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Some Applications of Complex Analysis to Fluids

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Abstract

This thesis investigates properties of the kinetic energy in two-dimensional travelling water waves over a flat bed, focusing on the two main categories: solitary and periodic waves. Using complex-analytic methods, we establish convexity and monotonicity properties of horizontal integral means of the kinetic energy. Furthermore, we determine the location of the extrema of kinetic energy, showing that these always occur at the free surface, specifically at wave crests and troughs. The convexity results on solitary waves are, to the best of our knowledge, new.

Abstract

Diese Arbeit befasst sich mit Eigenschaften der kinetischen Energie in zweidimensionalen Wasserwellen über einem flachen Boden, wobei der Schwerpunkt auf den beiden Hauptkategorien Solitärwellen und periodischen Wellen liegt. Mithilfe komplexanalytischer Methoden werden Konvexitäts- und Monotonieeigenschaften horizontaler Integralmittel der kinetischen Energie hergeleitet. Zudem wird die Lage der Extremstellen der kinetischen Energie bestimmt. Es zeigt sich, dass diese stets an der freien Oberfläche auftreten, insbesondere an Wellenbergen und -tälern. Die Konvexitätsergebnisse über Solitärwellen sind unseres Wissens nach neu.

Contents

1 Introduction

Water waves are disturbances that propagate along the free surface of a body of water, governed primarily by the interplay of gravity and surface tension. The study of travelling water waves, which are two-dimensional waves propagating in a fixed direction at constant speed without changing shape, in irrotational flow over a horizontal rigid bed, is a classical topic in fluid dynamics.

The irrotationality of the flow permits the application of results from harmonic analysis to reduce the problem's dimension by reformulating it solely on the surface, yielding an equivalent expression for the governing equations. However, certain characteristics of irrotational flows, such as the manner in which kinetic energy decays with depth, necessitate further investigation of their patterns beneath the surface.

Complex analysis provides a natural framework for studying harmonic and holomorphic functions that arise in potential flow theory, especially in two-dimensional irrotational flows. This thesis aims to demonstrate that methods from complex analysis can provide valuable insights into the behaviour of kinetic energy in travelling waves.

Travelling water waves typically fall into two principal categories: solitary waves and periodic waves. A solitary wave is a single, localized disturbance that maintains its shape while propagating at constant speed, and decays to the undisturbed state at spatial infinity; that is, the surface elevation approaches zero far away from the wave crest. Periodic waves, often referred to as Stokes waves, are waves of fixed shape that repeat at regular spatial intervals. These waves travel at constant speed without changing their form and are periodic in the horizontal direction. Unlike solitary waves, they do not decay at infinity, and their modelling involves using periodic boundary conditions.

The study of solitary and periodic water waves is not only of theoretical interest but also has significant practical applications. A soliton is a special type of solitary wave that maintains its shape and speed even after colliding with other solitons. Soliton theory appears across various branches of science, for example in modelling high temperature superconductors and energy transport in DNA, as well as in fibre-optic communications. The analysis of Stokes waves has important applications in oceanography, coastal and offshore engineering, and the modelling of wave–structure interactions. As the practical aspect is not the central concern of this thesis, we concentrate instead on the theoretical analysis of kinetic energy.

The thesis is divided into two main parts: a theoretical foundation and its application. In the theoretical part, we develop complex-analytic tools in two distinct settings. First, we study holomorphic functions on strips, presenting key results such as the Phragmén–Lindelöf principle, Hadamard three-lines and three-circles theorems, with the main result being Hardy–Ingham–Pólya theorem. Second, we explore holomorphic functions on annuli, with Hardy's convexity theorem as the principal result.

These theoretical tools are then employed in the second part of the thesis to analyse the kinetic energy in both solitary and periodic travelling waves. At the conclusion of this thesis, we will have demonstrated that data on velocity field of travelling waves collected along the free surface and at the flat bed supply reliable bounds for both the velocity and the kinetic energy at intermediate depths within the fluid domain, thereby considerably reducing the difficulty of measurement.

2 Theoretical Background in Complex Analysis

2.1 Convexity Theorems for Holomorphic Functions on Strips

In this section, we investigate the behaviour of holomorphic functions defined on vertical strips in the complex plane, with particular attention to their boundary behaviour and the convexity properties of their modulus along vertical lines. The beginning of the section follows the exposition in Section 6, Chapter 12 of Serge Lang's *Complex Analysis*[14], then we proceed to a key result taken from a paper of Hardy, Ingham and Pólya[12].

Theorem 2.1.1 (Phragmén-Lindelöf I, [14])**.** *Let f be a function that is continuous on the strip*

$$S_{[-\frac{\pi}{2},\frac{\pi}{2}]} = \{z = x + iy \in \mathbb{C} : -\frac{\pi}{2} \le x \le \frac{\pi}{2}\} \tag{2.1}$$

and holomorphic on the interior of S. Suppose that $|f(z)| \le 1$ on the boundary

$$\partial S_{[-\frac{\pi}{2},\frac{\pi}{2}]} = \{z = x + iy \in \mathbb{C} : x = \pm\frac{\pi}{2}\}$$

of $S_{[-\frac{\pi}{2},\frac{\pi}{2}]}$ and that there exists $0 < \alpha < 1$ and $C > 0$ such that

$$|f(z)| \le \exp(Ce^{\alpha|z|}) \tag{2.2}$$

for all $z \in S$ with $|z| \to \infty$. Then $|f| \le 1$ in S.

Proof. [14] Let $\alpha < \beta < 1$. For each $\epsilon > 0$ we construct an auxiliary function

$$g_\epsilon(z) = f(z)e^{-2\epsilon\cos(\beta z)}.$$

In order to estimate $|g_\epsilon(z)|$, we begin by computing the real part of the exponent of the exponential function:

$$\begin{aligned}\cos(\beta z) &= \frac{1}{2}(e^{i\beta z} + e^{-i\beta z})\\ &= \frac{1}{2}(e^{i\beta x}e^{-\beta y} + e^{-i\beta x}e^{\beta y}).\end{aligned}$$

Since $\Re(e^{\pm i\beta x}) = \cos(\beta x)$, we have

$$2\Re(\cos(\beta z)) = (e^{\beta y} + e^{-\beta y})\cos(\beta x).$$

Given $0 < \beta < 1$, it follows that $-\frac{\pi}{2} < \beta x < \frac{\pi}{2}$, thereby suggesting $\cos(\beta x)$ is non-negative, in the sense that there exists a $c_0 = \cos(\frac{\beta\pi}{2}) > 0$ such that $\cos(\beta x) > c_0$ for all $x \in [-\dfrac{\pi}{2}, \dfrac{\pi}{2}]$, and

$$2\Re(\cos(\beta z)) \geq c_0(e^{\beta y} + e^{-\beta y}). \tag{2.3}$$

With the restriction (2.2) on $|f(z)|$, we get

$$\begin{aligned} |g_\epsilon(z)| &= |f(z)|e^{-2\epsilon\Re\cos(\beta z)} \\ &\leq \exp(Ce^{\alpha|z|})\exp(-\epsilon c_0(e^{\beta y} + e^{-\beta y})) \\ &\leq \exp(Ce^{\alpha\sqrt{\frac{\pi^2}{4}+y^2}} - \epsilon c_0(e^{\beta y} + e^{-\beta y}), \end{aligned}$$

because $|z| = \sqrt{x^2 + y^2} \leq \sqrt{\frac{\pi^2}{4} + y^2}$.
Now we define

$$E(y) := Ce^{\alpha\sqrt{\frac{\pi^2}{4}+y^2}} - \epsilon c_0(e^{\beta y} + e^{-\beta y})$$

and aim to prove the claim that $E(y) \to -\infty$ as $|y| \to \infty$.
To demonstrate this, we factor out $e^{\beta|y|}$:

$$E(y) = e^{\beta|y|}(Ce^{\alpha\sqrt{\frac{\pi^2}{4}+y^2}-\beta|y|} - \epsilon c_0(e^{\beta(y-|y|)} + e^{-\beta(y+|y|)}).$$

Indeed, $\alpha\sqrt{\frac{\pi^2}{4} + y^2} - \beta|y| \sim (\alpha - \beta)|y|$ as $|y| \to \infty$. Based on the assumption that $\alpha < \beta$, $Ce^{\alpha\sqrt{\frac{\pi^2}{4}+y^2}-\beta|y|}$ converges to 0 as $|y| \to \infty$.

As for the latter part, if $y \to \infty$, then $y - |y| \to 0$ and $y + |y| \to \infty$. Conversely, if $y \to -\infty$, then $y - |y| \to -\infty$ and $y + |y| \to 0$. In either case, $e^{\beta(y-|y|)} + e^{-\beta(y+|y|)}$ converges to 1 as $|y| \to \infty$.

Therefore,

$$E(y) \sim -e^{\beta|y|} \to -\infty \text{ as } |y| \to \infty,$$

which implies that

$$|g_\epsilon(z)| \leq \exp(E(y)) \to 0. \tag{2.4}$$

Next we define the truncated rectangles of the strip with length L as

$$R_L = \{z = x + iy \in S : -L \leq y \leq L\}.$$

Recall that the maximum modulus principle (see [6]) asserts that if a function $f : \bar{U} \to \mathbb{C}$ is continuous on $\bar{U}$ and holomorphic in U, where U is a bounded connected open subset of $\mathbb{C}$, then f attains its maximum on ∂U (For a proof see [14]). Clearly, $g_\epsilon(z)$ satisfies the above conditions for $\bar{U} = R_L$, thus g_ϵ attains its maximum on ∂R_L.

On the vertical sides of R_L (i.e. $x = \pm\frac{\pi}{2}$) it is assumed in the theorem that $|f(z)| \leq 1$, and we know that $\Re(\cos(\beta z))$ is non-negative by (2.3), thus we obtain

$$|g_\epsilon(z)| = |f(\pm\frac{\pi}{2} + it)|e^{-2\epsilon\Re(\cos(\beta(\pm\frac{\pi}{2}+it)))} \leq 1.$$

On the horizontal sides of R_L (i.e. $y = \pm L$) we have as in (2.4) that $|g_\epsilon(z)| \to 0$ as $|y| \to \infty$. So g_ϵ is bounded by 1 on R_L for a sufficiently large L. By letting $L \to \infty$, we reach the conclusion that for all $z \in S$ it holds that $|g_\epsilon| \leq 1$, which in turn suggests that

$$|f(z)| \leq e^{2\epsilon\Re(\cos(\beta z))} \quad \forall z \in S. \tag{2.5}$$

Take $z \in R_L$. Since $\cos(\beta z)$ is bounded for all $z \in S$,the inequality (2.5) holds for all $\epsilon > 0$. By letting $\epsilon \searrow 0$ we ensure that $|f(z)| \leq 1$ for all $z \in R_L$. This concludes the proof of the theorem. □

Remark. The assumption $|f(z)| \leq 1$ can be replaced by an arbitrary bound $|f(z)| \leq M$, where $M > 0$, by applying theorem 2.1.1 to the function f/M.

The Phragmén–Lindelöf principle, which was introduced in 1884 by the Swedish mathematicians Edvard Phragmén and Ernst Leonard Lindelöf, can be viewed as a way to extend the maximum modulus principle to unbounded domains, in this case specifically to a strip. While the classical maximum modulus principle only guarantees maximum values on the boundaries of compact sets, the Phragmén-Lindelöf principle tells us that under suitable growth restrictions, the maximum modulus on the boundary still controls the function's bound in the interior.

In the previous theorem, the statement is only proved for a specific strip. We will see that it holds for arbitrary vertical strips through a change of variables. A similar statement also holds for sectors, though the proof is omitted here.

Definition (finite order)**.** [14] A function f on a strip

$$S_{[\sigma_1,\sigma_2]} = \{z = x + iy \in \mathbb{C} : \sigma_1 \leq x \leq \sigma_2\},$$

where $\sigma_1, \sigma_2 \in \mathbb{R}$, is of finite order on the strip if there exist constants $\lambda > 0$ and $C > 0$ such that

$$log|f(z)| \leq C|z|^\lambda \quad \text{for } |z| \to \infty,\ z \in S.$$

Theorem 2.1.2 (Phragmén-Lindelöf II, [14])**.** *Let f be a function that is continuous on the strip*

$$S_{[\sigma_1,\sigma_2]} = \{z = x + iy \in \mathbb{C} : \sigma_1 \leq x \leq \sigma_2\} \tag{2.6}$$

and holomorphic on the interior of S. Suppose that $|f(z)| \leq 1$ on the boundary

$$\partial S_{[\sigma_1,\sigma_2]} = \{z = x + iy \in \mathbb{C} : x = \sigma_1 \ or\ x = \sigma_2\}$$

of $S_{[\sigma_1,\sigma_2]}$ and f is of finite order on $S_{[\sigma_1,\sigma_2]}$. Then $|f(z) \leq 1$ on the strip $S_{[\sigma_1,\sigma_2]}$.[14]

Proof. [14] Notice that a linear change of variables

$$\varphi : S_{[-\frac{\pi}{2},\frac{\pi}{2}]} \to S_{[\sigma_1,\sigma_2]}, \quad \varphi(w) = aw + b = z,$$

enables us to map the strip $S_{[-\frac{\pi}{2},\frac{\pi}{2}]}$ from the previous theorem 2.1.1 to the strip $S_{[\sigma_1,\sigma_2]}$ defined in the present context. To determine the explicit form of the transformation, we identify the constants a and b by requiring that the boundaries of the original strip are mapped to the corresponding boundaries of the target strip. Specifically, we map the left boundary of $S_{[-\frac{\pi}{2},\frac{\pi}{2}]}$ to the left boundary of $S_{[\sigma_1,\sigma_2]}$. Setting $w = -\frac{\pi}{2}$, we impose $z = \varphi(-\frac{\pi}{2}) = \sigma_1$, which yields the equation

$$-\frac{\pi}{2}a + b = \sigma_1; \tag{2.7}$$

Analogously, mapping the right boundary $w = \frac{\pi}{2}$ to σ_2 gives the equation

$$\frac{\pi}{2}a + b = \sigma_2 \tag{2.8}$$

Subtracting the equation (2.7) from the equation (2.8) yields the expression

$$a = \frac{\sigma_2 - \sigma_1}{\pi},$$

and adding the two equations gives us the other constant

$$b = \frac{\sigma_1 + \sigma_2}{2}.$$

Substituting these values into the expression for $\varphi(w)$, we obtain the desired translation

$$\varphi(w) = \frac{\sigma_2 - \sigma_1}{\pi} w + \frac{\sigma_1 + \sigma_2}{2}.$$

Next we will show that the composition $g := f \circ \varphi : S_{[-\frac{\pi}{2},\frac{\pi}{2}]} \to \mathbb{C}$ satisfies all the conditions outlined in theorem 2.1.1. The continuity and holomorphicity of the composition of such functions are evident. Since $\varphi(w)$ maps the boundary of $S_{[-\frac{\pi}{2},\frac{\pi}{2}]}$ onto the boundary of $S_{[\sigma_1,\sigma_2]}$, and on the boundary of $S_{[\sigma_1,\sigma_2]}$ it holds that $|f(w)| \leq 1$, it follows that $|f(\varphi(w))| \leq 1$ on the boundary of $S_{[-\frac{\pi}{2},\frac{\pi}{2}]}$ as well. Therefore, it only remains to verify the growth condition to complete the argument. The fact that f is of finite order indicates that

$$\begin{aligned}
|g(z)| &= |f(\varphi(w)| \\
&\leq C_1 e^{|aw+b|^\lambda} \\
&= C_1 e^{(|a||w|+|b|)^\lambda} \\
&= e^{Ce^{\alpha|w|}},
\end{aligned}$$

because

$$C_1 e^{(|a||w|+|b|)^\lambda} = C_1 e^{(|a||w|+|b|)^\lambda + ln(C_1)} =: e^{A(w)},$$

and

$$A(w) \leq Ce^{\alpha|z|}, \quad \text{for } C > 0, \alpha \in (0,1).$$

Consequently, $|f \circ \varphi| \leq 1$ by theorem 2.1.1 and finally $|f| \leq 1$ on $S_{[-\frac{\pi}{2},\frac{\pi}{2}]}$, which proves the theorem. $\square$

Remark. As with the previous theorem, the assumption $|f(z)| \leq 1$ can be replaced by an arbitrary bound $|f(z)| \leq M$ where $M > 0$, by applying theorem 2.1.2 to the function f/M.

