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Weak-Star Compactness and the Neyman–Pearson Lemma

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Abstract

Deutsch. Hypothesentests stellen ein zentrales Instrument in den meisten Sozial- und Naturwissenschaften dar. Trotz ihrer Verbreitung sind die meisten nicht-trivialen Resultate zu Tests in endlichen Stichproben nur schwer zugänglich und setzen in der Regel zumindest grundlegende Kenntnisse der maßtheoretischen Wahrscheinlichkeitstheorie sowie der Funktionalanalysis voraus. Das Hauptanliegen dieser Arbeit besteht darin, ausgewählte Resultate über UMP Tests aus Lehmann & Romanos *Testing Statistical Hypotheses* darzulegen und zu beweisen. Der Schwerpunkt soll hierbei auf der Charakterisierung der schwach-*-Topologie auf L^∞ sowie einiger ihrer wesentlichen Eigenschaften liegen, um einen kürzeren Beweis einer Erweiterung des Neyman–Pearson-Lemmas zu ermöglichen.

Kapitel 1 führt Hypothesentests in einem entscheidungstheoretischen Kontext ein und enthält einige hilfreiche Resultate aus der Maßtheorie. In Kapitel 2 wird das klassische Resultat von Neyman & Pearson vorgestellt und bewiesen, während Kapitel 3 eine Abhandlung über Verteilungen mit Monotone Likelihood Ratio und den Satz von Karlin–Rubin betrachtet. Kapitel 4 behandelt die schwach-*-Topologie und präsentiert zwei Versionen des Satzes von Banach–Alaoglu zur schwach-*-Kompaktheit der Einheitskugel. In Kapitel 5 wird anhand dieser Resultate eine Erweiterung des Neyman–Pearson-Lemmas unter mehreren Nebenbedingungen bewiesen. Kapitel 6 stellt schließlich den Zusammenhang zwischen Hypothesentests und Konfidenzintervallen her.

Abstract

English. Hypothesis testing is an important tool in most modern social and natural sciences. Despite its prevalence, most non-basic results on finite-sample testing are quite inaccessible and require at least some measure-theoretic probability and functional analysis. The main goal of this thesis is to state and prove some results on uniformly most powerful tests from Lehmann & Romano's *Testing Statistical Hypothesis*. The focus lies on characterizing the weak-star topology on L^∞ with some of its properties to give a more concise proof on an extension of the Neyman–Pearson Lemma.

In Chapter 1 the framework of decision-theoretic hypothesis testing is introduced, along with some useful auxiliary results from measure theory. In Chapter 2 the well-known result by Neyman & Pearson is given, while Chapter 3 provides a treatment on monotone likelihood ratios and the Karlin–Rubin Theorem. Chapter 4 introduces the weak-star topology and two versions of the Banach–Alaoglu theorem on weak-star compactness of the unit ball. In Chapter 5, these results are used to prove an extension of the Neyman–Pearson Lemma with multiple side conditions. Chapter 6 establishes the connection between hypothesis testing and confidence bounds.

Contents

Chapter 1

Introduction

Consider a sample $X \sim \mathbb{P}_\theta \in \mathcal{P}$, where $\mathcal{P}$ is an identifiable class of probability measures with parameter space $\Theta \subset \mathbb{R}^p$ and X takes values in $\mathcal{X} \subset \mathbb{R}^n$. We want to test

$$\mathbb{P}_\theta \in H_0 \text{ vs. } \mathbb{P}_\theta \in H_1,$$

or equivalently by identifiability

$$\theta \in \Theta_0 \text{ vs. } \theta \in \Theta_1,$$

where H_0, H_1 partition $\mathcal{P}$ and Θ_0, Θ_1 partition Θ. We consider a measurable test function

$$\phi : \mathcal{X} \to [0, 1]$$

and aim to solve the constrained optimization problem

$$\sup_{\theta \in \Theta_1} \mathbb{E}_\theta \, \phi(X), \text{ s.t. } \mathbb{E}_\theta \, \phi(X) \leq \alpha \text{ for all } \theta \in \Theta_0$$

for some predetermined $\alpha \in [0, 1]$. That is, we want to maximize the probability of correctly rejecting while keeping the probability of falsely rejecting below α.

Definition 1.1. *A test function ϕ is said to be of* size *or* level $\alpha \in [0, 1]$ *for testing $\theta \in \Theta_0$ vs. $\theta \in \Theta_1$ if*

$$\mathbb{E}_\theta \, \phi(X) = \int \phi(X) \, d\mathbb{P}_\theta \leq \alpha \textit{ for all } \theta \in \Theta_0.$$

Definition 1.2. *Let ϕ be a test function. The* power function $\beta : \Theta \to [0, 1]$ *of ϕ is defined via $\beta(\theta) := \mathbb{E}_\theta \, \phi(X)$.*

Definition 1.3. *A test ϕ^* is said to be* uniformly most powerful (UMP) *for testing $\theta \in \Theta_0$ vs. $\theta \in \Theta_1$ at size α if ϕ^* is of size α and*

$$\mathbb{E}_\theta\, \phi^*(X) \geq \mathbb{E}_\theta\, \phi(X) \text{ for all } \theta \in \Theta_1$$

for any test ϕ of size α.

Definition 1.4. *Let $L : (\Theta \times [0,1]) \to [0, \infty)$ be a loss function. Then the* L-risk of a test ϕ under $\mathbb{P}_\theta$ *is defined as*

$$R_{L,\theta}(\phi) := \mathbb{E}_\theta\, L(\theta, \phi(X)) = \int L(\theta, \phi(X))\; d\mathbb{P}_\theta.$$

Definition 1.5. *A class of tests $\mathcal{C}$ is said to be* essentially complete *for L, if for any test function ϕ there exists $\phi' \in \mathcal{C}$ such that*

$$R_{L,\theta}(\phi') \leq R_{L,\theta}(\phi) \text{ for all } \theta \in \Theta.$$

$\mathcal{C}$ is said to be minimal essentially complete *for L if for any $\phi \in \mathcal{C}$ one can find $\theta \in \Theta$ and $\phi' \in \mathcal{C}$ such that*

$$R_{L,\theta'}(\phi') < R_{L,\theta'}(\phi),$$

i.e. $\mathcal{C}$ does not contain an essentially complete proper subclass.

Definition 1.6. *Let $(\mathcal{X}, \mathcal{A})$ be a measurable space and $A \in \mathcal{A}$. The* indicator function on A *is defined as $\mathbb{1}_A : \mathcal{X} \to \mathbb{R}$ with*

$$\mathbb{1}_A(x) := \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{if } x \notin A \end{cases}$$

and is trivially $\mathcal{A} - \mathcal{B}(\mathbb{R})$-measurable.

It is sometimes useful to consider only test functions of the form $\psi : \mathcal{X} \to \{0,1\}$. Given that the underlying probability space is large enough to support $U \sim \mathcal{U}([0,1])$ independent of X, it is possible to always find such a "non-randomized" test function with equal expectation (and hence power and risk) to a randomized test function.

Lemma 1.7. *Let $\phi : \mathcal{X} \to [0,1]$ be measurable and suppose that the probability space on which a $\mathcal{X}$-valued random variable X is defined is large enough to define $U \sim \mathcal{U}([0,1])$ independent of X. Then there exists a randomized test $\psi : \mathcal{X} \to \{0,1\}$ with*

$$\mathbb{E}_\theta\, \phi(X) = \mathbb{E}_\theta\, \psi(X) \text{ for all } \theta \in \Theta.$$

Proof. Let $R := \big\{(x,u) \in \mathcal{X} \times [0,1] : u \le \phi(x)\big\}$ and set $\psi(x) := \mathbb{1}_R(x,U)$. Then

$$\begin{aligned}\mathbb{E}_\theta\, \psi(X) &= \mathbb{P}_\theta(U \le \phi(X)) \\ &= \mathbb{E}_\theta\left[\mathbb{E}_\theta\left[\mathbb{1}_R(X,U) \mid \sigma(X)\right]\right] \\ &= \mathbb{E}_\theta\left[\mathbb{E}_\theta\left[\mathbb{1}_R(x,U)\right]\big|_{x=X}\right] \\ &= \mathbb{E}_\theta\left[\mathbb{P}\left(U \le \phi(x)\right)|_{x=X}\right] = \mathbb{E}_\theta\, \phi(X)\end{aligned}$$

by the properties of conditional expectation. □

Definition 1.8. *Let $(\mathcal{X}, \mathcal{A})$ be a measurable space and $f : \mathcal{X} \to \mathbb{R}$ be $\mathcal{A} - \mathcal{B}(\mathbb{R})$-measurable. Denote the* support *of f by*

$$\operatorname{supp}(f) := \{x \in \mathcal{X} : f(x) \neq 0\} \in \mathcal{A}.$$

Notice that this is different from the usual topological definition of the closed support*, which is the closure of the above set.*

Lemma 1.9. *Let $(\mathcal{X}, \mathcal{A}, \mu)$ be a σ-finite measure space and let $S \in \mathcal{A}$. Let $f : \mathcal{X} \to \mathbb{R}$ be $\mathcal{A} - \mathbb{B}(\mathbb{R})$-measurable. If $f(x) > 0$ for all $x \in S$, then*

$$\int_S f \; d\mu = 0 \implies \mu(S) = 0.$$

Proof. Let $S_n := \{x \in S : f(x) \ge 1/n\}$. Then by monotonicity of the integral

$$0 \le \mu(S_n) = \int_{S_n} 1 \; d\mu \le \int_{S_n} nf \; d\mu \le \int_S nf \; d\mu = 0 \text{ for all } n \ge 1.$$

Also, $S_1 \subset S_2 \subset \ldots \subset S = \bigcup_{n\ge1} S_n$ and by σ-finiteness of μ and continuity from below $\mu(S) = \lim_{n\to\infty} \mu(S_n) = 0$. □

Chapter 2

The Neyman–Pearson Lemma

A famous result is given for $\mathcal{P} = \{\mathbb{P}_0, \mathbb{P}_1\} \ll \mu$ with $\mathbb{P}_0 \neq \mathbb{P}_1$ and the test $H_0 : \mathbb{P}_0$ vs. $H_1 : \mathbb{P}_1$.

