MASTERARBEIT | MASTER'S THESIS

Titel | Title

The Razumov Stroganov Correspondence Revisited

verfasst von | submitted by

Christian Theiner BSc

angestrebter akademischer Grad | in partial fulfilment of the requirements for the degree of

Master of Science (MSc)

Wien | Vienna, 2025

Studienkennzahl lt. Studienblatt | Degree programme code as it appears on the student record sheet:	UA 066 821
Studienrichtung lt. Studienblatt | Degree programme as it appears on the student record sheet:	Masterstudium Mathematik
Betreut von | Supervisor:	Dr. Florian Schreier-Aigner BSc MSc

Abstract (eng)

The Razumov Stroganov correspondence states, that cylindrical loop percolations and fully packed loops have the same probability distribution with respect to their noncrossing matchings.
We will proof the correspondence by first giving the probabilities for cylindrical loop percolations by using properties of Markov chains from probability theory. Then in order to give probabilities for fully packed loops, we look at Temperley-Lieb operators and define the quasi TL-operators, which act on fully packed loops, similarly as Temperley-Lieb operators do on noncrossing matchings.

Abstract (deu)

Die Razumov Stroganov Korrespondenz besagt, dass Cylindrical Loop Percolations und Fully Packed Loops die gleiche Wahrscheinleichkeitsverteilung aufweisen, wenn man die von ihnen induzierten Noncrossing Matchings betrachtet.
Wir werden die Korrespondenz beweisen, indem wir zunächst die Wahrscheinlichkeiten für Cylindrical Loop Percolations mithilfe von Eigenschaften der Markov Ketten aus der Wahscheinlichkeitstheorie bestimmen. Um anschließend die Wahrscheinlichkeiten für Fully Packed Loops zu ermitteln, betrachten wir Temperley-Lieb Operatoren und definieren die Quasi TL-Operatoren, welche auf Fully Packed Loops ähnlich wirken wie Temperley-Lieb Operatoren auf Non Crossing Matchings.

Contents

1 Introduction

In this thesis we give a full overview over the Razumov Stroganov correspondence. We will start by introducing the basics of Markov chains. Hence there is no previous knowledge necessary other than basics in probability theory and graph theory. Everything else needed to understand the correspondence and the proof given in this paper, will be covered. One chapter is fully commited to cylindical loop percolations and another one to fully packed loops. The final chapter gives the proof for the Razumov Stroganov correspondence.

Definition. A *noncrossing matching* π of size n is a partition of $\{1, \ldots, 2n\}$ in blocks of size two, such that there exists no pair of blocks $\{i, j\}, \{k, l\}$ with $i < k < j < l$. We denote the set of all noncrossing matchings of size n with $\mathcal{NC}_n$.

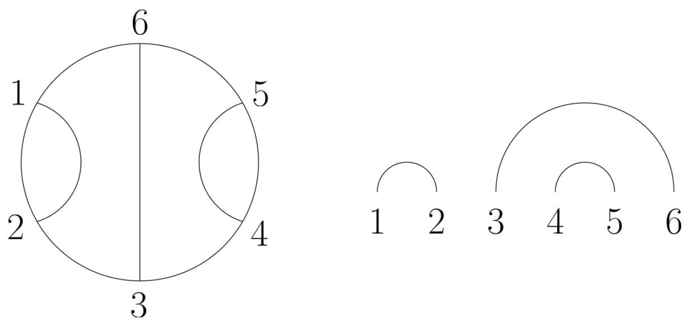


Figure 1: Two different visualizations of the noncrossing matching $\{1,2\}\{3,6\}\{4,5\} =$ ()(()) of size 3.

Note that none of the lines in a visualization of a noncrossing matching do intersect each other, which explains the name noncrossing matching.

Definition. A *Dyck path* of size n is a path on the $\mathbb{Z}^2$ grid, starting at (0,0) and ending at $(0, 2n)$. In between the path either performs a $(1,1)$ step or a $(1,-1)$ step with the rule, that it always has to stay above the x-axis.

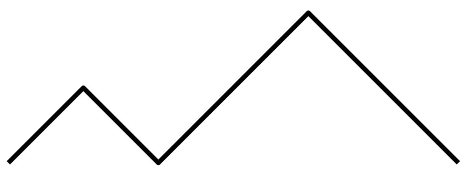


Figure 2: The Dyck path associated to ()(()).

We will sometimes use the notation of parenthesis words for noncrossing matchings, which is obtained directly from the visualization with the nested arcs. At each label, where an arc goes to the right connecting the label with another label to the right, we put an opening parenthesis '(', at labels, where the arc goes to the left, connecting it with a label of lower numerical value, we write ')'. An example for this is displayed in Figure 2.

With this notation we easily get a bijection between noncrossing matchings and Dyck paths, by making a $(1,1)$ step for each opening bracket '(' and a $(1,-1)$ step for each closing bracket ')'. So

$$|\mathcal{NC}_n| = C_n = \frac{1}{n+1}\binom{2n}{n},$$

where C_n is the n-th Catalan number.

Definition. The map $\rho : \mathcal{NC}_n \to \mathcal{NC}_n$ defined by

$$\rho(\pi)(m) = \pi(m+1) - 1,$$

where we identify 0 with $2n$ and 1 with $2n+1$, is called *rotation*.

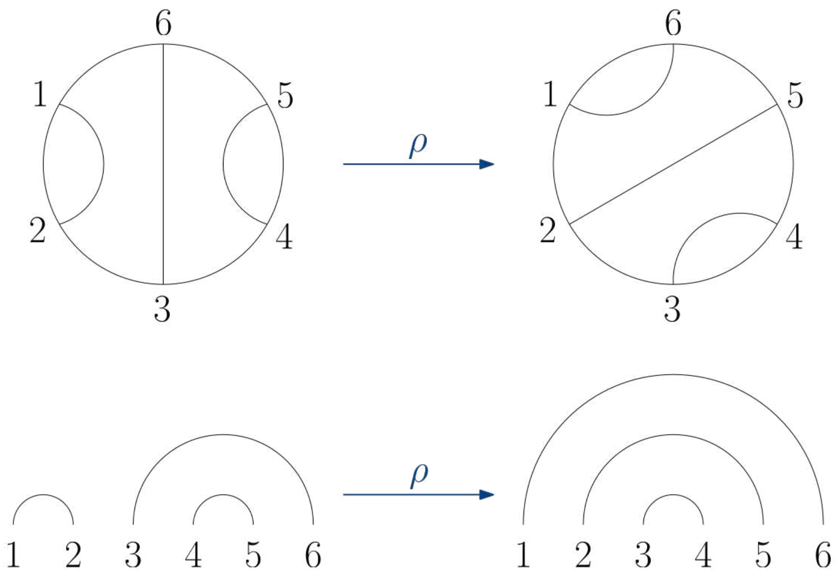
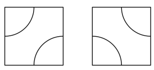


Figure 3: The visualization of the application of ρ on ()(()).

Definition. A cylindrical loop percolation is a semi-infinite cylinder with $2n$ columns and a tiling consisting of the two plaquettes  .

Definition. A fully packed loop is a graph with $n \times n$ vertices, $2n$ external edges and interior edges such that each vertice has degree two.

These two brief definitions in combination with Figure 4 shall give the reader a first impression, how the two major objects of this thesis look like. Their formal definitions will follow in their respective chapters.

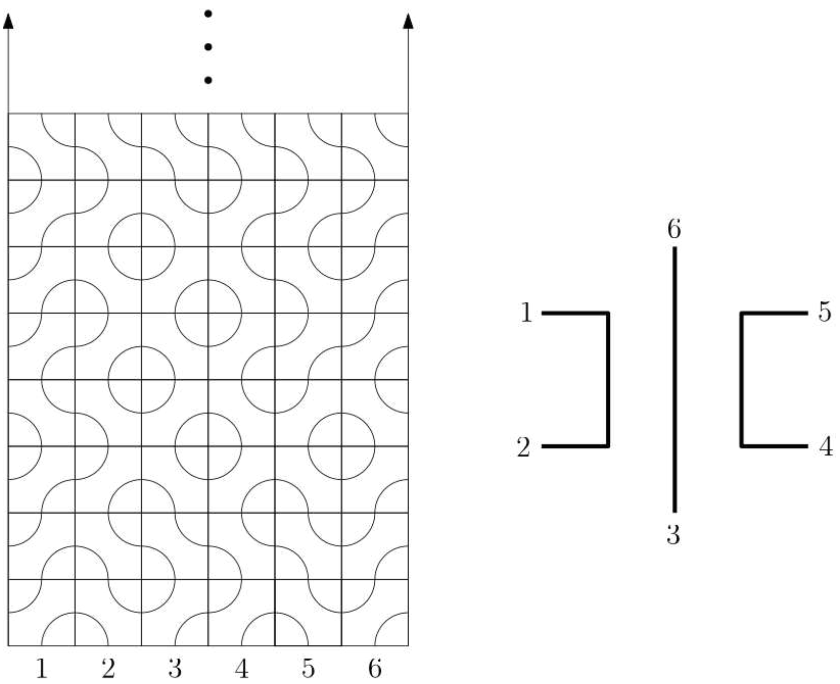


Figure 4: On the left we have an example of a cylindrical loop percolation. On the right an example of a fully packed loop. Exact definitions are given in their respective chapters. Both objects connect two numbers with each other, via the paths, found in that object. In this particular case, the pairing these two objects induce, is the noncrossing matching from Figure 1.

Cylindrical loop percolations and fully packed loops both induce noncrossing matchings. As Razumov and Stroganov studied those objects, they conjectured similar behaviours of the two objects regarding noncrossing matchings and formulated the following conjecture 1999 in [4].

Theorem 1.1. *Fix a $\pi \in \mathcal{NC}_n$. Now randomly generate a cylindrical loop percolation and a fully packed loop.*

The two generated objects have the same probability to induce π.

Cantini and Sportiello were the first, to prove that theorem 2011 in [1]. Three years later, they published a refined proof in [2]. Inspired by that paper, we want to revisit that topic and give a new simplified proof. In Chapter 2 we fully commit ourselves to probability theory. We define what a Markov chain is and learn their basic properties, to make use of them in later sections. In Chapter 3 we define the cylindrical loop percolation and learn about its probabilities, which noncrossing matching it induces, using the results from Chapter 2. Then we introduce Temperley-Lieb operators and show, how to use them, to find a simpler way for our probabilities.
In Chapter 4 the definition of a fully packed loop is given. After introducing some basic properties, we look at the Wieland gyration and look at a slightly different interpretation of it, in order to give the proof of the Razumov Stroganov conjecture in Chapter 5.

2 Markov Chains

Before we dedicate ourselves to combinatorics, we need to learn the basics of Markov chains, in order to use those results on combinatorial objects later on.

2.1 Definition and Simple Properties

Definition. A *stochastic process* $(X_n, n \geq 0)$ with values in S is a sequence of random variables taking values in S.
We call S the *state space* of the stochastic process. If there are also functions $P_n : S \times S \to [0, 1]$, such that for every $n \geq 0$ and every $x_n, x_{n+1} \in S$,

$$\mathbb{P}(X_{n+1} = x_{n+1} | X_n = x_n) = P_n(x_n, x_{n+1}), \tag{2.1}$$

we call the process a *Markov chain*.

The equation (2.1) means, given that the process is at state x_n after n steps, the probability, that it lands at state x_{n+1} in the next step, is given by the function P_n evaluated at $(x_n,\, x_{n+1})$. Hence with Markov chains the next step only depends on the current state x_n and the time n, whereas in general we would expect it to depend on all the prior states $x_0, \ldots, x_n$.
We will only look at finite S.

Definition. We call a Markov chain *time-homogeneous*, if $P_m = P_n$ for $m, n \geq 0$.
We then write P instead of P_n.

We will only look at time-homogenous Markov chains in this thesis.
One can interpret P as a matrix A, indexed by the states of S. Then $A_{x,y} = P(x, y)$ and we will sometimes write $A(x, y)$ instead of $A_{x,y}$. This matrix is called *transition matrix.* therefore the row $(A(x, x_1), \ldots, A(x, x_k))$ contains all the probabilities for the chain to jump from x to x_i. Note, that $x = x_j$ for some $j \in \{1, \ldots, k\}$.
Since $X_{n+1} \in S$ for all $n \geq 0$, we know that $\sum_{i=1}^{k} A(x, x_i) = 1$ in each row of the matrix.

Definition. The *diagram* of a given Markov chain is its visualization as a weighted directed graph $G = (V, E)$, where $V = S$. For the edge $e : x \to y$, its weight is defined as $A_{x,y}$.

Example 2.1. *Consider a reduced deck of cards. Instead of the whole 52 cards, we just take a king, a queen and a jack and look at their order in this deck of 3 cards. Hence all possible words of length 3 containing each K, Q and J exactly once gives our state* *space* $S = \{KQJ, KJQ, QKJ, QJK, JKQ, JQK\}$*. Each step in our Markov chain shall be a very simplified move of shuffling: We take out the middle card and put it to the left border with a probability of* $\frac{1}{3}$ *or to the right border with a probability of* $\frac{2}{3}$*. Then we get the transition matrix*

$$A \quad = \quad \begin{array}{c|cccccc} & KQJ & KJQ & QKJ & QJK & JKQ & JQK \\ KQJ & 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 \\ KJQ & \frac{2}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 \\ QKJ & \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 & 0 \\ QJK & 0 & 0 & \frac{2}{3} & 0 & 0 & \frac{1}{3} \\ JKQ & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} \\ JQK & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \end{array} .$$

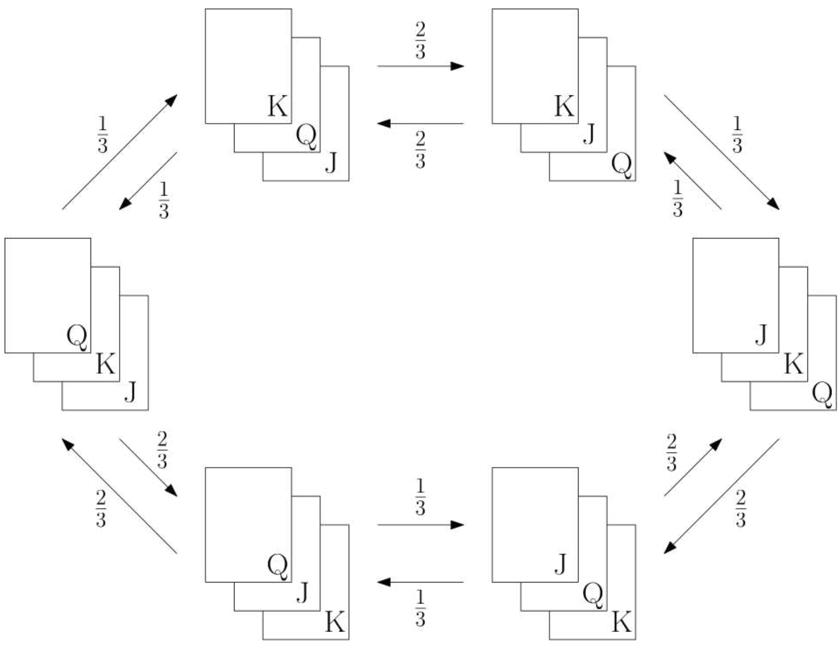


Figure 5: The Diagram of Example 2.1.

2.2 Starting Distribution and Stationary Distribution

The transition matrix gives us full information, how a Markov chain behaves, if we know, where it currently is. However sometimes we do not know that exactly, but rather have a distribution, how likely the chain is at a certain state. This information however turns out to be quite powerful itself.

