DISSERTATION | DOCTORAL THESIS

Titel | Title

Topics in the Theory of Localized Frames

verfasst von | submitted by

Lukas Köhldorfer BSc MSc

angestrebter akademischer Grad | in partial fulfilment of the requirements for the degree of

Doktor der Naturwissenschaften (Dr.rer.nat.)

Wien | Vienna, 2025

Studienkennzahl lt. Studienblatt | Degree programme code as it appears on the student record sheet:	UA 796 605 405
Dissertationsgebiet lt. Studienblatt | Field of study as it appears on the student record sheet:	Mathematik
Betreut von | Supervisor:	Mag. Dr. Karlheinz Gröchenig Mag. Dr. Peter Balazs Privatdoz.

Acknowledgements

First of all, I would like to thank my supervisor Peter Balazs. Peter opened the doors to science for me and is one of the reasons why I did a PhD in the first place. He already supervised me during my Master's thesis and, as PI of the LoFT-project, provided financial support to me for 3,5 years, making it financially possible for me doing research on fascinating topics. Although Peter is the director of the ARI and supervises several other PhD students besides me (for a while he supervised five students at the same time!), he always gave me his time and advice without hesitation. His endless patience and humorous nature always made our meetings uncomplicated and a lot of fun. In all this time I was able to learn a lot from Peter, both mathematically and in a more general professional sense.

Next, I would like to thank my co-supervisor Karlheinz Gröchenig. I have known Charly as a teaching professor for almost ten years and have never taken more courses (Functional Analysis, PDE, Advanced PDE, Topics in Harmonic Analysis, Seminar on Harmonic Analysis x2 (also Compressive Sensing and Approximation Theory without examination)) with any other professor during my entire studies. His way of teaching mathematical concepts and ideas has influenced and inspired me mathematically over the years. As my co-supervisor, he always took time for me and his answers to tricky mathematical questions always hit the mark.

I would also like to thank Daniel Freeman. He was my host during my research stay at Saint Louis University and went out of his way to make sure I had a good time here. I took up a lot of his time during my stay and am grateful to Dan for his endless patience and sacrifice to explain his math ideas to me. At this point I would also like to thank Brody Johnson. He was not only my office neighbor at Saint Louis University, but also like a second host who made my stay during those two months even more fun.

Special thanks also go to my PhD sibling Daniel Haider. Dani and I have been fellow students since (at least) summer term 2013 and have been through thick and thin since my time at ARI, becoming close friends. He encouraged me to do many things and, during my two-month stay in Saint Louis, was not only my understanding roommate, but also a great enrichment to my life. His many pieces of advice and his positive, jolly nature made my time at ARI and my first experience in the US considerably more valuable.

Last but not least, I would like to thank my fiance Nicole. Thank you for your love, your endless patience, your understanding of my absent-mindedness while mathematical ideas were still swirling around in my head, and thank you for being such a great support to me during all this time!

Abstract

This doctoral thesis consists of three papers in frame theory. Frames are countable families of vectors in a separable Hilbert space, which enable a bounded, linear, and stable reconstruction of any vector in that space from its frame coefficients. A natural generalization of frames are operator-valued frames, which analogously provide perfect reconstruction from linear higher-rank measurements. In this thesis, we study (operator-valued) frames whose elements satisfy an additional incoherence property called localization. The quality of localization is measured by the off-diagonal decay of the (operator-valued) Gram matrix of the underlying (operator-valued) frame. More precisely, such a frame is called localized, if its associated Gram matrix belongs to some suitable inverse-closed matrix algebra.

The first paper is devoted to the study of algebras of operator-valued matrices, which are inverse-closed in the Banach algebra of bounded operators acting on the Bochner space of square-summable Hilbert space-valued sequences. In particular, we study operator-valued versions of weighted Schur-type algebras, the Jaffard algebra and generalizations of it, the Baskakov-Gohberg-Sjöstrand algebra, and their anisotropic versions, and show that each of these matrix algebras is inverse-closed and, in fact, a symmetric Banach algebra.

With the matrix algebras from the first paper in mind, we introduce an abstract localization concept for operator-valued frames in the second paper. We prove that intrinsic localization of an operator-valued frame is preserved by its canonical dual. We show that for any localized operator-valued frame, there is a whole family of associated (quasi-) Banach spaces attached to it. We prove that the series associated with perfect reconstruction of the given operator-valued frame converges not only in the underlying Hilbert space but also in each of these spaces. Finally, we apply our theory to irregular Gabor g-frames.

In the third paper, we prove the equivalence of the frame property of a given suitably localized sequence in a Hilbert space and nine other conditions that do not involve an inequality. This result is applied in the context of shift-invariant spaces, where we obtain new conditions for stable sets of (irregular) sampling.

Kurzfassung

Diese Doktorarbeit setzt sich aus drei Arbeiten in der Frame-Theorie zusammen. Frames sind abzählbare Familien von Vektoren in einem separablen Hilbertraum, die es ermöglichen, jeden Vektor dieses Raums auf lineare, stetige und stabile Art und Weise aus seinen Frame-Koeffizienten zu rekonstruieren. Eine natürliche Verallgemeinerung von Frames sind operatorwertige Frames, die gleichermaßen eine perfekte Rekonstruktion aus linearen Messungen höheren Ranges ermöglichen. In dieser Arbeit untersuchen wir (operatorwertige) Frames, deren Elemente eine zusätzliche Inkohärenzeigenschaft erfüllen, die als Lokalisierung bezeichnet wird. Der Grad der Lokalisierung wird durch das Abklingverhalten der Einträge der zugrundeliegenden (operatorwertigen) Gram-Matrix von der Hauptdiagonale weg gemessen. Genauer gesagt, heißt ein Frame lokalisiert, wenn seine zugehörige Gram-Matrix zu einer passenden invers-abgeschlossenen Matrixalgebra gehört.

Der erste Artikel widmet sich der Untersuchung von Algebren operatorwertiger Matrizen, die in der Banach-Algebra der beschränkten Operatoren auf dem Bochner-Raum quadratisch summierbarer Hilbertraum-wertiger Folgen enthalten und invers-abgeschlossen sind. Insbesondere untersuchen wir operatorwertige Versionen von gewichteten Schur-Algebren, der Jaffard-Algebra und einige ihrer Verallgemeinerungen, die Baskakov-Gohberg-Sjöstrand-Algebra, sowie all ihre anisotropen Versionen, und zeigen, dass jede dieser Matrixalgebren invers-abgeschlossen und insbesondere auch eine symmetrische Banach-Algebra ist.

Im zweiten Artikel wird ein Lokalisierungskonzept für operatorwertige Frames eingeführt. Wir beweisen, dass operatorwertige Frames ihre Lokalisierung auf ihr kanonisch duales Frame vererben. Wir zeigen, dass zu jedem lokalisierten operatorwertigen Frame eine ganze Familie von zugehörigen (quasi-)Banachräumen existiert. Wir beweisen, dass die Rekonstruktionsformel bezüglich des gegebenen operatorwertigen Frames nicht nur im zugrundeliegenden Hilbertraum, sondern auch in jedem dieser eingeführten Räume gültig ist. Schließlich wenden wir unsere Theorie auf irreguläre Gabor-g-Frames an.

Im dritten Artikel beweisen wir die Äquivalenz der Frame-Eigenschaft einer gegebenen, geeignet lokalisierten Folge in einem Hilbertraum mit neun weiteren Bedingungen, die ohne Ungleichung formuliert sind. Dieses Ergebnis wenden wir in der Theorie der verschiebungsinvarianten Räume an, woraus wir neue Bedingungen für das stabile (unregelmäßige) Abtasten in diesen Räumen gewinnen.

List of Papers

This thesis comprises the following three papers:

Paper 1

L. Köhldorfer and P. Balazs. Wiener pairs of Banach algebras of operator-valued matrices. *Journal of Mathematical Analysis and Applications*, 549(2):129525, 2025. https://doi.org/10.1016/j.jmaa.2025.129525

Contribution: Lukas Köhldorfer is the main author of the paper and the main contributor to its content.

Paper 2

L. Köhldorfer and P. Balazs. Localization of operator-valued frames. *Submitted.* ArXiV: https://arxiv.org/pdf/2503.24170

Contribution: Lukas Köhldorfer is the main author of the paper and the main contributor to its content.

Paper 3

P. Balazs, L. Köhldorfer and M. Speckbacher. Localized frames without inequalities. *Submitted.* ArXiV: https://arxiv.org/pdf/2506.02862

Contribution: Peter Balazs, Lukas Köhldorfer and Michael Speckbacher contributed equally to the content of the paper.

In the course of my PhD studies, I also (co-)authored the following publications, which are not included in this thesis and only mentioned here:

Book chapters

L. Köhldorfer, P. Balazs., P. Casazza, S. Heineken, C. Hollomey, P. Morillas, and M. Shamsabadhi. *A Survey of Fusion Frames in Hilbert Spaces,* chapter 21. Springer Nature Switzerland, 2023. https://link.springer.com/chapter/10.1007/978-3-031-41130-4_11.

Conference Papers

L. Köhldorfer and P. Balazs. On the inverse-closedness of operator-valued matrices with polynomial off-diagonal decay, 2025. *Submitted.* https://arxiv.org/abs/2501.09603.

N. Hauschka, P. Balazs, and L. Köhldorfer. Banach distribution spaces for a Hilbert space, 2025. *Submitted.* https://arxiv.org/abs/2502.07378.

Contents

Part I.

Introduction

1. Frames in Hilbert space

Breaking down complicated objects into simpler and easier-to-understand building blocks is a long-established strategy in mathematics. In a separable Hilbert space $\mathcal{H}$ the most common building block systems are orthonormal bases, which are defined as a countable family of pairwise orthonormal vectors whose linear span is dense in $\mathcal{H}$. Any such orthonormal basis $(e_k)_{k\in X}$ satisfies Parseval's identity, that is,

$$\|f\|^2 = \sum_{k\in X} |\langle f, e_k\rangle|^2 \qquad (\forall f \in \mathcal{H}),$$

and provides the possibility to represent any vector via

$$f = \sum_{k\in X} \langle f, e_k\rangle e_k \qquad (\forall f \in \mathcal{H}),$$

where the coefficients $(\langle f, e_k\rangle)_{k\in X}$ are unique. In principle, orthonormal bases are a practical tool to better explain and understand phenomena in Hilbert space. However, orthonormal bases are highly sensitive to even smallest perturbations and therefore of little use in many other theoretical and practical problems which require a certain degree of stability and flexibility of the system used. One particularly successful generalization of orthonormal bases are *frames* [39, 35, 28], which allow non-orthogonal, redundant and stable representations of any vector in $\mathcal{H}$.

A *frame* in a separable Hilbert space $\mathcal{H}$ is a countable family $(\varphi_k)_{k\in X}$ of vectors in $\mathcal{H}$, for which there exist two positive constants $A \leq B$ such that

$$A\|f\|^2 \leq \sum_{k\in X} |\langle f, \varphi_k\rangle|^2 \leq B\|f\|^2 \qquad (\forall f \in \mathcal{H}). \tag{1.1}$$

The constants A and B are called *frame bounds*. A frame is called *tight*, if its frame bounds can be chosen to be equal, and it is called a *Parseval frame*, if $A = B = 1$ can be chosen as frame bounds. A quick comparison with Parseval's equation for orthonormal bases reveals that the defining frame inequalities (1.1) are a relaxed version of the latter; hence, orthonormal bases are Parseval frames (but the converse is not true [28]). Whenever the upper, but not necessarily the lower inequality in (1.1) is satisfied for some prescribed $B > 0$ and all $f \in \mathcal{H}$, then $(\varphi_k)_{k\in X}$ is called a *Bessel sequence* with *Bessel bound* B. For a comprehensive introduction to frames and many other related types of sequences in Hilbert space, we refer the reader to [28].

In order to formulate and understand many frame-theoretic phenomena, it is fruitful to consider the following canonical operators associated with arbitrary given countable families $\varphi := (\varphi_k)_{k\in X}$ and $\psi := (\psi_k)_{k\in X}$ of vectors in $\mathcal{H}$:

- The *synthesis operator*

$$D_\varphi : \mathrm{dom}(D_\varphi) \subseteq \ell^2(X) \longrightarrow \mathcal{H}, \qquad D_\varphi(c_k)_{k\in X} = \sum_{k\in X} c_k\varphi_k,$$

  where $\mathrm{dom}(D_\varphi) = \Big\{(c_k)_{k\in X} \in \ell^2(X) : \sum_{k\in X} c_k\varphi_k \in \mathcal{H}\Big\}$.

- The *analysis operator*

$$C_\varphi : \mathrm{dom}(C_\varphi) \subseteq \mathcal{H} \longrightarrow \ell^2(X), \qquad C_\varphi f = (\langle f, \varphi_k\rangle)_{k\in X},$$

  where $\mathrm{dom}(C_\varphi) = \Big\{f \in \mathcal{H} : (\langle f, \varphi_k\rangle)_{k\in X} \in \ell^2(X)\Big\}$.

- The *frame operator*

$$S_\varphi : \mathrm{dom}(S_\varphi) \subseteq \mathcal{H} \longrightarrow \mathcal{H}, \qquad S_\varphi f = \sum_{k\in X} \langle f, \varphi_k\rangle\varphi_k,$$

  where $\mathrm{dom}(S_\varphi) = \Big\{f \in \mathcal{H} : \sum_{k\in X}\langle f, \varphi_k\rangle\varphi_k \in \mathcal{H}\Big\}$.

- The *cross Gram matrix*

$$G_{\varphi,\psi} : \mathrm{dom}(G_{\varphi,\psi}) \subseteq \ell^2(X) \longrightarrow \ell^2(X), \qquad G_{\varphi,\psi}(c_k)_{k\in X} = \left(\sum_{l\in X}\langle \psi_l, \varphi_k\rangle c_l\right)_{k\in X},$$

  where $\mathrm{dom}(G_{\varphi,\psi}) = \Big\{(c_k)_{k\in X} \in \ell^2(X) : \big(\sum_{l\in X}\langle \psi_l, \varphi_k\rangle c_l\big)_{k\in X} \in \ell^2(X)\Big\}$.

- The *Gram matrix* G_φ, defined as above by $G_\varphi := G_{\varphi,\varphi}$.

For general sequences φ and ψ, these operators are possibly unbounded [9]. However, if φ and ψ both are Bessel sequences, then all these operators are bounded with $\mathrm{dom}(D_\varphi) = \mathrm{dom}(G_{\varphi,\psi}) = \ell^2(X)$, $\mathrm{dom}(C_\varphi) = \mathrm{dom}(S_\varphi) = \mathcal{H}$, $D_\varphi^* = C_\varphi$, $S_\varphi = D_\varphi C_\varphi$ and $G_{\varphi,\psi} = C_\varphi D_\psi$. In the following, we provide a collection of results, which sheds some more light on the hierarchy of the types of sequences in $\mathcal{H}$ discussed so far.

Theorem 1.1. *[28, 9] Let $\varphi = (\varphi_k)_{k\in X}$ be a countable family of vectors in $\mathcal{H}$. Then the following are equivalent.*

(i) φ is a Bessel sequence for $\mathcal{H}$.

(ii) The synthesis operator D_φ is well-defined and bounded on $\ell^2(X)$.

(iii) The analysis operator C_φ is well-defined and bounded on $\mathcal{H}$.

(iv) The frame operator S_φ is well-defined and bounded on $\mathcal{H}$.

(v) The Gram matrix G_φ defines a well-defined and bounded operator on $\ell^2(X)$.

Theorem 1.2. *[28, 9] Let $\varphi = (\varphi_k)_{k\in X}$ be a countable family of vectors in $\mathcal{H}$. Then the following are equivalent.*

(i) φ is a frame for $\mathcal{H}$.

(ii) The synthesis operator D_φ is well-defined on $\ell^2(X)$, bounded and surjective.

(iii) The analysis operator C_φ is well-defined on $\mathcal{H}$, bounded, injective and has closed range.

(iv) The frame operator S_φ is well-defined on $\mathcal{H}$, bounded and invertible.

(v) $\overline{span}(\varphi_k)_{k\in X} = \mathcal{H}$, the Gram matrix G_φ defines a well-defined and bounded operator on $\ell^2(X)$, and $G_\varphi|_{ran(C_\varphi)} : ran(C_\varphi) \longrightarrow ran(C_\varphi)$ is invertible.

Theorem 1.3. *[28, 9] Let $\varphi = (\varphi_k)_{k\in X}$ be a countable family of vectors in $\mathcal{H}$. Then the following are equivalent.*

(i) φ is an orthonormal basis for $\mathcal{H}$.

(ii) The synthesis operator $D_\varphi : \ell^2(X) \longrightarrow \mathcal{H}$ is unitary.

(iii) The analysis operator $C_\varphi : \mathcal{H} \longrightarrow \ell^2(X)$ is unitary.

(iv) $S_\varphi = \mathcal{I}_{\mathcal{B}(\mathcal{H})}$ and $\|\varphi_k\| = 1$ for all $k \in X$.

(v) $G_\varphi = \mathcal{I}_{\mathcal{B}(\ell^2(X))}$ and $\overline{span}(\varphi_k)_{k\in X} = \mathcal{H}$.

Of course, one may also consider sequences, whose associated frame-related operators satisfy other properties than the ones appearing above. A countable family $\varphi = (\varphi_k)_{k\in X}$ of vectors in $\mathcal{H}$ is called a *Riesz sequence*, if $D_\varphi : \ell^2(X) \longrightarrow \mathcal{H}$ is well-defined, bounded, injective, and has closed range. A Riesz sequence which additionally satisfies $\overline{\text{span}}(\varphi_k)_{k\in X} = \mathcal{H}$ is called a *Riesz basis*. Equivalently, $\varphi = (\varphi_k)_{k\in X}$ is called a Riesz basis, if $D_\varphi : \ell^2(X) \longrightarrow \mathcal{H}$ is well-defined, bounded and invertible.

As already mentioned in Theorem 1.2, whenever φ is a frame, its associated frame operator S_φ is invertible (and, in fact, strictly positive). Hence S_φ can be composed with its inverse (and vice versa), yielding the *frame reconstruction* formulae

$$f = \sum_{k\in X} \langle f, \widetilde{\varphi}_k\rangle \varphi_k = \sum_{k\in X} \langle f, \varphi_k\rangle \widetilde{\varphi}_k \qquad (\forall f \in \mathcal{H}), \tag{1.2}$$

where $\widetilde{\varphi}_k = S_\varphi^{-1}\varphi_k$ $(k \in X)$. The family $\widetilde{\varphi} := (\widetilde{\varphi}_k)_{k\in X}$ is also a frame and called the *canonical dual frame*. In case φ is a *redundant* frame, that is, a frame which is not a Riesz basis, then the kernel of the synthesis operator is non-trivial. As a consequence, one can show that there exist infinitely many other frames ψ so that

$$f = \sum_{k\in X} \langle f, \psi_k\rangle \varphi_k = \sum_{k\in X} \langle f, \varphi_k\rangle \psi_k \qquad (\forall f \in \mathcal{H}). \tag{1.3}$$

Such a frame ψ is called a *dual frame* of φ.

One of the biggest advantages of a (redundant) frame compared to an orthonormal basis or a Riesz basis is, that in case a frame vector is lost, then the resulting family might still be a frame and reconstruction might still be possible. In stark contrast to the latter, if given an orthonormal basis or Riesz basis, and one of its elements is unavailable, then there is no way to still recover every vector from the underlying Hilbert space. In particular, while reconstruction via Riesz bases or orthonormal bases is unique, frames provide an infinite number of options of dual frames used for the reconstruction of vectors. These (and other) properties of frames make them an indisputably useful tool for a variety of applications, such as signal processing [8], compressed sensing [88] and many more, see [28] and the numerous references therein. On the other hand, the study of frames led to many deep theoretical results, such as versions of the Balian-Low theorem [67, 16] or the Feichtinger conjecture. The latter turned out to be equivalent to the Kadison-Singer problem from the theory of C*-algebras (and a number of other profound problems in functional analysis) [21] and was finally solved in 2015 by Marcus, Spielmann, and Srivastava [85].

2. Localized Frames

We have already mentioned in the previous section that whenever $\varphi := (\varphi_k)_{k\in X}$ and $\psi := (\psi_k)_{k\in X}$ are Bessel sequences in a Hilbert space $\mathcal{H}$, then their associated cross Gram matrix $G_{\varphi,\psi}$ defines a bounded operator on $\ell^2(X)$. Boundedness on $\ell^2(X)$ of $G_{\varphi,\psi} = [\langle \psi_l, \varphi_k\rangle]_{k,l\in X}$ (viewed as a scalar matrix) is equivalent to some weak form of off-diagonal decay of the matrix entries $\langle \varphi_l, \varphi_k\rangle$, as described, e.g., in [84, Theorem 4.16]. The idea of localization is to impose stronger off-diagonal decay conditions on a given (cross) Gram matrix.

2.1. A motivating example: Gabor frames

In order to properly motivate the concept of localized frames, we restrict our attention to the Hilbert space $L^2(\mathbb{R}^d)$ of square-integrable measurable functions on $\mathbb{R}^d$ and consider a concrete example from time-frequency analysis. For a comprehensive introduction to time-frequency analysis we refer the reader to [49].

The fundamental operators in time-frequency analysis are the *translation operator* (or *time shift*) T_x and the *modulation operator* (or *frequency shift*) M_ω defined by

$$T_x f(t) = f(t-x) \quad \text{and} \quad M_\omega f(t) = e^{2\pi i\omega\cdot t} f(t) \qquad (x,\omega\in\mathbb{R}^d).$$

Here, $\omega\cdot t$ denotes the scalar product of ω and t in $\mathbb{R}^d$. Their composition $\pi(z) = M_\omega T_x$, where $z=(x,w)\in\mathbb{R}^{2d}$, is called a *time-frequency shift* by z. Each of the operators T_x, M_ω and $\pi(z)$ is unitary on $L^2(\mathbb{R}^d)$. One of the central objects used for the analysis of the time-frequency content of a function f is the *short-time Fourier transform* (STFT). For a fixed function g called a *window function* or simply *window*, the STFT of a function f with respect to the window g is (formally) defined by

$$V_g f(x,w) = \int_{\mathbb{R}^d} f(t)\overline{g(t-x)} e^{-2\pi i\omega\cdot t}\, dt \qquad (x,w\in\mathbb{R}^d).$$

Setting $z=(x,w)\in\mathbb{R}^{2d}$, we may rewrite the above to

$$V_g f(z) = \langle f, \pi(z)g\rangle \qquad (z\in\mathbb{R}^{2d})$$

whenever the bracket is well-defined. In particular, if both f and g belong to $L^2(\mathbb{R}^d)$, then $V_g f(z) = \langle f,\pi(z)g\rangle_{L^2(\mathbb{R}^d)}$ is precisely the L^2-inner product of f with a by z time-frequency shifted version of g. Moreover, for a fixed window $g\in L^2(\mathbb{R}^d)$, the STFT defines an operator $V_g : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^{2d})$ with

$$\|V_g f\|_{L^2(\mathbb{R}^{2d})} = \|g\|_{L^2(\mathbb{R}^d)}\|f\|_{L^2(\mathbb{R}^d)}. \tag{2.1}$$

Thus, in case $\|g\|_{L^2(\mathbb{R}^d)} = 1$, the STFT is an isometry $V_g : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^{2d})$. In a frame-theoretic language, (2.1) says that for non-zero $g \in L^2(\mathbb{R}^d)$, the system $(\pi(z)g)_{z\in\mathbb{R}^{2d}}$ constitutes a continuous tight frame for $L^2(\mathbb{R}^d)$ with tight frame bound $\|g\|_{L^2(\mathbb{R}^d)}$ [28]. As a consequence from abstract principles, one may obtain a continuous reconstruction of any $f \in L^2(\mathbb{R}^d)$ via

$$f = \frac{1}{\|g\|}\int_{\mathbb{R}^{2d}} \langle f, \pi(z)g\rangle \pi(z)g\, dz = \frac{1}{\|g\|}\int_{\mathbb{R}^{2d}} V_g f(z)\pi(z)g\, dz, \tag{2.2}$$

where the integral is to be understood in the weak sense. Although this reconstruction procedure perfectly describes an L^2-function as a continuous superposition of the family $(\pi(z)g)_{z\in\mathbb{R}^{2d}}$, it certainly lacks applicability for real-world problems, where only discrete (and, in fact, finite-dimensional) information can be processed. Therefore, we are interested in discretized versions of the STFT and the reconstruction procedure (2.2), which leads to Gabor frames.

For a discrete set $\Gamma \subset \mathbb{R}^{2d}$ and a window $g \in L^2(\mathbb{R}^d)$, the countable family $\mathcal{G}(\Gamma, g) := (\pi(z)g)_{z\in\Gamma}$ is called a *Gabor system*. In case $\mathcal{G}(\Gamma, g)$ is a frame for $L^2(\mathbb{R}^d)$, it is called a *Gabor frame*. By the discretization theorem for continuous frames [45] every continuous frame may be sampled to obtain a (discrete) frame. Hence, for every non-zero $g \in L^2(\mathbb{R}^d)$, there exists a discrete set $\Gamma \subset \mathbb{R}^{2d}$ so that $\mathcal{G}(\Gamma, g)$ is a Gabor frame. However, the proof of the discretization theorem for continuous frames is non-constructive, so it is apriori not clear which samples of the STFT result in a frame. In fact, determining pairs (Γ, g) of index sets Γ and widows g such that $\mathcal{G}(\Gamma, g)$ is a Gabor frame is a deep and hard problem and an active field of research until today [71, 73, 72, 63, 56, 40]. We will not touch this problem here but only quote the following result by Lyubarksi and Seip-Wallsten on the Gaussian window in the $d = 1$ case for later reference.

Theorem 2.1. *[83, 91, 92] Let $\varphi_0(t) = 2^{\frac{1}{4}}e^{-\pi t^2} \in L^2(\mathbb{R})$ be the normalized Gaussian and $a, b > 0$. Then $\mathcal{G}(a\mathbb{Z} \times b\mathbb{Z}, \varphi_0)$ is a frame for $L^2(\mathbb{R})$ if and only if $ab < 1$.*

The STFT allows for much more than a continuous superposition of signals f as in (2.2). If the window g has properties that go beyond its membership in $L^2(\mathbb{R}^d)$, its associated STFT V_g can capture finer analytical characteristics of a signal, such as smoothness or decay behavior. The central objects in this context are the modulation spaces defined below. We refer to [49] for a detailed introduction to modulation spaces and their properties.