Theorem 2.1 (Neyman–Pearson [5]). *Let $\mathbb{P}_0, \mathbb{P}_1$ be probability measures with densities p_0, p_1 w.r.t. a common dominating measure μ on a measurable space $(\mathcal{X}, \mathcal{A})$. Let $\alpha \in [0, 1]$.*

(i) For testing $\mathbb{P}_0$ vs. $\mathbb{P}_1$ there exists a test ϕ and a constant $k \in [0, \infty]$ such that

$$\phi(x) = \begin{cases} 1 & \text{if } p_1(x) > kp_0(x), \\ 0 & \text{if } p_1(x) < kp_0(x) \end{cases} \tag{2.1}$$

and

$$\mathbb{E}_0\, \phi(X) = \alpha. \tag{2.2}$$

(ii) If a test ϕ satisfies (2.1) for some $k \in [0, \infty]$ and (2.2), then it is most powerful for testing $\mathbb{P}_0$ vs. $\mathbb{P}_1$ at level α.

(iii) If a test ϕ^* is most powerful for $\mathbb{P}_0$ vs. $\mathbb{P}_1$ at level α, then it satisfies (2.1) μ-a.e. for some $k \in [0, \infty]$. Furthermore, if there is no test ψ with $\mathbb{E}_1\, \psi(X) = 1$ at level $\alpha' < \alpha$, then also

$$\mathbb{E}_0\, \phi^*(X) = \alpha.$$

Proof. (i) Consider the three cases $\alpha = 0, 1$ and $\alpha \in (0, 1)$.
$\boxed{\alpha = 0:}$ Set $k := \infty$ under the convention $0 \cdot \infty = 0$ and define

$$\phi(x) := \mathbb{1}_{\{p_1 > kp_0\}}(x),$$

which is of the form in (2.1) and since $p_0(x) > 0 \implies \phi(x) = 0$, one has

$$\mathbb{E}_0\, \phi(X) = \int \phi p_0 \, d\mu = \int_{\mathrm{supp}(p_0)} \phi p_0 \, d\mu = 0 = \alpha.$$

$\boxed{\alpha = 1:}$ Set $k := 0$ and define

$$\phi(x) := \mathbb{1}_{\{p_1 \geq k p_0\}}(x) = \mathbb{1}_{\mathcal{X}}(x) = 1,$$

which is of the form in (2.1). Then

$$\mathbb{E}_0\, \phi(X) = 1 = \alpha.$$

$\boxed{\alpha \in (0,1):}$ For $c \geq 0$ define

$$\begin{aligned}
\alpha(c) &:= \mathbb{P}_0\big(p_1(X) > c p_0(X)\big) \\
&= \int \mathbb{1}_{\{p_1 > c p_0\}} p_0 \; d\mu \\
&= \int_{\mathrm{supp}(p_0)} \mathbb{1}_{\{p_1/p_0 > c\}} p_0 \; d\mu \\
&= \mathbb{P}_0\left(\frac{p_1(X)}{p_0(X)} > c\right),
\end{aligned}$$

such that $\alpha(c)$ is non-increasing and right-continuous and

- $\lim_{c \to -\infty} \alpha(c) = 1$, $\lim_{c \to \infty} \alpha(c) = 0$,
- $\lim_{c' \nearrow c} \alpha(c') - \alpha(c) = \mathbb{P}_0\left(\frac{p_1(X)}{p_0(X)} = c\right)$.

Take k such that $\alpha(k) \leq \alpha \leq \lim_{c' \nearrow k} \alpha(c')$ and define

$$\phi(x) := \begin{cases} 1 & \text{if } p_1(x) > k p_0(x), \\ \dfrac{\alpha - \alpha(k)}{\lim_{c' \nearrow k} \alpha(c') - \alpha(k)} & \text{if } p_1(x) = k p_0(x), \\ 0 & \text{if } p_1(x) < k p_0(x). \end{cases}$$

Notice that ϕ is well-defined $\mathbb{P}_0$-a.s., since

$$\mathbb{P}_0\big(p_1(X) = k p_0(X)\big) > 0 \implies \lim_{c' \nearrow k} \alpha(c') - \alpha(k) > 0.$$

Furthermore, one has

$$\begin{aligned}
\mathbb{E}_0\, \phi(X) &= \frac{\alpha - \alpha(k)}{\lim_{c' \nearrow k} \alpha(c') - \alpha(k)} \mathbb{P}_0\big(p_1(X) = k p_0(X)\big) + \mathbb{P}_0\big(p_1(X) > k p_0(X)\big) \\
&= \alpha - \alpha(k) + \alpha(k) = \alpha.
\end{aligned}$$

(ii) Let ψ be any other test for $\mathbb{P}_0$ vs. $\mathbb{P}_1$ at level $\alpha \in [0,1]$. Then

$$\mathbb{E}_0\, \phi(X) - \mathbb{E}_0\, \psi(X) = \int (\phi - \psi) p_0 \; d\mu \geq 0.$$

Define $S_\pm := \{\phi \gtrless \psi\}$ such that

$$\begin{aligned} x \in S_+ &\implies \phi(x) > 0 \implies p_1(x) \geq k p_0(x), \\ x \in S_- &\implies \phi(x) < 1 \implies p_1(x) \leq k p_0(x). \end{aligned}$$

Therefore,

$$\int (\phi - \psi)(p_1 - k p_0) \; d\mu = \int_{S_+ \cup S_-} (\phi - \psi)(p_1 - k p_0) \; d\mu \geq 0,$$

or equivalently

$$\int (\phi - \psi) p_1 \; d\mu \geq k \int (\phi - \psi) p_0 \; d\mu.$$

By the fact that $k \in [0, \infty]$ it follows that

$$\mathbb{E}_1\, \phi(X) - \mathbb{E}_1\, \psi(X) = \int (\phi - \psi) p_1 \; d\mu \geq k \int (\phi - \psi) p_0 \; d\mu \geq 0.$$

(iii) Now let ϕ^* be most powerful for $\mathbb{P}_0$ vs. $\mathbb{P}_1$ at level α and let ϕ satisfy (2.1) for some $k \in [0, \infty]$. Define $S := \{\phi \neq \phi^*\} \cap \{p_1 \neq k p_0\}$. Suppose that $\mu(S) > 0$. Then similarly as above

$$\begin{aligned} 0 \geq \mathbb{E}_1\, \phi^*(X) - \mathbb{E}_1\, \phi(X) &\geq \int (\phi^* - \phi)(p_1 - k p_0) \; d\mu \\ &= \int_S (\phi^* - \phi)(p_1 - k p_0) \; d\mu < 0. \end{aligned}$$

The last inequality is a consequence of Lemma 1.9, since the integrand in the last integral is strictly negative on S. This is a contradiction, so $\mu(S) = 0$ and hence $\phi^* = \phi$ on the set $\{p_1 \neq k p_0\}$, as was to be shown. Now suppose $\mathbb{E}_0\, \phi^*(X) = \alpha' < \alpha$ and $\mathbb{E}_1\, \phi^*(X) < 1$. Then one could increase k until either $\mathbb{E}_0\, \phi^*(X) = \alpha$ or $\mathbb{E}_1\, \phi^*(X) = 1$. □

Corollary 2.2. *Let ϕ^* be most powerful for $\mathbb{P}_0$ vs. $\mathbb{P}_1$ at size $\alpha \in (0,1)$. Then*

$$\beta := \mathbb{E}_1\, \phi^*(X) > \alpha.$$

Proof. Notice that the test given by $\phi(x) := \alpha$ for all $x \in \mathcal{X}$ has size α such that $\alpha \leq \beta$ by the fact that ϕ^* is most powerful. If $\beta = 1$, then the result follows from $\alpha \in (0,1)$, so suppose for the sake of contradiction that $\alpha = \beta < 1$. Since ϕ^* and hence also ϕ are most powerful by Theorem 2.1, ϕ must satisfy (2.1) μ-a.e. Since $\alpha \notin \{0,1\}$, we must have $p_0 = k p_1$ μ-a.e. and hence $k = 1$ and $\mathbb{P}_0 = \mathbb{P}_1$, which contradicts identifiability. □

Chapter 3

Families with Monotone Likelihood Ratio

Another well-known result by Karlin and Rubin is given for families with monotone likelihood ratio, e.g. one-parameter exponential families.

Definition 3.1. *Consider a parametric family $\mathcal{P} = \{\mathbb{P}_\theta : \theta \in \Theta\}$, where $\Theta \subset \mathbb{R}$, and a σ-finite measure μ, such that $\mathcal{P} \ll \mu$ with respective densities $p_\theta : \mathcal{X} \to [0, \infty)$. We say that $\mathcal{P}$ has* monotone likelihood ratio *in a measurable function $T : \mathcal{X} \to \mathbb{R}$, if*

I. $\mathcal{P}$ is identifiable, i.e. $\mathbb{P}_\theta = \mathbb{P}_{\theta'} \iff \theta = \theta'$ and

II. for any $\theta < \theta'$ there exists a measurable, nondecreasing function $g_{\theta',\theta} : \mathbb{R} \to \mathbb{R}$ with

$$\frac{p_{\theta'}(x)}{p_\theta(x)} = g_{\theta',\theta}(T(x)) \text{ for all } x \in \operatorname{supp}(p_\theta).$$

A prominent example of families with monotone likelihood ratio are the one-parameter exponential families.

