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Exploration of Kinetic Models for Ant-Pheromone Interactions

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Abstract

This thesis discusses a model for the formation of ant trails, which arise through the reinforcement of paths due to pheromone markings. After presenting the motivation and introducing an individual-based model, kinetic equations consisting of a system of coupled partial differential equations are derived.
The existence and uniqueness of weak solutions for this system of equations is proven for both the original model and modifications thereof. In the final chapter, further modifications to the proposed model are discussed.

Kurzfassung

In dieser Arbeit wird ein Modell zur Entstehung von Ameisenstraßen, die sich durch Pheromonmarkierungen herausbilden, behandelt. Nach der Motivation und der Einführung eines individuenbasierten Modells werden kinetische Gleichungen abgeleitet, die aus einem System gekoppelter partieller Differentialgleichungen bestehen.
Die Existenz und Eindeutigkeit schwacher Lösungen für dieses Gleichungssystem wird sowohl für das ursprüngliche Modell als auch für Modifikationen bewiesen. Im letzten Kapitel werden weitere Abwandlungen des vorgeschlagenen Modells diskutiert.

Contents

1 Introduction

The formation of ant trails is only one of countless examples in which many individuals form a larger structure without a higher instance giving instructions, only by communicating with each other. In this case, communication does not happen directly; Ants deposit pheromones along their path and follow previously planted pheromone traces.
Trail pattern formation has been investigated in many different works and models; Ant displacement on a pre-existing pheromone trail and the role of ants' antennae [1], one-dimensional cellular automata [2] and the use of ODEs to study the decision-making process when several trails are available to choose from [3]. What is new in the model described by [4], on which this thesis is based, is the consideration that trails can emerge spontaneously, not on a lattice but in a time- and space-continuous setting.
This work relies on the essence of kinetic theory [5, Chapter 1]; the method of using multiple layers of description. First, the interacting mechanics of individuals are given (e.g. change of position over time), although it completely describes the system, this layer is not ideal; analytic results are difficult to obtain, and in the often occurring case of vast amounts of individuals, initial conditions for each individual are almost impossible to collect and numerics become impractical.
Therefore, a second layer of description is introduced, the kinetic equations. They describe the dynamics of the system using differential equations of distribution functions of the individuals' looked for properties (e.g. spatial distribution). The step from the first to the second level of description is attained via taking the mean-field limit, which is usually easy to obtain formally but hard to prove rigorously; during this process a loss of information is inevitable, which can cause a difference of behaviour of the two systems. The third layer consists of macroscopic equations characterizing average behaviour, for example, in this model the total amount of ants and pheromones.

2 The Model

In this chapter, we will introduce the model of [4] and highlight some of their results.

2.1 Individual-Based Model

This microscopic-scale ant-pheromone model considers individual ants that move in a certain direction at constant speed. After some time, they change direction; either due to random jumps or due to jumps following pheromone trails, which were previously placed by other ants. These pheromone particles are volatile and thus evaporate over time.
In this model, there are N ants, moving in a two-dimensional plane. The movement of each ant is described by its position $x_i \in \mathbb{R}^2$ and its orientation $\omega_i \in \mathbb{S}^1$. The pheromones are described similarly; with a position and orientation $(x_p, w_p) \in \mathbb{R}^2 \times \mathbb{S}^1$. Between directional jumps, an ant moves in the direction ω_i with constant speed c, so its dynamics are given by

$$\dot{x}_i = c\omega_i, \qquad \dot{\omega}_i = 0.$$

Jump times are randomly chosen according to a Poisson distribution. There are two different jump processes running simultaneously:
Random velocity jumps: The ant randomly changes direction, independently of its surroundings, the rate of this Poisson process is given by λ_r. Here, the new direction is chosen uniformly from $\mathbb{S}^1$.
Trail recruitment jump: The new direction is chosen uniformly from the pheromone orientations at the current position of the ant. The frequency of this process is given by $\lambda_p \rho_g$. Therefore, it is not constant but depends on the number of pheromones ρ_g at position x.
Furthermore, individuals place pheromones randomly, again determined by a Poisson distribution, with rate ν_d. The pheromones are placed according to the current position and orientation of the ant, that is, the new pheromone particle p placed at time t has coordinates $(x_p, w_p) = (x_i(t), w_i(t))$. The particles are volatile and their evaporation is modelled as a Poisson process with rate ν_e. Diffusion is not considered in this model.

2.2 Kinetic Model

From the individual-based model above one can formally deduce a kinetic model, describing the evolution of the ant distribution $f = f(x, \omega, t) \in \mathbb{R}$ and pheromone distribution $g = g(x, \omega, t) \in \mathbb{R}$ over time using partial differential equations:

$$\begin{aligned} \partial_t f + c\omega \cdot \nabla_x f &= \lambda_r \left(\frac{\rho_f}{2\pi} - f \right) + \lambda_p \left(\frac{g}{\rho_g} \rho_f - f \right) && (2.1) \\ &= \left(\frac{\lambda_r}{2\pi} + \lambda_p \frac{g}{\rho_g} \right) \rho_f - (\lambda_r + \lambda_p) f, \\ \partial_t g &= \nu_d f - \nu_e g, && (2.2) \\ f\big|_{t=0} &= f_0, \qquad g\big|_{t=0} = g_0, && (2.3) \end{aligned}$$

where

$$\rho_f(x,t) := \int_{\mathbb{S}^1} f(x, \omega, t) d\omega$$

is the local ant density, the total number of ants in position x at time t. The local pheromone density $\rho_g(x,t) = \int_{\mathbb{S}^1} g(x, \omega, t) d\omega$ is defined analogously.
Equation (2.2), which defines the dynamics of the distribution of pheromones, can easily be concluded; the first term on the right-hand side describes the deposition of pheromones by ants, and the second term characterizes pheromone evaporation.
To conclude equation (2.1) from the individual-based model, we start with the PDE

$$\partial_t f + c\omega \cdot \nabla_x f = Q(f),$$

where the left-hand side describes the motion of the ant with constant speed c in the direction ω. The right-hand side is given by an operator $Q(f)$, which governs the two jump processes and can be split into two parts, each for one type of jump:

$$Q(f) = Q_r(f) + Q_p(f).$$

Both operators on the right-hand side are of the form

$$Q_k(F)(x, \omega, t) = \lambda_k \int_{\mathbb{S}^1} P_k(\omega' \to \omega) f(x, \omega', t) - P_k(w \to \omega') f(x, \omega, t) d\omega',$$

where $k \in \{r, p\}$ and $P_k(\omega' \to \omega)$ is the probability that the process k jumps from angle ω' to ω. Choosing, as above, the uniform distribution on $\mathbb{S}^1$ for random velocity jumps $P_r(x, \omega, t) = 1/2\pi$ and the uniform distribution on the pheromones in the current position for trail recruitment jumps $P_p(x, \omega, t) = g(x, \omega, t)/\rho_g(x, t)$ gives equation (2.1).
Remark 1. It can be shown that the ant mass is preserved over time; by integrating w.r.t. x and ω, one gets

$$\begin{aligned} \int_{\mathbb{R}^2} \int_{\mathbb{S}^1} \partial_t f + c\omega \cdot \nabla_x f d\omega dx &= \int_{\mathbb{R}^2} \int_{\mathbb{S}^1} \lambda_r \left(\frac{\rho_f}{2\pi} - f \right) + \lambda_p \left(\frac{g}{\rho_g} \rho_f - f \right) d\omega dx \\ &= \int_{\mathbb{R}^2} \lambda_r \left(2\pi \frac{\rho_f}{2\pi} - \int_{\mathbb{S}^1} f dw \right) + \lambda_p \left(\frac{\rho_f}{\rho_g} \int_{\mathbb{S}^1} g d\omega - \int_{\mathbb{S}^1} f d\omega \right) \\ &= 0, \end{aligned}$$

and the integral $\int_{\mathbb{R}^2}\int_{\mathbb{S}^1} w \cdot \nabla_x f d\omega dx$ on the LHS is equal to zero by the divergence theorem, leaving

$$\int_{\mathbb{R}^2}\int_{\mathbb{S}^1} \partial_t f d\omega dx = 0.$$

This implies that, if f_0, g_0 are integrable, then f, g are integrable for all times.

Rescaling

In order to rescale the model, we first need to define new variables, namely,

$$x' = \varepsilon x, \qquad t' = \varepsilon t,$$

where $0 < \varepsilon \ll 1$ is the scaling parameter. This corresponds to zooming out in space and speeding up time. With this setup, the rescaled system of the functions $f_\varepsilon = f_\varepsilon(x', \omega, t'), g_\varepsilon = g_\varepsilon(x', \omega, t')$ reads as

$$\varepsilon(\partial_t f_\varepsilon + \omega \cdot \nabla_x f_\varepsilon) = \left(\frac{\lambda_r}{2\pi} + \lambda_p \frac{g_\varepsilon}{\rho_{g_\varepsilon}}\right) \rho_{f_\varepsilon} - (\lambda_r + \lambda_p) f_\varepsilon, \tag{2.4}$$

$$\partial_t g_\varepsilon = \nu_d f_\varepsilon - \nu_e g_\varepsilon, \tag{2.5}$$

$$f_\varepsilon\big|_{t=0} = f_{\varepsilon 0}, \qquad g_\varepsilon\big|_{t=0} = g_{\varepsilon 0}. \tag{2.6}$$

Remark 2. Note that for the sake of clarity, the primes have been skipped. Furthermore, we assume $c = 1$ for simplicity.

