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„On the Independence of Splitting and Unboundedness“

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# Abriss

In dieser Arbeit behandeln wir die Unabhängigkeit der Bounding Number $\mathfrak{b}$ und der Splitting Number $\mathfrak{s}$. Wir beginnen mit der Untersuchung der Auswirkungen von Mathias Forcing auf diese beiden Kardinalzahlen. In den beiden folgenden Kapiteln werden wir logarithmische Maße einführen und verwenden diese um ein Model für $\mathfrak{b} < \mathfrak{s}$ zu konstruieren. Nachdem wir eine andere Konstruktion dieses Models mittels Mathias Forcing relativiert zu einem bestimmten Ultrafilter besprechen, werden wir die Konsistenz von $\mathfrak{s} < \mathfrak{b}$ mit einer Iteration mit endlichem Träger von Hechler Forcing zeigen, um die Unabhängigkeit der Kardinalzahlen zu erhalten.

# Abstract

In this thesis, we study the independence of the bounding number $\mathfrak{b}$ and the splitting number $\mathfrak{s}$. We will begin by studying the effects of Mathias forcing on these two cardinal invariants. In the following two chapters, we will discuss logarithmic measures and use them to build a model to show the consistency of $\mathfrak{b} < \mathfrak{s}$. After discussing a different approach to construct this model using Mathias forcing with respect to a specific ultrafilter, we will show the consistency of $\mathfrak{s} < \mathfrak{b}$ via finite support iteration of Hechler forcing to obtain the independence of these cardinal invariants.

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# 1 Introduction

Cardinal characteristics and the relations between them are an important area of study in set theory. In this thesis, we will focus on two of them, the bounding and the splitting number. For a general reference on cardinal charactersitic, see [Bla09].
The bounding number was introduced by Rothberger in [Rot39]. For the contemporary definition recall that given two functions $f, g \in {}^{\omega}\omega$, $g$ is said to *dominate* $f$, denoted by $f \leq^* g$, if $f(n) \leq g(n)$ for all but finitely many $n \in \omega$, so there is $n_0 \in \omega$ such that for all $n \geq n_0$, $f(n) \leq g(n)$.

**Definition 1.1.** *A family of functions $\mathcal{B} \subseteq {}^{\omega}\omega$ is called unbounded, if there is no function in ${}^{\omega}\omega$ which dominates all members of $\mathcal{B}$.*
*The bounding number $\mathfrak{b}$ is the least size of an unbounded family, that is*

$$\mathfrak{b} := \min\{|\mathcal{B}| : \mathcal{B} \subseteq {}^{\omega}\omega \textit{ is unbounded}\}.$$

The splitting number was first introduced by Booth in [Boo74]. For the definition recall that for two infinite sets $x, y \in [\omega]^{\omega}$ we say that $y$ *splits* $x$ if both $x \cap y$ and $x \setminus y$ are infinite.

**Definition 1.2.** *A family $\mathcal{S} \subseteq [\omega]^{\omega}$ is called spitting, if for all $x \in [\omega]^{\omega}$ there is $y \in \mathcal{S}$ such that $y$ splits $x$.*
*The splitting number $\mathfrak{s}$ is the least size of a splitting family, that is*

$$\mathfrak{s} := \min\{|\mathcal{S}| : \mathcal{S} \subseteq [\omega]^{\omega} \textit{ is splitting}\}.$$

Both $\mathfrak{b}$ and $\mathfrak{s}$ are uncountable cardinals which are of size less than or equal to the continuum. We will show the consistency of $\mathfrak{b} < \mathfrak{s}$ and $\mathfrak{s} < \mathfrak{b}$, which means that these cardinals are independent.
In the second chapter, we will go through some general definitions and results which will be used in the later chapters. The focus mainly lies on iterated forcing constructions and preservation theorems.
Chapter 3 discusses the effects of Mathias forcing on the bounding and the splitting number. As we will see, a countable support iteration of Mathias forcing increases the size of the splitting number, which makes such iterations a candidate for the consistency of $\mathfrak{b} < \mathfrak{s}$. However, these iterations also increase the size of the bounding number, which makes countable support iterations of Mathias forcing unsuitable for the consistency of $\mathfrak{b} < \mathfrak{s}$.
The consistency of $\mathfrak{b} < \mathfrak{s}$ was first established by Shelah in [She84] where he introduced the notion of logarithmic measures to define the forcing notion $Q$. This poset is proper

and almost ${}^{\omega}\omega$-bounding, and a countable support iteration of length $\omega_2$ yields a model for $\mathfrak{b} = \omega_1 < \mathfrak{s} = \omega_2$. In the fourth and fifth chapter, we will follow [FS08] to show the more general result of $\mathfrak{b} = \kappa < \mathfrak{s} = \kappa^+$ which is shown using ccc posets $Q(C)$ which are suborders of $Q$. The most general inequality of $\mathfrak{b} = \kappa < \mathfrak{s} = \lambda$ was established in [FI10] using matrix iterations, but we will not cover this result in this thesis.
Chapter 6 discusses a connection between $Q(C)$ and Mathias forcing relativized to a specific ultrafilter $\mathcal{U}_C$ which was established in [FI10]. With this connection and the previous results about $Q(C)$, one can construct a model for $\mathfrak{b} < \mathfrak{s}$ with relativized Mathias forcing.
The consistency of $\mathfrak{s} < \mathfrak{b}$ was first shown by Balcar, Pelant and Simon in [BPS80]. We will follow [BD85] in the seventh chapter to show the consistency of $\mathfrak{s} < \mathfrak{b}$ via a finite support iteration of Hechler forcing.

# 2 Preliminaries

We will start by stating some basic definitions and theorems which will be used throughout the thesis.

## 2.1 Trees

We will need König's lemma in one proof, so for completeness we will state the relevant definitions and the lemma. More details can be found in [Kun13].

**Definition 2.1.** *A tree is a pair* $(T, \sqsubset)$ *where* $\sqsubset$ *is a strict partial order on* $T$ *and for each* $y \in T$ *the set* $\{x \in T : x \sqsubset y\}$ *is well-ordered by* $\sqsubset$*. Then define*

- $y \downarrow = \{x \in T : x \sqsubset y\}$ *and* $y \uparrow = \{x \in T : y \sqsubset x\}$
- $\operatorname{height}(y) = \operatorname{height}(y, T) \coloneqq \operatorname{type}(y \downarrow)$
- $\mathcal{L}_\alpha = \mathcal{L}_\alpha(T) \coloneqq \{y \in T : \operatorname{height}(y) = \alpha\}$*, which is the* $\alpha$*-th level of* $T$
- $\operatorname{height}(T)$ *is the least* $\alpha$ *such that* $\mathcal{L}_\alpha(T) = \emptyset$

**Definition 2.2.** *A path through a tree* $(T, \sqsubset)$ *is a chain* $C \subseteq T$ *such that* $\forall \alpha < \operatorname{height}(T) \, (C \cap \mathcal{L}_\alpha(T) \neq \emptyset)$*.*

Note that a tree of height $\omega$ may not have any paths through it. As an example consider the tree $T$ of all strictly decreasing sequences in $[\omega]^{<\omega}$. If $C$ is a path though $T$, then $\bigcup C : \omega \to \omega$ would be a strictly increasing function which is clearly impossible. However, the situation changes if each level of the tree is finite.

**Lemma 2.1** (König's Lemma)**.** *Let* $(T, \sqsubset)$ *be a tree such that* $\operatorname{height}(T) = \omega$ *and* $\mathcal{L}_n(T)$ *is finite for each* $n \in \omega$*. Then there is a path through* $T$*.*

*Proof.* We will choose a path through $T$ by recursion on $n$. As each level is non-empty and $\operatorname{height}(T) = \omega$, we can choose $x_0 \in \mathcal{L}_0(T)$ such that $x_0 \uparrow$ is infinite. As there are only finitely many nodes on level 1 there has to be an $x_1 \in \mathcal{L}_1(T)$ such that $x_1 \in x_0 \uparrow$ and $x_1 \uparrow$ is infinite. Repeat this process to obtain $\{x_n : n \in \omega\}$ where each $x_n \in \mathcal{L}_n(T)$, $x_n \uparrow$ is infinite and $x_{n+1} \in x_n \uparrow$. This is a path through $T$. $\square$

## 2.2 Forcing

We will use the method of forcing throughout this thesis to show various results for the independence of the bounding and the splitting number. We refer readers unfamiliar with the basics of forcing to [Kun13]. We will also follow the conventions from this book.
The following forcing notion will be used for the construction of a model for $\mathfrak{b} < \mathfrak{s}$.

**Definition 2.3.** *Let $\mathcal{A}$ be an infinite set of functions in ${}^{\omega}\omega$. Then $\mathbb{H}(\mathcal{A})$ is the forcing notion consisting of all pairs $(s, F)$, where $s \in {}^{<\omega}\omega$ and $F \in [\mathcal{A}]^{<\omega}$, with the extension relation $(s_1, F_1) \leq (s_2, F_2)$ if*

1. $s_2 \subseteq s_1$, $F_2 \subseteq F_1$
2. $\forall f \in F_2\, \forall k \in \mathrm{dom}(s_2) \setminus \mathrm{dom}(s_1)$ *we have* $s_1(k) \geq f(k)$.

Note that since for all $(s, F) \in \mathbb{H}(\mathcal{A})$, $s$ is countable and $F$ is a finite subset of $\mathcal{A}$, the forcing notion $\mathbb{H}(\mathcal{A})$ is of the same size as $\mathcal{A}$.

**Lemma 2.2.** *Let $\mathcal{A} \subseteq {}^{\omega}\omega$ be infinite. Then the partial order $\mathbb{H}(\mathcal{A})$ is $\sigma$-centered.*

*Proof.* Let $s \in {}^{<\omega}\omega$ and $(s, F_1), (s, F_2) \in \mathbb{H}(\mathcal{A})$. Then $(s, F_1 \cup F_2)$ is a common extension of $(s, F_1)$ and $(s, F_2)$. So $\{(s, F) : F \in [\mathcal{A}]^{<\omega}\}$ is centered and

$$\mathbb{H}(\mathcal{A}) = \bigcup_s \{(s, F) : F \in [\mathcal{A}]^{<\omega}\}.$$

Hence, $\mathbb{H}(\mathcal{A})$ is $\sigma$-centered.

□

**Lemma 2.3.** *Let $\mathcal{A}$ be an infinite family of reals. Then, for each $f \in \mathcal{A}$, the set*

$$D_f = \{(s, F) : (s, F) \in \mathbb{H}(\mathcal{A}), f \in F\}$$

*is dense in $\mathbb{H}(\mathcal{A})$.*

*Proof.* Fix $f \in \mathcal{A}$. Let $(s, F) \in \mathbb{H}(\mathcal{A})$ and we can assume that $f \notin F$ since otherwise $(s, F) \in D_f$. Then $(s, F \cup \{f\}) \in D_f$ and $(s, F \cup \{f\}) \leq (s, F)$, so $D_f$ is dense. □

**Lemma 2.4.** *Let $\mathcal{A}$ be an infinite family of reals. Then $\mathbb{H}(\mathcal{A})$ adds a real dominating $\mathcal{A}$.*

*Proof.* Let $G$ be a $\mathbb{H}(\mathcal{A})$-generic filter and define

$$f_G = \bigcup \{s : \exists F \in [\mathcal{A}]^{<\omega} : (s, F) \in G\}.$$

Let $f \in \mathcal{A}$ be arbitrary. By Lemma 2.3, the set $D_f$ is dense and so there is $(s, F) \in G \cap D_f$. Since $G$ is a filter, there is $(s', F') \in G$ such that $(s', F') \leq (s, F)$ and by definition of the extension relation we have $f(i) \leq f_G(i)$ for all $i > |s|$. □

### 2.2.1 Iterated Forcing

We will state the basic definitions and results about iterations of forcing notions, which we will use throughout the thesis. For more details and the proofs we refer again to [Kun13].
The general idea behind iterated forcing is to construct a generic extension of a model and then extending this new model once more using a second forcing notion. Since we want to build the final extension in the ground model, we need to name the second forcing notion.

**Definition 2.4.** *Let $\mathbb{P}$ be a forcing notion. A $\mathbb{P}$-name for a forcing notion is a triple of $\mathbb{P}$-names $(\dot{\mathbb{Q}}, \dot{\leq}_{\mathbb{Q}}, \dot{\mathbb{1}}_{\mathbb{Q}})$, such that $\dot{\mathbb{1}}_{\mathbb{Q}} \in \operatorname{dom}(\dot{\mathbb{Q}})$ and*

$$\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \left[\dot{\mathbb{1}}_{\mathbb{Q}} \in \dot{\mathbb{Q}} \wedge \dot{\leq}_{\mathbb{Q}} \textit{ is a pre-order on } \dot{\mathbb{Q}} \textit{ with largest element } \dot{\mathbb{1}}_{\mathbb{Q}}\right].$$

*We usually write $\dot{\mathbb{Q}}$ for $(\dot{\mathbb{Q}}, \dot{\leq}_{\mathbb{Q}}, \dot{\mathbb{1}}_{\mathbb{Q}})$.*

**Definition 2.5.** *Let $\mathbb{P}$ be a forcing notion and $\dot{\mathbb{Q}}$ a $\mathbb{P}$-name for a forcing notion. The two-step-iteration $\mathbb{P} * \dot{\mathbb{Q}}$ is a triple $(\mathbb{R}, \leq_{\mathbb{R}}, \mathbb{1}_{\mathbb{R}})$ such that*

1. $\mathbb{R} := \{(p, \dot{q}) \in \mathbb{P} \times \operatorname{dom}(\dot{\mathbb{Q}}) : p \Vdash_{\mathbb{P}} \dot{q} \in \dot{\mathbb{Q}}\}$
2. $\mathbb{1}_{\mathbb{R}} = (\mathbb{1}_{\mathbb{P}}, \dot{\mathbb{1}}_{\mathbb{Q}})$
3. $(p_1, \dot{q}_1) \leq_{\mathbb{R}} (p_2, \dot{q}_2)$ *iff* $p_1 \leq_{\mathbb{P}} p_2$ *and* $p_1 \Vdash_{\mathbb{P}} \dot{q}_1 \dot{\leq}_{\mathbb{Q}} \dot{q}_2$.

*Moreover, define $i : \mathbb{P} \to \mathbb{R}$ by $i(p) = (p, \dot{\mathbb{1}}_{\mathbb{Q}})$.*

**Lemma 2.5.** *Let $\mathbb{P}$ be a forcing notion, with $p_0, p_1 \in \mathbb{P}$ and let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a forcing notion with $\dot{q}_0, \dot{q}_1 \in \operatorname{dom}(\dot{\mathbb{Q}})$. Let $\mathbb{R} = \mathbb{P} * \dot{\mathbb{Q}}$ and we have*

1. *$\mathbb{R}$ is a forcing notion*
2. $p_0 \leq_{\mathbb{P}} p_1 \leftrightarrow i(p_0) \leq_{\mathbb{R}} i(p_1)$
3. $i(\mathbb{1}_{\mathbb{P}}) = \mathbb{1}_{\mathbb{R}}$
4. $p_0 \perp_{\mathbb{P}} p_1 \to (p_0, \dot{q}_0) \perp_{\mathbb{R}} (p_1, \dot{q}_1)$, *whenever* $(p_0, \dot{q}_0)$ *and* $(p_1, \dot{q}_1)$ *are in* $\mathbb{R}$
5. $p_0 \perp_{\mathbb{P}} p_1 \leftrightarrow (p_0, \dot{\mathbb{1}}_{\mathbb{Q}}) \perp_{\mathbb{R}} (p_1, \dot{q}_1)$, *whenever* $(p_1, \dot{q}_1)$ *is in* $\mathbb{R}$
6. $p_0 \perp_{\mathbb{P}} p_1 \leftrightarrow i(p_0) \perp_{\mathbb{R}} i(p_1)$
7. *$i$ is a complete embedding*

With the two-step-iteration $\mathbb{P} * \dot{\mathbb{Q}}$ we can view extending the ground model with $\mathbb{P}$ and then with $\mathbb{Q}$ is one generic extension.

**Theorem 2.1.** *Let $\mathbb{R} = \mathbb{P} * \dot{\mathbb{Q}}$ be a two-step-iteration in $M$ and $K$ be a $\mathbb{R}$-generic filter over $M$. Define $G = i^{-1}(K)$ and*

$$H = \{\dot{q}_G : \dot{q} \in \operatorname{dom}(\dot{\mathbb{Q}}) \wedge \exists p \in \mathbb{P}(p, \dot{q}) \in K\}.$$

*Then*

1. *$G$ is $\mathbb{P}$-generic over $M$,*
2. *$H$ is $\dot{\mathbb{Q}}_G$-generic over $M[G]$,*
3. *$K = G * H := \{(p, \dot{q}) \in \mathbb{R} : p \in G \wedge \dot{q}_G \in H\}$, and*
4. *$M[K] = M[G][H]$.*

The two-step-iteration can now be generalized to construct longer iterations.

**Definition 2.6.** *Let $\alpha$ be an ordinal. An iterated forcing construction is a pair of sequences*

$$(\langle(\mathbb{P}_\xi, \leq_\xi, \mathbb{1}_\xi) : \xi \leq \alpha\rangle, \langle(\dot{\mathbb{Q}}_\xi, \dot{\leq}_{\mathbb{Q}_\xi}, \dot{\mathbb{1}}_{\mathbb{Q}_\xi}) : \xi < \alpha\rangle)$$

*satisfying the following properties:*

1. *Each $(\mathbb{P}_\xi, \leq_\xi, \mathbb{1}_\xi)$ is a forcing notion.*
2. *Each $(\dot{\mathbb{Q}}_\xi, \dot{\leq}_{\mathbb{Q}_\xi}, \dot{\mathbb{1}}_{\mathbb{Q}_\xi})$ is a $\mathbb{P}_\xi$-name for a forcing notion.*
3. *Each $p \in \mathbb{P}_\xi$ is a sequence $\langle \dot{q}_\mu : \mu < \xi\rangle$, where each $\dot{q}_\mu \in \operatorname{dom}(\dot{\mathbb{Q}}_\mu)$. For each $\mu < \xi$, we denote $\dot{q}_\mu$ with $(p)_\mu$.*
4. *If $\xi < \eta$ and $p \in \mathbb{P}_\eta$, then $p \restriction \xi \in \mathbb{P}_\xi$.*
5. *If $\xi < \eta$, $p \in \mathbb{P}_\xi$ and $p'$ is the $\eta$-sequence with $p' \restriction \xi = p$ and $(p')_\mu = \dot{\mathbb{1}}_{\mathbb{Q}_\mu}$ for every $\xi \leq \mu < \eta$, then $p' \in \mathbb{P}_\eta$. Define $i^\eta_\xi(p) = p'$, which gives a map $i^\eta_\xi : \mathbb{P}_\xi \to \mathbb{P}_\eta$.*
6. *$\mathbb{1}_\xi$ is the sequence $\langle \dot{\mathbb{1}}_{\mathbb{Q}_\mu} : \mu < \xi\rangle$.*
7. *If $p = \langle \dot{q}_\mu : \mu < \xi\rangle \in \mathbb{P}_\xi$ and $p' = \langle \dot{q}'_\mu : \mu < \xi\rangle \in \mathbb{P}_\xi$, define $p \leq_\xi p'$ iff*

$$p \restriction \mu \Vdash_{\mathbb{P}_\mu} (\dot{q}_\mu \leq_{\mathbb{Q}_\mu} \dot{q}'_\mu) \text{ for all } \mu < \xi.$$

8. *If $\xi + 1 \leq \alpha$, then $\mathbb{P}_{\xi+1}$ is the set of all sequences $p^\frown \dot{q}$ such that $p \in \mathbb{P}_\xi, \dot{q} \in \dot{\mathbb{Q}}_\xi$ and $p \Vdash_{\mathbb{P}_\xi} \dot{q} \in \dot{\mathbb{Q}}_\xi$.*

Note that with the definition above, we have $\mathbb{P}_{\xi+1} \cong \mathbb{P}_\xi * \dot{\mathbb{Q}}_\xi$, whenever $\xi + 1 \leq \alpha$. However, the definition does not specify $\mathbb{P}_\eta$ when $\eta$ is a limit ordinal.
We will usually just write $\langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi : \xi \leq \alpha\rangle$ for an iterated forcing construction as in the definition above.