In the Phragmén-Lindelöf principle, our focus lay on the asymptotic behaviour of holomorphic functions for large t. Building on this idea, the Hadamard three-lines theorem offers a more precise quantitative estimate for bounded holomorphic functions on vertical strips. Indeed, it establishes a convexity property for the logarithm of the supremum norm of the function along vertical lines within the strip, based solely on the boundary values. In this sense, it can be regarded as a refinement of the Phragmén–Lindelöf principle.

Theorem 2.1.3 (Hadamard Three-Lines Theorem, [14])**.** *Let f be holomorphic on the strip*

$$S_{[a,b]} = \{z = x + iy \in \mathbb{C} : a \leq x \leq b\}.$$

For each x let $M_f(x) = M(x) = \sup_{y\in\mathbb{R}} |f(x + iy)|$. *Then* $logM(x)$ *is a convex function of x. [14]*

Proof. [14] A function $f(x)$ is convex on the interval [a,b] if and only if its difference quotient is monotonically increasing. This condition can be expressed formally as

$$\frac{f(x) - f(a)}{x - a} \leq \frac{f(b) - f(x)}{b - x},$$

which can be algebraically rearranged as follows:

$$\begin{aligned} (f(x) - f(a))(b - x) &\leq (f(b) - f(x))(x - a) \\ (b - a)f(x) &\leq f(b)(x - a) + f(a)(b - x) \\ f(x) &\leq f(b)\frac{x - a}{b - a} + f(a)\frac{b - x}{b - a}. \end{aligned}$$

Applying this expression to the function $logM(x)$, we obtain the condition for convexity

$$logM(x) \text{ is convex} \iff logM(x) \leq \frac{b - x}{b - a} logM(a) + \frac{x - a}{b - a} logM(b). \tag{2.9}$$

Exponentiating both sides of this inequality yields an equivalent multiplicative form, which we aim to prove:

$$M(x)^{b-a} \leq M(a)^{b-x} M(b)^{x-a}. \tag{2.10}$$

We first consider the case where $M(a) = M(b) = 1$, which represents a special case of the Phragmén-Lindelöf Principle II (2.1.2). For $x = a$, we have $|f(a+iy)| \leq M(a) = 1$ for all $y \in \mathbb{R}$, and similarly, the same inequality holds for $x = b$. Consequently, $|f(z)| \leq 1$ on the boundary of the strip $S_{[a,b]}$. Since f is holomorphic and bounded in the strip $S_{[a,b]}$ under the assumption that $M(a) = M(b) = 1$,it follows from theorem 2.1.2 that $|f(z)| \leq 1$ throughout the strip $S_{[a,b]}$, which implies $M(x) \leq 1$. Taking the logarithm on both sides yields $logM(x) \leq 1$. Because $logM(a) = logM(b) = 0$, we recover the convexity condition

$$logM(x) \leq 0 = \frac{b-x}{b-a} logM(a) + \frac{x-a}{b-a} logM(b).$$

To generalize this result, define the auxiliary function

$$h(z) := M(a)^{\frac{b-z}{b-a}} M(b)^{\frac{z-a}{b-a}}.$$

It is immediately apparent that h is entire, has no zeros, and that $\frac{1}{h}$ is bounded on the strip $S_{[a,b]}$. We now analyze the behavior of h on the boundary. For $z = a + iy$, we compute

$$\begin{aligned}
|h(a+iy)| &= |M(a)^{\frac{b-(a+iy)}{b-a}} M(b)^{\frac{(a+iy)-a}{b-a}}| \\
&= |M(a)^{1-\frac{iy}{b-a}} M(b)^{\frac{iy}{b-a}}| \\
&= M(a)|M(a)^{-\frac{iy}{b-a}}||M(b)^{\frac{iy}{b-a}}| \\
&= M(a),
\end{aligned}$$

as both exponents are purely imaginary and thus have modulus one. By the same argument, $|h(b+iy)| = M(b)$ for all $y \in \mathbb{R}$.
Next, consider the quotient f/h. On the boundary, we have

$$M_{f/h}(a) = \sup_{y\in\mathbb{R}} |\frac{f(a+iy)}{h(a+iy)}| = \sup_{y\in\mathbb{R}} \frac{|f(a+iy)|}{M(a)} = \frac{\sup |f(a+iy)|}{M(a)} = \frac{M(a)}{M(a)} = 1.$$

In a similar manner, we deduce that $M_{f/h}(b) = 1$. By the special case established above, we derive $|f/h| \leq 1$ throughout the strip $S_{[a,b]}$, whence $|f| \leq |h|$, or equivalently,

$$|f(z)| \leq M(a)^{\frac{b-z}{b-a}} M(b)^{\frac{z-a}{b-a}}$$

for all $z \in S_{[a,b]}$.
Taking the supremum over the vertical lines and raising both sides to the power $b - a$ yields the inequality (2.10), thus completing the proof. □

While the Hadamard three-lines theorem concerns holomorphic functions on vertical strips, there exists a closely related result known as the Hadamard three-circles theorem, which applies to holomorphic functions on annuli. Despite being set in different domains, the two theorems share a structural similarity: both describe the logarithmic convexity of the maximum modulus of a holomorphic function along curves (vertical lines in the strip and concentric circles in the annulus).

Although the three-circle theorem belongs to the setting of annuli, which will be discussed in the next section, we include it here due to its close conceptual and methodological connection to the three-lines theorem. The idea of both results is essentially the same: the modulus of a holomorphic function cannot behave too wildly between two boundaries if it is controlled on those boundaries. Indeed, a transformation from the strip to the annulus shows that the three-circles theorem is essentially the three-lines theorem expressed in a different geometric setting.

Theorem 2.1.4 (Hadamard Three-Circle Theorem, [14])**.** *Let $f(z)$ be a holomorphic function on the annulus*

$$A_{[\alpha,\beta]} = \{z \in \mathbb{C} : \alpha \leq |z| \leq \beta\}. \tag{2.11}$$

Let

$$M(r) = \sup_{|z|=r} |f(z)|.$$

Then $logM(r)$ is a convex function of $log(r)$. More precisely,

$$log(\beta/\alpha)logM(r) \leq log(\beta/r)logM(\alpha) + log(r/\alpha)logM(\beta).[14] \tag{2.12}$$

Proof. [14] To prove this theorem, we transform the annulus A into a strip and apply the Hadamard three-lines theorem 2.1.3 to obtain the desired result. Let us introduce the change of variables $s = log(z)$. This substitution maps the annulus A onto the vertical strip

$$S_{[a,b]} = \{s = \sigma + it \in \mathbb{C} : a \leq \sigma \leq b\},$$

where $e^a = \alpha$ and $e^b = \beta$. Now define the function

$$f^*(s) = f(e^s).$$

Since $f(z)$ is holomorphic on the annulus A, it follows that the function f^* is holomorphic on the strip $S_{[a,b]}$. Therefore, according to the Hadamard three-lines theorem 2.1.3, the maximum modulus for $f^*(s)$

$$M_{f^*} = M^*(\sigma) = \sup_t |f^*(\sigma + it)|$$

defines a convex function of the real variable σ. In particular, the logarithm of the maximum modulus satisfies the convexity condition

$$logM^*(\sigma) \leq \frac{b-\sigma}{b-a} logM^*(a) + \frac{\sigma - a}{b-a} logM^*(b), \tag{2.13}$$

as discussed in (2.9).
We now revert to the original variable. Since $f^*(s) = f(e^s)$, we have

$$M^*(\sigma) = M(e^\sigma),$$

so by setting $r = e^\sigma$, we obtain the identity

$$logM^*(\sigma) = logM(e^\sigma) = logM(r).$$

Taking the logarithm on both sides of the strip yields $a = log(\alpha)$ and $b = log(\beta)$. Substituting these into the convexity condition (2.13) yields

$$logM(r) \leq \frac{log\beta - logr}{log\beta - log\alpha} logM(\alpha) + \frac{logr - log\alpha}{log\beta - log\alpha} logM(\beta)$$

Multiplying both sides by $log(\frac{\beta}{\alpha})$, this inequality becomes

$$log(\frac{\beta}{\alpha}) logM(r) \leq log(\frac{\beta}{r}) logM(\alpha) + log(\frac{r}{\alpha}) logM(\beta),$$

which is precisely the assertion in the theorem. This concludes the proof. □

Both the three-lines and three-circles theorems were formulated at the turn of the 20th century by the French mathematician Jacques Hadamard, although the proof was later completed by Harald Bohr and Edmund Landau[13]. Hadamard made foundational contributions to the theory of entire and holomorphic functions and his work in this area laid the groundwork for much of modern complex analysis. These two theorems remain central tools, particularly in applications to harmonic analysis and partial differential equations.

In preparation for the proof of our main result, we first revisit fundamental facts about subharmonic functions, along with some key results from Jensen's work.

Definition. A continuous function $f : \Omega \subset \mathbb{C} \to \mathbb{R} \cup \{-\infty\}$ is called subharmonic, if and only if for any $z \in \Omega$ there exists $R > 0$ such that $B(z, R)$ is contained in Ω and

$$f(z) \leq \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta}) d\theta$$

for every $r < R$.[9]

Intuitively, a subharmonic function is bounded at each point by the average of its values along any surrounding circle centered at that point. Aside from this local sub-mean-value property, the subharmonicity of a real-valued continuous function f on Ω is characterized by the following equivalent property: For every domain $B \subseteq \Omega$ with $\bar{B} \subseteq \Omega$, and for every function $u(z)$ that is harmonic in B and continuous in $\bar{B}$, if $f(z) \leq u(z)$ on ∂B, then the inequality $f(z) \leq u(z)$ holds throughout B. In particular, if $u(z)$ is a harmonic function in B that agrees with a subharmonic function $f(z)$ on the boundary ∂B, then $f(z) \leq u(z)$ in B, since $f - u$ is subharmonic in B itself.[9]

Theorem 2.1.5 (Jensen's Formula[6]). *Let f be a holomorphic function on the domain Ω and $\bar{B}(z,r) \subseteq \Omega$. Let $\{a_k\}_k \subseteq B(z,r)$ be the zeros of f counted with multiplicity. Then*

$$\frac{1}{2\pi}\int_0^{2\pi} log|f(z+re^{i\theta})|d\theta = log|f(z)| + \sum_k log(\frac{r}{|a_k|}). \tag{2.14}$$

Proof. Without loss of generality assume $z = 0$. This is justified by the translation invariance of Jensen's formula; indeed, we may consider the function $g(w) = f(z+w)$, which is holomorphic on $B(0,r)$ if and only if f is holomorphic on $B(z,r)$, and the zeros of f in $B(z,r)$ correspond exactly to the zeros of g in $B(0,r)$. To prove the formula, we decompose the function f into a non-vanishing holomorphic function and finite product of monomials, and verify the formula separately for each component.

Step 1.

Assume that f does not vanish on $\bar{B}(0,r)$. Then, by continuity, f does not vanish on some larger disk $B(0,r')$ for $r' > r$. In this case, there exists a holomorphic function g on $B(0,r')$ that satisfies $f = e^g$, implying $|f| = e^{\Re(g)}$ and $\log|f| = \Re(g)$. By Cauchy's integral formula, we have

$$g(0) = \frac{1}{2\pi i}\int_{|z|=r} \frac{g(z)}{z}dz = \frac{1}{2\pi}\int_0^{2\pi} g(re^{i\theta})d\theta.$$

Taking the real part on both sides gives us the desired identity.

Step 2.

Let $f(z) = z - w$, where $w \in B(0,r)\backslash\{0\}$. We wish to show that the following equation holds:

$$\begin{aligned}\log|w| &= \frac{1}{2\pi}\int_0^{2\pi} \log|re^{i\theta} - w|d\theta + \log(\frac{|w|}{r}) \\ &= \frac{1}{2\pi}\int_0^{2\pi} (\log r + \log|e^{i\theta} - w/r|)d\theta + \log|w| - \log r,\end{aligned}$$

or, equivalently,

$$\int_0^{2\pi} \log|e^{i\theta} - w/r| = 0$$

Setting $a := w/r$, observe that $|a| < 1$ and $|e^{i\theta} - a| = |1 - e^{-i\theta}a|$. Thus, after a change of variable, the claim reduces to

$$\int_0^{2\pi} \log|1 - e^{-i\theta}a|d\theta = 0.$$

The function $1 - za$ does not vanish in a neighbourhood of $\bar{B}(0,1)$, so there is a holomorphic function g defined there such that $1 - za = e^{g(z)}$. Consequently,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\log|1-e^{-i\theta}a|d\theta &= \frac{1}{2\pi}\int_0^{2\pi}\Re(g(re^{i\theta}))d\theta\\ &= \Re(\frac{1}{2\pi i}\int_{|z|=1}\frac{g(z)}{z}dz)\\ &= \Re(g(0)) = 0.\end{aligned}$$

The second-to-last step follows from Cauchy's integral formula, and the final equality from the fact that $e^{\Re(g(0))} = |1 - 0a| = 1$.

Step 3.

Let f be as in the assumptions, and write

$$f(z) = \prod_{j=1}^{N} f_j(z)\cdot g(z),$$

where g is non-vanishing on $\bar{B}(0,r)$ and holomorphic in Ω, and each $f_j(z) = z - z_j$. Applying the results from the previous two steps, we obtain

$$\begin{aligned}\log|f(0)| &= \sum_{j=1}^{N}\log|f_j(0)| + \log|g(0)|\\ &= \sum_{j=1}^{N}(\frac{1}{2\pi}\int_0^{2\pi}\log|f_j(re^{i\theta})|d\theta + \log(\frac{|z_j|}{r})) + \frac{1}{2\pi}\int_0^{2\pi}\log|g(re^{i\theta})|d\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|d\theta + \sum_{j=1}^{N}\log(\frac{|z_j|}{r}),\end{aligned}$$

which concludes the proof. □

Theorem 2.1.6 (Jensen's inequality[16]). *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex function and $X : U \subseteq \mathbb{R}^n \to \mathbb{R}$ a continuous function. Then*

$$\phi(\mathbb{E}(X) \leq \mathbb{E}(\phi(X)). \tag{2.15}$$

Lemma 2.1.7. *Let $p > 0$ and $\Omega \subseteq \mathbb{C}$ an open set. If $f : \Omega \to \mathbb{C}$ is a holomorphic function, then $|f(z)|^p$ is subharmonic.*

Proof. Let $f(z)$ be a holomorphic function on Ω. The goal of the first part of the proof is to show that $log|f(z)|$ is a subharmonic function. This is a consequence of Jensen's formula 2.14.

First, consider the case where f is zero-free in the closed disk $\bar{B}(z,r) \subseteq \Omega$. In this setting, Jensen's formula simplifies to

$$\frac{1}{2\pi}\int_0^{2\pi} log|f(z+re^{i\theta})|d\theta = log|f(z)|.$$

This is precisely the mean value property that characterizes harmonic functions. Consequently, $log|f(z)|$ is harmonic in the disk $B(z,r)$. Since every harmonic function is subharmonic as well, we conclude that $log|f(z)|$ is subharmonic in this region.