3 Rescaled Model

3.1 Macroscopic Model

To obtain the macroscopic equations of our model (2.4) - (2.6), we assume that the ant and pheromone distributions $f_\varepsilon, g_\varepsilon$ converge as $\varepsilon \to 0$:

$$\begin{aligned} f_\varepsilon &\xrightarrow{\varepsilon \to 0} f, \\ g_\varepsilon &\xrightarrow{\varepsilon \to 0} g. \end{aligned}$$

Dividing (2.4) by ε, we get

$$\partial_t f_\varepsilon + \omega \cdot \nabla_x f_\varepsilon = \frac{1}{\varepsilon}\left(\left(\frac{\lambda_r}{2\pi} + \lambda_p \frac{g_\varepsilon}{\rho_{g_\varepsilon}}\right)\rho_{f_\varepsilon} - (\lambda_r + \lambda_p) f_\varepsilon\right) =: Q(f_\varepsilon, g_\varepsilon), \tag{3.1}$$

$$\partial_t g_\varepsilon = \nu_d f_\varepsilon - \nu_e g_\varepsilon, \tag{3.2}$$

$$f_\varepsilon\big|_{t=0} = f_{\varepsilon 0}, \qquad g_\varepsilon\big|_{t=0} = g_{\varepsilon 0}. \tag{3.3}$$

Assuming convergence as $\varepsilon \to 0$, (2.4) gives $Q(f,g) = 0$, which enables us to find another expression for f, the so-called consistency relationship, depending only on g, ρ_f and ρ_g:

$$\begin{aligned} 0 = Q(f,g) &= \left(\frac{\lambda_r}{2\pi} + \lambda_p \frac{g}{\rho_g}\right)\rho_f - (\lambda_r + \lambda_p) f \\ \Leftrightarrow f &= \frac{1}{\lambda_r + \lambda_p}\left(\frac{\lambda_r}{2\pi} + \lambda_p \frac{g}{\rho_g}\right)\rho_f. \end{aligned} \tag{3.4}$$

The calculations for mass preservation in Remark 1 show that the RHS of (3.1) is already equal to zero when only integrating w.r.t. ω, thus we get

$$\begin{aligned} 0 &= \partial_t \rho_{f_\varepsilon} + \nabla_x \cdot \int_{\mathbb{S}^1} \omega f_\varepsilon d\omega \\ \xrightarrow{\varepsilon \to 0} 0 &= \partial_t \rho_f + \frac{\lambda_p}{\lambda_r + \lambda_p} \nabla_x \cdot \left(\frac{J_g}{\rho_g}\rho_f\right), \end{aligned}$$

using (3.4). Analogously, if (3.1) is multiplied by ω before integrating, we get

$$\partial_t J_{f_\varepsilon} + \nabla_x \cdot \int_{\mathbb{S}^1} \omega \otimes \omega f_\varepsilon d\omega = \frac{1}{\varepsilon}\left(\lambda_p \frac{J_{g_\varepsilon}}{\rho_{g_\varepsilon}} \rho_{f_\varepsilon} - (\lambda_r + \lambda_p) J_{f_\varepsilon}\right),$$

where $J_f := \int_{\mathbb{S}^1} \omega f d\omega$ defines the average ant orientation and $J_g = \int_{\mathbb{S}^1} \omega g d\omega$ similarly defines the average pheromone orientation.

Remark 3.

- Due to the term $\nabla_x \cdot \int_{\mathbb{S}^1} \omega \otimes \omega f_\varepsilon d\omega$, the equation for J_{f_ε} cannot be closed. However, letting $\varepsilon \to 0$, we get
$$J_f = \frac{\lambda_p}{\lambda_r + \lambda_p} \rho_f \frac{J_g}{\rho_g}.$$
An alternative way to derive this equation is by multiplying (3.4) by ω and then integrating w.r.t. ω.

- The notation $\omega \otimes \omega$ denotes the matrix
$$\omega \otimes \omega = \begin{pmatrix} {\omega_1}^2 & \omega_1^\perp \cdot \omega_2 \\ \omega_2^\perp \cdot \omega_1 & {\omega_2}^2 \end{pmatrix}.$$

By integrating (3.2) w.r.t. ω and integrating the same equation after multiplying with ω, we obtain

$$\begin{aligned}
\partial_t \rho_{g_\varepsilon} &= \nu_d \rho_{f_\varepsilon} - \nu_e \rho_{g_\varepsilon}, \\
\partial_t J_{g_\varepsilon} &= \nu_d \int_{\mathbb{S}^1} \omega f_\varepsilon d\omega - \nu_e J_{g_\varepsilon} \\
&= \frac{\lambda_p}{\lambda_p + \lambda_r} \nu_d J_{g_\varepsilon} \frac{\rho_{f_\varepsilon}}{\rho_{g_\varepsilon}} - \nu_e J_{g_\varepsilon} \\
&= \left(\frac{\lambda_p}{\lambda_r + \lambda_p} \nu_d \frac{\rho_{f_\varepsilon}}{\rho_{g_\varepsilon}} - \nu_e \right) J_{g_\varepsilon}.
\end{aligned}$$

To determine the evolution of the pheromone orientation, we compute $\partial_t (g_\varepsilon / \rho_{g_\varepsilon})$:

$$\begin{aligned}
\partial_t \left(\frac{g_\varepsilon}{\rho_{g_\varepsilon}} \right) &= \frac{(\partial_t g_\varepsilon)\rho_{g_\varepsilon} - g_\varepsilon \partial_t(\rho_{g_\varepsilon})}{{\rho_{g_\varepsilon}}^2} \qquad (3.5) \\
&= \frac{(\nu_d f_\varepsilon - \nu_e g_\varepsilon)\rho_{g_\varepsilon} - g_\varepsilon(\nu_d \rho_{f_\varepsilon} - \nu_e \rho_{g_\varepsilon})}{{\rho_{g_\varepsilon}}^2} \\
&= \frac{\nu_d}{{\rho_{g_\varepsilon}}^2} (f_\varepsilon \rho_{g_\varepsilon} - g_\varepsilon \rho_{f_\varepsilon}) \\
&= \nu_d \frac{\rho_{f_\varepsilon}}{\rho_{g_\varepsilon}} \left(\frac{f_\varepsilon}{\rho_{f_\varepsilon}} - \frac{g_\varepsilon}{\rho_{g_\varepsilon}} \right) \\
&= \nu_d \frac{\rho_{f_\varepsilon}}{\rho_{g_\varepsilon}} \left(\frac{1}{\lambda_r + \lambda_p} \left(\frac{\lambda_r}{2\pi} + \lambda_p \frac{g_\varepsilon}{\rho_{g_\varepsilon}} \right) - \frac{g_\varepsilon}{\rho_{g_\varepsilon}} \right) \\
&= \nu_d \frac{\rho_{f_\varepsilon}}{\rho_{g_\varepsilon}} \frac{\lambda_r}{\lambda_r + \lambda_p} \left(\frac{1}{2\pi} - \frac{g_\varepsilon}{\rho_{g_\varepsilon}} \right),
\end{aligned}$$

which means that $g_\varepsilon / \rho_{g_\varepsilon}$ will converge to $1/2\pi$, i.e. the uniform distribution, as $t \to \infty$. It follows that also f_ε converges to a uniform distribution for long times.

3.2 Hilbert Expansion

In this section, we will attempt, but fail, to prove or disprove the consistency relationship (3.4), and with this the assumption that $f_\varepsilon, g_\varepsilon$ converge as $\varepsilon \to 0$. For this, we follow the ansatz of [6, Chapter 1.5.2]. We assume that the functions $f_\varepsilon, g_\varepsilon$ can be written as power series:

$$f_\varepsilon = \sum_{n\geq 0} \varepsilon^n f_n,$$
$$g_\varepsilon = \sum_{n\geq 0} \varepsilon^n g_n,$$

and that f_n, g_n are differentiable w.r.t. t, x. By substituting in (2.4) and (2.5), the rescaled model reads as

$$\begin{aligned}
\varepsilon &\left(\partial_t \sum_{n\geq 0} \varepsilon^n f_n + \omega \cdot \nabla_x \sum_{n\geq 0} \varepsilon^n f_n\right) \\
&= \left(\frac{\lambda_r}{2\pi} + \lambda_p \frac{\sum_{n\geq 0} \varepsilon^n g_n}{\int_{\mathbb{S}^1} \sum_{n\geq 0} \varepsilon^n g_n d\omega}\right) \int_{\mathbb{S}^1} \sum_{n\geq 0} \varepsilon^n f_n d\omega - (\lambda_r + \lambda_p) \sum_{n\geq 0} \varepsilon^n f_n \\
&= \left(\frac{\lambda_r}{2\pi} + \lambda_p \frac{1}{\rho_{g_0}} \left(1 - \frac{\varepsilon \rho_{g_1}}{\rho_{g_0}} + O\left(\varepsilon^2\right)\right) \sum_{n\geq 0} \varepsilon^n g_n\right) \int_{\mathbb{S}^1} \sum_{n\geq 0} \varepsilon^n f_n d\omega \\
&\quad - (\lambda_r + \lambda_p) \sum_{n\geq 0} \varepsilon^n f_n, \\
\partial_t \sum_{n\geq 0} \varepsilon^n g_n &= \nu_d \sum_{n\geq 0} \varepsilon^n f_n - \nu_e \sum_{n\geq 0} \varepsilon^n g_n.
\end{aligned} \tag{3.6}$$

Here we used the convergence of the geometric series for sufficiently small ε to find

$$\begin{aligned}
\frac{1}{\rho_g} &= \frac{1}{\rho_{g_0}\left(1 - \left(-\frac{\varepsilon\rho_{g_1} + \varepsilon^2 \rho_{g_2} + \dots}{\rho_{g_0}}\right)\right)} \\
&= \frac{1}{\rho_{g_0}} \sum_{n\geq 0} \left(-\frac{\varepsilon\rho_{g_1} + \varepsilon^2 \rho_{g_2} + \dots}{\rho_{g_0}}\right)^n \\
&= \frac{1}{\rho_{g_0}} \left(1 - \frac{\varepsilon g_1}{\rho_{g_0}} + O\left(\varepsilon^2\right)\right).
\end{aligned}$$

Remark 4. Note that g also needs to be expanded, otherwise equation (3.6) would give $g_n = 0$ for all $n \geq 1$.

