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Strong Multiplicity One for Cuspidal Automorphic Representations of GL(n)

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Sigrid Schrøder Jensen BSc

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Abstract

This thesis provides a self-contained exposition of Piatetski-Shapiro's proof of the strong multiplicity one theorem for cuspidal automorphic representations of $\mathrm{GL}(n, \mathbb{A})$, where $\mathbb{A}$ is the adele ring of an algebraic number field. The theorem asserts that two cuspidal automorphic representations whose local components are isomorphic at all but finitely many places must be equal. Central to the proof is the study of Whittaker models for local and global representations. We establish the uniqueness of global Whittaker models for irreducible admissible representations of $\mathrm{GL}(n, \mathbb{A})$, leveraging the tensor product theorem and local uniqueness results. The thesis extends proofs from Bump's *Automorphic Forms and Representations*—originally demonstrated for $n = 2$—to arbitrary $n \geq 2$. Chapters 2–3 introduce foundational material on local representations, while Chapters 4–5 develop the theory of automorphic forms, cusp forms, automorphic representations, and global Whittaker models. The culmination in Chapter 6 presents the proof of the strong multiplicity one theorem, emphasizing the interplay between local and global methods.

Zusammenfassung

Diese Arbeit enthält eine eigenständige Darstellung von Piatetski-Shapiros Beweis des "Strong multiplicity one"-Theorems für kuspidale automorphe Darstellungen von $\mathrm{GL}(n, \mathbb{A})$, wobei $\mathbb{A}$ der Adelring eines algebraischen Zahlkörpers ist. Das Theorem besagt, dass zwei kuspidale automorphe Darstellungen, deren lokale Komponenten an allen bis auf eine endliche Anzahl an Stellen isomorph sind, gleich sein müssen. Das Kernstück des Beweises besteht darin, die Whittakermodelle für lokale und globale Darstellungen zu studieren. Wir zeigen die Eindeutigkeit globaler Whittakermodelle für irreduzible zulässige Darstellungen von $\mathrm{GL}(n, \mathbb{A})$ unter Verwendung des Tensorproduktsatzes und lokaler Eindeutigkeitsergebnisse. Die Arbeit verallgemeinert Beweise aus Bumps *Automorphic Forms and Representations*, die ursprünglich nur für $n = 2$ vollzogen wurden, auf beliebige $n \geq 2$. Kapitel 2–3 präsentieren grundlegendes Material zu lokalen Darstellungen, mit dessen Hilfe in Kapitel 4–5 die Theorie der automorphen Formen, Spitzformen, automorphen Darstellungen und globalen Whittakermodelle etabliert wird. Das letzte Kapitel behandelt den Höhepunkt dieser Arbeit – den Beweis des "Strong multiplicity one"-Theorems, welcher das Zusammenspiel zwischen lokalen und globalen Methoden unterstreicht.

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Contents

Introduction

In a 1979 paper titled "Multiplicity One Theorems" [PS79], Piatetski-Shapiro proves two theorems regarding the multiplicity of irreducible admissible representations of $\mathrm{GL}_n(\mathbb{A})$ in the space of cusp forms on $\mathrm{GL}_n(\mathbb{A})$, where $\mathbb{A}$ is the adele ring of a global field F. The first theorem is called the multiplicity one theorem, and it builds upon a result by Piatetski-Shapiro and his co-author Gel'fand from thirteen years prior, that the space of cusp forms decomposes as a countable direct sum of irreducible admissible representations of $\mathrm{GL}_n(\mathbb{A})$, each occuring with finite multiplicity. The constituents of this direct sum are called cuspidal automorphic representations. The multiplicity one theorem asserts that the multiplicity of an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A})$ in the space of cusp forms is not only finite, but equal to one. In other words, if two cuspidal automorphic representations are isomorphic, then they must be equal. The multiplicity one theorem for $n = 2$ was proven in 1970 by Jacquet and Langlands, and extended to $n \geq 2$ in the years 1974–1975 independently by Shalika, Piatetski-Shapiro, and Gel'fand and Kazhdan. The second theorem is called the strong multiplicity one theorem, named as such since it is a stronger version of the multiplicity one theorem. It relates to the core property of representations of $\mathrm{GL}_n(\mathbb{A})$, that an irreducible admissible representation π is isomorphic to an infinite tensor product of local irreducible admissible representations π_v, ranging over all places v of the global field F, and each local component π_v is unique up to isomorphism. This result is referred to as the tensor product theorem in this thesis. So, while the multiplicity one theorem states that two cuspidal automorphic representations π and π' whose local components π_v and π'_v are isomorphic *at all places* must be equal, the statement of the strong multiplicity one theorem is that two cuspidal automorphic representations whose local components are isomorphic *at all but a finite number of places* must be equal. This was first due to Piatetski-Shapiro in his 1979 paper, but in his original proof of the strong multiplicity one theorem, Piatetski-Shapiro needed to further assume that the local components are isomorphic at all Archimedean places. The theorem without this assumption was later proven by Jacquet and Shalika in 1981, employing different methods from those of Piatetski-Shapiro.

The goal of this thesis is to give a complete account of Piatetski-Shapiro's proof of the strong multiplicity one theorem. To this end, we give detailed introductions to all notions mentioned above. Central to the proofs of both multiplicity one and strong multiplicity one is the study of Whittaker models for both local and global representations. Essentially, both proofs rely on the uniqueness of the Whittaker model for irreducible admissible representations of $\mathrm{GL}_n(\mathbb{A})$, which in turn, because of the tensor produtct theorem, is a direct consequence of the uniqueness of the Whittaker model for a local irreducible admissible representation. Therefore, in the chapters whose purpose is to introduce local representations, we do so not only with the goal of being able to define representations of $\mathrm{GL}_n(\mathbb{A})$ and state the tensor product theorem, but also with the goal of defining the notion of a Whittaker model and stating the theorem of uniqueness of the Whittaker model for each type of local representation; Archimedean and non-Archimedean.

A primary reference used throughout the thesis is the book by Bump from 1997 titled "Automorphic forms and representations" [Bum97], and the methodology and naming conventions in this thesis are almost entirely similar to those from there. While many proofs of significant theorems in Bump's book are primarily demonstrated for the case where $n = 2$ and occasionally for $F = \mathbb{Q}$, this thesis extends proofs included from Bump's book particularly the proof of uniqueness of global Whittaker models, to generalizations for $n \geq 2$ and any algebraic number field F. Another significant reference are the lecture notes by Cogdell from [CKM04], which guided the development of this thesis. However, Cogdell's lecture notes are not directly cited in the text, since they serve more as a survey of the field of automorphic representations and L-functions.

Chapter 1 is a short preliminary chapter whose purpose is to fix notation for important matrix groups and introduce results for integrals over quotient spaces that is used throughtout the rest of the thesis.

Chapter 2 and Chapter 3 are dedicated to the study of the two types of local representations that appear as components for an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A})$. They are called local Archimedean representations and local non-Archimedean representations, since the type only depends on whether the respective place of the global field F is Archimedean or non-Archimedean. Local non-Archimedean representations are smooth representations of locally profinite groups and are covered in Chapter 2, while local Archimedean representations are $(\mathfrak{g}, K)$-modules for real Lie groups and are covered in Chapter 3. These two types of representations are mostly unrelated, and for this reason, each type is dedicated its own chapter. However, the themes of study are identical; irreducible representations, the Hecke algebra, and Whittaker models are notions and properties explored for both. It was our intention to place greater focus on the non-Archimedean setting, and as a result, Chapter 2 requires fewer prerequisites, and provides a more self-contained account of the theory needed to prove the strong multiplicity one theorem, while Chapter 3 assumes substantial familiarity with Lie theory and functional analysis, and rather serves to fix notation and list needed results within the larger field, without further exploration.

Chapter 4 is an introduction to automorphic forms, cusp forms, and automorphic representations, with the primary goal of enabling an understanding of the statements of the multiplicity one theorems. Key concepts—such as the ring of adeles, the locally compact group $\mathrm{GL}_n(\mathbb{A})$, representations of $\mathrm{GL}_n(\mathbb{A})$, and the global Hecke algebra—are defined in this chapter.

Chapter 5 presents a detailed investigation of the global Whittaker model, in preparation for the proof of the strong multiplicity theorem. The global Whittaker model is explored from two perspectives: first, by studying Whittaker functions for cusp forms with the help of the adelic Fourier transform, and second, by examining the relationship between the global Whittaker model for an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A})$ and the local Whittaker models of its local components, through purely algebraic methods.

Chapter 6 represents the culmination of the thesis, where the strong multiplicity one theorem for cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A})$ is proven.

We assume basic knowledge of algebraic number theory (e.g. the material covered in Chapter I–V in [Lan94]), as well as familiarity with Haar measures on locally compact groups. As mentioned above, in Chapter 3 we assume substantial familiarity with Lie theory and functional analysis. However, readers without these prerequisites should still be able to follow the rest of the thesis by taking the representation theory at the Archimedean places for granted. Although all notions related to representations are defined in the thesis, the text assumes some familiarity with representation theory for either finite or infinite groups. To ensure accessibility to readers without a background in algebraic geometry, we avoid terminology from the theory of algebraic groups. Instead, we frame the discussion purely in terms of group theory and topology, and rely on explicit descriptions of matrix groups, which suffices for our purposes.

Chapter 1

Preliminaries

This thesis focuses almost exclusively on representations of the general linear group over various fields and rings, primarily local fields and the ring of adeles. For this reason, the first section of this chapter establishes the naming conventions and notations for all matrix groups appearing in the thesis. The second section addresses the definition and properties of quotient measures on the quotient space of two locally compact groups, a tool that is essential for studying Whittaker functions of cusp forms in Chapter 5.

1.1 Structure of the general linear group

For a commutative ring R, the general linear group of dimension n is the group of invertible $n \times n$ matrices with entries in R with group operation given by matrix multiplication, and it is denoted $\mathrm{GL}_n(R)$. In this section we omit the argument for the base ring, and simply write GL_n (and similarly for the names of subgroups) when making a statement that holds for any base ring. If R is a topological ring, then we equip the set of $n \times n$-matrices $\mathrm{Mat}_{n\times n}(R) \simeq R^{n^2}$ with the product topology, and under this topology, the determinant map

$$\det : \mathrm{Mat}_{n\times n}(R) \to R$$

is continuous. We endow $\mathrm{GL}_n(R)$ with the topology induced by the map

$$\mathrm{GL}_n(R) \to \mathrm{Mat}_{(n+1)\times(n+1)}(R) : g \mapsto \begin{pmatrix} g & 0 \\ 0 & \det(g)^{-1} \end{pmatrix},$$

with respect to which one can show $\mathrm{GL}_n(R)$ is a topological group. This topology might not coincide with the subspace topology induced by $\mathrm{Mat}_{n\times n}(R)$, but it does coincide if inversion on $R^\times$ endowed with the subspace topology from R is continuous. If moreover $R^\times$ is open in R, e.g. if R is a field, then $\mathrm{GL}_n(R)$ is open in $\mathrm{Mat}_{n\times n}(R)$, as $\mathrm{GL}_n(R)$ equals the preimage of $R^\times$ under the determinant map, i.e.

$$\mathrm{GL}_n(R) = \det^{-1}(R^\times).$$

We give the following names to subgroups of GL_n: the centre $Z(\mathrm{GL}_n)$ is denoted by Z_n, and note that if the base ring is a field F, then $Z_n(F)$ consists of scalar matrices

$$\begin{pmatrix} a & & \\ & \ddots & \\ & & a \end{pmatrix},$$

where $a \in F^\times$. A partition $\alpha = (n_1, \dots, n_r)$ of n is a sequence of positive integers such that $n_1 + \cdots + n_r = n$, and it is often simply written as $\alpha \vdash n$. Given a partition $\alpha \vdash n$, we define three different subgroups, each consisting of block matrices of dimensions given by α, that are denoted as follows: the *standard parabolic subgroup* P_α consisting of upper triangular block matrices, i.e.

$$P_\alpha = \left\{ \begin{pmatrix} g_1 & * & \cdots & * \\ & g_2 & \ddots & \vdots \\ & & \ddots & * \\ & & & g_r \end{pmatrix} \middle| \; g_i \in \mathrm{GL}_{n_i} \right\} \subseteq \mathrm{GL}_n;$$

the *standard Levi subgroup* M_α consisting of diagonal block matrices, i.e.

$$M_\alpha = \left\{ \begin{pmatrix} g_1 & & & \\ & g_2 & & \\ & & \ddots & \\ & & & g_r \end{pmatrix} \middle| \; g_i \in \mathrm{GL}_{n_i} \right\} \subseteq \mathrm{GL}_n;$$

and the unipotent radical of P_α is denoted U_α, and consists of upper triangular block matrices with identity matrices on the diagonal, i.e.

$$U_\alpha = \left\{ \begin{pmatrix} I_{n_1} & * & \cdots & * \\ & I_{n_2} & \ddots & \vdots \\ & & \ddots & * \\ & & & I_{n_r} \end{pmatrix} \right\} \subseteq \mathrm{GL}_n .$$

Note that $P_\alpha = M_\alpha U_\alpha$ is a semidirect product where U_α is normalised by M_α.

In later chapters, we also treat representations of subgroups where we impose that the last row is the elementary row vector $(0, \dots, 0, 1)$. The *mirabolic subgroup* of GL_n is denoted by P_n and is the largest subgroup satisfying this restriction, i.e.

$$P_n = \left\{ \begin{pmatrix} * & \cdots & * & * \\ \vdots & \ddots & \vdots & \vdots \\ * & \cdots & * & * \\ 0 & \cdots & 0 & 1 \end{pmatrix} \right\} \subseteq \mathrm{GL}_n .$$

We denote by Y_n the subgroup of P_n given by

$$Y_n = U_{(n-1,1)} = \left\{ \begin{pmatrix} I_{n-1} & * \\ 0 & 1 \end{pmatrix} \right\} \subseteq P_n,$$

and note that $Y_n(R) \cong (R^{n-1}, +)$. When considering GL_{n-1} as a subgroup of GL_n, it is always via the embedding

$$\mathrm{GL}_{n-1} \hookrightarrow \begin{pmatrix} \mathrm{GL}_{n-1} & 0 \\ 0 & 1 \end{pmatrix},$$

implying that GL_{n-1} is a subgroup of P_n. With this in mind, it is clear that $P_n = Y_n \mathrm{GL}_{n-1}$ is a semidirect product where GL_{n-1} normalises Y_n. By the *maximal unipotent subgroup*, denoted by N_n, we mean the subgroup consisting of upper triangular matrices with ones on the diagonal, i.e.

$$N_n = U_{(1,\dots,1)} = \left\{ \begin{pmatrix} 1 & * & \cdots & * \\ & 1 & \ddots & \vdots \\ & & \ddots & * \\ & & & 1 \end{pmatrix} \right\} \subseteq \mathrm{GL}_n .$$

Given an additive character $\psi : R \to \mathbb{C}^\times$ for a topological ring R (which the reader should recall is a continuous group homomorphism with the multiplicative group $\mathbb{C}^\times$ as its codomain), we always consider it as a character of $N_n(R)$ in the following way: as ψ is additive and the projection of the first upper diagonal onto R^{n-1}, i.e. the map

$$N_n(R) \to R^{n-1} : \begin{pmatrix} 1 & x_1 & \cdots & * \\ & 1 & \ddots & \vdots \\ & & \ddots & x_{n-1} \\ & & & 1 \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix},$$

is a continuous surjective group homomorphism, the map

$$\begin{pmatrix} 1 & x_1 & \cdots & * \\ & 1 & \ddots & \vdots \\ & & \ddots & x_{n-1} \\ & & & 1 \end{pmatrix} \mapsto \psi(x_1 + \cdots + x_{n-1})$$

is a well-defined character of $N_n(R)$. We also denote this character by ψ.

1.2 Haar measures and quotient space measures

This section serves as a recollection of the results related to the general Haar measure for locally compact groups that are used in the following chapters, while also fixing the notation related to integration. Throughout this section, G always denotes a locally compact Hausdorff topological group, which is also what is meant by a "locally compact group". Let $C_c(G)$ denote the space of continuous compactly supported functions $f : G \to \mathbb{C}$. We assume that the reader is familiar with the concept of a Radon measure and a Haar measure, and the existence and uniqueness of a Haar measure for a locally compact group. Throughout this section and for the following chapters, if not otherwise stated, "the Haar measure" for a general group always refers to a *right* Haar measure.

Definition 1.1 (Unimodular)**.** The group G is called *unimodular* if any right Haar measure is also a left Haar measure.

Recall that we have a unique continuous group homomorphism $\Delta_G : G \to \mathbb{R}_{>0}$ called *the module* of G, such that

$$\Delta_G(a) \int_G f(g)\, dg = \int_G f(a^{-1}g)\, dg \tag{1.1}$$

for any $f \in C_c(G)$, $a \in G$ and Haar measure dg, and note that G is unimodular if and only if $\Delta_G(G) = \{1\}$.

Let $H \subseteq G$ be a closed subgroup, and let dg and dh denote the Haar measures of G and H. For $f \in C_c(G)$, let f^H denote the function given by

$$f^H(g) = \int_H f(hg)\, dh$$

for $g \in G$.

The orbit space $H\backslash G$ is again locally compact and Hausdorff, but it is not necessarily a group, and hence does not carry a Haar measure. The goal is, regardless, to construct a unique measure on $H\backslash G$ which mimics the qualities of a Haar measure, specifically its invariance under right translation by G, and which still satisfies the analogue of the following well-known formula, when H is further assumed to be normal.

Proposition 1.2. *Suppose $H \lhd G$ is a closed normal subgroup, and let Haar measures on two of the three groups G, H, and $H\backslash G$ be given. Then there exists a unique Haar measure on the third group such that*

$$\int_G f(g)\,dg = \int_{H\backslash G}\int_H f(h\overline{g})\,dh\,d\overline{g},$$

for all $f \in C_c(G)$.

The proof of this (for left Haar measures and right quotients) can be found e.g. in [Rei68, §3.3].

In general, for a closed subgroup $H \subseteq G$, a G-invariant Radon measure on $H\backslash G$ with a similar formula need not exist. However, a sufficient condition to ensure this is to assume that both G and H are unimodular, which is the case for most of the groups we work with in the following chapters. The following theorem is a combination of Lemma 7.9 and Theorem 7.10 in [KL06] (but once again trivially modified to consider right Haar integrals and left quotients), and it establishes the desired measure on quotients in this setting.

Theorem 1.3. *Let G be unimodular and $H \subseteq G$ be a closed unimodular subgroup, and let dg and dh be the Haar measures on G and H. The map*

$$C_c(G) \to C_c(H\backslash G) : f \mapsto f^H$$

is well-defined and surjective, and for positive functions, $f = 0$ if and only if $f^H = 0$. Furthermore, with respect to dg and dh, there is a unique Radon measure $\overline{\mu}$ on $H\backslash G$ such that

$$\int_G f(g)\,dg = \int_{H\backslash G} f^H(\overline{g})\,d\overline{\mu}(\overline{g})$$

for all $f \in C_c(G)$.

The measure $\overline{\mu}$ on $H\backslash G$ is called *the quotient measure*, and in the following chapters, whenever the measures on H and G are clear and we write $\int_{H\backslash G}$, the integration is with respect to the quotient measure. For non-empty open sets $U \subseteq H\backslash G$, the quotient measure is positive and right-G-invariant, i.e.

$$\overline{\mu}(Ug) = \overline{\mu}(U)$$

for all $g \in G$.

In later chapters, we need to apply a version of Theorem 1.3 where all three integrals are over quotient spaces, and with functions in $L^1(G)$ that are not necessarily compactly supported. To this end, we require G to additionally be second countable. This is the subject of the next theorem, which is a combination of Theorem 7.12, Lemma 7.13, and Corollary 7.14 in [KL06].

Theorem 1.4. *Let G be a second countable unimodular locally compact group, let H and K be closed subgroups of G, and fix Haar measures on all three and on $K \cap H$.*

(i) *If $K \subseteq H$ and both are unimodular, then*

$$\int_{K\backslash G} f(g)\,dg = \int_{H\backslash G}\int_{K\backslash H} f(hg)\,dh\,dg$$

for all $f \in L^1(K\backslash G)$.

(ii) *If KH is closed and unimodular and $K \cap H$ is unimodular, then the functional*

$$C_c(KH) \to \mathbb{C} : f \mapsto \int_{(K\cap H)\backslash H} \int_K f(kh)\, dk\, dh$$

is a Haar integral on KH, and the induced Haar measure on KH satisfies

$$\int_{K\backslash G} f(g)\, dg = \int_{KH\backslash G} \int_{(K\cap H)\backslash H} f(hg)\, dh\, dg$$

for all $f \in L^1(K\backslash G)$.

Evidently, in order to apply these previous theorems, it is important to determine when groups are unimodular. The following proposition contains some standard results regarding unimodularity.

Proposition 1.5. *Let G be a locally compact group.*

(i) *If G is abelian, compact, or discrete, then G is unimodular.*
(ii) *If $H \lhd G$ is a closed normal subgroup, then $\Delta_H = \Delta_G|_H$. In particular, if G is unimodular and H is a closed normal subgroup, then H is unimodular.*
(iii) *The module Δ_G factors through the quotient by the centre $Z(G)\backslash G$, and $\Delta_{Z(G)\backslash G} = \Delta_G|_{Z(G)\backslash G}$. In particular, G is unimodular if and only if $Z(G)\backslash G$ is unimodular.*
(iv) *If G is nilpotent, then G is unimodular.*

Proof. Part (i) is Proposition 7.7 in [KL06, p. 73], and part (ii) can be found in [Rei68, §3.6].

To prove part (iii), note first that it follows from (ii) that $\Delta_G(z) = 1$ for all $z \in Z(G)$, since the centre is normal, and the centre of a Hausdorff topological group is closed, and thus Δ_G factors through $Z(G)\backslash G$. Let $a \in Z(G)\backslash G$ and $f \in C_c(G)$. By Proposition 1.2,

$$\begin{aligned}
\int_{Z(G)\backslash G} f^{Z(G)}(a^{-1}g)\, dg &= \int_{Z(G)\backslash G} \int_{Z(G)} f(za^{-1}g)\, dz\, dg \\
&= \int_{Z(G)\backslash G} \int_{Z(G)} f(a^{-1}zg)\, dz\, dg \\
&= \int_G f(a^{-1}g)\, dg.
\end{aligned}$$

Applying (1.1) to both sides yields

$$\Delta_{Z(G)\backslash G}(a) \int_{Z(G)\backslash G} f^{Z(G)}(g)\, dg = \Delta_G(a) \int_G f(g)\, dg = \Delta_G(a) \int_{Z(G)\backslash G} f^{Z(G)}(g)\, dg.$$

Since $f \mapsto f^{Z(G)}$ is a surjection onto $C_c(Z(G)\backslash G)$, we now have that $\Delta_{Z(G)\backslash G}(a) = \Delta_G(a)$ for all $a \in Z(G)\backslash G$.

To prove part (iv), note that if G is nilpotent, its upper central series terminates, i.e. there exists $n \geq 1$ such that

$$1 = Z_0 \lhd Z_1 \lhd Z_2 \lhd \cdots \lhd Z_n = G,$$

where Z_{i+1} satisfies $Z_i\backslash Z_{i+1} = Z\left(Z_i\backslash G\right)$ for all $i \geq 1$. By (iii), $Z_i\backslash G$ is unimodular if and only if $Z_{i+1}\backslash G$ is unimodular. Clearly $Z_n\backslash G = 1$ is unimodular, and thus after applying (iii) iteratively from $i = n-1$ to $i = 0$, it is clear that $Z_0\backslash G = G$ is unimodular. □

Chapter 2

Local non-Archimedean representations

The purpose of this chapter is to study the representations that appear as local components of an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A})$ at non-Archimedean places. These are smooth representations of locally profinite groups of the form $\mathrm{GL}_n(F)$, where F is a non-Archimedean local field.

Section 2.1 provides a brief overview of non-Archimedean local fields and their characters. Sections 2.2 and 2.3 offer a general introduction to smooth representations of locally profinite groups, reviewing smooth induction and studying the Hecke algebra of a locally profinite group and its connection to smooth representations, and both sections are based on the first chapter of [BH06], where many additional details can be found. The remaining sections focus specifically on smooth representations of $\mathrm{GL}_n(F)$, and they are all based on [BZ76], a pivotal work by Bernstein and Zelevinsky on smooth representations of reductive p-adic groups. Section 2.4 explores the structural properties of $\mathrm{GL}_n(F)$. Section 2.5 provides an overview of parabolic induction and cuspidal representations for $\mathrm{GL}_n(F)$. Sections 2.6 and 2.8 investigates Whittaker models for smooth representations of $\mathrm{GL}_n(F)$, and Section 2.7 discusses the related Kirillov models, which in this thesis are only considered for local non-Archimedean representations.

The proofs of the main theorems in Sections 2.5–2.7 are omitted in this thesis, with full details available in [BZ76].

2.1 Non-Archimedean local fields

Recall that a non-Archimedean local field is a field that is complete with respect to a discrete valuation and whose residue field is finite. In this chapter, F always denotes a non-Archimedean local field. We denote by $\mathcal{O}_F$ (or simply $\mathcal{O}$) the ring of integers of F, and the maximal ideal of $\mathcal{O}_F$ is denoted by $\mathfrak{p}$. We let $q = |\mathcal{O}_F/\mathfrak{p}|$ denote the cardinality of the residue field of F, and let $v_F : F^\times \to \mathbb{Z}$ denote the normalized valuation of F. We define an absolute value $|\cdot|_F : F \to \mathbb{R}_{\geq 0}$ on F, given by

$$|x|_F = \begin{cases} 0 & \text{if } x = 0 \\ q^{-v_F(x)} & \text{if } x \in F^\times. \end{cases}$$

Since $\mathcal{O}$ is a discrete valuation ring, it is a principal ideal domain, and thus we define a *prime element* ϖ of F to be an element $\varpi \in \mathcal{O}$ such that $\varpi\mathcal{O} = \mathfrak{p}$. The fractional ideals

$$\mathfrak{p}^m = \varpi^m\mathcal{O} = \{x \in F \mid |x|_F \leq q^{-m}\}$$

are compact open subgroups of the additive group F and form a basis of open neighbourhoods of $0 \in F$. One can show that for every $x \in F^\times$, there exists a unique pair $(r, u) \in \mathbb{Z} \times \mathcal{O}^\times$ such that $x = \varpi^r u$.

Recall that a character of a topological group G is a continuous group homomorphism $G \to \mathbb{C}^\times$. We call a character of G *unitary* if its image is contained in the circle group $\mathbb{S}^1 = \{z \in \mathbb{C}^\times \mid |z|_\infty = 1\}$, and we denote by $\widehat{G}$ the group of unitary characters of G.

Definition 2.1.

(i) Let $\psi : F \to \mathbb{C}^\times$ be an additive character of F. The *level* of ψ is the least integer d such that $\mathfrak{p}^d \subseteq \ker \psi$.

(ii) Let $\omega : F^\times \to \mathbb{C}^\times$ be a multiplicative character of $F^\times$. We say that ω is *unramified* if it is trivial on $\mathcal{O}^\times$, i.e. if $\omega(\mathcal{O}^\times) = \{1\}$.

Note that the circle group $\mathbb{S}^1$ is the unique maximal compact subgroup of $\mathbb{C}^\times$. As the image of a compact subgroup under a continuous group homomorphism is again a compact subgroup, and F is the union of its compact open subgroups, the image of any additive character $\psi : F \to \mathbb{C}^\times$ must be contained in $\mathbb{S}^1$. This implies that all additive characters of F are elements of $\widehat{F}$.

2.2 Smooth representations

This section serves as a brief introduction to general smooth representations of locally profinite groups and the related notions. It is based on the first chapter of [BH06], where all omitted details can also be found. Recall that a *locally profinite group* is a Hausdorff topological group G with the property that every neighbourhood of the identity contains a compact open subgroup of G. Recall furthermore that both F and $\mathrm{GL}_n(F)$ indeed are locally profinite, and that the subgroups

$$K_0 = \mathrm{GL}_n(\mathcal{O}) \qquad \text{and} \qquad K_m = I_n + \mathfrak{p}^m \operatorname{Mat}_{n\times n}(\mathcal{O}), \quad m \geq 1$$

are compact and open, and form a fundamental system of open neighbourhoods of the identity in $\mathrm{GL}_n(F)$. As closed subsets of compact spaces are compact, any closed subgroup of a locally profinite group is itself a locally profinite group.

A *smooth representation* of a locally profinite group G is a pair (π, V), where V is a $\mathbb{C}$-vector space and $\pi : G \to \mathrm{Aut}_{\mathbb{C}}(V)$ is a group homomorphism to the automorphism group of V, such that for all $v \in V$ there exists a compact open subgroup K of G that fixes v, i.e. $\pi(k)v = v$ for all $k \in K$. As usual, the hom-space $\mathrm{Hom}_G(\pi_1, \pi_2)$ of two smooth representations consists of linear maps $f : V_1 \to V_2$ that intertwine with the G-action, i.e.

$$f \circ \pi_1(g) = \pi_2(g) \circ f$$

for all $g \in G$. The class of smooth G-representations forms an abelian category, which is denoted by $\mathrm{Rep}(G)$. A *subrepresentation* of π is a subspace W of V that is closed under the action of G, i.e. $\pi(g)W \subseteq W$ for all $g \in G$. Naturally, W itself is also a smooth representation of G under the action of π restricted to W. For a compact open subgroup K of G we denote by V^K the space of all K-fixed vectors in V. The representation $(\pi|_K, V^K)$ is clearly a smooth representation of K that is isomorphic to the trivial representation of K. An important fact is that the functor $V \mapsto V^K$ is exact (see [BH06, §2.3 Corollary 1] for a proof).

Definition 2.2. Let (π, V) be a smooth representation of G.

(i) We call π *irreducible* if it has no proper non-zero subrepresentations.
(ii) We call π *admissible* if $\dim_{\mathbb{C}}(V^K) < \infty$ for all compact open subgroups K of G.

A character χ of G is a one-dimensional representation of G on the vector space $V = \mathbb{C}$, because $\mathrm{Aut}_{\mathbb{C}}(\mathbb{C}) \cong \mathbb{C}^\times$ via the map $f \mapsto f(1)$. Since χ is continuous, the kernel of χ is open, and thus there exists a compact open subgroup K such that $K \subseteq ker(\chi)$, hence every $v \in \mathbb{C}$ is fixed by K, implying that the representation $(\chi, \mathbb{C})$ is smooth. In general, every one-dimensional smooth representation of G is isomorphic to a character of G, and thus the two notions are used interchangeably.

In the setting of smooth representations of locally profinite groups, whose cardinality might be (and for us will be) infinite, we also have a version of Schur's lemma from the representation theory of finite groups, establishing that the hom-space of endomorphisms of irreducible representations is one-dimensional.

Theorem 2.3 (Schur's lemma)**.** *Let G be a locally profinite group such that G/K is countable for all compact open subgroups K. If (π, V) is an irreducible representation of G, then*

$$\mathrm{Hom}_G(\pi, \pi) = \mathbb{C}.$$

One can show that the assumption on G is equivalent to G/K being countable for *some* compact open subgroup K, which we show in Section 2.4 is the case for the groups that are of interest later in the chapter. We therefore assume for the remainder of this chapter that a locally profinite group G satisfies this property, although this might not be true in general.

An immediate consequence of Schur's lemma is that if $f \in \mathrm{Hom}_G(\pi_1, \pi_2)$ is a non-zero G-homomorphism of two irreducible representations, then f is an isomorphism, and it is unique up to scalar multiplication.

Another simple consequence of Schur's lemma is that for an irreducible representation (π, V) of G, the centre Z of G acts on V via a character $\omega_\pi : Z \to \mathbb{C}^\times$, i.e.

$$\pi(z)v = \omega_\pi(z)v$$

for all $v \in V$ and $z \in Z$. The character ω_π is unique and only depends on the isomorphism class of π, and we call it the *central character* of π.

For a closed subgroup H of G and a smooth representation (σ, W) of H, we define a space $L(G, \sigma)$ of functions $f : G \to W$ satisfying

(i) $f(hg) = \sigma(h)f(g)$ for all $h \in H$ and $g \in G$, and
(ii) there exists a compact open subgroup K of G such that $f(gk) = f(g)$ for all $g \in G$ and $k \in K$.

We define $S(G, \sigma) \subseteq L(G, \sigma)$ to be the space of all functions $f \in L(G, \sigma)$ that furthermore satisfy

(iii) f is *compactly supported modulo H*, i.e. there exists a compact subset $C \subseteq G$ such that $\operatorname{supp} f \subseteq HC$.

In general, we denote by ρ the action of right translation by G, i.e. if f is a function on G, $\rho(g)$ maps f to the function $\rho(g)f : g_0 \mapsto f(g_0 g)$. Both $L(G, \sigma)$ and $S(G, \sigma)$ are closed under the action of ρ, and are thus representations of G, and one can easily check that they are indeed smooth representations. The representations $(\rho, L(G, \sigma))$ and $(\rho, S(G, \sigma))$

of G are said to be *smoothly induced* by σ and *compactly induced* by σ, respectively. We write

$$\operatorname{Ind}_H^G \sigma = (\rho, L(G,\sigma)) \qquad \text{and} \qquad c\text{-}\operatorname{Ind}_H^G \sigma = (\rho, S(G,\sigma)).$$

Both Ind_H^G and c-Ind_H^G are functors $\operatorname{Rep}(H) \to \operatorname{Rep}(G)$, and they are additive and exact. Since $S(G,\sigma) \subseteq L(G,\sigma)$, there is a natural G-homomorphism $c\text{-}\operatorname{Ind}_H^G \sigma \hookrightarrow \operatorname{Ind}_H^G \sigma$ which is an isomorphism if and only if $H\backslash G$ is compact. If this is the case, the functor c-Ind, and thus Ind, also preserves admissibility, as a consequence of the following lemma.

Lemma 2.4. *Let H be a closed subgroup of G and (σ, W) a smooth representation of H. Let K be a compact open subgroup of G, and let $\mathcal{G}$ be a set of representatives of $H\backslash G/K$. For each $g \in \mathcal{G}$, fix a basis $\mathcal{W}_g$ of the vector space $W^{H\cap gKg^{-1}}$. For $g \in \mathcal{G}$ and $w \in \mathcal{W}_g$, define $f_{g,w} \in (c\text{-}\operatorname{Ind}_H^G \sigma)^K$ to be the unique function with* $\operatorname{supp} f_{g,w} = HgK$ *and $f_{g,w}(g) = w$. Then the set*

$$\{f_{g,w} \mid g \in \mathcal{G}, w \in \mathcal{W}_g\}$$

forms a basis for $(c\text{-}\operatorname{Ind}_H^G(\sigma))^K$.

Proof. Let $f \in (c\text{-}\operatorname{Ind}_H^G(\sigma))^K$. If $f(g) \neq 0$ for some $g \in G$, then $HgK \subseteq \operatorname{supp} f$, hence $\operatorname{supp} f$ is a union of double cosets of the form HgK for $g \in \mathcal{G}$. As f has compact support modulo H, this must be a finite union. So f decomposes into a finite sum of functions $f_g \in (c\text{-}\operatorname{Ind}_H^G(\sigma))^K$ with $\operatorname{supp} f_g = HgK$ for $g \in \mathcal{G}$. Such a function f_g is completely determined by its value at g, say $w_0 = f_g(g)$. This value is an element of $W^{H\cap gKg^{-1}}$, as for any $gkg^{-1} \in H \cap gKg^{-1}$,

$$\sigma(gkg^{-1})w_0 = f_g(gkg^{-1}g) = f_g(g) = w_0.$$

Since $\mathcal{W}_g$ forms a basis for $W^{H\cap gKg^{-1}}$, f_g must be a linear combination of $f_{g,w}$'s for $w \in \mathcal{W}_g$. Thus f is a linear combination of elements in $\{f_{g,w} \mid g \in \mathcal{G}, w \in \mathcal{W}_g\}$, and as this set is clearly linearly independent over $\mathbb{C}$ and f was arbitrary, it must form a basis for $(c\text{-}\operatorname{Ind}_H^G(\sigma))^K$. $\square$

A fundamental property of smooth induction is the *Frobenius Reciprocity.* Denote by $\operatorname{Res}_H : \operatorname{Rep}(G) \to \operatorname{Rep}(H)$ the functor $\pi \mapsto \pi|_H$.

Theorem 2.5 (Frobenius Reciprocity)**.** *The functor Ind_H^G is right adjoint to Res_H, i.e. for all smooth representations π of G and σ of H there is a canonical isomorphism*

$$\operatorname{Hom}_G(\pi, \operatorname{Ind}_H^G \sigma) \cong \operatorname{Hom}_H(\pi|_H, \sigma),$$

which depends functorially on π and σ.

2.3 The Hecke algebra

In this section we define the Hecke algebra of a locally profinite group G, and establish the connection between modules of the Hecke algebra of G and smooth representations of G.

Let us first define the general notion of an *idempotented algebra*, a concept used throughout the thesis.

Definition 2.6 (Idempotented algebra)**.** Let Ω be an algebraically closed field. An *idempotented Ω-algebra* is a pair $(\mathcal{H}, \mathcal{E})$, where $\mathcal{H}$ is an associative Ω-algebra and $\mathcal{E}$ is a set of idempotent elements in $\mathcal{H}$ that satisfy the following two properties:

(i) for each pair $e_1, e_2 \in \mathcal{E}$, there exists an element $e_0 \in \mathcal{E}$ such that $e_1 e_0 = e_0 e_1 = e_1$ and $e_2 e_0 = e_0 e_2 = e_2$; and
(ii) for each $f \in \mathcal{H}$, there exists an element $e \in \mathcal{E}$ such that $fe = ef = f$.

We call the elements of $\mathcal{E}$ the *elementary idempotents* of $(\mathcal{H}, \mathcal{E})$.

We equip $\mathcal{E}$ with a partial ordering denoted by $\geq$, where we write $e \geq e'$ if $e'e = ee' = e'$ for each pair $e, e' \in \mathcal{E}$. Definition 2.6(i) implies that $\mathcal{E}$ is a directed set with respect to this ordering. We often simply write $\mathcal{H}$ if the set of elementary idempotents is clear.

For an idempotent element $e \in \mathcal{H}$, $e\mathcal{H}e$ is a subalgebra of $\mathcal{H}$ with unit e, and is denoted by $\mathcal{H}[e]$. Similarly, if M is a (left) $\mathcal{H}$-module, then $e \cdot M$ is a $\mathcal{H}[e]$-module, and is denoted by $M[e]$. If $(\mathcal{H}, \mathcal{E})$ is an idempotented Ω-algebra and M is a $\mathcal{H}$-module, then we call M *smooth* if

$$M = \bigcup_{e \in \mathcal{E}} M[e],$$

and additionally *admissible* if each $M[e]$ is finite-dimensional as an Ω-vector space. In this thesis, modules are called *irreducible* if they have no non-trivial proper submodules. Such modules are also sometimes called *simple* in the literature.

Let us now consider the Hecke algebra of G, which is our first example of an idempotented algebra. For the remainder of Section 2.3 we assume that G is unimodular, as the group $\mathrm{GL}_n(F)$ indeed is (see Proposition 2.12), and fix a Haar measure μ on G.