Definition. In order to know, where the Markov chain starts at x_0, there is given a *starting distribution* $\lambda^0 : S \to [0,1]$ with

$$\sum_{x \in S} \lambda^0(x) = 1.$$

Since we only look at finite S, we will mostly view λ^0 as a row vector $\lambda^0 = (\lambda^0_x)_{x \in S}$, indexed by the elements of S, allowing matrix multiplication with A.

Lemma 2.2. *Let $n \geq 0$. For a Markov chain with starting distribution λ^0 and transition matrix A holds $\mathbb{P}(X_n = y) = \lambda^n(y)$, where*

$$\lambda^n := \lambda^0 \cdot A^n.$$

Proof. Let $y \in S$ be arbitrary.

$$\mathbb{P}(X_1 = y) = \sum_{x \in S} \lambda^0(x) \cdot A_{x,y} = (\lambda^0 \cdot A)(y) = \lambda^1(y).$$

That shows the statement for $n = 1$. For $n > 1$ do the same, using λ^{n-1} instead of λ^0. $\square$

We once again view λ^n as a vector indexed by the elements of S.

Definition. A function $\mu : S \to [0,1]$ is called *invariant measure*, if for all $y \in S$:

$$\sum_{x \in S} \mu_x \cdot A_{x,y} = \mu_y.$$

Hence μ is the left eigenvector of the transition Matrix A with eigenvalue 1.

Definition. An invariant measure μ is called *stationary distribution* if

$$\sum_{x \in S} \mu_x = 1.$$

Therefore, once a Markov chain hits a stationary distribution, it remains there forever, which explains the name.

Proposition 2.3. *Let $(X_n, n \geq 0)$ be a Markov chain with starting distribution λ^0. Assume the limit*

$$\lambda^\infty(x) := \lim_{n \to \infty} \mathbb{P}(X_n = x)$$

exists. Then λ^∞ is a stationary distribution.

Proof. Since we know $\mathbb{P}(X_n = x) \to \lambda^\infty(x)$ for $n \to \infty$, we also have $\mathbb{P}(X_{n+1} = y) \to \lambda^\infty(y)$ for $n \to \infty$.
Additionally we know

$$\mathbb{P}(X_{n+1} = y) = \sum_{x \in S} \lambda^n(x) \cdot A(x, y).$$

Now in the limit $n \to \infty$, we get

$$\lambda^\infty(y) = \mathbb{P}(X_{n+1} = y) = \sum_{x \in S} \lambda^\infty(x) \cdot A_{x,y},$$

which is precisely the definition of a stationary distribution. □

We will see in Lemma 2.4, that stationary distributions are unique. From now on, we denote stationary distributions by λ^∞.

Definition. Let $x, y \in S$ be two elements in a statespace of a Markov chain with transition matrix A.
We say x *leads to* y and write $x \to y$, if

$$A^n(x, y) > 0$$

for some $n \geq 0$. If $x \to y$ and $y \to x$, we say that x *communicates with* y and write $x \leftrightarrow y$.

From the point of combinatorics, we state $x \to y$ iff we can find a path from x to y in the diagram of the Markov chain.

Definition. Let X be a Markov chain with state space S. The partition of S given by $x \leftrightarrow y$, splits S into so called *communicating classes*. If a Markov chain has only a single communicating class, we call the chain *irreducible*.

Note, that Example 2.1 is an irreducible Markov chain. From now on, we will only look at irreducible Markov chains.

Lemma 2.4. *Let X be an irreducible Markov chain with finite state space S and transition matrix A.*
There are infiniteley many invariant measures, given by the family $(r\mu)_{r\in\mathbb{R}}$.
Therefore there is one unique stationary distribution λ^∞.

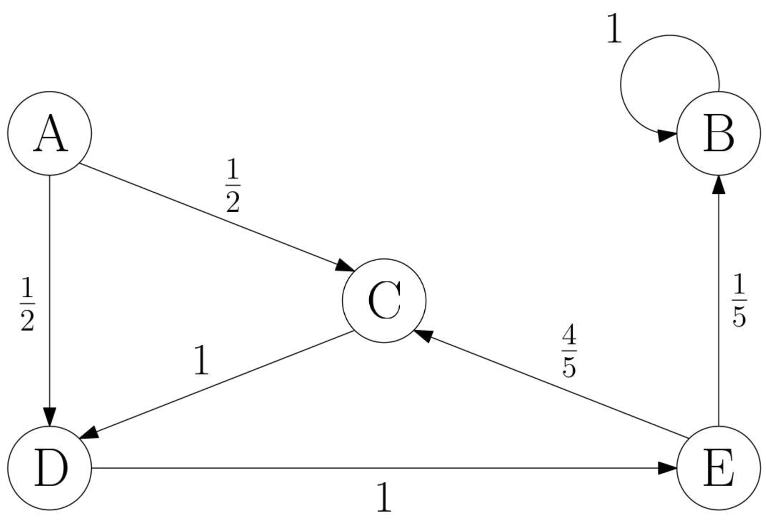


Figure 6: An example for a Markov chain with communicating classes $\{A\}\{B\}\{C,D,E\}$.

Proof. The identity

$$(\mu_1, \mu_2, \dots, \mu_n) \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n-1} & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n-1} & a_{n,n} \end{pmatrix} = (\mu_1, \mu_2, \dots, \mu_n)$$

with $A = (a_{i,j})_{1 \leq i,j \leq n}$ gives the system of equations

$$\begin{aligned} \mu_1 &= \mu_1 a_{1,1} + \mu_2 a_{2,1} + \dots + \mu_{n-1} a_{n-1,1} + \mu_n a_{n,1} \\ \mu_2 &= \mu_1 a_{1,2} + \mu_2 a_{2,2} + \dots + \mu_{n-1} a_{n-1,2} + \mu_n a_{n,2} \\ &\vdots \\ \mu_n &= \mu_1 a_{1,n} + \mu_2 a_{2,n} + \dots + \mu_{n-1} a_{n-1,n} + \mu_n a_{n,n} \end{aligned}.$$

However as we know, that all rows in A must sum up to 1, we can rewrite the last column of A, which changes the bottom equation ofthe above system to

$$\mu_n = \mu_1(1 - a_{1,1} - a_{1,2} - \dots - a_{1,n-1}) + \dots + \mu_n(1 - a_{n,1} - a_{n,2} - \dots - a_{n,n-1}).$$

By adding each other equation to that one we get

$$\mu_1 + \dots + \mu_n = \mu_1 + \dots + \mu_n,$$

meaning that we have an under-determined system of equations. Hence we have infinitely many solutions for μ. However each solution can be gained by taking another one and multiplying it with a scalar. That is because all μ_i are dependent of each other. Once a single μ_i is fixed, all the others resolve directly from it by the system of equations. If there were j's such that μ_{i_j} do not depend on μ_i, they would be in different communicating classes of the Markov chain then μ_i. This can not happen, as an irreducible Markov chain consists of one single communicating class. We then have

$$\lambda^\infty = \frac{\mu}{|\mu|},$$

for any non trivial solution μ, making λ^∞ unique. □

Example 2.5. *We look at Example 2.1 with the 3 cards. The identity $\mu A = \mu$ then gives the system of equations*

$$\begin{aligned}
\mu_{KQJ} &= \frac{2}{3}\mu_{KJQ} + \frac{1}{3}\mu_{QKJ}\\
\mu_{KJQ} &= \frac{2}{3}\mu_{KQJ} + \frac{1}{3}\mu_{JKQ}\\
\mu_{QKJ} &= \frac{1}{3}\mu_{KQJ} + \frac{2}{3}\mu_{QJK}\\
\mu_{QJK} &= \frac{2}{3}\mu_{QKJ} + \frac{1}{3}\mu_{JQK}\\
\mu_{JKQ} &= \frac{1}{3}\mu_{KJQ} + \frac{2}{3}\mu_{JQK}\\
\mu_{JQK} &= \frac{1}{3}\mu_{QJK} + \frac{2}{3}\mu_{JKQ}
\end{aligned}$$

where on the left hand side we got each entry of μ exactly once. On the right hand side the fractions of each entry of μ add up to one, giving us the underdetermined system of equations as we had in the prior proof. Now we can multiply the equations and add them to each other to arrive at

$$\mu = (x, x, x, x, x, x).$$

Hence to get a stationary distribution, we have to set $x = \frac{1}{6}$.
It is not surprising, that all states have the same probability in our stationary distribution, due to the symmetry of our Markov chain.

2.3 Aperiodicity and Long Term Behaviour

In this subsection we want to predict, at which state a Markov chain is after an infinite number of steps. That is the result we will need to apply to our combinatorial objects.

After introducing aperiodicity, which is needed for this prediction, we use a method called coupling, to show that a stationary distribution gives us the probability at which state a Markov chain is after an infinite number of steps.

Definition. A state x in a state space S of a Markov chain is called *aperiodic* if there exists an n_0 large enough such that

$$A^n(x,x) > 0,$$

for all $n \geq n_0$.

Lemma 2.6. *Let y be an aperiodic state of an irreducible, finite Markov chain. Then for all states $x, z \in S$ there is an m_0 such that*

$$A^n(x,z) > 0$$

for all $n \geq m_0$.

Proof. Since we only look at irreducible Markov chains, we know that there are $k, l \geq 0$ such that

$$A^k(x,y) > 0 \quad \text{and} \quad A^l(y,z) > 0.$$

As y is aperiodic, we know that there is an n_0 such that

$$A^n(y,y) > 0$$

for all $n \geq n_0$. Hence if we take $m = k + n + l$ we get

$$A^m(x,z) \quad = \quad A^{k+n+l}(x,z) \quad \geq \quad A^k(x,y) \cdot A^n(y,y) \cdot A^l(y,z) \quad > \quad 0.$$

This holds for all $m \geq m_0 := n_0 + k + l$. □

Therefore a single state of a Markov chain is aperiodic if and only if all states are aperiodic. In that case we call the Markov chain and the transition matrix *aperiodic*.

This attribute is quite important when talking about long term behaviour of a Markov chain. Take another look at Example 2.1 of shuffling 3 cards. Each state has precisely 2 neighbours. Since those are arranged in a circle and are an even number of states, the Markov chain is not aperiodic. If $X_0 = KQJ$ we know that $X_n \neq KQJ$ for all odd n. Then we already know, that the limit $\lim_{n\to\infty} \mathbb{P}(X_n = KQJ)$ can not exist, because Proposition 2.3 told us, that this limit would be given by the stationary distribution λ^∞, which we calculated already. However we know, that the probability would be 0 for all odd n's.

Definition. In probability theory an event is said to happen *almost surely*, if it has probability 1 to happen.

Note that the above definition does not ensure the event to happen.

Example 2.7. *Let $n \in \mathbb{N}$ be picked randomly, by using the uniform distribution on $\mathbb{N}$. Then $n \neq 42$ almost surely.*

Definition. Let $(X_n, n \geq 0)$ be a stochastic process with values in a state space S. We call the random variable T with values in $\{0, 1, \dots\} \cup \{\infty\}$ a *stopping time* if for every $n \geq 0$ the event $T = n$ is only determined by $X_0, \dots, X_n$. So there is a function $F_n : S^{n+1} \to \{0, 1\}$ such that the two following random variables are equal almost surely:

$$1_{\{T=n\}} = F_n(X_0, \dots, X_n).$$

Hence wether $T = n$ can solely be determined by the past and the present and does not depend on the future at all.

Definition. Let $X = (X_n, n \geq 0)$ be a Markov chain with state space S, transition matrix A and starting distribution λ^0 and let $Y = (Y_n, n \geq 0)$ be another Markov chain with the same state space and transition matrix but with starting distribution μ^0 such that Y is independent of X, meaning that the events of Y do not affect X and the other way around. The stochastic process

$$Z_n := (X_n, Y_n)$$

then is a Markov chain called *product chain* with state space $S^2 = S \times S$ and transition matrix

$$B((x_1, y_1), (x_2, y_2)) := A(x_1, x_2) \cdot A(y_1, y_2)$$

and starting distribution

$$\nu^0(x, y) := \lambda^0(x) \cdot \mu^0(y).$$

Remark 2.8. *The above definition needs some arguments. First of all, we have to argue,* *that Z is indeed a Markov chain itself. Therefore we look at the probability*

$$\begin{aligned}\mathbb{P}(Z_{n+1} = (x_{n+1}, y_{n+1}) | Z_n = (x_n, y_n)) \\ = \mathbb{P}(X_{n+1} = x_{n+1}, Y_{n+1} = y_{n+1} | X_n = x_n, Y_n = y_n) \\ = \mathbb{P}(X_{n+1} = x_{n+1} | X_n = x_n)\mathbb{P}(Y_{n+1} = y_{n+1} | Y_n = y_n) \\ = A(x_n, x_{n+1}) \cdot A(y_n, y_{n+1}).\end{aligned}$$

The above equation shows, that Z is Markov and that its transition matrix is indeed B as given by the definition.
If we now look at its sarting distribution

$$\nu^0(x, y) = \mathbb{P}(X_0 = x, Y_0 = y) = \mathbb{P}(X_0 = x)\mathbb{P}(Y_0 = y) = \lambda^0(x) \cdot \mu^0(y),$$

we see, that it is given by the product of the starting distributions of X and Y, as claimed in the definition.

Lemma 2.9. *Let X and Y be two independent, irreducible, aperiodic, Markov chains over S with transitionmatrix A and starting distribution λ^0 or μ^0 respectively. Then the product chain $Z_n = (X_n, Y_n)$ with transition matrix B is also irreducible and aperiodic.*

Proof. Since X and Y are both irreducible and aperiodic, we know that for all $x_1, x_2, y_1, y_2 \in S$

$$A^n(x_1, x_2) > 0 \qquad \textit{and} \qquad A^n(y_1, y_2) > 0$$

for n large enough. Therefore choose an n, large enough to fulfill the above, to get

$$B^n((x_1, y_1), (x_2, y_2)) = A^n(x_1, x_2) \cdot A^n(y_1, y_2) > 0$$

which shows that Z is irreducible and aperiodic. $\square$

Theorem 2.10. *Let A be an aperiodic transition matrix for the state space S and let λ^∞ be a stationary distribution. Let X be a Markov chain with transition matrix A and arbitrary starting distribution λ^0. Then for every $x \in S$*

$$\mathbb{P}(X_n = x) \to \lambda^\infty(x)$$

for $n \to \infty$.

The following proof resulted from [3].