Definition 3.2. *Suppose that $\mathcal{P} = \{\mathbb{P}_\theta : \theta \in \Theta\} \ll \mu$, where $\Theta \subset \mathbb{R}$, μ is a σ-finite measure on some measurable space $(\mathcal{X}, \mathcal{A})$ and $\mathbb{P}_\theta$ has μ-density p_θ. We say that $\mathcal{P}$ is a* one-parameter exponential family*, if there exist functions $h, T : \mathcal{X} \to \mathbb{R}$ and $\eta, \kappa : \Theta \to \mathbb{R}$ such that one can write*

$$p_\theta(x) = h(x) \exp\{\eta(\theta)T(x) - \kappa(\theta)\}.$$

In particular, one has $h(x) \geq 0$ for all $x \in \mathcal{X}$ and $\operatorname{supp}(p_\theta)$ is independent of θ.

Lemma 3.3. *Let $\mathcal{P} \ll \mu$ be an identifiable, one-parameter exponential family with η strictly monotone increasing. Then $\mathcal{P}$ has monotone likelihood ratio in T.*

Proof. Let $\theta' > \theta$ and write

$$\begin{aligned}\frac{p_{\theta'}(x)}{p_\theta(x)} &= \frac{h(x)\exp\{T(x)\eta(\theta') - \kappa(\theta')\}}{h(x)\exp\{T(x)\eta(\theta) - \kappa(\theta)\}} \\ &= \exp\{\kappa(\theta) - \kappa(\theta')\}\exp\{T(x)[\eta(\theta') - \eta(\theta)]\}.\end{aligned}$$

Now $x \mapsto \exp(x)$ is strictly monotone increasing and $\eta(\theta') > \eta(\theta)$. □

Remark 3.4. *Notice the explicit assumption on identifiability. Under the above assumptions, $\mathcal{P}$ is identifiable if and only if $T(X)$ is not a.s. constant, since for $\theta \neq \theta'$*

$$\begin{aligned}& p_\theta = p_{\theta'} \quad \mu\text{-a.e.} \\ \iff & T(X)[\eta(\theta) - \eta(\theta')] = \kappa(\theta) - \kappa(\theta') \quad \mathbb{P}_\theta - a.s. \text{ for all } \theta \in \Theta \\ \iff & T(X) = \frac{\kappa(\theta) - \kappa(\theta')}{\eta(\theta) - \eta(\theta')} \quad \mathbb{P}_\theta - a.s. \text{ for all } \theta \in \Theta\end{aligned}$$

by strict monotonicity of η and the fact that $\operatorname{supp}(p_\theta)$ *does not depend on θ. A treatment of the case $p \geq 2$ can for example be found in [2] (Theorem 1.13).*

Theorem 3.5 (Karlin–Rubin [3])**.** *Let $\mathcal{P}$ be a parametric family with monotone likelihood ratio in T. Let $\theta_0 \in \Theta$.*

(i) For testing $H : \theta \leq \theta_0$ vs. $K : \theta > \theta_0$ there exists a uniformly most powerful test given by

$$\phi(x) = \begin{cases} 1 & \text{if } T(x) > k, \\ \gamma & \text{if } T(x) = k, \\ 0 & \text{if } T(x) < k, \end{cases} \tag{3.1}$$

where γ, k are constants such that $E_{\theta_0}\phi(X) = \alpha$.

(ii) For any $\theta' \in \Theta$ the test determined by (3.1) is UMP for $H : \theta \leq \theta'$ vs $K : \theta > \theta'$ at size $\alpha' := \mathbb{E}_{\theta'}\,\phi(X)$.

(iii) The power function $\beta(\theta) = \mathbb{E}_\theta\,\phi(X)$ of ϕ is strictly increasing for all $\theta \in \Theta$ with $\beta(\theta) \in (0,1)$.

(iv) For all $\theta < \theta_0$ the test given by (3.1) satisfies

$$\phi \in \operatorname*{arginf}_{\psi \in \mathcal{V}} \mathbb{E}_\theta\,\psi(X),$$

where $\mathcal{V} = \{\psi : \mathcal{X} \to [0,1] \mid \mathbb{E}_{\theta_0}\,\psi(X) = \alpha\}$.

Proof. (i) Let $\Theta \ni \theta_1 > \theta_0$ be arbitrary and consider testing $\theta = \theta_0$ vs. $\theta = \theta_1$. Then for all $x \in \operatorname{supp}(p_{\theta_0})$ one has $r(x) = p_{\theta_1}(x)/p_{\theta_0}(x) = g_{\theta_1,\theta_0}(T(x))$. Now let $\alpha_T(c) := \mathbb{P}_{\theta_0}(T(X) > c)$ and, similar to the proof of Theorem 2.1, let k be such that

$$\alpha_T(k) \le \alpha \le \lim_{c \nearrow k} \alpha_T(c)$$

and let

$$\gamma := \begin{cases} \dfrac{\alpha - \alpha_T(k)}{\lim\limits_{c \nearrow k} \alpha_T(c) - \alpha_T(k)} & \text{if } \lim\limits_{c \nearrow k} \alpha_T(c) \neq \alpha_T(k), \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathbb{E}_{\theta_0} \phi(X) = \alpha$ and ϕ is well-defined $\mathbb{P}_{\theta_0}$-a.s. Furthermore

$$\begin{aligned} g_{\theta_1,\theta_0}(T(x)) &= \frac{p_{\theta_1}(x)}{p_{\theta_0}(x)} > g_{\theta_1,\theta_0}(k) \implies T(x) > k \implies \phi(x) = 1, \\ g_{\theta_1,\theta_0}(T(x)) &= \frac{p_{\theta_1}(x)}{p_{\theta_0}(x)} < g_{\theta_1,\theta_0}(k) \implies T(x) < k \implies \phi(x) = 0 \end{aligned}$$

and by Theorem 2.1 (ii) ϕ is most powerful for $\theta = \theta_0$ v. $\theta = \theta_1$ at size α. The values of k and γ do not depend on the choice of θ_1 and ϕ is hence uniformly most powerful for $\theta = \theta_0$ vs. $\theta > \theta_0$ at size α. Now consider the following nested classes:

$$\mathcal{C}_1 := \{\psi \mid \forall \theta \le \theta_0 : \mathbb{E}_\theta \psi(X) \le \alpha\} \subset \{\psi \mid \mathbb{E}_{\theta_0} \psi(X) \le \alpha\} =: \mathcal{C}_2.$$

Then we have already shown that $\mathbb{E}_\theta \phi(X) \ge \mathbb{E}_\theta \psi(X)$ for all $\theta > \theta_0$ and all $\psi \in \mathcal{C}_2$. By the above inclusion this power inequality must also hold on the smaller class $\mathcal{C}_1$.
(ii) Set $\alpha' := \beta(\theta')$ and repeat the above arguments.
(iii) By Corollary 2.2 and (ii) above

$$\mathbb{E}_{\theta_1} \phi(X) = \beta(\theta_1) > \alpha = \mathbb{E}_{\theta_0} \phi(X) = \beta(\theta_0)$$

for any $\theta_1 > \theta_0$ with $\beta(\theta_1) < 1$ and $\beta(\theta_0) > 0$.
(iv) Analogous to the proof of Theorem 2.1 one can show that the test given by $1 - \phi$ minimizes $\mathbb{E}_{\theta_1} \psi(X)$ among all tests ψ with $\mathbb{E}_{\theta_0} \psi(X)) = \alpha$ (important: not $\le \alpha$!) for all $\theta_1 < \theta_0$. As in (i), the values of k and γ do not depend on θ_1. The rest is shown exactly as (i) above. $\square$

Remark 3.6. *Following the reasoning in the proof of (iv) one easily finds a UMP test for testing $\theta \ge \theta_0$ vs. $\theta < \theta_0$ at size α.*

Theorem 3.7. *Let $\mathcal{P}$ be a parametric family with monotone likelihood ratio in T and let $L : \Theta \times [0,1] \to [0,\infty)$ be a loss function satisfying*

$$L(\theta,1) > L(\theta,0) \textit{ for } \theta < \theta_0,$$
$$L(\theta,1) < L(\theta,0) \textit{ for } \theta > \theta_0.$$

Then the class

$$\mathcal{D} := \{\phi : \mathcal{X} \to [0,1] \mid \phi \textit{ is of the form in } (2.1)\}$$

is essentialy complete for L. If additionaly $\operatorname{supp}(p_\theta)$ *does not depend on θ, then $\mathcal{D}$ is also minimal essentially complete for L.*

Proof. Let $\phi : \mathcal{X} \to [0,1]$ be any test. By Lemma 1.7 we may assume that there exists a randomized test $\psi : \mathcal{X} \to \{0,1\}$ with

$$\begin{aligned} R_{L,\theta}(\phi) = R_{L,\theta}(\psi) &= \int [\psi L(\theta,1) + (1-\psi)L(\theta,0)]\, p_\theta \, d\mu \\ &= L(\theta,0) + [L(\theta,1) - L(\theta,0)] \int \psi p_\theta \, d\mu. \end{aligned}$$

Hence for any two tests ϕ, ϕ' it holds that

$$\begin{aligned} R_{L,\theta}(\phi') - R_{L,\theta}(\phi) &= R_{L,\theta}(\phi') - R_{L,\theta}(\psi) \\ &= \{L(\theta,1) - L(\theta,0)\}\{\mathbb{E}_\theta\, \phi'(X) - \mathbb{E}_\theta\, \psi(X)\} \\ &= \{L(\theta,1) - L(\theta,0)\}\{\mathbb{E}_\theta\, \phi'(X) - \mathbb{E}_\theta\, \phi(X)\}. \end{aligned}$$

