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Marginally outer trapped surfaces in cosmological spacetimes: Examples and singularity theorems

Carl Rossdeutscher

Abstract

This thesis explores aspects of marginally outer trapped surfaces (MOTS) in cosmological spacetimes, i.e. spacetimes with compact Cauchy surfaces. We first prove an instability result for MOTS in a class of spacetimes containing a timelike conformal Killing field with a 'temporal function'. We then focus on de Sitter spacetime, where we establish an existence result for a family of marginally outer trapped tubes (MOTTs) with constant mean curvature sections of sufficiently high genus. In the second part, we extend a cosmological singularity result by Galloway and Ling in several respects. First, we relax their assumption on the Cauchy surface from being strictly 2-convex to 2-convex, now allowing for the important case of time-symmetric Cauchy surfaces, leading to a characterization of spacetimes whose Cauchy surfaces — or finite covers thereof — are surface bundles over S^1 with totally geodesic fibers. Next, in the presence of a $U(1)$ isometry, we further weaken the 2-convexity requirement. Finally, these results are strengthened in special cases where the Cauchy surface is Haken, non-prime, or nonorientable.

Zusammenfassung

Diese Dissertation untersucht Aspekte von marginal nach außen gefangenen Flächen (MOTS) in kosmologischen Raumzeiten, d.h. Raumzeiten mit kompakten Cauchy-Flächen. Zunächst beweisen wir ein Instabilitätsresultat für MOTS in einer Klasse von Raumzeiten, die ein zeitartiges konformes Killing-Vektorfeld mit einer 'Zeitfunktion' enthalten. Anschließend konzentrieren wir uns auf die de-Sitter-Raumzeit, wo wir ein Existenzresultat für eine Familie von marginal nach außen gefangenen Röhren (MOTTs) mit CMC-Schnitten von hinreichend hohem Genus etablieren. Im zweiten Teil erweitern wir ein kosmologisches Singularitätstheorem von Galloway und Ling in mehrerer Hinsicht. Zunächst schwächen wir ihre Annahme über die Cauchy-Fläche von streng 2-konvex auf 2-konvex ab, wodurch nun der wichtige Fall von zeitsymmetrischen Cauchy-Flächen zugelassen wird, was zu einer Charakterisierung von Raumzeiten führt, deren Cauchy-Flächen — oder endliche Überlagerungen davon — Flächenbündel über S^1 mit total geodätischen Fasern sind. Als nächstes schwächen wir in Anwesenheit einer $U(1)$-Isometrie die 2-Konvexitätsanforderung weiter ab. Schließlich werden diese Resultate in speziellen Fällen verstärkt, in denen die Cauchy-Fläche Haken, nicht-prim oder nicht-orientierbar ist.

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Introduction

Although the concept of a black hole — a region in spacetime from which not even light can escape — was first described as part of the Schwarzschild solution to Einstein's field equations in the early 20th century, the general mathematical framework for black holes was only developed by Roger Penrose in 1965. In his seminal paper *Gravitational collapse and space-time singularities* [1], Penrose introduces the notion of a so called trapped surface, which he defines as a 'closed, spacelike two-surface T^2 with the property that the two systems of null geodesics which meet T^2 orthogonally converge locally in future directions at T^2'. Intuitively, this means that if we emit light rays 'outward' from such a surface, the area of the surface decreases as we follow the light rays. In the borderline case where the area remains constant to first order, the surface is referred to as a marginally outer trapped surface (MOTS), where 'outer' refers to a specified outward direction.

In some sense, one can view MOTS therefore as the spacetime analogue to minimal surfaces, critical points of the area functional. This is especially apparent in time-symmetric slices where MOTS coincide with minimal surfaces. Furthermore, for many results on minimal surfaces, similar results exist for MOTS. For example, Eichmair famously proved a solution to the Plateau problem for MOTS [2]. However, this is only to say that there are some similarities, and many results used to study minimal surfaces in Riemannian geometry do not translate directly to a MOTS in Lorentzian geometry.

In the following decades after Penrose's groundbreaking work, (marginally) trapped surfaces played a fundamental role in mathematical relativity. From singularity theorems to spacetime positive mass theorems to the study of black hole dynamics and gravitational collapse, they became a cornerstone in understanding the geometry and physics of spacetime. Trapped surfaces not only provided key insights into the inevitability of singularities under certain conditions but also served as a crucial tool in understanding results like the Penrose inequality, the topology of black holes and quasi-local definitions of mass. Their importance extends to numerical relativity, where they are, for example, used to track black hole formation and evolution in simulations.

In this thesis, we focus on two key aspects of MOTS. We will go briefly into more detail on each of the included papers in the following.

First, we aim at investigating the time evolution of MOTS, with a particular focus on their behavior in de Sitter spacetime. Penrose's 'Weak Cosmic Censorship Conjecture' roughly states that singularities are hidden from distant observers by the event horizon of a black hole. However, since the definition of a black hole relies on knowledge of the entire future evolution of spacetime, it is more practical to replace this global concept with a quasi-local one that captures the essential features of a black hole and can be used to prove its existence. A standard approach, therefore, is to use MOTS, and more specifically their time evolution, as quasi-local replacements for black holes. The study of the time evolution of MOTS was particularly pioneered by Andersson, Mars, Metzger, and Simon in a series of influential papers (see e.g. [3] [4] [5]).

Notably, the concept of stability of MOTS was first introduced by Newman [6], who derived a linear elliptic stability operator for MOTS, analogous to the stability operator for minimal surfaces in Riemannian geometry. This definition of stability is equivalent to the definition by Andersson, Mars, and Simon [4] based on the sign of the principal eigenvalue, i.e., the eigenvalue with the smallest real part. Under a certain stability assumption it can be shown that MOTS present in an initial slice of a spacetime can be propagated to adjacent slices. In this case, they form a so-called Marginally Outer Trapped Tube (MOTT), which is a 3-surface foliated by MOTS. Moreover, in a foliated spacetime with suitable barriers on each leaf, there is always an outermost MOTT which is smooth and strictly stable except at finitely many "jump times". While stability often leads to well-defined and favorable consequences, instability tends to be more subtle and challenging to handle.

The first paper [7] included in this thesis focuses primarily on the (in)stability of MOTS and the construction of MOTTs in de Sitter-type spacetimes. By de Sitter-type spacetimes, we refer to spacetimes with compact Cauchy surfaces that satisfy the null energy condition and admit a future-directed timelike conformal Killing field as well as a temporal function. Of course, as the name suggests, de Sitter spacetime serves as the prime example of this class. In particular, we prove that spacetimes in this class do not admit stable MOTS. As a result, classical theorems that require (strictly) stable MOTS cannot be used to construct MOTTs. Nevertheless, it is still possible to construct MOTTs in de Sitter spacetime. Notably, by employing a spherical slicing of de Sitter space, one can explicitly construct spherical and toroidal MOTS. Furthermore, by leveraging a recent result [8] on foliations of high-genus constant mean curvature (CMC) surfaces on the 3-sphere, we are able to construct a MOTT with high-genus slices in de Sitter spacetime. The result in [8] employs the so-called DPW method to construct CMC surfaces from holomorphic potentials on Riemann surfaces by solving a monodromy problem. This is achieved using an implicit function theorem argument, where the parameter is inversely proportional to the genus. Consequently, the original result applies only to surfaces of 'sufficiently high' genus, as determined by the implicit function theorem. However, recently after the publication of [7], the same authors strengthened their result to include all genera ≥ 3. As a result, our construction extends to genus ≥ 3 as well, leaving genus 2 MOTTs as the only unresolved case.

The second aspect of MOTS investigated is their role in singularity theorems. The second paper [9] of this thesis focuses on this connection. Stability is not only relevant to the time evolution of MOTS; it can also be used to extend singularity theorems à la Penrose. Classical Penrose-type singularity theorems rely on the existence of trapped surfaces. Under certain stability assumptions, MOTS can be deformed into trapped surfaces, thereby providing a pathway to establish a singularity theorem (see e.g. [5]).

While Penrose-type singularity theorems usually assume that the underlying spacetime is spatially noncompact, Hawking-type singularity theorems assume spatial compactness. The classic Hawking theorem assumes the strong energy condition, but Galloway and Ling have demonstrated Hawking-type singularity theorems for $(3+1)$-dimensional spacetimes that satisfy the weaker null energy condition, under an assumption on the extrinsic curvature that guarantees the existence of a trapped surface [10]. They conclude then that the spacetime is incomplete or the Cauchy surface is a spherical space.

That there is a strong relation between singularities and the topology of spacetime is also captured in the famous results of Gannon and Lee under an asymptotic flatness assumption [11][12]. These results were later extended to higher dimensions by Costa e Silva, Minguzzi [13][14] and to low regularity by Schinnerl and Steinbauer [15]. Similarly, Galloway [16] used a topological assumption together with a condition on the extrinsic geometry of the Cauchy surface to ensure the existence of a closed trapped surface. This allows an application of the Penrose singularity theorem to conclude incompleteness. Meanwhile in [10], Galloway and Ling only assumed that the extrinsic curvature is strictly 2-convex (c.f. [17]), meaning that the sum of the two smallest eigenvalues of the extrinsic curvature tensor is positive, and made no additional assumptions on the closed Cauchy surface.

The main idea of Galloway and Ling's proof in [10] was to use the well-studied classification of closed 3-manifolds [18], as well as the positive resolution of the surface subgroup conjecture [19] and the classical results on the existence of minimal immersions by Schoen-Yau [20] and Sacks-Uhlenbeck [21]. We remark here that Greg Galloway recently pointed out a second, simpler proof of their theorem using the positive resolution of the virtual $b_1 > 0$ conjecture (see also Remark 5 in the second paper of this thesis).

A natural question to ask is whether one can prove a singularity theorem by weakening the assumptions on the extrinsic curvature. This is precisely the content of the second paper [9] of this thesis, where we assumed the extrinsic curvature to be only 2-convex, meaning the sum of the two smallest eigenvalues is also allowed to vanish, allowing the case of time-symmetric Cauchy surfaces. The first result we show under these assumptions in [9] is that the spacetime is past geodesically incomplete or the Cauchy surface is either a spherical space or finitely covered by a surface bundle over the circle with totally geodesic fibers. While the condition on the extrinsic curvature is still rather restrictive, excluding cases such as the family of so-called (t, ϕ)-symmetric data, we are able to relax the signature requirements for Cauchy surfaces invariant under the cyclic isometry group $U(1)$ in Theorem 2. Moreover, by assuming an additional topological condition, i.e., that the Cauchy surface is either a Haken manifold, or non-prime, or nonorientable, both main theorems can be strengthened (c.f. Props. 1-3 and Remark 4). These results are related to the Gannon-Lee type results mentioned earlier in that they both have additional topological assumptions rather than purely geometrical ones as Theorems 1 and 2.

The main idea behind our results is to find a suitable minimal surface with a neighborhood foliated by CMC surfaces that can be lifted to a noncompact cover. Assuming a geodesically complete spacetime and applying Penrose's singularity theorem, one can then conclude that the CMC surfaces are totally geodesic because of our assumption of 2-convexity. By applying a compactness theorem, the foliation can then be extended globally.

Perspectives of a singularity theorem in dimensions > 3

Let us conclude this introduction with a short note on the second paper in this thesis. As we have already discussed, the requirements of [10] and our paper are that the spacetime is $(3 + 1)$-dimensional. While this is a reasonable physical assumption one might ask the question of how this result extends to higher dimensions out of mathematical interest. Of course, the requirement that the Cauchy surface is a 3-manifold was not solely chosen because it is physically reasonable but also because it allows us to draw from the rich theory of 3-manifolds. While a general extension to higher dimensions does not currently exist, we highlight certain conditions, potential approaches, and challenges associated with such an extension at least for dimensions $\leq (7+1)$.

For simplicity, in the following we will only discuss an extension of Theorem 1 in [10]. Since the proof of the theorem relied heavily on the Penrose singularity theorem, a natural condition on the Cauchy surface is to have infinite fundamental group (which is why spherical Cauchy surfaces lead to no conclusions in dimensions 3).

Currently, no extension of the virtual $b_1 > 0$ conjecture exists for dimensions $n \geq 4$, nor is such an extension to be expected. However, it is known [22] that closed almost non-negatively curved manifolds have fundamental group that is almost nilpotent, meaning they contain a finite index nilpotent subgroup. Since such a subgroup has non-trivial abelianization, it follows that $b_1 \neq 0$.

Thus unsurprisingly, hyperbolic manifolds are once again the primary obstacle to extending the conjecture to higher dimensions. But not all hope is lost. That said, note that the condition of virtual $b_1 > 0$ is more restrictive than necessary for our purposes. Our arguments only require the existence of a closed minimal surface in some noncompact cover of the manifold.

Although this is still a rather complicated problem, in recent years there have been considerable advances toward answering the question of the existence of closed minimal surfaces in noncompact manifolds, in particular by Gromov [23] and Song [24], among others. Gromov calls manifolds that admit a strictly mean convex Morse function Plateau-Stein. He then shows that for manifolds 'thick at infinity' (a condition satisfied by noncompact covers of closed manifolds), this is equivalent to the fact that there exists no closed minimal hypersurface [23]. As a result, $\mathbb{H}^n$ for example does not contain any closed minimal hypersurface. Other examples of noncompact hyperbolic manifolds that do not contain a closed minimal hypersurface are certain types of Schottky manifolds (see in particular Lemma 4.1 in [25]) or the cover of a compact hyperbolic manifold M^n given by $\langle \gamma \rangle \subset \pi_1(M)$ for any $\gamma \in \pi_1(M)$. Nevertheless, we can provide conditions under which noncompact hyperbolic manifolds contain closed minimal hypersurfaces. A Plateau-Stein manifold with proper Morse function is connected at infinity (see [23]). Thus, a natural condition, though not a necessary one, would be to assume the existence of a cover of the manifold with more than one end. Anderson has already shown that if a hyperbolic manifold has a compact convex core and exactly two ends, then it contains a closed minimal hypersurface (see [26, Corollary 3.3]). Since a cover cannot have cusps, this result can be extended to hyperbolic manifolds with a compact convex core and at least two ends by adapting the ideas of Coskunuzer [25] on the existence of closed minimal surfaces in hyperbolic 3-manifolds. In particular, we use the fact that having three or more ends implies a non-trivial $(n-1)$-homology group of the compact convex core. Note that Coskunuzer is able to prove a much stronger result for 3-manifolds due to the topological tameness of hyperbolic 3-manifolds with finitely generated fundamental group.

The existence of such a cover is an open problem in dimensions greater than three. A sufficient condition for the existence of such a cover is having a convex, hyperbolic submanifold (in the sense of [27, Def. 8.3.1]) with two or more boundary components in the base manifold or a finite cover of it. This condition would be satisfied if there existed a totally geodesic hypersurface.

For geometrically infinite covers, it becomes even more challenging to establish conditions for the existence of closed minimal hypersurfaces, as the topological tameness of the ends cannot be utilized in dimensions greater than three.

Although these challenges highlight the complexity of extending topological singularity theorems beyond dimension 3+1, they also show strong connections between problems in differential geometry, topology, and general relativity that need to be further investigated.

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Marginally outer trapped tubes in de Sitter spacetime

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Abstract

We prove two results which are relevant for constructing marginally outer trapped tubes (MOTTs) in de Sitter spacetime. The first one (Theorem 1) holds more generally, namely for spacetimes satisfying the null convergence condition and containing a timelike conformal Killing vector with a "temporal function". We show that all marginally outer trapped surfaces (MOTSs) in such a spacetime are unstable. This prevents application of standard results on the propagation of stable MOTSs to MOTTs. On the other hand, it was shown recently, Charlton et al. (minimal surfaces and alternating multiple zetas, arXiv:2407.07130), that for every sufficiently high genus, there exists a smooth, complete family of CMC surfaces embedded in the round 3-sphere $\mathbb{S}^3$. This family connects a Lawson minimal surface with a doubly covered geodesic 2-sphere. We show (Theorem 2) by a simple scaling argument that this result translates to an existence proof for complete MOTTs with CMC sections in de Sitter spacetime. Moreover, the area of these sections increases strictly monotonically. We compare this result with an area law obtained before for holographic screens.

Keywords CMC surface · CMC foliation · de Sitter spacetime · Stability of marginally outer trapped surface

Mathematics Subject Classification 53C42 · 58J60 · 58J70 · 83C20

1 Introduction

Our general setting is a smooth n-dimensional ($n \geq 3$), oriented and time-oriented Hausdorff manifold $(\mathcal{M}, g)$ with a smooth metric of signature $(-, +, ...+)$. In Sect. 2, we require the Ricci tensor Ric_g to satisfy the null convergence condition $\mathrm{Ric}_g(\ell, \ell) \geq 0$ for all null vectors ℓ, while in Sect. 3, we restrict ourselves to de Sitter spacetime

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(dS). We first recall well-known key definitions to also fix our notation. Note that we use the "physics" convention throughout, in particular in Definition 1 where most mathematics references (e.g. [11]) define the mean curvature by $\widehat{H} = H/\dim \mathcal{U}$.

Definition 1 Let $\mathcal{U}$ be a hypersurface (i.e. a submanifold of codimension one) embedded in any semi-Riemannian manifold $\mathcal{N}$. The **mean curvature** H of $\mathcal{U}$ is the trace of its second fundamental form.