**Definition 2.7.** *Let $\langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi : \xi \leq \alpha\rangle$ be an iterated forcing construction.*

1. *Let $p \in \mathbb{P}_\xi$. The support of $p$ is the set*

$$\mathrm{supt}(p) := \{\mu < \xi : (p)_\mu \neq \dot{\mathbb{1}}_{\mathbb{Q}_\mu}\}.$$

2. *We say that $\mathbb{P}_\alpha$ is a finite support iteration, if for each limit $\eta \leq \alpha$, the support of all conditions in $\mathbb{P}_\eta$ is finite.*

3. *Similarly, we say that $\mathbb{P}_\alpha$ is a countable support iteration, if for each limit $\eta \leq \alpha$, the support of all conditions in $\mathbb{P}_\eta$ is countable.*

If $\mathbb{P}_\alpha$ is a finite support iteration and $\eta \leq \alpha$ is a limit ordinal, then $\mathbb{P}_\eta$ can be viewed as the direct limit of all $\mathbb{P}_\xi$ for $\xi < \eta$.
We will mostly work with finite support iterations of ccc forcing notions. One important property we will need is that these iterations are again ccc.

**Theorem 2.2.** *Let $\langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi : \xi \leq \alpha \rangle$ be a finite support iteration such that for each $\xi < \alpha$*

$$\mathbb{1}_\xi \Vdash_{\mathbb{P}_\xi} (\dot{\mathbb{Q}}_\xi \textit{ is ccc}).$$

*Then $\mathbb{P}_\alpha$ is ccc.*

### 2.2.2 Preservation Theorems

In this section, we will show some general preservation theorems for finite support iterations of ccc forcing notions which we will need for Chapter 5. The results in this sections are from [FS08].

**Lemma 2.6.** *Let $\kappa$ be a cardinal of uncountable cofinality and let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha \leq \kappa \rangle$ be a finite support iteration of ccc forcing notions. Then every real added by $V^{\mathbb{P}_\kappa}$ is already added at some initial stage of the iteration of countable cofinality. So*

$${}^\omega\omega \cap V^{\mathbb{P}_\kappa} = \bigcup \{{}^\omega\omega \cap V^{\mathbb{P}_\alpha} : \alpha < \kappa, cf(\alpha) = \omega\}.$$

*Proof.* Let $\dot{f}$ be a $\mathbb{P}_\kappa$-name for a function in ${}^\omega\omega$. We can assume that $\dot{f}$ is of the form

$$\dot{f} = \bigcup \{\langle \langle i, j^i_p \rangle, p \rangle : i \in \omega, p \in \mathcal{A}_i, j^i_p \in \omega\},$$

where each $\mathcal{A}_i$ is a maximal antichain in $\mathbb{P}_\kappa$. Every $p \in \mathcal{A}_i$ has finite support, as $\mathbb{P}$ is a finite support iteration and so, for each $i \in \omega$, we can define

$$\alpha^p_i = \max\{\mathrm{supt}(p)\} \text{ and } \alpha_i = \sup\{\alpha^p_i : p \in \mathcal{A}_i\}.$$

As $\mathbb{P}_\kappa$ is ccc, $\mathcal{A}_i$ is countable and so each $\alpha_i$ is of countable cofinality, which implies $\alpha_i < \kappa$. Finally, let

$$\alpha := \sup\{\alpha_i : i \in \omega\}.$$

Then $\alpha$ is of countable cofinality, so $\alpha < \kappa$ and $\dot{f}$ is a $\mathbb{P}_\alpha$-name. □

**Definition 2.8.** *A family $\mathcal{H} \subseteq {}^{\omega}\omega$ is called $<^*$-directed, if for every subfamily $\mathcal{H}' \subseteq \mathcal{H}$ with $|\mathcal{H}'| < |\mathcal{H}|$, there is $h \in \mathcal{H}$ such that $h' <^* h$ for all $h' \in \mathcal{H}'$.*

**Remark 2.1.** *Note that $<^*$-directed families are preserved under ccc forcing notions, so if $\mathcal{H} \subseteq {}^{\omega}\omega$ is $<^*$-directed and $\mathbb{P}$ is a ccc forcing notion, then $(\mathcal{H}$ is $<^*$ directed$)^{V^{\mathbb{P}}}$.*

**Theorem 2.3.** *Let $\mathcal{H} \subseteq {}^{\omega}\omega$ be an unbounded family such that every countable subset of $\mathcal{H}$ is dominated by an element of $\mathcal{H}$. Let $\alpha$ be a cardinal of countable cofinality and $\langle \mathbb{P}_\gamma, \dot{\mathbb{Q}}_\gamma : \gamma \leq \alpha \rangle$ be a finite support iteration such that*

$$\forall \gamma < \alpha : (\mathbb{1} \Vdash_{\mathbb{P}_\gamma} (\dot{\mathbb{Q}}_\gamma \textit{ is ccc}) \textit{ and } \mathbb{1} \Vdash_{\mathbb{P}_\gamma} (\check{\mathcal{H}} \textit{ is unbounded})).$$

*Then $\mathbb{1} \Vdash_{\mathbb{P}_\alpha} (\check{\mathcal{H}}$ is unbounded$)$.*

*Proof.* Suppose towards a contradiction that there is a $\mathbb{P}_\alpha$-generic filter $G$ and in $V[G]$ there is $f \in {}^{\omega}\omega$ which dominates $\mathcal{H}$. Let $\dot{f}$ be a $\mathbb{P}_\alpha$-name for $f$ and let $\{\alpha_n\}_{n\in\omega}$ be increasing and cofinal sequence in $\alpha$. For each $n \in \omega$, let $G_{\alpha_n} = G \cap \mathbb{P}_{\alpha_n}$ and let $f_n$ be a function in $V[G_{\alpha_n}]$ such that for each $i \in \omega$,

$$f_n(i) = j \iff \exists q \in \mathbb{P}_\alpha (q \restriction \alpha_n \in G_{\alpha_n} \text{ and } q \Vdash_{\mathbb{P}_\alpha} \dot{f}(i) = \check{j}).$$

Then, since $\mathbb{1} \Vdash_{\mathbb{P}_\gamma} (\check{\mathcal{H}}$ is unbounded$)$ for all $\gamma < \alpha$, there exists $h_n \in \mathcal{H}$ for each $n \in \omega$ such that

$$V[G_{\alpha_n}] \models (h_n \not\leq^* f_n).$$

By Theorem 2.2, $\mathbb{P}_\alpha$ is ccc and so there is a family $\mathcal{C} \in [\mathcal{H}]^{\omega} \cap V$ such that $\{h_n : n \in \omega\} \subseteq \mathcal{C}$ and as $\mathcal{C}$ is a countable subset of $\mathcal{H}$ in $V$, there is a function $h \in \mathcal{H} \cap V$ such that $\mathcal{C} \leq^* h$. In particular, there is $k_n \in \omega$ for each $n \in \omega$ such that

$$\forall i \geq k_n (h_n(i) \leq h(i)).$$

By assumption, $V[G] \models \mathcal{H} \leq^* f$, so there are $p \in G$ and $k \in \omega$ such that

$$\forall i \geq k (p \Vdash \check{h}(i) \leq \dot{f}(i)).$$

Now fix $\alpha_n$ such that $\operatorname{supt}(p) \subseteq \alpha_n$. Since $V[G_{\alpha_n}] \models h_n \not\leq^* f_n$, we have

$$V[G_{\alpha_n}] \models \exists i > \max\{k_n, k\} (f_n(i) < h_n(i)).$$

So there is $i > \max\{k_n, k\}$ and a condition $p' \in G_{\alpha_n}$ such that

$$p' \Vdash \dot{f}_n(i) < \check{h}_n(i),$$

where $\dot{f}_n$ is a $\mathbb{P}_{\alpha_n}$-name for $f_n$. By definition of $f_n$ there is a condition $q \in \mathbb{P}_\alpha$ such that $q \restriction \alpha_n \in G_{\alpha_n}$ and

$$q \Vdash_{\mathbb{P}_\alpha} \dot{f}_n(i) = \dot{f}(i).$$

Since $p \restriction \alpha_n, p', q \restriction \alpha_n \in G$, they have a common extension $q' \in \mathbb{P}_\alpha$. But then

$$q' \Vdash_{\mathbb{P}_\alpha} \dot{f}_n(i) = \dot{f}(i) < \check{h}_n(i) \leq \check{h}(i) \leq \dot{f}(i)$$

which contradicts our assumption. □

**Lemma 2.7.** *Let $\mathbb{C}$ denote the Cohen forcing notion and let $\mathcal{H} \subseteq {}^{\omega}\omega$ be an unbounded family. Then*

$$\mathbb{1} \Vdash_{\mathbb{C}} \check{\mathcal{H}} \text{ is unbounded.}$$

*Proof.* Let $\dot{f}$ be a $\mathbb{C}$-name for a function in ${}^{\omega}\omega$. We will show that there is $h \in \mathcal{H}$ such that $\mathbb{1} \Vdash \check{h} \not\leq^* \dot{f}$. For each $p \in \mathbb{C}$, let

$$g_p(i) = \min\{j : \exists q \leq p(q \Vdash \dot{f}(i) = j)\}.$$

The set $\{g_p : p \in \mathbb{C}\}$ is countable, hence there is $g \in {}^{\omega}\omega \cap V$ such that for every $p \in \mathbb{C}$, $g_p \leq^* g$. So for each $p \in \mathbb{C}$ we can find $m_p \in \omega$ such that

$$\forall i \geq m_p(g_p(i) \leq g(i)).$$

$\mathcal{H}$ is unbounded and so there is $h \in \mathcal{H}$ such that $h$ is not dominated by $g$, so the set

$$A = \{i \in \omega : g(i) < h(i)\}$$

is infinite. To see that $h$ is not dominated by $f$, it is sufficient to show that

$$\mathbb{1} \Vdash \exists^{\infty} i \in \check{A}(\dot{f}(i) \leq \check{g}(i)).$$

Let $p \in \mathbb{C}$ and suppose that there is no $q \leq p$ such that $q \Vdash \exists^{\infty} i \in \check{A}(\dot{f}(i) \leq \check{g}(i))$. That means that there is $m \in \omega$ such that

$$\forall i \in A, i > m(p \Vdash \check{g}(i) < \dot{f}(i)).$$

Let $i \in A$ with $i > \max\{m, m_q\}$. Further, let $q'$ be an extension of $q$ and $j \in \omega$ be such that $j = g_q(i)$ and $q' \Vdash \dot{f}(i) = \check{j}$. Then

$$q' \Vdash \dot{f}(i) = \check{g}_q(i) \leq \check{g}(i) < \dot{f}(i),$$

which is a contradiction. □

**Corollary 2.1.** *Let $\mathcal{H} \subseteq {}^{\omega}\omega$ be unbounded and let $\mathbb{C}(\kappa)$ be the forcing notion for adding $\kappa$ Cohen reals. Then*

$$\mathbb{1} \Vdash_{\mathbb{C}(\kappa)} \check{\mathcal{H}} \text{ is unbounded.}$$

*Proof.* By Lemma 2.7, $\mathcal{H}$ remains unbounded in $V^{\mathbb{C}}$. By Lemma 2.6, new reals only get added at initial stages of the iteration with countable cofinality, and so the assertion follows from Theorem 2.3. □

**Lemma 2.8.** *Let $\mathcal{H} \subseteq {}^{\omega}\omega$ be an unbounded, $<^*$-directed family of size $\kappa$ and $\mathbb{P}$ be a forcing notion with $|\mathbb{P}| < \kappa$. Then*

$$\mathbb{1} \Vdash_{\mathbb{P}} \check{\mathcal{H}} \text{ is unbounded.}$$

*Proof.* Let $\dot{f}$ be a $\mathbb{P}$-name for a function in ${}^{\omega}\omega$. For each $p \in \mathbb{P}$ and $i \in \omega$, let

$$g_p(i) = \min\{j : \exists q \leq p(q \Vdash \dot{f}\,(i) = \check{j})\}.$$

As $(\mathcal{H}$ is unbounded$)^V$, there is a function $h_p \in \mathcal{H} \cap V$ for each $p \in \mathbb{P}$, which is not dominated by $g_p$. As $|\{h_p : p \in \mathbb{P}\}| < \kappa = |\mathcal{H}|$, there is $h \in \mathcal{H} \cap V$ that dominates every $h_p$. So for each $p \in \mathbb{P}$ there is $n_p \in \omega$ such that

$$\forall i \geq n_p \left(h_p\,(i) \leq h\,(i)\right).$$

Now suppose that $p \in \mathbb{P}$ is such that

$$p \Vdash \mathcal{H} \leq^* \dot{f}.$$

So there is $p_0 \leq p$ and $n_0 \in \omega$ such that

$$\forall i \geq n_0(p_0 \Vdash \check{h}\,(i) \leq \dot{f}\,(i)).$$

Let $i > \max\{n_0, n_p\}$ be such that $g_{p_0}\,(i) < h_{p_0}\,(i)$ and let $q \leq p$ such that $q \Vdash g_{p_0}\,(i) = \dot{f}\,(i)$. Then

$$q \Vdash \dot{f}\,(i) = g_{p_0}\,(i) < h_{p_0}\,(i) \leq h(i) \leq \dot{f}\,(i)\,,$$

which is a contradiction. □

**Lemma 2.9.** *Let $\kappa$ be a regular uncountable cardinal and let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \kappa \rangle$ be a finite support iteration of ccc forcing notions. Then every family $\mathcal{A} \subseteq V\,[G] \cap {}^{\omega}\omega$, where $G$ is a $\mathbb{P}_\kappa$-generic filter, with $|\mathcal{A}| < \kappa$, is obtained at some proper initial segment of the iteration.*

*Proof.* Every $f \in \mathcal{A}$ has a $\mathbb{P}_\kappa$-name $\dot{f}$. By Lemma 2.6, every $\dot{f}$ is a $\mathbb{P}_{\alpha_f}$-name, where $\alpha_f$ is an ordinal with countable cofinality. Let

$$\alpha = \sup\{\alpha_f : f \in \mathcal{A}\}.$$

Then $\mathcal{A} \subseteq V\,[G \cap \mathbb{P}_\alpha] \cap {}^{\omega}\omega$ and $\mathrm{cf}\,(\alpha) \leq |\mathcal{A}| < \kappa$, so $\alpha < \kappa$. □

# 3 $\mathfrak{b}$ and $\mathfrak{s}$ in Mathias Forcing

In this chapter, we will see how the bounding and the splitting number behave in iterations of Mathias forcing. To obtain a model in which $\mathfrak{b} < \mathfrak{s}$ holds, one needs to increase the size of the splitting number. As we will see, this can be achieved using a countable support iteration of Mathias forcing. However, such an iteration also increases the size of the bounding number, which makes it unsuitable to show the consistency of $\mathfrak{b} < \mathfrak{s}$.

**Definition 3.1.** *1. Mathias forcing, denoted by* $\mathbb{M}$*, is the forcing notion consisting of all pairs* $(s, A) \in [\omega]^{<\omega} \times [\omega]^{\omega}$*, where* $\max(s) < \min(A)$*. The extension relation is defined as follows:* $(s_0, A_0) \leq (s_1, A_1)$ *iff*

   *a)* $s_0$ *is an end-extension of* $s_1$*,*

   *b)* $s_0 \setminus s_1 \subseteq A_1$*, and*

   *c)* $A_0 \subseteq A_1$*.*

*2. Mathias forcing with respect to a filter* $\mathcal{F} \subseteq [\omega]^{\omega}$*, denoted by* $\mathbb{M}(\mathcal{F})$*, consists of all pairs* $(s, A) \in [\omega]^{<\omega} \times \mathcal{F}$ *with* $\max(s) < \min(A)$ *and the extension relation as defined as for Mathias forcing.*

Note that $\mathbb{M}(\mathcal{F})$ is $\sigma$-centered and hence ccc, while $\mathbb{M}$ only satisfies the $\aleph_2$ chain condition.

**Lemma 3.1.** *The following sets are dense in* $\mathbb{M}$*:*

1. *For each* $n \in \omega$*,*
$$D_n := \{(s, A) \in \mathbb{M} : \exists m > n(m \in s)\}.$$

2. *For each* $A \in [\omega]^{\omega}$*,*
$$D_A := \{(s, B) \in \mathbb{M} : B \subseteq A \text{ or } B \subseteq A^c\}.$$

3. *For each* $f \in [\omega]^{<\omega} \cap V$*,*
$$D_f := \{(s, A) \in \mathbb{M} : \forall \ell \in \omega\, (A(\ell) \geq f(|s| + \ell))\}.$$

*Proof.* 1. Let $n \in \omega$ and $(s, A) \in \mathbb{M}$ be arbitrary. Since $A$ is infinite, there is some $m > n$ such that $m \in A$, and hence also $m > \max(s)$. Then we have

$$(s \cup \{m\}, A \setminus (m + 1)) \in D_n,$$

and this condition is an extension of $(s, A)$, so $D_n$ is dense for each $n \in \omega$.
2. Let $A \in [\omega]^\omega$ and $(s, B) \in \mathbb{M}$ be arbitrary. As both $A$ and $B$ are infinite, at least one of the sets $A \cap B$ and $A^c \cap B$ will be infinite. If $A \cap B$ is infinite, then

$$(s, A \cap B) \in D_A \text{ and } (s, A \cap B) \leq (s, B)\,,$$

so $D_A$ is dense. If $A^c \cap B$ is infinite, consider $(s, A^c \cap B)$ instead of $(s, A \cap B)$.
3. Let $f \in {}^\omega\omega \cap V$ and $(s, A) \in \mathbb{M}$ be arbitrary. Now define $A' \in [\omega]^\omega$ by setting the $\ell$-th element of $A'$ to be $A'(\ell) = A(k_\ell)$, with

$$k_\ell := \min\{k \in \omega : k \geq \ell \wedge A(k) \geq f\,(|s| + \ell)\}$$

for $\ell \in \omega$. Then $(s, A') \in D_f$ and $(s, A') \leq (s, A)$. □

Using these dense sets, we can now show that Mathias forcing adds a real which is not split by the ground model reals.

**Lemma 3.2.** *Mathias forcing adds a real not split by the ground model reals.*

*Proof.* Let $G$ be a $\mathbb{M}$-generic filter and consider

$$U_G := \bigcup\{s | \exists A\,(s, A) \in G\}.$$

First, note that $U_G \in [\omega]^\omega$. This is the case since for each $n \in \omega$, the set $D_n$ from Lemma 3.1 is dense.
Now, to see that $U_G$ is unsplit by the ground model reals, consider for each $A \in [\omega]^\omega$ the set $D_A$ defined in Lemma 3.1. Since $D_A$ is dense, there is some $(s, B) \in G$ such that $B \subseteq A$ or $B \subseteq A^c$. We will show that $U_G \subseteq^* B$.
Pick any $(s_0, A_0) \in G$ and since $G$ is a filter, there is $(s'_1, A'_1) \in G$ such that $(s'_1, A'_1)$ is a common extension of $(s, B)$ and $(s_0, A_0)$. In particular, we have $s'_1 \supseteq s_0$ and $s'_1 \setminus s \subseteq B$. To ensure that we get a proper superset of the finite parts, let $n = \max(s_0)$. Then there is $(s_1, A_1) \in G \cap D_n$ which extends $(s'_1, A'_1)$, and so is a common extension of $(s, B)$ and $(s_0, A_0)$. Proceed inductively to find a common extension $(s_{i+1}, A_{i+1}) \in G$ of $(s, B)$ and $(s_i, A_i)$ such that $s_{i+1} \supsetneq s_i$ and $s_{i+1} \setminus s \subseteq B$.
So we have $U_G \subseteq^* B$ and since $B \subseteq A$ or $B \subseteq A^c$, $U_G \cap A$ or $U_G \setminus A$ is finite. Hence $U_G$ is not split by $A$. □

**Lemma 3.3.** *Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha \leq \omega_2 \rangle$ be a countable support iteration of Mathias forcing, so for all $\alpha < \omega_2$ we have $\mathbb{1}_{\mathbb{P}_\alpha} \Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{Q}}_\alpha = \dot{\mathbb{M}}$. Then*

$$V^{\mathbb{P}_{\omega_2}} \models \mathfrak{s} = \aleph_2.$$

*Proof.* First note that $\mathbb{M}$ is proper and $\aleph_2$-cc, so $\mathbb{P}_{\omega_2}$ is also proper and $\aleph_2$-cc (for more details see [Abr10]). Hence it preserves cardinals. Let $\mathcal{A} \subseteq [\omega]^\omega \cap V^{\mathbb{P}_{\omega_2}}$ with $|\mathcal{A}| < \aleph_2$. Since $\mathbb{M}$ has the $\aleph_2$-cc and a name for a subset of $\omega$ is represented by $\aleph_0$ many antichains, there are $\aleph_1$ many conditions. Thus, there is $\alpha < \omega_2$ such that $\mathcal{A} \subseteq [\omega]^\omega \cap V^{\mathbb{P}_\alpha}$. Then, by Lemma 3.2, $\mathbb{Q}_\alpha$ adds a real not split by $\mathcal{A}$. So $\mathcal{A}$ is not splitting over $V^{\mathbb{P}_{\alpha+1}}$, hence

$$V^{\mathbb{P}_{\omega_2}} \models \mathfrak{s} = \mathfrak{c} = \aleph_2.$$

□

So Mathias can be used to increase the splitting number. However, Mathias forcing also adds a real which dominates all ground model reals. This means that a countable support iteration will also increase the bounding number and hence such an iteration does not produce a model for $\mathfrak{b} < \mathfrak{s}$.