Proof. Let Y be a Markov chain with transition matrix A and starting distribution λ^∞. Since λ^∞ is a stationary distribution, we get

$$\lambda^n = \lambda^\infty, \qquad \forall n \geq 0.$$

Let $z \in S$ be arbitrary. Look at

$$T := T_{(z,z)} = \inf\{n \geq 0 : X_n = Y_n = z\}.$$

Hence T is the first time, that both chains are at state z at the same time. Note that T is a stopping time for the product chain $Z = (X, Y)$. Via Lemma 2.9, we know that Z is aperiodic and therefore $\mathbb{P}(Z_n = (x, y), Z_{n+1} = (z, z)) > 0$ for $x, y, z \in S$ and n large enough. Hence $T < \infty$ almost surely. Now define the chain

$$X'_n := \begin{cases} X_n & \text{if } n \leq T \\ Y_n & \text{if } n > T. \end{cases}$$

Since T is a stopping time, X' is again a Markov chain with transition matrix A and starting distribution λ^0, which is the same starting distribution as the one of X. Therefore $\mathbb{P}(X_n = x) = \mathbb{P}(X'_n = x)$ for all $n \geq 0$ and all $x \in S$. And since $X'_n = Y_n$ for $n \geq T$ we know

$$\begin{aligned} \mathbb{P}(X_n = x) &= \mathbb{P}(X_n = x, n < T) + \mathbb{P}(X_n = x, n \geq T) \\ &= \mathbb{P}(X_n = x, n < T) + \mathbb{P}(Y_n = x, n \geq T), \end{aligned}$$

which we use to state the inequality

$$\begin{aligned} |\mathbb{P}(X_n = x) - \mathbb{P}(Y_n = x)| &\leq \mathbb{P}(X_n = x, n < T) + \mathbb{P}(Y_n = x, n < T) \\ &\leq 2\mathbb{P}(n < T). \end{aligned}$$

If we now let $n \to \infty$ the right side becomes 0 since we know almost surely that $T < \infty$, as stated before. We also know, that $\mathbb{P}(Y_n = x)$ converges to λ^∞. This implies

$$|\mathbb{P}(X_n = x) - \lambda^\infty(x)| \to 0,$$

for $n \to 0$. This proves the theorem since x was arbitrary. □

The above technique of modifying the trajectory of X with an independent chain Y is called *coupling* and T is the *coupling time*.

Therefore given that a Markov chain fulfills certain properties, we can let it run for an infinite number of steps without tracking them and still know the probabilities for its current state. This is still possible even if we do not know the starting distribution λ^0 of the Markov chain.

3 Halfinfinite Cylinders

This chapter follows Section 1.4 from [5] and contains a slightly more detailed version of the proof given in Appendix B of [5]. We introduce the cylindrical loop percolation and look at some properties. We view it as a Markov chain and with the help of the Temperley-Lieb operators, calculate the probability of a randomly generated cylinder to induce a certain noncrossing matching.

3.1 Introduction to cylindrical loop percolations

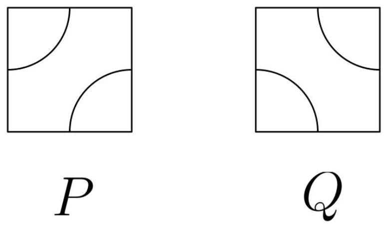


Figure 7: The two types of plaquettes P and Q, used in the construction of cylindrical loop percolation.

Definition. The *cylindrical loop percolation* of size n is defined in the following way. Take the halfinfinite strip $[0, 2n] \times [0, \infty)$ and perform a square-tiling on it with the the tiles P and Q, called plaquettes, shown in Figure 7. Then identifiy the left boundary with the right one, which turns the strip into the surface of a halfinfinite cylinder.

From now on, when we talk about cylinders, we assume, that it is a cylinder with the above properties.
We gain a probability space by equipping P and Q with probabilities p and q, where for each tile T we have

$$\mathbb{P}(T = P) = p, \qquad \mathbb{P}(T = Q) = 1 - p =: q$$

where $0 < p < 1$ and therefore also $0 < q < 1$. We will only look at situations, where p is fixed and does not vary from tile to tile.
Label the bottom of each column from 1 to $2n$. The arcs on our plaquettes give 3 different kinds of paths in the cylinder: interior loops, paths that start and end at the bottom of the cylinder, connecting 2 lables with each other and lastly paths starting at the bottom of the cylinder, going upwards infinitely long without going back down to the bottom. Let us assume, that we have no paths of the third kind, then the paths of the second

kind induce a noncrossing matching of size n, as shown in Figure 8. The paths of the cylinder can never intersect, since our plaquettes P and Q simply give no possibilities for intersections.

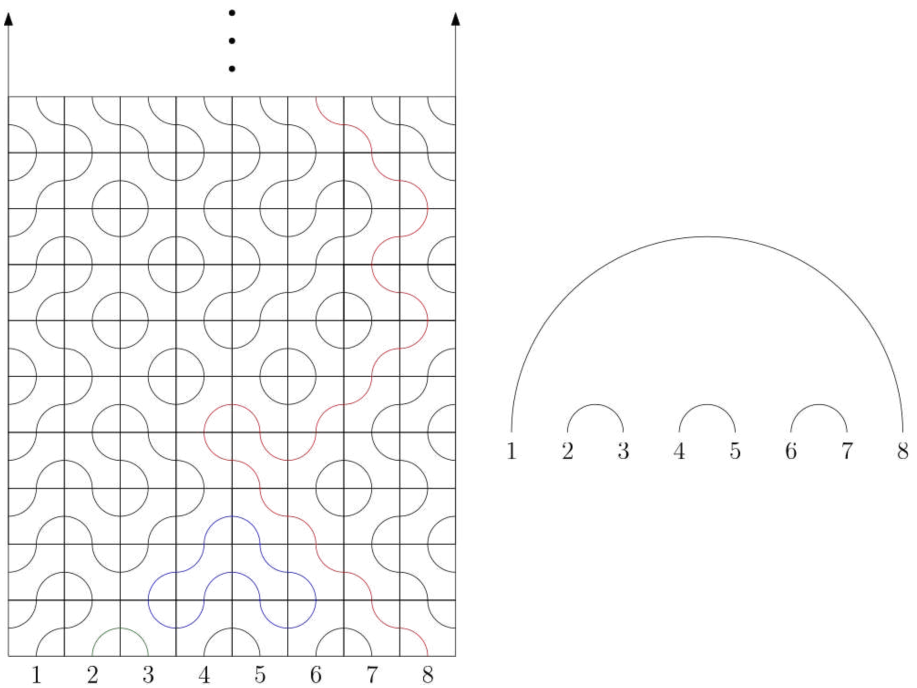


Figure 8: On the left we have a cylindrical loop percolation of size 4. An example of an interior loop is coloured in blue. Highlighted in green is a path, that goes from the bottom to the bottom. The red path starts at the bottom and moves out of the frame and might therefore be a path of the third kind. If however the path connects with the path starting from label 1 somewhere above our frame, we get the induced noncrossing matching of size 4 displayed at the right.

It turns out that the above assumption is true almost surely. However this needs a little argumentation.
Once the cylinder has a row, where P and Q alternate as shown in Figure 9, it is guaranteed, that each path, starting from the bottom boundary returns back down and finally ends at another label, since those rows seperate the paths below them with the paths above them. We call such a row *bouncing row*.

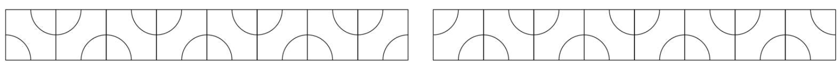


Figure 9: The two kinds of bouncing rows, forcing each path below it, to stay below.

Lemma 3.1. *Let C be a Cylinder of size n. The probability, that C contains at least one bouncing row is 1.*

Proof. The probability, that a specific row has the wanted alteranting structure is

$$2p^n q^n$$

which lies between 0 and 1. Hence the probability that a specific row is anything but a bouncing row is

$$1 - 2p^n q^n =: x$$

and again lies between 0 and 1. Now let us assume C has no bouncing rows at all. Since C has an infinite number of rows, the probability of that is

$$\lim_{k\to\infty} x^k = 0. \qquad \square$$

3.2 Cylinders as Markov chains

We now use the properties of a Markov chain, learned in the first chaper. We view the cylinder itself as a Markov chain. We choose $\mathcal{NC}_n$ as the state space S. A step of the Markov chain is represented by adding a random row to the bottom of the cylinder.

Definition. Let us denote a row of $2n$ plaquettes as a vector $\vec{a} \in \{0,1\}^{2n}$, where each 0 corresponds to a Q-plaquette and each 1 to a P-plaquette.
We then define the function $f_{\vec{a}} : \mathcal{NC}_n \to \mathcal{NC}_n$ as

$$f_{\vec{a}}(\pi) := \pi',$$

where π' is gained by plugging $\vec{a}$ below π as shown in Figure 10.

Lemma 3.2. *The so gained Markov chain is time-homogenous, aperiodic and irreducible with finite state space S.*

Proof. At first, we show, that the Markov chain is *time-homogenous.* Adding a new bottom row at step $n+1$ does not differ from adding a row at step n, making the chain time-homogenous.

We got a *finite state space* since we chose S to be $\mathcal{NC}_n$. This is a finite set with $|S| = \frac{1}{n+1}\binom{2n}{n}$. Showing *aperiodicity* requires a little more work from us. Given any state π, we can always find a row, which sends π to itself. To construct such a row, we define an algorithm.

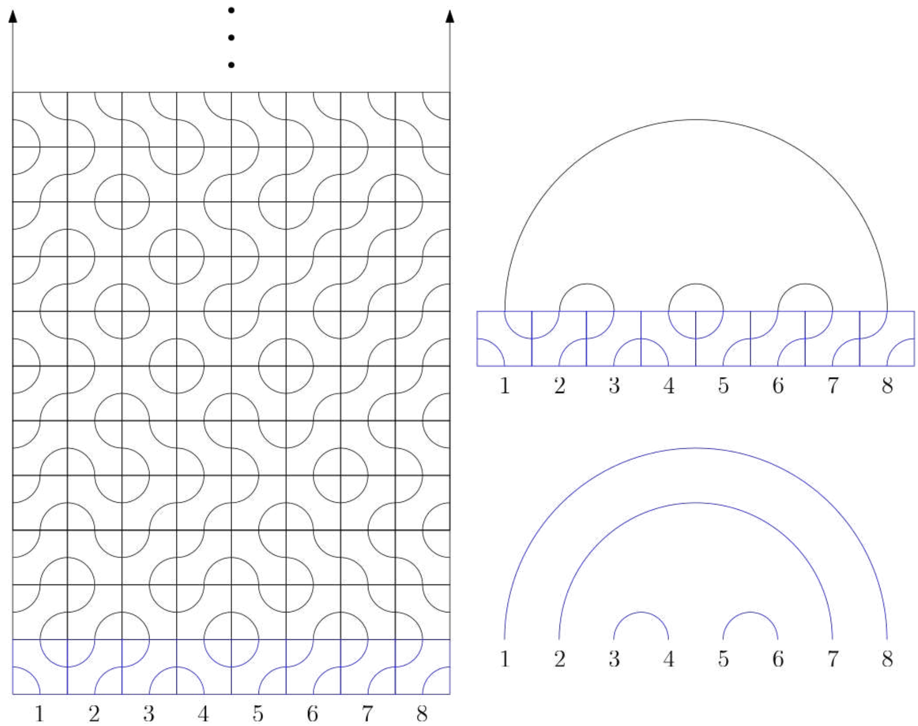


Figure 10: Adding a new bottom row to the cylinder to induce a new noncrossing matching.

Step A For each arc in the noncrossing matching, that connects two neighbours with each other, place a PQ combination directly under that arc to ensure a connection of those neighbours in the outcoming cylinder.

Step B For each arc in the noncrossing matching, that encloses labels which where already processed by prior steps of the algorithm, put a P on the left side and a Q on the right side.

Repeat Step B until the row is fully constructed.

This algorithm is shown in Figure 11.

This makes all communicating classes aperiodic.

Proving *irreducibility* works similar to showing aperiodicity. Regardless of the current state, we can arrive at any given π after a finite number of steps. We basically do the same as we did with aperiodicity. But this time adding a new row for each new step, copying the entries of the prior rows, giving us the following algorithm.

Step A' For each arc in the wanted noncrossing matching, that connects two neighbours

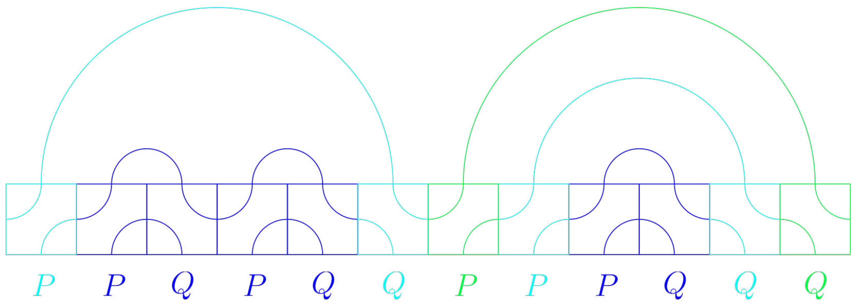


Figure 11: This figure gives an example of the construction of a row, which fixes a noncrossing matching by adding the row below. The row is constructed via the algorithm explained in the proof of Lemma 3.2 Step A is coloured in dark blue. Light blue gives the first iteration of Step B and the repetition of Step B is highligthed in green.

with each other, place a PQ combination directly under that arc to ensure a connection of those neighbours in the outcoming cylinder. Fill the empty cells of the row arbitrarily and add the row to the cylinder.

Step B' Copy the row of the prior step. For each arc in the wanted noncrossing matching, that encloses labels which where already processed by prior steps of the algorithm, put a P on the left side and a Q on the right side and add the row to the cylinder.

Repeat Step B' until all arcs of the wanted noncrossing matching where dealt with, in any step of the algorithm.

This algorithm is shown in Figure 12.

With that, all states in the Markov chain are connected to each other, making the chain irreducible. $\square$

Since this Markov chain fulfills all prerequisites of Theorem 2.10, we can use the theorem to find a stationary distribution λ^∞.
For $\pi_1, \pi_2 \in \mathcal{NC}_n$ and transition matrix A_p, the entries are

$$A_p(\pi_1, \pi_2) = \sum_{\vec{a} \in \{0,1\}^{2n}} \mathbb{P}(R = \vec{a}) \cdot 1_{\{f_{\vec{a}}(\pi_1) = \pi_2\}},$$

where R is the row to add and $\mathbb{P}(R = \vec{a}) = p^{\#P\text{'s in } R} \cdot q^{\#Q\text{'s in } R}$.
Since there are 2^{2n} many different rows, it get's quite pesky to compute those probabilities, if we look at a large n. That is why we will look for another way to compute a stationary distribution.
From now on we will write A instead of A_p, to make notation easier. The transition

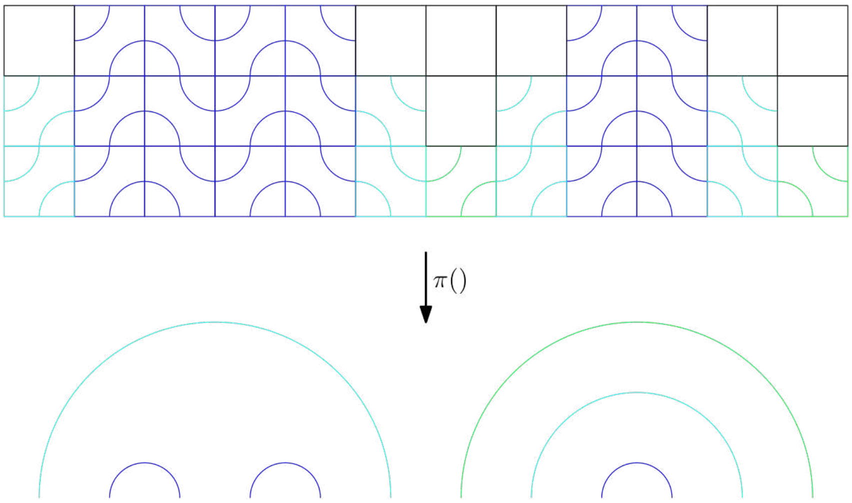


Figure 12: This figure gives an example of the construction of a finite number of rows, which induce an arbitrary fixed noncrossing matching. The construction is done via the algorithm explained in the proof of Lemma 3.2. The PQ pairs constructed in Step A' are coloured in dark blue. Those of Step B' are light blue. Highlighted in green are the PQ pairs constructed in the repetition of step B'. The black squares are the arbitrarily filled squares, which have no effect on the final outcome.

matrix remains dependent of p, but we will see, that our final result is not affected by the choice of p.