Definition 2 Let $\mathcal{F}$ be a spacelike surface of codimension two embedded in $\mathcal{M}$ with future directed, orthogonal null vectors $\ell^{\pm}$, and denote the null expansions of the corresponding families of emanating geodesics by $\theta^{\pm} = tr_{\mathcal{F}}(\nabla \ell^{\pm})$.

- A **Marginally outer trapped surface (MOTS)** or just **Marginal surface** $\mathcal{F}$ is defined by $\theta^{+} = 0$. Here, ℓ^{+} defines the "outer" direction; the latter is ill-defined for minimal surfaces for which $\theta^{-} = 0$ as well. (As to terminology cf. Remark 3 in Sect. 2).
- A **Marginally trapped surface (MTS)** is a MOTS with either $\theta^{-} < 0$ or $\theta^{-} > 0$ (we avoid the term "past trapped" or "antitrapped" for the latter case).
- A **Marginally outer trapped tube (MOTT)** $\mathcal{T}$ is a hypersurface foliated by MOTS.
- A **Marginally trapped tube (MTT)** or **holographic screen** is a hypersurface foliated by MTSs.

MOTS, MTS or trapped surfaces which satisfy $\theta^{+} < 0$ and $\theta^{-} < 0$ are key in singularity theorems which yield incomplete geodesics. However, in a typical collapse scenario, there is not only the (outermost) "key MOTS", but also a plethora of other, seemingly "spurious" MOTS (see e.g. [6, 22] and references therein). One motivation of this work is to better understand such spurious MOTS.

Apart from these collapse scenarios, MOTS are normally also abundant in non-singular spacetimes which do not satisfy all requirements for the singularity theorems. We mention two settings in which they are worth studying: Firstly, under additional assumptions, null geodesics emanating from MOTS have conjugate points while still being complete. This suggests a recent definition [21] of the "outward affine size" of a non-singular universe with a MOTS. Secondly, for asymptotically dS spacetimes, there are results on the "visibility" of future weakly trapped surfaces (defined by $\theta^{\pm} \leq 0$) from infinity [12].

A key property of MOTS is their (in)stability, cf. Definition 3 and Remark 2. Here, we recall important properties of strictly stable MOTSs: They have spherical topology, inherit all symmetries of the ambient space, and propagate along achronal MOTTs in a given spacelike foliation of spacetime [3, 4]. Somewhat weaker results hold in the non-strictly stable case. Moreover, stability also plays a role in a singularity theorem (cf. Theorem 7.1 and Remark 7.2 in [4]).

The present work has two somewhat independent aspects exposed in Sects. 2 and 3. In Sect. 2, we show the absence of stable MOTS in a class of spacetimes containing a timelike conformal Killing field with a "temporal function". The precise result is as follows:

Theorem 1 *Let $(\mathcal{M}, g)$ be an n-dimensional $(n \geq 3)$ spacetime satisfying the null convergence condition. Assume $(\mathcal{M}, g)$ admits a future directed timelike conformal Killing field ξ, i.e.*

$$\pounds_\xi g = 2\Psi g$$

where $\pounds_\xi$ is the Lie derivative and Ψ is a temporal function, i.e. its gradient is timelike and past directed. Then, $(\mathcal{M}, g)$ admits no stable MOTS.

In Sect. 3, we restrict ourselves to dS for which $\mathrm{Ric}_g = (n-1)\delta^2 g$, δ a positive constant. It can be globally written in the form

$$g_{\mathrm{dS}} = \frac{1}{\delta^2 \cos^2 \sigma}\left(-d\sigma^2 + g_{\mathbb{S}^{n-1}}\right) \tag{1}$$

where $\sigma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is taken to increase to the future, and $g_{\mathbb{S}^{n-1}}$ is the standard metric on the round unit sphere $\mathbb{S}^{n-1}$.

Choosing $\xi = \frac{\partial}{\partial\sigma}$ in Theorem 1, we show in Corollary 1 that all MOTSs in dS are unstable. Hence, the above-said results of [3, 4] on topology, symmetry and evolution to achronal MOTTs do not apply. To analyse MOTTs, we now restrict ourselves to 3+1 dimensions and focus on umbilic and constant curvature slicings of dS (with curvatures of all possible signs). It is easy to show (Lemma 2 in Sect. 3.2.1) that in this setting, MOTSs are automatically MTSs and (as hypersurfaces within the slices) surfaces of constant mean curvature (CMC). This allows, in particular, constructing a MTSs from certain families of CMC surfaces in either flat $\mathbb{R}^3$, the hyperbolic space $\mathbb{H}^3$ or the round $\mathbb{S}^3$. We consider the former two settings (in Sects. 3.2.2 and 3.2.3) as a "warmup" and focus on $\mathbb{S}^3$ (cf. Sect. 3.2.4).

In Sect. 3.2.4, we analyse, and then dS in terms of its "standard" umbilic slicing (1) by round three spheres $\mathbb{S}^3_\sigma$ of radius $\rho = \delta^{-1}\cos^{-1}\sigma$. In this slicing, MOTS with $\theta^\pm = 0$ correspond to CMC surfaces with mean curvature $H = \mp 2\delta \sin\sigma$, thus in particular minimal for $\sigma = 0$. The resulting MTSs are easily found explicitly in the case of spherical and toroidal MTS sections. In the former case, the MTSs are null surfaces with sections of constant area, while in the latter case, the MTSs are timelike and their area increases monotonically from the Clifford torus at $\sigma = 0$ to $\sigma \to \pm\frac{\pi}{2}$ (where it diverges).

On the other hand, knowledge of CMC surfaces of genus $\mathfrak{g}$ larger than one in $\mathbb{S}^3$ is rather rudimentary. However, for every $\mathfrak{g}$ sufficiently large ($\mathfrak{g} \gg 1$), Charlton et al. [11] recently proved existence of a complete family of CMC surfaces $f^{\mathfrak{g}}_\psi$, $\left(\psi \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)\right)$ embedded in the unit sphere $\mathbb{S}^3$. This family connects the well-known Lawson minimal surface $\xi_{1,\mathfrak{g}}$ (for $\psi = 0$) with a doubly covered geodesic sphere (for $\psi = \pm\frac{\pi}{4}$). Moreover, these authors show that the Willmore energy of these surfaces, defined by

$$W = A\left(\frac{H^2}{4} + 1\right), \tag{2}$$

where A is the area, increases strictly monotonically from the Lawson surfaces towards the limiting spheres.

After quoting and slightly extending the results of [11] as Theorem 3 in Sect. 3, we describe its straightforward application to dS. In fact, we only need to rescale the geometry from the $\mathbb{S}^3$ of unit radius to the spherical sections of dS whose radius blows up with the cosmological time σ as sketched above. Remarkably, this scaling entails that the Willmore energy of the CMC surfaces found in [11] translates just to the area of the MTS sections in the dS setting and yields the corresponding monotonicity statement. Thus, our adaptation of the results of [11] reads as follows.

Theorem 2 *Our setting is de Sitter spacetime in coordinates (1) for* $n = 4$*. For every* $\mathfrak{g} \in \mathbb{N}$, $\mathfrak{g} \gg 1$*, there exist a smooth function* $\sigma : (-\frac{\pi}{4}, \frac{\pi}{4}) \ni \psi \mapsto \sigma(\psi) \in (-\sigma_m, \sigma_m)$ *and a smooth family of conformal CMC embeddings* $\mathcal{F}^{\mathfrak{g}}_{\psi} : \mathcal{R}^{\mathfrak{g}} \to \mathbb{S}^3_{\sigma}$ *of a Riemann surface* $\mathcal{R}^{\mathfrak{g}}$ *of genus* $\mathfrak{g}$ *into the round 3-sphere* $\mathbb{S}^3_{\sigma}$ *of radius* $\rho_{\sigma} = \delta^{-1}\cos^{-1}\sigma$ *such that*

1. $\mathcal{F}^{\mathfrak{g}}_0$ *is the Lawson surface* $\xi_{1,\mathfrak{g}}$ *of genus* $\mathfrak{g}$*.*
2. *For* $\psi \to \pm\frac{\pi}{4}$, $\mathcal{F}^{\mathfrak{g}}_{\psi}$ *smoothly converges to a doubly covered geodesic 2-sphere with* $2\mathfrak{g}+2$ *branch points; the family* $\mathcal{F}^{\mathfrak{g}}_{\psi}$ *cannot be extended in* ψ *in the space of immersions.*
3. $\mathcal{F}^{\mathfrak{g}}_{\psi} = \mathcal{F}^{\mathfrak{g}}_{-\psi}$ *up to reparametrization and isometries of* $\mathbb{S}^3_{\sigma}$*, and accordingly the constant mean curvatures satisfy* $\mathcal{H}^{\mathfrak{g}}_{\psi} = -\mathcal{H}^{\mathfrak{g}}_{-\psi}$*.*
4. $\mathcal{H}^{\mathfrak{g}}_{\psi}$ *decreases strictly monotonically from zero at* $\psi = 0$ *to a minimum* $\mathcal{H}^{\mathfrak{g}}_{\psi_m} = -2\delta\sin\sigma^{\mathfrak{g}}_m$ *at some* $\psi = \psi_m$*, from which it increases strictly monotonically to zero for* $\psi \to \frac{\pi}{4}$*. The monotonicity behaviour of* $\mathcal{H}^{\mathfrak{g}}_{\psi}$ *for* $\psi \in (-\frac{\pi}{4}, 0)$ *is then determined by property 3.*
5. *The area of* $\mathcal{F}^{\mathfrak{g}}_{\psi}$ *increases strictly monotonically from* $\psi = 0$ *towards both* $\psi = \pm\frac{\pi}{4}$ *where it takes the values* $A_{\pm\pi/4} = 8\pi\delta^{-2}$*.*

This theorem has the following obvious Corollary:

Corollary 1 *For every* $\mathfrak{g} \gg 1$*, the smooth family of conformal embeddings constructed in Theorem 2 determines a smooth MTT whose MTS sections have properties 1–5.*

In Sect. 4, we note that the monotonicity of the area of the MTS sections of MTTs (holographic screens) is the key ingredient in their thermodynamic interpretation along the lines of the “second law”. We briefly discuss the area law established in [8].

2 A class of spacetimes with no stable MOTSs

We start with definitions, setting out from Definition 2. We normalize the null vectors to satisfy $\langle \ell^+, \ell^- \rangle = -2$. Capital letters $(A, B, ..)$ denote indices on objects on $\mathcal{F}$. The induced metric on $\mathcal{F}$ is denoted by j or j_{AB}, while its Levi-Civita derivative by D or D_A and $\Delta_j := D_A D^A$ is the Laplacian and R_j the scalar curvature. The null second fundamental form of $\mathcal{F}$ with respect to ℓ^+ is denoted by k^+ (k^+_{AB}) and the torsion one form s (s_A) is defined by

$$s(X) = -\frac{1}{2}\langle \ell^-, \nabla_X \ell^+ \rangle \quad \forall \text{ vector fields } X \text{ on } \mathcal{F}.$$

Remark 1 The motivation and inspiration for the following definition and the results on (in)stability of MOTS (cf. Sects. 3–5 of [3]) come from corresponding material on minimal surfaces of codimension 1 in manifolds with a Riemannian metric. We recall, however, that in the minimal surface case, stability can be defined via the second variation of the area functional. This is no longer the case for MOTS, which entails substantial differences (cf. Remark 2).

The key definitions for this section read as follows.

Definition 3 *(Stability)* We denote the Einstein tensor by $\mathrm{Ein}_g = \mathrm{Ric}_g - \frac{1}{2}\mathrm{R}_g g$.

- The stability operator L of a MOTS $\mathcal{F}$ is defined for smooth functions w by

$$L(w) = -\Delta_j w + 2s^A D_A w + \frac{1}{2}\Big(\mathrm{R}_j - \mathrm{Ein}_g(\ell^+, \ell^-) - 2s^A s_A + 2D_A s^A\Big) w \tag{3}$$

- L admits a principal eigenvalue λ, which is the eigenvalue with smallest real part. This eigenvalue is necessarily real and its eigenspace is one-dimensional and spanned by an everywhere positive function ϕ, called *principal eigenfunction*.
- The MOTS $\mathcal{F}$ is called strictly stable (resp. stable, marginally stable or unstable) provided $\lambda > 0$ ($\lambda \geq 0$, $\lambda = 0$ or $\lambda < 0$).

Remark 2 The stability operator is relevant for calculating the variation $\delta_\xi \theta^+$ of the expansion θ^+ with respect to an arbitrary vector field ξ normal to $\mathcal{F}$. In Lemma 3.1 of [3], it was shown that

$$\delta_\xi \theta^+ = L(u) - \frac{1}{2} B Z \tag{4}$$

where the functions u, B and Z are defined on $\mathcal{F}$ by $u := \langle \xi, \ell^+ \rangle$, $B := -\langle \xi, \ell^- \rangle$ and $Z := k^+_{AB} k_+^{AB} + \mathrm{Ein}_g(\ell^+, \ell^+)$.

The "stability operator L_v in the direction of a normal vector v" introduced in Definition 3.1 of [3] is related to (4) as follows:

$$\delta_\xi \theta^+ = L_v(u) \ \text{ for } \xi = u\, v \tag{5}$$

In particular, $L(u)$ as defined in (3) agrees with $L_v(u)$ for $v = -\frac{1}{2}\ell^-$.

The stability operator of a stable MOTS satisfies a maximum principle, cf. Lemma 4.2 of [3]:

Lemma 1 *Let $\mathcal{F}$ be a MOTS and L the corresponding stability operator. Let λ be the principal eigenvalue and $\phi > 0$ the principal eigenfunction. Let ψ be a smooth solution of $L\psi = f$ with $f \geq 0$. Then,*

(i) If $\lambda = 0$, then $f = 0$ and $\psi = C\phi$ for some constant C.
(ii) If $\lambda > 0$ and f is not identically zero, then $\psi > 0$.
(iii) If $\lambda > 0$ and $L\psi = 0$, then $\psi = 0$.

From this lemma, we can prove the following general result on the nonexistence of stable MOTS:

Proposition 1 *Let $\mathcal{F}$ be a MOTS in an n-dimensional ($n \geq 3$) spacetime $(\mathcal{M}, g)$ satisfying the null convergence condition. Assume that there exists a future causal vector field ξ along $\mathcal{F}$ which is not everywhere proportional to ℓ^+ and such that $\delta_\xi \theta^+ \geq 0$. Then, $\mathcal{F}$ cannot be strictly stable. Moreover, if $\delta_\xi \theta^+$ is positive somewhere, then $\mathcal{F}$ cannot be marginally stable either.*

Proof The basis $\{\ell^+, \ell^-\}$ is future directed, so ξ being future causal implies $u \leq 0$ and $B \geq 0$. Moreover, u is not identically zero, because ξ is not everywhere proportional to ℓ^+. Together with Equ. (4) and the hypotheses of the proposition, this implies $L(u) \geq 0$. If $\mathcal{F}$ is strictly stable (i.e. $\lambda > 0$), items (ii) and (iii) of Lemma 1 imply that u is either strictly positive or identically zero, which is a contradiction. This proves the first claim of the proposition. Finally, if $\mathcal{F}$ is marginally stable, then item (i) of the Lemma yields $\delta_\xi \theta^+ + \frac{1}{2} BZ = 0$, which cannot happen if $\delta_\xi \theta^+$ is positive somewhere. □

We are now ready to prove Theorem 1 stated in the Introduction. We use Greek indices on spacetime objects and sum over repeatedly occurring ones.

Proof of Theorem 1 In Corollary 1 of [10], the following general identity is proved for variations of θ^+ of a MOTS along an arbitrary vector field ξ in terms of the deformation tensor $a(\xi) := \pounds_\xi g$ of ξ

$$\delta_\xi \theta^+ = -\frac{1}{4}\theta^- a(\xi)_{\alpha\beta} \ell_+^\alpha \ell_+^\beta - a(\xi)_{\alpha\beta} e_A^\alpha e_B^\beta k_+^{AB} + j^{AB} e_A^\alpha e_B^\beta \ell_+^\nu \left(\frac{1}{2} \nabla_\nu a(\xi)_{\alpha\beta} - \nabla_\alpha a(\xi)_{\beta\nu} \right) \qquad (6)$$

where $\{e_A^\alpha\}$ is any basis of tangent vectors to $\mathcal{F}$. Assume now that ξ is a conformal Killing vector, so that $a(\xi) = 2\Psi g$. Inserting into (6) yields

$$\delta_\xi \theta^+ = j^{AB} e_A^\alpha e_B^\beta \ell_+^\nu \left(\nabla_\nu \Psi g_{\alpha\beta} - 2\nabla_\alpha \Psi g_{\beta\nu} \right) = (n-2)\ell_+^\nu \nabla_\nu \Psi.$$

Since ℓ^+ is future directed and Ψ is a temporal function, we have $\delta_\xi \theta^+ > 0$, and the result is a consequence of Proposition 1. □

Remark 3 Recall that MOTS were defined in Definition 2 by the vanishing of (at least) one of the null expansions. As the "outer" amendment apparently plays no role, neither in that Definition nor in the discussion above, the simpler notion "marginal surface" (introduced by Hayward [18]) would suffice. To indicate some benefit of the "MOTS" terminology, we recall that in Definition 3.1 of [3], stability of the MOTS with respect to any direction v was defined by requiring that the operator $L_v(w)$ (Eq. (5)) has nonnegative lowest eigenvalue. This is equivalent to nonnegativity of either side of (5) provided ℓ^+ points in the same direction as $\xi = u\, v$, viz. $u = \langle \xi, \ell^+ \rangle \geq 0$. Calling these directions "outer" (or "inner") is just more efficient terminology than some "... points in the same (or opposite) direction..." amendment. The next section provides corresponding "unstable examples".