**Lemma 3.4.** *Mathias forcing adds a real dominating all ground model reals.*

*Proof.* As before, let $G$ be a $\mathbb{M}$-generic filter and let

$$U_G := \{s | \exists A (s, A) \in G\} \in [\omega]^\omega .$$

First note that for $f \in {}^\omega\omega \cap V$, the set $D_f$ defined in Lemma 3.1 is a dense subset of $\mathbb{M}$. Now, let $(s, A) \in G$ and note that $U_G \setminus s \in A$ and $s$ is an initial segment of $U_G$. To see this, inductively construct $(s_i, A_i) \in G$ with $s_{i+1} \supsetneqq s_i$ and $s_{i+1} \setminus s \subseteq A$, as in the proof of Lemma 3.2.
So, for any $(s, A) \in G \cap D_f$, we have that $U_G \setminus s \subseteq A$ and $s$ is an initial segment of $U_G$. By denoting the enumeration function of $U_G$ with $f_G$ we get

$$f_G\left(|s| + \ell\right) \geq A\left(\ell\right) \geq f\left(|s| + \ell\right)$$

and so $f \leq^* f_G$. □

**Lemma 3.5.** *Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha \leq \omega_2 \rangle$ be a countable support iteration of Mathias forcing. Then*

$$V^{\mathbb{P}_{\omega_2}} \models \mathfrak{b} = \aleph_2.$$

*Proof.* Let $\mathcal{F} \subseteq {}^\omega\omega \cap V^{\mathbb{P}_{\omega_2}}$ be a family of functions with $|\mathcal{F}| < \aleph_2$. By counting antichains, there is $\alpha < \omega_2$ such that $\mathcal{F} \subseteq {}^\omega\omega \cap V^{\mathbb{P}_\alpha}$. Then, by Lemma 3.4, Mathias forcing adds a real at stage $\alpha + 1$ which dominates all reals in $V^{\mathbb{P}_\alpha}$. So $\mathcal{F}$ is not unbounded and therefore

$$V^{\mathbb{P}_{\omega_2}} \models \mathfrak{b} = \aleph_2.$$

□

# 4 Logarithmic Measures

In this chapter, we will discuss the notion of logarithmic measures, which was introduced by Shelah in [She84] to show the consistency of $\mathfrak{b} = \omega_1 < \mathfrak{s} = \omega_2$. We will follow [FS08] to show the more general result of $\mathfrak{b} = \kappa < \mathfrak{s} = \kappa^+$ in the following two chapters.
We will start this chapter by looking at basic properties of logarithmic measures before using them to define forcing notions in later parts of this chapter.

**Definition 4.1.** *Let $s \subseteq \omega$ and $h : [s]^{<\omega} \to \omega$. The function $h$ is called a logarithmic measure if for every $A \in [s]^{<\omega}$, with $h(A) > 0$, and for every $A_0, A_1$ with $A = A_0 \cup A_1$, we have*

$$h(A_0) \geq h(A) - 1 \text{ or } h(A_1) \geq h(A) - 1.$$

*Whenever $s$ is a finite set and $h$ is a logarithmic measure on $s$, the pair $x = (s, h)$ is called a finite logarithmic measure.*
*The value $h(s) = \|x\|$ is called the level of $x$.*

**Definition 4.2.** *Let $(h, s)$ be a finite logarithmic measure. A set $e \subseteq s$ is called $h$-positive, if $h(e) > 0$.*

**Lemma 4.1.** *Let $h$ be a logarithmic measure with $h(A_0 \cup ... \cup A_{n-1}) \geq \ell + 1$. Then there is some $j \in n$ such that $h(A_j) \geq n - j$.*

*Proof.* By definition of a logarithmic measure, $h(A_0 \cup ... \cup A_{n-1}) \geq \ell + 1$ implies

$$h(A_0) \geq \ell \text{ or } h(A_1 \cup ... \cup A_{n-1}) \geq \ell.$$

If $h(A_0) \geq \ell$ we are done. Otherwise we have $h(A_1 \cup ... \cup A_{n-1}) \geq \ell$ and so

$$h(A_1) \geq \ell - 1 \text{ or } h(A_2 \cup ... \cup A_{n-1}) \geq \ell - 1.$$

Repeating this procedure will yield $A_j$ with $h(A_j) \geq \ell - j$ for some $j \in n$. □

**Definition 4.3.** *Let $P$ be an upwards closed family of finite subsets of $\omega$. Then $P$ induces a logarithmic measure $h$ on $[\omega]^{<\omega}$ which is defined inductively on $|e|$ for $e \in [\omega]^{<\omega}$ by*

1. *$h(e) \geq 0$ for all $e \in [\omega]^{<\omega}$*
2. *$h(e) > 0$ if and only if $e \in P$*
3. *for $\ell \geq 1$, $h(e) \geq \ell + 1$ if and only if*
   a) *$e \in P$,*

*b)* $|e| > 1$ *and,*

*c) for all* $e_0, e_1 \subseteq e$ *with* $e = e_0 \cup e_1$*, we have* $h(e_0) \geq \ell$ *or* $h(e_1) \geq \ell$

*Then define*

$$h(e) := \max\{\ell \in \omega : h(e) \geq \ell\}.$$

*The elements of* $P$ *are called positive sets and* $h$ *is said to be induced by the positive sets* $P$*.*

**Definition 4.4.** *An induced logarithmic measure* $h$ *is said to be atomic if there is a singleton which is* $h$*-positive, i.e. there is* $n \in \omega$ *such that* $h(\{n\}) > 0$*.*

**Remark 4.1.** *Unless stated otherwise, we will assume that all logarithmic measures are non-atomic from now on.*

**Lemma 4.2.** *Let* $h$ *be an induced logarithmic measure and let* $e \in [\omega]^{<\omega}$ *be such that* $h(e) \geq \ell$*. Then* $h(a) \geq \ell$ *for all* $a \supseteq e$*.*

*Proof.* We prove this via induction on $|e|$.
If $|e| = 1$, then $h(e) = 0$ since we assume that the measure is non-atomic, and the claim follows from the definition of induced logarithmic measures.
Now let $|e| = n$ with $h(e) \geq \ell$ and assume that the claim holds for all sets of size smaller than $n$. Assume that the claim does not hold for $e$ and let $a \supseteq e$ be a set of minimal size such that $h(a) < \ell$. Then, by definition of the logarithmic measure, there are $a_0, a_1 \subseteq a$ such that

$$a = a_0 \cup a_1,\ h(a_0) < \ell - 1 \text{ and } h(a_1) < \ell - 1.$$

But then $e = (a_0 \cap e) \cup (a_1 \cap e)$, and so

$$h(a_0 \cap e) \geq \ell - 1 \text{ or } h(a_1 \cap e) \geq \ell - 1.$$

We either must have $|a_0 \cap e|, |a_1 \cap e| < n$ or $a_i \cap e = e$ for $i = 0$ or $i = 1$. By the inductive hypothesis, we cannot have $|a_i \cap e| < |e|$ with $h(a_i \cap e) \geq \ell - 1$ but $h(a_i) < \ell - 1$ for $i \in \{0, 1\}$. So assume $a_0 \cap e = e$. This would mean that we have found $a_0 \subset a$ such that $e \subseteq a_0$ and $h(a_0) < \ell$, which contradicts the minimality of the size of $a$. □

**Lemma 4.3.** *Let* $h$ *be a logarithmic measure induced by an upwards closed family of finite subsets of* $\omega$ *and let* $\ell \geq 1$*. Let* $A \subseteq \omega$ *be such that it contains no set of measure strictly greater than* $\ell$*. Then there are* $a_0, a_1 \subseteq A$ *such that* $A = a_0 \cup a_1$ *and neither of* $a_0$ *and* $a_1$ *contains a set of measure* $\geq \ell$*.*

*Proof.* We can assume that $A$ is infinite, as the assertion follows from the definition of the induced logarithmic measure if $A$ is finite.
For each $k \in \omega$, define $A_k := A \cap k$ and consider the family $T$ of all functions

$$f : m \to \bigcup_{k \leq m} \mathcal{P}(A_k) \times \mathcal{P}(A_k),$$

where $m \in \omega$, such that for each $k$

$$f(k) = (a_0^k, a_1^k) \in \mathcal{P}(A_k) \times \mathcal{P}(A_k),$$

with

$$a_0^k \cup a_1^k = A_k,\ h(a_0^k), h(a_1^k) \ngeq \ell \text{ and } a_0^k \subseteq a_0^{k+1}, a_1^k \subseteq a_1^{k+1}\ \forall k \in m.$$

Then $T$ together with the end-extension relation forms a tree.

**Claim.** *Each level of $T$ is non-empty.*

*Proof of claim.* Consider an arbitrary $m \in \omega$. Then $A_m$ is a finite set and by assumption we have $h(A_m) \ngeq \ell + 1$. By definition of $h$ there are $a_0^m, a_1^m \subseteq A_m$ such that

$$A_m = a_0^m \cup a_1^m \text{ and } h(a_0^m), h(a_1^m) \ngeq \ell.$$

Then we have that $a_0^{m-1} = A_{m-1} \cap a_0^m$ and $a_1^{m-1} = A_{m-1} \cap a_1^m$ are both of measure not greater or equal than $\ell$, by Lemma 4.2 and we have $A_{m-1} = a_0^{m-1} \cup a_1^{m-1}$. So we can inductively construct sequences

$$\langle a_0^k : k \leq m \rangle \text{ and } \langle a_1^k : k \leq m \rangle$$

such that for every $k \leq m$

$$a_0^k \cup a_1^k = A_k,\ h(a_0^k), h(a_1^k) \ngeq \ell \text{ and } a_0^k \subseteq a_0^{k+1}, a_1^k \subseteq a_1^{k+1}\ \forall k \in m.$$

So the function $f : m \to \bigcup_{k \leq m} \mathcal{P}(A_k) \times \mathcal{P}(A_k)$ defined by $f(k) = (a_0^k, a_1^k)$ is in the $m$-th level of $T$. □

Now we can apply König's Lemma 2.1 to $T$ which gives an infinite branch through $T$. Let

$$f : \omega \to \bigcup_{k \in \omega} \mathcal{P}(A_k) \times \mathcal{P}(A_k)$$

with

$$f(k) = (a_0^k, a_1^k)$$

be such an infinite branch. Now let

$$a_0 = \bigcup_{k \in \omega} a_0^k \text{ and } a_1 = \bigcup_{k \in \omega} a_1^k.$$

By construction of $T$ we have $A = a_0 \cup a_1$ and both $a_0$ and $a_1$ contain no set of measure greater or equal to $\ell$. The last assertion holds since if $x \subseteq a_0$ is finite, then there is $k \in \omega$ such that $x \subseteq a_0^k$ and since $h(a_0^k) \ngeq \ell$, we have $h(x) \ngeq \ell$. The same applies to $a_1$. □

**Lemma 4.4.** *Let $h$ be the logarithmic measure induced by some upwards closed family of finite subsets of $\omega$. Suppose that for every $n \in \omega$ and every partition of $\omega$ into $n$ sets, $\omega = A_0 \cup ... \cup A_{n-1}$, there is some $j \in n$ such that $A_j$ contains a positive set.*
*Then, for every $k, n \in \omega$ and every partition of $\omega$ into $n$ sets as above, there is some $j \in n$ such that $A_j$ contains a set of measure at least $k$.*

*Proof.* We prove this via induction on $k$.
For $k = 1$ the assertion just follows from the assumptions.
Now suppose that we have proved the lemma for $k$ and that it does not hold for $k + 1$. So there is some $n \in \omega$ for which there is a partition

$$\omega = A_0 \cup ... \cup A_{n-1}$$

such that no $A_j$ contains a set of measure greater or equal to $k + 1$. Then by Lemma 4.3, for every $j \in n$ there are $A_j^0$ and $A_j^1$ such that $A_j = A_j^0 \cup A_j^1$ and $A_j^0, A_j^1$ contain no set of measure greater or equal to $k$. But then

$$\omega = A_0^0 \cup A_0^1 \cup ... \cup A_{n-1}^0 \cup A_{n-1}^1$$

is a partition of $\omega$ where no $A_j^i$, for $j \in n$ and $i \in 2$, contains a set of measure greater or equal to $k$, which contradicts the assumption for $k$. □

## 4.1 Pure Conditions

We will now use the notion of logarithmic measures defined in the previous section to define the forcing notion $Q$ and ccc suborders $Q(C)$.

**Definition 4.5.** *Let $Q$ be the partial order of all pairs $(u, T)$ with $u \in [\omega]^{<\omega}$ and $T = \langle (s_i, h_i) : i \in \omega \rangle$ is a sequence of finite logarithmic measures such that*

1. $\max(u) < \min(s_0)$
2. $\max(s_i) < \min(s_{i+1})$ *for all* $i \in \omega$
3. $\langle h_i(s_i) : i \in \omega \rangle$ *is unbounded.*

*Given a condition $p = (u, T)$, the finite part $u$ is called the stem of $p$ and $T$ is called the pure part.*
*If $u = \emptyset$ then $(\emptyset, T)$ is called a pure condition and usually denoted by $T$.*
*Given a pure condition $T$, we define*

$$\operatorname{int}(T) := \bigcup \{s_i : i \in \omega\}.$$

*If $u_1 = u_2$ and $(u_2, T_2)$ extends $(u_1, T_1)$, then $(u_2, T_2)$ is called a pure extension of $(u_1, T_1)$. Let $(u_1, T_1), (u_2, T_2) \in Q$, where $T_\ell = \langle (s_i^\ell, h_i^\ell) : i \in \omega \rangle$ and for $\ell = 1, 2$. The extension relation $(u_2, T_2) \leq (u_1, T_1)$, is defined by:*

1. *$u_2$ is an end-extension of $u_1$ and* $u_2 \setminus u_1 \subseteq \operatorname{int}(T_1)$
2. $\operatorname{int}(T_2) \subseteq \operatorname{int}(T_1)$ *and there is an infinite sequence $\langle B_i : i \in \omega \rangle$ of finite subsets of $\omega$ such that:*
   a) $\max(u_2) < \min(s^1_{\min(B_0)})$,

*b)* $\max(B_i) \leq \min(B_{i+1})$ *for each* $i \in \omega$*, and*

*c)* $s_i^2 \subseteq \bigcup\{s_j^1 : j \in B_i\}$ *for each* $i \in \omega$*.*

*3. for every* $e \subseteq s_i^2$ *such that* $h_i^2(e) > 0$ *there is* $j \in B_i$ *such that* $h_j^1(e \cap s_j^1) > 0$*.*

**Remark 4.2.** *Note that if* $(u,T)$ *is a condition in* $Q$*, then* $(u, \mathrm{int}(T))$ *is a condition in Mathias forcing* $\mathbb{M}$*. Moreover, if* $(u_1,T_1),(u_2,T_2) \in Q$ *with* $(u_2,T_2) \leq_Q (u_1,T_1)$*, then* $(u_2, \mathrm{int}(T_2)) \leq_{\mathbb{M}} (u_1, \mathrm{int}(T_1))$*.*

**Definition 4.6.** *For a pure condition* $T = \langle t_i : i \in \omega\rangle$ *and for every* $k \in \omega$*, let*

$$i_T(k) \coloneqq \min\{i \in \omega : k < \min(\mathrm{int}(t_i))\}$$

*and let*

$$T \setminus k = T_{i_T(k)} = \langle t_i : i \geq i_T(k)\rangle.$$

*If* $u \in [\omega]^{<\omega}$*, let* $T \setminus u \coloneqq T_{i_T(\max(u))}$ *and* $(u,T) = (u, T\setminus u) \in Q$*.*

The poset $Q$ was introduced in [She84] to show the consistency of $\mathfrak{b} = \omega_1 < \mathfrak{s} = \omega_2$. $Q$ is proper and almost ${}^{\omega}\omega$-bounding, so the ground model reals remain unbounded in countable support iterations. At the same time, these iterations increase the splitting number for the same reason as we have seen with the Mathias forcing in chapter 3. So a countable support iteration of $Q$ of length $\omega_2$ will yield $\mathfrak{b} = \omega_1 < \mathfrak{s} = \omega_2$. For a detailed argument see [Abr10].
We will focus on the more general inequality of $\mathfrak{b} = \kappa < \mathfrak{s} = \kappa^+$ for which we will need specific suborders of $Q$.

**Definition 4.7.** *Let* $\mathcal{F}$ *be a family of pure conditions. Then define the suborder* $Q(\mathcal{F})$ *of* $Q$ *by*

$$Q(\mathcal{F}) \coloneqq \{(u,T) \in Q : \exists R \in \mathcal{F}\,(R \leq T)\}.$$

**Definition 4.8.** *Let* $C$ *be a family of pure conditions. We will call* $C$ *a centred family of pure conditions (or simply a centered family), if for all* $X, Y \in C$ *there is* $Z \in C$ *such that* $Z$ *is a common extension of* $X$ *and* $Y$*.*

Note that the partial order $Q(C)$ is the upwards closure of the family $\{(u,T) : T \in C\}$ with respect to the order in $Q$.

**Definition 4.9.** *Let* $C$ *be a centered family of pure conditions. A pure condition* $T$ *is said to be compatible with the family* $C$*, if it is compatible with every element of* $C$*.*

**Lemma 4.5.** *Let* $C$ *be a centered family of pure conditions. Then the partial order* $Q(C)$ *is* $\sigma$*-centered.*

*Proof.* For any $u \in [\omega]^{<\omega}$, define

$$Q_u(C) \coloneqq \{(u,T) \in Q(C)\}.$$

Then $Q_u(C)$ is centered. To see this, take $(u,T),(u,R) \in Q_u(C)$. By definition of $Q(C)$, there are $T', R' \in C$ such that $T' \leq T$ and $R' \leq R$. As $C$ is a centered family of pure conditions, there is $Z \in C$ such that $Z \leq T', R'$. Then $(u,Z)$ is a common extension of $(u,T)$ and $(u,R)$, so $Q_u(C)$ is centered.
Since we have

$$Q(C) = \bigcup_{u \in [\omega]^{<\omega}} Q_u(C),$$

$Q(C)$ can be written as a countable union of centered sets, so it is $\sigma$-centered. □

In particular, $Q(C)$ is ccc for all centered families $C$, whereas $Q$ is not ccc.