Now, every degree of ε gives two equations:

$$\begin{aligned} O\left(\varepsilon^0\right): \qquad 0 &= \left(\frac{\lambda_r}{2\pi} + \lambda_p \frac{g_0}{\rho_{g_0}}\right)\rho_{f_0} - (\lambda_r + \lambda_p) f_0, \\ \partial_t g_0 &= \nu_d f_0 - \nu_e g_0, \end{aligned} \tag{3.7}$$

$$\begin{aligned} O\left(\varepsilon^1\right): \partial_t f_0 + \omega \cdot \nabla_x f_0 &= \left(\frac{\lambda_r}{2\pi} + \lambda_p \frac{g_0}{\rho_{g_0}}\right)\rho_{f_1} - \lambda_p \frac{g_0}{{\rho_{g_0}}^2}\rho_{g_1}\rho_{f_0} + \frac{\lambda_p}{\rho_{g_0}} g_1 \rho_{f_0} - (\lambda_r + \lambda_p) f_1, \\ \partial_t g_1 &= \nu_d f_1 - \nu_e g_1. \end{aligned} \tag{3.8}$$

We immediately see that f_0 fulfils the compatibility equation:

$$f_0 = \frac{1}{\lambda_r + \lambda_p}\left(\frac{\lambda_r}{2\pi} + \lambda_p \frac{g_0}{\rho_{g_0}}\right)\rho_{f_0} =: H(f_0). \tag{3.9}$$

Now, we need to show that the limits f, g as $\varepsilon \to 0$ satisfy the above equation. This is a fixed-point problem: given λ_r, λ_p, g, is there an f s.t. $H(f) = f$? To solve this, we can use the Banach fixed-point theorem [7, Chapter 4].

Theorem 3.2.1 (Banach fixed-point theorem)**.** *Let $H(f): X \to X$ be a contraction on a complete metric space, i.e. there is a constant $c \in (0,1)$ s.t.*

$$\|H(\varphi) - H(\alpha)\|_X \leq c\|\varphi - \alpha\|_X$$

for all $\varphi, \alpha \in X$. Then H has a unique fixed point.

To apply the Banach fixed-point theorem, we need a non-empty, complete metric space (M, d), and then show that H is a contraction. Therefore, we use the space (4.7). Let $(\varphi, g), (\alpha, g) \in X_{M,\delta,T}$, then

$$\begin{aligned} \|H(\varphi) - H(\alpha)\|_\infty &= \frac{1}{\lambda_r + \lambda_p}\left\|\left(\frac{\lambda_r}{2\pi} + \lambda_p \frac{g}{\rho_g}\right)\rho_\varphi - \left(\frac{\lambda_r}{2\pi} + \lambda_p \frac{g}{\rho_g}\right)\rho_\alpha\right\|_\infty \\ &\leq \frac{1}{\lambda_r + \lambda_p}\left(\lambda_r\|\varphi - \alpha\|_\infty + 2\pi\lambda_p\left\|\frac{g}{\rho_g}\right\|_\infty \|\varphi - \alpha\|_\infty\right) \\ &\leq \frac{1}{\lambda_r + \lambda_p}\|\varphi - \alpha\|_\infty\left(\lambda_r + 2\pi\lambda_p\frac{M}{\delta}\right). \end{aligned}$$

We obtain analogously to the derivation of the macroscopic equations in the previous chapter from the above equations (3.7) - (3.9) the following:

$$0 = \partial_t \rho_{f_0} + \frac{\lambda_p}{\lambda_r + \lambda_p} \nabla_x \cdot \left(\frac{\rho_{f_0}}{\rho_{g_0}} J_{g_0} \right), \tag{3.10}$$

$$J_{f_0} = \frac{\lambda_p}{\lambda_r + \lambda_p} \rho_{f_0} \frac{J_{g_0}}{\rho_{g_0}}, \tag{3.11}$$

$$\partial_t \rho_{g_0} = \nu_d \rho_{f_0} - \nu_e \rho_{g_0}, \tag{3.12}$$

$$\partial_t J_{g_0} = \left(\nu_d \frac{\lambda_p}{\lambda_r + \lambda_p} \frac{\rho_{f_0}}{\rho_{g_0}} - \nu_e \right) J_{g_0}, \tag{3.13}$$

$$\partial_t J_{f_0} + \nabla_x \cdot \int \omega \otimes \omega f_0 d\omega = \lambda_p \frac{1}{\rho_{g_0}} \rho_{f_0} J_{g_1} + \lambda_p \frac{J_{g_0}}{\rho_{g_0}} \rho_{f_1} - \lambda_p \frac{J_{g_0}}{{\rho_{g_0}}^2} \rho_{g_1} \rho_{f_0} - (\lambda_r + \lambda_p) J_{f_1}.$$

Now, finding a different expression containing $\partial_t J_{f_0}$ could assist in disproving the consistency relationship (3.4). Using (3.10) - (3.13) we get

$$\begin{aligned}
\partial_t J_{f_0} &= \frac{\lambda_p}{\lambda_r + \lambda_p} \partial_t \left(\frac{\rho_{f_0}}{\rho_{g_0}} J_{g_0} \right) \\
&= \frac{\lambda_p}{\lambda_r + \lambda_p} \left(\partial_t (\rho_{f_0}) \frac{J_{g_0}}{\rho_{g_0}} + \rho_{f_0} \partial_t \left(\frac{J_{g_0}}{\rho_{g_0}} \right) \right) \\
&= \frac{\lambda_p}{\lambda_r + \lambda_p} \left(\partial_t (\rho_{f_0}) \frac{J_{g_0}}{\rho_{g_0}} + \rho_{f_0} \frac{\partial_t (J_{g_0}) \rho_{g_0} - J_{g_0} \partial_t (\rho_{g_0})}{{\rho_{g_0}}^2} \right) \\
&= - \left(\frac{\lambda_p}{\lambda_r + \lambda_p} \right)^2 \nabla_x \cdot \left(\frac{\rho_{f_0}}{\rho_{g_0}} J_{g_0} \right) \frac{J_{g_0}}{\rho_{g_0}} \\
&\quad + \frac{\rho_{f_0}}{{\rho_{g_0}}^2} \frac{\lambda_p}{\lambda_r + \lambda_p} \left(\frac{\nu_d \lambda_p}{\lambda_r + \lambda_p} \frac{\rho_{f_0}}{\rho_{g_0}} J_{g_0} - \nu_e J_{g_0} \right) \rho_{g_0} \\
&\quad - \frac{\rho_{f_0}}{{\rho_{g_0}}^2} \frac{\lambda_p}{\lambda_r + \lambda_p} J_{g_0} (\nu_d \rho_{f_0} - \nu_e \rho_{g_0}) \\
\Leftrightarrow \partial_t J_{f_0} + \nabla_x \cdot (J_{f_0}) \frac{J_{f_0}}{\rho_{f_0}} &= \frac{\lambda_p}{\lambda_r + \lambda_p} \frac{\rho_{f_0}}{\rho_{g_0}} \nu_d (J_{f_0} \rho_{g_0} - J_{g_0} \rho_{f_0}),
\end{aligned}$$

where the RHS is not zero, unless $\lambda_r = 0, \lambda_p \neq 0$ or $\lambda_r \neq 0, \lambda_p = 0$. Now, *if* this last equation were equal to zero, we would obtain the two equations along with the following condition:

$$\begin{aligned}
\partial_t J_{f_0} + \nabla_x \cdot \int_{\mathbb{S}^1} \omega \otimes \omega f d\omega &= 0, \\
\partial_t J_{f_0} + (\nabla_x \cdot J_{f_0}) \frac{J_{f_0}}{\rho_{f_0}} &= 0, \\
\Leftrightarrow \nabla_x \cdot \int_{\mathbb{S}^1} \omega \otimes \omega f d\omega &= (\nabla_x \cdot J_{f_0}) \frac{J_{f_0}}{\rho_{f_0}}.
\end{aligned}$$

Even then, we cannot draw any more conclusions since we do not have enough information to compare the two sides of the last equation.

4 Existence of Solutions

Remark 5. From now on, we abuse notation by ignoring ε in (3.1) and writing f, g instead of $f_\varepsilon, g_\varepsilon$. This can be interpreted as fixing an arbitrary $0 < \varepsilon \ll 1$ and then showing that for this now constant, thus for the wanted results not critical ε, there exists a solution.