Now we consider the case in which $f(z)$ has zeros inside the disk $B(z,r)$. Let $\{a_k\}_k \subseteq B(z,r)$ denote the set of such zeros of f. Since each zero a_k lies strictly inside the disk, we have $|a_k| < r$, and hence each term $log(\frac{r}{|a_k|}) > 0$. The sum in Jensen's formula on the right hand side is thus strictly positive, which means

$$\frac{1}{2\pi}\int_0^{2\pi} log|f(z+re^{i\theta})|d\theta > log|f(z)|,$$

This inequality is the defining condition for subharmonicity. Combining both cases, we conclude that $log|f(z)|$ is subharmonic regardless of the presence of zeros.

In the second part of the proof we aim to show that the subharmonicity of such a function is preserved under composition with a convex increasing function. Let $u : \Omega \to [-\infty, \infty)$ be a subharmonic function and $\phi : \mathbb{R} \to \mathbb{R}$ be convex and increasing. We wish to demonstrate that the composition $\phi \circ u$ is also subharmonic with the aid of Jensen's inequality. To this end, fix a point $z \in \Omega$ and $r > 0$ such that $\bar{B}(z,r) \subseteq \Omega$. Consider a continuous function $X : [0,2\pi] \to \mathbb{R}, X(\theta) = u(z+re^{i\theta})$, which is real-valued based on the assumption on u. Applying Jensen's inequality yields

$$\begin{aligned}\phi(\mathbb{E}(X)) &= \phi(\frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta})d\theta)\\ &\leq \frac{1}{2\pi}\int_0^{2\pi} \phi(u(z+re^{i\theta}))d\theta.\end{aligned}$$

On the other hand, since $u(z)$ is subharmonic, it satisfies

$$u(z) \leq \frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta})d\theta.$$

Combining this with the monotonicity of ϕ, we have

$$\phi(u(z)) \leq \phi(\frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta})d\theta).$$

Joining both inequalities above yields

$$\phi(u(z)) \leq \frac{1}{2\pi}\int_0^{2\pi} \phi(u(z+re^{i\theta}))d\theta,$$

which establishes the subharmonicity of $\phi \circ u$.
To conclude the proof, we apply this general result to our original setting. Observe that $log|f(z)|$ is subharmonic and $\phi(t) = e^{pt}$ is convex and increasing for any $p > 0$, so the composition $\phi(log|f(z)|) = |f(z)|^p$ is subharmonic as well. □

Remark. Combining the preceding lemma with the definition of a subharmonic function, we obtain the following inequality for a holomorphic function $f : \Omega \to \mathbb{C}$

$$|f(z_0)|^p \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(z_0 + re^{i\theta})|^p d\theta, \tag{2.16}$$

where $z_0 \in \Omega$ is arbitrary and $r > 0$ is such that $B(z_0, r) \subseteq \Omega$. This result is known as Hardy's theorem[11]. The proof in the case $p \geq 1$ is elementary: for $p = 1$, it follows directly from the Cauchy integral formula applied to $f(z_0)$; the extension to $p \geq 1$ then follows by applying Hölder's inequality.

Theorem 2.1.8 (Hardy-Ingham-Pólya, [12])**.** *Suppose that $f(z)$ is holomorphic and satisfies the Phragmén-Lindelöf condition (PL) appropriate to $S_{(a,b)}$*

$$f(z) = O(e^{e^{|k|y}}), \quad \textit{where } 0 < k < \frac{\pi}{\beta - \alpha} \tag{2.17}$$

uniformly in the strip $S_{(a,b)}$, and that $|f(z)|$ is continuous in $S_{[a,b]}$. In addition, suppose that the integral

$$J(x) = \int_{-\infty}^{\infty} |f(x + iy)|^p dy \tag{2.18}$$

is convergent, when $x = \alpha$ or $x = \beta$. Then

1. *$J(x)$ is convergent for $\alpha < x < \beta$,*
2. *$J(x) \leq \max\{J(\alpha), J(\beta)\}$,*
3. *$\log J(x)$ is a convex function of x,so that*

$$J(x) \leq J(\alpha)^{\frac{\beta - x}{\beta - \alpha}} J(\beta)^{\frac{x - \alpha}{\beta - \alpha}}. \tag{2.19}$$

Proof. The proof of this convexity theorem for integrals follows an approach analogous to that used in the Phragmén-Lindelöf theorem 2.1.1 and shall be divided into three steps. In order to circumvent the complications arising from the growth condition, we initially place an additional restriction on the function under consideration and prove the first two statements within this framework. We subsequently discard this restriction by constructing a family of auxiliary functions that satisfy the imposed restriction and converge to the original function. Last we complete the argument by proving the convexity property.

Step 1.

We begin with proving (i) and (ii) under the supplementary assumption that the function $f(z)$ exhibits rapid decay, i.e. it is very small in the distant region of $S_{[a,b]}$. More precisely, we impose the condition

$$|f(z)| < Ke^{-|y|} \tag{2.20}$$

for some constant $K > 0$. Here the convergence of $J(x)$ follows immediately. We estimate $J(x)$:

$$\begin{aligned} J(x) &= \int_{-\infty}^{\infty} |f(x+iy)|^p dy \\ &\leq K \int_{-\infty}^{\infty} e^{-|y|p} dy \\ &= K(\int_{-\infty}^{0} e^{yp} dy + \int_{0}^{\infty} e^{-yp} dy) \\ &= \frac{2}{p} \end{aligned}$$

To establish the continuity of $J(x)$ at an arbitrary point $x_0 \in [\alpha, \beta]$, let $(x_n)_n \in [\alpha, \beta]$ be a sequence such that $x_n \to x_0$ as $n \to \infty$. We examine the limit of the sequence $J(x_n)$. Since $f(z)$ is holomorphic on the interior of the strip and continuous in $S_[a, b]$, the integrand converges pointwise:

$$\lim_{n\to\infty} |f(x_n + iy)|^p = |f(x_0 + iy)|^p \quad \forall y \in \mathbb{R}.$$

Moreover, the decay condition (2.20) guarantees that for all $n \in \mathbb{N}$, the integrand is dominated by an integrable function independent of n:

$$|f(z)| \leq K^p e^{-p|y|}.$$

By the Lebesgue Dominated Convergence Theorem, we may interchange the limit and the integral

$$\begin{aligned} \lim_{n\to\infty} J(x_n) &= \lim_{n\to\infty} \int_{-\infty}^{\infty} |f(x_n + iy)|^p dy \\ &= \int_{-\infty}^{\infty} \lim_{n\to\infty} |f(x_n + iy)|^p dy \\ &= \int_{-\infty}^{\infty} |f(x_0 + iy)|^p dy \\ &= J(x_0) \end{aligned}$$

Therefore, $J(x)$ is continuous at x_0. Since x_0 is arbitrary, the continuity extends to the entire interval.

We proceed to prove (ii) by contradiction. Fix an arbitrary point $x_0 \in (\alpha, \beta)$ and we want to show that $J(x_0)$ is smaller than the maximum value of $J(x)$ on the boundary of the interval. To derive a contradiction, we assume the contrary.

Let $r > 0$ be such that the disks centered at $z_0 = x_0 + iy$ with radius r are contained in the strip $S_{[\alpha,\beta]}$; that is to say, we require $\alpha < x_0 - r < x_0 + r < \beta$. By Hardy's theorem 2.16 the value of $|f|^p$ at the center is bounded by the average of its value along the boundary:

$$|f(z_0)|^p \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(z_0 + re^{i\theta})|^p d\theta. \tag{2.21}$$

Now we consider the possibility that $J(x)$ may attain a local maximum at some point $x_0 \in (\alpha, \beta)$.

We employ the previous result to calculate $J(x_0)$. Specifically, we integrate with respect to y and use Fubini's theorem to interchange the order of integration:

$$\begin{aligned}
J(x_0) &= \int_{-\infty}^{\infty} |f(x_0 + iy)|^p dy \\
&\leq \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x_0 + iy + re^{i\theta})|^p d\theta dy \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} |f(x_0 + r\cos\theta + i\underbrace{(y + rsin\theta)}_{=:\tilde{y}})|^p dy d\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{\int_{-\infty}^{\infty} |f(x_0 + r\cos\theta + i\tilde{y})| d\tilde{y}}_{J(x_0 + r\cos\theta)} d\theta \\
&= \frac{1}{\pi} \int_{0}^{\pi} J(x_0 + r\cos\theta) d\theta, \quad \text{since } J \text{ is an even function.}
\end{aligned}$$

Assuming that $J(x)$ attains a maximum at x_0, we aim to show that $J(x)$ cannot attain a local maximum at any other point inside a disk centered at x_0, unless it is constant. Rearranging the above inequality yields:

$$\begin{aligned}
0 &\leq \frac{1}{\pi} \int_{0}^{\pi} J(x_0 + r\cos\theta) d\theta - J(x_0) \\
&= \frac{1}{\pi} \int_{0}^{\pi} J(x_0 + r\cos\theta) d\theta - \frac{1}{\pi} \int_{0}^{\pi} J(x_0) d\theta \\
&= \frac{1}{\pi} \int_{0}^{\pi} J(x_0 + r\cos\theta) - J(x_0) d\theta.
\end{aligned}$$

However, if $J(x_0)$ is indeed a local maximum, then

$$\frac{1}{\pi} \int_{0}^{\pi} \underbrace{J(x_0 + r\cos\theta) - J(x_0)}_{\leq 0} d\theta \leq 0.$$

Define

$$g_r(\theta) := J(x_0 + r\cos\theta) - J(x_0),$$

Then g_r is continuous, satisfies $g_r(\theta) \leq 0$, and, combining both previous inequalities, its integral over $[0, \pi]$ vanishes:

$$\frac{1}{\pi}\int_0^{\pi} g_r(\theta)d\theta = 0,$$

so $g_r \equiv 0$ holds for all appropriate r. Consequently,

$$J(x_0) = J(x_0 + r\cos\theta) \quad \forall\theta \in [0, \pi]$$

holds for any $r' < r$, which means that $J(x)$ is a constant function that takes the value $J(x_0)$ on $[x_0 - r, x_0 + r]$.

The function $\cos(\theta)$ attaining all values in the interval between $[-1, 1]$ as θ ranges over $[0, 2\pi]$ demonstrates that $J(x_0 \pm r) = J(x_0)$, which implies that this result holds for any $r' < r$, hence J is constant on $[x_0 - r, x_0 + r]$.

Now we want to show that $J(x)$ is constant on the entire interval $[\alpha, \beta]$ by discussing the connectedness of the set

$$A = \{a \in (\alpha, \beta) : J(a) = J(x_0)\}$$

The set A is closed by the continuity of $J(x)$ and open in (α, β) by an earlier argument. Since any point $a \in A$ satisfies $J(a) = J(x_0)$ and, as $J(x_0)$ is assumed to be a maximum, the same reasoning implies that $J(x)$ is constant in an open neighbourhood of a, that is to say J is locally constant around every point in A, so A will contain an open interval centered at a. Since A is both open and closed in an connected interval, it is either empty or the entire interval (α, β). If A is the entire interval, then $J(x) = J(x_0)$ in the interior of the interval and by the continuity of J also on the boundary,so $J(x)$ is a constant function; If A is empty, then $J(x)$ does not attain its maximum in the interior of the interval, hence the maximum must occur on the boundary. This proves (ii).

Step 2.

Now we dispense of the assumption made in (2.20) and proceed as in the proof of the Phragmén-Lindelöf theorem 2.1.1, in which we multiply the function $f(z)$, which may undergo rapid growth, by a factor that decays sufficiently fast. This strategy enables us to apply step 1 to a new function that satisfies the decay condition, by a limiting procedure we obtain the desired result. To compensate for the rapidly growing exponent of (2.17), we subtract $e^{k^*|y|}$ with $k^* > k$, such that the difference converges to 0 faster than $-|y|$.

In order to construct an analytic function that provides such decay, we take the exponential of the cosine function. The modulus of this function contains the hyperbolic cosine, as the following computation demonstrates:

$$\begin{aligned}\Re(\cos(z)) &= \Re(\frac{e^{iz}+e^{-iz}}{2}) \\ &= \frac{1}{2}\Re(e^{-y}e^{ix}+e^{y}e^{-ix}) \\ &= \frac{1}{2}(e^{-y}\cos(x)+e^{y}\cos(x)) \\ &= \cosh(y)\cos(x).\end{aligned}$$

The hyperbolic cosine contained in the real part of $\cos(z)$ enables the construction of a decaying exponential factor to control the growth. Let $k < k^* < \frac{\pi}{\beta-\alpha}$ and define

$$F_\epsilon(z) := f(z)\cdot\exp(-\epsilon\cos(k^*(z-a))), \quad \text{where } a := \frac{\alpha+\beta}{2}. \tag{2.22}$$

The function satisfies the decay condition (2.20) and $|F_\epsilon(z)| \leq |f(z)|$ for all $\epsilon > 0$.

Since a is the midpoint of the interval (α, β), we have for any $z \in (\alpha, \beta)$ that

$$k^*|z-a| \leq k^*\frac{\beta-\alpha}{2} < \frac{\pi}{2},$$

by our choice of k^*. This constraint ensures that $\cos(k^*(z-a))$ stays away from 0. In particular, we define

$$\delta := \cos(k^*\cdot\frac{\beta-\alpha}{2}) > 0,$$

such that

$$\cos(k^*(x-a)) \geq \delta.$$

Using our previous computation of the real part of $\cos(z)$, we get

$$\Re(\cos(k^*(z-a)) = \cosh(k^*y)\cos(k^*(x-a)) \geq \delta\underbrace{\cosh(k^*y)}_{\geq 1} \geq \delta.$$

Therefore, the modulus of $F_\epsilon(z)$ satisfies

$$|F_\epsilon(z)| = |f(z)|e^{-\epsilon\Re(\cos(k^*(z-a)))} \leq |f(z)|e^{-\epsilon\delta} \leq |f(z)|.$$

Moreover, we observe that for $0 < \epsilon_2 < \epsilon_1$

$$|F_{\epsilon_1}(z)| \leq |F_{\epsilon_2}.(z)| \leq |f(z)|, \tag{2.23}$$

meaning that for every fixed z, the family $\{|F_\epsilon(z)|\}$ increases monotonically as $\epsilon \searrow 0$ and

$$\lim_{\epsilon\searrow 0}|F_\epsilon(z)| = |f(z)|,$$

so the family converges pointwise and increasingly to $|f(z)|$. This allows for the use of the Monotone Convergence Theorem in the next step.

It remains to verify that $F_\epsilon(z)$ satisfies (2.20). To this end, we estimate the modulus of $F_\epsilon(z)$ using our previous calculations:

$$\begin{aligned} |F_\epsilon(z)| &\le K_1 \exp(e^{k|y|}) \cdot e^{-\epsilon \Re(\cos(k^*(z-a)))} \\ &= K_1 \exp(\underbrace{e^{k|y|} - \epsilon \cosh(k^* y) \cos(k^*(x-a))}_{:=E(y)}), \end{aligned} \tag{2.24}$$

where the first exponential term bounds $|f(z)|$ due to its prescribed growth (2.17) and the latter factor the modulus of the limiting function.

We further estimate the exponent $E(y)$:

$$\begin{aligned} E(y) &= e^{k|y|} - \epsilon(\frac{e^{k^* y} + e^{-k^* y}}{2} \underbrace{\cos(k^*(x-a))}_{\ge \delta > 0}) \\ &\le e^{k|y|} - \frac{\epsilon\delta}{2}(e^{k^* y} + e^{-k^* y}) \end{aligned} \tag{2.25}$$

Heuristically, in the case $y \ge 0$,the term $e^{k^* y}$ dominates both $e^{k|y|} = e^{ky}$ and $e^{-k^* y}$ as $y \to \infty$. In particular, the contribution from $e^{-k^* y}$ is negligible, while the negative term $-\frac{\epsilon\delta}{2} e^{k^* y}$ eventually outweighs the positive term e^{ky}. As a result, $E(y) \to -\infty$ and for sufficiently large y, we obtain a linear upper bound $E \le -\tilde{k} y$, where $\tilde{k} > 0$ is some constant.