Let $\varphi_0(t) = 2^{\frac{d}{4}}e^{-\pi t\cdot t}$ denote the normalized Gaussian in $L^2(\mathbb{R}^d)$ and $\nu_s(z) = (1 + |z|)^s$ denote the polynomial weight on $\mathbb{R}^{2d}$ $(s \geq 0)$. Then $M^1_{\nu_s}(\mathbb{R}^d)$ is defined as the space of all $f \in L^2(\mathbb{R}^d)$ such that

$$\|f\|_{M^1_{\nu_s}(\mathbb{R}^d)} := \|V_{\varphi_0} f\|_{L^1_{\nu_s}(\mathbb{R}^d)} = \int_{\mathbb{R}^{2d}} |V_{\varphi_0} f(z)|\nu_s(z)\, dz < \infty. \tag{2.3}$$

For a ν_s-*moderate* weight m, that is, a function $m : \mathbb{R}^{2d} \longrightarrow (0, \infty)$ satisfying

$$m(z + z') \leq Cm(z)\nu_s(z') \qquad (\forall z, z' \in \mathbb{R}^{2d})$$

for some uniform constant $C > 0$, and for $p \in [1, \infty]$, the *modulation space* $M^p_m(\mathbb{R}^d)$ is defined as the space of all elements f in the anti-linear dual space $(M^1_{\nu_s}(\mathbb{R}^d))'$ for which

$$\|f\|_{M^p_m(\mathbb{R}^d)} := \|V_{\varphi_0} f\|_{L^p_m(\mathbb{R}^d)} < \infty. \tag{2.4}$$

Some modulation spaces describe other well-known function spaces, such as the space $S(\mathbb{R}^d)$ of Schwartz-functions or its topological dual $S'(\mathbb{R}^d)$, the space of tempered distributions, namely,

$$S(\mathbb{R}^d) = \bigcap_{s \geq 0} M^\infty_{\nu_s}(\mathbb{R}^d) \qquad \text{and} \qquad S'(\mathbb{R}^d) = \bigcup_{s \geq 0} M^\infty_{1/\nu_s}(\mathbb{R}^d). \tag{2.5}$$

While $S(\mathbb{R}^d)$ is not a Banach space and $S'(\mathbb{R}^d)$ not even a Frechét space [17], each modulation space $M^p_m(\mathbb{R}^d)$ is, in fact, a Banach space with respect to the norm defined in (2.4), which makes working with these objects much simpler. Moreover, the Gaussian $\varphi_0 \in M^1_{\nu_s}(\mathbb{R}^d)$ does not play a special role in the definition of modulation spaces. In fact, replacing φ_0 in (2.3) by any other non-zero $g \in M^1_{\nu_s}(\mathbb{R}^d)$ leads to the same spaces $M^p_m(\mathbb{R}^d)$ with equivalent norms. The space $S_0(\mathbb{R}^d) = M^1(\mathbb{R}^d) := M^1_{\nu_0}(\mathbb{R}^d)$ is known as *Feichtinger's algebra* [42] and not only a Banach space, but even a Banach algebra with respect to both convolution and point-wise multiplication, and contains $S(\mathbb{R}^d)$ as a norm-dense subspace. One can show that $S_0(\mathbb{R}^d)$ is invariant under both the Fourier transform and actions of time-frequency shifts, and that $S_0(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) \subset S_0'(\mathbb{R}^d)$ (such a triple is called a *Banach-Gelfand triple* [41]). The pair $(S_0(\mathbb{R}^d), S_0'(\mathbb{R}^d))$ is widely considered as *the* appropriate test function-/distribution space pair for time-frequency analysis and applications to quantum physics [36].

In analogy to the L^2-setting described earlier, one may ask whether the definition of a modulation space, given by the L^p-norm of the STFT, can alternatively be described via discrete means. Again, such a characterization is indeed possible using frames and measuring the ℓ^p_m-norm of the associated frame coefficients. For simplicity, we state the following result in the $d = 1$ case.

Theorem 2.2. *Let $a, b > 0$ with $ab < 1$ and set $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$. Let $S_\mathcal{G}$ denote the frame operator associated with the Gabor frame $\mathcal{G} = \mathcal{G}(\Lambda, \varphi_0)$ for $L^2(\mathbb{R})$ (c.f. Theorem 2.1). Then, for each $1 \leq p \leq \infty$, $S_\mathcal{G}$ is bounded and invertible on $M^p_m(\mathbb{R})$ and the frame reconstruction formula*

$$f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)\varphi_0 \rangle S_\mathcal{G}^{-1} \pi(\lambda)\varphi_0 = \sum_{\lambda \in \Lambda} \langle f, S_\mathcal{G}^{-1} \pi(\lambda)\varphi_0 \rangle \pi(\lambda)\varphi_0$$

holds true for all $f \in M^p_m(\mathbb{R})$ with unconditional convergence (resp. unconditional weak-convergence for $p = \infty$). Furthermore, for $1 \leq p < \infty$, $M^p_m(\mathbb{R})$ is the completion of the space*

$$\{ f \in L^2(\mathbb{R}) : (\langle f, \pi(\lambda)\varphi_0 \rangle)_{\lambda \in \Lambda} \in \ell^p_m(\Lambda) \}$$

with respect to the norm $\|(\langle \cdot, \pi(\lambda)\varphi_0 \rangle)_{\lambda \in \Lambda}\|_{\ell^p_m(\Lambda)}$ and

$$\|f\|_{M^p_m(\mathbb{R})} \asymp \|(\langle f, \pi(\lambda)\varphi_0 \rangle_{\lambda \in \Lambda}\|_{\ell^p_m(\Lambda)} \asymp \|(\langle f, S_\mathcal{G}^{-1} \pi(\lambda)\varphi_0 \rangle_{\lambda \in \Lambda}\|_{\ell^p_m(\Lambda)}.$$

Theorem 2.2 was proven in various degrees of generality, see e.g. [49, Thm. 13.2.1, Cor. 12.2.6] or [57, Thm. 4.2]. The decisive argument for showing the invertibility of the Gabor frame operator is an application of Wiener's Lemma (or one of its many non-commutative generalizations) [55].

One conceptually particularly simple proof of Theorem 2.2 can be found in [44, Thm. 4.1], where the authors use the polynomial off-diagonal decay of the Gram matrix $G_{\mathcal{G}}$ to their advantage. Indeed, since $\varphi_0 \in S(\mathbb{R}^d)$, we have, by (2.5), that $\varphi_0 \in M^{\infty}_{\nu_s}(\mathbb{R}^d)$ for all $s \geq 0$, hence

$$\begin{aligned}
|[G_{\mathcal{G}}]_{\lambda,\mu}| &= |\langle \pi(\mu)\varphi_0, \pi(\lambda)\varphi_0 \rangle| \\
&= |(V_{\varphi_0}\pi(\mu)\varphi_0)(\lambda)| \\
&= |V_{\varphi_0}\varphi_0(\lambda-\mu)| \\
&= |V_{\varphi_0}\varphi_0(\lambda-\mu)|(1+|\lambda-\mu|)^s(1+|\lambda-\mu|)^{-s} \\
&\leq \|\varphi_0\|_{M^{\infty}_{\nu_s}(\mathbb{R}^d)}(1+|\lambda-\mu|)^{-s} \qquad (\forall \lambda,\mu \in \Lambda, \forall s \geq 0),
\end{aligned}$$

where the covariance property of the STFT [49, Lemma 3.1.3] is used in the third step. This means that the Gram matrix $G_{\mathcal{G}}$ is contained in the Jaffard class $\mathcal{J}_s$ of scalar matrices $A = [A_{\lambda,\mu}]_{\lambda,\mu\in\Lambda}$ for which

$$\sup_{\lambda,\mu\in\Lambda} |A_{\lambda,\mu}|(1+|\lambda-\mu|)^s < \infty.$$

For s large enough, $\mathcal{J}_s$ is a unital Banach *-algebra of matrices. The Wiener-type lemma applied in this setting is Jaffard's Lemma [70], which states that whenever a matrix $A \in \mathcal{J}_s$ (s sufficiently large) is invertible as an operator in $\mathcal{B}(\ell^2(\Lambda))$, its bounded inverse is not only contained in $\mathcal{B}(\ell^2(\Lambda))$ but also in $\mathcal{J}_s$. Furthermore, $\mathcal{J}_s$ is contained in $\mathcal{B}(\ell^p_m)$ for every $1 \leq p \leq \infty$ and every ν_s-moderate weight m; hence, $G_{\mathcal{G}}$ acting on ℓ^p_m-sequences is well-defined, and it is possible to define a whole class of associated Banach spaces in terms of weighted ℓ^p-norms of the frame coefficients. After studying the properties of the operators $C_{\mathcal{G}}$ and $D_{\mathcal{G}}$ on these spaces it becomes clear that they can be identified with the modulation spaces.

The frame $\mathcal{G}$ is actually an example of a (polynomially) localized frame and the above proof sketch is an application of the general theory of localized frames and provides an initial insight into its definitions and techniques used.

2.2. Localized frames: An overview to the abstract concept

A frame is called localized, if its associated Gram matrix satisfies a certain prescribed off-diagonal decay, such as polynomial decay as presented in the previous subsection. The decisive properties used in the proof of [44, Thm. 4.1] are the algebra property of $\mathcal{J}_s$ and Jaffard's Lemma. This point of view motivated the authors in [44] to consider the following type of matrix algebras for the study of abstract localized frames.

Definition 2.3. *[44, Section 2.3] A unital Banach *-algebra $\mathcal{A} = \mathcal{A}(X)$ of $\mathbb{C}$-valued matrices $A = [A_{k,l}]_{k,l\in X}$ is called a* solid spectral matrix algebra*, or simply* spectral algebra*, if*

(A1) $\mathcal{A} \subset \mathcal{B}(\ell^2(X))$, i.e., each $A \in \mathcal{A}$ defines a bounded operator on $\ell^2(X)$.

(A2) $\mathcal{A}$ is inverse-closed *in $\mathcal{B}(\ell^2(X))$, i.e., if $A \in \mathcal{A}$ is invertible in $\mathcal{B}(\ell^2(X))$, then its bounded inverse $A^{-1} \in \mathcal{B}(\ell^2(X))$ is contained in $\mathcal{A}$ as well.*

(A3) $\mathcal{A}$ is solid*, i.e., if $A = [A_{k,l}]_{k,l\in X} \in \mathcal{A}$ and $|B_{k,l}| \leq |A_{k,l}|$ for all $k, l \in X$, then $B = [B_{k,l}]_{k,l\in X} \in \mathcal{A}$ and $\|B\|_{\mathcal{A}} \leq \|A\|_{\mathcal{A}}$.*

Property (A2) is equivalent to equality $\sigma_{\mathcal{A}}(A) = \sigma_{\mathcal{B}(\ell^2(X))}(A)$ of the spectrum of A in $\mathcal{A}$ and the spectrum of A in $\mathcal{B}(\ell^2(X))$ for all $A \in \mathcal{A}$ and can be viewed as a non-commutative version of Wiener's lemma for infinite matrices [55]. We explicitly emphasize that here, the inverse-closedness of $\mathcal{A}$ in $\mathcal{B}(\ell^2(X))$ is required in the definition of a spectral algebra, whereas in concrete examples of such $\mathcal{A}$, this property typically is a deep theorem that is not easy to obtain and requires sophisticated arguments and techniques for its proof.

Example 2.4. *Let $X \subset \mathbb{R}^d$ be a* relatively separated *set, that is,*

$$\sup_{z\in\mathbb{R}^d} |X \cap (z + [0,1]^d)| < \infty. \tag{2.6}$$

Then the following classes of matrices are examples of spectral algebras.
(1) *For every $s > d$, the* Jaffard algebra *$\mathcal{J}_s = \mathcal{J}_s(X)$ of matrices $A = [A_{k,l}]_{k,l\in X}$ for which*

$$\sup_{k,l\in X} |A_{k,l}|(1 + |k - l|)^s < \infty$$

is a spectral algebra [70], see also [82] for a comparably short and accessible proof.
(2) *For certain weights ν the* Schur algebra *$\mathcal{S}^1_\nu = \mathcal{S}^1_\nu(X)$ of matrices $A = [A_{k,l}]_{k,l\in X}$ for which*

$$\max\left\{\sup_{k\in X}\sum_{l\in X} |A_{k,l}|\nu(k-l)\,,\, \sup_{l\in X}\sum_{k\in X} |A_{k,l}|\nu(k-l)\right\} < \infty$$

is a spectral algebra [58, 99].
(3) *For certain weights ν the* Baskakov-Gohberg-Sjöstrand algebra *$\mathcal{C}_\nu = \mathcal{C}_\nu(\mathbb{Z}^d)$ of matrices $A = [A_{k,l}]_{k,l\in\mathbb{Z}^d}$ for which*

$$\sum_{l\in\mathbb{Z}^d} \sup_{k\in\mathbb{Z}^d} |A_{k,k-l}|\nu(l) < \infty$$

is a spectral algebra [11, 47, 94].
(4) *Anisotropic versions of each of the aforementioned algebras are further examples of spectral algebras [75, Section 6].*

With the above examples of spectral algebras in mind, we finally give the definition of a localized frame in an abstract Hilbert space.

Definition 2.5. *[44, Definition 3] Let $\mathcal{A}$ be a spectral algebra. Two countable families $\psi = (\psi_k)_{k\in X}$ and $\varphi = (\varphi_k)_{k\in X}$ in $\mathcal{H}$ are called* mutually$\mathcal{A}$-localized, *in short $\psi \sim_{\mathcal{A}} \varphi$, if their associated cross Gram matrix $G_{\psi,\varphi}$ is contained in $\mathcal{A}$. Similarly, ψ is called (intrinsically) $\mathcal{A}$-localized if $\psi \sim_{\mathcal{A}} \psi$, i.e., $G_\psi \in \mathcal{A}$.*

From (A1) and Theorem 1.1 it follows immediately that every intrinsically localized sequence φ is automatically a Bessel sequence. In case φ is an intrinsically $\mathcal{A}$-localized frame, then its canonical dual frame $\widetilde{\varphi}$ possesses the same localization and it also holds $\widetilde{\varphi} \sim_{\mathcal{A}} \varphi$. The proof of this result is based on the identities $G_{\widetilde{\varphi}} = G_\varphi^\dagger$ and $G_{\varphi,\widetilde{\varphi}} = G_\varphi G_\varphi^\dagger$, where $G_\varphi^\dagger$ denotes the Moore-Penrose pseudo-inverse of G_φ [28], as well as the algebra property of $\mathcal{A}$ and the fact that spectral algebras are not only inverse-closed, but also pseudo-inverse closed [44, Cor. 3.5].

Theorem 2.6. *[44, Thm. 3.6] Let φ be a frame in $\mathcal{H}$ and $\mathcal{A}$ be a spectral algebra. If $\varphi \sim_{\mathcal{A}} \varphi$ then also $\widetilde{\varphi} \sim_{\mathcal{A}} \widetilde{\varphi}$ and $\widetilde{\varphi} \sim_{\mathcal{A}} \varphi$.*

As already hinted at in Theorem 2.2, another remarkable feature of localized frames is that the reconstruction formula (1.2) converges not only in the underlying Hilbert space $\mathcal{H}$, but also in a whole family of associated (quasi-)Banach spaces $\mathcal{H}^p_\omega(\varphi^d, \varphi)$. Depending on the given frame φ, these spaces are well-known function spaces such as modulation spaces [44, 60] or shift-invariant spaces [51]. In the abstract setting, the following minimal requirements need to be fulfilled in order to obtain a well-defined space $\mathcal{H}^p_\omega(\varphi^d, \varphi)$ via Definition 2.8:

(i) $\varphi = (\varphi_k)_{k\in X}$ is a frame and $\varphi^d = (\varphi^d_k)_{k\in X}$ a dual frame of φ.

(ii) The cross Gram matrix $G_{\varphi^d,\varphi}$ defines a bounded operator on $\ell^p_\omega(X)$.

If φ is an intrinsically $\mathcal{A}$-localized frame with canonical dual frame $\widetilde{\varphi}$, Theorem 2.6 guarantees that $G_{\widetilde{\varphi},\varphi} \in \mathcal{A}$. Furthermore, each of the spectral algebras listed in Example 2.4 is contained in $\mathcal{B}(\ell^p_\omega(X))$ for each $1 \le p \le \infty$ (in the case $\mathcal{A} = \mathcal{J}_s$ even for certain exponents $p < 1$) and every weight ω which is ν-moderate (where ν is the weight built into the definition of the corresponding spectral algebra). Hence, conditions (i) and (ii) are certainly satisfied in these cases. For abstract spectral algebras, we need to turn the boundedness conditions on $\ell^p_\omega(X)$ into a definition.

Definition 2.7. *[44] Let $\mathcal{A}$ be a spectral algebra and $0 < p_0 \le 1$. A weight function $\omega : \mathbb{R}^d \longrightarrow (0,\infty)$ is called* $(\mathcal{A}, p_0)$-admissible, *if every $A \in \mathcal{A}$ defines a bounded operator on $\ell^p_\omega(X)$ for every $p \in (p_0, \infty] \cup \{1\}$.*

Definition 2.8. *[44, Definition 5] Let $\mathcal{A}$ be a spectral algebra, ω be an $(\mathcal{A}, p_0)$-admissible weight, and (φ^d, φ) be a pair of dual frames for $\mathcal{H}$ such that $\varphi^d \sim_{\mathcal{A}} \varphi$. Let $\mathcal{H}^{00}(\varphi) := D_\varphi(c_{00}(X)) = \left\{\sum_{k\in X} c_k\varphi_k : c \in c_{00}(X)\right\}$ be the space of finite linear combinations of elements from φ. For $p \in (p_0, \infty) \cup \{1\}$ the* co-orbit space *$\mathcal{H}^p_\omega(\varphi^d, \varphi)$ is defined as the completion of $\mathcal{H}^{00}(\varphi)$ with respect to the norm $\|f\|_{\mathcal{H}^p(\varphi^d,\varphi)} := \|C_{\varphi^d} f\|_{\ell^p_\omega(X)}$. The space $\mathcal{H}^0_\omega(\varphi^d, \varphi)$ is defined as the completion of $\mathcal{H}^{00}(\varphi)$ with respect to the norm $\|f\|_{\mathcal{H}^0(\varphi^d,\varphi)} := \|C_{\varphi^d} f\|_{\ell^\infty_\omega(X)}$.*

It can be shown that whenever ω is $(\mathcal{A}, p_0)$-admissible, the weight $1/\omega$ is $(\mathcal{A}, 1)$-admissible and an interpolation argument also yields that the constant weight 1 is $(\mathcal{A}, 1)$-admissible [81]. In this case, the spaces $\mathcal{H}^p_{1/\omega}(\varphi^d, \varphi)$ and $\mathcal{H}^p(\varphi^d, \varphi) := \mathcal{H}^p_1(\varphi^d, \varphi)$ are defined analogously. There is also a natural, though rather technical, definition of $\mathcal{H}^\infty_\omega(\varphi^d, \varphi)$. We will not mention the details here in the introduction and refer instead to [7, 81, 66]. Alternatively, $\mathcal{H}^\infty_\omega(\varphi^d, \varphi)$ can be defined as the anti-linear topological dual space of $\mathcal{H}^1_{1/\omega}(\varphi^d, \varphi)$ [7].

The spaces $\mathcal{H}^p_\omega(\varphi^d, \varphi)$ are well-defined non-trivial Banach spaces for $1 \leq p \leq \infty$ and $p = 0$, and quasi-Banach spaces for $p_0 < p < 1$. Furthermore, it is immediately clear from the definition that the analysis operator $C_{\varphi^d} : \mathcal{H}^{00}(\varphi) \longrightarrow c_{00}(X)$ extends via density to an isometry $C_{\varphi^d} : \mathcal{H}^p_\omega(\varphi^d, \varphi) \to \ell^p_\omega(X)$ $(p \in (p_0, \infty) \cup \{0, 1\})$. For the same values of p the synthesis operator $D_\varphi : c_{00}(X) \longrightarrow \mathcal{H}^{00}(\varphi)$ extends to a bounded surjective operator $\ell^p_\omega(X) \longrightarrow \mathcal{H}^p_\omega(\varphi^d, \varphi)$ [44]. For $p = \infty$ the same properties hold, but the analysis and synthesis operators are defined a bit differently [7].

Theorem 2.9. *[44] Let $\mathcal{A}$ be a spectral algebra, ω be an $(\mathcal{A}, p_0)$-admissible weight, φ be a frame and φ^d be a dual frame of φ such that $\varphi^d \sim_\mathcal{A} \varphi$. Then for each $p \in (p_0, \infty) \cup \{0, 1\}$ it holds*

$$f = \sum_{k \in X} \langle f, \varphi^d_k \rangle \varphi_k \qquad (\forall f \in \mathcal{H}^p_\omega(\varphi^d, \varphi)), \tag{2.7}$$

with unconditional convergence.

Finally, the definition of $\mathcal{H}^p_\omega(\varphi^d, \varphi)$ indicates that it is highly dependent on the dual frame pair (φ^d, φ). However, if (ψ^d, ψ) is another pair of dual frames satisfying $\psi^d \sim_\mathcal{A} \psi$, and if $\psi^d \sim_\mathcal{A} \varphi$ and $\varphi^d \sim_\mathcal{A} \psi$, then one may use the above theorem to show that $\mathcal{H}^p_\omega(\varphi^d, \varphi) = \mathcal{H}^p_\omega(\psi^d, \psi)$ with equivalent norms [81, Thm. 3.25]. In case φ is an intrinsically $\mathcal{A}$-localized frame and $\varphi^d = \widetilde{\varphi}$ its canonical dual frame, then $\mathcal{H}^p_\omega(\widetilde{\varphi}, \varphi) = \mathcal{H}^p_\omega(\varphi, \widetilde{\varphi})$ and $\|C_{\widetilde{\varphi}} \cdot \|_{\ell^p_\omega(X)} \asymp \|C_\varphi \cdot \|_{\ell^p_\omega(X)}$ [44]. Thus, every intrinsically localized frame is automatically a *Banach-frame* [48, 29, 22] for the spaces $\mathcal{H}^p_\omega(\widetilde{\varphi}, \varphi)$).

3. Operator-valued frames

We have already mentioned the great versatility of frames in both theory and applications. However, despite its great usefulness in many situations, frame theory also has its limitations. In various applications, such as distributed and parallel processing [26, 23], wireless sensor networks [69], packet encoding [19, 20, 27], visual and hearing systems [90], or geophones in geophysics [32], the linear reconstruction of vectors from *higher-rank measurements* is required. This lead to the study of fusion frames and further generalizations of frames such as operator-valued frames.

A countable family $T = (T_k)_{k \in X}$ of bounded operators $T_k \in \mathcal{B}(\mathcal{H})$ on some separable Hilbert space $\mathcal{H}$ is called an *operator-valued frame* (originally called *generalized frame*, or simply *g-frame*) [102], if there exist positive constants $A \leq B$, such that

$$A\|f\|^2 \leq \sum_{k \in X} \|T_k f\|^2 \leq B\|f\|^2 \qquad (\forall f \in \mathcal{H}). \tag{3.1}$$

For brevity, we will often call an operator-valued frame a *g-frame* and refer to the constants A and B as the *lower* and *upper g-frame bound*, respectively. We call $(T_k)_{k \in X}$ a *g-Bessel sequence*, whenever the upper (but not necessarily the lower) inequality in (3.1) is satisfied for some prescribed $B > 0$ and all $f \in \mathcal{H}$.

In order to make the relation to frames clearer, let ψ_k be some vector in $\mathcal{H}$. Then

$$|\langle f, \varphi_k \rangle|^2 = \big\langle \langle f, \varphi_k \rangle \varphi_k, f \big\rangle = \|\varphi_k\|^2 \left\langle \left\langle f, \frac{\varphi_k}{\|\varphi_k\|} \right\rangle \frac{\varphi_k}{\|\varphi_k\|}, f \right\rangle = \|T_k f\|^2,$$

where $T_k = \|\varphi_k\| P_{\mathrm{span}\{\psi_k\}}$ and $P_{\mathrm{span}\{\psi_k\}}$ denotes the orthogonal projection of $\mathcal{H}$ onto $\mathrm{span}\{\psi_k\}$. Therefore, comparing (1.1) with (3.1), we see that a countable collection $(\varphi_k)_{k \in X}$ is a frame with frame bounds $A \leq B$ if and only if $(T_k)_{k \in X}$ is a g-frame with g-frame bounds $A \leq B$. Consequently, frames may be regarded as a special case of g-frames. Further examples of g-frames are

- *Fusion frames* [24, 26, 76], where $T_k = v_k P_k$, with $v_k > 0$ and P_k being an orthogonal projection onto an arbitrary closed subspace V_k of $\mathcal{H}$.
- *Time-frequency localization operators* for $\mathcal{H} = L^2(\mathbb{R}^d)$ [37], where T_k is given by

$$T_k f = \int_{\mathbb{R}^{2d}} \sigma(z - k) V_g f(z) \pi(z) g \, dz,$$

  and σ is a bounded and non-negative function on $\mathbb{R}^{2d}$.
- *Gabor g-frames* for $\mathcal{H} = L^2(\mathbb{R}^d)$ [95], where $T_k = \pi(k) T \pi(k)^*$, with $T \in \mathcal{B}(L^2(\mathbb{R}^d))$ being some suitable fixed (window) operator.

As for frames, the main feature of g-frames is a linear and stable reconstruction of vectors $f \in \mathcal{H}$. In contrast to classical frames, g-frames allow for a reconstruction from higher-rank measurements. Therefore, a natural choice for the analysis of the g-frame measurements $(T_k f)_{k\in X}$ is the Bochner space of $\mathcal{H}$-valued square-summable sequences

$$\ell^2(X;\mathcal{H}) = \left\{ (h_k)_{k\in X} : h_k \in \mathcal{H}\, (\forall k \in X), \sum_{k\in X} \|h_k\|^2 < \infty \right\}.$$

Unsurprisingly, the definition of the canonical frame-related operators naturally extends to the g-frame setting. If $T := (T_k)_{k\in X}$ is a g-Bessel sequence in $\mathcal{H}$, then the following operators are well-defined and bounded [102]:

- The *synthesis operator* $D_T : \ell^2(X;\mathcal{H}) \longrightarrow \mathcal{H}$, defined by
$$D_T(f_k)_{k\in X} = \sum_{k\in X} T_k^* f_k,$$
- The *analysis operator* $C_T : \mathcal{H} \longrightarrow \ell^2(X;\mathcal{H})$, defined by
$$C_T f = (T_k f)_{k\in X},$$
- The *g-frame operator* $S_T := D_T C_T : \mathcal{H} \longrightarrow \mathcal{H}$, given by
$$S_T f = \sum_{k\in X} T_k^* T_k f.$$
- The *g-Gram matrix* $G_T := C_T D_T : \ell^2(X;\mathcal{H}) \longrightarrow \ell^2(X;\mathcal{H})$.

These operators share the same properties with their frame-theoretic pendants. More precisely [102], T being a g-Bessel sequence with bound B implies that D_T and C_T are adjoint to one another and $\|D_T\|_{\mathcal{B}(\ell^2(X;\mathcal{H}),\mathcal{H})} = \|C_T\|_{\mathcal{B}(\mathcal{H},\ell^2(X;\mathcal{H}))} \leq \sqrt{B}$. Furthermore, both S_T and G_T are self-adjoint and bounded by B in this case. If T is a g-frame, then C_T is bounded, injective and has closed range, D_T is bounded and surjective, and S_T is bounded, self-adjoint, (strictly) positive and invertible. The invertibility of S_T yields the possibility of *g-frame reconstruction* via

$$f = \sum_{k\in X} T_k^* T_k S_T^{-1} f = \sum_{k\in X} S_T^{-1} T_k^* T_k f \qquad (\forall f \in \mathcal{H}). \tag{3.2}$$

The family $\widetilde{T} = (\widetilde{T}_k)_{k\in X} := (T_k S_T^{-1})_{k\in X}$, which appears in the reconstruction process (3.2), is again a g-frame and called the *canonical dual g-frame*. More generally [80], a g-frame $(T_k^d)_{k\in X}$ satisfying

$$f = \sum_{k\in X} T_k^* T_k^d f = \sum_{k\in X} (T_k^d)^* T_k f \qquad (\forall f \in \mathcal{H}) \tag{3.3}$$

is called a *dual g-frame* of $(T_k)_{k\in X}$.

