MASTERARBEIT | MASTER'S THESIS

Titel | Title

Realizing asymptotic couples in Hardy fields

verfasst von | submitted by

Clemens Kinn BSc

angestrebter akademischer Grad | in partial fulfilment of the requirements for the degree of

Master of Science (MSc)

Wien | Vienna, 2025

Studienkennzahl lt. Studienblatt | Degree programme code as it appears on the student record sheet:	UA 066 821
Studienrichtung lt. Studienblatt | Degree programme as it appears on the student record sheet:	Masterstudium Mathematik
Betreut von | Supervisor:	Univ.-Prof. Matthias Aschenbrenner PhD

Abstract

The set of germs of differentiable real-valued functions forms a ring with the induced addition and multiplication of functions. Hardy fields are subfields of this ring which are closed under taking derivatives and come with a natural ordering. Moreover, every element of a Hardy field has a limit in the extended real line. To compare different sizes of asymptotic growth of germs we equip Hardy fields with the standard valuation. One can associate to each Hardy field its asymptotic couple, which is a pair consisting of the value group of the Hardy field together with a map induced by the logarithmic derivative. These objects can also be defined purely algebraically, and this raises the question which asymptotic couples come from Hardy fields. Rosenlicht gave a partial explicit answer to a relative question: when does an extension of an asymptotic couple of a Hardy field also come from a corresponding Hardy field extension? He showed that this is possible under the assumption that the value groups are of finite rank. We give a self-contained treatment of this result and construct Hardy fields for asymptotic couples of Hardy type with small derivation which are of countable rank.

Abriss

Die Menge der Keime differenzierbarer reellwertiger Funktionen bildet einen Ring mit der punktweisen Addition und Multiplikation. Hardykörper sind Teilkörper dieses Rings, die unter Ableitungen abgeschlossen sind und eine natürliche Ordnung besitzen. Jeder Keim eines Hardykörpers hat einen eigentlichen oder uneigentlichen Grenzwert und um unterschiedliche Größenordnungen des asymptotischen Wachstums von Keimen zu vergleichen, werden Hardykörper mit der durch die Ordnung entstehenden natürlichen Bewertung ausgestattet. Jedem Hardykörper kann sein asymptotisches Paar zugeordnet werden, das aus der Wertegruppe des Hardykörpers zusammen mit einer durch die logarithmische Ableitung induzierten Abbildung besteht. Diese Objekte können auch rein algebraisch definiert werden, was die Frage aufwirft, welche asymptotischen Paare von Hardykörpern stammen. Rosenlicht gab eine teilweise explizite Antwort auf die Frage, unter welchen Bedingungen eine Erweiterung eines asymptotischen Paares eines Hardykörpers ebenfalls von einer Hardykörper Erweiterung stammt. Er zeigte, dass dies unter der Annahme möglich ist, dass die Wertegruppen von endlichem Rang sind. Diese Arbeit enthält eine ausführliche Erläuterung dieses Resultats und zeigt, dass asymptotische Paare von abzählbarem Rang durch Hardykörper realisiert werden.

Contents

0.1 Motivation and Introduction

The theory of Hardy fields originates in the early 20th century from the work of G.H. Hardy, who formalized the comparison of real valued functions based on their growth rates at infinity. His book *Orders of Infinity* [11] built upon the earlier *Infinitärcalcül* of Paul du Bois-Reymond, who first introduced a rigorous method to classify functions by their asymptotic behavior. Hardy's work led to the consideration of fields of germs of real valued functions at infinity closed under differentiation. These objects are now known as Hardy fields, which were introduced by Bourbaki. The structure of Hardy fields was further developed by Boshernitzan, Bourbaki, Rosenlicht and many more (for literature see [7]). Together, these developments shaped Hardy fields into a natural setting to analyze the asymptotics of real valued functions using differential algebra, valuation theory and model theory. Nowadays, Hardy fields also occur via definable functions in o-minimal structures, and in the study of transseries.

To develop the modern framework, we consider two pillars independently: the algebraic notion of asymptotic couples and the analytic notion of Hardy fields. Ordered abelian groups play an important role in both approaches which is why the beginning of Chapter 1 is dedicated to them. They are the fundamental building blocks for asymptotic structures in valuation theory. To connect the two pillars, we prove Rosenlicht's finite rank extension theorem for asymptotic couples of Hardy fields [12]. A natural question to ask is whether this result generalizes to countable rank and which extension problems can be solved. As shown in the study of simple extensions of H-couples [5], it is not uncommon to encounter infinite rank extensions when new elements are adjoined to an asymptotic couple. Also, closed asymptotic couples are not of finite rank and arise naturally from Hardy fields that are Liouville closed. Such extensions are beyond the scope of Rosenlicht's original theorem. Although our primary objective is to analyze the asymptotic behavior of real valued functions, our approach requires minimal reliance on classical analysis. Instead, the tools we employ belong to differential algebra, valuation theory, and the study of ordered algebraic structures developed in [4].

Only a basic understanding of analysis and algebra is required. All necessary results are either proven within the thesis or cited when their proofs are too involved, ensuring accessibility for readers with a standard mathematical background. Most of the notions come from [4].

The first two chapters of the thesis discuss ordered abelian groups, asymptotic couples and Hardy fields, aiming towards proving the extension theorem of Rosenlicht which is done in Chapter 3. Chapter 4 starts with an absolute version of realizing countable rank asymptotic couples in Hardy fields and to conclude, we state some generalizations and obstructions of Rosenlicht's result forming the first step to a relative extension theorem for asymptotic couples of countable rank.

I want to thank Matthias Aschenbrenner for supervising this thesis and Allen Gehret as well as Benjamin Riff for helpful discussions.

The set of germs $\mathcal{C}^{<\infty}$ of functions $f : \mathbb{R} \to \mathbb{R}$ such that for each $n \geq 1$, $f|_{(t_n,\infty)}$ is n times differentiable for some $t_n \in \mathbb{R}$ forms a differential ring with the induced addition, multiplication and derivation of functions. A Hardy field K is a differential subfield of $\mathcal{C}^{<\infty}$ and comes with a natural ordering given by

$$f > 0 :\iff \exists t_0 \in \mathbb{R} : \forall t > t_0 : f(t) > 0.$$

Hence every element of a Hardy field has a limit in the extended real line and does not oscillate infinity often. We denote the real closure of K by K^{rc}, which is also a Hardy field (Theorem 2.3.5). To compare different sizes of asymptotic growth of germs we equip Hardy fields with the standard valuation $v : K^\times \to \Gamma, f \mapsto [f]$ where $\Gamma := [K^\times]$ is equipped with the 'reverse' ordering

$$[f] \leq [g] :\iff |g| \leq n\,|f| \text{ for some } n \geq 1.$$

Here,

$$[f] := \{g \in K : |g| \leq n\,|f| \wedge |f| \leq n\,|g| \text{ for some } n \geq 1\}$$

is the archimedean class of f and $[K^\times]$ is the set of archimedean classes of $K^\times$. A similar definition also gives rise to archimedean classes of Γ and the rank of Γ is defined to be the cardinality of non-zero archimedean classes of Γ. One can associate to each Hardy field its asymptotic couple (Γ, ψ) where $\psi : \Gamma^{\neq} \to \Gamma, v(f) \mapsto v(f'/f)$ is a well defined map (Corollary 2.2.2). Such an asymptotic couple of a Hardy field has the following properties (see definition below):

- ψ is a convex valuation on Γ, so (Γ, ψ) is an asymptotic couple of H-type;
- Γ is of size at most continuum;
- (Γ, ψ) is of Hardy type, so ψ is equivalent to the standard valuation on Γ;
- (Γ, ψ) has small derivation.

Hence any asymptotic couple coming from a Hardy field is actually an H-asymptotic couple of Hardy type with small derivation. In general, asymptotic couples can also be defined purely algebraically without any connection to Hardy fields:

Definition. Let Γ be an abelian group equipped with a total ordering $\leq$ such that for all $\alpha, \beta, \gamma \in \Gamma$:

$$\alpha \leq \beta \implies \alpha + \gamma \leq \beta + \gamma,$$

and $\psi : \Gamma^{\neq} \to \Gamma$ be a map such that for all $\alpha, \beta \in \Gamma$, $\alpha, \beta \neq 0$:

(i) $\alpha + \beta \neq 0 \implies \psi(\alpha + \beta) \geq \min\{\psi(\alpha), \psi(\beta)\}$;

(ii) $k \in \mathbb{Z}^{\neq} \implies \psi(k\alpha) = \psi(\alpha)$;

(iii) $\alpha > 0 \implies \psi(\beta) < \alpha + \psi(\alpha)$.

Then (Γ, ψ) is called an *asymptotic couple*. If ψ satisfies

$$0 < \alpha \leq \beta \implies \psi(\alpha) \geq \psi(\beta)$$

for all $\alpha, \beta \neq 0$, then (Γ, ψ) is of *H-type* and if in addition

$$[\alpha] \neq [\beta] \implies \psi(\alpha) \neq \psi(\beta)$$

for all $\alpha, \beta \neq 0$, then (Γ, ψ) is of *Hardy type.* If $(\mathrm{id} + \psi)(\Gamma^{>}) \subseteq \Gamma^{>}$, then (Γ, ψ) has *small derivation.*

This raises the question which asymptotic couples come from Hardy fields and whether the list above are sufficient conditions. In particular we are interested in the following realization and extension problem. *We always assume $\mathbb{R}$ is contained in any Hardy field when considering the following questions* which can be achieved by passing to a Hardy field extension.

$$\begin{tikzcd} K \arrow[d, dashed, leftrightarrow] \\ (\Gamma, \psi) \end{tikzcd} \qquad \begin{tikzcd} K_0 \arrow[r, dashed, "\subseteq"] \arrow[d, leftrightarrow] & K \arrow[d, dashed, leftrightarrow] \\ (\Gamma_0, \psi_0) \arrow[r, "\subseteq"] & (\Gamma, \psi) \end{tikzcd}$$

(1) Realization problem (left): For a given asymptotic couple (Γ, ψ), is there a Hardy field K with asymptotic couple isomorphic to (Γ, ψ)?

(2) Extension problem (right): Given a Hardy field K_0 with asymptotic couple (Γ_0, ψ_0) and an asymptotic couple extension (Γ, ψ) of (Γ_0, ψ_0), is there a Hardy field extension K of K_0 realizing (Γ, ψ) as its asymptotic couple?

Example. Let (Γ, ψ) be the divisible hull of (Γ_0, ψ_0). Then the extension problem is solvable by taking $K = K_0^{rc}$.

Rosenlicht [12] showed that the extension problem is always solvable under the necessary assumptions above and the additional property that Γ is of finite archimedean rank in which case $\psi(\Gamma^{\neq})$ is finite. We give a self-contained treatment of this result in Chapter 3 after developing the tools needed (Chapters 1 and 2). In Chapter 4 we partially generalize Rosenlicht's theorem: the realization problem is solvable under the necessary assumptions above where additionally Γ_0 is of countable archimedean rank as well as divisible. It is time to state the main results of the thesis. In Theorem 3.0.1 (Rosenlicht's finite rank extension) there is more information about how to obtain such a Hardy field K of the extension problem.

Theorem 0.1.1. *There is a chain of Hardy field extensions $K_0 \subseteq K_1 \subseteq \cdots \subseteq K_{2n+1} = K$ of K_0 where $n := \operatorname{rank} \Gamma - \operatorname{rank} \Gamma_0$, $\Gamma_j := v(K_j^{\times})$ and $n_j := \operatorname{rank} \Gamma_j$ such that:*

(i) If j is even, then $K_{j+1} \subseteq K_j(\{y_i^s\}_{(i,s) \in A})^{rc}$ for some $A \subseteq [n_j] \times \mathbb{R}$ and $y_1, \ldots, y_{n_j} \in K_j^{>}$ such that $v(y_1), \ldots, v(y_{n_j}) > 0$ and $[v(y_1)] > \cdots > [v(y_{n_j})] > 0$. In this case, $\operatorname{rank} \Gamma_{j+1} = \operatorname{rank} \Gamma_j$;

(ii) If j is odd, then $K_{j+1} = K_j(\exp u)$ or $K_{j+1} = K_j(\log u)$ for some $u \in K_j^>$ such that $v(u) < 0$. In this case, $\operatorname{rank}\Gamma_{j+1} = 1 + \operatorname{rank}\Gamma_j$.

In the countable rank case, we have the following positive answer to the realization problem (Corollary 4.2.6).

Theorem 0.1.2. *Every H-couple over $\mathbb{Q}$ of Hardy type with small derivation which has countable rank is realized by a Hardy field.*

In the end, we conjecture the following case of the extension problem for (Γ, ψ) of countable rank.

Conjecture 0.1.3. *Let (Γ, ψ) be an H-couple over $\mathbb{Q}$ of Hardy type of countable rank with asymptotic integration and small derivation, and (Γ_0, ψ_0) a subcouple of (Γ, ψ) over $\mathbb{Q}$ with no gap. Suppose K_0 is a Hardy field with H-couple (Γ_0, ψ_0) over $\mathbb{Q}$. Then there is a Hardy field extension K of K_0 with H-couple over $\mathbb{Q}$ isomorphic to (Γ, ψ) over (Γ_0, ψ_0).*

A problem which does not occur in the finite rank case is the existence of gaps that might prohibit the solvability of the extension problem. A gap in an H-couple is responsible for having two canonical different non-isomorphic H-asymptotic couple extensions only one of which can be realized by a Hardy field extension. We hope to bypass this problem by assuming no gap in (Γ_0, ψ_0) and (Γ, ψ), but on the way, gaps can still be created and need to be dealt with. This is also one reason why we assume that $\mathbb{R} \subseteq K_0$ to avoid dealing with pre-H-fields.

1 Ordered abelian groups and asymptotic couples

This chapter is devoted to the fundamentals of ordered abelian groups and asymptotic couples. We analyze convex subgroups and convex valuations on ordered abelian groups including the standard valuation using the notion of archimedean classes. Furthermore, we consider quotients of ordered abelian groups by convex subgroups and give an explicit description of ordered abelian groups of finite rank $n \in \mathbb{N}$ as subgroups of the lexicographically ordered $\mathbb{R}^n$ (Hölder). After introducing H-couples and establishing basic properties, it is observed that quotients can also be equipped with a unique H-couple structure. We end this chapter with describing an analogue of Hölder's theorem for H-couples of finite rank.

1.1 Ordered abelian groups

In this section, we give a basic introduction to some algebraic concepts in the presence of an ordering. Our treatment and many notations are based on [4]. Elements from abelian groups are usually denoted by greek letters $\alpha, \beta, \gamma, \delta$ whereas m and n range over natural numbers $\mathbb{N} := \{0, 1, \ldots\}$. Orderings on sets are always considered to be total unless specified otherwise. We start with developing the basic theory of ordered abelian groups.

Definition. Let Γ be an abelian group (written additively) together with a total ordering $\leq$ on Γ such that for all α, β, γ,

$$\alpha \leq \beta \implies \alpha + \gamma \leq \beta + \gamma.$$

Then Γ is called an *ordered abelian group.*

Note that any subgroup Γ_0 of an ordered abelian group Γ is an ordered abelian group with the induced ordering of Γ, called an *ordered subgroup* of Γ. In this case, we say that Γ is an *ordered abelian group extension* of Γ_0.

Example. We equip the additive abelian groups $\mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ with the usual orderings making them ordered abelian groups. More generally, we consider $\mathbb{R}^n$ as an ordered abelian group equipped with componentwise addition and the *lexicographic ordering*

$$(a_1, \ldots, a_n) < (b_1, \ldots, b_n) \quad :\Longleftrightarrow \quad a_1 = b_1, \ldots, a_m = b_m, a_{m+1} < b_{m+1} \text{ for some } m.$$

By definition, $\mathbb{R}^0$ is a singleton set.

Example. Let Γ be an ordered abelian group and Γ_0 an ordered subgroup of Γ. For any $A \subseteq \Gamma$, the intersection of all ordered subgroups of Γ containing $\Gamma_0 \cup A$ is the smallest ordered subgroup of Γ containing $\Gamma_0 \cup A$, called the *ordered subgroup of* Γ *generated by* A over Γ_0.

Definition. Let Γ_1, Γ_2 be ordered abelian groups. A *morphism of ordered abelian groups* $h : \Gamma_1 \to \Gamma_2$ is a group morphism such that for all α, β,

$$\alpha \leq \beta \implies h(\alpha) \leq h(\beta).$$

We call h an *embedding (isomorphism) of ordered abelian groups* if additionally h is injective (bijective) respectively.

Suppose Γ_0 is a subgroup of both Γ_1 and Γ_2. Then we say that an embedding h of ordered abelian groups is *over* Γ_0 if $h(\alpha) = \alpha$ for all $\alpha \in \Gamma_0$.

Example. Let $\Gamma_1, \Gamma_2, \Gamma_3$ be ordered abelian groups and $h_i : \Gamma_i \to \Gamma_{i+1}$ be morphisms of ordered abelian groups for $i = 1, 2$. Then $h_2 \circ h_1 : \Gamma_1 \to \Gamma_3$ is a morphism of ordered abelian groups.

Let Γ *be an ordered abelian group.* We use the following convenient notations:

- $\Gamma^{\neq} := \Gamma \setminus \{0\}$;
- $\Gamma^{>} := \{\alpha \in \Gamma : \alpha > 0\}$ and $\Gamma^{\geq} := \{\alpha \in \Gamma : \alpha \geq 0\}$;
- $\Gamma^{<} := \{\alpha \in \Gamma : \alpha < 0\}$ and $\Gamma^{\leq} := \{\alpha \in \Gamma : \alpha \leq 0\}$;
- $|\alpha| := \max\{\alpha, -\alpha\}$.

Occasionally, we also use $\Gamma^{>\beta}, \Gamma^{<\beta}, \Gamma^{\geq\beta}, \Gamma^{\leq\beta}$ with a similar meaning, replacing 0 by β in the above convention.

If A and B are subsets of some totally ordered set S, write $A < B$ if for all $a \in A$ and $b \in B$ there holds $a < b$. As usual, if $A = \{a\}$, write $a < B$, and similarly if B is a singleton. Call A *downward closed* in S if for every $s \in S$ and $a \in A$, if $s \leq a$ then $s \in A$. Define the *downward closure* of A in S as $A^{\downarrow} := \{s \in S : s \leq a \text{ for some } a \in A\}$. Also, B is called *upward closed* in S if $S \setminus B$ is downward closed. For a downward closed set B in S we have $B < S \setminus B$, so B determines a *cut* in S. If a in a totally ordered set containing S satisfies $B < a < S \setminus B$, then a is said to *realize the cut* B in S.

1.1.1 Divisible hulls of ordered abelian groups

We recall a property that an ordered abelian group might have.

Definition. Call Γ *divisible* if for all $n \neq 0$ and α, there is some β with $n\beta = \alpha$.

Since $0 < |\alpha|$ implies $0 < n\,|\alpha|$ for every $\alpha \neq 0$, Γ is torsion free, but Γ might not be divisible. Since Γ is a $\mathbb{Z}$-module, we can define the *divisible hull*

$$\mathbb{Q}\Gamma := \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma = \left\{ \frac{1}{n}\alpha : \quad n \geq 1, \alpha \in \Gamma \right\}$$

of Γ viewing $\mathbb{Q}\Gamma$ as a divisible abelian group. Note that $\mathbb{Q}\Gamma$ is a $\mathbb{Q}$-vector space in a natural way. There is a unique ordering of $\mathbb{Q}\Gamma$ that makes the inclusion $\Gamma \to \mathbb{Q}\Gamma$ with $\gamma \mapsto 1 \otimes \gamma$ an ordered abelian group embedding, hence we can identify Γ as an ordered subgroup of $\mathbb{Q}\Gamma$ via the image of this map. In this spirit, $\mathbb{Q}\Gamma$ is the smallest divisible ordered abelian group extension of Γ such that any ordered abelian group embedding of Γ into some divisible ordered abelian group extends to $\mathbb{Q}\Gamma$.

Here is our first extension result for ordered abelian groups.

Lemma 1.1.1. *Suppose Γ is divisible and let Γ_1 and Γ_2 be divisible ordered abelian group extensions of Γ. If $\alpha \in \Gamma_1$ realizes the same cut D in Γ as $\beta \in \Gamma_2$, then there is an isomorphism of ordered abelian groups $\Gamma \oplus \mathbb{Q}\alpha \cong \Gamma \oplus \mathbb{Q}\beta$ over Γ sending α to β.*

Proof. As Γ is divisible we have $q\alpha, q\beta \notin \Gamma$ for all $q \in \mathbb{Q}^{\neq}$. Hence $\Gamma + \mathbb{Q}\alpha = \Gamma \oplus \mathbb{Q}\alpha$ and the same holds for β. Thus we have an isomorphism of groups $\Gamma \oplus \mathbb{Q}\alpha \to \Gamma \oplus \mathbb{Q}\beta$ sending α to β and it remains to show that this map is order preserving.

Claim. For all γ and $q \in \mathbb{Q}$ we have $\quad \gamma + k\alpha >_{\Gamma_1} 0 \quad \Longleftrightarrow \quad \gamma + k\beta >_{\Gamma_2} 0.$

For $q = 0$ this is trivial, so let $q \neq 0$. We distinguish two cases. If $q > 0$, then

$$\gamma + q\alpha > 0 \iff \alpha > -\gamma/q \iff -\gamma/q \in D \iff \beta > -\gamma/q \iff \gamma + k\beta > 0,$$

and if $q < 0$, then

$$\gamma + q\alpha > 0 \iff \alpha < -\gamma/q \iff -\gamma/q \notin D \iff \beta < -\gamma/q \iff \gamma + k\beta > 0,$$

using that α and β realize the same cut in Γ. $\square$

1.1.2 Archimedean rank in ordered abelian groups

We introduce an equivalence relation on Γ by setting

$$\alpha \sim \beta \quad :\Longleftrightarrow \quad |\alpha| \leq n\,|\beta| \wedge |\beta| \leq n\,|\alpha| \text{ for some } n \geq 1.$$

The equivalence class of α with respect to $\sim$ is called the *archimedean class* of α, denoted by $[\alpha]$. The set $[\Gamma]$ of archimedean classes of Γ becomes an ordered set with the ordering

$$[\alpha] < [\beta] \quad :\Longleftrightarrow \quad n\,|\alpha| < |\beta| \text{ for all } n,$$

which makes the map $\Gamma \to [\Gamma]$, $\alpha \mapsto [\alpha]$ order preserving. Note that then $[\alpha] \geq [\beta]$ holds iff there is some $n \geq 1$ such that $|\beta| \leq n\,|\alpha|$. We have $[0] = \{0\}$, which is the smallest archimedean class.

Lemma 1.1.2. *The following properties hold for all α:*

(i) $0 < \alpha \leq \beta \implies [\alpha] \leq [\beta]$;

(ii) $k \in \mathbb{Z}^{\neq} \implies [k\alpha] = [\alpha]$;

(iii) $[\alpha_1 + \cdots + \alpha_n] \leq \max\{[\alpha_1], \ldots, [\alpha_n]\}$ *with equality if* $[\alpha_i] > [\alpha_j]$ *for some* i *and all* $j \neq i$.

Proof. By definition, (i) holds and $|\alpha| \leq |k\alpha| = |k|\,|\alpha|$ gives (ii). For (iii), observe that

$$|\alpha_1 + \cdots + \alpha_n| \leq |\alpha_1| + \cdots + |\alpha_n| \leq n \max\{|\alpha_1|, \ldots, |\alpha_n|\}. \tag{1.1.1}$$

Suppose $[\alpha_i] > [\alpha_j]$ for all $j \neq i$ and let $\alpha := \alpha_1 + \cdots + \alpha_n$ and $\gamma := \alpha - \alpha_i$. Then $2\,|\gamma| \leq 2n \max_{j \neq i}\{|\alpha_j|\} < |\alpha_i|$ by (1.1.1). So

$$|\alpha_i| < 2(|\alpha_i| - |\gamma|) \leq 2\,|\alpha_i + \gamma|$$

by the reverse triangle inequality. □

Lemma 1.1.3. *Let* $h : \Gamma_1 \to \Gamma_2$ *be an embedding of ordered abelian groups. Then* $[\alpha] \leq [\beta]$ *if and only if* $[h(\alpha)] \leq [h(\beta)]$. *In particular,* h *preserves archimedean classes.*

Definition. We define the *archimedean rank* (or just *rank*) of Γ to be the cardinality of the ordered set $[\Gamma^{\neq}] := \{[\alpha] : \alpha \in \Gamma^{\neq}\}$. The rank of Γ is denoted by $\operatorname{rank}\Gamma$ and if $\operatorname{rank}\Gamma = 1$, then Γ is called *archimedean*.

The following definition is inspired by valuation bases for ordered valued vector spaces ([4], Section 2.3).

Definition. Let I be an index set and $(\alpha_i)_{i \in I} \in \Gamma^{>}$ be a sequence of representatives of $[\Gamma^{\neq}]$ such that

(i) $[\alpha_i] > [\alpha_j]$ for $i < j$;

(ii) $\bigcup_{i \in I} [\alpha_i] = \Gamma^{\neq}$.

Then $(\alpha_i)_i$ are called *archimedean representatives* of Γ.

Note that if $(\alpha_i)_{i \in I}$ are archimedean representatives of Γ, then $\alpha_i > \alpha_j > 0$ for all $i < j$, $[\Gamma^{\neq}] = \{[\alpha_i]\}_{i \in I}$, so for every $\alpha \neq 0$ there is some i such that $[\alpha] = [\alpha_i]$.

Corollary 1.1.4. *Every isomorphism of ordered abelian groups preserves archimedean representatives.*

We collect some useful facts concerning divisible hulls and archimedean classes.

Lemma 1.1.5. *We have* $[\mathbb{Q}\Gamma] = [\Gamma]$. *Hence* $\operatorname{rank}\mathbb{Q}\Gamma = \operatorname{rank}\Gamma$ *and any archimedean representatives of* Γ *are archimedean representatives of* $\mathbb{Q}\Gamma$.

Proof. If $\alpha \in \mathbb{Q}\Gamma$, then $n\alpha \in \Gamma$ for some $n \neq 0$, hence $[\alpha] = [n\alpha] \in [\Gamma]$. □

Lemma 1.1.6. *There holds* $\dim_{\mathbb{Q}} \mathbb{Q}\Gamma \geq \operatorname{rank}\Gamma$.

Proof. Take archimedean representatives $(\alpha_i)_i$ of Γ. It suffices to show that

$$\sum_{i \in I} q_i \alpha_i = 0 \implies \forall i \in I : q_i = 0$$

where $q_i \in \mathbb{Q}$, $q_i \neq 0$ only finitely often. If there is some $k \in I$ with $q_k \neq 0$, find the smallest $k \in I$ with this property. Then by Lemma 1.1.2

$$[\alpha_k] > \left[\sum_{i \neq k} q_i \alpha_i\right] = [-q_k \alpha_k] = [\alpha_k],$$

a contradiction. Hence $(\alpha_i)_i$ are $\mathbb{Q}$-linearly independent yielding $\dim_{\mathbb{Q}} \mathbb{Q}\Gamma \geq \operatorname{rank} \mathbb{Q}\Gamma = \operatorname{rank} \Gamma$ by Lemma 1.1.5. $\square$

Definition. The quantity $\operatorname{rank}_{\mathbb{Q}} \Gamma := \dim_{\mathbb{Q}} \mathbb{Q}\Gamma$ is called the *rational rank of* Γ.

Remark. Archimedean representatives $(\alpha_i)_i$ of Γ are $\mathbb{Q}$-linearly independent in $\mathbb{Q}\Gamma$ as shown in the proof of Lemma 1.1.6. In general, archimedean representatives do not form a $\mathbb{Q}$-basis for $\mathbb{Q}\Gamma$ as the example $\Gamma := \mathbb{Q} + \sqrt{2}\mathbb{Q} \subseteq \mathbb{R}$ which satisfies $\operatorname{rank} \Gamma = 1 < 2 = \dim_{\mathbb{Q}} \mathbb{Q}\Gamma$ shows.

1.1.3 Convex subgroups of ordered abelian groups

The ordering of Γ gives rise to the notion of convexity for subsets of Γ. A set $B \subseteq \Gamma$ is called *convex* in Γ if for all $\alpha \leq \beta \leq \gamma$,

$$\alpha, \gamma \in B \quad \implies \quad \beta \in B.$$

In particular, an ordered subgroup Δ of Γ where Δ is convex is called a *convex subgroup* of Γ. In this case, every element $\alpha \in \Gamma$ for which there is some $\beta \in \Delta$ with $|\alpha| \leq n\beta$ for some n is already contained in Δ. Every ordered abelian group has the trivial convex subgroups $\{0\}$ and Γ. Note that the intersection of any family of convex subgroups of Γ is a convex subgroup of Γ.

Example. For any $A \subseteq \Gamma$ we get the *convex subgroup of* Γ *generated by* A as an intersection of all convex subgroups containing A, denoted by $\operatorname{conv}_\Gamma(A)$ or simply $\operatorname{conv}(A)$. In more explicit terms,

$$\operatorname{conv}(A) = \{\alpha \in \Gamma | \quad \exists \beta \in A : |\alpha| \leq n\,|\beta| \text{ for some } n\}.$$

Every convex subgroup $\Delta \supseteq A$ satisfies $\Delta \supseteq \operatorname{conv}(A)$, hence $\operatorname{conv}(A)$ is the smallest convex subgroup of Γ containing A.

If Δ is a convex subgroup of Γ and $\alpha \in \Delta$, then $[\alpha] \subseteq \Delta$. To see this, if any $\alpha \neq 0$ is contained in Δ, then every $\beta \in \Gamma$ with $|\beta| \leq n\,|\alpha|$ is contained in Δ, i.e. $\bigcup_{[\beta] \leq [\alpha]} [\beta] \subseteq \Delta$. In particular, $\Delta = \bigcup_{\alpha \in \Delta} [\alpha]$ and $[\Delta]$ is downward closed in $[\Gamma]$, so $[\Delta]$ is a cut in $[\Gamma]$.

Moreover, we can identify the set of nonempty cuts in $[\Gamma]$ with the set of convex subgroups of Γ via the order preserving bijection

$$D \in \{\text{nonempty cuts of } [\Gamma]\} \mapsto \bigcup_{[\alpha]\in D} [\alpha] \in \{\text{convex subgroups of } \Gamma\}$$

viewing both sets as totally ordered sets by inclusion. For Γ of finite rank this correspondence yields that there are $\operatorname{rank}\Gamma$ many non-zero convex subgroups of Γ totally ordered by inclusion. If $\operatorname{rank}\Gamma = n$, we refer to the inclusion ordered set of convex subgroups of Γ as the *maximal chain of convex subgroups* of Γ and write $\Gamma = \Delta_0 \supset \cdots \supset \Delta_n = \{0\}$. Note that $\operatorname{rank}\Delta_i = n - i$ and if $\alpha_1, \dots, \alpha_n$ are archimedean representatives of Γ, then $\alpha_i \in \Delta_{i-1} \setminus \Delta_i$ for all i.

Example. The lexicographically ordered abelian group $\mathbb{R}^n$ has rank n, where the maximal chain of convex subgroups $\Delta_0 \supset \cdots \supset \Delta_n$ is given by

$$\Delta_k := \{0\}^k \times \mathbb{R}^{n-k}.$$

Lemma 1.1.7. *If* $\operatorname{rank}\Gamma = n$ *and* $\Gamma = \Delta_0 \supset \cdots \supset \Delta_n = \{0\}$ *is the maximal chain of convex subgroups of* Γ*, then*

$$\mathbb{Q}\Gamma = \mathbb{Q}\Delta_0 \supset \cdots \supset \mathbb{Q}\Delta_n = \{0\}$$

is the maximal chain of convex subgroups of $\mathbb{Q}\Gamma$ *and* $\mathbb{Q}\Delta_i \cap \Gamma = \Delta_i$ *for all* i*.*

Proof. By Lemma 1.1.5 it suffices to note that each $\mathbb{Q}\Delta_i$ is convex and $\mathbb{Q}\Delta_i \neq \mathbb{Q}\Delta_j$ for $i \neq j$. Let $\beta \in \mathbb{Q}\Delta_i$, $\alpha \in \mathbb{Q}\Gamma$ with $0 < \alpha < \beta$. Then $0 < m\alpha < m\beta \in \Delta_i$ for some $m \neq 0$ such that $m\alpha \in \Gamma$, hence $m\alpha \in \Delta_i$ by convexity of Δ_i. This implies $\alpha \in \mathbb{Q}\Delta_i$, so $\mathbb{Q}\Delta_i$ is convex. It is left to show that $\mathbb{Q}\Delta_i \setminus \mathbb{Q}\Delta_j \neq \emptyset$ for $i < j$. Take some $\alpha \in \Delta_i \setminus \Delta_j$. Then $n\,|\beta| < |\alpha|$ for all n and $\beta \in \Delta_j$. If $\alpha \in \mathbb{Q}\Delta_j$, then $m\alpha \in \Delta_j$ for some m, a contradiction. The last claim follows from convexity of Δ_i in Γ. □

The following fact is needed later.

Lemma 1.1.8. *If* $h : \Gamma \to \tilde{\Gamma}$ *is a morphism of ordered abelian groups, then* $\ker h$ *is convex.*

Proof. Let $\alpha, \gamma \in \ker h$, then each $\beta \in \Gamma$ such that $\alpha \leq \beta \leq \gamma$ satisfies

$$0 = h(\alpha) \leq h(\beta) \leq h(\gamma) = 0,$$

so $\beta \in \ker h$. □

1.1.4 Quotients of ordered abelian groups

For any convex subgroup $\Delta \subseteq \Gamma$, we have the surjective group morphism

$$\tau_\Delta : \Gamma \to \Gamma/\Delta, \quad x \mapsto x + \Delta$$

with kernel Δ. In the following, we add an ordering on Γ/Δ which is compatible with the ordering on Γ.