Example 1 (*The Friedman–Lemaitre–Robinson–Walker Universe* (*FLRW*)) Before focusing on dS in the next section, we apply Theorem 1 to n-dim. FLRW, viz.

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2 \right) \tag{7}$$

where $d\Omega^2$ is the spherical metric in $n-2$ dimensions and $\kappa \in \{-1, 0, 1\}$.

The requirements that Ψ is a temporal function for the conformal Killing field $\xi = a\partial_t$ and the null convergence condition translate into the following conditions on the scale factor $a(t)$, respectively

$$\ddot{a} > 0, \qquad \ddot{a} \leq \frac{\dot{a}^2 + \kappa}{a}. \tag{8}$$

Here, dot is the derivative with respect to t. If conditions (8) hold, all MOTS are unstable.

We remark in this context that MOTS in FLRW was investigated, e.g., in [16, 20]. In the latter paper, it was found that in closed ($\kappa = 1$) 4-dim. FLRW, a certain family of MOTS called CMC Clifford tori are always unstable. In the next section, we will analyse systematically families of CMC-MOTS in umbilic slices of dS.

3 MOTSs and MOTTs in de Sitter spacetime

3.1 Instability of MOTS

The following easy consequence of Theorem 1 was already sketched in the Introduction.

Corollary 2 *All MOTS in de Sitter spacetime are unstable.*

Proof In terms of the coordinates (1), $\xi = \frac{\partial}{\partial \sigma}$ is a future directed, timelike conformal Killing vector field satisfying

$$\pounds_\xi \, g_{\mathrm{dS}} = 2 \tan \sigma \; g_{\mathrm{dS}}.$$

Since tan is an increasing function and σ is a temporal function, so is $\Psi = \tan \sigma$. Hence, Theorem 1 implies the claim. □

Remark 4 There is a certain analogy between de Sitter spacetime on which we focus in this section and round spheres, in the sense that both are maximally symmetric spaces of constant positive curvature. Recalling also the analogy between the stability definitions for minimal surfaces and MOTS (cf. Remark 1), Corollary 2 can be seen as a counterpart to a result of [23] (namely case $p = n - 1$ of Thm 5.1.1) that all minimal surfaces of codimension 1 on the round $\mathbb{S}^n$ are unstable. However, there is no obvious analogy between the proofs, and we do not make any attempt of establishing one here.

3.2 MOTS in umbilic slicings

From now on, we restrict ourselves to dS in 3+1 dimensions. Our aim is to locate MOTTs in the three umbilic and constant curvature slicings of de Sitter, namely flat and hyperboloidal and spherical. Our focus will be on the latter case.

3.2.1 Preliminaries

We consider a spacelike hypersurface $(\mathcal{N}, h, K)$ with metric h, scalar curvature ${}^3R$, covariant derivative ${}^3\nabla$, extrinsic curvature K (w.r.t. a future-pointing unit normal N) and mean curvature $tr_h K$ (i.e. again with "physics convention", cf. Definition 1). The constraints take the form

$$R_h - |K|_h^2 + (tr_h K)^2 = 6\delta^2 \tag{9}$$

$$div_h \left(K - (tr_h K)h\right) = 0. \tag{10}$$

Lemma 2 *Let $(\mathcal{N}, h, K)$ be an umbilic slice in de Sitter, i.e. the extrinsic curvature satisfies $K = \beta h$ for some function β. Then, β is constant on $\mathcal{N}$. Furthermore, any MOTS $\mathcal{F} \subset \mathcal{N}$ with $\theta^+ = 0$ is a MTS and CMC as well, with mean curvature $H = div_h X = -2\beta$ where X is a unit outward normal vector to $\mathcal{F}$ in $\mathcal{N}$, i.e. $\langle X, \ell^+\rangle > 0$.*

Proof The constancy of β follows from the constraint (10). For the second statement, we use the normalization $\langle \ell^+, \ell^-\rangle = -2$ and the decomposition $\ell^\pm = N \pm X$. This induces the following decomposition of $\theta^\pm$

$$\theta^\pm = \pm H + tr_j K = \pm H + 2\beta \tag{11}$$

which implies the assertion, with $\theta^- = 4\beta$ on the MOTS $\theta^+ = 0$. □

Remark 5 In the following subsections, we consider as examples umbilic slicings $(\mathcal{N}, h, K)$ of constant curvature (i.e. h is flat, spherical or hyperboloidal) in dS. In this special situation, a recent result of [7] implies instability of all MOTS $\mathcal{F}$ contained in such slices. The argument goes as follows. The spaces $\mathcal{N}$ in question are homogeneous in the sense that their isometry groups act transitively. In particular, for a given MOTS $\mathcal{F}$, any pair of points $p \in \mathcal{F}$ and $q \notin \mathcal{F}$ can be joined by a group element ρ, i.e. $q = \rho(p)$. Hence, the Killing vector ξ of $\mathcal{N}$ tangent to this orbit cannot be everywhere tangent to $\mathcal{F}$. Now by Theorem 1.6 of [7] or by Theorem 8.1 of [3], $\mathcal{F}$ is either unstable or marginally stable, with ξ nowhere tangent to $\mathcal{F}$ in the latter case. But this case is ruled out by Theorem 5.1 of [7], since by Lemma 2, $H = -2\beta =$ const. on an umbilic slice.

While the results of [7] also cover other situations than the aforementioned one (as discussed in that paper), there is not much further overlap with the general settings considered in Proposition 1 and Corollary 2. Of course, the latter results apply to umbilic slices.

These slicings are conveniently defined in terms of the dS hyperboloid

$$-x_0^2 + x_\alpha x_\alpha = \delta^{-2} \qquad \alpha, \beta \in \{1, 2, 3, 4\}. \tag{12}$$

embedded in 5d Minkowski space

$$ds^2 = -dx_0^2 + dx_\alpha dx_\alpha \tag{13}$$

Above and henceforth, repeated indices are summed over. A useful reference for the slicings considered is Sect. 4 of [17].

3.2.2 Flat slicing

In terms of the coordinate transformation

$$x_0 = \delta^{-1} \sinh \delta t + \frac{\delta}{2} r^2 e^{\delta t} \tag{14}$$

$$x_1 = \delta^{-1} \cosh \delta t - \frac{\delta}{2} r^2 e^{\delta t} \tag{15}$$

$$x_i = e^{\delta t} y_i \qquad (i \in \{2, 3, 4\}) \qquad r^2 = y_i y_i \tag{16}$$

and upon introducing polar coordinates on $\mathbb{S}^2$, the induced metric on the hyperboloid (12) reads

$$ds^2 = -dt^2 + e^{2\delta t}[dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\phi^2)] \quad r \in [0, \infty),\ \vartheta \in [0, \pi],\ \phi \in [0, 2\pi] \tag{17}$$

Clearly, the t =const. [$t \in (-\infty, \infty)$] slicing is flat and umbilic (with $\beta = \delta$) but covers only half of dS as described by (1).

In order to determine the MOTS in these slices, we recall that closed CMC surfaces embedded in flat 3-space must necessarily be round spheres [2].

A sphere of radius $r = R$ in a slice $t = T$ obviously has the induced metric

$$ds_R^2 = e^{2\delta T} R^2 (d\vartheta^2 + \sin^2\vartheta d\phi^2). \tag{18}$$

We now choose the counter-intuitive label *"outgoing"* for the normal vector X pointing towards *decreasing* r, viz.:

$$X = -e^{-\delta T} \frac{\partial}{\partial r}. \tag{19}$$

We find for the mean curvature w.r.t. X

$$H = div_h X = -\frac{2}{R} e^{-\delta T}. \tag{20}$$

It is only with this choice that (11) can be solved to determine the location of a marginally *outer* trapped tube with MOTS sections $\theta^+ = 0$, namely:

$$e^{-\delta T} = \delta R \tag{21}$$

in consistency with Definition 2. We note that $H = -2\delta = const.$ along the MOTT.

3.2.3 Hyperboloidal slicing

We perform the coordinate transformation

$$x_0 = \delta^{-1} \sinh \delta t \cosh r \tag{22}$$

$$x_1 = \delta^{-1} \cosh \delta t \tag{23}$$

$$x_i = \delta^{-1} z_i \sinh \delta t \sinh r \qquad (i \in \{2, 3, 4\}) \qquad z_i z_i = 1 \tag{24}$$

which yields for the induced metric on the hyperboloid (12)

$$ds^2 = -dt^2 + \delta^{-2} \sinh^2(\delta t)[dr^2 + \sinh^2 r(d\vartheta^2 + \sin^2 \vartheta d\phi^2)] \tag{25}$$

where $t \in (-\infty, \infty)$, $r \in [0, \infty)$ and $\vartheta \in [0, \pi]$, $\phi \in [0, 2\pi)$. This slicing is umbilic with $\beta = \delta \coth \delta T$ for $t = T$ =const.; again it covers only part of dS given by (1), cf. Section 4.4.2 of [17].

Here, we restrict ourselves to calculating round CMC spheres; as to more general shapes cf. Remark 6. Then, the discussion becomes quite similar to the flat case of the previous subsection. The induced metric on slices $t = T$ =const., $r = R$ =const. reads

$$ds_R^2 = \delta^{-2} \sinh^2(\delta T) \sinh^2 R(d\vartheta^2 + \sin^2 \vartheta d\phi^2)$$

We now take as *outgoing* unit normal vector to the $R = const$ slices

$$X = -\delta \sinh^{-1}(\delta T) \frac{\partial}{\partial r}$$

which yields for the mean curvatures of the MOTSs,

$$H = div_h X = -2\delta \sinh^{-1}(\delta T) \coth R. \tag{26}$$

It then follows from (11) that the MOTT with spherical sections are determined by the equation

$$\cosh \delta T = \coth R. \tag{27}$$

For the mean curvatures of the MOTS sections of these MOTTs, we obtain

$$H = -2\delta \cosh R \leq -2\delta \tag{28}$$

Remark 6 By the maximum principle, the mean curvature of a closed CMC surface embedded in the unit hyperboloid $\mathbb{H}^3$ must satisfy either $H \geq 0$ or $H \leq -2$ (cf. e.g. [13]), in consistency with (28). We also recall the "Lawson correspondence" [13, 19], which is a locally bijective relation between CMC surfaces in 3-dim space forms. In particular, CMC surfaces in the unit hyperboloid $\mathbb{H}^3$ with $H \leq -2$ have corresponding

CMC or minimal surfaces in flat space or in the unit sphere $\mathbb{S}^3$. We restrict the detailed discussion to the latter case.

3.2.4 Complete spherical slicing

Introducing the coordinates

$$x_0 = \delta^{-1} \sinh \delta t \tag{29}$$

$$x_\alpha = \delta^{-1} z_\alpha \cosh \delta t \qquad (\alpha \in \{1, 2, 3, 4\}) \qquad z_\alpha z_\alpha = 1, \tag{30}$$

we obtain

$$ds^2 = -dt^2 + \delta^{-2} \cosh^2(\delta t)[d\tau^2 + \sin^2 \tau (d\vartheta^2 + \sin^2 \vartheta d\phi^2)] \qquad \tau \in [0, \pi] \tag{31}$$

This is equivalent to (1) where the time coordinates are related by $\tan \frac{\sigma}{2} = \tanh \frac{\delta t}{2}$. We proceed the discussion in terms of (1).

The extrinsic and mean curvatures of the surfaces $\sigma = const$ read

$$K = (\delta \sin \sigma)\, h \qquad tr_h K = 3\delta \sin \sigma. \tag{32}$$

Therefore, $\beta = \delta \sin \sigma$ in the notation of Lemma 2. In contrast to the previous slicings Sects. 3.2.2 and 3.2.3, we can now solve (11) for either sign which yields

$$H = \mp 2\delta \sin \sigma. \tag{33}$$

3.2.4.1 Spherical MOTS

In this and subsection 3.2.4.2, our presentation largely follows [5, Sect. 5] and [9, Ch. 5].

In polar coordinates, (1) takes the form

$$ds^2 = \delta^{-2} \cos^{-2} \sigma \left[d\tau^2 + \sin^2 \tau \left(d\vartheta^2 + \sin^2 \vartheta d\varphi^2\right)\right]; \quad \tau, \vartheta \in [0, \pi], \varphi \in [0, 2\pi) \tag{34}$$

For round 2-spheres $\tau = const.$, we obtain from (11):

$$\theta^{\pm} = 2\delta[\pm \cos \sigma \cot \tau + \sin \sigma]. \tag{35}$$

The MOTSs given by $\tau_0^{\pm} = \pm\sigma_0 + \frac{\pi}{2}$ are spheres with radius δ^{-1}.

Figure 1 shows, via a conformal rescaling ("Einstein cylinder", conformal factor $\delta^{-2} \cos^{-2} \sigma$), the MOTTs $\mathcal{T}^{\pm}$ determined by the two families of MOTSs. $\mathcal{T}^{\pm}$ are null surfaces. The φ and ϑ directions are suppressed; hence, spheres reduce to two points; the dots at $\sigma = 0$ correspond to the equators on $\mathbb{S}^3$, while the conical convergence towards $\sigma = \pm\frac{\pi}{2}$ is due to the conformal factor. The drawing is sketchy (i.e. not a faithful result of calculation).

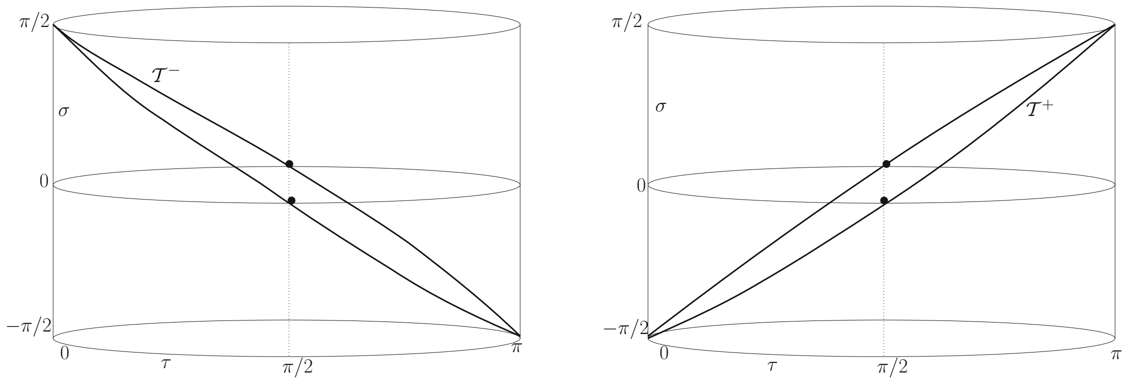

Fig. 1 MOTTs $\mathcal{T}^-$ (left) and $\mathcal{T}^+$ (right) with spherical CMC sections

3.2.4.2 Toroidal MOTS

We recall the toroidal foliation of $\mathbb{S}^3$ in terms of the following coordinates:

$$ds^2 = \delta^{-2}\cos^{-2}\sigma\,(d\tau^2 + \sin^2\tau\, d\gamma^2 + \cos^2\tau\, d\xi^2); \quad \tau \in [0, \frac{\pi}{2}], \quad \gamma, \xi \in [0, 2\pi) \tag{36}$$

On a torus $\mathcal{F}$ given by $\tau = const.$, we find from (11):

$$\theta^\pm = 2\delta[\pm\cos\sigma\cot 2\tau + \sin\sigma] \tag{37}$$

which formally differs just by a factor of 2 in the cot-term from the spherical case (35). This entails that the MOTSs determined by $\theta^\pm = 0$ and given via $\tau = \tau^\pm$ on some slice $\sigma = \sigma_0$ are now tori located at $\tau_0^\pm = \pm\frac{\sigma_0}{2} + \frac{\pi}{4}$. As above, the MOTS exist for all $\sigma \in (-\frac{\pi}{2}, \frac{\pi}{2})$ but *now they form timelike MOTTs.* The area of its toroidal sections

$$\frac{A}{4\pi} = |\frac{\sin\tau\cos\tau}{\delta^2\cos^2\sigma}| = |\frac{\sin 2\tau}{2\delta^2\cos^2\sigma}| = |\frac{\sin(\pm\sigma + \pi/2)}{2\delta^2\cos^2\sigma}| = \frac{1}{2\delta^2\cos\sigma} \tag{38}$$

diverges when $\sigma \to \pm\frac{\pi}{2}$ and approaches the minimum $\frac{1}{2\delta^2}$ at $\sigma = 0$.