Part II.

Discussion of the main results

The three papers comprising this thesis can be divided into two main themes:

(i) Introduction and development of the theory of localized operator-valued frames.

(ii) Characterization of the frame property of a given localized sequence.

The main object of study in theme (i) is the Gram matrix G_T associated with a g-frame $(T_k)_{k\in X}$. As stated in the introduction, $G_T = C_T D_T$ is given by the composition of the synthesis and the analysis operator associated with $(T_k)_{k\in X}$. An equivalent description of G_T is given by the $\mathcal{B}(\mathcal{H})$-valued matrix $[T_k T_l^*]_{k,l\in X}$, since the action of G_T on any $(g_l)_{l\in X} \in \ell^2(X;\mathcal{H})$ is precisely given by the matrix-vector multiplication of $[T_k T_l^*]_{k,l\in X}$ with $(g_l)_{l\in X}$ viewed as column vector. In view of this observation, it is natural to define localization of an operator-valued frame via G_T belonging to some spectral algebra $\mathcal{A} \subset \mathcal{B}(\ell^2(X;\mathcal{H}))$ of $\mathcal{B}(\mathcal{H})$-valued matrices, as done for frames. The introduction and development of such a localization theory for operator-valued frames is the content of the second paper. However, in order to fill this theory with life and apply it, an arsenal of such localization algebras $\mathcal{A}$ is needed. In the first paper, we consider several classes $\mathcal{B}(\mathcal{H})$-valued matrices and prove that they are spectral algebras. Along the way, we also derive some examples of spectral algebras that are new also in the scalar-valued setting.

Theme (ii) is covered in the third paper. Here, we consider countable families $(\psi_k)_{k\in X}$ of vectors in a Hilbert space $\mathcal{H}$ that are mutually localized with some fixed localized reference Riesz basis $(\varphi_k)_{k\in X}$. We characterize the frame property of such $(\psi_k)_{k\in X}$ in terms of the mapping properties of its frame-related operators acting on or mapping into the co-orbit spaces $\mathcal{H}^p(\widetilde{\varphi},\varphi)$ associated to the reference system $(\varphi_k)_{k\in X}$. This abstract characterization is applied to shift-invariant spaces, where we obtain new conditions for stable sets of sampling in these spaces.

4. Wiener pairs of Banach algebras of operator-valued matrices

The main definition of the first paper is the following.

Definition 4.1. *A nested pair $\mathcal{A} \subseteq \mathcal{B}$ of two unital Banach algebras $\mathcal{A}$ and $\mathcal{B}$ with common identity is called a* Wiener pair *[86, 87], if $\mathcal{A}$ is* inverse-closed *in $\mathcal{B}$, that is, if it holds*

$$A \in \mathcal{A} \text{ and } \exists A^{-1} \text{ in } \mathcal{B} \qquad \Longrightarrow \qquad A^{-1} \in \mathcal{A}.$$

The standard example of a (non-trivial) Wiener pair is $\mathcal{A}(\mathbb{T}) \subseteq C(\mathbb{T})$, where $C(\mathbb{T})$ is the space of continuous functions on the torus and $\mathcal{A}(\mathbb{T})$ the subclass of all continuous functions that possess an absolutely convergent Fourier series. The inverse-closedness of $\mathcal{A}(\mathbb{T})$ in $C(\mathbb{T})$ is precisely the classical version of Wiener's Lemma [105]. Since it is often easier to verify the invertibility of an element $A \in \mathcal{A} \subseteq \mathcal{B}$ in the larger algebra $\mathcal{B}$, than verifying its invertibility in the smaller algebra $\mathcal{A}$, $\mathcal{A}$ being inverse-closed in $\mathcal{B}$ is a fairly strong property and may serve as an immensively usefool tool in certain situations. In particular, Wiener's Lemma and its many generalizations, see, e.g., [5, 6, 10, 11, 13, 14, 15, 18, 34, 33, 47, 52, 58, 70, 79, 93, 94, 100, 101], appear in a vast number of applications such as in numerical analysis [30, 62, 97, 98], approximation theory [31, 64], time-frequency and wavelet analysis [31, 70, 93], sampling theory [1, 2, 5, 31], pseudo differential operators and differential equations [14, 52, 65, 94], or frames [1, 3, 4, 7, 44, 51, 50].

The main contribution of the first paper is a list of several new examples of Wiener pairs $\mathcal{A} \subseteq \mathcal{B}(\ell^2(X; \mathcal{H}))$, where (except for the algebra $\mathcal{C}_\nu$ and its anisotropic variation) X can be any relatively separated index set in $\mathbb{R}^d$. Besides their appearance in the theory of localized operator-valued frames (discussed in the second paper), we are convinced that our new examples of such Wiener pairs provide a useful arsenal of techniques for operator theory [38], the study of Fourier series of operators [43], or quantum harmonic analysis [104, 36].

The first paper is structured as follows. Case by case, we consider a class $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$-valued matrices and prove that it is a Banach space with respect to component-wise addition (in $\mathcal{B}(\mathcal{H})$) and a norm that is built into the definition of the class $\mathcal{A}$. We also examine for which classes of weight functions and values of p it holds $\mathcal{A} \subset \mathcal{B}(\ell^p_\omega(X; \mathcal{H}))$, which is relevant for the definition of an $(\mathcal{A}, p_0)$-admissible weight (see the second paper or Definition 2.7). For each such class $\mathcal{A}$, we obtain $\mathcal{A} \subset \mathcal{B}(\ell^2(X; \mathcal{H}))$ as a special case. As a consequence, we may define a multiplication in $\mathcal{A}$ by matrix multiplication (corresponding to the composition of operators in $\mathcal{B}(\ell^2(X; \mathcal{H}))$) and an involution in $\mathcal{A}$ via $([A_{k,l}]_{k,l\in X})^* := [A^*_{k,l}]^t_{k,l\in X}$ (which corresponds to taking the adjoint in $\mathcal{B}(\ell^2(X; \mathcal{H}))$). As a next step, we show that $\mathcal{A}$ is a unital Banach *-algebra with respect to the unit element $\operatorname{diag}(\mathcal{I}_{\mathcal{B}(\mathcal{H})})_{k\in X}$. Finally, we show the inverse-closedness of $\mathcal{A}$ in $\mathcal{B}(\ell^2(X;\mathcal{H}))$, which is the main theorem regarding the class $\mathcal{A}$.

For each class $\mathcal{A}$ introduced, proving its inverse-closedness in $\mathcal{B}(\ell^2(X;\mathcal{H}))$ is by far the most difficult property to achieve. In principal, inverse-closedness is an algebraic property. However, since $\mathcal{B}(\ell^2(X;\mathcal{H}))$ is a symmetric Banach *-algebra, a powerful result by Hulanicki allows for an analytic treatment of the verification of inverse-closedness.

Proposition 4.2 (Hulanicki's Lemma)**.** *[68] Let $\mathcal{A} \subseteq \mathcal{B}$ be a pair of unital Banach *-algebras with common identity, and suppose that $\mathcal{B}$ is symmetric. Then the following are equivalent:*

(i) $\mathcal{A}$ *is inverse-closed in* $\mathcal{B}$*.*

(ii) $r_{\mathcal{A}}(A) = r_{\mathcal{B}}(A) \qquad (\forall A = A^* \in \mathcal{A})$.

(iii) $r_{\mathcal{A}}(A) \leq r_{\mathcal{B}}(A) \qquad (\forall A = A^* \in \mathcal{A})$.

Moreover, if one of these conditions holds, then $\mathcal{A}$ is also symmetric.

Hulanicki's Lemma together with Gelfand's formula for the spectral radius

$$r_{\mathcal{A}}(A) = \lim_{n\to\infty} \|A^n\|_{\mathcal{A}}^{\frac{1}{n}}$$

are the key tools for our analysis. Some proofs and techniques crucial for the first paper can be adapted directly from the scalar-valued case. On the other hand, we had to overcome a number of issues, which are not present at all in the scalar-valued setting: First, when applying a $\mathcal{B}(\mathcal{H})$-valued matrix $[A_{k,l}]_{k,l\in X}$ to an $\mathcal{H}$-valued sequence $(g_l)_{l\in X}$, their respective entries belong to different spaces, whereas in the scalar-valued setting, both the matrix entries and the sequence entries are scalars. Secondly, the norm in $\mathcal{B}(\mathcal{H})$ is no longer multiplicative as in the case $\mathcal{B}(\mathbb{C}) = \mathbb{C}$ and we also do not have $\|A_{k,l}g_l\| = \|A_{k,l}\|\|g_l\|$ in general. These differences may seem innocent, but at some places they are not. For example, due to this reason, the operator norms on ℓ^∞ and ℓ^1 are no longer given by the largest row or column sum of a matrix. This suddenly makes it much more difficult to prove certain properties that are obvious in the scalar-valued case, such as the Banach space property of the Schur class.

Before we turn to the specific types of algebras considered in the first paper, we need to discuss various classes of weight functions. For more details on weight functions and their relevance in Banach algebra theory, we refer the reader to [54].

A *weight function*, or simply *weight*, is a positive function $\nu : \mathbb{R}^d \longrightarrow (0,\infty)$. It is called *submultiplicative*, if

$$\nu(x + x') \leq \nu(x)\nu(x') \qquad (\forall x, x' \in \mathbb{R}^d),$$

and *symmetric* if

$$\nu(x) = \nu(-x) \qquad (\forall x \in \mathbb{R}^d).$$

A weight m is called ν*-moderate* if there exists some constant $C > 0$ such that

$$m(x + x') \leq C m(x)\nu(x') \qquad (\forall x, x' \in \mathbb{R}^d).$$

We say that ν satisfies the *GRS-condition* (Gelfand-Raikov-Shilov condition [46]), if

$$\lim_{n\to\infty}(\nu(nz))^{\frac{1}{n}} = 1 \qquad (\forall z \in \mathbb{R}^d).$$

Typical examples of weights are the functions $\nu(x) = e^{\alpha|x|^\beta}(1+|x|)^s$. As noted in [14, Example 5.4], these weights are always symmetric, and they are submultiplicative whenever $0 \leq \beta \leq 1$, $\alpha, s \geq 0$. Under these conditions, ν satisfies the GRS-condition if and only if $0 \leq \beta < 1$.

Particularly relevant for the discussion of Schur-type algebras are *admissible* weights, which should not be confused with $(\mathcal{A}, p_0)$-admissible weights discussed in the second paper.

Definition 4.3. *[58] A weight function ν on $\mathbb{R}^d$ is called* admissible*, if*

(a) ν is of the form $\nu(x) = e^{\rho(\|x\|)}$, where $\rho : [0,\infty) \longrightarrow [0,\infty)$ is a continuous and concave function with $\rho(0) = 0$, and $\| \,.\, \|$ any norm on $\mathbb{R}^d$.

(b) ν satisfies the GRS-condition.

Admissible weights are submultiplicative, symmetric, and satisfy $\nu(0) = 1$ and $\nu \geq 1$ on $\mathbb{R}^d$. Examples of admissible weights are weights of the form

$$\nu(x) = e^{\alpha|x|^\beta}(1+|x|)^s(\log(e+|x|))^t)$$

with $s \geq 0$, $\alpha > 0$, $\beta \in (0,1)$ and $t \geq 0$.

4.1. Weighted Schur-type algebras

Let $X \subseteq \mathbb{R}^d$ be a relatively separated and ν be a weight. The *weighted Schur class* $\mathcal{S}^1_\nu = \mathcal{S}^1_\nu(X)$ is defined as the space of all $\mathcal{B}(\mathcal{H})$-valued matrices $A = [A_{k,l}]_{k,l\in X}$ for which

$$\|A\|_{\mathcal{S}^1_\nu} := \max\left\{\sup_{k\in X}\sum_{l\in X}\|A_{k,l}\|\nu(k-l)\,,\,\sup_{l\in X}\sum_{k\in X}\|A_{k,l}\|\nu(k-l)\right\} \tag{4.1}$$

is finite. It is easy to see that (4.1) defines a norm on $\mathcal{S}^1_\nu$.

We first present a vector-valued version of Schur's test [49].

Proposition 4.4. *For every ν-moderate weight m it holds $\mathcal{S}^1_\nu \subset \mathcal{B}(\ell^p_m(X;\mathcal{H}))$ for all $1 \leq p \leq \infty$. In particular, $\mathcal{S}^1_\nu \subset \mathcal{B}(\ell^2(X;\mathcal{H}))$.*

Proposition 4.5. *For every submultiplicative and symmetric weight ν, $(\mathcal{S}^1_\nu, \|\cdot\|_{\mathcal{S}^1_\nu})$ is a unital Banach *-algebra.*

The main result of this section is the inverse-closedness of $\mathcal{S}^1_\nu$ in $\mathcal{B}(\ell^2(X;\mathcal{H}))$ for admissible weights ν that satisfy a weak growth condition. As a first step, we prove this result for slowly growing polynomial weights. This result is a vector-valued version of Barnes' Lemma [10], see also [58, Lemma 5].

Theorem 4.6. *Let $\nu_\delta(x) = (1+|x|)^\delta$ for some $\delta \in (0,1]$. Then*

$$r_{\mathcal{S}^1_{\nu_\delta}}(A) = r_{\mathcal{S}^1}(A) = r_{\mathcal{B}(\ell^2(X;\mathcal{H}))}(A) \qquad (\forall A = A^* \in \mathcal{S}^1_{\nu_\delta}).$$

In particular, $\mathcal{S}^1_{\nu_\delta}$ is inverse-closed in $\mathcal{B}(\ell^2(X;\mathcal{H}))$ and $\mathcal{S}^1_{\nu_\delta}$ is a symmetric Banach algebra.

Employing the remaining properties of an admissible weight that have not been used so far, we may adapt ideas from [58] to obtain the following result.

Theorem 4.7. *Let ν be an admissible weight satisfying the weak growth condition*

$$\nu(x) \geq C(1+|x|)^\delta \qquad \text{for some } \delta \in (0,1], C > 0.$$

Then

$$r_{\mathcal{S}^1_\nu}(A) = r_{\mathcal{B}(\ell^2(X;\mathcal{H}))}(A) \qquad (\forall A = A^* \in \mathcal{S}^1_\nu).$$

In particular, $\mathcal{S}^1_\nu$ is inverse-closed in $\mathcal{B}(\ell^2(X;\mathcal{H}))$ and $\mathcal{S}^1_\nu$ is a symmetric Banach algebra.

We note, that the weak growth condition in Theorem 4.7 is essential. In fact, Tessera proved in [103] that the unweighted scalar-valued Schur algebra $\mathcal{S}^1(\mathbb{N})$ is *not* inverse-closed in $\mathcal{B}(\ell^p(\mathbb{N};\mathbb{C}))$ for each $1 < p < \infty$. In fact, $\mathcal{S}^1(\mathbb{N})$ is not even a symmetric Banach algebra [74].

4.2. Polynomial and more general off-diagonal decay conditions

Let $X \subset \mathbb{R}^d$ be relatively separated and ν be an admissible weight. Then we define $\mathcal{J}_\nu = \mathcal{J}_\nu(X)$ to be the space of all $\mathcal{B}(\mathcal{H})$-valued matrices $A = [A_{k,l}]_{k,l\in X}$ for which

$$\|A\|_{\mathcal{J}_\nu} := \sup_{k,l\in X} \|A_{k,l}\| \cdot \nu(k-l) < \infty. \tag{4.2}$$

It is easy to see that (4.2) indeed defines a norm on $\mathcal{J}_\nu$ and that $(\mathcal{J}_\nu, \|\,.\,\|_{\mathcal{J}_\nu})$ is a Banach space.

Lemma 4.8. *Let ν be an admissible weight. If*

$$\sup_{k\in X} \sum_{l\in X} \nu(k-l)^{-1} < \infty \tag{4.3}$$

then $\mathcal{J}_\nu$ is contained in $\mathcal{S}^1$ and consequently also in $\mathcal{B}(\ell^2(X;\mathcal{H}))$. If additionally $\exists C > 0$ such that

$$\sum_{n\in X} \nu(k-n)^{-1}\nu(n-l)^{-1} < C\nu(k-l) \qquad (\forall k,l \in X) \tag{4.4}$$

*then $\mathcal{J}_\nu$ is a unital *-algebra.*

The polynomial weight $\nu_s(x) = (1+|x|)^s$ on $\mathbb{R}^d$ with $s > d$ is admissible and satisfies conditions (4.3) and (4.4). In this case, we call $\mathcal{J}_s := \mathcal{J}_{\nu_s}$ the Jaffard algebra.

The norm $\|\,.\,\|_{\mathcal{J}_\nu}$ is not submultiplicative. This obstacle can be circumvented by working with the equivalent norm

$$|||A|||_{\mathcal{J}_\nu} := \sup_{\substack{B\in\mathcal{J}_\nu \\ \|B\|_{\mathcal{J}_\nu}=1}} \|A\cdot B\|_{\mathcal{J}_\nu} \qquad (A \in \mathcal{J}_\nu).$$

Next, we consider the $\mathcal{B}(\mathcal{H})$-valued version of the Banach algebra $\mathcal{B}_{u,s}$ from [58]. Let $s > 0$, u be an admissible weight and ν be the admissible weight defined by $\nu(x) = u(x)(1+|x|)^s$. Then we define $\mathcal{B}_{u,s} = \mathcal{B}_{u,s}(X)$ to be the Banach space $\mathcal{B}_{u,s} := \mathcal{S}^1_u \cap \mathcal{J}_\nu$ equipped with the norm

$$\|A\|_{\mathcal{B}_{u,s}} := 2^s\|A\|_{\mathcal{S}^1_u} + \|A\|_{\mathcal{J}_\nu}. \tag{4.5}$$

It can be shown that $\mathcal{B}_{u,s}$ is a unital Banach *-algebra and $\mathcal{B}_{u,s} \subseteq \mathcal{S}^1_u \subseteq \mathcal{B}(\ell^2(X;\mathcal{H}))$. Moreover, if $s > d$ then $\mathcal{J}_\nu \subseteq \mathcal{S}^1_u$ and thus $\mathcal{B}_{u,s} = \mathcal{J}_\nu$ with equivalent norms.

The main theorem of this section is the following analog of [58, Thm. 10]).

Theorem 4.9. *Assume that u is an admissible weight, let $\delta, s > 0$, $u \geq \nu_\delta$ and set $\nu = u\nu_s$. Then*

$$r_{\mathcal{B}_{u,s}}(A) = r_{\mathcal{B}(\ell^2(X;\mathcal{H}))}(A) \qquad (\forall A = A^* \in \mathcal{B}_{u,s}). \tag{4.6}$$

In particular, $\mathcal{B}_{u,s}$ is inverse-closed in $\mathcal{B}(\ell^2(X;\mathcal{H}))$ and $\mathcal{B}_{u,s}$ is a symmetric Banach algebra.

Corollary 4.10. *For every $s > d$, the Jaffard algebra $(\mathcal{J}_s, |||\,.\,|||_{\mathcal{J}_s})$ is inverse-closed in $\mathcal{B}(\ell^2(X;\mathcal{H}))$. In particular, $(\mathcal{J}_s, |||\,.\,|||_{\mathcal{J}_s})$ is a symmetric Banach algebra whenever $s > d$.*

4.3. The Baskakov-Gohberg-Sjöstrand algebra

In this section we consider Baskakov-Gohberg-Sjöstrand algebra $\mathcal{C}_\nu$ of convolution-dominated matrices. Many of the results of this section are either not new, or are simple adaptations of the scalar-valued case. They have been included in the first publication to give the reader an overview of the techniques used.

For a weight function ν on $\mathbb{R}^d$, let $\mathcal{C}_\nu$ be the space of $\mathcal{B}(\mathcal{H})$-valued matrices $A = [A_{k,l}]_{k,l\in\mathbb{Z}^d}$ for which

$$\|A\|_{\mathcal{C}_\nu} := \sum_{l\in\mathbb{Z}^d} \sup_{k\in\mathbb{Z}^d} \|A_{k,k-l}\|\nu(l) < \infty. \tag{4.7}$$

It is straightforward to show that (4.7) defines a norm on $\mathcal{C}_\nu$.

Proposition 4.11. *Let m be a ν-moderate weight. Then each $A \in \mathcal{C}_\nu$ defines a bounded operator on $\mathcal{B}(\ell^p_m(X;\mathcal{H}))$ for each $1 \leq p \leq \infty$. In particular, $\mathcal{C}_\nu \subseteq \mathcal{B}(\ell^2(X;\mathcal{H}))$.*

Proposition 4.12. *[77] For any submultiplicative and symmetric weight ν, $\mathcal{C}_\nu$ is a unital Banach *-algebra.*

In case ν is submultiplicative, symmetric and satisfies the GRS-condition, one can show that $\mathcal{C}_\nu \subseteq \mathcal{B}(\ell^2(\mathbb{Z}^d;\mathcal{H}))$ is a Wiener pair. The key ingredient for the proof of the latter is a generalization of the classical Wiener's Lemma [105] to its analogue on absolutely convergent Fourier series with coefficients in a (possibly non-commutative) Banach algebra by Bochner and Phillips [18, Theorem 1].

Theorem 4.13. *[12, 77] Let ν be a submultiplicative and symmetric weight satisfying the GRS-condition. Then $\mathcal{C}_\nu \subseteq \mathcal{B}(\ell^2(\mathbb{Z}^d;\mathcal{H}))$ forms a Wiener pair. In particular, $\mathcal{C}_\nu$ is a symmetric Banach algebra.*

4.4. Anisotropic decay conditions

In this section, we will derive further examples of Wiener pairs from a given Wiener pair $\mathcal{A} \subseteq \mathcal{B}(\ell^2(X;\mathcal{H}))$, where we assume $\mathcal{A}$ to be *solid*, meaning that if $A = [A_{k,l}]_{k,l\in X} \in \mathcal{A}$ and $B = [B_{k,l}]_{k,l\in X}$ is a $\mathcal{B}(\mathcal{H})$-valued matrix such that $\|B_{k,l}\| \le \|A_{k,l}\|$ for all $k,l \in X$, then $B \in \mathcal{A}$ and $\|B\|_\mathcal{A} \le \|A\|_\mathcal{A}$.

The proof the main theorem of this section is based on ideas from [64]. The key tools for its proof are (sets of commuting) derivations on $\mathcal{A}$ and their associated derivation subalgebras of $\mathcal{A}$ and the fact that commutators of matrices define derivations.

Theorem 4.14. *Let $\mathcal{A} \subseteq \mathcal{B}(\ell^2(X;\mathcal{H}))$ be a Wiener pair and assume that $\mathcal{A}$ is solid. Then for every $\alpha = (\alpha_j)_{j=1}^d \in \mathbb{N}_0^d$, the class of $\mathcal{B}(\mathcal{H})$-valued matrices $A = [A_{k,l}]_{k,l\in X}$ satisfying*

$$\left\| \left[\prod_{j=1}^d (1+|k_j-l_j|)^{\alpha_j} A_{k,l} \right]_{k,l\in X} \right\|_\mathcal{A} < \infty \tag{4.8}$$

*is a solid unital Banach *-algebra, which is inverse-closed in $\mathcal{B}(\ell^2(X;\mathcal{H}))$ and in particular a symmetric Banach algebra.*

Since all of our previously considered examples of inverse-closed sub-algebras $\mathcal{A}$ of $\mathcal{B}(\ell^2(X;\mathcal{H}))$ are solid, the above theorem applies to each of these cases. As a consequence, we obtain the following results.

Corollary 4.15. *For every $s > d$ and $\alpha = (\alpha_j)_{j=1}^d \in \mathbb{N}_0^d$, the class of $\mathcal{B}(\mathcal{H})$-valued matrices $[A_{k,l}]_{k,l\in X}$ satisfying the anisotropic polynomial decay condition*

$$\|A_{k,l}\| \le C(1+|k-l|)^{-s} \prod_{j=1}^d (1+|k_j-l_j|)^{-\alpha_j} \qquad (\forall k,l \in X)$$

*is a solid unital Banach *-algebra which is inverse-closed in $\mathcal{B}(\ell^2(X;\mathcal{H}))$ and thus symmetric.*

Corollary 4.16. *Let ν be an admissible weight in the sense of Definition 4.3 which satisfies the weak growth condition*

$$\nu(x) \ge C(1+|x|)^\delta \qquad \textit{for some } \delta \in (0,1], C > 0.$$

Then for each $\alpha = (\alpha_j)_{j=1}^d \in \mathbb{N}_0^d$, the class of $\mathcal{B}(\mathcal{H})$-valued matrices $[A_{k,l}]_{k,l\in X}$ satisfying the anisotropic Schur-type conditions

$$\sup_{k\in X} \sum_{l\in X} \prod_{j=1}^{d} (1+|k_j - l_j|)^{\alpha_j} \|A_{k,l}\| \nu(k-l) < \infty,$$
$$\sup_{l\in X} \sum_{k\in X} \prod_{j=1}^{d} (1+|k_j - l_j|)^{\alpha_j} \|A_{k,l}\| \nu(k-l) < \infty$$

*is a solid unital Banach *-algebra which is inverse-closed in $\mathcal{B}(\ell^2(X;\mathcal{H}))$ and thus symmetric.*

Corollary 4.17. *For every submultiplicative and symmetric weight satisfying the GRS-condition and every $\alpha = (\alpha_j)_{j=1}^d \in \mathbb{N}_0^d$, the class of $\mathcal{B}(\mathcal{H})$-valued matrices $[A_{k,l}]_{k,l\in\mathbb{Z}^d}$ satisfying the condition*

$$\sum_{l\in\mathbb{Z}^d} \prod_{j=1}^{d} (1+|l_j|)^{\alpha_j} \sup_{k\in\mathbb{Z}^d} \|A_{k,k-l}\| \nu(l) < \infty$$

*is a solid unital Banach *-algebra which is inverse-closed in $\mathcal{B}(\ell^2(\mathbb{Z}^d;\mathcal{H}))$ and thus symmetric.*

To the best of our knowledge, the Wiener pairs from Corollaries 4.16 and 4.17 are new even in the scalar-valued setting.

5. Localization of operator-valued frames

In the second paper, we study the concept of localized g-frames. The two main features of localized g-frames are: (1) intrinsic localization of a g-frame is preserved under canonical duality, and (2) to every mutually localized pair of dual g-frames, there is a whole chain of associated co-orbit spaces attached to it and g-frame reconstruction is also valid in each of those spaces.

At the core of this research are the properties of a spectral algebra.