Lemma 1.1.9. *There is a unique ordering on* Γ/Δ *making* Γ/Δ *an ordered abelian group and* τ_Δ *a morphism of ordered abelian groups.*

Proof. We define

$$\alpha + \Delta \leq \beta + \Delta :\iff \alpha - \beta \leq \delta \text{ for some } \delta \in \Delta.$$

First, we check that the definition does not depend on representatives. If $\alpha' + \Delta = \alpha + \Delta$ and $\beta' + \Delta = \beta + \Delta$, then $\alpha' - \alpha \in \Delta$ as well as $\beta - \beta' \in \Delta$. Thus $\alpha + \Delta \leq \beta + \Delta$ implies

$$\alpha' - \beta' = (\alpha' - \alpha) + (\alpha - \beta) + (\beta - \beta') \leq \delta'$$

for some $\delta' \in \Delta$, so $\alpha' + \Delta \leq \beta' + \Delta$.

Next we show that $\leq$ is a total ordering on Γ/Δ. Of course, $\alpha + \Delta \leq \alpha + \Delta$ holds since $\alpha - \alpha \leq 0 \in \Delta$. If $\alpha + \Delta \leq \beta + \Delta$ and $\beta + \Delta \leq \gamma + \Delta$, then $\alpha - \gamma = (\alpha - \beta) + (\beta - \gamma) \leq \delta_1 + \delta_2 \in \Delta$ for some $\delta_1, \delta_2 \in \Delta$, hence $\alpha + \Delta \leq \gamma + \Delta$. For $\alpha + \Delta \leq \beta + \Delta$ and $\beta + \Delta \leq \alpha + \Delta$ we get $\alpha - \beta \in \operatorname{conv} \Delta = \Delta$ and hence $\alpha + \Delta = \beta + \Delta$. If $\alpha - \beta > \Delta$, then $\beta - \alpha \leq 0 \in \Delta$, so $\leq$ is total.

It is left to verify the axiom for ordered abelian groups.

Suppose $\alpha + \Delta \leq \beta + \Delta$, then for any $\gamma \in \Gamma$, we get $(\alpha + \gamma) - (\beta + \gamma) = \alpha - \beta \leq \delta$, so

$$(\alpha + \Delta) + (\gamma + \Delta) = (\alpha + \gamma) + \Delta \leq (\beta + \gamma) + \Delta = (\beta + \Delta) + (\gamma + \Delta).$$

For the last part, observe that monotonicity of the group morphism τ_Δ follows from

$$\alpha \leq \beta \implies \alpha - \beta \leq 0 \in \Delta \implies \alpha + \Delta \leq \beta + \Delta$$

yielding that τ_Δ is indeed a morphism of ordered abelian groups. Uniqueness is clear as $\delta < \alpha$ for all $\delta \in \Delta$ implies $\Delta = \delta + \Delta \tilde{\leq} \alpha + \Delta$ for any ordering $\tilde{\leq}$ making τ_Δ an ordered abelian group morphism. $\square$

We want to analyze the induced ordering on the set $[\Gamma/\Delta] = \{[\alpha + \Delta] : \alpha \in \Gamma\}$.

Corollary 1.1.10. *The ordering of* $[\Gamma/\Delta]$ *can be described in terms of* Γ *by*

$$[\alpha + \Delta] \leq [\beta + \Delta] \quad \iff \quad |\alpha| \leq n\,|\beta| + \delta \text{ for some } n \text{ and } \delta \in \Delta. \tag{1.1.2}$$

So $[\alpha + \Delta] > [\beta + \Delta]$ *iff* $|\alpha| - n\,|\beta| > \delta$ *for all* n *and* $\delta \in \Delta$.

Proof. Note that $[\alpha + \Delta] > [\beta + \Delta]$ by definition means

$$|\alpha| + \Delta = |\alpha + \Delta| > n\,|\beta + \Delta| = n\,|\beta| + \Delta$$

for all n, so $|\alpha| - n\,|\beta| > \delta$ for each n and $\delta \in \Delta$. $\square$

Corollary 1.1.11. *The rank formula* $\operatorname{rank} \Gamma = \operatorname{rank} \Gamma/\Delta + \operatorname{rank} \Delta$ *holds.*

Proof. We claim that for $\alpha, \beta \in \Gamma \setminus \Delta$, the equivalence

$$[\alpha] \leq [\beta] \iff [\alpha + \Delta] \leq [\beta + \Delta]$$

holds. Taking $\delta = 0$ in (1.1.2) yields the forward implication. Suppose we have $|\alpha| \leq m\,|\beta| + \delta$ for some m and $\delta \in \Delta$, but $|\alpha| > n\,|\beta|$ for all n. Then

$$n\,|\beta| < |\alpha| \leq m\,|\beta| + \delta,$$

so $n = m + 1$ yields $\beta \in \Delta$ by convexity of Δ, contradicting the assumption.

Hence the map $[\Gamma \setminus \Delta] \to [\Gamma/\Delta^{\neq}]$, $[\alpha] \mapsto [\alpha + \Delta]$ is an order preserving bijection and

$$\operatorname{rank}\Gamma = \left|[\Gamma^{\neq}]\right| = |\{[\alpha] : \alpha \in \Gamma \setminus \Delta\}| + \left|[\Delta^{\neq}]\right| = \operatorname{rank}\Gamma/\Delta + \operatorname{rank}\Delta$$

yields the rank formula. □

Remark. If $\operatorname{rank}\Gamma = n$ and $\Gamma = \Delta_0 \supset \cdots \supset \Delta_n = \{0\}$ is the maximal chain of convex subgroups of Γ, we get the quotient morphisms $\tau_i := \tau_{\Delta_i}$ with kernel Δ_i for all i. Here, $\tau_i(\Delta_{i-1}) = \Delta_{i-1}/\Delta_i$ is archimedean by Corollary 1.1.11 using $\operatorname{rank}\Delta_i = n - i$.

1.1.5 Ordered abelian groups of finite rank

The following theorem ([4], Lemma 2.4.3) by Hölder characterizes archimedean ordered abelian groups.

Theorem 1.1.12. *Suppose $\Gamma \neq \{0\}$ is archimedean and $\gamma \in \Gamma^{>}$. Then there is a unique embedding $\iota : \Gamma \to \mathbb{R}$ of ordered abelian groups such that $\iota(\gamma) = 1$.*

Proof. If Γ has a smallest positive element γ_0, then $\Gamma \cong \mathbb{Z}\gamma_0$ and $n\gamma_0 = \gamma$ for some $n \geq 1$. So the map $k\gamma_0 \mapsto k/n$ for $k \in \mathbb{Z}$ is an embedding of ordered abelian groups which satisfies $\iota(\gamma) = 1$.

Suppose Γ has no smallest positive element. Given $\beta \in \Gamma$, consider the following subset of $\mathbb{Q}$ defined by

$$Q_\beta := \left\{ \frac{k}{n} : \quad k \in \mathbb{Z}, n \geq 1, n\beta \leq k\gamma \right\}.$$

Note that Q_β is upward closed in $\mathbb{Q}$ and both Q_β and $\mathbb{Q} \setminus Q_\beta$ are non-empty by archimedeanity of Γ. Hence $\mathbb{Q} \setminus Q_\beta$ defines a cut in $\mathbb{R}$. By completeness of $\mathbb{R}$, there exists $\inf Q_\beta \in \mathbb{R}$ and hence we have a map $\iota : \Gamma \to \mathbb{R}$ with $\iota(\beta) := \inf Q_\beta$. We claim that ι is an embedding of ordered abelian groups. To see this, we calculate

$$\iota(\alpha + \beta) = \inf Q_{\alpha+\beta} = \inf Q_\alpha + \inf Q_\beta = \iota(\alpha) + \iota(\beta)$$

where the second equality follows from

$$n(\alpha + \beta) \leq k\gamma \iff \exists r \in \mathbb{Z} : \quad n\alpha \leq r\gamma \quad \wedge \quad n\beta \leq (k - r)\gamma.$$

We also have $\iota(-\beta) = \inf Q_{-\beta} = \sup \mathbb{Q} \setminus Q_{-\beta} = -\inf Q_\beta = -\iota(\beta)$ since for $n\beta \neq k\gamma$,

$$-\frac{k}{n} \in \mathbb{Q} \setminus Q_{-\beta} \iff n(-\beta) > -k\gamma \iff n\beta < k\gamma \iff \frac{k}{n} \in Q_\beta.$$

To show monotonicity and injectivity of ι, fix $\alpha \leq \beta$ and observe that then $Q_\alpha \supseteq Q_\beta$, implying $\iota(\alpha) = \inf Q_\alpha \leq \inf Q_\beta = \iota(\beta)$. If in addition $\iota(\alpha) = \iota(\beta)$, then $\iota(\beta - \alpha) = 0$ and thus $\inf Q_{\beta-\alpha} = 0$ which implies $0 \leq n(\beta - \alpha) \leq \gamma$ for all $n \neq 0$. Thus $\alpha = \beta$ by the archimedean property of Γ, showing injectivity of ι. □

The following result generalizes Hölder's Theorem to ordered abelian groups of arbitrary finite rank. Denote the standard basis of $\mathbb{R}^n$ by $(e_i)_{i=1,\dots,n}$.

Theorem 1.1.13. *Suppose* $\operatorname{rank}\Gamma = n$. *Then there exists an embedding of ordered abelian groups* $\iota : \Gamma \to \mathbb{R}^n$. *In this case,* $\iota(\Delta_k) \subseteq \{0\}^k \times \mathbb{R}^{n-k}$ *for* $0 \leq k \leq n$ *where* $\Delta_0 \supset \cdots \supset \Delta_n$ *is the maximal chain of convex subgroups of* Γ. *Furthermore, if* $\alpha_1, \dots, \alpha_n$ *are archimedean representatives of* Γ, *then* ι *can be chosen so that* $\iota(\alpha_i) = e_i$ *for all* i.

Proof. First, we reduce to the case where Γ is divisible by passing to the divisible hull of Γ. Every embedding $h : \mathbb{Q}\Gamma \to \mathbb{R}^n$ of ordered abelian groups restricts to an embedding $\Gamma \to \mathbb{R}^n$.

By Lemma 1.1.7, the maximal chain of convex subgroups of $\mathbb{Q}\Gamma$ is $\Gamma = \mathbb{Q}\Delta_0 \supset \cdots \supset \mathbb{Q}\Delta_n$ where $h(\mathbb{Q}\Delta_i) \cap h(\Gamma) = h(\Delta_i)$ and each $\mathbb{Q}\Delta_i$ is a $\mathbb{Q}$-subspace of $\mathbb{Q}\Gamma$ with $\alpha_i \in \Delta_{i-1} \setminus \Delta_i$ for all i.

We prove the theorem by induction on $n := \operatorname{rank}\Gamma$. The case $n = 0$ is trivial and $n = 1$ is covered by Theorem 1.1.12. Suppose that Γ is divisible of rank $n > 1$ and the statement holds for all ordered abelian groups of rank less than n. Take a vector space complement $\Delta_1^\perp$ of Δ_1 in Γ such that $\Delta_1^\perp \oplus \Delta_1 = \Gamma$ and consider Γ/Δ_1, which is an archimedean ordered abelian group by Lemma 1.1.9 and Corollary 1.1.11. Every $\gamma \in \Gamma$ can be uniquely written as $\gamma = \gamma^\top + \gamma^\perp$ where $\gamma^\perp \in \Delta_1^\perp$ and $\gamma^\top \in \Delta_1$ and we have $(\alpha - \beta)^\perp = \alpha^\perp - \beta^\perp$. Here, $\Delta_1^\perp$ can be chosen such that $\alpha_1^\perp = \alpha_1$ by extending a basis of Δ_1 including α_1.

Our goal is to identify Γ/Δ_1 with the ordered subgroup $\Delta_1^\perp$ of Γ. This can be done via the ordered abelian group isomorphism $\gamma + \Delta_1 \mapsto \gamma^\perp$. First, we ensure this definition does not depend on representatives. If $\alpha + \Delta_1 = \beta + \Delta_1$, then

$$\alpha - \beta \in \Delta_1 \implies \alpha^\perp - \beta^\perp = (\alpha - \beta)^\perp = 0.$$

On the other hand, $\alpha^\perp = \beta^\perp$ implies $\alpha + \Delta_1 = \beta + \Delta_1$ by reversing the argument, so this map is injective. To get surjectivity, note that for any $\delta \in \Delta_1^\perp$ we have $\delta + \Delta_1 \mapsto \delta$ as $\delta^\perp = \delta$. Finally, we show that ordering is preserved. For $\alpha + \Delta_1 < \beta + \Delta_1$ we get $\beta - \alpha > \Delta_1$ and $(\beta - \alpha)^\perp = (\beta - \alpha) - (\beta - \alpha)^\top > 0$ gives $\alpha^\perp < \beta^\perp$.

Using the inductive hypothesis, we can embed Δ_1 into $\mathbb{R}^{n-1}$ via ι_{n-1} with $\iota_{n-1}(\alpha_{i+1}) = e_i$ for all $i = 1, \ldots, n-1$. By Theorem 1.1.12, there is an embedding $\iota : \Delta_1^{\perp} \cong \Gamma/\Delta_1 \to \mathbb{R}$ such that $\iota(\alpha_1^{\perp}) = 1$. Then ι and ι_{n-1} can be glued together forming

$$\iota_n := \iota \oplus \iota_{n-1} : \quad \Delta_1^{\perp} \oplus \Delta_1 \to \mathbb{R} \oplus \mathbb{R}^{n-1}, \gamma \mapsto (\iota(\gamma^{\perp}), \iota_{n-1}(\gamma^{\top})).$$

Observe that $\iota_n : \Gamma \to \mathbb{R}^n$ and $\iota_n(\alpha_i) = e_i$ for $1 \leq i \leq n$ by construction.

We claim that ι_n is an embedding of ordered abelian groups. It is clear that ι_n is a group embedding, so it is left to show that ι_n is order preserving. If $\alpha \in \Delta_1^{>}$, then $\alpha^{\perp} = 0$ so $\iota_n(\alpha) = (0, \iota_{n-1}(\alpha)) > 0$. On the other hand, $\alpha > \Delta_1$ gives $\alpha^{\perp} > 0$, so $\iota(\alpha^{\perp}) > 0$.

By the inductive assumption applied to Δ_1, we get $\iota_{n-1}(\Delta_k) \subseteq \{0\}^{k-1} \times \mathbb{R}^{n-k}$ for $1 \leq k \leq n$ since the maximal chain of convex subgroups of Δ_1 is $\Delta_1 \supset \cdots \supset \Delta_n$. This implies $\iota_n(\Delta_k) \subseteq \{0\}^k \times \mathbb{R}^{n-k}$ by virtue of $\alpha^{\perp} = 0$ for every $\alpha \in \Delta_1$ giving $\iota_n(\alpha) = (0, \iota_{n-1}(\alpha)) \in \{0\} \times \mathbb{R}^{n-1}$. □

Remark. Theorem 1.1.13 realizes abstract ordered abelian groups of rank n as ordered subgroups of $\mathbb{R}^n$ and gives concrete meaning to the maximal chain of convex subgroups and the quotient embeddings. There are more general versions of this result using Hahn products of $\mathbb{R}$, known as *Hahn Embedding Theorems* [4].

1.2 H-asymptotic couples

It is useful to start with convex valuations on ordered abelian groups to introduce a notion of size for elements of Γ.

Definition. A *convex valuation* on Γ is a map $v : \Gamma \to S_\infty$, where S_∞ is a totally ordered set with largest element ∞, satisfying for all α and β:

(i) $v(\alpha) = \infty \iff \alpha = 0$;

(ii) $v(-\alpha) = v(\alpha)$;

(iii) $v(\alpha + \beta) \geq \min\{v(\alpha), v(\beta)\}$;

(iv) $0 < \alpha \leq \beta \implies v(\alpha) \geq v(\beta)$.

Call (Γ, v) a *convex valued ordered abelian group* if v is a convex valuation on Γ.

Remark. The following properties hold.

(1) If $v(\alpha) < v(\beta)$, then

$$v(\alpha) = v(\alpha + \beta - \beta) \geq \min\{v(\alpha + \beta), v(-\beta)\} \geq \min\{v(\alpha), v(\beta)\} = v(\alpha),$$

hence $v(\alpha + \beta) = \min\{v(\alpha), v(\beta)\}$.

(2) If $v(\alpha_1) < v(\alpha_i)$, $2 \leq i \leq n$, then $v(\alpha_1 + \cdots + \alpha_n) = v(\alpha_1)$ by setting $\beta := \alpha_2 + \cdots + \alpha_n$ and observing that $v(\beta) \geq \min\{v(\alpha_2), \ldots, v(\alpha_n)\} > v(\alpha_1)$.

(3) We have $v(k\alpha) = v(\alpha)$ for all α and $k \in \mathbb{Z}^{\neq}$. By (ii) it suffices to show this for $\alpha > 0$ and $k > 0$. Axiom (iv) gives $v(k\alpha) \leq v(\alpha)$ as $0 < \alpha \leq k\alpha$. If $v(k\alpha) \neq v(\alpha)$, take $k > 1$ minimal with this property. Then

$$v(\alpha) = v((k-1)\alpha) = v(k\alpha - \alpha) = \min\{v(k\alpha), v(\alpha)\},$$

so $v(k\alpha) > v(\alpha)$, a contradiction.

Example. Take $S_\infty := [\Gamma]$ ordered by the reverse ordering

$$[\alpha] \leq [\beta] \quad :\Longleftrightarrow \quad |\beta| \leq n\,|\alpha| \text{ for some } n,$$

with $\infty = [0]$. The map $v : \Gamma \to [\Gamma]$, $\alpha \mapsto [\alpha]$ yields a convex valuation on Γ by Lemma 1.1.2, called the *standard valuation* on Γ.

We are now ready to introduce H-asymptotic couples, which are essentially convex valued ordered abelian groups with additional properties that are motivated by their origin from Hardy fields or more generally H-asymptotic fields. H-asymptotic couples and their extensions coming from Hardy fields are the main object of interest in the next chapters. We describe an explicit way to think of H-asymptotic couples of finite rank as we did with ordered abelian groups in Theorem 1.1.13. After introducing H-fields and Hardy fields, we explain how every H-field gives rise to an H-asymptotic couple.

Definition. Let $\psi : \Gamma^{\neq} \to \Gamma$ be a map such that for all $\alpha, \beta \neq 0$:

(HC1) $\alpha + \beta \neq 0 \implies \psi(\alpha + \beta) \geq \min\{\psi(\alpha), \psi(\beta)\}$;

(HC2) $k \in \mathbb{Z}^{\neq} \implies \psi(k\alpha) = \psi(\alpha)$;

(HC3) $\alpha > 0 \implies \psi(\beta) < \alpha + \psi(\alpha)$;

(HC4) $0 < \alpha \leq \beta \implies \psi(\alpha) \geq \psi(\beta)$.

Then (Γ, ψ) is called an *H-asymptotic couple*, or *H-couple* for short. If ψ satisfies in addition

$$[\alpha] \neq [\beta] \implies \psi(\alpha) \neq \psi(\beta)$$

for all $\alpha, \beta \neq 0$, then (Γ, ψ) is of *Hardy type.* The H-couple $(\{0\}, \psi)$ where ψ is the empty map is called *trivial.* The *rank* of (Γ, ψ) is defined to be $\operatorname{rank} \Gamma$. If $(\mathrm{id} + \psi)(\Gamma^{>}) \subseteq \Gamma^{>}$, then (Γ, ψ) has *small derivation.*

Let (Γ, ψ) be an H-couple.

Remark. The map ψ makes Γ a convex valued ordered abelian group, viewing $\psi : \Gamma \to \Gamma_\infty$ with $\psi(0) := \infty$. The above axioms still hold with the conventions $\infty + \alpha := \alpha + \infty := \infty$ for all $\alpha \in \Gamma_\infty$.

Since the image of ψ often plays an important role, we define $\Psi := \{\psi(\gamma) : \gamma \in \Gamma^{\neq}\}$. Furthermore, we let $\alpha^\dagger := \psi(\alpha)$ and $\alpha' := \alpha + \alpha^\dagger$ for $\alpha \neq 0$. These notations are chosen coherent with their analytic counterparts discussed later. The following examples come from [12].

Example. Let $\gamma \in \Gamma$. Then the constant map $\psi_\gamma : \Gamma^{\neq} \to \Gamma, \alpha \mapsto \gamma$ yields an H-couple (Γ, ψ_γ) which is not of Hardy type if $\operatorname{rank}\Gamma \geq 2$. Also, the *shift* $(\Gamma, \psi + \psi_\gamma)$ and $(\Gamma, \max\{\psi, \psi_\gamma\})$ are H-couples and shifts preserve Hardy type. If $(\Gamma, \tilde{\psi})$ is another H-couple, then $(\Gamma, \min\{\psi, \tilde{\psi}\})$ is an H-couple.

In the following, we prove some basic properties of (Γ, ψ).

Lemma 1.2.1. *The map ψ is constant on archimedean classes of Γ inducing a surjection $[\psi] : [\Gamma^{\neq}] \to \Psi$ determined by its values on any archimedean representatives of Γ. If Γ is of Hardy type, then $[\psi]$ is bijective.*

Proof. Let $\alpha, \beta \in \Gamma^{\neq}$. If $0 < |\alpha| \leq n\,|\beta|$ and $0 < |\beta| \leq n\,|\alpha|$ for some n, then $\psi(\alpha) = \psi(|\alpha|) \geq \psi(n\,|\beta|) = \psi(\beta)$ using (HC2) and (HC4). By symmetry we have $\psi(\beta) \geq \psi(\alpha)$ and so $\psi(\beta) = \psi(\alpha)$. Hence $[\psi]$ is well defined. Let (α_i) be archimedean representatives of Γ. Then $[\alpha] = [\alpha_i]$ for some i, hence $\psi(\alpha) = \psi(\alpha_i)$. By definition, this map is injective if (Γ, ψ) is of Hardy type. □

Corollary 1.2.2. *We have $|\Psi| \leq \operatorname{rank}(\Gamma)$ with equality if (Γ, ψ) is of Hardy type.*

The next technical result ([4], Lemma 6.5.3) is used to extend the H-couple structure of (Γ, ψ) to $\mathbb{Q}\Gamma$.

Lemma 1.2.3. *Let $\alpha, \beta \in \Gamma^{\neq}$. Then $n(\psi(\beta) - \psi(\alpha)) < |\alpha|$ for all n.*

Proof. By (HC2) we can assume that $\alpha, \beta > 0$. If $\psi(\beta) \leq \psi(\alpha)$, the statement holds trivially, so suppose $\psi(\beta) > \psi(\alpha)$. If there is some n for which the conclusion fails, then $n > 1$ by (HC3). For the smallest such n we have

$$0 \leq \gamma := n\psi(\beta) - n\psi(\alpha) - \alpha < \psi(\beta) - \psi(\alpha).$$

Another application of (HC3) yields

$$\psi(\beta) < \psi(\beta) - \psi(\alpha) + \psi(\psi(\beta) - \psi(\alpha)),$$

so $\psi(\alpha) < \psi(n\psi(\beta) - n\psi(\alpha))$ which gives $n\psi(\beta) - n\psi(\alpha) \neq \alpha$ and

$$\psi(\gamma) = \psi(n\psi(\beta) - n\psi(\alpha) - \alpha) = \psi(\alpha)$$

by (HC1). But now $\psi(\beta) < \gamma + \psi(\gamma) = \gamma + \psi(\alpha)$ due to (HC3) again, contradicting $\gamma < \psi(\beta) - \psi(\alpha)$. □

We analyze the map $\partial : \gamma \mapsto \gamma' : \Gamma^{\neq} \to \Gamma$. The following fact is Lemma 6.5.4 in [4].

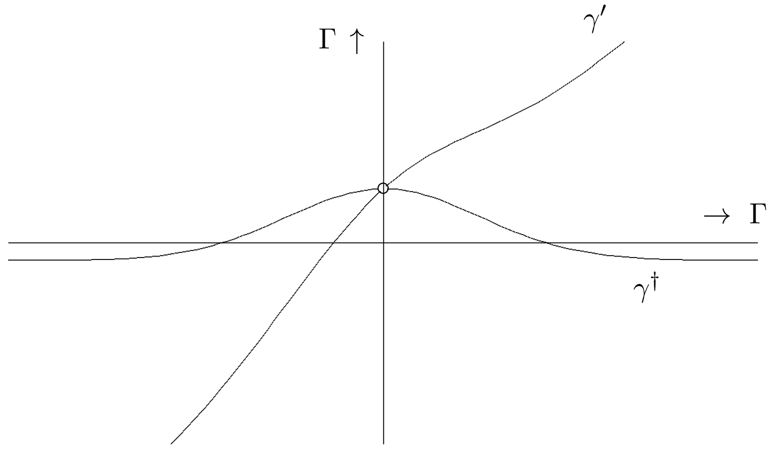
Lemma 1.2.4. *The map ∂ is strictly increasing.*

Proof. Let $\alpha, \beta \in \Gamma^{\neq}$. If $\psi(\alpha) = \psi(\beta)$, then $\alpha < \beta \implies \alpha + \psi(\alpha) < \beta + \psi(\beta)$. Otherwise $\psi(\alpha) \neq \psi(\beta)$. If $0 < \alpha < \beta$ then $\psi(\alpha) > \psi(\beta)$ by (HC4), so $\psi(\beta - \alpha) = \psi(\beta)$ and we get

$$\psi(\beta) + \beta - \alpha = \psi(\beta - \alpha) + \beta - \alpha > \psi(\alpha)$$

using (HC3). This yields $\alpha + \psi(\alpha) < \beta + \psi(\beta)$. By (HC2), it immediately follows that $-\alpha + \psi(-\alpha) < \beta + \psi(\beta)$. Adding $2\psi(\beta) < 2\psi(\alpha)$ to $\alpha + \psi(\alpha) < \beta + \psi(\beta)$ gives $-\beta + \psi(-\beta) < -\alpha + \psi(-\alpha)$. From this we get $-\beta + \psi(-\beta) < \alpha + \psi(\alpha)$. □

In a nutshell, ψ is constant on archimedean classes, symmetric and decreasing on $\Gamma^{>}$ whereas ∂ is strictly increasing and has the intermediate value property on both $\Gamma^{>}$ and $\Gamma^{<}$ ([4], Lemma 9.2.14). The following figure from [4] visualizes conceptually the behavior of ∂ and ψ.



Another technical result ([4], Theorem 9.2.1) will be useful later.

Lemma 1.2.5. *Suppose Ψ has a maximum and let $\gamma_0 := \max \Psi$. Then for every α we have*

$$\alpha \neq \gamma_0 \iff \text{there is some } \beta \in \Gamma^{\neq} \text{ such that } \psi(\beta) + \beta = \alpha.$$

Moreover, such β is uniquely determined.

Proof. For uniqueness, suppose $\beta_1, \beta_2 \in \Gamma^{\neq}$ such that $\beta_1 < \beta_2$ with $\psi(\beta_1) + \beta_1 = \psi(\beta_2) + \beta_2$. So $\psi(\beta_2) < \psi(\beta_1)$ and by (HC3),

$$\psi(\beta_1) < \psi(\beta_2 - \beta_1) + \beta_2 - \beta_1 = \psi(\beta_2) + \beta_2 - \beta_1$$

gives the contradiction $\psi(\beta_1) + \beta_1 < \psi(\beta_2) + \beta_2$.

($\Leftarrow$): Suppose that there is some $\beta \in \Gamma^{\neq}$ such that $\psi(\beta) + \beta = \gamma_0$. By (HC3), if $\beta > 0$, then $\gamma_0 < \psi(\beta) + \beta = \gamma_0$, which is false, hence $\beta < 0$. But then

$$\psi(\beta) = \gamma_0 - \beta > \gamma_0 \geq \psi(\beta)$$

yields a contradiction.

($\Rightarrow$): ([4], Theorem 9.2.1) $\square$

Remark. Fixing $\alpha \neq \gamma_0$, the solution to $\alpha = \beta + \psi(\beta)$ can be explicitly written as $\beta := \alpha - \psi(\gamma_0 - \alpha)$ as shown in [1].

1.2.1 Subcouples and morphisms of H-couples

We continue with a discussion of morphisms and substructures of H-couples. *Let* $(\tilde{\Gamma}, \tilde{\psi})$ *be an* H*-couple.*

Definition. A *morphism of* H*-couples* $h : (\Gamma, \psi) \to (\tilde{\Gamma}, \tilde{\psi})$ is an ordered abelian group morphism $\Gamma \to \tilde{\Gamma}$ which in addition satisfies

$$\alpha \notin \ker h \implies h(\psi(\alpha)) = \tilde{\psi}(h(\alpha)).$$

If h is injective (resp. bijective), then h is an *embedding* (resp. *isomorphism*) of H-couples. Assume (Γ_0, ψ_0) is an H-couple where Γ_0 is an ordered subgroup of both Γ and $\tilde{\Gamma}$ with $\psi_0 = \psi|_{\Gamma_0} = \psi|_{\tilde{\Gamma}_0}$. If $h(\gamma_0) = \gamma_0$ for all $\gamma_0 \in \Gamma_0$, then h is called a morphism of H-couples *over* (Γ_0, ψ_0).

Definition. An H-couple (Γ_0, ψ_0) where Γ_0 is an ordered subgroup of Γ and the natural inclusion $\Gamma_0 \hookrightarrow \Gamma$ is an embedding of H-couples is called a *subcouple* of (Γ, ψ). We also say that (Γ, ψ) *extends* (Γ_0, ψ_0). If for every $\gamma \in \Gamma^{\neq}$ there is some $\gamma_0 \in \Gamma_0$ such that $\psi(\gamma - \gamma_0) > \psi(\gamma)$, then we call the extension *immediate*.

Note that if (Γ_0, ψ_0) is a subcouple of (Γ, ψ), then $\psi_0 = \psi|_{\Gamma_0}$ and we can characterize subcouples of (Γ, ψ) as those pairs (Γ_0, ψ_0) such that Γ_0 is an ordered subgroup of Γ with $\psi(\Gamma_0^{\neq}) \subseteq \Gamma_0$ and $\psi_0 = \psi|_{\Gamma_0}$.

Lemma 1.2.6. *There is a unique extension* $\psi_{\mathbb{Q}}$ *of* ψ *to* $\mathbb{Q}\Gamma^{\neq}$ *such that* $(\mathbb{Q}\Gamma, \psi_{\mathbb{Q}})$ *is an* H*-couple extending* (Γ, ψ)*. If* (Γ, ψ) *is of Hardy type, then so is* $(\mathbb{Q}\Gamma, \psi_{\mathbb{Q}})$*.*

Proof. We define $\psi_{\mathbb{Q}}(q\gamma) := \psi(\gamma)$ for all $q \in \mathbb{Q}^{\times}$ and $\gamma \in \Gamma^{\neq}$. Then (HC1), (HC2) and (HC4) follow from the corresponding axioms for (Γ, ψ). Let $\alpha > 0, \beta \neq 0$ and $p, q \in \mathbb{Q}^{>}$, $p = m/n$. From Lemma 1.2.3 we get

$$n(\psi(\beta) - \psi(m\alpha)) < m\alpha,$$

so $\psi_{\mathbb{Q}}(q\beta) = \psi(\beta) < p\alpha + \psi_{\mathbb{Q}}(p\alpha)$, showing (HC3).

The values of any extension $\tilde{\psi}$ of ψ to $\mathbb{Q}\Gamma^{\neq}$ such that $(\mathbb{Q}\Gamma, \tilde{\psi})$ is an H-couple are determined by archimedean representatives of Γ by Lemmas 1.2.1 and 1.1.5, hence such extension is unique. We have $\Psi = \Psi_{\mathbb{Q}}$, so Hardy type is preserved. $\square$

Remark. Since $\psi_{\mathbb{Q}}$ is unique, we drop the index and write $(\mathbb{Q}\Gamma, \psi)$ instead of $(\mathbb{Q}\Gamma, \psi_{\mathbb{Q}})$.

As with ordered abelian groups, we get besides the ordering a natural H-couple structure on quotients by convex subgroups.

Lemma 1.2.7. *Let Δ be a convex subgroup of Γ. Then there is a unique map $\psi_\Delta : \Gamma/\Delta^{\neq} \to \Gamma/\Delta$ making $(\Gamma/\Delta, \psi_\Delta)$ an H-couple such that τ_Δ is a morphism of H-couples.*

Proof. By Lemma 1.1.9, Γ/Δ becomes an ordered abelian group such that the map $\tau_\Delta : \Gamma \to \Gamma/\Delta$ is a morphism of ordered abelian groups. Define $\psi_\Delta(\gamma + \Delta) := \psi(\gamma) + \Delta$ for all $\gamma \notin \Delta$. We need to show that the definition does not depend on the choice of representatives. Let $\alpha, \beta \notin \Delta$ such that $\psi(\alpha) < \psi(\beta)$. Our goal is to get

$$\alpha - \beta \in \Delta \implies \psi(\alpha) - \psi(\beta) \in \Delta.$$

We calculate

$$\psi(\alpha - \beta) = \psi(\alpha) < \psi(\beta) < \psi(\alpha - \beta) + |\alpha - \beta|\,.$$

Thus $0 < \psi(\beta) - \psi(\alpha) = \psi(\beta) - \psi(\alpha - \beta) < |\alpha - \beta| \in \Delta$ and by convexity of Δ we get $\psi(\alpha) - \psi(\beta) \in \Delta$.

The next step is to show that $(\Gamma/\Delta, \psi_\Delta)$ is an H-couple. Let $\alpha, \beta \notin \Delta$.