Figure 2 shows again the Einstein cylinder with two dimensions suppressed; each MOTS $\mathcal{T}^\pm$ reduces to 4 points; the Clifford torus is indicated by 4 dots at $\sigma = 0$. The

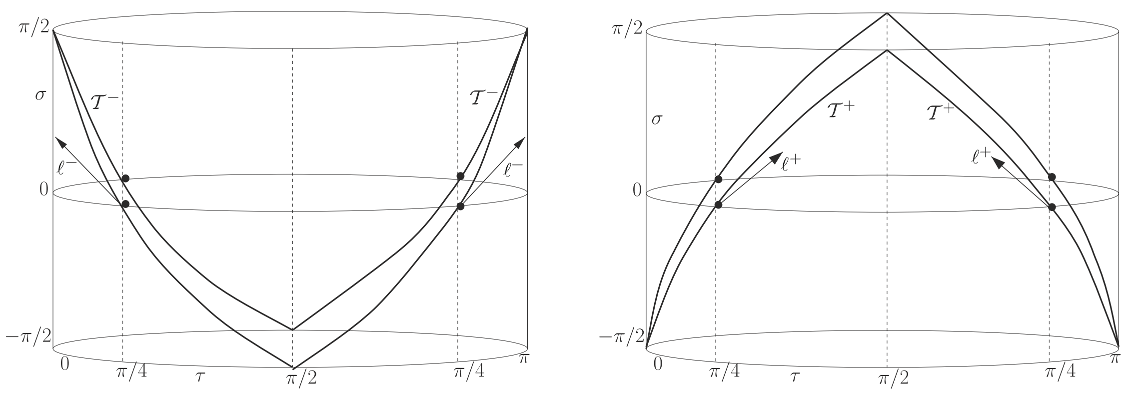

Fig. 2 MOTTs $\mathcal{T}^-$ (left) and $\mathcal{T}^+$ (right) with toroidal CMC sections

MOTSs are timelike, which is visualized by the null vectors $\ell^{\pm}$. As before, this figure is also sketchy.

3.2.4.3 MOTS of higher genus

We now turn to MOTSs and MOTTs of higher genus $\mathfrak{g}$. A natural strategy is to set out from a minimal surface at the time-symmetric slice $\sigma = 0$, as such surfaces also have played special roles in the cases of genus 0 and 1. We would like to show existence of their evolution to MOTTs, and determine the causal character of the latter as well as the evolution of the area of their MOTS sections. Unfortunately, for genus $\mathfrak{g} > 1$, explicit expressions for the embedding functions and other geometric quantities seem not to be available. Moreover, instability of the MOTS prevents us from applying the general results of [3, 4].

Following [11], we adopt here the following strategy. It is known that any CMC surface in $\mathbb{R}^3$ or $\mathbb{S}^3$ gives a solution to a certain Lax pair; vice versa, one can construct CMC surfaces as solutions to certain Lax pairs using the Sym-Bobenko formula. Making use of this, the DPW method [14] is a way of constructing CMC surfaces from holomorphic potentials on Riemann surfaces by solving a certain Cauchy problem. To ensure that the resulting CMC surface is well-defined, one has to solve in addition the so-called monodromy problem. In this way, the authors of [11] construct families of CMC embeddings on $\mathbb{S}^3$ for every sufficiently high genus $\mathfrak{g}$; the latter restriction comes from the use of an implicit function theorem argument near $t = 0$, where $t = \frac{1}{2\mathfrak{g}+2}\mathcal{K}^{-1/2}$, and the quantity $\mathcal{K}$ arises in the monodromy construction, cf. Prop. 9 of [11].

We present in Theorem 3 below a combination of Theorem 1 and a part of Theorem 2 of [11]; item 4. also contains a slight extension of the aforementioned results. Moreover, we change the parametrisation from φ to $\psi = \varphi - \frac{\pi}{4}$ which simplifies the presentation.

Theorem 3 (Theorems 1 and 2 of [11]) *Our setting is the 3-dim. unit sphere $\mathbb{S}^3$. For every $\mathfrak{g} \in \mathbb{N}$ sufficiently large, there exists a smooth family of conformal CMC embeddings $f_\psi^{\mathfrak{g}} : \mathcal{R}^{\mathfrak{g}} \to \mathbb{S}^3$ from a Riemann surface with genus g and parameter $\psi \in (-\frac{\pi}{4}, \frac{\pi}{4})$ satisfying*

1. *$f_0^{\mathfrak{g}}$ is the Lawson surface $\xi_{1,\mathfrak{g}}$ of genus $\mathfrak{g}$.*
2. *For $\psi \to \pm\frac{\pi}{4}$, the embedding $f_\psi^{\mathfrak{g}}$ smoothly converges to a doubly covered geodesic 2-sphere with $2\mathfrak{g} + 2$ branch points, i.e. the family $f_\psi^{\mathfrak{g}}$ cannot be extended in the parameter ψ in the space of immersions.*
3. *$f_\psi^{\mathfrak{g}} = f_{-\psi}^{\mathfrak{g}}$ up to reparametrization and (orientation reversing) isometries of $\mathbb{S}^3$, and accordingly, the constant mean curvatures satisfy $H_\psi^{\mathfrak{g}} = -H_{-\psi}^{\mathfrak{g}}$.*
4. *The mean curvature $H_\psi^{\mathfrak{g}}$ of $f_\psi^{\mathfrak{g}}$ decreases strictly monotonically from zero at $\psi = 0$ to some minimal value $H_{\psi_m}^{\mathfrak{g}}$ for $\psi = \psi_m$ from which it increases strictly monotonically to zero for $\psi \to \frac{\pi}{4}$. The monotonicity behaviour for $\psi \in (-\frac{\pi}{4}, 0)$ then follows from property 3.*
5. *The Willmore energy Equ. (2) of $f_\psi^{\mathfrak{g}}$ increases strictly monotonically from $\psi = 0$ towards both $\psi \to \pm\frac{\pi}{4}$ where $W_{\pm\pi/4} = 8\pi$.*

Proof Items 1–3 and 5 follow as in the proofs of Thms. 1 and 2 of [11]; to show item 4, we note that from Proposition 34 of [11], the mean curvature $H_\psi^{\mathfrak{g}} = H(t(\mathfrak{g}), \psi)$

with $t(\mathfrak{g}) = \frac{1}{2\mathfrak{g}+2}\mathcal{K}^{-1/2}(t,\psi)$ is smooth and its Taylor expansion in t takes the form

$$H(t,\psi) = 4t\cos 2\psi \ln\left[\tan(\psi + \frac{\pi}{4})\right] + O(t^2). \tag{39}$$

Here, $\mathcal{K}$ depends on t itself but such that $\mathcal{K}(t,\psi) = 1 + O(t^2)$ near $t = 0$, cf. Sect. 6.3 of [11]. A calculation shows that at the values $\psi = \psi_0$ and $\psi = -\psi_0$, $H_0(\psi) = (\partial H/\partial t)(0,\psi)$ takes on its unique non-degenerate minimum and maximum, respectively, i.e.

$$\frac{dH_0}{d\psi}(\pm\psi_0) = 0 \qquad \frac{d^2H_0}{d\psi^2}(\pm\psi_0) \neq 0, \tag{40}$$

where $\pm\psi_0$ are given implicitly by

$$(\sin 2\psi_0)\ln\left[\tan\left(\psi_0 + \frac{\pi}{4}\right)\right] = 1. \tag{41}$$

Moreover, there hold the monotonicity properties

$$\frac{dH_0}{d\psi} > 0 \qquad \forall\, \psi \in \left[-\frac{\pi}{4}, -\psi_0\right) \text{ and } \psi \in \left(\psi_0, \frac{\pi}{4}\right] \tag{42}$$

$$\frac{dH_0}{d\psi} < 0 \qquad \forall\, \psi \in (-\psi_0, \psi_0) \tag{43}$$

We now observe that there exists a $\psi_m(\psi,t) = \psi_0 + O(t)$ such that, for sufficiently small t, properties (40), (42) and (43) hold for $H(t,\psi)$ as well, with ψ_0 replaced by ψ_m everywhere. Away from the zeros of H which are at $\psi = 0, \pm\frac{\pi}{4}$ (cf. Prop. 18 of [11]), this follows from smoothness and from the non-degeneracy (40), while in a neighbourhood of the zeros of H, the assertion holds by virtue of Prop. 34 of [11]. □

Remark 7 The convergence properties of the CMC surfaces for $\psi \to \pm\frac{\pi}{4}$ (point 2 of Theorem 3) are explained in Prop. 26 of [11].

We are now ready to prove Theorem 2 stated in the Introduction.

Proof of Theorem 2 We define the conformal rescaling $\mathcal{I}_\sigma : \mathbb{S}^3 \to \mathbb{S}^3_\sigma$ from the unit sphere to the sphere of radius $\rho = \delta^{-1}\cos^{-1}\sigma$. This clearly induces a scaling for all embedded surfaces; we note that the mean curvature of such surfaces changes from H to $\mathcal{H}_\sigma = \delta H\cos\sigma$. We now define the embedding $\mathcal{F}^{\mathfrak{g}}_\psi : \mathcal{R}^{\mathfrak{g}} \to \mathbb{S}^3_\sigma$ by $\mathcal{F}^{\mathfrak{g}}_\psi = \mathcal{I}_{\sigma(\psi)} \circ f^{\mathfrak{g}}_\psi$ where $\sigma(\psi)$ is constructed as follows. We recall from (33) that any CMC surface with mean curvature $\mathcal{H}_\sigma$ on $\sigma =$ const. corresponds to a MTS if $\mathcal{H}_\sigma = \mp 2\delta\sin\sigma$. In particular, if the mean curvatures $H^{\mathfrak{g}}_\psi$ of the CMC surfaces $f^{\mathfrak{g}}_\psi$ constructed in Theorem 3 satisfy

$$H^{\mathfrak{g}}_\psi = \frac{\mathcal{H}_\sigma}{\delta\cos\sigma} = \mp 2\tan\sigma, \tag{44}$$

they build up MTTs, which serves to define $\sigma(\psi)$.

In order to prove the final statement 5. on monotonicity of the area, we show that the Willmore functional W_σ (2) built from the rescaled quantities is just the area of the embedded MOTS. To see this, we note that at time σ, the rescaled area is $\mathcal{A}_\sigma = A\delta^{-2}\cos^{-2}\sigma$, while the rescaled mean curvature is $\mathcal{H}_\sigma = \delta H \cos\sigma$. Inserting in (2), we obtain $W_\sigma = \delta^2 A_\sigma$ which finishes the proof. □

Remark 8 It is stated in Theorem 1 in [11] that "the moduli space of genus $\mathfrak{g}$ CMC surfaces is one-dimensional at the Lawson surface $\xi_{1,\mathfrak{g}}$". This is not to be understood in the sense that the MTTs constructed in Theorem 2 are unique, as also indicated in point 3. of that Theorem. In fact, isometries of $\mathbb{S}^3$ which are not tangent to $\mathcal{F}^{\mathfrak{g}}_\psi$ (and which exist, cf. Remark 5) will "move around" $\mathcal{F}^{\mathfrak{g}}_\psi$ at any parameter value ψ.

Remark 9 We recall that there are no non-trivial isometries of $\mathcal{F}^{\mathfrak{g}}_\psi$ in the case $\mathfrak{g} \geq 2$ in which the Euler number $\chi = 2(1-\mathfrak{g})$ is negative. Assuming the contrary, the Poincaré–Hopf theorem implies that χ is also the sum of all indices of the (isolated) zeros of the corresponding Killing field. But if an isometry in 2 dim. has fixed points, it is necessarily a rotation in a neighbourhood. Hence, all Killing indices are $+1$ which implies that χ is nonnegative. This restricts the topology of $\mathcal{F}^{\mathfrak{g}}_\psi$ to the sphere or the torus if it has isometries.

Remark 10 We note that the conformal rescaling $\mathcal{I}_\sigma : \mathbb{S}^3 \to \mathbb{S}^3_\sigma$ involved in passing from Theorems 3 to 2 is precisely the inverse of the one used to construct the Penrose diagrams in Figs. 1 and 2 of dS. Hence, a Penrose-type diagram arises just by directly "stacking" the family of CMC surfaces $f^{\mathfrak{g}}_\psi$ constructed in Theorem 3. We recall, however, that the embedding parameter ψ is not a monotonic function of the cosmological time σ—rather, the constructed MOTT "turns around" at $\psi = \pm\psi_m$.

4 Discussion

Note that monotonicity of area holds along MTTs with MTS sections of spherical, toroidal and high-genus topology in the complete spherical slicing of dS as described in Sect. 3.2.4. Except for spherical MTS whose area stays constant, the area increase is even strictly monotonic upon moving away from the time-symmetric surface. In view of the extensively discussed correspondence between area and entropy, our MTTs are thus candidates for satisfying the "second law" of thermodynamics in an appropriate setting.

Such settings arise in attempts of understanding the "information paradox" of black holes, and in attempts of making sense of the idea of AdS/CFT correspondence (cf. e.g. [15] and references therein). In the latter context, MTTs run under the name "holographic screens". In fact an area theorem has been obtained in [8] (cf. Theorem IV.3) and [1] for so-called "regular holographic screens" whose definition consists of four somewhat subtle conditions (cf. Definition II.8 of [8]). We have not been able to extract the required information from our Theorem 2 to check these requirements; in particular, the causal character of the constructed MTT is unclear except that they must be spacelike near the turning points $\pm\psi_m$ where the mean curvatures achieve their extrema $\pm\mathcal{H}^{\mathfrak{g}}_{\psi_m}$ (cf. item 4 of Theorem 2). In order to obtain monotonicity of area of the MTS (item 5 of Theorem 2), we are thus bound to the argument involving their rescaled Willmore energy (item 5 of Theorem 3).

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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Rigidity aspects of a cosmological singularity theorem

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Abstract

Improving a singularity theorem in General Relativity by Galloway and Ling we show the following (cf. Theorem 1): If a globally hyperbolic spacetime M satisfying the null energy condition contains a closed, spacelike Cauchy surface $(V, g, \mathcal{K})$ (with metric g and extrinsic curvature $\mathcal{K}$) which is 2-convex (meaning that the sum of the lowest two eigenvalues of $\mathcal{K}$ is non-negative), then either M is past null geodesically incomplete, or V is a spherical space, or V or some finite cover $\tilde{V}$ is a surface bundle over the circle, with totally geodesic fibers. Moreover, (cf. Theorem 2) if $(V, g, \mathcal{K})$ admits a $U(1)$ isometry group with corresponding Killing vector ξ, we can relax the convexity requirement in terms of a decomposition of $\mathcal{K}$ with respect to the directions parallel and orthogonal to ξ. Finally, (cf. Propositions 1-3) in the special cases that V is either non-orientable, or non-prime, or an orientable Haken manifold with vanishing second homology, we obtain stronger statements in both Theorems without passing to covers.

1 Introduction

The classic singularity theorems due to Hawking and Penrose [1] yield geodesic incompleteness in different settings, namely "cosmological" and "black hole" ones. In the former setting one requires a compact Cauchy surface and a condition on its second fundamental form, while the latter setting assumes non-compact data containing a trapped or marginally trapped surface. In either case, convergence ("energy-") conditions on the spacetime Ricci tensor Ric are also necessary. In the sequel we will only need the null energy condition (NEC), namely $\mathrm{Ric}(X,X) \geq 0$ for all null vectors X.

In the intervening decades, Hawking's and Penrose's results have been substantially modified and strengthened in various directions. In particular, Galloway and the first-named author obtained the following result. All manifolds and fields are smooth (C^∞) throughout the present paper.

Theorem 0. *(Thm. 1 in [2] and the Remark in Sec. 4 of [3])*

(1) Let M be a (3+1) dimensional, globally hyperbolic spacetime which satisfies the NEC and has a closed spacelike Cauchy surface V.

(2) Assume V is strictly 2-convex, i.e. the sum of the smallest two eigenvalues of the future second fundamental form $\mathcal{K}$ is positive.

Then at least one of the following scenarios applies:

(i) M is past null geodesically incomplete, or

(ii) V is a spherical space.

We recall that a *spherical space* V is by definition diffeomorphic to a quotient of the 3-sphere $\mathbb{S}^3$, $V = \mathbb{S}^3/\Gamma$, where Γ is isomorphic to a subgroup of $SO(4)$.

Remark 0. The above formulation indicates that conclusions (i) and (ii) do not exclude each other. A "truncated" de Sitter spacetime provides an example in which both conclusions hold.

Condition (2) on $\mathcal{K}$ in Theorem 0 is rather restrictive and obviously excludes even the interesting time-symmetric case. The purpose of the present work is to relax this condition. To state our results we fix some notation, almost all of which coincides with [2].

We denote the induced metric on V by g and the second fundamental form by $\mathcal{K}$ (in order to distinguish it from $K(\pi, 1)$ manifolds). Let u be the future directed timelike unit normal to V, and refer $\mathcal{K}$ to this direction. We next consider a two-sided embedding of a surface $\Sigma \subset V$. We choose some "outward" direction with unit normal ν on Σ and denote by H the corresponding outward mean curvature of Σ, cf. Figure 1.