Now let $\theta_0 \in \Theta$ be arbitrary and set $\alpha := \mathbb{E}_{\theta_0} \phi(X) \in [0,1]$. By Theorem 3.5 (i) there exists a UMP test $\phi^* \in \mathcal{D}$ for $\theta \le \theta_0$ vs. $\theta > \theta_0$ at size α. Together with Theorem 3.5 (iv) this yields

$$\begin{aligned} \theta < \theta_0 &\implies \mathbb{E}_\theta\, \phi^*(X) \le \mathbb{E}_\theta\, \phi(X), \\ \theta = \theta_0 &\implies \mathbb{E}_\theta\, \phi^*(X) = \alpha = \mathbb{E}_\theta\, \phi(X), \\ \theta > \theta_0 &\implies \mathbb{E}_\theta\, \phi^*(X) \ge \mathbb{E}_\theta\, \phi(X) \end{aligned}$$

and hence $R_{L,\theta}(\phi^*) \le R_{L,\theta}(\phi)$ for all $\theta \in \Theta$. This shows that $\mathcal{D}$ is essentially complete. For the second part let $0 \le \alpha < \alpha' \le 1$ and let $\phi, \phi' \in \mathcal{D}$ with $\mathbb{E}_{\theta_0} \phi(X) = \alpha$ and $\mathbb{E}_{\theta_0} \phi'(X) = \alpha'$, which exist by Theorem 3.5 (i). Then ϕ is also of size α' and by Theorem 3.5 (iii) we have

$$\mathbb{E}_\theta\, \phi'(X) \ge \mathbb{E}_\theta\, \phi(X) \text{ for all } \theta > \theta_0.$$

Suppose that there exists some $\theta > \theta_0$ with $\mathbb{E}_\theta\, \phi'(X) = \mathbb{E}_\theta\, \phi(X)$. Then

$$\int (\phi' - \phi)p_\theta \; d\mu = \int\limits_{\mathrm{supp}(p_\theta)} (\phi' - \phi)p_\theta \; d\mu = 0,$$

which implies $\phi(x) = \phi'(x)$ for all $x \in \mathrm{supp}(p_\theta)$ and therefore also for all $x \in \mathrm{supp}(p_{\theta_0})$. Hence $\alpha = \alpha'$, a contradiction, and we must have

$$\mathbb{E}_\theta\, \phi'(X) > \mathbb{E}_\theta\, \phi(X) \text{ for all } \theta > \theta_0.$$

In a similar manner, the tests $1 - \phi', 1 - \phi \in \mathcal{D}$ are UMP for testing $\theta \geq \theta_0$ vs. $\theta < \theta_0$ at sizes $1 - \alpha', 1 - \alpha$ respectively and by an analogous argument we get

$$\mathbb{E}_\theta\, \phi(X) > \mathbb{E}_\theta\, \phi'(X) \text{ for all } \theta < \theta_0.$$

By the properties of L it follows that

$$\begin{aligned} R_{L,\theta}(\phi') &< R_{L,\theta}(\phi) \text{ for all } \theta > \theta_0, \\ R_{L,\theta}(\phi') &> R_{L,\theta}(\phi) \text{ for all } \theta < \theta_0, \end{aligned}$$

showing that $\mathcal{D}$ is minimal essentially complete. $\square$

Chapter 4

The Weak-Star Topology on L^∞

This chapter introduces basic concepts and properties of the weak-star topology necessary for the subsequent chapter. Some of the following results will be referenced for brevity. The respective proofs can for example be found in Chapter 3 of [1].

Definition 4.1. *Let $(X, \|\cdot\|_X)$ be a normed vector space over $\mathbb{R}$. The* continuous dual *on X is defined as $X' := \{\lambda : X \to \mathbb{R} \mid \lambda \text{ linear and continuous}\}$. The strong topology on X' is normed through*

$$\lambda \mapsto \|\lambda\|_{X'} := \sup\{|\lambda(x)| : x \in X, \|x\|_X \leq 1\}.$$

Lemma 4.2. *Let $(X, \|\cdot\|_X)$ be a normed vector space over $\mathbb{R}$ with continuous dual X'. For $E \subset X$ and $r > 0$ set*

$$U_{E,r} := \left\{\lambda \in X' \;\middle|\; \forall x \in E : |\lambda(x)| < r\right\}.$$

Then

$$\tau^* := \left\{U \subset X' \;\middle|\; \forall \lambda \in U : \exists E \subset X, E \text{ finite}, \exists r > 0 : \lambda + U_{E,r} \subset U\right\}$$

is a topology and called the weak-star topology *on X'.*

Proof. See e.g. Proposition 3.12 in [1]. □

Lemma 4.3. *Let $(\mathcal{X}, \mathcal{A}, \mu)$ be a σ-finite measure space. Then there exists a finite measure ρ on $(\mathcal{X}, \mathcal{A})$ with $\mu \ll \rho$ and $\rho \ll \mu$.*

Proof. By σ-finiteness of μ there exists a partition $(\mathcal{X}_n)_{n\geq 1}$ of $\mathcal{X}$ with $\mathcal{X}_n \in \mathcal{A}$ and $0 < \mu(\mathcal{X}_n) < \infty$ for all $n \geq 1$. Now set

$$\rho(A) := \sum_{n=1}^{\infty} 2^{-n} \frac{\mu(A \cap \mathcal{X}_n)}{\mu(\mathcal{X}_n)} \text{ for all } A \in \mathcal{A}.$$

Then $\rho(\mathcal{X}) = 1, \rho(\emptyset) = 0$ and σ-additivity follows from Fubini's theorem. In particular

$$\mu(A) = 0 \iff \forall n \geq 1 : \mu(A \cap \mathcal{X}_n) = 0 \iff \rho(A) = 0,$$

showing $\mu \ll \rho$ and $\rho \ll \mu$. $\square$

In particular, we get $L^p(\mathcal{X}, \mathcal{A}, \mu) \simeq L^p(\mathcal{X}, \mathcal{A}, \rho)$ for all $p \in [1, \infty]$ via the respective Radon–Nikodym derivatives.

Theorem 4.4. *Let $(\mathcal{X}, \mathcal{A}, \mu)$ be a σ-finite measure space. Then $(L^1(\mu))'$ is isometrically isomorphic to $L^\infty(\mu)$.*

Proof. By Lemma 4.3 it suffices to consider the case where μ is finite. Let $\lambda \in (L^1(\mu))'$ and define the set function

$$\rho(A) := \lambda\left(\mathbb{1}_A\right) \text{ for all } A \in \mathcal{A}.$$

We aim to show that ρ is a finite signed measure. By linearity of λ it holds that

$$\rho(\emptyset) = \lambda\left(\mathbb{1}_\emptyset\right) = 0.$$

Since every continuous, linear functional is bounded, one gets $|\lambda(\mathbb{1}_A)| \leq M \cdot \mu(A)$ for some $M > 0$ and all $A \in \mathcal{A}$, so it remains to show σ-additivity. Let $A = \bigcup_{n\geq 1} A_n$ with $A_n \in \mathcal{A}$ disjoint for $n \geq 1$. By linearity of λ, for all $N \geq 1$,

$$\rho\left(\mathbb{1}_A\right) = \sum_{n=1}^{N} \rho(A_n) + \rho\left(\bigcup_{n=N+1}^{\infty} A_n\right).$$

By continuity of μ it holds that

$$\rho\left(\bigcup_{n=N+1}^{\infty} A_n\right) \leq M \cdot \mu\left(\bigcup_{n=N+1}^{\infty} A_n\right) \xrightarrow[N\to\infty]{} 0$$

and hence ρ is σ-additive. Let ρ_+, ρ_- be the Jordan decomposition of ρ and $\mathcal{X}_+, \mathcal{X}_-$ be a Hahn decomposition of $\mathcal{X}$. Then for all $A \in \mathcal{A}$,

$$\rho_+(A) = \rho(\mathcal{X}_+ \cap A) \leq M \cdot \mu(\mathcal{X}_+ \cap A) \leq M \cdot \mu(A)$$

and hence $\rho^+ \ll \mu$ (and $\rho_- \ll \mu$ by the same argument). Let ϕ_+, ϕ_- be the respective Radon–Nikodym derivatives. Then by setting $\phi = \phi_+ - \phi_- \in L^1(\mu)$ one gets

$$\forall A \in \mathcal{A} : \lambda(\mathbb{1}_A) = \rho(A) = \int \mathbb{1}_A \phi \, d\mu.$$

Define $\lambda_\phi(f) := \int \phi f \, d\mu$ for all $f \in L^1(\mu)$. Then by linearity of λ and the integral for every $\lambda \in (L^1(\mu))'$ there exists $\phi \in L^1(\mu)$ such that $\lambda(f) = \lambda_\phi(f)$ for all simple functions f. By continuity of λ and the integral it follows that $\lambda = \lambda_\phi$ on all of $L^1(\mu)$. It remains to show that $\phi \in L^\infty(\mu)$. Notice that, for any simple function f,

$$\left| \int f\phi \, d\mu \right| = |\lambda(f)| \leq M \cdot \|f\|_{L^1}.$$

By taking $f := \operatorname{sgn}(\phi)\mathbb{1}_{\{|\phi| \geq c\}} \in L^1(\mu)$ one gets

$$c \cdot \mu(\{|\phi| \geq c\}) \leq \int_{\{|\phi| \geq c\}} |\phi| \, d\mu \leq M \cdot \mu(\{|\phi| \geq c\})$$

and thus $\mu(\{|\phi| \geq c\}) = 0$ for all $c > M$. Hence it must hold that $\|\phi\|_{L^\infty} \leq M$. Furthermore, the extremal case of Hölder's inequality yields

$$\|\phi\|_{L^\infty} = \sup\{\lambda_\phi(f) : \|f\|_{L^1} \leq 1\} = \sup\{\lambda(f) : \|f\|_{L^1} \leq 1\} = \|\lambda\|_{((L^1(\mu))'},$$

so the map $\lambda \mapsto \phi$ is an isometry. Finally, $\lambda_\phi(f) \equiv 0$ implies $\phi = 0$ and hence this map is one-to-one. By linearity and continuity of the integral operator it is also onto and thus defines a bijective isometry between $((L^1(\mu))', \|\cdot\|_{((L^1(\mu))'})$ and $(L^\infty(\mu), \|\cdot\|_{L^\infty})$. □

Lemma 4.5. *Let $(X, \|\cdot\|_X)$ be a normed vector space. Then the weak-star topology τ^* on X' is the coarsest topology such that all maps*

$$T_x : X' \ni \lambda \mapsto \lambda(x) \in \mathbb{R}, x \in X$$

are continuous from (X', τ^*) to $\mathbb{R}$.