(HC1): If $\alpha + \beta \notin \Delta$, then

$$\begin{aligned}\psi_\Delta((\alpha + \Delta) + (\beta + \Delta)) &= \psi_\Delta((\alpha + \beta) + \Delta) = \psi(\alpha + \beta) + \Delta \\ &\geq \min\{\psi(\alpha), \psi(\beta)\} + \Delta = \min\{\psi(\alpha) + \Delta, \psi(\beta) + \Delta\} \\ &= \min\{\psi_\Delta(\alpha + \Delta), \psi_\Delta(\beta + \Delta)\}\end{aligned}$$

where we used that τ_Δ is order preserving.

(HC2): For all $k \in \mathbb{Z}^{\neq}$,

$$\psi_\Delta(k(\alpha + \Delta)) = \psi_\Delta(k\alpha + \Delta) = \psi(k\alpha) + \Delta = \psi(\alpha) + \Delta = \psi_\Delta(\alpha + \Delta).$$

(HC3): If $\alpha > \Delta$, then

$$\begin{aligned}\psi(\beta) < \psi(\alpha) + \alpha &\implies \psi(\beta) + \Delta \leq (\psi(\alpha) + \alpha) + \Delta \\ &\implies \psi_\Delta(\beta + \Delta) \leq \psi_\Delta(\alpha + \Delta) + (\alpha + \Delta).\end{aligned}$$

We have $2(\psi(\beta) - \psi(\alpha)) < \alpha$ by Lemma 1.2.3, hence

$$2(\psi_\Delta(\beta + \Delta) - \psi_\Delta(\alpha + \Delta)) \leq \alpha + \Delta.$$

If $\psi_\Delta(\beta + \Delta) = \psi_\Delta(\alpha + \Delta) + (\alpha + \Delta)$, then

$$2\alpha + \Delta = 2(\alpha + \Delta) = 2(\psi_\Delta(\beta + \Delta) - \psi_\Delta(\alpha + \Delta)) \leq \alpha + \Delta,$$

but $\alpha > \Delta$ gives a contradiction.

(HC4): If $0 < \alpha + \Delta < \beta + \Delta$, then $0 < \alpha < \beta$ and $\psi(\alpha) \geq \psi(\beta)$, so

$$\psi_\Delta(\alpha + \Delta) = \psi(\alpha) + \Delta \geq \psi(\beta) + \Delta = \psi_\Delta(\beta + \Delta)$$

using that τ_Δ is order preserving.

Uniqueness follows from

$$\psi(\alpha) + \Delta = \tau_\Delta(\psi(\alpha)) = \tilde{\psi}(\tau_\Delta(\alpha)) = \tilde{\psi}(\alpha + \Delta)$$

for all $\alpha \notin \Delta$ and any H-couple $(\Gamma/\Delta, \tilde{\psi})$ satisfying the assumptions. □

Remark. If (Γ, ψ) has small derivation, then so does $(\Gamma/\Delta, \psi_\Delta)$ ([4], Lemma 9.2.24). Hardy type is not preserved as for example in $\mathbb{R}^3$ we will show that if ψ has values $(0,0,1), (0,0,2), (0,0,3)$ on archimedean representatives, this would give a valid H-couple structure to $\mathbb{R}^3$ and for $\Delta = \{0\}^2 \times \mathbb{R}$ we get $|\Psi_\Delta| = 1 < 2 = \operatorname{rank} \Gamma/\Delta$.

Let $h : (\Gamma, \psi) \to (\tilde{\Gamma}, \tilde{\psi})$ be a morphism of H-couples. Then $\Delta := \ker h$ is a convex subgroup of Γ by Lemma 1.1.8 and we view $(\Gamma/\Delta, \psi_\Delta)$ as an H-couple as defined in Lemma 1.2.7 such that τ_Δ is a morphism of H-couples. We have the following decomposition of h [12]:

Proposition 1.2.8. *There is an embedding $h_\Delta : (\Gamma/\Delta, \psi_\Delta) \to (\tilde{\Gamma}, \tilde{\psi})$ of H-couples such that*

$$h = h_\Delta \circ \tau_\Delta.$$

$$\begin{tikzcd}
(\Gamma, \psi) \arrow[r, "\tau_\Delta"'] \arrow[rr, bend left, "h"] & (\Gamma/\Delta, \psi_\Delta) \arrow[r, "h_\Delta"'] & (\tilde{\Gamma}, \tilde{\psi})
\end{tikzcd}$$

Proof. Define $h_\Delta : \Gamma/\Delta \to \tilde{\Gamma}$ by $\alpha + \Delta \mapsto h(\alpha)$. This definition does not depend on representatives and makes h_Δ an injective map since

$$\alpha - \beta \in \Delta \iff h(\alpha - \beta) = 0 \iff h(\alpha) = h(\beta).$$

It remains to show that h_Δ is a morphism of H-couples. For $\alpha \notin \Delta$ we get

$$h_\Delta(\psi_\Delta(\alpha + \Delta)) = h_\Delta(\psi(\alpha) + \Delta) = h(\psi(\alpha)) = \tilde{\psi}(h(\alpha)) = \tilde{\psi}(h_\Delta(\alpha + \Delta)).$$

Finally, $h(\gamma) = h_\Delta(\gamma + \Delta) = h_\Delta(\tau_\Delta(\gamma)) = h_\Delta \circ \tau_\Delta(\gamma)$ for each $\gamma \in \Gamma$. □

1.2.2 H-couples of finite rank

Based on [12], we describe H-couple structures on $\mathbb{R}^n$ and show that a similar embedding property as in Theorem 1.1.13 holds for H-couples of finite rank. Indices i, j, k, l range over $\{1, \ldots, n\}$. Let (e_i) be the standard basis of $\mathbb{R}^n$ and $\psi : (\mathbb{R}^n)^{\neq} \to \mathbb{R}^n$ a map such that $(\mathbb{R}^n, \psi)$ is an H-couple. Note that $e_1, \ldots, e_n$ are archimedean representatives of $\mathbb{R}^n$. The following discussion only uses that $(e_i)_k = 0$ for $k < i$ and $(e_i)_i > 0$. Any other archimedean representatives of $\mathbb{R}^n$ would also do the job. We derive certain necessary properties of ψ.

By Lemma 1.2.1, we find that ψ is uniquely determined by its values on (e_i). If we denote $\psi_i := \psi(e_i) = (M_{i1}, \ldots, M_{in})$, we can collectively represent (ψ_i) in the matrix M with ψ_i in the i-th row so that $M_{ij} = \psi(e_i)_j$. We denote the i-th row of M as M_i, so $M_i = \psi_i$. There are two observations which follow from Lemma 1.2.1 and the axioms for H-couples:

(i) $\psi_i \geq \psi_j$ for all $i > j$ by (HC4);

(ii) $\psi_i < \psi_j + \epsilon e_j$ for all i, j and $\epsilon > 0$ by (HC3).

So $0 \le \psi_i - \psi_j < \epsilon e_j$ for $i > j$. This forces $M_{ik} = M_{jk}$ for $k = 1, \dots, j$, so

$$M = (\psi_i) = \begin{pmatrix} M_{11} & M_{12} & M_{13} & \cdots & M_{1,n-1} & M_{1n} \\ M_{11} & M_{22} & M_{23} & \cdots & M_{2,n-1} & M_{2n} \\ M_{11} & M_{22} & M_{33} & \cdots & M_{3,n-1} & M_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ M_{11} & M_{22} & M_{33} & \cdots & M_{n-1,n-1} & M_{n-1,n} \\ M_{11} & M_{22} & M_{33} & \cdots & M_{n-1,n-1} & M_{nn} \end{pmatrix}. \tag{1.2.1}$$

We want to give matrices of this form a special name.

Definition. A matrix $M \in \mathbb{R}^{n \times n}$ is called *admissible* if for all $i \ge j$:

(A1) $M_i \ge M_j$;

(A2) $M_{ij} = M_{jj}$.

The matrix M in (1.2.1) is admissible. The map which assigns to $A \in \mathrm{Gl}_n(\mathbb{R})$ the endomorphism $x \mapsto xA$ of $\mathbb{R}^n$ is an isomorphism of $\mathrm{Gl}_n(\mathbb{R})$ onto the group of automorphisms of the $\mathbb{R}$-linear space $\mathbb{R}^n$. We identify matrices as automorphisms in this way.

Lemma 1.2.9. *The group of $\mathbb{R}$-linear automorphisms of the ordered abelian group $\mathbb{R}^n$ can be described by*

$$\mathcal{U} := \{A \in \mathrm{Gl}_n(\mathbb{R}) : \quad A_{ii} > 0 \text{ for all } i \text{ and } A_{ik} = 0 \text{ for all } i > k\}.$$

Proof. We show that for any $A \in \mathcal{U}$, the map $x \mapsto xA$ is order preserving. Note that by linearity it suffices to check that $0 < x$ implies $0 < xA$. If $0 < x$, then $x_1 = \cdots = x_{k-1} = 0 < x_k$ for some $k \ge 1$. This yields $xA_l = \sum_{i=1}^n A_{il}x_i = \sum_{i=1}^l A_{il}x_i = 0$ for $l < k$ and $xA_k = \sum_{i=1}^k A_{ik}x_i = A_{kk}x_k > 0$, so $0 < xA$. It is clear that $\mathcal{U}$ forms a subgroup of automorphisms of $\mathbb{R}^n$. By Lemma 1.1.3, any $\mathbb{R}$-linear automorphism A of $(\mathbb{R}^n, \le)$ has to preserve archimedean classes, hence $A \in \mathcal{U}$. □

Lemma 1.2.10. *If M is admissible and $A \in \mathcal{U}$, then MA is admissible.*

Proof. Let $i \ge j$.

(A1): If $M_i = M_j$, then $MA_i = MA_j$. From $M_i > M_j$ it follows that $M_{ik} = M_{jk}$ for $k < m$ and $M_{im} > M_{jm}$ for some $m \le n$. Hence $\sum_{l=1}^k M_{il}A_{lk} = \sum_{l=1}^k M_{jl}A_{lk}$ for $k < m$ and $\sum_{l=1}^m M_{il}A_{lm} > \sum_{l=1}^m M_{jl}A_{lm}$ since $A_{mm} > 0$. This shows

$$MA_i = \left(\sum_{l=1}^k M_{il}A_{lk}\right)_k > \left(\sum_{l=1}^k M_{jl}A_{lk}\right)_k = MA_j.$$

(A2): We have

$$MA_{ij} = \sum_{l=1}^{j} M_{il}A_{lj} = \sum_{l=1}^{j} M_{ll}A_{lj} = \sum_{l=1}^{j} M_{jl}A_{lj} = MA_{jj}.$$

Hence MA is admissible. □

Proposition 1.2.11. *Let M be an admissible matrix. Then M induces an H-couple $(\mathbb{R}^n, \psi_M)$ where $\psi_M(e_i) = M_i$.*

Proof. Let $\alpha, \beta \in \mathbb{R}^n \setminus \{0\}$ and i, j be such that $[\alpha] = [e_i]$, $[\beta] = [e_j]$.

(HC1): If $\alpha + \beta \neq 0$, then $[\alpha + \beta] \leq \max\{[\alpha], [\beta]\} = \max\{[e_i], [e_j]\}$ by Lemma 1.1.2. Hence if $[\alpha + \beta] = [e_l]$, then $l \geq \min\{i, j\}$ and using (A2) we get

$$\psi_M(\alpha + \beta) = \psi_M(e_l) = M_l \geq \min\{M_i, M_j\} = \min\{\psi_M(\alpha), \psi_M(\beta)\}.$$

(HC2): We have $[r\alpha] = [e_i]$ for all $r \in \mathbb{R}^\times$, thus $\psi_M(r\alpha) = M_i = \psi_M(\alpha)$.

(HC3): Suppose $\alpha > 0$. If $i \geq j$, then $\psi_M(\alpha) = M_i \geq M_j = \psi_M(\beta)$ by (A1) and hence $\psi_M(\beta) < \psi_M(\alpha) + \alpha$. If $i < j$, then $M_{ik} = M_{kk} = M_{jk}$ for $k \leq i$ by (A2), which implies $[M_j - M_i] < [e_i] = [\alpha]$. Thus $\psi_M(\beta) < \alpha + \psi_M(\alpha)$.

(HC4): Suppose $0 < \alpha < \beta$. Then $i \geq j$, so $\psi_M(\alpha) = M_i \geq M_j = \psi_M(\beta)$ by (A1). □

Lemma 1.2.12. *Let $A \in \mathcal{U}$. Then $x \mapsto xA$ yields an isomorphism of H-couples $(\mathbb{R}^n, \psi_M) \to (\mathbb{R}^n, \psi_{MA})$.*

Proof. By Lemma 1.1.3, A preserves archimedean classes. So $[x] = [xA] = [e_i]$ for some i and

$$\begin{aligned}\psi_{MA}(xA) = \psi_{MA}(e_i) = MA_i &= \left(\sum_{l=1}^{n} M_{il}A_{lk}\right)_k \\ &= M_iA = \psi_M(e_i)A = \psi_M(x)A\end{aligned}$$

yields the claim. □

With this flexibility, it is desirable to get an even simpler description of H-couple structures on $\mathbb{R}^n$. This motivates the following definition.

Definition. Let M be admissible and $(\mathbb{R}^n, \psi_M)$ be the H-couple of Proposition 1.2.11. Suppose for all i and $k > i$:

(i) $M_{ii} \in \{-1, 0, 1\}$;

(ii) $M_{ii} \neq 0 \implies M_{ik} = 0$.

Then $(\mathbb{R}^n, \psi_M)$ is said to be in *normal form*.

Let M be an admissible matrix. We show that the H-couple $(\mathbb{R}^n, \psi_M)$ can be brought into normal form via some suitable automorphism $A \in \mathcal{U}$. For $l = 1, \dots, n$ define the following matrices A_l recursively. Let $M^l := MA_1 \cdots A_{l-1}$ for each step l. For $M^l_{ll} = 0$, set $A_l := \mathrm{id}_n$. If $M^l_{ll} \neq 0$, we define

$$A_l := \frac{1}{M^l_{ll}} \begin{pmatrix} M^l_{ll} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & M^l_{ll} & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \mathrm{sign}(M^l_{ll}) & -M^l_{l,l+1} & \cdots & -M^l_{ln} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & M^l_{ll} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & M^l_{ll} \end{pmatrix} \in \mathcal{U}.$$

This yields $M^{l+1}_{ll} \in \{1, -1\}$ and $M^{l+1}_{lk} = 0$ for $k > l$. In the end, $M^{n+1} = MA$ where $A := A_1 \cdots A_n$ satisfies $MA_{ii} \in \{-1, 0, 1\}$ and if $MA_{ii} \neq 0$, then $MA_{ik} = 0$ for $k > i$. Thus $(\mathbb{R}^n, \psi_{MA})$ is a normal form of $(\mathbb{R}^n, \psi_M)$.

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^n \\ \mathbb{R}^n \arrow[u, "\psi_M"] \arrow[r, "A"] & \mathbb{R}^n \arrow[u, "\psi_{MA}"'] \end{tikzcd}$$

Remark. Observe that two admissible matrices M and $\tilde{M}$ have the same normal forms if and only if there is some $A \in \mathcal{U}$ such that $\tilde{M} = MA$.

Example. For $n = 3$, let $\psi_1 = (3, 1, 5), \psi_2 = (3, 2, 2), \psi_3 = (3, 2, 4)$, then

$$M = \begin{pmatrix} 3 & 1 & 5 \\ 3 & 2 & 2 \\ 3 & 2 & 4 \end{pmatrix}$$

and rescaling by the factors

$$A_1 = \begin{pmatrix} 1/3 & -1/3 & -5/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}$$

yields a normal form

$$MA = MA_1A_2A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

using the automorphism

$$x \mapsto x \begin{pmatrix} 1/3 & -1/3 & -4/3 \\ 0 & 1 & 3/2 \\ 0 & 0 & 1/2 \end{pmatrix}.$$

The normal form of the example has only $+1$ appearing on the diagonal. This H-couple appears in dimension n from Hardy fields of finitely iterated logarithms as we show at the end of the next chapter.

Remark. Note that normal forms are not unique. To give an example in $\mathbb{R}^2$ consider

$$M := \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, A := \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \in \mathcal{U}, \text{ and } MA = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}.$$

Here, the admissible matrices M and MA are different normal forms of M.

Here is an analogue of Theorem 1.1.13 for H-couples.

Theorem 1.2.13. *If* $\operatorname{rank}\Gamma = n$*, then* (Γ, ψ) *is isomorphic to some subcouple of* $(\mathbb{R}^n, \psi_M)$ *of rank n for some admissible M in normal form.*

Proof. Take archimedean representatives $\alpha_1, \dots, \alpha_n$ of Γ. By Theorem 1.1.13 there is an embedding $\iota : \mathbb{Q}\Gamma \hookrightarrow \mathbb{R}^n$ of ordered abelian groups such that $\iota(\alpha_i) = e_i$ for all i.

Claim. The matrix $M = (\iota(\psi(\alpha_i)))_i$ is admissible.

Let $i \geq j$. Then $[\alpha_i] \leq [\alpha_j]$ and so $\psi(\alpha_i) \geq \psi(\alpha_j)$ using (HC4).

(A1): Since ι is order preserving, $M_i = \iota(\psi(\alpha_i)) \geq \iota(\psi(\alpha_j)) = M_j$.

(A2): Observe that by (HC3) we have $\psi(\alpha_j) \leq \psi(\alpha_i) < \psi(\alpha_j) + \alpha$ for all $\alpha > 0$ such that $[\alpha] = [\alpha_j]$. Hence $0 \leq \iota(\psi(\alpha_i)) - \iota(\psi(\alpha_j)) = M_i - M_j < \iota(\alpha)$. Working in $(\mathbb{Q}\Gamma, \psi)$, we get $[M_i - M_j] < [e_j]$ by considering

$$0 \leq \iota(\psi_i) - \iota(\psi_j) \leq \inf\{\iota(\alpha) : \alpha \in \mathbb{Q}\Gamma^{>}, [\alpha] = [\alpha_j]\} \leq \frac{1}{n} e_j$$

for all n. So $M_{ij} - M_{jj} = 0$ for $i \geq j$.

By Proposition 1.2.11, M gives rise to an H-couple $(\mathbb{R}^n, \psi_M)$ such that $\psi_M(e_i) = M_i = \iota(\psi(\alpha_i))$. As $\{\psi_M(e_i) : i = 1, \dots, n\} \subseteq \iota(\Gamma)$, $(\iota(\Gamma), \psi_M)$ is a subcouple of $(\mathbb{R}^n, \psi_M)$. Then $\iota : \Gamma \to \iota(\Gamma)$ is by definition an isomorphism of H-couples. Take $A \in \mathcal{U}$ such that $(\mathbb{R}^n, \psi_{MA})$ is in normal form. By Lemma 1.2.12 we have an isomorphism of H-couples $(\mathbb{R}^n, \psi_M) \to (\mathbb{R}^n, \psi_{MA})$. Hence we can identify (Γ, ψ) as a subcouple of $(\mathbb{R}^n, \psi_{MA})$, which is in normal form. $\square$

Corollary 1.2.14. *Suppose* $\operatorname{rank}\Gamma = n > 0$*. Then there is some* $\beta \in \Gamma^{\neq}$ *such that* $\psi(\beta) \in \mathbb{Z}\beta$*. In particular,* $\mathbb{Z}\beta$ *is the underlying ordered abelian group of a subcouple of* (Γ, ψ)*.*

Proof. Embed (Γ, ψ) into $(\mathbb{R}^n, \psi_M)$ for some admissible M in normal form via Theorem 1.2.13. In case $M_{ii} = 0$ for all i we get that $\max\Psi = 0 \in \mathbb{Z}\beta$ for any β such that $\psi(\beta) = \max\Psi$. Otherwise let i be minimal such that $M_{ii} \neq 0$. Then $\psi_M(\psi_M) = \psi_M$, so the preimage β of ψ_M under the embedding does the job. $\square$

Remark. If $0 \neq \max\Psi$, we can also use Lemma 1.2.5 to get some β with $\beta + \psi(\beta) = 0$ and by [1] such β is given by $\psi(\max\Psi)$ which is coherent with Corollary 1.2.14.

2 Hardy fields

In this chapter, we turn to the field-theoretic perspective, introducing orderings, valuations, and derivations on fields. Together, these different features interact to form the structure of an H-field. We explain how every H-field naturally gives rise to an H-couple, and how Hardy fields containing $\mathbb{R}$ can be viewed as H-fields. It is established that the H-couple of a Hardy field is of Hardy type and has small derivation. Finally, we develop some extension results for Hardy fields, studying their real closures and considering solutions of first-order differential equations over Hardy fields which are shown to generate Hardy fields.

2.1 Ordered valued differential fields

The formal objects of our approach to ordered asymptotic differential algebra are H-fields. Building some theory of H-fields lays the groundwork for introducing Hardy fields. Following [4], we are interested in the interactions between different features of H-fields: derivations, orderings and valuations, which are dealt with in this section.

Let K be a field of characteristic 0. Then the prime subfield of K is $\mathbb{Q}$. Write K^{ac} for the algebraic closure of K and $K^{\times} := K \setminus \{0\}$ for the group of units of K. To emphasize the possible functional nature of the objects under consideration, f, g, h range over elements of K. We start with formalizing derivations on K.

Definition. A *derivation on* K is a map $\partial : K \to K$ such that for all f, g:

(i) $\partial(f + g) = \partial(f) + \partial(g)$;

(ii) $\partial(f \cdot g) = \partial(f)g + f\partial(g)$.

Call (K, ∂) a *differential field* if ∂ is a derivation on K.

We usually refer to K as a differential field without mentioning ∂. Moreover, introduce notations $f' := \partial(f)$ and $f^{(n)} := \partial(f^{(n-1)})$ with $f^{(0)} := f$. For $f \neq 0$, denote the *logarithmic derivative* $f^{\dagger} := f'/f$ of f. (This comes from $(\log g(t))' = g'(t)/g(t)$ for continuously differentiable positive functions g.)

Example. For $g \neq 0$ we have the quotient rule

$$f'g - fg' = (fg^{-1}g)'g - fg' = (fg^{-1})'g^2.$$

Hence if R is an integral domain equipped with a map ∂ satisfying (i) and (ii) such that $f' \in R$ for all $f \in R$, then its fraction field $\mathrm{Frac}(R)$ of R is a differential field with the unique derivation extending ∂.

Remark. We have the logarithmic derivative identity $(fg)^\dagger = f^\dagger + g^\dagger$ for $f, g \neq 0$ as

$$(fg)^\dagger = \frac{(fg)'}{fg} = \frac{f'g + fg'}{fg} = \frac{f'}{f} + \frac{g'}{g} = f^\dagger + g^\dagger.$$

If K is a differential field and $f \in K$ such that $\partial(f) = 0$, then f is called a *constant*. The collection of all constants $C_K := \{f \in K : \partial(f) = 0\}$ is a subfield of K, called the *field of constants* of K. If K is clear from the context, we write $C = C_K$.

Definition. Suppose $L \supseteq K$ is an extension of differential fields and $y \in L$. Then the differential subfield of L generated by y over K is $K\langle y\rangle := K(y, y', y'', \ldots)$.

Next, we describe how to add an ordering to K.

Definition. Let $\leq$ be a total ordering on K such that for all f, g, h:

(i) $f \leq g \implies f + h \leq g + h$;

(ii) $f \leq g \wedge 0 \leq h \implies f \cdot h \leq g \cdot h$.

Then $(K, \leq)$ is called an *ordered field*.

Example. Take any integral domain R and equip it with a total ordering $\leq$ satisfying (i) and (ii). Then its fraction field $\mathrm{Frac}(R)$ is an ordered field with a unique ordering $\tilde{\leq}$ extending $\leq$. This ordering is given by

$$f_1/g_1 \mathrel{\tilde{\leq}} f_2/g_2 \qquad \Longleftrightarrow \qquad f_1 g_2 \leq f_2 g_1$$

for all $f_1, f_2, g_1, g_2 \in R$ with $g_1, g_2 > 0$.

It is clear that algebraically closed fields cannot be ordered fields since for $i := \sqrt{-1}$ we have $i^2 = -1 < 0$. However, in the world of ordered fields there is a natural replacement for being algebraically closed:

Definition. Suppose K is an ordered field. Then K is called *real closed* if

(i) $\forall f : \exists g : \quad f \geq 0 \implies f = g^2$;

(ii) $\forall f_1, \ldots, f_{2n} : \exists g : \quad g^{2n+1} + f_{2n} g^{2n} + \cdots + f_1 g + f_0 = 0$ for all n.

Remark. Real closed ordered fields can be characterized in different ways ([4], Theorem 3.5.4) from which we need the following version. If K is an ordered field, then K is real closed if and only if $K(i)$ is algebraically closed. A *real closure* of K is an algebraic ordered field extension of K which is real closed. By Artin-Schreier ([4], Theorem 3.5.9), K has a real closure which is unique up to unique isomorphism over K. We denote the real closure of K by K^{rc}. Every ordered field embedding $K \to L$ into some real closed ordered field L extends uniquely to an ordered field embedding $K^{rc} \to L$.

The last objective is to introduce a notion of size for elements of K similar as we did for ordered abelian groups. For this, we define a map on the multiplicative group of K taking values in some ordered abelian group Γ to compare different functions by their values in Γ. Negative valuations in Γ correspond to large elements in K.

Definition. A *valuation* on K is a map $v : K^\times \to \Gamma$ into some ordered abelian group Γ (written additively) such that for all $f, g \neq 0$:

(i) $v(fg) = v(f) + v(g)$;

(ii) $f + g \neq 0 \implies v(f + g) \geq \min\{v(f), v(g)\}$.

Remark. By (i), the image $v(K^\times)$ is a subgroup of Γ, called the *value group of* K and we always assume $\Gamma = v(K^\times)$. We set $v(0) := \infty > \Gamma$.

For the following discussion we refer to [4], Chapter 3.

Definition. Let A be a subring of K such that for all $f \neq 0$:

$$f \in A \quad \vee \quad f^{-1} \in A. \tag{2.1.1}$$

Then A is called a *valuation ring* of K.

Note that any valuation ring of K is a local ring with unique maximal ideal $A \setminus A^\times$.
Let v be a valuation on K. The set

$$\mathcal{O}_v := \{f \in K : v(f) \geq 0\}$$

is a valuation ring of K. We drop the index v if the valuation is clear from the context. The unique maximal ideal of $\mathcal{O}$ is $\mathcal{o} := \{f \in K : v(f) > 0\}$. Valuation rings and valuations are closely related: Every valuation gives rise to its valuation ring $\mathcal{O}$ and conversely every valuation ring A of K is the valuation ring of some valuation. A natural valuation with valuation ring A is

$$v_A : K^\times \to \Gamma, f \mapsto fA^\times$$

where Γ is an additive copy of the multiplicative group $K^\times/A^\times$ equipped with the ordering

$$fA^\times \geq gA^\times :\iff f/g \in A.$$

Two valuations v_1 and v_2 have the same valuation ring if and only if there is an isomorphism $\iota : v_1(K^\times) \to v_2(K^\times)$ of ordered abelian groups such that $v_2 = \iota \circ v_1$. In this case, v_1 and v_2 are called *equivalent.*

Call $(K, \mathcal{O})$ a *valued field* if $\mathcal{O}$ is a valuation ring of K. If $\mathcal{O}$ is clear from the context, we just refer to the valued field K. Let $\operatorname{res}(K) := \mathcal{O}/\mathcal{o}$ be the *residue field* of K.

Suppose K is a valued field. If $L \supseteq K$ is a valued field such that $\mathcal{O}_L \supseteq \mathcal{O}$ and $\mathcal{O}_L \cap K = \mathcal{O}$, then L is a *valued field extension* of K. For a valued field extension L of K, Γ_L is an ordered abelian group extension of Γ_K and $\operatorname{res}(L)$ is a field extension of $\operatorname{res}(K)$. Recall that the *transcendence degree* $\operatorname{trdeg}(L|K)$ of a field extension L over K is the cardinality of a maximal K-algebraically independent set of elements of L if finite, otherwise $\operatorname{trdeg}(L|K) := \infty$. There is a useful inequality relating K, L and their valuations, called the Zariski-Abhyankar inequality ([4], Corollary 3.1.11):

Lemma 2.1.1. *Let L be a valued field extension of K. Then*

$$\operatorname{trdeg}(L|K) \geq \operatorname{rank}_{\mathbb{Q}}(v(L^\times)/v(K^\times)) + \operatorname{trdeg}(\operatorname{res}(L)|\operatorname{res}(K)).$$

Proof. We give a quick proof sketch for this. It is enough to show that any $x_1, \dots, x_n \in \mathcal{O}_L$ such that $x_1 + o_L, \dots, x_n + o_L \in \operatorname{res}(L)$ are algebraically independent over $\operatorname{res}(K)$ and $y_1, \dots y_m \in L^\times$ with $v(y_1), \dots, v(y_m)$ $\mathbb{Z}$-linearly independent over $v(K^\times)$ are K-algebraically independent. Note that $x_i \in \mathcal{O}_L^\times$, so $v(x_i) = 0$ for all i. To avoid too much notation, Einstein summation convention is used for i, j, k, l. We need to check that any polynomial $P = f_{k,l}^{i,j} X_i^k Y_j^l \in K[X, Y]^{\neq}$ in $n + m$ variables satisfies

$$P(x_1, \dots, x_n, y_1, \dots, y_m) \neq 0.$$

This follows from

$$v(P(x_1, \dots, x_n, y_1, \dots, y_m)) = \min_{j_0, l_0} \{ \min_{i_0, k_0} \{ v(f_{k_0, l_0}^{i_0, j_0}) \} + l_0 v(y_{j_0}) \},$$

what we show next. Fixing j_0 and l_0 such that $y_{j_0}^{l_0}$ appears in P, divide P by some coefficient $f_{k_0,l_0}^{i_0,j_0}$ of smallest valuation to get new coefficients $\tilde{f}_{k,l_0}^{i,j_0} \in \mathcal{O}$ and so $\tilde{f}_{k,l_0}^{i,j_0} x_i^k \in \mathcal{O}_L$. If $\tilde{f}_{k,l_0}^{i,j_0} x_i^k \in o_L$, then applying the residue map yields $(\tilde{f}_{k,l_0}^{i,j_0} + o_L)(x_i + o_L)^k = o_L$ in $\operatorname{res}(L)$, contradicting algebraic independence of $\{x_i + o_L\}_i$ over $\operatorname{res}(K)$. So $v(\tilde{f}_{k,l_0}^{i,j_0} x_i^k) = 0$ and $v(f_{k,l_0}^{i,j_0} x_i^k) = v(f_{k_0,l_0}^{i_0,j_0}) = \min_{i,k} \{ v(f_{k,l_0}^{i,j_0}) \}$ after multiplying with $f_{k_0,l_0}^{i_0,j_0}$ again.

By $\mathbb{Z}$-linear independence of $\{v(y_i)\}_i$ over $v(K^\times)$, for all $(j_0, l_0) \neq (\tilde{j}_0, \tilde{l}_0)$ such that $y_{j_0}^{l_0}$ and $y_{\tilde{j}_0}^{\tilde{l}_0}$ appear in P we have

$$v(f_{k,l_0}^{i,j_0} x_i^k y_{j_0}^{l_0}) = \min_{i,k} \{ v(f_{i,j_0}^{k,l_0}) \} + l_0 v(y_{j_0}) \neq \min_{i,k} \{ v(f_{i,\tilde{j}_0}^{k,\tilde{l}_0}) \} + \tilde{l}_0 v(y_{\tilde{j}_0}) = v(f_{k,\tilde{l}_0}^{i,\tilde{j}_0} x_i^k y_{\tilde{j}_0}^{\tilde{l}_0}),$$

and so

$$v(f_{k,l}^{i,j} x_i^k y_j^l) = \min_{j_0, l_0} \{ \min_{i_0, k_0} \{ v(f_{k_0,l_0}^{i_0,j_0}) \} + l_0 v(y_{j_0}) \} \in v(L^\times).$$

This implies $P(x_1, \dots, x_n, y_1, \dots, y_m) \neq 0$. $\square$

Valuations can also be expressed using relational language which compare the asymptotic behavior of elements of K.

Definition. A *dominance relation* on K is a binary relation $\preccurlyeq$ such that for all f, g, h:

- $1 \not\preccurlyeq 0$;
- $f \preccurlyeq f$;
- $f \preccurlyeq g \wedge g \preccurlyeq h \implies f \preccurlyeq h$;
- $f \preccurlyeq g \vee g \preccurlyeq f$;

- $f \preccurlyeq g \iff fh \preccurlyeq gh$ for $h \neq 0$;
- $f \preccurlyeq h \wedge g \preccurlyeq h \implies f + g \preccurlyeq h$.

There is a one-to-one correspondence between valuation rings and dominance relations ([4], Section 3.1). Every dominance relation gives rise to the valuation ring

$$\mathcal{O} := \{f \in K : f \preccurlyeq 1\}$$

of K and every valuation induces the dominance relation $f \preccurlyeq g :\iff v(f) \geq v(g)$ on K. There are several other useful asymptotic relations:

- $f \prec g \quad :\iff \quad f \preccurlyeq g \wedge g \not\preccurlyeq f$;
- $f \asymp g \quad :\iff \quad f \preccurlyeq g \wedge g \preccurlyeq f$;
- $f \sim g \quad :\iff \quad f - g \prec g$.

Here, $f \sim g$ implies $f \asymp g$, and both of these relations are equivalence relations on K.

2.1.1 Convex valuations on ordered fields

Suppose K is an ordered field. As with ordered abelian groups, we introduce convex valuations on K, which are valuations respecting the ordering on K. The valuation ring of a convex valuation is a convex subring of K, and each convex subring of K gives rise to a convex valuation as we show next.