A step of our Markov chain is executed by adding a new bottom row to the cylinder. Since the cylinder already has an infinite number of rows, we can picture each cylinder as the state of the Markov chain after an infinite number of steps. Hence the stationary distribution λ^∞ gives us the probability of any cylinder to induce the noncrossing matching $\pi \in \mathcal{NC}_n$.

Example 3.3. *Let C be a cylinder of size 3 and let $p = q = \frac{1}{2}$. Hence for each cell the two types of plaquettes P and Q have the same probability to appear in that cell. This automatically means, that all possible rows have the same probability. Since there are $2^{2\cdot 3}$ differernt rows, each row has probability $\frac{1}{2^6}$.*

Lets order the 5 noncrossing matchings of size 3 the way like shown in Figure 13. Then for the i-th row in the transition matrix A the j-th entry is

$$\frac{\#\text{rows that send } \pi_i \text{ to } \pi_j}{\#\text{possible rows}}.$$

Since we have $2^6 = 64$ possible rows and 5 different noncrossing matchings, we would *have to look at* $64 \cdot 5 = 320$ *cases to know all the entries of* A.

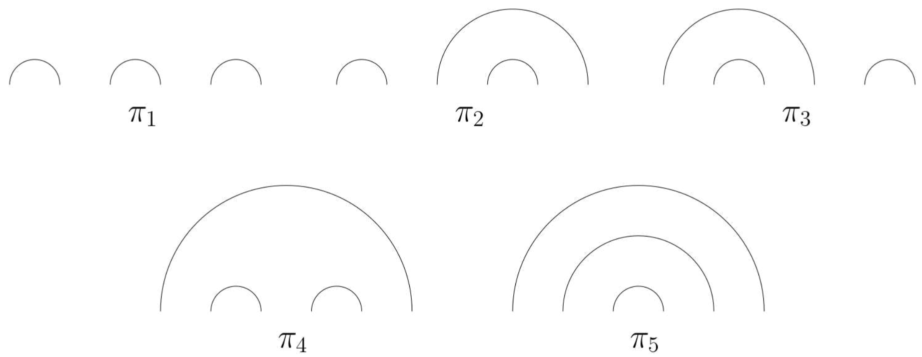


Figure 13: The 5 noncrossing matchings of size 3.

However note, that if we got $\pi, \rho(\pi) \in \mathcal{NC}_n$, both must have the same probability to be induced by a randomly generated cylinder, since we can apply ρ to the cylinder by pushing each plaquette one to the left and put the leftmost column into the last column. Since this is a bijection,

$$\#\text{cylinders inducing } \pi = \#\text{cylinders inducing } \rho(\pi).$$

Therefore even for $n = 3$ it already gets quite pesky to find all the entries of A, due to the number of possible rows we can create with the two plaquettes P and Q. So we will look for a different way to calculate the stationary distribution λ^∞ of the cylinder.

3.3 Temperley-Lieb Operators

In this chapter we display an easier way to compute the probabilities, which noncrossing matching is induced by a rondomly generated cylindric loop percolation. In order to do so, we introduce the Temperley-Lieb operators and define a new cylinder using them as building blocks and finally prove, that those two cylinders have the same stationary distribution.

Definition. The *Temperley-Lieb operator* $e_j : \mathcal{NC}_n \to \mathcal{NC}_n$ of size n is given by

$$e_k(\pi)(m) = \begin{cases} \pi(m) & \text{if } m \notin \{k, k+1, \pi(k), \pi(k+1)\}, \\ k+1 & \text{if } m = k, \\ k & \text{if } m = k+1, \\ \pi(k+1) & \text{if } m = \pi(k), \\ \pi(k) & \text{if } m = \pi(k+1), \end{cases}$$

where $k+1$ is interpreted as 1 if $k = 2n$.

This definition seems pretty complicated and might lead to confusion. Lets put it in words, to show, that we are actually working with a quite simple function:
The Temperley-Lieb operator e_k *connects* k *with* $k+1$ *in the noncrossing matching* π.
If those were already connected in π, *nothing happens. Otherwise, connecting* k *with* $k+1$ *sets loose two numbers, which we then connect to each other.*
However it is even easier understood graphically as displayed in Figure 14.

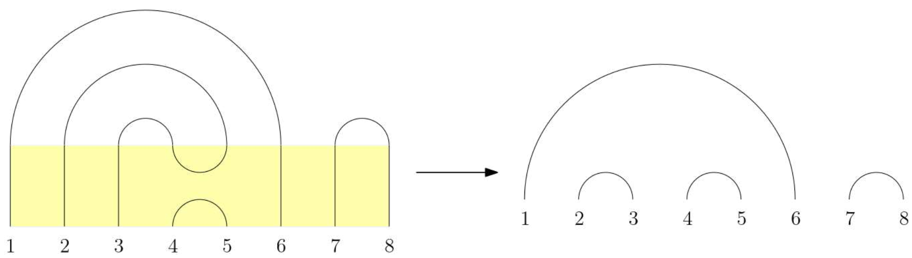


Figure 14: In yellow we have the graphic interpretation of the Temperley-Lieb operator e_4, which is a strip containing $n-2$ straight lines and two arcs. One connecting the labels 4 and 5, and the other mirrored on the opposite side. Applying a Temperley-Lieb operator to a noncrossing matching, means putting the strip below it, which extends and redirects the paths, resulting in new connections between the labels, giving us a new noncrossing matching. In the figure, e_4 is applied to the noncrossing matching ((()))(), which gives (()())().

Lemma 3.4. *The Temperley-Lieb operators fulfill the equations*

- $e_k e_k = e_k$,
- $e_k e_{k+1} e_k = e_k$,
- $e_k e_{k-1} e_k = e_k$,
- $e_k e_j = e_j e_k \quad$ *for* $|k-j| \geq 2$.

Proof. We perform this proof sheer by graphic demonstration, displayed in Figure 15. $\square$

Definition. Taking Temperley-Lieb operators as a basis with $\mathbb{C}$ as a scalar field using composition with the above properties from Lemma 3.4 as multiplication of the basis vectors gives an algebra called *Temperley-Lieb algebra.*

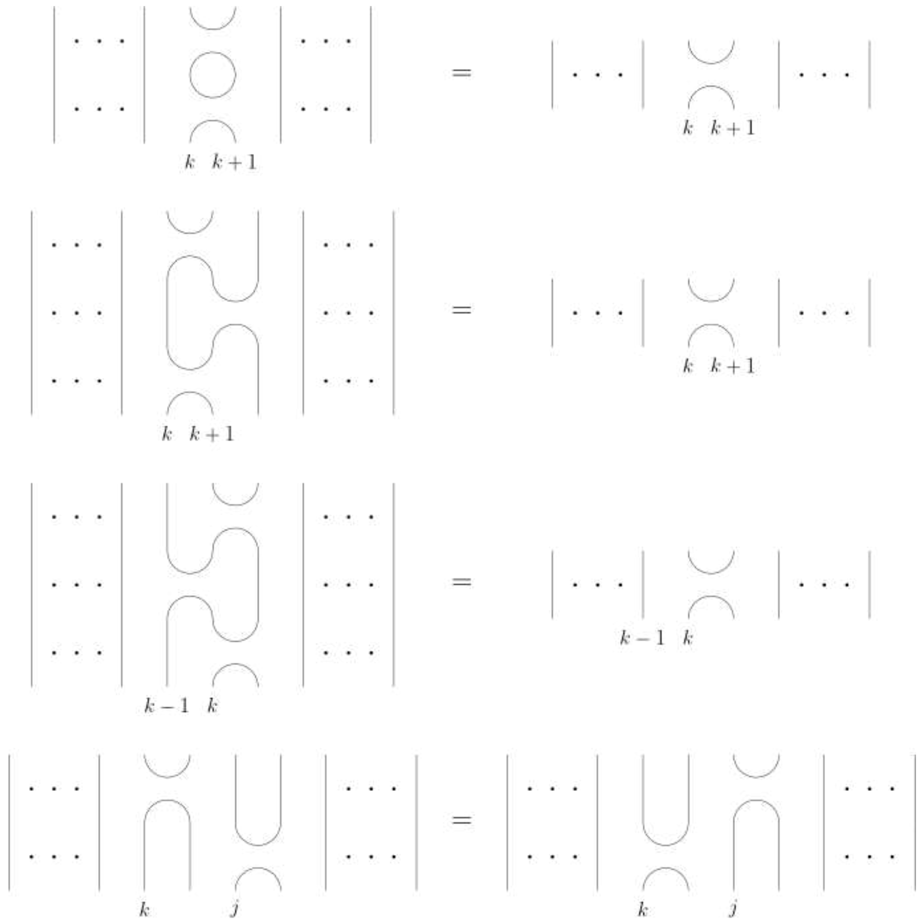


Figure 15: This figure gives a visualization of the equations, stated in Lemma 3.4.

Definition. We define the semi-infinite *TL-cylinder* of size n like the cylinder in the previous subsection, but this time the rows aren't filled with plaquettes but the rows are different Temperley-Lieb operators of size n. One can also view it as the infinite word $(e_{k_0}e_{k_1}e_{k_2}\dots)$.

We gain a probability space, by equipping each Temperley-Lieb operator with a probability. Here we just want to work with the uniform distribution, so for each row R of a TL-cylinder of size n we have

$$\mathbb{P}(R = e_i) = \frac{1}{2n},$$

for $i \in \{1, \dots, 2n\}$. TL-cylinders of size n once again induce a noncrossing matching of size n. The argument, that this always happens is similar as for the cylinder with plaquettes. However this time, we do not have bouncing rows, but bouncing row bundles, gained by a combination of n consecutive rows. Examples for such rows are given at Figure 16. The probability for n consecutive rows to form a bouncing row bundle is greater than one. Since we have an infinite number of rows, such a bundle will appear anywhere in the cylinder almost surely.

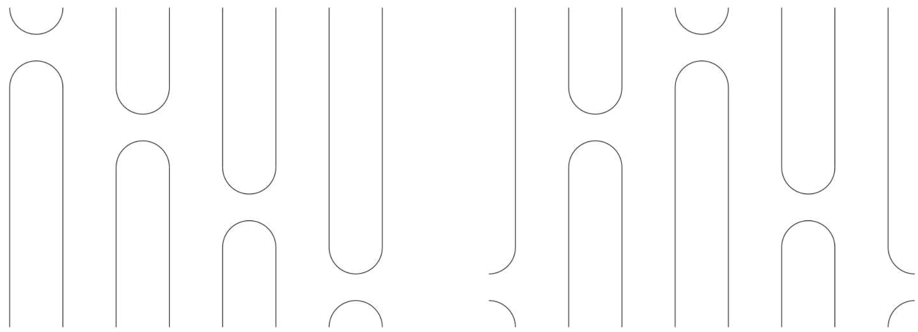


Figure 16: These are two examples of bouncing row bundles for TL-cylinders of size 4. On the left we have the bundle $e_1e_3e_5e_7$ and on the right $e_8e_6e_2e_4$. In order to get a bouncing row bundle, you need that the n consecutive Temperley-Lieb operators e_{i_j} either all fulfill, that i_j's are odd or that they all are even. Additionally, they all need to be different.

Once again we can interpret a TL-cylinder as a Markov chain with state space $\mathcal{NC}_n$.

Theorem 3.5. *Define the matrix H indexed with elements of $\mathcal{NC}_n$ by*

$$H_{\pi,\pi'} = 2n \cdot \delta_{\pi,\pi'} - \#\{1 \leq k \leq 2n : e_k(\pi) = \pi'\}. \tag{3.1}$$

Then for the stationary distribution λ^∞ of the Markov chain gained from the cylindrical loop percolation holds

$$\lambda^\infty H = \vec{0},$$

where $\vec{0}$ is the zero vector of dimension C_n.

Proof. At first we look at a different way to express the entries of the transition matrix A of the Markov chain gained from the cylindrical loop percolation

$$A(\pi,\pi') = \sum_{\substack{\vec{a}\in\{0,1\}^{2n} \\ f_{\vec{a}}(\pi)=\pi'}} p^{\sum_i a_i} \cdot (1-p)^{\sum_i (1-a_i)},$$

where $\pi, \pi' \in \mathcal{NC}_n$ and $\vec{a} = (a_1, \ldots, a_{2n})$.

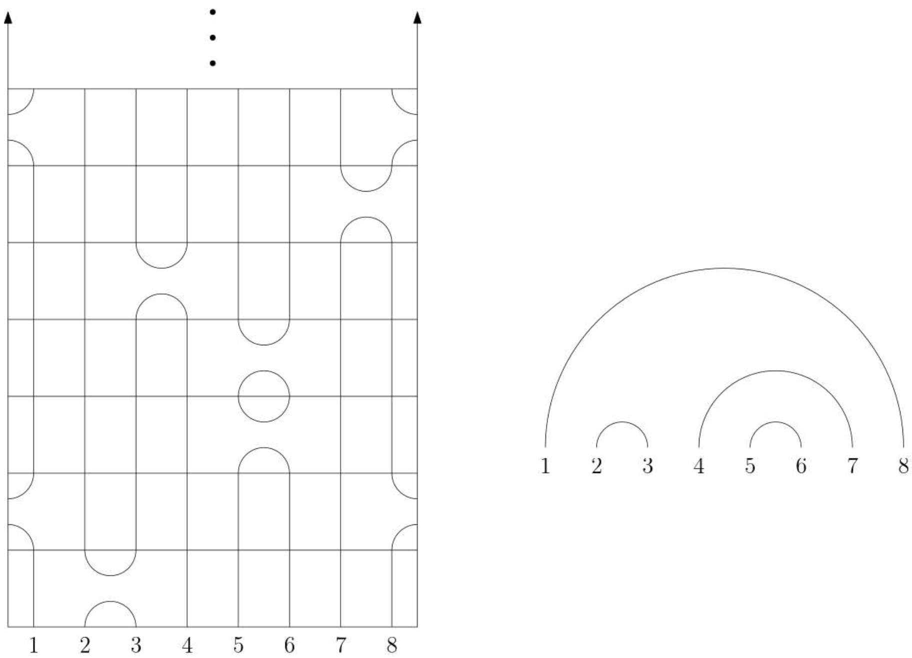


Figure 17: An example of a TL-cylinder of size 4 and its induced noncrossing matching.