A similar argument holds in the case $y < 0$. Here, the term $e^{-k^* y}$ dominates over $e^{k|y|} = e^{-ky}$, while $e^{k^* y}$ becomes insignificant as $y \to -\infty$, and as before we obtain $E(y) \le -\tilde{k}|y|$ for sufficiently large $|y|$.

We multiply the inequality in (2.25) by -1 and factor out $e^{k^*|y|}$ from the resulting expression. Note that the term $e^{k^* y} + e^{-k^* y}$ is both symmetric and real, so taking the modulus does not alter its value:

$$-e^{k|y|} + \frac{\epsilon\delta}{2}(e^{k^*|y|} + e^{-k^*|y|}) = e^{k^*|y|}(\underbrace{-e^{(k-k^*)|y|}}_{\to 0} + \frac{\epsilon\delta}{2}(1 - \underbrace{e^{-2k^*|y|}}_{\to 0})$$

As $|y| \to \infty$, both $e^{(k-k^*)|y|} \to 0$ and $e^{-2k^*|y|} \to 0$, so the expression in the parenthesis is represented by the constant term $\frac{\epsilon\delta}{2}$. It is easy to find a constant $\tilde{k} > 0$ to form a lower linear bound, such that

$$\frac{\epsilon\delta}{2} e^{k^*|y|} \ge \tilde{k}|y| \quad \forall y \in \mathbb{R},$$

which implies

$$-e^{k|y|} + \frac{\epsilon\delta}{2}(e^{k^*|y|} + e^{-k^*|y|}) \ge \tilde{k}|y|.$$

Therefore, substituting this back into (2.24) yields

$$|F_\epsilon(z)| \leq K_1 e^{-\tilde{k}|y|} \quad \forall y \in \mathbb{R},$$

which confirms that $F_\epsilon(z)$ satisfies the decay condition (2.20). This enables the application of (ii) to $F_\epsilon(z)$, yielding

$$J_{F_\epsilon}(x) \leq max\{J_{F_\epsilon}(\alpha), J_{F_\epsilon}(\beta)\},$$

that is,

$$\int_{-\infty}^{\infty} |F_\epsilon(x+iy)|^p dy \leq max\{\int_{-\infty}^{\infty} |F_\epsilon(\alpha+iy)|^p dy, \int_{-\infty}^{\infty} |F_\epsilon(\beta+iy)|^p dy\}. \tag{2.26}$$

Recall that

$$|F_\epsilon(z)| \nearrow |f(z)| \quad \text{as } \epsilon \searrow 0$$

for each fixed z. Since $|F_\epsilon(z)|^p$ increases pointwise to $|f(z)|^p$, we may apply the Monotone Convergence Theorem to obtain

$$\begin{aligned}
\lim_{\epsilon \searrow 0} \int_{-\infty}^{\infty} |F_\epsilon(x+iy)|^p dy &= \int_{-\infty}^{\infty} \lim_{\epsilon \searrow 0} |F_\epsilon(x+iy)|^p dy \\
&= \int_{-\infty}^{\infty} |f(x+iy)|^p dy \\
&= J(x)
\end{aligned}$$

Taking the limit of the inequality (2.26) and combining it with the above computation, we obtain (ii) for $f(z)$ without the decay condition.

Step 3.

In the final and most straightforward step we prove the convexity. The central idea is to construct an auxiliary function that equalizes the values of $J(\alpha)$ and $J(\beta)$. This approach yields a sharp bound instead of a maximum of two quantities, from which we deduce the convexity property for $f(z)$. Define the auxiliary function

$$g(z) = e^{\frac{kz}{p}} f(z),$$

where the constant $k \in \mathbb{R}$ is chosen to satisfy

$$e^{k\alpha} J(\alpha) = e^{k\beta} J(\beta).$$

We notice that

$$|g(z)|^p = e^{k\Re(z)} |f(z)|^p,$$

hence the associated integral becomes

$$J_g(\alpha) = e^{k\alpha} \int_{-\infty}^{\infty} |f(\alpha+iy)|^p dy = e^{k\alpha} J(\alpha),$$

while an analogous computation yields

$$J_g(\beta) = e^{k\beta} \int_{-\infty}^{\infty} |f(\beta + iy)|^p dy = e^{k\beta} J(\beta).$$

By our choice of k, these two expressions are equal.

It is obvious that if $f(z)$ satisfies the hypothesis of the theorem, then so does $g(z)$, because the growth of the multiplicative factor $e^{\frac{kz}{p}}$ does not interfere with the growth condition imposed on $f(z)$, thus we can apply (i) and (ii) to $g(z)$ and get

$$\begin{aligned} e^{kx} J(x) &= J_g(x) \\ &\leq max\{J_g(\alpha), J_g(\beta)\} \\ &= J_g(\alpha) \\ &= e^{k\alpha} J(\alpha). \end{aligned}$$

Note that the condition

$$e^{k\alpha} J(\alpha) = e^{k\beta} J(\beta)$$

implies

$$e^{k(\alpha-\beta)} = \frac{J(\beta)}{J(\alpha)},$$

and solving for k, we get an explicit expression

$$k = \frac{1}{\alpha - \beta} log \frac{J(\beta)}{J(\alpha)}.$$

By rearranging the terms and plugging in the above result, we get

$$\begin{aligned} J(x) &\leq e^{k(\alpha - x)} J(\alpha) \\ &= e^{k(\alpha-\beta)\frac{x-\alpha}{\beta-\alpha}} J(\alpha) \\ &= (\frac{J(\beta)}{J(\alpha)})^{\frac{x-\alpha}{\beta-\alpha}} J(\alpha) \\ &= J(\beta)^{\frac{x-\alpha}{\beta-\alpha}} J(\alpha)^{1-\frac{x-\alpha}{\beta-\alpha}} \\ &= J(\beta)^{\frac{x-\alpha}{\beta-\alpha}} J(\alpha)^{\frac{\beta-x}{\beta-\alpha}}, \end{aligned}$$

which establishes the logarithmic convexity of $J(x)$ and thus completes the proof of (iii) and thereby the theorem. □

Remark. Using the rotation $z \mapsto iz$, we are able to transform a vertical strip to a horizontal strip, allowing us to apply the Hardy-Ingham-Pólya theorem 2.1.8 to solve problems on horizontal strips as well. The theorem can be reformulated as following:

Suppose that $f(z)$ is holomorphic and satisfies the Phragmén-Lindelöf condition (PL) appropriate to the strip $S = \{x + iy : y_1 < y < y_2\}$

$$f(z) = O(e^{e^{|k|x}}), \quad \text{where } 0 < k < \frac{\pi}{y_2 - y_1} \tag{2.27}$$

uniformly in the strip S, and that $|f(z)|$ is continuous in S. In addition, suppose that the integral

$$J(y) = \int_{-\infty}^{\infty} |f(x + iy)|^p dx \tag{2.28}$$

is convergent, when $y = y_1$ or $y = y_2$. Then

1. $J(y)$ is convergent for $y_1 < x < y_2$,
2. $J(y) \leq \max\{J(y_1), J(y_2)\}$,
3. $\log J(y)$ is a convex function of y,so that

$$J(y) \leq J(y_1)^{\frac{y_2 - y}{y_2 - y_1}} J(y_2)^{\frac{y - y_1}{y_2 - y_1}}. \tag{2.29}$$

2.2 Convexity Theorems for Holomorphic Functions on Annuli

In this section, we investigate holomorphic functions defined between concentric circles. Studying how these functions behave within the annulus and along its boundaries reveals important convexity properties of their maximum modulus, analogous to the behaviour observed for holomorphic functions on strips.

A central result in this scenario is Hardy's convexity theorem, which establishes a logarithmic convexity property for the maximum modulus of a holomorphic function as the radius varies between the inner and outer boundaries of the annulus.

This theorem can be regarded as an extension of the Hadamard three-circles theorem, delivering a more profound understanding of how the growth of holomorphic functions is constrained by their boundary behaviour.

Theorem 2.2.1. *[9] Let $g : A_{(R_1,R_2)} \to \mathbb{R}$ be a subharmonic function on the annulus*

$$A_{(R_1,R_2)} = \{z \in \mathbb{C} : R_1 < |z| < R_2\}.$$

Define the function

$$m(r) = \frac{1}{2\pi}\int_0^{2\pi} g(re^{i\theta})d\theta, \quad R_1 < r < R_2.$$

Then $m(r)$ is a convex function of $\log(r)$*.*

Proof. Let r_1, r_2 be such that $R_1 < r_1 < r_2 < R_2$ and let $u(z)$ denote the harmonic function harmonic on the annulus $A_{(r_1.r_2)}$ that satisfies

$$u(z) = g(z) \quad \text{for } |z| = r_1 \text{ and } |z| = r_2.$$

In other words, $u(z)$ solves the Dirichlet problem on $A_{(r_1.r_2)}$ with boundary condition given by $g(z)$. Such a function exists according to Theorem 12 in [10] and can be expressed via Green functions.

Since the subharmonic function g and the harmonic function u agree on both boundary circles, the maximum principle for subharmonic functions implies that

$$g(z) \leq u(z) \quad \text{for all } z \in A_{(r_1,r_2)}.$$

Therefore, we obtain

$$\begin{aligned} m(r) &= \frac{1}{2\pi}\int_0^{2\pi} g(re^{i\theta})d\theta \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})d\theta \\ &=: \tilde{m}(r), \quad r_1 \leq r \leq r_2, \end{aligned}$$

Now we wish to show that $\tilde{m}(r)$ assumes the form

$$\begin{aligned}\tilde{m}(r) &:= \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})d\theta \\ &= c(r_1)\log(r) + b(r_1)\end{aligned} \tag{2.30}$$

for some constants $c(r_1), b(r_1)$ depending on r_1.

Since u is harmonic on the annulus $A_{(r_1.r_2)}$, we have $\Delta u = 0$ there. In polar coordinates, the Laplacian becomes

$$\begin{aligned}\Delta u &= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \\ &= \frac{1}{r}\partial_r(ru_r) + \frac{1}{r^2}u_{\theta\theta} \\ &= 0.\end{aligned}$$

Here $u_r(r,\theta)$ denotes $\dfrac{\partial}{\partial r}u(re^{i\theta})$.

To see the claim (2.30), we integrate Δu over the annulus $A_{(r_1,r)}$:

$$\begin{aligned}0 &= \iint_{A_{(r_1.r)}} \Delta u dA \\ &= \int_0^{2\pi}\int_{r_1}^{r}\left(\frac{1}{\rho}\partial_\rho(\rho u_\rho) + \frac{1}{\rho^2}u_{\theta\theta}\right)(\rho,\theta)\cdot \rho d\rho d\theta \\ &= \int_0^{2\pi}\int_{r_1}^{r}\left(\partial_\rho(\rho u_\rho) + \frac{1}{\rho}u_{\theta\theta}\right)(\rho,\theta) d\rho d\theta\end{aligned}$$

This integral decomposes as the sum of two terms:

$$I_1 + I_2 := \int_0^{2\pi}\int_{r_1}^{r}\partial_\rho(\rho u_\rho)(\rho,\theta)d\rho d\theta + \int_{r_1}^{r}\frac{1}{\rho}\int_0^{2\pi} u_{\theta\theta}(\rho,\theta)d\theta d\rho.$$

so we can compute them separately. For the second term, since u_θ is $2\pi-$periodic in θ, it follows that

$$\int_0^{2\pi} u_{\theta\theta}(\rho,\theta)d\theta = \left[u_\theta(\rho,\theta)\right]_0^{2\pi} = 0.$$

Thus the second term vanishes and the entire integral reduces to I_1:

$$\begin{aligned} I_1 &= \int_0^{2\pi} \int_{r_1}^{r} \partial_\rho(\rho u_\rho)(\rho,\theta) d\rho d\theta \\ &= \int_0^{2\pi} \big[\rho u_\rho(\rho,\theta)\big]_{r_1}^{r} d\theta \\ &= \int_0^{2\pi} (r u_r(r,\theta) - r_1 u_r(r_1,\theta)) d\theta \\ &= 0 \end{aligned}$$

Hence, we find that

$$\int_0^{2\pi} r u_r(r,\theta) d\theta = \int_0^{2\pi} r_1 u_r(r_1,\theta) d\theta =: 2\pi c \quad \forall r \in (r_1, r_2).$$

Now recall that

$$\tilde{m}(r) := \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta}) d\theta,$$

so differentiating with respect to r yields

$$\tilde{m}'(r) := \frac{1}{2\pi} \int_0^{2\pi} u_r(r,\theta) d\theta.$$

Multiplying both sides by r gives

$$r\tilde{m}'(r) = r_1 \tilde{m}'(r_1) =: c,$$

and thus

$$\tilde{m}'(r) = \frac{c}{r}.$$

Integrating this differential equation yields

$$\tilde{m}(r) = \int \frac{c}{r} dr = c\log(r) + b,$$

where the constant of integration is

$$b := \tilde{m}(r_1) - c\log(r_1) = \frac{1}{2\pi} \int_0^{2\pi} u(r_1 e^{i\theta}) d\theta - c\log(r_1).$$

Hence, as claimed in (2.30),

$$\tilde{m}(r) = c\log(r) + b.$$

Now we prove the convexity of $m(r)$ using the claim (2.30). The function $\log(r)$ is strictly increasing and concave on $(0,\infty)$, so for any $r \in (r_1, r_2)$, there exists $\alpha \in (0,1)$ such that

$$\log(r) = \alpha \log(r_1) + (1-\alpha)\log(r_2).$$

It follows that

$$\begin{aligned}
m(r) \leq \tilde{m}(r) &= c\log(r) + b \\
&= c(\alpha\log(r_1) + (1-\alpha)\log(r_2)) + b \\
&= \alpha c\log(r_1) + c\log(r_2) - \alpha c\log(r_2) + b + \alpha b - \alpha b \\
&= \alpha(c\log(r_1) + b) + (1-\alpha)(c\log(r_2) + b) \\
&= \alpha\tilde{m}(r_1) + (1-\alpha)\tilde{m}(r_2)
\end{aligned}$$

Since $\tilde{m}(r_1) = m(r_1)$ and $\tilde{m}(r_2) = m(r_2)$ by construction (as $u = g$ on the boundary circles), we conclude that

$$m(r) \leq \alpha m(r_1) + (1-\alpha)m(r_2).$$

This proves the convexity of $m(r)$ in $\log(r)$. □

Theorem 2.2.2 (Hardy's convexity theorem, [9]). *Let f be an analytic function on the annulus $A_{(R_1,R_2)} = \{z \in \mathbb{C} : R_1 < |z| < R_2\}$, and let $0 < p \leq \infty$. Define the integral means*

$$\begin{aligned}
M_p(r,f) &= \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}}, \quad 0 < p < \infty; \\
M_\infty(r,f) &= \sup_{0\leq\theta<2\pi} |f(re^{i\theta})|.
\end{aligned}$$

Then $\log M_p(r,f)$ is a convex function of $\log(r)$ on the interval (R_1,R_2). That is, for any $r_1, r_2 \in (R_1.R_2)$ and any $\alpha \in (0,1)$, if

$$\log(r) = \alpha\log(r_1) + (1-\alpha)\log(r_2),$$

then

$$\log M_p(r,f) \leq \alpha\log M_p(r,f) + (1-\alpha)\log M_p(r,f),$$

or equivalently,

$$M_p(r,f) \leq [M_p(r,f)]^\alpha [M_p(r,f)]^{1-\alpha}.$$

Notice that in the special case $p = \infty$, the theorem reduces to Hadamard's three circle theorem 2.1.4. Therefore, it only remains to show the case $p < \infty$.