**Remark 4.3.** *From now on, we will assume that all centered families are closed with respect to final sequences, so if $C$ is a centered family and $T \in C$, then $T \setminus v \in C$ for every $v \in [\omega]^{<\omega}$.*

**Definition 4.10.** *Let $C$ and $C'$ be centered families. $C'$ is said to extend $C$, if $C \subseteq Q(C')$.*
*If $C'$ extends $C$ and there is $R \in Q(C')$ such that $T \leq R$ for all $T \in C'$, then $C'$ extends $C$ below $R$.*

**Lemma 4.6.** *Two conditions in $Q(C)$ are compatible as conditions in $Q(C)$ if and only if they are compatible in $Q$.*

*Proof.* Let $(u,T)$ and $(v,R)$ be two conditions in $Q(C)$ which are compatible in $Q$ and let $(w,Z) \in Q$ be their common extension. By definition of the extension relation in $Q$, $w$ is a common end-extension of $u$ and $v$, so either $u$ is an end-extension of $v$ or $v$ is an end-extension of $u$.
Without loss of generality, assume that $u$ is an end-extension of $v$. Then, by definition of the extension relation and $w \supseteq u$,

$$u \setminus v \subseteq w \setminus v \subseteq \operatorname{int}(R).$$

Since $(u,T)$ and $(v,R)$ are in $Q(C)$, there are pure conditions $T', R'$ in $C$ such that $T' \leq T$ and $R' \leq R$. Since the family $C$ is centered, there is a pure condition $Z' \in C$ which is a common extension of $T'$ and $R'$. Then $(u,Z') \in Q(C)$ and it is a common extension of $(u,T)$ and $(v,R)$.
The other direction is clear as $Q(C)$ is a suborder of $Q$. □

## 4.2 Restricting Pure Conditions

In this section, we will restrict pure conditions to subsets of $\omega$ and see when these restrictions are again pure conditions. This will allow us to find pure extensions of a condition such that the integers of the extension are contained in one set of a finite partition of $\omega$.

**Lemma 4.7.** *Let* $(x, h)$ *be a finite logarithmic measure with* $h(x) \leq n$*. Then there are* $x_0, ..., x_{2^n-1}$ *such that* $h(x_i) = 0$ *for all* $i \in 2^n$ *and*

$$x = \bigcup\{x_i : i \in 2^n\}.$$

*Proof.* The proof is by induction on $n$.
Let $n = 1$, so $h(x) \leq 1$. Then by definition of a logarithmic measure, there are $x_0, x_1 \subseteq x$ such that $x = x_0 \cup x_1$, $h(x_0) \ngeq 1$ and $h(x_1) \ngeq 1$, which shows the claim.
Now suppose that we have shown the claim for every measure with level at most $n$ for $n \geq 2$. Let $(x, h)$ be a logarithmic measure of level $n + 1$. By definition, there are sets $x_0, x_1 \subseteq x$ such that

$$x = x_0 \cup x_1, h(x_0) \leq n \text{ and } h(x_1) \leq n.$$

Using the inductive hypothesis on $x_0$ and $x_1$ we get

$$x_\ell = \bigcup\{x_\ell^i : i \in 2^n\} \text{ for } \ell \in 2,$$

where $h\left(x_\ell^i\right) = 0$ for all $i \in 2^n$. Hence $x$ can be presented as a union of $2^{n+1}$ sets, all of which have measure 0. □

**Lemma 4.8.** *Let* $T = \langle(s_i, h_i) : i \in \omega\rangle$ *be a pure condition. If* $A \in [\omega]^\omega$ *is such that the sequence* $\langle h_i(s_i \cap A) : i \in \omega\rangle$ *is bounded, then the condition* $T$ *has no pure extension* $R$ *with* $\text{int}(R) \subseteq A$.

*Proof.* Assume towards a contradiction that there is a pure condition $R = \langle(x_i, g_i) : i \in \omega\rangle$ in $Q$ which extends $T$ and such that $\text{int}(R) \subseteq A$. Then, by definition of the extension relation, there is a sequence $\langle B_i : i \in \omega\rangle \subseteq [\omega]^{<\omega}$ such that

$$x_i \subseteq \bigcup\{s_j : j \in B_i\} \ \forall i \in \omega.$$

Since $\langle h_i(s_i \cap A) : i \in \omega\rangle$ is bounded, there is $N \in \omega$ such that

$$h_i(s_i \cap A) \leq N \ \forall i \in \omega.$$

$R$ is a pure condition, and so the sequence $\langle g_i(x_i) : i \in \omega\rangle$ is unbounded, which means that there is $\ell \in \omega$ such that $g_\ell(x_\ell) \geq 2^N + 1$. Since $\text{int}(R) \subseteq A$ and $R \leq T$, we have

$$x_\ell = x_\ell \cap A \subseteq \bigcup\{s_j \cap A : j \in B_\ell\}.$$

Moreover, for each $j \in B_\ell$, we have $h_j(s_j \cap A) \leq N$ and so by Lemma 4.7 there is a family of sets $\{s_j^n : n \in 2^N\}$ such that $s_j \cap A = \bigcup\{s_j^n : n \in 2^N\}$ and for each $n \in 2^N$, $h_j(s_j^n) = 0$. For each $n \in 2^N$ let

$$a_n = x_\ell \cap (\bigcup\{s_j^n : j \in B_\ell\}).$$

Then $x_\ell = \bigcup\{a_n : n \in 2^N\}$ and since we assumed that $h_\ell(x_\ell) \geq 2^N + 1$, there is $n \in 2^N$ by Lemma 4.1 such that

$$g_\ell(a_n) \geq 2^N + 1 - m \geq 1.$$

Hence there is $j \in B_\ell$ with $h_j(a_n \cap s_j) > 0$. However, we have $s_j^n = a_n \cap s_j$ and so $h_j(s_j^n) > 0$ which is a contradiction to $h_j(s_j^n) = 0$. □

**Definition 4.11.** *For a pure condition $T = \langle (s_i, h_i) : i \in \omega \rangle$ and $A \subseteq \omega$, define*

$$T \restriction A := \langle (s_i \cap A, h_i \restriction \mathcal{P}(s_i \cap A)) : i \in \omega \rangle.$$

Note that given a pure condition $T$ and $A \subseteq \omega$, the restriction $T \restriction A$ is not necessarily a pure condition, as the sequence of logarithmic measures may not be unbounded.

**Lemma 4.9.** *Let $T = \langle (s_i, h_i) : i \in \omega \rangle$ be a pure condition and $A_0, ..., A_{n-1}$ a finite partition of $\omega$. Then there is $j \in n$ such that $T \restriction A_j$ is a pure condition.*

*Proof.* Suppose this is not the case. Then the level of the logarithmic measures of each $j \in n$ is bounded. So for every $j \in n$, there is $M_j \in \omega$ such that

$$h_i(s_i \cap A_j) \leq M_j \; \forall i \in \omega.$$

Let $M = \max_{j \in n}(M_j)$. Since $T$ is a pure condition, the levels of the logarithmic measures $(s_i, h_i)$ is unbounded, so there is $i \in \omega$ such that

$$h_i(s_i) \geq M + n.$$

For every $j \in n$, let $s_i^j = s_i \cap A_j$. Then $s_i$ is partitioned into the $n$ sets $s_i^0, ..., s_i^{n-1}$ and so by Lemma 4.1 there is $j \in n$ such that

$$h_i(s_i^j) \geq h_i(s_i) - j = M + n - j \geq M + 1 > M_j,$$

which is a contradiction to the assumption. □

**Lemma 4.10.** *Let $R$ and $T$ be pure conditions with $R \leq T$ and let $A \in [\omega]^\omega$ be such that $R \restriction A$ and $T \restriction A$ are pure conditions. Then $R \restriction A \leq T \restriction A$.*

*Proof.* Let $T = \langle (s_i, h_i) : i \in \omega \rangle$ and $R = \langle (r_i, g_i) : i \in \omega \rangle$. Since $R$ is a pure extension of $T$, there is a sequence $\langle B_i : i \in \omega \rangle \subseteq [\omega]^{<\omega}$ such that

$$r_i \subseteq \bigcup\{s_j : j \in B_i\} \; \forall i \in \omega.$$

In particular,

$$r_i \cap A \subseteq \bigcup\{s_j \cap A : j \in B_i\} \; \forall i \in \omega.$$

Let $e \subseteq r_i \cap A$ be such that $h_i(e) \geq 0$. Then there is $j \in B_i$ such that $g_j(e \cap s_j) > 0$ since $R \leq T$. Finally, since $e \subseteq A$, we have $e \cap s_j = e \cap s_j \cap A$, and so $R \restriction A$ is a pure extension of $T \restriction A$. □

**Lemma 4.11.** *Let $C$ be a centered family, $T$ a pure condition compatible with $C$ and $\omega = A_0 \cup ... \cup A_{n-1}$ a finite partition of $\omega$. Then there is $j \in n$ such that $T \restriction A_j$ is a pure condition compatible with $C$.*

*Proof.* Assume that the claim does not hold. Let $I \subseteq n$ be the set of all indices $j \in n$ such that $T \restriction A_j$ is a pure condition in $Q$. $I$ is non-empty by Lemma 4.9. By assumption, for every $j \in I$, there is $T_j \in C$ such that

$$T_j \perp T \restriction A_j.$$

Since $I$ is finite and $C$ is centered, there is $X \in C$ which is a common extension of all $T_j$ for $j \in I$. Since $X$ and $T$ are in $C$, they have a common extension $R \in Q$. Again, using Lemma 4.9 there is $i \in n$ such that $R \restriction A_i$ is a pure condition. Moreover, by Lemma 4.10 $R \restriction A_i \leq T \restriction A_i$. So $T \restriction A_i$ has pure extension such that $\mathrm{int}(R \restriction A_i) \subseteq A_i$. Then, by Lemma 4.8, $T \restriction A_i$ is a pure condition, so $i \in I$. Also

$$R \restriction A_i \leq R \leq X \leq T_i$$

and hence $R \restriction A_i$ is a common extension of $T_i$ and $T \restriction A_i$, which is a contradiction. □

## 4.3 Good $Q(C)$-Names for Reals

In this section, we will define the notion of good $Q(C)$-names, which will be used throughout the remainder of this chapter and in the next chapter.

**Remark 4.4.** *We will use the fact that for some forcing notion $\mathbb{P}$ and $f \in V^{\mathbb{P}} \cap {}^{\omega}\omega$, $f$ has a $\mathbb{P}$-name of the form*

$$\dot{f} = \bigcup \{\langle\langle i, j_p^i \rangle, p\rangle : p \in \mathcal{A}_i, i \in \omega, j_p^i \in \omega\},$$

*where $\mathcal{A}_i = \mathcal{A}_i^{\dot{f}}$ is a maximal antichain in $\mathbb{P}$ for each $i \in \omega$.*

**Definition 4.12.** *Let $C$ be a centered family of pure conditions. We say that a $Q(C)$-name $\dot{f}$ for a real is a good $Q(C)$-name, if $\dot{f}$ is a $Q(C')$-name for every centered family $C'$ extending $C$.*

**Remark 4.5.** *As each $Q(C)$-name $\dot{f}$ is decided by maximal antichains $\mathcal{A}_i^{\dot{f}}$ for each $i \in \omega$, $\dot{f}$ is a good $Q(C)$-name if and only if, for each $i \in \omega$, $\mathcal{A}_i^{\dot{f}}$ remains a maximal antichain in $Q(C')$ for every centered family $C'$ extending $C$.*

**Lemma 4.12.** *Let $C$ be a centered family of pure conditions and let $\dot{f}$ be a $Q(C)$-name for a real. Then the following are equivalent:*

1. *$\dot{f}$ is a good $Q(C)$-name for a real*
2. *$\dot{f}$ is a good $Q(C')$-name for every centered family $C'$ extending $C$ with $|C'| = |C|$*

*Proof.* The implication from 1. to 2. follows directly from the definition of a good $Q(C)$-name.
For the other direction, we will show that if $\dot{f}$ is not a good $Q(C)$-name, then there is some centered family $C''$ extending $C$ with $|C''| = |C|$ and such that $\dot{f}$ is not a $Q(C'')$-name.
So let $C'$ be a centered family extending $C$ with $|C'| > |C|$ such that $\dot{f}$ is not a $Q(C')$-name for a real. Then there is a condition $p = (u, T) \in Q(C')$ and some $i \in \omega$ such that $p$ is incompatible with all elements of $\mathcal{A}_i^{\dot{f}}$ in $Q(C')$. We will inductively construct a centered family $C' \subseteq Q(C'')$ such that $C \cup \{T\} \subseteq C''$ and $|C''| = |C|$. Let $C_0 = C \cup \{T\}$. Then $C_0 \subseteq Q(C')$ and so all $X, Y \in C_0$ have a common extension in $Q(C')$, which we will denote by $Z_{X,Y}$. Now let

$$C_0' = \{Z_{X,Y} : X, Y \in C_0\} \text{ and } C_1 = C_0 \cup C_0'.$$

Suppose that $C_n = C_{n-1} \cup C_{n-1}'$ has already been defined and $C_n \subseteq Q(C')$. Then all $X, Y \in C_{n-1}$ have a common extension $Z_{X,Y} \in Q(C')$. Let

$$C_n' = \{Z_{X,Y} : X, Y \in C_{n-1}\} \text{ and } C_{n+1} = C_n \cup C_n'.$$

Finally, let

$$C'' := \bigcup_{n \in \omega} C_n.$$

Then $C''$ is a centered family such that

$$C' \subseteq Q\left(C''\right), \; C \cup \{T\} \subseteq C'' \text{ and } \left|C''\right| = |C|.$$

Since $C \cup \{T\} \subseteq C''$, we have $p \in Q(C'')$. Moreover, $p$ is incompatible with all elements of $\mathcal{A}_i^{\dot{f}}$ in $Q(C')$, hence they are also incompatible in $Q$ by Lemma 4.6. Using the same Lemma, we also have that $p$ is incompatible with all elements in $\mathcal{A}_i^{\dot{f}}$ in $Q(C'')$ and so $\mathcal{A}_i^{\dot{f}}$ is not maximal in $Q(C'')$, hence $\dot{f}$ is not a $Q(C'')$-name. □

**Corollary 4.1.** *Let $C$ be a centered family of pure conditions and let $\dot{f}$ be a $Q(C)$-name for a real. If $\dot{f}$ is not a good $Q(C')$-name for some centered family $C'$ extending $C$, then there is a centered family $C''$ extending $C$, with $|C''| = |C|$ and such that $\dot{f}$ is not a $Q(C'')$-name for a real.*

## 4.4 Extending Centered Families

In this section, we will define the poset $Q_{fin}$ of finite sequences of strictly increasing logarithmic measures and suborders $\mathbb{P}(T)$ which will be used to construct extensions of arbitrary centered families.

**Definition 4.13.** *Let $Q_{fin}$ denote the partial order of all finite sequences $\overline{r} = \langle r_0, ...r_n \rangle$, where $n \in \omega$, for every $i \in n+1$, $r_i = (s_i, h_i)$ is a finite logarithmic measure and for every $i \in n$*

$$\max(s_i) < \min(s_{i+1}) \text{ and } h_i(s_i) < h_{i+1}(s_{i+1}).$$

*The level of the sequence $\overline{r}$ is the level of the highest measure $r_n$ and is denoted by $\|\overline{r}\|$. For $\overline{r}_1, \overline{r}_2 \in Q_{fin}$ define $\overline{r}_1 \leq \overline{r}_2$ if $\overline{r}_2$ is an initial segment of $\overline{r}_1$.*

**Definition 4.14.** *Let $\overline{r} = \langle r_0, ..., r_{n-1} \rangle \in Q_{fin}$, where for every $i \in n$, $r_i = (s_i, h_i)$. Then $\overline{r}$ extends a pure condition $T = \langle t_i : i \in \omega \rangle$, where $t_i = (x_i, g_i)$, denoted by $\overline{r} \leq T$, if*

1. $\operatorname{int}(\overline{r}) = \bigcup\{s_i : i \in n\} \subseteq \operatorname{int}(T)$
2. *there is a sequence $\langle B_0, ..., B_{n-1} \rangle$ of finite subsets of $\omega$ such that*
   a) $\max(B_i) < \min(B_{i+1})$ *for all $i \in n-1$, and*
   b) $s_i \subseteq \bigcup\{x_j : j \in B_i\}$ *for all $i \in n$*
3. *for every $i \in n$ and $e \subseteq s_i$ such that $h_i(e) > 0$, there is $j \in B_i$ such that $g_j(e \cap x_j) > 0$.*

*A single finite logarithmic measure $r = (s, h)$ extends a pure condition $T$, if the sequence $\langle r \rangle$ extends $T$.*

**Definition 4.15.** *Let $T$ be a pure condition. Then*

$$\mathbb{P}(T) := \{\overline{r} \in Q_{fin} : \overline{r} \leq T\}$$

*defines a suborder of $Q_{fin}$.*

**Lemma 4.13.** *Let $T \in Q$ be a pure condition.*

1. *For all $k \in \omega$ the set*
$$E_k := \{\overline{r} \in \mathbb{P}(T) : |\overline{r}| \geq k\}$$
   *is dense in $\mathbb{P}(T)$.*
2. *For every pure condition $X$ compatible with $T$ and every $n \in \omega$, the set*
$$D_T(X, n) := \{\overline{r} \in \mathbb{P}(T) : \exists r_j \in \overline{r}\,(r_j \leq X \text{ and } \|r_j\| \geq n)\}$$
   *is dense in $\mathbb{P}(T)$.*

*Proof.* Let $\overline{r} = \langle r_0, ..., r_{m-1} \rangle \in \mathbb{P}(T)$, where $T = \{(s_i, h_i) : i \in \omega\}$, with $|\overline{r}| = m < k$ and let $\langle B_0, ..., B_{m-1} \rangle \subseteq [\omega]^{<\omega}$ witness $\overline{r} \leq T$. Then let $\ell := \max(B_{m-1}) + 1$ and define $r_m = (s_\ell, h_\ell)$. We have that

$$\overline{r}^\frown\langle r_m \rangle \in Q_{fin},\ \overline{r}^\frown\langle r_m \rangle \leq \overline{r} \text{ and } \overline{r}^\frown\langle r_m \rangle \leq T,$$

which is witnessed by $\langle B_0, ..., B_{m-1}, \{\ell\}\rangle$ and so $\overline{r}^\frown\langle r_m\rangle \in \mathbb{P}(T)$. Repeating this procedure, we get an extension of $\overline{r}$ of length $k$ in finitely many steps and so $E_k$ is dense.
To see that $D_T(X, n)$ is dense, let $\overline{r} \in \mathbb{P}(T)$. Since $T$ and $X$ are compatible, $T \setminus \operatorname{int}(\overline{r})$ and $X$ are also compatible. So there is a finite logarithmic measure $z$ with

$$\|z\| > \max\{\|\overline{r}\|, n\}$$

which is a common extension of $X$ and $T \setminus \operatorname{int}(\overline{r})$. Then $\overline{r}^\frown\langle z\rangle \leq \overline{r}$ and it is in $D_T(X, n)$. □

**Corollary 4.2.** *Let $C$ be a centered family of pure conditions, $T$ a pure condition compatible with $C$ and $G$ a $\mathbb{P}(T)$-generic filter. Then there is a centered family $C'$ in $V[G]$ extending $C$ below*

$$R_G = \bigcup G = \langle r_i : i \in \omega\rangle$$

*(and so below $T$) which has the same cardinality as $C$.*

*Proof.* $R_G$ is a pure condition of strictly increasing finite logarithmic measures by Lemma 4.13. Since $D_T(X, n) \subseteq \mathbb{P}(T)$ from Lemma 4.13 is dense for every $X \in C$ and $n \in \omega$, we have $G \cap D_T(X, n) \neq \emptyset$. Therefore, the sequence

$$R_G \wedge X := \langle r_i \in R_G : r_i \leq X\rangle$$

is infinite and a common extension of $R_G$ and $X$. If $Y \leq X$, then $R_G \wedge Y \leq R_G \wedge X$ and so

$$C' := \{R_G \wedge X : X \in C\}$$

is a centered family below $R_G$ and with $|C'| = |C|$. □

## 4.5 Preprocessed Conditions

We will define the notion of a preprocessed condition in this section and show when pure conditions have preprocessed extensions. This notion will be used in coming sections to define forcing posets.

