k = \emptyset$.

<u>Case II:</u> $F(\alpha) = \{\sigma^\alpha_x, \sigma^\alpha_y\}$ is a pair of $\mathbb{P}_\alpha$-names for reals in $L[G_\alpha]$ such that $\sigma^\alpha_x <_L \sigma^\alpha_y$ (i.e., $x = (\sigma^\alpha_x)^G <_\alpha y = (\sigma^\alpha_y)^G$). In this case define $\dot{\mathbb{Q}}^1_\alpha$ to be a

$\mathbb{P}_\alpha * \dot{\mathbb{Q}}^0_\alpha$-name for $\mathbb{Q}^1_\alpha = \mathbb{K}^0_\alpha * \mathbb{K}^1_\alpha * \mathbb{K}^2_\alpha$ as defined in the proof of Theorem 3.3.35, however modifying the definition of $\mathbb{K}^0_\alpha$ by taking closed bounded subsets of $\omega_1 \setminus S^1_{\alpha+k}$, for $k \in \Delta(x_\alpha * y_\alpha)$.

This completes the definition of $\mathbb{P} = \mathbb{P}_{\omega_2}$. Note that for all $\alpha < \omega_2$, $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for either a proper, a $T_1 \cup T_2$-proper, or a $T_0 \cup T_2$-proper forcing notion, and in each case $\dot{\mathbb{Q}}_\alpha$ strongly preserves the tightness of $\mathcal{A}_1$. Therefore $\mathbb{P}$ is T_2-proper and so preserves ω_1 as well as the tightness of $\mathcal{A}_1$. Let G be $\mathbb{P}$-generic over L. For each $\alpha < \omega_2$, $\mathbb{Q}_\alpha = \mathbb{Q}$ adds a real not split by the ground model reals, so $(\mathfrak{s} = \aleph_2)^{L[G]}$, as in Theorem 3.2.32. For cofinally many $\alpha < \omega_2$, $\mathbb{Q}_\alpha$ adds an infinite $a_\alpha \subseteq \omega$ such that $\mathcal{A}_2 = \{a_\alpha \mid \alpha < \omega_2\}$ is a tight mad family with a Π^1_2 definition in $L[G]$, as in Theorem 3.4.20. Moreover the wellorder $<_G = \bigcup_{\alpha<\omega_2}$ can be shown to be a Δ^1_3 wellorder of the reals of $L[G]$, as in Theorem 3.3.35. Then altogether we have that $L[G]$ witnesses the conclusions of the theorem. □

3.6 Concluding remarks and questions

Weakly ω^ω-bounding iterations

Proposition 3.4.15 about the iterand $\mathbb{K}_\alpha$ shows that the ω_2-length iteration of the in the proof of Theorem 3.4.20 is weakly ω^ω-bounding. However, this does not immediately imply $\mathbb{K}_\alpha$ is almost ω^ω-bounding, and in general the following appears to be open:

Question 3.6.1. *If a countable support iteration of proper forcings is weakly ω^ω-bounding, is each iterand an almost ω^ω-bounding forcing notion?*

In particular, it is still open if the Friedman-Zdomskyy poset is almost ω^ω-bounding. Given a positive answer to this, however, could yield a proof of the relative consistency of $\mathfrak{b} < \mathfrak{a} = \mathfrak{s} = \aleph_2$, with a Π^1_2 tight witness for $\mathfrak{a}$, in the following way.