The two past directed null normals to Σ read $\ell^\pm = -u \pm \nu$, with corresponding null second fundamental forms $\chi^\pm = \mathcal{P}_\Sigma(\nabla \ell^\pm)$ (where ∇ is the covariant derivative on M and $\mathcal{P}_\Sigma$ the projection onto Σ) and null expansions $\theta^\pm = \mathrm{tr}_\Sigma(\chi^\pm)$ (where tr_Σ denotes the trace on Σ). Marginally inner (outer) trapped surfaces (MITS, MOTS) are defined by $\theta^- = 0 \quad (\theta^+ = 0)$ where $\theta^\pm$ refer to the inward (outward) null directions $\ell^\pm$. We also recall the general decomposition

$$\theta^\pm = -\,\mathrm{tr}_\Sigma\, \mathcal{K} \pm H. \tag{1.1}$$

We can now state our main results.

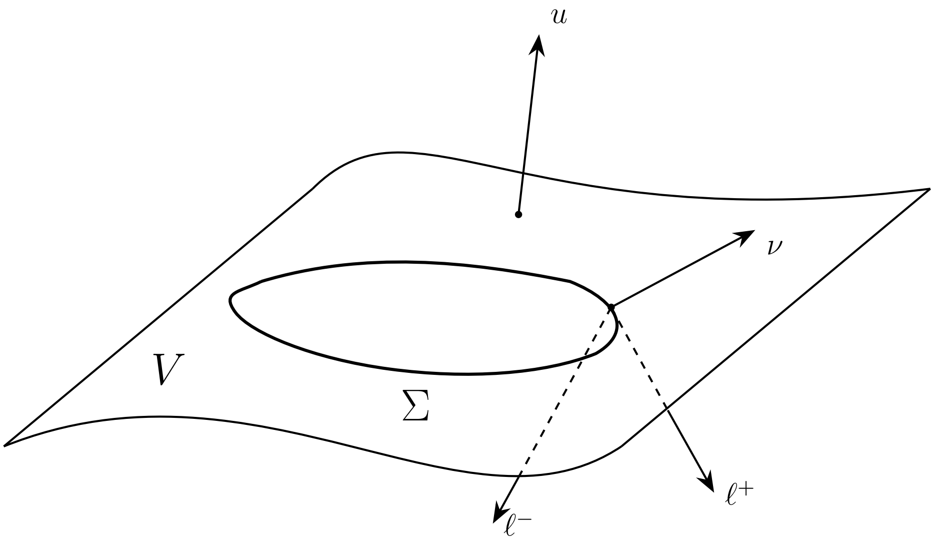

Figure 1: The basic geometric setup

Theorem 1.

(1) Let M be a (3+1)-dimensional, globally hyperbolic spacetime which satisfies the NEC and has a closed spacelike Cauchy surface V.

(2) Assume V is 2-convex, i.e. the sum of the smallest two eigenvalues of the future second fundamental form $\mathcal{K}$ is nonnegative.

Then at least one of the following scenarios applies:

(i) M is past null geodesically incomplete, or

(ii) V is a spherical space, or

(iii) V or a finite cover $\tilde{V}$ thereof is a surface bundle over the circle $\mathbb{S}^1$, where the fibers Σ are totally geodesic within V (or the finite cover $\tilde{V}$) and are MOTS/MITS within M (or the corresponding spacetime cover $\tilde{M}$).

Remark 1. Extending Remark 0, the "Misner-identified" Taub-NUT space (cf. Example 2 in Section 4) is inextensible as a globally hyperbolic spacetime. It satisfies requirements (1) and (2), and both (*i*) and (ii) apply. Moreover, spacetimes with data of type (iii) can be geodesically complete or incomplete, as Examples 3, 4, 6, 8 show.

There are important special cases in which we can strengthen the statements of Theorem 1. In the following three propositions, we obtain rigidity without having to pass to a cover. To state these results we recall that a *Haken manifold* is a P^2-irreducible compact 3-manifold which contains an embedded two-sided *incompressible surface*. By the latter we mean a non-simply connected closed two-sided immersion $f\colon \Sigma \to V$ with $f_*\colon \pi_1(\Sigma) \to \pi_1(V)$ being injective. Furthermore, by a *semibundle* we mean V is the union of two twisted I-bundles over a surface S. By fibers of a semibundle we refer to the fibers of $V \setminus (S \sqcup S)$, which is a surface bundle over an open interval with fibers Σ which double cover S.

Proposition 1. *Under the assumptions of Theorem 1, if V is an orientable Haken manifold with $H_2(V) = 0$, then at least one of the following scenarios applies:*

(i) M is past null geodesically incomplete, or

(ii) V has semibundle structure, where the fibers Σ are totally geodesic within V and MOTS/MITS within M.

An example where (ii) in Proposition 1 holds is when V is the flat Hantzsche-Wendt manifold (see Example 8 in Section 4). Then V is orientable, Haken, and $H_2(V) = 0$. The Lorentzian product $M = \mathbb{R} \times V$ is complete and V satisfies conclusion (ii) of Proposition 1. Note that this is a stronger statement than obtained from Theorem 1, since we do not have to go to a cover.

A similar statement can be made if V is nonorientable.

Proposition 2. *Under the assumptions of Theorem 1, if V is nonorientable, then at least one of the following scenarios applies:*

(i) M is past null geodesically incomplete, or

(ii) V is diffeomorphic to a surface bundle over the circle $\mathbb{S}^1$, where the fibers Σ are totally geodesic within V and MOTS/MITS within M.

Below the proof of Proposition 2 in section 3, we give three examples where (ii) applies.

Recall that a three-manifold V is *prime* if it cannot be written as the connected sum of two three-manifolds A and B with neither of them being the three-sphere.

Proposition 3. *Under the assumptions of Theorem 1, if V is non-prime, then at least one of the following scenarios applies:*

(i) M is past null geodesically incomplete, or

(ii) V is diffeomorphic to $\mathbb{RP}^3 \# \mathbb{RP}^3$, where the fibers $\Sigma \cong \mathbb{S}^2$ are totally geodesic within V and MOTS/MITS within M.

An example where (ii) of Proposition 3 applies is the Nariai spacetime (see Example 4 in Section 4) quotiented out by the double covering $\mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{RP}^3 \# \mathbb{RP}^3$ on the spatial slices.

Unfortunately, the condition of 2-convexity assumed above still excludes a lot of interesting cases. However, in the presence of the cyclic isometry group $U(1)$ we can indeed weaken the signature requirements on $\mathcal{K}$ in terms of its decomposition with respect to the Killing vector ξ corresponding to the $U(1)$ symmetry. Let $\|\xi\|$

be the norm of ξ, $\zeta = \|\xi\|^{-1}\xi$ the normalisation and ζ^* its dual. We next define the projection $\mathcal{P} = \mathcal{I} - (\zeta \otimes \zeta^*)$ (a $(1,1)$- tensor) onto the 2-space orthogonal to ξ, where $\mathcal{I}$ is the identity. Further, we denote by $\mathcal{K}(\xi,\xi)$ (a function) and $\mathcal{K}^\perp = \mathcal{K}(\perp,\perp) = \mathcal{P}^T\mathcal{K}\mathcal{P}$ (a $(1,1)$-tensor), the projections of $\mathcal{K}$ parallel and orthogonal to ξ.

In the sequel we will use the shorthand "data" for the set $(V, g, \mathcal{K})$, still without assuming any field equations. We obtain the following result.

Theorem 2.

(1) Let M be a (3+1)-dimensional globally hyperbolic spacetime which satisfies the NEC and has a closed spacelike Cauchy surface V. Suppose further that the data $(V, g, \mathcal{K})$ are invariant under the cyclic isometry group $U(1)$ with corresponding Killing vector ξ.

(2) Furthermore, suppose that

$$\mathcal{K}(\xi,\xi) + \mu\|\xi\|^2 \geq 0 \tag{1.2}$$

where μ is the smaller eigenvalue of $\mathcal{K}^\perp$, i.e. $\mu \leq \widehat{\mu}$ for the eigenvalues $\mu, \widehat{\mu}$.

Then at least one of the following scenarios applies:

(i) M is past null geodesically incomplete, or

(ii) V is a spherical space, or

(iii) V or a finite cover $\tilde{V}$ thereof is a surface bundle over the circle $\mathbb{S}^1$, where the fibers Σ are $U(1)$-symmetric and totally geodesic with Euler number $\chi(\Sigma) \geq 0$[1] within V (or the finite cover $\tilde{V}$). Moreover, theses fibers are MOTS/MITS within M (or the corresponding spacetime cover $\tilde{M}$), or

(iv) V or a finite cover $\tilde{V}$ is diffeomorphic to a surface bundle over the circle $\mathbb{S}^1$ on which the given isometry acts. Moreover, the fibers Σ are minimal within V or $\tilde{V}$.

Remark 2. In the presence of the $U(1)$ isometry, 2-convexity indeed implies (1.2) and is thus more restrictive. This can be seen as follows: Assume condition (1) from the above theorem with normalized Killing vector ζ and that V is 2-convex. Let $\lambda_1 \leq \lambda_2 \leq \lambda_3$ be the eigenvalues of $\mathcal{K}$ and let $\mu \leq \widehat{\mu}$ be the eigenvalues of $\mathcal{K}^\perp$. Then $\mathcal{K}(\zeta,\zeta) + \mu = \operatorname{tr}\mathcal{K} - \operatorname{tr}\mathcal{K}^\perp + \mu = \lambda_1 + \lambda_2 + \lambda_3 - \widehat{\mu} \geq \lambda_2 + \lambda_3 \geq 0$, where we used the Eigenvalue Interlace Theorem [4] to conclude that $\lambda_1 - \widehat{\mu} \geq 0$. Therefore, condition (1.2) is satisfied.

Examples reveal that condition (2) of Theorem 2 is in fact significantly less restrictive than 2-convexity. In particular, while 2-convexity excludes the family of so-called (t-ϕ)-symmetric data, the latter are all admitted by (1.2) (cf. Section 4 for details).

[1]Specifically, Σ is either the torus or the 2-sphere if orientable or the Klein bottle or the two-dimensional projective space if nonorientable.

Remark 3. Apart from the ambiguities mentioned in Remarks 0 and 1 we mention the following: It is clear that (iii) and (iv) are exclusive with respect to a given Killing field ξ, since it cannot be both tangent to the base and to the fibers. However, when the data $(V, g, \mathcal{K})$ enjoy a symmetry group containing $U(1) \times U(1)$, (iii) and (iv) can both apply with respect to the different factors; we refer to Examples 3 and 6 in Section 4.

Remark 4. There are Propositions 1, 2 and 3 in the setting of Theorem 2 *mutatis mutandis*, more precisely:

- Points (ii) in the propositions could be reformulated as in (iii) of Theorem 2.
- A further alternative (iii) has to be added in each proposition corresponding to point (iv) of Theorem 2.

This paper is organized as follows. The subsequent Section 2 contains three auxiliary results, while the cores of the proofs of the results stated above are given in Section 3. As mentioned earlier the final Section 4 consists of examples.

We conclude here with sketching the strategy of our proofs. For simplification, we assume that V is not a spherical space, and we pass to a finite, orientable cover $\tilde{V}$ whenever necessary or useful. Our key prerequisite is the prime decomposition of orientable 3-manifolds, see (3.1) below. Moreover, from the positive resolution of the virtual positive b_1-conjecture [5], it follows that V or a finite cover $\tilde{V}$ has vanishing second homology H_2. As is well-known, this implies that V (or $\tilde{V}$) contains an embedded, non-separating, two-sided minimal surface Σ of least area. By a known Lemma (Lemma 0), it follows that a neighborhood of Σ can be foliated by CMC surfaces Σ_t. To prove Theorem 1 we now invoke the convexity requirement on $\mathcal{K}$ which, via (1.1), yields control on the signs of the null expansions $\theta^{\pm}$ on each Σ_t. Together with the null energy condition, this allows an application of Penrose's singularity theorem on a suitable infinite cover of V (or $\tilde{V}$). This gives past null geodesic incompleteness except for the degenerate cases stated in (ii), (iii) of Theorem 1. In case (iii) compactness theorems allow us to extend the local folitaion of Lemma 1 to all of V.

The proof of Theorem 2 deviates from above as follows: Setting out from the local CMC foliation provided by Lemma 0, Lemma 2 shows that the given isometry is either everywhere tangent to all leaves, or nowhere tangent to any leaf. In the former case, the less restrictive convexity condition (1.2) now takes the role of 2-convexity and yields analogous results as before, while in the latter case we end up with the additional alternative (iv) of Theorem 2.

We choose not to sketch the proofs of Propositions 1-3 since they are highly technical.

Remark 5. We remark that Theorem 1 can be proven in a different way, namely by using the positive resolution of the virtual Haken conjecture and Proposition 1. In fact, this detour was taken by the authors originally. We are grateful to Greg

Galloway for bringing to our attention the positive resolution of the virtual positive b_1 conjecture which greatly simplifies the proof. A similar simplification also applies to the proof of the main result in [2], reformulated as Theorem 0 above.

Apart from our main reference [2], the recent paper [3] provided some inspiration for the present work as well. In particular there is some overlap between Theorems 1 and 2 above and Theorem 9 in [3]. While the latter amounts to an extension of Theorem 0 in the case $H_2(V) \neq 0$, in this paper we also address the more subtle case $H_2(V) = 0$.

2 Preliminaries

Here we collect some preliminary results.

To begin with, recall that for a minimal surface Σ the *stability operator* $\mathcal{L}(\psi)$ for $\psi \in C^\infty(\Sigma)$ is given by

$$\delta_{\psi\nu} H = \mathcal{L}_\nu(\psi) = -\Delta_\Sigma \psi - \frac{\psi}{2}\left(R_V - R_\Sigma + \mathcal{C}_\Sigma(t \otimes t)\right). \tag{2.1}$$

In (2.1) R_V and R_Σ are the Ricci scalars of the manifolds in the subscripts, t is the trace-free part of the extrinsic curvature and $\mathcal{C}_\Sigma$ is a contraction (over both indices).

A minimal surface Σ is called *stable (strictly stable, marginally stable)* if the lowest eigenvalue λ of $\mathcal{L}(\psi)$ satisfies $\lambda \geq 0$ ($\lambda > 0$, or $\lambda = 0$, respectively).

The following lemma is a standard application of the inverse function theorem. For a proof of the more involved $\lambda = 0$ case, see the proof of Thm. 2.38 in [6] or [3, Appendix A].

Lemma 0. *Let Σ be an embedded stable minimal surface within a Riemannian manifold V. Then there is a neighborhood U of Σ within V, diffeomorphic to $(-\epsilon, \epsilon) \times \Sigma$, such that each leaf $\Sigma_t := \{t\} \times \Sigma$ has constant mean curvature.*

Lemma 1. *Under the assumptions of Theorem 1, assume further that*

(1) M is past null geodesically complete,

(2) V contains an embedded, two-sided, minimal surface Σ,

(3) There exists a noncompact cover $p\colon \tilde{V} \to V$ and a lift $\tilde{\Sigma}$ of Σ which separates $\tilde{V}$ into two disjoint noncompact open submanifolds.

Then there exists a neighborhood U of Σ in V which is diffeomorphic to $(-\epsilon, \epsilon) \times \Sigma$ and each $\Sigma_t = \{t\} \times \Sigma$ is totally geodesic within V.

Proof. The following facts will be used to prove this Lemma as well as Lemma 2. First, there is a spacetime cover $P\colon \tilde{M} \to M$ such that $p = P|_{\tilde{V}}$, $\tilde{V}$ is a Cauchy surface for $\tilde{M}$, and the second fundamental form on $\tilde{V}$ is the pullback $\tilde{\mathcal{K}} = P^*\mathcal{K}$, see [2, Lem. 4]. Moreover, the null convergence condition also "lifts", and M is null geodesically complete if and only if $\tilde{M}$ is. Lastly, if a surface Σ' embedded in V is constructed via a foliation from Σ within V, then Σ' lifts to some $\tilde{\Sigma}'$, which, like $\tilde{\Sigma}$, also separates $\tilde{V}$ into two noncompact ends.

Let λ denote the principal eigenvalue for the stability operator $\mathcal{L}$ of Σ within V. We show that $\lambda = 0$. Let $\phi > 0$ be the corresponding principal eigenfunction. Consider the variation Σ_t given by $x \mapsto \exp_x\big(t\phi(x)\nu(x)\big)$ for $x \in \Sigma$. Then

$$\frac{\partial H}{\partial t}\bigg|_{t=0} = \mathcal{L}(\phi) = \lambda\phi.$$

If $\lambda < 0$, then $H(t) < 0$ for small $t > 0$. By the 2-convexity of $\mathcal{K}$, it follows that $\theta^+(t) < 0$ for these small t, cf. (1.1). Therefore an application of Penrose's singularity theorem (Thm. 7.1 of [7]) applied to the outer trapped surface $\tilde{\Sigma}_t$ within $\tilde{M}$ implies past null geodesic incompleteness within $\tilde{M}$ and hence also in M, which is a contradiction. Similarly, if $\lambda > 0$, then $H(t) < 0$ for some values $t < 0$. Again, an application of Penrose's singularity theorem in the cover yields a contradiction. Thus $\lambda = 0$.

By Lemma 0 there is a neighborhood U diffeomorphic to $(-\epsilon, \epsilon) \times \Sigma$ such that each leaf Σ_t has constant mean curvature. We distinguish two cases.

1.) $\mathbf{H_t \neq 0}$ **for some t**. If $H_t > 0$, then $\theta_t^- < 0$. If $H_t < 0$, then $\theta_t^+ < 0$. In either case, we can apply Penrose's singularity theorem in the cover since both components of $\tilde{V} \setminus \tilde{\Sigma}_t$ are noncompact, a contradiction.