Proof. See e.g. the first Definition in Section 3.4 in [1]. □

Corollary 4.6. *Let $(\mathcal{X}, \mathcal{A}, \mu)$ be a σ-finite measure space. Then the maps*

$$L^\infty(\mu) \ni \phi \mapsto \lambda_f(\phi) = \int \phi f \, d\mu \in \mathbb{R}$$

are weak-star continuous for every $f \in L^1(\mu)$ by Lemma (4.5).

Lemma 4.7. *Let $(X, \|\cdot\|_X)$ be a normed vector space and $(\phi_n)_{n\geq 1} \subset X'$. Then*

$$\phi_n \xrightarrow[n\to\infty]{w^*} \phi \in X' \iff \forall x \in X : \phi_n(x) \xrightarrow[n\to\infty]{} \phi(x).$$

Proof. See e.g. Proposition 3.13 in [1]. □

In particular, for a σ-finite measure space $(\mathcal{X}, \mathcal{A}, \mu)$ and $(\phi_n)_{n\geq 1} \subset L^\infty(\mu)$ one gets $\phi_n \xrightarrow{w^*} \phi \iff \forall f \in L^1(\mu) : \int f\phi_n \, d\mu \longrightarrow \int f\phi \, d\mu$.

Definition 4.8. *Let (X, τ) be a topological space and $E \subset X$. We say that E is* compact *with respect to τ, if every τ-open cover of E has a finite subcover. We say that E is* sequentially compact *with respect to τ, if every sequence $(x_n)_{n\geq 1} \subset E$ has a subsequence that converges with respect to τ.*

If (X, τ) is metrizable, then any set E is compact if and only if it is sequentially compact.

Theorem 4.9 (Banach–Alaoglu–Bourbaki)**.** *Let X be a Banach space. Then $B' = \{\lambda \in X' : \|\lambda\|_{X'} \leq 1\}$ is weak-star compact in X'.*

Proof. See e.g. Theorem 3.16 in [1]. □

Corollary 4.10. *Let X be a separable Banach space. Then B' is weak-star sequentially compact in X'.*

Proof. By Theorem 3.28 in [1], B' is metrizable in the weak-star topology on X'. By Theorem 4.9 B' is weak-star compact and by metrizability compactness is equivalent to sequential compactness. □

Lemma 4.11. *Let $(X, \|\cdot\|_X)$ be a Banach space with continuous dual $(X', \|\cdot\|_{X'})$ and let $(\lambda_n)_{n\geq 1} \subset X'$ be a sequence such that $\lambda_n \xrightarrow[n\to\infty]{w^*} \lambda \in X'$. Then*

$$\|\lambda\|_{X'} \leq \liminf_{n\to\infty} \|\lambda_n\|_{X'}.$$

Proof. Let $\varepsilon > 0$. Then by definition of $\|\cdot\|_{X'}$ there exists $x_\varepsilon \in X$ with $\|x_\varepsilon\|_X \leq 1$ such that

$$|\lambda(x_\varepsilon)| + \varepsilon > \|\lambda\|_{X'}.$$

By Lemma 4.7 it holds that $\lambda_n(x) \to \lambda(x)$ for all $x \in X$ and hence $|\lambda_n(x_\varepsilon)| \to |\lambda(x_\varepsilon)|$. Therefore $\exists N_{x,\varepsilon} : \forall n \geq N_{x,\varepsilon} : \big||\lambda_n(x_\varepsilon)| - |\lambda(x_\varepsilon)|\big| < \varepsilon$ and

$$\forall n \geq N_{x,\varepsilon} : |\lambda_n(x_\varepsilon)| + 2\varepsilon > \|\lambda\|_{X'}.$$

Hence for all $\varepsilon > 0$ it holds that

$$\liminf_{n\to\infty} \|\lambda_n\|_{X'} + 2\varepsilon \geq \liminf_{n\to\infty} |\lambda_n(x_\varepsilon)| + 2\varepsilon \geq \|\lambda\|_{X'}$$

and the result follows for $\varepsilon \searrow 0$. □

Chapter 5

Neyman–Pearson with Multiple Constraints

Lemma 5.1. *Let $(\mathcal{X}, \mathcal{A}, \mu)$ be a finite measure space and let $\mathcal{A} = \sigma(\mathcal{M})$ for some algebra of sets $\mathcal{M} \subset 2^{\mathcal{X}}$. Then*

$$\forall \varepsilon > 0, \forall A \in \mathcal{A} : \exists M \in \mathcal{M} : \mu(A \triangle M) < \varepsilon.$$

Proof. Let

$$\mathcal{E} := \left\{ A \in \mathcal{A} \;\middle|\; \forall \varepsilon > 0, \exists M \in \mathcal{M} : \mu(A \triangle M) < \varepsilon \right\} \supset \mathcal{M}.$$

We aim to show that $\mathcal{E}$ is a σ-algebra.

- Since $\mathcal{M}$ is an algebra, one has $\mathcal{X} \in \mathcal{M}$ and hence $\mathcal{X} \in \mathcal{E}$.
- Let $A \in \mathcal{E}$ and $\varepsilon > 0$. Let $M \in \mathcal{M}$ be such that $\mu(A \triangle M) < \varepsilon$. Then

  $$\mu(A^c \triangle M^c) = \mu(A \triangle M) < \varepsilon$$

  and since $M^c \in \mathcal{M}$, one gets $A^c \in \mathcal{E}$.
- We first show that $\mathcal{E}$ is closed under taking finite unions. Let $A_1, A_2 \in \mathcal{E}$, $\varepsilon > 0$ and $M_1, M_2 \in \mathcal{M}$ such that $\mu(A_1 \triangle M_1), \mu(A_2 \triangle M_2) < \varepsilon/2$. Then

  $$\begin{aligned} \mu\big((A_1 \cup A_2) \triangle (M_1 \cup M_2)\big) &\leq \mu\left((A_1 \triangle M_1) \cup (A_2 \triangle M_2)\right) \\ &\leq \mu(A_1 \triangle M_1) + \mu(A_2 \triangle M_2) < \varepsilon, \end{aligned}$$

  where $M_1 \cup M_2 \in \mathcal{M}$. Hence $A_1 \cup A_2 \in \mathcal{E}$. Now let $A_1, A_2, \ldots \in \mathcal{E}$ be disjoint and $\varepsilon > 0$. Let $M_1, M_2, \ldots \in \mathcal{M}$ be such that $\mu(A_n \triangle M_n) < 2^{-(n+1)} \cdot \varepsilon$ for all $n \geq 1$. Now let $N \geq 1$ be such that $\mu\left(\bigcup_{n>N} A_n\right) < \varepsilon/2$. Now

  $$\mu\left(\left(\bigcup_{n\geq 1} A_n\right) \triangle \left(\bigcup_{n=1}^{N} M_n\right)\right) \leq \mu\left(\bigcup_{n=1}^{N} (A_n \triangle M_n)\right) + \mu\left(\bigcup_{n>N} A_n\right) < \varepsilon,$$

where the first inequality follows from the laws of set operations and subadditivity.

Hence $\mathcal{E} \subset \mathcal{A}$ is a σ-algebra and $\mathcal{A} = \sigma(\mathcal{M}) \subset \sigma(\mathcal{E}) = \mathcal{E}$. Thus $\mathcal{A} = \mathcal{E}$, as was to be shown. $\square$

Remark 5.2. *If $\mathcal{M} \subset 2^{\mathcal{X}}$ is an arbitrary countable family of sets, then the algebra generated by $\mathcal{M}$, given by*

$$\sigma_0(\mathcal{M}) := \bigcap_{\substack{\mathcal{D} \text{ algebra} \\ \mathcal{M} \subseteq \mathcal{D}}} \mathcal{D} = \bigcup_{n \geq 1} \bigcup_{m_i \geq 1} \left\{ M \subset \mathcal{X} \;\middle|\; M = \bigcup_{i=1}^{n} \bigcap_{j=1}^{m_i} M_{ij}, M_{ij} \in \mathcal{M} \text{ or } M_{ij}^c \in \mathcal{M} \right\},$$

is again countable.