4.1 Original Model

To show that there is a (unique) solution to our model (3.1) - (3.3), we assume a slightly different problem, replacing f and g in the PDE containing the respectively other time derivative, with functions φ, ψ, which will not be further described here. This is done to avoid a coupled system of differential equations and instead have two uncoupled equations.

$$\partial_t f + \omega \cdot \nabla_x f = \left(\frac{\lambda_r}{2\pi} + \lambda_p \frac{\varphi}{\rho_\varphi}\right) \rho_f - (\lambda_r + \lambda_p) f =: Lf(\varphi) - af, \tag{4.1}$$

$$\partial_t g = \nu_d \psi - \nu_e g, \tag{4.2}$$

$$f\big|_{t=0} = f_0, \qquad g\big|_{t=0} = g_0. \tag{4.3}$$

To prove that the system (4.1) - (4.3) has a unique solution, we will show that there are unique solutions for each of the differential equations and then demonstrate that, for a non-empty complete metric space X, an appropriate function $\Phi : X \to X$, mapping φ, ψ to said solutions, is a contraction. Thus, we can use the Banach fixed-point theorem to show that there is a unique solution to the original model.

Lemma 4.1.1. *The solution of* (4.2) *is given by*

$$g = g_0 e^{-t\nu_e} + \int_0^t e^{-\nu_e(t-s)} \nu_d \psi(s) ds.$$

Proof. Equation (4.2) is an ordinary differential equation whose solution can be obtained by separation of variables. □

Now, for the uniqueness of this solution, recall the Picard-Lindelöf Theorem (see, e.g. [8]).

Theorem 4.1.2 (Picard-Lindelöf)**.** *Let $\mathcal{L} : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ be a continuous function which satisfies the Lipschitz condition*

$$|\mathcal{L}(y_1, t) - \mathcal{L}(y_2, t)| \leq M|y_1 - y_2| \qquad \forall y_1, y_2 \in \mathbb{R}^n, t \in [0,T]$$

for some $M > 0$. Then, for any $y_0 \in \mathbb{R}^n$ there is a unique solution $g : [0,T] \to \mathbb{R}^n$ of the initial value problem

$$\partial_t g(t) = \mathcal{L}(g(t), t),$$
$$g\big|_{t=0} = g_0.$$

Corollary 4.1.3. *If ψ is continuous with respect to time in $[0,T]$, then there exists a unique solution for (4.2) for all $(x,\omega,t) \in \mathbb{R}^2 \times \mathbb{S}^1 \times [0,T]$.*

Proof. To be able to apply the Picard-Lindelöf theorem, we need to show Lipschitz-continuity of $\mathcal{L}$, which is easy to see:

$$\begin{aligned}|\mathcal{L}(y_1,t) - \mathcal{L}(y_2,t)| &= |\nu_d\psi(x,\omega,t) - \nu_e y_1 - (\nu_d\psi(x,\omega,t) - \nu_e y_2)| \\ &= \nu_e|y_1 - y_2|,\end{aligned}$$

as well as continuity of $\mathcal{L}$ w.r.t. $(y,t) \in U \times [0,T]$:

$$\begin{aligned}|\mathcal{L}(y,t) - \mathcal{L}(y_0,t_0)| &= |\nu_d\psi(x,\omega,t) - \nu_e y - (\nu_d\psi(x,\omega,t_0) - \nu_e y_0)| \\ &\le \nu_d|\psi(x,\omega,t) - \psi(x,\omega,t_0)| + \nu_e|y - y_0|,\end{aligned}$$

which gives the continuity condition on ψ as asked for in the corollary. □

Now, for the solution of (4.1), we first need a definition of weak solutions of a PDE and further the Duhamel integral, see [5, Chapter 4.4].

Definition. We call $f(x,\omega,t)$ a weak solution of

$$\partial_t f + \omega \cdot \nabla_x f = \int_{\mathbb{S}^1} k(x,\omega,\widetilde{\omega},t) f(x,\omega,t) d\widetilde{\omega} - af =: Kf - af, \tag{4.4}$$

with k, a being given functions, if for all $(x,\omega,t) \in \mathbb{R}^2 \times \mathbb{S}^1 \times [0,T]$ and s such that $t+s \in [0,T]$, the map $s \mapsto f(x+\omega s,\omega,t+s)$ is continuously differentiable w.r.t. s and

$$\partial_s f(x+\omega s,\omega,t+s) = Kf(x+\omega s,\omega,t+s) - af(x+\omega s,\omega,t+s)$$

is satisfied.

Remark 6. The definition does not make a statement about differentiability w.r.t. x, so the weak solution may not satisfy the equation (4.4) in the classical sense, but when multiplied with any test function $\xi \in C_0^\infty(\mathbb{R}^2 \times \mathbb{S}^1 \times [0,T])$ and integrated w.r.t. all variables x,ω,t, the solution fulfils

$$\begin{aligned}\int_{\mathbb{R}^2}\int_{\mathbb{S}^1}\int_0^T (\partial_t f + \omega \cdot \nabla_x f)\xi dt d\omega dx &= -\int_{\mathbb{R}^2}\int_{\mathbb{S}^1}\int_0^T f(\partial_t \xi + \omega \cdot \nabla_x \xi) dt d\omega dx \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{S}^1}\int_0^T (Kf - af)\xi dt d\omega dx.\end{aligned}$$

Theorem 4.1.4 (Duhamel Integral)**.** *If*

$$\begin{aligned}f_0 &\in C_b(\mathbb{R}^2 \times \mathbb{S}^1), \\ 0 \le a &\in C_b(\mathbb{R}^2 \times \mathbb{S}^1 \times [0,T]), \\ 0 \le k &\in C_b(\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1 \times [0,T]),\end{aligned} \tag{4.5}$$

$$\sup_{x\in\mathbb{R}^2,\omega\in\mathbb{S}^1,t\in[0,T]} \left| \int_{\widetilde{\omega}\in\mathbb{S}^1} k(x,\omega,\widetilde{\omega},t) d\widetilde{\omega} \right| < \infty, \tag{4.6}$$

then there exists a unique weak solution to (4.4)*, implicitly given by*

$$f(x,\omega,t) = f_0(x-\omega t,\omega)\exp\left(-\int_0^t a(x-\omega(t-s),\omega,t)ds\right)$$
$$+\int_0^t Kf(x-\omega(t-s),\omega,s)\exp\left(-\int_s^t a(x-\omega(t-\tau),\omega,\tau)ds\right).$$

Corollary 4.1.5. *If*

$$f_0 \in C_b(\mathbb{R}^2 \times \mathbb{S}^1),$$
$$0 \le \varphi \in C_b(\mathbb{R}^2 \times \mathbb{S}^1 \times [0,T]),$$
$$0 < \inf_{x\in\mathbb{R}^2, t\in[0,T]} \rho_\varphi,$$

then there exists a unique weak solution of (4.1)*, given by*

$$f(x,\omega,t) = f_0(x-\omega t,\omega)e^{-t(\lambda_r+\lambda_p)}$$
$$+\int_0^t e^{-(t-s)(\lambda_r+\lambda_p)} Lf(\varphi)(s,x-\omega(t-s),\omega)ds.$$

Proof. In our case, $a = \lambda_r + \lambda_p \ge 0$ is constant, thus $0 \le a \in C_b$ is always given, and k is independent of $\widetilde{w}$, so conditions (4.5) and (4.6) are equivalent and fulfilled if $0 \le \varphi \in C_b(\mathbb{R}^2 \times \mathbb{S}^1 \times [0,T])$ and $\inf_{x,t}\rho_\varphi > 0$. □

Now, by finding an appropriate contraction from a non-empty complete metric space into itself, we can use the Banach fixed-point theorem to complete the proof of existence and uniqueness of solutions of (4.1) - (4.3) as well as (3.1) - (3.3). Let $M, \delta, T > 0$, and

$$X_{M,\delta,T} = \left\{ (\varphi,\psi) \in (C_b(\mathbb{R}^2 \times \mathbb{S}^1 \times [0,T]))^2 : \begin{array}{ccc} \inf_{x,t}\rho_\varphi & \ge & \delta > 0 \\ \psi & \ge & 0 \\ \|\varphi\|_\infty, \|\psi\|_\infty & \le & M \end{array} \right\} \tag{4.7}$$

be said space with the norm $\|(\varphi,\psi)\|_{X_{M,\delta,T}} = \|\varphi\|_\infty + \|\psi\|_\infty$. First, note that C_b with the infinity norm is a Banach space, thus C_b^2 with the norm as above is also a Banach space. We obtain this result from [9, Theorem 3.1.2]: Let X be a metric space, if $f_n \in C(X)$ converges locally uniformly in X, which means that for every point $x_0 \in X$ there exists a neighbourhood $x \in U \subseteq X$ in which f_n converges uniformly, then the limit $f = \lim_{n\to\infty} f_n$ is also continuous, i.e. $f \in C(X)$. Due to the other conditions, $X_{M,\delta,T}$ is no longer a vector space, but still a non-empty (consider a constant function $\delta < h < M$, then $(h,h) \in X_{M,\delta,T}$) complete metric space. Completeness w.r.t. the boundary conditions is outlined as follows: suppose $\{h_n\}_{n\in\mathbb{N}} \subset C_b$ a Cauchy sequence, if $h := \lim_{n\to\infty} h_n$ with $\|h\|_\infty > M$, then there is a m, s.t. $\|h_m\|_\infty > M$, a contradiction.

Remark 7. The need for these rather specific conditions in the definition of $X_{M,\delta,T}$ is given by a variety of reasons:

- Certainly, to make a plausible biological model, $\varphi, \psi \ge 0$ is needed.