2.) $\mathbf{H_t \equiv 0}$ $\forall$ **t.** We distinguish two more cases.

a.) $tr_{\Sigma_t}\mathcal{K} \not\equiv 0$**.** By the 2-convexity of $\mathcal{K}$, both null expansions satisfy $\theta_t^\pm \leq 0$ and hence $\tilde{\theta}_t^\pm \leq 0$ but are not identically vanishing. Hence for either choice, Lemma 5.2 of [8] yields a deformation $\tilde{\Sigma}'$ of $\tilde{\Sigma}_t$ such that (at least) the chosen $\tilde{\theta}'$ satisfies $\tilde{\theta}' < 0$ on $\tilde{\Sigma}'$. As above Penrose's theorem in $\tilde{M}$ contradicts past null geodesic completeness within M.

b.) $tr_{\Sigma_t}\mathcal{K} \equiv 0$**.** Now obviously $\tilde{\theta}_t^\pm \equiv 0$ for both signs. In terms of the null second fundamental forms $\tilde{\chi}_t^\pm$ defined in the Introduction, we first assume that at least one of

$$\tilde{W}_t^\pm = |\tilde{\chi}_t^\pm|^2 + \tilde{\mathrm{Ric}}(\tilde{\ell}_t^\pm, \tilde{\ell}_t^\pm), \tag{2.2}$$

say $\tilde{W}_t^+$, does not vanish identically on $\tilde{\Sigma}_t$. By virtue of Raychaudhuri's equation and another application of Lemma 5.2 in [8] we find surfaces $\tilde{\Sigma}'$ near the past light cone of $\tilde{\Sigma}_t$ with $\tilde{\theta}'^+ < 0$. Another application of Penrose's theorem in the cover $\tilde{M}$ and projecting down to M contradicts past null geodesic completeness within M. Therefore both (+)-terms on the r.h.s. of (2.2) have to vanish individually. Clearly, the case $\tilde{W}_t^-$ proceeds analogously. Thus the family Σ_t is totally geodesic within V since its second fundamental form is given by $\mathcal{K} = \chi_t^+ - \chi_t^-$.

□

Lemma 2. *Under the assumptions of Theorem 2, assume that*

(1) M is past null geodesically complete,

(2) V contains an embedded, two-sided, minimal surface Σ which is least area in its homology class.

(3) There exists a noncompact cover $p\colon \tilde{V} \to V$ and a lift $\tilde{\Sigma}$ of Σ which separates $\tilde{V}$ into two disjoint noncompact open submanifolds.

Then either (i) or (ii) holds.

(i) There exists a neighborhood U of Σ in V which is diffeomorphic to $(-\epsilon, \epsilon) \times \Sigma$ and each $\Sigma_t = \{t\} \times \Sigma$ is totally geodesic with ξ tangent to it. Moreover, Σ or its orientable double cover $\tilde{\Sigma}$ has spherical or toroidal topology.

(ii) V is diffeomorphic to a surface bundle over the circle $\mathbb{S}^1$ on which the given isometry acts; the fibers are minimal.

Proof. Σ being least area implies non-negativity of the second variation of area and therefore stability. Applying Lemma 0 provides a smooth CMC foliation Σ_t. All subsequent statements apply to sufficiently small one-sided neighborhoods of $\Sigma = \Sigma_0$; w.l.o.g. we choose $t \geq 0$, and we take the "outward" direction – in particular the normal ν to the leaves – pointing towards increasing t.

As in Lemma 1 we distinguish two main cases, but now subtleties in point 1. require due attention.

1.) $\mathbf{H_t \neq 0}$ for some $\mathbf{t \neq 0}$. We observe that by a mean value argument there is a neighborhood of some s such that for all Σ_t in that neighborhood $H_t > 0$ and strictly monotonically increasing in the outward direction.

The next step is to decompose the given Killing vector ξ into the normal and tangential parts to Σ_s, viz. $\xi = \xi^\perp + \xi^\parallel = \psi\nu + \xi^\parallel$ where s is the point selected above and $\psi \in C^\infty$ is the "lapse" of the foliation. Now the key step is to show that ξ is tangent to all Σ_t in the neighborhood of s i.e. $\xi^\perp = \psi\nu \equiv 0$.

We first observe that this tangency necessarily holds on a strictly stable Σ as the following calculation (cf. e.g. Prop. 2.12 in [6]) shows

$$0 = \delta_\xi H = \delta_{\psi\nu} H = \mathcal{L}_\nu \psi. \tag{2.3}$$

Here we have used that the variation of H along the Killing vector ξ vanishes on Σ, and definition (2.1) of the stability operator $\mathcal{L}_\nu$. But if $\psi \not\equiv 0$, (2.3) implies that ψ is an eigenfunction of $\mathcal{L}_\nu$ with eigenvalue zero, which contradicts strict stability.

In the general case where Σ is just least area rather than strictly stable we proceed as follows. We assume again $\psi \not\equiv 0$ and arrive at a contradiction.

Choosing a CMC surface Σ_s as above, namely with $H_s > 0$ and strictly monotonically increasing outward, the flow corresponding to the isometry generates a family $(\Sigma_{s,u})_{u\in(-\epsilon,\epsilon)}$ of surfaces around $\Sigma_{s,0} = \Sigma_s$ which all have the same mean curvature H_s as Σ_s, see Figures 2 and 3.

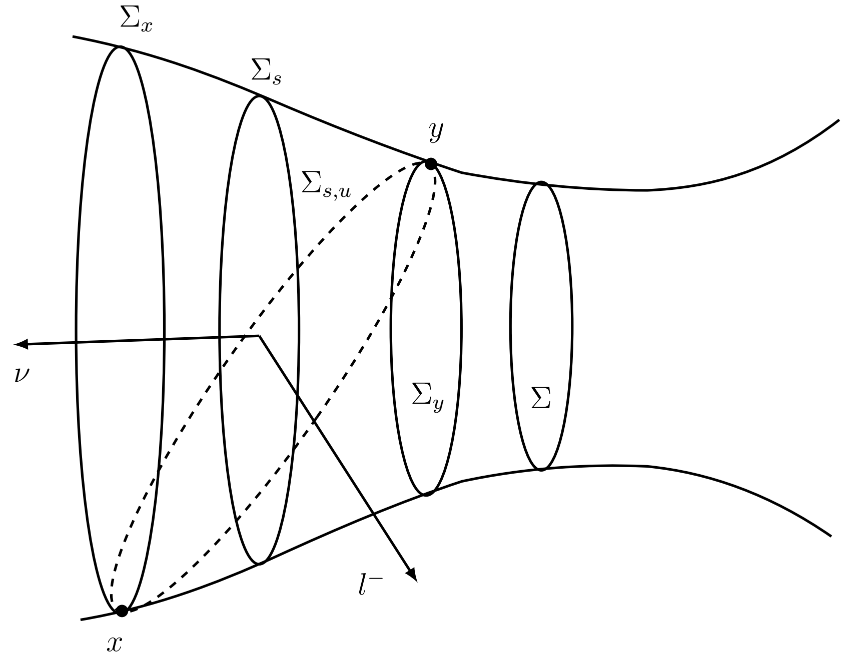

Figure 2: The setup in case 1.)

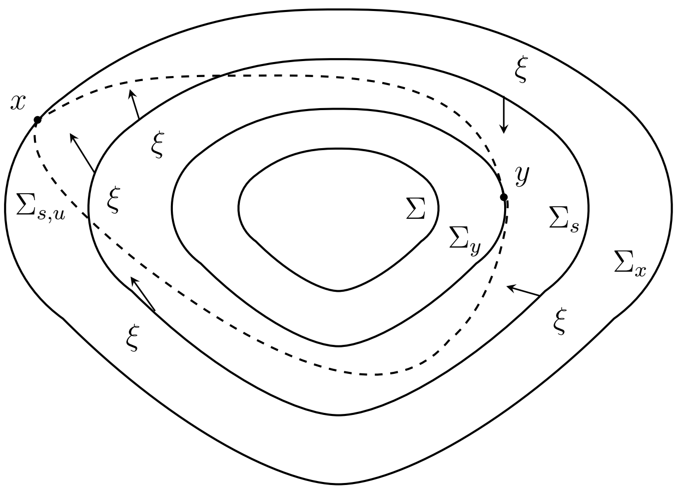

Figure 3: The setup in case 1.)

Now each $\Sigma_{s,u}$ intersects the leaves of the given CMC foliation in a neighborhood of Σ_s such that for $|u|$, $|v|$ small enough, $\Sigma_{s,u}$ touches some Σ_x from the inside at some point x such that $0 < H_s < H_x$, while $\Sigma_{s,v}$ touches some Σ_y

from the outside at some point y with $0 < H_y < H_s$. We note that the points x and y always lie to the exterior and interior of Σ_s, respectively. This fact entails that, if $\xi^\perp$ changes direction on Σ_s both points lie on the same surface $\Sigma_{s,u} = \Sigma_{s,v}$ for some $u = v$; this case is depicted in Figures 2 and 3.

In any case, and at either point this contradicts the maximum principle for the quasilinear mean curvature operators acting on the graph functions of the respective pairs of surfaces near their touching points (cf. Thm. 2.4. of [9]) unless the family $(\Sigma_{s,u})$ agrees with Σ_s, a contradiction. In particular $\psi \equiv 0$ as claimed.

Finally, we are in the position to invoke condition (2) of Theorem 2. Using the notation introduced in Section 1 we write $\zeta = \xi/\|\xi\|$ for the normalized Killing vector and $\mathcal{K}^\perp = \mathcal{K}(\perp,\perp) = \mathcal{P}^T\mathcal{K}\mathcal{P}$ where $\mathcal{P}$ is the projection orthogonal to ξ.

Let $\mathcal{O}$ be the orthogonal matrix which diagonalizes $\mathcal{K}^\perp$, i.e. $\bar{\mathcal{K}}^\perp = \mathcal{O}\mathcal{K}^\perp\mathcal{O}^T = \mathrm{diag}(\mu, \widehat{\mu})$, where we choose $\mu \leq \widehat{\mu}$. We also introduce an orthonormal basis (ζ, ρ) of the tangent space to Σ_s, define $\bar{\rho} = \mathcal{O}\rho$ and denote the components of $\bar{\rho}$ by $\bar{\rho}_1, \bar{\rho}_2$.

We obtain

$$\begin{aligned} \|\xi\|^2 \operatorname{tr}_{\Sigma_s} \mathcal{K} &= \|\xi\|^2 \left(\mathcal{K}(\zeta,\zeta) + \mathcal{K}^\perp(\rho,\rho)\right) = \mathcal{K}(\xi,\xi) + \|\xi\|^2 \bar{\mathcal{K}}^\perp(\bar{\rho},\bar{\rho}) = \\ &= \mathcal{K}(\xi,\xi) + \|\xi\|^2(\mu\bar{\rho}_1^2 + \widehat{\mu}\bar{\rho}_2^2) \geq \mathcal{K}(\xi,\xi) + \|\xi\|^2\mu \end{aligned} \tag{2.4}$$

since $\bar{\rho}$ is also a unit vector.

Therefore, from condition (1.2) of Theorem 2, $\operatorname{tr}_{\Sigma_s}\mathcal{K} \geq 0$, and thus (1.1) implies that $\theta_s^- < 0$. As in the proof of Lemma 1 this property is inherited by the cover, i.e. $\tilde{\theta}_s^- < 0$, and application of Penrose's singularity theorem (with respect to the non-compact end in the $-\nu$-direction from $\tilde{\Sigma}$) contradicts the required null geodesic completeness within M.

2.) $\mathbf{H_t \equiv 0\ \forall\ t}$**.** Now Σ is marginally stable and each Σ_t is minimal. There are two more cases to distinguish.

a.) For all leaves Σ_t**,** $\xi^\perp = \psi\nu \equiv 0$.

Using calculation (2.4) and condition (1.2) of Theorem 2, we now conclude $\operatorname{tr}_{\Sigma_t}\mathcal{K} \geq 0$, and thus (1.1) implies that for every minimal surface Σ_t, we have $\theta^\pm \leq 0$.

From here on, we can continue with the reasoning of the proof of point 2 of Lemma 1. This yields conclusion (i) except for the statement on the topology of Σ.

To show the latter we recall the relations $\chi = 2(1-\mathfrak{g})$ and $\chi = 2 - \mathfrak{g}$ between genus $\mathfrak{g}$ and Euler number χ for orientable and non-orientable 2-surfaces, respectively. The Euler number is also the sum of all indices of the (isolated) zeros of ξ, by virtue of the Poincaré-Hopf theorem. The zeros of ξ are necessarily isolated since Σ is two-dimensional [10, Thm. 8.1.5], and the index of an isolated zero of a Killing vector field is $+1$ since they're rotations in a neighborhood of the zero. This restricts the topology of an orientable surface Σ (or of its orientable double cover $\tilde{\Sigma}$) to the sphere or the torus, and to the projective plane or the Klein bottle for a non-orientable Σ.

b.) There exists a leaf Σ such that $\xi^\perp = \psi\nu \not\equiv 0$.

Now calculation (2.3) does imply that the lapse ψ is an eigenfunction of $\mathcal{L}_\nu$ with eigenvalue zero. By stability, ψ is the unique (up to a constant) principal eigenfunction of $\mathcal{L}$ which has constant sign. Reversing the direction of ξ if convenient yields $\psi > 0$. Since the given isometry forms a $U(1)$ group this implies conclusion (ii), i.e. a foliation by disjoint leaves Σ_t which are all isometric to each other. Note that requirement (2) of Theorem 2 is not used in this case. Accordingly, we do not obtain totally geodesic leaves in general.

□

Remark 6. The fixed point set of an isometry on a 3 dimensional manifold is either empty or a (possibly disconnected) one-dimensional subset, cf. [11], Thm. II.5.3.(1). In the preceding Lemma, the case of the free action corresponds to the toroidal case of (i) or to case (ii), while the one-dimensional set of fixed points are the circles formed by the axes points of the leaves in case (i). See Examples 3–8 in Section 4.

3 Proofs of the main results

In connection with the subsequent proof, we recall the *prime decomposition* of closed 3-manifolds: each such orientable 3-manifold V can be decomposed uniquely into a finite connected sum of primes V_i, viz. [12]

$$V = V_1 \# V_2 \# \cdots \# V_k, \tag{3.1}$$

where each prime is either a spherical space, diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$, or a $K(\pi, 1)$-manifold.

Proof of Theorems 1 and 2. Assume that V is not a spherical space and M is past null geodesically complete. We can assume V is orientable by passing to the orientable double cover if necessary.

We claim that V or a finite cover thereof has nontrivial second homology. To see this, recognize that by the prime decomposition of three-manifolds, V is homeomorphic to $V_1 \# \cdots \# V_k$, where each V_i is a prime manifold. Consequently, $H_2(V) = H_2(V_1) \oplus \cdots \oplus H_2(V_k)$. If some V_i is topologically $\mathbb{S}^1 \times \mathbb{S}^2$, then $H_2(V) \neq 0$. If some V_i is aspherical (i.e., the universal cover of V_i is contractible), then V_i has a finite cover with nonvanishing second homology by the positive resolution of the virtual $b_1 > 0$

conjecture [5]; in this case, V also has a finite cover with nontrivial second homology (see [2, Lem. 6]). Lastly, if each V_i is a spherical space, then the arguments below the proof of [2, Lem. 6] show the desired result. This proves the claim.

Thus we can assume $H_2(V) \neq 0$. From well-known results in geometric measure theory, V contains an embedded, nonseparating, two-sided, minimal, least-area surface Σ.

We first prove Theorem 1 by showing that conclusion (iii) holds. Since Σ is two-sided and nonseparating, we can cut V along Σ to produce a compact manifold W with two boundary components – each one isometric to Σ. We construct a covering $\tilde{V}$ of V by gluing $\mathbb{Z}$ copies of this compact manifold W end to end. See e.g. the proof of Prop. 5 of [2]. It's clear that the hypotheses of Lemma 1 are satisfied for some particular copy of Σ in the cover $\tilde{V}$. Thus there is a neighborhood of Σ within V, diffeomorphic to $(-\epsilon, \epsilon) \times \Sigma$, such that its leaves Σ_t are totally geodesic for all t. Using the traced Gauss equation on each Σ_t, the compactness of V implies that the scalar curvature of the Σ_t stays uniformly bounded. Therefore, by standard results (e.g. the "Compactness Theorem" in Sect. 2 of [13]), the foliation has a limit surface Σ_ϵ, which is embedded, compact, and minimal. Moreover Σ_ϵ is two-sided: if not, then Σ would have separated V. Therefore, by applying Lemma 1 again, there is a totally geodesic foliation near Σ_ϵ. The maximum principle implies that Σ_ϵ is diffeomorphic to Σ. A continuity argument now shows that $W \approx [0,1] \times \Sigma$ and each Σ_t is totally geodesic. Thus V is a surface bundle over $\mathbb{S}^1$ with totally geodesic fibers.

To prove Theorem 2, we proceed as above but invoke Lemma 2 instead of Lemma 1. (To this end, note that the minimal surface constructed above is least area.) If (i) of Lemma 2 holds, then we proceed analogously as in the previous paragraph. If (ii) holds, then we obtain (iv) of Theorem 2 immediately.