Definition 5.1. *Let $\mathcal{A} = \mathcal{A}(X)$ be a unital Banach *-algebra of $\mathcal{B}(\mathcal{H})$-valued matrices $A = [A_{k,l}]_{k,l\in X}$. Then $\mathcal{A}$ is called a* solid spectral matrix algebra*, if*

*(1.) $\mathcal{A} \subseteq \mathcal{B}(\ell^2(X;\mathcal{H}))$, meaning that the inclusion map $\iota : \mathcal{A} \longrightarrow \mathcal{B}(\ell^2(X;\mathcal{H}))$ is a unital *-monomorphism (i.e. a faithful representation).*

(2.) $\mathcal{A}$ is inverse-closed *in $\mathcal{B}(\ell^2(X;\mathcal{H}))$, i.e., if $A \in \mathcal{A}$ defines an invertible operator in $\mathcal{B}(\ell^2(X;\mathcal{H}))$, then its inverse $A^{-1} \in \mathcal{B}(\ell^2(X;\mathcal{H}))$ is contained in $\mathcal{A}$ as well,*

(3.) $\mathcal{A}$ is solid*: if $A = [A_{k,l}]_{k,l\in X} \in \mathcal{A}$ and if $B = [B_{k,l}]_{k,l\in X}$ is a $\mathcal{B}(\mathcal{H})$-valued matrix such that $\|B_{k,l}\| \leq \|A_{k,l}\|$ for all $k,l \in X$, then $B \in \mathcal{A}$ and $\|B\|_{\mathcal{A}} \leq \|A\|_{\mathcal{A}}$.*

For brevity reason, we will simply call such an algebra $\mathcal{A}$ a spectral algebra.

All algebras discussed in the first paper are spectral algebras. With these examples in mind, we give the central definition of the second article.

Definition 5.2. *Let $\mathcal{A} = \mathcal{A}(X)$ be a spectral algebra and $T = (T_k)_{k\in X}$ be a $\mathcal{B}(\mathcal{H})$-valued family indexed by X. We say that T is* intrinsically $\mathcal{A}$-localized*, in short $T \sim_{\mathcal{A}} T$, if its associated g-Gram matrix $G_T = [T_k T_l^*]_{k,l\in X}$ belongs to $\mathcal{A}$. If $U = (U_k)_{k\in X}$ is another $\mathcal{B}(\mathcal{H})$-valued family indexed by X, then we call U and T* mutually $\mathcal{A}$-localized*, in short $U \sim_{\mathcal{A}} T$, if the cross g-Gram matrix $G_{U,T} = [U_k T_l^*]_{k,l\in X}$ belongs to $\mathcal{A}$.*

A first concrete example of an intrinsically localized g-frame are g-Gabor systems for $L^2(\mathbb{R}^d)$. As already mentioned in the introduction, these are g-frames of the form $\mathcal{G}(T,X) = (\pi(k)T\pi(k)^*)_{k\in X}$, where $T \in \mathcal{B}(L^2(\mathbb{R}^d))$ is a fixed (window) operator [95].

Theorem 5.3. *Let $X \subset \mathbb{R}^{2d}$ be relatively separated, $s > 2d$, and $T \in \mathcal{B}(L^2(\mathbb{R}^d))$ be an operator from the space $\mathcal{B}_{\nu_s\otimes\nu_s}$ (c.f. [95]), that is, T of the form*

$$T = \sum_{n\in\mathbb{N}} \varphi_n \otimes \psi_n,$$

where $\{\varphi_n\}_{n\in\mathbb{N}}$ *and* $\{\psi_n\}_{n\in\mathbb{N}}$ *are sequences in* $M^1_{\nu_s}(\mathbb{R}^d)$ *with*

$$\sum_{n\in\mathbb{N}} \|\varphi_n\|_{M^1_{\nu_s}(\mathbb{R}^d)} \|\psi_n\|_{M^1_{\nu_s}(\mathbb{R}^d)} < \infty.$$

Then the g-Gabor system $\mathcal{G} = \mathcal{G}(T, X)$ *is intrinsically* $\mathcal{J}_s$*-localized. In particular,* $\mathcal{G}(T, X)$ *is a g-Bessel sequence for* $L^2(\mathbb{R}^d)$*.*

Sufficiency conditions for the window operator T guaranteeing the frame property of $\mathcal{G}(T, X)$ are given in [95].

Theorem 5.3 was the starting point of this line of research. This result made us confident that a localization concept for g-frames, as defined in Definition 5.2, is meaningful, and also served as an encouragement to study $\mathcal{B}(\mathcal{H})$-valued versions of the Jaffard algebra and other known examples of spectral algebras. Only later did we find out that the special case $\mathcal{A} = \mathcal{C}_\nu(\mathbb{Z}^d)$ of Definition 5.2 already appeared in [78, Definition 5.6].

The main contribution of this article is a systematic study of localized g-frames as defined above. Conceptually, the theory of localized g-frames is very similar to the theory of localized (vector) frames. The main difficulties were in dealing with the technical details of the space $\mathcal{H}^\infty_\omega(T^d, T)$ (and closing some small gaps in the literature also regarding the scalar-valued case) and other subtleties regarding the vector-/operator-valued setting, which are not present in the scalar-valued setting (such as in the duality theorem for the spaces $\mathcal{H}^p_\omega(T^d, T)$). We believe that the theory developed here will be relevant to operator theory [38], the study of Fourier series of operators [43], or quantum harmonic analysis [36, 104].

5.1. Intrinsic localization and duality

The proofs of the main results of this section are based on the properties of a spectral algebra and the following lemma.

Lemma 5.4. *Let* $T = (T_k)_{k\in X}$ *be a g-frame and* $\widetilde{T} = (T_k S_T^{-1})_{k\in X}$ *its canonical dual g-frame. Then* G_T *has closed range and*

$$G_{\widetilde{T}} = G_T^\dagger \qquad \text{and} \qquad G_{T,\widetilde{T}} = G_T G_T^\dagger. \tag{5.1}$$

Theorem 5.5. *Let* $T = (T_k)_{k\in X}$ *be an intrinsically* $\mathcal{A}$*-localized g-frame. Then the following hold:*

(i) The canonical dual g-frame $\widetilde{T} = (T_k S_T^{-1})_{k\in X}$ *is intrinsically* $\mathcal{A}$*-localized.*

(ii) T *and* $\widetilde{T}$ *are mutually* $\mathcal{A}$*-localized.*

Theorem 5.6. *The relation* $\sim_\mathcal{A}$ *is symmetric. Furthermore,* $\sim_\mathcal{A}$ *is an equivalence relation on the set of all intrinsically* $\mathcal{A}$*-localized g-frames indexed by* X*.*

5.2. Operator-valued frames and associated (quasi-) Banach spaces

Definition 5.7. *Let $\mathcal{A}$ be a spectral algebra and $0 < p_0 \leq 1$. A weight function ω is called* $(\mathcal{A}, p_0)$-admissible, *if every $A \in \mathcal{A}$ defines a bounded operator $\ell^p_\omega(X; \mathcal{H}) \longrightarrow \ell^p_\omega(X; \mathcal{H})$ for every $p \in (p_0, \infty] \cup \{1\}$.*

For each spectral algebra $\mathcal{A}$ discussed in the first paper, every ν-moderate weight ω is $(\mathcal{A}, 1)$-admissible, where ν is the weight built into the definition of $\mathcal{A}$. In the case of the Jaffard algebra $\mathcal{J}_s(X)$ ($X \subset \mathbb{R}^d$), for every $r \geq 0$ so that $s > d + r$, any ν_r-moderate weight ω is $(\mathcal{J}_s, \frac{d}{s-r})$-admissible, where $\nu_r(z) = (1 + |z|)^r$ is the polynomial weight on $\mathbb{R}^d$ [82].

Proposition 5.8. *Let $\mathcal{A}$ be a spectral algebra and ω be $(\mathcal{A}, p_0)$-admissible. Then both $1/\omega$ and the trivial weight 1 are $(\mathcal{A}, 1)$-admissible.*

Definition 5.9. *Let $\mathcal{A}$ be a spectral algebra, ω be an $(\mathcal{A}, p_0)$-admissible weight, $T = (T_k)_{k\in X}$ be a g-frame and $T^d = (T^d_k)_{k\in X}$ be a dual g-frame of T such that $T^d \sim_{\mathcal{A}} T$. Then for each $p \in (p_0, \infty) \cup \{1\}$ we define $\mathcal{H}^p_\omega(T^d, T)$ as the completion of $\mathcal{H}^{00}(T) = D_T(c_{00}(X; \mathcal{H}))$ with respect to the (quasi-)norm*

$$\| \cdot \|_{\mathcal{H}^p_\omega(T^d,T)} = \|C_{T^d} \cdot \|_{\ell^p_\omega(X;\mathcal{H})} = \|(T^d_k \cdot)_{k\in X}\|_{\ell^p_\omega(X;\mathcal{H})}.$$

Furthermore we analogously define $\mathcal{H}^0_\omega(T^d, T)$ as the completion of $\mathcal{H}^{00}(T)$ with respect to the norm $\| \,.\, \|_{\mathcal{H}^0_\omega(T^d,T)} = \|C_{T^d} \,.\, \|_{\ell^\infty_\omega(X;\mathcal{H})}$.

The key point for the analysis of the spaces $\mathcal{H}^p_\omega(T^d, T)$ is the following: Since $D_T \in \mathcal{B}(\ell^2(X; \mathcal{H}), \mathcal{H})$ and $D_T(\ell^2(X; \mathcal{H})) = \mathcal{H}$ for every g-frame T, $\mathcal{H}^{00}(T)$ is a dense subspace of $\mathcal{H}$, because $\ell^{00}(X; \mathcal{H})$ is a dense subspace of $\ell^2(X; \mathcal{H})$. Therefore, g-frame reconstruction is valid in $\mathcal{H}^{00}(T)$, which, by definition, is dense in $\mathcal{H}^p_\omega(T^d, T)$ (w.r.t the norm $\| \cdot \|_{\mathcal{H}^p_\omega(T^d,T)}$). Using the boundedness of $G_{T^d,T}$ on the spaces $\ell^p_\omega(X; \mathcal{H})$, we can also extend g-frame reconstruction, as well as the definition of the operators C_{T^d} and D_T to the spaces $\mathcal{H}^p_\omega(T^d, T)$ and $\ell^p_\omega(X; \mathcal{H})$ respectively.

In the article, we also give the definition of the space $\mathcal{H}^\infty_\omega(T^d, T)$. In this case, the definition of C_{T^d} and D_T is different and we do not argue directly via density, but rather via a type of weak*-density. We avoid the tedious technicalities of the space $\mathcal{H}^\infty_\omega(T^d, T)$ and its discussion here and refer instead directly to the research article included later.

If not explicitly stated otherwise, all results listed below also hold in the case $p = \infty$ (in a certain sense).

Proposition 5.10. *Let $\mathcal{A}$ be a spectral algebra, ω be an $(\mathcal{A}, p_0)$-admissible weight, $T = (T_k)_{k\in X}$ be a g-frame and $T^d = (T^d_k)_{k\in X}$ be a dual g-frame of T such that $T^d \sim_{\mathcal{A}} T$. Then $\mathcal{H}^p_\omega(T^d, T)$ is a Banach space for each $p \in [1, \infty) \cup \{0\}$, and a quasi-Banach space for $p_0 < p < 1$. Moreover, C_{T^d} extends via density to an isometry $\mathcal{H}^p_\omega(T^d, T) \longrightarrow \ell^p_\omega(X; \mathcal{H})$ for $p \in (p_0, \infty) \cup \{0, 1\}$.*

Proposition 5.11. *Let $\mathcal{A}$ be a spectral algebra, ω be an $(\mathcal{A}, p_0)$-admissible weight, $T = (T_k)_{k\in X}$ be a g-frame and $T^d = (T_k^d)_{k\in X}$ be a dual g-frame of T such that $T^d \sim_{\mathcal{A}} T$. Then, for each $p \in (p_0, \infty] \cup \{0, 1\}$, the synthesis operator*

$$\begin{aligned} D_T : \ell^p_\omega(X; \mathcal{H}) &\longrightarrow \mathcal{H}^p_\omega(T^d, T), \\ (g_l)_{l\in X} &\mapsto \sum_{l\in X} T_l^* g_l \end{aligned} \tag{5.2}$$

is bounded and surjective and the series (5.2) converges unconditionally in $\mathcal{H}^p_\omega(T^d, T)$. Furthermore, for all $p \in (p_0, \infty) \cup \{0, 1\}$,

$$\|f\|_{\mathcal{H}^p_\omega} \asymp \inf\{\|g\|_{\ell^p_\omega(X;\mathcal{H})} : g \in \ell^p_\omega(X; \mathcal{H}), f = D_T g\}.$$

Theorem 5.12. *Let $\mathcal{A}$ be a spectral algebra, ω be an $(\mathcal{A}, p_0)$-admissible weight, $T = (T_k)_{k\in X}$ be a g-frame and $T^d = (T_k^d)_{k\in X}$ be a dual g-frame of T such that $T^d \sim_{\mathcal{A}} T$. Then for each $p \in (p_0, \infty) \cup \{0, 1\}$ it holds*

$$f = \sum_{k\in X} T_k^* T_k^d f \qquad (\forall f \in \mathcal{H}^p_\omega(T^d, T))$$

with unconditional convergence in $\mathcal{H}^p_\omega(T^d, T)$.

Using the latter theorem, we are able to give an alternative description of the spaces $\mathcal{H}^p_\omega(T^d, T)$. The next result is only valid for $p < \infty$.

Proposition 5.13. *Let $\mathcal{A}$ be a spectral algebra, ω be an $(\mathcal{A}, p_0)$-admissible weight, $T = (T_k)_{k\in X}$ be a g-frame and $T^d = (T_k^d)_{k\in X}$ be a dual g-frame of T such that $T^d \sim_{\mathcal{A}} T$. Then for each $p \in (p_0, \infty) \cup \{0, 1\}$, $\mathcal{H}^p_\omega(T^d, T)$ is the completion of the space*

$$V^p_\omega(T^d, T) = \{f \in \mathcal{H} : C_{T^d} f \in \ell^p_\omega(X; \mathcal{H})\}$$

with respect to $\|\,.\,\|_{\mathcal{H}^p_\omega(T^d,T)}$.

For exponents $p < \infty$ and weights ω chosen in such a way that $\ell^p_\omega(X; \mathcal{H})$ is continuously embedded in $\ell^2(X; \mathcal{H})$, we obtain an even simpler description of the (quasi-)Banach spaces associated with localized g-frames.

Corollary 5.14. *Let $\mathcal{A}$ be a spectral algebra, ω be an $\mathcal{A}$-admissible weight, $T = (T_k)_{k\in X}$ be a g-frame and $T^d = (T_k^d)_{k\in X}$ be a dual g-frame of T such that $T^d \sim_{\mathcal{A}} T$. Let $p < \infty$ and ω be so that $\ell^p_\omega(X; \mathcal{H})$ is continuously embedded in $\ell^2(X; \mathcal{H})$. Then*

$$\mathcal{H}^p_\omega(T^d, T) = V^p_\omega(T^d, T).$$

So far, the spaces $\mathcal{H}^p_\omega(T^d, T)$ seem to be heavily dependent on the choice of the dual g-frame pair (T^d, T). However, $\mathcal{H}^p_\omega(T^d, T)$ is independent of the choice of the dual g-frame pair (T^d, T) as long as we stay in the same localization class $\sim_{\mathcal{A}}$. The proof of the following result relies on (almost) all previously discussed results.

Theorem 5.15. *Let $\mathcal{A}$ be a spectral algebra and ω be an $(\mathcal{A}, p_0)$-admissible weight. Let $T = (T_k)_{k\in X}$ and $U = (U_k)_{k\in X}$ be g-frames and assume that $T^d = (T_k^d)_{k\in X}$ is a dual g-frame of T such that $T^d \sim_{\mathcal{A}} T$, and that $U^d = (U_k^d)_{k\in X}$ is a dual g-frame of U such that $U^d \sim_{\mathcal{A}} U$. If $T^d \sim_{\mathcal{A}} U$ and $U^d \sim_{\mathcal{A}} T$, then, for each $p \in (p_0, \infty) \cup \{0, 1\}$,*

$$\mathcal{H}^p_\omega(T^d, T) = \mathcal{H}^p_\omega(U^d, U)$$

with equivalent norms.

Via the above theorem, we can extend Theorem 2.9 as follows if T and T^d also are intrinsically localized.

Theorem 5.16. *Let $\mathcal{A}$ be a spectral algebra and ω be an $(\mathcal{A}, p_0)$-admissible weight. Let $T = (T_k)_{k\in X}$ be an intrinsically $\mathcal{A}$-localized g-frame and assume that $T^d = (T_k^d)_{k\in X}$ is an intrinsically $\mathcal{A}$-localized dual g-frame of T such that $T^d \sim_{\mathcal{A}} T$. Then, for each $p \in (p_0, \infty) \cup \{0, 1\}$,*

$$\mathcal{H}^p_\omega(T^d, T) = \mathcal{H}^p_\omega(T, T^d)$$

with equivalent norms and

$$f = \sum_{k\in X} T_k^* T_k^d f = \sum_{k\in X} (T_k^d)^* T_k f \qquad (\forall f \in \mathcal{H}^p_\omega(T^d, T))$$

with unconditional convergence in $\mathcal{H}^p_\omega(T^d, T)$.

We explicitly emphasize that the above result contains the statement

$$\|(T_k^d f)_{k\in X}\|_{\ell^p_\omega(X;\mathcal{H})} \asymp \|(T_k f)_{k\in X}\|_{\ell^p_\omega(X;\mathcal{H})}.$$

Corollary 5.17. *Let $\mathcal{A}$ be a spectral algebra and ω be an $(\mathcal{A}, p_0)$-admissible weight. If $T = (T_k)_{k\in X}$ is an intrinsically $\mathcal{A}$-localized g-frame, then for each $p \in (p_0, \infty) \cup \{0, 1\}$,*

$$\mathcal{H}^p_\omega(\widetilde{T}, T) = \mathcal{H}^p_\omega(T, \widetilde{T})$$

with equivalent norms and

$$f = \sum_{k\in X} T_k^* T_k S_T^{-1} f = \sum_{k\in X} S_T^{-1} T_k^* T_k f \qquad (\forall f \in \mathcal{H}^p_\omega(\widetilde{T}, T))$$

with unconditional convergence.

Finally, we present the duality theorem. Note that the spaces $\mathcal{H}^q_{1/\omega}(T, T^d)$ that occur below are well-defined by Proposition 5.8.

Theorem 5.18. *Let $\mathcal{A}$ be a spectral algebra, ω be an $(\mathcal{A}, 1)$-admissible weight, $T = (T_k)_{k\in X}$ be an intrinsically $\mathcal{A}$-localized g-frame and $T^d = (T_k^d)_{k\in X}$ be a dual g-frame of T such that $T^d \sim_{\mathcal{A}} T$. Then*

$$\big(\mathcal{H}^p_\omega(T^d, T)\big)^* \simeq \mathcal{H}^q_{1/\omega}(T, T^d),$$

for all $p \in [1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$ and for $(p, q) = (0, 1)$. In particular,

$$\mathcal{H}^\infty_\omega(T^d, T) \simeq \big(\mathcal{H}^0_\omega(T^d, T)\big)^{**}.$$

6. Localized Frames without inequalities

In the third paper, we characterize the frame property of a suitably localized sequence in a Hilbert space $\mathcal{H}$ via nine other equivalent conditions on certain mapping properties of the canonical frame-related operators acting on or mapping into the co-orbit spaces associated to a fixed reference Riesz basis.

The main contribution of the third paper is twofold. First, we provide a characterization of abstract localized frames similarly as described above. As emphasized in the introduction, the frame property of a given sequence is often very hard to verify. Here, we present a list of nine additional new options that may simplify this task in certain situations. Second, we apply our characterization to shift-invariant spaces and obtain new conditions for stable sets of sampling in these spaces, thereby illustrating the usefulness of our characterization in terms of a first non-trivial example.

6.1. Main result

Let $\mathcal{H}^p(\varphi) := \mathcal{H}^p(\widetilde{\varphi}, \varphi)$, $\mathrm{Ran}_p(C_\psi) := C_\psi(\mathcal{H}^p(\varphi))$ and $\mathrm{Ran}_p(D_\psi) := D_\psi(\ell^p(X))$, where $\psi = (\psi_k)_{k\in X}$ is some countable family of vectors in $\mathcal{H}$. Here, in the definition of a spectral algebra, we require the additional property $\mathcal{A} \subset \mathcal{B}(\ell^p)$ for all $1 \leq p \leq \infty$, since all concrete examples of such $\mathcal{A}$ do so anyway.

Theorem 6.1. *Let $\mathcal{A}$ be a spectral algebra and $\varphi = (\varphi_k)_{k\in X}$ be an intrinsically $\mathcal{A}$-localized Riesz basis in $\mathcal{H}$. Let $\psi = (\psi_k)_{k\in X}$ be mutually $\mathcal{A}$-localized with respect to φ, and let $\omega = (\omega_k)_{k\in X}$ be given by*

$$\omega_k = \sum_{l\in X} \langle \psi_l, \varphi_k \rangle S_\varphi^{-1/2} \varphi_l, \qquad k \in X.$$

Then the following are equivalent.

(1) ψ is a frame for $\mathcal{H}$.

(2) S_ψ is invertible on $\mathcal{H}^1(\varphi)$.

(3) S_ψ is invertible on $\mathcal{H}^\infty(\varphi)$.

(4) $C_\psi : \mathcal{H}^\infty(\varphi) \to \ell^\infty(X)$ is injective and $\mathrm{Ran}_\infty(C_\psi)$ as well as $\mathrm{Ran}_\infty(D_\psi)$ are closed.

(5) $D_\psi : \ell^1(X) \to \mathcal{H}^1(\varphi)$ is surjective and $\mathrm{Ran}_1(C_\psi)$ is closed.

(6) $D_\omega : \ell^\infty(X) \to \mathcal{H}^\infty(\varphi)$ is injective and $\mathrm{Ran}_\infty(C_\omega)$ as well as $\mathrm{Ran}_\infty(D_\omega)$ are closed.

(7) $C_\omega : \mathcal{H}^1(\varphi) \to \ell^1(X)$ is surjective and $\mathrm{Ran}_1(D_\omega)$ is closed.

(8) G_ω is invertible on $\ell^1(X)$.

(9) G_ω is invertible on $\ell^\infty(X)$.

(10) ω is a Riesz sequence in $\mathcal{H}$.

In particular, if any of the above conditions is satisfied, then $\mathcal{H}^p(\psi) = \mathcal{H}^p(\varphi)$ with equivalent norms for all $1 \le p \le \infty$.

The above theorem was motivated by the article [53] on fourteen equivalent conditions *without inequalities* for a Gabor system over a lattice with window $0 \ne g \in M^1(\mathbb{R}^d)$ being a frame. The main result of [53] is that any such Gabor system $\mathcal{G} = \mathcal{G}(g, \Lambda)$ is a frame for $L^2(\mathbb{R}^d)$ if and only if the associated frame related operators satisfy certain mapping properties, for example, if the analysis operator $C_\mathcal{G} : M^\infty(\mathbb{R}^d) \to \ell^\infty(\Lambda)$ is injective. The proof of the latter relies on deep results for Gabor systems over lattices, such as the Ron-Shen duality principle [89], Wiener's Lemma for twisted convolutions [57], and a result on linear independence of time-frequency shifts [53]. As already mentioned in the Introduction, some of the tools and objects that appear in the Gabor setting are also present in the theory of abstract localized frames, which led to an investigation of this phenomenon in the abstract setting.

Although the proof ideas for Theorem 6.1 are to some extent similar to the ones of the main theorem in [53], Theorem 6.1 is in fact *not* a generalization of the latter. In the following, we point out some similarities and some major differences between the two.

First, as mentioned in the Introduction, the modulation spaces $M^p(\mathbb{R}^d)$ are intrinsically defined via the STFT and the Gaussian (or any other Schwartz function). In the abstract setting, however, the definition and many relevant properties of the spaces $\mathcal{H}^p(\varphi)$ depend on the frame property of some explicitly given localized frame φ. Therefore, we relate the family ψ to a fixed localized reference frame φ generating the co-orbit spaces $\mathcal{H}^p(\varphi)$, instead of working with the spaces $\mathcal{H}^p(\psi)$ directly. However, on the other hand, it be difficult, if not even impossible for a Gabor frame to be localized with respect to a localized Riesz basis due to several versions of the Balian-Low theorem [49, 59], see also [51, Section 5] for a discussion on that matter. Second, several closed range conditions appear in our setting but not in [53, Theorem 3.1]. In the third paper, we argue, why each of these closed range conditions is in fact necessary in our setting, since otherwise, the statement is simply not true. For Gabor frames, however, these conditions are automatically satisfied.

In the following, we provide an overview on the tools and techniques used in the proof of Theorem 6.1. The chain of implications that we prove is depicted in Figure 6.1.

(a) The family ω is a so-called *R-dual of type III* [96] and may be regarded as a substitute of the Ron-Shen dual [89] of a Gabor-frame. However, an R-dual associated with a Gabor frame on a lattice does not coincide with its Ron-Shen dual in general, see [25, 96]. In order to guarantee that ψ is a frame if and only if ω is a Riesz sequence, we need to assume that the reference frame φ is in fact a Riesz basis.

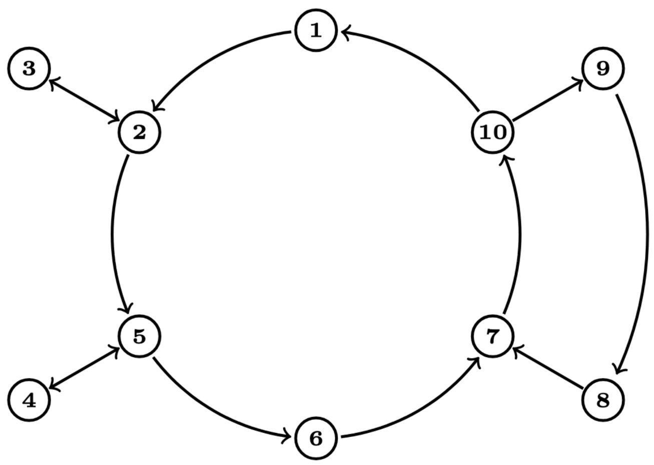

Figure 6.1.: Chain of implications for the proof of Theorem 6.1

(b) Since inverse-closedness of $\mathcal{A} \subseteq \mathcal{B}(\ell^2(X))$ is equivalent to the equality $\sigma_{\mathcal{A}}(A) = \sigma_{\mathcal{B}(\ell^2(X))}(A)$ of the spectrum of A in $\mathcal{A}$ and the spectrum of A in $\mathcal{B}(\ell^2(X))$ for all $A \in \mathcal{A}$ [55], we have as a consequence, that the holomorphic functional calculi with respect to $\mathcal{A}$ and $\mathcal{B}(\ell^2(X))$ coincide. Using this to our advantage we show that $\omega \sim_{\mathcal{A}} \varphi$.

(c) Via a series of auxiliary results, we show that all frame-related operators associated with ψ and ω are bounded on $\mathcal{H}^p(\varphi)$ or $\ell^p(X)$ respectively. These results are based on the assumption $\psi \sim_{\mathcal{A}} \varphi$ and the property $\omega \sim_{\mathcal{A}} \varphi$ from (b).