- To avoid division by zero or a blow-up of φ/ρ_φ, we need $\inf_{x,t} \rho_\varphi > 0$, but a strict inequality leads to non-completeness of the space; considering e.g. $f_n = 1/n \in X_{M,\delta,T}$ but $f = \lim_{n\to\infty} f_n \equiv 0 \notin X_{M,\delta,T}$. So $\inf_{x,t} \rho_\varphi \geq \delta > 0$ for some δ is needed.
- When estimating (4.11), we need to show $\left\| \frac{\varphi}{\rho_\varphi} - \frac{\alpha}{\rho_\alpha} \right\|_\infty \leq c\|\varphi - \alpha\|_\infty$, which requires a uniform bound on $\varphi : \|\varphi\|_\infty \leq M$. But since g, the first entry of $\Phi(\varphi, \psi)$, needs to fulfil $\|g\|_\infty \leq M$ again and depends on ψ, we also need $\|\psi\|_\infty \leq M$.

Define the function

$$\begin{aligned}
&\Phi : X_{M,\delta,T} \to X_{M,\delta,T},\\
&\Phi(\varphi,\psi)(x,\omega,t) = (g,f)(x,\omega,t)\\
&\quad := \begin{pmatrix} g_0(x,\omega)e^{-t\nu_e} + \int_0^t e^{-(t-s)\nu_e}\nu_d \psi(x,\omega,s)ds \\ f_0(x-\omega t,\omega)e^{-t(\lambda_r+\lambda_p)} + \int_0^t e^{-(t-s)(\lambda_r+\lambda_p)} Lf(\varphi)(x-\omega(t-s),\omega,s)ds \end{pmatrix},
\end{aligned}$$

where f, g are the solutions of (4.1) - (4.3). So Φ maps (φ, ψ) into the previously established solutions of the uncoupled differential equations. First, we need to validate that Φ in fact maps $X_{M,\delta,T}$ back to $X_{M,\delta,T}$. The next condition is continuity, which is given, if f_0, g_0 are continuous w.r.t. (x, ω). Next, let us show boundedness:

$$\begin{aligned}
\|\Phi(\varphi,\psi)\|_{X_{M,\delta,T}} &= \|g\|_\infty + \|f\|_\infty \\
&=: I_1 + I_2 \\
&\leq \|g_0\|_\infty + \frac{\nu_d}{\nu_e}\|\psi\|_\infty + \|f_0\|_\infty,
\end{aligned} \tag{4.8}$$

where we can estimate I_1 and I_2 by

$$\begin{aligned}
I_1 &= \left\| g_0(x,\omega)e^{-t\nu_e} + \int_0^t e^{-(t-s)\nu_e}\nu_d\psi(x,\omega,s)ds \right\|_\infty \\
&\leq \|g_0\|_\infty + \frac{\nu_d}{\nu_e}\|\psi\|_\infty, \\
I_2 &= \Bigg\| f_0(x-\omega t,\omega)e^{-t(\lambda_r+\lambda_p)} \\
&\qquad + \int_0^t e^{-(t-s)(\lambda_r+\lambda_p)} Lf(\varphi)(x-\omega(t-s),\omega,t)ds \Bigg\|_\infty \\
&\leq \|f_0\|_\infty + T\left(\lambda_r + 2\pi\lambda_p \left\|\frac{\varphi}{\rho_\varphi}\right\|_\infty\right)\|f\|_\infty \\
\Leftrightarrow \|f\|_\infty &\leq \frac{\|f_0\|_\infty}{1 - T\left(\lambda_r + 2\pi\lambda_p \frac{\|\varphi\|_\infty}{\delta}\right)} \\
\Rightarrow I_2 = \|f\|_\infty &\leq \|f_0\|_\infty \left(1 - \frac{T\left(\lambda_r + 2\pi\lambda_p\frac{\|\varphi\|_\infty}{\delta}\right)}{1 - T\left(\lambda_r + 2\pi\lambda_p\frac{\|\varphi\|_\infty}{\delta}\right)}\right) = \|f_0\|_\infty,
\end{aligned}$$

where we used the definition of L and

$$\sup_{x,\omega,t} \int_0^t e^{-(t-s)c} = \sup_{x,\omega,t} \frac{1-e^{-tc}}{c} \le \frac{1}{c} \tag{4.9}$$

in the last equation. Therefore, provided that f_0, g_0, ψ are bounded (whereas f_0 already needs to be bounded to be able to use the Duhamel formula), $\|\Phi(\varphi,\psi)\|_{X_{M,\delta,T}}$ is again bounded. In addition to boundedness, we need non-negativeness of f, which is given by the definition of Φ and the non-negativity of φ, ψ. Furthermore, we also need $\inf_{x,t} \rho_g \ge \delta$:

$$\begin{aligned}\rho_g &= \int_{\mathbb{S}^1} \left(g_0(x,\omega)e^{-t\nu_e} + \int_0^t e^{-(t-s)\nu_e}\nu_d \psi(x,\omega,s)ds \right) d\omega \\ &= \rho_{g_0}(x)e^{-t\nu_e} + \int_0^t e^{-(t-s)\nu_e}\nu_d \rho_\psi(x,s)ds,\end{aligned}$$

so, for some time $t > 0$ the infimum is sufficiently large. Finally, we need to show uniform boundedness of f, g; in the estimates of I_2, we already saw that $\|f\|_\infty \le \|f_0\|_\infty$, implying that $\|f_0\|_\infty \le M$ is sufficient. For g, consider another estimate:

$$\begin{aligned}\|g\|_\infty &\le \|g_0\|_\infty + \nu_d \left(\frac{1-e^{-T\nu_e}}{\nu_e} \right) \|\psi\|_\infty \\ &\le \|g_0\|_\infty + \frac{\nu_d}{\nu_e}(1-e^{-T\nu_e})M,\end{aligned}$$

which satisfies the requirement if $\|g_0\|_\infty \le M/2$ and T being small enough such that $\frac{\nu_d}{\nu_e}(1-e^{-T\nu_e}) \le M/2$. Next, we address the contraction property of Φ: Let $(\varphi,\psi), (\alpha,\beta) \in X_{M,\delta,T}$, f_ψ, g_φ and f_β, g_α be the solutions of (4.1) - (4.3), with φ, ψ or α, β inserted, respectively. Then

$$\begin{aligned}\|\Phi(\varphi,&\psi) - \Phi(\alpha,\beta)\|_{X_{M,\delta,T}} = \|g_\varphi - g_\alpha\|_\infty + \|f_\psi - f_\beta\|_\infty \qquad (4.10)\\ &= \sup_{x,\omega,t} \Bigg| g_0(x,\omega)e^{-t\nu_e} + \int_0^t e^{-(t-s)\nu_e}\nu_d\psi(x,\omega,s)ds \\ &\qquad - \left(g_0(x,\omega)e^{-t\nu_e} + \int_0^t e^{-(t-s)\nu_e}\nu_d\beta(x,\omega,s)ds \right) \Bigg| \\ &+ \sup_{x,\omega,t} \Bigg| f_0(x-\omega t,\omega)e^{-t(\lambda_r+\lambda_p)} + \int_0^t e^{-(t-s)(\lambda_r+\lambda_p)} L f_\psi(\varphi)(x-\omega(t-s),\omega,t)ds \\ &\qquad - \left(f_0(x-\omega t,\omega)e^{-t(\lambda_r+\lambda_p)} + \int_0^t e^{-(t-s)(\lambda_r+\lambda_p)} L f_\beta(\alpha)(x-\omega(t-s),\omega,t)ds \right) \Bigg| \\ &=: J_1 + J_2 \\ &\le cT(\|\psi-\beta\|_\infty + \|\alpha-\varphi\|_\infty) \\ &= cT\|(\varphi,\psi)-(\alpha,\beta)\|_{X_{M,\delta,T}},\end{aligned}$$

where

$$c = \max\left\{ \nu_d, \frac{1}{1-T\left(\lambda_r - 2\pi\lambda_p\frac{M}{\delta}\right)} \frac{8\pi^2 M}{\delta} \right\}.$$

To prove the last inequality, we compute the following:

$$\begin{aligned} J_1 &= \left\| \int_0^t e^{-(t-s)\nu_e} \nu_d \psi(x,\omega,s) ds - \int_0^t e^{-(t-s)\nu_e} \nu_d \beta(x,\omega,s) ds \right\|_\infty \\ &\le \sup_{x,\omega,t} \int_0^t e^{-(t-s)\nu_e} \nu_d |\psi - \beta| ds \\ &\le \nu_d T \|\psi - \beta\|_\infty, \end{aligned}$$

and similarly,

$$\begin{aligned} J_2 &\le \sup_{x,\omega,t} \int_0^t e^{-(t-s)(\lambda_r+\lambda_p)} |(Lf_\psi(\varphi) - Lf_\beta(\alpha))(x-\omega(t-s),\omega,t)| ds \\ &\le T \left\| \left(\frac{\lambda_r}{2\pi} + \lambda_p \frac{\varphi}{\rho_\varphi} \right) \rho_{f_\psi} - \left(\frac{\lambda_r}{2\pi} + \lambda_p \frac{\alpha}{\rho_\alpha} \right) \rho_{f_\beta} \right\|_\infty \\ &\le T \left(\lambda_r \|f_\psi - f_\beta\|_\infty + \lambda_p \left\| \frac{\varphi}{\rho_\varphi} \rho_{f_\psi} - \frac{\alpha}{\rho_\alpha} \rho_{f_\beta} \right\|_\infty \right) \\ &\le T \left(\lambda_r \|f_\psi - f_\beta\|_\infty + 2\pi\lambda_p \left\| \frac{\varphi}{\rho_\varphi} - \frac{\alpha}{\rho_\alpha} \right\|_\infty \|f_\psi\|_\infty \right. \\ &\qquad \left. + 2\pi\lambda_p \left\| \frac{\alpha}{\rho_\alpha} \right\|_\infty \|f_\psi - f_\beta\|_\infty \right) \end{aligned}$$