In [She84], Shelah shows the consistency of both $\mathfrak{b} = \mathfrak{a} < \mathfrak{s}$ as well as $\mathfrak{b} < \mathfrak{a} = \mathfrak{s}$ (see [She84, Main Theorem 1.15(2)]). The latter is achieved by a modification of the original forcing $\mathbb{Q}$ which we have studied in Section 3.2 (see Definition 3.2.17). This modification is to first add ω_1-many Cohen reals to a model V of CH, obtaining a model $V_1 = V[\langle r_i : i < \omega_1\rangle]$, and in this latter model there exists a partial order $\mathbb{Q}[I]$ which is proper, almost ω^ω-bounding, and adds a real almost disjoint from every element of a given mad family $\mathcal{A} \in V_1$. Another characterization of the forcing iteration yielding $\mathfrak{b} = \mathfrak{a} < \mathfrak{s}$ is done in [BR14]. This partial order of Shelah was integrated in the construction of a Δ^1_3 wellorder with countable support and produced Theorem 3 of [FF10], showing $\mathfrak{b} < \mathfrak{a} = \mathfrak{s}$ is consistent with a Δ^1_3 wellorder of the reals. That the initial segments of the iteration were almost ω^ω-bounding followed from the fact that Sacks coding $C(Y)$ is ω^ω-bounding.

Recall the definability of the Δ^1_3 wellorder and the definability of the Π^1_2 tight mad family are both achieved by similar coding methods (namely, coding a certain pattern of stationarity/nonstationarity into an ω-block of the fixed sequence $\vec{S}$), and the primary difference between the two constructions is the type of generic real added. Whereas in the former construction of [FF10] the generic reals were Sacks reals, in the latter the generic reals can be seen as a form of Mathias real. More specifically, when a_α is a $\mathbb{K}_\alpha$-generic real, a_α arises as a result of the forcing notion of almost disjoint coding; this forcing can be considered a form of restricted Mathias forcing, that is, a modification of Mathias forcing (see the discussion at the beginning of Section 3.2) where the second coordinate of the conditions are taken to be elements of a filter.

This relative consistency result would answer Question 18 of [FZ10]:

Question 3.6.2. *Is $\mathfrak{b} < \mathfrak{a}$ consistent with a Π^1_2 (tight) mad family of size $\mathfrak{a}$?*

Separating parallel notions of tightness

Theorem 3.4.20, Theorem 3.5.1, and Theorem 3.5.2 give instances in which $|\mathrm{spec}(\mathfrak{a})| \geq 2$ and every $\kappa \in \mathrm{spec}(\mathfrak{a})$ is witnessed by a mad family with an optimal projective definition. This was nontrivially obtained, in the sense that we had to actively ensure the existence of an optimal projective witness for each κ in the spectrum, using preservation arguments. There are other instances in which optimal projective witnesses for multiple values in the spectrum associated to a cardinal characteristic is almost immediate. For example, consider the cardinal

$$\mathfrak{a}_e = \min\{|\mathcal{F}| \mid \mathcal{F} \subseteq \omega^\omega,\ \mathcal{F} \text{ is maximal eventually different}\},$$

where a family $\mathcal{F} \subseteq \omega^\omega$ is *eventually different* if $f(n) = g(n)$ for all but finitely many $n \in \omega$, for every distinct $f, g \in \mathcal{F}$. Such a family is *maximal eventually different* (med) if it is maximal with respect to inclusion. Shelah and Horowitz [HS24] have shown that there always exists a Borel maximal eventually different of size $\mathfrak{c}$, and a model of $\aleph_1 = \mathfrak{a}_e = \mathfrak{d} < \mathfrak{c}$ with a coanalytic witness to $\mathfrak{a}_e$ is given in [FS21a, Theorem 8]; therefore in this model both $\aleph_1, \mathfrak{c} \in \mathrm{spec}(\mathfrak{a}_e)$ have optimal projective witnesses.[10] Fischer and Switzer [FS21a] introduce a parallel notion of tightness for eventually different families as well as a notion of strong preservation of tightness of eventually different families. It is shown in [FS21a, Theorem 6.1] that Miller forcing strongly preserves the tightness of eventually different families; recall Miller forcing also strongly preserves tightness in the case of mad families. However it is not the case that the class of forcings strongly preserving tightness of eventually different families coincides with the class of forcings preserving tightness of almost disjoint families, and the forcing notion $\mathbb{Q}$ of Definition 3.2.17 witnesses this. Indeed, the analogous version of Theorem 3.5.1 for eventually different families cannot hold, by the following ZFC result:

Theorem 3.6.3. $\mathfrak{s} \leq \mathrm{non}(\mathcal{M}) \leq \mathfrak{a}_\mathfrak{e}$

The latter inequality follows from the combinatorial characterization of $\mathrm{non}(\mathcal{M})$ given by Bartoszyński and Judah; see also [FS21a, Fact 1.2]. The former inequality can be found in [Bla93].