(c) Using the identification $(\mathcal{H}^1(\varphi))' = \mathcal{H}^\infty(\varphi)$ and the duality relations $D'_\psi = C_\psi, C'_\psi = D_\psi, G'_\psi = G_\psi, S'_\psi = S_\psi$ (by (c), analogous relations hold for the operators associated with ω), we prove several of the implications depicted in Figure 6.1 by arguing via duality, properties of adjoint operators and the closed range theorem for Banach spaces.

(d) At a certain point we use an interpolation result to obtain the Riesz sequence property of ω.

6.2. Application to shift-invariant spaces and sampling

Let T_k denote the translation operator on $L^p(\mathbb{R}^d)$ $(k \in \mathbb{R}^d)$, given by $T_k f(t) = f(t-k)$. For a generator $\varphi \in L^2(\mathbb{R}^d)$, let $V^p(\varphi)$ be the *shift-invariant space* defined by

$$V^p(\varphi) = \left\{ \sum_{k \in \mathbb{Z}^d} c_k T_k \varphi : c \in \ell^p(\mathbb{Z}^d) \right\} \qquad (1 \leq p \leq \infty).$$

Assuming that

(i) φ is continuous;

(ii) $|\varphi(x)| \lesssim (1+|x|)^{-s}$ for some $s > d$; and

(iii) $(T_k\varphi)_{k\in\mathbb{Z}^d}$ forms a Riesz basis for $V^2(\varphi)$,

then $V^p(\varphi)$ is a well-defined closed subspace of $L^p(\mathbb{R}^d)$ $(1 \le p \le \infty)$ and $V^2(\varphi)$ is a reproducing kernel Hilbert space with kernel K_x^φ, i.e.

$$f(x) = \langle f, K_x^\varphi \rangle \qquad (\forall f \in V^2(\varphi), \forall x \in \mathbb{R}^d). \tag{6.1}$$

It was shown in [51] that the Riesz basis $(T_k\varphi)_{k\in\mathbb{Z}^d}$ is intrinsically $\mathcal{J}_s(\mathbb{Z}^d)$-localized, that its associated co-orbit spaces $\mathcal{H}^p\big((T_k\varphi)_{k\in\mathbb{Z}^d}\big)$ coincide with the shift-invariant spaces $V^p(\varphi)$ $(1 \le p \le \infty)$ and that

$$|\langle K_x^\varphi, T_l\varphi\rangle| = |\varphi(x-l)| \lesssim (1+|x-l|)^{-s} \qquad (\forall x \in \mathbb{R}^d, \forall l \in \mathbb{Z}^d). \tag{6.2}$$

Let $X = (x_k)_{k\in\mathbb{Z}^d} \subset \mathbb{R}^d$ be discrete. If $\psi = (\psi_k)_{k\in\mathbb{Z}^d} := (K_{x_k}^\varphi)_{k\in\mathbb{Z}^d}$ forms a frame for $V^2(\varphi)$ with frame bounds $A \le B$, then X is called a *stable set of sampling* for $V^2(\varphi)$, since, due to (6.1), the frame inequality takes the form

$$A\|f\|^2_{L^2(\mathbb{R}^d)} \le \sum_{k\in\mathbb{Z}^d} |f(x_k)|^2 \le B\|f\|^2_{L^2(\mathbb{R}^d)} \qquad (\forall f \in V^2(\varphi))$$

in this case. Similarly, X is called a *stable set of sampling* for $V^p(\varphi)$ $(1 \le p \le \infty)$, if there exist constants $A_p \le B_p$ such that

$$A_p\|f\|_{L^p(\mathbb{R}^d)} \le \|(f(x_k))_{k\in\mathbb{Z}^d}\|_{\ell^p(\mathbb{Z}^d)} \le B_p\|f\|_{L^p(\mathbb{R}^d)} \qquad (\forall f \in V^p(\varphi)).$$

In case $X = (x_k)_{k\in\mathbb{Z}^d} \subset \mathbb{R}^d$ is relatively separated and satisfies the mild perturbation condition

$$|x_k - k| \le C, \qquad \text{for some } C > 0 \text{ and every } k \in \mathbb{Z}^d. \tag{6.3}$$

then we show in the article that ψ and $(T_k\varphi)_{k\in\mathbb{Z}^d}$ are mutually $\mathcal{J}_s(\mathbb{Z}^d)$-localized. Therefore, all the assumptions of Theorem 6.1 are met in this setting, and we obtain ten equivalent conditions for such a set X being a stable set of sampling. In the following, we reformulate only a part of the ten conditions in Theorem 6.1.

Theorem 6.2. *Assume that $\varphi \in L^2(\mathbb{R}^d)$ satisfies the conditions (i)–(iii), and let $(x_k)_{k\in\mathbb{Z}^d} \subset \mathbb{R}^d$ be relatively separated. Let G_ω be the autocorrelation matrix with entries given by*

$$[G_\omega]_{k,n} = \sum_{l\in\mathbb{Z}^d} \varphi(x_l - k)\overline{\varphi(x_l - n)} \qquad (k, n \in X)$$

and assume that condition (6.3) holds. Then the following are equivalent.

(a) $(x_k)_{k\in\mathbb{Z}^d}$ is a stable set of sampling for $V^2(\varphi)$.

(b) $(x_k)_{k\in\mathbb{Z}^d}$ *is a stable set of sampling for* $V^\infty(\varphi)$ *and* $D_\psi : \ell^\infty \to V^\infty(\varphi)$ *has closed range.*

(c) G_ω *is invertible on* $\ell^1(\mathbb{Z}^d)$*.*

(d) G_ω *is invertible on* $\ell^\infty(\mathbb{Z}^d)$*.*

(e) G_ω *is invertible on* $\ell^2(\mathbb{Z}^d)$*.*

In [61] several other equivalent conditions for a relatively separated set $(x_k)_{k\in\mathbb{Z}} \subset \mathbb{R}$ being a stable set of sampling for $V^2(\varphi)$ were proven. In fact, there it was shown that the closed range condition of $D_\psi : \ell^\infty \to V^\infty(\varphi)$ from (b) can actually be omitted (at least in the case $d = 1$). In the language of [61], the assumptions (i)−(iii) guarantee that φ is contained in the Wiener amalgam space $W_0 = W(C, \ell^1)$ and has *stable integer shifts*, and the authors employ a non-commutative version of Wiener's lemma ([60, Proposition 8.1]) in their analysis. In contrast to the latter, we assume the mild perturbation condition (6.3) in order to guarantee localization with respect to the Jaffard algebra $\mathcal{J}_s(\mathbb{Z}^d)$. In particular, the Wiener-type lemma that we use is Jaffard's Lemma, i.e. the inverse-closedness of $\mathcal{J}_s(\mathbb{Z}^d)$ in $\mathcal{B}(\ell^2(\mathbb{Z}^d))$. We also remark that the autocorrelation matrix G_ω is given by the product $G_\omega = P_X(\varphi)^* P_X(\varphi)$, where $P_X(\varphi) = (\varphi(x_l - k))_{l,k\in\mathbb{Z}}$ is the matrix appearing in [61, Theorem 3.1]. According to [61, Theorem 3.1] (in the case $d = 1$), stable sampling is equivalent to $P_\Gamma(\varphi) : \ell^\infty(\mathbb{Z}^d) \to \ell^\infty(\Gamma)$ being bounded and injective for *all* weak limits Γ of integer translates of $(x_k)_{k\in\mathbb{Z}}$. In contrast, conditions (c)−(e) from above only concern *one* specific operator G_ω and do not involve weak limits. To the best of our knowledge, these invertibility conditions on the autocorrelation matrix G_ω are new.

Bibliography

[1] A. Aldroubi, A. Baskakov, and I. Krishtal. Slanted matrices, Banach frames, and sampling. *Journal of Functional Analysis*, 255(7):1667 – 1691, 2008.

[2] N. Atreas, J. Benedetto, and C. Karanikas. Local sampling for regular wavelet and Gabor expansions. *Sampling Theory in Signal and Image Processing*, 2:2–24, 01 2003.

[3] R. Balan, P. Casazza, C. Heil, and Z. Landau. Density, overcompleteness, and localization of frames. I. Theory. *Journal of Fourier Analysis and Applications*, 12:105–143, 04 2006.

[4] R. Balan, P. G. Casazza, C. Heil, and Z. Landau. Density, overcompleteness, and localization of frames. II. Gabor systems. *Journal of Fourier Analysis and Applications*, 12(3):307–344, Jun 2006.

[5] R. Balan, J. Christensen, I. Krishtal, K. Okoudjou, and J. Romero. Multi-window Gabor frames in amalgam spaces. *Mathematical Research Letters*, 21(1):55–69, 2014.

[6] R. Balan and I. Krishtal. An almost periodic noncommutative Wiener's Lemma. *Journal of Mathematical Analysis and Applications*, 370:339–349, 10 2010.

[7] P. Balazs and K. Gröchenig. A guide to localized frames and applications to Galerkin-like representations of operators. In I. Pesenson, H. Mhaskar, A. Mayeli, Q. T. L. Gia, and D.-X. Zhou, editors, *Frames and Other Bases in Abstract and Function Spaces*, Applied and Numerical Harmonic Analysis series (ANHA). Birkhauser/Springer, 2017.

[8] P. Balazs, N. Holighaus, T. Necciari, and D. Stoeva. Frame Theory for Signal Processing in Psychoacoustics. In R. Balan, J. J. Benedetto, W. Czaja, and K. Okoudjou, editors, *Excursions in Harmonic Analysis Vol. 5,*, pages 225–268. Springer, International Publishing, 2017.

[9] P. Balazs, D. Stoeva, and J.-P. Antoine. Classification of General Sequences by Frame-Related Operators. *Sampl. Theory Signal Image Process.*, 10(2):151–170, 2011.

[10] B. A. Barnes. The spectrum of integral operators on Lebesgue spaces. *Journal of Operator Theory*, 18(1):115–132, 1987.

[11] A. G. Baskakov. Wiener's theorem and the asymptotic estimates of the elements of inverse matrices. *Funct. Anal. Appl.*, 24(3):222–224, 1990.

[12] A. G. Baskakov. Asymptotic estimates for the entries of the matrix of inverse operators and harmonic analysis. *Sibirian Mathematical Journal*, 38(1):10–22, 1997.

[13] A. G. Baskakov and I. A. Krishtal. Memory estimation of inverse operators. *Journal of Functional Analysis*, 267(8):2551–2605, 2014.

[14] A. G. Baskakov and I. A. Krishtal. Spectral properties of an operator polynomial with coefficients in a Banach algebra. In *Frames and harmonic analysis. AMS special session on frames, wavelets and Gabor systems and special session on frames, harmonic analysis, and operator theory, North Dakota State University, Fargo, ND, USA, April 16–17, 2016. Proceedings*, pages 93–114. Providence, RI: American Mathematical Society (AMS), 2018.

[15] A. G. Baskakov, I. A. Krishtal, and N. B. Uskova. Method of similar operators in harmonious Banach spaces, 2024.

[16] J. Benedetto, C. Heil, and D. Walnut. Differentiation And The Balian-Low Theorem. *Journal of Fourier Analysis and Applications*, 1:355–402, 01 1995.

[17] E. Berge. A brief introduction to the Feichtinger algebra $\mathbf{S}_0(\mathbb{R})$. 2021.

[18] S. Bochner and R. S. Phillips. Absolutely Convergent Fourier Expansions for Non-Commutative Normed rings. *Annals of Mathematics*, 43(3):409–418, 1942.

[19] B. Bodmann. Optimal linear transmission by loss-insensitive packet encoding. *Applied and Computational Harmonic Analysis*, 22:274–285, 2007.

[20] B. G. Bodmann, D. W. Kribs, and V. I. Paulsen. Decoherence-Insensitive Quantum Communications by Optimal C^*-Encoding. *IEEE Transactions on Information Theory*, 53(12):4738–4749, 2007.

[21] P. Casazza, O. Christensen, A. Lindner, and R. Vershynin. Frames and the Feichtinger conjecture. *Proceedings of the American Mathematical Society*, 133, 04 2005.

[22] P. Casazza, O. Christensen, and D. T. Stoeva. Frame expansions in separable Banach spaces. *J. Math. Anal. Appl.*, 307(2):710 – 723, 2005.

[23] P. Casazza and G. Kutyniok. Robustness of fusion frames under erasures of subspaces and local frame vectors. *Contemporary Math*, 464:149–160, 2008.

[24] P. G. Casazza and G. Kutyniok. Frames of subspaces. *Cont. Math.*, 2004.

[25] P. G. Casazza, G. Kutyniok, and M. C. Lammers. Duality principles in frame theory. *J. Fourier Anal. Appl.*, 10(4):383–408, 2004.

[26] P. G. Casazza, G. Kutyniok, and S. Li. Fusion frames and distributed processing. *Appl. Comput. Harmon. Anal.*, 254(1):114–132, 2008.

[27] P. G. Casazza, G. Kutyniok, S. Li, and C. J. Rozell. Modeling sensor networks with fusion frames. *Proceed. of SPIE - Wavelets XII*, 6701:11910–11913, 2007.

[28] O. Christensen. *An Introduction to Frames and Riesz Bases*. Birkhäuser, 2016.

[29] O. Christensen and D. T. Stoeva. p-frames in separable Banach spaces. *Advances in Computational Mathematics*, 18(2-4):117–126, 2003.

[30] O. Christensen and T. Strohmer. The finite section method and problems in frame theory. *Journal of Approximation Theory*, 133(2):221–237, 2005.

[31] E. Cordero and K. Gröchenig. Localization of frames II. *Applied and Computational Harmonic Analysis*, 17:29–47, 2004.

[32] M. S. Craig and R. L. Genter. Geophone array formation and semblance evaluation. *Society of Exploration Geophysicists*, 71, 01 2006.

[33] P. A. Dabhi and K. B. Solanki. On Hulanicki and Barnes lemmas for p-Banach algebras. In *Proc. Indian Acad. Sci. Math.*, volume 132, pages pp. Paper No. 73, 13, 2022.

[34] P. A. Dabhi and K. B. Solanki. Vector valued Beurling algebra analogues of Wiener's theorem. *Indian Journal of Pure and Applied Mathematics*, pages 1–14, 09 2023.

[35] I. Daubechies, A. Grossmann, and Y. Meyer. Painless non-orthogonal expansions. *Journal of Mathematical Physics*, 27:1271–1283, 1986.

[36] M. A. de Gosson. *Quantum Harmonic Analysis. An Introduction.* De Gruyter, Berlin, Boston, 2021.

[37] M. Dörfler, H. G. Feichtinger, and K. Gröchenig. Time-frequency partitions for the Gelfand triple (S_0, L^2, S_0'). *Mathematica Scandinavica*, 98:81–96, 01 2006.

[38] R. G. Douglas. *Banach Algebra Techniques in Operator Theory.* Springer, 1998.

[39] R. J. Duffin and A. C. Schaeffer. A class of nonharmonic Fourier series. *Trans. Amer. Math. Soc.*, 72:341–366, 1952.

[40] M. Faulhuber, I. Shafkulovska, and I. Zlotnikov. On the frame property of Hermite functions and exploration of their frame sets. *Journal of Fourier Analysis and Applications*, 31, 04 2025.

[41] H. Feichtinger, F. Luef, and E. Cordero. *Banach Gelfand Triples for Gabor Analysis*, volume 1949, pages 1–33. Springer, 08 2008.

[42] H. G. Feichtinger. On a new segal algebra. *Monatshefte für Mathematik*, 92:269–290, 1981.

[43] H. G. Feichtinger and W. Kozek. *Quantization of TF lattice-invariant operators on elementary LCA groups*, pages 233–266. Birkhäuser Boston, Boston, MA, 1998.

[44] M. Fornasier and K. Gröchenig. Intrinsic localization of frames. *Constr. Approx.*, 22(3):395–415, 2005.

[45] D. Freeman and D. Speegle. The discretization problem for continuous frames. *Adv. Math.*, 345:784 – 813, 2019.

[46] I. Gelfand, D. Raikov, and G. Shilov. *Commutative Normed Rings*. Chelsea Publishing Company, Bronx, New York, 1964.

[47] I. Gohberg, M. Kaashoek, and H. Woerdeman. The band method for positive and strictly contractive extension problems: An alternative version and new applications. *Integral Equations Operator Theory*, 12:343–382, 01 1989.

[48] K. Gröchenig. Describing functions: atomic decompositions versus frames. *Monatsh. Math.*, 112(3):1–41, 1991.

[49] K. Gröchenig. *Foundations of Time-Frequency Analysis*. Birkhäuser, Boston, 2001.

[50] K. Gröchenig. Localized frames are finite unions of Riesz sequences. *Adv. Comput. Math.*, 18:149–157, 02 2003.

[51] K. Gröchenig. Localization of frames, Banach frames, and the invertibility of the frame operator. *Journal of Fourier Analyis and Applications*, 10(2):105–132, 2004.

[52] K. Gröchenig. Time-frequency analysis of Sjöstrand's class. *Rev. Mat. Iberoam.*, 22(2):703–724, 2006.

[53] K. Gröchenig. Gabor frames without inequalities. *Int. Math. Res. Not. IMRN*, 2007(23):ID rnm111, 21, 2007.

[54] K. Gröchenig. *Weight functions in time-frequency analysis*, chapter 15, pages 343–366. American Mathematical Society, 11 2007.

[55] K. Gröchenig. *Wiener's lemma: Theme and variations. An introduction to spectral invariance and its applications.*, chapter 5, pages 175 – 234. Applied and Numerical Harmonic Analysis. Birkhäuser, 2010.

[56] K. Gröchenig. The mystery of Gabor frames. *Journal of Fourier Analysis and Applications*, 20:865–895, 08 2014.

[57] K. Gröchenig and M. Leinert. Wiener's lemma for twisted convolution and Gabor frames. *J. Amer. Math. Soc.*, 17(1):1–18, 2004.

[58] K. Gröchenig and M. Leinert. Symmetry and inverse-closedness of matrix algebras and functional calculus for infinite matrices. *Trans. Amer. Math. Soc.*, 358(6):2695–2711, 2006.

[59] K. Gröchenig and E. Malinnikova. Phase space localization of Riesz bases for $L^2(\mathbb{R}^d)$. *Rev. Mat. Iberoam.*, 29:115, 02 2013.

[60] K. Gröchenig, J. Ortega-Cerdà, and J. L. Romero. Deformation of Gabor systems. *Adv. Math.*, 277:388–425, 2015.

[61] K. Gröchenig, J.-L. Romero, and J. Stöckler. Sampling theorems for shift-invariant spaces, Gabor frames, and totally positive functions. *Invent. Math.*, 211:1119–1148, 2018.

[62] K. Gröchenig, Z. Rzeszotnik, and T. Strohmer. Convergence analysis of the finite section method and Banach algebras of matrices. *Integr. Equ. Oper. Theory*, 67(2):183–202, 2010.

[63] K. Gröchenig and J. Stöckler. Gabor frames and totally positive functions. *Duke Mathematical Journal*, 162(6):1003–1031, 2013.

[64] K. Gröchenig and A. Klotz. Noncommutative Approximation: Inverse-Closed Subalgebras and Off-Diagonal Decay of Matrices. *Constr Approx*, 32:429–466, 2010.

[65] K. Gröchenig and J. Toft. The range of localization operators and lifting theorems for modulation and bargmann-fock spaces. *Transactions of the American Mathematical Society*, 365, 10 2010.

[66] N. Hauschka, P. Balazs, and L. Köhldorfer. Banach distribution spaces for a Hilbert space. https://arxiv.org/abs/2502.07378, 2025.

[67] E. Hernández and G. Weiss. *A First Course on Wavelets.* CRC Press, Boca Raton – New York – London – Tokyo, 1996.

[68] A. Hulanicki. On the spectrum of convolution operators on groups with polynomial growth. *Invent. Math.*, 17:135–142, 1972.

[69] S. Iyengar and R. B. Eds. *Distributed Sensor Networks.* Chapman & Hall/CRC CRC Press, Baton Rouge, 2005.

[70] S. Jaffard. Propriétés des matrices "bien localisées" preé de leur diagonale et qualques applications. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 7(5):461–476, 1990.

[71] A. Janssen. Some Weyl-Heisenberg frame bound calculations. *Indagationes Mathematicae*, 7(2):165–183, 1996.

[72] A. Janssen. On generating tight Gabor frames at critical density. *Journal of Fourier Analysis and Applications*, 9:175–214, 03 2003.

[73] A. Janssen and T. Strohmer. Hyperbolic secants yield Gabor frames. *Applied and Computational Harmonic Analysis*, 12(2):259–267, 2002.

[74] J. W. Jenkins. Symmetry and nonsymmetry in the group algebras of discrete groups. *Pacific Journal of Mathematics*, 32:131–145, 1970.

[75] L. Köhldorfer and P. Balazs. Wiener pairs of Banach algebras of operator-valued matrices. *Journal of Mathematical Analysis and Applications*, 549(2):129525, 2025.

[76] L. Köhldorfer, P. Balazs, P. Casazza, S. Heineken, C. Hollomey, P. Morillas, and M. Shamsabadhi. *A Survey of Fusion Frames in Hilbert Spaces*, chapter 21. Springer Nature Switzerland, 2023.

[77] I. Krishtal. Wiener's lemma: Pictures at an exhibition. *Revista de la Unión Matemática Argentina*, 52(2):61–79, 01 2011.

[78] I. A. Krishtal. Wiener's lemma and memory localization. *Journal of Fourier Analysis and Applications*, 17(4):674–690, August 2011.

[79] V. Kurbatov. Some algebras of operators majorized by a convolution. *Funct. Differ. Equ.*, 8:323–333, 01 2001.

[80] G. Kutyniok, V. Paternostro, and F. Philipp. The effect of perturbations of frame sequences and fusion frames on their duals. *Oper. Matrices*, 11:301–336, 2017.

[81] L. Köhldorfer and P. Balazs. Localization of operator-valued frames. https://arxiv.org/abs/2503.24170, 2025.

[82] L. Köhldorfer and P. Balazs. On the inverse-closedness of operator-valued matrices with polynomial off-diagonal decay. https://arxiv.org/abs/2501.09603, 2025.

[83] Y. Lyubarskii. *Frames in the Bargmann space of entire functions*, pages 167–180. American Mathematical Society, Providence, 10 1992.

[84] I. J. Maddox. *Infinite Matrices of Operators*. Lecture Notes in Mathematics. Springer, Berlin, 1980.

[85] A. Marcus, D. Spielman, and N. Srivastava. Interlacing Families II: Mixed Characteristic Polynomials and the Kadison-Singer Problem. *Annals of Mathematics*, 182:327–350, 2015.

[86] M. Naimark. *Normed Rings*. Woters-Noordhoff, 1970.

[87] M. Naimark. *Normed Algebras: By M.A. Naimark*. Wolters-Noordhoff series of monographs and textbooks on pure and applied mathematics. Wolters-Noordhoff, 1972.

[88] H. Rauhut, K. Schnass, and P. Vandergheynst. Compressed sensing and redundant dictionaries. *IEEE Trans. Inform. Theory*, 54(5):2210–2219, 2008.

[89] A. Ron and Z. Shen. Weyl-Heisenberg frames and Riesz bases in $L^2(\mathbb{R}^d)$. *Duke Math. J.*, 89(2):237–282, 1997.

[90] C. Rozell and D. Johnson. Analyzing the robustness of redundant population codes in sensory and feature extraction systems. *Neurocomputing*, 69(10–12):1215–1218, 2006.

[91] K. Seip. Density theorems for sampling and interpolation in the Bargmann-Fock space I. *Bulletin of the American Mathematical Society*, 26:91–106, 05 1992.

[92] K. Seip and R. Wallstén. Density theorems for sampling and interpolation in the Bargmann-Fock space II. *Journal für die reine und angewandte Mathematik*, 429:107–114, 1992.

[93] C. Shin and Q. Sun. Stability of Localized Operators. *Journal of Functional Analysis*, 256, 11 2008.

[94] J. Sjöstrand. Wiener type algebras of pseudodifferential operators. *Séminaire Équations aux dérivées partielles (Polytechnique)*, pages 1–19, 1994-1995.

[95] E. Skrettingland. On gabor g-frames and fourier series of operators, 2020.

[96] D. T. Stoeva and O. Christensen. On R-duals and the duality principle in Gabor analysis. *J. Fourier Anal. Appl.*, 21:383–400, 2015.

[97] T. Strohmer. Rates of convergence for the approximation of dual shift-invariant systems in $\ell^2(\mathbb{Z})$. *Journal of Fourier analysis and applications*, 5(6):599–616, 1999.

[98] T. Strohmer. Four short stories about toeplitz matrix calculations. *Linear Algebra and its Applications*, 343-344:321–344, 2002. Special Issue on Structured and Infinite Systems of Linear equations.

[99] Q. Sun. Wiener's lemma for infinite matrices. *Trans. Amer. Math. Soc.*, 359(7):3099–3123, 2007.

[100] Q. Sun. Wiener's lemma for infinite matrices. *Trans. Amer. Math. Soc.*, 359(7):3099–3123, 2007.

[101] Q. Sun. Wiener's Lemma for Infinite Matrices II. *Constructive Approximation*, 34:209–235, 10 2011.

[102] W. Sun. G-frames and g-Riesz bases. *Journal of Mathematical Analysis and Applications*, 322(1):437 – 452, 2006.

[103] R. Tessera. The inclusion of the Schur algebra in $B(\ell^2)$ is not inverse-closed. *Monatshefte für Mathematik*, 164:115–118, 2009.

[104] R. Werner. Quantum harmonic analysis on phase space. *Journal of Mathematical Physics*, 25(5):1404–1411, 05 1984.

[105] N. Wiener. Tauberian theorems. *Annals of Mathematics*, 33(1):1–100, 1932.

Part III.

Publications

7. Paper 1

Bibliographic information:
L. Köhldorfer and P. Balazs. Wiener pairs of Banach algebras of operator-valued matrices. *Journal of Mathematical Analysis and Applications*, 549(2):129525, 2025. https://doi.org/10.1016/j.jmaa.2025.129525

Contribution:
Lukas Köhldorfer is the main author of the paper and the main contributor to its content.

J. Math. Anal. Appl. 549 (2025) 129525

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Regular Articles

Wiener pairs of Banach algebras of operator-valued matrices

Lukas Köhldorfer *, Peter Balazs

Acoustics Research Institute, Dominikanerbastei 16, Vienna, 1090, Austria

ARTICLE INFO

Article history:
Received 23 July 2024
Available online 26 March 2025
Submitted by A. Sims

Keywords:
Wiener pair
Inverse-closed
Spectral invariance
Matrix algebra
Operator-valued matrix

ABSTRACT

In this article we consider several new examples of Wiener pairs $\mathcal{A} \subseteq \mathcal{B}$, where $\mathcal{B} = \mathcal{B}(\ell^2(X;\mathcal{H}))$ is the Banach algebra of bounded operators acting on the Hilbert space-valued Bochner sequence space $\ell^2(X;\mathcal{H})$ and $\mathcal{A} = \mathcal{A}(X)$ is a Banach algebra consisting of operator-valued matrices indexed by some relatively separated set $X \subset \mathbb{R}^d$. In particular, we consider $\mathcal{B}(\mathcal{H})$-valued versions of the Jaffard algebra, of certain weighted Schur-type algebras, of Banach algebras which are defined by more general off-diagonal decay conditions than polynomial decay, of weighted versions of the Baskakov-Gohberg-Sjöstrand algebra, and of anisotropic variations of all of these matrix algebras, and show that they are inverse-closed in $\mathcal{B}(\ell^2(X;\mathcal{H}))$. In addition, we obtain that each of these Banach algebras is symmetric.