□

Proof of Proposition 1. Assume M is past null geodesically complete. The main result in [14] and Theorem 5.1 in [15] yield an incompressible, two-sided, least area immersion $f\colon \Sigma \to V$ of genus $g \geq 1$. (Here two-sided means that there is a global normal vector field defined on f.) This immersion is either embedded or double covers an embedded one-sided surface K. We discuss these cases in turn.

I. f is an incompressible, two-sided least area embedding. By general covering space theory, there is a covering $p\colon \tilde{V} \to V$ such that $\pi_1(\tilde{V}) \cong \pi_1(\Sigma)$. By the lifting criterion, f lifts to $F\colon \Sigma \to \tilde{V}$ which is a two-sided minimal embedding (in fact it is least area in its homotopy class by Theorem 3.4 in [15]). It must be that Σ separates $\tilde{V}$. If not, then there is a loop in $\tilde{V}$ which traverses Σ once and therefore generates an element in $\pi_1(\tilde{V}) \setminus \pi_1(\Sigma)$, contradicting the construction of $\tilde{V}$[2].

[2]This is a consequence of a more general fact. In short: If Σ were nonseparating in $\tilde{V}$, then $\pi_1(\tilde{V}) \cong A*_{\pi_1(\Sigma)}$ (see e.g. [16, Proposition 1.2]) but then $|\pi_1(\tilde{V}) : \pi_1(\Sigma)| = \infty$, a contradiction to our assumptions.

Since Σ separates $\tilde{V}$, we have that $\tilde{V} \setminus \Sigma = U \sqcup W$ with $\partial U = \partial W = \Sigma$. Applying the Seifert-van Kampen theorem, we get the following commutative diagram

$$\begin{tikzcd}
 & \pi_1(\bar{U}) \arrow[d, "j_1"] \arrow[dr, "\phi_1"] & \\
\pi_1(\Sigma) \arrow[ur, "i_1"] \arrow[r, "i_*"] \arrow[dr, "i_2"'] & \pi_1(\tilde{V}) & \pi_1(\bar{U}) * \pi_1(\bar{W}) / \overline{\{i_1(g) i_2(g)^{-1}, g \in \pi_1(\Sigma)\}} \arrow[l, "k"'] \\
 & \pi_1(\bar{W}) \arrow[u, "j_2"] \arrow[ur, "\phi_2"'] &
\end{tikzcd}$$

By construction and Seifert-van Kampen i_* and k are isomorphisms. But then i_1 and i_2 are injective and j_1 and j_2 are surjective. This means in particular that Σ is incompressible in $\bar{U}$, $\bar{W}$ and those manifolds are irreducible (see also [17]). From the injectivity of i_1 and i_2 we can conclude that ϕ_1 and ϕ_2 are injective (Thm. 11.67 in [18]) and thus j_1 and j_2 must be isomorphisms. (See 17.2 in [19].)

Aiming at a contradiction, assume $\bar{U}$ is compact. Then $\pi_1(\bar{U}) \cong \pi_1(\Sigma)$ together with Thm. 10.2 in [20] implies that $\bar{U}$ is a trivial I-bundle over Σ. This entails that $\bar{U}$ has two disjoint boundary components. But from above, Σ is separating and $\tilde{V}$ has no boundary, which is a contradiction. It follows that $\bar{U}$ is noncompact, and the same reasoning applies to $\bar{W}$ as well.

With these arguments at hand we are now in a situation where we can apply Lemma 1. Thus there exists a neighborhood B of Σ in V such that B is diffeomorphic to $(-\epsilon, \epsilon) \times \Sigma$ and $\Sigma_t = \{t\} \times \Sigma$ is totally geodesic within V and a MOTS within M. As in the previous case, we can apply a compactness theorem to obtain an embedded, compact, totally geodesic limit surface Σ_ϵ.

If Σ_ϵ is two-sided, we can apply Lemma 1 to again obtain a local foliation $\Sigma_\epsilon \times (-\delta, \delta)$ for some $\delta > 0$. But this foliation intersects B and thus by the maximum principle for minimal surfaces, we have $\Sigma_\epsilon \cong \Sigma$. This extends B to $B_\delta \cong \Sigma \times (-\epsilon, \epsilon + \delta)$. We can apply the compactness theorem to obtain a limit surface $\Sigma_{\epsilon+\delta}$. Aiming at a contradiction, assume all limit surfaces are two-sided, we can repeat these arguments indefinitely. By an open and closed argument we get a contradiction to the compactness of V.

Thus, there are nonorientable limit surfaces Σ_1 and Σ_2. By gluing two copies of $V \setminus (\Sigma_1 \cup \Sigma_2)$ along their two boundary components, we get a double cover V' of V. The boundary components of $V \setminus (\Sigma_1 \cup \Sigma_2)$ are diffeomorphic to Σ (this can be seen via [20, Thm. 10.5] or another maximum principle argument within V'). Thus, V' is a surface bundle over $\mathbb{S}^1$ with fiber Σ. Hence Σ_1, Σ_2 are double covered by Σ. Subsequently, $\Sigma_1 \cong \Sigma_2$ by the classification theorem of closed surfaces. Therefore, by construction, V has a semibundle structure, meaning it is the union of two twisted I-bundles over $\Sigma_1 \cong \Sigma_2$ glued along their common boundary Σ. That is, $V = V_1 \cup V_2$ where $V_i \cong \Sigma_i \tilde{\times} I$.

II. f **double covers a one-sided surface** K . We can go to the double cover $\tilde{V}$ such that the lift $\tilde{K}$ of K is a connected, orientable, separating, incompressible embedded minimal surface. Thus by our previous arguments $\tilde{V}$ has a semibundle structure. Furthermore by construction both components of $\tilde{V} \setminus \tilde{K}$ are diffeomorphic to $V \setminus K$. But then V is diffeomorphic to $(\tilde{K}\tilde{\times}I \setminus \tilde{K}) \cup K$, i.e. it itself has a semibundle structure (in particular $\tilde{K}$ is totally geodesic in V).

□

Proof of Proposition 2. Assume M is past null geodesically complete. There are three separate cases to consider:

I. V **is reducible.** We first show that V has to be prime. Let $\tilde{V}$ be its orientable double cover. Apply Theorem 1 to the corresponding covering spacetime $\tilde{M}$. Therefore $\tilde{V}$ is finitely covered by a surface bundle over $\mathbb{S}^1$ – call it $\tilde{V}'$. We see that $\tilde{V}'$ is prime: If the genus of the fibers is ≥ 1, then its universal cover is $\mathbb{R}^3$ and hence irreducible. This contradicts the assumed reducibility of V, [21, Prop. 1.6]. If the genus is zero, then $\tilde{V}'$ is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^2$ and hence prime.

To finish the argument, we use the following fact [12, p. 7], which we prove below. Fact: The only nonprime manifold covered by a prime manifold is $\mathbb{RP}^3 \# \mathbb{RP}^3$ double covered by $\mathbb{S}^1 \times \mathbb{S}^2$.

Since $\mathbb{RP}^3 \# \mathbb{RP}^3$ is orientable and V is not, the fact implies that V is prime. Therefore V is homeomorphic to $\mathbb{S}^1 \tilde{\times} \mathbb{S}^2$ by reducibility. Then $\tilde{V}$ is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^2$. Theorem 1 implies that $\tilde{V}$ is foliated by totally geodesic two-spheres. This foliation maps to a foliation of totally geodesic two-spheres within V.

Proof of fact: Suppose $V = V_1 \# V_2$ is not prime and it's covered by an orientable prime manifold – call it $\tilde{V}$. Since $\tilde{V}$ is reducible, it follows that $\tilde{V}$ is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^2$ or $\mathbb{S}^1 \tilde{\times} \mathbb{S}^2$. In either case, $\mathbb{Z}$ is a subgroup of $\pi_1(V)$ with finite index, so by Lemma 11.4 in [20], $\pi_1(V)$ contains a finite normal subgroup K with $\pi_1(V)/K$ isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}_2 * \mathbb{Z}_2$. K must be the trivial subgroup, otherwise it contradicts Lemma 11.2 in [20]. Thus $\pi_1(V) = \mathbb{Z}_2 * \mathbb{Z}_2$. Then by Thm. 7.1 in [20], along with the positive resolution of the elliptization conjecture, it follows that V_1 and V_2 are homeomorphic to $\mathbb{RP}^3$.

II. V **is irreducible and** $\pi_2(V) \neq 0$**.** Then V contains an embedded 2-sided projective plane $\mathbb{RP}^2$ by the projective plane theorem (see [22, Thm. 1.1] or [20, Thm. 4.12]). It follows that there exists an embedded stable least area surface Σ homeomorphic to $\mathbb{RP}^2$ [23, Prop. 2.3]. Moreover, Σ is necessarily 2-sided [24, Lem. 2]. Note that $0 \neq [\Sigma] \in \pi_2(V)$ since Σ is incompressible in V by Proposition 2.1 in [23]. By Lemma 2.1 in [25] the fundamental group of the orientable double cover $\tilde{V}$ of V is torsion-free. Since $\tilde{V}$ is reducible (by the sphere theorem), part I of this proof shows that $\tilde{V}$ is either $\mathbb{S}^1 \times \mathbb{S}^2$ or $\mathbb{RP}^3 \# \mathbb{RP}^3$ but only the former has torsion-free fundamental group. Furthermore [25, Lem. 2.1] also implies that V is homotopic to $V' := \mathbb{S}^1 \times \mathbb{RP}^2$.

Since $\pi_2(V') \cong \pi_2(\mathbb{RP}^2)$ there is a homotopy from Σ in V to a nonseparating embedded fiber $\Sigma' \cong \mathbb{RP}^2$ in V'. Therefore there is a nontrivial loop in c' in V' which intersects Σ' transversally only once. We remark that transversality is a generic condition and thus by the homotopy from V to V', this loop is homotopic to a nontrivial loop c in V which intersects Σ transversally only once (mod 2) (see also Chapters 2.3 and 2.4 in [26], in particular exercise 2 in section 2.4). Thus Σ is nonseparating.

Therefore we can pass to a noncompact cover and use arguments[3] as in Theorem 1 to deduce that V is a surface bundle over $\mathbb{S}^1$ with totally geodesic fibers diffeomorphic to $\mathbb{RP}^2$.

III. V **is irreducible and** $\pi_2(V) = 0$**.** We claim and prove in the next paragraph that there is a two-sided least area incompressible nonseparating embedding of Σ into V (with $\Sigma \neq \mathbb{S}^2, \mathbb{RP}^2$). Then we can apply the arguments of the proof of Theorem 1 to conclude that V is a surface bundle over a circle.

We now prove the claim. First, since V is nonorientable, by [20, Lemmas 6.6 and 6.7] it contains a two-sided, incompressible, nonseparating embedding $g\colon \Sigma \to V$. Moreover, V contains no two-sided $\mathbb{RP}^2$ since $\pi_2(V) = 0$; therefore it's P^2-irreducible and hence a Haken manifold. There exists a two-sided incompressible least area immersion $f\colon \Sigma \to V$ homotopic to g [15, Thm. 3.1]. Then either f is an embedding or it double covers a one-sided surface K and $g(\Sigma)$ bounds a submanifold of V which is a twisted I-bundle over a surface isotopic to K [15, Thm. 5.1]. However, the latter case contradicts the fact that g is nonseparating. Lastly, since g is nonseparating and f is homotopic to g, it follows that f is also nonseparating since the intersection number mod 2 is preserved under homotopy (see the corollary on page 79 of [26]).

$\square$

We give examples where (ii) of Proposition 2 applies for each of the three scenarios in the proof. For I, consider the Nariai spacetime (Example 4 in Section 4) quotiented by a twist on the spatial slices yielding the orientatable double covering $\mathbb{S}^1 \times \mathbb{S}^2 \to \mathbb{S}^1 \tilde{\times} \mathbb{S}^2$. Similarly, for II, quotient the Nariai spacetime via its spatial slices $\mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{RP}^2$. For III, consider a flat vacuum spacetime (Example 8 in Section 4) with spatial topology $\mathbb{S}^1 \times \mathbb{K}$ where $\mathbb{K}$ is the Klein bottle.

Proof of Proposition 3. Assume that M is past null geodesically complete. From the first part of the proof of Proposition 2, we know that V is homeomorphic to $\mathbb{RP}^3 \# \mathbb{RP}^3$.

[3] Note that the construction of the cover presented in the proof of Theorem 1 only used that Σ was nonseparating and two-sided; orientability was not used.

This implies that there is an embedding $g\colon \mathbb{S}^2 \to V$ which separates and generates a nontrivial element of $\pi_2(V)$ [21, Prop. 3.7 and 3.10]. Then, by [27, Thm. 7] and [28, Thm. 5.8], g is homotopic to a non-trivial minimal immersion $f\colon \mathbb{S}^2 \to V$ which is either embedded or double covers an embedded projective plane. In the former case we can apply the same arguments as in the previous results to obtain a totally geodesic fibration within V where the fibers are MOTS/MITS within M. In the latter case we can lift f to a double cover V', which is also homeomorphic to $\mathbb{RP}^3 \# \mathbb{RP}^3$, to find a minimal embedding $f'\colon \mathbb{S}^2 \to V$ homotopic to f.

□

Remark 7. The proof of Theorems 1 and 2 relied on the positive resolution of the "virtual $b_1 > 0$ conjecture" to obtain a minimal genus $g \geq 1$ surface Σ embedded within a noncompact cover $\tilde{V}$ of V. Although not needed for this paper, we remark another way in which one can obtain a minimal embedding. If V is a 3-dimensional closed, irreducible, orientable manifold with $|\pi_1(V)| = \infty$, then the positive resolution of the surface subgroup conjecture [29] along with results from [14] shows that there is a genus $g \geq 1$ surface Σ and a least area immersion $f\colon \Sigma \to V$. Consider the noncompact covering $p\colon \tilde{V} \to V$ as in the proof Proposition 1 and the lift $F\colon \Sigma \to \tilde{V}$. Since V is irreducible, so is $\tilde{V}$ ($\pi_2(\tilde{V}) = 0$ by the homotopy lifting property of coverings). Therefore F is a homotopy equivalence. Then, by Thm. 2.1 in [15], it follows that $F\colon \Sigma \to \tilde{V}$ is an embedding.

4 Discussion, Examples

While Theorems 1 and 2 cover a wide range of manifolds, their motivation stems from General Relativity, i.e. from solutions of Einstein's equations. In the following examples we restrict ourselves to the vacuum case, i.e. $\mathrm{Ric} = \Lambda g$, $\Lambda \geq 0$; in fact we take $\Lambda > 0$ except for Examples 2, 8 and 9.

The topology of V is spherical in Examples 1 and 2, and $\mathbb{S}^2 \times \mathbb{S}^1$ in Examples 3–7. While the $\mathbb{S}^2$-factors are round in Examples 3–5, they are only axially symmetric in Examples 6 and 7, but with additional (t-ϕ)-symmetries in the data, as defined below. As to the $\mathbb{S}^1$-direction, Examples 3, 4 and 6 enjoy the corresponding continuous symmetry. In fact these three examples have symmetry groups containing $U(1) \times U(1)$ and satisfy (1.2) with respect to both Killing fields. They thus provide examples for the simultaneous appearance of cases (iii) and (iv) in Theorem 2, cf. Remark 3. The corresponding spacetimes M have trivial (Example 3) rather subtle (Example 4) or largely unknown (in)completeness structures (Example 6, Remark 8). On the other hand, in Examples 5 and 7 there is no symmetry in the $\mathbb{S}^1$-direction, and the area of the $\mathbb{S}^2$-sections has minima and maxima (i.e. strictly stable or strictly unstable extrema). This entails horizons and incomplete geodesics according to cases (i) in Theorems 1 and 2. Finally we consider locally flat and hyperbolic $K(\pi, 1)$ geometries in Examples 8 and 9.

Our Theorems 1 and 2 are admittedly of limited value for predicting the singularity structure of a spacetime from initial data. However, in Examples 4 and

9 geodesic incompleteness was (or would be) non-trivial to obtain by other means. In any case, the value of our results rather lies in providing a classification of the general case, i.e. without symmetries and beyond solutions of Einstein's equations. We nevertheless believe that all subsequent examples are useful to illustrate the applicability of our results and some methods of the proofs.

Example 1 (Spherical slices in de Sitter spacetime)**.**
We consider de Sitter spacetime given by the manifold $\mathbb{R} \times \mathbb{S}^3$ and metric given by

$$\begin{aligned} ds^2 &= -dt^2 + \frac{3}{\Lambda}\cosh^2(\sqrt{\frac{\Lambda}{3}}t)[d\tau^2 + \sin^2\tau(d\vartheta^2 + \sin^2\vartheta d\phi^2)] \\ & \quad t \in (-\infty, \infty) \quad \tau, \theta \in [0, \pi) \quad \phi \in [0, 2\pi). \end{aligned} \tag{4.1}$$

The extrinsic curvature of the round spheres $\mathbb{S}^3$ at $t = const.$ reads

$$\mathcal{K} = \frac{1}{2}\frac{d}{dt}g = g\sqrt{\frac{\Lambda}{3}}\cosh\sqrt{\frac{\Lambda}{3}}t\ \sinh\sqrt{\frac{\Lambda}{3}}t. \tag{4.2}$$

Theorem 0 applies to all round spheres $\mathbb{S}^3$ for all $t > 0$. (or quotients thereof), while Theorem 1 applies for all $t \geq 0$. The well-known geodesic completeness is consistent with the spherical topology of the Cauchy surfaces.