Lemma 5.3. *Let $(\mathcal{X}, \mathcal{A}, \mu)$ be a σ-finite measure space and assume that $\mathcal{A}$ is countably generated, i.e. there exists a countable family $\mathcal{M} \subset 2^{\mathcal{X}}$ such that $\mathcal{A} = \sigma(\mathcal{M})$. Then $L^p(\mathcal{X}, \mathcal{A}, \mu)$ is separable for all $p \in [1, \infty)$.*

Proof. By Lemma 4.3 we can assume w.l.o.g. that μ is finite. Let $f \in L^p(\mathcal{X}, \mathcal{A}, \mu)$ and let $\varepsilon > 0$. By construction of the integral there exists a simple function $g = \sum_{i=1}^n \alpha_i \mathbb{1}_{A_i}$, with some $\alpha_i \in \mathbb{R}, A_i \in \mathcal{A}$, such that $\|f - g\|_{L^p} < \varepsilon$. Let $q_i \in \mathbb{Q}, i = 1, \ldots, m$ be such that

$$|\alpha_i - q_i| < \frac{\varepsilon}{2n(\mu(\mathcal{X}))^{1/p}}.$$

By Lemma 5.1 and Remark 5.2 there exist $M_i \in \sigma_0(\mathcal{M}), i = 1, \ldots, n$ such that

$$\mu(M_i \triangle A_i) < \left(\frac{\varepsilon}{2n|q_i|} \right)^p.$$

By setting $h := \sum_{i=1}^n q_i \mathbb{1}_{M_i}$ we get

$$\begin{aligned}
\|f - h\|_{L^p} &\leq \|f - g\|_{L^p} + \|g - h\|_{L^p} \\
&< \varepsilon + \left\| \sum_{i=1}^n \alpha_i \mathbb{1}_{A_i} - q_i \mathbb{1}_{M_i} \right\|_{L^p} \\
&\leq \varepsilon + \sum_{i=1}^n \left\| (\alpha_i - q_i) \mathbb{1}_{A_i} \right\|_{L^p} + \left\| q_i (\mathbb{1}_{A_i} - \mathbb{1}_{M_i}) \right\|_{L^p} \\
&\leq \varepsilon + \sum_{i=1}^n |\alpha_i - q_i| (\mu(\mathcal{X}))^{1/p} + |q_i| (\mu(A_i \triangle M_i))^{1/p} < 2\varepsilon.
\end{aligned}$$

The third inequality follows by adding and subtracting $q_i \mathbb{1}_{A_i}$ in the sum and from the triangle inequality. The fourth inequality follows from $\|\mathbb{1}_{A_i}\|_{L^p}^p = \mu(A_i) \leq \mu(\mathcal{X})$ for $i = 1, \ldots, n$. Setting

$$\mathcal{D} := \bigcup_{n\geq 1} \bigcup_{q\in\mathbb{Q}^n} \bigcup_{(M_1,\ldots,M_n)\in(\sigma_0(\mathcal{M}))^n} \left\{ f \;\middle|\; f = \sum_{i=1}^{n} q_i \mathbb{1}_{M_i} \right\},$$

which is countable, the result follows. □

Lehmann and Romano give the following theorem in [4, Chapter 3.6].

Theorem 5.4. *Consider a σ-finite measure space $(\mathcal{X}, \mathcal{A}, \mu)$, where $\mathcal{A}$ is countably generated. Let $\phi_0 : \mathcal{X} \to [0,1]$ be a test function and $f_1, \ldots, f_m, f_{m+1} \in L^1(\mu)$. For $i = 1, \ldots, m$ set*

$$c_i := \int \phi_0 f_i \; d\mu \in \mathbb{R}$$

and define

$$\mathcal{C} := \left\{ \phi : \mathcal{X} \to [0,1] \;\middle|\; \int \phi f_i \; d\mu = c_i \text{ for } i = 1, \ldots, m \right\}.$$

(i) There exists $\phi^* \in \mathcal{C}$ with

$$\int \phi^* f_{m+1} \; d\mu \geq \int \phi f_{m+1} \; d\mu \text{ for all } \phi \in \mathcal{C}.$$

(ii) Suppose $\phi^* \in \mathcal{C}$ is of the form

$$\phi^*(x) = \begin{cases} 1 & \text{if } f_{m+1}(x) > \sum_{i=1}^{m} k_i f_i(x), \\ 0 & \text{if } f_{m+1}(x) < \sum_{i=1}^{m} k_i f_i(x) \end{cases} \tag{5.1}$$

for some $k_1, \ldots, k_m \in \mathbb{R}$. Then $\phi^* \in \operatorname{argmax}_{\phi\in\mathcal{C}} \int \phi f_{m+1} \; d\mu$.

(iii) Suppose $\phi^* \in \mathcal{C}$ is of the form in equation (5.1) with $k_1, \ldots, k_m \geq 0$. Then $\phi^* \in \operatorname{argmax}_{\phi\in\tilde{\mathcal{C}}} \int \phi f_{m+1} \; d\mu$, where

$$\tilde{\mathcal{C}} := \left\{ \phi : \mathcal{X} \to [0,1] \;\middle|\; \int \phi f_i \; d\mu \leq c_i \text{ for } i = 1, \ldots, m \right\} \supset \mathcal{C}.$$

Proof. (i) Let $B = \{\phi \in L^\infty(\mu) : \|\phi\|_{L^\infty} \leq 1\} \supset \mathcal{C}$, which is weak-star sequentially compact in $L^\infty(\mu)$ by Corollary 4.10. If we can show that $\mathcal{C}$ is weak-star sequentially closed in B, then $\mathcal{C}$ is also weak-star sequentially compact. Notice that $\mathcal{C} \neq \emptyset$ by construction. Let $(\phi_n)_{n\geq 1} \subset \mathcal{C}$ such that $\phi_n \xrightarrow[n\to\infty]{w^*} \phi$. By Lemma 4.11

$$\|\phi\|_{L^\infty} \leq \liminf_{n\to\infty} \|\phi_n\|_{L^\infty} \leq 1,$$

so $\phi \in [-1, 1]$ μ-a.e. By Lemma 4.3 we can assume w.l.o.g. that μ is finite such that $\mathbb{1}_{\{\phi<0\}} \in L^1(\mu)$ and thus

$$\int\limits_{\{\phi<0\}} \phi_n \, d\mu \longrightarrow \int\limits_{\{\phi<0\}} \phi \, d\mu \leq 0.$$

By non-negativity of ϕ_n for all $n \geq 1$ it follows that

$$0 \geq \int\limits_{\{\phi<0\}} \phi \, d\mu = \limsup_{n\to\infty} \int\limits_{\{\phi<0\}} \phi_n \, d\mu \geq 0,$$

so we must have $\int_{\{\phi<0\}} \phi \, d\mu = 0$ and hence $\mu(\{\phi < 0\}) = 0$. Furthermore,

$$\int \phi f_i \, d\mu = \lim_{n\to\infty} \int \phi_n f_i \, d\mu = c_i \text{ for } i = 1, \ldots, m$$

by the fact that $f_1, \ldots, f_m \in L^1(\mu)$. Now define

$$\phi'(x) := \phi(x) \cdot \mathbb{1}_{\{x\in\mathcal{X}:\phi(x)\in[0,1]\}}$$

such that $\phi_n \xrightarrow[n\to\infty]{w^*} \phi' \in \mathcal{C}$ and $\mu(\{\phi \neq \phi'\}) = 0$. This shows that $\mathcal{C}$ is weak-star sequentially closed. Since $\mathcal{C}$ is non-empty,

$$-\infty < s := \sup_{\phi\in\mathcal{C}} \int \phi f_{m+1} \, d\mu \leq \|f_{m+1}\|_{L^1} < \infty$$

and hence for every $n \geq 1$ there must exist $\phi_n \in \mathcal{C}$ with

$$s - 1/n < \int \phi_n f_{m+1} \, d\mu \leq s.$$

This yields a sequence $\phi_n \in \mathcal{C}, n \geq 1$ with

$$\int \phi_n f_{m+1} \, d\mu \to \sup_{\phi\in\mathcal{C}} \int \phi f_{m+1} \, d\mu.$$

By weak-star sequential compactness of $\mathcal{C}$ there exists a subsequence $(\phi_{n_k})_{k\geq 1} \subset \mathcal{C}$ that converges in weak-star to a limit $\phi^* \in \mathcal{C}$, so in particular

$$\begin{aligned}\int \phi^* f_{m+1} \, d\mu &= \lim_{k\to\infty} \int \phi_{n_k} f_{m+1} \, d\mu \\ &= \lim_{n\to\infty} \int \phi_n f_{m+1} \, d\mu \\ &= \sup_{\phi\in\mathcal{C}} \int \phi f_{m+1} \, d\mu.\end{aligned}$$

(ii) Let $\phi^* \in \mathcal{C}$ be as in (5.1) and let $\phi \in \mathcal{C}$. Then

$$\int (\phi^* - \phi) f_i \, d\mu = 0, \text{ for } i = 1, \ldots, m.$$

Denote $S_\pm := \{\phi^* \gtrless \phi\}$ such that (cf. Proof of Theorem 2.1)

$$x \in S_+ \implies f_{m+1}(x) \geq \sum_{i=1}^m k_i f_i(x),$$
$$x \in S_- \implies f_{m+1}(x) \leq \sum_{i=1}^m k_i f_i(x).$$

By the convention $0 \cdot \infty = 0$ one gets

$$\begin{aligned}
\int (\phi^* - \phi) f_{m+1} \, d\mu &= \int (\phi^* - \phi) f_{m+1} \, d\mu - \sum_{i=1}^m k_i \int (\phi^* - \phi) f_i \, d\mu \\
&= \int (\phi^* - \phi) \left(f_{m+1} - \sum_{i=1}^m k_i f_i \right) \, d\mu \\
&= \int_{S_+ \cup S_-} (\phi^* - \phi) \left(f_{m+1} - \sum_{i=1}^m k_i f_i \right) \, d\mu \geq 0.
\end{aligned}$$

(iii) Similarly to (ii) one has

$$\int (\phi^* - \phi) f_i \, d\mu \geq 0, \text{ for } i = 1, \ldots, m$$

and from $k_1, \ldots, k_m \geq 0$ it follows that

$$\begin{aligned}
\int (\phi^* - \phi) f_{m+1} \, d\mu &\geq \int (\phi^* - \phi) f_{m+1} \, d\mu - \sum_{i=1}^m k_i \int (\phi^* - \phi) f_i \, d\mu \\
&= \int_{S_+ \cup S_-} (\phi^* - \phi) \left(f_{m+1} - \sum_{i=1}^m k_i f_i \right) \, d\mu \geq 0.
\end{aligned}$$

$\square$

Remark 5.5. *One can drop the assumption on $\mathcal{A}$ being countably generated. By Theorem 4.9 $\mathcal{C}$ is weak-star compact in $L^\infty(\mu)$. By Corollary 4.6 the map $\lambda_{f_{m+1}} : \mathcal{C} \to \mathbb{R}$ with $\phi \mapsto \int \phi f_{m+1} \, d\mu$ is weak-star continuous and hence achieves a finite minimum and maximum on $\mathcal{C}$.*

Proposition 5.6. *Suppose the assumptions of Theorem 5.4 hold and let*

$$M := \left\{ u \in \mathbb{R}^m \;\middle|\; \exists \phi : \mathcal{X} \to [0,1] : u_i = \int \phi f_i \, d\mu \text{ for } i = 1, \dots, m \right\}.$$

Then

(i) M is closed and convex.