Proof. Consider the auxiliary function

$$g(z) = |z|^\lambda |f(z)|^p, \quad \lambda \in \mathbb{R},$$

defined on the annulus $A_{(R_1,R_2)}$. We aim to show that this function is subharmonic on $A_{(R_1,R_2)}$, if f is analytic.

To see this, Recall that a function g is subharmonic at z_0, if and only if for each $z_0 \in A_{(R_1,R_2)}$ there exists a $\delta > 0$ such that $B(z_0,\delta) \subseteq A_{(R_1,R_2)}$ and

$$g(z_0) \leq \frac{1}{2\pi}\int_0^{2\pi} g(z_0 + re^{i\theta})d\theta \quad \text{for all } r < \delta.$$

We consider two cases. If $f(z_0) = 0$ or $z_0 = 0$, then

$$g(z_0) = 0 \leq \frac{1}{2\pi}\int_0^{2\pi} |z_0|^\lambda |f(z_0 + re^{i\theta})|^p d\theta,$$

which satisfies the subharmonicity condition. It remains to show the case $f(z_0) \neq 0$ and $z_0 \neq 0$.

If $f(z_0) \neq 0$, since f is analytic, by continuity there exists a $\delta' > 0$ such that $f(z)$ is non-vanishing on the small disk $B(z_0,\delta')$, implying that $\frac{f'}{f}$ is analytic on $B(z_0,\delta')$ and possesses a primitive. Consequently, f admits an analytic logarithm on $B(z_0,\delta')$, i.e. there exists an analytic function F on $B(z_0,\delta')$ such that $f(z) = e^{F(z)}$, and thus $f(z)^p = e^{pF(z)}$. Since $p \in \mathbb{R}$, it follows that $|f(z)^p| = |f(z)|^p$.

Similarly, since $z_0 \neq 0$, there is a branch of logarithm that is analytic on $B(z_0,\delta')$ for δ' sufficiently small, so that $z^\lambda = e^{\lambda \log z}$.

Hence, the product $z^\lambda f(z)$ is analytic on $B(z_0,\delta')$, and its modulus $|z^\lambda||f(z)|^p$ is therefore subharmonic.

By Theorem 2.2.1,

$$\begin{aligned} m_g(r) &= \frac{1}{2\pi}\int_0^{2\pi} |re^{i\theta}|^\lambda |f(re^{i\theta})|^p d\theta \qquad (2.31)\\ &= r^\lambda \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta \\ &= r^\lambda M_p^p(r,f) \end{aligned}$$

is a convex function of $\log(r)$. Given $R_1 < r_1 < r_2 < R_2$, choose $\lambda \in \mathbb{R}$ such that

$$m_g(r_1) = m_g(r_2) =: K,$$

which is equivalent to the condition

$$r_1^\alpha M_p^p(r_1,f) = r_2^\alpha M_p^p(r_2,f).$$

Now let $\alpha \in (0,1)$ and define $r := r_1^\alpha \cdot r_2^{1-\alpha}$. Since $m_g(r)$ is a convex function of $\log(r)$, we have

$$\begin{aligned} m_g(r) &\leq \alpha m_g(r_1) + (1-\alpha)m_g(r_2) \\ &= \alpha K + (1-\alpha)K \\ &= K. \end{aligned}$$

Using the previous computations (2.31), we get

$$r^{\lambda} M_p^p(r, f) \le K.$$

On the other hand, by the choice of λ, we also have

$$\begin{aligned} K &= K^{\alpha} \cdot K^{(1-\alpha)} \\ &= r_1^{\alpha\lambda} \big[M_p^p(r_1, f))\big]^{\alpha} \cdot r_2^{(1-\alpha)\lambda} \big[M_p^p(r_2, f)\big]^{1-\alpha} \\ &= r^{\lambda} \big[M_p^p(r_1, f))\big]^{\alpha} \big[M_p^p(r_2, f)\big]^{1-\alpha}. \end{aligned}$$

Combining the inequalities, we conclude that

$$M_p^p(r, f) \le \big[M_p^p(r_1, f))\big]^{\alpha} \big[M_p^p(r_2, f)\big]^{1-\alpha},$$

which is precisely the desired statement. This completes the proof □

Remark. Notice that $\log M_p(r_1, f)$ is a convex function of $\log r$, if and only if $\log M_p^x(r_1, f)$ is a convex function of $\log r$ for all $x \in \mathbb{R}_+$.

A result to extend a holomorphic function analytically that we employ repeatedly in the next sections is stated below.

Essentially, under certain conditions, a holomorphic function on an open set contained in the upper half plane can be extended to the open set that is symmetric with respect to the real axis by reflecting it across it. This theorem is a symmetry tool that doubles the domain of a function while preserving analyticity. In our setting, as we will see in the next chapter, the horizontal flat bed serves as a natural choice for the axis of reflection; indeed, we shall use this result to enable the application of maximum modulus principle.

Let Ω^+ be a connected open set in the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) \ge 0\}$ and $\Omega^- = \{z \in \mathbb{C} : \bar{z} \in \Omega^+\}$ be its reflection across the real axis. If f is an analytic function on Ω^+, then the function f^- on Ω^- given by $\overline{f(\bar{z})}$ is also analytic, and $g(z) = f(z) - \overline{f(\bar{z})}$ is analytic on Ω^+. Since Ω^+ is connected, its boundary $\partial\Omega^+$ must contain an open interval of the real axis. If f is real on $\Omega_0 := \Omega^+ \cap \mathbb{R} = \{z \in \Omega^+ : \Im(z) = 0\}$, then $g(x) \equiv 0$ on Ω_0. However, Ω_0 has a limit point in Ω^+, so $f(z) = \overline{f(\bar{z})}$ for all $z \in \Omega^+$. Now we use this equation to extend a function defined on Ω^+. [6]

Theorem 2.2.3 (Schwarz Reflection Principle, [6])**.** *Let Ω^+, Ω^- and Ω_0 be defined as above. If $f : \Omega^+ \cup \Omega_0 \to \mathbb{C}$ is a continuous function that is analytic in Ω^+, and if $f(x)$ is real on Ω_0, then there is an analytic function $g : \Omega^+ \cup \Omega_0 \cup \Omega^- \to \mathbb{C}$ is given by*

$$g(z) = \begin{cases} f(z), & z \in \Omega^+, \\ \overline{f(\bar{z})}, & z \in \Omega^-. \end{cases} \tag{2.32}$$

We shall also use Morera's theorem in our further considerations.

Theorem 2.2.4 (Morera's theorem, [16]). *Suppose g is a continuous function on an open subset $\Omega \subseteq \mathbb{C}$ such that*

$$\int_{\partial\triangle} g(z)dz = 0$$

for every closed triangle in Ω, then g is holomorphic in Ω

3 Applications to Water Waves

3.1 Preliminaries

3.1.1 Modelling Water Waves

In preparation for the application, we state the simplifying physical assumptions and derive from them the governing equations of the motion. This section follows Chapter 2 of [3].

In order to develop a model for travelling water waves, we adopt a set of simplifying assumptions that preserve the essential physical features while rendering the problem mathematically accessible. Water waves in the real world are influenced by numerous factors such as viscosity, variable density, and bottom topography, which often exert only a minor impact on the wave motion of interest. By neglecting or idealising these effects, we reduce the problem to its fundamental components, allowing the application of tools from harmonic and complex analysis without compromising the physical relevance of our results.

- We model water as a continuous medium. an idealization consistent with everyday observation of water.
- Although in reality the density of water is influenced by factors including but not limited to temperature, depth, and salinity, their impact is minimal, typically causing density variations of less than 0.5%. Consequently, it is accurate to regard water as a homogeneous substance.
- Viscosity is neglected, since the inviscid approximation is sufficiently precise for time scales of hours.
- The motion of the air above the surface influences the water through pressure. Assuming this pressure to be constant decouples the motion of the water from that of the air, thereby allowing us to disregard the latter. Any disturbance of the water surface induced by air motion then corresponds to a negligible change in the atmospheric pressure, which we denote by a constant value P_{atm}.
- We restrict ourselves to a rigid flat bed for the sake of simplicity and ignore the impact of bottom topography.
- The progression of water surface disturbances is regulated by the balance between a restoring force and the system's inertia. We focus on gravity waves, which occur

when gravity restores the water that has been displaced from its equilibrium state. The influence of surface tension is insignificant for waves higher than a few cm.

- We focus our attention on two-dimensional flows that propagate at constant speed in a fixed direction. In such a wave train, the wave crest is the highest point above the undisturbed water level, and the lowest point below the surface is known as the wave trough.

3.1.2 The Governing Equations

We now derive the governing equations for water waves and discuss the boundary conditions. We suppose that at time t = 0, a disturbance is generated on the surface of the still water, for instance by the wind, and our goal is to determine the subsequent motion of the water wave.

The Equation of Mass Conservation

Consider a volume of water V with constant density ρ and bounded by a C^1- surface S, so that surface integrals make sense. Let $\vec{n}$ be the unit normal vector pointing outward from S and $\vec{u} \in C^1$ be the velocity of water at each point, such that the water velocity field is smooth enough to take derivatives and do calculus operations. Then the outward velocity component across S is given by $\vec{u} \cdot \vec{n}$. Note that the tangential component of $\vec{u}$ parallel to S does not contribute to flow across boundary. At each point of S, the mass flux, that is the rate at which mass flows through a unit area, is given by $\rho(\vec{u} \cdot \vec{n})$. Integrating over S yields the rate at which mass leaves V is

$$\int_S \rho\vec{u} \cdot \vec{n} dS,$$

where $\vec{u} \cdot \vec{n} < 0$ occurs, when water flows outward, because mass leaves the system, and $\vec{u} \cdot \vec{n} > 0$ when water flows inward.

Recall that the mass in V is given by

$$M(t) = \int_V \rho(\vec{x}, t) d\vec{x}.$$

Since ρ is constant everywhere and does not change with time, we have

$$M(t) = \rho \cdot (\text{volume of V}),$$

and

$$\tfrac{d\rho}{dt} = 0. \tag{3.1}$$

Consequently, the rate of change of mass in V is

$$\frac{d}{dt} \int_V \rho d\vec{x} = \int_V \frac{d\rho}{dt} d\vec{x} = 0,$$

Physically, this means that mass is neither created nor destroyed anywhere in the water; any change in the mass of V comes solely from flow across the boundary S. If water flows out across the surface, we lose mass; if water flows in, we gain mass. So the change of mass in V is simply determined by the relation change of mass = (mass inflow - mass outflow). According to mass conservation, this is equal to zero.

The rate of change of mass is given by

$$\frac{dM(t)}{dt} = -\text{outflow rate}$$

The minus sign is there because if more flows out, mass inside decreases. Therefore, the rate at which mass leaves through S is zero as well:

$$\frac{d}{dt}\int_V \rho d\vec{x} = -\int_S \rho\vec{u}\cdot\vec{n} dS = 0.$$

Recall the divergence theorem:

Theorem 3.1.1 (Divergence Theorem, [10]). *Let $\vec{u} \in C^1(V)$, where $V \subseteq \mathbb{R}^n$ be a bounded open subset, and S the C^1-boundary of V. Then*

$$\int_V \nabla\cdot\vec{u} d\vec{x} = \int_S \vec{u}\cdot\vec{n} dS$$

Applying the divergence theorem, this is equivalent to

$$\int_V \nabla\cdot\vec{u} d\vec{x} = 0.$$

Since this holds for any volume V, we obtain the equation of mass conservation

$$\nabla\cdot\vec{u} = 0. \tag{3.2}$$

The Equation of Motion: Euler's Equation

In hydrodynamics, two kinds of forces are relevant:

- Body forces (external forces): These act on every particle of the fluid in the same way irrespective of their position and originate from sources external to the fluid. A classical example is the gravity.
- Internal forces (local forces): These are the forces that particles of the fluid exert on each other and obey Newton's third law, whereby for every action there is an equal and opposite reaction.

In the context of gravity waves, the relevant forces can be categorized as follows:

- The predominant body force is due to gravity. Using Cartesian coordinates $\vec{x} = (x, y, z)$ with z pointing upward, the gravity force per unit mass is given by $\vec{F} = (0, 0, -g)$, where $g > 0$ is the gravitational acceleration.
- In an ideal fluid, which is a continuously distributed inviscid medium, the internal forces are attributed to the pressure P, which acts only in the normal direction to the surface, with no tangential component.

The setting remains the same: consider a volume V of water bounded by a C^1-surface S. We assume that the velocity $\vec{u}$ and the pressure P are of class C^1 throughout the fluid domain.

Note that the product of density ρ and force per unit mass $\vec{F}$ is the force per unit volume, thus the total body force on V is obtained by integrating it over V:

$$\int_V \rho \vec{F} d\vec{x}.$$

The pressure on V from outside is

$$-\int_S P\vec{n} dS.$$

The minus sign is there because $\vec{n}$ is the outward normal vector, but pressure pushes inward.

Hence the total force consisting of both body forces and local forces acting on the water in V is given by

$$\int_V \rho \vec{F} d\vec{x} - \int_S P\vec{n} dS.$$

Applying the divergence theorem 3.1.1 to the second term, this expression can be rewritten as

$$\int_V (\rho \vec{F} - \nabla P) d\vec{x}.$$

Newton's second law states that the total force $\vec{F}$ on V is equal to the rate of change of momentum of V. The momentum per particle is $m\vec{u}$, where m is the mass of a particle. Since mass is the product of density and volume, the mass of a particle can be represented as ρdV, while the acceleration of that particle is $\frac{D\vec{u}}{Dt}$. Thus we obtain the total rate of change of momentum of V by integrating over V:

$$\int_V \rho \frac{D\vec{u}}{Dt} d\vec{x}.$$

So by Newton's second law,

$$\int_V \rho \frac{D\vec{u}}{Dt} d\vec{x} = \int_V (\rho \vec{F} - \nabla P) d\vec{x}. \tag{3.3}$$

Here, the left-hand side represents the total mass times the acceleration of the fluid particles in V, where $\frac{D}{Dt}$ denotes the total differentiation following the particular particle. Indeed, if a particle has velocity $\vec{u}(\vec{x},t)$, its trajectory $\vec{x}(t)$ satisfies $\frac{d\vec{x}}{dt} = \vec{u}(\vec{x},t)$, and thus the acceleration of the particle is given by

$$\frac{d^2\vec{x}}{dt^2} = \frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u} = \frac{D\vec{u}}{Dt}$$

as in (3.3). Since the choice of V in (3.3) is arbitrary, the integrands must be equal, and we deduce the Euler equation of motion by inserting $\vec{F} = (0,0,-g)$:

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\nabla P + (0,0,-g). \tag{3.4}$$

The pressure P appears as the fourth unknown in addition to the three components of the velocity $\vec{u}$. The nonlinearity of the Euler equations arises from the term $\vec{u}\cdot\nabla\vec{u}$, which prevents the application of the principle of superposition and significantly complicates their analysis.

The Boundary Conditions

In our setting, the fluid domain is bounded by two distinct surfaces: the rigid, horizontal flat bed and the free water surface, assuming the fluid extends infinitely in the horizontal directions. The kinematic boundary conditions, which are called this way because they do not involve the action of forces, express the fact that fluid particles initially located on a boundary remain there at all subsequent times.