$$\begin{aligned} \Leftrightarrow J_2 = \|f_\psi - f_\beta\|_\infty &\le \frac{T}{1 - T\left(\lambda_r - 2\pi\lambda_p \frac{M}{\delta}\right)} 2\pi \left\| \frac{\varphi}{\rho_\varphi} - \frac{\alpha}{\rho_\alpha} \right\|_\infty \|f_\psi\|_\infty \qquad (4.11) \\ &\le \frac{T}{1 - T\left(\lambda_r - 2\pi\lambda_p \frac{M}{\delta}\right)} 2\pi \left\| \frac{(\varphi - \alpha)\rho_\alpha + \alpha(\rho_\alpha - \rho_\varphi)}{\rho_\alpha \rho_\varphi} \right\|_\infty \|f_\psi\|_\infty \\ &\le \frac{T}{1 - T\left(\lambda_r - 2\pi\lambda_p \frac{M}{\delta}\right)} \frac{2\pi}{\delta^2} (\|\varphi - \alpha\|_\infty \|\rho_\alpha\|_\infty + \|\alpha\|_\infty \|\rho_\alpha - \rho_\varphi\|_\infty) \|f_\psi\|_\infty \\ &\le \frac{T}{1 - T\left(\lambda_r - 2\pi\lambda_p \frac{M}{\delta}\right)} \frac{8\pi^2 M^2}{\delta^2} \|\varphi - \alpha\|_\infty. \end{aligned}$$

In conclusion, for a sufficiently short time T, (4.10) is therefore well defined and a contraction and the Banach fixed-point theorem can be applied. In total, we have the following theorem.

Theorem 4.1.6 (Unique existence of solutions of (3.1) - (3.3)). *Let $(\varphi,\psi),(g,f) \in X_{M,\delta,T}$ and $\|f_0\|_\infty \le M, \|g_0\|_\infty \le M/2$, then there exists a time $0 < \tau \le T$, for which Φ is well defined and there exists a unique solution of (3.1) - (3.3) in $X_{M,\delta,\tau}$.*

4.2 Vicsek-Version

In order to investigate a model in which ant trails might emerge, and to avoid some technical details in choosing a non-empty complete metric space like in (4.7), we can modify our model (2.1) - (2.3) by changing the distribution with which the trail recruitment jump chooses a new direction. Here we chose the von Mises distribution; then the probability that an ant changes its angle by $\omega \in \mathbb{S}^1$ is given by

$$M_{\Omega_g}(\omega) = \frac{\exp(\Omega_g \cdot \omega)}{\displaystyle\int_{\mathbb{S}^1} \exp(\Omega_g \cdot \widetilde{\omega}) d\widetilde{\omega}},$$
$$\Omega_g = \frac{J_g}{|J_g|}.$$

Conducting the same rescaling as before and assuming again that $f_\varepsilon \xrightarrow{\varepsilon \to 0} f, g_\varepsilon \xrightarrow{\varepsilon \to 0} g$ results in the model

$$\partial_t f + \omega \cdot \nabla_x f = \left(\frac{\lambda_r}{2\pi} + \lambda_p M_{\Omega_g}\right) \rho_f - (\lambda_r + \lambda_p) f =: Q(f,g), \tag{4.12}$$
$$\partial_t g = \nu_d f - \nu_e g, \tag{4.13}$$
$$f\big|_{t=0} = f_0, \qquad g\big|_{t=0} = g_0, \tag{4.14}$$

with $Q(f,g) = 0$. With the same computations as in Remark 1 and Chapter 3.1, we then get mass preservation, as well as a similar consistency relationship

$$0 = Q(f,g) = \left(\frac{\lambda_r}{2\pi}\rho_f + \lambda_p M_{\Omega_g}\right) - (\lambda_r + \lambda_p) f$$
$$\Leftrightarrow f = \frac{1}{\lambda_r + \lambda_p}\left(\frac{\lambda_r}{2\pi} + \lambda_p M_{\Omega_g}\right)\rho_f,$$

and macroscopic equations

$$\partial_t\left(\frac{g}{\rho_g}\right) = \nu_d \frac{\rho_f}{\rho_g}\frac{\lambda_p}{\lambda_r + \lambda_p}\left(\frac{1}{2\pi} - \left(\frac{\lambda_r + \lambda_p}{\lambda_p} - \frac{\lambda_p}{\lambda_r} M_{\Omega_g}\frac{\rho_g}{g}\right)\right),$$
$$\partial_t \rho_f + \frac{\lambda_p}{\lambda_r + \lambda_p}\nabla_x \cdot \Omega_g = 0,$$
$$J_f = \frac{\lambda_p}{\lambda_r + \lambda_p}\rho_f \Omega_g,$$
$$\partial_t \rho_g = \nu_d \rho_f - \nu_e \rho_g,$$
$$\partial_t J_g = \nu_d \frac{\lambda_p}{\lambda_r + \lambda_p}\Omega_g - \nu_e J_g.$$

In this case, the asymptotic behaviour as $t \to \infty$ is less transparent than in the original model. We now proceed to discuss the existence of solutions. Following the same approach as in the previous chapter, we start by examining a slightly modified problem involving the functions φ, ψ.

$$\partial_t f + \omega \cdot \nabla_x f = \left(\frac{\lambda_r}{2\pi} + \lambda_p M_{\Omega_\varphi}\right)\rho_f - (\lambda_r + \lambda_p) f =: Sf(\varphi) - af, \tag{4.15}$$

$$\partial_t g = \nu_d \psi - \nu_e g, \tag{4.16}$$

$$f\big|_{t=0} = f_0, \qquad g\big|_{t=0} = g_0. \tag{4.17}$$

This model is similar to the original one, so we expect similar results. However, the von Mises distribution was deliberately chosen to avoid technicalities, which will allow us to choose a different space.

Lemma 4.2.1. *If ψ is continuous with respect to time in $[0,T]$, then there exists a unique solution for (4.16) for all $(x,\omega,t) \in \mathbb{R}^2 \times \mathbb{S}^1 \times [0,T]$ and the solution g is given by*

$$g = g_0 e^{-t\nu_e} + \int_0^t e^{-\nu_e(t-s)} \nu_d \psi(s) ds.$$

Proof. See Lemma 4.1.1 and Corollary 4.1.3. □

Lemma 4.2.2. *If*

$$f_0 \in C_b(\mathbb{R}^2 \times \mathbb{S}^1),$$
$$0 \le \varphi \in C_b(\mathbb{R}^2 \times \mathbb{S}^1 \times [0,T]),$$
$$J_\varphi(x,t) \neq 0 \qquad \forall x \in \mathbb{R}, t \in [0,T],$$

then there exists a unique weak solution of (4.15), given by

$$f(x,\omega,t) = f_0(x - \omega t, \omega) e^{-t(\lambda_r + \lambda_p)} + \int_0^t e^{-(t-s)(\lambda_r+\lambda_p)} Sf(\varphi)(s, x - \omega(t-s), \omega) ds.$$

Proof. As in the previously used model, $\lambda_r + \lambda_p$ is constant and thus bounded and continuous. We further need $\frac{\lambda_r}{2\pi} + \lambda_p M_{\Omega_\varphi}$ to be bounded and continuous. Continuity is given if φ is continuous. By the definition of the von Mises distribution, it is always bounded:

$$\begin{aligned} \|M_{\Omega_\varphi}\|_\infty &= \sup_{x,\omega,t} \left| \frac{e^{\frac{J_\varphi}{|J_\varphi|}\cdot\omega}}{\int_{\mathbb{S}^1} e^{\frac{J_\varphi}{|J_\varphi|}\cdot\widetilde{\omega}} d\widetilde{\omega}} \right| \\ &= \sup_{x,\omega,t} \left| \frac{e^{\frac{J_\varphi}{|J_\varphi|}\cdot\omega}}{2\pi I_0(1)} \right|, \end{aligned}$$

where $\frac{J_\varphi}{|J_\varphi|}, \omega \in \mathbb{S}^1$, and $I_0(x)$ is the modified Bessel function. In order for Ω_φ to be well defined, J_φ must not be equal to 0 for all $(x,t) \in \mathbb{R}^2 \times [0,T]$. □

The next step is again to find a complete metric space Y and a contraction mapping from Y to Y. Let

$$Y_T = C_b(\mathbb{R}^2 \times \mathbb{S}^1 \times [0,T])^2,$$

with the norm $\|(\varphi,\psi)\|_{Y_T} = \|\varphi\|_\infty + \|\psi\|_\infty$, which is now even a Banach space. The function Φ is analogously defined:

$$\Phi : Y_T \to Y_T,$$
$$\Phi(\varphi,\psi) = \begin{pmatrix} g_0(x,\omega)e^{-t\nu_e} + \int_0^t e^{-(t-s)\nu_e}\nu_d\psi(x,\omega,s)ds \\ f_0(x-\omega t,\omega)e^{-t(\lambda_r+\lambda_p)} + \int_0^t e^{-(t-s)(\lambda_r+\lambda_p)}Sf(\varphi)(x-\omega(t-s),\omega,s)ds \end{pmatrix}.$$

Theorem 4.2.3 (Unique existence of solutions of (4.12) - (4.14)). *Let $(\varphi,\psi),(g,f)\in Y_T$ and $\inf_x |J_{g_0}| > \delta$ for some $\delta > 0$, then there exists a time $0<\tau\le T$, for which Φ is well defined and there exists a unique weak solution of (4.12) - (4.14) in Y_τ.*