(ii) If $c \in \operatorname{relint}(M)$, then there exists $\phi^{**} \in \mathcal{C}$ satisfying (5.1) for some $k_1, \dots, k_m$.

(iii) For any $\phi' \in \mathcal{C}$,

$$\int \phi' f_{m+1} \, d\mu = \sup_{\phi \in \mathcal{C}} \int \phi f_{m+1} \, d\mu \implies \phi' \text{ satisfies (5.1) } \mu\text{-a.e. for } k_1, \dots, k_m.$$

Proof. (i) In $\mathbb{R}^m$ closedness is equivalent to sequential closedness. It is easily seen that $\lambda(\|f_1\|_{L^1}, \dots, \|f_m\|_{L^1}) \in M$ for all $\lambda \in [0,1]$, so in particular $M \neq \emptyset$. Hence let $(u_n)_{n \geq 1} \subset M$ be a convergent sequence in M. Then, if $u = (u^{(1)}, \dots, u^{(m)}) = \lim_{n \to \infty} u_n$, one has

$$u^{(i)} = \lim_{n \to \infty} \int \phi_n f_i \, d\mu$$

for some sequence $(\phi_n)_{n \geq 1} \subset \mathcal{C}$. By weak-star sequential compactness of $\mathcal{C}$ (shown in Theorem 5.4 (i)) there exists a convergent subsequence $(\phi_{n_k})_{k \geq 1} \subset \mathcal{C}$ with limit $\phi_\infty \in \mathcal{C}$ and in particular

$$\int \phi_\infty f_i \, d\mu = \lim_{k \to \infty} \int \phi_{n_k} f_i \, d\mu = u^{(i)}$$

such that $u \in M$. Convexity of M is easily seen by linearity of the integral and the fact that for $\lambda \in [0,1]$ and test functions ϕ, ψ the function $\lambda\phi + (1-\lambda)\psi$ is again a test function.

We now prove (ii) and (iii). In the following let d denote the Euclidean distance on $\mathbb{R}^m$ or $\mathbb{R}^{m+1}$ and let $U_r(u)$ denote the open ball w.r.t. d with radius r around a point u in $\mathbb{R}^m$ or $\mathbb{R}^{m+1}$. By the same reasoning as in (i) the set

$$N := \left\{ u \in \mathbb{R}^{m+1} \;\middle|\; \exists \phi : \mathcal{X} \to [0,1] : u_i = \int \phi f_i \, d\mu \text{ for } i = 1, \dots, m+1 \right\}$$

is closed and convex and $(u_1, \dots, u_{m+1}) \in N \implies (u_1, \dots, u_m) \in M$. Now let $(c_1, \dots, c_m) \in \operatorname{relint}(M)$ and denote $I := \{t \in \mathbb{R} : (c_1, \dots, c_m, t) \in N\}$. This set is closed and convex in $\mathbb{R}$ and hence a closed interval, i.e. $I = [c^*_{m+1}, c^{**}_{m+1}]$. Distinguish the following two cases.

$\boxed{\text{First case: } c^*_{m+1} < c^{**}_{m+1}}$ Notice that $c^{**} = (c_1, \ldots, c_m, c^{**}_{m+1})$ must be a boundary point of N, so there exists a hyperplane supporting N in c^{**}. That is, there exists $k \in \mathbb{R}^{m+1}$ such that for the hyperplane

$$H = \left\{ v \in \mathbb{R}^{m+1} \;\middle|\; \langle v, k \rangle = \langle c^{**}, k \rangle \right\}$$

one has $\langle u, k \rangle \geq \langle c^{**}, k \rangle$ for all $u \in N$. We first show that $(c_1, \ldots, c_m, c_{m+1}) \in \operatorname{relint}(N)$ for all $c_{m+1} \in (c^*_{m+1}, c^{**}_{m+1})$. First notice that $0 \in M \subset \operatorname{aff}(M)$, so $\operatorname{aff}(M) = \operatorname{span}(M)$. Let $r_1, r_2 > 0$ be such that

$$U_{r_1}(c) \cap \operatorname{span}(M) \subset M,$$
$$(c_{m+1} - r_2, c_{m+1} + r_2) \subset [c^*_{m+1}, c^{**}_{m+1}].$$

Set $r := \frac{\min(r_1, r_2)}{\sqrt{m+1}}$ and let

$$z := (z_1, \ldots, z_m, z_{m+1}) \in \operatorname{span}(N) \cap U_r\big((c_1, \ldots, c_{m+1})\big).$$

Then it is easy to check that $(z_1, \ldots, z_m) \in \operatorname{span}(M)$ and $z_{m+1} \in [c^*_{m+1}, c^{**}_{m+1}]$. Also

$$d\big((z_1, \ldots, z_m), c\big) \leq d\big(z, (c_1, \ldots, c_{m+1})\big) < r \leq r_1,$$
$$|z_{m+1} - c_{m+1}| \leq \sum_{i=1}^{m+1} |z_i - c_i| \leq \sqrt{m+1} \cdot d\big(z, (c_1, \ldots, c_{m+1})\big) < r_2,$$

so $(z_1, \ldots, z_m) \in M$ and $z_{m+1} \in [c^*_{m+1}, c^{**}_{m+1}]$ and hence $z \in N$. This shows that $(c_1, \ldots, c_{m+1}) \in \operatorname{relint}(N)$ and hence $(c_1, \ldots, c_{m+1}) \notin H$. But, since $c^{**} \in H$, it must hold that $k_{m+1} \neq 0$ and by scaling the hyperplane normal vector accordingly one can take $k_{m+1} = -1$. This yields

$$u_{m+1} - \sum_{i=1}^m k_i u_i \leq c^{**}_{m+1} - \sum_{i=1}^m k_i c_i \text{ for all } u \in N.$$

Since $c^{**} \in N$, there must exist a test function ϕ^{**} such that $c_i = \int \phi^{**} f_i \, d\mu$ for $i = 1, \ldots, m$ and $c^{**}_{m+1} = \int \phi^{**} f_{m+1} \, d\mu$. By the above inequality

$$\int \phi \left(f_{m+1} - \sum_{i=1}^m k_i f_i \right) \, d\mu \leq \int \phi^{**} \left(f_{m+1} - \sum_{i=1}^m k_i f_i \right) \, d\mu$$

for all test functions ϕ. Now notice that the integral

$$\int \phi \left(f_{m+1} - \sum_{i=1}^m k_i f_i \right) \, d\mu$$

is trivially maximized by any test function ϕ satisfying

$$\phi(x) = \begin{cases} 1 & \text{if } f_{m+1}(x) > \sum_{i=1}^{m} k_i f_i(x), \\ 0 & \text{if } f_{m+1}(x) < \sum_{i=1}^{m} k_i f_i(x), \end{cases}$$

so

$$\int (\phi^{**} - \phi) \left(f_{m+1} - \sum_{i=1}^{m} k_i f_i \right) \, d\mu = 0.$$

The fact that $\phi^{**} = \phi$ on the set $\{f_{m+1} \neq \sum_{i=1}^{m} k_i f_i\}$ now follows exactly as in the proof of Theorem 2.1 (iii). This shows (ii) and (iii).

$\boxed{\text{Second case: } c^*_{m+1} = c^{**}_{m+1}}$ Let $M \ni c' = (c'_1, \ldots, c'_m) \neq c$. If we can show that there exists a unique value c'_{m+1} such that $(c'_1, \ldots, c'_m, c'_{m+1}) \in N$, then, by the fact that N is convex and $0 \in N$, it follows that $N \subset H'$ for a hyperplane

$$H' = \left\{ u \in \mathbb{R}^{m+1} \;\middle|\; \langle u, k' \rangle = 0 \right\},$$

where $k' \in \mathbb{R}^{m+1}$. In particular this hyperplane cannot be parallel to the vector $(0, \ldots, 0, 1) \in \mathbb{R}^{m+1}$ and so $k'_{m+1} \neq 0$ and one can again set $k'_{m+1} = -1$. Hence for all test functions ϕ,

$$\int \phi \left(f_{m+1} - \sum_{i=1}^{m} k'_i f_i \right) \, d\mu = 0.$$

By taking $S_\pm := \{f_{m+1} \gtrless \sum_{i=1}^{m} k'_i f_i\}$ and setting $\phi_\pm := \mathbb{1}_{S_\pm}$ it follows from Lemma 1.9 that $f_{m+1} = \sum_{i=1}^{m} k_i f_i$ μ-a.e. and both (ii) and (iii) hold vacuously. Hence it suffices to show that for each $c' \in M$ there is a unique c'_{m+1} such that $(c'_1, \ldots, c'_m, c'_{m+1}) \in N$. Suppose $(c'_1, \ldots, c'_m, \overline{c}_{m+1}), (c'_1, \ldots, c'_m, \underline{c}_{m+1}) \in N$ with $\overline{c}_{m+1} \neq \underline{c}_{m+1}$. Let $\varepsilon > 0$ be such that $U_\varepsilon(c) \cap \operatorname{span}(M) \subset M$ and set

$$c'' = (c''_1, \ldots, c''_m) := c + \frac{\varepsilon}{2\|c - c'\|} \cdot (c - c').$$

Then $c'' \in M$ and

$$c = \left(\frac{\varepsilon}{2\|c - c'\| + \varepsilon} \right) \cdot c' + \left(\frac{2\|c - c'\|}{2\|c - c'\| + \varepsilon} \right) \cdot c'',$$

so in particular $c \in \operatorname{int}(\operatorname{conv}(c', c''))$. Let c''_{m+1} be such that $(c''_1, \ldots, c''_m, c''_{m+1}) \in N$ (which must always exist). Then

$$\operatorname{conv}((c'_1, \ldots, c'_m, \overline{c}_{m+1}), (c'_1, \ldots, c'_m, \underline{c}_{m+1}), (c''_1, \ldots, c''_m, c''_{m+1})) \subset N.$$

In particular, the points

$$\big(c_1, \ldots, c_m, \lambda\overline{c}_{m+1} + (1-\lambda)c''_{m+1}\big) \neq \big(c_1, \ldots, c_m, \lambda\underline{c}_{m+1} + (1-\lambda)c''_{m+1}\big)$$

must both be in N for $\lambda := \frac{\varepsilon}{2\|c-c'\|+\varepsilon}$, which is a contradiction. $\square$

Remark 5.7. *Notice that* $c \in \operatorname{int}(M) \implies c \in \operatorname{relint}(M)$*.*

Chapter 6

Confidence Bounds

Suppose we want to find a lower bound $\theta_L = \theta_L(X)$ for the parameter $\theta \in \Theta \subset \mathbb{R}$, which solves the constrained optimization problem

$$\begin{aligned} \text{minimize } & \mathbb{P}_\theta(\theta_L(X) \leq \theta') \text{ for all } \theta' < \theta, \\ \text{s.t. } & \mathbb{P}_\theta(\theta_L(X) \leq \theta) \geq 1 - \alpha \text{ for all } \theta \in \Theta \end{aligned}$$

for a given confidence level $1 - \alpha \in [0, 1]$.