Define $C_c^\infty(G)$ as the space of locally constant functions $f : G \to \mathbb{C}$ with compact support. This is a $\mathbb{C}$-vector space with the usual pointwise addition and scalar multiplication. Note that since G is locally profinite, and $f \in C_c^\infty(G)$ is locally constant and of compact support, there exist compact open subgroups K_1 and K_2 such that $f(gk_1) = f(g)$ for all $g \in G$ and $k_1 \in K_1$, and $f(k_2 g) = f(g)$ for all $g \in G$ and $k_2 \in K_2$. Taking $K = K_1 \cap K_2$, we find that all functions $f \in C_c^\infty(G)$ are finite linear combinations of characteristic functions of double cosets of the form KgK, for some compact open subgroup K.

We define *convolution* on $C_c^\infty(G)$ to be the bilinear operator $*$, given by

$$*(f_1, f_2) = f_1 * f_2 : G \to \mathbb{C}$$

where

$$(f_1 * f_2)(g) = \int_G f_1(x) f_2(x^{-1}g)\, d\mu(x).$$

Convolution is associative—a pivotal insight—and distributivity and compatibility with scalars follow from the integral having those properties. Thus, $C_c^\infty(G)$ endowed with convolution forms a $\mathbb{C}$-algebra. It is called the *Hecke algebra* of G, and we denote it by $\mathcal{H}_G$. Note that the structure of $\mathcal{H}_G$ depends on the choice of Haar measure μ, but it is unique up to isomorphism.

For a compact open subgroup K, we denote by $e_K \in \mathcal{H}_G$ the function given by

$$e_K(x) = \begin{cases} \mu(K)^{-1} & \text{if } x \in K \\ 0 & \text{otherwise,} \end{cases}$$

i.e. $\mu(K)^{-1}$ times the characteristic function of K. One can show that e_K is an idempotent, and that a function $f \in \mathcal{H}_G$ satisfies $e_K * f = f$ if and only if $f(kg) = f(g)$ for all $g \in G$

and $k \in K$, and satisfies $f * e_K = f$ if and only if $f(gk) = f(g)$ for all $g \in G$ and $k \in K$. This implies that functions $f \in \mathcal{H}_G[e_K] = e_K * \mathcal{H}_G * e_K$ are finite linear combinations of characteristic functions of double cosets of the form KgK. Let

$$\mathcal{E}_G = \{e_K \mid K \text{ compact open subgroup of } G\}.$$

From the previous characterization of the elements of $\mathcal{H}_G$, we see that the Hecke algebra must be a union of its subalgebras $\mathcal{H}_G[e_K]$, running over all compact open subgroups K of G, and thus the pair $(\mathcal{H}_G, \mathcal{E}_G)$ satisfies (ii) in the definition of an idempotented $\mathbb{C}$-algebra (Definition 2.6). Given two idempotents $e_{K_1}, e_{K_2} \in \mathcal{E}_G$, it is clear that $e_{K_1 \cap K_2} \geq e_{K_1}$ and $e_{K_1 \cap K_2} \geq e_{K_2}$, hence $(\mathcal{H}_G, \mathcal{E}_G)$ also satisfies (i), and thus form an idempotented $\mathbb{C}$-algebra.

Given a smooth representation (π, V), we want to relate it to a smooth $\mathcal{H}_G$-module, that is, we want to define an action of $\mathcal{H}_G$ on V via the action of π. For a function $f \in \mathcal{H}_G$, we define

$$\pi(f)v = \int_G f(g)\pi(g)v \, d\mu(g).$$

To see that $\pi(f)v$ is a well-defined element of V, note that for a compact open subgroup K that fixes v and under which f is right translation invariant (which indeed exists), (2.3) can be rewritten as

$$\pi(f)v = \mu(K) \sum_{g \in G/K} f(g)\pi(g)v,$$

which is a finite sum since f is compactly supported. It is now also simple to see that the operator $\pi(e_K)$ is the unique K-projection $V \to V^K$ for all compact open subgroups. This action of $\mathcal{H}_G$ on V leads to the following theorem for the Hecke algebra.

Theorem 2.7. *Let (π, V) be a smooth representation of G. The $\mathbb{C}$-vector space V endowed with the operation*

$$\mathcal{H}_G \times V \to V : (f, v) \mapsto \pi(f)v$$

forms a smooth $\mathcal{H}_G$-module. If (π', V') is another smooth representation of G, then

$$\operatorname{Hom}_G(V, V') = \operatorname{Hom}_{\mathcal{H}_G}(V, V').$$

Conversely, if M is a smooth $\mathcal{H}_G$-module, there exists a unique G-homomorphism $\pi : G \to \operatorname{Aut}_{\mathbb{C}}(M)$, such that (π, M) forms a smooth representation of G, and

$$f \cdot m = \pi(f)m$$

for all $f \in \mathcal{H}_G$ and $m \in M$.

Proof. For the direct part of the theorem, see [BH06, §4.2 Proposition 1], and for the converse, see [BH06, §4.2 Proposition 2]. □

From this theorem, we see that the notions of a smooth G-representation and a smooth $\mathcal{H}_G$-module are equivalent and interchangeable, and note that the notions of irreducibility and admissibility for G-representations similarly correspond to those for smooth $\mathcal{H}_G$-modules.

Example 2.8. Let $L(G)$ denote the space of functions $\varphi : G \to \mathbb{C}$ that are right translation invariant by some compact open subgroup K, as in part (ii) of the definition of the space $L(G, \sigma)$. Then $(\rho, L(G))$ forms a smooth representation of G, which implies that $L(G)$ is a smooth $\mathcal{H}_G$-module under the action $(f, \varphi) \mapsto \rho(f)\varphi$ for $f \in \mathcal{H}_G$ and $\varphi \in L(G)$. For an element $f \in \mathcal{H}_G$, let $\check{f} \in \mathcal{H}_G$ denote the function $g \mapsto f(g^{-1})$. Then $\rho(f)$ is is just convolution by $\check{f}$ on the right: indeed

$$\begin{aligned}(\rho(f)\varphi)(g) &= \int_G f(x)\,(\rho(x)\varphi)(g)\,d\mu(x)\\ &= \int_G \varphi(gx)f(x)\,d\mu(x)\\ &= \int_G \varphi(y)f(g^{-1}y)\,d\mu(y)\\ &= \int_G \varphi(y)\check{f}(y^{-1}g)\,d\mu(y)\\ &= \varphi * \check{f}(g),\end{aligned}$$

where the third equality follows from the substitution $y = gx$, which leaves the Haar measure invariant. Thus $\mathcal{H}_G$ acts on $L(G)$ by convolution on the right, and φ is right translation invariant by the compact open subgroup K if and only if φ is fixed by the elementary idempotent $e_K = \widecheck{e_K}$, i.e. $\rho(e_K)\varphi = \varphi * e_K = \varphi$, or, equivalently, if and only if $\varphi \in L(G)[e_K]$.

2.4 Structure of $\mathrm{GL}_n(F)$

We now turn our attention to the specifics of the representation theory for $\mathrm{GL}_n(F)$, where F is a non-Archimedean local field. For the remainder of this chapter, $G = \mathrm{GL}_n(F)$ denotes the general linear group of F and $K = \mathrm{GL}_n(\mathcal{O}_F)$ denotes the general linear group of the ring of integers $\mathcal{O}_F$ in F, which is the maximal compact subgroup of G.

Let us first examine three different decompositions: the Levi decomposition, the Iwasawa decomposition, and the Cartan decomposition.

Definition 2.9. Let $G = \mathrm{GL}_n(F)$.

(i) A *Levi subgroup* M of G is a subgroup of G which is conjugate to the standard Levi subgroup $M_\alpha(F)$ (cf. Section 1.1) for some partition $\alpha \vdash n$.
(ii) A *parabolic subgroup* P of G is a subgroup of G which is conjugate to the standard parabolic subgroup $P_\alpha(F)$ (cf. Section 1.1) for some partition $\alpha \vdash n$.
(iii) Let P be a parabolic subgroup of G and let $g \in G$ such that $g^{-1}Pg = P_\alpha(F)$. $M = gM_\alpha(F)g^{-1}$ is called *the Levi factor* of P, and $U = gU_\alpha(F)g^{-1}$ is called *the unipotent radical* of P.
(iv) The factorization $P = MU$ is called *the Levi decomposition* of P.

Proposition 2.10 (Iwasawa decomposition)**.** *Let P be a parabolic subgroup of G. Then*

$$G = PK = KP,$$

and thus $P\backslash G$ is compact.

Proof. Let $B = P_{(1,\ldots,1)}(F)$, and note that $B \subseteq P_\alpha(F)$ for any $\alpha \vdash n$. It is enough to show that $G = KB$: indeed, if $G = KB$, then for any $g \in G$, we can find elements $k \in K$ and $b \in B$ such that $g = kb$. Take $g \in G$ such that $P = gP_\alpha g^{-1}$ for some $\alpha \vdash n$. Then it follows that

$$KP = K(gP_\alpha(F)g^{-1}) \supseteq K(gBg^{-1}) = KkbBb^{-1}k^{-1} = KBk^{-1} = Gk^{-1} = G.$$

Taking the inverse on both sides yields

$$G = KP = PK.$$

The proposition now follows from [Bum97, Proposition 4.5.2]. □

As a parabolic subgroup P of G is a closed subgroup of G, the induction functor Ind_P^G is well-defined. As $P\backslash G$ is compact, it follows that $c\text{-}\mathrm{Ind}_P^G = \mathrm{Ind}_P^G$.

Recall that we assume in Section 2.2 that the locally profinite group G satisfies that G/K' is countable for some compact open subgroup K'. In order to show that this is the case for $G = \mathrm{GL}_n(F)$ and $K' = K$, we use the following proposition, of which it is an easy consequence.

Proposition 2.11 (Cartan decomposition)**.** *Let ϖ be a prime element of F. The matrices*

$$\begin{pmatrix} \varpi^{m_1} & & & \\ & \varpi^{m_2} & & \\ & & \ddots & \\ & & & \varpi^{m_n} \end{pmatrix} \qquad m_i \in \mathbb{Z},\ \ m_1 \geq m_2 \geq \cdots \geq m_n,$$

then form a set of representatives of the double quotient space $K\backslash G/K$.

Proof. For $n = 2$, this is the content of [Bum97, Proposition 4.6.2], whose proof uses the structure theorem for finitely generated modules over a PID, and which can also easily be generalised to $n \geq 2$. Another proof for the Cartan decomposition, in a much more general setting, can be found in [KP23, §5.2]. □

An additional assumption that we make on the group in Section 2.3 is that it is unimodular. This is in general true for reductive groups, however, for the sake of being self-contained, we include a proof for the case of $\mathrm{GL}_n(F)$ within our current framework.

Proposition 2.12. *G is unimodular.*

Proof. Recall from Section 1.2 that $\Delta_G : G \to \mathbb{R}_{>0}$ denotes the module of G, and that it is a continuous group homomorphism.

Firstly, we wish to show that Δ_G is trivial on the subgroup $KZ(G)SL_n(F)$. The maximal compact subgroup K is closed in G, so it follows from Proposition 1.5(ii) that $\Delta_G|_K = \Delta_K$, and hence Δ_G is trivial on K, as K is a compact group and thus unimodular. In the proof of Proposition 1.5(iii) we prove that $Z(G') \subseteq \ker \Delta_{G'}$ for any locally compact group G', and this must then in particular hold for G. As G is the general linear group over an infinite field, the commutator subgroup $[G, G]$ coincides with $SL_n(F)$, and since $\mathbb{R}_{>0}$ is commutative, the value of a commutator under Δ_G is 1, thus Δ_G is also trivial on $SL_n(F)$.

Secondly, we wish to show that the subgroup $H = KZ(G)SL_n(F)$ has finite index in G. Let $\varpi^{\mathbb{Z}}$ denote the subgroup $\{\varpi^r \mid r \in \mathbb{Z}\}$ of $F^\times$, and recall that $F^\times = \mathcal{O}^\times \varpi^{\mathbb{Z}}$ is a direct product. Furthermore, let $\varpi^{\mathbb{Z}} I_n$ denote the subgroup

$$\varpi^{\mathbb{Z}} I_n = \left\{ \begin{pmatrix} \varpi^r & & \\ & \ddots & \\ & & \varpi^r \end{pmatrix} \middle|\ r \in \mathbb{Z} \right\} \subseteq G.$$

Clearly $\varpi^{\mathbb{Z}} I_n \subseteq Z(G)$, and

$$\det\left(K\left(\varpi^{\mathbb{Z}} I_n\right) SL_n(F)\right) = \mathcal{O}^\times \varpi^{n\mathbb{Z}}.$$

The determinant map induces an isomorphism

$$G \big/ SL_n(F) \cong F^\times = \mathcal{O}^\times \varpi^{\mathbb{Z}}$$

and under this isomorphism we then get

$$K \left(\varpi^{\mathbb{Z}} I_n\right) SL_n(F) \big/ SL_n(F) \cong \mathcal{O}^\times \varpi^{n\mathbb{Z}}.$$

As $\mathcal{O}^\times \varpi^{\mathbb{Z}} / \mathcal{O}^\times \varpi^{n\mathbb{Z}} \cong \mathbb{Z}/n\mathbb{Z}$, we have

$$G \big/ K \left(\varpi^{\mathbb{Z}} I_n\right) SL_n(F) \cong \mathbb{Z}/n\mathbb{Z}.$$

Thus $K\left(\varpi^{\mathbb{Z}} I_n\right) SL_n(F)$ has finite index in G, and since $\varpi^{\mathbb{Z}} I_n$ is contained in $Z(G)$, H must have finite index as well.

It follows directly that $\Delta_G(G) = \{1\}$: indeed, since $H \subseteq \ker \Delta_G$, Δ_G factors through the finite group G/H, hence the image of G is a finite subgroup of $\mathbb{R}_{>0}$. But the only finite subgroup of $\mathbb{R}_{>0}$ is the trivial one, implying that $\Delta_G(G) = \{1\}$, and thus G must be unimodular. □

Let us fix a Haar measure on G, which we will refer to as the usual measure on G in subsequent chapters: let μ be any Haar measure. Then the n^2-fold product measure $dg = \prod_{i,j} d\mu(g_{i,j})$ defines a Haar measure on $\mathrm{Mat}_{n\times n}(F)$, and

$$dg^* = \frac{dg}{|\det g|_F^n}.$$

defines a Haar measure on G. We define the *usual measure on* G to be $\mathrm{vol}_{dg^*}(K)^{-1}$ times the measure dg^*. Note that it does not depend on the choice of μ. Due to the previous proposition, it is automatically both left and right invariant. It is the unique Haar measure on G where $\mathrm{vol}(K) = 1$.

Proposition 2.13. *Let $\mathcal{H}_G$ be the Hecke algebra of G, defined with the usual measure. Then $\mathcal{H}_G[e_K]$ for $K = \mathrm{GL}_n(\mathcal{O})$ is commutative.*

Proof. The determinant is invariant under taking the transpose, and thus the measure dg^* on G is as well. Furthermore, since G is unimodular, dg^* is also invariant under inverses. This implies that the map $f \mapsto {}^\iota f$ on $\mathcal{H}_G$ where ${}^\iota f(g) = f(g^T)$ is an involution: indeed

$$\begin{aligned} {}^\iota f_2 * {}^\iota f_1(g) &= \int_G f_2(x^T) f_1((x^{-1}g)^T)\, dx^* \\ &= \int_G f_2(y^{-1}g^T) f_1(y)\, dy^* \\ &= {}^\iota(f_1 * f_2)(g), \end{aligned}$$

where we substitute $y = g^T(x^{-1})^T$.

The $\mathbb{C}$-algebra $\mathcal{H}_G[e_K]$ has a basis consisting of characteristic functions of double cosets of the form KgK. The Cartan decomposition implies that these functions are invariant under taking the transpose, and thus the involution restricts to the identity map on $\mathcal{H}_G[e_K]$, implying that $\mathcal{H}_G[e_K]$ is commutative. □

We say that a character χ of G is *unramified* if $\chi(K) = \{1\}$. Similarly, a character of a Levi subgroup M is called *unramified* if $M \cap K$ is contained in its kernel. Note that any character χ of G is of the form $\omega \circ \det$ for some character ω of $F^\times$, since χ, as Δ_G in the proof of Proposition 2.12, factors through $G/SL_n(F)$.

2.5 Parabolic induction and cuspidal representations

This section serves as a brief overview of the definition and some important properties of parabolic induction, a central tool in the field of representations of p-adic reductive groups. We furthermore give a short introduction to cuspidal representations and state two important properties of irreducible representations. This section is based on [BZ76] unless otherwise stated.

Let $P = MU$ be a parabolic subgroup of G, where M is its Levi factor and U its unipotent radical. We view smooth representations of M as smooth representations of P by asserting that they are trivial on U, and to this end construct the following functor: for a smooth representation (σ, W) of M, we define $\operatorname{Inf}\sigma$ to be the smooth representation on W of P given by

$$(\operatorname{Inf}\sigma)(mu) = \sigma(m),$$

and we call it the *inflation* of σ. It is clear that $\operatorname{Inf} : \operatorname{Rep}(M) \to \operatorname{Rep}(P)$ is a functor of smooth representations of M, and that it is additive and exact. It is also clear that the inflation of an admissible representation remains admissible.

We define *parabolic induction* to be the composition of inflation from M to P and smooth induction from P to G, and we denote the functor by $\operatorname{Ind}_{M,P}^G$. I.e. we define $\operatorname{Ind}_{M,P}^G : \operatorname{Rep}(M) \to \operatorname{Rep}(G)$ to be the composition

$$\operatorname{Ind}_{M,P}^G = \operatorname{Ind}_P^G \circ \operatorname{Inf}.$$

As both functors Ind_P^G and Inf are additive and exact, so is parabolic induction. Since $P\backslash G$ is compact, it follows from Lemma 2.4 that c-Ind_P^G ($= \operatorname{Ind}_P^G$) preserves admissibility. Therefore, both Ind_P^G and Inf preserve admissibility, implying that this must also be the case for $\operatorname{Ind}_{M,P}^G$.

Parabolic induction allows us to define the notion of a cuspidal representation of $\mathrm{GL}_n(F)$.

Definition 2.14. An irreducible representation π of G is called *cuspidal* if there exists no proper parabolic subgroup $P = MU$ and no irreducible representation σ of M such that π is a subrepresentation of $\operatorname{Ind}_{M,P}^G \sigma$.

There are other equivalent definitions for cuspidal representations (see [BZ76, §3.18] and [BZ76, §3.21]), and although these are beyond our current scope, it is important to note that one of the alternative definitions implies cuspidal representations are admissible (see [BZ76, §2.41] and [BZ76, §3.26]).

Suppose M_α is a standard Levi subgroup of G, with $\alpha = (n_1, \ldots, n_r) \vdash n$. Then M_α is naturally isomorphic to the direct product $\prod_{i=1}^r G_{n_i}$, and we thus have a notion of a parabolic subgroup of M_α, and thus also a natural notion of a cuspidal representation of M_α, as a natural extension of those for G. We can now state the following important result about cuspidal representations.

Theorem 2.15 ([BZ76, §3.19]). *If π is an irreducible representation of G, then there is a standard parabolic subgroup $P = MU$ and a cuspidal representation σ of M such that π is isomorphic to a subrepresentation of $\operatorname{Ind}_{M,P}^G \sigma$.*

Although we do not prove this theorem, we give a short proof of one of its most important consequences:

Corollary 2.16. *Every irreducible representation of G is admissible.*

Proof. Let (π, V) be an irreducible representation of G. It follows from Theorem 2.15 that there exists a standard parabolic subgroup $P = MU$ and a cuspidal representation σ of M such that

$$\pi \hookrightarrow \operatorname{Ind}_{M,P}^{G} \sigma.$$

Since cuspidal representations are admissible, and parabolic induction preserves admissibility, the G-representation $\operatorname{Ind}_{M,P}^{G} \sigma$ must be admissible. Since π is embedded in $\operatorname{Ind}_{M,P}^{G} \sigma$, $V^{K'}$ must be embedded in $(\operatorname{Ind}_{M,P}^{G} \sigma)^{K'}$ for all compact open subgroups K' of G, and thus $V^{K'}$ must also be finite-dimensional. □

An important class of smooth representations is the class of *unramified* representations. A smooth representation (π, V) of G is called *unramified* if there exists a non-zero K-fixed vector, i.e. if $V^K \neq \{0\}$. A non-zero vector in V^K is called *spherical*. Note that when viewing a character $\omega : G \to \mathbb{C}^\times$ of G as a one-dimensional smooth representation, the two notions of being unramified agree. In Section 4.6 we show the tensor product theorem, which states that an irreducible admissible representation of $GL_n(\mathbb{A})$ decomposes into a restricted tensor product of irreducible local representations, where almost all components are irreducible unramified representations.

The following result about the dimension of V^K is another consequence of the previous theorem, together with the Borel-Matsumoto theorem, which implies that if a cuspidal representation has a non-zero vector that is fixed by the Iwahori subgroup I, which consists of matrices in K that reduce to upper triangular matrices in $\mathrm{GL}_n(\mathcal{O}/\mathfrak{p})$, then it must be an unramified character (see [Cas80, §2.6 Proposition]).

Proposition 2.17. *If (π, V) is an irreducible representation of G, then the space V^K of K-fixed vectors is at most one-dimensional.*

Proof. By Theorem 2.15 there exist a parabolic subgroup $P = MU$ and a cuspidal representation (σ, W) of M such that $\pi \hookrightarrow \operatorname{Ind}_{M,P}^{G}(\sigma)$. The functor $\operatorname{Rep}(G) \to \operatorname{Rep}(K) : V \mapsto V^K$ is exact, hence

$$V^K \hookrightarrow \left(\operatorname{Ind}_{M,P}^{G}(\sigma)\right)^K .$$

By the Iwasawa decomposition we have that $G = PK$, and applying Lemma 2.4 in this simple setting implies that the linear map

$$\left(\operatorname{Ind}_{M,P}^{G}(\sigma)\right)^K \to W : f \mapsto f(1)$$

is an isomorphism onto its image. Hence $(\operatorname{Ind}_{M,P}^{G}(\sigma))^K \cong W^{K\cap M}$ as vector spaces, and this induces an embedding

$$V^K \hookrightarrow W^{K\cap M}$$

of vector spaces. Since the Iwahori subgroup of G is contained in K, it follows from the Borel-Matsumoto theorem that a cuspidal M-representation has non-zero $(K \cap M)$-fixed vectors if and only if it is an unramified character of M, so either $W^{K\cap M} = 0$ or $W^{K\cap M}$ is one-dimensional. In both cases it follows from the previous inclusion that V^K must also be at most one-dimensional. □

2.6 Whittaker models

A Whittaker model is a realization of an irreducible representation of $G = \mathrm{GL}_n(F)$ in an induced representation of a certain type of character of $N_n(F)$. The term was introduced by Jacquet in [Jac66] for irreducible representations of SL_2 over any local field, and in

[JL70], Jacquet and Langlands proved uniqueness of the Whittaker model for irreducible representations of $\mathrm{GL}_2(F)$. This was later extended to $\mathrm{GL}_n(F)$ by Gel'fand and Kazhdan in [GK75] and [GK72]. Whittaker models are highly significant in the theory of automorphic forms.

Recall from Section 1.1 that $N = N_n(F)$ is the maximal unipotent subgroup of G consisting of upper triangular matrices with ones on the diagonal. Furthermore, recall that an additive character of F is also considered a character of N, by applying it to the sum of the entries on the first upper diagonal.

Definition 2.18 (ψ-generic)**.** Let (π, V) be a smooth representation of G and let ψ be a non-trivial additive character of F. The representation π is called *ψ-generic* if

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_N^G \psi) \neq 0.$$

The theorem of uniqueness of the Whittaker model of an irreducible representation of G is also sometimes called the local multiplicity one theorem, as it pertains to the multiplicity of an irreducible representation in $\mathrm{Ind}_N^G \psi$ for some non-trivial additive character ψ of F, which in fact is at most one.

Theorem 2.19 ([BZ76, §5.16, Theorem])**.** *Let F be a non-Archimedean local field, and let (π, V) be an irreducible representation of $G = \mathrm{GL}_n(F)$. Let $N = N_n(F)$ and let ψ be an additive character of F. Then*

$$\dim_{\mathbb{C}}(\mathrm{Hom}_G(\pi, \mathrm{Ind}_N^G \psi)) \leq 1.$$

The proof of this theorem is beyond the scope of this thesis, as it requires a significant amount of technical machinery. However, the key idea behind the proof is to show that the algebra $\mathrm{End}_G(\mathrm{Ind}_N^G \psi)$ of G-endomorphisms on $\mathrm{Ind}_N^G \psi$ is commutative, which then implies that $\dim_{\mathbb{C}}(\mathrm{End}_G(\mathrm{Ind}_N^G \psi)) = 1$ and all subrepresentations of Ind_N^G must have multiplicity one. To see that $\mathrm{End}_G(\mathrm{Ind}_N^G \psi)$ is commutative, one uses the Bruhat decomposition $G = NWDN$, where W is the set of permutation matrices and D is the diagonal matrices in G, and moreover $W \times D$ forms a complete set of unique representatives of the double quotient $N\backslash G/N$, With this decomposition, one is able to contruct an involution i on $\mathrm{End}_G(\mathrm{Ind}_N^G \psi)$ under which all elements are invariant. Then for $A_1, A_2 \in \mathrm{End}_G(\mathrm{Ind}_N^G \psi)$, one has $A_1 \circ A_2 = i(A_1 \circ A_2) = i(A_2) \circ i(A_1) = A_2 \circ A_1$.

Suppose (π, V) is a ψ-generic irreducible representation of G. It then follows from Theorem 2.19 that there exists a non-zero G-homomorphism $A : V \to \mathrm{Ind}_N^G(\psi)$, which is unique up to a scalar factor. Since π is irreducible, A is injective, and thus A is an isomorphism onto its image. By the uniqueness of A, this image in $L(G, \psi)$ is the unique subrepresentation of $\mathrm{Ind}_N^G(\psi)$ isomorphic to π. It is now appropriate to introduce the following definition.

Definition 2.20 (Whittaker model)**.** Let (π, V) be a ψ-generic irreducible representation of G and let $A : V \to \mathrm{Ind}_N^G \psi$ be a non-zero G-homomorphism. The image $A(V)$ is called *the (ψ-)Whittaker model* of π with respect to ψ, and is denoted by $\mathcal{W}(\pi, \psi)$. Consequently, $(\pi, V) \cong (\rho, \mathcal{W}(\pi, \psi))$.

2.7 Kirillov models

In this section, we explore the representations of the mirabolic subgroup $P = P_n(F)$, which consists of the $(n \times n)$-matrices with bottom row $(0, \ldots, 0, 1)$. One can study a

P-representation π inductively, as it can be decomposed into a subrepresentation π' related to $P_{n-1}(F) \subseteq P$ and a quotient representation π/π' related to $\mathrm{GL}_{n-1}(F) \subseteq P$, which was first discovered by Gel'fand and Kazhdan (see [GK72]). This section is based on [BZ76, §5], and serves as a short overview of the deep results from there that are related to Whittaker models of irreducible representations of G, which are used in the proof for strong multiplicity one in the end of the thesis.

As the maximal unipotent subgroup $N = N_n(F)$ is also a subgroup of P, we can define the *standard representation* $\tau_P = \mathrm{Ind}_N^P(\psi)$ of P, along with its restriction $\tau_P^0 = c\text{-}\mathrm{Ind}_N^P(\psi)$ to $S(P,\psi)$. As in Definition 2.18, a P-representation π is called ψ-generic, if it satisfies

$$\mathrm{Hom}_P(\pi, \tau_P) \neq 0.$$

From the standard embedding $S(P,\psi) \hookrightarrow L(P,\psi)$ it is clear that τ_P^0 is a subrepresentation of τ_P, but using the induction technique by Gel'fand and Kazhdan, one can show (as in [BZ76, §5.15, Proposition b)]) that

$$\mathrm{Hom}_P(\tau_P^0, \tau_P) \cong \mathbb{C},$$

and thus that the standard embedding is the only embedding of τ_P^0 in τ_P. With the same techniques applied to a ψ-generic P-representation π, one is also able to construct a certain "completely ψ-generic" subrepresentation of π, and show it is isomorphic to copies of τ_P^0. These two results then combine to show the following.

Proposition 2.21 ([BZ76, §5.15, Proposition c)])**.** *Suppose $\pi \in \mathrm{Rep}(P)$ is a non-zero subrepresentation of τ_P. Then τ_P^0 is a subrepresentation of π.*

In order to prove strong multiplicity one later, we want to apply these results from the theory of P-representations to Whittaker models of generic G-representations. The following important result is the mechanism that allows us to do so. It was first conjectured by Gel'fand and Kazhdan in [GK72], and later proven by Bernstein and Zelevinsky in [BZ77].

Theorem 2.22. *Let π be a ψ-generic irreducible representation of G and let $\mathcal{W}$ be its ψ-Whittaker model. The map*

$$\mathcal{K} : \mathcal{W} \to L(P,\psi) : W \mapsto W|_P$$

is an injective P-homomorphism.

Proof. This result follows from Theorem 4.9 in [BZ77] due to the following: if $\mathcal{W}_0 = \ker \mathcal{K}$, then $\mathcal{W}_0$ is a P-subrepresentation of the irreducible G-representation $\mathcal{W}$. But by Theorem 2.19, $\mathcal{K}$ is unique up to multiplying with a scalar, hence

$$\mathrm{Hom}_P(\mathcal{W}_0, \tau_P) = 0,$$

and thus $\mathcal{W}_0$ is not a ψ-generic representation of P. By [BZ77, §4.9 Theorem], a non-ψ-generic P-subrepresentation of a ψ-generic irreducible G-representation must be trivial. It then follows that $\mathcal{W}_0 = 0$, implying $\mathcal{K}$ is injective. □

We call the image of the ψ-Whittaker model of π under $\mathcal{K}$ the *ψ-Kirillov model* of π. Proposition 2.21 reveals the importance of the Kirillov model, namely that all ψ-Kirillov models, each belonging to one ψ-generic irreducible representation of G, share a common subrepresentation, specifically τ_P^0. This, together with the uniqueness of the Whittaker models, is the underlying mechanism that allows us to prove the Strong Multiplicity One theorem for cuspidal automorphic representations. The following corollary restates this result in the manner in which it is used in Chapter 5.

Corollary 2.23. *Let (π, V) be a ψ-generic irreducible representation of $G = \mathrm{GL}_n(F)$, and let $\mathcal{W} \subseteq \mathrm{Ind}_N^G(\psi)$ be its Whittaker model with respect to the additive character ψ of F. For every $f \in c\text{-}\mathrm{Ind}_N^P(\psi)$ there exists an element $W \in \mathcal{W}$ such that $W|_P = f$.*

Proof. Let $\mathcal{W}_P$ be the image of $\mathcal{W}$ under the restriction map $\mathcal{K}$ in Theorem 2.22. From the definition of τ_P, it follows that $\mathcal{W}_P$ is a subrepresentation of τ_P. Proposition 2.21 now implies that τ_P^0 in turn is a subrepresentation of $\mathcal{W}_P$. Thus, any element $f \in S(P, \psi)$, which is the representation space associated to τ_P^0, is also an element of $\mathcal{W}_P$. As the map $\mathcal{K}$ is injective, there must exist an element $W \in \mathcal{W}$ such that $f = \mathcal{K}(W) = W|_P$. □

2.8 Whittaker models for unramified representations

In this section we study Whittaker models for unramified representations, with the goal of showing the existence of a K-fixed element W° in the Whittaker model of an unramified representation, that satisfies $W^\circ(1) = 1$. We call such an element the *spherical element* of a Whittaker model.

Proposition 2.24. *Let (π, V) be an unramified and ψ-generic irreducible representation, where ψ is of level 0, and let $\mathcal{W}$ be the ψ-Whittaker model of π. If $W \in \mathcal{W}^K$ is non-zero, then $W(1) \neq 0$.*

In order to prove this we need the following lemma: let $\underline{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ be a sequence of integers, and define $\varpi^{\underline{m}}$ to be the diagonal matrix in G of the form

$$\varpi^{\underline{m}} = \begin{pmatrix} \varpi^{m_1} & & \\ & \ddots & \\ & & \varpi^{m_n} \end{pmatrix}.$$

Lemma 2.25. *Let $W \in \mathcal{W}^K$. For every $\underline{m} \in \mathbb{Z}^n$ with $m_n \geq 0$, there exists a constant $c_{\underline{m}} \in \mathbb{C}$ such that $W(\varpi^{\underline{m}}) = c_{\underline{m}} W(1)$.*

Let us first show how Proposition 2.24 follows from this, and return to the proof of the lemma afterwards.

Proof of Proposition 2.24. By the Iwasawa decomposition for the standard parabolic subgroup $P_{(1,\ldots,1)}(F) = NM_{(1,\ldots,1)}(F)$, we have that $G = NM_{(1,\ldots,1)}(F)K$. Let $\varpi^{\mathbb{Z}^n}$ denote the set $\{\varpi^{\underline{m}} \mid \underline{m} \in \mathbb{Z}^n\}$. As $F^\times = \varpi^{\mathbb{Z}}\mathcal{O}^\times$ is a direct product,

$$M_{(1,\ldots,1)}(F)K = \varpi^{\mathbb{Z}^n}K,$$

and thus $G = N\varpi^{\mathbb{Z}^n}K$. Similarly,

$$\mathrm{GL}_{n-1}(F) = N_{n-1}(F)\varpi^{\mathbb{Z}^{n-1}}\,\mathrm{GL}_{n-1}(\mathcal{O}).$$

By Theorem 2.22, the restriction map $\mathcal{W} \to L(P_n(F), \psi) : W \mapsto W|_{P_n(F)}$ is injective, so a non-zero function $W \in \mathcal{W}^K$ is not the zero function on $P_n(F)$. Let

$$\begin{pmatrix} g & x \\ & 1 \end{pmatrix} \in P_n(F)$$

where $g \in \mathrm{GL}_{n-1}(F)$, be an element such that $W\left(\begin{smallmatrix} g & x \\ & 1 \end{smallmatrix}\right) \neq 0$. Since

$$W\begin{pmatrix} g & x \\ & 1 \end{pmatrix} = W\left(\begin{pmatrix} I_{n-1} & x \\ & 1 \end{pmatrix}\begin{pmatrix} g & \\ & 1 \end{pmatrix}\right) = \psi\begin{pmatrix} I_{n-1} & x \\ & 1 \end{pmatrix} W\begin{pmatrix} g & \\ & 1 \end{pmatrix},$$

we also get that $W\begin{pmatrix} g & \\ & 1 \end{pmatrix} \neq 0$, so a non-zero function $W \in \mathcal{W}^K$ is also not the zero function on $\mathrm{GL}_{n-1}(F) \subseteq G$.

As

$$W\begin{pmatrix} u\varpi^{\underline{m}}k & \\ & 1 \end{pmatrix} = \psi\begin{pmatrix} u & \\ & 1 \end{pmatrix} W\begin{pmatrix} \varpi^{\underline{m}} & \\ & 1 \end{pmatrix}$$

for any $u \in N_{n-1}(F)$, $\underline{m} \in \mathbb{Z}^{n-1}$, and $k \in \mathrm{GL}_{n-1}(\mathcal{O})$, this implies that elements W of $\mathcal{W}^K$ are fully determined on $\mathrm{GL}_{n-1}(F) \subseteq G$ by their values at $\varpi^{\underline{m}}$ for $\underline{m} \in \mathbb{Z}^n$ with $m_n = 0$. By Lemma 2.25, $W(\varpi^{\underline{m}})$ with $m_n \geq 0$ is in turn fully determined by $W(1)$. Consequently, an element $W \in \mathcal{W}^K$ with $W(1) = 0$ must be the zero function on $\mathrm{GL}_{n-1}(F)$, and by Theorem 2.22 the zero function on all of G. □

Proof of Lemma 2.25. This proof is based on [Shi76], but with a slight modification, so that we do not need to assume the existence of an element $W \in \mathcal{W}^K$ with $W(1) = 1$ as is the case in [Shi76]. We give the modified proof in detail.

Let us by Theorem 2.7 regard $(\rho, \mathcal{W})$ as a smooth $\mathcal{H}_G$-module. As K-fixed vectors of a G-representation remain K-fixed under the action of $\mathcal{H}_G[e_K]$, $\rho(\mathcal{H}_G[e_K])\mathcal{W}^K = \mathcal{W}^K$. Since by Proposition 2.17 $\dim_{\mathbb{C}}(\mathcal{W}^K) = 1$, we have a $\mathbb{C}$-algebra homomorphism

$$\lambda : \mathcal{H}_G(K) \to \mathbb{C}$$

such that $\rho(f)W = \lambda(f)W$ for all $f \in \mathcal{H}_G[e_K]$ and $W \in \mathcal{W}^K$. For all $1 \leq i \leq n$ let

$$\underline{m}^i = (\overbrace{1, \dots, 1}^{i}, 0, \dots, 0) \in \mathbb{Z}^n,$$

let $f_i \in \mathcal{H}_G[e_K]$ denote the characteristic function of the double coset $K\varpi^{\underline{m}^i}K$, and let $\lambda_i = \lambda(f_i)$. Note that $\lambda_n \neq 0$, as $K\varpi^{(1,\dots,1)}K = \varpi^{(1,\dots,1)}K$, which implies that

$$\lambda_n W(g\varpi^{(-1,\dots,-1)}) = W(g)$$

for all $W \in \mathcal{W}^K$ and $g \in G$, but there exists at least one pair W and g such that $W(g) \neq 0$.

By the sublemma in [Shi76, p. 181], the double coset $K\varpi^{\underline{m}^i}K$ has the following decomposition into disjoint unions:

$$K\varpi^{\underline{m}^i}K = \bigcup_{\underline{\varepsilon} \in I_i} \bigcup_{x \in N_n(\mathcal{O})/N(\underline{\varepsilon})} x\varpi^{\underline{\varepsilon}}K, \tag{2.1}$$

where $I_i = \{\underline{\varepsilon} \in \{0,1\}^n \mid \varepsilon_1 + \dots + \varepsilon_n = i\}$, and $N(\underline{\varepsilon}) = N_n(\mathcal{O}) \cap \varpi^{\underline{\varepsilon}}K\varpi^{-\underline{\varepsilon}}$. One proves this by using the Bruhat decomposition for $K = \mathrm{GL}_n(\mathcal{O})$.

$N(\underline{\varepsilon})$ consists of upper triangular matrices x with ones on the diagonal and entries x_{ij} for $i > j$ which are either in $\mathcal{O}$ or in $\mathfrak{p}$ depending on $\underline{\varepsilon}$. Using a counting argument, and the fact that $q = [\mathcal{O} : \mathfrak{p}]$, one finds that

$$[N_n(\mathcal{O}) : N(\underline{\varepsilon})] = q^{in - \frac{i(i-1)}{2} - \sum_{j=1}^n j\varepsilon_j}, \qquad \underline{\varepsilon} \in I_i$$

for all $1 \leq i \leq n$.

Let us denote elements $x \in N_n(F)$ in the following way:

$$x = (x_{ij})_{i>j} = \begin{pmatrix} 1 & x_{12} & \cdots & x_{1n} \\ & \ddots & \ddots & \vdots \\ & & \ddots & x_{n-1,n} \\ & & & 1 \end{pmatrix}.$$

Then $\varpi^{\underline{m}}(x_{ij})_{i>j}\varpi^{-\underline{m}} = \left(x_{ij}\varpi^{m_i-m_j}\right)_{i>j}$ for all $\underline{m} \in \mathbb{Z}^n$.