**Definition 4.16.** *Let $C$ be a centered family of pure conditions, $\dot{f}$ a good $Q(C)$-name for a real, $k, i \in \omega$ and $T$ a pure condition in $Q(C)$ such that $k < \min(\operatorname{int}(T))$. Then $T$ is called preprocessed for $\dot{f}(i), k, C$ if the following holds:*
*For every $v \subseteq k$, if there is a centered family $C'$ extending $C$ with $|C'| = |C|$, a pure condition $T' \in Q(C')$ and $q \in \mathcal{A}_i^{\dot{f}}$ such that $T' \leq T$ and $(v, T') \leq q$, then there is $p \in \mathcal{A}_i^{\dot{f}}$ such that $(v, T) \leq p$.*

**Lemma 4.14.** *Let $C$ be a centered family, $\dot{f}$ a good $Q(C)$-name for a real, $i, k \in \omega$ and $T \in Q(C)$ a pure condition which is preprocessed for $\dot{f}(i), k, C$. Further, let $C'$ be a centered family extending $C$ with $|C'| = |C|$. Then, any $T' \in Q(C')$ which is a pure extension of $T$ is preprocessed for $\dot{f}(i), k, C'$.*

*Proof.* Let $C''$ be any centered family which extends $C'$ and is of the same size as $C'$. Then, in particular, $C''$ also extends $C$ and both families are of the same size. Let $T'' \in Q(C'')$ be a pure condition extending $T'$, such that there are $q \in \mathcal{A}_i^{\dot{f}}$ and $v \subseteq k$ with $(v, T'') \leq q$.
Since $T''$ extends $T'$, it is also a pure extension of $T$ and as $T$ is preprocessed for $\dot{f}(i), k, C$ and $(v, T'') \leq q$, there is some $p \in \mathcal{A}_i^{\dot{f}}$ such that $(v, T) \leq p$. Now, since $T' \leq T$, we also have $(v, T') \leq p$ and so $T'$ is preprocessed for $\dot{f}(i), k, C'$. □

**Corollary 4.3.** *Let $C$ be a centered family, $\dot{f}$ be a good $Q(C)$-name, let $C'$ be a centered family extending $C$ with $|C'| = |C|$ and let $T \in Q(C)$ be a pure condition. If $T$ is preprocessed for $\dot{f}(i), k, C$, for some $i, k \in \omega$, then $T$ is preprocessed for $\dot{f}(i), k, C'$.*

**Lemma 4.15.** *Let $C$ be a centered family and $T$ be a pure condition in $Q(C)$. Further, let $\dot{f}$ be a good $Q(C)$-name for a real and $i, k \in \omega$.*
*Then there is a centered family $C'$ extending $C$, such that $|C'| = |C|$, and a pure extension $T' \in Q(C')$ of $T$ such that $T'$ is preprocessed for $\dot{f}(i), k, C'$.*

*Proof.* We will inductively construct a centered family $C'$ and a pure condition $T' \in Q(C')$ which extends $T$ and is preprocessed for $\dot{f}(i), k, C'$. Let $v_0, ..., v_n$ be an enumeration of all subsets of $k$.
For the base case consider the condition $(v_0, T \setminus k)$. If there is a centered family $C_0'$ extending $C$ with $|C_0'| = |C|$ and a pure condition $T_0' \in Q(C_0')$ such that $T_0' \leq T \setminus k$ and $(v_0, T_0') \leq p_0$, for some $p_0 \in \mathcal{A}_i^{\dot{f}}$, let $T_0 = T_0'$ and $C_0 = C_0'$. Otherwise let $T_0 = T$ and $C_0 = C$.
Now suppose that $T_{j-1}$ and $C_{j-1}$ have already been constructed and consider $(v_j, T_{j-1})$ and $C_{j-1}$. If there is a centered family $C_j'$ extending $C_{j-1}$ with $|C_j'| = |C_{j-1}|$ and a pure condition $T_j' \in Q(C_j')$ extending $T_{j-1}$ and there is $p_j \in \mathcal{A}_i^{\dot{f}}$ such that $(v_j, T_j') \leq p_j$, then let $T_j = T_j'$ and $C_j = C_j'$. Otherwise let $T_j = T_{j-1}$ and $C_j = C_{j-1}$.
Finally, let $T' = T_n$ and $C' = C_n$. It is left to show that $T'$ is preprocessed for $\dot{f}(i), k, C'$. So let $v$ be an arbitrary subset of $k$, $C''$ be a centered family extending $C'$ with $|C''| = |C'|$ and $T'' \in Q(C'')$ be a pure condition which extends $T$. Since $v$ is a subset of $k$, there is $j \in n+1$ such that $v = v_j$. By construction, $C'$ extends $C_{j-1}$ and so $C''$ extends $C_{j-1}$ and we have $T'' \leq T' \leq T_{j-1}$. Then, if there is some $p \in \mathcal{A}_i^{\dot{f}}$ such that $(v, T'') \leq p$, we have chosen $C_j$ and $T_j \in Q(C_j)$ in our construction such that $(v_j, T_j) \leq p_j$ for some $p_j \in \mathcal{A}_i^{\dot{f}}$. As $T'$ is an extension of $T$ and $v_j = v$, we have $(v, T') \leq p_j$ and so $T'$ is preprocessed for $\dot{f}(i), k, C'$. □

**Corollary 4.4.** *Let $C$ be a centered family, $T$ a pure condition in $Q(C)$, $\dot{f}$ a good $Q(C)$-name for a real and $k \in \omega$.*
*Then there are a centered family $C'$ extending $C$ with $|C'| = |C|$ and a pure condition $T' \in Q(C')$ such that $T' \leq T$ and $T'$ is preprocessed for $\dot{f}(i), k, C'$, for every $i \in k+1$.*

*Proof.* Using Lemma 4.15, we can find a centered family $C_0$ which extends $C$, with $|C_0| = |C|$, and a pure condition $T_0 \in Q(C_0)$ such that $T_0 \leq T \setminus k$ and $T_0$ is preprocessed

for $\dot{f}(0), k, C_0$.
Starting with $T_0$ and $C_0$, we can build two finite sequences

$$\langle C_i : i \leq k \rangle \text{ and } \langle T_i : i \leq k \rangle,$$

using Lemma 4.15, such that $C_{i+1}$ is a centered family extending $C_i$ and $|C_{i+1}| = |C_i| = |C|$ for each $i \in k$ and each $T_i$ is a pure condition in $Q(C_i)$, which is preprocessed for $\dot{f}(i), k, C_i$, and $T_{i+1} \leq T_i$ for all $i \in k$.
Let $T' = T_k$ and $C' = C_k$. By construction, $C' \leq C$, $|C'| = |C|$ and $T' \in Q(C')$ with $T' \leq T_i \leq T$ for each $i \in k+1$. Finally, for each $i \in k+1$ we have that $T'$ is preprocessed for $\dot{f}(i), k, C'$ by Lemma 4.14. □

## 4.6 Generic Preprocessed Conditions

Using results from the previous section, we can now show that there are infinite sequences of preprocessed conditions with desired properties and use them to define a partial order.

**Lemma 4.16.** *Let $C$ be a centered family of pure conditions, $\dot{f}$ a good $Q(C)$-name for a real and let $T$ be a pure condition in $Q(C)$.*
*Then there is a centered family $C'$ extending $C$ with $|C'| = |C|$ and a sequence of pure conditions $\langle T_n : n \in \omega \rangle$ in $Q(C')$, such that*

1. $T_0 \leq T$ *and* $\forall n \geq 1 \, (T_n \leq T_{n-1})$
2. $\forall n \in \omega \forall i \leq n$, *$T_n$ is preprocessed for* $\dot{f}(i), n, C'$.

*Proof.* We will inductively build the sequence $\langle T_n : n \in \omega \rangle$. Using Lemma 4.15 we get a centered family $C_0$ extending $C$ with $|C_0| = |C|$ and a pure condition $T_0 \in Q(C_0)$ which extends $T$ and is preprocessed for $\dot{f}(0)$, 0, $C_0$.
Now suppose that we have defined a centered family $C_n$ extending $C_{n-1}$ with $|C_n| = |C_{n-1}|$ and $T_n \in Q(C_n)$ such that $T_n \leq T_{n-1}$ and for each $i \leq n$, $T_n$ is preprocessed for $\dot{f}(i)$, $n$, $C_n$. By Corollary 4.4 we can find a centered family $C_{n+1}$ extending $C_n$ with $|C_{n+1}| = |C_n|$ and a pure condition $T_{n+1} \in Q(C_{n+1})$ which extends $T_n$ and such that $T_{n+1}$ is preprocessed for $\dot{f}(i)$, $k$, $C_{n+1}$ for all $i \leq n+1$.
Finally, let

$$C' = \bigcup_{n \in \omega} C_n.$$

Then $C'$ is a centered family extending $C$ with $|C'| = |C|$ and $\langle T_n : n \in \omega \rangle \subseteq Q(C')$. For every $n \in \omega$ and $i \leq n$, $T_n$ is preprocessed for $\dot{f}(i), n, C_n$ by construction of the sequence. As $C'$ extends $C_n$ and the families have the same size, Corollary 4.3 gives for each $i \leq n$ that $T_n$ is preprocessed for $\dot{f}(i), n, C'$. □

**Remark 4.6.** *Note that the sequence $\langle T_n : n \in \omega \rangle$ from Lemma 4.16 is not uniquely determined by $T$.*

*Until the end of this section, we will fix a centered family $C$, a good $Q(C)$-name $\dot{f}$ for a real, a pure condition $T \in Q(C)$ and a sequence*

$$\tau = \langle T_n : n \in \omega \rangle$$

*of pure conditions contained in $Q(C)$ which satisfies Lemma 4.16.*

**Definition 4.17.** *Define $\mathbb{P}_\tau(C, T, \dot{f})$ to be the suborder of $\mathbb{P}(T)$ which consists of all finite sequences $\overline{r} = \langle r_0, ..., r_{n-1} \rangle$, $n \in \omega$ such that $r_0 \leq T_0$ and for all $i \in \{1, ..., n-1\}$, $r_i \leq T_{j_i}$, where $j_i = \max(\text{int}(r_{i-1}))$.*

**Lemma 4.17.** *The following sets are dense in $\mathbb{P}_\tau(C, T, \dot{f})$:*

1. *For all $k \in \omega$*
$$E_k = \{\overline{r} \in \mathbb{P}_\tau(C, T, \dot{f}) : |\overline{r}| \geq k\}.$$
2. *For all $X \in C$, $n \in \omega$,*
$$D_\tau(X, n) = \{\overline{r} \in \mathbb{P}_\tau(C, T, \dot{f}) : \exists r_j \in \overline{r}\, (r_j \leq X \text{ and } \|r_j\| \geq n)\}.$$

*Proof.* Let $k \in \omega$ and $\overline{r} = \langle r_0, ..., r_{\ell-1} \rangle \in \mathbb{P}_\tau(C, T, \dot{f})$ with $|r| < k$. We can find an arbitrary finite logarithmic measure $r_\ell$ such that $r_\ell \leq T_{j_\ell}$, where $j_\ell = \max(\text{int}(r_{\ell-1}))$. Then $\overline{r}^\frown \langle r_\ell \rangle \in E_{\ell+1}$ and $\overline{r}^\frown \langle r_\ell \rangle \leq \overline{r}$. Repeating this finitely many times will result in a sequence $\overline{r}'$ with $\overline{r}' \in E_k$ and $\overline{r}' \leq \overline{r}$.
To see that $D_\tau(X, n)$ is dense for $X \in C$ and $n \in \omega$, let $\overline{r} = \langle r_0, ..., r_{\ell-1} \rangle \in \mathbb{P}_\tau(C, T, \dot{f})$. Let $j_\ell = \max(\text{int}(r_{\ell-1}))$. Since $T$ and $X$ are compatible, $T_{j_\ell} \setminus \text{int}(\overline{r})$ is compatible with $X$, and so there is finite logarithmic measure $r_\ell$ with

$$\|r_\ell\| \geq \max\{\|\overline{r}\|, n\}$$

and which is a common extension of $T_{j_\ell} \setminus \text{int}(\overline{r})$ and $X$. Then $\overline{r}^\frown \langle r_\ell \rangle \leq \overline{r}$ and $\overline{r}^\frown \langle r_\ell \rangle \in D_\tau(X, n)$. □

**Corollary 4.5.** *Let $G$ be a $\mathbb{P}_\tau(C, T, \dot{f})$-generic filter and let*

$$R_G = \bigcup G = \langle r_i : i \in \omega \rangle.$$

1. *Then $R_G$ is a pure condition of strictly increasing logarithmic measures such that for every $n \in \omega$, the condition $R_n = \langle r_i : i \geq n \rangle$ is a pure extension of $T_{j_n}$, where $j_n = \max(\text{int}(r_{n-1}))$.*
2. *In $V[G]$ there is a centered family $C'$ extending $C$ below $R_G$ (and hence below $T$) such that $|C'| = |C|$.*
*Then, in particular, $R_n \setminus x$ is preprocessed for $\dot{f}(n)$, $\max(x)$, $C'$ for all $n \in \omega$ and $x \in [\text{int}(R_n)]^{<\omega}$.*

*Proof.* By Lemma 4.17, $R_G$ is a pure condition of strictly increasing finite logarithmic measures such that for each $n \in \omega$, $R_n$ is a pure extension of $T_{j_n}$.
For the second part of the assertion, note that for each $X \in C$, the sequence

$$R_G \wedge X \coloneqq \langle r_i : r_i \leq X \rangle$$

is infinite, since $G \cap D_\tau (X, n) \neq \emptyset$ for all $n \in \omega$ and it is a common extension of $R_G$ and $X$. For $X \leq Y$ we have $R_G \wedge X \leq R_G \wedge Y$ and therefore the family

$$C' \coloneqq \{R_G \wedge X : X \in C\}$$

is centered. So $C'$ extends $C$ below $R_G$ and $|C'| = |C|$. □

## 4.7 Induced Logarithmic Measures

In this section, we will study the logarithmic measure induced by specific families of sets and we will see that these logarithmic measures take arbitrarily high values.

**Definition 4.18.** *Let $\mathcal{M}$ be the ideal of meager subsets of the real line. Moreover, let* $\operatorname{cov}(\mathcal{M})$ *be the minimal size of a family of meager sets which covers the real line.*

**Definition 4.19.** *Let $\kappa$ be an uncountable cardinal. The $MA_{countable}(\kappa)$ is the statement that for every countable partial order $\mathbb{P}$ and every family $\mathcal{D}$ of dense subsets of $\mathbb{P}$, there is a filter $G \subseteq \mathbb{P}$ that meets all sets in $\mathcal{D}$, so $\forall D \in \mathcal{D} : (G \cap D) \neq \emptyset$.*

Note that for every regular uncountable $\kappa$, we have $\operatorname{cov}(\mathcal{M}) \geq \kappa$ if and only if $MA_{countable}(\kappa)$.

**Lemma 4.18.** *Let $C$ be a centered family of pure conditions with $|C| < \operatorname{cov}(\mathcal{M})$ and $\dot{f}$ a good $Q(C)$-name for a real. Further, let $n \in \omega$ and $T = \langle (s_i, h_i) : i \in \omega \rangle$ a pure condition in $Q(C)$ such that for all $x \in [\operatorname{int}(T)]^{<\omega}$, $T \setminus x$ is preprocessed for $\dot{f}(n)$, $\max(x)$, $C$.*
*For $v \in [\omega]^{<\omega}$, let $\mathcal{P}_v(C, T, \dot{f}(n))$ be the family of all $x \in [\operatorname{int}(T)]^{<\omega}$ such that*

1. *there is $i \in \omega$ such that $h_i(x \cap s_i) > 0$, and*
2. *there are $w \subseteq x$ and $p \in \mathcal{A}_n^{\dot{f}}$ such that $(v \cup w, T \setminus x) \leq p$.*

*Then the logarithmic measure induced by $\mathcal{P}_v(C, T, \dot{f}(n))$ takes arbitrarily high values.*

*Proof.* We will show the assertion using Lemma 4.4, so we need to show that for any arbitrary partition $\omega = A_0 \cup ... \cup A_{M-1}$, there is some $j \in M$ such that $A_j$ contains a set which is positive with respect to the logarithmic measure induced by $\mathcal{P}_v(C, T, \dot{f}(n))$. Fix a partition of $\omega$ as above. Then, by Lemma 4.11, there is a pure condition $T'$ which extends $T$, is compatible with $C$ and such that there is some $j \in M$ with $\operatorname{int}(T') \subseteq A_j$. By $|C| < \operatorname{cov}(\mathcal{M})$ and Corollary 4.2, there is a centered family $C'$ extending $C$ with $|C'| = |C|$, and a pure condition

$$R = \langle r_i : i \in \omega \rangle \in Q(C'),$$

where the $r_i = (s_i^R, h_i^R)$ are finite logarithmic measures of strictly increasing levels, such that $R \leq T' \leq T$. As $\dot{f}$ is a good $Q(C)$-name, it is also a $Q(C')$-name and so $\mathcal{A}_n^{\dot{f}}$ is a maximal antichain in $Q(C')$. Hence there is some condition $(v', R') \in Q(C')$ which is a common extension of $(v, R)$ and some $q \in \mathcal{A}_n^{\dot{f}}$. Since $(v', R') \leq (v, R)$ we have that $v'$ is an end-extension of $v$ and $w := v' \setminus v \subseteq \text{int}(R)$. As $w$ is finite, there is a finite subsequence $\langle r_i : a \leq i \leq b \rangle$ in $R$ such that

$$w \subseteq x := \bigcup_{i=a}^{b} \text{int}(r_i).$$

We have that

$$x \subseteq \text{int}(R) \subseteq \text{int}(T') \subseteq A_j,$$

so we need to show that $x \in \mathbb{P}_v(C, T, \dot{f}(n))$. We can assume that $w \neq \emptyset$ and so there is $k \in [a, b]$ such that

$$0 < \|r_k\| = h_i^R(s_i^R) = h_i^R(s_i^R \cap x).$$

Since $R \leq T$, there is $i \in \omega$ such that $h(x \cap s_i) > 0$ by definition of the extension relation, and so 1. holds.

For 2. observe that $v \cup w \subseteq \max(x)$, $R' \leq T \setminus x$ and $(v \cup w, R') \leq q$, where $q \in \mathcal{A}_n^{\dot{f}}$. As $T \setminus x$ is preprocessed for $\dot{f}(n), \max(x), C$, there is $p \in \mathcal{A}_n^{\dot{f}}$ such that $(v \cup w, T \setminus x) \leq p$. □

**Corollary 4.6.** *Let $C$ be a centered family of pure conditions with $|C| < \text{cov}(\mathcal{M})$ and $\dot{f}$ a good $Q(C)$-name for a real. Further, let $n, k \in \omega$ and $T = \langle (s_i, h_i) : i \in \omega \rangle \in Q(C)$ a pure condition such that for all $x \in [\text{int}(T)]^{<\omega}$, $T \setminus x$ is preprocessed for $\dot{f}(n)$, $\max(x)$, $C$. Let $\mathcal{P}_k(C, T, \dot{f}(n))$ be the family of all $x \in [\text{int}(T)]^{<\omega}$ such that*

1. *there is $i \in \omega$ such that $h_i(s_i \cap x) > 0$, and*
2. *$\forall v \subseteq k \, \exists w \subseteq x \, \exists p \in \mathcal{A}_n^{\dot{f}}$ such that $(v \cup w, T \setminus x) \leq p$.*

*Then the logarithmic measure induced by $\mathcal{P}_k(C, T, \dot{f}(n))$ takes arbitrarily high values.*

*Proof.* We will show that the logarithmic measure takes arbitrarily high values using Lemma 4.4, so fix an arbitrary partition $\omega = A_0 \cup ... \cup A_{M-1}$. Proceed as in the proof of Lemma 4.18 to obtain a pure condition $T'$ which extends $T$ such that $\text{int}(T') \subseteq A_j$ for some $j \in M$, a centered family $C'$ extending $C$ and

$$R = \langle r_i : i \in \omega \rangle \in Q(C'),$$

where $r_i = (s_i^R, h_i^R)$ are logarithmic measures of increasing levels, such that $R \leq T' \leq T$. Then, in particular, $R \setminus x \leq T \setminus x$ for all $x \in [\text{int}(T)]^{<\omega}$ and hence, by Lemma 4.14, $R \setminus x$ is preprocessed for $\dot{f}(n), \max(x)$ and $C'$.