Now we differentiate this equation with respect to p

$$\frac{d}{dp}A(\pi,\pi') = \sum_{\substack{\vec{a}\in\{0,1\}^{2n}\\ f_{\vec{a}}(\pi)=\pi'}} \left(\left(\sum_i a_i\right)\cdot p^{\sum_i (a_i)-1}\right)\cdot(1-p)^{\sum_i(1-a_i)}$$
$$- p^{\sum_i a_i}\left(\left(\sum_i (1-a_i)\right)\cdot(1-p)^{\sum_i(1-a_i)-1}\right).$$

Furthermore we set $p=0$ in the above equation to get

$$\left(\frac{d}{dp}A(\pi,\pi')\right)\Big|_{p=0} = \sum_{\substack{\vec{a}\in\{0,1\}^{2n}\\ f_{\vec{a}}(\pi)=\pi'}} \left(\left(\sum_i a_i\right)\cdot 0^{\sum_i (a_i-1)}\right)\cdot 1^{\sum_i(1-a_i)}$$
$$- 0^{\sum_i a_i}\left(\left(\sum_i (1-a_i)\right)\cdot 1^{\sum_i(1-a_i)-1}\right).$$

For most of the vectors $\vec{a}$ this sum will be zero. For the first summand, $0^{\sum_i a_i}$ is zero except for $\vec{a}=\vec{0}$. For the second summand, $0^{\sum_i (a_i-1)}$ is zero except for each $\vec{a}$, where exactly one $a_j = 1$. This gives

$$\left(\frac{d}{dp}A(\pi,\pi')\right)\Big|_{p=0} = -2n \cdot \delta_{f_{\vec{0}}(\pi),\pi'} + \#\{1 \le k \le 2n : f_{\vec{a_k}}(\pi) = \pi'\}, \tag{3.2}$$

where $\vec{a}_k = (0,\ldots,0,1,0,\ldots,0)$ with the 1 at the k-th position.
Now take a look at Figure 18. We see, that $f_{\vec{0}}$ moves each end of the paths from the noncrossing matching one to the right. Hence it performs a rotation on the noncrossing matching and we get

$$f_{\vec{0}}(\pi) = \rho^{-1}(\pi). \tag{3.3}$$

We also see, that $f_{\vec{a}_k}$ has a similar effect, moving most of the paths one to the right. However additionally the single P plaquette forces a connection between the labels k and $k+1$, whilst also connecting the two labels, which where at first connected to the labels $k-1$ and k respectively. This reminds us of the Temperley-Lieb operators. Therefore via a combination of the rotation and a Temperley-Lieb operator, we can formulate

$$f_{\vec{a}_k} = \rho^{-1} \circ e_{k-1} = e_k \circ \rho^{-1}. \tag{3.4}$$

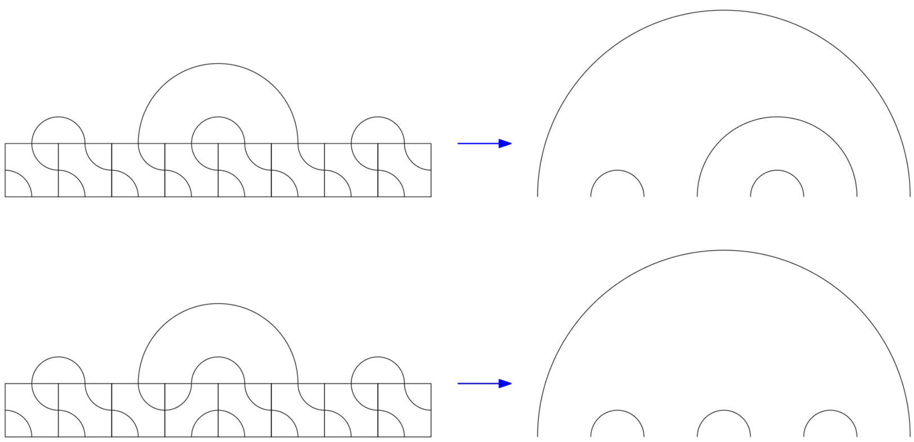


Figure 18: Two types of plaquette rows acting on a noncrossing matching. On top $f_{\vec{0}}$ and on the bottom $f_{\vec{a}_4}$.

Let

$$A' := \left(\frac{d}{dp}A\right)\Big|_{p=0},$$

then we get

$$\lambda^\infty A' = \lambda^\infty \left(\frac{d}{dp}A\right)\Big|_{p=0} = \left(\frac{d}{dp}\lambda^\infty A\right)\Big|_{p=0} = \left(\frac{d}{dp}\lambda^\infty\right)\Big|_{p=0} = 0.$$

If we now express ρ as a matrix, indexed by noncrossing matchings

$$R_{\pi,\pi'} = \delta_{\rho(\pi),\pi'},$$

we observe

$$\lambda^\infty R = \lambda^\infty,$$

since rotation doesn't change the probability of a cylinder.
Using (3.1), (3.2), (3.3) and (3.4) we can conclude

$$\begin{aligned} -R_{\pi,\pi'}A'(\pi,\pi') &= -A'(\rho(\pi),\pi') \\ &= 2n\cdot\delta_{f_{\vec{0}}(\rho(\pi)),\pi'} - \#\{1\le k\le 2n : f_{\vec{a_k}}(\rho(\pi)) = \pi'\} \\ &= 2n\cdot\delta_{\rho^{-1}(\rho(\pi)),\pi'} - \#\{1\le k\le 2n : e_k(\rho^{-1}(\rho(\pi))) = \pi'\} \\ &= 2n\cdot\delta_{\pi,\pi'} - \#\{1\le k\le 2n : e_k(\pi) = \pi'\} = H_{\pi,\pi'}. \end{aligned}$$

With that identity we can deduce

$$\lambda^\infty H = \lambda^\infty(-RA') = -\lambda^\infty A' = 0.$$ □

Corollary 3.6. *As a direct consequence of Theorem 3.5, we obtain*

$$M := \mathbb{I} - \frac{1}{2n}H,$$

is a transition matrix of a Markov chain with $\lambda^\infty M = \lambda^\infty$.

Proof. Just use the definition of M in the above equation to get

$$\lambda^\infty M = \lambda^\infty(\mathbb{I} - \frac{1}{2n}H) = \lambda^\infty - \frac{1}{2n}\lambda^\infty H = \lambda^\infty - \vec{0} = \lambda^\infty.$$

Since all rows of M sum up to 1, M is a transition matrix. □

The Markov chain, which has M as its transition matrix is actually the TL-cylinder viewed as a Markov chain. Therefore λ^∞ actually gives us the probabilities, which $\pi \in \mathcal{NC}_n$ is induced by a randomly generated TL-cylinder. Since we took the distribution

λ^∞ directly from the equation

$$\lambda^\infty A = \lambda^\infty,$$

it is also the distribution for a randomly generated cylinder with plaquettes. Theorem 3.5 also showed, that the choice of the probabilities p and $1-p$ of the plaquettes P and Q, don't affect the final probability which noncrossing matching is induced by the cylinder. In order to find λ^∞ easier, we simplify the entries of M

$$M_{\pi,\pi'} = \frac{\#\{1 \le k \le 2n : e_k(\pi) = \pi'\}}{2n}.$$

The stationary distribution is therefore given by the system of equations

$$\lambda_\pi^\infty = \frac{1}{2n} \sum_{k=1}^{2n} \sum_{\substack{\pi' \in \mathcal{NC}_n \\ e_k(\pi')=\pi}} \lambda_{\pi'}^\infty. \tag{3.5}$$

Example 3.7. *Let $n = 3$.*
With Figure 19 we easily get the transition matrix

$$M \quad = \quad (m_{\pi_i,\pi_j})_{1\le i,j\le 5} \quad = \quad \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{6} & \frac{1}{2} & 0 & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{2} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix},$$

where the entry at the i-th row and j-th column gives the probability, that our cylinder jumps from induced noncrossing matching π_i to π_j, by adding a new bottom row.
We already said, that for all $\pi, \pi' \in \mathcal{NC}_n$ with $\rho^k(\pi) = \pi'$ and $k \in \mathbb{N}$, the probability of a cylinder inducing π must be the same as for π', since rotating everything one to the left does not change the probability. Hence for such π and π' we have

$$\lambda^\infty(\pi) = \lambda^\infty(\pi').$$

Since that gives us an equivalence relation, we only have to determine $\lambda_{\pi_i}^\infty$ for each equivalence class.
In this case there are the two equivalence classes $[\pi_1, \pi_4, \pi_5]$ and $[\pi_2, \pi_3]$. There are 3 Temperley-Lieb operators in Figure 19 going from π_2 to itself and respectively 2 Temperley-Lieb operators going from each noncrossing matching π_1, π_4 and π_5 to π_2, so we have the

equation

$$\lambda^\infty_{\pi_2} \quad = \quad \frac{1}{2}\lambda^\infty_{\pi_2} + \frac{1}{3}\lambda^\infty_{\pi_1} + \frac{1}{3}\lambda^\infty_{\pi_4} + \frac{1}{3}\lambda^\infty_{\pi_5} \quad = \quad \frac{1}{2}\lambda^\infty_{\pi_2} + 3\frac{1}{3}\lambda^\infty_{\pi_1}.$$

Additionally the properties of the stationary distribution gives us

$$1 = 3\lambda^\infty_{\pi_1} + 2\lambda^\infty_{\pi_2}$$

these two equations are solved by

$$\lambda^\infty_{\pi_1} = \frac{1}{7} \qquad \textit{and} \qquad \lambda^\infty_{\pi_2} = \frac{2}{7},$$

resulting in the stationary distribution

$$\lambda^\infty = \left(\frac{1}{7}, \frac{2}{7}, \frac{2}{7}, \frac{1}{7}, \frac{1}{7}\right).$$

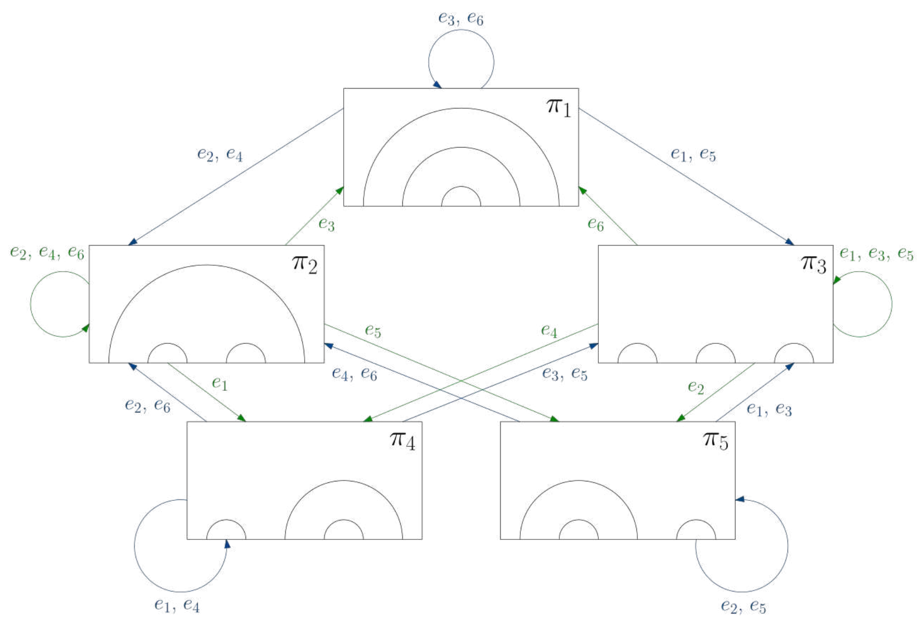


Figure 19: This is the diagram of the noncrossing matchings of size 3 and their relations in terms of Temperley-Lieb operators. The edges are coloured only for readability. The ones starting from π_2 or π_3 are both green since $\pi_2 = \rho(\pi_3)$. The others are uniformly blue as π_1, π_4 and π_5 can also be gained from each other by rotation.

4 Fully Packed Loops

In this chapter we take a closer look at fully packed loops. First of all we are giving the definition and prove their bijection to alternating sign matrices of same size. Later on we are looking at the Wieland gyration and ultimately use it, to define an equivalent of Temperley-Lieb operators acting on fully packed loops, which we need for our proof of the Razumov Stroganov correspondence.

4.1 Introduction and Bijection with ASMs

Definition. Take a graph with $n \times n$ vertices arranged in a square. For each vertex construct 2 horizontal and 2 vertical edges, connecting them with their direct neighbours. For the vertices on the boundary some of these edges do not connect them with other vertices but instead just point outwards the graph. We call them external edges. Now we perform a *bicolouring* of the edges, such that the topmost edge of the external edges on the left of the square is coloured black and the other external edges are coloured in an alternating manner black and grey. The rest of the edges are coloured in a way, such that each vertex lies on two black and two grey edges. Taking only the black edges, gives you a graph, where each vertex has degree 2. The so gained graph is called *fully packed loop* (FPL). The same is possible with the grey edges. We denote the set of FPLs of size n with external edges as the black graph with $\mathcal{FPL}_n^+$ and the other one with $\mathcal{FPL}_n^-$.

Remark 4.1. *Via the bicolouring there is a natural bijection between $\mathcal{FPL}_n^+$ and $\mathcal{FPL}_n^-$ by taking the grey edges ionstead of the black or the other way around.*

From now on, we will focus on $\mathcal{FPL}_n^+$ and therefore drop the $+$ for readability, however we will make use of $\mathcal{FPL}_n^-$ later on.
For an FPL $F \in \mathcal{FPL}_n$ label the $2n$ external edges with $\{1, \dots, 2n\}$, starting at the topmost left edge and then going counterclockwise. Since every vertex has order 2, F gives us paths, that connect two labels with each other. Therefore F induces a noncrossing matching $\pi \in \mathcal{NC}_n$ and we write

$$\pi(F) = \pi.$$

We denote the number of FPLs of size n with A_n, and the number of FPLs inducing π with A_π.

Definition. An *alternating sign matrix* (ASM) of size n is an $n \times n$ matrix with entries in $\{-1, 0, 1\}$, such that the nonzero entries alternate between -1 and 1 in each row and

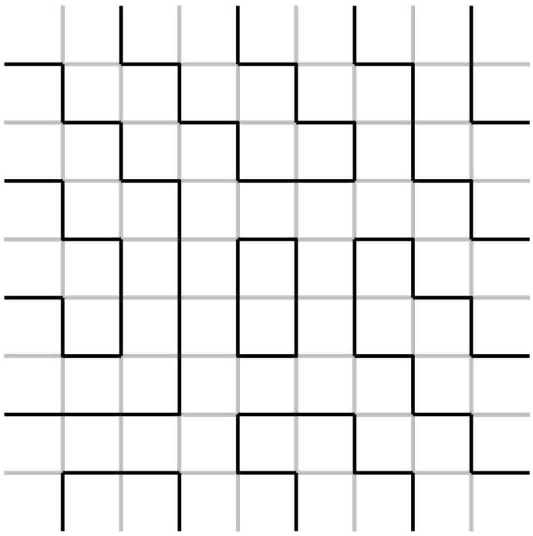


Figure 20: Example of a bicolouring of size 8

each column. Additionally the entries of each row and each column must sum up to 1. We denote the set of all ASMs of size n with $\mathcal{ASM}_n$.

Example 4.2.

$$\begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{matrix}$$

This is an example for an ASM of size 5.