Let us fix a non-zero K-fixed element $W \in \mathcal{W}^K$. Suppose $\underline{m} \in \mathbb{Z}^n$ is such that $m_1 \geq \cdots \geq m_n$ is not satisfied, i.e. $m_k < m_{k+1}$ for some $1 \leq k \leq n-1$. The level of $\psi : F \to \mathbb{C}^\times$ is 0, so we must be able to find $y \in \mathcal{O}^\times$ such that $\psi(y\varpi^{m_k-m_{k+1}}) \neq 1$. Then for $x = (x_{ij})_{i>j} \in N_n(\mathcal{O})$ with $x_{k,k+1} = y$ and $x_{ij} = 0$ otherwise,

$$\begin{aligned}
W(\varpi^{\underline{m}}) &= W(\varpi^{\underline{m}}x) \\
&= W\big((\varpi^{\underline{m}}x\varpi^{-\underline{m}})\varpi^{\underline{m}}\big) \\
&= \psi(\varpi^{\underline{m}}x\varpi^{-\underline{m}})W(\varpi^{\underline{m}}) \\
&= \psi(y\varpi^{m_k-m_{k+1}})W(\varpi^{\underline{m}}),
\end{aligned} \tag{2.2}$$

so $W(\varpi^{\underline{m}}) = 0$.

Now suppose $\underline{m} \in \mathbb{Z}^n$ is such that $m_1 \geq \cdots \geq m_n$. Then $\varpi^{\underline{m}}x\varpi^{-\underline{m}} \in N_n(\mathcal{O})$ for all $x \in N_n(\mathcal{O})$. By using the decomposition of $K\varpi^{\underline{m}^i}K$ in (2.1), we get for $1 \leq i \leq n$ that

$$\begin{aligned}
\lambda_i W(\varpi^{\underline{m}}) &= \int_G W(\varpi^{\underline{m}}g)f_i(g)\,dg \\
&= \sum_{\underline{\varepsilon}\in I_i}\ \sum_{x\in N_n(\mathcal{O})/N(\underline{\varepsilon})} \int_K W(\varpi^{\underline{m}}x\varpi^{\underline{\varepsilon}}k)\,dk \\
&= \sum_{\underline{\varepsilon}\in I_i}\ \sum_{x\in N_n(\mathcal{O})/N(\underline{\varepsilon})} W(\varpi^{\underline{m}}x\varpi^{\underline{\varepsilon}}) \\
&= \sum_{\underline{\varepsilon}\in I_i}\ \sum_{x\in N_n(\mathcal{O})/N(\underline{\varepsilon})} W\big(\underbrace{(\varpi^{\underline{m}}x\varpi^{-\underline{m}})}_{\in N_n(\mathcal{O})}\varpi^{\underline{m}+\underline{\varepsilon}}\big) \\
&= \sum_{\underline{\varepsilon}\in I_i}\ \sum_{x\in N_n(\mathcal{O})/N(\underline{\varepsilon})} \underbrace{\psi(\varpi^{\underline{m}}x\varpi^{-\underline{m}})}_{=1}\, W\big(\varpi^{\underline{m}+\underline{\varepsilon}}\big) \\
&= \sum_{\underline{\varepsilon}\in I_i} [N_n(\mathcal{O}) : N(\underline{\varepsilon})]\, W\big(\varpi^{\underline{m}+\underline{\varepsilon}}\big) \\
&= \sum_{\underline{\varepsilon}\in I_i} q^{in-\frac{i(i-1)}{2}-\sum_{j=1}^n j\varepsilon_j}\, W\big(\varpi^{\underline{m}+\underline{\varepsilon}}\big).
\end{aligned}$$

Let $w(\underline{m}) = q^{\sum_{j=1}^n (n-j)m_j}W(\varpi^{\underline{m}})$. Then the previous calculation implies that

$$q^{\frac{i(i-1)}{2}}\lambda_i w(\underline{m}) = \sum_{\underline{\varepsilon}\in I_i} w(\underline{m}+\underline{\varepsilon}).$$

So $w : \mathbb{Z}^n \to \mathbb{C}$ is a solution to the following system of difference equations for a function $y : \mathbb{Z}^n \to \mathbb{C}$:

$$\begin{cases} q^{\frac{i(i-1)}{2}}\lambda_i y(\underline{m}) = \sum_{\underline{\varepsilon}\in I_i} y(\underline{m}+\underline{\varepsilon}) \quad \text{for } 1 \leq i \leq n & \text{if } m_1 \geq \cdots \geq m_n, \\ y(\underline{m}) = 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

Let us consider this system defined for functions $y : \mathbb{Z}^n_{\geq 0} \to \mathbb{C}$. Such a system has a unique solution up to a constant factor, or a single unique solution given an initial condition $y(0,\ldots,0) = y_0$. In fact, we can find a non-zero solution in the form of a Schur polynomial.

Let $s_\mu(X_1,\ldots,X_n)$ denote the Schur polynomial of an integer partition $\mu = (\mu_1,\ldots,\mu_n)$, $\mu_1 \geq \cdots \geq \mu_n \geq 0$, of length n. Let $e_i(X_1,\ldots,X_n)$ denote the elementary symmetric polynomial of degree i in n variables. Then by the dual version of Pieri's rule (see e.g. [Sta23, p. 375])

$$e_i s_\mu = \sum_\nu s_\nu$$

summed over all partitions ν of length n obtained by adding 1 in i different entries of μ. Thus, if $a = (a_1, \dots, a_n) \in \mathbb{C}^n$ such that $e_i(a) = q^{\frac{i(i-1)}{2}} \lambda_i$ for $1 \le i \le n$, then $s_{\underline{m}}(a)$ as a complex function in $\underline{m} \in \mathbb{Z}^n_{\geq 0}$, which is zero when $\underline{m}$ is not an integer partition, satisfies the system (2.3) for $y : \mathbb{Z}^n_{\geq 0} \to \mathbb{C}$. But by Vieta's formulas, $(-1)^i e_i(a)$ are the coefficients of the complex monic univariate polynomial with roots $a_1, \dots, a_n$. Thus, we let $a_1, \dots, a_n$ be the n complex roots (possibly with repetition) of the polynomial

$$X^n + \sum_{i=1}^{n} (-1)^i q^{\frac{i(i-1)}{2}} \lambda_i X^{n-i} = \prod_{i=1}^{n} (X - a_i).$$

Since $\lambda_n \neq 0$, we get that none of the roots $a_1, \dots, a_n$ are zero. Thus $s_{\underline{m}}(a)$ is not the zero function, and it satisfies the system (2.3) with initial condition $y(0, \dots, 0) = 1$. Since $w|_{\mathbb{Z}^n_{\geq 0}}$ is also a solution to the system (2.3), we have that $w(\underline{m}) = w(0, \dots, 0) s_{\underline{m}}(a)$ for $\underline{m} \in \mathbb{Z}^n_{\geq 0}$ (where we do not exclude the possibility that $w(0, \dots, 0) = 0$). Thus for all $\underline{m} \in \mathbb{Z}^n$ with $m_n \geq 0$ there exists a $c_{\underline{m}} \in \mathbb{C}$, namely

$$c_{\underline{m}} = \begin{cases} q^{\sum_{j=1}^{n} (j-n) m_j} s_{\underline{m}}(a) & \text{if } m_1, \dots, m_{n-1} \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

such that $W(\varpi^{\underline{m}}) = c_{\underline{m}} W(1)$. □

We state an easy corollary of Proposition 2.24 that is used later, when considering global Whittaker models.

Corollary 2.26. *Let (π, V) and $\mathcal{W}$ be as in Proposition 2.24. Then there exists a unique element $W^\circ \in \mathcal{W}^K$ with $W^\circ(K) = \{1\}$. We call this element the* spherical *element of $\mathcal{W}$.*

Proof. By Proposition 2.17 V^K is one-dimensional and thus $\mathcal{W}^K$ is one-dimensional. By Proposition 2.24 $W(1) \neq 0$ for all non-zero elements $W \in \mathcal{W}^K$, hence we have a unique non-zero element $W^\circ \in \mathcal{W}^K$ such that $W^\circ(1) = 1$. Now, as W° is fixed by K,

$$W^\circ(k) = W^\circ(1) = 1$$

for all $k \in K$. □

Chapter 3

Local Archimedean representations

This chapter provides an overview of the notions and results related to the representations that appear as local components of an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A})$ at Archimedean places. These components are $(\mathfrak{g}, K)$-modules of the real Lie group $\mathrm{GL}_n(\mathbb{R})$ or $\mathrm{GL}_n(\mathbb{C})$, where $\mathfrak{g}$ is the Lie algebra and K is a maximal compact subgroup. The purpose of Sections 3.1 and 3.2 is to define such modules. Sections 3.3–3.5 introduce the Hecke algebra for $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$, and Section 3.6 concludes with a brief introduction to Whittaker models of $(\mathfrak{g}, K)$-modules. The two primary sources for this chapter are [Bum97, Chapter 2] and [KV95, Chapter I], where all omitted details can be found.

In this chapter, F denotes a local Archimedean field, meaning F is either $\mathbb{R}$ or $\mathbb{C}$. We denote by G the general linear group $\mathrm{GL}_n(F)$ of F. This is a real Lie group that is connected if $F = \mathbb{C}$ and disconnected (with two connected components) if $F = \mathbb{R}$. We fix a maximal compact subgroup K in G, which is the orthogonal group $O(n)$ if $F = \mathbb{R}$ and the unitary group $U(n)$ if $F = \mathbb{C}$.

Let $\mathfrak{g}$ denote the Lie algebra of G and let $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ denote the complexified Lie algebra of G. We denote by $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$, and by $\mathcal{Z}(\mathfrak{g})$ the centre of $\mathcal{U}(\mathfrak{g})$. Let $\mathfrak{k}$ denote the Lie algebra of K, and note that $\mathfrak{k}_{\mathbb{C}}$ is a subalgebra of $\mathfrak{g}_{\mathbb{C}}$.

The results in this chapter are generally valid when G is a real Lie group with finitely many connected components and K is a maximal compact subgroup of G. However, for our purposes in later chapters, we focus exclusively on the cases $G = \mathrm{GL}_n(\mathbb{R})$ and $G = \mathrm{GL}_n(\mathbb{C})$, as well as one additional case, where G is a product of copies of $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$. To avoid broadening the discussion in subsequent sections to groups other than $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$, we take it for granted that the theory presented here can be extended to finite products of $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$, as required for the third case.

3.1 Representsions of Lie groups

This section serves as a brief introduction to only the few notions from the representation theory of Lie groups that are necessary in this thesis. This is, of course, a rich topic, and a much more thorough introduction can be found in Chapter 2 of [Bum97], along with foundational works such as [War72] for a more general setting.

Recall that a *Fréchet space* is a locally convex metrizable complete Hausdorff topological vector space. A finite-dimensional complex vector space V is always assumed to carry the unique vector space topology such that it is isomorphic (as topological vector spaces) to the Euclidean space $\mathbb{C}^N$ where $N = \dim_{\mathbb{C}}(V)$. In particular, finite-dimensional complex vector spaces are Fréchet spaces.

Definition 3.1. A *representation* of G is a pair (π, V) where V is a Fréchet space over $\mathbb{C}$ and $\pi : G \to \mathrm{Aut}_{\mathbb{C}}(V)$ is a group homomorphism such that the map

$$G \times V \to V : (g, v) \mapsto \pi(g)v$$

is continuous.

We define a representation on K in the same manner, i.e. by replacing G with K in the above definition. We often only use either π or V to denote the representation (π, V). The homomorphisms of two G-representations (π, V) and (π', V') are continuous linear maps $f : V \to V'$ that intertwine with the G-actions, i.e.

$$f \circ \pi(g) = \pi'(g) \circ f$$

for all $g \in G$. A *subrepresentation* of a G-representation (π, V) is a closed subspace $W \subseteq V$ that is stable under the action of π. We write (π, W) and understand that here the action π is the restricted action $\pi(g)|_W$ for all $g \in G$.

Definition 3.2. Let (π, V) be a representation of G. We call a vector $v \in V$ *smooth* if the map

$$G \to V : g \mapsto \pi(g)v$$

is smooth. We denote by V^∞ the space of all smooth vectors in V. If $V = V^\infty$ we call (π, V) a *smooth* representation of G.

The reader should recall that a function $f : M \to V$ on a finite-dimensional smooth manifold M with values in a Fréchet space V is smooth if for all continuous linear maps $F : V \to \mathbb{R}$ the function

$$F \circ f : M \to \mathbb{R}$$

is smooth in the usual sense for real-valued functions on smooth manifolds.

Given a G-representation (π, V), we can define an action of $\mathfrak{g}$ on V^∞ given by

$$\pi(X)v = \frac{d}{dt}\pi(\exp(tX))v\big|_{t=0}$$

for $X \in \mathfrak{g}$ and $v \in V^\infty$, and one can show that $\pi(X)v$ is again a smooth vector, and that

$$\pi([X, Y]) = \pi(X)\pi(Y) - \pi(Y)\pi(X)$$

on V^∞. Thus the linear map $\pi : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(V^\infty)$ extends uniquely to a unital $\mathbb{C}$-algebra homomorphism $\pi : \mathcal{U}(\mathfrak{g}) \to \mathrm{End}_{\mathbb{C}}(V^\infty)$. We call $\pi : \mathcal{U}(\mathfrak{g}) \to \mathrm{End}_{\mathbb{C}}(V^\infty)$ the *differential* of the G-representation (π, V).

We now turn our attention to representations of K.

Definition 3.3. Let V be a complex vector space, and let $\pi : K \to \mathrm{Aut}_{\mathbb{C}}(V)$ be a group homomorphism. If a vector $v \in V$ satisfies that $\dim_C\langle\pi(K)v\rangle < \infty$ and that the map

$$\begin{gathered} K \times \langle\pi(K)v\rangle \to \langle\pi(K)v\rangle \\ (k, w) \mapsto \pi(k)w \end{gathered}$$

is continuous, then we say v is *K-finite.* We denote by V_K the space of all K-finite vectors in V, and note that (π, V_K) forms a representation of K. If $V = V_K$, we say that (π, V) is *K-finite.*

Similarly, for a representation (π, V) of G, we define V_K to be the subspace

$$V_K = \{v \in V \mid \dim_{\mathbb{C}}(\pi(K)v) < \infty\},$$

and call its elements the K-finite vectors of (π, V). (Here we have no additional continuity condition as this is already imposed in the definition of a G-representation.) It is a classic result that V_K is dense in V, but V_K is in general not stable under the action of G.

Let $\widehat{K}$ denote the set of isomorphism classes of irreducible finite-dimensional representations of K. For $\gamma \in \widehat{K}$ we let $\chi_\gamma : K \to \mathbb{C}$ denote the character of γ, which the reader should recall is the function $\chi_\gamma(k) = \mathrm{Tr}(\gamma(k))$. The character χ_γ is independent on the choice of representative of the isomorphism class. It is a standard result (see e.g. [KV95, Proposition 1.18]) that a K-finite representation (π, V) decomposes as a (possibly infinite) direct sum of finite-dimensional irreducible K-subrepresentations, in particular,

$$V = \bigoplus_{\gamma \in \widehat{K}} V(\gamma)$$

where $V(\gamma)$, called the *γ-isotopic component*, is the sum of all irreducible K-subrepresentations of (π, V) that are isomorphic to γ. In fact, $V(\gamma)$ is equal to a (possibly infinite) direct sum of finite-dimensional irreducible K-subrepresentations of V that are isomorphic to γ, and we call the number of summands in this direct sum the *multiplicity* of γ in (π, V).

3.2 $(\mathfrak{g}, K)$-modules

With representations of G and K defined, as well as the differential of a G-action, we are ready to define a $(\mathfrak{g}, K)$-module. Stricly, it is not a representation, although we often refer to it as such, but instead a pair of $\mathcal{U}(\mathfrak{g})$- and K-actions on the same space, which in some sense agree with each other.

Definition 3.4 ($(\mathfrak{g}, K)$-module)**.** A $(\mathfrak{g}, K)$-*module* is a complex vector space V together with a unital algebra homomorphism $\pi : \mathcal{U}(\mathfrak{g}) \to \mathrm{End}_{\mathbb{C}}(V)$ and a group homomorphism $\pi : K \to \mathrm{Aut}_{\mathbb{C}}(V)$ such that

(i) (π, V) is K-finite as a K-representation;
(ii) the differentiated version of the K-action is the restriction of the $\mathcal{U}(\mathfrak{g})$-action to $\mathfrak{k}$, i.e.
$$\frac{d}{dt}\pi(\exp(tY)v\big|_{t=0} = \pi(Y)v$$
for all $Y \in \mathfrak{k}$ and $v \in V$; and
(iii) the homomorphisms π respect the adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}_{\mathbb{C}}(\mathfrak{g})$ extended to $\mathcal{U}(\mathfrak{g})$ in the sense that
$$\pi(k)\pi(X)\pi(k^{-1})v = \pi\left(\mathrm{Ad}(k)X\right)v$$
for all $k \in K$, $X \in \mathcal{U}(\mathfrak{g})$ and $v \in V$.

A homomorphism between two $(\mathfrak{g}, K)$-modules (π_1, V_1) and (π_2, V_2) is a linear map $f : V_1 \to V_2$ that intertwines with both the $\mathcal{U}(\mathfrak{g})$-action and K-action. A $(\mathfrak{g}, K)$-*submodule* of a $(\mathfrak{g}, K)$-module (π, V) is a subspace $W \subseteq V$ that is simultaneously stable under the K-action and under the $\mathcal{U}(\mathfrak{g})$-action. We call (π, V) *irreducible* if it has no proper non-trivial $(\mathfrak{g}, K)$-submodules. We say (π, V) has *finite length* if every descending chain $V \supsetneq V_1 \supsetneq V_2 \supsetneq \dots$ of $(\mathfrak{g}, K)$-submodules of (π, V) has finite length. We call (π, V) *finitely* *generated* if there exists a finite-dimensional subspace $W \subseteq V$ such that W is K-stable, and $\pi(\mathcal{U}(\mathfrak{g}))W = V$. We say a $(\mathfrak{g}, K)$-module (π, V) is $\mathcal{Z}(\mathfrak{g})$*-finite* if there exists an ideal $\mathcal{J} \subseteq \mathcal{Z}(\mathfrak{g})$ of finite co-dimension that annihilates V, i.e. such that $\pi(\mathcal{J})v = 0$ for all $v \in V$.

We also have a notion of admissibility for $(\mathfrak{g}, K)$-modules: since a $(\mathfrak{g}, K)$-module (π, V) is in particular a K-finite K-representation, the γ-isotopic component $V(\gamma)$ for $\gamma \in \widehat{K}$ is well-defined (but is not necessarily a $(\mathfrak{g}, K)$-submodule). We call (π, V) *admissible* if $\dim_{\mathbb{C}}(V(\gamma)) < \infty$ for all $\gamma \in \widehat{K}$, or in other words, if the multiplicity of γ in (π, V) as a K-representation is finite for all $\gamma \in \widehat{K}$.

One of the advantanges of working with $(\mathfrak{g}, K)$-modules in comparison to smooth G-representations is that irreducibility implies admissibility, so for $(\mathfrak{g}, K)$-modules we have the analogous result of Corollary 2.16 for irreducible smooth representations of locally profinite groups. This is a corollary to the following theorem, which in [Vog81] is accredited to Harish-Chandra.

Theorem 3.5 ([Vog81, Corollary 5.4.15])**.** *Let (π, V) be a $(\mathfrak{g}, K)$-module. Then the following are equivalent.*

(i) (π, V) *has finite length.*
(ii) (π, V) *is finitely generated and $\mathcal{Z}(\mathfrak{g})$-finite.*
(iii) (π, V) *is admissible and $\mathcal{Z}(\mathfrak{g})$-finite.*
(iv) (π, V) *is admissible and finitely generated.*

Corollary 3.6. *Every irreducible $(\mathfrak{g}, K)$-module is admissible.*

Proof. This follows as irreducible $(\mathfrak{g}, K)$-modules in particular are of finite length. □

The study of $(\mathfrak{g}, K)$-modules is essentially of algebraic nature; we have not required a topology on the complex vector space V in Definition 3.4. There is an important connection between representations of G and $(\mathfrak{g}, K)$-modules, which is the motivation for considering $(\mathfrak{g}, K)$-modules in the first place, as they serve as a tool for studying representations of G purely in algebraic terms. Given a G-representation (π, V), one can show that the subspace V_K^∞ of V, consisting of K-finite smooth vectors in V, is stable under both the differential $\mathcal{U}(\mathfrak{g})$-action of π as well as the restricted action $\pi|_K$, and these are compatible in such a way that (π, V_K^∞) forms a $(\mathfrak{g}, K)$-module. This is the *underlying* $(\mathfrak{g}, K)$*-module* of the G-representation (π, V).

3.3 Distributions of compact support

In Section 3.5 we define the Hecke algebra of G, which resides in the space of distributions on G of compact support. The purpose of this section is to study this space of distributions, and to introduce the relevant notation needed in later sections.

The algebraic structure of the space of compactly supported distributions on a real Lie group is simpler to describe for groups that are unimodular, so we first need to establish this fact for G.

Proposition 3.7. *G is unimodular.*

Proof. Let μ denote the usual Lebesgue measure on $\mathbb{R}$, and μ^n the product measure on $\mathbb{R}^n$. Then it is known from measure theory that for any measurable set $A \subseteq \mathbb{R}^n$ and any linear transform $T : \mathbb{R}^n \to \mathbb{R}^n$, the image $T(A)$ has measure $|\det(T)|\mu^n(A)$. Let μ^{n^2} denote the product measure on $\mathrm{Mat}_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$. It then follows that for any measurable set $A \subseteq \mathrm{Mat}_{n\times n}(\mathbb{R})$ and $g \in \mathrm{GL}_n(\mathbb{R})$,

$$\mu^{n^2}(gA) = |\det(g)|^n \mu^{n^2}(A) \qquad \text{and} \qquad \mu^{n^2}(Ag) = |\det(g)|^n \mu^{n^2}(A)$$

hence the measure

$$\frac{d\mu^{n^2}(g)}{|\det(g)|^n}$$

is a left and right Haar measure on $\mathrm{GL}_n(\mathbb{R})$, proving it is unimodular.

Given an isomorphism $\mathbb{C}^n \cong \mathbb{R}^{2n}$ of real vector spaces that respects the usual Lebesgue measure η on $\mathbb{C}$, we have the exact same construction as before. Here, the real linear map $T' : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ that corresponds to a complex linear map $T : \mathbb{C}^n \to \mathbb{C}^n$ under this isomorphism, satisfies $|\det(T')| = |\det(T)|^2$, and thus

$$\frac{d\eta^{n^2}(g)}{|\det(g)|^{2n}}$$

defines a left and right Haar measure on $\mathrm{GL}_n(\mathbb{C})$ as before. □

Let us fix a Haar measure dx on G for the remainder of the chapter: it is the unique Haar measure such that $\mathrm{vol}_{dx}(K) = 1$. This measure also induces a fixed measure on K, the normalized Haar measure on K. Note that K is unimodular since it is compact (see Proposition 1.5(i)).

We denote by $C^\infty(G)$ the space of smooth functions $f : G \to \mathbb{C}$ (viewing G as a smooth manifold), and denote by $C_c^\infty(G)$ the space of smooth functions of compact support. It is a standard result that we can equip $C^\infty(G)$ with a collection of seminorms such that it becomes a Fréchet space (see e.g. [KV95, §B.1]). Let $\mathcal{E}'(G)$ denote the continuous dual of $C^\infty(G)$, i.e. the space of continuous linear functionals $T : C^\infty(G) \to \mathbb{C}$. This is the space of *distributions with compact support* on G. We denote the operation of a distribution $T \in \mathcal{E}'(G)$ on a smooth function $f \in C^\infty(G)$ by

$$\langle T, f\rangle, \qquad Tf, \qquad \text{or} \qquad \int_G f(x)\, dT(x).$$

The support of a distribution $T \in \mathcal{E}'(G)$ is defined as the complement of the union of all open subsets $U \subseteq G$ such that $Tf = 0$ for all $f \in C_c^\infty(U)$, and we denote it by $\mathrm{supp}(T)$. (Here we view $C_c^\infty(U)$ as the subspace of $C^\infty(G)$ consisting of compactly supported functions whose support is contained in U.) One can easily show that T vanishes on $C_c^\infty(\mathrm{supp}(T)^c)$, and that the support $\mathrm{supp}(T_1 + T_2)$ of the sum of two distributions $T_1, T_2 \in \mathcal{E}'(G)$ is contained in the union $\mathrm{supp}(T_1) \cup \mathrm{supp}(T_2)$ of the support of each distribution. Moreover, the zero distribution $0 \in \mathcal{E}'(G)$ is the only element of empty support. The following proposition is the justification for calling elements of $\mathcal{E}'(G)$ distributions with compact support.

Proposition 3.8. *If $T \in \mathcal{E}'(G)$ then $\mathrm{supp}(T)$ is compact.*

Proof. The proof relies on the fact that G is a second-countable Hausdorff locally compact topological space. This implies that G is exhaustible by compact sets, i.e. there exists a sequence $\{K_i\}_{i=1}^\infty$ of compact subsets of G such that K_i is contained in the interior of K_{i+1} for all i, and such that $G = \bigcup_{i=1}^\infty K_i$. Now, assume, for the sake of contradiction, that $\mathrm{supp}(T)$ is not compact. Then $\mathrm{supp}(T) \nsubseteq K_i$ for any i, so for each i we can choose $f_i \in \mathbb{C}_c^\infty(K_i^c)$ such that $Tf_i \neq 0$. Let $h_i \in C_c^\infty(K_i^c)$ be the function $(Tf_i)^{-1} f_i$, such that $Th_i = 1$. $h_i|_{K_i} = 0$, and thus clearly $h_i \to 0$ as $i \to \infty$. But $Th_i \to 1$ as $i \to \infty$, and thus T cannot be continuous, contradicting the claim. □

The space $\mathcal{E}'(G)$ carries a bilinear operation

$$\mathcal{E}'(G) \times \mathcal{E}'(G) \to \mathcal{E}'(G) : (T, S) \mapsto T * S$$

called *convolution*, where the operation $T * S$ is given by

$$\langle T * S, f \rangle = \int_G \int_G f(xy)\, dT(x)\, dS(y).$$

Here, one uses Fubini's theorem for distributions to show that this is a well-defined element of $\mathcal{E}'(G)$. As for in the non-Archimedean case, convolution is associative (see [KV95, Proposition 1.8]), so $\mathcal{E}'(G)$ becomes an associative $\mathbb{C}$-algebra with convolution as its bilinear product.

For an element $g \in G$, we denote by δ_g the distribution that evaluates at g, or equivalently, the distribution coming from the Dirac measure at g. That is,

$$\delta_g f = \int_G f(x)\, d\delta_g(x) = f(g).$$

Is is easy to see that δ_1 is an identity element with respect to convolution, so $\mathcal{E}'(G)$ is a unital associative algebra.

Let λ and ρ be the actions of G on $C^\infty(G)$ given by left and right translation respectively, that is, for a function $f \in C^\infty(G)$

$$\begin{aligned} &\lambda(g)f : x \mapsto f(g^{-1}x) \\ &\rho(g)f : x \mapsto f(xg). \end{aligned} \tag{3.1}$$

Both $(\lambda, C^\infty(G))$ and $(\rho, C^\infty(G))$ are smooth representations of G. We use the same letters for the differentiated versions of the G-actions, which give rise to representations of $\mathfrak{g}$, and by extension representations of $\mathcal{U}(\mathfrak{g})$. E.g.

$$(\rho(X)f)(g) = \frac{d}{dt} f(g \exp(tX))\big|_{t=0} \tag{3.2}$$

for $X \in \mathfrak{g}$, $f \in C^\infty(G)$ and $g \in G$. We also have a natural notion of a left and right regular action of G on $\mathcal{E}'(G)$ given by

$$\begin{aligned} \langle \lambda(g)T, f \rangle &= \langle T, \lambda(g^{-1})f \rangle \\ \langle \rho(g)T, f \rangle &= \langle T, \rho(g^{-1})f \rangle, \end{aligned}$$

and one can show $\mathcal{E}'(G)$ is stable under both actions. Note that

$$\lambda(g)T = \delta_g * T \qquad \text{and} \qquad \rho(g)T = T * \delta_{g^{-1}}.$$

Given a compactly supported distribution $T \in \mathcal{E}'(G)$, by Fubini's theorem (see e.g. [KV95, Theorem B.20]) the functions

$$\begin{aligned} C^\infty(G \times G) &\to C^\infty(G) \\ f &\mapsto T(f(\_\_, y)) = \int_G f(x, y)\, dT(x) \\ f &\mapsto T(f(x, \_\_)) = \int_G f(x, y)\, dT(y) \end{aligned}$$

are well-defined and continuous. This implies that we have the following action of $\mathcal{E}'(G)$ on $C^\infty(G)$: for a function f on G, we denote by $\check{f}$ the function $g \mapsto f(g^{-1})$, and for a distribution $T \in \mathcal{E}'(G)$ we denote by $\check{T}$ the distribution $f \mapsto T(\check{f})$. Given $T \in \mathcal{E}'(G)$, $f \in C^\infty(G)$, and $x \in G$, define $(f * T)(x)$ as the value

$$f * T(x) = T(\rho(x^{-1})\check{f}) = \int_G f(xy^{-1})\, dT(y).$$

As $(x, y) \mapsto f(xy^{-1})$ is an element of $C^\infty(G \times G)$, it follows from Fubini's theorem that $f * T \in C^\infty(G)$. We call $f * T$ the *convolution* of f with T, and thus $\mathcal{E}'(G)$ acts on $C^\infty(G)$ by right convolution. Given a smooth function $h \in C_c^\infty(G)$ of compact support, it defines a distribution $h\,dx$ in $\mathcal{E}'(G)$ given by

$$\langle h\,dx, f\rangle = \int_G f(x)h(x)\,dx.$$

For such distributions, convolution with a function $f \in C^\infty(G)$ is nothing more than the usual convolution for smooth functions; indeed,

$$\begin{aligned} f * (h\,dy)(x) &= \langle h\,dy, \rho(x^{-1})\check{f}\rangle \\ &= \int_G f(xy^{-1})h(y)\,dy \\ &= \int_G f(y')h((y')^{-1}x)\,dy' \\ &= (f * h)(x), \end{aligned}$$

where the substitution $y' = xy^{-1}$ in the third equation is possible because G is unimodular.

We also make note of another action of $\mathcal{E}'(G)$ on $C^\infty(G)$, which we denote by ρ: for $T \in \mathcal{E}'(G)$ and $f \in C^\infty(G)$ define $\rho(T)f$ to be the function

$$\rho(T)f(x) = T(\lambda(x^{-1}f) = \int_G f(xy)\,dT(y). \tag{3.3}$$

As with convolution, $\rho(T)f$ is again a smooth function on G. In fact, it is simply convolution with $\check{T}$:

$$f * \check{T}(x) = \int_G \left(\rho(x^{-1})\check{f}\right)(y)\,d\check{T}(y) = \int_G \left(\rho(x^{-1})\check{f}\right)(y^{-1})\,dT(y) = \int_G f(xy)\,dT(y).$$

We can identify $\mathcal{U}(\mathfrak{g})$ with the distributions in $\mathcal{E}'(G)$ that are supported at the identity, by associating an element $u \in \mathcal{U}(\mathfrak{g})$ with the distribution

$$\langle \partial(u), f\rangle = (\rho(u)f)(1)$$

that is the evaluation at 1 of the left-G-invariant differential operator attached to u. We define an antiautomorphism on $\mathfrak{g}$, which we call the *transpose*, given by the map $X \mapsto X^t$, where

$$X^t = -X.$$

We can extend it to an antiautomorphism on $\mathcal{U}(\mathfrak{g})$ by requiring that $(uv)^t = v^t u^t$ for all $u, v \in \mathcal{U}(\mathfrak{g})$. Then one can show that

$$\partial(uv) = \partial(u) * \partial(v), \quad \widecheck{\partial(u)} = \partial(u^t), \quad \text{and} \quad \rho(u)f = f * \partial(u^t) = \rho(\partial(u))f \tag{3.4}$$

for all $u, v \in \mathcal{U}(\mathfrak{g})$ and $f \in C^\infty(G)$. We also define left and right regular actions of $\mathcal{U}(\mathfrak{g})$ on $\mathcal{E}'(G)$ given by

$$\begin{aligned} \langle \lambda(u)T, f\rangle &= \langle T, \lambda(u^t)f\rangle \\ \langle \rho(g)T, f\rangle &= \langle T, \rho(u^t)f\rangle, \end{aligned}$$

and one can show $\mathcal{E}'(G)$ is again stable under both actions.

3.4 The Hecke algebra of K

We now want to define the Hecke algebra of K, as it plays an important role in describing the Hecke algebra of G. It is a two-sided ideal of $\mathcal{E}'(K)$, and thus an associative $\mathbb{C}$-algebra without a unit. To define it, we need the following proposition.

Proposition 3.9 ([KV95, Proposition 1.31])**.** *Let $\mathcal{E}'(K)_{K,\lambda}$ and $\mathcal{E}'(K)_{K,\rho}$ denote the spaces of K-finite vectors of the regular K-representations $(\lambda, \mathcal{E}'(K))$ and $(\rho, \mathcal{E}'(K))$ respectively. Then $\mathcal{E}'(K)_{K,\lambda}$ (resp. $\mathcal{E}'(K)_{K,\rho}$) consists of all distributions of the form $f dk$, where $f \in C^\infty(K)$ is a smooth left-K-finite (resp. right-K-finite) function on K. Moreover, every smooth function $f \in C^\infty(K)$ is left-K-finite if and only if it is right-K-finite, and consequently $\mathcal{E}'(K)_{K,\lambda} = \mathcal{E}'(K)_{K,\rho}$.*

We simply call distributions *K-finite* if they are K-finite with respect to both the left and right regular action of K. We can now define the *Hecke algebra* of K to be the subspace

$$\mathcal{H}_K = \mathcal{E}'(K)_{K,\lambda} = \mathcal{E}'(K)_{K,\rho} = \{K\text{-finite distributions on } K\}.$$

One can show that this is an associative algebra with convolution, and a two-sided ideal of $\mathcal{E}'(G)$.

We want to specify a set $\mathcal{E}_{\mathcal{H}_K}$ of elementary idempotents in $\mathcal{H}_K$, in order to show that $(\mathcal{H}_K, \mathcal{E}_{\mathcal{H}_K})$ forms an idempotented $\mathbb{C}$-algebra cf. Definition 2.6. For $\gamma \in \widehat{K}$ define $e_\gamma \in \mathcal{E}'(K)$ to be the distribution given by

$$e_\gamma = \dim(\gamma) \check{\chi}_\gamma \, dk.$$

One can show that e_γ is an idempotent in $\mathcal{H}_K$, that

$$e_\gamma * T = T * e_\gamma \tag{3.5}$$

for all $T \in \mathcal{E}'(K)$, and moreover that the map

$$T \mapsto e_\gamma * T$$

is the projection map $\mathcal{H}_K \to \mathcal{H}_K(\gamma)$ of the K-finite K-representation $(\lambda, \mathcal{H}_K)$ to its γ-isotopic component. This is a consequence of the Schur orthogonality relations for compact Lie groups. For a finite subset $A \subseteq \widehat{K}$ we define the distribution $e_A \in \mathcal{H}_K$ to be the sum

$$e_A = \sum_{\gamma \in A} e_\gamma.$$

As $\mathcal{H}_K = \bigoplus_{\gamma \in \widehat{K}} \mathcal{H}_K(\gamma)$, it is clear that for every $T \in \mathcal{H}_K$, there exists a finite set $A \subseteq \widehat{K}$ such that

$$e_A * T = T. \tag{3.6}$$

and furtheremore, for any pair of finite sets $A, B \subseteq \widehat{K}$,

$$e_A * e_B = e_{A \cap B}. \tag{3.7}$$

This implies that e_A is an idempotent. It is also immediate that (3.5) still holds for e_A. Let $\mathcal{E}_{\mathcal{H}_K} = \{e_A \mid A \subseteq \widehat{K} \text{ finite}\}$ denote the set of such idempotents in $\mathcal{H}_K$. Equations (3.5) and (3.7) imply that for any pair $e_A, e_B \in \mathcal{E}_{\mathcal{H}_K}$ there exists an element $e_C \in \mathcal{E}_{\mathcal{H}_K}$ (namely $e_{A \cup B}$) such that

$$e_A * e_C = e_C * e_A = e_A \qquad \text{and} \qquad e_B * e_C = e_C * e_B = e_B,$$

and thus $\mathcal{E}_{\mathcal{H}_K}$ forms a directed set with the partial ordering $e_A \geq e_B$ if and only if $e_A * e_B = e_B * e_A = e_A$, satisfying condition (i) in the definition of an idempotented algebra (Definition 2.6). Equations (3.5) and (3.6) imply $\mathcal{E}_{\mathcal{H}_K}$ satisfies condition (ii), hence $(\mathcal{H}_K, \mathcal{E}_{\mathcal{H}_K})$ forms an idempotented $\mathbb{C}$-algebra.

The reader should recall that we define the notion of a *smooth* module over an idempotented algebra in Section 2.3, as well as the properties *irreducible* and *admissible* for such modules.

Given a K-finite representation (π, V) of K, we want to relate it to a smooth $\mathcal{H}_K$-module, just like we did in Section 2.3 for smooth modules over the Hecke algebra of a locally profinite group. For a distribution $T \in \mathcal{H}_K$, let $f \in C^\infty(K)$ be such that $T = f\,dk$ (which we know exists by Proposition 3.9). We define the action $\pi(T)$ on V by

$$\pi(T)v = \int_K f(k)\pi(k)v\,dk. \tag{3.8}$$

One can show (see e.g. [KV95, Proposition 1.24]) that the operator $\pi(e_A)$ is the projection map $V \to V(\gamma_1) \oplus \cdots \oplus V(\gamma_n)$ to the direct sum of the γ_i-isotopic components of (π, V) for $\gamma_i \in A = \{\gamma_1, \ldots, \gamma_n\}$. This implies the direct part of the following proposition.

Proposition 3.10 ([KV95, Theorem 1.57])**.** *Let (π, V) be a K-finite representation of K. The vector space V endowed with the operation*

$$\mathcal{H}_K \times V \to V : (T, v) \mapsto \pi(T)v$$

forms a smooth $\mathcal{H}_K$-module. If (π', V') is another K-finite representation of K, then

$$\mathrm{Hom}_K(V, V') = \mathrm{Hom}_{\mathcal{H}_K}(V, V').$$

Conversely, if M is a smooth $\mathcal{H}_K$-module, there exists a unique K-homomorphism $\pi : K \to \mathrm{Aut}_{\mathbb{C}}(M)$ such that (π, M) forms a K-finite representation of K, and

$$T \cdot m = \pi(T)m$$

for all $T \in \mathcal{H}_K$ and $m \in M$.

We note that a K-finite representation of K is admissible if and only if it is admissible as a smooth $\mathcal{H}_K$-module.