Let $S(\vec{x},t) = 0$ be the C^1-surface, where $S : \mathbb{R}^4 \to \mathbb{R}$ is a C^k function ($k \in \mathbb{N}$) with non-vanishing spatial gradient: $(S_x, S_y, S_z) \neq \vec{0} \in \mathbb{R}^3$. Consider an arbitrary particle $\vec{x_0}$ at time $t = t_0$ on the free surface. Its subsequent location $\vec{x}(t)$ is given by the unique solution of the differential equation $\vec{x}'(t) = \vec{u}(\vec{x}(t),t)$ with initial data $\vec{x}(t_0) = \vec{x_0}$, which satisfies $S(\vec{x}(t),t) = 0$ for $t \geq t_0$. Differentiating with respect to t shows that a necessary condition for this invariance is $\frac{DS}{Dt} = 0$ along the surface. If $S(\vec{x}(t),t) = 0$, its total derivative with respect to t must be zero as well. Applying the chain rule, we get

$$\frac{d}{dt}S(\vec{x}(t),t) = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial x}\frac{dx}{dt} + \frac{\partial S}{\partial y}\frac{dy}{dt} + \frac{\partial S}{\partial z}\frac{dz}{dt} = 0$$

The velocity of the particle is the fluid velocity,

$$\frac{d}{dt}\vec{x}(t) = (\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}) = \vec{u}(\vec{x}(t),t) = (u_1, u_2, u_3),$$

so the above equation becomes

$$S_t + u_1 S_x + u_2 S_y + u_3 S_z = 0.$$

The total differential of S is thus

$$\frac{DS}{Dt} = S_t + \vec{u} \cdot \nabla S = 0.$$

This condition is also sufficient, as it ensures the flow is everywhere tangential to the time-dependent free surface: the absence of a normal velocity component prevents particles from crossing it.

Physically, when water is in contact with a rigid surface (e.g. the seabed) or another fluid (e.g. air) it does not mix with, the velocity of the water normal to the surface must match the velocity of the surface itself in the normal direction in order to preserve the contact; the defining property of the surface is that water does not cross it. At any given point on that surface, the surface itself might be moving, as the free surface changes shape over time, and the fluid at that point has its own velocity. If the fluid moves faster in the normal direction than the surface, it would "pierce through" it. If it moves slower, the surface would leave the fluid behind and thus break contact.

If $\vec{n}$ is the normal unit vector at a point $\vec{P}$ on the surface of contact, $\vec{u} = (u_1, u_2, u_3)$ is the water velocity, and $\vec{q}$ is the velocity at the point $\vec{P}$, then the normal velocity of the fluid is $\vec{u} \cdot \vec{n}$, and the normal velocity of the surface is $\vec{q} \cdot \vec{n}$. So the no-crossing condition is expressed as

$$\vec{q} \cdot \vec{n} = \vec{u} \cdot \vec{n}$$

or

$$(\vec{u} - \vec{q}) \cdot \vec{n} = 0$$

For a surface given by $S(x, y, z, t) = 0$, this implies

$$\vec{u} \cdot \vec{n} = \frac{u_1 S_x + u_2 S_y + u_3 S_z}{\sqrt{S_x^2 + S_y^2 + S_z^2}}$$

and

$$\vec{q} \cdot \vec{n} = -\frac{S_1}{\sqrt{S_x^2 + S_y^2 + S_z^2}}.$$

The equation $\frac{DS}{Dt} = 0$ therefore expresses precisely that the relative velocity between the water and the surface is everywhere tangential

For the free surface $z = h(x, y, t)$, the kinematic boundary condition becomes

$$u_3 = h_t + u_1 h_x + u_2 h_y \quad \text{on} \quad z = h(x, y, t), \tag{3.5}$$

The surface $z = h(x, y, t)$ can be written as the zero-level set of the function

$$S(x, y, z, t) = z - h(x, y, t) = 0.$$

Computing the derivatives of S yields

$$\begin{aligned}\frac{DS}{Dt} &= S_t + u_1 S_x + u_2 S_y + u_3 S_z \\ &= (-h_t) + u_1(-h_x) + u_2(-h_y) + u_3 \cdot 1 \\ &= 0.\end{aligned}$$

Rearranging the terms gives us the statement.

Similarly, on the rigid horizontal flat bed $z = -d$ the kinematic boundary condition reduces to

$$u_3 = 0 \quad \text{on} \quad z = -d. \tag{3.6}$$

The atmosphere exerts its influence on the water surface in the form of pressure. The dynamic boundary condition at the free surface specifies that the pressure equals the constant atmospheric pressure P_{atm}:

$$P = P_{atm} \quad \text{on} \quad z = h(x, y, t), \tag{3.7}$$

which reflects the external actions on the free surface. At the bottom we do not encounter this type of condition, as no interactions occur there. This decouples the water motion from the air motion, which we ignore from now on. While any disturbance of the free surface is a consequence of some sort of motion of the air, we may still approximate the air pressure by its undisturbed state, since the air pressure variations due to wave motion are negligible compared to P_{atm} owing to the low density of air.

The water-wave problem for gravity waves is thus described by the governing equations for water waves consisting of the Euler's equation (3.4), the equation of mass conservation (3.2) and the boundary conditions ((3.5)–(3.7). A distinctive feature in water-wave theory is that the free surface is unknown a priori and must be determined as part of the solution, making this a free boundary problem: the wave shape and the motion of the fluid are correlated.

3.2 General Theory of 2D Water Waves

We begin by summarizing the discussion from the previous section. The governing equations for two-dimensional water waves with the velocity field (u, v) consist of the equation of mass conservation

$$u_x + v_y = 0, \qquad -d \le y \le \eta(x,t)\,, \tag{3.8}$$

together with the Euler equation

$$\begin{cases} u_t + uu_x + vu_y = -\dfrac{1}{\rho}\, p_x\,, \\ v_t + uv_x + vv_y = -\dfrac{1}{\rho}\, p_y\, - g\,, \end{cases} \qquad -d \le y \le \eta(x,t)\,, \tag{3.9}$$

These equations are complemented by the following boundary conditions.

The dynamic boundary condition

$$p = p_{atm} \quad \text{on} \quad y = \eta(x,t)\,, \tag{3.10}$$

decouples the water motion from the motion of the air above the free surface $y = \eta(x,t)$.

The kinetic boundary conditions

$$v = \eta_t + u\eta_x \quad \text{on} \quad y = \eta(x,t)\,, \tag{3.11}$$

$$v = 0 \quad \text{on} \quad y = -d\,, \tag{3.12}$$

ensure that particles on the free surface $y = \eta(x,t)$ and the flat bed $y = -d$ are remain there at all times.

Here ρ is the water's constant density (about 1000 kg/m^3), g is the constant acceleration of gravity and p_{atm} is the constant atmospheric pressure at sea level (about 10^5 kg/ms^2); see the discussion in [3].

We further impose the irrotationality condition

$$u_y - v_x = 0\,, \qquad -d \le y \le \eta(x,t)\,. \tag{3.13}$$

From the mass conservation (3.8) and the irrotationality condition (3.13) we see that the pair $(u, -v)$ satisfies the Cauchy-Riemann equations, thus the complex-valued function $f = u + i(-v)$ is analytic in the fluid domain.

Definition. The kinetic energy in the water waves is given by

$$E(x,y,t) = \frac{1}{2}(u^2 + v^2) = \frac{1}{2}|f|^2.$$

Our aim is to establish relationships between the L^p-means of the kinetic energy $E = (u^2 + v^2)/2$ at different depths. Specifically, we consider the L^p-means of the kinetic energy at each instant t on horizontal lines below the trough line

$$\mu(t) = \inf_{x\in\mathbb{R}} \eta(x,t). \tag{3.14}$$

We introduce the notations

$$\mathcal{M}_p(E,y) = \int_{-\infty}^{\infty} E(x,y)^p dx, \quad p \geq 1, \tag{3.15}$$

and

$$\mathcal{M}_\infty(E,t,y) = \sup_{x\in\mathbb{R}} E(x,y,t), \tag{3.16}$$

where $y \in [-d, \mu(t)]$.

Theorem 3.2.1. *Let $p \in (0,\infty]$. If $\mathcal{M}_p(E,t,y_j) < \infty$ for $j = 1,2$ and $y_1 < y_2 < \mu(t)$, then $\mathcal{M}_p(E,t,y) < \infty$ for all $y \in [y_1, y_2]$. Moreover, the function $y \mapsto \log \mathcal{M}_p(E,t,y)$ is convex at each instant $t \geq 0$.*

Proof. We first show the convexity and separate the proof in two cases.

Case 1: $p = \infty$.

In a previous section we proved the Hadamard three-lines theorem 2.1.3 on a vertical strip $S_{[a,b]}$, of which the same statement on a horizontal strip is a direct consequence. The function f is holomorphic on the horizontal strip within the fluid domain $S_{fluid} = \{z = x+iy : y_1 \leq y \leq y_2\}$, since it satisfies the Cauchy-Riemann equations. Define the map

$$\varphi(z) = -iz.$$

If $z = x + iy \in S_{fluid}$, then

$$\varphi(z) = -i(x+iy) = y - ix,$$

so $\Re(\varphi(z)) = y \in [y_1, y_2]$ and $\Im(\varphi(z)) = -x \in \mathbb{R}$. Hence φ is a biholomorphism from the horizontal strip S_{fluid} to the vertical strip $S'_{fluid} = \{w = u+iv : y_1 \leq u \leq y_2\}$ with the inverse $\varphi^{-1}(w) = iw$.

Define

$$g(w) := f(\varphi^{-1}(w)) = f(iw). \tag{3.17}$$

Then g is holomorphic on the vertical strip S'_{fluid}, thus the Hadamard three-lines theorem applies to g, meaning that for $u \in [y_1, y_2]$,

$$M_g(u) = \sup_{v\in\mathbb{R}} |g(u+iv)| = \sup_{v\in\mathbb{R}} |f(i(u+iv)| = \sup_{v\in\mathbb{R}} |f(-v+iu)|.$$

Substituting $x = -v$, which runs over $\mathbb{R}$ as v does, we obtain

$$M_g(u) = \sup_{x\in\mathbb{R}} |f(x+iu)| = M_f(u),$$

which means that for every $u \in [y_1, y_2]$, the supremum of g on the vertical line $\Re(w) = u$ and the supremum of f on the horizontal line $\Im(z) = u$ are identical. By the Hadamard three-lines theorem applied to g, we know that $\log M_g(u)$ is convex on $[y_1, y_2]$. Since $M_g(u) = M_f(u)$, this implies that $\log M_f(u)$ is convex on $[y_1, y_2]$ as well. Reverting to the original variable, we conclude

$$y \mapsto \log(\sup_{x\in\mathbb{R}} |f(x+iy)|) = M_f(y)$$

is convex on $[y_1, y_2]$.

Next we extend this result to the log-convexity of the kinetic energy. We know that

$$\mathcal{M}_\infty(E,t,y) = \sup_{x\in\mathbb{R}} \frac{1}{2}|f|^2 = \frac{1}{2}\sup_{x\in\mathbb{R}} |f|^2.$$

Since $|f|^2 \geq 0$, we have

$$\sup |f|^2 = (\sup |f|)^2 = M_f(y)^2,$$

which implies

$$\mathcal{M}_\infty(E,t,y) = \frac{1}{2}M_f(y)^2.$$

Therefore,

$$\log \mathcal{M}_\infty(E,t,y) = \log\frac{1}{2} + 2\log M_f(y)$$

is a convex function, since a convex function multiplied by a positive constant and then shifted by an additive constant remains convex.

The finiteness of $\mathcal{M}_\infty(E,t,y)$ follows immediately from the convexity of $M_f(y)$. By assumption,

$$\mathcal{M}_\infty(E,t,y_j) = \sup_x E(x,y_j,t) = \sup_x \frac{1}{2}|f(x,y_j)|^2 = \frac{1}{2}(\sup_x |f(x,y_j)|)^2 < \infty$$

for $j = 1, 2$. Therefore,

$$M_f(y_j) = \sup_x |f(x,y_j)| < \infty.$$

By the convexity of $y \mapsto M_f(y)$, for any $y = (1-\alpha)y_1 + \alpha y_2$ with $\alpha \in [0,1]$, we have

$$M_f(y) \leq M_f(y_1)^{(1-\alpha)} M_f(y_2)^\alpha < \infty.$$

Consequently,

$$\mathcal{M}_\infty(E,t,y) = \frac{1}{2}M_f(y)^2 < \infty$$

for all $y \in [y_1, y_2]$.

Case 2: $p < \infty$.

Since

$$|f|^2 = u^2 + v^2 = 2E,$$

for any fixed t and y we have

$$E^p = (|f|^2/2)^p = 2^{-p}|f|^{2p}.$$

Put

$$J_{2p}(y) := \int_{-\infty}^{\infty} |f(x+iy)|^{2p} dx.$$

Then by the identity above

$$\mathcal{M}_p(E,t,y) = 2^{-p} \int_{-\infty}^{\infty} |f(x+iy)|^{2p} dx = 2^{-p} J_{2p}(y).$$

We now verify all the hypotheses of the Hardy-Ingham-Pólya theorem 2.1.8.

It is given in the assumption that f is holomorphic and $|f|$ is continuous on the strip S.

To describe a physically realistic circumstance, we assume the boundedness of the velocity components u and v. Hence $|f|$ is bounded and satisfies (PL) for any admissible k.

Since

$$\mathcal{M}_p(E,t,y) = 2^{-p} J_{2p}(y)$$

and

$$\mathcal{M}_p(E,t,y_j) < \infty \quad \text{for } j = 1, 2,$$

$J_{2p}(y_1)$ and $J_{2p}(y_2)$ are finite.

So all the hypotheses of the theorem hold for f with exponent $2p$. Consequently, $J_{2p}(y)$ is finite for every $y \in [y_1, y_2]$, and $\log J_{2p}(y)$ is convex on $[y_1, y_2]$.

In particular,

$$J_{2p}(y) \le J_{2p}(y_1)^{\frac{y_2 - y}{y_2 - y_1}} J_{2p}(y_2)^{\frac{y - y_1}{y_2 - y_1}}.$$

To conclude the proof, we transfer the statement to $\mathcal{M}_p$. Since $\mathcal{M}_p(E,t,y) = 2_{2p}^{-p} J(y)$, we have

$$\log \mathcal{M}_p(E,t,y) = \log J_{2p}(y) - p \log 2,$$

so $\log \mathcal{M}_p(E,t,y)$ is a convex function. Specifically,

$$\mathcal{M}_p(E,t,y) \le \mathcal{M}_p(E,t,y_1)^{\frac{y_2 - y}{y_2 - y_1}} \mathcal{M}_p(E,t,y_2)^{\frac{y - y_1}{y_2 - y_1}}.$$

Plugging in $\mathcal{M}_p(E,t,y) = 2^{-p}J_{2p}(y)$, we obtain

$$\mathcal{M}_p(E,t,y) \leq \left(2^{-p}J_{2p}(y_1)\right)^{\frac{y_2-y}{y_2-y_1}} \left(2^{-p}J_{2p}(y_2)\right)^{\frac{y-y_1}{y_2-y_1}} < \infty,$$

since $J_{2p}(y)$ is finite. □

3.3 Travelling Waves

In the travelling wave framework, the velocities u and v, the free surface η and the pressure p depend on the variables x and t in the form such that they are functions of $x - ct$, where c denotes the constant propagation speed of the wave. We consider "classical", i.e. smooth solutions to the governing equations.

This property is particularly convenient, as it permits a change of variables to the moving frame, thereby removing the explicit time–dependence from the governing equations (3.8)-(3.13).

Let

$$\xi = x - ct,$$

so that

$$\frac{du}{dt} = \frac{du}{d\xi}\frac{d\xi}{dt} = -c\frac{du}{d\xi}.$$

Substituting into the governing equations so that they become

$$u_x + v_y = 0, \qquad -d \leq y \leq \eta(x)\,, \tag{3.18}$$

$$\begin{cases} (u-c)u_x + vu_y = -\dfrac{1}{\rho}\, p_x\,, \\ (u-c)v_x + vv_y = -\dfrac{1}{\rho}\, p_y \, - g\,, \end{cases} \qquad -\, d \leq y \leq \eta(x)\,, \tag{3.19}$$

$$p = p_{atm} \quad \text{on} \quad y = \eta(x)\,, \tag{3.20}$$

$$v = (u-c)\eta_x \quad \text{on} \quad y = \eta(x)\,, \tag{3.21}$$

$$v = 0 \quad \text{on} \quad y = -d\,, \tag{3.22}$$

$$u_y - v_x = 0\,, \qquad -d \leq y \leq \eta(x)\,. \tag{3.23}$$

A result of Amick and Toland in 1981 establishes that such smooth steady flows admit no stagnation points; that is, the horizontal velocity satisfies

$$u < c \quad \text{on the fluid domain} \;\; -d \leq y \leq \eta(x),$$

see [2].