1. Introduction

From a historical point of view, both the title and the content of this article originate from *Wiener's Lemma on absolutely convergent Fourier series* [63], which states that if a continuous function $f \in C(\mathbb{T})$ on the torus $\mathbb{T}$ admits an absolutely convergent Fourier series and is nowhere vanishing, then the function $1/f$ admits an absolutely convergent Fourier series as well. In more abstract terms, Wiener's Lemma can be rephrased to a certain relation between the Banach algebra $\mathcal{A}(\mathbb{T})$ of absolutely convergent Fourier series and the Banach algebra $C(\mathbb{T})$ of continuous functions on $\mathbb{T}$. Indeed, $f \in \mathcal{A}(\mathbb{T}) \subseteq C(\mathbb{T})$ being nowhere vanishing is equivalent to $f \in \mathcal{A}(\mathbb{T})$ being invertible in the larger Banach algebra $C(\mathbb{T})$, and Wiener's Lemma guarantees that these two assumptions already imply that the continuous inverse $1/f \in C(\mathbb{T})$ of f is contained in $\mathcal{A}(\mathbb{T})$ as well, i.e. that f is also invertible in the smaller Banach algebra $\mathcal{A}(\mathbb{T})$.

Today, the above theme manifests itself in various kinds of situations and degrees of abstraction. The general definition is the following. If $\mathcal{A} \subseteq \mathcal{B}$ is a nested pair of two (possibly non-commutative) Banach algebras $\mathcal{A}$ and $\mathcal{B}$ with common identity, then $\mathcal{A}$ is called *inverse-closed* in $\mathcal{B}$ [8], if

$$A \in \mathcal{A} \text{ and } \exists A^{-1} \text{ in } \mathcal{B} \qquad \Longrightarrow \qquad A^{-1} \in \mathcal{A}. \tag{1}$$

* Corresponding author.
E-mail address: lukas.koehldorfer@oeaw.ac.at (L. Köhldorfer).

https://doi.org/10.1016/j.jmaa.2025.129525
0022-247X/

In this case we say that $\mathcal{A} \subseteq \mathcal{B}$ is a *Wiener pair* [51,52]. As in the example of the Wiener pair $\mathcal{A}(\mathbb{T}) \subseteq C(\mathbb{T})$, it is often easier to verify the invertibility of an element $A \in \mathcal{A} \subseteq \mathcal{B}$ in the larger algebra $\mathcal{B}$, than to verify its invertibility in the smaller algebra $\mathcal{A}$. Consequently, $\mathcal{A}$ being inverse-closed in $\mathcal{B}$ is a fairly strong property and thus immensely useful in a huge number of situations. In particular, Wiener's Lemma and its many generalizations, see, e.g., [5,6,9,10,13–16,21,22,29,32,38,42,46,53,54,58,59], appear in a vast number of applications such as in numerical analysis [19,34,56,57], approximation theory [20,37], time-frequency and wavelet analysis [20,42,53], sampling theory [1,2,5,20], pseudo differential operators and differential equations [14,32,39,54], or frames [1,3,4,7,27,31,35]. For more mathematical background and historical remarks on Wiener's Lemma and its variations, we refer the reader to the survey [33], which is one of the main inspirations of this work.

Our motivation for writing this article comes from studying localization properties of g-frames, which are a generalization of frames [18] to families of operators satisfying a frame-like inequality. More precisely, a countable family $T = (T_k)_{k\in X}$ of bounded operators $T_k \in \mathcal{B}(\mathcal{H})$ on some separable Hilbert space $\mathcal{H}$ is called a *g-frame* [60], if there exist positive constants $0 < A \leq B < \infty$, such that

$$A\|f\|^2 \leq \sum_{k\in X} \|T_k f\|^2 \leq B\|f\|^2 \qquad (\forall f \in \mathcal{H}). \tag{2}$$

Every g-frame $(T_k)_{k\in X}$ allows for a linear, bounded and stable reconstruction of any vector $f \in \mathcal{H}$ from the vector-valued family $(T_k f)_{k\in X} \in \ell^2(X;\mathcal{H})$, where

$$\ell^2(X;\mathcal{H}) = \left\{(g_k)_{k\in X} : g_k \in \mathcal{H}\ (\forall k \in X), (\|g_k\|)_{k\in X} \in \ell^2(X)\right\}.$$

In addition to their capability to achieve perfect reconstruction of vectors in $\mathcal{H}$, g-frames might satisfy other important properties that do not follow solely from the existence of the g-frame bounds A and B. For example, in [55], g-frames for $\mathcal{H} = L^2(\mathbb{R}^d)$ of the form $(\pi(k)T\pi(k)^*)_{k\in X}$, called *Gabor g-frames*, were considered, where $X = \Lambda \subseteq \mathbb{R}^{2d}$ is a full-rank lattice, $T \in \mathcal{B}(L^2(\mathbb{R}^d))$ is some suitable window operator, and $\pi(k) = M_{k_2}T_{k_1}$ denotes a time-frequency shift by $k = (k_1, k_2) \in \mathbb{R}^{2d}$ in the time-frequency plane [30]. Via Fourier methods for periodic operators, which ultimately depend on the group structure of the lattice Λ, Skrettingland proved in [55] several results for this class of g-frames, which are typically concluded from suitable localization properties of a given frame in some abstract Hilbert space $\mathcal{H}$ [27]. We are convinced that – in analogy to localized frames [27] – these localization-type results do not rely on Fourier methods or even on the group structure of the index set X, but are a consequence of the intrinsic localization properties of the underlying g-frame $(T_k)_{k\in X}$, which are measured by the decay of the operator norms $\|T_kT_l^*\|$ of the entries $T_kT_l^* \in \mathcal{B}(\mathcal{H})$ of the g-Gram matrix $G_T = [T_kT_l^*]_{k,l\in X} \in \mathcal{B}(\ell^2(X;\mathcal{H}))$. In fact, an article on g-frames, whose localization quality is measured by its g-Gram matrix belonging to some suitable Banach algebra $\mathcal{A}$, which is inverse-closed in the Banach algebra $\mathcal{B} = \mathcal{B}(\ell^2(X;\mathcal{H}))$ of bounded operators acting on the Hilbert space $\ell^2(X;\mathcal{H})$ (see also [45, Definition 5.7]), is currently in preparation [48].

In view of the above motivation, the aim of this article is to prove several new examples of Wiener pairs $\mathcal{A} \subseteq \mathcal{B}(\ell^2(X;\mathcal{H}))$, where $\mathcal{A} = \mathcal{A}(X)$ is a Banach algebra of $\mathcal{B}(\mathcal{H})$-valued matrices indexed by some relatively separated set $X \subset \mathbb{R}^d$. Beside the above-mentioned possibility to generalize the concept of intrinsically localized frames from [27] to the g-frame setting [48], we believe that our new examples of such Wiener pairs provide a useful arsenal of techniques for operator theory [24], the study of Fourier series of operators [26], or quantum harmonic analysis [23,62].

The paper is structured as follows. In Section 2 we make our notation precise and collect some preliminary facts which we will need later on. Each of the later sections is devoted to defining one specific class $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$-valued matrices and showing that $\mathcal{A}$ is indeed a Banach algebra which is inverse-closed in $\mathcal{B}(\ell^2(X;\mathcal{H}))$. In particular, we prove these results for certain weighted Schur-type algebras $\mathcal{S}^1_\nu(X)$ (see Section 3), for the Jaffard algebra $\mathcal{J}_s(X)$, which describes polynomial decay of order s away from the diagonal and the

Banach algebras $\mathcal{J}_\nu(X)$ or $\mathcal{B}_{u,s}(X)$ which are defined by more general off-diagonal decay conditions than polynomial decay (see Section 4), for weighted versions of the Baskakov-Gohberg-Sjöstrand algebra $\mathcal{C}_v(\mathbb{Z}^d)$ of convolution dominated matrices (see Section 5), and for anisotropic variations of all of these Banach algebras (see Section 6). In particular, we prove that all these Banach algebras are symmetric.

It should be pointed out that the Banach algebras $\mathcal{S}^1_\nu(\mathbb{Z}^d)$, $\mathcal{J}_s(\mathbb{Z}^d)$ and $\mathcal{C}_\nu(\mathbb{Z}^d)$, and their inverse-closedness in $\mathcal{B}(\ell^2(\mathbb{Z}^d;\mathcal{H}))$ were already mentioned in the excellent overview article [44]. However, we would like to emphasize that the corresponding proofs of those results in [44] are on many occasions rather outlined than given in detail, and at the same time heavily depend on Fourier methods and other Banach algebraic methods which rely on the group structure of the index set $\mathbb{Z}^d$. Except for the Baskakov-Gohberg-Sjöstrand algebra $\mathcal{C}_v(\mathbb{Z}^d)$ and its anisotropic variation, all Banach algebras that are considered in the current article, however, are indexed by some arbitrary relatively separated index set $X \subset \mathbb{R}^d$ which does not necessarily possess a group structure. Investigating these more general matrix algebras seems only natural to us, since this will allow a systematic treatment of more general classes of localized g-frames, including irregular Gabor g-frames. However, it should be noted that while the index set X is not necessarily a group, the group structure of $\mathbb{R}^d$ is still inherently present in the problem due to the way matrix multiplication is defined. As noted in [14, Remark 5.6], one could also consider continuous analogs of the algebras $\mathcal{S}^1_\nu(\mathbb{Z}^d)$, $\mathcal{J}_s(\mathbb{Z}^d)$ and $\mathcal{C}_\nu(\mathbb{Z}^d)$ (see, e.g., [14]), which are defined via a bounded uniform partition of unity (BUPU) indexed by $\mathbb{Z}^d$; and since the definition of each of the aforementioned algebras involves a *balanced weight* (see Subsection 2.2), one may argue similarly as in [14, Lemma 5.5] and deduce some of our results from the continuous setting.

In this article, we pursue a more self-contained presentation by working with the definitions directly and employing different proof methods than the Fourier-based approaches mentioned in [44]. In particular, our analysis is based on various kinds of Banach algebraic techniques, such as *Hulanicki's Lemma* (see Proposition 2.1), *Brandenburg's trick* [17], a vector-valued version of *Barnes' Lemma* [9] and adapted techniques from [38], and approaches using derivation algebras [37].

2. Notation and preliminary results

Throughout these notes, all the index sets considered are countable. We denote the cardinality of a set X by $|X|$. For $x \in \mathbb{R}^d$, the symbol $|x|$ denotes the Euclidean norm on $\mathbb{R}^d$ and $|x|_\infty$ the maximum norm on $\mathbb{R}^d$. The standard scalar product of x and y in $\mathbb{R}^d$ is denoted by $x \cdot y = \sum_{j=1}^d x_j y_j$. The symbol $\mathbb{N}$ denotes the set of positive integers $\{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$ denotes the set of non-negative integers. As usual, for multi-indices $\alpha = (\alpha_j)_{j=1}^d, \beta = (\beta_j)_{j=1}^d \in \mathbb{N}_0^d$ we write $|\alpha| = \sum_{j=1}^d \alpha_j$, $\binom{\alpha}{\beta} = \prod_{j=1}^d \binom{\alpha_j}{\beta_j}$, $x^\alpha = x_1^{\alpha_1} \dots x_d^{\alpha_d}$ for $x \in \mathbb{R}^d$, and $\beta \le \alpha$ if and only if $\beta_j \le \alpha_j$ for all $1 \le j \le d$. We write δ_{ij} for the Kronecker-delta.

$\mathcal{H}$ is always a separable Hilbert space. The domain, kernel and range of an operator T is denoted by $\mathcal{D}(T)$, $\mathcal{N}(T)$ and $\mathcal{R}(T)$ respectively. The identity element on a given space X is denoted by $\mathcal{I}_X$. The set of bounded operators between two (quasi-)normed spaces X and Y is denoted by $\mathcal{B}(X, Y)$ and we set $\mathcal{B}(X) := \mathcal{B}(X, X)$.

2.1. Banach algebras

We first recall some basic concepts from Banach algebra theory.

A *Banach algebra* is a (complex) Banach space $(\mathcal{A}, \|\,.\,\|_\mathcal{A})$ equipped with a bilinear map $\cdot : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$, called *multiplication*, which is associative and satisfies $\alpha(A \cdot B) = A \cdot (\alpha B) = (\alpha A) \cdot B$ as well as $\|A \cdot B\|_\mathcal{A} \le \|A\|_\mathcal{A} \|B\|_\mathcal{A}$ for all $A, B \in \mathcal{A}, \alpha \in \mathbb{C}$. We call $\mathcal{A}$ *unital* if $\mathcal{A}$ contains a neutral element $\mathcal{I}_\mathcal{A}$ with respect to multiplication. An element A of a unital Banach algebra $\mathcal{A}$ is called *invertible*, if there exists an element $A^{-1} \in \mathcal{A}$ such that $A \cdot A^{-1} = A^{-1} \cdot A = \mathcal{I}_\mathcal{A}$. An involution on $\mathcal{A}$ is a continuous conjugate linear map $^* : \mathcal{A} \longrightarrow \mathcal{A}$ satisfying $(A^*)^* = A$ and $(A \cdot B)^* = B^* \cdot A^*$ for all $A, B \in \mathcal{A}$. If $\mathcal{A}$ possesses an involution which is an isometry, then $\mathcal{A}$ is called a *Banach *-algebra*.

Let $\mathcal{A}$ be a unital Banach algebra. For any A in $\mathcal{A}$, the *resolvent set* of A is defined by $\rho_{\mathcal{A}}(A) := \{\lambda \in \mathbb{C} : \lambda\mathcal{I}_{\mathcal{A}} - A \text{ is invertible}\}$. The *spectrum* of A is the compact set defined by $\sigma_{\mathcal{A}}(A) := \mathbb{C} \setminus \rho_{\mathcal{A}}(A)$. The *spectral radius* of A is defined by $r_{\mathcal{A}}(A) = \max\{|\lambda| : \lambda \in \sigma_{\mathcal{A}}(A)\}$. By *Gelfand's formula*, the spectral radius of any $A \in \mathcal{A}$ can be computed via

$$r_{\mathcal{A}}(A) = \lim_{n\to\infty} \|A^n\|_{\mathcal{A}}^{\frac{1}{n}}. \tag{3}$$

If $\mathcal{A}$ and $\mathcal{B}$ are two unital Banach algebras (Banach *-algebras) with a common identity, then we write $\mathcal{A} \subseteq \mathcal{B}$, if $\mathcal{A}$ is contained in $\mathcal{B}$ and if the inclusion map from $\mathcal{A}$ into $\mathcal{B}$ is an algebra homomorphism (*-homomorphism). If $\mathcal{A}$ and $\mathcal{B}$ are two unital Banach algebras with common identity, then [33]

$$\mathcal{A} \subseteq \mathcal{B} \qquad \Longrightarrow \qquad \sigma_{\mathcal{B}}(A) \subseteq \sigma_{\mathcal{A}}(A) \quad (\forall A \in \mathcal{A}). \tag{4}$$

Such a nested pair $\mathcal{A} \subseteq \mathcal{B}$ of unital Banach algebras with common identity is called a *Wiener pair* if $\mathcal{A}$ is *inverse-closed* in $\mathcal{B}$, see (1). One can show [33] that a nested pair $\mathcal{A} \subseteq \mathcal{B}$ of unital Banach algebras with common identity forms a Wiener pair if and only if $\sigma_{\mathcal{A}}(A) = \sigma_{\mathcal{B}}(A)$ for all $A \in \mathcal{A}$. This is the reason why the term *spectral invariance* is often used synonymously when discussing inverse closedness [33].

Verifying whether $\mathcal{A} \subseteq \mathcal{B}$ forms a Wiener pair is often a rather subtle challenge. However, if $\mathcal{A}$ and $\mathcal{B}$ are Banach *-algebras and $\mathcal{B}$ is *symmetric*, which means that $\sigma_{\mathcal{B}}(A^*A) \subseteq [0,\infty)$ for all $A \in \mathcal{B}$, a powerful trick by Hulanicki allows an analytical treatment of this task.

Proposition 2.1 *(Hulanicki's Lemma). [40] Let $\mathcal{A} \subseteq \mathcal{B}$ be a pair of unital Banach *-algebras with common identity, and suppose that $\mathcal{B}$ is symmetric. Then the following are equivalent:*

(i) *$\mathcal{A}$ is inverse-closed in $\mathcal{B}$.*
(ii) $r_{\mathcal{A}}(A) = r_{\mathcal{B}}(A) \qquad (\forall A = A^* \in \mathcal{A})$.
(iii) $r_{\mathcal{A}}(A) \leq r_{\mathcal{B}}(A) \qquad (\forall A = A^* \in \mathcal{A})$.

Moreover, if one of these conditions holds, then $\mathcal{A}$ is also symmetric.

We will also need some facts from derivation algebras. Let $\mathcal{A}$ be a symmetric Banach algebra and $\delta : \mathcal{D}(\delta) = \mathcal{D}(\delta, \mathcal{A}) \longrightarrow \mathcal{A}$ be a closed (possibly unbounded) linear operator, where the domain $\mathcal{D}(\delta) = \mathcal{D}(\delta, \mathcal{A})$ is an arbitrary subspace of $\mathcal{A}$. Such an operator δ is called a *derivation on* $\mathcal{A}$, if the Leibniz rule holds true on $\mathcal{D}(\delta)$, i.e., if

$$\delta(A \cdot B) = A \cdot \delta(B) + \delta(A) \cdot B \qquad (\forall A, B \in \mathcal{D}(\delta)). \tag{5}$$

If $\mathcal{A}$ is involutive, we additionally assume that the domain $\mathcal{D}(\delta)$ is invariant under involution (i.e., $A^* \in \mathcal{D}(\delta)$ whenever $A \in \mathcal{D}(\delta)$) and that $\delta(A^*) = \delta(A)^*$ for all $A \in \mathcal{D}(\delta)$. Equipped with the graph norm

$$\|A\|_{\mathcal{D}(\delta)} = \|A\|_{\mathcal{A}} + \|\delta(A)\|_{\mathcal{A}}, \tag{6}$$

$\mathcal{D}(\delta)$ then forms a (not necessarily unital) symmetric Banach (*-)algebra whenever $\mathcal{A}$ is a symmetric unital Banach (*-)algebra [37, Theorem 3.4].

2.2. Weight functions

A *weight function* on $\mathbb{R}^d$, or simply a *weight*, is a continuous and positive function $\nu : \mathbb{R}^d \longrightarrow (0,\infty)$. In the sections ahead, we will consider weighted versions of various kinds of matrix algebras. The following classes of weight functions are fundamental for the study of these matrix algebras.

A weight ν is called *submultiplicative*, if

$$\nu(x + x') \leq \nu(x)\nu(x') \qquad (\forall x, x' \in \mathbb{R}^d),$$

and *symmetric* if

$$\nu(x) = \nu(-x) \qquad (\forall x \in \mathbb{R}^d).$$

A weight m is called *ν-moderate* if there exists some constant $C > 0$ such that

$$m(x + x') \leq Cm(x)\nu(x') \qquad (\forall x, x' \in \mathbb{R}^d).$$

A weight ν is called *balanced*, if there exist $a, b \in (0, \infty)$, such that

$$a \leq \inf_{x' \in [0,1]^d} \frac{\nu(x + x')}{\nu(x)} \leq \sup_{x' \in [0,1]^d} \frac{\nu(x + x')}{\nu(x)} \leq b \qquad (\forall x \in \mathbb{R}^d).$$

We say that ν satisfies the *GRS-condition* (Gelfand-Raikov-Shilov condition [28]), if

$$\lim_{n \to \infty} (\nu(nz))^{\frac{1}{n}} = 1 \qquad (\forall z \in \mathbb{R}^d).$$

Note that any submultiplicative weight ν is balanced and ν-moderate and that any submultiplicative and symmetric weight ν satisfies $\nu \geq 1$ on $\mathbb{R}^d$.

Typical examples of weights are the functions $\nu(x) = e^{\alpha|x|^\beta}(1 + |x|)^s$. As noted in [14, Example 5.4], these weights ν are balanced whenever $0 \leq \beta \leq 1$ and submultiplicative, if additionally $\alpha, s \geq 0$. Under these conditions, ν satisfies the GRS-condition if and only if $0 \leq \beta < 1$. More general weight functions are given by mixtures of the form

$$\nu(x) = e^{\alpha|x|^\beta}(1 + |x|)^s(\log(e + |x|))^t).$$

For $s \geq 0$, $\alpha > 0$, $\beta \in (0, 1)$ and $t \geq 0$, these weights are examples of *admissible weights*, as defined below, and will play an important role in this article.

Definition 2.2. [38] A weight function ν on $\mathbb{R}^d$ is called *admissible*, if

(a) ν is of the form

$$\nu(x) = e^{\rho(\|x\|)} \qquad (x \in \mathbb{R}^d), \tag{7}$$

where $\rho : [0, \infty) \longrightarrow [0, \infty)$ is a continuous and concave function with $\rho(0) = 0$ and $\| \, . \, \|$ any norm on $\mathbb{R}^d$,

(b) ν satisfies the GRS-condition.

The assumptions on ρ imply that ρ is subadditive. Together with the properties of a norm, we immediately see, that admissible weights are submultiplicative, balanced, symmetric and satisfy $\nu(0) = 1$. In particular, $\nu \geq 1$ on $\mathbb{R}^d$.

We refer the reader to the article [36] for a comprehensive overview on various classes of weight functions and their relevance in Banach algebra theory.

2.3. Relatively separated index sets

A countable set $X \subset \mathbb{R}^d$ is called *relatively separated* if

$$\sup_{x \in \mathbb{R}^d} |X \cap (x + [0,1]^d)| < \infty. \tag{8}$$

We collect the following preparatory result for later reference.

Lemma 2.3. *Let $X \subset \mathbb{R}^d$ be a relatively separated set.*

(a) *[31, Lemma 1] For any $s > d$, there exists a constant $C = C(s) > 0$ such that*

$$\sup_{x \in \mathbb{R}^d} \sum_{k \in X} (1 + |x - k|)^{-s} = C < \infty$$

(b) *[31, Lemma 2 (a)] For any $s > d$, there exists a constant $C = C(s) > 0$ such that*

$$\sum_{n \in X} (1 + |k - n|)^{-s} (1 + |l - n|)^{-s} \leq C(1 + |k - l|)^{-s} \qquad (\forall k, l \in X).$$

2.4. Bochner sequence spaces

Occasionally, our analysis will be based on properties on Banach space-valued ℓ^p-spaces, defined as follows. If B is a Banach space, X a countable index set, and $\nu = (\nu_k)_{k \in X}$ a family of positive numbers (e.g. samples of a weight function as in Subsection 2.2), then for each $p \in [1, \infty]$ the Bochner sequence space $\ell^p(X; B)$ [41, Chapter 1] is defined by

$$\ell^p_\nu(X; B) := \{(Y_k)_{k \in X} : Y_k \in B \ (\forall k \in X), (\|Y_k\|_B \cdot \nu_k)_{k \in X} \in \ell^p(X)\}.$$

In case $\nu_k = 1$ for all $k \in X$, we write $\ell^p_\nu(X; B) = \ell^p(X; B)$.

The Bochner sequence spaces $\ell^p_\nu(X; B)$ share many properties with the classical ℓ^p-spaces. In particular (see [41, Chapter 1]), $\ell^p_\nu(X; B)$ is a Banach space with respect to the norm

$$\|(Y_k)_{k \in X}\|_{\ell^p_\nu(X;B)} = \|(\|Y_k\|_B \cdot \nu_k)_{k \in X}\|_{\ell^p(X)}$$

for every $1 \leq p \leq \infty$. If $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the dual space $(\ell^p_\nu(X; B))^*$ of $\ell^p_\nu(X; B)$ is isometrically isomorphic to $\ell^q_{1/\nu}(X; B^*)$. Hence, from now on we identify the latter with $(\ell^p_\nu(X; B))^*$. Moreover, Riesz-Thorin interpolation holds. For later reference, we state the following special case of [41, Theorem 2.2.1].

Proposition 2.4 *(Riesz-Thorin interpolation). Let $A \in \mathcal{B}(\ell^1_\nu(X; B))$ and $A \in \mathcal{B}(\ell^\infty_\nu(X; B))$. Then $A \in \mathcal{B}(\ell^p_\nu(X; B))$ for every $1 < p < \infty$ and*

$$\|A\|_{\mathcal{B}(\ell^p_\nu(X;B))} \leq \max\{\|A\|_{\mathcal{B}(\ell^1_\nu(X;B))}, \|A\|_{\mathcal{B}(\ell^\infty_\nu(X;B))}\} \qquad (\forall 1 \leq p \leq \infty).$$

2.5. The Banach algebra $\mathcal{B}(\ell^2(X; \mathcal{H}))$

Of special interest to us in this article is the Hilbert space $\ell^2(X; \mathcal{H})$ and the Banach algebra $\mathcal{B}(\ell^2(X; \mathcal{H}))$ of bounded operators acting on it.

Assume that for each $k, l \in X$ we are given a bounded operator $A_{k,l} \in \mathcal{B}(\mathcal{H})$. Then

$$A(f_l)_{l\in X} := \Big(\sum_{l\in X} A_{k,l} f_l\Big)_{k\in X} \tag{9}$$

defines a linear operator $A : \mathcal{D}(A) \subseteq \ell^2(X;\mathcal{H}) \longrightarrow \ell^2(X;\mathcal{H})$, where

$$\mathcal{D}(A) = \left\{ (f_l)_{l\in X} \in \ell^2(X;\mathcal{H}) : \sum_{k\in X} \Big\| \sum_{l\in X} A_{k,l} f_l \Big\|^2 < \infty \right\}. \tag{10}$$

If we view elements $(f_l)_{l\in X}$ from $\ell^2(X;\mathcal{H})$ as (possibly infinite) column vectors, then we can represent the operator A by the matrix

$$A = \begin{bmatrix} \ddots & \vdots & \vdots & \\ \dots & A_{k,l} & A_{k,l+1} & \dots \\ \dots & A_{k+1,l} & A_{k+1,l+1} & \dots \\ & \vdots & \vdots & \ddots \end{bmatrix}, \tag{11}$$

because (9) precisely corresponds to the formal matrix multiplication of A with $(f_l)_{l\in X}$.