Example 3.11. For the K-finine representation $(\rho, C^\infty(K)_K)$ of K on the left- and right-K-finite smooth functions on K, the equivalent $\mathcal{H}_K$-action on $C^\infty(K)_K$ for a distribution $T \in \mathcal{H}_K$ is the action $\rho(T)$ we define in Section 3.3, that is given by right convolution with $\check{T}$.

3.5 The Hecke algebra of G

We are now ready to define the *Hecke algebra* of G: it is the subalgebra of $\mathcal{E}'(G)$ consisting of K-finite distributions on G with support in K, and we denote it by $\mathcal{H}_G$. We view $\mathcal{H}_K \subseteq \mathcal{H}_G$ in the following way: let a distribution $T \in \mathcal{H}_K$ operate on $f \in C^\infty(G)$ via

$$\langle T, f\rangle_{C^\infty(G)} = \langle T, f|_K\rangle_{C^\infty(K)}.$$

and thus T viewed in $\mathcal{E}'(G)$ is a distribution on G with support in K. It is easy to show that the K-finite property of an element in $\mathcal{H}_K$ implies it is also K-finite as a distribution on G.

If $T \in \mathcal{H}_K$ and $u \in \mathcal{U}(\mathfrak{g})$, then $T * \partial(u)$ is a distribution on G with support K. It is clear that it is K-finite under the left regular action of K, as

$$\lambda(k)(T * \partial(u)) = \delta_k * (T * \partial(u)) = (\delta_k * T) * \partial(u) = (\lambda(k)T) * \partial(u).$$

It is also K-finite under the right regular action of K, because it is in general true that a distribution on G with support in K is right-K-finite if and only if it is left-K-finite (see [KV95, Proposition 1.83]). So we get a bilinear map

$$\begin{aligned} \mathcal{H}_K \times \mathcal{U}(\mathfrak{g}) &\to \mathcal{H}_G \\ (T, u) &\mapsto T * \partial(u) \end{aligned}$$

which by the universal property of the tensor product of $\mathbb{C}$-vector spaces defines a linear map

$$\mathcal{H}_K \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{g}) \to \mathcal{H}_G \tag{3.9}$$

We get from (3.4) that

$$(\rho(v^t)T) * \partial(u) = \big((T * \partial(v)\big) * \partial(u) = T * \big(\partial(v) * \partial(u)\big) = T * \partial(vu),$$

and thus if we view $\mathcal{H}_K$ as a right $\mathcal{U}(\mathfrak{k})$-module with the action $(T, v) \mapsto \rho(v^t)T$ and $\mathcal{U}(\mathfrak{g})$ as a left $\mathcal{U}(\mathfrak{k})$-module by means of the usual multiplication in $\mathcal{U}(\mathfrak{g})$, the linear map in (3.9) descends to a linear map

$$\mathcal{H}_K \bigotimes_{\mathcal{U}(\mathfrak{k})} \mathcal{U}(\mathfrak{g}) \to \mathcal{H}_G : T \otimes u \mapsto T * \partial(u) \tag{3.10}$$

of $\mathbb{C}$-vector spaces. By [KV95, Corollary 1.71] this is in fact a vector space isomorphism, and as a map from right to left, it sends the left action of $\mathcal{E}'(K)$ on $\mathcal{H}_G$ to the left action on $\mathcal{H}_K$ (left action being that of left convolution) and the right action of $\mathcal{U}(\mathfrak{g})$ on $\mathcal{H}_G$ (namely $\rho(u)$) to the right action on $\mathcal{U}(\mathfrak{g})$ (right multiplication). It is possible to equip $\mathcal{H}_K \bigotimes_{\mathcal{U}(\mathfrak{k})} \mathcal{U}(\mathfrak{g})$ with a bilinear product such that the isomorphism in (3.10) becomes a $\mathbb{C}$-algebra isomorphism, but some work is required to define it (see [KV95, Proposition 1.80]), and we do not need its description. Similarly, one can also show that the map $(u, T) \mapsto \partial(u) * T$ induces a $\mathbb{C}$-vector space isomorphism $\mathcal{U}(\mathfrak{g}) \bigotimes_{\mathcal{U}(\mathfrak{k})} \mathcal{H}_K \cong \mathcal{H}_G$, but once again work is required to translate the presentation of an element of $\mathcal{H}_G$ between the two isomorphisms.

Since $\mathcal{H}_G$ is both left- and right-K-finite, we can apply Proposition 3.10 to both K-representations $(\lambda, \mathcal{H}_G)$ and $(\rho, \mathcal{H}_G)$, and thus for each $S \in \mathcal{H}_G$ we find $e_A, e_B \in \mathcal{E}_{\mathcal{H}_K}$ such that $e_A * S = S$ and $S * e_B = S$. Since $\mathcal{E}_{\mathcal{H}_K}$ is a directed set with respect to its partial ordering, we can find $e_C \in \mathcal{E}_{\mathcal{H}_K}$ such that $e_C \geq e_A$ and $e_C \geq e_B$, hence

$$S * e_C = e_C * S = S.$$

Viewing $\mathcal{E}_{\mathcal{H}_K} \subseteq \mathcal{H}_G$, we have that the pair $(\mathcal{H}_G, \mathcal{E}_{\mathcal{H}_K})$ also forms an idempotented $\mathbb{C}$-algebra, and thus we also call the elements of $\mathcal{E}_{\mathcal{H}_K}$ the *elementary idempotents* of $\mathcal{H}_G$.

We are now ready to relate $(\mathfrak{g}, K)$-modules to smooth $\mathcal{H}_G$-modules. For a $(\mathfrak{g}, K)$-module (π, V) and a distribution $S = T * \partial(u) \in \mathcal{H}_G$, define the action $\pi(S)$ on V by

$$\pi(S)v = \pi(T * \partial(u))v = \pi(T)\big(\pi(u)v\big),$$

where $\pi(T)$ is defined for the K-action as in (3.8).

Theorem 3.12 ([KV95, Theorem 1.117])**.** *Let (π, V) be a $(\mathfrak{g}, K)$-module. The vector space V endowed with the operation*

$$\mathcal{H}_G \times V \to V : (S, v) \mapsto \pi(S)v$$

forms a smooth $\mathcal{H}_G$-module. If (π', V') is another $(\mathfrak{g}, K)$-module, then

$$\mathrm{Hom}_{\mathfrak{g},K}(V, V') = \mathrm{Hom}_{\mathcal{H}_G}(V, V').$$

Conversely, if M is a smooth $\mathcal{H}_G$-module, there exists a unique algebra homomorphism $\pi : \mathcal{U}(\mathfrak{g}) \to \mathrm{End}_{\mathbb{C}}(M)$ and a unique group homomorphism $\pi : K \to \mathrm{Aut}_{\mathbb{C}}(M)$ such that (π, M) forms a $(\mathfrak{g}, K)$-module, and

$$S \cdot m = \pi(S)m$$

for all $S \in \mathcal{H}_G$ and $m \in M$. Moreover, (π, V) is admissible as a $(\mathfrak{g}, K)$-module if and only if it is admissible as a smooth $\mathcal{H}_G$-module.

Example 3.13. The space $C^\infty(G)_{K,\rho}$ of K-finite vectors of the smooth representation $(\rho, C^\infty(G))$ of G is the underlying $(\mathfrak{g}, K)$-module of $(\rho, C^\infty(G))$. So, $(\rho, C^\infty(G)_{K,\rho})$ forms a $(\mathfrak{g}, K)$-module, where the K-action ρ is simply right translation and the $\mathcal{U}(\mathfrak{g})$-action ρ is differentiation as in (3.2). The equivalent $\mathcal{H}_G$-action on $C^\infty(G)_{K,\rho}$ from Theorem 3.12 is the restriction to $\mathcal{H}_G$ of the action ρ of $\mathcal{E}'(G)$ on $C^\infty(G)$ from (3.3), that is, the action $\rho(S)$ of an element $S \in \mathcal{H}_G$ is simply right convolution with $\check{S}$. Thus the pair $(\rho, C^\infty(G)_{K,\rho})$ makes sense both as a $(\mathfrak{g}, K)$-module and a smooth $\mathcal{H}_G$-module, and the two interpretations are equivalent.

3.6 Whittaker models

As with local non-Archimedean representations, we also have a notion of a Whittaker model for irreducible $(\mathfrak{g}, K)$-modules. However, in order for the Whittaker model to be unique, if it exists, we need to impose a condition on the growth of the functions in a Whittaker model.

Definition 3.14. We say a smooth function $f \in C^\infty(G)$ is of *moderate growth* if there exist constants $C \in \mathbb{R}_{>0}$ and $N \in \mathbb{N}$ such that

$$|f(g)|_{\mathbb{C}} \leq C \|g\|_\infty^N$$

for all $g \in G$, where

$$\|g\|_\infty = \max_{1 \leq i,j \leq n} \left\{ \left|g_{ij}\right|_{\mathbb{C}}, \left|(g^{-1})_{ij}\right|_{\mathbb{C}} \right\}.$$

Let us fix a non-trivial unitary continuous group homomorphism $\psi : F \to \mathbb{S}^1$. This is a unitary character of the additive group $(F, +)$, and should not be confused with the earlier notion of a character χ_γ of a K-representation $\gamma \in \widehat{K}$. Let $N_n(F)$ be the maximal unipotent subgroup of G, and recall from Section 1.1 that we also consider ψ as character of $N_n(F)$ by applying it to the sum of the first upper diagonal.

Definition 3.15 (Whittaker function on G)**.** We call a smooth function $W \in C^\infty(G)$ a *ψ-Whittaker function* if it satisfies the following conditions:

(i) $W(mg) = \psi(m)W(g)$ for all $m \in N_n(G)$ and $g \in G$, and
(ii) W is of moderate growth.

Since the space of ψ-Whittaker functions consists of smooth functions on G, we can act on the space via the action ρ of the Hecke algebra $\mathcal{H}_G$, or equivalently (as we saw in Example 3.13), we can act on the space via right translation of K and via the differentiated action ρ of $\mathcal{U}(\mathfrak{g})$, although, for our purposes, we do not argue that the space of Whittaker functions is stable under such actions.

Theorem 3.16. *Let (π, V) be an irreducible $(\mathfrak{g}, K)$-module. There exists at most one space $\mathcal{W}(\pi, \psi)$ of ψ-Whittaker functions such that $\mathcal{W}(\pi, \psi)$ is stable under the actions ρ of K and $\mathcal{U}(\mathfrak{g})$ (cf.* (3.1) *and* (3.2)*), and such that $(\rho, \mathcal{W}(\pi, \psi))$ is isomorphic to (π, V) as $(\mathfrak{g}, K)$-modules. Moreover, every function in $\mathcal{W}(\pi, \psi)$ is analytic.*

Remark 3.17. This is originally due to [Sha74, Theorem 3.1], where Shalika proved an equivalent statement for unitary irreducible admissible representations of G. In [Cas89], Casselman proves that the category of his notion of "$(\mathfrak{g}, K)$-Harish-Chandra modules" (which irreducible $(\mathfrak{g}, K)$-modules are objects of) and the category of smooth representations of G of "moderate growth" and finite length are equivalent. In [CHM00, Theorem 9.2] he, together with his co-authors, proves the equivalent statement of Theorem 3.16 for irreducible smooth representations of G of moderate growth and finite length, which implies our statement of the theorem. Both Shalika and Casselman-Hecht-Miličić prove that the space of certain continuous functionals on the smooth vectors of a unitary irreducible admissible representation of G (resp. an irreducible smooth representation of G of moderate growth and finite length), called "Whittaker functionals", is one-dimensional, and it is from the definition of the Fréchet topology on the smooth vectors of a unitary representation of G (resp. Fréchet topology on the representation space of a smooth representation of G) and the fact that Whittaker functionals are continuous, that the K-finite vectors, via a Whittaker functional, define functions of moderate growth. (See e.g. Lemma 8.3.1 in [JPSS79] in the case of unitary representations.)

Remark 3.18. Due to the fact that we also have an Iwasawa decomposition (see Proposition 2.10) for $\mathrm{GL}_n(F)$ in the Archimedean case, it is possible to show that in the case $n = 2$, our definition of moderate growth of Whittaker functions is equivalent to the one found in [JL70, Theorem 5.13 and Theorem 6.3], that is, functions $f \in C^\infty(\mathrm{GL}_2(F))$ that satisfy

$$f\begin{pmatrix} a & \\ & 1 \end{pmatrix} = O\big(|a|_{\mathbb{C}}^N\big)$$

as $|a|_{\mathbb{C}} \to \infty$, for some positive number N. It is, however, not trivial to show, and heavily relies on the description of irreducible $\mathcal{H}_G$ modules specific to the case $G = \mathrm{GL}_2(F)$.

Definition 3.19 (Whittaker model)**.** Let (π, V) be an irreducible $(\mathfrak{g}, K)$-module. If the space $\mathcal{W}(\pi, \psi)$ in Theorem 3.16 exists, we call it the *ψ-Whittaker model* of (π, V). If (π, V) has a ψ-Whittaker model, then we say (π, V) is *ψ-generic.*

Chapter 4

Automorphic forms and automorphic representations

This chapter transitions to the global setting, with the primary objective of defining cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A})$. Its purpose is to provide a comprehensive overview of the foundational theory, while the core contributions of this thesis—concerning global Whittaker models and the proof of the strong multiplicity one theorem—are developed in the final two chapters.

We begin in Sections 4.1–4.3 with a study of the ring of adeles $\mathbb{A}$, including its topology, key approximation theorems, and a discussion of its additive and multiplicative characters. We also introduce the one-dimensional Fourier expansion for adelic periodic functions. In Section 4.4, we turn our attention to the adele group $\mathrm{GL}_n(\mathbb{A})$, examining its topology, approximation properties, and fixing notation for its essential subgroups. Section 4.5 is devoted to defining the algebraic notion of a representation of $\mathrm{GL}_n(\mathbb{A})$. In Section 4.6, we introduce the global Hecke algebra of $\mathrm{GL}_n(\mathbb{A})$. By establishing a correspondence between smooth modules of the global Hecke algebra and representations of $\mathrm{GL}_n(\mathbb{A})$—analogous to the local cases in Chapters 2 and 3—we lay the groundwork for proving the tensor product theorem for representations of $\mathrm{GL}_n(\mathbb{A})$. Finally, Sections 4.7–4.8 focus on defining the spaces of automorphic forms and cusp forms on $\mathrm{GL}_n(\mathbb{A})$. These spaces serve as the representation spaces for automorphic representations and cuspidal automorphic representations, which are treated in Section 4.9.

4.1 The ring of adeles

This section introduces the ring of adeles and the group of ideles, along with their topology and results concerning the principal adeles, as well as strong and weak approximation for the ring of adeles. This section is based on chapter VI in [Lan64], unless otherwise stated.

Let us fix an algebraic number field F for the remainder of the chapter. To F we associate a *set of places*, where a *place* is usually denoted by v. Each place v is associated with an equivalence class of absolute values, and we call v *infinite* or *Archimedean* if the associated absolute values are Archimedean, and similarly we call v *finite* or *non-Archimedean* if the associated absolute values are non-Archimedean. We denote by S_∞ the set of all infinite places of F. The reader should be familiar with the fact that there are a finite number of infinite places, as an Archimedean absolute value is equivalent to one induced by one of the embeddings of F into $\mathbb{R}$ or $\mathbb{C}$, as well as the fact that there are countably infinitely many finite places, as a non-Archimedean absolute value is equivalent to a $\mathfrak{p}$-adic absolute value for some prime ideal $\mathfrak{p}$ of the ring of integers $\mathcal{O}_F$ of F.

We write F_v for the completion of F with respect to an absolute value associated with v, and note that this completion is independent of the choice of equivalent absolute values. The reader should recall that if v is infinite, F_v is an Archimedean local field, i.e. equal to either $\mathbb{R}$ or $\mathbb{C}$, depending on the embedding of F associated with v, and if v is finite, F_v is a non-Archimedean local field. We equip each F_v with a local absolute value $|\cdot|_v : F_v \to \mathbb{R}_{\geq 0}$, given as in Chapter 2 if v is finite, and given as the usual absolute value $|\cdot|_{\mathbb{C}}$ on $\mathbb{C}$ if v is infinite, which in both cases is the extension of an absolute value on F associated with v. For simplicity, we denote the ring of integers of F_v by $\mathcal{O}_v$ in the case where v is finite.

We define the *ring of adeles* $\mathbb{A}$ of F to be the restricted direct product of the F_v's, where v ranges over all places, with respect to the $\mathcal{O}_v$'s, i.e. $\mathbb{A}$ is the topological ring with underlying set

$$\mathbb{A} = \left\{ (x_v)_v \in \prod_v F_v \;\middle|\; x_v \in \mathcal{O}_v \text{ for almost all } v \notin S_\infty \right\},$$

whose topology is generated by the family that consists of all sets of the form

$$\prod_{v \in S} U_v \times \prod_{v \notin S} \mathcal{O}_v,$$

where S is a finite set of places containing S_∞, and U_v is open in F_v for all $v \in S$. Since each F_v is a locally compact field and $\mathcal{O}_v$ is compact in F_v for all finite places, one can show that $\mathbb{A}$ is indeed a locally compact ring, with componentwise addition and multiplication.

Analogously, we define the *idele group* $\mathbb{A}^\times$ of F to be the restricted direct product of the $F_v^\times$'s with respect to the $\mathcal{O}_v^\times$'s, and one should note that this choice of topology on $\mathbb{A}^\times$ does not coincide with that of its subspace topology in $\mathbb{A}$ (since taking inverses is not continuous in the given topology of $\mathbb{A}$). Once again, one can show $\mathbb{A}^\times$ is a locally compact group with componentwise multiplication.

As the ring of integers $\mathcal{O}_F$ of F is a Dedekind domain, every fractional ideal $\mathfrak{a}$ has a unique factorization

$$\mathfrak{a} = \prod_{\mathfrak{p}} \mathfrak{p}^{r_\mathfrak{p}},$$

ranging over all prime ideals $\mathfrak{p}$ of $\mathcal{O}_F$, where $r_\mathfrak{p} \in \mathbb{Z}$ for all $\mathfrak{p}$, and $r_\mathfrak{p}$ is equal to 0 for almost all $\mathfrak{p}$. Thus, every element $\alpha \in F$ has $|\alpha|_v \leq 1$ for almost all finite places v. This implies that under the natural embeddings $F \hookrightarrow F_v$, resulting from the completion at each place v, α is embedded in $\mathcal{O}_v$ at almost all finite places. and as a result, the diagonal map

$$F \to \mathbb{A} : \alpha \mapsto (\alpha)_v = (\alpha, \alpha, \dots)$$

gives a natural embedding $F \hookrightarrow \mathbb{A}$. We call the image of F under this map the *space of principal adeles*, and we will always associate it with F itself, and in this way view F as a subset of $\mathbb{A}$ (and when considered as additive groups, view F as a subgroup of $\mathbb{A}$). Of course, for $\alpha \in F^\times$, $|\alpha|_v = 1$ for almost all finite places v, and thus we also get an embedding $F^\times \hookrightarrow \mathbb{A}^\times$ via the same diagonal embedding map. As before, this embedding allows us to view $F^\times$ as a multiplicative subgroup of $\mathbb{A}^\times$.

For a finite set of places S, we define F_S to be the direct product

$$F_S = \prod_{v \in S} F_v,$$

and we define the *S-adeles*, denoted by $\mathbb{A}_S$, to be the projection of $\mathbb{A}$ onto the direct product

$$\prod_{v \notin S} F_v,$$

or equivalently, as the restricted direct product of the F_v's for $v \notin S$ with respect to the $\mathcal{O}_v$'s for the finite places $v \notin S$. For $S = S_\infty$ we call the S_∞-adeles the *finite adeles*, and denote them by $\mathbb{A}_f = \mathbb{A}_{S_\infty}$. For a finite set of places $S \supset S_\infty$, we define the *S-integral adeles*, denoted by $\mathbb{A}(S)$, as the direct product

$$\mathbb{A}(S) = \prod_{v \in S} F_v \times \prod_{v \notin S} \mathcal{O}_v,$$

and we simply call $\mathbb{A}(S_\infty)$ the *integral adeles*, and denote them by $\mathbb{A}(\infty)$.

We have the following three fundamental results on the principal adeles.

Theorem 4.1.

(i) *The additive group F is a discrete subgroup of $\mathbb{A}$, and the multiplicative group $F^\times$ is a discrete subgroup of $\mathbb{A}^\times$.*

(ii) *The quotient group $F \backslash \mathbb{A}$ is compact.*

(iii) *Let $N = [F : \mathbb{Q}]$ and $\omega_1, \ldots, \omega_N$ be an integral basis for $\mathcal{O}_F$. Let $\mathcal{F}_\infty$ be the fundamental parallelepiped of F_{S_∞} consisting of the vectors $\sum_{i=1}^N t_i \omega_i$ for $t_i \in [0, 1)$. Then*

$$\mathcal{F} = \mathcal{F}_\infty \times \prod_{v \notin S_\infty} \mathcal{O}_v$$

is a fundamental domain for $F \backslash \mathbb{A}$.

Proof. This is the contents of Theorem 1, 2, and 3 in [Lan64, VI]. But, it is also easy to see that (i) (for F) and (ii) follow from (iii); indeed, if we assume (iii) has been proven, the parallelepiped $\mathcal{F}_\infty^\circ = \left\{ \sum_{i=1}^N t_i \omega_i \,\middle|\, t_i \in (-1, 1) \right\}$ of F_{S_∞} is open, and since $\mathcal{F}$ is a fundamental domain for $F \backslash \mathbb{A}$, the open subset

$$\mathcal{F}_\infty^\circ \times \prod_{v S_\infty} \mathcal{O}_v \subseteq \mathbb{A}$$

contains only the identity element $0 \in F$, and thus F is discrete in $\mathbb{A}$. Furthermore, the parallelepiped $\mathcal{F}_\infty^c = \left\{ \sum_{i=1}^N t_i \omega_i \,\middle|\, t_i \in [0, 1] \right\}$ is a compact subset of F_{S_∞}, and since $\mathcal{F}_\infty \subseteq \mathcal{F}_\infty^c$ and $\mathcal{F}$ is a fundamental domain for $F \backslash \mathbb{A}$, we get that also

$$F + \left(\mathcal{F}_\infty^c \times \prod_{v \notin S_\infty} \mathcal{O}_v \right) = \mathbb{A},$$

and since the product in the parentheses is a compact subset of $\mathbb{A}$, and F is discrete, F must be cocompact in $\mathbb{A}$. □

Although the embedding $F \hookrightarrow \mathbb{A}$ is discrete, any diagonal embedding of F into either finite direct products of its completions or the S-adeles for $S \neq \emptyset$ is dense. This means that in such spaces, elements are well-approximated by elements of F, explaining why these two results are called *weak approximation* and *strong approximation*, respectively.

Theorem 4.2 (Weak approximation for $\mathbb{A}$)**.** *For each non-empty finite set of places S, the diagonal embedding $F \hookrightarrow F_S$ is dense.*

Theorem 4.3 (Strong approximation for $\mathbb{A}$)**.** *For each non-empty finite set of places S, the diagonal embedding $F \hookrightarrow \mathbb{A}_S$ is dense.*

Theorem 4.2 is a reformulation of [Lan62, I §1, Theorem 1] in the context of our framework, but it is a general theorem for any field, and is the analogue of the Chinese remainder theorem for absolute values. Theorem 4.3 is naturally a much stronger result and its proof is more involved. A proof can be found in [CF67, II §15]. In Section 4.4, we will see that the idele group $\mathbb{A}^\times$ has weak approximation as well, as the locally compact group $\mathrm{GL}_1(\mathbb{A})$ is exactly $\mathbb{A}^\times$.

4.2 Adelic characters

As usual, a character of a topological group G is a continuous group homomorphism $\chi : G \to \mathbb{C}^\times$, and a *unitary* character is a character whose codomain is the circle group $\mathbb{S}^1 = \{z \in \mathbb{C}^\times \mid |z|_{\mathbb{C}} = 1\}$. The *trivial* character is the character $\mathbb{1} : G \to \mathbb{C}^\times$ that maps everything to $1 \in \mathbb{C}^\times$.

A character $\chi : \mathbb{A} \to \mathbb{C}^\times$ of the additive group of $\mathbb{A}$ will often be called *additive* in order to distinguish it from a character $\mathbb{A}^\times \to \mathbb{C}^\times$ on the ideles (and *multiplicative* vice-versa) even though the codomain in both cases is a multiplicative group. A non-trivial character $\omega : \mathbb{A}^\times \to \mathbb{C}^\times$ of the ideles is called a *Hecke character* if it is trivial on $F^\times$, i.e. if $\omega(\alpha) = 1$ for all $\alpha \in F^\times$. For Hecke characters, we will often write $\omega : F^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$ although we strictly still consider thier domain as $\mathbb{A}^\times$ rather than the quotient group $F^\times\backslash\mathbb{A}^\times$.

Given a character $\chi : \mathbb{A} \to \mathbb{C}^\times$, the natural inclusion $i_v : F_v \hookrightarrow \mathbb{A}$ at any place v induces a character $\chi_v = \chi \circ i_v$ on F_v. Similarly a character $\omega : \mathbb{A}^\times \to \mathbb{C}^\times$ induces the characters $\omega_v : F_v^\times \to \mathbb{C}^\times$.

Proposition 4.4.

(i) *For an additive character $\chi : \mathbb{A} \to \mathbb{C}^\times$, the local character χ_v is trivial on $\mathcal{O}_v$ for almost all non-Archimedean places.*

(ii) *For a multiplicative character $\omega : \mathbb{A}^\times \to \mathbb{C}^\times$, the local character ω_v is trivial on $\mathcal{O}_v^\times$ (i.e. unramified) for almost all non-Archimedean places.*

Proof. Both statements follow from the fact that characters of locally profinite groups have open kernels. The additive character χ of $\mathbb{A}$ induces a character χ_f of $\mathbb{A}_f$, which is a locally profinite group, so $\ker(\chi_f)$ must be open in $\mathbb{A}_f$, and contain a basic open set of the form

$$\prod_{v\in S} U_v \times \prod_{v\notin S} \mathcal{O}_v$$

for some finite set S of non-Archimedean places and open subsets $U_v \subseteq F_v$ for $v \in S$. Thus $\chi_v(\mathcal{O}_v) = \{1\}$ for all $v \notin S$.

Similarly for the multiplicative character ω of $\mathbb{A}^\times$, the kernel $\ker(\omega_f)$ must contain a basic open set of the form

$$\prod_{v\in S} U_v \times \prod_{v\notin S} \mathcal{O}_v^\times$$

for some finite set S of non-Archimedean places and open subsets $U_v \subseteq F_v^\times$ for $v \in S$, so $\omega_v(\mathcal{O}_v^\times) = \{1\}$ for all $v \notin S$. □

This allows us to decompose a character $\chi : \mathbb{A} \to \mathbb{C}^\times$ into its induced local components

$$\chi((x_v)_v) = \prod_v \chi_v(x_v),$$

as the product is actually finite. We write $\chi = \otimes_v \chi_v$ and think of it as a "restricted" tensor product, as almost all non-Archimedean components χ_v are characters that are trivial on

$\mathcal{O}_v$. Similarly, for a multiplicative character $\omega : \mathbb{A}^\times \to \mathbb{C}^\times$ we write $\omega = \otimes_v \omega_v$, which is now also well-defined.

Example 4.5. The *adelic norm* $\|\cdot\|_{\mathbb{A}} : \mathbb{A}^\times \to \mathbb{R}_{>0}$, defined as the product

$$\|(x_v)_v\|_{\mathbb{A}} = \prod_v |x_v|_v,$$

which is finite for all $(x_v)_v \in \mathbb{A}^\times$, as $\|x_v\|_v = 1$ at almost all non-Archimedean places, is a Hecke character.

Proposition 4.6. *If a Hecke character $\omega : F^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$ is trivial at almost all places, i.e. $\omega_v = \mathbb{1}$ for almost all v, then $\omega = \mathbb{1}$.*

Proof. Suppose S is a finite set of places such that $\omega_v = \mathbb{1}$ for all $v \notin S$. The map $\omega_S = \bigotimes_{v\in S} \omega_v$ given by

$$\omega_S((x_v)_{v\in S})) = \prod_{v\in S} \omega_v(x_v)$$

is a character of $F_S^\times = \prod_{v\in S} F_v^\times$. Since ω is trivial on $F^\times \hookrightarrow \mathbb{A}^\times$, ω_S must be trivial on $F^\times \hookrightarrow F_S^\times$. But since S is finite, it follows from weak approximation that $F^\times$ is dense in $F_S^\times$ (see Theorem 4.2), and thus ω_S is trivial on all of $F_S^\times$, from which it must follow that $\omega = \mathbb{1}$. □

We now turn our attention to additive characters $\chi : \mathbb{A} \to \mathbb{C}^\times$ that are trivial on F. As with Hecke characters, we will often use the short-hand notation $\chi : F\backslash\mathbb{A} \to \mathbb{C}^\times$. Since $F\backslash\mathbb{A}$ is compact and the circle group $\mathbb{S}^1$ is the maximal compact subgroup of $\mathbb{C}^\times$, the image of any character $\chi : F\backslash\mathbb{A} \to \mathbb{C}^\times$ must lie in $\mathbb{S}^1$, meaning χ is unitary. We denote the set of all characters of $\mathbb{A}$ that are trivial on F by $\widehat{F\backslash\mathbb{A}}$. It is naturally a group with multiplication, and we call it the *dual group of* $F\backslash\mathbb{A}$.

Theorem 4.7 ([Wei95, §IV-2, Theorem 3])**.** *Let χ be a non-trivial character of $\mathbb{A}$ which is trivial on F. For $\alpha \in F$, let $\chi_\alpha : F\backslash\mathbb{A} \to \mathbb{C}^\times$ be the map given by $\chi_\alpha(x) = \chi(\alpha x)$. Then the map*

$$F \to \widehat{F\backslash\mathbb{A}} : \alpha \mapsto \chi_\alpha$$

is an isomorphism of groups.

The following proposition is given as a corollary of the previous theorem in [Wei95], although some work is still needed to bridge the gap, and therefore we also state this without including a proof.

Proposition 4.8 ([Wei95, §IV-2, Corollary 1])**.** *If $\chi : F\backslash\mathbb{A} \to \mathbb{C}^\times$ is non-trivial, then its local components χ_v are non-trivial at all places, and χ_v has level 0 (cf. Definition 2.1) at almost all non-Archimedean places.*

4.3 Adelic fourier expansion

In this section we briefly introduce the results we need from the theory of one-dimensional adelic Fourier transform. For this, we need to fix a Haar measure on $\mathbb{A}$, and define the class of smooth functions on which we apply the transform.

Firstly, we fix an additive Haar measure dx on $\mathbb{A}$, namely the measure such that the volume of the fundamental domain $\mathcal{F}$ for $F\backslash\mathbb{A}$ in Theorem 4.1(iii) is equal to one. More explicitly, given measures on each local field F_v such that the volume of $\mathcal{O}_v$ is equal to one for almost all non-Archimedean places, we can define their product measure on $\mathbb{A}(S)$ for each finite set of places $S \supseteq S_\infty$, as this is a locally compact group of which almost all factors are compact of local volume 1. There is then a unique measure on $\mathbb{A}$ which induces this product measure on each space $\mathbb{A}(S)$. Our measure dx on $\mathbb{A}$ is then the measure constructed in this way, with the local measures given as in [Lan64, VII, §1]. The fact that $\mathcal{F}$ has volume 1 under dx is the content of Proposition 7 in [Lan64, VII, §6].

One can show the following transformation formula for multiplying the integrand by an idele $a \in \mathbb{A}^\times$, which in fact holds for any additive adelic Haar measure:

$$d(ax) = \|a\|_{\mathbb{A}} dx.$$

Similarly, if we endow the n-dimensional $\mathbb{A}$-vector space $\mathbb{A}^n$ with the n-fold product measure $dx = \prod_{i=1}^n dx_i$, we have the following transformation formula for multiplying the integrand $x = (x_1, \ldots, x_n)$ with a matrix $g \in \mathrm{GL}_n(\mathbb{A})$:

$$d(gx) = \| \det(g)\|_{\mathbb{A}} dx.$$

This also implies that multiplication by elements in $\mathrm{GL}_n(F)$ leaves the measure on $\mathbb{A}^n$ invariant, as the adelic norm of elements in $F^\times$ is equal to one.

Definition 4.9 (Smooth periodic adelic function)**.** A continuous function $\varphi : \mathbb{A} \to \mathbb{C}$ is called a *smooth adelic function* if it satisfies

(i) for all $x_f \in \mathbb{A}_f$ the function $\varphi(\_\_, x_f)$ on F_∞ is smooth in the usual sense for a complex-valued function on the direct product of copies of $\mathbb{R}$, and
(ii) for all $x_\infty \in F_\infty$ the function $\varphi(x_\infty, \_\_)$ on $\mathbb{A}_f$ is locally constant.

We denote the space of such functions by $C^\infty(\mathbb{A})$. If a smooth adelic function $\varphi : \mathbb{A} \to \mathbb{C}$ further satisfies

(iii) $\varphi(\alpha + x) = \phi(x)$ for all $x \in \mathbb{A}$ and $\alpha \in F$,

we call it *periodic*. We denote the space of all smooth periodic adelic functions by $C^\infty(F\backslash\mathbb{A})$.

The compactness of $F\backslash\mathbb{A}$ allows us to use the theory from the Fourier transform on compact abelian groups to study smooth periodic functions. Adelic Fourier transform is a rich topic in and of itself, but for ease of exposition, we only include the following central theorem on adelic Fourier expansion. A detailed introduction on adelic Fourier transform can be found in [Wei95], most specifically in chapter VII-2.

Theorem 4.10 (Adelic Fourier expansion)**.** *Let $\varphi \in C^\infty(F\backslash\mathbb{A})$ be a smooth periodic adelic function. For $\chi \in \widehat{F\backslash\mathbb{A}}$, define $\widehat{\varphi}_\chi \in \mathbb{C}$ as the value*

$$\widehat{\varphi}_\chi = \int_{F\backslash\mathbb{A}} \varphi(x)\chi(x)^{-1}\, dx.$$

We have the following expansion:

$$\varphi(x) = \sum_{\chi \in \widehat{F\backslash\mathbb{A}}} \widehat{\varphi}_\chi \chi(x),$$

where the sum converges absolutely for all $x \in \mathbb{A}$.

Weil considers the Fourier transform of certain *standard functions* of vector spaces over $\mathbb{A}$, and from here we get, amongst other things, that our measure introduced on $\mathbb{A}$ indeed is a self-dual measure, such that the inversion formula holds not only up to the multiplication of a constant scalar, but exactly. To consider the Fourier transform and inversion of smooth periodic adelic functions, one can follow the construction in [GH11, §1.6–1.8] for the field $F = \mathbb{Q}$, and generalize it to general algebraic number fields. The space of "adelic Bruhat-Schwartz functions" is larger than that of Weil's "standard functions" on $\mathbb{A}$, but the theory presented in [Wei95] related to standard functions also holds for the larger space of Bruhat-Schwartz functions in the cases we need, as evidenced in [GH11] (once again, generalized to a general algebraic number field F).

In later chapters, we apply an extended version of Theorem 4.10 and the notion of a smooth periodic adelic function, for products $\mathbb{A}^n$ of finite numbers of copies of $\mathbb{A}$. It is an easy exercise to show that one can simply replace F with F^n and $\mathbb{A}$ with $\mathbb{A}^n$ in Definition 4.9 and Theorem 4.10 to obtain such an extension.

4.4 The adele group $\mathrm{GL}_n(\mathbb{A})$

Since $\mathbb{A}$ is a topological ring, $\mathrm{GL}_n(\mathbb{A})$ (cf. Section 1.1) is a topological group. It is canonically isomorphic to the following restricted direct product: the set

$$\left\{ (g_v)_v \in \prod_v \mathrm{GL}_n(F_v) \;\middle|\; g_v \in \mathrm{GL}_n(\mathcal{O}_v) \text{ for almost all } v \notin S_\infty \right\}$$

endowed with the topology generated by the family that consists of all sets of the form

$$\prod_{v \in S} U_v \times \prod_{v \notin S} \mathrm{GL}_n(\mathcal{O}_v),$$

where S is a finite set of places containing S_∞ and $U_v \subseteq \mathrm{GL}_n(F_v)$ is open for all $v \in S$. The isomorphism is easy to realize: it is the map

$$\begin{aligned} \mathrm{GL}_n(\mathbb{A}) &\to {\prod_v}' \mathrm{GL}_n(F_v) \\ \big((g_v^{i,j})_v\big)_{1 \le i,j \le n} &\mapsto \big((g_v^{i,j})_{1 \le i,j \le n}\big)_v \end{aligned}$$

that maps a matrix with entries consisting of tuples from the restricted product $\prod_v' F_v$ to a tuple of matrices. The map is well-defined because the determinant map commutes with projection onto the v'th component of $\mathrm{GL}_n(\mathbb{A})$ and $\mathbb{A}^\times$, respectively. (Note that $\mathrm{GL}_1(\mathbb{A})$ is exactly $\mathbb{A}^\times$.) As with the adeles, since $\mathrm{GL}_n(F_v)$ is locally compact at all places and $\mathrm{GL}_n(\mathcal{O}_v)$ is compact in $\mathrm{GL}_n(F_v)$ at all finite places, one can show that $\mathrm{GL}_n(\mathbb{A})$ is a locally compact group.

The diagonal embedding $F \hookrightarrow \mathbb{A}$ induces a diagonal embedding $\mathrm{GL}_n(F) \hookrightarrow \mathrm{GL}_n(\mathbb{A})$, and as with the ideles, we identify $\mathrm{GL}_n(F)$ with its image under this embedding, and view it as a subgroup of $\mathrm{GL}_n(\mathbb{A})$. It is a discrete subgroup, which follows from F being discrete in $\mathbb{A}$ (see Theorem 4.1).

Let $G = \mathrm{GL}_n$ and let S be a finite set of places. We let $G(F_S)$ denote the finite direct product

$$G(F_S) = \prod_{v \in S} G(F_v).$$

Furthermore, we define $G(\mathbb{A}_S)$ to be the projection of $G(\mathbb{A})$ on the direct product

$$\prod_{v \notin S} G(F_v).$$

If $S = S_\infty$ is the set of all infinite places, then we simply denote $G(F_{S_\infty})$ by $G(F_\infty)$, and $G(\mathbb{A}_{S_\infty})$ by $G(\mathbb{A}_f)$. Note that we will often view $G(F_S)$ and $G(\mathbb{A}_S)$ as subgroups of $G(\mathbb{A})$ by inserting the identity matrix I_n in the remaining entries.