Proof. Showing that Φ maps from Y_T to Y_T is done analogously as before, only the norm is different now. For the contraction property, we need $\|M_{\Omega_\varphi} - M_{\Omega_\alpha}\|_\infty \le \|\varphi-\alpha\|_\infty$. For this we refer to [10, Proposition 2.11], the rest can again be computed analogously as in the previous chapter. □

4.3 Generalization of Lemma 4.1.6

The similarity of proofs for the existence and uniqueness of solutions in the two models discussed above raises the question if similar results can be obtained not only for specific models but for a set of models. To investigate this, let us consider a more general model with $k_g, a, \nu_d, \nu_e \ge 0$:

$$\partial_t f(x,\omega,t) + \omega\cdot\nabla_x f(x,\omega,t) = \int_{\mathbb{S}^1} k_g(x,\omega,\widetilde{\omega},t)f(x,\widetilde{\omega},t)d\widetilde{\omega} - a(x,\omega,t)f(x,\omega,t) \tag{4.18}$$
$$=: Jf(g) - af,$$
$$\partial_t g(x,\omega,t) = \nu_d(x,\omega,t)f(x,\omega,t) - \nu_e(x,\omega,t)g(x,\omega,t), \tag{4.19}$$
$$f\big|_{t=0} = f_0, \qquad g\big|_{t=0} = g_0, \tag{4.20}$$

with $\int_{\mathbb{S}^1} k_g f d\widetilde{\omega}$ describing the ants changing their angle to ω while af does the opposite. Note that the subscript in k_g empathizes the dependency of k on g and ν_d, ν_e are not constants but functions describing the deposition and evaporation of pheromones. As usual, we define a similar problem, using the functions φ, ψ:

$$\partial_t f(x,\omega,t) + \omega\cdot\nabla_x f(x,\omega,t) = \int_{\mathbb{S}^1} k_\varphi(x,\omega,\widetilde{\omega},t)f(x,\widetilde{\omega},t)d\widetilde{\omega} - a(x,\omega,t)f(x,\omega,t) \tag{4.21}$$
$$=: Jf(\varphi) - af,$$
$$\partial_t g(x,\omega,t) = \nu_d(x,\omega,t)\psi(x,\omega,t) - \nu_e(x,\omega,t)g(x,\omega,t), \tag{4.22}$$
$$f\big|_{t=0} = f_0, \qquad g\big|_{t=0} = g_0. \tag{4.23}$$

Now, let us start by applying the Picard-Lindelöf and Duhamel theorem.

Lemma 4.3.1. *If ψ is continuous w.r.t. t in $[0,T]$ and $\|\nu_d\|_\infty, \|\nu_e\|_\infty < \infty$, then there exists a unique solution for (4.22) for all $(x,\omega,t) \in \mathbb{R}^2 \times \mathbb{S}^1 \times [0,T]$.*

Proof. Direct application of the Theorem 4.1.2. □

This result can be generalized even more by using one of many existence and uniqueness theorems which can be found in the literature, e.g., see [11].

Lemma 4.3.2. *The criteria for the existence and uniqueness of solutions of (4.21) are exactly those of Theorem 4.1.4.*

Now what is left to be done, is to find a non-empty complete metric space $Z_T \subseteq C_b(\mathbb{R}^2 \times \mathbb{S}^1 \times [0,T])$, where the function Φ maps Z_T into Z_T, where Φ is defined as usual:

$$\Phi : Z_T \to Z_T,$$

$$\Phi(\varphi,\psi) = \begin{pmatrix} g_0(x,\omega)\exp\left(-\int_0^t \nu_e(x,\omega,s)ds\right) \\ \qquad + \int_0^t \exp\left(-\int_s^t \nu_e(x,\omega,\tau)d\tau\right)\nu_d(x,\omega,t)\psi(x,\omega,s)ds, \\ f_0(x-\omega t,\omega)\exp\left(-\int_0^t a(x-\omega(t-s),\omega,t)ds\right) \\ \qquad + \int_0^t Jf(x-\omega(t-s),\omega,s)\exp\left(-\int_s^t a(x-\omega(t-\tau),\omega,\tau)ds\right) \end{pmatrix},$$

which can of course not be done explicitly for all functions k, a, ν_e, ν_d, so we need to restrict the problem to the functions where such a space exists. Reexamining the proof of Theorem 4.1.6, one finds that the operator J needs to fulfil

$$\|Jf(\varphi) - Jf(\alpha)\|_{Z_T} \leq c\|\varphi - \alpha\|_{Z_T}. \tag{4.24}$$

A new requirement is that a, ν_e need to be bounded from below, since they are no longer constant, specifically, we require $\inf_{x,\omega,t} a, \inf_{x,\omega,t} \nu_e > 0$. This ensures that the estimate (4.9) can be used. With these preceding lemmata, we get conditions for the existence of solutions for both problems stated at the beginning of this chapter.

Theorem 4.3.3. *Let Lemma 4.3.1, Theorem 4.1.4 and $\inf_{x,\omega,t} a, \inf_{x,\omega,t} \nu_e > 0$ be satisfied, if there exists a non-empty complete metric space Z_T where (4.24) is fulfilled and $(\varphi,\psi), (g,f) \in Z_T$, then there exists a time $0 < \tau \leq T$, for which Φ is well defined and there exists a unique weak solution of (4.18) - (4.20) in Z_τ.*

Remark 8. The additional, explicit conditions such as $\inf_{x,t} \rho_\varphi \geq \delta, J_\varphi \neq 0$ in the previous models were necessary to ensure boundedness and well definedness of the posed question.

5 Exploring other Models

In this section, we will introduce and study modifications of (3.1) - (3.3). The depth of analysis of these models varies depending on the perceived biological relevance of the resulting models. The computations for the macroscopic equations are very similar to the original model, so the details will be left as an exercise to the reader. Furthermore, mass conservation can also be shown for the following models using the same integrations.

5.1 No Evaporation

Assuming there is no evaporation or other decay of the pheromones, the differential equations change to

$$\begin{aligned} \partial_t f + \omega \cdot \nabla_x f &= \left(\frac{\lambda_r}{2\pi} + \lambda_p \frac{g}{\rho_g} \right) \rho_f - (\lambda_r + \lambda_p) f =: Q_1(f, g), \\ \partial_t g &= \nu_d f, \\ f\big|_{t=0} &= f_0, \qquad g\big|_{t=0} = g_0. \end{aligned}$$

With $Q_1(f, g) = 0$, we get the same consistency relation

$$f = \frac{1}{\lambda_r + \lambda_p} \left(\frac{\lambda_r}{2\pi} + \lambda_p \frac{g}{\rho_g} \right) \rho_f.$$

Since the only difference in the dynamics is the changed pheromone evaporation, we get almost the same equations; only the evaporation terms are missing. And, since in the computation of $\partial_t(g/\rho_g)$ in (3.5) the evaporation terms cancel out, we observe the same result here, namely that the local pheromone orientation converges to the uniform distribution. In total, we get

$$\begin{aligned} 0 &= \partial_t \rho_f + \frac{\lambda_p}{\lambda_r + \lambda_p} \nabla_x \cdot \left(J_g \frac{\rho_f}{\rho_g} \right), \\ J_f &= \frac{\lambda_p}{\lambda_r + \lambda_p} \rho_f \frac{J_g}{\rho_g}, \\ \partial_t \rho_g &= \nu_d \rho_f, \\ \partial_t J_g &= \nu_d \frac{\lambda_p}{\lambda_r + \lambda_p} \frac{\rho_f}{\rho_g} J_g, \\ \partial_t \left(\frac{g}{\rho_g} \right) &= \nu_d \frac{\rho_f}{\rho_g} \frac{1}{\lambda_r + \lambda_p} \left(\frac{1}{2\pi} - \frac{g}{\rho_g} \right). \end{aligned}$$

5.2 No Noise

Now, assuming there are no random jumps, the dynamics change to

$$
\begin{aligned}
\partial_t f + \omega \cdot \nabla_x f &= \lambda_p \left(\frac{g}{\rho_g} \rho_f - f \right) =: Q_2(f,g), \\
\partial_t &= \nu_d f - \nu_e g, \\
f\big|_{t=0} = f_0, &\qquad g\big|_{t=0} = g_0.
\end{aligned}
$$

To derive our consistency relationship, we use $Q_2(f,g) = 0$ once more and then

$$
f = \rho_f \frac{g}{\rho_g}
$$

follows. The pheromone orientation shows different dynamics:

$$
\begin{aligned}
\partial_t \left(\frac{g}{\rho_g} \right) &= \frac{(\partial_t g)\rho_g - g \partial_t \rho_g}{{\rho_g}^2} \\
&= \frac{(\nu_d f - \nu_e g)\rho_g - g(\nu_d \rho_f - \nu_e \rho_g)}{{\rho_g}^2} \\
&= \nu_d \frac{\frac{\rho_f}{\rho_g} g \rho_g - g \rho_f}{\rho_g^2} \\
&= 0.
\end{aligned}
$$

This implies that the relative amount of pheromones pointing in one direction does not change over time and that the convergence to the uniform distribution is solely driven by uniform, random noise. The dynamics of the local densities and average orientations change to

$$
\begin{aligned}
0 &= \partial_t \rho_f + \nabla_x \cdot \left(\rho_f \frac{J_g}{\rho_g} \right), \\
J_f &= \rho_f \frac{J_g}{\rho_g}, \\
\partial_t \rho_g &= \nu_d \rho_f - \nu_e \rho_g, \\
\partial_t J_g &= J_g \left(\nu_d \frac{\rho_f}{\rho_g} - \nu_e \right).
\end{aligned}
$$