Lemma 4.3. *Let A be an ASM of size n.*

(1) *Each row and each column of A has an odd number of nonzero entries.*

(2) *The first nonzero entry in each row and each column is 1.*

(3) *The boundary rows and columns consist of a single 1 and $(n-1)$ 0's.*

Proof. If A would have an even number of nonzero entries in a row or column, they would sum up to an even number. This can not be, since the entries of each row and each column must sum up to 1, proving (1).
If the first nonzero entry in a row or column would be -1, the entries would sum up to -1, as the nonzero entries must alternate between 1 and -1. This again would be a

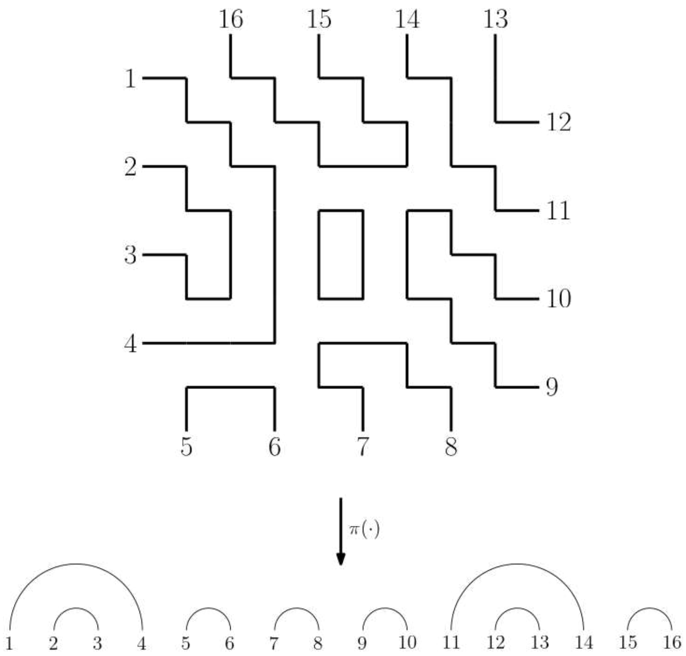


Figure 21: The noncrossing matching induced by the FPL of size 8 we gained from the bicolouring of Figure 20.

contraditction to the fact, that all entries in each row and coulumn must add up to 1, proving (2).
The boundary rows and columns can not be all 0's, because their entries need to sum up to 1. Let us assume w.l.o.g. that the first row had more nonzero entries than a single 1. Then there would be a j, such that the j-th entry of the first row is -1. But then the first entry of the j-th column would be -1, which is a contradiction to (2), proving (3). □

Definition. Let $\gamma : \mathcal{ASM}_n \to \mathcal{FPL}_n$ such that each entry in the ASM A represents a vertex in the FPL F. At each zero the two edges of the corresponding vertex form a 90° corner and at each nonzero entry they form a straight line. We call the vertices where the latter occurs *refinement positions* (rp). An rp in an FPL can either be *vertical* or *horizontal.*

For γ to be a bijection we need an equivalent of Lemma 4.3 for FPLs.

Lemma 4.4. *Let F be an FPL of size n.*

(1) *Each row and each column of F has an odd number of refinement positions.*

(2) *If F has a refinement position in the i-th row at the j-th position, the parity of the* *number of refinement positions in the i-th row up to this point must coincide with the parity of the number of refinement positions in the j-th column up to this point.*

(3) *The boundary rows and columns contain exactly one refinement position.*

Proof. (1): Each row consists of refinement positions and sequences of consecutive 90° corners. In a sequence of corners each second horizontal edge is chosen. If the chosen edges connect an even position with an odd position (reading from left to right), we call the sequence ϕ, if it connects odd to even, we call it ψ. Within a row, each rp forces a change from ϕ to ψ and vice versa.

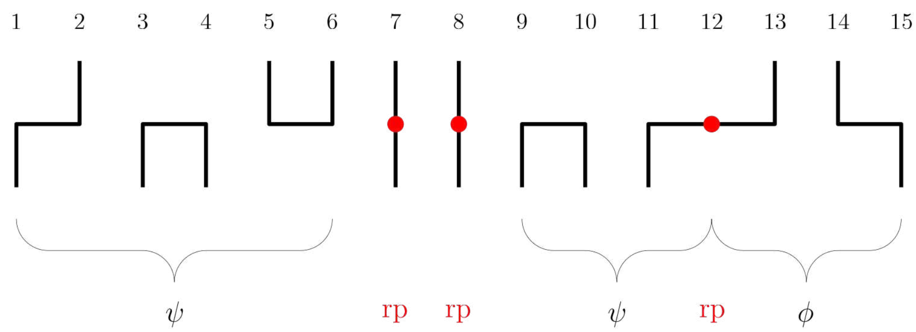


Figure 22: Example of an even numbered row in an FPL of size 15.

Let us look at a row without external edges in an FPL of odd size, as in Figure 22. Then assume, that we do not have an rp at position 1, so the row starts with ψ. If we have no rp at position n, the row ends with ϕ. In order to start with ψ and end with ϕ, there must be an odd number of rp in between. If there would be an rp at position 1 or respectively position n, the row instead starts with ϕ or respectively ends with ψ, keeping our argument.
The cases for other parity of n and other external conditions, work the same.
For columns we can do the same, by going from top to bottom instead of left to right.

(3): Due to (1), we must have at least one rp at the boundary. Once we fixed that rp, all edges to the left and to the right are fixed, as each vertex must have degree two. Since the external edges are fixed, the forced edges give sequences of consecutive corners and no additional refinement positions. Figure 23 displays how those forced edges look like.

(2): Let us look at a row and a column both without an external edge. For this to happen, we need an evenly numbered row and an oddly numbered column. The rp at their intersection must either be horizontal or vertical. If it is horizontal both the row and the column must land at the intersection with ϕ. Since both started with ψ, both

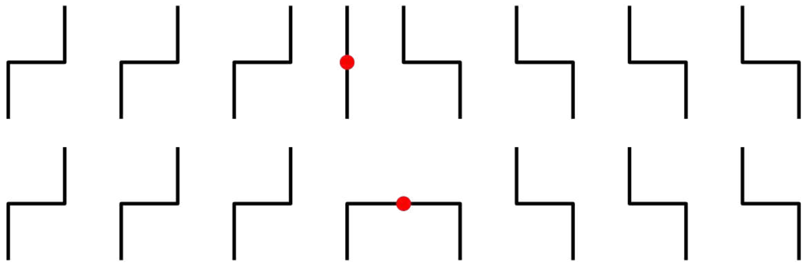


Figure 23: Here we have 2 examples of a bottom row in an FPL of size 15. The upper one has a vertical rp at the bottom row and the lower one a horizontal rp.

must have an odd number of refinement positions prior to the intersection. If the rp at the intersection is vertical, both must land there with ψ, meaning that both must have an even number of refinement positions prior to the intersection. This is displayed at Figure 24.
Looking instead at a row with an external edge, changes the number of steps for the column by an odd number and also changes the row to start from ψ instead of ϕ. Since we used an argument about parity, this still works. The same is true, when looking at a column with an external edge. $\square$

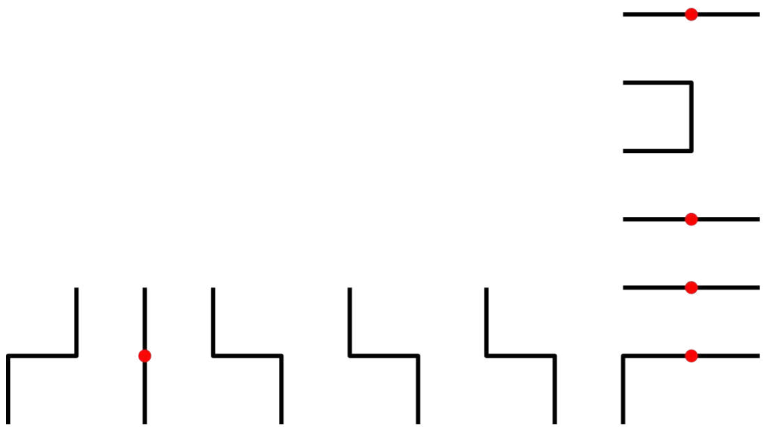


Figure 24: Example of an intersection of an evenly numbered row and an oddly numbered column in an FPL.

Is γ well defined and gives us a bijection between $\mathcal{ASM}_n$ and $\mathcal{FPL}_n$? Take another look at Figure 23. At the boundary, knowing where the refinement position is, already forces all other edges in that row or column respectively. That is because of the already fixed external edges and the fact, that each vertex has degree 2. This just gives us a smaller square to look at with already fixed external edges. The way these edges are arranged does not fulfill the definition of an FPL, but knowing where these edges are, is enough to once again force the next rows and columns, by knowing the rp's. If we keep going like this, the not yet defined square gets smaller from step to step until the whole FPL is constructed. Hence we never actually had a choice, where to put an edge, making γ well defnied. The inverse function γ^{-1} constructs the ASM by placing a nonzero entry in the ASM at each rp of the FPL and zeros everywhere else.

$$\begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{matrix} \xrightarrow{\gamma}$$

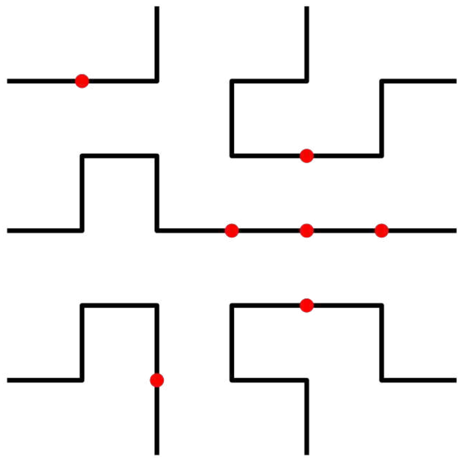


Figure 25: FPL gained, by applying γ to the ASM of Example 4.2.

Since it will play a primary role, from now on, when we mention a refinement position of an FPL, we are always talking about the unique one at the bottom of the FPL.

4.2 Wieland Gyration

In [6] Benjamin Wieland defined the following operation on FPLs, that performs a rotation similar to ρ operating on noncrossing matchings. This operation will be of great importance in the next subsection and plays a primary role in our proof of the Razumov Stroganov conjecture.

Definition. Let $G_\pm$ be a function going from $\mathcal{FPL}_n^\pm$ to $\mathcal{FPL}_n^\mp$. The operation $G_\pm$ then changes the choice of edges in certain squares of the grid. If the square contains two parallel edges, they stay the same. In each other case, take the complement choice of edges as shown in Figure 26. Performing this action on squares, which lie on the boundary, adds edges which are not part of the definition of FPLs. We still proceed in that way and simply remove those edges not belonging to the bicolouring afterwards.
The squares we are doing this change at, are chosen in an checkerboard manner. For G_+ we start with the topmost square on the very left, for G_- we take the complementary

choice of squares.

The *Wieland gyration* then is $W = G_- \circ G_+$.

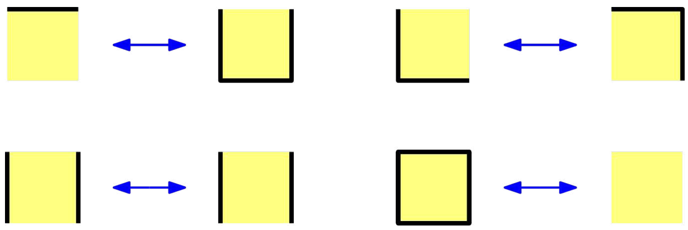


Figure 26: All possible operations on squares via G_+ or G_- up to rotation.

Note that W goes from $\mathcal{FPL}_n^+$ to $\mathcal{FPL}_n^+$ and is a bijection.

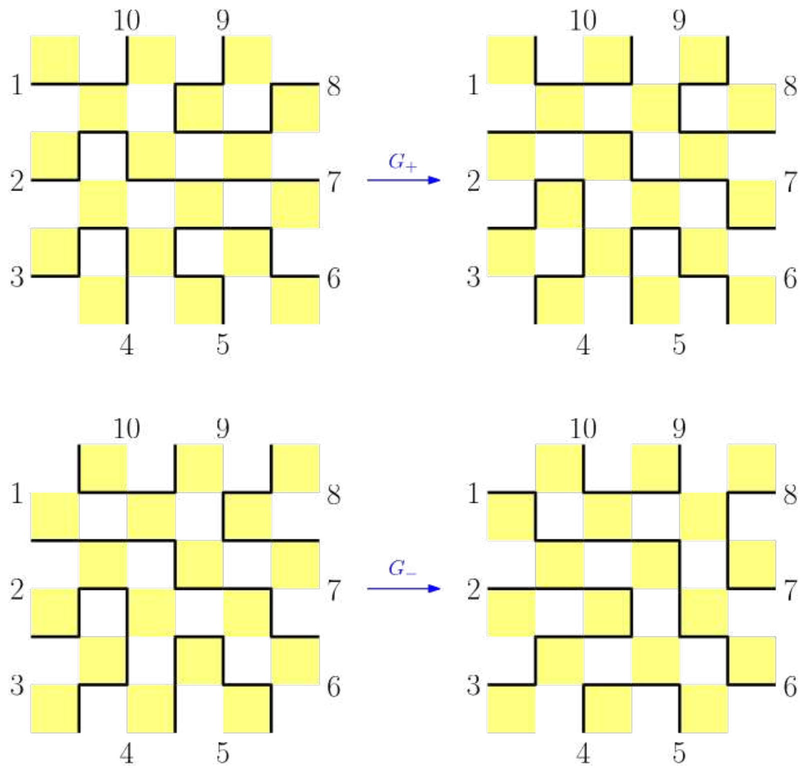


Figure 27: Example of a Wieland Gyration acting on an FPL of size 5. The yellow squares are the ones where the operations of Figure 26 are performed.

Lemma 4.5. *Let $F \in \mathcal{FPL}_n$. Then*

$$\pi(W(F)) = \rho(\pi(F)).$$

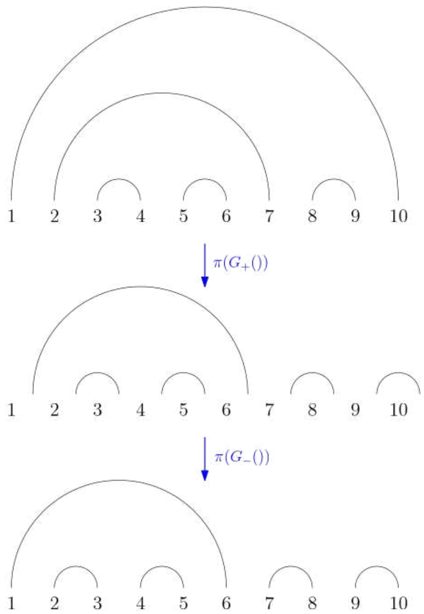


Figure 28: This figure shows how the induced noncrossing matching changes during the Wieland Gyration of Figure 27. The object in the middle actually does not represent a noncrossing matching. We do not give a definition for that object since we will not need it furthermore, as its only purpose is helping to display, what is happening at the Wieland gyration.

Proof. Let's look at an interior square, changed by G^+ or G^- respectively. We assume, that the vertices v and v' lie on that square and are connected via edges on that square but also both having an edge pointing outwards that square. Then G^+ or G^- respectively changes that square in a way, such that v and v' are still connected within that square. On the boundary squares, we got something similar, but instead of having two connected vertices, we got a path from a vertex v to an external edge. Then $G_\pm$ changes that path towards the next external edge clockwise, resulting in a rotation of the noncrossing matching. $\square$

Like Cantini and Sportiello did in [2], we will use $G_\pm$ in a little different aspect. Firstly we will not just look at $W = G_+ \circ G_-$ but look at the *half gyrations* G_+, G_- on their own. Additionally we will rotate the labels clockwise alongside the boundary edges, instead of keeping them still, as we did with the Wieland gyration. We denote the functions, which rotate the labels alongside the edges with $\tilde{G}_+$ and $\tilde{G}_-$ respecitvely.

$$
\begin{tikzcd}
\mathcal{FPL}_n^+ \arrow[r, "W"] \arrow[d, "\pi(\cdot)"'] & \mathcal{FPL}_n^+ \arrow[d, "\pi(\cdot)"] \\
\mathcal{NC}_n \arrow[r, "\rho"'] & \mathcal{NC}_n
\end{tikzcd}
$$

Figure 29: The commuting diagram of W, ρ and $\pi(\cdot)$.