Proposition 4.11 (Weak approximation for $\mathrm{GL}_n(\mathbb{A})$)**.** *For each non-empty finite set of places S, the diagonal embedding $G(F) \hookrightarrow G(F_S)$ is dense.*

Proof. This is an easy consequence of weak approximation for $\mathbb{A}$ (see Theorem 4.2) together with the fact that the topology of G over a topological field coincides with the subspace topology from $\mathrm{Mat}_{n\times n}$, while also being open in $\mathrm{Mat}_{n\times n}$. This implies that any open subset $U \subseteq G(F_S)$ is also open in $\mathrm{Mat}_{n\times n}(F_S)$. Since $\mathrm{Mat}_{n\times n}(\mathbb{A}) \cong \mathbb{A}^{n^2}$, weak approximation for $\mathbb{A}$ implies that $\mathrm{Mat}_{n\times n}(F)$ is dense in $\mathrm{Mat}_{n\times n}(F_S)$. Thus the intersection $U \cap \mathrm{Mat}_{n\times n}(F)$ is non-empty, but as this intersection is contained in $G(F)$, the intersection $U \cap G(F)$ must also be non-empty. □

We let K denote the following compact subgroup of $\mathrm{GL}_n(\mathbb{A})$:

$$K = \prod_v K_v, \qquad \text{where} \qquad K_v = \begin{cases} O(n) & \text{if } v \text{ is a real place} \\ U(n) & \text{if } v \text{ is a complex place} \\ \mathrm{GL}_n(\mathcal{O}_v) & \text{if } v \text{ is a finite place.} \end{cases}$$

We denote the projection of K in $\mathrm{GL}_n(F_\infty)$ as K_∞, and the projection of K in $\mathrm{GL}_n(\mathbb{A}_f)$ as K_f. We let $\mathfrak{g}_\infty$ denote the Lie algebra of $\mathrm{GL}_n(F_\infty)$, which we identify with the finite direct sum $\bigoplus_{v\in S_\infty} \mathfrak{g}_v$, where $\mathfrak{g}_v$ denotes the Lie algebra of $\mathrm{GL}_n(F_v)$.

Theorem 4.12 (Strong approximation)**.**

(i) *$SL_n(\mathbb{A})$ has strong approximation, i.e. the diagonal embedding*

$$SL_n(F) \hookrightarrow SL_n(\mathbb{A}_S)$$

is dense for all non-empty finite sets of places $S \supseteq S_\infty$.

(ii) *If $N \subseteq \mathrm{GL}_n$ is a unipotent subgroup, then $N(\mathbb{A})$ has strong approximation.*

(iii) *The quotient space*

$$\mathrm{GL}_n(F)\,\mathrm{GL}_n(F_\infty)\backslash \mathrm{GL}_n(\mathbb{A}) / K_f$$

is finite, and its cardinality is equal to the class number of F.

(iv) *For each compact open subset L of $\mathrm{GL}_n(\mathbb{A}_f)$, there exists a finite subset $C \subseteq \mathrm{GL}_n(\mathbb{A}_f)$ such that*

$$\mathrm{GL}_n(\mathbb{A}) = \bigcup_{c\in C} \mathrm{GL}_n(F)\left(\mathrm{GL}_n(F_\infty)\cdot c \cdot L\right),$$

where the right-hand side is a disjoint union.

Proof. Part (i) is the content of the theorem in [Hum80, §14.3]. Part (ii) is Corollary 2.5 in [Bor63]. Part (iii) in the case $n = 1$ can be found in [Lan64, IV, §3], and can be extended to the general case using the strong approximation for SL_n in part (i). This follows from Proposition 2.7 in [Bor63] where $G = \mathrm{GL}_n$, $N = SL_n$ and H_A is the subgroup given by embedding $\mathbb{A}^\times$ in $\mathrm{GL}_n(\mathbb{A})$, e.g. as diagonal matrices with ones in the rest of the diagonal entries. Part (iv) immediately follows from part (iii). □

We fix a Haar measure dg on $\mathrm{GL}_n(\mathbb{A})$ to be used throughout the remainder of this thesis, and we always assume closed subgroups of $\mathrm{GL}_n(\mathbb{A})$ to have the Haar measure induced by dg, unless otherwise stated. We choose dg in the following manner, similarly to how we fixed an additive Haar measure on $\mathbb{A}$: for all v, let dg_v denote the normalized Haar measure on $\mathrm{GL}_n(F_v)$ such that $\mathrm{vol}_{dg_v}(K_v) = 1$. All local groups $\mathrm{GL}_n(F_v)$ are unimodular (see Proposition 2.12 and Proposition 3.7), so dg_v is both a left and right Haar measure at all places. Note that at non-Archimedean places, dg_v is what we called the usual measure in Chapter 2. For each finite set of places S, the product measure $\prod_v dg_v$ on the group

$$G(S) = \prod_{v\in S} \mathrm{GL}_n(F_v) \times \prod_{v\notin S} K_v \subseteq \mathrm{GL}_n(\mathbb{A})$$

is a well-defined left and right Haar measure, as almost all factors are compact of local volume 1. There is a unique Haar measure on $\mathrm{GL}_n(\mathbb{A})$ which induces this product measure on $G(S)$ for all finite sets of places, and we choose this as our fixed measure dg. As $\mathrm{GL}_n(\mathbb{A})$ is the union of all $G(S)$, dg is also both a left and right Haar measure, and thus $\mathrm{GL}_n(\mathbb{A})$ must be unimodular.

4.5 Representations of $\mathrm{GL}_n(\mathbb{A})$

In the remaining sections of this chapter, we focus on representations of $\mathrm{GL}_n(\mathbb{A})$. We define the notion in this section, while we prove an equivalent description as a smooth module over the global Hecke algebra in the following section. The definitions and notation in the remainder of the chapter is based on those from Section 3.3 in [Bum97], unless stated otherwise.

As noted in the introduction to Chapter 3, we can extend the theory in Chapter 3 to the real Lie group $\mathrm{GL}_n(F_\infty)$, since it is a finite product of copies of $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$. In this case, the maximal compact subgroup considered is simply the product K_∞. Thus, we can define the notion of a $(\mathfrak{g}_\infty, K_\infty)$-module cf. Definition 3.4, on which both $\mathcal{U}(\mathfrak{g}_\infty)$ and K_∞ act. This leads to the following global definition, which in some sense is the global equivalent to the local concept of $(\mathfrak{g}, K)$-modules.

Definition 4.13 ($\mathrm{GL}_n(\mathbb{A})$-representation)**.** Let V be a complex vector space such that, together with an action π_∞, the pair (π_∞, V) forms a $(\mathfrak{g}_\infty, K_\infty)$-module, while simultaneously, together with an action π_f, the pair (π_f, V) forms a smooth $\mathrm{GL}_n(\mathbb{A}_f)$-representation, i.e. $\pi_f : \mathrm{GL}_n(\mathbb{A}_f) \to \mathrm{Aut}_{\mathbb{C}}(V)$ is a group homomorphism where for each vector $\xi \in V$, there exists a compact open subgroup L of $\mathrm{GL}_n(\mathbb{A}_f)$, such that ξ is fixed by L, i.e. $\pi_f(l)\xi = \xi$ for all $l \in L$. Assume furthermore that the two actions π_∞ and π_f commute, i.e. that $\pi_\infty(u_\infty)$ and $\pi_0(k_\infty)$ commute with $\pi_f(g_f)$ for all $u_\infty \in \mathcal{U}(\mathfrak{g}_\infty)$, $k_\infty \in K_\infty$, and $g_f \in \mathrm{GL}_n(\mathbb{A}_f)$. We write π to denote both actions on V, and call (π, V) a *representation* of $\mathrm{GL}_n(\mathbb{A})$.

One should note, that a $\mathrm{GL}_n(\mathbb{A})$-representation (π, V) does not form a representation of $\mathrm{GL}_n(\mathbb{A})$ in the usual sense.

A *subrepresentation* of a representation (π, V) of $\mathrm{GL}_n(\mathbb{A})$ is a subspace $W \subseteq V$ that is simultaneously stable under both actions, and we write (π, W) although we think of the actions as being restricted to W. A *subquotient* of (π, V) is the quotient V_1/V_2 of two subrepresentations $V_2 \subseteq V_1 \subseteq V$, and we write $(\pi, V_1/V_2)$, where π is the induced action on the representatives. A $\mathrm{GL}_n(\mathbb{A})$-representation is called *irreducible* if it has no proper non-zero subrepresentations. A *homomorphism* between to representations (π, V) and (π', V') of $\mathrm{GL}_n(\mathbb{A})$ is a map $f : V \to V'$ that is simultaneously a $(\mathfrak{g}_\infty, K_\infty)$-module homomorphism and a $\mathrm{GL}_n(\mathbb{A}_f)$-homomorphism, and we define isomorphisms analogously.

Let (π, V) be a representation of $\mathrm{GL}_n(\mathbb{A})$. V is K_∞-finite, by the definition of a $(\mathfrak{g}_\infty, K_\infty)$-module (see Definition 3.4), which together with the fact that the action π_f of $\mathrm{GL}_n(\mathbb{A})$ is smooth, implies that (π, V) is K-finite; indeed, for each $\xi \in V$, there exists a compact open subgroup L of $\mathrm{GL}_n(\mathbb{A}_f)$ such that L fixes ξ under the action π_f. But as L has finite index in K_f, the set $\pi_f(K_f)\xi$ spans a finite-dimensional subspace of V.

For a compact open subgroup L of $\mathrm{GL}_n(\mathbb{A}_f)$, let us denote the set of L-fixed vectors in V under the $\mathrm{GL}_n(\mathbb{A}_f)$-action π_f by V^L. Since π_∞ and π_f commute, (π_∞, V^L) forms a K_∞-subrepresentation of (π, V) which by the definition of a $(\mathfrak{g}_\infty, K_\infty)$-module is K_∞-finite (cf. Definition 3.3). Thus, for each $\gamma \in \widehat{K_\infty}$ (the set of isomorphism classes of irreducible finite-dimensional representations of K_∞), we may form the γ-isotopic component $V^L(\gamma)$ of (π_∞, V^L) (see end of Section 3.1).

Definition 4.14 (Admissible $\mathrm{GL}_n(\mathbb{A})$-representation)**.** Let (π, V) be a $\mathrm{GL}_n(\mathbb{A})$-representation. We say (π, V) is *admissible* if $\dim_{\mathbb{C}}\left(V^L(\gamma)\right) < \infty$ for all $\gamma \in \widehat{K_\infty}$ and all compact open subgroups L of $\mathrm{GL}_n(\mathbb{A}_f)$.

Remark 4.15. This is not the usual definition of admissibility for representations of $\mathrm{GL}_n(\mathbb{A})$ found in the literature, but in order to avoid discussing representations of K, we make use of this equivalent definition. A proof that our definition is equivalent to the one in e.g. [Bum97] can be found in [Gro23, Lemma 10.42].

An important theorem for irreducible admissible representations of $\mathrm{GL}_n(\mathbb{A})$ is that they decompose into products of irreducible admissible local representations at each place of F. In order to state this theorem, we first need to consider the notion of restricted tensor products of (infinitely many) vector spaces (here, always assumed to be over $\mathbb{C}$).

Let $(V_v)_{v\in\Sigma}$ be a sequence of vector spaces for an infinite index set Σ, such that for all v's outside a given finite subset $S_0 \subseteq \Sigma$, we are given a distinguished non-zero vector $\xi_v^\circ \in V_v$. The family

$$\mathcal{S} = \{S \subseteq \Sigma \mid S_0 \subseteq S \text{ and } S \text{ finite}\}$$

is a directed set with respect to inclusion. For each pair $S, S' \in \mathcal{S}$ where $S \subseteq S'$, we define the homomorphism

$$\lambda_{S,S'} : \bigotimes_{v\in S} V_v \to \bigotimes_{v\in S'} V_v$$

to be the linear extension of the map

$$\underset{v\in S}{\otimes} \xi_v \mapsto \Big(\underset{v\in S}{\otimes} \xi_v\Big) \otimes \Big(\underset{v\in S'\setminus S}{\otimes} \xi_v^\circ\Big)$$

on pure tensors. The finite tensor products $\bigotimes_{v\in S} V_v$ together with the maps $\lambda_{S,S'}$ form a direct system over $\mathcal{S}$, and we define the *restricted tensor product* of $(V_v)_{v\in\Sigma}$ with respect to the ξ_v°'s, denoted $\bigotimes'_{v\in\Sigma} V_v$, to be the direct limit

$$\varinjlim_{S\in\mathcal{S}} \bigotimes_{v\in S} V_v.$$

The natural inclusion maps $\lambda_S : \bigotimes_{v\in S} V_v \to \bigotimes'_{v\in\Sigma} V_v$ are injective, and the image of an element $\xi = \otimes_{v\in S}\, \xi_v \in \bigotimes_{v\in S} V_v$ under λ_S is denoted by $\otimes_{v\in\Sigma}\, \xi_v$, where $\xi_v = \xi_v^\circ$ for all $v \in \Sigma \setminus S$. Elements of this form in $\bigotimes'_{v\in\Sigma} V_v$ is called *pure tensors*, and we view $\bigotimes'_{v\in\Sigma} V_v$ as the space spanned by these, i.e. spanned by tensors $\otimes_{v\in\Sigma}\, \xi_v$ where $\xi_v = \xi_v^\circ$ for almost all v's.

Although the precise definition of $\bigotimes'_{v\in\Sigma} V_v$ depends on the choice of S_0 and the ξ_v°'s, if the family of ξ_v°'s is changed for only a finite number of v's, the resulting restricted tensor product is naturally isomorphic to $\bigotimes'_{v\in\Sigma} V_v$.

If for each $v \in \Sigma$ we are given a locally compact group G_v and for almost all $v \in \Sigma$ we are given a compact open subgroup K_v of G_v, then we may form the restricted direct product $G = \prod'_{v\in\Sigma} G_v$ with respect to the K_v's. Moreover, if at each $v \in \Sigma$ we are also given an (algebraic) representation (π_v, V_v) of G_v, such that for almost all v's, there exists a non-zero K_v-fixed element $\xi_v^\circ \in V_v$, then we may define the *restricted tensor product of the representations* (π_v, V_v) to be the (algebraic) representation $(\otimes_{v\in\Sigma} \pi_v, \bigotimes'_{v\in\Sigma} V_v)$ of G on the restricted tensor product of the V_v's with respect to the ξ_v°'s, where $\otimes_{v\in\Sigma} \pi_v : G \to \mathrm{Aut}_{\mathbb{C}}\left(\bigotimes'_{v\in\Sigma} V_v\right)$ is defined at each $g = (g_v)_{v\in\Sigma} \in G$ to be the linear extension of the map

$$\underset{v\in\Sigma}{\otimes} \xi_v \mapsto \underset{v\in\Sigma}{\otimes} \pi_v(g_v)\xi_v$$

on pure tensors $\otimes_{v\in\Sigma} \xi_v \in \bigotimes'_{v\in\Sigma} V_v$ Note that the right-hand side is a well-defined pure tensor, since at almost all v, we have that g_v is an element of K_v and ξ_v is equal to ξ_v°, which is K_v-fixed.

We are now ready to state the following theorem for representations of $\mathrm{GL}_n(\mathbb{A})$.

Theorem 4.16 (Tensor product theorem)**.** *Suppose (π, V) is an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A})$. Then, there exists an irreducible $(\mathfrak{g}_v, K_v)$-module (π_v, V_v) at each Archimedean place v and an irreducible smooth $\mathrm{GL}_n(F_v)$-representation (π_v, V_v) at each non-Archimedean place v satisfying the following: for almost all finite places, π_v is unramified and there exists a non-zero K_v-fixed vector $\xi_v^\circ \in V_v^{K_v}$ such that (π, V) is isomorphic to the restricted tensor product of the representations (π_v, V_v) with respect to the distinguished elements ξ_v°. The representations (π_v, V_v) are uniquely determined up to isomorphism at all places, and we wrie $\pi \cong \otimes_v \pi_v$.*

Given an irreducible admissible representation π of $\mathrm{GL}_n(\mathbb{A})$, we call the representations π_v in its tensor product decomposition $\pi \cong \otimes_v \pi_v$, the *local components* or the *local representations* of π, and comparably sometimes refer to π as a *global representation.* If v is an Archimedean place, we call π_v *Archimedean* and if v is a non-Archimedean place, we call π_v *non-Archimedean.* Of course, at Archimedean places, the local representations (π_v, V_v) are not representations of $\mathrm{GL}_n(F_v)$, but $(\mathfrak{g}_v, K_v)$-modules, however we still refer to them as local Archimedean representations.

In order to prove the tensor product theorem, which is done at the end of the subsequent section, we will introduce the global Hecke algebra, which in turn has a similar decomposition theorem for its smooth modules, and then show that an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A})$ is equivalent to an irreducible admissible module over the global Hecke algebra.

4.6 The global Hecke algebra

In this section we first consider restricted tensor products of infinite families of idempotented algebras and smooth modules over them. We do not include any proofs, and instead refer to [Bum97, §3.4] for a much more detailed introduction and discussion, which in turn is based on [Fla79]. This allows us to define the Hecke algebra of $\mathrm{GL}_n(\mathbb{A})$, which we call the global Hecke algebra, and similar to the local settings, we show that the notion of a representation of $\mathrm{GL}_n(A)$ is equivalent to that of a smooth global Hecke algebra module.

Let Ω be an algebraically closed field, and let $(\mathcal{H}_v, \mathcal{E}_v)_{v\in\Sigma}$ be a sequence of idempotented Ω-algebras (cf. Definition 2.6) for an infinite index set Σ (for our purposes, this will be the set of places of our number field F), such that for all but finitely many v's, we specify a distinguished elementary idempotent $e_v^\circ \in \mathcal{E}_v$. Let $\mathcal{H}$ be the restricted tensor product $\bigotimes'_{v\in\Sigma} \mathcal{H}_v$ of the $\mathcal{H}_v$'s with respect to the e_v°'s. The Ω-vector space $\mathcal{H}$ can naturally be given the structure of an Ω-algebra, as in the finite case, and we can form a set $\mathcal{E}$ of elementary idempotents of $\mathcal{H}$ by taking pure tensors of the form $\otimes_{v\in\Sigma} e_v$, where $e_v \in \mathcal{E}_v$ is an elementary idempotent for all v, and $e_v = e_v^\circ$ is the distinguished elementary idempotent for almost all v. Then one can show that $(\mathcal{H}, \mathcal{E})$ is an idempotented Ω-algebra.

Let M_v be an irreducible admissible $\mathcal{H}_v$-module for all $v \in \Sigma$, and assume additionally that $\mathcal{H}_v[e_v^\circ]$ is a commutative algebra for almost all v. Then the $\mathcal{H}_v[e_v^\circ]$-module $M_v[e_v^\circ]$ is either zero- or one-dimensional over Ω (see [Bum97, Proposition 3.4.5]). We therefore further assume that $M_v[e_v^\circ]$ is one-dimensional for almost all v, and that we are given a distinguished non-zero element $m_v^\circ \in M_v[e_v^\circ]$ at those places. We define the restricted tensor product M of the M_v's to be the restricted tensor product $\bigotimes'_{v\in\Sigma} M_v$ (over Ω) with respect to the m_v°'s. One can show that M is an $\mathcal{H}$-module, and that M is naturally isomorphic to the restricted tensor product over the M_v's for any choice of distinguished elements $m_v^\circ \in M_v[e_v^\circ]$ (i.e. not only finite changes of the distinguished elements are allowed). The assumptions on commutativity and one-dimensionality are necessary conditions for the following important result.

Theorem 4.17 ([Bum97, Theorem 3.4.4])**.** *Let $(\mathcal{H}_v, \mathcal{E}_v)_{v\in\Sigma}$ be a sequence of idempotented Ω-algebras as given above, with specified distinguished elementary idempotents $e_v^\circ \in \mathcal{E}_v$ for almost all v, such that $\mathcal{H}_v[e_v^\circ]$ is commutative, and let $\mathcal{H}$ be their restricted tensor product with respect to the e_v°'s. For each $v \in \Sigma$, let there be given an irreducible admissible $\mathcal{H}_v$-module M_v, and suppose we are given a distinguished non-zero element $m_v^\circ \in M_v[e_v^\circ]$ for almost all v. Then $\bigotimes'_{v\in\Sigma} M_v$ with respect to the m_v°'s is an irreducible admissible $\mathcal{H}$-module, and every irreducible admissible $\mathcal{H}$-module is of this type, with uniquely determined $\mathcal{H}_v$-modules M_v, up to isomorphism.*

Let G_v denote the group $\mathrm{GL}_n(F_v)$ for all places v of F, and let $(\mathcal{H}_{G_v}, \mathcal{E}_{G_v})$ denote the local Hecke algebra of G_v as defined in Section 2.3 if v is non-Archimedean, or as defined in Section 3.5 if v is Archimedean. Note that the Haar measure on G_v with respect to which the local Hecke algebra is considered, is the Haar measure normalized such that $\mathrm{vol}(K_v) = 1$. If v is non-Archimedean, define the distinguished elementary idempotent $e_v^\circ \in \mathcal{E}_{G_v}$ as the elementary idempotent e_{K_v}, which is the characteristic function of K_v. We define the *global Hecke algebra* as the restricted tensor product $\mathcal{H} = \bigotimes'_v \mathcal{H}_{G_v}$ with respect to the distinguished elementary idempotents e_v°. Note that it follows from Proposition 2.13 that $\mathcal{H}_{G_v}[e_v^\circ]$ is commutative at all non-Archimedean places.

We can also write the global Hecke algebra as the tensor product $\mathcal{H} = \mathcal{H}_\infty \bigotimes \mathcal{H}_f$, where $\mathcal{H}_\infty = \bigotimes_{v\in S_\infty} \mathcal{H}_{G_v}$ is the finite tensor product of the $\mathcal{H}_{G_v}$'s for v Archimedean, and $\mathcal{H}_f = \bigotimes'_{v\notin S_\infty} \mathcal{H}_{G_v}$ is the restricted tensor product of the $\mathcal{H}_{G_v}$'s for v non-Archimedean with respect to the distinguished elementary idempotents e_v°.

Let us first consider the finite component $\mathcal{H}_f$. The locally compact group $\mathrm{GL}_n(\mathbb{A}_f)$ is totally disconnected, so if we can show that $\mathrm{GL}_n(\mathbb{A}_f)/L$ is countable for all compact open subgroups L (or equivalently for at least *one* such L) and that $\mathrm{GL}_n(\mathbb{A}_f)$ is unimodular, then we can apply the theory discussed in Section 2.2 and 2.3 to $\mathrm{GL}_n(\mathbb{A}_f)$, in order to get a notion of a Hecke algebra of $\mathrm{GL}_n(\mathbb{A}_g)$.

Proposition 4.18. $\mathrm{GL}_n(\mathbb{A}_f)/K_f$ *is countable.*

Proof. $\mathrm{GL}_n(\mathbb{A}_f)$ is a union of the direct products

$$\mathrm{GL}_n(\mathbb{A}_f(S)) = \prod_{v\in S} G_v \times \prod_{v\notin S\cup S_\infty} K_v$$

ranging over all finite sets of non-Archimedean places S, and for each such S, the quotient space

$$\mathrm{GL}_n(\mathbb{A}_f(S))/K_f \cong \prod_{v\in S} G_v/K_v$$

is countable, as it is a finite product of countable sets. We now have

$$\mathrm{GL}_n(\mathbb{A}_f)/K_f = \bigcup_{S \text{ finite}} \mathrm{GL}_n(\mathbb{A}_f(S))/K_f,$$

which is a countable union of countable sets and must itself be countable. □

We fix a Haar measure dg_f on $\mathrm{GL}_n(\mathbb{A}_f)$, constructed analogously to the fixed Haar measure on $\mathrm{GL}_n(\mathbb{A})$: it is the unique measure that restricts to the product measure $\prod_{v\notin S_\infty} dg_v$ of the usual measures on G_v on the spaces $\prod_{v\in S} G_v \times \prod_{v\notin S\cup S_\infty} K_v$ for all finite sets of non-Archimedean places S. As with the fixed measure on $\mathrm{GL}_n(\mathbb{A})$, this is a left and right Haar measure, so $\mathrm{GL}_n(\mathbb{A}_f)$ is unimodular. Let $\mathcal{H}_{\mathrm{GL}_n(\mathbb{A}_f)}$ denote the Hecke algebra of $\mathrm{GL}_n(\mathbb{A}_f)$ (cf. Section 2.3), defined with respect to the fixed measure dg_f. Note that $\mathrm{vol}_{dg_f}(K_f) = 1$.

Proposition 4.19. *The map*

$$\bigotimes_{v\notin S_\infty} f_v \mapsto \left(\mathrm{GL}_n(\mathbb{A}_f) \to \mathbb{C} : (g_v)_{v\notin S_\infty} \mapsto \prod_{v\notin S_\infty} f_v(g_v)\right) \tag{4.1}$$

of pure tensors in $\mathcal{H}_f$ extends linearly to a $\mathbb{C}$-algebra isomorphism $\mathcal{H}_f \xrightarrow{\sim} \mathcal{H}_{\mathrm{GL}_n(\mathbb{A}_f)}$. Under this isomorphism the set

$$\mathcal{E}_f = \left\{ \bigotimes_{v\notin S_\infty} e_v \;\middle|\; e_v \in \mathcal{E}_{G_v} \text{ for all } v \notin S_\infty \text{ and } e_v = e_v^\circ \text{ for a.a. } v \notin S_\infty \right\}$$

of elementary idempotents of $\mathcal{H}_f$ corresponds to the set

$$\mathcal{E}_{\mathrm{GL}_n(\mathbb{A}_f)} = \{e_L \mid L \text{ compact open subgroup of } \mathrm{GL}_n(\mathbb{A}_f)\}$$

of elementary idempotents in $\mathcal{H}_{\mathrm{GL}_n(\mathbb{A}_f)}$.

Proof. For an element $g_f \in \mathrm{GL}_n(\mathbb{A}_f)$, we will always denote the v'th entry in g_f by g_v, i.e. $g_f = (g_v)_{v\notin S_\infty}$. Given a finite set of non-Archimedean places S, let $\mathcal{H}_S = \bigotimes_{v\in S} \mathcal{H}_{G_v}$, and let the map $\phi_S : \mathcal{H}_S \to \mathcal{H}_{\mathrm{GL}_n(\mathbb{A}_f)}$ be defined by

$$\bigotimes_{v\in S} f_v \mapsto \left(g_f \mapsto \prod_{v\in S} f_v(g_v) \cdot \prod_{v\notin S\cup S_\infty} e_v^\circ(g_v)\right).$$

This map is well-defined; it is clear that for a pure tensor $f_S = \otimes_{v\in S} f_v$, the map $\phi_S(f_S)$ has support

$$\prod_{v\in S} \mathrm{supp}(f_v) \times \prod_{v\notin S\cup S_\infty} K_v,$$

which is compact in $\mathrm{GL}_n(\mathbb{A}_f)$. It is also clear that $\phi_S(f_S)$ is locally constant, since f_v is locally constant on G_v for all $v \in S$.

It follows from the properties of multiplication in $\mathbb{C}$ and the universal property of finite tensor products, that ϕ_S is a $\mathbb{C}$-linear map. Because the Haar measure on $\mathrm{GL}_n(\mathbb{A}_f)$ is given by the product measure of the measures on G_v that the Hecke algebras $\mathcal{H}_{G_v}$ are with respect to, it is also simple to see that ϕ_S respects the bilinear products of $\mathcal{H}_S$ and $\mathcal{H}_{\mathrm{GL}_n(\mathbb{A}_f)}$, i.e. that

$$\phi_S\Big(\bigotimes_{v\in S} f_v * h_v\Big) = \phi_S\Big(\bigotimes_{v\in S} f_v\Big) * \phi_S\Big(\bigotimes_{v\in S} h_v\Big).$$

Thus, ϕ_S is a $\mathbb{C}$-algebra homomorphism. It is injective, as $\phi_S(f_S)$ is the zero function if and only if f_v is the zero function on G_v for some $v \in S$, which implies that f_S is the zero tensor in $\mathcal{H}_S$.

Now, to the direct system $\langle \mathcal{H}_S, \lambda_{S,S'} \rangle$ of $\mathbb{C}$-algebras over the directed set

$$\mathcal{S} = \{S \mid S \text{ is a finite set of non-Archimedean places}\},$$

where the maps $\lambda_{S,S'}$ are given as in Section 4.5, we have a family of $\mathbb{C}$-algebra homomorphisms $\phi_S : \mathcal{H}_S \to \mathcal{H}_{\mathrm{GL}_n(\mathbb{A}_f)}$ for $S \in \mathcal{S}$, such that $\phi_S = \phi_{S'} \circ \lambda_{S,S'}$ whenever $S \subseteq S'$. By the universal property of the direct limit, there exists a unique $\mathbb{C}$-algebra homomorphism $\phi : \mathcal{H}_f \to \mathcal{H}_{\mathrm{GL}_n(\mathbb{A}_f)}$ such that $\phi \circ \lambda_S = \phi_S$ for all $S \in \mathcal{S}$, where λ_S are the natural injections. Thus ϕ is the map in (4.1) on pure tensors, and ϕ is injective as each ϕ_S is.

It remains to show that ϕ is surjective. The underlying space of $\mathcal{H}_{\mathrm{GL}_n(\mathbb{A}_f)}$ is the space $C_c^\infty(\mathrm{GL}_n(\mathbb{A}_f))$, consisting all complex-valued locally constant functions on $\mathrm{GL}_n(\mathbb{A}_f)$ with compact support. Thus, every element of $\mathcal{H}_{\mathrm{GL}_n(\mathbb{A}_f)}$ is a finite linear combination of characteristic functions of double cosets of the form Lg_fL, where L is a compact open subgroup of $\mathrm{GL}_n(\mathbb{A}_f)$ and $g_f \in \mathrm{GL}_n(\mathbb{A}_f)$, and it is enough to show that such characteristic functions are in the image of ϕ. Let f be such a characteristic function of a double coset Lg_fL. The subgroup L is of the form

$$\prod_{v\in S} L_v \times \prod_{v\notin S\cup S_\infty} K_v$$

for some finite set of non-Archimedean places S and compact open subgroups $L_v \subsetneq K_v$ for $v \in S$. Since $g_v \in K_v$ for almost all v, there exists a finite set of non-Archimedean places $S' \supseteq S$ such that

$$Lg_fL = \prod_{v\in S'} L_v g_v L_v \times \prod_{v\notin S'\cup S_\infty} K_v.$$

Thus, if we let f_v denote the characteristic function of $L_v g_v L_v$ for all $v \in S'$, and $f_v = e_v^\circ$ for all $v \notin S' \cup S_\infty$, then

$$f(x_f) = \prod_{v\notin S_\infty} f_v(x_v) = \phi\Big(\bigotimes_{v\notin S_\infty} f_v\Big)(x_f),$$

implying that f is in the image of ϕ. We have now proven that $\phi : \mathcal{H}_f \xrightarrow{\sim} \mathcal{H}_{\mathrm{GL}_n(\mathbb{A}_f)}$ is an isomorphism of $\mathbb{C}$-algebras.

It is clear from the last calculation, and from the fact that the measure on $\mathrm{GL}_n(\mathbb{A}_f)$ is the product measure of the usual measures on G_v, that the set of elementary idempotents in $\mathcal{H}_f$ corresponds to the set of the functions $e_L \in \mathcal{H}_{\mathrm{GL}_n(\mathbb{A}_f)}$, ranging over all compact open subgroups L of $\mathrm{GL}_n(\mathbb{A}_f)$, where e_L is $\mathrm{vol}(L)^{-1}$ times the characteristic function of L. $\square$

We now want to consider the infinite component $\mathcal{H}_\infty = \bigotimes_{v\in S_\infty} \mathcal{H}_{G_v}$ of the global Hecke algebra in a similar manner. Let $\mathcal{H}_{\mathrm{GL}_n(F_\infty)}$ denote the Hecke algebra of the real Lie group $\mathrm{GL}_n(F_\infty)$ cf. Section 3.5.

Proposition 4.20. $\mathcal{H}_{\mathrm{GL}_n(F_\infty)} \cong \bigotimes_{v\in S_\infty} \mathcal{H}_{G_v}$.

Proof (sketch). Let $\mathcal{H}_{K_\infty}$ denote the Hecke algebra of K_∞ cf. Section 3.4. Then

$$\mathcal{H}_{\mathrm{GL}_n(F_\infty)} \cong \mathcal{H}_{K_\infty} \bigotimes_{\mathcal{U}(\mathfrak{k}_\infty)} \mathcal{U}(\mathfrak{g}_\infty). \tag{4.2}$$

Since $\mathfrak{g}_\infty \cong \bigoplus_{v\in S_\infty} \mathfrak{g}_v$, we have that $\mathcal{U}(\mathfrak{g}_\infty) \cong \bigotimes_{v\in S_\infty} \mathcal{U}(\mathfrak{g}_v)$.

$\mathcal{H}_{K_\infty}$ is defined as the space of left- and right-K_∞-finite distributions on K of compact support, which, due to Proposition 3.9, is equal to the space of distributions of the form $f\,dk_\infty$, where $f \in C^\infty(K_\infty)$ is left- and right-K_∞-finite. But, this is also saying that $\mathcal{H}_{K_\infty}$ is isomorphic to the space of finite complex measures on K_∞ (see e.g. the discussion after the proof of Lemma 7.4 in [Kna88]), which, since finite measures factor along products, implies that $\mathcal{H}_{K_\infty} \cong \bigotimes_{v\in S_\infty} \mathcal{H}_{K_v}$, where the map from right to left is given by Fubini's theorem for distributions.

In Chapter 3 we did not give the details of how to equip the tensor product on the right-hand side of (4.2) with a bilinear product, such that the isomorphism of $\mathbb{C}$-vector spaces in (4.2) becomes a $\mathbb{C}$-algebra isomorphism. But with such a description (e.g. using Lemma 1.113 in [KV95]), one can show that

$$\left(\bigotimes_{v\in S_\infty} \mathcal{H}_{K_v}\right) \bigotimes_{\mathcal{U}(\mathfrak{k}_\infty)} \left(\bigotimes_{v\in S_\infty} \mathcal{U}(\mathfrak{g}_v)\right) \cong \bigotimes_{v\in S_\infty} \left(\mathcal{H}_{K_v} \bigotimes_{\mathcal{U}(\mathfrak{k}_v)} \mathcal{U}(\mathfrak{g}_v)\right),$$

and hence, by combining with (4.2), we get the desired tensor product decomposition of $\mathcal{H}_{\mathrm{GL}_n(F_\infty)}$. □

We can now describe the action of the global Hecke algebra $\mathcal{H}$ on V, for a given representation (π, V) of $\mathrm{GL}_n(\mathbb{A})$: in Section 2.3, for a smooth $\mathrm{GL}_n(\mathbb{A}_f)$-representation (π, V), we define an action of $\mathcal{H}_{\mathrm{GL}_n(\mathbb{A}_f)}$ on V via π, which is denoted $\pi(f)$ for $f \in \mathcal{H}_{\mathrm{GL}_n(\mathbb{A}_f)}$. In Section 3.5, for a $(\mathfrak{g}_\infty, K_\infty)$-module (π, V), we similarly define an action of $\mathcal{H}_{\mathrm{GL}_n(F_\infty)}$ on V via π, which is denoted $\pi(S)$ for $S \in \mathcal{H}_{\mathrm{GL}_n(F_\infty)}$. Thus, given a representation (π, V) of $\mathrm{GL}_n(\mathbb{A})$, we can define an action of $\mathcal{H}$ on V via the actions π in the following manner: for $S \otimes f \in \mathcal{H}_{\mathrm{GL}_n(F_\infty)} \bigotimes \mathcal{H}_{\mathrm{GL}_n(\mathbb{A}_f)}$ define

$$\pi(S \otimes f)v = \pi(S)\big(\pi(f)v\big).$$

This extends linearly to an action of $\mathcal{H}_{\mathrm{GL}_n(F_\infty)} \bigotimes \mathcal{H}_{\mathrm{GL}_n(\mathbb{A}_f)}$, which through the isomorphism

$$\mathcal{H} = \mathcal{H}_\infty \bigotimes \mathcal{H}_f \cong \mathcal{H}_{\mathrm{GL}_n(F_\infty)} \bigotimes \mathcal{H}_{\mathrm{GL}_n(\mathbb{A}_f)},$$

we have by Proposition 4.19 and Proposition 4.20, gives an action $\pi(T)$ for $T \in \mathcal{H}$ on V. We are now ready to state the analogue of Theorem 2.7 for local non-Archimedean representations and Theorem 3.12 for local Archimedean representations in the global setting.

Theorem 4.21. *Let (π, V) be a representation of $\mathrm{GL}_n(\mathbb{A})$. The vector space V endowed with the operation*

$$\begin{aligned} \mathcal{H} \times V &\to V \\ (T, v) &\mapsto \pi(T)v \end{aligned}$$

forms a smooth $\mathcal{H}$-module. If (π', V') is another representation of $\mathrm{GL}_n(\mathbb{A})$, then

$$\mathrm{Hom}_{\mathrm{GL}_n(\mathbb{A})}(V, V') = \mathrm{Hom}_{\mathcal{H}}(V, V').$$

Conversely, if M is a smooth $\mathcal{H}$-module, then there exist a unique algebra homomorphism $\pi : \mathcal{U}(\mathfrak{g}_\infty) \to \mathrm{End}_{\mathbb{C}}(M)$, a unique group homomorphism $\pi : K_\infty \to \mathrm{Aut}_{\mathbb{C}}(M)$, and a unique group homomorphism $\pi : \mathrm{GL}_n(\mathbb{A}_f) \to \mathrm{Aut}_{\mathbb{C}}(M)$ such that (π, M) forms a representation of $\mathrm{GL}_n(\mathbb{A})$, and

$$T \cdot m = \pi(T)m$$

for all $T \in \mathcal{H}$ and $m \in M$. Moreover, (π, V) is admissible as a $\mathrm{GL}_n(\mathbb{A})$-representation if and only if it is admissible as an $\mathcal{H}$-module.

Proof. The direct part of the theorem follows directly from the direct parts of Theorem 2.7 and Theorem 3.12, as well as the equivalence of the notion of admissibility, since $V^L(\gamma) = V[e_\gamma \otimes e_L]$ for all $\gamma \in \widehat{K_\infty}$ and compact open subgroups L of $\mathrm{GL}_n(\mathbb{A}_f)$.

For the converse, suppose M is a smooth $\mathcal{H}$-module, which we through the isomorphism $\mathcal{H} \cong \mathcal{H}_{\mathrm{GL}_n(F_\infty)} \bigotimes \mathcal{H}_{\mathrm{GL}_n(\mathbb{A}_f)}$ view as a smooth $\mathcal{H}_{\mathrm{GL}_n(F_\infty)} \bigotimes \mathcal{H}_{\mathrm{GL}_n(\mathbb{A}_f)}$-module. Let us define an action σ_∞ of $\mathcal{H}_{\mathrm{GL}_n(F_\infty)}$ on M and an action σ_f of $\mathcal{H}_{\mathrm{GL}_n(\mathbb{A}_f)}$ on M in the following way: for $m \in M$, let $e = e_\infty \otimes e_f$ be an elementary idempotent in $\mathcal{H}_{\mathrm{GL}_n(F_\infty)} \bigotimes \mathcal{H}_{\mathrm{GL}_n(\mathbb{A}_f)}$ such that $m \in M[e_\infty \otimes e_f]$. For $S \in \mathcal{H}_{\mathrm{GL}_n(F_\infty)}$ and $f \in \mathcal{H}_{\mathrm{GL}_n(\mathbb{A}_f)}$, define

$$\sigma_\infty(S)m = (S \otimes e_f) \cdot m \qquad \text{and} \qquad \sigma_f(f)m = (e_\infty \otimes f) \cdot m,$$

and note that due to the partial ordering of elementary idempotents, this definition is independent of the choice of elementary idempotent e such that $m \in M[e]$. It is clear that (σ_∞, M) forms a smooth $\mathcal{H}_{\mathrm{GL}_n(F_\infty)}$-module and that (σ_f, M) forms a smooth $\mathcal{H}_{\mathrm{GL}_n(\mathbb{A}_f)}$-module. It now follows from the converse parts of Theorem 2.7 and Theorem 3.12, that there exist a unique algebra homomorphism $\pi_\infty : \mathcal{U}(\mathfrak{g}_\infty) \to \mathrm{End}_{\mathbb{C}}(M)$, a unique group homomorphism $\pi_\infty : K_\infty \to \mathrm{Aut}_{\mathbb{C}}(M)$, and a unique group homomorphism $\pi_f : \mathrm{GL}_n(\mathbb{A}_f) \to \mathrm{Aut}_{\mathbb{C}}(M)$ such that (π_∞, M) forms a $(\mathfrak{g}_\infty, K_\infty)$-module and (π_f, M) forms a smooth $\mathrm{GL}_n(\mathbb{A}_f)$-representation, and such that

$$\pi_\infty(S)m = \sigma_\infty(S)m \qquad \text{and} \qquad \pi_f(f)m = \sigma_f(f)m$$

for all $S \in \mathcal{H}_{\mathrm{GL}_n(F_\infty)}$, $f \in \mathcal{H}_{\mathrm{GL}_n(\mathbb{A}_f)}$, and $m \in M$. It now only remains to show that the actions π_∞ and π_f commute, and that $\pi_\infty(S)\big(\pi_f(f)m\big) = (S \otimes f) \cdot m$.