Example 2 (The Taub-NUT spacetime [30] [31])**.**
This 2-parameter family of $\Lambda = 0$ solutions can be written as

$$ds^2 = -U^{-1}dt^2 + 4\ell^2 U(d\psi + \cos\theta d\phi)^2 + (t^2 + \ell^2)(d\theta^2 + \sin^2\theta d\phi^2)$$

where $U(t) = -1 + \frac{2(mt+\ell^2)}{t^2+\ell^2}$ and $m, \ell \in \mathbb{R}$.

The $t =$ const. surfaces are diffeomorphic to $\mathbb{S}^3$, which is parametrized by the Euler angles $\psi \in [0, 4\pi);\ \theta \in [0, \pi),\ \phi \in [0, 2\pi)$. The "Taub-region" $U > 0$ is given by $m - \sqrt{m^2 + \ell^2} < t < m + \sqrt{m^2 + \ell^2}$.

Selecting now (V, g_{ij}) to be the surface $t = 0$, the extrinsic curvature reads

$$\mathcal{K} = \frac{\sqrt{U}}{2}\frac{\partial}{\partial t}g = 4m \begin{pmatrix} 1 & 0 & \cos\theta \\ 0 & 0 & 0 \\ \cos\theta & 0 & \cos^2\theta \end{pmatrix} \cong 4m \begin{pmatrix} 1 + \cos^2\theta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\cong$ denotes diagonalisation. This matrix is obviously 2-convex which allows application of Theorem 1. While the Taub region is known to be null geodesically incomplete (as a maximally extended globally hyperbolic spacetime), V is obviously a spherical space, cf. Remark 1.

Example 3 ($\mathbb{S}^2 \times \mathbb{S}^1$ slices in Nariai spacetime)**.**
The Nariai spacetime is a geodesically complete solution which reads

$$\begin{aligned} ds^2 &= -dt^2 + \frac{1}{\Lambda}\left(\cosh^2\sqrt{\Lambda}t\, d\alpha^2 + d\vartheta^2 + \sin^2\vartheta d\phi^2\right) \\ & \quad t \in (-\infty, \infty) \quad \alpha \in [0, L),\ L \in \mathbb{R}, \quad \theta \in [0, \pi) \quad \phi \in [0, 2\pi). \end{aligned} \tag{4.3}$$

The $t = const.$ slices have topology $\mathbb{S}^2 \times \mathbb{S}^1$; we call the $t = 0$ slice the "Nariai torus" henceforth. We obtain for the only non-vanishing component of the extrinsic curvature $\mathcal{K}$

$$\mathcal{K}_{\alpha\alpha} = \frac{1}{2}\frac{d}{dt}g_{\alpha\alpha} = \Lambda^{-1/2}\cosh\sqrt{\Lambda}t\ \sinh\sqrt{\Lambda}t. \tag{4.4}$$

Therefore, $\mathcal{K}$ is positive semidefinite for all $t \geq 0$. Now Theorem 1 applies with consequence (iii), while Theorem 2 also applies, with conclusion (iii) with respect to the Killing vector $\partial/\partial\phi$ but conclusion (iv) with respect to $\partial/\partial\alpha$. In this case, we need not pass to a finite cover to obtain the rigidity results.

Example 4 (Products of constant curvature metrics)**.**
Fajman and Kröncke [32] have analyzed spacetimes where the spatial sections are products of Einstein spaces. In 3 spatial dimensions their ansatz takes the form [32, Thm. 1.6]

$$ds^2 = -dt^2 + \frac{b(t)^2}{\Lambda}d\alpha^2 + \frac{a(t)^2}{\widehat{\Lambda}}[d\vartheta^2 + \sin^2\vartheta d\phi^2] \tag{4.5}$$

where $\widehat{\Lambda}$ is a constant. The authors consider the initial value problem with $a(0) = 1$ and $[db/dt](0) = 0$. Then the constraint implies $\widehat{\Lambda} \leq \Lambda$. While $\widehat{\Lambda} = \Lambda$ reduces to the Nariai case $a \equiv 1$, $\widehat{\Lambda} < \Lambda$, yields incompleteness in the sense of diverging curvature scalars.

As to our results, at $t = 0$ the extrinsic curvature

$$\mathcal{K} = \frac{1}{2}\frac{d}{dt}g = \widehat{\Lambda}^{-1}\frac{da}{dt}\,\mathrm{diag}[0, 1, \sin^2\vartheta] \tag{4.6}$$

is strictly 2-convex. (The degeneracy at $\vartheta = 0$ is of course an artifact of the polar coordinates.) Since this $t = 0$ section of (4.5) is obviously not a spherical space, Theorem 0 yields null geodesic incompleteness which is consistent with the findings of Thm. 1.6. in [32]. In this example Theorems 1 and 2 are inconclusive as (4.5) is also consistent with consequences (iii) and (iv).

Example 5 (Time-symmetric slices in Kottler spacetime)**.**
In the well-known Kottler (also Schwarzschild de Sitter) spacetime we restrict ourselves to the time-symmetric slices. They are conformal to the Nariai torus with metric

$$ds^2 = r^2\left(\frac{dr^2}{\sigma} + d\vartheta^2 + \sin^2\vartheta d\phi^2\right) = r(\alpha, m, \Lambda)^2\left(d\alpha^2 + d\vartheta^2 + \sin^2\vartheta d\phi^2\right). \tag{4.7}$$

Here $m \in \mathbb{R}$, $m < 1/(3\sqrt{\Lambda})$, $\sigma = (r^2 - 2mr - \Lambda r^4/3)$.

The positive zeros $r = r_b$ and $r = r_c$ of σ give the black hole and the cosmological horizons. The coordinates are obviously related via

$$\frac{d\alpha}{dr} = \sigma^{-\frac{1}{2}} \qquad r(\alpha = 0) = r_b. \tag{4.8}$$

We have to require that the period $P(m,\Lambda)$ defined below "fits" on the Nariai torus, which restricts its size L via

$$P(m,\Lambda) = 2\int_{r_b}^{r_c} \sigma^{-\frac{1}{2}} dr = \frac{L}{j} \qquad (j \in \mathbb{N}). \tag{4.9}$$

Thus the Cauchy surface contains j pairs of black hole and cosmological horizons, and (4.9) entails a relation between the parameters m, L and j. As is well-known and suggested by the "horizon" terminology, the spacetime is (future and past) null geodesically incomplete. Thus we are in case (i) of Theorems 1 and 2; the other cases are ruled out since $\partial/\partial\alpha$ is not Killing and the foliation is not totally geodesic.

In connection with the following two examples, we revisit the notion of (t-ϕ)-symmetric data. For data with Killing field ξ we first recall some definitions from Section 1 (before Thm 2): $\|\xi\|$ is the norm of ξ, $\zeta = \|\xi\|^{-1}\xi$ its normalisation and the $(1,1)-$ tensor $\mathcal{P} = \mathcal{I} - (\zeta \otimes \zeta^*)$ projecting onto the 2-space orthogonal to ξ, where $\mathcal{I}$ is the identity. We denote the components of $\mathcal{K}$ with respect to this decomposition by $\mathcal{K}(\zeta,\zeta)$, $\mathcal{K}(\zeta,\perp) = \zeta^*\mathcal{K}\mathcal{P}$ and $\mathcal{K}^\perp = \mathcal{K}(\perp,\perp) = \mathcal{P}^T\mathcal{K}\mathcal{P}$.

The following definition seems to appear first in [33] (eqs. (D5) and (D6)). There (and elsewhere) it is used in connection with axially symmetric data, (i.e. the isometry has fixed points) which we do not require.

Definition ("(t-ϕ)-symmetric" data)**.**
Initial data $(V, g, \mathcal{K})$ are called "(t-ϕ)-symmetric" if they are $U(1)$ symmetric with Killing field ξ such that $\mathcal{K}(\xi,\xi)$ and $\mathcal{K}(\perp,\perp)$ vanish identically.

This definition captures the idea that the spacetime should be invariant under the simultaneous change of the direction of time and rotation: Such changes reverse the signs of $\mathcal{K}$ and ξ, respectively, and invariance requires that all products containing an odd number of these tensors have to vanish. We are thus left with $\kappa = \mathcal{K}(\zeta,\perp)$ as the only non-vanishing components. We further observe that the eigenvalues of $\mathcal{K}$ are $\{-c, 0, c\}$ where c is the norm of the 2-vector κ. In particular, the data are maximal. Now this class is obviously admitted by condition (1.2) and therefore covered by Theorem 2, but incompatible with 2-convexity as required in Theorem 1 unless $\mathcal{K}$ vanishes identically.

Example 6 ("Squashed Nariai" data)**.**
As in Example 5 we consider data for Einstein's equations with $\Lambda > 0$ conformal to the Nariai torus but now with conformal factor depending on ϑ rather than α, viz.

$$ds^2 = \varphi(\vartheta, J, \Lambda)^4 \left(d\alpha^2 + d\vartheta^2 + \sin^2\vartheta d\phi^2\right). \tag{4.10}$$

Here the constant J is called the angular momentum—as the name suggests the data contain a corresponding "momentum" $\mathcal{K}(\vartheta, J, \Lambda)$. We sketch the proof that such $\mathcal{K}$ and φ exist, which is rather intricate, cf. [34, 35] for details.

We first recall the (suitably adapted) conformal method for solving the constraints. We assume we are given an arbitrary "seed manifold" $(\widehat{V}, \widehat{g})$ together with a trace-free, divergence-free tensor $\widehat{\mathcal{K}}$. We look for initial data solving the constraints

$$\mathrm{div}_g \mathcal{K} = 0 \qquad {}^g R = 2\Lambda + \mathcal{C}_g(\mathcal{K} \otimes \mathcal{K}) \tag{4.11}$$

where ${}^g R$ is the scalar curvature of (V, g) and $\mathcal{C}_g$ is a double contraction. From the conformal behavior of the scalar curvature it is clear that

$$\mathcal{K} = \varphi^{-2}\widehat{\mathcal{K}} \qquad g = \varphi^4 \widehat{g} \tag{4.12}$$

solve (4.11) provided φ satisfies the "Lichnerowicz"-equation on $(\widehat{V}, \widehat{g})$,

$$\left(\widehat{\Delta} - \frac{1}{8}\widehat{R}\right)\varphi = -\frac{1}{4}\Lambda\varphi^5 - \frac{\mathcal{C}_{\widehat{g}}(\widehat{\mathcal{K}} \otimes \widehat{\mathcal{K}})}{8\varphi^7}\,. \tag{4.13}$$

We now choose for $(\widehat{V}, \widehat{g})$ the Nariai torus and for $\widehat{\mathcal{K}}$ the symmetric, trace-free tensor

$$\widehat{\mathcal{K}} = 3J\Lambda^{5/2}\,\aleph \otimes_s \Phi \tag{4.14}$$

where $\aleph = \partial/\partial\alpha$ and $\Phi = \partial/\partial\phi$ are the orthogonal Killing vectors on the torus, and $\otimes_s$ is the symmetrized tensor product. This tensor is indeed divergence-free, i.e. $\mathrm{div}_{\widehat{g}}\,\widehat{\mathcal{K}} = 0$.

It remains to solve (4.13) with the input from above. From results by Premoselli [36], there is a $J_{\max} \in \mathbb{R}_0^+$ such that for each $J < J_{\max}$, there exists a unique smooth, positive solution φ_J of (4.13) which is stricty stable (in the sense that the lowest eigenvalue of the linearisation of (4.13) at φ_J is positive). Moreover, for $J = J_{\max}$ there is a unique marginally stable solution (i.e. the lowest eigenvalue vanishes).

Each of these solutions φ_J now leads to a solution of the constraints $(V, g, \mathcal{K})$ which (due to their stability) share the $U(1) \times U(1)$ symmetry of the coefficients in the Lichnerowicz equation (4.13), cf. e.g. [34, Prop. 2]. This implies that the metric indeed takes the form (4.10), which is a totally geodesic foliation. Moreover, it is obvious from (4.14) that the data are both (t-ϕ)-symmetric as well as $(t-\alpha)$ symmetric as defined above. The constant J is the "Komar" angular momentum of the data with respect to the Killing field $\partial/\partial\phi$.

We now observe that our Theorem 2 does apply (w.r.t. both Killing vectors Φ and $\aleph$) but does not allow to draw any conclusions on (in)completeness of the evolved spacetime.

Remark 8. More generally, there has been studied the case that a compact and connected two-dimensional Lie group acts effectively on V (cf. [37] for a review). This reduces to the so-called Gowdy class in the case of vanishing "twist constants", and the latter vanish automatically for Λ-vacuum data and topology $\mathbb{S}^2 \times \mathbb{S}^1$. Geodesic (in)completness for this class of data has been studied in [38, 39], mainly by numerical methods. It would be interesting to investigate the applicability of our Theorem 2 in this context.

Example 7 (Kerr-de Sitter).
This is a well-known two-parameter family of stationary, axially symmetric black hole spacetimes. In Boyer-Lindquist coordinates the induced metric on maximal slices reads

$$ds^2 = \rho^2 \left(\frac{dr^2}{\sigma} + \frac{d\vartheta^2}{\chi} \right) + \frac{\sin^2\vartheta}{\kappa^2\rho^2} \left[\chi(r^2+a^2)^2 - \sigma a^2 \sin^2\vartheta \right] d\phi^2, \tag{4.15}$$

where m, a and $\kappa = 1 + \Lambda a^2/3$ are constants and

$$\sigma = (r^2+a^2)\left(1 - \frac{\Lambda r^2}{3}\right) - 2mr, \quad \rho^2 = r^2 + a^2\cos^2\vartheta, \quad \chi = 1 + \frac{\Lambda a^2 \cos^2\vartheta}{3}. \tag{4.16}$$

The constants m and a satisfy bounds given by the extreme solutions, but also "absolute" bounds in terms of Λ alone, cf. e.g. [40, 41] for details. The data are (t, ϕ)-symmetric. As in the Kottler case of Example 5 we restrict ourselves to the region $r \in [r_b, r_c]$ outside the horizons where $\sigma \geq 0$. At $\sigma = 0$ the metric (4.15) is regularized by replacing r by α defined via (4.8) but with σ from (4.16). While the solution is no longer conformal to the Nariai torus, we can still fit the metric on a $\mathbb{S}^2 \times \mathbb{S}^1$ manifold of length L provided the period $P(m, a, \Lambda)$ satisfies

$$P(m, a, \Lambda) = 2\int_{r_b}^{r_c} \sigma^{-\frac{1}{2}} dr = \frac{L}{j} \qquad j \in \mathbb{N} \tag{4.17}$$

and we still obtain an array of j pairs of horizons [42]. In consistency with well-known facts and in analogy with Example 5, our Theorem 2 yields null geodesic incompleteness via point (i).

Example 8 (Flat vacuum initial data sets).
Let V be a closed flat three-dimensional manifold equipped with time-symmetric vacuum initial data for the Einstein equations. The corresponding maximal globally hyperbolic vacuum development is the Lorentzian product $\mathbb{R} \times V$, which is clearly geodesically complete. There are 10 such data sets of the form $V = \mathbb{E}^3/\Gamma$ where Γ is a discrete and fixed-point free symmetry group of the Euclidean space $\mathbb{E}^3$. There are six orientable ones $G_1, \ldots, G_6$ and four non-orientable ones $B_1, \ldots, B_4$ (see the classification in [43].) For each one Theorem 0 does not apply, but Theorem 1 does. If V is either the flat torus $G_1 = \mathbb{T}^3$ or one of other 4 possibilities with $H_2(V, \mathbb{Z}) \neq 0$, then Theorem 1 (iii) applies directly to V (without going to a cover). On the other hand, G_6 (i.e., the Hantzsche–Wendt manifold), has $H_2(G_6, \mathbb{Z}) = 0$ and is not a surface bundle over the circle. Now Theorem 1 (iii) still applies to a cover. Indeed, if one applies the construction in the proof of Theorem 1 to G_6, then one obtains its double cover G_2. However, as mentioned in the introduction, Proposition 1 applies directly to G_6 (i.e. without invoking a cover).

Example 9 (Hyperbolic initial data sets).
Let $(V, g, \mathcal{K})$ be an initial data set for the vacuum Einstein equations with or without

a cosmological constant. Assume g is a hyperbolic metric on a closed 3-manifold V and $\mathcal{K}$ is 2-convex. Then Theorem 1 implies that the resulting maximal globally hyperbolic development is past null geodesically incomplete. For if this was not the case, then there would be a three-manifold with a hyperbolic metric which is a surface bundle over $\mathbb{S}^1$ containing totally geodesic fibers. But this is impossible; see [44, Thm. 3.3] and the discussion on mapping tori below it.

It would be interesting to know if there exists a Riemannian metric h on a 3-manifold V with hyperbolic structure such that (V, h) embeds into a spacetime (M, g) where Theorem 1 (iii) applies. Example 9 shows that h cannot be taken to be the hyperbolic metric.

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