5.3 Time delayed Pheromone Evaporation

When we assume that pheromones do not immediately start to evaporate, only after a time τ, then g depends on another variable; the age $h \in \mathbb{R}_+$, giving $g = g(x, \omega, h, t)$. The PDEs describing these dynamics are now

$$\begin{aligned}
\partial_t f + \omega \cdot \nabla_x f &= \left(\frac{\lambda_r}{2\pi} + \lambda_p \frac{g}{\rho_g} \right) \rho_f - (\lambda_r + \lambda_p) f =: Q_3(f, g), \\
\partial_t g + \frac{1}{\tau} \partial_h g &= \nu_d f - \nu_e g \chi_{\{h > \tau\}}, \\
\partial_t h &= \frac{1}{\tau}, \\
f\big|_{t=0} &= f_0, \qquad g\big|_{t=0} = g_0, \qquad h\big|_{t=0} = h_0.
\end{aligned}$$

where χ is the characteristic function. Using $Q_3(f, g) = 0$ once more, we can get the consistency relationship

$$f = \frac{1}{\lambda_r + \lambda_p} \left(\frac{\lambda_r}{2\pi} + \lambda_p \frac{g}{\rho_g} \right) \rho_f.$$

The dynamics of the macroscopic equations are given by

$$\begin{aligned}
0 &= \partial_t \rho_f + \frac{\lambda_p}{\lambda_r + \lambda_p} \nabla_x \cdot \left(J_g \frac{\rho_f}{\rho_g} \right), \\
J_f &= \frac{\lambda_p}{\lambda_r + \lambda_p} \rho_f \frac{J_g}{\rho_g}, \\
\partial_t \rho_g + \frac{1}{\tau} \partial_h \rho_g &= \nu_d \rho_f - \nu_e \rho_g \chi_{\{h > \tau\}}, \\
\partial_t J_g + \frac{1}{\tau} \partial_h J_g &= \left(\nu_d \frac{\lambda_p}{\lambda_r + \lambda_p} \frac{\rho_f}{\rho_g} - \nu_e \chi_{\{h > \tau\}} \right) J_g.
\end{aligned}$$

The change of the relative amount of pheromones $\partial_t \left(\frac{g}{\rho_g} \right)$ is computed via

$$\begin{aligned}
\partial_t \left(\frac{g}{\rho_g} \right) &= \frac{(\partial_t g) \rho_g - g \partial_t \rho_g}{{\rho_g}^2} \\
&= \frac{\left(-\frac{1}{\tau} \partial_h g + \nu_d f - \nu_e g \chi_{\{h > \tau\}} \right) \rho_g}{\rho_g^2} \\
&\quad - \frac{g \left(-\frac{1}{\tau} \partial_h \rho_g + \nu_d \rho_f - \nu_e \rho_g \chi_{\{h > \tau\}} \right)}{{\rho_g}^2} \\
&= -\frac{1}{\tau} \frac{(\partial_h g) \rho_g - g (\partial_h \rho_g)}{{\rho_g}^2} + \nu_d \frac{\rho_f}{\rho_g} \frac{\lambda_r}{\lambda_r + \lambda_p} \left(\frac{1}{2\pi} - \frac{g}{\rho_g} \right) \\
\Leftrightarrow \partial_t \left(\frac{g}{\rho_g} \right) + \frac{1}{\tau} \partial_h \left(\frac{g}{\rho_g} \right) &= \nu_d \frac{\rho_f}{\rho_g} \frac{\lambda_r}{\lambda_r + \lambda_p} \left(\frac{1}{2\pi} - \frac{g}{\rho_g} \right).
\end{aligned}$$

5.4 Randomness on $S \subseteq \mathbb{S}^1$

Here we assume that the new direction of a random jump is not chosen from the whole unit circle, but uniformly from the interval $[\omega-a,\omega+a]$, where ω is the current orientation and $0 \leq a \leq \pi$ determines the width from which the new angle is chosen. This gives the PDE system

$$\begin{aligned}
\partial_t f + \omega \cdot \nabla_x f &= \lambda_r \left(\frac{1}{2a} \int_{-a}^{a} f(x, R_\varepsilon \omega, t) d\varepsilon - f \right) + \lambda_p \left(\rho_f \frac{g}{\rho_g} - f \right) \\
&= \frac{\lambda_r}{2a} f_a + \lambda_p \frac{g}{\rho_g} - (\lambda_r + \lambda_p) f =: Q_4(f,g), \\
\partial_t g &= \nu_d f - \nu_e g, \\
f\big|_{t=0} &= f_0, \qquad g\big|_{t=0} = g_0.
\end{aligned}$$

where R_ε is the rotation by degree ε and $f_a := \int_{-a}^{a} f(x, R_\varepsilon \omega, t) d\varepsilon$. The consistency relationship is received via $Q_4(f,g) = 0$ and is given by

$$f = \frac{1}{\lambda_r + \lambda_p} \left(\frac{\lambda_r}{2a} f_a + \lambda_p \rho_f \frac{g}{\rho_g} \right).$$

Next, the dynamics of g/ρ_g over time are computed:

$$\partial_t \left(\frac{g}{\rho_g} \right) = \nu_d \frac{\rho_f}{\rho_g} \frac{\lambda_r}{\lambda_r + \lambda_p} \left(\frac{1}{2a} \frac{f_a}{\rho_f} - \frac{g}{\rho_g} \right),$$

so the pheromone orientation relaxes to the current average ant distribution in the angle interval $[\omega - a, \omega + a]$.

Remark 9. With $a = 0$, this model is the same as that discussed in Chapter 5.2. To check if this model agrees with the model without random noise, one can choose $a = 0$ to see that $\partial_t(g/\rho_g) = 0$. It is important to note that $\frac{f_a}{2a}$ converges to $\int f(x, \omega - \varepsilon)\delta_0(\varepsilon) d\varepsilon$, where δ_0 is the delta distribution. With $a = \pi$ it becomes the original model again.

The macroscopic dynamics are given by

$$\begin{aligned}
0 &= \partial_t \rho_f + \frac{1}{\lambda_r + \lambda_p} \nabla_x \cdot \left(\frac{\lambda_r}{2a} \int_{\mathbb{S}^1} \omega f_a d\omega + \lambda_p \frac{\rho_f}{\rho_g} J_g \right) \\
&= \partial_t \rho_f + \frac{1}{\lambda_r + \lambda_p} \nabla_x \cdot \left(\lambda_r \frac{\sin(a)}{a} J_f + \lambda_p \frac{\rho_f}{\rho_g} J_g \right), \\
J_f &= \lambda_r \frac{\sin(a)}{a} J_f + \lambda_p \rho_f \frac{J_g}{\rho_g}, \\
\partial_t \rho_g &= \nu_d \rho_f - \nu_e \rho_g, \\
\partial_t J_g &= \nu_d \int_{\mathbb{S}^1} \omega f d\omega - \nu_e J_g \\
&= \frac{\lambda_r \nu_d}{\lambda_r + \lambda_p} \frac{\sin(a)}{a} J_f + J_g \left(\frac{\lambda_p \nu_d}{\lambda_r + \lambda_p} \frac{\rho_f}{\rho_g} - \nu_e \right).
\end{aligned}$$

Here, $\int_{\mathbb{S}^1} \omega f_a d\omega = 2J_f \sin(a)$ is shown by

$$\begin{aligned}
\int_{\mathbb{S}^1} \omega \int_{-a}^{a} f(x, R_\varepsilon \omega, t) d\varepsilon d\omega &= \int_{-a}^{a} \int_{\mathbb{S}^1} \omega f(x, R_\varepsilon \omega, t) d\omega d\varepsilon \\
&= \int_{-a}^{a} \int_{\mathbb{S}^1} R_\varepsilon^{-1}(u) f(x, u, t) du d\varepsilon \\
&= \int_{-a}^{a} \int_{\mathbb{S}^1} R_\varepsilon^{-1} \left(u f(x, u, t)\right) du d\varepsilon \\
&= \int_{-a}^{a} R_\varepsilon^{-1} \left(\int_{\mathbb{S}^1} u f(x, u, t) du \right) d\varepsilon \\
&= \int_{-a}^{a} R_\varepsilon^{-1} J_f d\varepsilon \\
&= 2J_f \int_0^a \cos(\theta) d\theta \\
&= 2J_f \sin(a),
\end{aligned}$$

and

$$\begin{aligned}
\int_{-a}^{a} R_\varepsilon^{-1} J_f d\varepsilon &= \int_0^a (R_\varepsilon^{-1} + R_\varepsilon) J_f d\varepsilon \\
&= \int_0^a \frac{2 \left\langle \begin{pmatrix} \cos(\varepsilon) J_{f_x} \\ \cos(\varepsilon) J_{f_y} \end{pmatrix}, \begin{pmatrix} J_{f_x} \\ J_{f_y} \end{pmatrix} \right\rangle \begin{pmatrix} J_{f_x} \\ J_{f_y} \end{pmatrix}}{\sqrt{J_{f_x}^2 + J_{f_y}^2}} \\
&= \int_0^a \frac{2 \cos(\varepsilon)(x^2 + y^2) \begin{pmatrix} J_{f_x} \\ J_{f_y} \end{pmatrix}}{\sqrt{J_{f_x}^2 + J_{f_y}^2}} \\
&= \int_0^a 2 \cos(\varepsilon) J_f.
\end{aligned}$$

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