In order to show that π_∞ and π_f commute, it is enough to show that the Hecke algebra actions σ_∞ and σ_f commute. For $m \in M$, $S \in \mathcal{H}_{\mathrm{GL}_n(F_\infty)}$ and $f \in \mathcal{H}_{\mathrm{GL}_n(\mathbb{A}_f)}$, it follows from the partial ordering of elementary idempotents, that there exist $e^0_\infty \in \mathcal{E}_{\mathrm{GL}_n(F_\infty)}$ and $e^0_f \in \mathcal{E}_{\mathrm{GL}_n(\mathbb{A}_f)}$ such that $S \in \mathcal{H}_{\mathrm{GL}_n(F_\infty)}[e^0_\infty]$, $f \in \mathcal{H}_{\mathrm{GL}_n(\mathbb{A}_f)}[e^0_f]$, and $m \in M[e^0_\infty \otimes e^0_f]$. Thus, $\sigma_\infty(S)m$ and $\sigma_f(f)m$ are elements of $M[e^0_\infty \otimes e^0_f]$, and

$$\begin{aligned}
\sigma_\infty(S)\big(\sigma_f(f)m\big) &= (S \otimes e^0_f) \cdot (\sigma_f(f)m) \\
&= (S \otimes e^0_f) \cdot \big((e^0_\infty \otimes f) \cdot m\big) \\
&= \big((S * e^0_\infty) \otimes (e^0_f * f)\big) \cdot m \\
&= \big((e^0_\infty * S) \otimes (f * e^0_f)\big) \cdot m \\
&= (e^0_\infty \otimes f) \cdot \big((S \otimes e^0_f) \cdot m\big) \\
&= (e^0_\infty \otimes f) \cdot (\sigma_\infty(S)m) \\
&= \sigma_f(f)\big(\sigma_\infty(S)m\big).
\end{aligned} \tag{4.3}$$

Moreover, it follows from line (4.3) in the previous calculation, that

$$\pi_\infty(S)\big(\pi_f(f)m\big) = \sigma_\infty(S)\big(\sigma_f(f)m\big) = \big((S * e_\infty^0) \otimes (e_f^0 * f)\big) \cdot m = (S \otimes f) \cdot m.$$

Thus, if we identify both π_∞ and π_f with π, then (π, M) forms a representation of $\mathrm{GL}_n(\mathbb{A})$, and π is the unique representation of $\mathrm{GL}_n(\mathbb{A})$ on M such that $\pi(T)m = T \cdot m$ for all $T \in \mathcal{H}$ and $m \in M$. $\square$

As promised above, the tensor product theorem (Theorem 4.16) for irreducible admissible representations of $\mathrm{GL}_n(\mathbb{A})$ is now an easy consequence of the decomposition theorem for irreducible admissible modules over idempotented algebras (Theorem 4.17).

Proof of Theorem 4.16. Let (π, V) be an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A})$. Then by Theorem 4.21, we can view (π, V) as an irreducible admissible $\mathcal{H}$-module. Since $\mathcal{H} = \bigotimes_v' \mathcal{H}_{G_v}$ with respect to the distinguished elementary idempotents $e_v^\circ \in \mathcal{E}_{G_v}$ at the non-Archimedean places, and $\mathcal{H}_{G_v}[e_v^\circ]$ is commutative at all non-Archimedean places (see Proposition 2.13), it follows from Theorem 4.17 that at all places v, there exists an irreducible admissible $\mathcal{H}_{G_v}$-module V_v and at almost all non-Archimedean places v there exists a non-zero distinguished element $\xi_v^\circ \in V[e_v^\circ]$, such that V, as an $\mathcal{H}$-module, is isomorphic to the restricted tensor product $\bigotimes_v' V_v$ with respect to the ξ_v°'s. Let $A : V \xrightarrow{\sim} \bigotimes_v' V_v$ denote this $\mathcal{H}$-isomorphism. For v Archimedean, it follows from Theorem 3.12 that there exists a unique algebra homomorphism $\pi_v : \mathcal{U}(\mathfrak{g}_v) \to \mathrm{End}_{\mathbb{C}}(V_v)$ and a unique group homomorphism $\pi_v : K_v \to \mathrm{Aut}_{\mathbb{C}}(V_v)$ such that (π_v, V_v) forms an irreducible admissible $(\mathfrak{g}_v, K_v)$-module and $\pi_v(S_v)\xi_v = S_v \cdot \xi_v$ for all $S_v \in \mathcal{H}_{G_v}$ and $\xi_v \in V_v$. For v non-Archimedean, it follows from Theorem 2.7 that there exists a unique group homomorphism $\pi_v : G_v \to \mathrm{Aut}_{\mathbb{C}}(V_v)$ such that (π_v, V_v) forms an irreducible admissible G_v-representation and $\pi_v(f_v)\xi_v = f_v \cdot \xi_v$ for all $f_v \in \mathcal{H}_{G_v}$ and $\xi_v \in V_v$. Thus, A is an $\mathcal{H}$-isomorphism between (π, V) and $(\otimes_v \pi_v, \bigotimes_v' V_v)$, and by Theorem 4.21 this is equivalent to A being an isomorphism of representations of $\mathrm{GL}_n(\mathbb{A})$.

It remains to show that the non-zero distinguished vectors $\xi_v^\circ \in V_v[e_v^\circ]$ at almost all non-Archimedean places are K_v-fixed under π_v. As $\pi_v(e_v^\circ)$ is the K_v-projection $V_v \to V_v^{K_v}$ for the smooth G_v-representation (π_v, V_v), it is clear that $V_v[e_v^\circ] = V_v^{K_v}$, and thus $\pi_v(k_v)\xi_v^\circ = \xi_v^\circ$ for all $k_v \in K_v$. Hence π_v is unramified at almost all non-Archimedean places. $\square$

4.7 Automorphic forms

In this section we introduce automorphic forms, the space of which will serve as an underlying vector space for automorphic representations. In order to do so, we first need to define the properties *smooth*, *K-finite*, *$\mathcal{Z}(\mathfrak{g}_\infty)$-finite*, and *moderate growth* for functions $\mathrm{GL}_n(\mathbb{A}) \to \mathbb{C}$.

Smoothness of complex-valued functions on $\mathrm{GL}_n(\mathbb{A})$ is defined analogously to that of complex-valued functions on $\mathbb{A}$ (see Definition 4.9).

Definition 4.22 (Smooth)**.** We say a continuous function $\varphi : \mathrm{GL}_n(\mathbb{A}) \to \mathbb{C}$ is *smooth* if

(i) for all $y_f \in \mathrm{GL}_n(\mathbb{A}_f)$ the function $\varphi(\_\_ \cdot y_f)$ on $\mathrm{GL}_n(F_\infty)$ is smooth in the usual since for a complex-valued function on smooth manifolds, and
(ii) for all $x_\infty \in \mathrm{GL}_n(F_\infty)$ the function $\varphi(x_\infty \cdot \_\_)$ on $\mathrm{GL}_n(\mathbb{A}_f)$ is locally constant.

We let $C^\infty(\mathrm{GL}_n(\mathbb{A}))$ denote the $\mathbb{C}$-vector space of all smooth functions $\mathrm{GL}_n(\mathbb{A}) \to \mathbb{C}$.

Note that the right translate of a smooth function $\mathrm{GL}_n(\mathbb{A}) \to \mathbb{C}$ by an element in K is again smooth.

Definition 4.23 (K-finite)**.** We say a smooth function $\varphi \in C^\infty(\mathrm{GL}_n(\mathbb{A}))$ is *K-finite* if right translation by elements of K span a finite-dimensional subspace of $C^\infty(\mathrm{GL}_n(\mathbb{A})$, i.e.

$$\dim_{\mathbb{C}} \langle \varphi(\_\_ \cdot k) \mid k \in K \rangle < \infty.$$

Remark 4.24. Since $\mathrm{GL}_n(\mathbb{A}_f)$ is the restricted direct product of $\mathrm{GL}_n(F_v)$ for non-Archimedean v's with respect to the K_v's, and since every compact open subgroup of $\mathrm{GL}_n(\mathbb{A}_f)$ has finite index in $K_f = \prod_{v \notin S_\infty} K_v$, one can show that if $\varphi_f : \mathrm{GL}_n(\mathbb{A}_f) \to \mathbb{C}$ is locally constant and K_f-finite, there exists a compact open subgroup L of $\mathrm{GL}_n(\mathbb{A}_f)$ such that φ_f is right-L-invariant, i.e. $\varphi_f(y_f l) = \varphi_f(y_f)$, for all $y_f \in \mathrm{GL}_n(\mathbb{A}_f)$ and $l \in L$.

As in Chapter 3, we have an action of $\mathfrak{g}_\infty$ on $C^\infty(\mathrm{GL}_n(\mathbb{A}))$ given by the differentiated right translation, i.e.

$$(\rho(X)\varphi)(g) = \frac{d}{dt}\varphi(g\exp(tX))\big|_{t=0}$$

for $X \in \mathfrak{g}_\infty$ and $g \in \mathrm{GL}_n(\mathbb{A})$. This action once again extends to the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_\infty)$ of $\mathfrak{g}_\infty$. Let $\mathcal{Z}(\mathfrak{g}_\infty)$ denote the centre of $\mathcal{U}(\mathfrak{g}_\infty)$.

Definition 4.25 ($\mathcal{Z}(\mathfrak{g}_\infty)$-finite)**.** We call a smooth function $\varphi \in C^\infty(\mathrm{GL}_n(\mathbb{A}))$ *$\mathcal{Z}(\mathfrak{g}_\infty)$-finite* if φ lies in a finite-dimensional $\mathcal{Z}(\mathfrak{g}_\infty)$-invariant subspace of $C^\infty(\mathrm{GL}_n(\mathbb{A}))$, i.e. if

$$\dim_{\mathbb{C}} \langle \rho(z)\varphi \mid z \in \mathcal{Z}(\mathfrak{g}_\infty) \rangle < \infty.$$

This definition is equivalent to the existence of an ideal $\mathcal{J} \subseteq \mathcal{Z}(\mathfrak{g}_\infty)$, of finite codimension, that annihilates φ.

Definition 4.26. Let $g \in \mathrm{GL}_n(\mathbb{A})$. We define the *height* $\|g\|$ of $g = \left((g_v^{i,j})_{1\le i,j\le n}\right)_v$ as the product

$$\|g\| = \prod_v \max_{1\le i,j\le n} \left\{ \left|g_v^{i,j}\right|_v, \left|(g^{-1})_v^{i,j}\right|_v \right\}.$$

Since for non-Archimedean places, $|g_v^{i,j}|_v \le 1$ for all $g_v = (g_v^{i,j})_{1\le i,j\le n} \in \mathrm{GL}_n(\mathcal{O}_v)$ and $|\cdot|_v$ is ultrametric, the factor

$$\max_{1\le i,j\le n} \left\{ \left|g_v^{i,j}\right|_v, \left|(g^{-1})_v^{i,j}\right|_v \right\} = 1$$

for $g_v \in \mathrm{GL}_n(\mathcal{O}_v)$. This implies that $\|g\|$ is well-defined, as the product in its definition is finite.

We also have the following product formula for the height function.

Proposition 4.27 ([MW95, I.2.2(ii)])**.** *There exists a positive constant $c > 0$ such that $\|gh\| \le c\|g\|\|h\|$ for all $g, h \in \mathrm{GL}_n(\mathbb{A})$.*

We can now define moderate growth for functions on $\mathrm{GL}_n(\mathbb{A})$ analogously to Definition 3.14 for functions on $\mathrm{GL}_n(F_v)$ for F_v Archimedean.

Definition 4.28 (Moderate growth)**.** We say a function $\varphi : \mathrm{GL}_n(\mathbb{A}) \to \mathbb{C}$ is of *moderate growth* if there exist a $C \in \mathbb{R}_{>0}$ and an $N \in \mathbb{N}$ such that

$$|\varphi(g)|_{\mathbb{C}} \le C\|g\|^N$$

for all $g \in \mathrm{GL}_n(\mathbb{A})$.

Definition 4.29 (Automorphic form)**.** A smooth function $\varphi \in C^\infty(\mathrm{GL}_n(\mathbb{A}))$ is called an *automorphic form* if the following holds:

(i) φ is left-$\mathrm{GL}_n(F)$-invariant, i.e. $\varphi(\gamma g) = \varphi(g)$ for all $\gamma \in \mathrm{GL}_n(F)$ and $g \in \mathrm{GL}_n(\mathbb{A})$.
(ii) φ is K-finite.
(iii) φ is $\mathcal{Z}(\mathfrak{g}_\infty)$-finite.
(iv) φ is of moderate growth.

We denote by $\mathcal{A}(\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}))$ the space of all automorphic forms.

Given a Hecke character $\omega : F^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$, which we view as a character of the centre $Z_n(\mathbb{A})$, we denote by $\mathcal{A}(\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}), \omega)$ the space of all automorphic forms $\varphi \in \mathcal{A}(\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}))$ satisfying

$$\varphi(zg) = \omega(z)\varphi(g)$$

for all $z \in Z_n(\mathbb{A})$ and $g \in \mathrm{GL}_n(\mathbb{A})$.

We will often simply write $\mathcal{A}$ instead of $\mathcal{A}(\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}))$ and $\mathcal{A}(\omega)$ instead of $\mathcal{A}(\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}), \omega)$, as we will not consider automorphic forms over different adele groups in this thesis.

4.8 Cusp forms

Let $\alpha = (n_1, \dots, n_r) \vdash n$ be an ordered partition of n, i.e. a tuple of positive integers such that $n_1 + \cdots + n_r = n$, and let U_α denote the standard unipotent radical with respect to α (cf. Section 1.1), that is,

$$U_\alpha = \left\{ \begin{pmatrix} I_{n_1} & * & \cdots & * \\ & I_{n_2} & \ddots & \vdots \\ & & \ddots & * \\ & & & I_{n_r} \end{pmatrix} \right\} \subseteq \mathrm{GL}_n .$$

Note that U_α is nilpotent; indeed, for the lower central series

$$U_\alpha = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots ,$$

where $G_{i+1} = [G_i, U_\alpha]$, we have that G_i is the subgroup of matrices whose entries in the j'th upper block-diagonal are 0 for all $1 \le j \le i$. Thus the lower central series stabilizes at $G_r = \{I_n\}$. U_α is also unipotent as the name would suggest, as every element is an upper triangular matrix with ones on the diagonal.

Proposition 4.30. *The quotient space $U_\alpha(F)\backslash U_\alpha(\mathbb{A})$ is compact for all partitions $\alpha \vdash n$.*

Proof. This is true in general for unipotent subgroups of GL_n (see e.g. [Bor63, Theorem 5.6]). It essentially follows from the strong approximation for unipotent groups (see Theorem 4.12(ii)) and the compactness of $U(F)\backslash U(F_\infty)$ for any unipotent subgroup $U \subseteq \mathrm{GL}_n$. □

Since $U_\alpha(\mathbb{A})$ and $U_\alpha(F)$ are nilpotent, it follows from Proposition 1.5(iv) that they are unimodular. As F is discrete in $\mathbb{A}$ and thus closed, $U_\alpha(F)$ is closed in $U_\alpha(\mathbb{A})$. Thus, it follows from Theorem 1.3 that we have a quotient measure on $U_\alpha(F)\backslash U_\alpha(\mathbb{A})$. By Proposition 4.30 this measure is necessarily a finite measure, and any continuous function $f : U_\alpha(\mathbb{A}) \to \mathbb{C}$ such that $f(\gamma u) = f(u)$ for all $\gamma \in U_\alpha(F)$ and $u \in U_\alpha(\mathbb{A})$ can be viewed as a function in $C_c(U_\alpha(F)\backslash U_\alpha(\mathbb{A}))$.

Definition 4.31 (Cusp form)**.** An automorphic form $\varphi \in \mathcal{A}(\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}))$ is called a *cusp form* if

$$\int_{U_\alpha(F)\backslash U_\alpha(\mathbb{A})} \varphi(ug)\, du = 0 \qquad \forall g \in \mathrm{GL}_n(\mathbb{A}),\, \forall \alpha \vdash n. \tag{4.4}$$

We denote the space of cusp forms by $\mathcal{A}_0(\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}))$. Any continuous function $\varphi : \mathrm{GL}_n(\mathbb{A}) \to \mathbb{C}$ satisfying (4.4) is called *cuspidal*.

Remark 4.32. The condition in Definition 4.31 is equivalent to the integral over the quotient of the unipotent radical of any proper parabolic subgroup of GL_n being equal to zero, instead of just the standard parabolic subgroups (which form a set of representatives of the conjugacy classes of parabolic subgroups), and therefore, this is often used as the definition instead. One can also show that it is sufficient for all integrals over the unipotent radicals of maximal standard proper parabolic subgroups to be equal to zero, i.e. for partitions $\alpha = (n_1, n_2) \vdash n$ of length two, in order for all integrals over the unipotent radical of any proper parabolic subgroup of GL_n to be equal to zero.

For a Hecke character $\omega : F^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$, we denote the space of cuspidal elements of $\mathcal{A}(\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}), \omega)$ by $\mathcal{A}_0(\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}), \omega)$. As with automorphic forms, we will often abbreviate $\mathcal{A}_0(\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}))$ to $\mathcal{A}_0$ and $\mathcal{A}_0(\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}), \omega)$ to $\mathcal{A}_0(\omega)$.

4.9 Automorphic representations

We want to define an action of the global Hecke algebra $\mathcal{H}$ on the space of automorphic forms. From Example 2.8 we have that $(\rho_f, L(\mathrm{GL}_n(\mathbb{A}_f)))$ forms a smooth $\mathcal{H}_{\mathrm{GL}_n(\mathbb{A}_f)}$-module, where $L(\mathrm{GL}_n(\mathbb{A}_f))$ is the space of complex-valued functions on $\mathrm{GL}_n(\mathbb{A}_f)$ that are right translation invariant by some compact open subgroup of $\mathrm{GL}_n(\mathbb{A}_f)$, and the action ρ_f is given by the following convolution:

$$\rho_f(h_f)\varphi(g_f) = \varphi * \widecheck{h_f}(g_f) = \int_{\mathrm{GL}_n(\mathbb{A}_f)} \varphi(g_f x_f) h_f(x_f)\, dx_f \tag{4.5}$$

for $\varphi \in L(\mathrm{GL}_n(\mathbb{A}_f))$. (Recall that $\check{f} : g \mapsto f(g^{-1})$.) From Example 2.8 we also get that $\varphi * e_L = \varphi$ if and only if φ is right-L-invariant, where L is a compact open subgroup of $\mathrm{GL}_n(\mathbb{A}_f)$, and it is clear that $e_L = \widecheck{e_L}$.

From Example 3.13 we have that $(\rho_\infty, C^\infty(\mathrm{GL}_n(F_\infty))_{K_\infty,\rho})$ forms a smooth $\mathcal{H}_{\mathrm{GL}_n(F_\infty)}$-module, where $C^\infty(\mathrm{GL}_n(F_\infty))_{K_\infty,\rho})$ is the space of smooth functions on $\mathrm{GL}_n(F_\infty)$ that are right-K_∞-finite, and the action ρ_∞ is similarly given by

$$\rho_\infty(T_\infty)\varphi(g_\infty) = \varphi * \check{T}_\infty(g_\infty) = \int_{\mathrm{GL}_n(F_\infty)} \varphi(g_\infty x_\infty)\, dT_\infty(x_\infty). \tag{4.6}$$

We also have that $\varphi \in C^\infty(\mathrm{GL}_n(F_\infty))$ is right-K_∞-finite if and only if there exists an elementary idempotent $e_A \in \mathcal{E}_{\mathcal{H}_{\mathrm{GL}_n(F_\infty)}}$ such that $\varphi * \widecheck{e_A} = \varphi$.

Let $T_\infty \in \mathcal{H}_{GL_n(F_\infty)}$ and $h_f \in \mathcal{H}_{GL_n(\mathbb{A}_f)}$. Let us use the notation for distributions from Section 3.3 and denote by T_f the distribution $h_f\, dg_f$ on $\mathrm{GL}_n(\mathbb{A}_f)$. Then by Fubini's theorem, $T_\infty \otimes T_f$ is a distribution on $\mathrm{GL}_n(F_\infty) \times \mathrm{GL}_n(\mathbb{A}_f) = \mathrm{GL}_n(\mathbb{A})$, and

$$\begin{aligned} &\int_{GL_n(\mathbb{A})} \varphi(x)\, d(T_\infty \otimes T_f)(x) \\ &= \int_{GL_n(F_\infty)} \int_{GL_n(\mathbb{A}_f)} \varphi(x_\infty x_f)\, dT_f(x_f)\, dT_\infty(x_\infty) \end{aligned}$$

$$= \int_{GL_n(\mathbb{A}_f)} \int_{GL_n(F_\infty)} \varphi(x_\infty x_f)\, dT_\infty(x_\infty)\, dT_f(x_f)$$

for all $\varphi \in C^\infty(\mathrm{GL}_n(\mathbb{A}))$. The actions ρ_∞ and ρ_f define the action $\rho = \rho_\infty \otimes \rho_f$ of the global Hecke algebra $\mathcal{H} \cong \mathcal{H}_{\mathrm{GL}_n(F_\infty)} \bigotimes \mathcal{H}_{\mathrm{GL}_n(\mathbb{A}_f)}$, where, for a pure tensor $T = T_\infty \otimes T_f$, $\rho(T)$ is the operator

$$\begin{aligned}
\rho(T)\varphi(g) &= \rho_\infty(T_\infty)\big(\rho_f(T_f)\varphi\big)(g) \\
&= \int_{GL_n(F_\infty)} \int_{GL_n(\mathbb{A}_f)} \varphi(g x_\infty x_f)\, dT_f(x_f)\, dT_\infty(x_\infty) \\
&= \int_{GL_n(\mathbb{A})} \varphi(gx)\, d(T_\infty \otimes T_f)(x) \\
&= \varphi * \check{T}(g).
\end{aligned}$$

This linearly extends to all of $\mathcal{H}$, hence the action $\rho(T)$ of $\mathcal{H}$ is again right convolution by $\check{T}$. We get from Examples 2.8 and 3.13 that the space of functions on $\mathrm{GL}_n(\mathbb{A})$ which are smooth and right-K_∞-finite in the infinite components and right-L-invariant for some compact open subgroup L of $\mathrm{GL}_n(\mathbb{A}_f)$ in the finite components, is stable under the $\mathcal{H}$-action ρ. By Remark 4.24, this just means that the subspace $C^\infty(\mathrm{GL}_n(\mathbb{A}))_K$ of all K-finite elements of $C^\infty(\mathrm{GL}_n(\mathbb{A}))$ is stable under ρ. Moreover, $C^\infty(\mathrm{GL}_n(\mathbb{A}))_K$ is a smooth module of $\mathcal{H}$, since we get from the previous calculations that $\varphi \in C^\infty(\mathrm{GL}_n(\mathbb{A}))$ is K-finite if and only if there exists an elementary idempotent $e = e_A \otimes e_L$ with $e_A \in \mathcal{E}_{\mathcal{H}_{\mathrm{GL}_n(F_\infty)}}$ and L a compact open subgroup of $\mathrm{GL}_n(\mathbb{A}_f)$, such that $\varphi * \check{e} = \varphi$.

The following theorem is the motivation for defining the space of automorphic forms as we did, because it states that $\mathcal{Z}(\mathfrak{g}_\infty)$-finite elements of $C^\infty(\mathrm{GL}_n(\mathbb{A}))_K$ of moderate growth generate admissible $\mathcal{H}$-modules. It is a reformulation of results found in [BJ79, §4].

Theorem 4.33.

(i) *The spaces $\mathcal{A}$ and $\mathcal{A}_0(\omega)$ are stable under the action ρ of the global Hecke algebra $\mathcal{H}$, and both $\mathcal{A}$ and $\mathcal{A}_0$ are $\mathcal{H}$-modules.*

(ii) *For each $\varphi \in \mathcal{A}$, the space $\varphi * \mathcal{H}$ is an admissible $\mathcal{H}$-module.*

It follows from Theorem 4.33 that an irreducible $\mathcal{H}$-subquotient of the space $\mathcal{A}$ (or $\mathcal{A}_0(\omega)$) is always admissible. We recall from Theorem 4.21 that smooth $\mathcal{H}$-modules are naturally equivalent to representations of $\mathrm{GL}_n(\mathbb{A})$, so we get that $(\rho, \mathcal{A})$ and $(\rho, \mathcal{A}_0(\omega))$ form representations of $\mathrm{GL}_n(\mathbb{A})$. This fact, together with Theorem 4.33 allows us to make the following definition.

Definition 4.34 (Automorphic representation)**.**

(i) An irreducible representation (π, V) of $\mathrm{GL}_n(\mathbb{A})$ is called an *automorphic representation* if it is isomorphic to a subquotient of $(\rho, \mathcal{A})$.

(ii) A *cuspidal automorphic representation* (π, V) is an irreducible $\mathrm{GL}_n(\mathbb{A})$-subrepresentation of $(\rho, \mathcal{A}_0(\omega))$ for some unitary Hecke character $\omega : F^\times \backslash \mathbb{A}^\times \to \mathbb{S}^1$.

Let us briefly expand on the $\mathrm{GL}_n(\mathbb{A})$-action ρ on $\mathcal{A}$. The $\mathrm{GL}_n(\mathbb{A}_f)$-action on $\mathcal{A}$ is equivalent to the $\mathcal{H}_{\mathrm{GL}_n(\mathbb{A}_f)}$-action ρ_f in (4.5) and the $(\mathfrak{g}_\infty, K_\infty)$-action on $\mathcal{A}$ is equivalent to the $\mathcal{H}_{\mathrm{GL}_n(F_\infty)}$-action ρ_∞ in (4.6). From Examples 2.8 and 3.13, it follows that $\mathrm{GL}_n(\mathbb{A}_f)$ and K_∞ acts on $\mathcal{A}$ by right translation, i.e.

$$(\rho(k_\infty g_f)\varphi)(x) = \varphi(x k_\infty g_f)$$

for all $k_\infty \in K_\infty$, $g_f \in \mathrm{GL}_n(\mathbb{A}_f)$, and $\varphi \in \mathcal{A}$, while $\mathcal{U}(\mathfrak{g}_\infty)$ acts on $\mathcal{A}$ by differentiated right translation, i.e.

$$(\rho(X)\varphi)(x) = \frac{d}{dt}\varphi(x\exp(tX))\big|_{t=0}$$

for all $X \in \mathfrak{g}_\infty$ and $\varphi \in \mathcal{A}$, as expected.

If a cuspidal automorphic representation (π, V) is a subrepresentation of $\mathcal{A}_0(\omega)$ for the unitary Hecke character ω, then we call ω the *central character* of π, and we denote it by ω_π. As it is a character of $\mathbb{A}^\times$, we get a decomposition $\omega_\pi = \otimes_v \omega_{\pi,v}$ into characters $\omega_{\pi,v} : F_v^\times \to \mathbb{C}^\times$. Since $\mathrm{GL}_n(\mathbb{A}_f)$ acts via right translation, the character $\omega_{\pi,v} = \omega_{\pi_v}$ is simply the central character of the local representation π_v of π at the non-Archimedean places, as defined in Section 2.2, where we view $\omega_{\pi,v}$ as a character on $Z_n(F_v) \cong F_v^\times$.

In the literature, cuspidal automorphic representations are not always required to have a unitary central character, as we require here. Our motivation for this restricted definition comes from a key result that applies to this narrower class.

Theorem 4.35. *Let ω be a unitary Hecke character. The space $\mathcal{A}_0(\omega)$ is the direct sum of irreducible representations of* $\mathrm{GL}_n(\mathbb{A})$*, and each isomorphism class occurs with finite multiplicity.*

This theorem is ultimately due to Gel'fand and Piatetski-Shapiro (see [GGPS69, §3.4, Theorem 4]) who considered a different space of cusp forms (namely cuspidal functions with unitary central character in $L^2(\mathrm{GL}_n(F)Z_n(\mathbb{A})\backslash \mathrm{GL}_n(\mathbb{A}))$), but it is not an easy consequence. This is briefly discussed in [BJ79, §4.6], but a much more thorough discussion can be found in Chapter 13 in [Gro23] (although the results here are not immediately applicable to our setting either). For $n = 2$, this is the exact statement of Proposition 10.9 in [JL70].

In fact, we have the stronger result, that the multiplicity of an irreducible representation of $\mathrm{GL}_n(\mathbb{A})$ in the space $\mathcal{A}_0(\omega)$, is not only finite, but equal to one.

Theorem 4.36 (Multiplicity one)**.** *Let (π, V) and (π', V') be two irreducible* $\mathrm{GL}_n(\mathbb{A})$*-subrepresentations of $(\rho, \mathcal{A}_0(\omega))$. If $\pi \cong \pi'$, then $(\pi, V) = (\pi', V')$.*

We will return to the proof of this theorem in Chapter 6, where we will prove the strong multiplicity one theorem, of which Theorem 4.36 is an immediate consequence.

Chapter 5

Global Whittaker models

In this chapter, we examine global Whittaker models for irreducible admissible representations of $\mathrm{GL}_n(\mathbb{A})$. The primary goal is to prove the existence and uniqueness of a global Whittaker model for cuspidal automorphic representations—a result that plays a vital role in the proof of the strong multiplicity one theorem. In Section 5.1, we define the notion of a Whittaker function attached to a cusp form, which can be viewed as a type of Fourier inversion. We then prove that cusp forms admit a type of Fourier expansion in terms of their Whittaker functions. In Section 5.2, we define the notion of a global Whittaker model and use local uniqueness results to establish the uniqueness of such a model for all irreducible admissible representations of $\mathrm{GL}_n(\mathbb{A})$, if it exists. Finally, we demonstrate that the Whittaker functions constructed in the first section form a global Whittaker model for cuspidal automorphic representations, thereby proving existence in this case.

5.1 Global Whittaker functions

The purpose of this section is to introduce the notion of a Whittaker function of a cusp form. In Section 5.2, we show that the Whittaker functions of the elements in a cuspidal automorphic representation (π, V), form a global Whittaker model for (π, V). Thus, this section lays the groundwork for proving the existence of a Whittaker model for cuspidal automorphic representations.

Cf. Section 1.1, we let N_n denote the maximal unipotent subgroup of GL_n, consisting of upper triangular matrices with ones on the diagonal, and recall that we also view additive characters of $F\backslash\mathbb{A}$ as characters on $N_n(\mathbb{A})$ that are trivial on $N_n(F)$. Let $\psi \in \widehat{F\backslash\mathbb{A}}$ be a fixed non-trivial character for the remainder of this section.

Definition 5.1 (Whittaker function)**.** The (ψ-)Whittaker function of a cusp form $\varphi \in \mathcal{A}_0$ is the function $W_\varphi : \mathrm{GL}_n(\mathbb{A}) \to \mathbb{C}$ given by

$$W_\varphi(g) = \int_{N_n(F)\backslash N_n(\mathbb{A})} \varphi(mg)\psi(m)^{-1}\,dm.$$

The Whittaker function of any cusp form is a well-defined element of $C^\infty(\mathrm{GL}_n(\mathbb{A}))$, since $N_n(F)\backslash N_n(\mathbb{A})$ is compact by Proposition 4.30.

Proposition 5.2. *Let $\varphi \in \mathcal{A}_0$ and let W_φ be its Whittaker function. Then*

$$W_\varphi(mg) = \psi(m)W_\varphi(g)$$

for all $g \in \mathrm{GL}_n(\mathbb{A})$ and $m \in N_n(\mathbb{A})$.

Proof. Let $g \in \mathrm{GL}_n(\mathbb{A})$ and $m \in N_n(\mathbb{A})$. Then

$$\begin{aligned}\psi(m)W_\varphi(g) &= \int_{N_n(F)\backslash N_n(\mathbb{A})} \varphi(xg)\psi(xm^{-1})^{-1}\,dx \\ &= \int_{N_n(F)\backslash N_n(\mathbb{A})} \varphi(x'mg)\psi(x')^{-1}\,dx' \\ &= W_\varphi(mg),\end{aligned}$$

where the second step is a change of variables with $x = x'm$, and the equality follows from the right-invariance of the Haar integral. □

If we think of W_φ as a sort of Fourier transform of φ, it is natural to ask whether we also have a Fourier expansion of φ in terms of W_φ. This is the content of the following central theorem on Whittaker functions of cusp forms.

Theorem 5.3 (Fourier expansion of cusp forms)**.** *Let $\varphi \in \mathcal{A}_0$ and let W_φ be its Whittaker function. Then φ has the following Fourier expansion*

$$\varphi(g) = \sum_{\gamma \in N_{n-1}(F)\backslash \mathrm{GL}_{n-1}(F)} W_\varphi\left(\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix} g\right),$$

where the sum converges absolutely for all $g \in \mathrm{GL}_n(\mathbb{A})$ and uniformly on each compact subset of $\mathrm{GL}_n(\mathbb{A})$.

The remainder of the section is spent on proving this theorem. We follow the proof of Piatetski-Shapiro in [PŠ75], while elaborating on the arguments given there.

First, we want to prove a similar statement for functions on the mirabolic subgroup P_n of $\mathrm{GL}_n(\mathbb{A})$, using induction on n. Recall from Section 1.1 that the mirabolic group P_n is the subgroup

$$P_n = \left\{\begin{pmatrix} * & \cdots & * & * \\ \vdots & \ddots & \vdots & \vdots \\ * & \cdots & * & * \\ 0 & \cdots & 0 & 1\end{pmatrix}\right\} \subseteq \mathrm{GL}_n,$$

and note that $N_n \subseteq P_n$. We will also consider its subgroup

$$Y_n = U_{(n-1,1)} = \left\{\begin{pmatrix} I_{n-1} & * \\ 0 & 1\end{pmatrix}\right\} \subseteq P_n.$$

We identify $Y_n(\mathbb{A})$ with $\mathbb{A}^{n-1}$ and $Y_n(F)$ with F^{n-1}, and write elements of Y_n both as matrices and as $(n-1)$-dimensional vectors. Clearly the quotient group $Y_n(F)\backslash Y_n(\mathbb{A})$ is isomorphic to $(F\backslash\mathbb{A})^{n-1}$.

Let us denote by $\mathcal{S}_n$ the set of all functions ϕ on $P_n(\mathbb{A})$ satisfying the following three conditions.

(S1) $\phi \in C^\infty(P_n(\mathbb{A}))$ (cf. Definition 4.22 where GL_n is replaced by P_n),
(S2) $\phi(\gamma p) = \phi(p)$ for all $\gamma \in P_n(F)$ and $p \in P_n(\mathbb{A})$, and
(S3) $\int_{U_\alpha(F)\backslash U_\alpha(\mathbb{A})} \phi(up)\,du = 0$ for all $p \in P_n(\mathbb{A})$ and $\alpha \vdash n$.

The *Whittaker function* of an element of $\mathcal{S}_n$ is defined analogously to that of a cusp form, but we also need to consider the following *partial χ-Whittaker function* $w_{\phi,\chi}$ of a function $\phi \in \mathcal{S}_n$ and a character $\chi : (F\backslash\mathbb{A})^{n-1} \to \mathbb{C}^\times$:

$$w_{\phi,\chi}(p) = \int_{Y_n(F)\backslash Y_n(\mathbb{A})} \phi\left(\begin{pmatrix} I_{n-1} & y \\ 0 & 1\end{pmatrix} p\right)\chi(y)^{-1}\,dy,$$

and write $w_\phi = w_{\phi,\psi'}$, where $\psi' : (F\backslash\mathbb{A})^{n-1} \to \mathbb{C}^\times$ is the character given by $\psi'(y_1, \ldots, y_{n_1}) = \psi(y_{n-1})$ for the fixed character ψ of $F\backslash\mathbb{A}$. We call w_ϕ the *partial Whittaker function* of ϕ.

The goal is to prove the equivalent statement of Theorem 5.3 for functions in $\mathcal{S}_n$, and the first step is the following lemma.

Lemma 5.4. *Let $\phi \in \mathcal{S}_n$. We have the expansion*

$$\phi(p) = \sum_{\gamma \in P_{n-1}(F)\backslash \mathrm{GL}_{n-1}(F)} w_\phi\left(\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix} p\right),$$

where the sum converges absolutely for all $p \in P_n(\mathbb{A})$ and uniformly on compact subsets of $P_n(\mathbb{A})$.