Now fix an enumeration $v_0, ..., v_{L-1}$ of all subsets of $k$. Then, for each $i \in L$, we can find $x_i \in \mathcal{P}_{v_i}(C', R, \dot{f}(n))$ by Lemma 4.18. Now let

$$x := \bigcup \{x_i : i \in L\}$$

and we will show that $x \in \mathcal{P}_k(C, T, \dot{f}(n))$.
For each $i \in L$ we have $x_i \in \mathcal{P}_{v_i}(C', R, \dot{f}(n))$ and so there is $j_i \in \omega$ such that $h^R_{j_i}(s^R_{j_i} \cap x_i) > 0$. By definition of $R \leq T$ there is $\ell \in \omega$ such that $h_\ell(s_\ell \cap x_i) > 0$ and so

$$h_\ell(s_\ell \cap x) \geq h_\ell(s_\ell \cap x_i) > 0.$$

To see that 2. holds as well, let $v \subseteq k$. Then $v = v_i$ for some $i \in L$. Since $x_i \in \mathcal{P}_{v_i}(C', R, \dot{f}(n))$, there are $w_i \subseteq x_i$ and $q_i \in \mathcal{A}^{\dot{f}}_n$ such that $(v_i \cup w_i, R \setminus x_i) \leq q_i$, and hence

$$(v_i \cup w_i, R \setminus x) \leq q_i.$$

Since $R \leq T$, $C'$ is extends $C$ with $|C'| = |C|$ and $T \setminus x$ is preprocessed for $\dot{f}(n)$, $\max(x)$ and $C$, there is $p \in \mathcal{A}^{\dot{f}}_n$ such that

$$(v \cup w, T \setminus x) \leq p.$$

So $x \in \mathcal{P}_k(C, T, \dot{f}(n))$ and since

$$x \subseteq \mathrm{int}(R) \subseteq \mathrm{int}(T') \subseteq A_j,$$

the logarithmic measure induced by $\mathcal{P}_k(C, T, \dot{f}(n))$ takes arbitrarily high values. $\square$

# 5 $\mathfrak{b} = \kappa < \mathfrak{s} = \kappa^+$

This chapter builds on results from the previous chapter, and in the final chapter we will give the construction for a model in which $\mathfrak{b} = \kappa < \mathfrak{s} = \kappa^+$ holds. This chapter follows [FS08].

## 5.1 Good Extensions

**Remark 5.1.** *For this section, fix a centered family $C$ with $|C| < \mathrm{cov}(\mathcal{M})$, a good $Q(C)$-name $\dot{f}$ and a pure condition $T = \langle t_i : i \in \omega \rangle \in Q(C)$ such that for all $n \in \omega$ and $x \in [\mathrm{int}(T_n)]^{<\omega}$, where $T_n = \langle t_i : i \geq n \rangle$, $T \setminus x$ is preprocessed for $\dot{f}(n)$, $\max(x)$, $C$.*

**Definition 5.1.** *Let $\mathbb{P}(C, T, \dot{f})$ be the suborder of $\mathbb{P}(T)$ of all sequences $\overline{r} = \langle r_0, ..., r_{n-1} \rangle$ such that for all $i \in n$, all $v \subseteq i$ and all $r_i$-positive $s \subseteq \mathrm{int}(r_i)$, there are $w \subseteq s$ and $p \in \mathcal{A}_i^{\dot{f}}$ such that $(v \cup w, T \setminus s) \leq p$.*

**Lemma 5.1.** *For every $k \in \omega$ the set*

$$E_k(C, T, \dot{f}) = \{\overline{r} \in \mathbb{P}(C, T, \dot{f}) : |\overline{r}| \geq k\}$$

*is dense in $\mathbb{P}(C, T, \dot{f})$.*

*Proof.* Let $\overline{r} = \langle r_0, ..., r_{m-1} \rangle \in \mathbb{P}(C, T, \dot{f})$ with $m < k$. We will construct an extension of $\overline{r}$ which is in $E_k(C, T, \dot{f})$.
Since $\overline{r} \leq T$, we have $i_T(\max(\mathrm{int}(\overline{r}))) \geq m$ and so

$$T' := T \setminus \mathrm{int}(\overline{r}) \leq T_m.$$

By Lemma 4.14 and our choice of $T$, $T' \setminus x$ is preprocessed for $\dot{f}(m), \max(x), C$, for all $x \in [\mathrm{int}(T')]^{<\omega}$. Then by Corollary 4.6 and $|C| < \mathrm{cov}(\mathcal{M})$, the logarithmic measure $h$ induced by the family $\mathcal{P}_m(C, T', \dot{f}(m))$ takes arbitrarily high values, hence

$$\exists x \in \mathcal{P}_m(C, T', \dot{f}(m)) : \ h(x) > \|r_{m-1}\|.$$

Let $r_m = (x, h \restriction \mathcal{P}(x))$. We will show that $\overline{r}^\frown\langle r_m \rangle \in \mathbb{P}(C, T, \dot{f})$.
Let $v \subseteq m$ and $s \subseteq \mathrm{int}(r_m)$ with $h(s) > 0$. By the definition of $h$, there is $w \subseteq s$ and $p \in \mathcal{A}_m^{\dot{f}}$ such that $(v \cup w, T' \setminus s) \leq p$. Since $s \subseteq x \in [\mathrm{int}(T \setminus \mathrm{int}(\overline{r}))]^{<\omega}$, we have $T' \setminus s = T \setminus s$ and hence

$$(v \cup w, T \setminus s) \leq p.$$

So $\overline{r}^\frown\langle r_m \rangle$ is a condition in $E_{m+1}(C, T, \dot{f})$ which extends $\overline{r}$. By repeating this process finitely many times, we get an extension of $\overline{r}$ in $E_k(C, T, \dot{f})$. $\square$

**Lemma 5.2.** *For every $X \in C$ and $n \in \omega$, the set*

$$D_{X,n}(C, T, \dot{f}) = \{\overline{r} \in \mathbb{P}(C, T, \dot{f}) : \exists r_j \in \overline{r}\, (r_j \le X \text{ and } \|r_j\| \ge n)\}$$

*is dense in $\mathbb{P}(C, T, \dot{f})$.*

*Proof.* Let $\overline{r} = \langle r_0, ..., r_{m-1}\rangle \in \mathbb{P}(C, T, \dot{f})$ and we will construct an extension of $\overline{r}$ which is in $D_{X,n}(C, T, \dot{f})$ for any $X \in C$ and $n \in \omega$.
Since $\overline{r} \le T$, we have $i_T\,(\max(\mathrm{int}(\overline{r}))) \ge m$ and so

$$T' \coloneqq T \setminus \mathrm{int}(\overline{r}) \le T_m.$$

Since $X, T' \in Q\,(C)$, each condition has an extension in $C$, and as $C$ is centered, there is $Y \in C$ such that $Y \le X, T'$. Then, by the choice of $T$ and Lemma 4.14, for every $x \in [\mathrm{int}\,(Y)]^{<\omega}$, $Y$ is preprocessed for $\dot{f}\,(m)$, $\max(x)$ and $C$. By Corollary 4.6 and $|C| < \mathrm{cov}\,(\mathcal{M})$, the logarithmic measure $h$ induced by $\mathcal{P}_m(C, Y, \dot{f}\,(m))$ takes arbitrarily high values, and so

$$\exists x \subseteq \mathcal{P}_m(C, Y, \dot{f}(m)) : h(x) > \max\{\|r_{m-1}\|, n\}.$$

Let $r_m = (x, h \restriction \mathcal{P}\,(x))$. We will show that $\overline{r}^\frown\langle r_m\rangle \in \mathbb{P}(C, T, \dot{f})$. Let $v \subseteq m$ and $s \subseteq \mathrm{int}\,(r_m)$ with $h\,(s) > 0$. By the definition of $h$, there is $w \subseteq s$ and $q \in \mathcal{A}_m^{\dot{f}}$ such that

$$(v \cup w, Y \setminus s) \le q.$$

Since $T \setminus s$ is preprocessed for $\dot{f}\,(m)$, $\max\,(s)$ and $C$, and $Y \setminus s \le T \setminus s$, there is $p \in \mathcal{A}_m^{\dot{f}}$ such that $(v \cup w, T \setminus s) \le p$. Hence $\overline{r}^\frown\langle r_m\rangle \in \mathbb{P}(C, T, \dot{f})$ and, since $\|r_m\| \ge n$ and $\overline{r}^\frown\langle r_m\rangle \le X$, we have $\overline{r}^\frown\langle r_m\rangle \in D_{X,n}(C, T, \dot{f})$. □

**Corollary 5.1.** *Let $G$ be a filter in $\mathbb{P}(C, T, \dot{f})$ which meets $D_{X,n}(C, T, \dot{f})$ and $E_k(C, T, \dot{f})$ for all $X \in C$ and $n, k \in \omega$. Let*

$$R_G = \bigcup G = \langle r_i : i \in \omega\rangle.$$

1. *Then $R_G$ is a pure condition of finite logarithmic measures of strictly increasing levels such that $\forall i \in \omega\, \forall v \subseteq i\, \forall s \in \mathrm{int}\,(r_i)$ which are $r_i$-positive, there is $w \subseteq s$ and $p \in \mathcal{A}_i^{\dot{f}}$ such that $(v \cup w, R_G \setminus s) \le p$.*
2. *Furthermore, there is a centered family $C'$ extending $C$ below $R_G$ (and so below $T$) such that $|C'| = |C|$.*

*Proof.* By assumption, $G$ meets all $E_k(C, T, \dot{f})$ and $D_{X,n}(C, T, \dot{f})$ and so $R_G$ is a pure condition of finite logarithmic measures of strictly increasing levels. Let $i \in \omega$, $v \subseteq i$ and $s \in \mathrm{int}\,(r_i)$ be $r_i$-positive. By definition of $\mathbb{P}(C, T, \dot{f})$, there is $w \subseteq s$ and $p \in \mathcal{A}_i^{\dot{f}}$ such that $(v \cup w, T \setminus s) \le p$. Since $R_G \le T$, we have $(v \cup w, R_G \setminus s) \le p$.

For the second part of the assertion, note that since $G$ meets $D_{X,n}(C, T, \dot{f})$ for all $X \in C$ and $n \in \omega$, the sequence

$$R_G \wedge X \coloneqq \langle r_i : r_i \leq X \rangle$$

is infinite and a common extension of $R_G$ and $X$. For any $Y \in Q$ with $X \leq Y$ we have $R_G \wedge X \leq R_G \wedge Y$, and so the family

$$C' \coloneqq \{R_G \wedge X : X \in C\}$$

is a centered family which extends $C$ below $R_G$ and with $|C'| = |C|$. $\square$

## 5.2 Preserving unbounded $<^*$-directed Families

In this section, we will show that any centered family has an extension which preserves the unboundedness of an unbounded $<^*$-directed family (see Definition 2.8) of size $\mathrm{cov}(\mathcal{M})$.

**Theorem 5.1.** *Let $\kappa$ be a regular uncountable cardinal with* $\mathrm{cov}\,(\mathcal{M}) = \kappa$ *and let $\mathcal{H} \subseteq {}^{\omega}\omega$ be an unbounded, $<^*$-directed family with $|\mathcal{H}| = \kappa$. Further, let $C$ be a centered family with $|C| < \kappa$, $\dot{f}$ a good $Q\,(C)$-name for a real and $T \in Q\,(C)$.*
*Then there is a centered family $C'$ extending $C$ such that $|C'| = |C|$, a pure condition $R \in Q\,(C')$ which extends $T$ and a real $h \in \mathcal{H}$ such that for every centered family $C''$ extending $C'$, for every $a \in [\omega]^{<\omega}$*

$$(a, R) \Vdash_{Q(C'')} \exists^{\infty} i \in \omega(\dot{f}\,(i) < h\,(i)).$$

*Proof.* By $|C| < \mathrm{cov}\,(\mathcal{M})$ and Corollary 4.5 there is a centered family $C_1$, with $|C_1| = |C|$ that extends $C$ below $T$ and such that there is $T_1 = \langle t^1_i : i \in \omega\rangle \in Q\,(C_1)$ with $T_1 \leq T$ and, for each $n \in \omega$ and $x \in \left[\mathrm{int}\left(T_1 \setminus \mathrm{int}\left(t^1_{n-1}\right)\right)\right]^{<\omega}$, $T_1 \setminus x$ is preprocessed for $\dot{f}\,(n)\,, \max(x)$ and $C_1$. Since $|C_1| < \mathrm{cov}\,(\mathcal{M})$ there is a filter $G \subseteq \mathbb{P}(C_1, T_1, \dot{f})$ which meets $E_k(C_1, T_1, \dot{f})$ and $D_{X,n}(C_1, T_1, \dot{f})$ for all $X \in C_1$ and $k, n \in \omega$. By Corollary 5.1,

$$T_2 \coloneqq \bigcup G = \langle r_i : i \in \omega\rangle$$

is a pure condition of finite logarithmic measures of strictly increasing levels that extends $T_1$ and such that for all $i \in \omega$, all $v \subseteq i$ and for each $r_i$-positive $s \subseteq \mathrm{int}(r_i)$, there is $w \subseteq s$ and $p \in \mathcal{A}^{\dot{f}}_i$ such that $(v \cup w, T_2 \setminus s) \leq p$. Define

$$g(i) \coloneqq \max\{k \in \omega : \exists v \subseteq i, w \subseteq \mathrm{int}(r_i), p \in \mathcal{A}^{\dot{f}}_i\,((v \cup w, T_2) \leq p \text{ and } p \Vdash \check{k} = \dot{f}(i))\}.$$

We can assume that $g$ is non-decreasing, otherwise let $g\,(i) = \max\{g\,(j) : j \leq i\}$. For every $X \in C_1$ let $J_X \coloneqq \{i : r_i \leq X\}$ and

$$F_X\,(\ell) \coloneqq g\,(J_X\,(i+1)) \text{ iff } \ell \in (J_X\,(i)\,, J_X\,(i+1)]\,,$$

where $J_X(m)$ denotes the $m$-th element of $J_X$. As $\mathcal{H}$ is unbounded, there is $h_X \in \mathcal{H}$ such that $h_X \not\leq^* F_X$ for each $X \in C_1$. Since $|\{h_X : X \in C_1\}| < \kappa$, there is $h \in \mathcal{H}$ with $h_X \leq^* h$ for all $X \in C_1$. We can assume that $h$ is non-decreasing. Then, for each $X \in C_1$ we have $g(i) \leq F_X(i)$ for all $i \in \omega$ by definition of $F_X$. Further, $h \not\leq^* F_X$ since $h_X \not\leq^* F_X$ and $h_X \leq^* h$. Hence the set $J := \{i \in \omega : g(i) < h(i)\}$ is infinite.

**Claim.** *There are infinitely many $i \in J_X$ such that $F_X(i) < h(i)$.*

*Proof of claim.* Suppose that this is not the case, so for all but finitely many $i \in J_X$, we have $h(i) \leq F_X(i)$. Then there is $m \in \omega$ such that

$$\forall j \geq m : \; h(J_X(k)) \leq F_X(J_X(k)).$$

For each $\ell \in \omega \setminus J_X(m)$ there is $i \geq m$ such that $\ell \in (J_X(i), J_X(i+1)]$, since $\omega \setminus J_X(m) = \bigcup_{k \geq m} (J_X(k), J_X(k+1)]$. Then, since $h$ is non-decreasing, we have

$$h(\ell) \leq h(J_X(i+1)) \leq F_X(J_X(i+1)) = F_X(\ell).$$

But this contradicts $h \not\leq^* F_X$. □

By definition of $J_X$ and $F_X$, we have $F_X(j) = g(j)$ for all $j \in J_X$ and since $h \not\leq^* F_X$, the set $I_X := J \cap J_X$ is infinite. Let $R = \langle r_i : i \in J \rangle$, then for each $X \in C_1$,

$$R \wedge X = \langle r_i : i \in I_X \rangle = \langle r_i \in R : r_i \leq X \rangle$$

is a common extension of $R$ and $X$. If $X \leq Y$, then $J_X \subseteq J_Y$, so $I_X \subseteq I_Y$ and hence

$$C' := \{R \wedge X : X \in C_1\}$$

is a centered family which extends $C_1$ below $R$ with $|C'| = |C_1| = |C|$.
Now let $C''$ be an arbitrary centered family which extends $C'$. Fix $a \in [\omega]^{<\omega}$, $k \in \omega$ and let $(b, R') \in Q(C'')$ be an arbitrary extension of $(a, R)$. Since $J$ is infinite, there is $i \in J$ such that $i > k$, $b \subseteq i$ and the set

$$s := \operatorname{int}(R') \cap \operatorname{int}(r_i)$$

is $r_i$-positive. Then, by our choice of $T_2$, there are $w \subseteq s$ and $p \in \mathcal{A}_i^{\dot{f}}$ with $(b \cup w, T_2 \setminus s) \leq p$. Since $R' \setminus s \leq R \setminus s \leq T_2 \setminus s$, we have $(b \cup w, R' \setminus s) \leq p$ and $(b \cup w, R' \setminus s) \leq (b, R')$. Let $j \in \omega$ be such that $p \Vdash \check{j} = \dot{f}(i)$. By definition of $g$, $j \leq g(i)$ and since $i \in J$, $g(i) < h(i)$ and so

$$\left(b \cup w, R' \setminus s\right) \Vdash_{Q(C'')} \dot{f}(i) = \check{j} \leq \check{g}(i) < \check{h}(i).$$

But since $(b, R') \leq (a, R)$ was arbitrary as well as $k \in \omega$, we have

$$(a, R) \Vdash_{Q(C'')} \exists^\infty i \in \omega (\dot{f}(i) < \check{h}(i)).$$

□

## 5.3 Adding an unsplit Real

Using Theorem 5.1, we can construct a centered family that preserves the unboundedness of directed families of size $\kappa$ and adds a new real which is not split by the ground model reals.