Lemma 2.1.2. *Let A be a convex subring of K. Then A is a valuation ring of K.*

Proof. This follows from the observation that $[0, 1] \subseteq A$ and for all f there holds $|f| \leq 1$ or if $|f| > 1$, then $\left|f^{-1}\right| < 1$. □

Lemma 2.1.3. *Let v be a valuation on K, then the following conditions are equivalent:*

(i) $\mathcal{O}_v$ is convex;

(ii) $\forall f, g : \quad 0 < f \leq g \implies v(f) \geq v(g)$.

Proof. (ii) $\implies$ (i): Suppose $g \in \mathcal{O}$ and $f \neq 0$ such that $|f| \leq |g|$. Then by (ii), $v(f) = v(|f|) \geq v(|g|) = v(g) \geq 0$, hence $f \in \mathcal{O}$, so $\mathcal{O}$ is convex.

(i) $\implies$ (ii): Since $\mathcal{O}$ is convex, we get that if $0 < f/g \leq 1$, then $f/g \in \mathcal{O}$, hence $v(f/g) \geq 0$ showing $v(f) \geq v(g)$. □

Definition. A valuation on K that satisfies the equivalent conditions of Lemma 2.1.3 is called *convex.*

Every ordered field can be equipped with a natural convex valuation, the standard valuation of its underlying additive ordered abelian group:

Lemma 2.1.4. *Let $v : K^\times \to \Gamma$ be the standard valuation on the additive group of K and define $[f] + [g] := [fg]$ for $f, g \neq 0$. Then the ordered set Γ is an ordered abelian group and v a convex valuation on K with valuation ring $\mathcal{O} = \operatorname{conv} \mathbb{Q}$, the convex hull of $\mathbb{Q}$ in K.*

Proof. By definition of $+$, the multiplicative group structure of $K^\times$ induces the additive group structure on Γ. This does not depend on representatives since for all f_1, g_1, f_2, g_2 such that $|f_1| \leq n' |f_2| \leq n |f_1|$ and $|g_1| \leq m' |g_2| \leq m |g_1|$ for some n, n', m, m' we have that

$$|f_1 g_1| \leq n'm' |f_2 g_2| \leq nm |f_1 g_1| .$$

Recall that Γ is equipped with the reverse ordering. If $[f] \leq [g]$ in Γ, then

$$|g| \leq n |f| \implies |gh| \leq n |fh| ,$$

so $[fh] \leq [gh]$ in Γ. Thus Γ is an ordered abelian group.

Next, we show that v is a convex valuation on K. Let $f, g, h \neq 0$. By definition we have $v(fg) = [fg] = [f] + [g] = v(f) + v(g)$ and if $f + g \neq 0$, then $v(f + g) = [f + g] \geq \min\{[f], [g]\} = \min\{v(f), v(g)\}$ in Γ by Lemma 1.1.2. The valuation ring of v is $\mathcal{O} = \{f \in K : |f| \leq n \text{ for some } n\} = \operatorname{conv} \mathbb{Q}$ which is convex, so v is convex. □

Corollary 2.1.5. *Let $\preccurlyeq$ be a dominance relation on K which satisfies*

$$\{f \in K : f \preccurlyeq 1\} = \operatorname{conv} \mathbb{Q}.$$

Then the valuation v associated to $\preccurlyeq$ is equivalent to the standard valuation.

Proof. By Lemma 2.1.4, the standard valuation has the same valuation ring as v, hence by the correspondence between valuations and valuation rings, v is equivalent to the standard valuation. □

Remark. Any other convex valuation $\tilde{v}$ on K satisfies $0 < v(f) \leq v(g) \implies \tilde{v}(f) \leq \tilde{v}(g)$ if v is the standard valuation as $\tilde{v}$ is constant on archimedean classes of $K^\times$.

2.2 H-fields and their H-couples

Let K be an ordered valued differential field with value group Γ which is a differential field equipped with an ordering and a valuation ring where no interaction between derivation, ordering and valuation is assumed. We change that in the next definition, adding some natural expected properties.

Definition. We call K an *H-field* if for all f:

(H1) $f > C \implies f' > 0$;

(H2) $\mathcal{O} = \operatorname{conv} C$ and $\mathcal{O} = C + \scriptstyle\mathcal{O}$.

If $\partial(\scriptstyle\mathcal{O}) \subseteq \scriptstyle\mathcal{O}$, then K is said to have *small derivation.*

Suppose K is an H-field.

Remark. Note that when C is archimedean, then $\mathcal{O}$ is the valuation ring of the standard valuation on K. By (H2), the ordering and differentiation on K already determine the valuation on K which is always convex and if K has small derivation, then $\partial(\mathcal{O}) \subseteq \mathcal{o}$.

We want to derive a version of L'Hôpitals rule for K. In the proof we need the following property, referred to as K being *pre-differentially valued.*

Lemma 2.2.1. *If $f \preccurlyeq 1$ and $0 \neq g \not\asymp 1$, then $f' \prec g^\dagger$.*

Proof. By (H2), $f \in \mathcal{O}$ yields some $c_0 \in C$ such that $v(c_0 - f) > 0$. We can assume $g > C$ since $g^\dagger \asymp (1/g)^\dagger \asymp (-g)^\dagger \asymp (-1/g)^\dagger$ and by (H1) we get $g' > 0$. Also, $0 < c - f \asymp 1$ for all $c \in C^{>c_0}$, so $(c-f)g > C$, hence

$$(c-f)g' - f'g = ((c-f)g)' > 0 \implies cg' - fg' > f'g \implies c - f > f'/g^\dagger$$

by (H1). Similarly, for $c \in C^{<c_0}$ we get $0 < f - c \asymp 1$, thus $(f-c)g > C$, and so

$$(f-c)g' + f'g = ((f-c)g)' > 0 \implies fg' - cg' > -f'g \implies c - f < f'/g^\dagger.$$

Hence $C^< < f - c_0 + f'/g^\dagger < C^>$ yields $f - c_0 + f'/g^\dagger \prec 1$, which gives $v(f'/g^\dagger) \geq \min\{v(f - c_0 + f'/g^\dagger), v(c_0 - f)\} > 0$, so $f'/g^\dagger \prec 1$. □

The next statement is L'Hôpitals rule for K.

Corollary 2.2.2. *If $f, g \neq 0$ and $f \asymp g \not\asymp 1$, then $f'/g' \sim f/g$. Hence $f' \asymp g'$.*

Proof. First, we note that $g' \neq 0$ by (H2). By Lemma 2.2.1 $(f/g)' \prec g^\dagger$. Thus

$$\frac{f'}{g'} - \frac{f}{g} = \frac{f'g - fg'}{gg'} = \frac{(f/g)'}{g^\dagger} \prec 1 \asymp \frac{f}{g},$$

which yields the first claim. Moreover, $f'/g' \sim f/g \asymp 1$ gives $f' \asymp g'$. □

Here is another useful fact about K.

Lemma 2.2.3. *If $f, g \neq 0$ such that $f \prec g \prec 1$, then $f^\dagger \succcurlyeq g^\dagger$.*

Proof. We can assume that $f, g > 0$. It follows that $g/f > C$ and $1/g > C$, hence $(g/f)' > 0$ and $-g^\dagger/g = (1/g)' > 0$ by (H1). This implies $(g/f)^\dagger = (g/f)'(f/g) > 0$, and thus

$$0 > g^\dagger = \left(f\frac{g}{f}\right)^\dagger = f^\dagger + \left(\frac{g}{f}\right)^\dagger > f^\dagger.$$

Now $0 < -g^\dagger < -f^\dagger$ implies $g^\dagger \preccurlyeq f^\dagger$ by convexity of v. □

There is additional structure on the value group of K. We can associate to K a map $\psi : \Gamma^{\neq} \to \Gamma$ defined by $\psi(v(f)) := v(f^\dagger)$ for $0 \neq f \not\asymp 1$. Corollary 2.2.2 ensures that the definition of ψ does not depend on the choice of f:

$$1 \not\asymp f \asymp g \implies f'/g' \sim f/g \implies f^\dagger = f'/f \sim g'/g = g^\dagger.$$

The main bridge between H-couples and H-fields is the next theorem.

Theorem 2.2.4. *Let $\psi : \Gamma^{\neq} \to \Gamma$ where $\psi(v(f)) := v(f^\dagger)$. Then (Γ, ψ) is an H-couple.*

Proof. We show that the axioms for H-couples hold. For this, let $f, g \neq 0$ such that $v(f), v(g) \neq 0$. In showing the axioms, we implicitly use that v is surjective and $v(f) = v(-f) = -v(1/f)$.

(HC1): The logarithmic derivative identity yields

$$\begin{aligned}\psi(v(f) + v(g)) &= \psi(v(fg)) = v((fg)^\dagger) = v(f^\dagger + g^\dagger) \\ &\geq \min\{v(f^\dagger), v(g^\dagger)\} = \min\{\psi(v(f)), \psi(v(g))\}.\end{aligned}$$

(HC2): If $k \in \mathbb{Z}^{\neq}$, then $v(f^k) = kv(f)$ and $v(kf) = v(f)$, so

$$\psi(kv(f)) = \psi(v(f^k)) = v\left(\frac{(f^k)'}{f^k}\right) = v\left(\frac{kf^{k-1}f'}{f^k}\right) = v\left(k\frac{f'}{f}\right) = v(f^\dagger) = \psi(v(f)).$$

(HC3): From Lemma 2.2.1 it follows that if $v(f) > 0$, then

$$\psi(v(f)) + v(f) = v(f^\dagger f) = v(f') > v(g^\dagger) = \psi(v(g)).$$

(HC4): We have

$$0 < v(g) < v(f) \implies \psi(v(f)) = v(f^\dagger) \leq v(g^\dagger) = \psi(v(g))$$

using Lemma 2.2.3. □

Remark. Note that the H-couple of an H-field with small derivation has small derivation.

2.3 Hardy fields as H-fields

In this section we develop the basics of Hardy fields [4]. Our first task is to introduce the ring of germs at $+\infty$. Consider the set $\mathbb{R}^{\mathbb{R}}$ of all real valued functions $f : \mathbb{R} \to \mathbb{R}$ and define the equivalence relation

$$f =_e g \quad :\Longleftrightarrow \quad \exists t_0 \in \mathbb{R} : \forall t > t_0 : f(t) = g(t)$$

on $\mathbb{R}^{\mathbb{R}}$. The equivalence class f_e of f consists of all functions which are eventually equal to f. This way we emphasize only the asymptotic behavior of f, forgetting the values of f on any bounded interval of its domain. The set f_e is called the *germ* of f at $+\infty$. Let x denote the germ of the identity function $\mathrm{id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$. We say that a property holds for $f(t)$ *eventually* if there is some $t_0 \in \mathbb{R}$ such that the property holds for $f(t)$ when $t > t_0$. Some property holds for a germ if and only if it holds for any representative eventually. Since eventual properties are independent of representatives, we use the same letter f for functions and germs in statements which hold eventually.

Using this terminology we can define addition and multiplication of germs inducing a ring structure on the set of germs. If $f_1(t) = f_2(t)$ and $g_1(t) = g_2(t)$ eventually, then $f_1(t) + g_1(t) = f_2(t) + g_2(t)$ and $f_1(t)g_1(t) = f_2(t)g_2(t)$ eventually.

This gives rise to the *ring of germs* at $+\infty$, which is not a field since for example the germ of the function $t \mapsto \sin(t)$ has no inverse. In general, no non-zero function whose set of zeros is unbounded towards $+\infty$ can have an inverse. Denoting the class of n times differentiable functions as C^n (where C^0 means continuous), we get several interesting subrings of the ring of germs at $+\infty$. Define for each n the ring

$$\mathcal{C}^n := \{f \in \mathbb{R}^{\mathbb{R}}/_e| \quad \exists t_0 \in \mathbb{R} : f|_{(t_0,\infty)} \in C^n\}$$

of n-times differentiable germs. As the definition indicates, a germ is n-times differentiable iff some representative is C^n eventually. Note that a germ $f \in \mathcal{C}^{n+1}$ has a derivative $f' \in \mathcal{C}^n$ given by the germ of the derivative of some representative of f. (Taking derivatives is also independent of representatives, as $f(t) = g(t)$ eventually implies $f'(t) = g'(t)$ eventually.)

We restrict our attention even further to germs which admit for each n some representative which is C^n, so that we can differentiate this germ infinitely often. Formally, we define the ring $\mathcal{C}^{<\infty} := \bigcap_{n\in\mathbb{N}} \mathcal{C}^n$ of (infinitely often) differentiable germs and by construction, $(\mathcal{C}^{<\infty})' \subseteq \mathcal{C}^{<\infty}$. Note that the domain of a representative of a germ in $\mathcal{C}^{<\infty}$ may depend on the order of the desired differentiability, i.e. for each n we find some representative on some halfline (t_n, ∞) which is C^n, but it could in general be possible that $t_n \to \infty$ as $n \to \infty$. Hence we need to distinguish between $\mathcal{C}^{<\infty}$ and the ring $\mathcal{C}^\infty$ of germs which have some C^∞ representative. The same reasoning as above shows that $\mathcal{C}^{<\infty}$ is still not a field, hence we elevate this condition into the definition of our main object of asymptotic analysis:

Definition. A *Hardy field* is a differential subfield of $\mathcal{C}^{<\infty}$, i.e. a subring of $\mathcal{C}^{<\infty}$ closed under differentiation which happens to be a field.

Remark. The construction of a germ in $\mathcal{C}^{<\infty} \setminus \mathcal{C}^\infty$ which lives in a Hardy field can be found in for example [10].

Let K be a Hardy field. Then the prime field $\mathbb{Q}$ of K can be identified with germs of constant functions and the constant field C of K as a subfield of $\mathbb{R}$. By Zorn, every Hardy field is contained in a maximal Hardy field and there are $2^{\mathfrak{c}}$ many different maximal Hardy fields where $\mathfrak{c}$ is the cardinality of the continuum [6].

In the following, we analyze some basic properties of K. Note that by definition, K is a differential field and we naturally get an ordering and a valuation on K.

Ordering. To begin with, we observe that every germ $f \in K^\times$ has a multiplicative inverse, so $f(t) \neq 0$ holds eventually. By eventual continuity of some representative of f, either $f(t) > 0$ or $f(t) < 0$ eventually. Setting

$$g \leq h :\iff 0 \leq h(t) - g(t) \text{ eventually}$$

yields a total ordering on K. Equipped with this ordering, K becomes an ordered field. Furthermore, if f is not constant, then $f' > 0$ or $f' < 0$, so f is eventually monotone. So the limit $\lim_{t\to\infty} f(t)$ exists in $\mathbb{R}\cup\{\pm\infty\}$ and does not depend on the representative of f.

Valuation. In order to compare asymptotic behavior of germs we introduce a relation on K by

$$f \preccurlyeq g \quad :\Longleftrightarrow \qquad g \neq 0 \wedge \lim_{t\to\infty} \frac{f(t)}{g(t)} \in \mathbb{R}$$

and set by definition $0 \preccurlyeq 0$.

Lemma 2.3.1. *The relation $\preccurlyeq$ yields a dominance relation on K which induces the standard valuation on K.*

Proof. It can be checked that all properties of dominance relations are satisfied and the valuation ring induced by this dominance relation is given by $\mathcal{O} = \{f \in K : f \preccurlyeq 1\} = \{f \in K : \lim_{t\to\infty} f(t) \in \mathbb{R}\} = \operatorname{conv}\mathbb{Q}$. This valuation ring gives rise to the standard valuation by Corollary 2.1.5. $\square$

The description above makes K into an ordered valued differential field. We use the notions

$$f \preccurlyeq g \iff v(f) \geq v(g) \iff [f] \leq [g] \iff \exists n : |f| \leq n\,|g|$$

interchangeably. Here is the analytic meaning of the different asymptotic relations for $f, g \neq 0$:

$$\begin{aligned}
v(f) < v(g) \quad &\Longleftrightarrow \quad f \succ g \quad \Longleftrightarrow \quad \lim_{t\to\infty}\frac{f(t)}{g(t)} = \pm\infty;\\
v(f) > v(g) \quad &\Longleftrightarrow \quad f \prec g \quad \Longleftrightarrow \quad \lim_{t\to\infty}\frac{f(t)}{g(t)} = 0;\\
v(f) = v(g) \quad &\Longleftrightarrow \quad f \asymp g \quad \Longleftrightarrow \quad \lim_{t\to\infty}\frac{f(t)}{g(t)} \in \mathbb{R}^\times;\\
v(f-g) > v(f) \quad &\Longleftrightarrow \quad f \sim g \quad \Longleftrightarrow \quad \lim_{t\to\infty}\frac{f(t)}{g(t)} = 1.
\end{aligned}$$

Corollary 2.3.2. *If $K \supseteq \mathbb{R}$, then K is an H-field with constant field $\mathbb{R}$ and small derivation.*

Proof. We showed above that Hardy fields are ordered valued differential fields with valuation ring $\operatorname{conv}(\mathbb{Q})$ and constant field contained in $\mathbb{R}$, hence $C = \mathbb{R}$. It remains to check the axioms for H-fields.

(H1): Let $f > \mathbb{R}$. If $f'(t) \leq 0$ for all $t > t_0$, then $f(t) \leq f(t_0)$ for all $t > t_0$, contradiction. So $f'(t) > 0$ eventually.

(H2): Let $f \preccurlyeq 1$. As $C = \mathbb{R}$, we have $c := \lim_{t\to\infty} f(t) \in C$ and so $f - c \prec 1$.

Next we show that the derivation is small. For this let $f > 0$. If $f \preccurlyeq 1$, then $\lim_{t\to\infty} f(t) = c$ for some representative of f and $c \in \mathbb{R}$. If $f' \succcurlyeq 1$, then $\lim_{t\to\infty} |f'(t)| \geq \epsilon$ for some $\epsilon > 0$. But this gives $|f(t)| \geq \epsilon t + d$ eventually for some $d \in \mathbb{R}$. Now $f \succcurlyeq \epsilon x + d \asymp x \succ 1$ in $K(x)$, which is a Hardy field as we will show in Corollary 2.3.10, but this contradicts $f \preccurlyeq 1$. $\square$

2.3.1 H-couples of Hardy fields

Suppose $\mathbb{R} \subseteq K$. By Corollary 2.3.2, K is an H-field and as such, K has an H-couple (Γ, ψ) where $\Gamma := [K^{\times}]$ is the value group of K from the standard valuation equipped with the reverse ordering. Recall that K has valuation ring $\mathcal{O} = \operatorname{conv} C = \operatorname{conv} \mathbb{Q}$ and $\psi : \Gamma^{\neq} \to \Gamma$ is defined as $\psi(v(f)) := v(f^{\dagger})$ for all $0 \neq f \not\asymp 1$.

Lemma 2.3.3. *The H-couple (Γ, ψ) of K is of Hardy type.*

Proof. Let $f, g > 0$ such that $v(f), v(g) < 0$ and $v(f^{\dagger}) = v(g^{\dagger})$. The goal is to prove that $v(f) = v(g)$. In Corollary 2.3.10, we will show that $L := K(\log(f), \log(g))$ is a Hardy field extension of K. Working in L we have $v(\log(f)), v(\log(g)) < 0$ and

$$v(\log(f)') = v(f^{\dagger}) = v(g^{\dagger}) = v(\log(g)').$$

Translating this into an analytic statement gives

$$\lim_{t\to\infty} \log(f(t))' / \log(g(t))' \in \mathbb{R}^{\times}$$

and L'Hôpitals rule for real valued functions yields

$$\lim_{t\to\infty} \log(f(t)) / \log(g(t)) = \log(f(t))' / \log(g(t))' \in \mathbb{R}^{\times}.$$

Hence $0 > v(\log(f)) = v(\log(g))$, so $[\log(f)] = [\log(g)]$ in $[v(L^{\times})]$. It follows that $\log(f) \leq n \log(g)$ and $\log(g) \leq n \log(f)$ for some n which gives $0 < f \leq g^n$ and $0 < g \leq f^n$. Thus $v(g) \leq v(f^n) = nv(f)$ and $v(f) \leq v(g^n) = nv(g)$ by convexity of v, implying $[v(f)] = [v(g)]$. $\square$

Let (Γ, ψ) be the H-couple of K. The above shows that (Γ, ψ) is necessarily of Hardy type and has small derivation. Also, since K has size at most continuum, the same is true for Γ. But it is not clear whether a given H-couple of Hardy type with small derivation of size at most continuum arises as the H-couple of some Hardy field. This question is dealt with in the coming chapters. Before that, we describe an analytic intuition of the ψ map. The standard valuation on K does not distinguish between elements which are in constant multiple reach of each other. We can zoom out even further and compare multiplicatively the growth behavior of germs having the same ψ-value. The proof of Lemma 2.3.3 yields the following corollary.

Corollary 2.3.4. *For $f, g \succ 1$ we have*

$$\psi(v(f)) = \psi(v(g)) \quad \Longleftrightarrow \quad |g|^n \leq |f| \wedge |f|^n \leq |g| \ \textit{for some } n \geq 1.$$

This motivates the following asymptotic relations for $f, g \succ 1$:

$$\begin{aligned}
f \preceq\!\!\!\preceq g \quad &:\Longleftrightarrow \quad f^{\dagger} \preccurlyeq g^{\dagger} \quad \Longleftrightarrow \quad \psi(v(f)) \geq \psi(v(g));\\
f \succeq\!\!\!\succeq g \quad &:\Longleftrightarrow \quad f^{\dagger} \succcurlyeq g^{\dagger} \quad \Longleftrightarrow \quad \psi(v(f)) \leq \psi(v(g));\\
f \prec\!\!\!\prec g \quad &:\Longleftrightarrow \quad f^{\dagger} \prec g^{\dagger} \quad \Longleftrightarrow \quad \psi(v(f)) > \psi(v(g));\\
f \succ\!\!\!\succ g \quad &:\Longleftrightarrow \quad f^{\dagger} \succ g^{\dagger} \quad \Longleftrightarrow \quad \psi(v(f)) < \psi(v(g));\\
f \asymp\!\!\!\asymp g \quad &:\Longleftrightarrow \quad f^{\dagger} \asymp g^{\dagger} \quad \Longleftrightarrow \quad \psi(v(f)) = \psi(v(g)).
\end{aligned}$$

Suppose $f, g \succ 1$. If $f \asymp g$ then f and g are called *comparable* and Corollary 2.3.4 characterizes the *comparability class* $\{g \succ 1 : f \asymp g\}$ of f. The relation $\asymp$ is an equivalence relation on $K^{\succ} := K \setminus \mathcal{O}$, partitioning $K^{\succ}$ into its comparability classes. The map $f \mapsto v(f^{\dagger})$ for $f \succ 1$ induces a bijection between the set of comparability classes and Ψ. Also, $f \preccurlyeq g$ if an only if $|f| \leq |g|^n$ for some n ([4], Corollary 9.1.11). We call f *flatter* than g if $f \prec\!\!\prec g$ in which case $|f|^n < |g|$ for all n.

2.3.2 Extensions of Hardy fields

This section outlines some methods to extend K to a Hardy field $L \supseteq K$ and investigates when a germ $y \in \mathcal{C}^{<\infty}$ generates a Hardy field $K\langle y\rangle$ over K. The main result states that if a germ $y \in \mathcal{C}^1$ satisfies a first order differential equation over K, then $K\langle y\rangle = K(y)$ is a Hardy field extension of K. In particular, the constant field of K can be enlarged to contain $\mathbb{R}$ so that $K(\mathbb{R})$ is an H-field. For any Hardy field extension $L \supseteq K$ we have $v(L^{\times}) \supseteq v(K^{\times})$ and if $K \supseteq \mathbb{R}$ and $v(L^{\times}) = v(K^{\times})$, we call L an *immediate* extension of K. In this case, K and L have the same H-couple.

The polynomial ring over K is denoted by $K[Y]$ where Y is a single indeterminate. Likewise, we define the (differential) ring of differential polynomials $K\{Y\} := K[Y, Y', \ldots]$ in Y where $\partial(Y^{(n)}) = Y^{(n+1)}$. In particular, to show that there is a Hardy field extension of K containing some germ $y \in \mathcal{C}^{<\infty}$ amounts to showing that for every differential polynomial $P \in K\{Y\}^{\neq}$, if $P(y) \neq 0$, then $P(y)$ has to satisfy $P(y(t)) \neq 0$ eventually. Here $P(y)$ means plugging in $y^{(n)}$ for $Y^{(n)}$. Call a germ $y \in \mathcal{C}^0$ *algebraic* over K if it satisfies a polynomial equation over K. The collection of all algebraic germs over K is a real closure of K contained in $\mathcal{C}^{<\infty}$. When referring to a real closure K^{rc} of K, the latter is meant.

The first natural question is whether an algebraic germ over K lives in a Hardy field extension of K, which is answered in the next theorem due to A. Robinson ([7], Proposition 5.3.2).

Theorem 2.3.5. *The real closure K^{rc} of K is a Hardy field extension of K.*

Hence any algebraic germ over K generates a Hardy field over K.

Lemma 2.3.6. *If $y \in \mathcal{C}^{<\infty}$ is algebraic over K, then $v(y) \in \mathbb{Q}v(K^{\times})$. In particular, $v(y) \notin \mathbb{Q}v(K^{\times})$ implies that y is transcendental over K.*

Proof. By Theorem 2.3.5 there is some Hardy field extension of K containing y. Take some monic polynomial $P(Y) = Y^n - \sum_{i=0}^{n-1} g_i Y^i \in K[Y]^{\neq}$ such that $P(y) = 0$. Then $y^n = \sum_{i=0}^{n-1} g_i y^i$, hence $nv(y) \geq \min_{g_i \neq 0}\{v(g_i) + iv(y)\}$. If there are $i < j$ such that $v(g_i) + iv(y) = v(g_j) + jv(y)$, then $(j-i)v(y) = v(g_i/g_j)$. Otherwise $nv(y) = \min_{g_i \neq 0}\{v(g_i) + iv(y)\}$, so $(n-j)v(y) = v(g_j)$ for some j. In both cases it follows that $v(y) \in \mathbb{Q}v(K^{\times})$. $\square$

Lemma 2.3.7. *The value group of K^{rc} is $\mathbb{Q}v(K^{\times})$.*

Proof. As K^{rc} is real closed, $\sqrt{z} \in K^{rc}$ for every $z \in K^{>}$. Applying this repeatedly we get $z^{1/2^k} \in K^{rc}$ for all $k > 0$. We now show that $z^{1/n} \in K^{rc}$. If n is odd then $Y^n - z$

has a zero since K^{rc} is real closed. If n is even, write $n = 2^k m$ for some odd m. Then $Y^n - z = (Y^m - z^{1/2^k}) \prod_{i=1}^{k} (Y^{n/2^i} + z^{1/2^i})$ has a zero because $Y^m - z^{1/2^k} \in K^{rc}[Y]$ is a polynomial of odd degree.

As $v(z) = nv(z^{1/n})$ we get $\mathbb{Q}\Gamma \subseteq \Gamma^{rc}$. Any $z \in K^{rc}$ is algebraic over K, thus $v(z) \in \mathbb{Q}v(K^\times)$ by Lemma 2.3.6. $\square$

Lemma 2.3.8. *Let $\mathbb{R} \subseteq K \subseteq L$ be Hardy fields and $y \in L$ such that $K(y)$ is a Hardy field with $v(y) \notin \mathbb{Q}\Gamma$. Then y is transcendental over K and $v(\tilde{K}^\times) = v(K^\times) \oplus \mathbb{Q}v(y)$ for $\tilde{K} := K(y)^{rc}$.*

Proof. By Lemma 2.3.6, y is transcendental over K. It is clear that $v(K(y)^\times) \supseteq v(K^\times) \oplus \mathbb{Z}v(y)$ by observing $v(gy^k) = v(g) + kv(y)$ for $g \in K^\times$, $k \in \mathbb{Z}$. Using Lemma 2.1.1 (Zariski-Abyankahr) yields

$$1 = \operatorname{trdeg}(\tilde{K}|K) \geq \operatorname{rank}_{\mathbb{Q}}(v(\tilde{K}^\times)/v(K^\times)) + \operatorname{trdeg}(\operatorname{res}(\tilde{K})|\operatorname{res}(K)).$$

As $\operatorname{res}(\tilde{K}) = \operatorname{res}(K) = \mathbb{R}$ we have $v(\tilde{K}^\times) \subseteq \mathbb{Q}v(K^\times) \oplus \mathbb{Q}v(y)$ and L being real closed yields equality by Lemma 2.3.7. $\square$

The next statement shows that a solution of any first order linear differential equation over K generates a Hardy field over K.

Lemma 2.3.9. *Suppose $y \in \mathcal{C}^1$ satisfies*

$$y'(t) = f(t)y(t) + g(t) \tag{2.3.1}$$

eventually for $f, g \in K$. Then $K(y)$ is a Hardy field extending K.

Proof. If $y \in \mathcal{C}^1$ is algebraic over K, then by Theorem 2.3.5, $K(y)$ is a Hardy field. So suppose that K is real closed.

We show that $K[y]$ is closed under taking derivatives and that any polynomial $P(Y) \in K[Y]^{\neq}$ such that $P(y) \neq 0$ satisfies $P(y(t)) \neq 0$ eventually. Then $K[y]$ is an integral domain and by the quotient rule, its fraction field $K(y)$ is a Hardy field.

Fix $P(Y) \in K[Y]^{\neq}$ such that $P(y) \neq 0$.

It suffices to show that $Q(y(t)) \neq 0$ eventually for each monic irreducible factor Q of P. Let $\deg(Q) > 1$, then Q is the minimum polynomial of some $h = h_1 + h_2 i \in K^{ac} = K(i)$ for some $h_1, h_2 \in K, h_2 \neq 0$. So $Q(Y) = (Y - (h_1 + h_2 i))(Y - (h_1 - h_2 i)) = (Y - h_1)^2 + h_2^2$ which gives

$$Q(y(t)) = (y(t) - h_1(t))^2 + h_2(t)^2 \geq h_2(t)^2 > 0 \text{ eventually.}$$

Now focus on a factor $Q(Y) = Y - h$ of P with $h \in K$ and $Q(y) \neq 0$. Since $y \neq h$, we find for each $t \in \mathbb{R}$ some $t_1 > t$ with $y(t_1) \neq h(t_1)$. We need to show that there is some $t \in \mathbb{R}$ such that $y(t_1) \neq h(t_1)$ for all $t_1 > t$. Suppose not, then for all $t \in \mathbb{R}$, there is some $t_1 > t$ such that $y(t_1) = h(t_1)$.

Suppose $h' = fh + g$. Then by uniqueness of solutions to (2.3.1) with common initial value at some t_1 we get that $y(t) = h(t)$ for all $t \geq t_1$, contradiction. So $fh+g-h' \in K^\times$ and $y - h$ solves

$$(y-h)'(t) = (fy + g - h')(t) = f(t)(y-h)(t) + (fh + g - h')(t) \neq f(t)(y-h)(t)$$

eventually. Since $f, y - h \in \mathcal{C}^1$, either $(y-h)' > f(y-h)$ or $(y-h)' < f(y-h)$. Hence $y - h$ is either a super solution or a sub solution of $z' = fz$ in the sense of [13]. This equation has solution $z = 0$ which crosses $y - h$ since $y(t_1) - h(t_1) = 0 = z(t_1)$ for some large enough $t_1 \in \mathbb{R}$. Thus for $t > t_1$ either $y(t) - h(t) > z(t) = 0$ or $y(t) - h(t) < z(t) = 0$ ([13], Lemma 1.2).

It is left to show that the integral domain $K[y]$ is closed under taking derivatives. Writing $P(y) = \sum_{i=0}^n f_i y^i \in K[y]$ we define

$$P^\partial(y) := \sum_{i=0}^n (f_i)' y^i \quad \text{and} \quad P'(y) := \sum_{i=1}^n i f_i y^{i-1}$$

and calculate $(P(y))' = P^\partial(y) + P'(y)y' = P^\partial(y) + P'(y)(fy + g) \in K(y)$. $\square$

Corollary 2.3.10. *The subfields $K(\mathbb{R})$ and $K(x)$ of $\mathcal{C}^{<\infty}$ are Hardy fields. Moreover, for any $f \in K^>$ and $s \in \mathbb{R}$, we get Hardy fields $K(f^s)$, $K(\exp(f))$, $K(\log(f))$ and $K(g)$ where $g \in K$ such that $g' = f$.*

Proof. Note that $x' = 1$ and $c' = 0$ for any constant $c \in \mathbb{R}$ by definition. Let $s \in \mathbb{R}$ and $f \in K^>$, then $(f^s)' = sf^{s-1}f' = (sf^\dagger)f^s$, $g' = f$, $(\exp(f))' = f'\exp(f)$ and $(\log(f))' = f^\dagger$. Applying Lemma 2.3.9 yields the Hardy field extensions of K. $\square$

Remark. By Corollary 2.3.10 we can always assume that $\mathbb{R} \subseteq K$. Hence every Hardy field can be extended to an H-field by adjoining all real constants. For the existence of an H-couple, it is not necessary for K to contain $\mathbb{R}$. Every Hardy field is a so called *pre-H-field* which also has an H-couples associated to it [4].