Proof. Let $\chi : (F\backslash\mathbb{A})^{n-1} \to \mathbb{C}^\times$ be a character. For a given $p \in P_n(\mathbb{A})$, the function

$$f_p : \mathbb{A}^{n-1} \to \mathbb{C} : y \mapsto \phi\left(\begin{pmatrix}I_{n-1} & y\\ 0 & 1\end{pmatrix} p\right)$$

is a smooth function on $\mathbb{A}^{n-1}$ that satisfies $f_p(\upsilon + y) = f_p(y)$ for all $\upsilon \in F^{n-1}$, hence it is an $(n-1)$-dimensional smooth periodic adelic function (cf. Definition 4.9). It then follows immediately from the $(n-1)$-dimensional version of Theorem 4.10, that we have the expansion

$$\phi\left(\begin{pmatrix}I_{n-1} & y\\ 0 & 1\end{pmatrix} p\right) = \sum_{\chi \in \widehat{(F\backslash\mathbb{A})^{n-1}}} w_{\phi,\chi}(p)\chi(y), \tag{5.1}$$

since $\hat{f}_{p_\chi} = w_{\phi,\chi}(p)$ and $Y_n(F)\backslash Y_n(\mathbb{A}) \cong F^{n-1}\backslash\mathbb{A}^{n-1} \cong (F\backslash\mathbb{A})^{n-1}$. By Theorem 4.10, this sum is absolutely convergent for all $p \in P_n(\mathbb{A})$ and $y \in \mathbb{A}^{n-1}$, and because ϕ is continuous, one can show that the convergence is uniform on compact subsets of $P_n(\mathbb{A})$. This would not be hard to see if we had given a detailed account of the construction of the adelic Fourier transform, but as this is beyond the scope of the thesis, we leave this statement without proof. Note that

$$\begin{aligned}
w_{\phi,\chi}(p)\chi(x) &= \int_{Y_n(F)\backslash Y_n(\mathbb{A})} \phi\left(\begin{pmatrix}I_{n-1} & y\\ 0 & 1\end{pmatrix} p\right)\chi(y-x)^{-1}\,dy\\
&= \int_{Y_n(F)\backslash Y_n(\mathbb{A})} \phi\left(\begin{pmatrix}I_{n-1} & x+y'\\ 0 & 1\end{pmatrix} p\right)\chi(y')^{-1}\,dy'\\
&= \int_{Y_n(F)\backslash Y_n(\mathbb{A})} \phi\left(\begin{pmatrix}I_{n-1} & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}I_{n-1} & y'\\ 0 & 1\end{pmatrix} p\right)\chi(y')^{-1}\,dy'\\
&= w_{\phi,\chi}\left(\begin{pmatrix}I_{n-1} & x\\ 0 & 1\end{pmatrix} p\right),
\end{aligned}$$

where the second equality follows from the change of variables $y' = y - x$, leaving the Haar measure invariant, hence (5.1) can immediately be rewritten as

$$\phi\left(\begin{pmatrix}I_{n-1} & y\\ 0 & 1\end{pmatrix} p\right) = \sum_{\chi \in \widehat{(F\backslash\mathbb{A})^{n-1}}} w_{\phi,\chi}\left(\begin{pmatrix}I_{n-1} & y\\ 0 & 1\end{pmatrix} p\right),$$

and setting $y = 0$ yields

$$\phi(p) = \sum_{\chi \in \widehat{(F\backslash\mathbb{A})^{n-1}}} w_{\phi,\chi}(p), \tag{5.2}$$

where the sum is still absolutely convergent for all $p \in P_n(\mathbb{A})$ and uniformly convergent on compact subsets.

We now wish to change the indexing of the sum in order to have only the fixed w_ϕ in the summands. By Theorem 4.7, the map

$$F \xrightarrow{\sim} \widehat{F\backslash\mathbb{A}} : a \mapsto \big(x \mapsto \psi(ax)\big)$$

establishes an isomorphism $\widehat{F\backslash\mathbb{A}} \cong F$ since ψ is not the trivial character, and thus there is an isomorphism

$$F^{n-1} \xrightarrow{\sim} (\widehat{F\backslash\mathbb{A})^{n-1}} : a \mapsto \big(y \mapsto \psi(a^T y)\big)\,. \tag{5.3}$$

Given $a \in F^{n-1} \setminus \{0\}$, as it is not the zero vector, there exists a matrix $\gamma \in \mathrm{GL}_{n-1}(F)$ such that $a^T = \gamma^{(n-1)}$, where $\gamma^{(n-1)}$ denotes the last row of γ. (Its construction could for example be done by first padding the column above one non-zero entry in a^T with zeros and then entering the columns of the identity matrix I_{n-2} in the remaining spaces.) It is easy to see that $P_{n-1}(F)$ is the left stabilizer of the last row of a matrix in $\mathrm{GL}_{n-1}(F)$, and thus there is a one-to-one correspondence between $F^{n-1}\setminus\{0\}$ and $P_{n-1}(F)\backslash\,\mathrm{GL}_{n-1}(F)$, and the isomorphism in (5.3) further gives us a one-to-one correspondence between $(\widehat{F\backslash\mathbb{A})^{n-1}}\setminus\{\mathbb{1}\}$ and $P_{n-1}(F)\backslash\,\mathrm{GL}_{n-1}(F)$.

Now, given a non-trivial character $\chi : (F\backslash\mathbb{A})^{n-1} \to \mathbb{C}^\times$, let $\gamma \in P_{n-1}(F)\backslash\,\mathrm{GL}_{n-1}(F)$ such that $\chi(y) = \psi\left(\gamma^{(n-1)}y\right)$. Then

$$\begin{aligned}
w_\phi\left(\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}p\right) &= \int_{Y_n(F)\backslash Y_n(\mathbb{A})} \phi\left(\begin{pmatrix}I_{n-1} & y\\ 0 & 1\end{pmatrix}\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}p\right)\psi(y_{n-1})^{-1}\,dy\\
&= \int_{Y_n(F)\backslash Y_n(\mathbb{A})} \phi\left(\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}I_{n-1} & \gamma^{-1}y\\ 0 & 1\end{pmatrix}p\right)\psi(y_{n-1})^{-1}\,dy\\
&= \int_{Y_n(F)\backslash Y_n(\mathbb{A})} \phi\left(\underbrace{\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}}_{\in P_n(F)}\begin{pmatrix}I_{n-1} & \gamma^{-1}y\\ 0 & 1\end{pmatrix}p\right)\psi(y_{n-1})^{-1}\,dy\\
&= \int_{Y_n(F)\backslash Y_n(\mathbb{A})} \phi\left(\begin{pmatrix}I_{n-1} & \gamma^{-1}y\\ 0 & 1\end{pmatrix}p\right)\psi(y_{n-1})^{-1}\,dy,
\end{aligned}$$

where the last line follows from the left-invariance of $P_n(F)$ on ϕ. We now have a change of variables $y = \gamma y'$, i.e. the integrand in the domain $\mathbb{A}^{n-1}$ is multiplied with an element in $\mathrm{GL}_{n-1}(F)$. We saw in Section 4.3 that this leaves the measure on $\mathbb{A}^{n-1}$ invariant, from which it must also follow that the quotient measure on $Y_n(F)\backslash Y_n(\mathbb{A})$ is invariant under such transformations. Thus the lines of equations continue:

$$\begin{aligned}
&= \int_{Y_n(F)\backslash Y_n(\mathbb{A})} \phi\left(\begin{pmatrix}I_{n-1} & \gamma^{-1}\gamma y'\\ 0 & 1\end{pmatrix}p\right)\psi((\gamma y')_{n-1})^{-1}\,dy'\\
&= \int_{Y_n(F)\backslash Y_n(\mathbb{A})} \phi\left(\begin{pmatrix}I_{n-1} & y'\\ 0 & 1\end{pmatrix}p\right)\psi(\gamma^{(n-1)}y')^{-1}\,dy'\\
&= \int_{Y_n(F)\backslash Y_n(\mathbb{A})} \phi\left(\begin{pmatrix}I_{n-1} & y'\\ 0 & 1\end{pmatrix}p\right)\chi(y')^{-1}\,dy'\\
&= w_{\phi,\chi}(p).
\end{aligned}$$

Furthermore, note that for $\chi = \mathbb{1}$, the trivial character,

$$w_{\phi,\mathbb{1}}(p) = \int_{Y_n(F)\backslash Y_n(\mathbb{A})} \phi\,(yp)\,dy = 0,$$

since ϕ satisfies (S3).

The last two calculations, together with the correspondence between $\widehat{(F\backslash\mathbb{A})^{n-1}} \setminus \{\mathbb{1}\}$ and $P_{n-1}(F)\backslash \mathrm{GL}_{n-1}(F)$, finally allow us to rewrite (5.2) into the following expansion:

$$\phi(p) = \sum_{\gamma \in P_{n-1}(F)\backslash \mathrm{GL}_{n-1}(F)} w_\phi\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} p\right),$$

where the sum remains absolutely convergent for all $p \in P_n(\mathbb{A})$ and uniformly convergent on compact subsets. □

Remark 5.5. The subgroup

$$\left\{ \begin{pmatrix} \varrho & \upsilon \\ 0 & 1 \end{pmatrix} \,\middle|\, \varrho \in P_{n-1}(F),\, \upsilon \in F^{n-1} \right\} \subseteq P_n(F)$$

is contained in the stabilizer group of left translation of w_ϕ, since

$$\begin{aligned} w_\phi\left(\begin{pmatrix} \varrho & \upsilon \\ 0 & 1 \end{pmatrix} p\right) &= \int_{Y_n(F)\backslash Y_n(\mathbb{A})} \phi\left(\begin{pmatrix} I_{n-1} & y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \varrho & \upsilon \\ 0 & 1 \end{pmatrix} p\right) \psi(y_{n-1})^{-1}\, dy \\ &= \int_{Y_n(F)\backslash Y_n(\mathbb{A})} \phi\left(\underbrace{\begin{pmatrix} \varrho & \upsilon \\ 0 & 1 \end{pmatrix}}_{\in P_n(F)}\begin{pmatrix} I_{n-1} & \varrho^{-1}y \\ 0 & 1 \end{pmatrix} p\right) \psi(y_{n-1})^{-1}\, dy \\ &= \int_{Y_n(F)\backslash Y_n(\mathbb{A})} \phi\left(\begin{pmatrix} I_{n-1} & \varrho^{-1}y \\ 0 & 1 \end{pmatrix} p\right) \psi(y_{n-1})^{-1}\, dy \\ &= w_\phi(p), \end{aligned}$$

where the last equality follows from the same calculations as in the proof of Lemma 5.4. This additionally shows, in an alternative manner, that the choice of representative of a coset of $P_{n-1}(F)\backslash \mathrm{GL}_{n-1}(F)$ in the Fourier expansion in the previous lemma leaves the summand invariant.

We now fix a $p_0 \in P_n(\mathbb{A})$ and consider a function $\theta_{p_0} : P_{n-1}(\mathbb{A}) \to \mathbb{C}$ given by

$$\theta_{p_0}(q) = w_\phi\left(\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} p_0\right).$$

In order to use this function as a part of an induction step in the proof of Theorem 5.3, we need to show the following lemma.

Lemma 5.6. *The function θ_{p_0} is an element of $\mathcal{S}_{n-1}$ for all $p_0 \in P_n(\mathbb{A})$.*

Proof. Let $p_0 \in P_n(\mathbb{A})$ be fixed, and let us write $\theta = \theta_{p_0}$. From standard integration theory it is clear that $w_\phi \in C^\infty(P_n(\mathbb{A}))$, hence $\theta \in C^\infty(P_{n-1}(A))$, satisfying (S1).

For any $\varrho \in P_{n-1}(F)$ and $q \in P_{n-1}(\mathbb{A})$,

$$\theta(\varrho q) = w_\phi\left(\begin{pmatrix} \varrho q & 0 \\ 0 & 1 \end{pmatrix} p_0\right) = w_\phi\left(\begin{pmatrix} \varrho & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} p_0\right). \tag{5.4}$$

By Remark 5.5, the matrix $\left(\begin{smallmatrix} \varrho & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is an element of the stabilizer group of left translation of w_ϕ, hence it follows that the right-hand side of (5.4) is equal to

$$w_\phi\left(\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} p_0\right) = \theta(q),$$

and thus $\theta(\varrho q) = \theta(q)$, satisfying (S2).

In order to show (S3), let a partition $\alpha = (n_1, \ldots, n_r)$ of $n-1$ be given, and let U_α be the corresponding standard unipotent subgroup. We wish to show

$$\int_{U_\alpha(F)\backslash U_\alpha(\mathbb{A})} \theta(uq)\, du = 0$$

for all $q \in P_{n-1}(\mathbb{A})$.

First, we let R and R_0 denote the two subgroups

$$R = U_{(n_1,\ldots,n_r,1)} = \left\{ \begin{pmatrix} I_{n_1} & * & \cdots & * & * \\ & \ddots & \ddots & \vdots & \vdots \\ & & I_{n_{r-1}} & * & * \\ & & & I_{n_r} & * \\ & & & & 1 \end{pmatrix} \right\} \subseteq \mathrm{GL}_n \quad \text{and}$$

$$R_0 = U_{(n_1,\ldots,n_{r-1},n_r+1)} = \left\{ \begin{pmatrix} I_{n_1} & * & \cdots & * & * \\ & \ddots & \ddots & \vdots & \vdots \\ & & I_{n_{r-1}} & * & * \\ & & & I_{n_r} & 0 \\ & & & & 1 \end{pmatrix} \right\} \subseteq \mathrm{GL}_n\,.$$

Let us consider the following two factorizations of R, in order to use Theorem 1.4(ii) twice to split up integration over $R(F)\backslash R(\mathbb{A})$: we observe that $R = R_0 Y_n$, and that $R = Y_n U_\alpha$, where we consider U_α as a subgroup of R through the embedding

$$U_\alpha \hookrightarrow R : u \mapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}.$$

The latter product is a semidirect product, where $Y_n \cap U_\alpha = 1$ and Y_n is normal. This means that the quotient map by $Y_n(\mathbb{A})$ induces an isomorphism $U_\alpha(\mathbb{A}) \xrightarrow{\sim} Y_n(\mathbb{A})\backslash R(\mathbb{A})$, and we get the following isomorphism of quotient spaces:

$$R(F)Y_n(\mathbb{A})\backslash R(\mathbb{A}) = Y_n(\mathbb{A})U_\alpha(F)\backslash R(\mathbb{A}) \simeq U_\alpha(F)\backslash U_\alpha(\mathbb{A}). \tag{5.5}$$

The Haar measures on $Y_n(\mathbb{A}) \cong \mathbb{A}^{n-1}$ and $Y_n(F) \cong F^{n-1}$ are such that the usual measures on $\mathbb{A}$ and F are preserved. We fix Haar measures on $R(\mathbb{A})$, $R(F)$, $U_\alpha(\mathbb{A})$, and $U_\alpha(F)$ such that the homeomorphism in (5.5) preserves the quotient measures. Now by Theorem 1.4(ii) we have that

$$\begin{aligned}
\int_{R(F)\backslash R(\mathbb{A})} f(r)\, dr &= \int_{R(F)Y_n(\mathbb{A})\backslash R(\mathbb{A})} \int_{R(F)\cap Y_n(\mathbb{A})\backslash Y_n(\mathbb{A})} f(x\overline{r})\, dx\, d\overline{r} \\
&= \int_{U_\alpha(F)\backslash U_\alpha(\mathbb{A})} \int_{Y_n(F)\backslash Y_n(\mathbb{A})} f\left(x \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \right) dx\, du \\
&= \int_{U_\alpha(F)\backslash U_\alpha(\mathbb{A})} \int_{Y_n(F)\backslash Y_n(\mathbb{A})} f\left(\begin{pmatrix} I_{n-1} & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \right) dy\, du \\
&= \int_{U_\alpha(F)\backslash U_\alpha(\mathbb{A})} \int_{Y_n(F)\backslash Y_n(\mathbb{A})} f\left(\begin{pmatrix} u & y \\ 0 & 1 \end{pmatrix} \right) dy\, du,
\end{aligned}$$

for all $f \in L^1(R(F)\backslash R(\mathbb{A}))$. As R is additive in its $(n-1,n)$-entry, the function

$$\psi' : R(\mathbb{A}) \to \mathbb{C}^* : r \mapsto \psi(r_{n-1,n})$$

is a well-defined character trivial on $R(F)$, that is also trivial on $R_0(\mathbb{A})$.

We now observe that for $q \in P_{n-1}(\mathbb{A})$,

$$\begin{aligned}
&\int_{U_\alpha(F)\backslash U_\alpha(\mathbb{A})} \theta(uq)\,du \\
&= \int_{U_\alpha(F)\backslash U_\alpha(\mathbb{A})} \int_{Y_n(F)\backslash Y_n(\mathbb{A})} \phi\left(\begin{pmatrix} I_{n-1} & y \\ 0 & 1\end{pmatrix}\begin{pmatrix} uq & 0 \\ 0 & 1\end{pmatrix} p_0\right)\psi(y_{n-1})^{-1}\,dy\,du \\
&= \int_{U_\alpha(F)\backslash U_\alpha(\mathbb{A})} \int_{Y_n(F)\backslash Y_n(\mathbb{A})} \phi\left(\begin{pmatrix} u & y \\ 0 & 1\end{pmatrix}\begin{pmatrix} q & 0 \\ 0 & 1\end{pmatrix} p_0\right)\psi(y_{n-1})^{-1}\,dy\,du.
\end{aligned}$$

From our previously chosen Haar measures, we see that the double integral can be replaced by a single integral over $R(F)\backslash R(\mathbb{A})$:

$$\begin{aligned}
&= \int_{R(F)\backslash R(\mathbb{A})} \phi\left(r\begin{pmatrix} q & 0 \\ 0 & 1\end{pmatrix} p_0\right)\psi(r_{n-1,n})^{-1}\,dr \\
&= \int_{R(F)\backslash R(\mathbb{A})} \phi\left(r\begin{pmatrix} q & 0 \\ 0 & 1\end{pmatrix} p_0\right)\psi'(r)^{-1}\,dr.
\end{aligned}$$

We can now apply 1.4(ii) once again, with subgroups $R_0(\mathbb{A})$ and $R(F)$ of $R(\mathbb{A})$, where $R(F) \cap R_0(\mathbb{A}) = R_0(F)$:

$$\begin{aligned}
&= \int_{R(F)R_0(\mathbb{A})\backslash R(\mathbb{A})} \int_{R_0(F)\backslash R_0(\mathbb{A})} \phi\left(r_0\, r\begin{pmatrix} q & 0 \\ 0 & 1\end{pmatrix} p_0\right)\psi'(r_0\, r)^{-1}\,dr_0\,dr \\
&= \int_{R(F)R_0(\mathbb{A})\backslash R(\mathbb{A})} \int_{R_0(F)\backslash R_0(\mathbb{A})} \phi\left(r_0\, r\begin{pmatrix} q & 0 \\ 0 & 1\end{pmatrix} p_0\right)dr_0\,\psi'(r)^{-1}\,dr.
\end{aligned}$$

Since $\phi \in \mathcal{S}_n$, it satisfies (S3). As R_0 is a standard unipotent subgroup of GL_n, and the product $r\left(\begin{smallmatrix} q & 0 \\ 0 & 1\end{smallmatrix}\right)p_0$ is an element of $P_n(\mathbb{A})$ not depending on the integrand r_o, we must have that the inner integral in the last line must be equal to zero, and thus our initial integral of θ over $U_\alpha(F)\backslash U_\alpha(\mathbb{A})$ must be equal to zero. As α and q were arbitrary, θ must now also satisfy (S3). $\square$

The previous two lemmas enable us to prove the following last lemma, before we can prove Theorem 5.3. It is a simple rephrasing of Theorem 5.3 to our current setting with $\mathcal{S}_n$ as our function space.

Lemma 5.7. *Let $\phi \in \mathcal{S}_n$. Then ϕ has the Fourier expansion*

$$\phi(p) = \sum_{\delta \in N_n(F)\backslash P_n(F)} W_\phi(\delta p),$$

which converges absolutely for all $p \in P_n(\mathbb{A})$ and uniformly on compact subsets.

Proof. We will prove this by induction over n. For $n = 2$ the statement is equivalent to Lemma 5.4; indeed, for $n = 2$, w_ϕ is clearly equal to W_ϕ. Furthermore, in this case, the sum in Lemma 5.4 is indexed over $F^\times$, and the map

$$\begin{pmatrix} \gamma & v \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix}$$

on $P_2(F)$ with kernel $N_2(F)$ induces an isomorphism between $N_2(F)\backslash P_2(F)$ and the subgroup

$$\left\{ \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \,\middle|\, \gamma \in F^\times \right\}.$$

Due to Remark 5.5, $N_2(F)$ is contained in the stabilizer group of left translation of w_ϕ, and thus the summands in Lemma 5.7 do not depend on the choice of δ for each left coset.

For the induction step, let $n > 2$, and assume that the statement of Lemma 5.7 holds for $n-1$. Note that by Lemma 5.6, the function $\theta = \theta_{p_0}$ is an elemtent of $\mathcal{S}_{n-1}$, hence it satisfies the statement of the Lemma by assumption, i.e. it satisfies that

$$\theta(q) = \sum_{\varrho \in N_{n-1}(F)\backslash P_{n-1}(F)} W_\theta(\varrho q)$$

for all $q \in P_{n-1}(\mathbb{A})$, where

$$W_\theta(q) = \int_{N_{n-1}(F)\backslash N_{n-1}(\mathbb{A})} \theta(mq)\psi(m)^{-1}\, dm.$$

Recall that for $m \in N_n(\mathbb{A})$, $\psi(m)$ is the image of the sum of the first upper diagonal in m under $\psi : F\backslash\mathbb{A} \to \mathbb{C}^\times$.

Following the arguments in the last part of the proof of Lemma 5.6, where we proved that θ satisfies (S3), we see that for the partition $\alpha = (1, \dots, 1)$ of $n-1$, we have that $R = N_n$ and $U_\alpha = N_{n-1}$, so the integral over $N_n(F)\backslash N_n(\mathbb{A})$ splits up into an integral over $N_{n-1}(F)\backslash N_{n-1}(\mathbb{A})$ and an integral over $Y_n(F)\backslash Y_n(\mathbb{A})$ as before. This implies that for any $q \in P_{n-1}(\mathbb{A})$

$$\begin{aligned}
W_\theta(q) &= \int\limits_{N_{n-1}(F)\backslash N_{n-1}(\mathbb{A})} \theta(mq)\psi(m)^{-1}\, dm \\
&= \int\limits_{N_{n-1}(F)\backslash N_{n-1}(\mathbb{A})} \int\limits_{Y_n(F)\backslash Y_n(\mathbb{A})} \phi\left(\begin{pmatrix} I_{n-1} & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} mq & 0 \\ 0 & 1 \end{pmatrix} p_0 \right) \psi(y_{n-1})^{-1}\, dy\, \psi(m)^{-1}\, dm \\
&= \int\limits_{N_{n-1}(F)\backslash N_{n-1}(\mathbb{A})} \int\limits_{Y_n(F)\backslash Y_n(\mathbb{A})} \phi\left(\begin{pmatrix} m & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} p_0 \right) \psi\left(\begin{pmatrix} m & y \\ 0 & 1 \end{pmatrix} \right)^{-1} dy\, dm \\
&= \int\limits_{N_n(F)\backslash N_n(\mathbb{A})} \phi\left(m' \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} p_0 \right) \psi(m')^{-1}\, dm' \\
&= W_\phi\left(\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} p_0 \right).
\end{aligned}$$

Let $p \in P_n(\mathbb{A})$ be fixed, and given $\gamma \in \mathrm{GL}_{n-1}(F)$, let us denote by θ_γ the function θ_{p_0} where

$$p_0 = \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} p.$$

It now follows from the induction hypothesis, Lemma 5.4, and the previous calculation, that

$$
\begin{aligned}
\phi(p) &= \sum_{\gamma \in P_{n-1}(F) \backslash \mathrm{GL}_{n-1}(F)} w_\phi\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} p\right) \\
&= \sum_{\gamma \in P_{n-1}(F) \backslash \mathrm{GL}_{n-1}(F)} \theta_\gamma\left(I_{n-1}\right) \\
&= \sum_{\gamma \in P_{n-1}(F) \backslash \mathrm{GL}_{n-1}(F)} \sum_{\varrho \in N_{n-1}(F) \backslash P_{n-1}(F)} W_{\theta_\gamma}(\varrho) \\
&= \sum_{\gamma \in P_{n-1}(F) \backslash \mathrm{GL}_{n-1}(F)} \sum_{\varrho \in N_{n-1}(F) \backslash P_{n-1}(F)} W_\phi\left(\begin{pmatrix} \varrho & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} p\right) \\
&= \sum_{\gamma \in P_{n-1}(F) \backslash \mathrm{GL}_{n-1}(F)} \sum_{\varrho \in N_{n-1}(F) \backslash P_{n-1}(F)} W_\phi\left(\begin{pmatrix} \varrho\gamma & 0 \\ 0 & 1 \end{pmatrix} p\right) \\
&= \sum_{\gamma \in N_{n-1}(F) \backslash \mathrm{GL}_{n-1}(F)} W_\phi\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} p\right).
\end{aligned}
$$

All sums are absolutely convergent, since it follows from the induction hypothesis that the inner sums are, and from Lemma 5.4 that the outer sums are. Considering $\mathrm{GL}_{n-1}(F)$ as a subgroup of $P_n(F)$ through the identification $\gamma \mapsto \left(\begin{smallmatrix} \gamma & 0 \\ 0 & 1 \end{smallmatrix}\right)$, it is clear that

$$
\begin{aligned}
N_{n-1}(F) \backslash \mathrm{GL}_{n-1}(F) &= \left(N_n(F) \cap \mathrm{GL}_{n-1}(F)\right) \backslash \mathrm{GL}_{n-1}(F) \\
&\simeq N_n(F) \backslash N_n(F)\, \mathrm{GL}_{n-1}(F) \\
&= N_n(F) \backslash P_n(F).
\end{aligned}
$$

Thus, we can now finish the induction step, as this implies from the previous calculation that

$$
\phi(p) = \sum_{\delta \in N_n(F) \backslash P_n(F)} W_\phi(\delta p),
$$

and our manipulations preserve the absolute convergence. The convergence is also still uniform on compact subsets of $P_n(\mathbb{A})$. □

The proof of Theorem 5.3 is now very simple.

Proof of Theorem 5.3. Let $\varphi \in \mathcal{A}_0$ and $g \in \mathrm{GL}_n(\mathbb{A})$. The function

$$
\phi_g : P_n(\mathbb{A}) \to \mathbb{C} : p \mapsto \varphi(pg)
$$

is clearly an element of $\mathcal{S}_n$. For a compact set $C \subseteq \mathrm{GL}_n(\mathbb{A})$, the set $\{\phi_g \mid g \in C\}$ is a compact family of $\mathcal{S}_n$. By Lemma 5.7 we have that

$$
\phi_g(p) = \sum_{\delta \in N_n(F) \backslash P_n(F)} W_{\phi_g}(\delta p) = \sum_{\gamma \in N_{n-1}(F) \backslash \mathrm{GL}_{n-1}(F)} W_{\phi_g}\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} p\right),
$$

where the sum converges absolutely for all $p \in P_n(\mathbb{A})$ and uniformly on compact subsets. It is clear that $W_\varphi(pg) = W_{\phi_g}(p)$ for all $p \in P_n(\mathbb{A})$, so choosing $p = I_n$, we get that

$$\begin{aligned}\varphi(g) = \phi_g(I_n) &= \sum_{\gamma \in N_{n-1}(F)\backslash \operatorname{GL}_{n-1}(F)} W_{\phi_g}\left(\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}\right)\\ &= \sum_{\gamma \in N_{n-1}(F)\backslash \operatorname{GL}_{n-1}(F)} W_{\varphi}\left(\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix} g\right)\end{aligned}$$

and the convergence must be absolute and uniform for $g \in C$. □

5.2 Global Whittaker models

This section is dedicated to the study of global Whittaker models for representations of $\operatorname{GL}_n(\mathbb{A})$. We begin by defining this notion, which closely mirrors the notion of a Whittaker model for a local representation. Using the uniqueness of local Whittaker models, we then establish the uniqueness of global Whittaker models. Finally, we demonstrate that the Whittaker functions of cusp forms, introduced in the previous section, form Whittaker models for cuspidal automorphic representations.

This section is based on Section 3.5 in [Bum97]. While the treatment in [Bum97, §3.5] is restricted to the case $n = 2$, we generalize the notion of a global Whittaker model to $n \geq 2$, along with the proof of Theorems 3.5.4 and 3.5.5 in [Bum97]. The uniqueness of the global Whittaker model for general n was first proven by Shalika in [Sha74], using a different definition of a global Whittaker model than ours.

Let $G_v = \operatorname{GL}_n(F_v)$, let $\mathcal{H}_{G_v}$ denote the local Hecke algebra of G_v, and let $\mathcal{H} = \bigotimes_v' \mathcal{H}_{G_v}$ denote the global Hecke algebra.

Definition 5.8 (Global Whittaker model)**.** Let (π, V) be an irreducible admissible representation of $\operatorname{GL}_n(\mathbb{A})$ and let ψ be a non-trivial additive character of $F\backslash\mathbb{A}$. A *(global) Whittaker model* of (π, V) with respect to ψ is a space $\mathcal{W}$ that consists of smooth functions $W \in C^\infty(\operatorname{GL}_n(\mathbb{A}))$ that additionally satisfy

(i) $W(mg) = \psi(m)W(g)$ for all $g \in \operatorname{GL}_n(\mathbb{A})$ and $m \in N_n(\mathbb{A})$, and
(ii) $W|_{\operatorname{GL}_n(F_\infty)}$ is of moderate growth (cf. Definition 4.28),

such that $\mathcal{W}$ is stable under the action ρ of $\mathcal{H}$ and isomorphic to (π, V) as $\mathcal{H}$-modules.

Suppose (π, V) has a Whittaker model $\mathcal{W}$. By Theorem 4.21 $(\rho, \mathcal{W})$ must form a $\operatorname{GL}_n(\mathbb{A})$-representation isomorphic to (π, V), where ρ is the same right action as on automorphic forms (see Section 4.9). When we have fixed an $\mathcal{H}$-isomorphism (or equivalently a $\operatorname{GL}_n(\mathbb{A})$-isomorphism) $A : V \xrightarrow{\sim} \mathcal{W}$, we will often denote the image $A(\xi)$ of a vector $\xi \in V$ by W_ξ. In this notation, we have

$$W_{\pi(T)\xi} = \rho(T) W_\xi$$

for all $T \in \mathcal{H}$ and $\xi \in V$. We have that the $\mathcal{H}_f$-action ρ_f is equivalent to the $\operatorname{GL}_n(\mathbb{A}_f)$-action ρ_f, which acts by right translation, so

$$W_{\pi(g_f)\xi}(x) = W_\xi(x g_f)$$

for all $g_f \in \operatorname{GL}_n(\mathbb{A}_f)$, $x \in \operatorname{GL}_n(\mathbb{A})$, and $\xi \in V$.

Theorem 2.19 and Theorem 3.16 state that the Whittaker models of irreducible local representations are unique, if they exist. It should not be surprising that a global Whittaker model of $\pi = \otimes_v \pi_v$ with respect to $\psi = \otimes_v \psi_v$ is connected to the ψ_v-Whittaker models of each local representation π_v. Ultimately, it is this connection that allows us to show that the global Whittaker model is unique, if it exists.

Recall that for a local non-Archimedean representation π_v, if π_v is unramified (i.e. $V_v^{K_v} \neq 0$) and has a ψ_v-Whittaker model $\mathcal{W}_v$, and if ψ_v has level 0 (cf. Definition 2.1), then we have a unique element $W_v^\circ \in \mathcal{W}_v^{K_v}$ such that $W_v^\circ(K_v) = \{1\}$, which we called the spherical element of $\mathcal{W}_v$ (see Corollary 2.26).

Given an irreducible admissible $\mathrm{GL}_n(\mathbb{A})$-representation (π, V), it follows from the tensor product theorem (Theorem 4.16) that π_v is unramified at almost all non-Archimedean places. It follows from Proposition 4.8 that a non-trivial character ψ of $F\backslash\mathbb{A}$ has level 0 at almost all non-Archimedean places. Thus if π_v is ψ_v-generic at all places, then the spherical element W_v° in the ψ_v-Whittaker model $\mathcal{W}_v$ of π_v must exist at almost all non-Archimedean places.

We can now state the theorem of uniqueness of global Whittaker models.

Theorem 5.9 (Global uniqueness of Whittaker models)**.** *An irreducible admissible representation (π, V) of $\mathrm{GL}_n(\mathbb{A})$ has a Whittaker model $\mathcal{W}$ with respect to the non-trivial additive character ψ of $F\backslash\mathbb{A}$ if and only if each local representation (π_v, V_v) is ψ_v-generic. In this case, the Whittaker model $\mathcal{W}$ is unique, and if $\mathcal{W}_v$ is the Whittaker model of (π_v, V_v) with respect to ψ_v, then $\mathcal{W}$ consists of finite linear combinations of functions $W : \mathrm{GL}_n(\mathbb{A}) \to \mathbb{C}$ of the form*

$$W(g) = \prod_v W_v(g_v),$$

where $W_v \in \mathcal{W}_v$, and, at almost all non-Archimedean places, $W_v = W_v^\circ$ (the spherical element of $\mathcal{W}_v$).

We split the proof into two parts, one for each of the directions of the if-and-only-if-statement. Additionally, the second part is proven using the help of additional lemmas stated after part one of the proof of the theorem.

Proof, part one. (*existence locally $\Rightarrow$ existence globally.*) First assume that for an irreducible admissible representation (π, V) of $\mathrm{GL}_n(\mathbb{A})$ and a non-trivial character ψ of $F\backslash\mathbb{A}$, each local representation (π_v, V_v) is ψ_v-generic, i.e. at each place v there exists a local ψ_v-Whittaker model $\mathcal{W}_v$ of π_v (cf. Definition 3.19 for Archimedean places and Definition 2.20 for non-Archimedean places). Let $\xi_v^\circ \in V_v^{K_v}$ be the distinguished non-zero K_v-fixed element that we are given at almost all non-Archimedean places, such that (π, V) is isomorphic to the restricted tensor product of the (π_v, V_v)'s with respect to the distinguished elements ξ_v° (cf. Theorem 4.16). Without loss of generality, let $V = \bigotimes_v' V_v$.

At all places, we wish to fix an $\mathcal{H}_{G_v}$-isomorphism $V_v \xrightarrow{\sim} \mathcal{W}_v$. It follows from Theorem 2.19 and Theorem 3.16 that any such isomorphism is unique up to a constant scalar factor.

At non-Archimedean places where π_v is spherical and ψ_v is of level 0, we choose a G_v-isomorphism $V_v \xrightarrow{\sim} \mathcal{W}_v : \xi_v \mapsto W_{\xi_v}$ (which is also an $\mathcal{H}_{G_v}$-isomorphism) in the following way: As both $V_v^{K_v}$ and $\mathcal{W}_v^{K_v}$ are one-dimensional according to Proposition 2.17, and elements in $V_v^{K_v}$ are mapped to $\mathcal{W}_v^{K_v}$, there is exactly one isomorphism such that the image of ξ_v°, the distinguished non-zero K_v-fixed element, is the spherical element $W_v^\circ \in \mathcal{W}_v^{K_v}$, i.e. such that $W_{\xi_v^\circ} = W_v^\circ$, and this is the one we choose.

At the remaining places, we choose an $\mathcal{H}_{G_v}$-isomorphism $V_v \xrightarrow{\sim} \mathcal{W}_v : \xi_v \mapsto W_{\xi_v}$ arbitrarily.

For a pure tensor $\xi = \otimes_v \xi_v \in \bigotimes_v' V_v$, where $\xi_v = \xi_v^\circ$ for almost all v, define the function $W_\xi : \mathrm{GL}_n(\mathbb{A}) \to \mathbb{C}$ as the map

$$W_\xi(g) = \prod_v W_{\xi_v}(g_v), \tag{5.6}$$

which converges by construction, as almost all non-Archimedean factors are of the form $W_v^\circ(g_v)$ with $g_v \in K_v$, and thus are equal to one. Since the $\mathcal{H}_{G_v}$-isomorphisms at each place are intertwining operators, we naturally get for all pure tensors $T = \otimes_v T_v \in \mathcal{H}$ and $g \in \mathrm{GL}_n(\mathbb{A})$ that

$$W_{\pi(T)\xi}(g) = \prod_v W_{\pi_v(T_v)\xi_v}(g_v) = \prod_v \rho_v(T_v) W_{\xi_v}(g_v) = \rho(T) W_\xi(g).$$

The map $\xi \mapsto W_\xi$ of pure tensors in V extends naturally to an $\mathcal{H}$-homomorphism on all of V, and we denote the resulting image by $\mathcal{W}$, which thus consists of finite linear combinations of functions of the form (5.6).

It now remains to show that $\mathcal{W}$ is a global Whittaker model as defined in Definition 5.8. As $\mathcal{W}$ is the image of an $\mathcal{H}$-homomorphism, it is naturally a smooth $\mathcal{H}$-module (under the action ρ). Since V is irreducible, V is an irreducible $\mathcal{H}$-module, so this homomorphism must be injective, as $\mathcal{W} \neq \{0\}$, and thus V and $\mathcal{W}$ are isomorphic as $\mathcal{H}$-modules. From the definitions of the local Whittaker models, we also immediately get that every $W \in \mathcal{W}$ satisfies part (ii) in Definition 5.8, that is, $W(mg) = \psi(m)W(g)$ for all $g \in \mathrm{GL}_n(\mathbb{A})$ and $m \in N_n(\mathbb{A})$. Thus it remains to show every $W \in \mathcal{W}$ is smooth and of moderate growth at Archimedean places, but it is sufficient to show these properties for elements of the form (5.6).

Suppose $\xi \in V$ is a pure tensor. Let us first show that W_ξ is smooth (cf. Definition 4.22). To that end, let $y_f = (y_v)_{v \notin S_\infty} \in \mathrm{GL}_n(\mathbb{A}_f)$ be fixed and let us show that $W_\xi(\underline{\quad} \cdot y_f)$ is smooth as a function on $\mathrm{GL}_n(F_\infty)$. At all Archimedean places w, the function

$$\mathrm{GL}_n(F_\infty) \to \mathbb{C} : (x_v)_{v \in S_\infty} \mapsto W_{\xi_w}(x_w)$$

is smooth, since W_{ξ_v} is smooth as a function on G_w. The product of smooth functions is obviously smooth, hence

$$\begin{aligned} &\mathrm{GL}_n(F_\infty) \to \mathbb{C} \\ &x_\infty = (x_v)_{v \in S_\infty} \mapsto W_\xi(x_\infty y_f) = \prod_{w \in S_\infty} W_{\xi_w}(x_w) \cdot \prod_{v \notin S_\infty} W_{\xi_v}(y_v) \end{aligned} \tag{5.7}$$

is a smooth function on $\mathrm{GL}_n(F_\infty)$, since the product over finite places is constant.

Let now $x_\infty \in \mathrm{GL}_n(F_\infty)$ be fixed, and let $y_f \in \mathrm{GL}_n(\mathbb{A}_f)$ be an arbitrary element, and let us show that $W_\xi(x_\infty \cdot \underline{\quad})$ is locally constant at y_f. Let $S \supseteq S_\infty$ be the finite set of places such that $W_{\xi_v} = W_v^\circ$ for all $v \notin S$. At non-Archimedean places, the representation $(\rho_v, \mathcal{W}_v)$ of G_v is smooth, so at all places $v \in S \setminus S_\infty$ there exists a compact open subgroup $L_v \subseteq G_v$ such that $W_{\xi_v}(y_v l_v) = W_{\xi_v}(y_v)$ for all $l_v \in L_v$. The set

$$L = \prod_{v \in S \setminus S_\infty} L_v \times \prod_{v \notin S} K_v.$$

is a compact open subgroup of $\mathrm{GL}_n(\mathbb{A}_f)$, and $y_f L$ is an open neighbourhood of y_f. For $v \notin S$, $W_{\xi_v} = W_v^\circ$ is K_v-fixed. Thus for all $u_f \in y_f L$, there exist elements $l_v \in L_v$ for $v \in S \setminus S_\infty$ and elements $k_v \in K_v$ for $v \notin S$ such that

$$u_f = ((y_v l_v)_{v \in S \setminus S_\infty}, (y_v k_v)_{v \notin S}).$$

Now W_ξ agree on $x_\infty y_f$ and $x_\infty u_f$, as

$$\begin{aligned} W_\xi(x_\infty u_f) &= \prod_{v\in S_\infty} W_{\xi_v}(x_v)\cdot \prod_{v\in S\setminus S_\infty} W_{\xi_v}(y_v l_v)\cdot \prod_{v\notin S} W_v^\circ(y_v k_v) \\ &= \prod_{v\in S_\infty} W_{\xi_v}(x_v)\cdot \prod_{v\in S\setminus S_\infty} W_{\xi_v}(y_v)\cdot \prod_{v\notin S} W_v^\circ(y_v) \\ &= W_\xi(x_\infty y_f). \end{aligned}$$

So $W_\xi(x_\infty \cdot \_\_)$ is locally constant at y_f, and thus locally constant on all of $\mathrm{GL}_n(\mathbb{A}_f)$.