Due to irrotationality ($u_y = v_x$), Euler's equations are equivalent to

$$\begin{cases} A_x = 0, \\ A_y = 0, \end{cases}$$

where

$$A(x,y) = \frac{(u-c)^2+v^2}{2}(x,y) + gy + p(x,y), \quad -d < y < \eta(x,t).$$

Euler's equations are therefore equivalent to Bernoulli's law

$$\frac{(u-c)^2+v^2}{2} + gy + p = const. \tag{3.24}$$

3.3.1 Kinetic Energy Properties of Solitary Waves

John Scott Russell (9 May 1808 – 8 June 1882) was a Scottish civil engineer, naval architect and shipbuilder. He discovered the wave of translation, a phenomenon that laid the foundation for the modern study of solitons.

An interesting connection exists between Russell and Vienna: he designed the dome of the Rotunde, which was constructed for the 1873 Vienna World's Fair. At 108 meters in diameter, it was the largest dome in the world at the time and retained this distinction for nearly a century. Some consider it his greatest structural engineering achievement.

In 1834, while conducting experiments to determine the most efficient design for canal boats, he encountered a remarkable phenomenon that he called the wave of translation. In fluid dynamics the wave is now known as Russell's solitary wave. He described his discovery in a report as such:

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour (14 km/h), preserving its original figure some thirty feet (9 m) long and a foot to a foot and a half (30-45 cm) in height. Its height gradually diminished, and after a chase of one or two miles (2–3 km) I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation."[17]

At his residence, Russell constructed experimental wave tanks to investigate the properties of this wave. His findings revealed that:

- The waves are stable and are capable of travelling over long distances without dissipating, unlike normal waves, which tend to flatten out or break.
- The speed of the waves depends on their size, and the width depends on water depth.
- They do not merge like normal waves: if two waves collide, the large wave overtakes the smaller one, instead of the two combining.

Russell's experimental results stood in contradiction to the linear hydrodynamic theories used in the 19th century. George Biddell Airy and George Gabriel Stokes found it difficult to reconcile observations with the established mathematical descriptions of water waves.

Only with the more advanced modern computers in the 1960s did the significance of Russell's discovery started to attract attention. It became apparent that the concept of solitary waves, culminating in the development of the modern general theory of solitons, is nonlinear and cannot be captured by linear theory. Solitary waves have profound applications across multiple fields of science, including physics, electronics, biology and fibre optics.[8]

We now define solitary waves in a mathematically rigorous way.

Definition (solitary waves[5])**.** Solitary waves are travelling waves of elevation, with profiles that decay from their crest towards their undisturbed uniform-flow state. They are characterized by the additional assumptions:

$$\lim_{|x|\to\infty} u(x,y) = \lim_{|x|\to\infty} v(x,y) = 0 \quad \text{uniformly in} \quad y\,,$$

and

$$\lim_{|x|\to\infty} \eta(x) = 0\,.$$

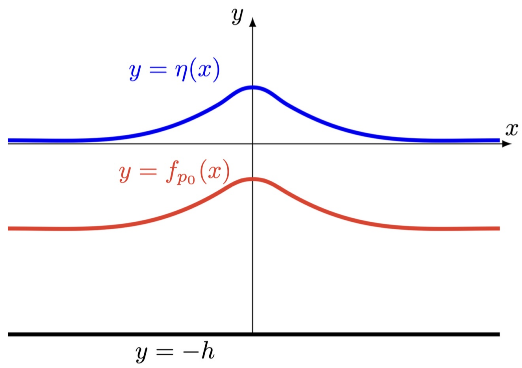

Figure 3.1: Profile of a solitary wave

The rigorous existence of these types of travelling waves is well-known and various properties of the wave profile were established analytically; Craig and Sternberg proved that the profile of solitary waves is symmetric with respect to the wave crest and monotonic on both sides of the crest,and exhibits exponential decay, which means $|\partial^\alpha u| \le Ce^{-c|x|}$ uniformly in y (see [7]). Hence we have $E^p \le C'e^{-2pc|x|}$, and integrating yields

$$\mathcal{M}_p(E,t,y) \le C' \int_{-\infty}^{\infty} e^{-2pc|x|}dx = C'\frac{2}{2pc} < \infty,$$

and this assumption in theorem 3.2.1 is automatically satisfied.

Without loss of generality we assume that the asymptotic water level μ lies on the real line, otherwise we shift the flat bed to achieve this.

We reformulate the preceding theorem for solitary waves specifically.

Theorem 3.3.1. *Let $p \in (0, \infty]$. Then $\mathcal{M}_p(E,t,y) < \infty$ for all $y \in [-d, 0]$; the function $y \mapsto \log \mathcal{M}_p(E,t,y)$ is convex; and the function $y \mapsto \mathcal{M}_p(E,t,y)$ is non-decreasing.*

Proof. The boundedness and convexity properties can be established in precisely the same manner as in the proof of the theorem 3.2.1. It remains, therefore, to show the monotonicity

Case 1: $p < \infty$.

Thanks to the uniform exponential decay of the velocity field, the Lebesgue Dominated Convergence Theorem permits differentiation under the integral sign. Thus, for $y = -d$, we compute

$$\begin{aligned}
\frac{d}{dy}\mathcal{M}_p(E,t,y)\Big|_{y=-d} &= \frac{1}{2^p}\frac{d}{dy}\int_{-\infty}^{\infty}(u^2(x,y,t)+v^2(x,y,t))^p dx\Big|_{y=-d} \\
&= \frac{p}{2^{p-1}}\int_{-\infty}^{\infty}(u^2+v^2)^{p-1}(uu_y(x,-d,t)+\underbrace{vv_y(x,-d,t)}_{=0 \text{ by } (3.6)})dx \\
&= \int_{-\infty}^{\infty}((u^2)^{p-1}uu_y(x,-d,t)+\underbrace{(v^2)^{p-1}uu_y(x,-d,t)}_{=0 \text{ by } (3.6)})dx \\
&= \int_{-\infty}^{\infty} u^{2p-1}u_y(x,-d,t)dx \\
&= \int_{-\infty}^{\infty} u^{2p-1}v_x(x,-d,t)dx \\
&= 0,
\end{aligned} \tag{3.25}$$

where the last equality follows from the boundary condition (3.6) and (3.7).

The convexity of $y \mapsto \log \mathcal{M}_p(E,y)$ implies immediately the convexity of $y \mapsto \mathcal{M}_p(E,y)$. Indeed, the composition of a convex non-decreasing function φ, such as the exponential function, and a convex function g is convex. Applying this observation to $\varphi(t) = e^t$ and $g(y) = \log \mathcal{M}_p(E,y)$ yields the desired convexity of $y \mapsto \mathcal{M}_p(E,y)$. By basic properties of convex functions, this in turn implies that the derivative $y \mapsto \frac{d}{dy}\mathcal{M}_p(E,y)$ is non-decreasing. Combined with relation (3.25), we conclude that $y \mapsto \mathcal{M}_p(E,y)$ is non-decreasing on $[-d, 0]$ for any $p \geq 1$.

case 2: $p = \infty$.

Applying the maximum modulus principle to the bounded domain

$$\mathcal{D}_R = \{(x, y) : |x| \leq R, -d \leq y \leq 0\}$$

and letting $R \to \infty$ shows that the supremum of f cannot be attained in the interior of the fluid domain.

Using the Schwarz reflection principle, we can reflect the function f across the flat bed $y = -d$, thereby embedding it into a larger symmetric domain. The Maximum Modulus Principle then implies that the supremum cannot be attained in the interior of this extended domain; consequently, it cannot occur on the bed $y = -d$, which now lies strictly inside the reflected domain. Thus, the supremum of f must occur on the free surface. It follows that $y \mapsto \mathcal{M}_p(E, t, y)$ is non-decreasing.

□

Definition. The total kinetic energy within the region $R_{y_0} = (-\infty, \infty) \times [-d, y_0]$ with $-d < y_0 < \mu$ is given by

$$E_T(y_0) = \iint_{R_{y_0}} E(x, y) dA(x, y) = \int_{-d}^{y_0} \int_{-\infty}^{\infty} E(x, y) dx dy.$$

Corollary 3.3.2. *The function $y_0 \mapsto E_T(y_0)$ is convex.*

Proof. We compute

$$E'_T(y_0) = \int_{-\infty}^{\infty} E(x, y_0) dy.$$

From previous theorem, $y_0 \mapsto E'_T(y_0)$ is non-decreasing. □

Remark. The kinetic energy with respect to the moving frame is

$$E_m = \frac{1}{2}((u - c)^2 + v^2).$$

Remark. All L^p norm integrals of E_m diverge, because $(u - c)^2 + v^2 \to c^2$ as $u, v \to 0$. We only consider the case $p = \infty$.

Lemma 3.3.3. *E_m decreases on the surface $y = \eta(x)$ as the elevation y increases.*

Proof. The assertion is an immediate consequence of

$$E_m(x, y) + gy = \text{constant} \quad \text{on} \quad y = \eta(x)$$

which follows from Bernoulli's law together with relation (3.20). □

Theorem 3.3.4. *In the setting of solitary waves of elevation, the minimum of the kinetic energy is attained at the wave crest while*

$$\sup_{(x,y)\in\mathcal{S}} E_m(x, y) = \sup_{y=\eta(x)} E_m(x, y) = \frac{|c|^2}{2} = \lim_{|x|\to\infty} E_m(x, \eta(x)) \tag{3.26}$$

is never attained.

Proof. Step 1: We begin by proving that $\sup|f|$ cannot be attained either in the interior of the fluid domain $\mathcal{S} = \{(x, y) : x \in \mathbb{R}, -d \leq y \leq \eta(x)\}$ or on the flat bed, and that although the free surface remains a possible location, the supremum is not attained there either.

case 1: Interior of the fluid domain.

For each $R > 0$, consider the truncated rectangle

$$\mathcal{D}_R = \{(x, y) : |x| \leq R, \ -d \leq y \leq \eta(x)\}. \tag{3.27}$$

The function f is analytic in the interior of $\mathcal{S}$, hence $|f|$ satisfies the maximum modulus principle. Therefore, for every $R > 0$, $\sup\limits_{z \in \mathcal{D}_R} |f(z)|$ cannot be attained at an interior point of the fluid domain $\mathcal{S}$. Letting $R \to \infty$, it follows that $\sup\limits_{x \in \mathcal{S}} |f(z)|$ is not attained in the interior of $\mathcal{S}$.

Case 2: Flat bed $y = -d$.

By condition (3.6), the real part of f vanishes along the flat bed $y = -d$. By the Schwarz reflection principle, f extends analytically to the reflected domain

$$\mathcal{D}'_R = \{(x, y) : |x| \leq R, \ -2d - \eta(x) \leq y \leq \eta(x)\}. \tag{3.28}$$

The maximum modulus principle applied to this extended analytic function implies that $\sup\limits_{z \in \mathcal{S}} |f(z)|$ cannot be attained in the interior of $\mathcal{D}'_R$, thus the supremum is not on the flat bed $y = -d$ either.

Case 3: Free surface $y = \eta(x)$.

Within the fluid domain, the function $f = v + i(u - c)$ is analytic in the interior and continuous up to the boundary. Since $|f|^2 = 2E_m$, we have

$$\sup_{y=\eta(x)} |f(x + iy)| = \sup_{y=\eta(x)} \sqrt{2E_m(x, y)}.$$

By Lemma 3.3.3, E_m increases along the free surface $\eta(x)$ as y decreases, and given $\lim\limits_{|x| \to \infty} \eta(x) = 0$, y is minimal when $|x| \to \infty$. This implies

$$\sup_{y=\eta(x)} |f(x + iy)| = \sup_{y=\eta(x)} \sqrt{2E_m(x, y)} = \lim_{|x| \to \infty} \sqrt{2E_m(x, \eta(x))} = |c|. \tag{3.29}$$

However, for every $|x| < \infty$ we have $|f(x + iy)| < |c|$, so the supremum is not attained along $y = \eta(x)$.

Combining all three cases above with the asymptotic behaviour $\lim_{|x|\to\infty} f(x+iy) = c$ uniformly in y, we obtain (3.26).

Step 2: Analysis of the minimum of E_m.

Since there are no stagnation points we have $u < c$. Hence, $f = v + i(u - c)$ is continuous on $\mathcal{S}$ and analytic in its interior, and f has no zeros, therefore $\frac{1}{f}$ is also analytic in $\mathcal{S}$.

From Lemma 3.3.3 it follows that the maximum of $\frac{1}{|f|}$, and equivalently, the minimum of $|f| = \sqrt{2E_m}$) along the free surface $y = \eta(x)$, is attained at the wave crest.

Denote

$$m := \min_{y=\eta(x)} |f(x+iy)|.$$

Since $\lim_{|x|\to\infty} |f(x+iy)| = |c|$, Lemma 3.3.3 ensures $m < |c|$. Moreover, by the asymptotic behavior of f, there exists $R_0 > 0$ such that

$$|f(x+iy)| > m, \quad \text{for all} \quad |x| \geq R_0 \quad \text{and} \quad -d \leq y \leq \eta(x). \tag{3.30}$$

For $R > R_0$, consider the truncated rectangle

$$\mathcal{D}_R = \{(x,y) : |x| \leq R, -d \leq y \leq \eta(x)\}.$$

The maximum of $\frac{1}{|f|}$ cannot occur on the lateral sides of $\mathcal{D}_R$, since $\frac{1}{|f|} = \frac{1}{m}$ at the crest but $\frac{1}{|f|} < \frac{1}{m}$ on the lateral sides $|x| = R$.

On the flat bed $y = -d$, we have $v = 0$, so the real part of $\frac{1}{f}$ vanishes on $y = -d$. By the Schwarz Reflection Principle, $\frac{1}{f}$ extends analytically below $y = -d$, and by the maximum modulus principle, its maximum cannot occur on $y = -d$. Therefore, the maximum of $\frac{1}{f}$ on $\mathcal{D}_R$ must occur at the wave crest. Letting $R \to \infty$ yields the desired conclusion: the minimum of $|f| = \sqrt{2E_m}$, and hence of E_m, is attained at the wave crest. □

Remark. The supremum of E_m on every horizontal line is ' attained' at infinity and is equal to $c^2/2$. The minimum of E_m on the fluid domain is attained at wave crest due to the previous theorem.

Theorem 3.3.5 (Moving Frame)**.** *In the setting of solitary waves,* $\inf_{x\in\mathbb{R}} E_m(x,y) = \min_{x\in\mathbb{R}} E_m(x,y)$ *is attained below the wave crest and the function* $y \mapsto \log \min_{x\in\mathbb{R}} E_m(x,y)$ *is concave and* $y \mapsto \min_{x\in\mathbb{R}} E_m(x,y)$ *is decreasing in* $y \in [-d, 0]$.

Proof. To establish that the minimum is attained below the wave crest, we first recall Bernoulli's law 3.24

$$E_m(x,y) + gy + p(x,y) = const.$$

For solitary waves of elevation, it is known (see [4]) that the pressure p achieves its maximum directly beneath the wave crest. This follows from the monotonicity of p along horizontal lines: as one moves horizontally towards the crest, the pressure increases, reaching its peak beneath the crest, and then decreases away from it.