Example. We describe a Hardy field with H-couple of finite rank. Define $\ell_0 := x$ and $\ell_{m+1} := \log \ell_m$ for $m \geq 0$. Then $K_0 := \mathbb{R}$ and $K_{n+1} := K_n(\ell_n) = \mathbb{R}(\{\ell_m\}_{m<n})$ are Hardy fields by Corollary 2.3.10. Denoting the values of ℓ_m by $-\epsilon_m < 0$, we get $\Gamma_n := v(K_n^\times) = \bigoplus_{m<n} \mathbb{Z}\epsilon_m$ since $[\ell_i] \neq [\ell_j]$ for $i \neq j$. Hence the H-couple (Γ_n, ψ_n) of K_n has rank n and can be embedded into $(\mathbb{R}^n, \psi_M)$ for some admissible M in normal form by Theorem 1.2.13. We have $\ell_0^\dagger = \ell_0^{-1}$ and $\ell_{m+1}^\dagger = \ell_m^\dagger / \ell_{m+1} = \ell_0^{-1} \cdots \ell_{m+1}^{-1}$ for $m \geq 0$, so $\psi_n(\epsilon_m) = \sum_{i \leq m} \epsilon_i$. Mapping $\epsilon_i \mapsto e_i$ for all i is an embedding of H-couples $(\Gamma_n, \psi_n) \hookrightarrow (\mathbb{R}^n, \psi_M)$ for

$$M = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}.$$

3 Realizing finite rank extensions of asymptotic couples in Hardy fields

Let (Γ, ψ) *be a nontrivial* H*-couple of Hardy type where* Γ *is of finite rank and* (Γ_0, ψ_0) *a nontrivial subcouple of* (Γ, ψ)*. Furthermore, let* $K_0 \supseteq \mathbb{R}$ *be a Hardy field with* H*-couple* (Γ_0, ψ_0)*.* Hence (Γ_0, ψ_0) has small derivation. In this chapter, we shall state and prove the theorem of Rosenlicht [12] solving the extension problem for H-couples of finite rank. For an ordered abelian subgroup Γ_i of Γ indexed by i, let $\Psi_i := \psi(\Gamma_i^{\neq})$. If Γ_i is the underlying ordered group of a subcouple of (Γ, ψ), then $\psi_i := \psi|_{\Gamma_i}$ and the H-couple is denoted by (Γ_i, ψ_i). Furthermore we abbreviate $[n] := \{1, \dots, n\}$.

Theorem 3.0.1. *There is a Hardy field extension* K *of* K_0 *with* H*-couple isomorphic to* (Γ, ψ)*. Moreover,* K *can be chosen in a way such that there is a chain of Hardy field extensions* $K_0 \subseteq K_1 \subseteq \cdots \subseteq K_{2n+1} = K$ *of* K_0 *where* $n := \operatorname{rank}\Gamma - \operatorname{rank}\Gamma_0$ *and* $n_j := \operatorname{rank} v(K_j^{\times})$ *such that:*

(i) If j *is even, then* $K_{j+1} \subseteq K_j(\{y_i^s\}_{(i,s)\in A})^{rc}$ *for some* $A \subseteq [n_j] \times \mathbb{R}$ *and* $y_1, \dots, y_{n_j} \in K_j^{>}$ *such that* $v(y_1), \dots, v(y_{n_j})$ *are archimedean representatives of* $v(K_j^{\times})$*. In this case,* $\operatorname{rank} v(K_{j+1}^{\times}) = \operatorname{rank} v(K_j^{\times})$*;*

(ii) If j *is odd, then* $K_{j+1} = K_j(\exp u)$ *or* $K_{j+1} = K_j(\log u)$ *for some* $u \in K_j^{>}$ *such that* $v(u) < 0$*. In this case,* $\operatorname{rank} v(K_{j+1}^{\times}) = 1 + \operatorname{rank} v(K_j^{\times})$*.*

$$\begin{tikzcd} K_0 \arrow[r, dashed, "\subseteq"] \arrow[d, leftrightarrow] & K \arrow[d, dashed, leftrightarrow] \\ (\Gamma_0, \psi_0) \arrow[r, "\subseteq"] & (\Gamma, \psi) \end{tikzcd}$$

Adding new archimedean classes analytically corresponds to adjoining exponentials and logarithms whereas real powers can be used to add new values in already present archimedean classes. The number $\operatorname{rank}\Gamma - \operatorname{rank}\Gamma_0$ describes exactly the number of archimedean classes of Γ missing in Γ_0.

Starting with the assumption that Γ_0 has the same rank as Γ, we adjoin germs of real powers of elements of K_0 to K_0 until Γ is exhausted. Then we turn to extensions where the rank differs by one and use logarithms or exponentials of germs of K_0 to realize the new archimedean class in the value group. Finally, we repeat the above $\operatorname{rank}\Gamma - \operatorname{rank}\Gamma_0$ many times to construct the sequence of Hardy field extensions from the theorem.

3.1 The case of equal rank

Assume $\operatorname{rank}\Gamma_0 = \operatorname{rank}\Gamma$ *and hence* $\Psi_0 = \Psi$. We show that (Γ_0, ψ_0) can be extended to a subcouple (Γ_1, ψ_1) of (Γ, ψ) that arises from a Hardy field K_1 such that $\mathbb{Q}\Gamma_1 \supseteq \Gamma$. To get K_1, we need to adjoin suitable real powers of elements of K_0 to K_0. Afterwards, we explain how to obtain a Hardy field extension $K \subseteq K_1^{rc}$ of K_1 having (Γ, ψ) as its asymptotic couple.

Theorem 3.1.1. *There is a subcouple* (Γ_1, ψ_1) *of* (Γ, ψ) *that extends* (Γ_0, ψ_0) *such that* $\mathbb{Q}\Gamma_1 \supseteq \Gamma$. *Moreover, the Hardy field*

$$K_1 := K_0(\{y_i^s\}_{(i,s)\in A})$$

has H*-couple* (Γ_1, ψ_1) *for some* $A \subseteq [n]\times\mathbb{R}$ *and* $y_1, \ldots, y_n \in K_0^>$ *such that* $v(y_1), \ldots, v(y_n)$ *are archimedean representatives of* Γ_0.

Proof. By assumption, any archimedean representatives of Γ_0 are archimedean representatives of Γ. Every ordered abelian group extension Γ_A of Γ_0 inside Γ gives rise to a subcouple (Γ_A, ψ_A) of (Γ, ψ) since $\Psi_A = \Psi \subseteq \Gamma_0 \subseteq \Gamma_A$. So any ordered abelian group embedding $\Gamma_0 \hookrightarrow \Gamma_A$ over Γ_0 is an embedding of H-couples $(\Gamma_0, \psi_0) \hookrightarrow (\Gamma_A, \psi_A)$ over (Γ_0, ψ_0).

Pick $y_1, \ldots, y_n \in K_0^>$ such that $v(y_1), \ldots, v(y_n)$ are archimedean representatives of Γ_0. We define $K_A := K_0(\{y_i^s\}_{(i,s)\in A})$ for any $A \subseteq [n]\times\mathbb{R}$ and denote its associated H-couple by (Γ_A, ψ_A). Note that K_A is a Hardy field extension of K_0 by Corollary 2.3.10 which is contained in a closure under powers of K_0. We only consider those K_A such that

(i) there is an embedding $\iota_A : \Gamma_A \to \Gamma$ over Γ_0 of ordered abelian groups,

(ii) Γ_A is the ordered abelian group generated by $\{v(y_i^s)\}_{(i,s)\in A}$ over Γ_0,

and collect them in the partial ordering $W := \{(K_A, \iota_A)_A\}$ of extensions of $(K_0, \mathrm{id}_{\Gamma_0})$ with

$$(K_A, \iota_A) \leq (K_{A'}, \iota_{A'}) \quad :\Longleftrightarrow \quad K_0 \subseteq K_A \subseteq K_{A'} \wedge \mathrm{id}_{\Gamma_0} \subseteq \iota_A \subseteq \iota_{A'}.$$

Note that W is non-empty since $(K_0, \mathrm{id}_{\Gamma_0}) \in W$. An increasing union of Hardy fields is a Hardy field, so every ordered chain of extensions has an upper bound in W given by the union $(\bigcup_A K_A, \bigcup_A \iota_A)$ of the chain, so Zorn's Lemma yields a maximal element. Take some $(K_1, \iota_1) \in W$ which has no proper extension in W. We claim that K_1 is our desired extension of K_0 with H-couple (Γ_1, ψ_1) such that $\Gamma \subseteq \mathbb{Q}\Gamma_1$. For each pair $(i, s) \in [n]\times\mathbb{R}$, one of the following has to be true:

(i) $y_i^s \in K_1$;

(ii) $v(K_1(y_i^s)^\times) \neq v(K_1^\times) + \mathbb{Z}v(y_i^s)$;

(iii) $v(K_1(y_i^s)^\times)$ cannot be embedded into Γ over Γ_0.

This holds because if (i), (ii) and (iii) would fail, then $K_1(y_i^s)$ is a proper extension of K_1 such that $v(K_1(y_i^s)^\times)$ embeds into Γ over Γ_0 and $v(K_1(y_i^s)^\times)$ is generated by $v(y_i^s)$ over $v(K_1^\times)$. But then $K_1(y_i^s) \in W$ contradicts the maximality of K_1.

In order to show that $\Gamma \subseteq \mathbb{Q}\Gamma_1$, we argue by contradiction. Suppose that $\Gamma \setminus \mathbb{Q}\Gamma_1 \neq \emptyset$. We want to find a pair $(d, c) \in [n] \times \mathbb{R}$ such that (i), (ii) and (iii) are violated.

Let $\Gamma = \Delta_0 \supset \cdots \supset \Delta_n = \{0\}$ be the maximal chain of convex subgroups of Γ where $v(y_i) \in \Delta_{i-1} \setminus \Delta_i$ for all i and let d be minimal such that $\Delta_d \subseteq \mathbb{Q}\Gamma_1$. Note that $d \geq 1$ by assumption and $\Delta_{d-1} \setminus \mathbb{Q}\Gamma_1 \neq \emptyset$.

Pick some $\alpha \in \Delta_{d-1} \setminus \mathbb{Q}\Gamma_1$. By the rank formula, Δ_{d-1}/Δ_d is archimedean, so Theorem 1.1.12 gives an embedding of ordered abelian groups $\iota : \Delta_{d-1}/\Delta_d \hookrightarrow \mathbb{R}$ such that $v(y_d) \mapsto 1$. Consider the ordered abelian group morphism $\tau : \Delta_{d-1} \to \mathbb{R}$, $\tau = \iota \circ \tau_d|_{\Delta_{d-1}}$ with domain Δ_{d-1} where $\tau_d : \Gamma \to \Gamma/\Delta_d$ is the quotient morphism. Then $\ker(\tau) = \Delta_d$ and $\tau(v(y_d)) = 1$. We show that d and $c := \tau(\alpha)$ do the job.

Claim 1: $\quad c \notin \mathbb{Q}\tau(\Gamma_1 \cap \Delta_{d-1})$.

If not, then $0 = nc - \tau(\delta) = \tau(n\alpha - \delta)$ for some $n \geq 1$ and $\delta \in \Gamma_1 \cap \Delta_{d-1}$. This gives $n\alpha - \delta \in \ker\tau = \Delta_d \subseteq \mathbb{Q}\Gamma_1$, so $\alpha \in \mathbb{Q}\Gamma_1$, a contradiction.

Claim 2: $\quad v((y_d^c)^k f) \neq 0$ for all $f \in K_1^\times, k \in \mathbb{Z}^{\neq}$. In particular, $y_d^c \notin K_1$.

Observe that $v(y_d^r) < 0$ for $r \in \mathbb{R}^{<}$ and $v(y_d^r) > 0$ for $r \in \mathbb{R}^{>}$ by examining $\lim_{t\to\infty} y_d(t)^r$ using $\lim_{t\to\infty} y_d(t) = 0$. Suppose $f \in K_1^\times$ and $k \in \mathbb{Z}^{\neq}$ with $0 = v((y_d^c)^k f) = v(f) + v(y_d^{kc})$. Then

$$-v(y_d^n) = v(y_d^{-n}) < v(y_d^{kc}) = -v(f) < v(y_d^n)$$

for $n > |kc|$. This gives

$$|v(f)| < v(y_d^n) = nv(y_d) \implies v(f) \in \operatorname{conv}(\Delta_{d-1}) = \Delta_{d-1}.$$

Hence $v(f) \in \operatorname{dom}\tau = \Delta_{d-1}$ and we calculate $\tau(v(f))$. By rational approximations, we show that $\tau(v(f)) = -kc$. For this, let $p, q \in \mathbb{Z}$, $q \neq 0$. If $p/q > kc$, then

$$\begin{aligned} v(f^q y_d^p) &= v(f^q) + v(y_d^p) = v(y_d^p) - v((y_d^{kc})^q) = v(y_d^{p-qkc}) > 0 \\ &\implies q\tau(v(f)) + p = \tau(v(f^q)) + p\tau(v(y_d)) = \tau(v(f^q y_d^p)) \geq 0 \\ &\implies \tau(v(f)) \geq -\frac{p}{q}. \end{aligned}$$

Similarly, if $p/q < kc$, then

$$v(f^q y_d^p) = v(y_d^{p-qkc}) < 0 \implies q\tau(v(f)) + p = \tau(v(f^q y_d^p)) \leq 0 \implies \tau(v(f)) \leq -\frac{p}{q}.$$

Now $f \in K_1^\times$ gives $v(f) \in \Gamma_1 \cap \Delta_{d-1}$ and $\mathbb{Q} \cap (-\infty, -kc) \leq \tau(v(f)) \leq \mathbb{Q} \cap (-kc, +\infty)$ implies $-kc = \tau(v(f)) \in \tau(\Delta_{d-1} \cap \Gamma_1)$, which contradicts Claim 1.

Claim 3: y_d^c is transcendental over K_1.

Suppose y_d^c is algebraic over K_1, then $v(y_d^c) \in \mathbb{Q}\Gamma_1$ by Lemma 2.3.6. But then $kv(y_d^c) \in \Gamma_1$ for some $k \in \mathbb{Z}^{\neq}$, so there is some $f \in K_1^\times$ such that $v((y_d^c)^k f) = kv(y_d^c) + v(f) = 0$, contradicting Claim 2.

Claim 4: The value group of $K_1(y_d^c)$ is given by $\Gamma_1 \oplus \mathbb{Z}v(y_d^c)$.

First, observe that $\Gamma_1 + \mathbb{Z}v(y_d^c) = \Gamma_1 \oplus \mathbb{Z}v(y_d^c)$ since if $v(f) + kv(y_d^c) = 0$ for some $f \in K_1^\times$ and $k \in \mathbb{Z}^{\neq}$, then $v((y_d^c)^k f) = 0$, contradicting Claim 2, so $\Gamma_1 \cap \mathbb{Z}v(y_d^c) = \{0\}$.

We clearly have $v(K_1(y_d^c)^\times) \supseteq \Gamma_1 \oplus \mathbb{Z}v(y_d^c)$ as $v((y_d^c)^k f) = v(f) + kv(y_d^c)$ for all $f \in K_1^\times$ and $k \in \mathbb{Z}$. By Claim 3, $K_1(y_d^c)$ is the field of fractions of polynomials in y_d^c and Claim 2 yields that any polynomial in y_d^c has a valuation dominant term: If $v((y_d^c)^k f) = v((y_d^c)^l g)$ for some $(g, l) \neq (f, k)$, then $v(f/g(y_d^c)^{k-l}) = 0$, a contradiction.

Claim 5: The map $\iota : \Gamma_1 \oplus \mathbb{Z}v(y_d^c) \to \Gamma_1 \oplus \mathbb{Z}\alpha$ such that $v(y_d^c) \mapsto \alpha$ is an isomorphism of ordered abelian groups over Γ_1.

Since $\alpha \in \Delta_{d-1} \setminus \mathbb{Q}\Gamma_1$ we get $\Gamma_1 + \mathbb{Z}\alpha = \Gamma_1 \oplus \mathbb{Z}\alpha$. By construction, ι is an isomorphism of groups over Γ_1. It remains to show that

$$v(fy_d^{kc}) = v(f) + kv(y_d^c) > 0 \implies v(f) + k\alpha > 0$$

for any $f \in K_1^\times$ and $k \in \mathbb{Z}$. If $k = 0$, this becomes trivial, so suppose $k \neq 0$.

Consider the order preserving morphism $\tau_{d-1} : \Gamma \to \Gamma/\Delta_{d-1}$ with kernel Δ_{d-1} as in Lemma 1.1.9. We show that $\tau_{d-1}(v(f)) \geq 0$ in Γ/Δ_{d-1}. Take $l \in \mathbb{Z}$ such that $l > ck$, then

$$v(f) + lv(y_d) = v(fy_d^l) > v(fy_d^{kc}) = v(f) + kv(y_d^c) > 0.$$

This yields that

$$\tau_{d-1}(v(f)) = \tau_{d-1}(v(f) + lv(y_d)) \geq \Delta_{d-1}$$

using $\tau_{d-1}(v(y_d)) = 0$.

If $v(f) \notin \Delta_{d-1}$, then $v(f) > \Delta_{d-1}$ and since $\alpha \in \Delta_{d-1}$ we get $v(f) + k\alpha > 0$.

Otherwise $v(f) \in \Delta_{d-1} = \operatorname{dom}\tau$. We approximate $\tau(v(f))$ by rationals to get $\tau(v(f)) \geq -kc$. Let $p/q > kc$ where $p, q \in \mathbb{Z}$, $q \neq 0$. Then

$$v(f^q y_d^p) = v(f^q) + v(y_d^p) > v(y_d^{-qkc}) + v(y_d^p) = v(y_d^{p-qkc}) > 0,$$

so

$$q\tau(v(f)) + p = q\tau(v(f)) + p\tau(v(y_d)) = \tau(v(f^q y_d^p)) \geq 0,$$

hence $\tau(v(f)) \geq -p/q$. Thus $\tau(v(f)) \geq \mathbb{Q} \cap (-kc, -\infty)$ and we get $\tau(v(f) + k\alpha) = \tau(v(f)) + kc \geq 0$. This implies $v(f) + k\alpha > 0$ since τ is a morphism of ordered abelian groups and $\Gamma_1 \cap \mathbb{Z}\alpha = \{0\}$.

By claim 2, 4 and 5, the pair (d, c) violates (i), (ii) and (iii). $\square$

Lemma 3.1.2. *Let K_1 be a Hardy field with H-couple (Γ_1, ψ_1), a subcouple of (Γ, ψ) extending (Γ_0, ψ_0) with $\Gamma \subseteq \mathbb{Q}\Gamma_1$. Suppose S is a multiplicative subgroup of $K_1^{>}$ such that $v(S) = \Gamma_1$ and define*

$$\sqrt{S} := \{f \in K_1^{rc} : 1 \not\asymp f > 0, f^n \in S \text{ for some } n > 0\}.$$

Then there is a Hardy field extension K of K_1 with asymptotic couple (Γ, ψ) such that $K = K_1(\tilde{S})$ for some $\tilde{S} \subseteq \sqrt{S}$.

Proof. From Lemma 2.3.7 we get $v((K_1^{rc})^{\times}) = \mathbb{Q}\Gamma_1 \supseteq \Gamma$. Let W be the inclusion ordered set of Hardy field extensions $K_A := K_1(\{f^{1/n} : 1 \not\asymp f > 0, (f, n) \in A\}) \subseteq K_1^{rc}$ of K_1 where $A \subseteq S \times \mathbb{Z}^{>}$ such that $\Gamma_1 \subseteq v(K_A^{\times}) \subseteq \Gamma$.

Since $K_1 \in W$ and every ordered chain in W has its increased union as an upper bound in W, we get a maximal element $K := K_{\tilde{A}}$ of W by Zorn. Then $K = K_1(\tilde{S})$ for $\tilde{S} := \{f^{1/n} : 1 \not\asymp f > 0, (f, n) \in \tilde{A}\} \subseteq \sqrt{S}$. We claim that $v(K^{\times}) = \Gamma$. If not, then there exists some $\alpha \in \Gamma \setminus v(K^{\times})$. But $\Gamma \subseteq \mathbb{Q}\Gamma_1$, so there is some minimal $n \geq 1$ such that $n\alpha \in \Gamma_1$. Let $f \in S$ such that $v(f) = n\alpha$ and by Corollary 2.3.10, $L := K(f^{1/n})$ is a (proper) Hardy field extension of K. We show that $L \in W$, which contradicts maximality of K. For this we need $v(L^{\times}) \subseteq \Gamma$. We have $n \leq [v(L^{\times}) : v(K^{\times})]$ since $i\alpha + \Gamma \neq j\alpha + \Gamma$ as $(j - i)\alpha \notin \Gamma$ for $0 \leq i < j < n$. Furthermore, $[L : K] \leq n$ since the minimal polynomial of $f^{1/n}$ over K divides $Y^n - f$. By the fundamental inequality ([4], Corollary 3.1.8), $n \leq [v(L^{\times}) : v(K^{\times})] \leq [L : K] \leq n$, hence $v(L^{\times}) = v(K^{\times}) + \mathbb{Z}\alpha \subseteq \Gamma$. □

3.2 The case of different rank

Assume $\operatorname{rank}\Gamma > \operatorname{rank}\Gamma_0$ *and hence* $\Psi \setminus \Psi_0 \neq \emptyset$. Our first goal is to show that there is a subcouple (Γ_1, ψ_1) of (Γ, ψ) extending (Γ_0, ψ_0) such that $\operatorname{rank}\Gamma_1 = 1 + \operatorname{rank}\Gamma_0$.

Theorem 3.2.1. *There exists a subcouple (Γ_1, ψ_1) of (Γ, ψ) extending (Γ_0, ψ_0) such that $\Psi_1 = \Psi_0 \cup \{\psi(\beta)\}$ for some $\beta \in \Gamma^{\neq}$ with $\psi(\beta) \notin \Psi_0$.*

Proof. We use induction on $n := |\Psi|$. If $n = 1$, then $(\{0\}, \emptyset)$ is the only subcouple of (Γ, ψ) with lesser rank. If $\max\Psi = 0$, take $\beta \neq 0$ such that $\psi(\beta) = 0 \in \mathbb{Z}\beta$. If $0 \neq \max\Psi$, then $\psi(\beta) = -\beta \in \mathbb{Z}\beta$ for some $\beta \neq 0$ by Lemma 1.2.5. In both cases, $\mathbb{Z}\beta$ is the underlying ordered abelian group of a subcouple of (Γ, ψ) extending $(\{0\}, \emptyset)$. This idea was independent of the ambient H-couple and works in general to extend the trivial H-couple. From now on, suppose that $n > 1$ and $0 < |\Psi_0| < n = \operatorname{rank}\Gamma$.

Case 1: There exists some $\beta \in \Gamma^{\neq}$ with $\psi(\beta) \in \Gamma_0 \setminus \Psi_0$.

Here, $\beta \notin \mathbb{Q}\Gamma_0$ as $[\mathbb{Q}\Gamma_0] = [\Gamma_0]$ by Lemma 1.1.5 and we claim that $\Gamma_1 := \Gamma_0 \oplus \mathbb{Z}\beta$ works. Since $\psi(\beta) \notin \Psi_0$ we have $\psi(\alpha + k\beta) = \min\{\psi(\alpha), \psi(\beta)\} \in \Psi_0 \cup \{\psi(\beta)\} \subseteq \Gamma_1$ for all $k \neq 0$ and $\alpha \neq 0$. This yields $\Psi_1 = \Psi_0 \cup \{\psi(\beta)\}$. Hence (Γ_1, ψ_1) is an H-couple with the desired property.

Case 2: $\max \Psi_0 < \max \Psi$ and for every $\gamma \in \Gamma^{\neq}$, $\psi(\gamma) \notin \Psi_0 \implies \psi(\gamma) \notin \Gamma_0$.

Recall from Lemma 1.2.5 that there does not exist any $\alpha \in \Gamma_0^{\neq}$ such that $\psi(\alpha) + \alpha = \max \Psi_0$, but as $\max \Psi_0 \neq \max \Psi$, there is a unique $\beta \in \Gamma^{\neq}$ such that $\psi(\beta) + \beta = \max \Psi_0$. Then $\beta \in \Gamma \setminus \Gamma_0$ and $\beta < 0$ as $|\beta| + \psi(\beta) > \max \Psi_0$ by (HC3). Hence $\psi(\beta) > \max \Psi_0$ and in particular $\psi(\beta) \notin \Psi_0$, which implies $\psi(\beta) \notin \Gamma_0$ by our assumption.

We claim that $\Gamma_1 := \Gamma_0 \oplus \mathbb{Z}\beta$ does the job. Since $\psi(\beta) \notin \Psi_0$ we have $\psi(\alpha + k\beta) = \min\{\psi(\alpha), \psi(\beta)\} \in \Psi_0 \cup \{\psi(\beta)\} \subseteq \Gamma_1$ for all $\alpha \neq 0$ and $k \neq 0$. Hence

$$\Psi_1 = \Psi_0 \cup \{\psi(\beta)\} = \Psi_0 \cup \{\max \Psi_0 - \beta\} \subseteq \Gamma_0 + \mathbb{Z}\beta = \Gamma_1.$$

Case 3: $\max \Psi_0 = \max \Psi$ and for every $\gamma \in \Gamma^{\neq}$, $\psi(\gamma) \notin \Psi_0 \implies \psi(\gamma) \notin \Gamma_0$.

We pass to a suitable quotient $(\Gamma/\Delta, \psi_\Delta)$ of (Γ, ψ) as in Lemma 1.2.7 to which the inductive hypothesis applies.

Fix $\gamma_0, \gamma_1 \in \Gamma^{>}$ such that $\psi(\gamma_0) = \max \Psi$ and $\psi(\gamma_1) = \max(\Psi \setminus \{\max \Psi\})$. So $[\gamma_0] \leq [\gamma]$ for all $\gamma \neq 0$. We enlarge Γ_0 to contain the smallest non-trivial convex subgroup $\Delta := \Delta_{n-1}$ of Γ. Then $\psi(\gamma_0) - \psi(\gamma_1) \in \Delta$ as $[\psi(\gamma_0) - \psi(\gamma_1)] < [\gamma_1]$ by Lemma 1.2.3, giving $[\psi(\gamma_0) - \psi(\gamma_1)] = [\gamma_0] \subseteq \Delta$. Let $\Gamma_2 := \Gamma_0 + \Delta$. By Lemma 1.2.3, every $\delta \in \Delta^{\neq}$ satisfies

$$n(\psi(\gamma_0) - \psi(\delta)) < |\delta| < n(\psi(\gamma_0) - \psi(\gamma_1))$$

for large enough n, hence $\psi(\delta) > \psi(\gamma_1)$, so $\psi(\delta) = \psi(\gamma_0)$. This yields

$$\psi(\alpha + \delta) \geq \min\{\psi(\alpha), \psi(\gamma_0)\} = \psi(\alpha)$$

for $\alpha \in \Gamma_0^{\neq}$ with equality for $\psi(\alpha) \neq \psi(\gamma_0)$ which implies $\Psi_2 = \Psi_0 \cup \{\psi(\delta)\} = \Psi_0$. In particular, $\max \Psi_2 = \max \Psi$.

If (Γ_2, ψ_2) satisfies Case 1, take $\beta \in \Gamma^{\neq}$ such that $\psi(\beta) \in \Gamma_2 \setminus \Psi_2$ and set $\Gamma_1 := \Gamma_2 + \mathbb{Z}\beta$ continuing as in Case 1. Otherwise (Γ_2, ψ_2) falls under Case 3.

By Lemma 1.2.7, $(\Gamma_\Delta, \psi_\Delta)$ where $\Gamma_\Delta := \Gamma/\Delta$ and $\psi_\Delta(\alpha + \Delta) = \psi(\alpha) + \Delta$ for all $\alpha \in \Gamma \backslash \Delta$ is an H-couple. We need to ensure that the following conditions are met to apply the inductive hypothesis to the pair $(\Gamma_\Delta, \psi_\Delta)$ extending $(\Gamma_{2\Delta}, \psi_{2\Delta}) := (\Gamma_2/\Delta, \psi_{2\Delta})$:

(1) $|\Psi_\Delta| < n$;

(2) $\Psi_\Delta \neq \Psi_{2\Delta}$.

For (1), note that $\psi(\gamma_1) + \Delta = \psi(\gamma_0) + \Delta$ as $\psi(\gamma_0) - \psi(\gamma_1) \in \Delta$. Hence $|\Psi_\Delta| < |\Psi| = n$.

To get (2), pick $\gamma \in \Gamma^{\neq}$ such that $\psi(\gamma) \notin \Psi_2$. Then $\psi(\gamma) \notin \Gamma_2$ by assumption and hence $\psi_\Delta(\gamma + \Delta) \notin \Psi_{2\Delta}$. (Otherwise $\psi(\gamma) \in \Psi_2 + \Delta \subseteq \Gamma_2$, a contradiction.)

Applying the inductive hypothesis as indicated yields a subcouple $(\Gamma_{3\Delta}, \psi_{3\Delta})$ of $(\Gamma_\Delta, \psi_\Delta)$ extending $(\Gamma_{2\Delta}, \psi_{2\Delta})$ such that $\Psi_{3\Delta} = \Psi_{2\Delta} \cup \{\psi_\Delta(\beta + \Delta)\}$ for some $\beta + \Delta \in \Gamma_{3\Delta} \setminus \Gamma_{2\Delta}$. Then $\Gamma_{3\Delta} = \Gamma_3/\Delta$ where $\Gamma_3 := \{\gamma \in \Gamma : \gamma + \Delta \in \Gamma_{3\Delta}\}$ is an ordered abelian subgroup of Γ such that $\Gamma_2 \subseteq \Gamma_3$.

We claim that (Γ_3, ψ_3) is a subcouple of (Γ, ψ). First note that $\psi(\Delta^{\neq}) = \{\psi(\gamma_0)\} \subseteq \Gamma_3$, so let $\gamma \in \Gamma_3 \setminus \Delta$. Then $\psi(\gamma) + \Delta = \psi_\Delta(\gamma + \Delta) \in \psi_{3\Delta} \subseteq \Gamma_{3\Delta}$, so $\psi(\gamma) \in \Gamma_3$ by definition of Γ_3. Hence $\Psi_3 \subseteq \Gamma_3$.

Moreover, $\Psi_{3\Delta} \neq \Psi_{2\Delta}$ gives $\Psi_3 \neq \Psi_2$. For any $\gamma \in \Gamma_3^{\neq}$, $\psi(\gamma) \notin \Psi_2 \implies \psi(\gamma) \notin \Gamma_2$, thus $\psi(\gamma) + \Delta \in \Psi_{3\Delta} \setminus \Psi_{2\Delta}$ replacing Γ by Γ_3 in the proof of (2). This implies $\psi(\gamma) + \Delta = \psi(\beta) + \Delta$. In particular, $\Psi_3 \setminus \Psi_2 \subseteq \psi(\gamma) + \Delta$.

Let $\Gamma_4 := \Gamma_2 + \mathbb{Z}\psi(\gamma)$ for some $\gamma \in \Gamma_3^{\neq}$ such that $\psi(\gamma) \in \Psi_3 \setminus \Psi_2$. Then $\Gamma_2 \subseteq \Gamma_4 \subseteq \Gamma_3$ and $\Psi_4 \subseteq \Psi_3 \subseteq \Gamma_4$, so (Γ_4, ψ_4) is an H-couple.

Now (Γ_4, ψ_4) is a subcouple of (Γ, ψ) extending (Γ_2, ψ_2) with $|\Psi_4| \leq |\Psi_2| + 1$. To see the latter, suppose that there are different $\psi(\alpha_1 + k_1\gamma^\dagger), \psi(\alpha_2 + k_2\gamma^\dagger) \in \Psi_4 \setminus \Psi_2$. Then $k_1, k_2 \neq 0$ and $k_1\alpha_2 - k_2\alpha_1 \neq 0$. Hence we get the contradiction

$$\begin{aligned}\psi(k_1\alpha_2 - k_2\alpha_1) &= \psi(k_1(\alpha_2 + k_2\gamma^\dagger) - k_2(\alpha_1 + k_1\gamma^\dagger)) \\ &= \min\{\psi(\alpha_2 + k_2\gamma^\dagger), \psi(\alpha_1 + k_1\gamma^\dagger)\} \notin \Psi_2.\end{aligned}$$

If $|\Psi_4| = |\Psi_2| + 1$, then taking $\Gamma_1 := \Gamma_4$ yields the desired subcouple (Γ_1, ψ_1) of (Γ, ψ) extending (Γ_0, ψ_0). If not, then $\Psi_4 = \Psi_2 \subset \Psi_3$ and we take $\Gamma_1 := \Gamma_4 + \mathbb{Z}\beta$ for some $\beta \in \Gamma_3^{\neq}$ such that $\psi(\beta) \notin \Psi_2$. Then $\psi(\beta) \in \Gamma_4 \setminus \Psi_4$ as $\Psi_3 \subseteq \Gamma_4$. By the same reasoning as in Case 1 we get $\Psi_1 = \Psi_0 \cup \{\psi(\beta)\}$, so (Γ_1, ψ_1) is the desired subcouple of (Γ, ψ) extending (Γ_0, ψ_0). $\square$

Let (Γ_1, ψ_1) *be as in Theorem 3.2.1.* Next, we refine (Γ_1, ψ_1) further to a subcouple (Γ_2, ψ_2) of (Γ, ψ) such that $\Psi_2 = \Psi_1$, which is then shown to be the H-couple of a Hardy field extension of K_0. We need to distinguish two cases. Either $\max \Psi_1 > \max \Psi_0$ or $\max \Psi_1 = \max \Psi_0$ treated in Lemmas 3.2.2 and 3.2.4 respectively.

Lemma 3.2.2. *Assume that* $\max \Psi_1 > \max \Psi_0$. *Then there is a unique* $\theta \in \Gamma_1^{<}$ *with* $\psi(\theta) + \theta = \max \Psi_0$. *Moreover,* $\Gamma_2 := \Gamma_0 \oplus \mathbb{Z}\theta$ *is the underlying ordered abelian group of a subcouple* (Γ_2, ψ_2) *of* (Γ_1, ψ_1) *such that* $\Psi_2 = \Psi_1$.