Let us now show that $W_\xi|_{\mathrm{GL}_n(F_\infty)}$ is of moderate growth. At Archimedean places, by Definition 3.15, W_{ξ_v} is of Archimedean moderate growth (cf. Definition 3.14), hence there exist a positive constant $C_v > 0$ and $N_v \in \mathbb{N}$ such that

$$\left|W_{\xi_v}(g_v)\right|_{\mathbb{C}} \le C_v \|g_v\|_\infty^{N_v}$$

for all $g_v \in G_v$. Let $N = \max_{v\in S_\infty}\{N_v\}$ and let $C = \prod_{v\in S_\infty} C_v \cdot \prod_{v\in S\setminus S_\infty} |W_{\xi_v}(1)|_{\mathbb{C}}$. Then, for all $g \in \mathrm{GL}_n(F_\infty) \subseteq \mathrm{GL}_n(\mathbb{A})$,

$$\begin{aligned} \left|W_\xi(g)\right|_{\mathbb{C}} &= \prod_{v\in S_\infty} \left|W_{\xi_v}(g_v)\right|_{\mathbb{C}} \cdot \prod_{v\in S\setminus S_\infty} \left|W_{\xi_v}(1)\right|_{\mathbb{C}} \cdot \prod_{v\notin S} \big|\underbrace{W_v^\circ(1)}_{=1}\big|_{\mathbb{C}} \\ &\le \prod_{v\in S_\infty} C_v \|g_v\|_\infty^{N_v} \cdot \prod_{v\in S\setminus S_\infty} \left|W_{\xi_v}(1)\right|_{\mathbb{C}} \\ &\le C\cdot \left(\prod_{v\in S_\infty} \|g_v\|_\infty\right)^N \\ &= C\cdot \|g\|^N, \end{aligned}$$

since all factors of the height $\|g\|$ of g at non-Archimedean places are equal to one. □

We now consider the opposite direction of Theorem 5.9. Let (π, V) be an irreducible admissible $\mathrm{GL}_n(\mathbb{A})$-representation, and suppose (π, V) has a Whittaker model $\mathcal{W}$ with respect to ψ. Without loss of generality, we again assume $(\pi, V) = (\otimes_v \pi_v, \bigotimes_v' V_v)$ with respect to distinguished non-zero K_v-fixed elements $\xi_v^\circ \in V_v^{K_v}$ at almost all non-Archimedean places.

By the definition of a global Whittaker model, there exists an $\mathcal{H}$-isomorphism $A : V \xrightarrow{\sim} \mathcal{W}$, and we denote by $W_\xi = A(\xi)$ the value of an element $\xi \in V$ under this isomorphism.

Lemma 5.10. *There exists a pure tensor $\xi^\circ \in V$ such that $W_{\xi^\circ}(1) = 1$.*

Proof. First, we wish to show that there exists an element $\xi \in V$ such that $W_\xi(1) \neq 0$. Let $\xi_1 \in V$ be a non-zero element. Then $W_{\xi_1} \neq 0$, and there must exist an element $g \in \mathrm{GL}_n(\mathbb{A})$ such that $W_{\xi_1}(g) \neq 0$. Let $g_\infty \in \mathrm{GL}_n(F_\infty) \subseteq \mathrm{GL}_n(\mathbb{A})$ and $g_f \in \mathrm{GL}_n(\mathbb{A}_f) \subseteq \mathrm{GL}_n(\mathbb{A}))$ such that $g = g_\infty g_f$, and let $\xi_2 = \pi(g_f)\xi_1$. Then

$$W_{\xi_2}(g_\infty) = W_{\pi(g_f)\xi_1}(g_\infty) = W_{\xi_1}(g_\infty g_f) = W_{\xi_1}(g) \neq 0.$$

Since (π_v, V_v) is irreducible at all Archimedean places, Theorem 3.5 implies V_v is $\mathcal{Z}(\mathfrak{g}_v)$-finite, hence (π, V) must be $\mathcal{Z}(\mathfrak{g}_\infty)$-finite, which implies that $\mathcal{W}$ is $\mathcal{Z}(\mathfrak{g}_\infty)$-finite. As $\mathcal{W}$ in particular is a $(\mathfrak{g}_\infty, K_\infty)$-module, it is also K_∞-finite.

Since W_{ξ_2} as a function on $\mathrm{GL}_n(F_\infty) \subseteq \mathrm{GL}_n(\mathbb{A})$ is not the zero function, and since K_∞ intersects every connected component of $\mathrm{GL}_n(F_\infty)$, there exists some $k_\infty \in K_\infty$ such that

$g_\infty k_\infty$ is an element of the connected component G^0_∞ of $\mathrm{GL}_n(F_\infty)$ at the identity element $1 \in \mathrm{GL}_n(F_\infty)$. Let $\xi_3 = \pi(k_\infty^{-1})\xi_2$. Then

$$W_{\xi_3}(g_\infty k_\infty) = W_{\pi(k_\infty)\xi_3}(g_\infty) = W_{\xi_2}(g_\infty) \neq 0.$$

Thus the function $W_{\xi_3}|_{G^0_\infty}$, restricted to the connected component $G^0_\infty \subseteq \mathrm{GL}_n(\mathbb{A})$ of $\mathrm{GL}_n(F_\infty)$ at the identity element, is a smooth, $K_\infty \cap G^0_\infty$-finite, and $\mathcal{Z}(\mathfrak{g}_\infty)$-finite function on G^0_∞ that is not the zero function. It follows from a general result from representation theory of connected Lie groups (see e.g. [Bor72, Proposition 3.14]) that $W_{\xi_3}|_{G^0_\infty}$ is analytic, and hence represented by a multi-variable Taylor series around the identity element in G^0_∞. But as $W_{\xi_3}|_{G^0_\infty}$ is not the zero function, some coefficient in this Taylor series must be non-zero, i.e. there exists $u_\infty \in \mathcal{U}(\mathfrak{g}_\infty)$ such that $\rho(u_\infty)W_{\xi_3}(1) \neq 0$. Let $\xi_4 = \pi(u_\infty)\xi_3$. Then

$$W_{\xi_4}(1) = W_{\pi(u_\infty)\xi_3}(1) = \rho(u_\infty)W_{\xi_3}(1) \neq 0,$$

so ξ_4 is the element that we wished to find. Since A is a linear map, we now immediately also have a pure tensor $\xi^\circ \in V$ such that $W_{\xi^\circ}(1) = 1$. □

The restricted tensor product $\bigotimes'_v V_v$ with respect to the components of $\xi^\circ = \otimes_v \xi^\circ_v$ is naturally isomorphic to V, as it involves changing a finite number of distinguished non-zero vectors, so without loss of generality, let $V = \bigotimes'_v V_v$ with respect to the ξ°_v's at all places.

At each place v, denote by i_v the inclusion map

$$i_v : V_v \hookrightarrow V : \xi_v \mapsto \xi_v \otimes \left(\underset{w \neq v}{\otimes} \xi^\circ_w \right),$$

where the last tensor product ranges over places w of F that are not equal to v. For a given local vector $\xi_v \in V_v$, define a function $W_{v,\xi_v} : \mathrm{GL}_n(F_v) \to \mathbb{C}$ as the map

$$W_{v,\xi_v}(g_v) = W_{i_v(\xi_v)}(g_v),$$

where on the right-hand side, the element $g_v \in \mathrm{GL}_n(F_v)$ is as usual viewed as an element of $\mathrm{GL}_n(\mathbb{A})$ by setting the entries at all other places equal to the identity matrix. Note that $W_{v,\xi^\circ_v}(1) = 1$.

Lemma 5.11. *At any place v, the map $A_v : \xi_v \mapsto W_{v,\xi_v}$ is an $\mathcal{H}_{G_v}$-isomorphism and the space $\mathcal{W}_v = \{W_{v,\xi_v} \mid \xi_v \in V_v\}$ forms a ψ_v-Whittaker model for π_v.*

Proof. Since A is an $\mathcal{H}$-isomorphism, it is clear that A_v must be a $\mathbb{C}$-linear map. Thus, to show it is an $\mathcal{H}_{G_v}$-homomorphism, it suffices to show it intertwines with π_v and ρ_v. Let $\xi_v \in V_v$ and let $T_v \in \mathcal{H}_{G_v}$. Keeping the notation of Section 4.6, at all places $w \neq v$, let $e_w \in \mathcal{E}_{G_w}$ be an idempotent such that $\pi_w(e_w)\xi^\circ_w = \xi^\circ_w$, and note that at almost all non-Archimedean places, we can pick $e_w = e^\circ_w = e_{K_w}$, which is the distinguished elementary idempotent in $\mathcal{H}_{G_w}$ with which the restricted tensor product $\mathcal{H} = \bigotimes'_w \mathcal{H}_{G_w}$ is with respect to. Let $T = T_v \otimes (\otimes_{w \neq v} e_w)$. Then $i_v(\pi_v(T_v)\xi_v) = \pi(T)i_v(\xi_v)$, and by the properties of $\mathcal{E}$ as a directed set, there exists $e_0 \in \mathcal{E}$ such that

$$T * e_0 = e_0 * T = T \qquad \text{and} \qquad \pi(e_0)i_v(\xi_v) = i_v(\xi_v),$$

from which it must follow from the second equation, that $\rho(e_0)W_{i_v(\xi_v)} = W_{i_v(\xi_v)}$. It is clear that ρ_v acts on $W_{i_v(\xi_v)}$ through only its v'th entry, and from the definition of ρ and the isomorphisms $\mathcal{H}_{\mathrm{GL}_n(F_\infty)} \cong \bigotimes_{v\in S_\infty} \mathcal{H}_{G_v}$ in Proposition 4.20 and $\mathcal{H}_{\mathrm{GL}_n(\mathbb{A}_f)} \cong \bigotimes'_{v\notin S_\infty} \mathcal{H}_{G_v}$ in Proposition 4.19, we get that the following manipulations are allowed:

$$\rho_v(T_v)W_{v,\xi_v} = \rho_v(T_v)W_{i_v(\xi)} = \rho_v(T_v)\rho(e_0)W_{i_v(\xi_v)} = \rho(T * e_0)W_{i_v(\xi_v)} = \rho(T)W_{i_v(\xi_v)}.$$

Thus for any $g_v \in G_v$,

$$\begin{aligned} W_{v,\pi_v(T_v)\xi_v}(g_v) &= W_{i_v(\pi_v(T_v)\xi_v)}(g_v) \\ &= W_{\pi(T)i_v(\xi_v)}(g_v) \\ &= (\rho(T)W_{i_v(\xi_v)})(g_v) \\ &= (\rho_v(T_v)W_{v,\xi_v})(g_v), \end{aligned}$$

and A_v must be an $\mathcal{H}_{G_v}$-homomorphism. Since (π, V) is irreducible, (π_v, V_v) is irreducible, and thus A_v must be either the zero map or an isomorphism. But since $W_{v,\xi_v^\circ}(1) = 1$, there is at least one non-zero element in the image of A_v, and thus it must follow that A_v is an $\mathcal{H}_{G_v}$-isomorphism.

It now remains to show that $\mathcal{W}_v = \{W_{v,\xi_v} \mid \xi_v \in V_v\}$ forms a ψ_v-Whittaker model for π_v. If $m_v \in N_n(F_v) \subseteq N_n(\mathbb{A})$ and $g_v \in G_v \subseteq \mathrm{GL}_n(\mathbb{A})$, then the following is an easy consequence of $\mathcal{W}$ being a global Whittaker model with respect to ψ:

$$\begin{aligned} W_{v,\xi_v}(m_v g_v) &= W_{i_v(\xi_v)}(m_v g_v) \\ &= \psi(m_v)W_{i_v(\xi_v)}(g_v) \\ &= \left(\psi_v(m_v) \cdot \prod_{w\neq v} \psi_w(1)\right) W_{i_v(\xi_v)}(g_v) \\ &= \psi_v(m_v)W_{v,\xi_v}(g_v). \end{aligned} \tag{5.8}$$

Now, suppose v is a non-Archimedean place. By Definition 2.20, we need to show that $W_{v,\xi_v} \in \mathrm{Ind}_{N_n(F_v)}^{G_v}(\psi_v)$ for all ξ_v. Let $\xi_v \in V_v$. Since we already showed in (5.8) that $W_{v,\xi_v}(m_v g_v) = \psi_v(m_v)W_{v,\xi_v}(g_v)$ for all $m_v \in N_n(F_v)$ and $g_v \in G_v$, we are only left with showing that there exists a compact open subgroup of G_v, which W_{v,ξ_v} is right invariant under. But since (π_v, V_v) is a smooth representation of G_v, and A_v is also a G_v-homomorphism (Theorem 2.7), $(\rho_v, \mathcal{W}_v)$ is also a smooth representation of G_v, where ρ_v is right translation, and thus by the definition of a smooth representation, there exists a compact open subgroup L_v of G_v such that $W_{v,\xi_v} \in \mathcal{W}_v^{L_v}$, i.e.

$$W_{v,\xi_v}(g_v l_v) = \rho_v(l_v)W_{v,\xi_v}(g_v) = W_{v,\xi_v}(g_v)$$

for all $l \in L_v$ and $g_v \in G_v$. So, $\mathcal{W}_v$ forms a ψ_v-Whittaker model for π_v, in the case v is non-Archimedean.

Suppose now that v is Archimedean. By Definition 3.19, in order to show $\mathcal{W}_v$ forms a ψ_v-Whittaker model, we need to show $\mathcal{W}_v$ consists of ψ_v-Whittaker functions (cf. Definition 3.15). Thus, we need to show that $W_{v,\xi_v} \in C^\infty(G_v)$ and that W_{v,ξ_v} is of Archimedean moderate growth (cf. Definition 3.14) for all $\xi_v \in V_v$, as we have already in (5.8) shown Definition 3.15(i) is satisfied. Let $\xi_v \in V_v$. It is clear from the fact that $W_{i_v(\xi_v)}$ is smooth as a function on $\mathrm{GL}_n(\mathbb{A})$, that $W_{i_v(\xi_v)}$ is smooth as a function on G_v when fixing all except the v'th entry, hence W_{v,ξ_v} must be smooth. It is clear to see that (adelic) moderate growth of $W_{i_v(\xi_v)}\big|_{\mathrm{GL}_n(F_\infty)}$ implies moderate growth of $W_{i_v(\xi_v)}\big|_{G_v}$, hence W_{v,ξ_v} is of Archimedean moderate growth. So, $\mathcal{W}_v$ also forms a ψ_v-Whittaker model for π_v in the case that v is Archimedean. $\square$

For a finite set of places S, we denote by V_S the finite tensor product $\bigotimes_{v\in S} V_v$. We view V_S as a subspace of V, being the one containing finite linear combinations of pure tensors $\xi = \otimes_v \xi_v$, where $\xi_v = \xi_v^\circ$ for all $v \notin S$.

Lemma 5.12. *For any finite set of places S, if $h \in \mathrm{GL}_n(F_S) \subseteq \mathrm{GL}_n(\mathbb{A})$ and if $\xi = \otimes_v \xi_v$ is a pure tensor in V_S, i.e. if $\xi_v = \xi_v^\circ$ for all $v \notin S$, then*

$$W_\xi(h) = \prod_{v\in S} W_{v,\xi_v}(h_v). \tag{5.9}$$

Proof. We will prove this using induction on the cardinality of S. If $S = \emptyset$, the only pure tensor in V_S we need to consider is ξ°, and $\mathrm{GL}_n(\mathbb{A}_S) = \{1\} \subseteq \mathrm{GL}_n(\mathbb{A})$, hence the left-hand side of (5.9) is equal to $W_{\xi^\circ}(1) = 1$, while the right-hand side is a product over an empty index set, and therefore also equal to 1. So the lemma holds for the base case.

Suppose now that Lemma 5.12 holds for a given arbitrary finite set of places S. We wish to show that for any given place $w \notin S$, Lemma 5.12 still holds for the finite set of places $S \cup \{w\}$. Let $\xi = \otimes_v \xi_v \in V_{S\cup\{w\}}$ be a pure tensor, and let $h \in \mathrm{GL}_n(F_{S\cup\{w\}}) \subseteq \mathrm{GL}_n(\mathbb{A})$. For each $\eta_w \in V_w$, define a function $\Omega_{\eta_w} : \mathrm{GL}_n(F_w) \to \mathbb{C}$ as the map

$$\Omega_{\eta_w}(g_w) = W_\eta\left(g_w, (h_v)_{v\neq w}\right),$$

where $\eta = \eta_w \otimes (\otimes_{v\neq w} \xi_v)$. We wish to show that the space $\mathcal{V}_w = \{\Omega_{\eta_w} \mid \eta_w \in V_w\}$ is either zero or forms a ψ_w-Whittaker model for π_w.

Let $B_w : \eta_w \mapsto \Omega_{\eta_w}$. By following the same arguments as in the proof of Lemma 5.11, where we replace v with w, $i_w(\xi_w)$ with η, and think of G_w as a subset of $\mathrm{GL}_n(\mathbb{A})$ by inserting h_v (instead of I_n) in the remaining places, it follows that B_w is an $\mathcal{H}_{G_w}$-homomorphism. Thus, by the irreducibility of π_w, $\mathcal{V}_w$ is either zero, or $(\rho_w, \mathcal{V}_w)$ is isomorphic to (π_w, V_w) as $\mathcal{H}_{G_w}$-modules. Suppose the latter is the case. Then we can continue following the arguments in the proof of Lemma 5.11 with the same replacements, and reach the conclusion that $\mathcal{V}_w$ is a ψ_w-Whittaker model. By uniqueness of the local Whittaker model (see Theorem 2.19 and Theorem 3.16), $\mathcal{V}_w$ must be equal to the ψ_w-Whittaker model $\mathcal{W}_w = \{W_{w,\eta_w} \mid \eta_w \in V_w\}$ in Lemma 5.11, and the isomorphism $B_w : \eta_w \mapsto \Omega_{\eta_w}$ is a scalar multiple of the isomorphism $A_w : \eta_w \mapsto W_{w,\eta_w}$. So, regardless of whether B_w is the zero map, there exists a constant $c \in \mathbb{C}$ such that $\Omega_{\eta_w} = cW_{w,\eta_w}$ for all $\eta_w \in V_w$.

For $\eta_w = \xi_w^\circ$, we get from the induction hypothesis with $(1, (h_v)_{v\neq w}) \in \mathrm{GL}_n(F_S) \subseteq \mathrm{GL}_n(\mathbb{A})$ and $\xi_w^\circ \otimes (\otimes_{v\neq w} \xi_v) \in V_S$ that

$$\prod_{v\in S} W_{v,\xi_v}(h_v) = W_{\xi_w^\circ \otimes(\otimes_{v\neq w}\xi_v)}(1, (h_v)_{v\neq w}) = \Omega_{\xi_w^\circ}(1) = cW_{w,\xi_w^\circ}(1) = c.$$

Now, for $\eta_w = \xi_w$, we have

$$W_\xi(h) = \Omega_{\xi_w}(h_w) = cW_{w,\xi_w}(h_w) = \left(\prod_{v\in S} W_{v,\xi_v}(h_v)\right) W_{w,\xi_w}(h_w) = \prod_{v\in S\cup\{w\}} W_{v,\xi_v}(h_v),$$

which concludes the induction step. □

Armed with the previous three lemmas, we are now ready to prove the remaining parts of Theorem 5.9.

Proof of Theorem 5.9, part two. (*existence globally ⇒ existence locally, and global uniqueness.*) Suppose (π, V) has a Whittaker model $\mathcal{W}$ with respect to ψ. We wish to show that all local representations of π are ψ_v-generic, and that $\mathcal{W}$ is equal to the space constructed in the proof of the first part of the theorem. Keeping with the notation from the previous lemmas, we get by Lemma 5.10 that there exists a pure tensor $\xi^\circ \in V$ such that $W_{\xi^\circ}(1) = 1$. Changing the restricted tensor product $V = \bigotimes_v' V_v$ to be with respect to the components of ξ°, it follows from Lemma 5.11 that at all places, the space $\mathcal{W}_v = \{W_{v,\xi_v} \mid \xi_v \in V_v\}$ forms a ψ_v-Whittaker model for π_v, and thus each local representation must be ψ_v-generic.

It remains to prove that

$$W_\xi(g) = \prod_v W_{v,\xi_v}(g_v)$$

for all $g \in \mathrm{GL}_n(\mathbb{A})$ and all pure tensors $\xi = \otimes_v \xi_v \in V$.

Let such g and ξ be given. Let $S \supseteq S_\infty$ be a finite set of places such that if $v \notin S$, then we have that $g_v \in K_v$, that $\xi_v = \xi_v^\circ$, and that $\xi_v^\circ \in V_v^{K_v}$. Thus for $v \notin S$,

$$W_{v,\xi_v}(g_v) = \rho_v(g_v) W_{v,\xi_v^\circ}(1) = W_{v,\pi_v(g_v)\xi_v^\circ}(1) = W_{v,\xi_v^\circ}(1) = 1. \tag{5.10}$$

Let us write g as the product $g = hg^\circ$, where $h \in \mathrm{GL}_n(F_S) \subseteq \mathrm{GL}_n(\mathbb{A})$ is the element such that $h_v = g_v$ for all $v \in S$ and $h_v = 1$ for all $v \notin S$, and thus $g^\circ \in K_f$. For $v \notin S$, ξ_v is K_v-fixed and $g_v^\circ \in K_v$, and for $v \in S$, we have $g_v^\circ = 1$, so it must hold that $\pi(g^\circ)\xi = \xi$. Since $g^\circ \in \mathrm{GL}_n(\mathbb{A}_f)$, right translation with g° on $\mathcal{W}$ intertwines with the $\pi(g^\circ)$-action on V, hence

$$W_\xi(g) = W_\xi(hg^\circ) = W_{\pi(g^\circ)\xi}(h) = W_\xi(h).$$

Since $\xi \in V_S$ and $h \in \mathrm{GL}_n(\mathbb{A}_S)$, we can apply Lemma 5.12 to the right-hand side, and thus

$$W_\xi(g) = \prod_{v \in S} W_{v,\xi_v}(h_v).$$

As $h_v = g_v$ for all $v \in S$, and we have from (5.10) that $W_{v,\xi_v}(g_v) = 1$ for all $v \notin S$, it now follows that

$$W_\xi(g) = \prod_v W_{v,\xi_v}(g_v),$$

which is what we wanted to show. □

We have now shown that if an irreducible admissible $\mathrm{GL}_n(\mathbb{A})$-representation has a global Whittaker model with respect to ψ, then it is unique, but it is not necessarily the case that all such representations have a global Whittaker model. One of the important qualities of cuspidal automorphic representations is that they all have a global Whittaker model, and their models consist of the Whittaker functions that we explored in Section 5.1.

Theorem 5.13. *Let (π, V) be a cuspidal automorphic representation and ψ a non-trivial character of $F\backslash\mathbb{A}$. Then the space*

$$\mathcal{W} = \{W_\varphi \mid \varphi \in V\}$$

consisting of ψ-Whittaker functions (cf. Definition 5.1) forms a global Whittaker model of π with respect to ψ.

Proof. We need to show that W_φ is smooth and satisfies (i) and (ii) in Definition 5.8 for all $\varphi \in V$, and show that $\mathcal{W}$ is stable under the action ρ of the global Hecke algebra $\mathcal{H}$ and isomorphic to (π, V) as $\mathcal{H}$-modules. We have already seen in Section 5.1 that W_φ is smooth, and it follows from Proposition 5.2 that W_φ satisfies Definition 5.8(i). It follows from the Fourier expansion of cusp forms (Theorem 5.3) that the $\mathbb{C}$-linear map $\varphi \mapsto W_\varphi$ on V is bijective, and consequently, if it intertwines with π and ρ, we immediately have that $(\rho, \mathcal{W})$ must be an $\mathcal{H}$-module that is isomorphic to (π, V). So, it remains to show that $W_\varphi|_{\mathrm{GL}_n(F_\infty)}$ is of moderate growth, and that $\varphi \mapsto W_\varphi$ is an intertwining operator.

Let $\varphi \in V$ be a cusp form, and let us use the moderate growth of φ to show W_φ is of moderate growth on $\mathrm{GL}_n(F_\infty)$. By Proposition 4.27, there exists a positive constant $c > 0$ such that $\|gh\| \leq c\|g\|\|h\|$ for all $g, h \in \mathrm{GL}_n(\mathbb{A})$. Since φ is of moderate growth, there exist $C > 0$ and $N \in \mathbb{N}$ such that $|\varphi(g)|_{\mathbb{C}} \leq C\|g\|^N$ for all $g \in \mathrm{GL}_n(\mathbb{A})$. Since $N_n(F)\backslash N_n(\mathbb{A})$ is compact by Proposition 4.30, there exists a compact subset $\Omega \subseteq N_n(\mathbb{A})$ such that $N_n(\mathbb{A}) = N_n(A)\Omega$. As $\|\cdot\|$ is bounded on compact sets, there exists $C_\Omega > 0$ such that $\|\omega\| \leq C_\Omega$ for all $\omega \in \Omega$. Thus, for all $g \in \mathrm{GL}_n(F_\infty)$ and $m \in N_n(\mathbb{A})$ there exist $\mu \in N_n(F)$ and $\omega \in \Omega$ such that $m = \mu\omega$ and by the left-$\mathrm{GL}_n(F)$-invariance of φ,

$$\big|\varphi(mg)\big|_{\mathbb{C}} = \big|\varphi(\mu\omega g)\big|_{\mathbb{C}} = \big|\varphi(\omega g)\big|_{\mathbb{C}} \leq C\|\omega g\|^N \leq Cc\|\omega\|^N\|g\|^N \leq CcC_\Omega^N\|g\|^N.$$

Let $C' = CcC_\Omega^N$, and note that it does not depend on g and m. Now for all $g \in \mathrm{GL}_n(F_\infty)$,

$$\begin{aligned}
\big|W_\varphi(g)\big|_{\mathbb{C}} &= \left|\int_{N_n(F)\backslash N_n(\mathbb{A})} \varphi(mg)\psi(m)^{-1}\,dm\right|_{\mathbb{C}} \\
&\leq \int_{N_n(F)\backslash N_n(\mathbb{A})} \big|\varphi(mg)\big|_{\mathbb{C}}\,dm \\
&\leq C'\|g\|^N \cdot \mathrm{vol}(N_n(F)\backslash N_n(\mathbb{A})) \\
&= C'\|g\|^N
\end{aligned}$$

where $|\psi(m)^{-1}|_{\mathbb{C}} = 1$ as ψ is unitary since $F\backslash\mathbb{A}$ is compact. Thus $W_\varphi|_{\mathrm{GL}_n(F_\infty)}$ is of moderate growth.

Let us now show that $\varphi \mapsto W_\varphi$ is an intertwining operator. Let $T \in \mathcal{H}$, and note that since (π, V) is a cuspidal automorphic representation, $\pi = \rho$. Then it is a simple consequence of Fubini's theorem, that

$$\begin{aligned}
(\rho(T)W_\varphi)(g) &= \int_{\mathrm{GL}_n(\mathbb{A})} W_\varphi(gx)\,dT(x) \\
&= \int_{\mathrm{GL}_n(\mathbb{A})}\int_{N_n(F)\backslash N_n(\mathbb{A})} \varphi(mgx)\psi(m)^{-1}\,dm\,dT(x) \\
&= \int_{N_n(F)\backslash N_n(\mathbb{A})}\int_{\mathrm{GL}_n(\mathbb{A})} \varphi(mgx)\,dT(x)\,\psi(m)^{-1}\,dm \\
&= \int_{N_n(F)\backslash N_n(\mathbb{A})} (\pi(T)\varphi)(mg)\,\psi(m)^{-1}\,dm \\
&= W_{\pi(T)\varphi}(g),
\end{aligned}$$

and thus $\varphi \mapsto W_\varphi$ is an $\mathcal{H}$-isomorphism. $\square$

Chapter 6

Strong multiplicity one

This chapter presents the proof of the strong multiplicity one theorem for cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A})$, bringing together the theory presented in earlier chapters. We begin by stating the theorem:

Theorem 6.1 (Strong multiplicity one)**.** *Let (π, V) and (π', V') be two cuspidal automorphic representations of $\mathrm{GL}_n(A)$. If their local components satisfy $\pi_v \cong \pi'_v$ at almost all places and at all Archimedean places, then $(\pi, V) = (\pi', V')$.*

This theorem was first proven for $n = 2$ independently by Miyake in [Miy71] and Casselman in [Cas73], both including the assumption at the Archimedean places. Both authors relied on the foundational contributions of Jacquet and Langlands in [JL70], where the multiplicity one theorem for $n = 2$ was established. Later, in [PS79], Piatetski-Shapiro extended the result to $n \geq 2$, and two years afterwards, in [JS81], Jacquet and Shalika proved the statement for unitary cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A})$, without the assumption at the Archimedean places.

The proof presented here follows the approach of [PS79] and the proof of Theorem 3.4.6 in [Bum97].

Let P_n denote the mirabolic subgroup of GL_n and let Z_n denote the centre of GL_n, cf. Section 1.1. We first need the following lemma.

Lemma 6.2. *$\mathrm{GL}_n(F)Z_n(\mathbb{A})P_n(\mathbb{A})$ is dense in $\mathrm{GL}_n(\mathbb{A})$.*

Proof. Let Q denote the $(n-1, 1)$ standard parabolic subgroup $P_{(n-1,1)} = P_n Z_n$ of $G = \mathrm{GL}_n$. We want to show that $G(F)Q(\mathbb{A})$ is dense in $G(\mathbb{A})$. Let U be a given open subset of $G(\mathbb{A})$. Suppose without loss of generality that U is of the form

$$\prod_{v \in S} U_v \times \prod_{v \notin S} K_v$$

for some finite subset of places S containing all Archimedean places, and open subsets $U_v \subseteq F_v$ for $v \in S$. We wish to show that the intersection

$$\left(\prod_{v \in S} U_v \times \prod_{v \notin S} K_v \right) \cap G(F)Q(\mathbb{A}) \tag{6.1}$$

is non-empty. By weak approximation of $G(\mathbb{A})$ (see Proposition 4.11), we have that $G(F) \hookrightarrow G(F_S)$ is dense, and thus there exists $\alpha \in G(F)$ such that

$$(\alpha)_{v \in S} = (\alpha, \ldots, \alpha) \in \prod_{v \in S} U_v.$$

Since Q is a standard parabolic subgroup, $G(F_v)$ has Iwasawa decomposition with respect to $Q(F_v)$ (see Proposition 2.10) at all non-Archimedean places, i.e. $G(F_v) = K_v Q(F_v)$, and thus for all $v \notin S$ there exists a $k_v \in K_v$ and a $q_v \in Q(F_v)$ such that $\alpha = k_v q_v$. Note that $q_v \in K_v$ for almost all $v \notin S$, as this is the case for α, hence the tuple

$$q = \big((1)_{v\in S}, (q_v^{-1})_{v\notin S}\big)$$

is a well-defined element of $Q(\mathbb{A})$, and thus $\alpha q \in G(F)Q(\mathbb{A})$. But by construction,

$$\alpha q = \big((\alpha)_{v\in S}, (\alpha q_v^{-1})_{v\notin S}\big) = ((\alpha)_{v\in S}, (k_v)_{v\notin S}) \in \prod_{v\in S} U_v \times \prod_{v\notin S} K_v,$$

so the intersection in (6.1) is non-empty. □

Proof of Theorem 6.1. Let (π, V) and (π', V') be two cuspidal automorphic representations, and suppose there exists a finite set of non-Archimedean places S, such that the local representations $\pi_v \cong \pi_v'$ are isomorphic for all $v \notin S$. In particular, $\pi_v \cong \pi_v'$ at all Archimedean places. Both V and V' are subspaces of $\mathcal{A}_0$, so to show that $V = V'$, it follows from the irreducibility of the representations, that it suffices to show that $V \cap V' \neq \{0\}$, i.e., to show the existence of a non-zero cusp form $\varphi \in V \cap V'$.

Let ψ be a non-trivial character of $F\backslash\mathbb{A}$. By the existence of a global ψ-Whittaker model for cuspidal automorphic representations (see Theorem 5.13) and the uniqueness of global Whittaker models for irreducible admissible representations of $\mathrm{GL}_n(\mathbb{A})$ (see Theorem 5.9), V (resp. V') has a unique global Whittaker model $\mathcal{W}$ (resp. $\mathcal{W}'$) that consists of the ψ-Whittaker functions of the elements of V (resp. V'). Thus, if $W \in \mathcal{W}$ is a non-zero Whittaker function, it follows from the Fourier expansion of cusp forms (see Theorem 5.3), that the function $\varphi_W : \mathrm{GL}_n(\mathbb{A}) \to \mathbb{C}$ given by

$$\varphi_W(g) = \sum_{\gamma\in N_{n-1}(F)\backslash \mathrm{GL}_{n-1}(F)} W\left(\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix} g\right), \tag{6.2}$$

is a non-zero cusp form in V, and similarly $\varphi_{W'} \in V'$ for $W' \in \mathcal{W}'$. Note that if $\varphi_W(g) = \varphi_{W'}(g)$ for all $g \in \mathrm{GL}_n(F)Z_n(\mathbb{A})P_n(\mathbb{A})$, it would follow from the continuity of φ_W and $\varphi_{W'}$ that they would agree on all of $G(\mathbb{A})$, since $\mathrm{GL}_n(F)P_n(\mathbb{A})Z_n(\mathbb{A})$ is dense in $\mathrm{GL}_n(\mathbb{A})$ by Lemma 6.2, and thus the proof would be finished. Let us therefore first construct two Whittaker functions $W \in \mathcal{W}$ and $W' \in \mathcal{W}'$ that agree on $P_n(\mathbb{A})$.

We denote by $\mathcal{W}_v$ (resp. $\mathcal{W}_v'$) the ψ_v-Whittaker model of π_v (resp. π_v'). Let W and W' be the functions given by

$$W(g) = \prod_v W_v(g_v) \qquad \text{and} \qquad W'(g) = \prod_v W_v'(g_v),$$

where the W_v's and W_v''s are constructed in the following manner: at almost all non-Archimedean places $v \notin S$, π_v is unramified (see Theorem 4.16) and ψ_v has level 0 (see Proposition 4.8), and at such places, we take W_v to be the spherical element $W_v^\circ \in \mathcal{W}_v^{K_v}$ such that $W_v^\circ(K_v) = \{1\}$ (which exists and is unique by Corollary 2.26). At the remaining places $v \notin S$, we choose $W_v \in \mathcal{W}_v$ to be non-zero. Since $\pi_v \cong \pi_v'$ for all $v \notin S$, it follows from local uniqueness (see Theorem 2.19 and Theorem 3.16) that the ψ_v-Whittaker models of π_v and π_v' are equal, and thus we choose $W_v' = W_v$ for all $v \notin S$. For $v \in S$, choose an arbitrary non-zero element $f_v \in c\text{-}\mathrm{Ind}_{N_n(F_v)}^{P_n(F_v)}(\psi_v)$ (cf. Section 2.2). By Corollary 2.23, it is possible to choose $W_v \in \mathcal{W}_v$ and $W_v' \in \mathcal{W}_v'$ such that

$$W_v|_{P_n(F_v)} = W_v'|_{P_n(F_v)} = f_v.$$

This concludes our construction of W and W', and it follows from the global uniqueness theorem (Theorem 5.9) that both are well-defined elements of $\mathcal{W}$ and $\mathcal{W}'$, respectively. Furthermore, it follows from the construction, that for any $p \in P_n(\mathbb{A})$,

$$W(p) = \prod_v W_v(p_v) = \prod_{v\in S} f_v(p_v) \cdot \prod_{v\notin S} W_v(p_v) = \prod_{v\in S} W'_v(p_v) \cdot \prod_{v\notin S} W'_v(p_v) = W'(p).$$

We now let $\varphi = \varphi_W$ and $\varphi' = \varphi_{W'}$ be the inverse Fourier transform from (6.2) of W and W', respectively. It is clear from (6.2) that $\varphi(p) = \varphi'(p)$ for all $p \in P_n(\mathbb{A})$, as W and W' agree on $P_n(\mathbb{A})$.

Let $\omega, \omega' : F^\times \backslash \mathbb{A}^\times \to \mathbb{S}^1$ be the central characters of π and π', respectively, that is, ω and ω' are unitary Hecke characters such that $V \subseteq \mathcal{A}_0(\omega)$ and $V' \subseteq \mathcal{A}_0(\omega')$. We have that $\omega = \otimes_v \omega_v$ and $\omega' = \otimes_v \omega'_v$, where ω_v and ω'_v are the central characters of π_v and π'_v, respectively, at all non-Archimedean places (cf. Section 2.2). Since $\pi_v \cong \pi'_v$ for all $v \notin S$, we have that $\omega_v = \omega'_v$ at all non-Archimedean places $v \notin S$. So, applying Proposition 4.6 to $\frac{\omega_\pi}{\omega_{\pi'}}$ yields that ω and ω' must be equal at all places. Thus, we have that

$$\varphi(zp) = \omega(z)\varphi(p) = \omega'(z)\varphi'(p) = \varphi'(zp)$$

for all $z \in Z_n(\mathbb{A})$ and $p \in P_n(\mathbb{A})$. Finally, since φ and φ' are both automorphic forms, and thus invariant under left translation by elements in $\mathrm{GL}_n(F)$, this implies that φ and φ' must be equal on the whole of $\mathrm{GL}_n(F)Z_n(\mathbb{A})P_n(\mathbb{A})$, and thus by the previous density argument, they must agree on all of $\mathrm{GL}_n(\mathbb{A})$, so $\varphi = \varphi' \in V \cap V'$, which concludes the proof. $\square$

Remark 6.3. The assumption that $\pi_v \cong \pi'_v$ at all Archimedean places arises because we have not introduced the notion of a Kirillov model for irreducible local Archimedean representations in this thesis, unlike the case for irreducible local non-Archimedean representations. An analogue of Corollary 2.23 in the Archimedean case was also unavailable to Piatetski-Shapiro at the time of his proof, but it was proven two years later in [JS81]. This analogue requires introducing induction functors for $(\mathfrak{g}, K)$-modules—a topic beyond the scope of this thesis. However, with this in hand, the same argument would show Theorem 6.1 without the assumption at the Archimedean places.

In [JS81], Jacquet and Shalika prove the strong multiplicity one theorem, without this restriction at Archimedean places, using alternative methods. Their approach involves analysing certain L-functions $L(s, \pi \times \pi')$ attached to the pair π, π', and showing that they have a pole at $s = 1$. However, their proof also relies on the uniqueness of local Whittaker models, existence of the Kirillov model for local generic representations, the uniqueness of global Whittaker models, the existence of a global Whittaker model for cuspidal automorphic representations, and the Fourier expansion of cusp forms, and therefore shares some similarities to the proof presented above. Specifically, Proposition (3.8) in [JS81] serves as the Archimedean analogue of our Corollary 2.23 in their setting.

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