A notion of tightness has also been introduced for another combinatorial family, maximal cofinitary groups, which give rise to the cardinal invariant $\mathfrak{a}_g$:

$$\mathfrak{a}_g = \min\{|\mathcal{G}| \mid \mathcal{G} \leq S_\infty,\ \mathcal{G} \text{ is a maximal cofinitary group}\}.$$

A subgroup $\mathcal{G}$ of the symmetric group $S_\infty = \{f \in \omega^\omega \mid f \text{ is a bijection}\}$ is called *cofinitary* if every $f \in \mathcal{G}$ which is not the identity function has only finitely many fixed points. A cofinitary group is *maximal (mcg)* if if it is maximal with respect to inclusion. Note that if $\mathcal{G}$ is a cofinitary group, it is an eventually different family of permutations. The recent work of Fischer, Schrittesser, and Schembecker [FSS25] defines the notion of a tight

[10]The definability here is optimal, since any Borel set is either countable or of size continuum.

cofinitary group, strengthening the notion of maximality, as well as a notion of preservation of tightness (see [FSS25, Definition 3]). They show that the Sacks coding satisfies this property in [FSS25, Theorem 17], obtaining the consistency of $\mathfrak{a}_g = \aleph_1 < \mathfrak{c} = \aleph_2$ with a Δ^1_3 wellorder and moreover a coanalytic tight cofinitary group of size $\aleph_1$; previously a coanalytic witness to $\mathfrak{a}_g = \aleph_1$ in a model of $\aleph_1 < \mathfrak{c}$ was constructed in [FST17]. Similar to the case of eventually different families, Horowitz and Shelah [HS25] have constructed a Borel maximal cofinitary group (under ZF), and thus there always exists an optimally definable mcg of size $\mathfrak{c}$. Therefore in the model of [FSS25], $\mathrm{spec}(\mathfrak{a}_g) = \{\aleph_1, \aleph_2\}$ is realized with optimal projective witnesses in a model with a Δ^1_3 wellorder.

The ZFC theorem above again precludes Shelah's forcing $\mathbb{Q}$ from satisfying this preservation notion, as $\mathrm{non}(\mathcal{M}) \leq \mathfrak{a}_g$. Taking these facts into account we can see Theorem 3.3.35 as separating the parallel combinatorial strengthenings for almost disjoint families, eventually different families, and cofinitary groups.

Coanalytic Spectra and Finite Support

While a model of $\aleph_2 < \mathfrak{b} = \mathfrak{c}$ and each $\kappa \in \mathrm{spec}(\mathfrak{a})$ has a Π^1_1 witness is by Brendle and Khomskii [BK13], their construction relies heavily on the preservation of splitting families and increases the size of $\mathfrak{c}$ by using Hechler forcing, which satisfies this last preservation property. Therefore their construction cannot be used answer either of the following.

Question 3.6.4. *Is it consistent with $\mathfrak{a} < \mathfrak{c}$ or even with $\mathfrak{a} < \mathfrak{s} = \mathfrak{c}$ that there exist coanalytic mad families of sizes $\mathfrak{a}$ and $\mathfrak{c}$?*

Because our above constructions used countable support iterations, our techniques can only yield models with $\mathfrak{c} = \aleph_2$. Naturally, then, we can ask:

Question 3.6.5. *Is it consistent that $|\mathrm{spec}(\mathfrak{a})| \geq 3$ and for each $\kappa \in \mathrm{spec}(\mathfrak{a})$ there exists a projective mad family of size κ with an optimal definition?*

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