Proof. Existence and uniqueness of $\theta \in \Gamma_1^{\neq}$ such that $\psi(\theta) + \theta = \max \Psi_0$ follows from Lemma 1.2.5 as $\max \Psi_0 \neq \max \Psi$. Note that $\theta \notin \Gamma_0$, because $\max \Psi_0 \neq \gamma + \psi(\gamma)$ for all $\gamma \in \Gamma_0^{\neq}$ by Lemma 1.2.5. Furthermore, $\psi(\theta) + |\theta| > \max \Psi_0$ by (HC3), so $\theta < 0$. This yields $\psi(\theta) > \max \Psi_0$ and we get $\psi(\theta) = \psi(\beta)$, so it follows that $[\theta] \notin [\Gamma_0]$, $\Psi_2 = \Psi_1$ and $\Gamma_0 + \mathbb{Z}\theta = \Gamma_0 \oplus \mathbb{Z}\theta$.

It remains to note that (Γ_2, ψ_2) is an H-couple. For this, observe that $\Psi_2 \subseteq \Psi_1 \subseteq \Gamma_2$ as $\Gamma_2 \subseteq \Gamma_1$ and $\psi(\beta) = \psi(\theta) = \max \Psi_0 - \theta \in \Gamma_2$. $\square$

Let (Γ_2, ψ_2) *be as in Lemma 3.2.2.* We now show that the extension of Lemma 3.2.2 can be realized by a Hardy field.

Theorem 3.2.3. *There is some* $u \in K_0^{>}$, $v(u) < 0$ *with* $v(u^\dagger) = \max \Psi_0$ *such that the Hardy field extension* $K_2 := K_0(\log u)$ *has* H*-couple* (Γ_2, ψ_2).

Proof. Let $u \in K_0$ be as indicated. We have

$$v(\log u) + \psi(v(\log u)) = v(\log u)' = v(u^\dagger) = \max \Psi_0,$$

which together with $v(\log u) < 0$ yields $\psi(v(\log u)) > \max \Psi_0$. Hence $[\log(u)] < [\Gamma_0^{\neq}]$ and $v(K_2^\times) = \Gamma_0 \oplus \mathbb{Z}v(\log(u))$. This also shows that θ and $v(\log(u))$ realize the same cut in $\mathbb{Q}\Gamma_0$, which is $\mathbb{Q}\Gamma_0^<$, so the isomorphism of groups $\Gamma_0 \oplus \mathbb{Z}\theta \to \Gamma_0 \oplus \mathbb{Z}v(\log u)$, $\theta \mapsto v(\log u)$ is order preserving by Lemma 1.1.1. It is left to show that the H-couple of K_2 is isomorphic to (Γ_2, ψ_2). This follows from the unique H-couple structure on Γ_2 given by the condition $\psi(\theta) + \theta = \max \Psi_0$. We have $\psi(\alpha + k\theta) = \min\{\psi(\alpha), \max \Psi_0 - \theta\}$ for $\alpha \neq 0$ and $k \neq 0$. □

We now turn to the case $\max \Psi_1 = \max \Psi_0$. Here, we first need to refine to an intermediate H-couple (Γ_3, ψ_3) with $\Psi_3 = \Psi_0$ so that $\psi(\beta) \in \Gamma_3 \setminus \Psi_3$. Afterwards we extend (Γ_3, ψ_3) to an H-couple (Γ_2, ψ_2) with $\Psi_2 = \Psi_1$ and realize (Γ_2, ψ_2) as a Hardy field also using the material of Section 3.1.

Lemma 3.2.4. *Assume that* $\max \Psi_1 = \max \Psi_0$*. Then there is a unique* $\alpha \in \Gamma_1^<$ *with* $\psi(\alpha) + \alpha = \psi(\beta)$ *such that* $\psi(\beta) \in \Gamma_0 + \mathbb{Z}\alpha$*. Moreover,* $\Gamma_3 := \Gamma_0 + \mathbb{Z}\alpha$ *is the underlying ordered abelian group of a subcouple* (Γ_3, ψ_3) *of* (Γ_1, ψ_1) *with* $\Psi_3 = \Psi_0$*.*

Proof. By Lemma 1.2.5, there is a unique $\alpha \in \Gamma_1^{\neq}$ with $\psi(\alpha) + \alpha = \psi(\beta)$ as $\psi(\beta) \neq \max \Psi_1$. From (HC3) we get $\psi(\alpha) + |\alpha| > \psi(\beta)$, so $\alpha < 0$. This yields $\psi(\alpha) > \psi(\beta)$ and so $\psi(\alpha) \in \Psi_0$. Now $\psi(\beta) = \psi(\alpha) + \alpha \in \Gamma_0 + \mathbb{Z}\alpha$, thus it remains to show $\Psi_3 = \Psi_0$ which also gives that (Γ_3, ψ_3) is an H-couple.

If there is some $\gamma \in \Gamma_0$ and $k \in \mathbb{Z}$ such that $\gamma + k\alpha \neq 0$ and $\psi(\gamma + k\alpha) \in \Psi_3 \setminus \Psi_0$, then $k \neq 0$ and $\gamma \neq 0$. This yields

$$\psi(\beta) = \psi(\gamma + k\alpha) \geq \min\{\psi(\gamma), \psi(\alpha)\}. \tag{3.2.1}$$

As $\psi(\alpha) > \psi(\beta)$, we get $\psi(\beta) \geq \psi(\gamma)$ and equality holds in (3.2.1). But $\psi(\beta) = \psi(\gamma)$ gives a contradiction to $\psi(\beta) \notin \Psi_0$. □

Example. Consider $\Gamma = \mathbb{R}^2$ endowed with the H-couple structure of Hardy type $\psi([e_1]) = (0, a)$, $\psi([e_2]) = (0, 1)$ with $a \in \mathbb{R}^{<1}$ from Section 1.2.2 (Note that $\begin{pmatrix} 0 & a \\ 0 & 1 \end{pmatrix}$ is admissible). Let $\Gamma_0 := \{0\} \times \mathbb{Z}$, $\Gamma_1 := \mathbb{Z} \times (\mathbb{Z} + \mathbb{Z}a)$ and $\beta := (1, 0)$. Then $\Psi_0 = \{(0, 1)\}$ and $\Psi_1 = \Psi_0 \cup \{\psi(\beta)\}$, hence we can apply Lemma 3.2.4 as $\psi(\beta) = (0, a) < (0, 1) = \max \Psi_0$. Here, defining $\alpha := (0, a - 1)$ by examining the condition $\psi(\alpha) + \alpha = (0, a)$ yields the intermediate H-couple (Γ_3, ψ_3) with underlying group $\Gamma_3 = \{0\} \times (\mathbb{Z} + \mathbb{Z}a)$.

Let (Γ_3, ψ_3) *be as in Lemma 3.2.4.* Since $\Psi_0 = \Psi_3$, Section 3.1 implies that (Γ_3, ψ_3) can be realized by a Hardy field extension K_3 of K_0 by adjoining real powers of elements of K_0 to K_0. The following Lemma gives an H-couple extension (Γ_2, ψ_2) of (Γ_3, ψ_3) such that $\Psi_2 = \Psi_1$ to be realized by a Hardy field extension K_2 of K_3.

Lemma 3.2.5. *There is some* $\theta \in \Gamma_1^<$ *with* $\psi(\theta) \in \Gamma_3 \setminus \Psi_3$ *such that* $\Gamma_2 := \Gamma_3 + \mathbb{Z}\theta$ *is the underlying ordered abelian group of a subcouple* (Γ_2, ψ_2) *of* (Γ_1, ψ_1) *with* $\Psi_2 = \Psi_1$*.*

Proof. Since $\psi(\beta) \in \Gamma_3 \setminus \Psi_3$, choose any $\theta \in \Gamma_1^<$ such that $\psi(\theta) \notin \Psi_3$ as discussed in Case 1 of the proof of Theorem 3.2.1. Then $\Psi_2 = \Psi_1$ and (Γ_2, ψ_2) is an H-couple. $\square$

We turn to realizing (Γ_2, ψ_2) as a Hardy field extension K_2 of K_3 with H-couple (Γ_2, ψ_2).

Theorem 3.2.6. *Let K_3 be a Hardy field with H-couple (Γ_3, ψ_3). Then there is some $u \in K_3^>$, $v(u) < 0$ with $\psi(v(u)) + v(u) = \psi(\theta)$ such that the Hardy field extension $K_2 := K_3(\exp u)$ of K_3 has H-couple isomorphic to (Γ_2, ψ_2).*

Proof. As $\psi(\theta) \in \Gamma_3 \setminus \Psi_3$, there is some $u \in K_3^>$ with $\psi(v(u)) + v(u) = \psi(\theta)$ by Lemma 1.2.5. By (HC3), $\psi(v(u)) + |v(u)| > \psi(\theta)$, hence $v(u) < 0$. By Corollary 2.3.10, K_2 is a Hardy field. We claim that the H-couple of K_2 is isomorphic to (Γ_2, ψ_2). Observe that

$$\psi(v(\exp u)) = v(u') = \psi(v(u)) + v(u) = \psi(\theta),$$

hence $v(\exp u) \notin \mathbb{Q}v(K_2^\times)$ and $v(K_2^\times) = \Gamma_3 \oplus \mathbb{Z}v(\exp u)$. Now $v(\exp u)$ and θ realize the same cut over $\mathbb{Q}\Gamma_3$ which is $\{\gamma \in \mathbb{Q}\Gamma_3^< : \psi(\gamma) < \psi(\theta)\}$, hence the group isomorphism $\Gamma_3 \oplus \mathbb{Z}\theta \to \Gamma_3 \oplus \mathbb{Z}v(\exp u)$, $\theta \mapsto v(\exp u)$ is order preserving by Lemma 1.1.1. Lastly, observe that the H-couple structure on Γ_2 is uniquely determined by $\psi(\alpha + k\theta) = \min\{\psi(\alpha), \psi(\theta)\}$ for $\alpha \neq 0$ and $k \neq 0$. $\square$

We put everything together to proof Theorem 3.0.1:

Proof. Starting with the H-couple (Γ_0, ψ_0) and the Hardy field K_0 we describe how to get K_1 and K_2. First, apply Theorem 3.2.1 to get a subcouple (Γ_1, ψ_1) of (Γ, ψ) extending (Γ_0, ψ_0) with $\operatorname{rank}\Gamma_1 = 1 + \operatorname{rank}\Gamma_0$. If $\max\Psi_1 > \max\Psi_0$, let $K_1 := K_0$. After applying Lemma 3.2.2, we get a Hardy field extension $K_2 := K_1(\log u)$ of K_0 for some $u \in K_1^>$ such that $v(u) < 0$ by Theorem 3.2.3 with H-couple isomorphic to (Γ_2, ψ_2) and $\operatorname{rank}\Gamma_2 = \operatorname{rank}\Gamma_1$. If $\max\Psi_1 = \max\Psi_0$, apply Lemma 3.2.4 and then Theorem 3.1.1 and Lemma 3.1.2 to get a Hardy field extension $K_1 := K_3$ of K_0 with H-couple (Γ_3, ψ_3). Then Theorem 3.2.6 gives a Hardy field extension $K_2 := K_3(\exp(u))$ of K_0 where $u \in K_1^>$, $v(u) < 0$ with H-couple (Γ_2, ψ_2).

Replace (Γ_0, ψ_0) by (Γ_2, ψ_2) and K_0 by K_2 and repeat this procedure shifting the indices by two. After doing this $\operatorname{rank}\Gamma - \operatorname{rank}\Gamma_0$ times, we end up with a Hardy field K_{2n} such that $\operatorname{rank} v(K_{2n}^\times) = \operatorname{rank}\Gamma$. Applying Theorem 3.1.1 and Lemma 3.1.2 gives the desired Hardy field extension $K = K_{2n+1}$ of K_0 with H-couple (Γ, ψ). $\square$

Here is an illustration of the proof. On the left side, Lemma 3.2.2 and Theorem 3.2.3 are used. On the right side, Theorem 3.1.1 and Lemma 3.1.2 as well as Lemma 3.2.4 and

Theorem 3.2.6 apply. In the diagram, $\gamma_0 := \max \Psi_0$ and $\gamma_1 := \max \Psi_1$.

$$
\begin{tikzcd}
K_0 \arrow[dd] \\
\\
K_2 := K_0(\log u)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (\Gamma_0, \psi_0) \arrow[ddl, "\gamma_1>\gamma_0"'] \arrow[dr, "\gamma_1=\gamma_0"] & \\
 & & (\Gamma_3, \psi_3) \arrow[d] \\
(\Gamma_2, \psi_2) & & (\Gamma_2, \psi_2)
\end{tikzcd}
\qquad
\begin{tikzcd}
K_0 \arrow[d] \\
K_3 \arrow[d] \\
K_2 := K_3(\exp u)
\end{tikzcd}
$$

4 Realizing further asymptotic couples in Hardy fields

In this chapter, we aim to prove the realization problem for divisible H-couples of countable rank. In order to achieve that, we need to find a different strategy as the approach of Chapter 3 relied on finiteness of the rank of Γ in several ways. For example Theorem 3.2.1 used induction on the cardinality of Ψ and that the set of convex subgroups of Γ is well-ordered. Also, the existence of $\max \Psi$ is used throughout various results. In general, the extension problem of so called ungrounded H-couples where Ψ does not have a maximum is not necessarily solvable as we will see in this chapter. In particular, if $[\Gamma]$ is countably infinite, then Ψ is infinite and could have no maximum. Exactly one of the following occurs ([4], Corollary 9.2.16):

(i) There is a unique γ such that $\Psi < \gamma < (\Gamma^{>})'$. Such γ is called a *gap*.

(ii) Ψ has a maximum. In this case, (Γ, ψ) is called *grounded*.

(iii) $(\Gamma^{\neq})' = \Gamma$. Then (Γ, ψ) is said to have *asymptotic integration*.

Remark. An H-field has a gap, is grounded or has asymptotic integration if its H-couple satisfies (i), (ii) or (iii) respectively.

Note that every H-couple of finite non-zero rank is grounded since $|\Psi| \leq \operatorname{rank} \Gamma$ by Corollary 1.2.2. For a given H-couple (Γ, ψ) it is possible to find an H-couple extension which is of a prescribed case ([4], Section 9.8). More precisely, if (Γ, ψ) has gap β, then there is a grounded extension $(\Gamma \oplus \mathbb{Z}\alpha, \psi^{\alpha})$ of H-couples with $\alpha' = \beta$. In this case, there are two options for such α, either $0 < \alpha < \Gamma^{>}$ or $\Gamma^{<} < \alpha < 0$. If (Γ, ψ) is grounded, then there is an extension $(\Gamma \oplus \mathbb{Z}\alpha, \psi^{\alpha})$ with $\alpha < 0$ and $\alpha' = \max \Psi$ where Ψ^{α} has new maximum $\psi^{\alpha}(\alpha) = \max \Psi - \alpha$. After countable iteration of adding larger values to Ψ we end up with an H-couple that has asymptotic integration. And if (Γ, ψ) has asymptotic integration, then it is possible to create a gap in an extension of (Γ, ψ).

4.1 H-couples over ordered fields

This discussion is based on [5]. Until now, we used ordered abelian groups as the underlying set of an H-couple. This perspective is too general for our purposes, hence we upgrade the definition of an H-couple introducing the concept of an H-couple over an ordered field, especially over $\mathbb{Q}$ when considering divisible H-couples. The underlying set of of an H-couple over an ordered field is an ordered vector space respecting the field ordering in the following sense.

Definition. Let C be an ordered field. An ordered abelian group Γ together with a map $C \times \Gamma \to \Gamma$ making Γ a vector space over C which satisfies

$$\forall \alpha \in \Gamma : \forall c \in C : \quad 0 \leq_\Gamma \alpha \wedge 0 \leq_C c \implies 0 \leq_\Gamma c \cdot \alpha$$

is called an *ordered vector space* over C.

Example. If Γ is a divisible ordered abelian group, then Γ naturally becomes an ordered vector space over $\mathbb{Q}$.

Let C be an ordered field and Γ an ordered vector space over C. As with ordered abelian groups, we consider morphisms and substructures. Suppose $\tilde{\Gamma}$ is an ordered vector space over C. A *morphism of ordered vector spaces* $\Gamma \to \tilde{\Gamma}$ is a linear map which is order preserving. Embeddings and isomorphisms are defined as usual. Any subspace Γ_0 of Γ is an ordered vector space over C with the induced ordering, making Γ_0 an *ordered C-subspace* of Γ and Γ an *extension* of Γ_0. Also, convex valuations on ordered abelian groups upgrade to ordered vector spaces over ordered fields:

Definition. Let $v : \Gamma \to S_\infty$ be a convex valuation on Γ viewed as an ordered abelian group. If $v(\alpha) = v(c\alpha)$ for all $\alpha \neq 0$ and $c \in C^\times$, call Γ an *convex valued ordered vector space* over C.

The following definition relativizes the notion of archimedean classes with respect to C and gives rise to a natural valuation on Γ.

Definition. The *C-archimedean class* of α is defined as

$$[\alpha]_C = \{\beta \in \Gamma : \quad c^{-1} |\alpha| < |\beta| < c |\alpha| \text{ for some } c \in C^{>1}\}.$$

The ordering of the set of C-archimedean classes $[\Gamma]_C$ is analogous to the ordering of standard archimedean classes:

$$[\alpha]_C < [\beta]_C \quad :\Longleftrightarrow \quad \forall c \in C^{>1} : c|\alpha| < |\beta|.$$

Equip $I_\infty := [\Gamma]_C$ with the reverse ordering making $[0]_C$ the largest element of I. Then $v : \Gamma \to I_\infty$, $\alpha \mapsto [\alpha]_C$ makes Γ an convex valued ordered vector space over C.

Suppose K is an ordered differential field. Viewing K as an ordered vector space over its constant field C we partition K into its C-archimedean classes. Analogous to the standard valuation, equip the set $S := [K^\times]_C$ with the reverse ordering and a group structure similar to Lemma 2.1.4 to get the convex *C-valuation* $v : K^\times \to S$, $f \mapsto [f]_C$ on K making K an ordered valued differential field. The valuation ring of K with respect to the C-valuation is $\mathcal{O}_v = \operatorname{conv} C$ as $f \leq c$ implies $v(f) \geq v(c) = 0$. If K is an H-field with valuation ring $\mathcal{O}$, then the valuation on K is equivalent to the C-valuation as $\mathcal{O} = \operatorname{conv} C$. Consider any H-field as a valued field with its natural C-valuation.

The following property guarantees a useful interaction between Γ and C.

Definition. If for any $\alpha, \beta \neq 0$, there exists $c \in C^\times$ such that

$$[\alpha]_C = [\beta]_C \implies [\alpha - c\beta]_C < [\alpha]_C,$$

then Γ is called a *Hahn space* over C.

Lemma 4.1.1. *If $C = \mathbb{R}$, then Γ is a Hahn space over C.*

Proof. Since $\mathbb{R}$ is archimedean we have $[.]_\mathbb{R} = [.]$. Suppose $\alpha, \beta \neq 0$ with $[\alpha] = [\beta]$. Then $[\alpha] = [\beta] = [-\alpha] = [-\beta]$ and we distinguish the cases where α and β have the same sign or the opposite sign.

Let $\alpha, \beta > 0$. Then there exists $c \in \mathbb{R}^{>1}$ with $c^{-1}\beta < \alpha < c\beta$, so

$$A := \{c \in \mathbb{R}^> : \alpha \geq c\beta\} \neq \emptyset$$

and $\mathbb{R}^> \setminus A \neq \emptyset$. Moreover, A is downward closed in $\mathbb{R}^>$ and let $\tilde{c} := \sup A \in \mathbb{R}^>$. We claim that $[\alpha - \tilde{c}\beta] < [\beta] = [\alpha]$.

To show this, let $d \in \mathbb{R}^>$. Then $\tilde{c} + d > \tilde{c}$, hence $\tilde{c} + d \in \mathbb{R} \setminus A$, so we get

$$\alpha < (\tilde{c} + d)\beta = \tilde{c}\beta + d\beta \implies \alpha - \tilde{c}\beta < d\beta.$$

If $\alpha, \beta < 0$, then $[\alpha - \tilde{c}\beta] = [-|\alpha| + \tilde{c}\,|\beta|] = [|\alpha| - \tilde{c}\,|\beta|] < [\beta]$.

Analogously, for $\beta < 0 < \alpha$ we get $[\alpha - \tilde{c}\beta] < [\beta]$ for $\tilde{c} := \inf\{c \in \mathbb{R}^< : \alpha \leq c\beta\} \in \mathbb{R}^<$. □

Example. Not every ordered vector space over $\mathbb{Q}$ is a Hahn space over $\mathbb{Q}$, for example $\Gamma := \mathbb{Q} + \sqrt{2}\mathbb{Q} \subseteq \mathbb{R}$ as an ordered $\mathbb{Q}$-subspace of $\mathbb{R}$. To see this, observe that $[\Gamma] = \{[0], [1]\}$ where $[0] = \{0\}$ and $[1] = \Gamma^{\neq}$, so $[1] = [\sqrt{2}]$ but for every $a \in \mathbb{Q}$ we have $[a - \sqrt{2}] \neq [0]$.

Let I be a totally ordered set. Recall that an element γ of a product $\prod_{i \in I} \Gamma_i$ of C-vector spaces Γ_i is a map $\gamma : I \to \bigcup_{i \in I} \Gamma_i$ such that $\gamma(i) \in \Gamma_i$ for each i. For $\gamma \in \prod_{i \in I} \Gamma_i$ define $\operatorname{supp}(\gamma) := \{i \in I : \gamma_i \neq 0\} \subseteq I$.

Example. Consider the C-subspace $H[I, C]$ of $\prod_{i \in I} C$ consisting of all maps $\gamma : I \to C$ such that $\operatorname{supp}(\gamma)$ is well-ordered in I. We equip $H[I, C]$ with the *Hahn valuation* $v : \gamma \mapsto \min \operatorname{supp}(\gamma)$ and the *Hahn ordering*

$$0 < \gamma \quad :\Longleftrightarrow \quad 0 < \gamma_{v(\gamma)}.$$

The convex valued ordered vector space $H[I, C]$ over C is called the *Hahn product* of C over I.

Remark. By Lemma 4.1.1, $H[I, \mathbb{R}]$ is a Hahn space over $\mathbb{R}$.

Let Γ be some Hahn space over C equipped with the C-valuation $v : \Gamma^{\neq} \to I$ where $I := [\Gamma^{\neq}]_C$ carries the reverse ordering. Then $H[I, C]$ has the same value set as Γ, namely I. By the Hahn embedding theorem for Hahn spaces over C ([4], Corollary 2.4.23), Γ can be viewed as an ordered convex valued C-subspace of $H[I, C]$ making $H[I, C]$ an immediate extension of Γ. Every Hahn product is spherically complete, thus $H[I, C]$ is a maximal immediate extension of Γ ([4], Corollary 2.2.7 and 2.3.2).

Next, we upgrade the definition of an H-couple using the additional assumption that the underlying ordered abelian group is an ordered vector space over C.

Definition. An *H-couple over C* is an H-couple (Γ, ψ) such that for all $\alpha \neq 0$, $c \in C^\times$ there holds $\psi(c\alpha) = \psi(\alpha)$. Such an H-couple (Γ, ψ) over C is of *Hahn type* if

$$\psi(\alpha) = \psi(\beta) \quad \Longrightarrow \quad \exists c \in C^\times : \psi(\alpha - c\beta) > \psi(\alpha).$$

We call (Γ_0, ψ_0) a *subcouple of* (Γ, ψ) *over* C if (Γ_0, ψ_0) is a C-subcouple of (Γ, ψ) and Γ_0 is a subspace of Γ.

Example. If Γ is divisible, then every H-couple (Γ, ψ) naturally becomes an H-couple over $\mathbb{Q}$. Hence the H-couple of any real closed Hardy field becomes an H-couple over $\mathbb{Q}$.

Remark. An H-couple (Γ, ψ) over $\mathbb{R}$ is of Hahn type if and only if the underlying H-couple (Γ, ψ) is of Hardy type.

We also adjust the notion of H-couple morphisms to H-couples over C.

Definition. Suppose (Γ, ψ) and $(\tilde{\Gamma}, \tilde{\psi})$ are H-couples over C. A map $h : \Gamma \to \tilde{\Gamma}$ is called a *morphism of H-couples over C* if h is a morphism of ordered vector spaces over C which in addition satisfies

$$\alpha \notin \ker h \implies h(\psi(\alpha)) = \tilde{\psi}(h(\alpha)).$$

If h is injective (bijective), then h is called an *embedding (isomorphism) of H-couples over C* respectively. For an embedding h of H-couples over C and (Γ_0, ψ_0) a subcouple of both (Γ, ψ) and $(\tilde{\Gamma}, \tilde{\psi})$ over C, we say that h is an embedding of H-couples *over* (Γ_0, ψ_0) if $h(\gamma) = \gamma$ for all $\gamma \in \Gamma_0$.

Let (Γ, ψ) be an H-couple over C and (Γ_0, ψ_0) a subcouple of (Γ, ψ) over C. To make sense of the H-couple generated by a subset of Γ inside (Γ, ψ) over C we need the following lemma.

Lemma 4.1.2. *Given a collection $\{(\Gamma_i, \psi_i)\}_{i \in I}$ of subcouples of (Γ, ψ) over C for some index set I, the intersection $\bigcap_{i \in I} \Gamma_i$ is the underlying C-vector space of a subcouple of (Γ, ψ) over C.*

Definition. If $B \subseteq \Gamma$, we define $(\Gamma_0\langle B \rangle, \psi|_{\Gamma_0\langle B \rangle})$ to be the intersection of all subcouples of (Γ, ψ) over C whose underlying vector spaces contain $\Gamma_0 \cup B$. It is therefore the smallest H-couple extension of (Γ_0, ψ_0) over C containing B, which is referred to as the *H-couple generated by* B over (Γ_0, ψ_0). In case $B = \{\beta\}$ for some $\beta \in \Gamma$, the H-couple extension is called *simple*, and $(\Gamma_0\langle \{\beta\} \rangle, \psi|_{\Gamma_0\langle \{\beta\} \rangle})$ is denoted by $(\Gamma_0\langle \beta \rangle, \psi_\beta)$.

4.1.1 H-fields with H-couples over C

Let K be an H-field with constant field C. The following property of K ensures that its H-couple becomes an H-couple over C.

Definition. Call K *closed under powers* if for every $f \in K^\times$ and $c \in C$, there exists $g \in K^\times$ such that $g^\dagger = cf^\dagger$. An extension L of K such that C_L is algebraic over C, and for each $y \in L$ there is a finite sequence $t_1, \dots, t_n$ such that $y \in K(t_1, \dots, t_n)$ and for each i, t_i is algebraic over $K(t_1, \dots, t_{i-1})$ or $t_i^\dagger = cf^\dagger$ for some $c \in C_{K(t_1,\dots,t_{i-1})}$ and $f \in K(t_1, \dots, t_{i-1})^{\neq}$ is called a *power extension.* A *closure under powers* of K is a real closed H-field power extension of K which is closed under powers.

Interestingly, K has at most two closures under powers, and K has exactly one closure under powers if and only if there is no gap coming from a power product, i.e. there exist no $c_1, \dots, c_n \in C$, $f_1, \dots, f_n \in K^>$ such that $\Psi < v(c_1 f_1^\dagger + \dots + c_n f_n^\dagger) < (\Gamma^>)'$ ([2], Proposition 8.10). In this case, $v(L^\times) = Cv(K^\times)$ for the unique closure under powers L of K ([2], Corollary 8.11).

Lemma 4.1.3. *If K is closed under powers, then $v(K^\times)$ becomes an ordered vector space over C declaring $c \cdot v(f) := v(g)$ if $g^\dagger = cf^\dagger$ for $f, g \in K^\times$. Moreover, the H-couple of K can be viewed as an H-couple over C.*

Especially, if $K \supseteq \mathbb{R}$ is a Hardy field, then iteratively adjoining all real powers yield a Hardy field extension of K by Corollary 2.3.10 closed under real powers [2]. Then we get an additional $\mathbb{R}$-vector space structure on the value group $v(K^\times)$ via $r \cdot v(f) := v(f^r)$ for each $f \in K$ and $r \in \mathbb{R}$ as $(f^r)^\dagger = rf^\dagger$. Thus $v(K^\times)$ is a Hahn space by Lemma 4.1.1 and by Lemma 4.1.3 the H-couple of K becomes an H-couple over $\mathbb{R}$ which is of Hahn type.

Besides closure under powers, there is another important closure property:

Definition. Call K *Liouville closed* if K is real closed and for every $f \in K$ there are $g, h \in K$, $h \neq 0$ such that $g' = f$ and $h^\dagger = f$.

Remark. The H-couple of a Liouville closed H-field is an H-couple over C, has asymptotic integration, and Ψ is downward closed. An H-couple with these properties is called *closed.*

A *Liouville extension* of K is an H-field extension L of K such that C_L is algebraic over C and for all $y \in L$ there are $t_1, \dots, t_n \in L$ with t_i algebraic over $K(t_1, \dots, t_{i-1})$ or $t_i' \in K(t_1, \dots, t_{i-1})$ or $t_i^\dagger \in K(t_1, \dots, t_{i-1})$ for all i such that $y \in K(t_1, \dots, t_n)$. A *Liouville closure* of K is a Liouville closed Liouville extension of K. As shown in [1], every H-field has at least one, but at most two Liouville closures. The number of Liouville closures and closures under power depends on whether during their construction a gap appears that needs to be integrated. If K is a Hardy field, then the Hardy-Liouville closure $\mathrm{Li}(K)$ which is the smallest real closed Hardy field extension closed under exponentials and integrals is unique and coincides with some Liouville closure of K.

In the following, we give a criterion for gap prevention and creation. Suppose K is ungrounded. We say that $s \in K$ *creates a gap* over K if $v(f)$ is a gap in some H-field extension $K(f)$ of K with $f \neq 0$ and $f^\dagger = s$. There is a way to see whether some element of K creates a gap. Call a sequence in K *well-indexed* if the set of indices is well-ordered and has no maximum. A well-indexed sequence $(f_\rho)_\rho$ *pseudoconverges* to $f \in K$ if $f - f_\sigma \prec f - f_\rho$ for $\sigma > \rho$ large enough.

A *logarithmic sequence* in K is a well-indexed sequence $(\ell_\rho)_\rho$ coinitial in $K^{>C}$ such that $\ell'_{\rho+1} \asymp \ell_\rho^\dagger$ and $\ell_{\rho+1} \prec \ell_\rho$ for all ρ. Note that in Hardy fields closed under logarithms, the sequence $\log_n$ where $\log_0 := x$ and $\log_{n+1} := \log(\log_n)$ is an initial segment of a logarithmic sequence. If there is an element $\log_\omega \in K^{>C}$ such that $\log_\omega \prec \log_n$ for all n, iterate the construction starting with $\log_\omega$ instead of $\log_0$ until the sequence is coinitial in $K^{>C}$ to end up with a logarithmic sequence. Then $(\lambda_\rho)_\rho$ with $\lambda_\rho := -\ell_\rho^{\dagger\dagger}$ is called a λ-*sequence* in K. All λ-sequences in K are equivalent, i.e. they have the same pseudolimits in every valued field extension of K. Call K λ-*free* if no λ-sequence in K has a pseudolimit in K. It can be shown that K is λ-free if and only if for every $f \in K$ there is some $g > \mathcal{O}$ such that $f - g^{\dagger\dagger} \succcurlyeq g^\dagger$. If K is λ-free, then no element of K creates a gap over K. On the other hand, if K has asymptotic integration and $v(K^\times)$ is divisible with $\lambda \in K$ a pseudolimit of $(\lambda_\rho)_\rho$ in K, then $s = -\lambda$ creates a gap over K ([4], Lemma 11.5.14). If K is λ-free, then every Liouville extension of K does not have a gap [9].

4.2 H-fields for given H-couples over $\mathbb{R}$

Let (Γ, ψ) *be an H-couple over* $\mathbb{R}$ which makes Γ a Hahn space over $\mathbb{R}$ by Lemma 4.1.1. For the construction of an H-field which realizes the H-couple (Γ, ψ), we use ordered Hahn fields [4] and adapt [2] to a more general setting.

Take a multiplicative copy $t^\Gamma := \{t^\gamma : \gamma \in \Gamma\}$ of Γ. We think of t^Γ as a group of monomials where t is infinitesimal, inducing the ordering $t^v < t^w$ if $v > w$ on t^Γ making $\gamma \mapsto t^\gamma$ an order-reversing group morphism. The notation should remind us of the usual properties of powers, like $t^v t^w = t^{v+w}$ and $t^0 = 1$. Let $\mathbb{R}((t^\Gamma))$ be the set of all formal sums $f := \sum_{\gamma \in \Gamma} f_\gamma t^\gamma$ where $\operatorname{supp}(f) := \{\gamma \in \Gamma : f_\gamma \neq 0\}$ is well-ordered in Γ. Equipped with addition and multiplication

$$f + g = \sum_{\gamma \in \Gamma} (f_\gamma + g_\gamma) t^\gamma, \qquad f \cdot g = \sum_{\gamma \in \Gamma} \left(\sum_{\alpha + \beta = \gamma} f_\alpha g_\beta \right) t^\gamma,$$

$\mathbb{R}((t^\Gamma))$ becomes a field. This is not immediately obvious. It needs to be checked that for every γ, the set $\{(\alpha, \beta) \in \operatorname{supp}(f) \times \operatorname{supp}(g) : \alpha + \beta = \gamma\}$ is finite and $\operatorname{supp}(fg) \subseteq \{\alpha + \beta : (\alpha, \beta) \in \operatorname{supp}(f) \times \operatorname{supp}(g)\}$ is well-ordered in Γ. To see the first assertion, use the fact that every totally ordered infinite set contains an increasing or a decreasing sequence. For the second part, use that a totally ordered set is well-ordered if and only if any sequence in it has a non-decreasing subsequence. Existence of multiplicative inverses can also be shown ([4], Lemma 3.1.3).