Definition 6.1. *A solution $\theta_L : \mathcal{X} \to \Theta$ to the above constrained optimization problem is called* uniformly most accurate lower confidence bound *at confidence level $1 - \alpha$.*

Definition 6.2. *Let $S : \mathcal{X} \to 2^\Theta$. S is a* family of confidence sets *at confidence level $1 - \alpha$, if the set $\{x \in \mathcal{X} : \theta \in S(x)\}$ is measurable and $\mathbb{P}_\theta(\theta \in S(X)) \geq 1 - \alpha$ for all $\theta \in \Theta$.*

Theorem 6.3.

(i) For $\theta_0 \in \Theta$ denote $A(\theta_0)$ the acceptance region of a size α, non-randomized test of $\theta = \theta_0$ vs. $\theta > \theta_0$. Set $S(x) := \{\theta \in \Theta : x \in A(\theta)\}$. Then S constitutes a family of confidence sets at confidence level $1 - \alpha$.

(ii) If the underlying test is UMP for $\theta = \theta_0$ vs. $\theta > \theta_0$ for all $\theta_0 \in \Theta$, then for each $\theta_0 \in \Theta$ S minimizes $\mathbb{P}_\theta(\theta_0 \in S(X))$ for all $\theta > \theta_0$ among all confidence sets at confidence level $1 - \alpha$.

Proof. (i) Notice that $\forall \theta \in \Theta : \big(\theta \in S(x) \iff x \in A(\theta)\big)$, so we get

$$\mathbb{P}_\theta(\theta \in S(X)) = 1 - \mathbb{P}_\theta(X \notin A(\theta)) \geq 1 - \alpha$$

by the fact that the underlying test is of size α.
(ii) Let S^* be any other family of confidence sets at confidence level $1-\alpha$ and set $A^*(\theta) := \{x \in \mathcal{X} : \theta \in S^*(x)\}$. Then

$$\mathbb{P}_{\theta_0}(X \in A^*(\theta_0)) = \mathbb{P}_{\theta_0}(\theta_0 \in S^*(X)) \geq 1-\alpha$$

and hence $A^*(\theta_0)$ defines the acceptance region of some size α test for $\theta = \theta_0$ vs. $\theta > \theta_0$. By the fact that the test corresponding to A is UMP we get

$$\mathbb{P}_\theta(X \in A^*(\theta_0)) \geq \mathbb{P}(X \in A(\theta_0)) \text{ for all } \theta > \theta_0.$$

□

Corollary 6.4. *Let $\Theta \subset \mathbb{R}$ and suppose $\mathcal{P} = \{\mathbb{P}_\theta : \theta \in \Theta\} \ll \mu$ with respective μ-density p_θ has monotone likelihood ratio in T. Assume that the cdf*

$$F_\theta(t) = \mathbb{P}_\theta(T(X) \leq t)$$

is continuous in t for all $\theta \in \Theta$ and continuous in θ for all $t \in \mathbb{R}$.

(i) There exists a uniformly most accurate lower confidence bound $\theta_L(X)$ for θ for each confidence level $1-\alpha \in [0,1]$.

(ii) Let $x \in \mathcal{X}$. If the equation $F_\theta(T(x)) = 1-\alpha$ has a solution $\theta^*(x)$ in θ, then this solution is unique and $\theta^*(x) = \theta_L(x)$.

Proof. (i) Let $\theta_0 \in \Theta$ be arbitrary and for $\alpha \in [0,1]$ define $c_\alpha : \Theta \to \mathbb{R}$ via

$$c_\alpha(\theta) := \inf\{t \in \mathbb{R} : 1 - F_\theta(t) \leq \alpha\}.$$

By continuity of F_θ in t one has $\mathbb{P}_{\theta_0}(T(X) > c_\alpha(\theta_0)) = \alpha$ and by Theorem 3.5 (ii) the test with rejection region $\{x \in \mathcal{X} : T(x) > c_\alpha(\theta_0)\}$ is UMP for $\theta = \theta_0$ vs. $\theta > \theta_0$ at size α.
Let $\theta_1 > \theta_0$. Then by construction

$$\alpha = \mathbb{P}_{\theta_0}(T(X) > c_\alpha(\theta_0)) = \mathbb{P}_{\theta_1}(T(X) > c_\alpha(\theta_1))$$

and by Corollary 2.2

$$\mathbb{P}_{\theta_1}(T(X) > c_\alpha(\theta_0)) > \mathbb{P}_{\theta_0}(T(X) > c_\alpha(\theta_0)).$$

Hence $\mathbb{P}_{\theta_1}(T(X) > c_\alpha(\theta_0)) > \mathbb{P}_{\theta_1}(T(X) > c_\alpha(\theta_1))$ and c_α is strictly increasing in $\theta > \theta_0$. For $x \in \mathcal{X}$ and $\theta \in \Theta$ set

$$\begin{aligned} A(\theta) &:= \{x \in \mathcal{X} : T(x) \leq c_\alpha(\theta)\}, \\ S(x) &:= \{\theta \in \Theta : x \in A(\theta)\} \end{aligned}$$

such that

$$x \in A(\theta) \iff \theta \in S(x) \iff T(x) \leq c(\theta).$$

It trivially holds that

$$\theta_0 < \inf\{\theta \in \Theta : T(x) \leq c_\alpha(\theta)\} \implies T(x) > c_\alpha(\theta_0),$$

so

$$T(x) \leq c_\alpha(\theta_0) \implies \theta_0 \geq \inf\{\theta \in \Theta : T(x) \leq c_\alpha(\theta)\}.$$

By the monotonicity of c_α

$$\theta_0 \geq \inf\{\theta \in \Theta : T(x) \leq c_\alpha(\theta)\} \implies T(x) \leq c_\alpha(\theta_0),$$

so set $\theta_L(x) := \inf\{\theta \in \Theta : T(x) \leq c_\alpha(\theta)\}$ such that $S(x) = \{\theta \in \Theta : \theta \geq \theta_L(x)\}$. Now $A(\theta_0)$ is an acceptance region of a size α test for $\theta = \theta_0$ vs. $\theta > \theta_0$ and by Theorem 6.3 (i) S is a family of confidence sets at confidence level $1 - \alpha$. By Theorem 3.5 (ii) the underlying test is UMP, so by Theorem 6.3 (ii) the family of confidence sets also minimizes $\mathbb{P}_\theta(\theta_0 \in S(X)) = \mathbb{P}_\theta(\theta_0 \geq \theta_L(X))$ for all $\theta > \theta_0$.
(ii) Let $\theta' < \theta$. As long as $\mathbb{P}_\theta(T(X) > t), \mathbb{P}_{\theta'}(T(X) > t) \in (0,1)$ it holds that

$$\begin{aligned} \mathbb{P}_\theta(T(X) > t) &= \mathbb{P}_\theta\big(g_{\theta',\theta}(T(X)) > g_{\theta',\theta}(t)\big) \\ &= \mathbb{P}_\theta\left(\frac{p_{\theta'}(X)}{p_\theta(X)} > g_{\theta',\theta}(t)\right) \\ &< \mathbb{P}_{\theta'}\left(\frac{p_{\theta'}(X)}{p_\theta(X)} > g_{\theta',\theta}(t)\right) \\ &\leq \mathbb{P}_{\theta'}\left(T(X) > t\right). \end{aligned}$$

The first equality and the last inequality follow from the monotonicity of $g_{\theta',\theta}$ and the strict inequality follows from Corollary 2.2. Hence $F_\theta(t)$ is strictly increasing in θ for all t as long as $F_\theta(t) \in (0,1)$ and the equation

$$F_\theta(t) = 1 - \alpha$$

can have at most one solution in θ. Let x be an observed value of X. If $\theta^*(x)$ is a solution to the above equation, then $\mathbb{P}_{\theta^*(x)}(T(X) > T(x)) = \alpha$ and by strict monotonicity of F_θ it must hold that $T(x) = c_\alpha(\theta^*(x))$. By monotonicity of c_α it holds that $\theta^*(x) \leq \theta_0 \iff c_\alpha(\theta^*(x)) \leq c_\alpha(\theta_0)$, so

$$\theta^*(x) \leq \theta_0 \iff T(x) \leq c_\alpha(\theta_0).$$

Using $T(x) \leq c_\alpha(\theta_0) \iff \theta_L(x) \leq \theta_0$ from above we see that

$$\theta^*(x) \leq \theta_0 \iff \theta_L(x) \leq \theta_0$$

and hence $\theta^*(x) = \theta_L(x)$. $\square$

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