**Lemma 5.3.** *Let $\kappa$ be a regular uncountable cardinal such that* $\mathrm{cov}(\mathcal{M}) = \kappa$ *and* $\forall\lambda < \kappa\left(2^\lambda \le \kappa\right)$. *Further, let $\mathcal{H} \subseteq {}^\omega\omega$ be an unbounded, $<^*$-directed family of size $\kappa$. Then there is a centered family $C$ of size $\kappa$ such that*

1. $\mathbb{1} \Vdash_{Q(C)} (\check{\mathcal{H}}$ *is unbounded)*
2. $Q(C)$ *adds a real not split by the ground model reals.*

*Proof.* We will construct the centered family $C$ via transfinite induction of length $\kappa$. Let $\mathcal{F} = \{\dot{f}_\alpha\}_{\alpha<\kappa}$ enumerate all names for reals in partial orders $Q(C')$ where $C'$ is a centered family with $|C'| < \kappa$. Also let $\mathcal{A} = \{A_{\alpha+1}\}_{\alpha<\kappa}$ enumerate $[\omega]^\omega$.
For the base case, take any pure condition $T \in Q$ and let $C_0 = \{T \setminus v : v \in [\omega]^{<\omega}\}$, which is a centered family of size less than $\kappa$.
Suppose $\alpha = \beta + 1$ is a successor and assume that we have already constructed $C_\beta$. Let $\dot{g}_{\beta+1}$ be the name with the least index in $\mathcal{F} \setminus \{\dot{g}_{\gamma+1}\}_{\gamma<\beta}$ which is a $Q(C_\beta)$-name for a real. Then $\dot{g}_{\beta+1}$ is either a good $Q(C_\beta)$-name for a real or it is not a good name.
If $\dot{g}_{\beta+1}$ is a good $Q(C_\beta)$-name, pick an arbitrary $T' \in Q(C_\beta)$. By Lemma 4.11 there is a pure extension $T''$ of $T'$ which is compatible with $C_\beta$ and we have

$$\mathrm{int}\left(T''\right) \subseteq A_{\beta+1} \text{ or } \mathrm{int}\left(T''\right) \subseteq A^c_{\beta+1}.$$

By $|C_\beta| < \mathrm{cov}(\mathcal{M})$ and Corollary 4.2 there is a centered family $C'_{\beta+1}$ which extends $C_\beta$ below $T''$ such that $|C'_{\beta+1}| = |C_\beta|$. Then by Theorem 5.1 there is a centered family $C_{\beta+1}$ extending $C'_{\beta+1}$ with $|C_{\beta+1}| = |C'_{\beta+1}|$ and a pure condition $T_{\beta+1} \in Q(C_{\beta+1})$ such that $T_{\beta+1} \le T''$ and for some $h_{\beta+1} \in \mathcal{H}$, every centered family $C''$ extending $C_{\beta+1}$ and all $a \in [\omega]^{<\omega}$

$$(a, T_{\beta+1}) \Vdash_{Q(C'')} \exists^\infty i \in \omega(\dot{g}_{\beta+1}(i) < \check{h}_{\beta+1}(i)).$$

Since $T_{\beta+1}$ extends $T''$, we also have

$$\mathrm{int}(T_{\beta+1}) \subseteq A_{\beta+1} \text{ or } \mathrm{int}(T_{\beta+1}) \subseteq A^c_{\beta+1}.$$

If $\dot{g}_{\beta+1}$ is not a good $Q(C_\beta)$-name, then by Corollary 4.1 there is a centered family $C'_{\beta+1}$ extending $C_\beta$ with $|C'_{\beta+1}| = |C_\beta|$ such that $\dot{g}_{\beta+1}$ is not a $Q(C'_{\beta+1})$-name for a real. By Lemma 4.11 there is a pure extension $T_{\beta+1}$ of $T'$ which is compatible with $C'_{\beta+1}$ such that

$$\mathrm{int}\left(T_{\beta+1}\right) \subseteq A_{\beta+1} \text{ or } \mathrm{int}\left(T_{\beta+1}\right) \subseteq A^c_{\beta+1}.$$

By $|C'_{\beta+1}| < \mathrm{cov}(\mathcal{M})$ and Corollary 4.2 there is a centered family $C_{\beta+1}$ extending $C'_{\beta+1}$ below $T_{\beta+1}$ such that $|C_{\beta+1}| = |C'_{\beta+1}|$.

If $\alpha$ is a limit, let $C_\alpha = \bigcup_{\beta<\alpha} C_\beta$. Then $|C_\alpha| < \kappa$ and $C_\alpha$ extends $C_\beta$ for each $\beta < \alpha$. Complete the construction by letting

$$C = \bigcup_{\alpha<\kappa} C_\alpha.$$

Then $|C| = \kappa$ and $C$ is a centered family which extends $C_\alpha$ for all $\alpha < \kappa$.
To see that 1. holds, let $\dot{f}$ be a $Q(C)$-name for a real, so it is of the form

$$\dot{f} = \bigcup \{\langle\langle i, j_p^i\rangle, p\rangle : i \in \omega, p \in \mathcal{A}_i^{\dot{f}}, j_p^i \in \omega\},$$

where $\mathcal{A}_i^{\dot{f}}$ is a maximal antichain in $Q(C)$ for each $i \in \omega$. For every $i \in \omega$ and $p \in \mathcal{A}_i^{\dot{f}}$ let

$$\alpha_i(p) = \min\{\gamma : p \in Q(C_\gamma)\}.$$

Since $\kappa$ is regular and uncountable,

$$\alpha_i := \sup\{\alpha_i(p) : p \in \mathcal{A}_i(\dot{f})\} < \kappa \text{ and } \alpha := \sup\{\alpha_i : i \in \omega\} < \kappa.$$

Moreover, $\alpha$ is minimal such that $\dot{f}$ is a $Q(C_\alpha)$-name for a real. So $\dot{f} \in \mathcal{F}$ and since $\dot{f}$ is a $Q(C_\beta)$-name for all $\beta \geq \alpha$, there is $\delta < \kappa$ such that $\dot{f}$ is the name with least index in $\mathcal{F} \setminus \{\dot{g}_{\gamma+1}\}_{\gamma<\delta}$ which is a $Q(C_\delta)$-name, so $\dot{f} = \dot{g}_{\delta+1}$.

**Claim.** *$\dot{f}$ is a good $Q(C_\delta)$-name.*

*Proof of claim.* Suppose that this is not the case. Then, by our construction, we would have chosen a centered family $C_{\delta+1}$ to be such that $\dot{f}$ is not a good $Q(C_{\delta+1})$-name. Then we could find $q \in Q(C_{\delta+1})$ and $i \in \omega$ such that $q$ is incompatible with all elements of $A_i^{\dot{f}}$ in $Q(C_{\delta+1})$. But then, by Lemma 4.6, $q$ would remain incompatible with all elements of $\mathcal{A}_i^{\dot{f}}$ in $Q$ and then they are also incompatible in $Q(C)$. So $\mathcal{A}_i^{\dot{f}} \cup \{q\}$ is an antichain in $Q(C)$, which contradicts the maximality of $\mathcal{A}_i^{\dot{f}}$. □

Hence $\dot{f}$ is a good $Q(C_\delta)$-name and then we have chosen $T_{\delta+1}$ and $C_{\delta+1}$ in our construction such that for all $a \in [\omega]^{<\omega}$

$$(a, T_{\delta+1}) \Vdash_{Q(C)} \exists^\infty i \in \omega(\dot{f}(i) < \check{h}_{\delta+1}(i)).$$

Finally, note that $\{(a, T_{\delta+1}) : a \in [\omega]^{<\omega}\}$ is predense in $Q(C)$ as any two conditions with equal stem are compatible. So we have

$$\Vdash_{Q(C)} h_{\delta+1} \not\leq^* \dot{f}.$$

To see that $Q(C)$ adds a real not split by the ground model reals, let $G$ be a $Q(C)$-generic filter and $\bigcup G = \{u : \exists T\,(u,T) \in G\}$. Then for all $\gamma \in \kappa$ the set

$$D_{\gamma+1} := \{(u,T) \in Q(C) : T \leq T_{\gamma+1}\}$$

is dense, because any two conditions with equal stem are compatible. Therefore, $G$ meets $D_{\gamma+1}$ and so $\bigcup G \subseteq^* \text{int}(T_{\gamma+1})$, which gives

$$\bigcup G \subseteq^* A_{\gamma+1} \text{ or } \bigcup G \subseteq^* A_{\gamma+1}^c.$$

So the set $\bigcup G$ is not split by the ground model reals. □

## 5.4 $\mathfrak{b} = \kappa < \mathfrak{s} = \kappa^+$

Finally, by applying Lemma 5.3 and the preservation theorems from section 2.2.2, we can construct a model which separates the bounding and the splitting numbers.

**Theorem 5.2** (GCH)**.** *Let $\kappa$ be a regular uncountable cardinal. Then there is a ccc generic extension in which $\mathfrak{b} = \kappa < \mathfrak{s} = \kappa^+$.*

*Proof.* Construct a model $V$ of $\mathfrak{b} = \mathfrak{c} = \kappa$ by a finite support iteration of length $\kappa$ of Hechler forcing. In $V$ we have $2^\lambda = \kappa$ for all $\lambda < \kappa$ and the family $\mathcal{H} = V \cap {}^\omega\omega$ is unbounded and $<^*$-directed. We will construct a finite support iteration of *ccc* forcing notions $\langle\langle \mathbb{P}_\alpha : \alpha \leq \kappa^+\rangle, \langle \dot{\mathbb{Q}}_\alpha : \alpha < \kappa^+\rangle\rangle$ as follows.
If $\alpha = \beta + 1$ is a successor and $\mathbb{P}_\beta$ has been defined, then:

1. Let $\dot{\mathbb{Q}}_\beta$ be a $\mathbb{P}_\beta$ name for the forcing notion $\mathbb{C}(\kappa)$ of adding $\kappa$ many Cohen reals and let $\mathbb{P}_\alpha = \mathbb{P}_\beta * \dot{\mathbb{Q}}_\beta$. Then
   a) $\mathcal{H}$ remains unbounded in $V^{\mathbb{P}_\alpha}$ by Corollary 2.1
   b) $\mathcal{H}$ is $<^*$-directed in $V^{\mathbb{P}_\alpha}$, by Remark 2.1 and since $\mathbb{P}_\alpha$ is ccc
   c) Since $\mathbb{P}_\alpha$ is ccc, it does not collapse cardinals and so $\forall \lambda < \kappa (2^\lambda \leq \kappa)$ in $V^{\mathbb{P}_\alpha}$
   d) $\operatorname{cov}(\mathcal{M}) = \kappa$ in $V^{\mathbb{P}_\alpha}$
2. Now we can apply Lemma 5.3 in $V^{\mathbb{P}_\alpha}$ to obtain a centered family $C$ of pure conditions such that $Q(C)$ preserves that $\mathcal{H}$ is unbounded and it adds a real not split by $V^{\mathbb{P}_\alpha} \cap [\omega]^\omega$. Let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for $Q(C)$ and $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$. Then
   a) $\mathcal{H}$ remains unbounded in $V^{\mathbb{P}_{\alpha+1}}$ by Lemma 5.3
   b) $\mathcal{H}$ is $<^*$-directed in $V^{\mathbb{P}_{\alpha+1}}$ by Remark 2.1 and since $\mathbb{P}_{\alpha+1}$ is ccc
   c) Since $\mathbb{P}_{\alpha+1}$ is *ccc*, $\forall \lambda < \kappa \left(2^\lambda \leq \kappa\right)$.
3. Let $\mathcal{A} \subseteq {}^\omega\omega$ be an unbounded family of size less than $\kappa$ in $V^{\mathbb{P}_{\alpha+1}}$. Let $\dot{\mathbb{Q}}_{\alpha+1}$ be a $\mathbb{P}_{\alpha+1}$-name for $\mathbb{H}(\mathcal{A})$ and let $\mathbb{P}_{\alpha+2} = \mathbb{P}_{\alpha+1} * \dot{\mathbb{Q}}_{\alpha+1}$. Then:
   a) $\mathcal{H}$ remains unbounded in $V^{\mathbb{P}_{\alpha+2}}$ by Lemma 2.8, since $|\mathbb{H}(\mathcal{A})| = |\mathcal{A}| < \kappa$
   b) $\mathcal{H}$ is $<^*$-directed in $V^{\mathbb{P}_{\alpha+2}}$ by Remark 2.1, since $\mathbb{H}(\mathcal{A})$ is $\sigma$-centered by Lemma 2.2, and hence it is ccc
   c) Since $\mathbb{P}_{\alpha+2}$ is ccc, $\forall \lambda < \kappa \left(2^\lambda \leq \kappa\right)$.
   d) $\mathcal{A}$ is bounded in $V^{\mathbb{P}_{\alpha+2}}$ by Lemma 2.4

If $\alpha$ is a limit, suppose that for every $\beta < \alpha$ we have defined a ccc forcing notion $\mathbb{P}_\beta$ and a $\mathbb{P}_\beta$-name $\dot{\mathbb{Q}}_\beta$ such that $\mathcal{H}$ is unbounded in $V^{\mathbb{P}_\beta}$ and $\Vdash_{\mathbb{P}_\beta} (\dot{\mathbb{Q}}_\beta$ is ccc$)$. Let $\mathbb{P}_\alpha$ be the finite support iteration of $\langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta : \beta < \alpha\rangle$. Then:

1. $\mathcal{H}$ remains unbounded in $V^{\mathbb{P}_\alpha}$ by Theorem 2.3
2. $\mathcal{H}$ is $<^*$-directed in $V^{\mathbb{P}_\alpha}$ by Remark 2.1 and since $\mathbb{P}_\alpha$ is ccc .

3. $\forall \lambda < \kappa \left(2^\lambda \leq \kappa\right)$ in $V^{\mathbb{P}_\alpha}$.

This completes the construction. Let $\mathbb{P} = \mathbb{P}_{\kappa^+}$. Then $\mathbb{P}$ is a ccc forcing notion and in $V^{\mathbb{P}}$ we have $2^\omega = \kappa^+$. Let $\mathcal{A} \subseteq [\omega]^\omega \cap V^{\mathbb{P}}$ be an arbitrary family of size less than $\kappa^+$. Then by Lemma 2.9 $\mathcal{A}$ is contained in some proper initial segment of the construction, so there is $\alpha < \kappa^+$ and a $\mathbb{P}$-generic filter $G$ such that $\mathcal{A} \subseteq V[G_\alpha]$, where $G_\alpha = G \cap \mathbb{P}_\alpha$. By the second step in the successor stage of the construction of $\mathbb{P}$, there is a real not split by $\mathcal{A}$ in $V[G_{\alpha+3}]$. So $V^{\mathbb{P}} \models \mathfrak{s} = \kappa^+$. By Theorem 2.3 and construction of $\mathbb{P}$, $\mathcal{H}$ remains unbounded in $V^{\mathbb{P}}$. As every family of reals in $V^{\mathbb{P}}$ of size less than $\kappa$ is obtained at some initial stage of the iteration, one can guarantee that any such family is bounded in $V^{\mathbb{P}}$ and so

$$V^{\mathbb{P}} \models \mathfrak{b} = \kappa < \mathfrak{s} = \kappa^+.$$

□

# 6 $\mathfrak{b} < \mathfrak{s}$ via relativized Mathias Forcing

As we have seen in Chapter 3, general Mathias forcing adds a dominating real and so increases the bounding number as well as the splitting number. However, relativized Mathias forcing may not add a dominating real. In [Can88] a characterization of ultrafilters was given such that $\mathbb{M}(\mathcal{U})$ does not add a dominating real. In this chapter we follow [FI10] to show that given an unbounded $<^*$-directed family $\mathcal{H}$, there is an ultrafilter $\mathcal{U}_{\mathcal{H}}$ such that $\mathbb{M}(\mathcal{U}_{\mathcal{H}})$ preserves the unboundedness of $\mathcal{H}$.
Recall that for a condition $(u, T) \in Q$ (definition 4.5) we have that $(u, \mathrm{int}(T))$ is a condition in the Mathias forcing notion $\mathbb{M}$. Moreover, if we have $(u_1, T_1), (u_2, T_2) \in Q$ with $(u_1, T_1) \leq_Q (u_2, T_2)$, then we have $(u_1, \mathrm{int}(T_1)) \leq_{\mathbb{M}} (u_2, \mathrm{int}(T_2))$.
The following Lemma shows that in fact, $Q(C)$ can be embedded into $\mathbb{M}(\mathcal{F})$ for a specific ultrafilter $\mathcal{F}$.

**Lemma 6.1.** *Let $C$ be a centered family of pure conditions in $Q$ and let*

$$\mathcal{F}_C \coloneqq \{A \in [\omega]^\omega : \exists T \in C\, (\mathrm{int}(T) \subseteq A)\}.$$

*Then $Q(C)$ is densely embedded into $\mathbb{M}(\mathcal{F}_C)$.*

*Proof.* Define a mapping $i$ from $Q(C)$ to $\mathbb{M}(\mathcal{F}_C)$ by

$$i : (u, T) \mapsto (u, \mathrm{int}(T)).$$

We will show that this is a dense embedding. As mentioned before, if $(u_1, T_1), (u_2, T_2) \in Q(C)$ with $(u_1, T_1) \leq_{Q(C)} (u_2, T_2)$, then

$$(u_1, \mathrm{int}(T_1)) = i((u_1, T_1)) \leq_{\mathbb{M}} (u_2, \mathrm{int}(T_2)) = i((u_2, T_2)),$$

so $i$ is order preserving.
To see that $i$ also preserves incompatibility, let $(u, T)$ and $(v, R)$ be incompatible conditions in $Q(C)$. If $u$ is not an end-extension of $v$ and $v$ is not an end-extension of $u$, then

$$(u, \mathrm{int}(T)) \perp_{\mathbb{M}(\mathcal{F}_C)} (v, \mathrm{int}(R)).$$

So without loss of generality, assume that $u$ is an end-extension of $v$. By definition of $Q(C)$, there are $T', R' \in C$ such that $T' \leq T$ and $R' \leq R$. As $C$ is centered, there is $Z \in C$ which is a common extension of $T'$ and $R'$. Now, if $u \setminus v \subseteq \mathrm{int}(R)$, then $(u, Z) \in Q(C)$ would be a common extension of $(u, T)$ and $(v, R)$, which contradicts our assumption. Therefore, we have $u \setminus v \subsetneq \mathrm{int}(R)$. Now suppose that there is some $(a, X) \in \mathbb{M}(\mathcal{F}_C)$ which is a common extension of $(u, \mathrm{int}(T))$ and $(v, \mathrm{int}(R))$. Then, in

particular, $a$ is an end-extension of $u$, so $u \setminus v \subseteq a \setminus v$, but also $a \setminus v \subseteq \text{int}(R)$, which contradicts $u \setminus v \subsetneq \text{int}(R)$.
Finally, to see that $i\,(Q(C))$ is a dense subset of $\mathbb{M}\,(\mathcal{F}_C)$, take any $(a, X) \in \mathbb{M}\,(\mathcal{F}_C)$. By definition of $\mathcal{F}_C$, there is $T \in C$ such that $\text{int}(T) \subseteq X$. In particular, this implies $\max(a) < \min(\text{int}(T))$, so $(a, T)$ is a condition in $Q(C)$ and $i\,((a, T)) \leq_{\mathbb{M}} (a, X)$. □

Using the previous lemma and results about $Q(C)$ from earlier chapters, one can now construct a specific ultrafilter for which relativized Mathias forcing preserves the unboundedness of an unbounded, $<^*$-directed family.

**Theorem 6.1.** *Let $\kappa$ be a regular cardinal such that $2^\lambda \leq \kappa$ for all $\lambda < \kappa$ and let $\text{cov}(\mathcal{M}) = \kappa$. Let $\mathcal{H} \subseteq {}^\omega\omega$ be an unbounded, $<^*$-directed family of size $\kappa$. Then there is an ultrafilter $\mathcal{U}_\mathcal{H}$ such that $\mathbb{M}\,(\mathcal{U}_\mathcal{H})$ preserves the unboundedness of $\mathcal{H}$.*

*Proof.* Let $\mathcal{H}$ be an unbounded directed family of size $\kappa$. Let $C = C_\mathcal{H}$ be the centered family constructed in the proof of Lemma 5.3. Now let

$$\mathcal{U}_\mathcal{H} := \mathcal{F}_C = \{X \in [\omega]^\omega : \exists T \in C\,(\text{int}(T) \subseteq X)\}$$

and by Lemma 6.1, $Q(C)$ is densely embedded into $\mathbb{M}(\mathcal{U}_\mathcal{H})$, so the two posets are forcing equivalent and hence $\mathbb{M}(\mathcal{U}_\mathcal{H})$ preserves that $\mathcal{H}$ is unbounded.
It remains to show that $\mathcal{U}_\mathcal{H}$ is an ultrafilter. Fix an enumerations $\{A_{\beta+1}\}_{\beta<\kappa}$ of $[\omega]^\omega$. Recall that $C$ is defined by $C = \bigcup_{\alpha<\kappa} C_\alpha$, where for each successor $\alpha = \beta + 1 < \kappa$, there is $D_\alpha$, with $D_\alpha = A_\alpha$ or $D_\alpha = A_\alpha^c$ such that

$$\forall X \in C_\alpha\,(\text{int}(X) \subseteq D_\alpha)\,.$$

Now consider any $A \in [\omega]^\omega$. Then, $A = A_{\beta+1}$ for some $\beta < \kappa$. Then, by the property above, each element of $C_{\beta+1}$ is a witness that $A \in \mathcal{U}_\mathcal{H}$ or $A^c \in \mathcal{U}_\mathcal{H}$. □

Since $\mathbb{M}\,(\mathcal{F}_C)$ adds a real not split by the ground model reals, we can use this fact together with Theorem 6.1 to build a model for $\mathfrak{b} < \mathfrak{s}$ by replacing $Q(C)$ with $\mathbb{M}(\mathcal{F}_C)$ in the second step of the construction in the proof of Theorem 5.2.