Definition. Let

$$G^l := \underbrace{\cdots \circ \tilde{G}_{\pm} \circ \tilde{G}_{\mp} \circ \tilde{G}_{\pm}}_{l \text{ operations}},$$

where we start with $\tilde{G}_+$, when $F \in \mathcal{FPL}_n^+$ and with $\tilde{G}_-$, when $F \in \mathcal{FPL}_n^-$ respectively. We then get

$$G^{-l} := \underbrace{\cdots \circ \tilde{G}_{\pm} \circ \tilde{G}_{\mp} \circ \tilde{G}_{\pm}}_{l \text{ operations}},$$

where we start with $\tilde{G}_+$, when $F \in \mathcal{FPL}_n^-$ and with $\tilde{G}_-$, when $F \in \mathcal{FPL}_n^+$ respectively.

As $\tilde{G}_+$ and $\tilde{G}_-$ are both self-inverse, we get

$$G^{-l} \circ G^l = \text{id}.$$

This makes $(\{G^l : l \in \mathbb{Z}\}, \circ)$ a group acting on $\mathcal{FPL}_n^+ \cup \mathcal{FPL}_n^-$.

Definition. Let $F \in \mathcal{FPL}_n^{\pm}$. We define $r : \mathcal{FPL}_n^{\pm} \to \{1, \dots, n\}$ as

$$r(F) := \text{position of rp in } F.$$

Additionally we define $r_l : \mathcal{FPL}_n^{\pm} \to \{1, \dots, n\}$ as

$$r_l(F) := \text{position of rp in } G^l(F).$$

Example 4.6. *In order to understand the above definded functions better, we look at an example. In Figure 27, we did not rotate the labels alongside the edges, but we can* *look at the position of rp anyways. Let us denote the FPL on the top left with F_1 the ones on the top right and the bottom left with F_2 and the one on the bottom right with F_3. We have $r(F_1) = 2, r(F_2) = 3$ and $r(F_3) = 3$. So after applying G_+ to F_1, rp moved one position to the right and G_- then does not move rp, but instead turns it from being vertical to being horizontal. So we have*

$$r(F_1) = 2, \quad r_1(F_1) = 3, \quad r_2(F_1) = 3.$$

We are now interested, how $r(F)$ is affected by G^l. We will see, that it is not necessery to know the whole fully packed loop configuration F, but it suffices to look at it locally around rp. There are actually just 4 different cases, as shown in Figure 30.

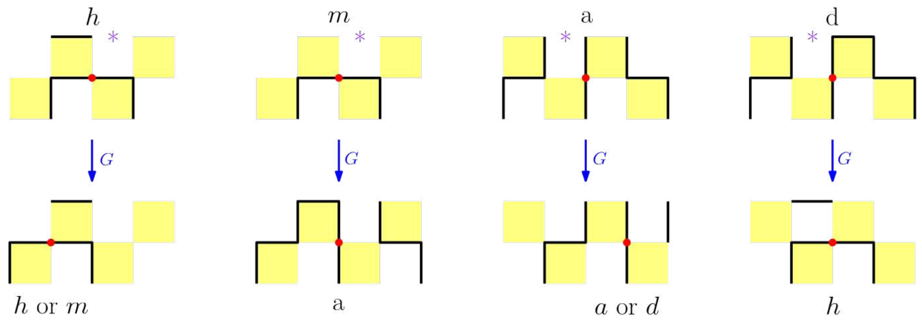


Figure 30: These are the 4 different configurations h, m, a and d, which can occur around r and how they change under G^1. The purple stars represent positions of the initial configurations, where it makes no difference if there are edges or not.

We see in Figure 30, that h initializes the refinement position to make a step to the left, so $r_1(F) = r(F) - 1$. In the same way a initializes a step to the right. Both m and d hold the refinement position at its place and instead change it from being vertical to being horizontal or the other way around. These results come from Lemma 5.1 of [2].
So $r(\cdot)$ can only decrease, while rp is horizontal and increase while rp is vertical. Figure 30 also shows that rp will always make at least one step after changing orientation. We call an FPL F to be of type h when it has configuration h around rp. We denote the types m, a and d analogously.

Lemma 4.7. *Let $F \in \mathcal{FPL}_n^-$ and j be the rightmost label in the bottom row. Then for some $i \in \{0, \dots, n-1\}$, the position of j coincides with $r_i(F)$.*

Proof. If $r_0(F) = n$, they already coincide and we are done. So now assume the opposite, then rp is left of j. With each step of G, j moves one step to the left. So at some point, rp is either one or two steps to the left of j. If it is one step to the left, we either have

configuration h or m, if it is two steps to the left, we can only have a or d.
Case m: In the next iteration of G, rp will not move, but will instead change from horizontal to vertical. In the same time j moves one step to the left.
Case a: In the next iteration of G, rp will make a step to the right, whilst j will make a step to the left, resolving in both landing on the same spot.
Case h: In the next iteration of G, rp will make one step to the left, leaving the distance between rp and j the same. We then are either in Case m or again in Case h. However once rp is at position 0, we can not have Case h anymore, since the boundary condition on the external edges of FPLs do not allow the configuration of h.
Case d: In the next iteration of G, rp will not move, but instead change from vertical to horizontal, letting us land in either Case m or Case h. □

Corollary 4.8. *Let $F \in \mathcal{FPL}_n^{\pm}$ and j be a label, which is not on the bottom of F. In $4n$ iterations of G, the position of j and rp coincide exactly once.*

Proof. We can find an $l \in \{1, \ldots, 3n\}$, such that j is the rightmost label in the bottom row of $G^l(F)$. Then we can apply Lemma 4.7. After they coincided once, they can not hit again right afterwards, because j keeps moving to the left, whilst rp can not instantly move to the left, as it was vertical in the moment of the hit. □

We can not ensure the same for labels on the bottom of F. After $4n$ iterations of G, all labels will have returned to their initial position and $\pi(F) = \pi(G^{4n}(F))$. However G^{4n} does not have to be the identity, even though it will be in some cases. However if $F \neq G^{4n}(F)$, then possibly $r_0(F) \neq r_{4n}(F)$. Example 4.11 shows a situation, where $G^{4n}(F) \neq F$.

Corollary 4.9. *Let $F \in \mathcal{FPL}_n^{\pm}$ and j be a label on the bottom of F.Then after $4n$ iterations of G there are three possible cases.*

- *In F, j is to the left of rp and in $G^{4n}(F)$, j is to the right of rp. Then j and r never coincided in $4n$ iterations of G.*
- *In F, j is weakly to the right of rp and in $G^{4n}(F)$, j is weakly to the left of rp. Then j and rp coincided twice in $4n$ iterations of G.*
- *In $G^{4n}(F)$, j is on the same side of rp, as it was in F. Then j and rp coincided once in $4n$ iterations of G.*

Proof. The amount of times j and rp coincide is equal to the number, that j and rp pass each other on their paths, as Lemma 4.7 tells us, that they can not pass each other without a hit. Since j performs one circle around F in $4n$ iterations of G, the realtive positions of j and rp before and after G^{4n}, gives us the number of times, they passed each other. $\square$

Corollary 4.10. *Let $F \in \mathcal{FPL}_n^{\pm}$ and $k \in \mathbb{Z}$ such that $G^{4n \cdot k}(F) = F$. Then for any label j, the number of times, j and rp coincided after $4n \cdot k$ iterations of G is k.*

Proof. Since $G^{4n \cdot k}(F) = F$, it must also hold, that $r_{4n \cdot k} = r_0$. Therefore the relative positions of j and rp are the same after $4n \cdot k$ iterations of G. Then the statement follows from Lemma 4.7 and Corollary 4.9 $\square$

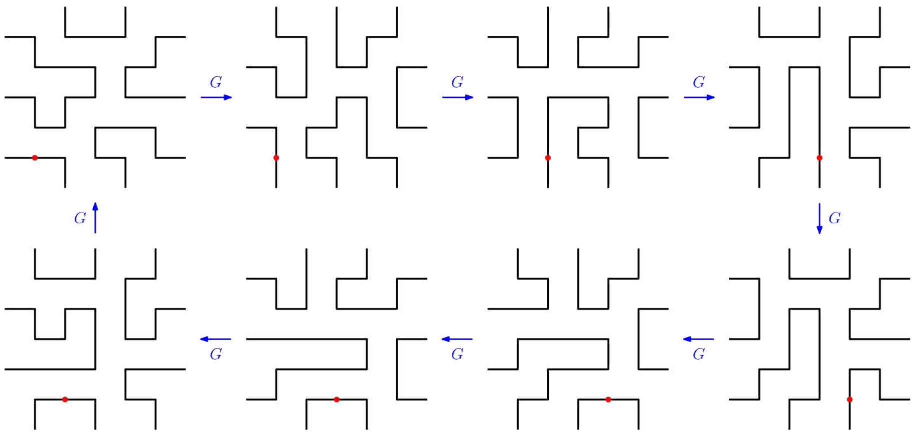


Figure 31: Example of an FPL of size 5, for which G^{4n} is not the identity.

Example 4.11. *In Figure 31 we see the orbit of an FPL of size 5. As this FPL has periodicity of size 8, $G^{4n} = G^{20} \neq id$, as $8 \nmid 20$. However $8 \mid 40$, therefore $G^{40}(F) = F$ for taking F to be one of the 8 FPLs from Figure 31.*

4.3 Quasi TL-operators

In [2] Cantini and Sportiello, introduced the maps $\tilde{e}_1$ and $\tilde{e}_{2n}$, which operate on fully packed loops, serving as an analogon to the Temperley-Lieb operators. In this chapter, we will take this idea two steps further, in order to formulate another proof of the Razumov-Stroganov conjecture.

Definition. Let $F \in \mathcal{FPL}_n^{\pm}$. We define the *momentary quasi TL-operator* $\tilde{e}$: $\mathcal{FPL}_n \to \mathcal{FPL}_n$ such that, if F has configuration a as displayed in Figure 30, we switch the square to the right of rp, consisting of two vertical edges, to two horizontal edges, creating an h

configuration, as displayed in Figure 32. If F has configuration h, m or d, we do nothing and $\tilde{e}$ acts as the identity.

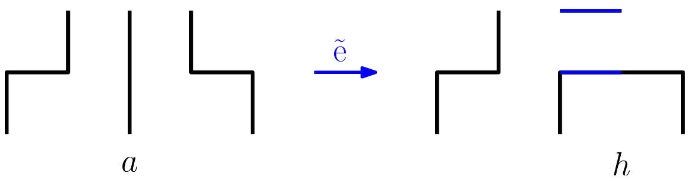


Figure 32: The *momentary quasi TL-operator* acting on an FPL with configuration a.

So if we have configuration a, $\tilde{e}$ connects the label i, which is at r, with the label $i+1$ and also the two labels with each other, which where connected to i or $i+1$ beforehand.

Remark 4.12. *For an FPL F with configuration m or d, the preimage $\tilde{e}^{-1}(F) = F$. FPLs with configuration a have no preimage and FPLs with configuration h have two preimages. One with configuration a and one with h where $\tilde{e}^{-1}(F) = F$.*
If one would want explicit preimages, to have an inverse function, one would have to reduce the sets like

$$\tilde{e} : \{F \in \mathcal{FPL}_n^{\pm} : F \text{ has vertical } r\} \to \{F \in \mathcal{FPL}_n^{\pm} : F \text{ has configuration } h \text{ or } d\} \quad (4.1)$$

.

Definition. Let $F \in \mathcal{FPL}_n$ and i any label of F. For $j \in \mathbb{N}$, define $\nu_j : \mathcal{FPL}_n \times \{1, \ldots, 2n\} \to \mathbb{N}$ such that,

$G^{\nu_j(F,i)}$is the j-th time, label i coincides with r in the orbit $(F, G(F), G^2(F), ...)$ of F.

Definition. We fix n. Let

$$\mathbf{k} := \min\{l \in \mathbb{N} : \forall F \in \mathcal{FPL}_n : G^{4n \cdot l}(F) = F\} = \mathrm{ord}(G^{4n}),$$

we then define the *quasi TL-operator* $\tilde{e}_{i,j} : \mathcal{FPL}_n \to \mathcal{FPL}_n$ as

$$\tilde{e}_{i,j} := G^{-\nu_j(F,i)} \circ \tilde{e} \circ G^{\nu_j(F,i)},$$

where $i \in \{1, \ldots, 2n\}$ and $j \in \{1, \ldots, \mathbf{k}\}$.

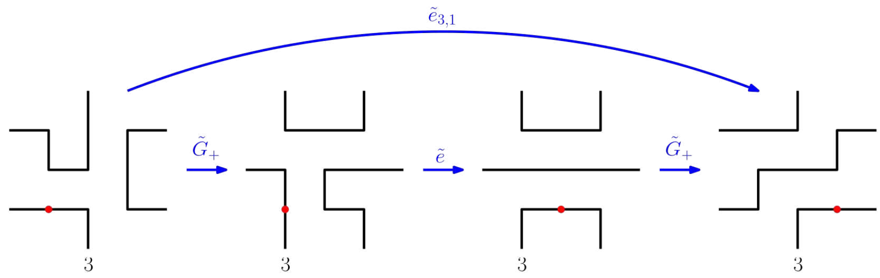


Figure 33: Example of the quasi TL-operator $\tilde{e}_{3,1}$ applied to an FPL of size 3. After one application of $\tilde{G}_+$ label 3 coincides with the refinement position. Therefore we can then already apply $\tilde{e}$ connecting the labels 3 and 4 with each other. Finally we rotate back with one application of $\tilde{G}_+$.

Lemma 4.13. *The below diagram commutes.*

$$\begin{tikzcd}
\mathcal{FPL}_n \arrow[r, "\tilde{e}_{i,j}"] \arrow[d, "\pi(\cdot)"'] & \mathcal{FPL}_n \arrow[d, "\pi(\cdot)"] \\
\mathcal{NC}_n \arrow[r, "e_i"'] & \mathcal{NC}_n
\end{tikzcd}$$

Proof. Let $F' \in \mathcal{FPL}_n$ with $\pi(F') = \pi'$ and $e_i(\pi') = \pi$. What happens, when we apply $\tilde{e}_{i,j}$ to F' for any j? The first part of the operator is $G^{\nu_j(F',i)}$ and does not change the noncrossing matching besides rotating it due to Lemma 4.5. Then $\tilde{e}$ connects label i with $i+1$ and the two labels that were connected to them with each other, unless $G^{\nu_j(F',i)}(F')$ has configuration d. In that case $\tilde{e}$ does nothing, as i and $i+1$ where already connected with each other. In that case $e_i(F') = F'$. Now $G^{-\nu_j(F',i)}$ again does not change the noncrossing matching besides rotating it back.
Therefore $\pi(\tilde{e}_{i,j}) = \pi$ and in terms of the induced noncrossing matching $\tilde{e}_{i,j}$ behaves as e_i does. Hence

$$e_i(\pi(F')) = \pi(\tilde{e}_{i,j}(F'))$$

and the diagram commutes. $\square$

However there is still an important difference between e_i and $\tilde{e}_{i,j}$. If i and $i+1$ are already connected in π, then $e_i(\pi) = \pi$, but if i and $i+1$ are connected in F, $\tilde{e}_{i,j}$ doesn't have to be the identity, as $\tilde{e}$ might change F after the rotations as displayed in Figure 34. Note, that in this setting $\tilde{e}_{i,j} = \mathrm{id}$, only if we have configuration d after $\nu_j(F,i)$ rotations. That is because $G^{\nu_j(F,i)}$ ensures, that r is vertical, since otherwise it could not coincide with any label. This means we have the situation explained in Remark 4.12. Hence we have an inverse function $\tilde{e}_{i,j}^{-1}$.