In fact, one can show that there is a one-to-one correspondence between bounded operators $A \in \mathcal{B}(\ell^2(X;\mathcal{H}))$ and $\mathcal{B}(\mathcal{H})$-valued matrices $[A_{k,l}]_{k,l\in X}$ which satisfy a certain domain condition. This fact was mentioned in [50], for the proof details, we refer to [47,49]. More precisely:

Proposition 2.5. *[47,49,50] For every bounded operator $A \in \mathcal{B}(\ell^2(X;\mathcal{H}))$ there exists a unique $\mathcal{B}(\mathcal{H})$-valued matrix $[A_{k,l}]_{k,l\in X}$ such that the action of A on any $(f_l)_{l\in X} \in \ell^2(X;\mathcal{H})$ is given by (9). Conversely, if $A = [A_{k,l}]_{k,l\in X}$ is $\mathcal{B}(\mathcal{H})$-valued matrix that satisfies $\mathcal{D}(A) = \ell^2(X;\mathcal{H})$, where $\mathcal{D}(A)$ is defined as in (10), then $A \in \mathcal{B}(\ell^2(X;\mathcal{H}))$.*

The above proposition motivates the notation

$$\mathbb{M}(A) = [A_{k,l}]_{k,l\in X}$$

and we call $\mathbb{M}(A)$ the *canonical matrix representation* of A [49].

Remark 2.6. Composing bounded operators A and B from $\mathcal{B}(\ell^2(X;\mathcal{H}))$ corresponds to matrix multiplication of their corresponding canonical matrix representations. In other words, if $A, B \in \mathcal{B}(\ell^2(X;\mathcal{H}))$, then $\mathbb{M}(AB) = \mathbb{M}(A) \cdot \mathbb{M}(B)$, where each entry $[AB]_{k,l} = \sum_{n\in X} A_{k,n} B_{n,l}$ of $\mathbb{M}(AB)$ converges in the operator norm topology with respect to $\mathcal{B}(\mathcal{H})$ [47,50]. In particular, this means that $\mathcal{B}(\ell^2(X;\mathcal{H}))$ forms a Banach algebra of $\mathcal{B}(\mathcal{H})$-valued matrices.

The result below on the canonical matrix representation of the adjoint A^* of $A \in \mathcal{B}(\ell^2(X;\mathcal{H}))$ was mentioned in [50]; its proof details can be found in [47].

Proposition 2.7. *[47,50] Let $A \in \mathcal{B}(\ell^2(X;\mathcal{H}))$ and $\mathbb{M}(A) = [A_{k,l}]_{k,l\in X}$ be its canonical matrix representation. Then the canonical matrix representation of the adjoint operator $A^* \in \mathcal{B}(\ell^2(X;\mathcal{H}))$ is given by the relation*

$$\mathbb{M}(A^*) = [A^*_{k,l}]^t_{k,l\in X} \tag{12}$$

where the exponent t denotes transposition.

3. Weighted Schur-type algebras

Let $X \subseteq \mathbb{R}^d$ be a relatively separated and ν be a weight. Then for $1 \leq p < \infty$ we define $\mathcal{S}_\nu^p = \mathcal{S}_\nu^p(X)$ to be the space of all $\mathcal{B}(\mathcal{H})$-valued matrices $A = [A_{k,l}]_{k,l \in X}$ for which

$$\max \left\{ \sup_{k \in X} \left(\sum_{l \in X} \|A_{k,l}\|^p \nu(k-l)^p \right)^{\frac{1}{p}}, \sup_{l \in X} \left(\sum_{k \in X} \|A_{k,l}\|^p \nu(k-l)^p \right)^{\frac{1}{p}} \right\} \tag{13}$$

is finite. For $\nu \equiv 1$ we abbreviate $\mathcal{S}^p := \mathcal{S}_1^p$.

It is easy to see that (13) defines a norm on $\mathcal{S}_\nu^p$. In fact, $\mathcal{S}_\nu^p$ is complete with respect to this norm, as shown below.

Proposition 3.1. *For every $1 \leq p < \infty$, $(\mathcal{S}_\nu^p, \| \, . \, \|_{\mathcal{S}_\nu^p})$ is a Banach space.*

Proof. Let $\{A^{(n)}\}_{n=1}^\infty$ be a Cauchy sequence in $\mathcal{S}_\nu^p$, i.e. for any given $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that for all $m, n \geq N$

$$\sup_{k \in X} \left(\sum_{l \in X} \|A_{k,l}^{(m)} - A_{k,l}^{(n)}\|^p \nu(k-l)^p \right)^{\frac{1}{p}} < \varepsilon \tag{14}$$

and

$$\sup_{l \in X} \left(\sum_{k \in X} \|A_{k,l}^{(m)} - A_{k,l}^{(n)}\|^p \nu(k-l)^p \right)^{\frac{1}{p}} < \varepsilon. \tag{15}$$

Step 1. Relations (14) and (15) imply that for each index pair $(k,l) \in X \times X$, $\{A_{k,l}^{(n)}\}_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{B}(\mathcal{H})$ and thus converges to some $A_{k,l} \in \mathcal{B}(\mathcal{H})$. Set $A = [A_{k,l}]_{k,l \in X}$. We will show that $A^{(n)} \to A$ in $\mathcal{S}_\nu^p$ and that $A \in \mathcal{S}_\nu^p$.

Step 2. For each $n \in \mathbb{N}$, set

$$\begin{aligned} \tilde{A}^{(n)} &:= \big(\tilde{A}_k^{(n)}\big)_{k \in X}, \quad \text{where} \quad \tilde{A}_k^{(n)} = \big(A_{k,l}^{(n)} \nu(k-l)\big)_{l \in X} \in \ell^p(X; \mathcal{B}(\mathcal{H})) \\ \tilde{\tilde{A}}^{(n)} &:= \big(\tilde{\tilde{A}}_l^{(n)}\big)_{l \in X}, \quad \text{where} \quad \tilde{\tilde{A}}_l^{(n)} = \big(A_{k,l}^{(n)} \nu(k-l)\big)_{k \in X} \in \ell^p(X; \mathcal{B}(\mathcal{H})). \end{aligned}$$

Then, relation (14) implies that $\{\tilde{A}^{(n)}\}_{n=1}^\infty$ is a Cauchy sequence in the Banach space $\ell^\infty(X; \ell^p(X; \mathcal{B}(\mathcal{H})))$ and thus converges to some $\tilde{A} = \big(\tilde{A}_k\big)_{k \in X} \in \ell^\infty(X; \ell^p(X; \mathcal{B}(\mathcal{H})))$. Analogously, (15) implies that $\{\tilde{\tilde{A}}^{(n)}\}_{n=1}^\infty$ is a Cauchy sequence in $\ell^\infty(X; \ell^p(X; \mathcal{B}(\mathcal{H})))$ and thus converges to some $\tilde{\tilde{A}} = \big(\tilde{\tilde{A}}_l\big)_{l \in X} \in \ell^\infty(X; \ell^p(X; \mathcal{B}(\mathcal{H})))$.

Step 3. We set

$$\begin{aligned} \tilde{A}_k &= \big(\tilde{A}_{k,l} \nu(k-l)\big)_{l \in X} \quad \text{for each } k \in X, \\ \tilde{\tilde{A}}_l &= \big(\tilde{\tilde{A}}_{k,l} \nu(k-l)\big)_{k \in X} \quad \text{for each } l \in X. \end{aligned}$$

Then, since $\tilde{A}^{(n)} \to \tilde{A}$ in $\ell^\infty(X; \ell^p(X; \mathcal{B}(\mathcal{H})))$, we have $\tilde{A}_k^{(n)} = \big(A_{k,l}^{(n)} \nu(k-l)\big)_{l \in X} \to \tilde{A}_k = \big(\tilde{A}_{k,l} \nu(k-l)\big)_{l \in X}$ in $\ell^p(X; \mathcal{B}(\mathcal{H}))$ for every $k \in X$. Similarly, we see that $\tilde{\tilde{A}}_l^{(n)} = \big(A_{k,l}^{(n)} \nu(k-l)\big)_{k \in X} \to \tilde{\tilde{A}}_l = \big(\tilde{\tilde{A}}_{k,l} \nu(k-l)\big)_{k \in X}$ in $\ell^p(X; \mathcal{B}(\mathcal{H}))$ for every $l \in X$. The latter two conclusions imply that $[\tilde{A}_k^{(n)}]_l = A_{k,l}^{(n)} \nu(k-l) \to [\tilde{A}_k]_l =$

$\tilde{A}_{k,l}\nu(k-l)$ in $\mathcal{B}(\mathcal{H})$ and that $[\tilde{\tilde{A}}_l^{(n)}]_k = A_{k,l}^{(n)}\nu(k-l) \to [\tilde{\tilde{A}}_l]_k = \tilde{\tilde{A}}_{k,l}\nu(k-l)$ in $\mathcal{B}(\mathcal{H})$ for each $(k,l) \in X \times X$. However, by Step 1 we have $A_{k,l}^{(n)} \to A_{k,l}$ in $\mathcal{B}(\mathcal{H})$, and therefore we must have $\tilde{A}_{k,l} = \tilde{\tilde{A}}_{k,l} = A_{k,l}$ for all $k,l \in X$. In particular, this implies that $\tilde{A}_k = (A_{k,l}\nu(k-l))_{l\in X}$ for each $k \in X$, and $\tilde{\tilde{A}}_l = (A_{k,l}\nu(k-l))_{k\in X}$ for each $l \in X$.

Step 4. Let $\varepsilon' > 0$ be arbitrary. Then $\tilde{A}^{(n)} \to \tilde{A}$ in $\ell^\infty(X;\ell^p(X;\mathcal{B}(\mathcal{H})))$ by Step 2. Combined with the observations made in Step 3, we know that there exists some $N' = N'(\varepsilon') \in \mathbb{N}$ such that for all $n \geq N'$

$$\sup_{k\in X}\left(\sum_{l\in X}\|A_{k,l}^{(n)} - A_{k,l}\|^p \nu(k-l)^p\right)^{\frac{1}{p}} < \varepsilon'. \tag{16}$$

Similarly, $\tilde{\tilde{A}}^{(n)} \to \tilde{A}$ in $\ell^\infty(X;\ell^p(X;\mathcal{B}(\mathcal{H})))$ implies that there exists some $N'' = N''(\varepsilon') \in \mathbb{N}$ such that for all $n \geq N''$

$$\sup_{l\in X}\left(\sum_{k\in X}\|A_{k,l}^{(n)} - A_{k,l}\|^p \nu(k-l)^p\right)^{\frac{1}{p}} < \varepsilon'. \tag{17}$$

Combining (16) with (17) yields that for all $n \geq \max\{N', N''\}$

$$\|A^{(n)} - A\|_{\mathcal{S}_\nu^p} < \varepsilon'. \tag{18}$$

Hence $\{A^{(n)}\}_{n=1}^\infty$ converges in $\mathcal{S}_\nu^p$ to A.

Step 5. By relation (18) we know that the $\mathcal{B}(\mathcal{H})$-valued matrix $A - A^{(n)}$ belongs to $\mathcal{S}_\nu^p$ for sufficiently large n. This implies that $A = (A - A^{(n)}) + A^{(n)} \in \mathcal{S}_\nu^p$ and the proof is complete. □

Remark 3.2. The definition of $\mathcal{S}_\nu^p$ canonically extends to the case $p = \infty$. In fact, we set $\mathcal{S}_\nu^\infty = \ell_u^\infty(X \times X;\mathcal{B}(\mathcal{H}))$, where $u(k,l) = \nu(k-l)$. In this case, the completeness of $\mathcal{S}_\nu^\infty$ follows immediately from the completeness of Bochner sequence spaces. In Section 4 these spaces will be denoted by $\mathcal{J}_\nu$ and discussed in more detail. In the case $\nu(x) = \nu_s(x) = (1+|x|)^s$ $(x \in \mathbb{R}^d)$ of a polynomial weight, $\mathcal{S}_{\nu_s}^\infty = \mathcal{J}_{\nu_s}$ equals the *Jaffard algebra* (see Section 4), whence the notation $\mathcal{J}_\nu$.

We will now focus on the case $p = 1$. In fact, for admissible weights ν (in the sense of Definition 2.2), we shall hereinafter prove step by step that $\mathcal{S}_\nu^1 \subseteq \mathcal{B}(\ell^2(X;\mathcal{H}))$ is a Wiener pair.

First, we show a generalization of Schur's test [30] to $\mathcal{B}(\mathcal{H})$-valued matrices.

Lemma 3.3. *Let ν be an admissible weight and m be ν-moderate. Then every $A \in \mathcal{S}_\nu^1$ defines (via (9)) a bounded operator on $\ell_m^p(X;\mathcal{H})$ for all $1 \leq p \leq \infty$ and it holds*

$$\|A\|_{\mathcal{B}(\ell_m^p(X;\mathcal{H}))} \leq C\|A\|_{\mathcal{S}_\nu^1} \qquad (\forall 1 \leq p \leq \infty),$$

where C is the constant arising from m being ν-moderate.

Proof. Let $A \in \mathcal{S}_\nu^1$ be arbitrary. Then

$$\begin{aligned}\|A\|_{\mathcal{B}(\ell_m^1(X;\mathcal{H}))} &= \sup_{\|f\|_{\ell_m^1(X;\mathcal{H})}=1}\sum_{k\in X}\left\|\sum_{l\in X}A_{k,l}f_l\right\| m(l+k-l) \\ &\leq C\sup_{\|f\|_{\ell_m^1(X;\mathcal{H})}=1}\sum_{k\in X}\sum_{l\in X}\|A_{k,l}\|\|f_l\|m(l)\nu(k-l)\end{aligned}$$

$$
\begin{aligned}
&= C \sup_{\|f\|_{\ell^1_m(X;\mathcal{H})}=1} \sum_{l\in X} \|f_l\| m(l) \sum_{k\in X} \|A_{k,l}\|\nu(k-l) \\
&\leq C\|A\|_{\mathcal{S}^1_\nu}. \qquad (19)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\|A\|_{\mathcal{B}(\ell^\infty_m(X;\mathcal{H}))} &= \sup_{\|f\|_{\ell^\infty_m(X;\mathcal{H})}=1} \sup_{k\in X} \left\| \sum_{l\in X} A_{k,l} f_l \right\| m(l+k-l) \\
&\leq C \sup_{\|f\|_{\ell^\infty_m(X;\mathcal{H})}=1} \sup_{k\in X} \sum_{l\in X} \|A_{k,l}\| \|f_l\| m(l)\nu(k-l) \\
&\leq C\|A\|_{\mathcal{S}^1_\nu}. \qquad (20)
\end{aligned}
$$

By Riesz-Thorin interpolation (Proposition 2.4), the statement follows. □

Remark 3.4. In the scalar-valued case, the reverse inequalities of (19) and (20) hold true in the case $m = \nu \equiv 1$ (and hence $C = 1$). Consequently, in the scalar-valued setting, the norm $\|A\|_{\mathcal{S}^1}$ is simply the larger of the operator norms of A on $\ell^1(X)$ and $\ell^\infty(X)$ respectively, which yields a shortcut to showing the completeness of $\mathcal{S}^1_\nu$ in this case. However, in the $\mathcal{B}(\mathcal{H})$-valued setting, this shortcut is no longer justifiable and proving the completeness of $\mathcal{S}^1_\nu$ is significantly more involved (see Proposition 3.1).

By Lemma 3.3, $\mathcal{S}^1_\nu$ is continuously embedded into $\mathcal{B}(\ell^2(X;\mathcal{H}))$. Thus, by the matrix calculus from Subsection 2.5, there is a canonical way to define a multiplication and an involution on $\mathcal{S}^1_\nu$. In fact, the following holds.

Proposition 3.5. *If ν is an admissible weight, then $\mathcal{S}^1_\nu$ is a unital Banach *-algebra with respect to matrix multiplication and involution as defined in (12).*

Proof. Let $A, B \in \mathcal{S}^1_\nu$. Then, for arbitrary $k \in X$ we have by the submultiplicativity of ν that

$$
\begin{aligned}
\sum_{l\in X} \|[AB]_{k,l}\| v(k-l) &\leq \sum_{l\in X}\sum_{n\in X} \|A_{k,n}\|\|B_{n,l}\| v(k-n)v(n-l) \\
&= \sum_{n\in X} \|A_{k,n}\| v(k-n) \sum_{l\in X} \|B_{n,l}\| v(n-l) \\
&\leq \|A\|_{\mathcal{S}^1_\nu}\|B\|_{\mathcal{S}^1_\nu}.
\end{aligned}
$$

Consequently,

$$
\sup_{k\in X} \sum_{l\in X} \|[AB]_{k,l}\| v(k-l) \leq \|A\|_{\mathcal{S}^1_\nu}\|B\|_{\mathcal{S}^1_\nu}.
$$

Similarly, we see that

$$
\sup_{l\in X} \sum_{k\in X} \|[AB]_{k,l}\| v(k-l) \leq \|A\|_{\mathcal{S}^1_\nu}\|B\|_{\mathcal{S}^1_\nu},
$$

hence

$$
\|AB\|_{\mathcal{S}^1_\nu} \leq \|A\|_{\mathcal{S}^1_\nu}\|B\|_{\mathcal{S}^1_\nu}.
$$

Thus, together with Proposition 3.1, we conclude that $\mathcal{S}^1_\nu$ is a Banach algebra. Next, since ν is a symmetric weight, the definition of the norm (13) implies that $\|A^*\|_{\mathcal{S}^1_\nu} = \|A\|_{\mathcal{S}^1_\nu}$ for all $A \in \mathcal{S}^1_\nu$ (where the involution is defined as in (12)). Moreover, since $\nu(0) = 1$, the identity $\mathcal{I}_{\mathcal{S}^1_\nu} = \mathcal{I}_{\mathcal{B}(\ell^2(X;\mathcal{H}))} = \mathrm{diag}[\mathcal{I}_{\mathcal{B}(\mathcal{H})}]_{k\in X}$ is contained in $\mathcal{S}^1_\nu$. $\square$

If ν is an admissible weight and if ν additionally satisfies a certain growth condition, then we can show that $\mathcal{S}^1_\nu \subseteq \mathcal{B}(\ell^2(X;\mathcal{H}))$ forms a Wiener pair. Hulanicki's lemma will be the key tool to verify the inverse-closedness of $\mathcal{S}^1_\nu$ in $\mathcal{B}(\ell^2(X;\mathcal{H}))$. In order to derive the desired inequality of spectral radii, we follow ideas of [9] and [38].

We start with a vector-valued version of part of [9, Lemma 4.6]. The proof techniques are basically the same as in [9]. Nevertheless, we provide the reader with the details adapted to our setting.

Lemma 3.6. *Let $\nu(x) = \nu_\delta(x) = (1 + |x|)^\delta$ for some $\delta \in (0, 1]$. Then*

$$r_{\mathcal{S}^1_\nu}(A) = r_{\mathcal{S}^1}(A) \qquad (\forall A \in \mathcal{S}^1_\nu).$$

Proof. For $0 < \varepsilon \le 1$, let $\mu_\varepsilon(x) = (1 + \varepsilon|x|)^\delta$. Then μ_ε is a submultiplicative weight. Combining Proposition 3.1 with the first part of the proof of Proposition 3.5 (where only the submultiplicativity of the corresponding weight is used) yields that $\mathcal{S}^1_{\mu_\varepsilon}$ is a Banach algebra. We also note that

$$\mu_\varepsilon \le \nu \le \varepsilon^{-\delta}\mu_\varepsilon.$$

This yields $\|A\|_{\mathcal{S}^1_\nu} \le \varepsilon^{-\delta}\|A\|_{\mathcal{S}^1_{\mu_\varepsilon}}$ and consequently

$$\|A^n\|^{\frac{1}{n}}_{\mathcal{S}^1_\nu} \le (\varepsilon^{-\delta})^{\frac{1}{n}}\|A^n\|^{\frac{1}{n}}_{\mathcal{S}^1_{\mu_\varepsilon}} \qquad (\forall A \in \mathcal{S}^1_\nu).$$

Using Gelfand's formula this implies that

$$r_{\mathcal{S}^1_\nu}(A) \le r_{\mathcal{S}^1_{\mu_\varepsilon}}(A) \le \|A\|_{\mathcal{S}^1_{\mu_\varepsilon}} \qquad (\forall A \in \mathcal{S}^1_\nu). \tag{21}$$

Next, let $A \in \mathcal{S}^1_\nu$. Then, since $t^\delta \le (1 + t)^\delta \le 1 + t^\delta$ for $t \ge 0$, we have

$$\begin{aligned}\sup_{k\in X}\sum_{l\in X}\|A_{k,l}\| &\le \sup_{k\in X}\sum_{l\in X}\|A_{k,l}\|(1 + \varepsilon|k - l|)^\delta \\ &\le \sup_{k\in X}\sum_{l\in X}\|A_{k,l}\|(1 + \varepsilon^\delta|k - l|^\delta) \\ &\le \sup_{k\in X}\sum_{l\in X}\|A_{k,l}\| + \varepsilon^\delta \sup_{k\in X}\sum_{l\in X}\|A_{k,l}\|(1 + |k - l|)^\delta.\end{aligned}$$

Interchanging the roles of k and l, we analogously obtain

$$\begin{aligned}\sup_{l\in X}\sum_{k\in X}\|A_{k,l}\| &\le \sup_{l\in X}\sum_{k\in X}\|A_{k,l}\|(1 + \varepsilon|k - l|)^\delta \\ &\le \sup_{l\in X}\sum_{k\in X}\|A_{k,l}\| + \varepsilon^\delta \sup_{l\in X}\sum_{k\in X}\|A_{k,l}\|(1 + |k - l|)^\delta,\end{aligned}$$

and consequently

$$\|A\|_{\mathcal{S}^1} \le \|A\|_{\mathcal{S}^1_{\mu_\varepsilon}} \le \|A\|_{\mathcal{S}^1} + \varepsilon^\delta\|A\|_{\mathcal{S}^1_\nu} \qquad (\forall A \in \mathcal{S}^1_\nu).$$

Combining the latter with the inequalities (21) gives us

$$r_{\mathcal{S}^1_\nu}(A) \leq \lim_{\varepsilon\to 0} \|A\|_{\mathcal{S}^1_{\mu_\varepsilon}} = \|A\|_{\mathcal{S}^1} \qquad (\forall A \in \mathcal{S}^1_\nu). \tag{22}$$

Finally, the inequality (22) implies that $r_{\mathcal{S}^1_\nu}(A)^n = r_{\mathcal{S}^1_\nu}(A^n) \leq \|A^n\|_{\mathcal{S}^1}$, hence $r_{\mathcal{S}^1_\nu}(A) \leq \|A^n\|_{\mathcal{S}^1}^{1/n} \to r_{\mathcal{S}^1}(A)$ (as $n \to \infty$). This implies $r_{\mathcal{S}^1_\nu}(A) \leq r_{\mathcal{S}^1}(A)$ for all $A \in \mathcal{S}^1_\nu$. On the other hand, we clearly have $\mathcal{S}^1_\nu \subseteq \mathcal{S}^1$, hence $r_{\mathcal{S}^1_\nu}(A) \geq r_{\mathcal{S}^1}(A)$ for all $A \in \mathcal{S}^1_\nu$ as a consequence of (4). This proves the desired equation. □

Our next result is a vector-valued version of *Barnes' Lemma* from [38, Lemma 5]. Note that additional estimates of the proof ideas from [9] are necessary to prove this result in the $\mathcal{B}(\mathcal{H})$-valued setting.

Theorem 3.7. *Let $\nu_\delta(x) = (1 + |x|)^\delta$ for some $\delta \in (0, 1]$. Then*

$$r_{\mathcal{S}^1_{\nu_\delta}}(A) = r_{\mathcal{S}^1}(A) = r_{\mathcal{B}(\ell^2(X;\mathcal{H}))}(A) \qquad (\forall A = A^* \in \mathcal{S}^1_{\nu_\delta}). \tag{23}$$

In particular, $\mathcal{S}^1_{\nu_\delta}$ is inverse-closed in $\mathcal{B}(\ell^2(X;\mathcal{H}))$ and $\mathcal{S}^1_{\nu_\delta}$ is a symmetric Banach algebra.

Proof. We abbreviate $\mathcal{B} = \mathcal{B}(\ell^2(X;\mathcal{H}))$.

By Lemma 3.6 it suffices to show the equation

$$r_{\mathcal{S}^1}(A) = r_{\mathcal{B}}(A) \qquad (\forall A = A^* \in \mathcal{S}^1_{\nu_\delta}). \tag{24}$$

Once we have established (24), the rest of the statement follows from Hulanicki's lemma (Proposition 2.1). Moreover, it suffices to show (24) for non-zero A, since otherwise the statement is trivial upon an application of Gelfand's formula.

Step 1. Fix some non-zero $A = A^* \in \mathcal{S}^1_{\nu_\delta}$. Since A is self-adjoint it holds

$$\|A\|_{\mathcal{S}^1} = \sup_{l\in X} \sum_{k\in X} \|A_{k,l}\|.$$

Thus, for any arbitrary but fixed $n \in \mathbb{N}$, we have

$$\|A^{n+1}\|_{\mathcal{S}^1} \leq \sup_{l\in X} \sum_{\substack{k\in X\\ |k-l|\leq 2^n}} \|[A^{n+1}]_{k,l}\| + \sup_{l\in X} \sum_{\substack{k\in X\\ |k-l|>2^n}} \|[A^{n+1}]_{k,l}\|. \tag{25}$$

Our strategy is to estimate each of the summands of the right-hand side of (25) separately, then combine those estimates and finally derive the desired equation (24) via Gelfand's formula.

Step 2. For each $l \in X$, let $B_{2^n}(l) = \{k \in X : |k - l| \leq 2^n\}$. We claim that there exists some $C_1 > 0$ such that

$$|B_{2^n}(l)| \leq C_1 2^{nd} \qquad (\forall l \in X). \tag{26}$$

To prove the claim, we first note that $B_{2^n}(l) \subseteq B^\infty_{2^n}(l) := \{k \in X : |k - l|_\infty \leq 2^n\}$. Since $X \subset \mathbb{R}^d$ is relatively separated, we have that $\gamma := \sup_{x\in\mathbb{R}^d} |X \cap (x + [0, 1]^d)|$ is finite. Furthermore, since $B^\infty_{2^n}(l)$ can be covered by $(2 \cdot 2^n)^d$ many translated unit cubes, we see that $|B^\infty_{2^n}(l)| \leq (2 \cdot 2^n)^d\gamma$. This proves the claim.