The map $v : \mathbb{R}((t^\Gamma))^\times \to \Gamma$, $f \mapsto \min \operatorname{supp}(f)$ is a valuation on $\mathbb{R}((t^\Gamma))$ with value group Γ, called the *Hahn valuation* on $\mathbb{R}((t^\Gamma))$. Equipped with this valuation, the valued field $\mathbb{R}((t^\Gamma))$ is called a *Hahn field*. The *Hahn ordering* on $\mathbb{R}((t^\Gamma))$ is defined by

$$f > 0 \quad :\Longleftrightarrow \quad f_{v(f)} > 0$$

for every $f \in \mathbb{R}((t^\Gamma))^\times$, which makes $\mathbb{R}((t^\Gamma))$ an *ordered Hahn field*.

In the following we define a derivation on $\mathbb{R}((t^\Gamma))$ to make it an H-field with H-couple (Γ, ψ). Viewing Γ embedded into its Hahn product of $\mathbb{R}$ over $I := [\Gamma^{\neq}]$ ordered in reverse, fix archimedean representatives $(e_i)_{i\in I}$ of Γ so that each $\gamma \in \Gamma$ can be written as $\gamma = \sum_{i\in I} \gamma_i e_i$ for some $\gamma_i \in \mathbb{R}$ where $\operatorname{supp}(\gamma) := \{i \in I : \gamma_i \neq 0\}$ is well-ordered in I. We say that e_i *occurs in* γ if $i \in \operatorname{supp}(\gamma)$, i.e $\gamma_i \neq 0$.

Define a map $\partial : \mathbb{R}((t^\Gamma)) \to \mathbb{R}((t^\Gamma))$ such that $f' := \partial(f) := \sum_{\gamma\in\Gamma} f_\gamma (t^\gamma)'$ for $f = \sum_{\gamma\in\Gamma} f_\gamma t^\gamma$ with

$$(t^\gamma)' := \partial(t^\gamma) := -\sum_{i\in I} \gamma_i t^{\psi(e_i)+\gamma}$$

if $\gamma = \sum_{i\in I} \gamma_i e_i$. We need to ensure that the definition makes sense and we claim that ∂ is a derivation on $\mathbb{R}((t^\Gamma))$. For this, note that

$$\operatorname{supp}((t^\gamma)') = \gamma + \{\psi(e_i) : i \in \operatorname{supp}(\gamma)\}$$

is well ordered since $i < j \iff \psi(e_i) < \psi(e_j)$. We also have to show that the family of $\{(t^\gamma)'\}_{\gamma\in\operatorname{supp}(f)}$ is summable, that is

(i) for each $\alpha \in \Gamma$, there are only finitely many $\gamma \in \operatorname{supp}(f)$ such that $\alpha \in \operatorname{supp}((t^\gamma)')$, i.e. $\alpha = \gamma + \psi(e_i)$ for only finitely many γ with e_i occurring in γ;

(ii) $\operatorname{supp}(f') \subseteq \bigcup_{\gamma\in\operatorname{supp}(f)} \operatorname{supp}((t^\gamma)')$ is well-ordered in Γ, i.e. the set of all $\gamma + \psi(e_i)$ is well-ordered where e_i occurs in $\gamma \in \operatorname{supp}(f)$.

To see (i), suppose that we had a sequence $(\gamma_n)_n$ of distinct elements in $\operatorname{supp}(f)$ such that $\gamma_n + \psi(e_{i_n}) = \alpha$ for some α and $i_n \in I$ where e_{i_n} occurs in γ_n. Since $\operatorname{supp}(f)$ is well-ordered, we can assume that $(\gamma_n)_n$ is strictly increasing. Then $\psi(e_{i_n}) < \psi(e_{i_m})$ for $n > m$ which gives $[e_{i_n}] > [e_{i_m}]$, so $(i_n)_n$ is strictly decreasing in I and by Lemma 1.2.3 we get

$$[\gamma_n - \gamma_m] = [\psi(e_{i_m}) - \psi(e_{i_n})] < [e_{i_n}].$$

This shows that e_{i_n} has to occur in γ_m since it occurs in γ_n, so every e_{i_n} occurs in γ_0. This contradicts the fact that $\operatorname{supp}(\gamma_0)$ is well-ordered, hence there cannot be a strictly decreasing sequence (γ_n) in $\operatorname{supp}(f)$.

For (ii), assume that we had a strictly decreasing sequence $(\gamma_n + \psi(e_{i_n}))_n$ in $\operatorname{supp}(f')$. We can assume that $(\gamma_n)_n \subseteq \operatorname{supp}(f)$ is non-decreasing as $\operatorname{supp}(f)$ is well-ordered. Then we get

$$0 \le \gamma_n - \gamma_m < \psi(e_{i_m}) - \psi(e_{i_n})$$

for $n > m$, which shows that $[e_{i_n}] > [e_{i_m}]$, so $(i_n)_n$ is a strictly decreasing sequence in I. But now Lemma 1.2.3 yields

$$[\gamma_n - \gamma_0] \leq [\psi(e_{i_0}) - \psi(e_{i_n})] < [e_{i_n}],$$

so each e_{i_n} occurs in γ_0, contradicting that $\mathrm{supp}(\gamma_0)$ is well-ordered in I.

That ∂ is linear follows from the definition, and the Leibniz rule is tedious to check evaluating both sides to

$$-\sum_{\gamma\in\Gamma}\sum_{i\in I}\sum_{\gamma_0+\gamma_1+\psi(e_i)=\gamma} f_{\gamma_0}g_{\gamma_1}(\gamma_0+\gamma_1)_i t^{\gamma}.$$

Up to now, $\mathbb{R}((t^\Gamma))$ is an ordered valued differential field. In the following, we show that $\mathbb{R}((t^\Gamma))$ is an H-field with constant field $\mathbb{R}$ and value group Γ.

Lemma 4.2.1. *The ordered Hahn field $\mathbb{R}((t^\Gamma))$ equipped with the derivation ∂ from above is an H-field closed under powers with H-couple (Γ,ψ) over $\mathbb{R}$.*

Proof. We verify the axioms for H-fields. Let $(e_i)_{i\in I}$ be archimedean representatives of Γ, $f = \sum_{\gamma\in\Gamma} f_\gamma t^\gamma$ and $\beta = \sum_{i\in I}\beta_i e_i$ such that $\beta = v(f) = v(f_\beta t^\beta)$, so $\psi(e_{i_0}) = \psi(\beta)$ for $i_0 := \min \mathrm{supp}(\beta)$.

(H1): Suppose $f > \mathbb{R}$, then $f' = -f_\beta \beta_{i_0} t^{\beta+\psi(\beta)}(1+\epsilon)$ with $\epsilon \prec 1$ which gives $f' > 0$ since $f_\beta > 0$ as $f > 0$ and $\beta_{i_0} < 0$ since $\beta = v(f) < 0$.

(H2): We have $\mathcal{O} = \{f \in \mathbb{R}((t^\Gamma)) : \mathrm{supp}(f) \subseteq \Gamma^{\geq}\} = \mathrm{conv}(\mathbb{Q})$ and $C = \mathbb{R}$ which is shown below. Moreover, $f = f - f_0 + f_0$ where $\mathrm{supp}(f - f_0) \subseteq \Gamma^{>}$ for $f \in \mathcal{O}$. Hence $f - f_0 \in \scriptstyle\mathcal{O}$ and $\mathcal{O} = \mathbb{R} \oplus \scriptstyle\mathcal{O}$.

To show that the H-couple of $\mathbb{R}((t^\Gamma))$ is (Γ,ψ), we want $v(f') = v(f) + \psi(v(f))$ to get $\psi(v(f)) = v(f^\dagger)$. We have

$$v(t^{e_i})^\dagger = v(-t^{\psi(e_i)}) = \psi(e_i)$$

and so

$$v(t^\beta)^\dagger = \min_{i\in\mathrm{supp}(\beta)} v(-\beta_i t^{\psi(e_i)}) = v(t^{e_{i_0}})^\dagger = \psi(e_{i_0}) = \psi(\beta).$$

This gives

$$v(f') = v(f_\beta(t^\beta)') = v(t^\beta) + v(t^\beta)^\dagger = \beta + \psi(\beta) = v(f) + \psi(v(f))$$

since $\mathrm{id} + \psi$ is increasing on $\Gamma^{\neq}$ and $\psi(e_i) < \psi(e_j)$ for $i < j$ with e_i, e_j occurring in γ.

Next, we argue that $\mathbb{R}$ is the constant field of $\mathbb{R}((t^\Gamma))$. It is clear that if $f = f_0 t^0$, then $f' = 0$. If $f \neq 0$ with $f' = 0$, then $f - f_0 = 0$ since otherwise, there is some minimal $\gamma \neq 0$ such that $f_\gamma \neq 0$ and $v(f) = \gamma$ which implies $v(f') = \gamma + \psi(\gamma) \neq \infty$, a contradiction.

If $f = at^\gamma(1+\epsilon)$ for $a \in \mathbb{R}$, $\epsilon \prec 1$, $\gamma = v(f)$, then for every $c \in \mathbb{R}$, $g_c := t^{c\gamma}(1+\epsilon)^c$ is an element of $\mathbb{R}((t^\Gamma))$ such that $g_c^\dagger = cf^\dagger$ ([2], Section 11), so $\mathbb{R}((t^\Gamma))$ is closed under powers. □

Remark. Note that the construction of the derivation depends on the choice of archimedean representatives of Γ, but the H-couple does not change with different archimedean representatives. Also, for an subcouple (Γ_0, ψ_0) of (Γ, ψ) over $\mathbb{Q}$ such that $\Psi \subseteq \Gamma_0$, the ordered Hahn field $\mathbb{R}((t^{\Gamma_0}))$ becomes an H-subfield of $\mathbb{R}((t^{\Gamma}))$ with H-couple (Γ_0, ψ_0) over $\mathbb{Q}$.

4.2.1 Hardy fields for H-couples of countable rank

All Hardy fields contain $\mathbb{R}$. In order to show an absolute version of a countable rank theorem, we need to allow a few black boxes. The next results can be found in [3] (7.2 and 6.1).

Theorem 4.2.2. *Suppose K is an H-field with small derivation and* $\operatorname{trdeg}(K|\mathbb{R})$ *is countable. Then K is isomorphic to some Hardy field.*

Theorem 4.2.3. *Let K be a Hardy field with $v(K^\times)$ of countable rank such that K is not λ-free, and suppose M is a maximal Hardy field that extends K. If L is an immediate H-field extension of K, then there is an embedding $L \hookrightarrow M$ over K.*

Another important ingredient ([4], Lemma 10.5.20) is a uniqueness criterion for exponential integration.

Lemma 4.2.4. *Let K be a real closed H-field and $s < 0$ such that for all $g \in K^\times$, $g^\dagger \neq s$ and $v(s - g^\dagger) \leq \psi(\gamma)$ for some $\gamma \neq 0$. Then $K(f)$ where f is transcendental over K with $f^\dagger = s$ is an H-field extension of K for a unique ordering and valuation such that $f > 0$. Moreover, $v(K(f)^\times) = v(K^\times) \oplus \mathbb{Z}v(f)$ and $v(f) > 0$.*

Here is an answer to the realization problem for H-couples of countable rank.

Theorem 4.2.5. *Let I be a countable totally ordered set and $(\hat{\Gamma}, \hat{\psi})$ where $\hat{\Gamma} := H[I, \mathbb{R}]$ be some H-couple of Hahn type over $\mathbb{R}$ with small derivation. Suppose $(e_i)_{i \in I}$ are archimedean representatives of $\hat{\Gamma}$ and consider the subcouple (Γ_0, ψ_0) of $(\hat{\Gamma}, \hat{\psi})$ generated by $(e_i)_{i \in I}$ over $\mathbb{Q}$. Then there is a Hardy field K_0 with H-couple (Γ_0, ψ_0) over $\mathbb{Q}$ and any extension (Γ, ψ) of (Γ_0, ψ_0) over $\mathbb{Q}$ which embeds into $(\hat{\Gamma}, \hat{\psi})$ over (Γ_0, ψ_0) arises from a Hardy field extension K of K_0.*

Proof. The ordered Hahn field $\mathbb{R}((t^{\hat{\Gamma}}))$ equipped with the derivation of Section 4.2 is closed under powers, has asymptotic couple $(\hat{\Gamma}, \hat{\psi})$ by Lemma 4.2.1 and therefore small derivation, and is maximal as a valued field, i.e. there is no proper valued field extension of $\mathbb{R}((t^{\hat{\Gamma}}))$ with residue field $\mathbb{R}$ and the same value group. This gives a framework where the following construction takes place.

Consider the H-subfield $\mathbb{R}\langle t^{\Gamma_0} \rangle$ of $\mathbb{R}((t^{\hat{\Gamma}}))$ with Γ_0 the underlying $\mathbb{Q}$-vector space of the H-couple generated by $(e_i)_{i \in I}$ over the trivial subcouple viewing $(\hat{\Gamma}, \hat{\psi})$ as an H-couple over $\mathbb{Q}$. Note that the maximal immediate extension $K_0 := \mathbb{R}((t^{\Gamma_0}))$ of $\mathbb{R}\langle t^{\Gamma_0}, \lambda \rangle$ is also an H-subfield of $\mathbb{R}((t^{\hat{\Gamma}}))$ and is real closed ([4], Corollary 3.5.19). If $\mathbb{R}\langle t^{\Gamma_0} \rangle$ is λ-free, then we extend $\mathbb{R}\langle t^{\Gamma_0} \rangle$ to an H-field $\mathbb{R}\langle t^{\Gamma_0}, \lambda \rangle$ generated by a pseudolimit limit $\lambda \in K_0$ of a

λ-sequence $(\lambda_\rho)_\rho$ in $\mathbb{R}\langle t^{\Gamma_0}\rangle$. Since $\mathbb{R}\langle t^{\Gamma_0}, \lambda\rangle$ has small derivation and $\operatorname{trdeg}(\mathbb{R}\langle t^{\Gamma_0}, \lambda\rangle|\mathbb{R})$ is countable as Γ_0 is countable, $\mathbb{R}\langle t^{\Gamma_0}, \lambda\rangle$ is isomorphic to some Hardy field H by Theorem 4.2.2 and we let M be some maximal Hardy field extending H. As $\mathbb{R}\langle t^{\Gamma_0}, \lambda\rangle$ is not λ-free, Theorem 4.2.3 implies that K_0 embeds into M over $\mathbb{R}\langle t^{\Gamma_0}, \lambda\rangle$ as well. Then the image $H_0 \subseteq M$ under this embedding is a Hardy field with H-couple (Γ_0, ψ_0).

Fix an immediate extension (Γ, ψ) of (Γ_0, ψ_0) over $\mathbb{Q}$, viewed as a subcouple of $(\hat{\Gamma}, \hat{\psi})$ over $\mathbb{Q}$ via the Hahn embedding theorem. The first inductive step is to adjoin $t^\gamma \in \mathbb{R}((t^{\hat{\Gamma}}))$ to K_0 where $\gamma \in \Gamma \setminus \Gamma_0$. Let $\gamma = \sum_{i\in I} \gamma_i e_i$ be the representation of γ with respect to the chosen archimedean representatives, then

$$s := (t^\gamma)^\dagger = -\sum_{i\in I} \gamma_i t^{\psi(e_i)} \in K_0.$$

If $s > 0$, replace γ by $-\gamma$ arranging $s < 0$. Then t^γ is transcendental over K_0 by Lemma 2.3.6 since $\gamma \notin \Gamma_0$ and $K_0(t^\gamma)$ is a differential field extension of K_0. For every $h \in K_0^\times$ we have $h^\dagger \neq s$ and $\gamma \neq v(h)$, so

$$v(s - h^\dagger) = v((t^\gamma/h)^\dagger) = \psi(v(t^\gamma/h)) = \psi(\gamma - v(h)) \in \hat{\Psi} = \Psi_0.$$

This enables the use of Lemma 4.2.4, supplying us with a unique ordering and valuation making $K_0(t^\gamma)$ an H-field extension of K_0 with value group $\Gamma_0 \oplus \mathbb{Z}\gamma$.

Let $g \in H_0$ be the germ identified with $s \in K_0$. Since maximal Hardy fields are Liouville closed by Lemma 2.3.9, $f := \exp \int g \in M$. If $f \in H_0$, then $s = f^\dagger$ but $h^\dagger \neq s$ for all $h \in H_0^\times$. So f is transcendental over H_0 as H_0 is real closed and $H_0(f)$ is a Hardy field extension of H_0 in M such that $f^\dagger = s$ by Corollary 2.3.10. By the uniqueness part of Lemma 4.2.4, we have an isomorphism $K_0(t^\gamma) \cong H_0(f)$ over $K_0 \cong H_0$, mapping t^γ to f. Taking real closures gives a Hardy field $H_0(f)^{rc} \cong K_0(t^\gamma)^{rc}$ with value group $\Gamma_0 \oplus \mathbb{Q}\gamma$ by Lemma 2.3.7.

Iterating the above procedure replacing Γ_0 by $\Gamma_1 := \Gamma_0 \oplus \mathbb{Q}\gamma$ and γ by $\tilde{\gamma} \in \Gamma \setminus \Gamma_1$ allows adjoining one element at a time. $\square$

$$\begin{tikzcd}
(\hat{\Gamma}, \hat{\psi}) & & \mathbb{R}((t^{\hat{\Gamma}})) & M \\
(\Gamma, \psi) \arrow[u] & & \mathbb{R}((t^{\Gamma})) \arrow[u] \arrow[ur] & \\
(\Gamma_0 \oplus \mathbb{Q}\gamma, \psi_\gamma) \arrow[u, dashed] & & K_0(t^\gamma)^{rc} \arrow[u, dashed] \arrow[r] & H_1 \arrow[uu] \\
(\Gamma_0, \psi_0) \arrow[u] & & K_0 \arrow[u] \arrow[r, "4.2.3"] & H_0 \arrow[u] \\
(\Gamma_0, \psi_0) \arrow[u] & & \mathbb{R}\langle t^{\Gamma_0}, \lambda\rangle \arrow[u] \arrow[r, "4.2.2"] & H \arrow[u]
\end{tikzcd}$$

Corollary 4.2.6. *Any H-couple (Γ, ψ) over $\mathbb{Q}$ of Hardy type of countable rank with small derivation is isomorphic to the H-couple of some Hardy field.*

Proof. By the Hahn embedding theorem ([4], Proposition 2.4.18), (Γ, ψ) can be embedded into $(\hat{\Gamma}, \hat{\psi})$ as an ordered convex valued $\mathbb{Q}$-vector space where $\hat{\Gamma} := H[\mathbb{R}, I]$ and $I := [\Gamma^{\neq}]$ equipped with the reverse ordering. Note that $[\hat{\Gamma}] = [\Gamma]$, so $\hat{\Psi} = \Psi$ and for $\alpha \in \hat{\Gamma}^{\neq}$ we have $\hat{\psi}(\alpha) = \psi(\alpha_0)$ for any $\alpha_0 \in \Gamma$ such that $[\alpha_0] = [\alpha]$. It remains to show that (HC3) holds. Take archimedean representatives $(e_i)_{i \in I}$ of $\hat{\Gamma}$ which are also archimedean representatives of Γ. Let $\alpha \in \hat{\Gamma}^{>}$ and $i \in I$ such that $[\alpha] = [e_i]$. Then $e_i \leq n\alpha$ for some $n \geq 1$, so by (HC3) for (Γ, ψ),

$$\hat{\Psi} = \Psi < e_i/n + e_i^{\dagger} \leq \alpha + \alpha^{\dagger}.$$

Hence $(\hat{\Gamma}, \hat{\psi})$ is an H-couple extension of (Γ, ψ). Now apply Theorem 4.2.5. □

4.3 Hardy fields for relative H-couple extensions

All Hardy fields contain $\mathbb{R}$. After considering the extension problem for different cases of the trichotomy stated in the beginning of the chapter, we turn to simple extensions of H-couples followed by an outlook of a general answer to the extension problem for H-couples over $\mathbb{Q}$ of countable rank. The general theme of this section is that given an extension of H-couples over $\mathbb{Q}$ and a real closed Hardy field realizing the small H-couple over $\mathbb{Q}$, we can find a Hardy field extension realizing the larger H-couple over $\mathbb{Q}$.

4.3.1 Cycling the trichotomy

Let (Γ, ψ) be a non-trivial H-couple over $\mathbb{Q}$ of Hardy type with small derivation. The plan for this section based on [5] is to analytically cycle through the trichotomy stated at the beginning of the chapter. Note that for any cut D in Γ, there is an ordered $\mathbb{Q}$-vector space extension $\Gamma^{\alpha} := \Gamma \oplus \mathbb{Q}\alpha$ of Γ such that α realizes D. This ordering is given by

$$\gamma + q\alpha > 0 \quad :\Longleftrightarrow \quad -\gamma/q \in D$$

for $q > 0$ and

$$\gamma + q\alpha > 0 \quad :\Longleftrightarrow \quad -\gamma/q \notin D$$

for $q < 0$. The following lemma is used throughout.

Lemma 4.3.1. *Let α be an element in some ordered $\mathbb{Q}$-vector space extension of Γ such that $[\alpha] \notin [\Gamma]$. Suppose $\beta = \gamma_0 + q_0\alpha \in \Gamma^{\alpha} \setminus \Psi$ so that if $q_0 \neq 0$, then $q_0 = -1$ and $\gamma_0 = \sup \Psi$. Furthermore, suppose that $\beta < (\Gamma^{>})'$ and $\gamma_1^{\dagger} > \beta > \gamma_2^{\dagger}$ for all $\gamma_1, \gamma_2 \in \Gamma^{\neq}$ with $[\gamma_1] < [\alpha] < [\gamma_2]$. Then there is a unique map $\psi^{\alpha} : (\Gamma^{\alpha})^{\neq} \to \Gamma^{\alpha}$ making $(\Gamma^{\alpha}, \psi^{\alpha})$ an H-couple extension of (Γ, ψ) over $\mathbb{Q}$ such that $\psi^{\alpha}(\alpha) = \beta$. This map is given by*

$$\psi^{\alpha}(\gamma + q\alpha) = \begin{cases} \psi(\gamma) & \text{if } q = 0 \\ \beta & \text{if } \gamma = 0 \\ \min\{\psi(\gamma), \beta\} & \text{otherwise.} \end{cases}$$

Moreover, for any $\tilde{\alpha}$ in some ordered $\mathbb{Q}$-vector space extension of Γ realizing the same cut as α in Γ, we have $(\Gamma^{\alpha}, \psi^{\alpha}) \cong (\Gamma^{\tilde{\alpha}}, \psi^{\tilde{\alpha}})$ over (Γ, ψ) such that $\alpha \mapsto \tilde{\alpha}$ and $\psi^{\tilde{\alpha}}(\tilde{\alpha}) = \gamma_0 + q\tilde{\alpha}$.

Proof. It is tedious to check the axioms for H-couples. The argument can be adopted from the proofs of Lemmas 9.8.2, 9.8.3 and 9.8.7 in [4]. Uniqueness follows from (HC1) and that $\beta \notin \Psi$.

We show that the map $h : (\Gamma^{\alpha}, \psi^{\alpha}) \to (\Gamma^{\tilde{\alpha}}, \psi^{\tilde{\alpha}})$ over (Γ, ψ) such that $\alpha \mapsto \tilde{\alpha}$ is an H-couple isomorphism. It is clear that h is an isomorphism of ordered $\mathbb{Q}$-vector spaces as α and $\tilde{\alpha}$ realize the same cut in Γ. Furthermore,

$$h(\psi^{\alpha}(\alpha)) = h(\beta) = \gamma_0 + q\tilde{\alpha} = \psi^{\tilde{\alpha}}(\tilde{\alpha}) = \psi^{\tilde{\alpha}}(h(\alpha)).$$

By uniqueness, this gives that $(\Gamma^{\alpha}, \psi^{\alpha}) \cong (\Gamma^{\tilde{\alpha}}, \psi^{\tilde{\alpha}})$. □

First, we want to address the limits to building Hardy field extensions realizing (Γ, ψ). It turns out that a gap in (Γ, ψ) can be closed in two ways in an H-couple extension of (Γ, ψ) referred to as a *fork on the road*, which is the main obstruction for building Hardy fields realizing H-couples.

Lemma 4.3.2. *Suppose (Γ, ψ) has gap β. Then there are two non-isomorphic H-couple extensions (Γ_1, ψ_1) and (Γ_2, ψ_2) of (Γ, ψ) over $\mathbb{Q}$ such that there are $\alpha_1 \in \Gamma_1^{<}$ and $\alpha_2 \in \Gamma_2^{>}$ with $\alpha_1' = \beta$ and $\alpha_2' = \beta$.*

Proof. Let $\Gamma_i := \Gamma \oplus \mathbb{Q}\alpha_i$ for $i = 1, 2$ be ordered $\mathbb{Q}$-vector space extensions of Γ such that $\Gamma^{<} < q\alpha_1 < 0 < q\alpha_2 < \Gamma^{>}$ for all $q \in \mathbb{Q}^{>}$. Note that $[\Gamma_i] = [\Gamma] \cup \{[\alpha_i]\}$ and $[\alpha_i] < [\Gamma^{\neq}]$. As $\beta - \alpha_i \in \Gamma_i \setminus \Psi$ and $\Psi < \beta - \alpha_i < (\Gamma^{>})'$, Lemma 4.3.1 shows that Γ_i has a unique H-couple structure $\psi_i : \Gamma_i^{\neq} \to \Gamma_i$ over $\mathbb{Q}$ extending (Γ, ψ) such that $\psi_i(\alpha_i) = \beta - \alpha_i$. It remains to show that (Γ_1, ψ_1) and (Γ_2, ψ_2) cannot be isomorphic over (Γ, ψ). Any isomorphism $h : (\Gamma_1, \psi_1) \to (\Gamma_2, \psi_2)$ of H-couples over $\mathbb{Q}$ over (Γ, ψ) has to send α_1 to $q\alpha_2$ for some $q \in \mathbb{Q}^{<}$ since h preserves ordering and archimedean classes by Lemma 1.1.3. But then

$$\psi(\alpha_2) + q\alpha_2 = h(\psi(\alpha_1) + \alpha_1) = h(\beta) = \beta = \psi(\alpha_2) + \alpha_2,$$

a contradiction. □

Remark. The lemma above shows that closing gaps might not be realized by an extension of Hardy fields. Let $K \supseteq \mathbb{R}$ be a Hardy field with H-couple (Γ, ψ) which has gap β and suppose $s \in K$ such that $v(s) = \beta$. Then the asymptotic behavior of a germ $g \not\asymp 1$ satisfying $v(g') = \beta$ generating a Hardy field over K is given by the asymptotic behavior of a germ f such that $f' = s$ with $f \not\asymp 1$. Hence either $v(g) > 0$ or $v(g) < 0$ in every Hardy field extension of K containing g, so only one of the described H-couple extensions of Lemma 4.3.2 can be realized by a Hardy field extension of K closing the gap ([4], Lemma 10.2.1 or 10.2.2, Corollary 10.5.10 or 10.5.11). However, one can construct two non-isomorphic H-field extensions of K closing the gap [1], which is one reason for K to have more than one Liouville closure as an H-field.

Suppose (Γ, ψ) is grounded. Then in contrast to the gap case, $\max \Psi$ integrates uniquely and the resulting extension is grounded again. This type of extension can be realized by Hardy fields. Let $\Gamma^\alpha := \Gamma \oplus \mathbb{Q}\alpha$ be an ordered $\mathbb{Q}$-vector space extension of Γ such that $\Gamma^< < q\alpha < 0$ for all $q \in \mathbb{Q}^>$. We have $[\Gamma^\alpha] = [\Gamma] \cup \{[\alpha]\}$ and $[\alpha] < [\Gamma^{\neq}]$. Since $\Psi < \max \Psi - \alpha < (\Gamma^>)'$, Lemma 4.3.1 gives a unique H-couple structure $\psi^\alpha : (\Gamma^\alpha)^{\neq} \to \Gamma^\alpha$ on Γ^α over $\mathbb{Q}$ satisfying $\psi^\alpha(\alpha) = \max \Psi - \alpha$.

If (Γ_1, ψ_1) is an H-couple extension of (Γ, ψ) over $\mathbb{Q}$ with an element $\alpha_1' = \max \Psi$, then $\alpha_1 < 0$ as $\alpha_1 \in \Gamma_1^>$ would imply $\Psi_1 < \alpha' = \max \Psi$ by (HC3), a contradiction. So $[\alpha_1] < [\Gamma^{\neq}]$ since $\Psi < \max \Psi - \alpha_1 = \psi(\alpha_1)$. This gives $\gamma < q\alpha_1 < 0$ for all $q \in \mathbb{Q}^>$ and $\gamma \in \Gamma^<$. Hence $(\Gamma^\alpha, \psi^\alpha)$ embeds into (Γ_1, ψ_1) over (Γ, ψ) with $\alpha \mapsto \alpha_1$ by Lemma 4.3.1.

Lemma 4.3.3. *If K is a real closed Hardy field with H-couple (Γ, ψ) over $\mathbb{Q}$ and $s \in K$ such that $v(s) = \max \Psi$, then the Hardy field extension $\tilde{K} := K(f)^{rc}$ of K for some $f \in \mathcal{C}^{<\infty}$ with $f' = s$ has H-couple isomorphic to $(\Gamma^\alpha, \psi^\alpha)$ over (Γ, ψ) with $v(f) \mapsto \alpha$ and any embedding of K into a real closed Hardy field L with an element $\tilde{f} \in L$ so that $\tilde{f}' = s$ extends to an embedding $\tilde{K} \hookrightarrow L$ where $f \mapsto \tilde{f}$.*

Proof. Note that $K(f)$ is a Hardy field by Corollary 2.3.10 and $f' = s$ gives $v(f') = v(s) = \max \Psi \notin (\Gamma^{\neq})'$, hence $f \notin K$ and so f is transcendental over K. Note that $v(f) < 0$, so there is no ambiguity in the asymptotic behavior of any anti derivative $f = \int s$ of s. By Lemma 2.3.8, $v(\tilde{K}) = \Gamma \oplus \mathbb{Q}v(f)$ for the Hardy field extension $\tilde{K} := K(f)^{rc}$ of K also using Theorem 2.3.5 and Lemma 2.3.7. Now setting (Γ_1, ψ_1) to be the H-couple of $\tilde{K}$ over $\mathbb{Q}$ as in the argument above yields that (Γ_1, ψ_1) is isomorphic to $(\Gamma^\alpha, \psi^\alpha)$.

For the next statement, let L be a real closed Hardy field such that $\tilde{f} \in L$ satisfies the assumption. Then $\tilde{f}$ is transcendental over the copy of K inside L and $K(\tilde{f})^{rc} \subseteq L$ has value group $\Gamma \oplus \mathbb{Q}v(\tilde{f}) \subseteq v(L^\times)$ by Lemma 2.1.1 (Zariski-Abhyankar), so the H-couple of $\tilde{K}$ embeds into the H-couple of L over (Γ, ψ). Moreover, $K(\tilde{f})^{rc} \cong \tilde{K}$, $f \mapsto \tilde{f}$ as H-fields ([4], Lemma 10.2.3, Corollary 10.5.12) which yields a copy of $\tilde{K}$ inside L. □

When applying the lemma above to the grounded H-couple (Γ, ψ) over $\mathbb{Q}$, the resulting H-couple $(\Gamma^\alpha, \psi^\alpha)$ over $\mathbb{Q}$ is still grounded with new maximum $\max \Psi^\alpha = \alpha^\dagger = \max \Psi - \alpha > \max \Psi$. Iterating this procedure yields grounded H-couples (Γ_n, ψ_n), where $\Gamma_{n+1} = \Gamma_n^{\alpha_{n+1}}$ with $\alpha_{n+1}' = \max \Psi_n$ as well as $\psi_{n+1} = \psi_n^{\alpha_{n+1}}$ for $\alpha_1 := \alpha$. Defining $\Gamma_\omega := \bigcup_n \Gamma_n$ and $\psi_\omega := \bigcup_n \psi_n$ yields an H-couple $(\Gamma_\omega, \psi_\omega)$ over $\mathbb{Q}$ with asymptotic integration, referred to as the *divisible asymptotic integration closure* of (Γ, ψ) [8]. Note that $(\Gamma_\omega, \psi_\omega)$ cannot have a gap β since β would have to appear at some finite stage (Γ_n, ψ_n) for some n. In each step, we can realize (Γ_n, ψ_n) as a Hardy field K_n by Lemma 4.3.3, hence $K_\omega := \bigcup_n K_n$ is a Hardy field with H-couple $(\Gamma_\omega, \psi_\omega)$ over $\mathbb{Q}$ and asymptotic integration. As a union of grounded Hardy fields, K_ω is $\boldsymbol{\lambda}$-free ([4], Corollary 11.6.4).

Another useful extension of H-couples over $\mathbb{Q}$ realized by Hardy fields is the following. Let D be a non-empty cut in $[\Gamma^{\neq}]$ and $\Gamma_\alpha := \Gamma \oplus \mathbb{Q}\alpha$ where $\alpha > 0$ and $[\alpha]$ realizes D. For $\beta \in \Gamma \setminus \Psi$ such that $\gamma_1^\dagger > \beta > \gamma_2^\dagger$ for all $\gamma_1, \gamma_2 \in \Gamma^{\neq}$ with $[\gamma_1] \in D, [\gamma_2] \notin D$, we get a unique H-couple structure $(\Gamma^\alpha, \psi^\alpha)$ from Lemma 4.3.1.