The physical intuition behind this behaviour is straightforward: the greater the column of water above a point, the larger the pressure over this point. Accordingly, the pressure is minimal in the far field, where the free surface is undisturbed, and maximal under the crest. Since E_m is linked to pressure via Bernoulli's equation, the minimum of E_m over x occurs where the maximum of pressure occurs, i.e. below the crest.

We now address the concavity property. The function

$$h := \frac{1}{f_m} = \frac{1}{(u-c) + i(-v)}$$

is analytic in the fluid domain $\mathcal{S}$, because the flow is free of stagnation ($u - c < 0$ throughout the fluid domain) i.e. the denominator of h has no zeros in the fluid domain.

For each horizontal line, we have

$$\sup |h| = \frac{1}{\inf |f_m|} = \frac{1}{\min |f_m|}.$$

Applying Hadamard's three-lines theorem to h on horizontal strips yields that the function

$$y \mapsto \log \tfrac{1}{\min |f_m|} = -\log \min |f_m|$$

is convex, hence $y \mapsto \log \min |f_m|$ is concave.

Taking the infimum of Bernoulli's law over x for fixed y gives

$$\inf_{x \in \mathbb{R}} E_m(x, y) = const - gy - \sup_{x \in \mathbb{R}} p(x, y).$$

As noted earlier, the supremum of the pressure is attained below the crest and takes the value $\max\limits_{x \in \mathbb{R}} p(x, y) = p(0, y)$. Consequently,

$$\min_{x \in \mathbb{R}} E_m(x, y) = const - gy - p(0, y).$$

From the concavity of $y \mapsto \log \min |f_m|$ established above, it follows that $y \mapsto \log \min E_m(x, y)$ is concave as well. Moreover, by the Schwarz reflection principle, $\sup\limits_{x \in \mathbb{R}} |h(x, y)|$ increases with the elevation of y, and then $\frac{1}{\min |f_m|}$ increases correspondingly, which implies that $\min |f_m|$ and thereby $\min E_m$ decreases with y.

$\square$

3.3.2 Kinetic Energy Properties of Stokes Waves

The following sections of this thesis follow the paper [5].

A periodic wave is a wave whose profile repeats at regular spatial intervals (i.e. wavelength) and propagates with constant shape, amplitude, and speed, so that every crest and trough follows the same path and timing. The term "periodic" refers to both spatial and temporal regularity: at any fixed point in space, the surface elevation and fluid velocity repeat after a fixed time period, and at any given instant in time, the wave pattern repeats along the direction of propagation.

Periodic waves are the most regular patterns found in natural water waves. They are usually observed when waves generated by distant storms enter calm water regions with a flat seabed, such as on the continental shelf or across the abyssal plains that cover nearly one-third of the Earth's surface.

In this study, we consider Stokes waves: they are smooth solutions which are periodic in the horizontal coordinate x with wavelength λ, whose profile is strictly monotonic between each crest and trough, and symmetric about the wave crest. Without loss of generality, the crest is located at $x = 0$, with troughs at $\pm\frac{\lambda}{2}$. The surface elevation $\eta(x)$ satisfies $\eta(x) = \eta(-x)$ for all $x \in \mathbb{R}$, increasing on $[-\frac{\lambda}{2}, 0]$ and decreasing on $[0, \frac{\lambda}{2}]$.

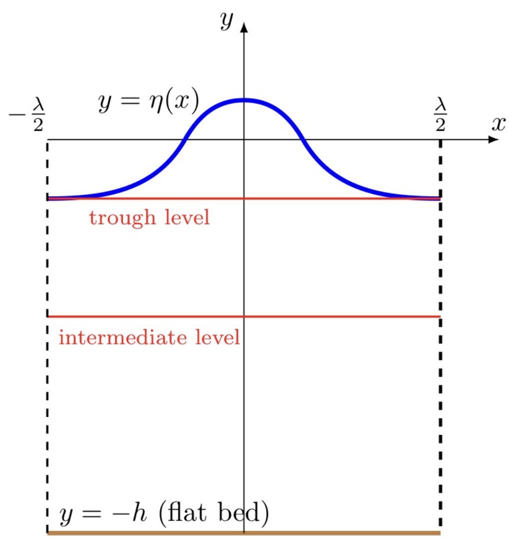

Figure 3.2: Profile of periodic waves

Periodic waves, and in particular Stokes waves, are widely used as idealised models in the analysis of wave motion and its associated energy properties.

Theorem 3.3.6. *At each $t \geq 0$ the function $y \mapsto \log \mathcal{M}_p(E, y)$ is convex on $[-d, \mu(t)]$ for all $1 \leq p \leq \infty$, where $\mu(t)$ denotes the trough level at the time t. Furthermore, the function $y \mapsto \mathcal{M}_p(E, y)$ is increasing on $[-d, \mu(t)]$ for any $t \geq 0$.*

Proof. Set

$$R = [0, \lambda] \times [-d, \mu(t)]$$

and $k = \frac{2\pi}{\lambda}$. We are allowed to leave out the time dependence of μ for the sake of simplicity in rest of this proof, if we fix a time t.

The function $f : R \to \mathbb{C}$, $f = v + iu$ is continuous on R and holomorphic on the interior of R, because it satisfies the Cauchy-Riemann equations by (3.2) and (3.7). Additionally, it satisfies $|f|^2 = 2E$.

We are able to transfer our statement from R to an annulus due to the periodicity of f, where we may employ Hardy's convexity theorem 2.21.

We consider the transformation $z \mapsto e^{-ikz}$ which maps R onto the annulus

$$A = \{z \in \mathbb{C} : e^{-kd} \leq |z| \leq e^{k\mu}\},$$

since $|z| = e^{ky}$ implies $e^{-kd} \leq |z| \leq e^{k\mu}$ by definition of R.

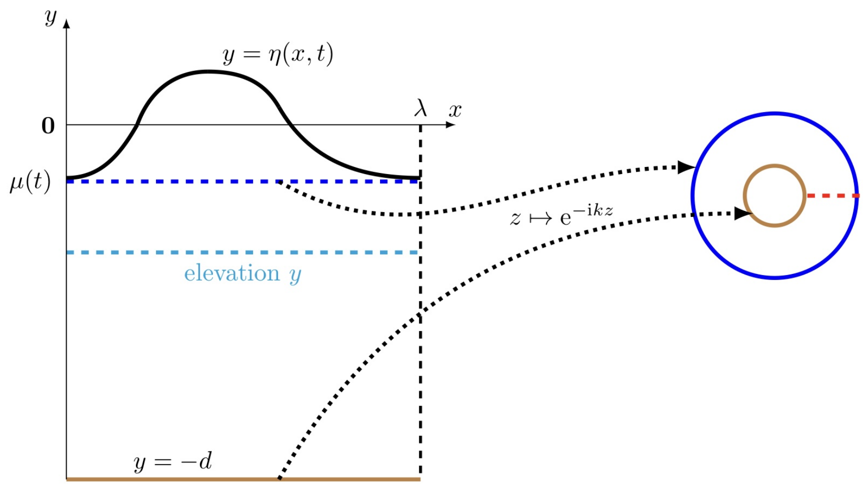

Figure 3.3: Transformation from R to annulus

The restriction of this transformation to

$$R \setminus \{\lambda\} \times [-d, \mu(t)]$$

is a bijection to the annulus, whose inverse is

$$z \mapsto \frac{i}{k} \log z.$$

Since f is periodic in the x-variable, the map

$$g : A \setminus [e^{-kd}, e^{k\mu}] \to \mathbb{C}$$

given by the reparametrization of f on the annulus minus the line segment (to avoid the branch cut)

$$g(z) = f\left(\frac{i}{k} \log z\right),$$

extends to a continuous function on the whole annulus A, despite the multi-valued nature of the complex logarithm, which is resolved by the periodicity of f.

Take two paths approaching the same point $z_0 \in A$ from different angles, then these paths correspond to values of $\arg z$ differing by 2π, which means their logarithms differ by $2\pi i$, thus

$$\frac{i}{k} \log z \mapsto \frac{i}{k}(\log z + 2\pi i) = \frac{i}{k} \log z - \frac{2\pi}{k} = \frac{i}{k} \log z - \lambda.$$

That is, the argument of f shifts exactly bt λ, which is the period of f in the x-variable, i.e.

$$f(x + n\lambda + iy) = f(x + iy).$$

Therefore,

$$f(\frac{i}{k}(\log z + 2\pi in)) = f(\frac{i}{k} \log z - \frac{2\pi n}{k}) = f(\frac{i}{k} \log z - n\lambda) = f(\frac{i}{k} \log z),$$

which shows that the multi-valuedness of $\log z$ us cancelled out by the periodicity of f, and the composition g is well-defined even across the branch cut. Hence, it is extended continuously to all of A, including the boundary circles.

The periodicity of f lets us glue together the sides $x = 0$ and $x = \lambda$ of the rectangle R to form a cylinder, then the exponential map unfolds the cylinder into an annulus. The periodicity ensures that when we circle around the annulus, i.e. change the $\arg z$ by 2π, $g(z)$ always returns to the same value, even though the branch of $\log z$ changes.

We apply Morera's Theorem 2.2.4 to see that g is analytic in A.
By the same arguments as above, potential discontinuities in $g(z)$ across the branch cut are avoided, because even though $\log z$ jumps, f is invariant under that jump.

Two types of triangles need to be considered. If the triangle is contained in $A \setminus [e^{-kd}, e^{k\mu}]$, its boundary avoids the branch cut completely, and $g(z)$ is already defined as an analytic function as the composition of an analytic function and $\log z$, so Morera's condition is satisfied.

In the nontrivial case, suppose $\partial\triangle$ intersects the branch cut. However, due to λ-periodicity of f, the values of $g(z)$ along the path still match up continuously, and Morera's condition is still satisfied. All in all, Morera's theorem ensures that g is continuous in the interior of A.

For $e^{-kd} \leq r \leq e^{\mu}$ and $p \geq 1$ we have

$$\begin{aligned} M_{2p}(g,r) :&= \int_0^{2\pi} |g(re^{i\theta})|^{2p} d\theta \\ &= \int_0^{2\pi} |f\left(\frac{i}{k}\log(re^{i\theta})\right)|^{2p} d\theta \\ &= \int_0^{2\pi} |f\left(\frac{i}{k}\log r - \frac{\theta}{k}\right)|^{2p} d\theta. \end{aligned} \tag{3.31}$$

Substituting $s = -\frac{\theta}{k}$ we obtain

$$\begin{aligned} M_{2p}(g,r) &= k\int_{-2\pi/k}^{0} |f\left(i\frac{\log r}{k} + s\right)|^{2p} ds \\ &= k\int_{-\lambda}^{0} |f\left(s + i\frac{\log r}{k}\right)|^{2p} ds \\ &= k\int_{0}^{\lambda} |f\left(s + i\frac{\log r}{k}\right)|^{2p} ds, \\ &= k\,2^p\int_{0}^{\lambda} E\left(s, \frac{\log r}{k}\right)^p ds, \\ &= k\,2^p \mathcal{M}_p(E, \frac{\log r}{k}), \end{aligned} \tag{3.32}$$

where the third equality follows from periodicity.

By a change of variables $y = \frac{\log r}{k}$, we obtain

$$\mathcal{M}_p(E,y) = \frac{1}{2^p k} M_{2p}(E, \frac{\log r}{k}), \quad -d \leq y \leq \mu. \tag{3.33}$$

By Hardy's convexity theorem 2.21, the map $r \mapsto \log M_{2p}(g,r)$ is a convex function of $\log r$ for $r \in [e^{-kd}, e^{k\mu}]$ and $p \geq 1$. By a change of variable $r = e^{ky}$, we know that $\log M_{2p}(g, e^{ky})$ is convex in $\log r = ky$, thus it is convex in y on $[-d,\mu]$. Therefore, by (3.33),

$$\log \mathcal{M}_p(E,y) = \log M_{2p}(g, e^{ky}) - \log(2^p k)$$

is convex. We deduce from (3.33) that for any $y \in [-d,\mu]$ and every $\alpha \in [0,1]$,

$$\mathcal{M}_p(E,y) \leq [M_p(E,y_1)]^{\alpha} \cdot [M_p(E,y_2)]^{1-\alpha}, \tag{3.34}$$

where $y = \alpha y_1 + (1 - \alpha y_2)$.

With this, the convexity assertion is proven. Next let us shoe that $y \mapsto \mathcal{M}_p(E, y)$ us non-decreasing on $[-d, \mu]$.

Observe that

$$\begin{aligned}
\frac{d}{dy}\mathcal{M}_p(E,y)\Big|_{y=-d} &= \frac{1}{2^p}\frac{d}{dy}\int_0^\lambda (u^2+v^2)^p dx\Big|_{y=-d} \\
&= \frac{p}{2^{p-1}}\int_0^\lambda (u^2(t,x,-d)+v^2(t,x,-d))^{p-1}(uu_y(t,x,-d)+vv_y(t,x,-d)dx \\
&= \frac{p}{2^{p-1}}\int_0^\lambda (u^2)^{p-1}uu_y dx\Big|_{y=-d} \\
&= \frac{p}{2^{p-1}}\int_0^\lambda u^{2p-1}v_x dx\Big|_{y=-d} \\
&= 0
\end{aligned} \tag{3.35}$$

in light of (3.6) and (3.7) by integration by parts.

The convexity of $y \mapsto \log \mathcal{M}_p(E, y)$ implies the convexity of $y \mapsto \mathcal{M}_p(E, y)$, since the composition of a convex non-decreasing function (e.g. the exponential function) with a convex function is still convex. Hence $y \mapsto \frac{d}{dy}\mathcal{M}_p(E, y)$ is non-decreasing and the derivative of $\mathcal{M}_p(E, y)$ at the bottom is zero due to (3.35). Together they guarantee that $\frac{d}{dy}\mathcal{M}_p(E, y)$ is non-negative, thus $y \mapsto \mathcal{M}_p(E, y)$ is non-decreasing on $[-d, \mu]$ for all $p \geq 1$.

□

Remark. The physical motivation of Theorem 3.3.6 is that the wave velocity is usually measured near the surface or near the bed. The above theorem allows us to extract from such data information the kinetic energy of the flow at intermediate depths.

Theorem 3.3.7. *In the setting of periodic travelling waves, the minima and the maxima of the kinetic energy with respect to the moving frame are attained on the surface. More precisely, the minima are attained at the wave crests while the maxima are attained at the wave troughs.*

Proof. Since there are no stagnation points, we have $f \neq 0$ throughout the fluid domain $\mathcal{S}$, hence $\frac{1}{f}$ is analytic in the interior of $\mathcal{S}$. By same argument as before, the maximum of $\frac{1}{|f|}$ and equivalently, the minimum of $|f|$, must be attained on the free surface $y = \eta(x)$. Thus, the minimum of the kinetic energy given by

$$\min E_m(x, y) = \min \tfrac{|f|^2}{2}$$

is attained on the free surface as well.

From Lemma 3.3.3, the distribution of extrema of the kinetic energy follows directly. The kinetic energy with respect to the moving frame on the free surface can be expressed in the form

$$E_m(x, y) = constant - gy \quad \text{for } y = \eta(x).$$

Since y is maximized at wave crests and minimized at wave troughs, it follows that the maximum kinetic energy occurs at wave troughs, where the free surface elevation η is minimal; the minimum kinetic energy occurs at wave crests, where η is maximal. This concludes the proof. □

Remark. An important aspect of Longuet-Higgins' work [15] was the investigation of the decay rate of kinetic energy with depth beneath a periodic wave (see also [1] for more recent developments in this direction). Theorem 3.3.7 suggests that, for periodic travelling waves, both the maximum and minimum values of the kinetic energy occur on the free surface. Consequently, measurements of the velocity field taken at the surface suffice to offer reliable data on kinetic energy throughout the entire fluid domain.

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