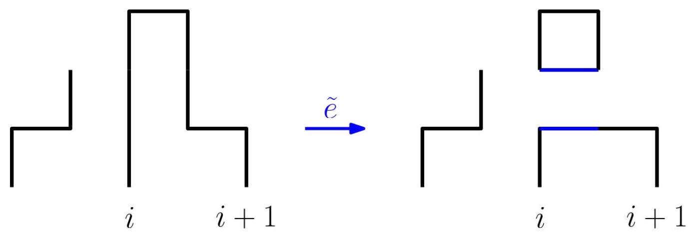


Figure 34: Example of rp and the configuration around it for an FPL F changed by $\tilde{e}$, although i and $i+1$ were already connected in F

Definition. Fix an FPL F of size n and look at its orbit

$$O_F := \{F, G(F), \dots, G^{4n\mathbf{k}-1}\}.$$

We write

$$H_F := \{F' \in O_F : F' \text{ is of type h}\}$$

and define M_F, A_F and D_F analogously.

Proposition 4.14. *Let $F \in \mathcal{FPL}_n$. Then*

$$|\{(\tilde{e}_{i,j}, F') : \tilde{e}_{i,j}(F') = F, F' \in \mathcal{FPL}_n\}| = 2n \cdot \boldsymbol{k}.$$

The proof shows, how to construct the inverse function $\tilde{e}_{i,j}^{-1}$ for i and j fixed.

Proof. As $\tilde{e}_{i,j}$ is only defined for $j \in \{1, \dots, \mathbf{k}\}$ and $G^{4n\mathbf{k}}(F) = F$, we only need to look at $\{G^l(F) : l \in \{0, \dots, 4n\mathbf{k}-1\}\}$.
Define

$$H := |H_F|, \quad M := |M_F|, \quad A := |A_F|, \quad D := |D_F|,$$

we then have

$$H = \#\text{ leftsteps of rp in } O_F, \quad A = \#\text{ rightsteps of rp in } O_F$$

while M gives the number of changes from horizontal to vertical rp and D the number of changes from vertical to horizontal rp. Since $G^{4n\mathbf{k}}(F) = F$ we get $r_{4n\mathbf{k}} = r_0$. In order for rp to be at the same position after an even number of iterations of G, it must have made the same amount of steps to the left as steps to the right and also has to have changed from vertical to horizontal as often as from horizontal to vertical. Hence we have

$$H = A \quad \text{and} \quad M = D$$

Since $4n\mathbf{k} = H + M + A + D$ we get

$$2n\mathbf{k} = H + D.$$

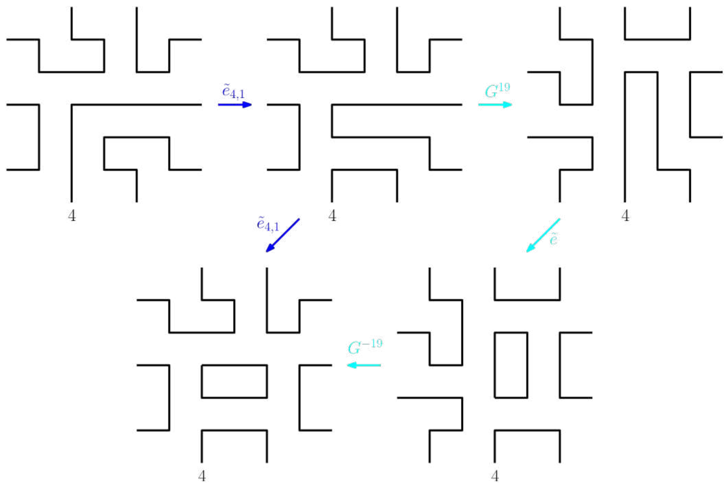
Remember that we can only find an explicit preimage of an FPL F regarding $\tilde{e}$, if F has configuration h or d. Therefore we get preimages of an FPL F regarding $\tilde{e}_{i,j}$, by applying $\tilde{e}$ at situations where $G^l(F)$ has configuration h or d for $l \in \{0, \ldots, 4n\mathbf{k} - 1\}$ and then applying G^{-l}. This gives us

$$|\{(\tilde{e}_{i,j}, F') : \tilde{e}_{i,j}(F') = F, F' \in \mathcal{FPL}_n\}| = H + D = 2n \cdot \mathbf{k}.$$

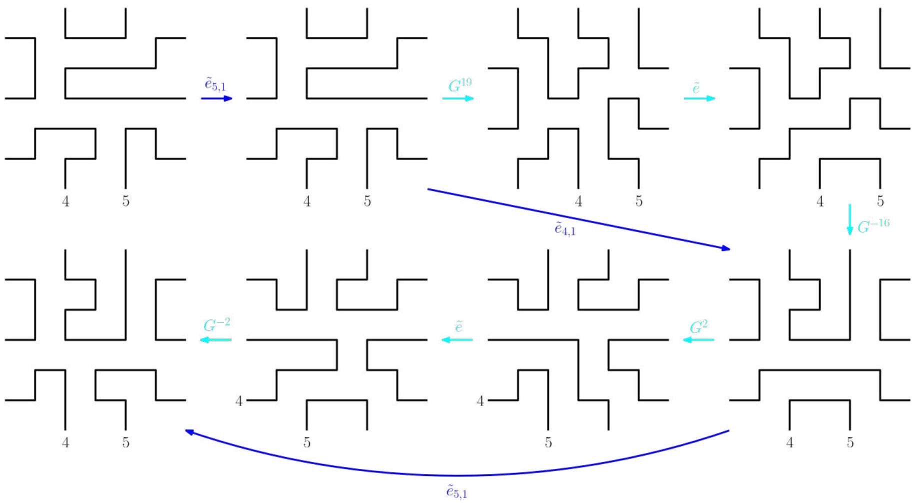
$\square$

In Lemma 3.4 we stated 4 identities for the Temperley-Lieb operators. For the quasi TL-operators none of these identities hold in general, as the upcoming counterexamples show.

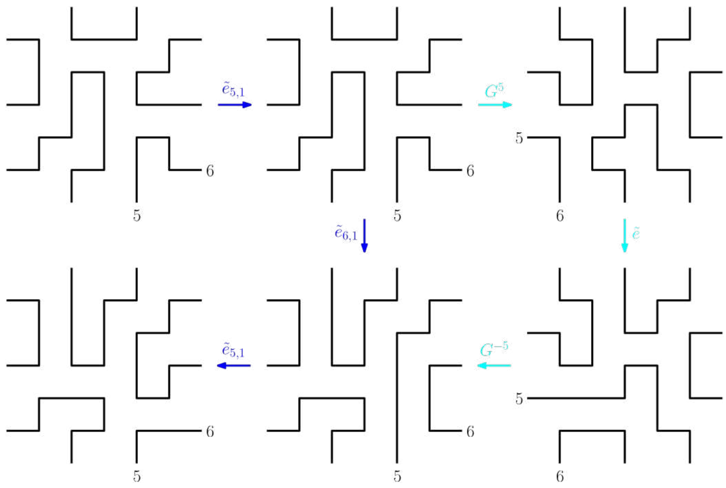
Example 4.15. *The first example shows, that $(\tilde{e}_{i,j})^2 = \tilde{e}_{i,j}$ does not hold in general. The middle FPL in the top row is gained after applying $\tilde{e}_{4,1}$ once. The left FPL on the bottom row is gained after a second application of $\tilde{e}_{4,1}$. Since those two differ, the identity does not hold in general.*



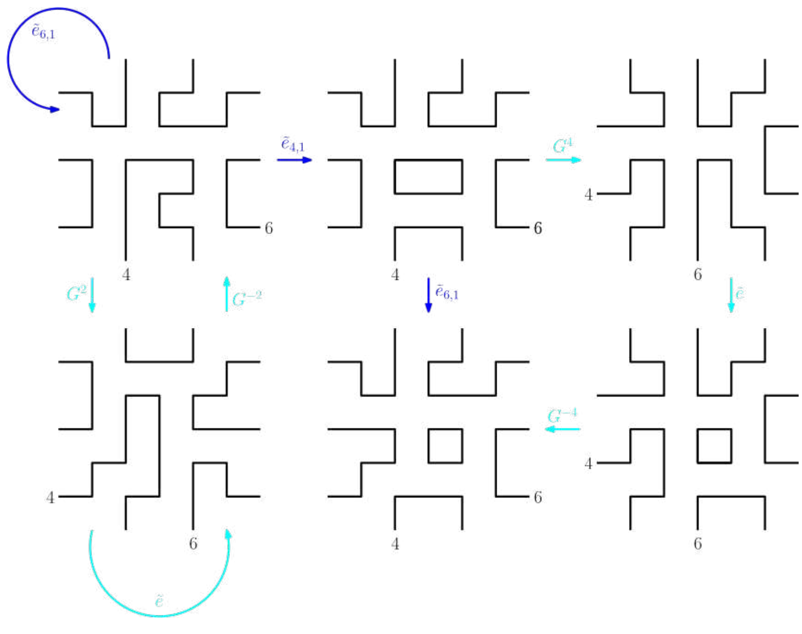
The next figure gives a counterexample for the identity $\tilde{e}_{i,j} \circ \tilde{e}_{i+1,j} \circ \tilde{e}_{i,j} = \tilde{e}_{i,j}$. For this FPL $\tilde{e}_{5,1}$ serves as the identity, since the FPL already has rp at label 5 with configuration d. Then after applying $\tilde{e}_{5,1} \circ \tilde{e}_{4,1}$, we get the leftmost FPL of the bottom row. As those FPLs do not coincide, we found a counterexample.



The upcoming example shows, that $\tilde{e}_{i,j} \circ \tilde{e}_{i-1,j} \circ \tilde{e}_{i,j} = \tilde{e}_{i,j}$ does not hold in general. Once again $\tilde{e}_{5,1}$ serves as the identity, as our initial FPL has rp at label 5 with configuration d. Then after applying $\tilde{e}_{6,1}$ and $\tilde{e}_{5,1}$, we arrive at the leftmost FPL on the bottom row, which differs from our initial FPL, proving the identity does not hold in general.



The last figure gives a counterexample for the identity $\tilde{e}_{l,j} \circ \tilde{e}_{i,j} = \tilde{e}_{i,j} \circ \tilde{e}_{l,j}$ for $|i-l| > 2$. Applying $\tilde{e}_{6,1}$ to the initial FPL dos not change it, because after applying G^2, rp coincides with label 6 with configuration d. Then after applying $\tilde{e}_{4,1}$ we land at the middle FPL on the top row. If we however apply $\tilde{e}_{4,1}$ first and $\tilde{e}_{6,1}$ afterwards, we arrive at the middle FPL on the bottom row. As these two are not the same, we found a counterexample.



5 Proof of the Razumov Stroganov Correspondence

Now we got all the tools we need, to formulate our proof of the Razumov Stroganov correspondence.
Since (3.5) gives the probability for a randomly generated cylinder to induce π, showing that for randomly generated FPLs F, F'

$$2n \cdot \mathbb{P}(F \ \textit{induces} \ \pi) = \sum_{i=1}^{2n} \sum_{\substack{\pi' \in \mathcal{NC}_n \\ e_i(\pi')=\pi}} \mathbb{P}(F' \ \textit{induces} \ \pi')$$

holds for FPLs, proves Theorem 1.1. For FPLs we can just rewrite $\mathbb{P}(F \ \textit{induces} \ \pi)$ by $\frac{A_\pi}{A_n}$ to formulate the next theorem.

Theorem 5.1. *For any $n \in \mathbb{N}$*

$$2n \cdot \frac{A_\pi}{A_n} = \sum_{i=1}^{2n} \sum_{\substack{\pi' \in \mathcal{NC}_n \\ e_i(\pi')=\pi}} \frac{A_{\pi'}}{A_n}$$

holds.

Proof. At first we can multiply by A_n to get

$$2n \cdot A_\pi = \sum_{i=1}^{2n} \sum_{\substack{\pi' \in \mathcal{NC}_n \\ e_i(\pi')=\pi}} A_{\pi'},$$

which is the equation, we want to show.

We will prove that equation by double counting. We fix $\pi \in \mathcal{NC}_n$. The set whose elements we want to count is

$$\{(\tilde{e}_{i,j}, F') : e_i(\pi(F')) = \pi\}. \tag{5.1}$$

The first way of counting those tuples is by starting with taking all FPLs F with $\pi(F) = \pi$. The number of these is A_π. For each F with this property, we get $2n\mathbf{k}$ tuples $(\tilde{e}_{i,j}, F')$ such that $\tilde{e}_{i,j}(F') = F$, as Proposition 4.14 taught us. We can look at $\tilde{e}_{i,j}(F') = F$ instead of $e_i(\pi(F')) = \pi$ due to Lemma 4.13. Therefore the amount of all tuples is given by

$$2n\mathbf{k} \cdot A_\pi.$$

Our second way of counting these tuples starts by taking all pairs (e_i, F') of Temperley-Lieb operators and FPLs such that $e_i(\pi(F')) = \pi$. The number of these pairs is given by

$$\sum_{i=1}^{2n} \sum_{\substack{\pi' \in \mathcal{NC}_n \\ e_i(\pi')=\pi}} A_{\pi'}.$$

Now for each of these pairs (e_i, F') we get $\mathbf{k}$ tuples $(\tilde{e}_{i,j}, F')$, as j ranges from 1 to $\mathbf{k}$, giving us the total amount of tuples

$$\mathbf{k} \cdot \sum_{i=1}^{2n} \sum_{\substack{\pi' \in \mathcal{NC}_n \\ e_i(\pi')=\pi}} A_{\pi'}.$$

These two ways of counting the elements of (5.1) gives us the equation

$$2n\mathbf{k} \cdot A_\pi = \mathbf{k} \cdot \sum_{i=1}^{2n} \sum_{\substack{\pi' \in \mathcal{NC}_n \\ e_i(\pi')=\pi}} A_{\pi'}.$$

Now by dividing both sides by $\mathbf{k}$, we arrive at the wanted equation. $\square$

References

[1] L. Cantini and A. Sportiello. Proof of the razumov-stroganov conjecture. *J. Combin. Theory Ser. A 118*, pages 1549–1574, 2011.

[2] L. Cantini and A. Sportiello. A one-parameter refinement of the razumov–stroganov correspondence. *J. Combin. Theory Ser. A 127*, pages 400–440, 2014.

[3] D. A. Levin, Y. Peres, and E. L. Wilmer. *Markov Chains and Mixing Times.* American Mathematical Society, Providence, 2017.

[4] Y. G. Razumov, A. V. ; Stroganov. Combinatorial nature of the ground-state vector of the o(1) loop model. *Theoret. Math. Phys., 138(3): 333–337*, pages 333–337, 2004.

[5] D. Romik. Connectivity Patterns in Loop Percolation I: the Rationality Phenomenon and Constant Term Identities. *Commun. Math. Phys. 330, 499–538*, pages 499–538, 2014.

[6] B. Wieland. A large dihedral symmetry of the set of alternating sign matrices. *Electron. J. Combin., 7:R37, 13pp.*, 2000.