Lemma 4.3.4. *Suppose K is a real closed Hardy field with H-couple (Γ, ψ) over $\mathbb{Q}$ and $s \in K^<$ such that $v(s) = \beta$. Then the Hardy field extension $\tilde{K} := K(f)^{rc}$ of K for some positive $f \in \mathcal{C}^{<\infty}$ with $f^\dagger = s$ has H-couple isomorphic to $(\Gamma^\alpha, \psi^\alpha)$ over (Γ, ψ) where $v(f) \mapsto \alpha$. Moreover, any embedding of K into a real closed Hardy field L with some $\tilde{f} \in L^>$ such that $\tilde{f}^\dagger = s$ extends to an embedding $\tilde{K} \hookrightarrow L$ where $f \mapsto \tilde{f}$.*

Proof. By Corollary 2.3.10, $K(f)$ is a Hardy field extension of K where $f := \exp(\int s)$. Observe that the choice of anti derivative of s does not matter for the asymptotic behavior of f. Now K is real closed, so f is transcendental over K and the value group of $\tilde{K} := K(f)^{rc}$ is $\Gamma \oplus \mathbb{Q}v(f)$ by Lemma 2.3.8. Note that $v(f^\dagger) = \beta \notin \Psi$, hence $[v(f)]$ and $[\alpha]$ realize the same cut in $[\Gamma^{\neq}]$. As $v(f) > 0$ and $\alpha > 0$, Lemma 4.3.1 yields that the H-couple of $\tilde{K}$ is isomorphic to $(\Gamma^\alpha, \psi^\alpha)$ over (Γ, ψ). Since $s \neq g^\dagger$ for all $g \in K^\times$ we have $v(s - g^\dagger) = \min\{\beta, v(g^\dagger)\} \leq \psi(\gamma)$ for $\gamma^\dagger > \beta$ and all $g \neq 0$ as D is non-empty. Applying Lemma 4.2.4 yields that any real closed Hardy field extension L of K with an element $\tilde{f} \in L^>$ such that $\tilde{f}^\dagger = s$ contains a copy of $\tilde{K}$. □

Remark. The lemma above also holds when replacing $f, \tilde{f} > 0$ by $f, \tilde{f} < 0$ as $v((-f)^\dagger) = v(f^\dagger) = \beta$.

Suppose (Γ, ψ) has asymptotic integration. Recall that (Γ, ψ) is said to be closed if Ψ is downward closed in Γ. An H-couple extension (Γ^c, ψ^c) of (Γ, ψ) over $\mathbb{Q}$ is said to be a *closure* of (Γ, ψ) if (Γ^c, ψ^c) is a closed H-couple over $\mathbb{Q}$ such that any embedding $(\Gamma, \psi) \hookrightarrow (\tilde{\Gamma}, \tilde{\psi})$ into a closed H-couple $(\tilde{\Gamma}, \tilde{\psi})$ over $\mathbb{Q}$ extends to an embedding $(\Gamma^c, \psi^c) \hookrightarrow (\tilde{\Gamma}, \tilde{\psi})$. If (Γ, ψ) is grounded or has asymptotic integration, then (Γ, ψ) has a closure. But if (Γ, ψ) has a gap, it has two 'closures' similar to the phenomenon of having two Liouville closures for H-fields. In the latter case, any embedding into a closed H-couple over $\mathbb{Q}$ extends to one of the 'closures'. In the following, we assume that (Γ, ψ) has a closure, hence (Γ, ψ) has no gap. Fix some closure (Γ^c, ψ^c) of (Γ, ψ).

Corollary 4.3.5. *If K is a real closed Hardy field with H-couple (Γ, ψ), then there is a Hardy field extension of K with H-couple isomorphic to some closed H-couple contained in (Γ^c, ψ^c).*

Proof. If K is grounded, take the divisible asymptotic integration closure which is realized by a Hardy field extension of K. Hence we can assume K has asymptotic integration. Apply Lemma 4.3.4 for every $\beta \in \Psi^\downarrow \setminus \Psi$ and let K_1 be the Hardy field with H-couple (Γ_1, ψ_1) such that $\Psi_1 = \Psi^\downarrow$ where the downward closure is taken in Γ. Iterate this procedure replacing K by K_1 to get for each $n > 1$ some K_n with H-couple (Γ_n, ψ_n) such that $\Psi_{n+1} = \Psi_n^\downarrow$ where the downward closure is taken in Γ_n. Then $K_\omega := \bigcup_n K_n$ is a Hardy field with closed H-couple. Since (Γ^c, ψ^c) is closed, this construction can be carried out in (Γ^c, ψ^c). □

4.3.2 Simple extensions of H-couples

Let (Γ, ψ) be an H-couple of Hardy type over $\mathbb{Q}$ and (Γ_0, ψ_0) a non-trivial subcouple of (Γ, ψ) over $\mathbb{Q}$ with asymptotic integration. We consider H-couple extensions of (Γ_0, ψ_0)

over $\mathbb{Q}$ by adjoining new elements from Γ. In particular, we analyze simple extensions of (Γ_0, ψ_0) over $\mathbb{Q}$ inside (Γ, ψ). Since the H-couple over $\mathbb{Q}$ generated by $\beta \in \Gamma \setminus \Gamma_0$ over Γ_0 has to contain the $\mathbb{Q}$-subspace of Γ generated by $\{\beta\} \cup \Gamma_0$, it follows that $\Gamma_0 + \mathbb{Q}\beta \subseteq \Gamma_0\langle\beta\rangle$ and now there are two possibilities. Either $\psi(\beta - \alpha) \in \Gamma_0 + \mathbb{Q}\beta$ for all $\alpha \in \Gamma_0$ or there is some $\alpha \in \Gamma_0$ such that $\psi(\beta - \alpha) \notin \Gamma_0 + \mathbb{Q}\beta$. In the latter case, we have to at least include the value $\psi(\beta - \alpha)$ to close off under ψ, hence $\Gamma_0 + \mathbb{Q}\beta + \mathbb{Q}\psi(\beta - \alpha) \subseteq \Gamma_0\langle\beta\rangle$. Replacing Γ_0, β with $\Gamma_0 + \mathbb{Q}\beta$, $\psi(\beta - \alpha)$ we again need to ensure to include all possible ψ-values and continuing that way, either we end up closing off under ψ at some point or we need to iterate the above indefinitely. The next theorem [5] characterizes simple extensions for H-couples over $\mathbb{Q}$ with asymptotic integration. For this, recall that a set $B \subseteq \Gamma$ is $\mathbb{Q}$-linearly independent over Γ_0 if $\{b + \Gamma_0 : b \in B\}$ is $\mathbb{Q}$-linearly independent in Γ/Γ_0. Equivalently, only trivial linear combinations of elements of B land inside Γ_0.

Theorem 4.3.6. *For any $\beta \in \Gamma \setminus \Gamma_0$, the H-couple extension $(\Gamma_0\langle\beta\rangle, \psi_\beta)$ of (Γ_0, ψ_0) over $\mathbb{Q}$ satisfies one of the following conditions.*

(a) $\psi(\Gamma_0 + \mathbb{Q}\beta^{\neq}) = \Psi_0$. *In this case,* $\Gamma_0\langle\beta\rangle = \Gamma_0 \oplus \mathbb{Q}\beta$.

(b) There are sequences $(\alpha_i)_{i\in\mathbb{N}}$ in Γ_0 and $(\beta_i)_{i\in\mathbb{N}}$ in $\Gamma^{\neq}$ with $(\beta_i)_{i\in\mathbb{N}}$ $\mathbb{Q}$-linearly independent over Γ_0 such that $\beta_0 = \beta - \alpha_0$, $\beta_{i+1} = \psi(\beta_i) - \alpha_{i+1}$ for all i and $\Gamma_0\langle\beta\rangle = \Gamma_0 \oplus \bigoplus_{i\in\mathbb{N}} \mathbb{Q}\beta_i$. In this case, $[\Gamma_0\langle\beta\rangle] = [\Gamma_0] \cup \{[\beta_i]\}_{i\in\mathbb{N}}$ and $[\beta_i] > [\beta_{i+1}]$.

(c) There are $(\alpha_i)_{i\leq n}$ in Γ_0 and $(\beta_i)_{i\leq n}$ in $\Gamma^{\neq}$ with $(\beta_i)_{i\leq n} \cup \{\psi(\beta_n)\}$ $\mathbb{Q}$-linearly independent over Γ_0 such that $\beta_0 = \beta - \alpha_0$, $\beta_{i+1} = \psi(\beta_i) - \alpha_{i+1}$ for $i < n$ and $\psi(\Gamma_0 + \mathbb{Q}\psi(\beta_n)^{\neq}) = \Psi_0$ so that $\Gamma_0\langle\beta\rangle = \Gamma_0 \oplus \bigoplus_{i\leq n} \mathbb{Q}\beta_i \oplus \mathbb{Q}\psi(\beta_n)$. In this case, $[\Gamma_0\langle\beta\rangle] = [\Gamma_0] \cup \{[\beta_i]\}_{i\leq n}$ and $[\beta_i] > [\beta_{i+1}]$.

(d) There are $(\alpha_i)_{i\leq n}$ in Γ_0 and $(\beta_i)_{i\leq n}$ in $\Gamma^{\neq}$ with $(\beta_i)_{i\leq n}$ $\mathbb{Q}$-linearly independent over Γ_0 such that $\beta_0 = \beta - \alpha_0$, $\beta_{i+1} = \psi(\beta_i) - \alpha_{i+1}$ for $i < n$ and $\psi(\beta_n) \in \Gamma_0 \setminus \Psi_0$ so that $\Gamma_0\langle\beta\rangle = \Gamma_0 \oplus \bigoplus_{i\leq n} \mathbb{Q}\beta_i$. In this case, $[\Gamma_0\langle\beta\rangle] = [\Gamma_0] \cup \{[\beta_i]\}_{i\leq n}$ and $[\beta_i] > [\beta_{i+1}]$.

Remark. In cases (b) and (d), the extension $(\Gamma_0\langle\beta\rangle, \psi_\beta)$ of (Γ_0, ψ_0) over $\mathbb{Q}$ still has asymptotic integration ([5], Lemmas 4.2, 4.4). But in cases (a) and (c) a gap might occur in $(\Gamma_0\langle\beta\rangle, \psi_\beta)$ ([5], Lemmas 3.4, 4.3), and if $|\beta_n| < \Gamma^{>}$ in case (c), then $(\Gamma_0\langle\beta\rangle, \psi_\beta)$ is grounded. In cases (b), (c) and (d), every member of the sequence (β_i) is of the same case as β. Examples of case (b) extensions are given in [6].

We aim to describe for each case a corresponding extension of Hardy fields realizing the extension under certain assumptions. This would give a strategy to answer the extension problem for divisible H-couples of countable rank. For case (a) we only conjecture the relative extension of Hardy fields in the next section, so we start with case (b) extensions where we require (Γ_0, ψ_0) to be closed and conjecture that this assumption could also be removed.

Lemma 4.3.7. *If $\beta \in \Gamma \setminus \Gamma_0$ generates an H-couple over $\mathbb{Q}$ of case (b) over Γ and K_0 is a Liouville closed Hardy field with closed H-couple (Γ_0, ψ_0) over $\mathbb{Q}$, then there is a Hardy field extension K of K_0 with H-couple isomorphic to $(\Gamma_0\langle\beta\rangle, \psi_\beta)$ over (Γ_0, ψ_0).*

Proof. This is just a sketch. A rigorous construction can be found in [6]. We demonstrate how to construct an H-field extension of K_0 with H-couple $(\Gamma_0\langle\beta\rangle, \psi_\beta)$ by defining a sequence $(y_i)_i$ using $(\alpha_i)_i$ and $(\beta_i)_i$ given from (b). First, we need to produce some candidate in an H-field extension for realizing β. Observe that the theory of Liouville closed H-fields can be axiomatized in the language of ordered valued differential fields $\{0, 1, -, +, \cdot, \partial, \leq, \preccurlyeq\}$. Hence by compactness we find some Liouville closed extension L of K_0 which realizes the type $\{x \preccurlyeq f : v(f) \leq \beta\} \cup \{x \succcurlyeq f : v(f) \geq \beta\}$. Take some realization $y \in L^>$ which gives that $v(y)$ and β realize the same cut in Γ. Then y generates the differential subfield $K_0\langle y\rangle$ of L over K_0. Let $K_y := K_0\langle y\rangle^{rc}$, so $\Gamma_1 := v(K_y^\times)$ is an ordered $\mathbb{Q}$-vector space making (Γ_1, ψ_1) a subcouple of the H-couple of L over $\mathbb{Q}$. Using that (Γ_0, ψ_0) is closed yields an isomorphism $\iota : (\Gamma_0\langle\beta\rangle, \psi_\beta) \to (\Gamma_0\langle v(y)\rangle, \psi_{v(y)})$ of H-couples over (Γ_0, ψ_0) such that $\beta \mapsto v(y)$ ([6], Lemma 8.12). Note that y is differentially transcendental over K_0 since otherwise, $\Gamma_0\langle v(y)\rangle$ would have finite dimension over Γ_0 as vector spaces over $\mathbb{Q}$ by Lemma 2.1.1 (Zariski-Abhyankar). Take $f_i \in K_0^>$ such that $v(f_i) = \alpha_i$ and set $y_0 := y/f_0$ and $y_{i+1} := y_i^\dagger/f_{i+1}$. Inductively, we have $y_i \neq 0$ and

$$\iota(\beta_0) = \iota(\beta - \alpha_0) = \iota(\beta) - \iota(\alpha_0) = v(y) - v(f_0) = v(y_0)$$

as well as

$$\iota(\beta_{i+1}) = \iota(\psi(\beta_i) - \alpha_{i+1}) = \psi(\iota(\beta_i)) - \iota(\alpha_{i+1}) = v(y_i^\dagger) - v(f_{i+1}) = v(y_{i+1})$$

for $i \geq 0$ assuming $\iota(\beta_i) = v(y_i)$. So $y_i \in L$ realizes β_i for each i.

Define $K_n := K(y_0, \ldots, y_n)$. Then K_n has value group

$$v(K_n^\times) = \Gamma_0 \oplus \mathbb{Z}v(y_0) \oplus \cdots \oplus \mathbb{Z}v(y_n)$$

as $y_0, \ldots, y_n$ are algebraically independent over K. So ι restricts to an isomorphism

$$\Gamma_0 \oplus \mathbb{Z}\beta_0 \oplus \cdots \oplus \mathbb{Z}\beta_n \to v(K_n^\times).$$

By Zariski-Abhyankar,

$$n = \operatorname{trdeg}(K_n|K_0) \geq \operatorname{rank}_{\mathbb{Q}}(v(K_n^\times)/\Gamma_0) + \operatorname{trdeg}(\operatorname{res}(K_n)|\operatorname{res}(K_0))$$

where $n = \operatorname{rank}_{\mathbb{Q}}(v(K_n^\times)/\Gamma_0)$ by the above. This gives that $\operatorname{res}(K_n)$ is algebraic over $\operatorname{res}(K)$ since $\operatorname{trdeg}(\operatorname{res}(K_n)|\operatorname{res}(K_0)) = 0$. But $\operatorname{res}(K)$ is real closed, so $\operatorname{res}(K_n) = \operatorname{res}(K)$. Now $K_0\langle y\rangle = \bigcup K_n$ has residue field $\operatorname{res}(K_0\langle y\rangle) = \operatorname{res}(K_0)$ and $C_{K_0} = C_{K_0\langle y\rangle}$. Hence $K_0\langle y\rangle^{rc}$ is an H-subfield of L with H-couple isomorphic to $(\Gamma\langle\beta\rangle, \psi_\beta)$ over $\mathbb{Q}$. To show that the sequence $(y_i)_i$ can be reverse engineered in a Hardy field is more involved ([6], Section 9). □

Next we derive analogous results for cases (c) and (d), where the sequences $(\alpha_i)_i$ and $(\beta_i)_i$ are finite. They differ in the first step of reverse engineering but when β_n is realized as a germ in a Hardy field, then Lemma 4.3.4 gives the rest. For simplicity, we denote the ψ map of any subcouple of $(\Gamma_0\langle\beta\rangle, \psi_\beta)$ over $\mathbb{Q}$ with the same letter ψ_β.

Lemma 4.3.8. *If $\beta \in \Gamma \setminus \Gamma_0$ generates an H-couple over $\mathbb{Q}$ of case (c) over Γ such that $|\beta_n| \geq \alpha$ for some $\alpha \in \Gamma_0^>$ and K_0 is a real closed Hardy field with H-couple $(\Delta, \psi|_\Delta)$ over $\mathbb{Q}$ where $\Delta := \Gamma_0 \oplus \mathbb{Q}\psi(\beta_n)$, then there is a Hardy field extension K of K_0 with H-couple isomorphic to $(\Gamma_0\langle\beta\rangle, \psi_\beta)$ over (Γ_0, ψ_0).*

Proof. Let $(\alpha_i)_i$ and $(\beta_i)_i$ be finite sequences as in case (c). Take $f_i \in K_0^>$ with $v(f_i) = \alpha_i$. We reverse engineer a finite sequence $(y_i)_i$ of germs living in a Hardy field extension of K_0 such that $y_0 = y/f_0$, $y_{i+1} = y_i^\dagger/f_{i+1}$ for $0 \leq i < n$ so that $v(y) = \beta$ as in case (b).

Take $y_{n+1} \in K_0$ such that $v(y_{n+1}) = \psi(\beta_n) \in \Delta \setminus \Gamma_0$. Note that in particular, $\beta_n^\dagger \notin \Psi_\Delta = \Psi_0$ and by assumption, $\beta_n^\dagger \in \Psi_0^\downarrow$. With $y_n := \exp\int y_{n+1}$, $K_1 := K_0(y_n)^{rc}$ is a real closed Hardy field with H-couple $(\Gamma_0 \oplus \mathbb{Q}v(y_{n+1}) \oplus \mathbb{Q}v(y_n), \psi_{K_1})$ isomorphic to $(\Delta \oplus \mathbb{Q}\beta_n, \psi_\beta)$ over (Γ_0, ψ_0) by Lemma 4.3.4.

We continue this way using the recursion satisfied by y_i and Lemma 4.3.4 to end up with a real closed Hardy field $K(y_{n+1}, \ldots, y_0)$ with H-couple isomorphic to $(\Gamma_0\langle\beta\rangle, \psi_\beta)$ over (Γ_0, ψ_0). We demonstrate one step. Suppose $L = K(y_{n+1}, \ldots, y_{i+1})$ has already been constructed for some $0 \leq i \leq n$ with H-couple $(\Gamma_0 \oplus \mathbb{Q}v(y_{n+1}) \oplus \cdots \oplus \mathbb{Q}v(y_{i+1}), \psi_L)$ isomorphic to $(\Gamma_0 \oplus \mathbb{Q}\beta_n^\dagger \oplus \cdots \oplus \mathbb{Q}\beta_{i+1}, \psi_\beta)$ over (Γ_0, ψ_0). Taking $y_i := \exp\int y_{i+1}f_{i+1}$ yields $y_{i+1} = y_i^\dagger/f_{i+1}$ and $v(y_i^\dagger) = v(y_{i+1}f_{i+1}) = \beta_{i+1} + \alpha_{i+1} \in \Gamma_L \setminus \Psi_L$ by $\mathbb{Q}$-linear independence of $\beta_i, \ldots, \beta_n, \beta_n^\dagger$ over Γ_0. Note that $\beta_i^\dagger < \beta_{i+1}^\dagger \in \Psi_L$, hence we can apply Lemma 4.3.4 to get the Hardy field extension $L_1 := L(y_i)^{rc}$ with H-couple $(\Gamma_0 \oplus \mathbb{Q}v(y_{n+1}) \oplus \cdots \oplus \mathbb{Q}v(y_i), \psi_{L_1})$ isomorphic to $(\Gamma_0 \oplus \mathbb{Q}\beta_n^\dagger \oplus \cdots \oplus \mathbb{Q}\beta_i, \psi_\beta)$ over (Γ_0, ψ_0). □

Remark. If case (a) extensions can be realized by Hardy fields and if Lemma 4.3.4 is true for $D = \emptyset$, then we can start with a Hardy field K_0 with H-couple (Γ_0, ψ_0) over $\mathbb{Q}$ instead of $(\Delta, \psi|_\Delta)$ and drop the assumption that $|\beta_n| \geq \alpha$ for some $\alpha \in \Gamma_0^>$.

Lemma 4.3.9. *If $\beta \in \Gamma \setminus \Gamma_0$ generates an H-couple over $\mathbb{Q}$ of case (d) over Γ and K_0 is a real closed Hardy field with H-couple (Γ_0, ψ_0), then there is a Hardy field extension K of K_0 with H-couple isomorphic to $(\Gamma_0\langle\beta\rangle, \psi_\beta)$ over (Γ_0, ψ_0).*

Proof. Let $(\alpha_i)_i$ and $(\beta_i)_i$ be finite sequences as in case (d). Analogous to case (c) we want to reverse engineer germs (y_i) realizing $(\beta_i)_i$. Take for each $0 \leq i \leq n$ some $f_i \in K_0^>$ with $v(f_i) = \alpha_i$. Since $\beta_n^\dagger \in \Gamma_0 \setminus \Psi_0$ and $(\Gamma_0\langle\beta\rangle, \psi_\beta)$ has asymptotic integration, applying Lemma 4.3.4 to $\beta_n^\dagger \in \Psi_0^\downarrow$ yields a real closed Hardy field $K_1 := K_0(\exp\int y_n)^{rc}$ with H-couple $(\Gamma_0 \oplus \mathbb{Q}v(y_n), \psi_{K_1})$ isomorphic to $(\Gamma_0 \oplus \mathbb{Q}\beta_n, \psi_\beta)$. Continuing as in case (c) using the recursion satisfied by y_i and Lemma 4.3.4 yields a real closed Hardy field $K(y_{n+1}, \ldots, y_0)$ with H-couple isomorphic to $(\Gamma_0\langle\beta\rangle, \psi_\beta)$ over (Γ_0, ψ_0). □

Remark. If (Γ_0, ψ_0) is closed, case (d) does not occur for any element in Γ.

Final remarks and informal ideas

Suppose (Γ, ψ) is of countable rank. First, we conjecture that gaps can be created in Hardy fields matching the creation of gaps for H-couples over $\mathbb{Q}$. For this, let K be a Hardy field with H-couple (Γ, ψ) over $\mathbb{Q}$ of countable rank with asymptotic integration.

Then (Γ, ψ) has a unique H-couple extension $(\Gamma^\alpha, \psi^\alpha)$ for $\Gamma^\alpha := \Gamma \oplus \mathbb{Q}\alpha$ over $\mathbb{Q}$ such that $\Psi < \alpha < (\Gamma^>)'$ ([5], Lemma 2.3). Call K $\boldsymbol{\omega}$-*free* if the pseudocauchy sequence $(\boldsymbol{\omega}_\rho)_\rho$ where $\boldsymbol{\omega}_\rho := -2\boldsymbol{\lambda}'_\rho - \boldsymbol{\lambda}^2_\rho$ does not have a pseudolimit in K.

Conjecture 4.3.10. *There is a real closed Hardy field extension $\tilde{K}$ of K with H-couple isomorphic to $(\Gamma^\alpha, \psi^\alpha)$ over (Γ, ψ).*

An idea for a proof might be as follows. If K is not $\boldsymbol{\lambda}$-free, take a pseudolimit $\boldsymbol{\lambda} \in K$ of a $\boldsymbol{\lambda}$-sequence, so that $-\boldsymbol{\lambda}$ creates a gap over K ([4], 11.5.14) and let $\boldsymbol{\gamma} := \exp \int -\boldsymbol{\lambda}$. Then the value group of $K(\boldsymbol{\gamma})^{rc}$ is $\Gamma \oplus \mathbb{Q}v(\boldsymbol{\gamma})$ and has gap $v(\boldsymbol{\gamma})$, so by uniqueness of the H-couple structure as described above, $K(\boldsymbol{\gamma})^{rc}$ has H-couple isomorphic to $(\Gamma^\alpha, \psi^\alpha)$ over (Γ, ψ).

We now turn to the case where K is $\boldsymbol{\lambda}$-free. Countability of $\operatorname{rank}\Gamma$ yields the existence of a countable $\boldsymbol{\lambda}$-sequence $(\boldsymbol{\lambda}_n)_n$ in K, some pseudolimit $\boldsymbol{\lambda}$ of which lives in a Hardy field extension of K ([6], Corollary 3.2). If K is $\boldsymbol{\omega}$-free, then $(\boldsymbol{\lambda}_n)_n$ is of differentially transcendental type ([4], Corollary 13.6.3), so $K\langle\boldsymbol{\lambda}\rangle$ is an immediate H-field extension of K ([4], Lemma 11.4.7, Corollary 11.4.13). This reduces to the case that K is not $\boldsymbol{\lambda}$-free.

It would remain to consider the case where K is $\boldsymbol{\lambda}$-free but not $\boldsymbol{\omega}$-free ([4], Sections 11 and 13).

Next we conjecture that case (a) extensions of Theorem 4.3.6 can also be realized by Hardy fields in hope of Conjecture 4.3.10.

Conjecture 4.3.11. *If $\beta \in \Gamma \setminus \Gamma_0$ generates an H-couple over $\mathbb{Q}$ of case (a) over Γ_0 and K_0 is a real closed Hardy field with H-couple (Γ_0, ψ_0) over $\mathbb{Q}$, then there is a Hardy field K extending K_0 with H-couple isomorphic to $(\Gamma_0\langle\beta\rangle, \psi_\beta)$ over (Γ_0, ψ_0).*

If $(\Gamma_0\langle\beta\rangle, \psi_\beta)$ has a gap, apply Conjecture 4.3.10. Otherwise $(\Gamma_0\langle\beta\rangle, \psi_\beta)$ has asymptotic integration. Since $\operatorname{rank}\Gamma$ is countable, β has countable type, i.e. the cofinality $\operatorname{cf}(\Gamma_0^{<\beta})$ and coinitiality $\operatorname{ci}(\Gamma_0^{>\beta})$ are countable. If additionally (Γ_0, ψ_0) is an H-couple over $\mathbb{R}$, then β is a pseudolimit of a countable divergent pseudocauchy sequence $(\gamma_n)_n$ in Γ_0 ([6], Lemma 8.2). Take $f_n \in K_0^>$ with $v(f_n) = \gamma_n$ and a maximal Hardy field extension M of K_0. Note that $(f_n^\dagger)_n$ is a pseudocauchy sequence in K_0 as for $n > m$ large enough,

$$v(f_n^\dagger - f_m^\dagger) = v(f_n/f_m)^\dagger = \psi(v(f_n) - v(f_m)) = \psi(\gamma_n - \gamma_m).$$

So there is a pseudolimit $s \in M$ of $(f_n^\dagger)_n$ ([6], Corollary 3.2). We hope that $K := K_0\langle f\rangle^{rc}$ for $f := \exp \int s$ could be our desired Hardy field with H-couple isomorphic to $(\Gamma_0\langle\beta\rangle, \psi_\beta)$ over (Γ_0, ψ_0). It is unclear whether K is an immediate extension of K_0.

A variant of the following conjecture might be provable.

Conjecture 4.3.12. *Suppose K_0 is a Hardy field with H-couple (Γ_0, ψ_0) over $\mathbb{Q}$. Then there is a Hardy field extension K of K_0 with H-couple over $\mathbb{Q}$ isomorphic to (Γ, ψ) over (Γ_0, ψ_0).*

For this discussion we assume Conjecture 4.3.11 and that case (b) extensions can be shown for non-closed H-couples over $\mathbb{Q}$. A reason why case (b) might go through without closed could be as follows. If K has H-couple (Γ, ψ) over $\mathbb{Q}$ which has a closure, then Corollary 4.3.5 gives a Hardy field extension K^c of K with H-couple isomorphic to some closure (Γ^c, ψ^c) of (Γ, ψ) and the element $\beta \in \Gamma \setminus \Gamma_0$ should still fall under case (b) over (Γ^c, ψ^c) and Lemma 4.3.7 applies. Then the produced element with value β adjoined to K instead of K^c might do the job. We would also rely on the assumption that Liouville closed can be reduced to real closed. Also, we assume that case (c) extensions work without the additional assumption $|\beta_n| \geq \alpha$ for some $\alpha \in \Gamma_0^>$, i.e. a gap can be made into the maximum of the extended ψ-set in Hardy fields.

Here may be some ingredients for a proof of Conjecture 4.3.12. First, we try to reduce to the case where Ψ_0 is cofinal in Ψ to avoid gaps and ensure asymptotic integration when applying the simple extensions. We implicitly use that we can always pass to the real closure of a Hardy field without changing the H-couple over $\mathbb{Q}$ and that the divisible asymptotic integration closure of H-couples not having gaps is also realized by Hardy fields. If Conjecture 4.3.10 holds, then the following extension problem could be solved for Hardy fields:

Let $(\Gamma_0 \oplus \mathbb{Q}\alpha_0 \oplus \bigoplus_n \mathbb{Q}\beta_n, \psi)$ be an H-couple extension of (Γ_0, ψ_0) such that

- $(\Gamma_0 \oplus \mathbb{Q}\beta_0, \psi^{\beta_0})$ where $\Psi_0 < \beta_0 < (\Gamma_0^>)'$ is the unique extension of (Γ_0, ψ_0) adding a gap ([5], Lemma 2.3).
- $(\Gamma_0 \oplus \mathbb{Q}\alpha_0 \oplus \mathbb{Q}\beta_0, (\psi^{\beta_0})^{\alpha_0})$ with $\beta_0 = \alpha_0'$ integrating the gap upwards or downwards.
- $(\Gamma_0 \oplus \mathbb{Q}\alpha_0 \oplus \bigoplus_n \mathbb{Q}\beta_n, \psi)$ is the unique divisible asymptotic integration closure of $(\Gamma_0 \oplus \mathbb{Q}\alpha_0 \oplus \mathbb{Q}\beta_0, (\psi^{\beta_0})^{\alpha_0})$ ([8], Lemma 4.8).

An extension with the properties above should be unique. To see this, let $(\Gamma_0 \oplus \mathbb{Q}\tilde{\beta}_0, \psi^{\tilde{\beta}_0})$ where $\tilde{\beta} = (-\alpha_0)'$. Then $(\Gamma_0 \oplus \mathbb{Q}\tilde{\beta}_0, \psi^{\tilde{\beta}_0}) \cong (\Gamma_0 \oplus \mathbb{Q}\beta_0, \psi^{\beta_0})$ since $\Psi_0 < \tilde{\beta}_0 < (\Gamma_0^>)'$ as well. Then $(\Gamma_0 \oplus \mathbb{Q}\alpha_0 \oplus \mathbb{Q}\beta_0, (\psi^{\beta_0})^{\alpha_0}) \cong (\Gamma_0 \oplus \mathbb{Q}\alpha_0 \oplus \mathbb{Q}\tilde{\beta}_0, (\psi^{\tilde{\beta}_0})^{\alpha_0})$ over (Γ_0, ψ_0) sending $\beta_0 \mapsto \tilde{\beta}_0$ and $\alpha_0 \mapsto -\alpha_0$.

Here, many elements of $(\Gamma_0 \oplus \mathbb{Q}\alpha_0 \oplus \mathbb{Q}\beta_0, \psi)$ could be considered gaps over (Γ_0, ψ_0), so the choice of integration of the gap might not matter as in one extension integrating upwards would give the same result as integrating downwards in another. In this sense, the gap is 'invisible' and the choice of integration does not matter on the H-couple side if the gap is not present in (Γ_0, ψ_0). Hence the gap can be integrated the way the Hardy field intends.

This way, we can iteratively add 'copies of $\mathbb{N}$' to the end of the ψ-set until Ψ is exhausted. Alternatively, one could conjecture the extension problem with H-couples as in ([8], Lemma 4.10) where a gap is made into the maximum of the ψ-set, which would also achieve the reduction to Ψ_0 cofinal in Ψ.

Then no simple extension can have a gap anymore and a Zorn argument might do the rest using that all cases of simple extensions can be realized by Hardy fields and the assumption on asymptotic integration can be achieved in each step.

References

[1] Matthias Aschenbrenner, Lou van den Dries. H-fields and their Liouville extensions. *Mathematische Zeitschrift*, 242(3):543–588, 2002.

[2] Matthias Aschenbrenner, Lou van den Dries. Liouville closed H-fields. *J. Pure Appl. Algebra*, 197(1-3):83–139, 2005.

[3] Matthias Aschenbrenner, Lou van den Dries. Analytic Hardy fields, 2024. `https://arxiv.org/abs/2311.07352`.

[4] Matthias Aschenbrenner, Lou van den Dries, Joris van der Hoeven. *Asymptotic Differential Algebra and Model Theory of Transseries*, volume 195 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2017.

[5] Matthias Aschenbrenner, Lou van den Dries, Joris van der Hoeven. Revisiting closed asymptotic couples. *Proc. Edinb. Math. Soc. (2)*, 65(2):530–555, 2022.

[6] Matthias Aschenbrenner, Lou van den Dries, Joris van der Hoeven. Filling gaps in Hardy fields. *J. Reine Angew. Math.*, 815:107–172, 2024.

[7] Matthias Aschenbrenner, Lou van den Dries, Joris van der Hoeven. *Maximal Hardy Fields*. 2025. `https://arxiv.org/abs/2304.10846`.

[8] Allen Gehret. The asymptotic couple of the field of logarithmic transseries. *J. Algebra*, 470:1–36, 2017.

[9] Allen Gehret. A tale of two Liouville closures. *Pacific Journal of Mathematics*, 290(1):41–76, 2017.

[10] Dmitry Gokhman. Functions in a Hardy field not ultimately c-infinity. *Complex Variables*, 32:1–6, 1997.

[11] Godfrey Harold Hardy. *Orders of infinity. The Infinitärcalcül of Paul du Bois-Reymond*, volume No. 12 of *Cambridge Tracts in Mathematics and Mathematical Physics*. Hafner Publishing Co., New York, 1971. Reprint of the 1910 edition.

[12] Maxwell Rosenlicht. On the value group of a differential valuation. II. *Amer. J. Math.*, 103(5):977–996, 1981.

[13] Gerald Teschl. *Ordinary differential equations and dynamical systems*, volume 140 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012.