# 7 $\mathfrak{s} = \omega_1 < \mathfrak{b}$

In the previous chapters, we have shown the consistency of $\mathfrak{b} < \mathfrak{s}$. To get the independence of these two cardinal invariants, it is left to show the consistency of $\mathfrak{s} < \mathfrak{b}$. This was first show by Balcar, Pelant and Simon in [BPS80]. We will follow [BD85] and show this result via a finite support iteration of Hechler forcing.

**Definition 7.1.** *The forcing notion $\mathbb{D}$ consists of pairs $(s, f)$ where $s \in {}^{<\omega}\omega$, $f \in {}^{\omega}\omega$, $s \subseteq f$ and $f$ is strictly increasing.*
*The extension relation is given by $(s, f) \leq (t, g)$ if $s \supseteq t$ and $f(n) \geq g(n)$ for all $n \in \omega$.*

**Lemma 7.1.** *1. For each $n \in \omega$, the set*

$$D_n \coloneqq \{(s, f) \in \mathbb{D} : n \in \operatorname{dom}(s)\}$$

*is dense in $\mathbb{D}$.*

*2. For each $g \in {}^{\omega}\omega$, the set*

$$D_g \coloneqq \{(s, f) \in \mathbb{D} : g \leq^* f\}$$

*is dense in $\mathbb{D}$.*

*Proof.* To see the first part, fix $n \in \omega$ and let $(s, f) \in \mathbb{D}$ with $n \notin \operatorname{dom}(s)$. Define $s' \in {}^{<\omega}\omega$ by

$$s'(k) = \begin{cases} s(k) & \text{for } k < |s| \\ f(k) & \text{for } |s| \leq k \leq n \end{cases}.$$

Then $(s', f) \in D_n$ and $(s', f) \leq (s, f)$, so $D_n$ is dense.
For 2. let $g \in {}^{\omega}\omega$ and $(s, f) \in \mathbb{D}$ be arbitrary. Define $f' \in {}^{\omega}\omega$ by

$$f'(n) = \begin{cases} s(n) & \text{for } n < |s| \\ \max\{f(n), g(n)\} & \text{for } n \geq |s| \end{cases}.$$

Then $(s, f') \in \mathbb{D}$ and for all $n \geq |s|$ we have $f'(n) \geq g(n)$, so $g \leq^* f'$ and hence $(s, f') \in D_g$. □

**Lemma 7.2.** $\mathbb{D}$ *adds a real which dominates all ground model reals.*

*Proof.* Let $G$ be a $\mathbb{D}$-generic filter and define

$$f_G \coloneqq \bigcup\{s \mid \exists f\, ((s, f) \in G)\}.$$

Then $f_G \in {}^\omega\omega$ since for each $n \in \omega$, the set $D_n$ from Lemma 7.1 is dense. Now let $g \in {}^\omega\omega \cap V$ be any real in the ground model. Since the set $D_g$ is dense, so there is $(s, f) \in D_g \cap G$ such that $g \leq^* f$. Since $s \subseteq f$ and $D_n$ is dense for each $n \in \omega$, we get $g \leq^* f_G$. □

**Lemma 7.3.** *Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha \leq \lambda \rangle$ be a finite support iteration $\mathbb{D}$, so*

$$\mathbb{1}_\alpha \Vdash_{\mathbb{P}_\alpha} (\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{D}})\ \forall \alpha < \lambda.$$

*Then*

$$V^{\mathbb{P}_\lambda} \models \mathfrak{b} = \lambda.$$

*Proof.* Let $\mathcal{D} \subseteq {}^\omega\omega \cap V^{\mathbb{P}_\lambda}$ be an arbitrary family of functions with $|\mathcal{D}| < \lambda$. As $\mathbb{D}$ has the countable chain condition, each real is added at some initial stage of the construction by Lemma 2.6. Hence there is an $\alpha < \lambda$ such that $\mathcal{D} \subseteq {}^\omega\omega \cap V^{\mathbb{P}_\alpha}$. But at stage $\alpha + 1$ of the iteration, a dominating real gets added by Lemma 7.2, so $\mathcal{D}$ is not unbounded in $V^{\mathbb{P}_{\alpha+1}}$ and so also not unbounded in $V^{\mathbb{P}_\lambda}$. □

**Definition 7.2.** *A sequence $\langle a_\xi : \xi < \lambda \rangle \subseteq [\omega]^\omega$ is called eventually narrow if*

$$\forall a \in [\omega]^\omega\, \exists \xi < \lambda \forall \eta > \xi\, (|a \setminus a_\eta| = \omega)\,.$$

*A sequence $\langle b_\xi : \xi < \lambda \rangle \subseteq [\omega]^\omega$ is called eventually splitting if*

$$\forall b \in [\omega]^\omega\, \exists \xi < \lambda \forall \eta > \xi\, (|b \cap b_\eta| = |b \setminus b_\eta| = \omega)\,.$$

Note that every eventually splitting sequence is in particular a splitting family and that a sequence $\langle a_\xi : \xi < \lambda \rangle$ is eventually narrow if and only if the sequence $\langle b_\xi : \xi < \lambda \rangle$, where $b_{2\xi} = a_\xi$ and $b_{2\xi+1} = \omega \setminus a_\xi$, is eventually splitting.

**Definition 7.3.** *Let $D$ be a dense open subset of $\mathbb{D}$. Inductively define the sequence $\langle D_\alpha : \alpha < \omega_1 \rangle$ by*

1. *$D_0 = \{s \in {}^{<\omega}\omega : \exists f\, (s, f) \in D\}$*
2. *$D_{\alpha+1} = \{s \in {}^{<\omega}\omega : \exists n \geq |s| \forall k \in \omega \exists t \in D_\alpha\, (s \subseteq t \wedge t(i) > k\ \forall i : |s| \leq i < n)\}$*
3. *If $\alpha$ is a limit ordinal, let $D_\alpha = \bigcup_{\beta<\alpha} D_\beta$*

Note that by using $t = s$ in the definition of $D_{\alpha+1}$ we get $D_\alpha \subseteq D_{\alpha+1}$ and since ${}^{<\omega}\omega$ is countable, there are only countably many indices such that $D_\alpha \subsetneq D_{\alpha+1}$.

**Lemma 7.4.** *Let $D \subseteq \mathbb{D}$ be a dense open set and let $\langle D_\alpha : \alpha < \omega_1 \rangle$ be defined as above. Then*

$$D_{\omega_1} = \{s \in {}^{<\omega}\omega : s \text{ is strictly increasing}\}.$$

*Proof.* Assume towards a contradiction that there is a strictly increasing $s \in {}^{<\omega}\omega$ which is not in $D_{\omega_1}$. We will call $t \in D_{\omega_1}$ a minimal extension of $s$ if $s \subseteq t$ and $t \restriction (|t| - 1) \notin D_{\omega_1}$.

**Claim.** *For each $n \in \omega$, there are only finitely many minimal extensions of $s$ of length $n$.*

*Proof of claim.* Let

$$T_n = \{t \in D_{\omega_1} : |t| = n \wedge t \text{ is a minimal extension of } s\}$$

and suppose that $T_n$ is infinite. As each $t \in D_{\omega_1}$ is strictly increasing, the set $\{t(n-1) : t \in T_n\}$ is infinite. Now define $i_n$ to be the minimum integer such that

$$|\{t(i_n) : t \in T_n\}| = \omega.$$

Then the set $\{t \restriction i_n : t \in T_n\}$ is finite since each $t \in T_n$ is increasing. So there must be an $u \in {}^{<\omega}\omega$ and $T'_n \subseteq T_n$ such that

$$\left|\{t(i_n) : t \in T'_n\}\right| = \omega \text{ and } t \restriction i_n = u \; \forall t \in T'_n.$$

Then, for any $k \in \omega$, there is $t \in T'_n$ such that $t(i_n) > k$ and since $t$ is increasing, $t(j) > k$ for all $i_n + 1 = |u| \le j < n$. Since $t \in D_{\omega_1} = \bigcup_{\alpha<\omega_1} D_\alpha$, there is $\gamma < \omega_1$ such that $t \in D_\gamma$ and then by definition of $D_{\gamma+1}$ we have $u \in D_{\gamma+1} \subseteq D_{\omega_1}$, contradicting the assumption that the elements of $T'_n$ are minimal extensions of $s$. □

Let $T_n$ be defined as in the proof of the claim. Let $f \in {}^\omega\omega$ be such that $f$ is increasing and

$$\forall n \ge |s| \left(f(n) > \max\{t(n) : t \in T_{n+1}\}\right).$$

As $D$ is a dense set, there is $(t, g) \in D$ such that $(t, g) \le (s, f)$. By definition of $D_0$, we have $t \in D_0$ and so there is some initial segment $t'$ of $t$ which must be a minimal extension of $s$. Then we have for $m = |t'|$ that $t' \in T_m$ and so

$$t(m-1) = t'(m-1) < f(m-1)$$

by definition of $f$, but this contradicts $(t, g) \le (s, f)$. □

**Theorem 7.1.** *An eventually narrow sequence remains eventually narrow in $V^{\mathbb{D}}$.*

*Proof.* Assume that the theorem does not hold, so there is an eventually narrow sequence $\langle a_\xi : \xi < \lambda\rangle$ in $V$, a $\mathbb{D}$-name $\dot{a}$ and $(s, f) \in \mathbb{D}$ such that

$$(s, f) \Vdash \forall \xi < \lambda \exists \eta > \xi \left(|\dot{a} \setminus a_\eta| < \omega\right).$$

Let $M$ be a countable model of ZFC large enough so that it contains $\mathbb{D}$, $\dot{a}$ and $Z_f(i)$ defined below. Since $M$ is countable, there is a $\xi < \lambda$ such that for all $a \in M \cap [\omega]^\omega$, $a \setminus a_\xi$ is finite. So for this $\xi < \lambda$ we can find $n_0 \in \omega$ such that

$$(s, f) \Vdash \forall i \ge n_0 \left(i \in \dot{a} \to i \in a_\xi\right). \tag{7.1}$$

Let $\dot{h}$ be a $\mathbb{D}$-name such that

$$\mathbb{1} \Vdash \dot{h} \text{ enumerates } \dot{a} \text{ in increasing order.} \tag{7.2}$$

For each $t \in {}^{<\omega}\omega$ with $t \subseteq f$ and $(t, f) \le (s, f)$ and each $i \ge n_0$, define

$$Z_t(i) := \{j \in \omega : \forall g \in {}^\omega\omega \exists (t', g') \le (t, g)((t', g') \Vdash \dot{h}(i) = j)\}.$$

**Claim.** *For each $t$ and $i$ as above, $Z_t(i)$ is non-empty.*

*Proof of claim.* Fix $i \geq n_0$ and define

$$D^i := \{p \in \mathbb{D} : \exists j \in \omega (p \Vdash \dot{h}(i) = j)\}.$$

As $D^i$ is dense open, we can build the sequence $\langle D^i_\alpha : \alpha < \omega_1 \rangle$ as in Definition 7.3. By Lemma 7.4 it is enough to show the claim for all $t \in D^i_\alpha$.
If $t \in D^i_0$, then there is $g \in {}^\omega\omega$ such that $(t, g) \in D^i$, and so there is $j \in \omega$ with $(t, g) \Vdash \dot{h}(i) = j$. Since $(t, g)$ is compatible with any $(t, \tilde{g}) \in \mathbb{D}$, we have $j \in Z_t(i)$.
Now assume that $t \in D^i_{\alpha+1} \setminus D^i_\alpha$ for some $\alpha$. By definition of $D^i_{\alpha+1}$, there is a sequence $\langle t_n : n \in \omega \rangle$ in $D^i_\alpha$ such that $|t_n| = \ell$ for all $n \in \omega$ and $t_n(|t|) \geq n$ for all all $n \in \omega$. We can assume that $\langle t_n : n \in \omega \rangle \in M$.

**Claim.** *There is some $j \in \omega$ such that $j \in Z_{t_n}(i)$ for infinitely many $t_n$.*

*Proof of claim.* Suppose that this is not the case. For each $n$ fix $j_n = \min(Z_{t_n}(i))$ and let $J = \{j_n : n \in \omega\}$. Since no $j_n$ belongs to infinitely many $Z_{t_n}(i)$, the set $J$ must be infinite. As $J$ is definable from $\langle t_n : n \in \omega \rangle$, we can take $J \in M$. By our choice of $a_\xi$, we then have that $J \setminus a_\xi$ is also infinite. Now choose $n$ large enough such that $j_n \geq n_0$, $n \geq f(\ell - 1)$ and $j_n \notin a_\xi$. Then $(t_n, f) \leq (t, f) \leq (s, f)$ and since $j_n \in Z_{t_n}(i)$, there is some $(u, g) \leq (t_n, f)$ such that

$$(u, g) \Vdash \dot{h}(i) = j_n.$$

But this contradicts (7.1). □

So we have that there is some $j \in \omega$ which belongs to infinitely many $Z_{t_n}(i)$. Then also $j \in Z_t(i)$. □

Since $Z_s(i) \neq \emptyset$ for each $i \geq n_0$, we can define

$$k_i = \min(Z_s(i))$$

for each $i \geq n_0$ and set $K = \{k_i : i \geq n_0\}$. Since $k_i \geq i$, the set $K$ must be infinite and we can take $K \in M$, so $|K \setminus a_\xi| = \omega$. So, in particular $K \setminus a_\xi \neq \emptyset$, hence there is $k_i \in K \setminus a_\xi$. Then there is $(s', g) \leq (s, f)$ such that $(s', g) \Vdash \dot{h}(i) = k_i$. However, by (7.2) and (7.1), this implies $k_i \in a_\xi$, which is a contradiction. □

**Theorem 7.2.** *Let $\langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta : \beta \leq \alpha \rangle$ be a finite support iteration of $\mathbb{D}$. Then any eventually narrow sequence $\langle a_\xi : \xi < \lambda \rangle$ remains eventually narrow in $V^{\mathbb{P}_\alpha}$.*

*Proof.* The proof proceeds by induction on $\alpha$. The successor case is dealt with in Theorem 7.1 so we only need to consider the limit step.
By Lemma 2.6 we can assume that $\mathrm{cf}(\alpha) = \omega$, as the iteration adds no new reals at stages with uncountable cofinalities. So suppose that $\mathrm{cf}(\alpha) = \omega$ and $\langle a_\xi : \xi < \lambda \rangle$ is not eventually narrow in $V^{\mathbb{P}_\alpha}$. Thus there is some $p \in \mathbb{P}_\alpha$ and a name $\dot{a}$ such that

$$p \Vdash ((\dot{a} \text{ is infinite}) \text{ and } (\forall \xi < \lambda \exists \eta > \xi\, (\dot{a} \setminus a_\eta \text{ is finite}))).$$

Let $\langle \alpha_n : n \in \omega \rangle$ be a cofinal sequence in $\alpha$ and let $G_\alpha$ be a $\mathbb{P}_\alpha$-generic filter over $V$ such that $p \in G_\alpha$. Now for each $\xi < \lambda$ fix a condition $p_\xi \in G_\alpha$ and $n_\xi \in \omega$ such that

$$p_\xi \Vdash \forall i \geq n_\xi \, (i \in \dot{a} \to i \in a_\xi) \, .$$

By our assumption, there will be a cofinal subset $B$ of $\lambda$ such that $p_\xi$ and $n_\xi$ are defined for all $\xi \in B$. As $\mathbb{P}_\alpha = \bigcup_{\beta<\alpha} \mathbb{P}_\beta$ there is $m \in \omega$ for each $\xi \in B$ such that $p_\xi \in G_{\alpha_m} = G_\alpha \cap \mathbb{P}_{\alpha_m}$. Thus there is a cofinal set $A \subseteq B$, which can be determined in $V[G_{\alpha_m}]$, and fixed $n, m \in \omega$ such that $\forall \xi \in A$, $p_\xi \in G_{\alpha_m}$ and $n_\xi = n$. However, we have

$$\mathbb{1} \Vdash \dot{a} \setminus n \subseteq \bigcap \{a_\xi \setminus n : \xi \in A\}.$$

Hence $a := \bigcap \{a_\xi \setminus n : \xi \in A\}$ is infinite, but $a \setminus a_\xi = \emptyset$, which contradicts the inductive hypothesis, namely that $\langle a_\xi : \xi < \lambda \rangle$ is eventually narrow in $V[G_{\alpha_m}]$. □

**Corollary 7.1.** *An eventually splitting sequence in $V$ remains eventually splitting in $V^{\mathbb{P}_\alpha}$.*

So an eventually splitting sequence of length $\kappa$ will remain eventually splitting in $V^{P_\alpha}$ and so we will have an upper bound for the splitting number in the model, namely $(\mathfrak{s} \leq \kappa)^{V^{P_\alpha}}$.

**Lemma 7.5.** *Let $\alpha$ be a cardinal with cf($\alpha$) $> \omega$. Let $\langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta : \beta \leq \alpha \rangle$ be a non-trivial finite support ccc iteration, so*

$$\mathbb{1} \Vdash_{\mathbb{P}_\beta} \exists \dot{q}_1, \dot{q}_2 \in \dot{\mathbb{Q}}_\beta (\dot{q}_1 \perp \dot{q}_2) \; \forall \beta < \alpha.$$

*Then there is an eventually splitting sequence of length at most cf($\alpha$) in $V^{\mathbb{P}_\alpha}$.*

*Proof.* First note that if $\langle \mathbb{P}_n, \dot{\mathbb{Q}}_n : n < \omega \rangle$ is a non-trivial finite support iteration of ccc forcing notions, then $\mathbb{P}_\omega$ adds Cohen reals over the ground model. To see this, let $q_n^0, q_n^1 \in \mathrm{dom}(\dot{\mathbb{Q}}_n)$ be such that

$$\mathbb{1} \Vdash_{\mathbb{P}_n} q_n^0 \perp_{\mathbb{Q}_n} q_n^1.$$

for each $n \in \omega$ and let $G_n$ be a $\mathbb{Q}_n$-generic filter. Then, the function $f : \omega \to 2$ defined by

$$f(n) = 0 \iff q_n^0 \in G_n$$

is a Cohen real.
Applying the above observation repeatedly, we find a sequence $\langle a_\xi : \xi < \mathrm{cf}(\alpha) \rangle$ such that for each $\beta < \alpha$, there is a $\xi < \mathrm{cf}(\alpha)$ such that $a_\xi$ is Cohen-generic over $V^{\mathbb{P}_\beta}$. As $\mathrm{cf}(\alpha) > \omega$, we have that for each set $a \in [\omega]^\omega \cap V^{\mathbb{P}_\alpha}$ there is a $\beta < \alpha$ such that $a \in [\omega]^\omega \cap V^{\mathbb{P}_\beta}$. So $\langle a_\xi : \xi < \mathrm{cf}(\alpha) \rangle$ is eventually splitting. □

**Theorem 7.3.** *Let $\lambda > \omega_1$ and let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha \leq \lambda \rangle$ be the finite support iteration of $\mathbb{D}$. Then we have $\mathfrak{s} = \omega_1 < \mathfrak{b} = \lambda$ in $V^{\mathbb{P}_\lambda}$.*

*Proof.* First note that by Lemma 7.3 we have that $(\mathfrak{b} = \lambda)^{V^{\mathbb{P}_\lambda}}$.
By Lemma 7.5, there is an eventually splitting sequence of length $\omega_1$ in $V^{\mathbb{P}_{\omega_1}}$. Then, by Corollary 7.1, the sequence remains eventually splitting in $V^{\mathbb{P}_\lambda}$. So we get

$$V^{\mathbb{P}_\lambda} \models \mathfrak{s} = \omega_1 < \mathfrak{b} = \lambda.$$

□

# 8 Open Problems

While the bounding and splitting number are independent, the following inequalities involving $\mathfrak{b}$, $\mathfrak{s}$ and the almost disjointness number $\mathfrak{a}$ are still open questions:

1. Is $\mathfrak{b} < \mathfrak{s} < \mathfrak{a}$ relatively consistent?
2. Is $\mathfrak{b} < \mathfrak{a} < \mathfrak{s}$ relatively consistent?
3. Is $\mathfrak{b} < \mathfrak{s} = \mathfrak{a}$ relatively consistent without assuming a measurable?

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