Step 3. We estimate the first summand of the right hand side of (25). Let $l \in X$ be arbitrary. Then

$$\sum_{\substack{k\in X\\ |k-l|\leq 2^n}} \|[A^{n+1}]_{k,l}\| = \sum_{\substack{k\in X\\ |k-l|\leq 2^n}} \left\| \sum_{m\in X} [A^n]_{k,m} A_{m,l} \right\| \cdot 1$$

$$\leq \left(\sum_{\substack{k\in X \\ |k-l|\leq 2^n}} \left\| \sum_{m\in X} [A^n]_{k,m} A_{m,l} \right\|^2 \right)^{\frac{1}{2}} \left(\sum_{\substack{k\in X \\ |k-l|\leq 2^n}} 1 \right)^{\frac{1}{2}}$$

$$\leq (C_1 2^{nd})^{\frac{1}{2}} \left(\sum_{\substack{k\in X \\ |k-l|\leq 2^n}} \left\| \sum_{m\in X} [A^n]_{k,m} A_{m,l} \right\|^2 \right)^{\frac{1}{2}} \tag{27}$$

by the Cauchy-Schwarz inequality and (26). Next, since the supremum

$$\|[A^{n+1}]_{k,l}\|^2 = \sup_{\|f\|_{\mathcal{H}}=1} \left\| \sum_{m\in X} [A^n]_{k,m} A_{m,l} f \right\|_{\mathcal{H}}^2$$

exists for each k, we know from the property of suprema that for each $\varepsilon^{(k)} > 0$, there exists some unit norm vector $f^{(k)} \in \mathcal{H}$, such that

$$\left\| \sum_{m\in X} [A^n]_{k,m} A_{m,l} \right\|^2 - \varepsilon^{(k)} < \left\| \sum_{m\in X} [A^n]_{k,m} A_{m,l} f^{(k)} \right\|^2 .$$

Since $A \in \mathcal{B}$ by Lemma 3.3 and $A \neq 0$ by assumption we can choose $\varepsilon^{(k)} = \|A\|_{\mathcal{B}}^{2n}(1+|k|)^{-(d+1)} > 0$ and may estimate the sum on the right hand side of (27) via

$$\sum_{\substack{k\in X \\ |k-l|\leq 2^n}} \left\| \sum_{m\in X} [A^n]_{k,m} A_{m,l} \right\|^2$$

$$\leq \|A\|_{\mathcal{B}}^{2n} \sum_{\substack{k\in X \\ |k-l|\leq 2^n}} (1+|k|)^{-(d+1)} + \sum_{\substack{k\in X \\ |k-l|\leq 2^n}} \left\| \sum_{m\in X} [A^n]_{k,m} A_{m,l} f^{(k)} \right\|^2$$

$$\leq \|A\|_{\mathcal{B}}^{2n} \sum_{k\in X} (1+|k|)^{-(d+1)} + \sum_{\substack{k\in X \\ |k-l|\leq 2^n}} \sum_{j\in X} \left\| \sum_{m\in X} [A^n]_{j,m} A_{m,l} f^{(k)} \right\|^2$$

$$\leq C_2 \|A\|_{\mathcal{B}}^{2n} + \sum_{\substack{k\in X \\ |k-l|\leq 2^n}} \sum_{j\in X} \left\| \sum_{m\in X} [A^n]_{j,m} A_{m,l} f^{(k)} \right\|^2 , \tag{28}$$

where we applied Lemma 2.3 for $s = d+1$ in the last estimate. Now, note that since each $f^{(k)} \in \mathcal{H}$ is unit norm and since $A \in \mathcal{S}^2$ (as a consequence of $\mathcal{S}^1_{\nu_\delta} \subseteq \mathcal{S}^1 \subseteq \mathcal{S}^2$) we have by an easy computation that $(A_{m,l} f^{(k)})_{m\in X} \in \ell^2(X;\mathcal{H})$ (for each $l \in X$). Thus, since $A^n \in \mathcal{B}$ by Lemma 3.3,

$$\sum_{j\in X} \left\| \sum_{m\in X} [A^n]_{j,m} A_{m,l} f^{(k)} \right\|^2 \leq \|A^n\|_{\mathcal{B}}^2 \|(A_{m,l} f^{(k)})_{m\in X}\|_{\ell^2(X;\mathcal{H})}^2$$

$$\leq \|A\|_{\mathcal{B}}^{2n} \|A\|_{\mathcal{S}^2}^2$$

holds for each k. Combining the latter estimate with (28) yields

$$\begin{aligned}\sum_{\substack{k\in X\\|k-l|\le 2^n}} \left\| \sum_{m\in X} [A^n]_{k,m} A_{m,l} \right\|^2 &\le C_2 \|A\|_{\mathcal{B}}^{2n} + \sum_{\substack{k\in X\\|k-l|\le 2^n}} \|A\|_{\mathcal{B}}^{2n} \|A\|_{\mathcal{S}^2}^2 \\ &= C_2\|A\|_{\mathcal{B}}^{2n} + |B_{2^n}(l)| \|A\|_{\mathcal{B}}^{2n} \|A\|_{\mathcal{S}^2}^2 \\ &\le C_2\|A\|_{\mathcal{B}}^{2n} + C_1 2^{nd} \|A\|_{\mathcal{B}}^{2n} \|A\|_{\mathcal{S}^2}^2, \end{aligned} \tag{29}$$

where we applied (26) in the last estimate. Combining (29) with (27) and taking the supremum over all $l \in X$ yields

$$\begin{aligned}\sup_{l\in X} \sum_{\substack{k\in X\\|k-l|\le 2^n}} \|[A^{n+1}]_{k,l}\| &\le \left(C_1C_2 2^{nd}\|A\|_{\mathcal{B}}^{2n} + C_1^2 4^{nd} \|A\|_{\mathcal{B}}^{2n}\|A\|_{\mathcal{S}^2}^2\right)^{\frac{1}{2}} \\ &\le (C_1C_2)^{\frac{1}{2}} 2^{\frac{nd}{2}} \|A\|_{\mathcal{B}}^n + C_1 2^{nd} \|A\|_{\mathcal{B}}^n \|A\|_{\mathcal{S}^2}, \end{aligned} \tag{30}$$

where we used the subadditivity of the square root function on $[0, \infty)$.

Step 4. We estimate the second summand of the right-hand side of (25). For $|k - l| > 2^n$ we have $(1 + |k - l|)^\delta > (1 + 2^n)^\delta > 2^{n\delta}$ and consequently $1 < 2^{-n\delta}\nu_\delta(k - l)$. Therefore,

$$\begin{aligned}\sup_{l\in X} \sum_{\substack{k\in X\\|k-l|> 2^n}} \|[A^{n+1}]_{k,l}\| &\le 2^{-n\delta} \sup_{l\in X} \sum_{\substack{k\in X\\|k-l|> 2^n}} \|[A^{n+1}]_{k,l}\|\nu_\delta(k-l) \\ &\le 2^{-n\delta} \|A^{n+1}\|_{\mathcal{S}^1_{\nu_\delta}}. \end{aligned}$$

Step 5. After combining the main estimates from Step 1, Step 3 and Step 4, we obtain that

$$\begin{aligned}&\|A^{n+1}\|_{\mathcal{S}^1}^{\frac{1}{n+1}} \\ &\le \left((C_1C_2)^{\frac{1}{2}} 2^{\frac{nd}{2}} \|A\|_{\mathcal{B}}^n + C_1 2^{nd}\|A\|_{\mathcal{B}}^n \|A\|_{\mathcal{S}^2} + 2^{-n\delta} \|A^{n+1}\|_{\mathcal{S}^1_{\nu_\delta}} \right)^{\frac{1}{n+1}} \\ &\le (C_1C_2)^{\frac{1}{2(n+1)}} 2^{\frac{nd}{2(n+1)}} \|A\|_{\mathcal{B}}^{\frac{n}{n+1}} + C_1^{\frac{1}{n+1}} 2^{\frac{nd}{n+1}} \|A\|_{\mathcal{B}}^{\frac{n}{n+1}} \|A\|_{\mathcal{S}^2}^{\frac{1}{n+1}} + 2^{\frac{-n\delta}{n+1}} \|A^{n+1}\|_{\mathcal{S}^1_{\nu_\delta}}^{\frac{1}{n+1}}, \end{aligned}$$

where we used the subadditivity of $f(x) = x^{\frac{1}{n+1}}$ on $[0, \infty)$ in the second estimate. Since n was arbitrary, the latter estimate holds for all $n \in \mathbb{N}$. Letting $n \to \infty$ yields via Gelfand's formula

$$r_{\mathcal{S}^1}(A) \le (2^{d/2} + 2^d)\|A\|_{\mathcal{B}} + 2^{-\delta} r_{\mathcal{S}^1_{\nu_\delta}}(A).$$

By Lemma 3.6 we have $r_{\mathcal{S}^1_\nu}(A) = r_{\mathcal{S}^1}(A)$. Hence the above estimate implies that

$$r_{\mathcal{S}^1}(A) \le (1 - 2^{-\delta})^{-1}(2^{d/2} + 2^d)\|A\|_{\mathcal{B}}. \tag{31}$$

Step 6. Finally, since A was chosen arbitrarily, (31) holds for all (non-zero) self-adjoint $A \in \mathcal{S}^1_{\nu_\delta}$. Therefore we have

$$r_{\mathcal{S}^1}(A) = r_{\mathcal{S}^1}(A^n)^{\frac{1}{n}} \le \left((1 - 2^{-\delta})^{-1}(2^{d/2} + 2^d)\right)^{\frac{1}{n}} \|A^n\|_{\mathcal{B}}^{\frac{1}{n}}$$

for all $n \in \mathbb{N}$ and all $A = A^* \in \mathcal{S}^1_{\nu_\delta}$. Letting $n \to \infty$ yields

$$r_{\mathcal{S}^1}(A) \le r_{\mathcal{B}}(A) \qquad (\forall A = A^* \in \mathcal{S}^1_{\nu_\delta}).$$

Conversely, by (4), $r_{\mathcal{S}^1}(A) \ge r_{\mathcal{B}}(A)$ holds true, since $\mathcal{S}^1_{\nu_\delta} \subseteq \mathcal{S}^1 \subseteq \mathcal{B}$. This completes the proof. $\square$

Following the ideas from [38], we can extend the above statement to a more general class of weights and obtain a broader class of Schur-type algebras which are inverse-closed in $\mathcal{B}(\ell^2(X;\mathcal{H}))$.

We need the following preparatory result.

Lemma 3.8. *[38, Lemma 8] For any unbounded admissible weight function ν there exists a sequence of admissible weights ν_n with the following properties:*

(a) $\nu_{n+1} \leq \nu_n \leq \nu$ *for all* $n \in \mathbb{N}$.
(b) *There exist* $c_n > 0$ *such that* $\nu \leq c_n \nu_n$ *for all* $n \in \mathbb{N}$.
(c) $\lim_{n\to\infty} \nu_n = 1$ *uniformly on compact sets of* $\mathbb{R}^d$.

In particular, all the weights ν_n are equivalent and

$$r_{\mathcal{S}^1_\nu}(A) = r_{\mathcal{S}^1_{\nu_n}}(A) \qquad (\forall A \in \mathcal{S}^1_\nu, \forall n \in \mathbb{N}).$$

The proof of the next lemma can be adapted almost word for word from the proof of [38, Lemma 9 (a)].

Lemma 3.9. *Let ν be an admissible weight function satisfying the weak growth condition*

$$\nu(x) \geq C(1+|x|)^\delta \qquad \textit{for some } \delta \in (0,1], C > 0.$$

Then, with ν_n is in Lemma 3.8, it holds

$$\lim_{n\to\infty} \|A\|_{\mathcal{S}^1_{\nu_n}} = \|A\|_{\mathcal{S}^1_\nu} \qquad (\forall A = A^* \in \mathcal{S}^1_\nu).$$

Having the previous two lemmata and Theorem 3.7 at hand, we can prove the following analogue of [38, Theorem 6, Corollary 7].

Theorem 3.10. *Let ν be an admissible weight satisfying the weak growth condition*

$$\nu(x) \geq C(1+|x|)^\delta \qquad \textit{for some } \delta \in (0,1], C > 0.$$

Then

$$r_{\mathcal{S}^1_\nu}(A) = r_{\mathcal{B}(\ell^2(X;\mathcal{H}))}(A) \qquad (\forall A = A^* \in \mathcal{S}^1_\nu). \tag{32}$$

In particular, $\mathcal{S}^1_\nu$ is inverse-closed in $\mathcal{B}(\ell^2(X;\mathcal{H}))$ and $\mathcal{S}^1_\nu$ is a symmetric Banach algebra.

Proof. We follow the arguments of [38, Lemma 9 (b)].

By the equivalence of the weights ν_n from Lemma 3.8 we have

$$r_{\mathcal{S}^1_\nu}(A)^m = r_{\mathcal{S}^1_\nu}(A^m) = r_{\mathcal{S}^1_{\nu_n}}(A^m) \leq \|A^m\|_{\mathcal{S}^1_{\nu_n}} \qquad (\forall A \in \mathcal{S}^1_\nu, \forall m, n \in \mathbb{N}).$$

By Lemma 3.9, this implies that

$$r_{\mathcal{S}^1_\nu}(A)^m \leq \lim_{n\to\infty} \|A^m\|_{\mathcal{S}^1_{\nu_n}} = \|A^m\|_{\mathcal{S}^1} \qquad (\forall A = A^* \in \mathcal{S}^1_\nu, \forall m \in \mathbb{N}).$$

Thus, taking m-th roots and letting $m \to \infty$ yields

$$r_{\mathcal{S}^1_\nu}(A) \leq r_{\mathcal{S}^1}(A) \qquad (\forall A = A^* \in \mathcal{S}^1_\nu).$$

However, since ν is weakly growing by assumption, we have $\mathcal{S}^1_\nu \subseteq \mathcal{S}^1_{\nu_\delta}$. Therefore, Theorem 3.7 implies that

$$r_{\mathcal{S}^1_\nu}(A) \leq r_{\mathcal{S}^1}(A) = r_{\mathcal{B}(\ell^2(X;\mathcal{H}))}(A) \qquad (\forall A = A^* \in \mathcal{S}^1_\nu).$$

Conversely, by (4), we have $r_{\mathcal{B}(\ell^2(X;\mathcal{H}))}(A) \leq r_{\mathcal{S}^1_\nu}(A)$ for all $A = A^* \in \mathcal{S}^1_\nu$ since $\mathcal{S}^1_\nu \subseteq \mathcal{B}(\ell^2(X;\mathcal{H}))$. Thus, (32) follows. Now, having established the desired equation of spectral radii, the rest of the theorem follows from Hulanicki's lemma (Proposition 2.1). $\square$

Remark 3.11. 1.) The weak growth condition in Theorem 3.10 is essential. In fact, by giving a concrete counterexample, Tessera proved in [61] that the unweighted scalar-valued Schur algebra $\mathcal{S}^1(\mathbb{N})$ is *not* inverse-closed in $\mathcal{B}(\ell^p(\mathbb{N};\mathbb{C}))$ for each $1 < p < \infty$. As emphasized in the same reference, $\mathcal{S}^1(\mathbb{N})$ is not even symmetric [43].

2.) Sun proved in [58] the inverse-closedness of the scalar-valued versions of $\mathcal{S}^p_\nu(X)$ in $\mathcal{B}(\ell^2(X;\mathbb{C}))$. We believe that the methods from [58] can be used to obtain analogous results in the $\mathcal{B}(\mathcal{H})$-valued setting. We leave a detailed investigation for future research.

4. Polynomial and more general off-diagonal decay conditions

After proving that the Schur algebra $\mathcal{S}^1_\nu$ is inverse-closed in $\mathcal{B}(\ell^2(X;\mathcal{H}))$, we can follow the ideas from [38] and derive more examples of Wiener pairs $\mathcal{A} \subseteq \mathcal{B}(\ell^2(X;\mathcal{H}))$. Most notably, we will consider a $\mathcal{B}(\mathcal{H})$-valued version of the Jaffard algebra and prove its inverse-closedness in $\mathcal{B}(\ell^2(X;\mathcal{H}))$.

Let $X \subset \mathbb{R}^d$ be relatively separated and ν be an admissible weight in the sense of Definition 2.2. Then we define $\mathcal{J}_\nu = \mathcal{J}_\nu(X)$ to be the space of all $\mathcal{B}(\mathcal{H})$-valued matrices $A = [A_{k,l}]_{k,l\in X}$ for which there exists $C > 0$ such that

$$\|A\|_{\mathcal{J}_\nu} := \sup_{k,l\in X} \|A_{k,l}\| \cdot \nu(k-l) \leq C. \tag{33}$$

Again, it is easy to see that (33) indeed defines a norm on $\mathcal{J}_\nu$ and that $(\mathcal{J}_\nu, \|\,.\,\|_{\mathcal{J}_\nu})$ is precisely the Bochner sequence space $\ell^\infty_u(X \times X; \mathcal{B}(\mathcal{H}))$, where $u(k,l) = \nu(k-l)$, and thus a Banach space.

The proof of the following lemma is analogous to the proof of [38, Lemma 1 (b)] and is therefore omitted.

Lemma 4.1. *Let ν be an admissible weight. If*

$$\sup_{k\in X} \sum_{l\in X} \nu(k-l)^{-1} < \infty \tag{34}$$

then $\mathcal{J}_\nu$ is contained in $\mathcal{S}^1$. If additionally $\exists C > 0$ such that

$$\sum_{n\in X} \nu(k-n)^{-1}\nu(n-l)^{-1} < C\nu(k-l) \qquad (\forall k,l \in X) \tag{35}$$

*then $\mathcal{J}_\nu$ is a unital *-algebra which satisfies $\mathcal{J}_\nu \subseteq \mathcal{B}(\ell^2(X;\mathcal{H}))$.*

Remark 4.2. Let ν_s be the polynomial weight $\nu_s(x) = (1+|x|)^s$ on $\mathbb{R}^d$ with decay parameter $s > d$. Then ν_s is admissible. Furthermore, by Lemma 2.3, ν_s satisfies conditions (34) and (35). Thus, Lemma 4.1 can be viewed as a generalization of Lemma 2.3. Moreover, in the case $X = \mathbb{Z}^d$, condition (34) is equivalent to $\nu^{-1} \in \ell^1(\mathbb{Z}^d)$ and condition (35) reads $\nu^{-1} * \nu^{-1} \leq C\nu^{-1}$ (i.e. ν^{-1} is *subconvolutive* [25]).

The norm $\|\,.\,\|_{\mathcal{J}_\nu}$ is not submultiplicative. This obstacle can be circumvented by working with the equivalent norm

$$|||A|||_{\mathcal{J}_\nu} := \sup_{\substack{B\in\mathcal{J}_\nu \\ \|B\|_{\mathcal{J}_\nu}=1}} \|A\cdot B\|_{\mathcal{J}_\nu} \qquad (A \in \mathcal{J}_\nu). \tag{36}$$

Next, we consider the $\mathcal{B}(\mathcal{H})$-valued version of the Banach algebra $\mathcal{B}_{u,s}$ from [38]. Let $s > 0$, u be an admissible weight and ν be the admissible weight defined by $\nu(x) = u(x)(1 + |x|)^s$. Then we define $\mathcal{B}_{u,s} = \mathcal{B}_{u,s}(X)$ to be the Banach space $\mathcal{B}_{u,s} := \mathcal{S}^1_u \cap \mathcal{J}_\nu$ equipped with the norm

$$\|A\|_{\mathcal{B}_{u,s}} := 2^s\|A\|_{\mathcal{S}^1_u} + \|A\|_{\mathcal{J}_\nu}. \tag{37}$$

The constant 2^s in (37) stems from the subadditivity of ν_s on $\mathbb{R}^d$ and plays an important role in the proof of the next lemma.

Lemma 4.3. *The Banach space $\mathcal{B}_{u,s}$ is a solid unital Banach *-algebra and $\mathcal{B}_{u,s} \subseteq \mathcal{S}^1_u \subseteq \mathcal{B}(\ell^2(X;\mathcal{H}))$. Moreover, if $s > d$ then $\mathcal{J}_\nu \subseteq \mathcal{S}^1_u$ and thus $\mathcal{B}_{u,s} = \mathcal{J}_\nu$ with equivalent norms.*

Proof. The proof of $\mathcal{B}_{u,s}$ being a Banach *-algebra and the moreover-part is exactly the same as the proof of [38, Lemma 2] after replacing absolute values with operator norms and applying Lemma 4.1 instead of [38, Lemma 1] (Lemma 2.3 is applied as well). The statement $\mathcal{B}_{u,s} \subseteq \mathcal{S}^1_u \subseteq \mathcal{B}(\ell^2(X;\mathcal{H}))$ is obviously true. □

Now we can prove the main theorem of this section. We will establish the proof details (which also appear in [38]) so that the reader can fully grasp the key ideas.

Theorem 4.4. *Assume that u is an admissible weight, let $\delta, s > 0$, $u \geq \nu_\delta$ and set $\nu = u\nu_s$. Then*

$$r_{\mathcal{B}_{u,s}}(A) = r_{\mathcal{B}(\ell^2(X;\mathcal{H}))}(A) \qquad (\forall A = A^* \in \mathcal{B}_{u,s}). \tag{38}$$

In particular, $\mathcal{B}_{u,s}$ is inverse-closed in $\mathcal{B}(\ell^2(X;\mathcal{H}))$ and $\mathcal{B}_{u,s}$ is a symmetric Banach algebra.

Proof. Once again, Hulanicki's Lemma (Proposition 2.1) is the key for proving spectral invariance. The crucial idea for deriving the desired inequality of spectral radii is a proof routine called Brandenburg's trick [17].

By submultiplicativity of u and subadditivity of ν_s, the weight $\nu = u\nu_s$ satisfies the inequality

$$\nu(x + y) \leq 2^s u(x)u(y)(\nu_s(x) + \nu_s(y)) = 2^s\,(\nu(x)u(y) + u(x)\nu(y))$$

on $\mathbb{R}^d$. This implies that for any $A, B \in \mathcal{B}_{u,s} \subseteq \mathcal{J}_\nu$

$$\begin{aligned}
\|AB\|_{\mathcal{J}_\nu} &= \sup_{k,l\in X} \Big\| \sum_{n\in X} A_{k,n}B_{n,l} \Big\| \nu(k - n + n - l) \\
&\leq 2^s \sup_{k,l\in X} \sum_{n\in X} \|A_{k,n}\|\|B_{n,l}\| \big(\nu(k - n)u(n - l) + u(k - n)\nu(n - l)\big) \\
&\leq 2^s(\|A\|_{\mathcal{J}_\nu}\|B\|_{\mathcal{S}^1_u} + \|B\|_{\mathcal{J}_\nu}\|A\|_{\mathcal{S}^1_u}).
\end{aligned}$$

Consequently,

$$\|A^{2n}\|_{\mathcal{J}_\nu} \leq 2^{s+1}\|A^n\|_{\mathcal{J}_\nu}\|A^n\|_{\mathcal{S}^1_u} \qquad (\forall n \in \mathbb{N}).$$

In particular, this yields the estimate

$$\begin{aligned}
\|A^{2n}\|_{\mathcal{B}_{u,s}} &= 2^s\|A^{2n}\|_{\mathcal{S}^1_u} + \|A^{2n}\|_{\mathcal{J}_\nu} \\
&\le 2^s\|A^n\|^2_{\mathcal{S}^1_u} + 2^{s+1}\|A^n\|_{\mathcal{J}_\nu}\|A^n\|_{\mathcal{S}^1_u} \\
&\le 2^{s+1}\|A^n\|_{\mathcal{S}^1_u}(2^s\|A^n\|_{\mathcal{S}^1_u} + \|A^n\|_{\mathcal{J}_\nu}) \\
&= 2^{s+1}\|A^n\|_{\mathcal{S}^1_u}\|A^n\|_{\mathcal{B}_{u,s}} \qquad (\forall n \in \mathbb{N}).
\end{aligned}$$

Taking $2n$-th roots and letting $n \to \infty$ yields via Gelfand's formula that

$$r_{\mathcal{B}_{u,s}}(A) \le \lim_{n\to\infty} 2^{\frac{s+1}{2n}}\|A^n\|^{\frac{1}{2n}}_{\mathcal{S}^1_u}\|A^n\|^{\frac{1}{2n}}_{\mathcal{B}_{u,s}} = r_{\mathcal{S}^1_u}(A)^{\frac{1}{2}} r_{\mathcal{B}_{u,s}}(A)^{\frac{1}{2}}$$

and consequently

$$r_{\mathcal{B}_{u,s}}(A) \le r_{\mathcal{S}^1_u}(A) \qquad (\forall A \in \mathcal{B}_{u,s}).$$

In particular, since $\mathcal{S}^1_u \subseteq \mathcal{S}^1_{\nu_\delta}$, we can apply Theorem 3.7 and obtain that

$$r_{\mathcal{B}_{u,s}}(A) \le r_{\mathcal{S}^1_u}(A) = r_{\mathcal{S}^1}(A) = r_{\mathcal{B}(\ell^2(X;\mathcal{H}))}(A) \qquad (\forall A = A^* \in \mathcal{B}_{u,s}).$$

Now, an application of Hulanicki's lemma (Proposition 2.1) yields the desired statement. □

Finally, we consider the polynomial weight ν_s for $s > d$ and call $\mathcal{J}_s := \mathcal{J}_{\nu_s}$ the *Jaffard algebra*. By Remark 4.2, and Lemma 4.3, the Jaffard algebra is (up to norm-equivalence via (36)) a unital Banach *-algebra whenever $s > d$. By combining Lemma 4.3 with Theorem 4.4, we obtain the following generalization of Jaffard's Lemma [42] to the $\mathcal{B}(\mathcal{H})$-valued setting.

Corollary 4.5 *(Jaffard's Lemma). For every $s > d$, the Jaffard algebra $(\mathcal{J}_s, |||\,.\,|||_{\mathcal{J}_s})$ is inverse-closed in $\mathcal{B}(\ell^2(X;\mathcal{H}))$. In particular, $(\mathcal{J}_s, |||\,.\,|||_{\mathcal{J}_s})$ is a symmetric Banach algebra whenever $s > d$.*

5. The Baskakov-Gohberg-Sjöstrand algebra

In this section, we discuss the $\mathcal{B}(\mathcal{H})$-valued *Baskakov-Gohberg-Sjöstrand algebra* and weighted versions of it [10,29,54]. This time we work with the relatively separated index set $X = \mathbb{Z}^d$. Note that some of the results appearing in this section have also been stated in [44]. Therefore, we will avoid some of the proof details and refer instead to [44] and other related references.

For a weight function ν on $\mathbb{R}^d$, let $\mathcal{C}_\nu$ be the space of $\mathcal{B}(\mathcal{H})$-valued matrices $A = [A_{k,l}]_{k,l\in\mathbb{Z}^d}$ for which

$$\|A\|_{\mathcal{C}_\nu} := \sum_{l\in\mathbb{Z}^d} \sup_{k\in\mathbb{Z}^d} \|A_{k,k-l}\|\nu(l) < \infty. \tag{39}$$

It is immediately clear, that (39) defines indeed a norm on $\mathcal{C}_\nu$. In fact

$$\|A\|_{\mathcal{C}_\nu} = \Big\| \Big(\big\|(\|A_{k,k-l}\|)_{k\in\mathbb{Z}^d}\big\|_{\ell^\infty(\mathbb{Z}^d)} \Big)_{l\in\mathbb{Z}^d} \Big\|_{\ell^1_\nu(\mathbb{Z}^d)},$$

hence $\mathcal{C}_\nu$ is isometrically isomorphic to the space $\ell^1_\nu(\mathbb{Z}^d;\ell^\infty(\mathbb{Z}^d;\mathcal{B}(\mathcal{H})))$ and thus a Banach space.

The following notation will be useful to us: For $A = [A_{k,l}]_{k,l\in\mathbb{Z}^d} \in \mathcal{C}_\nu$, let $d_A(l)$ be the supremum of the operator norms of the entries of A along its l-th side diagonal, that is,

$$d_A(l) = \sup_{k\in\mathbb{Z}^d} \|A_{k,k-l}\| \qquad (l \in \mathbb{Z}^d). \tag{40}$$

The supremum $d_A(l)$ exists for each l and, in fact, $\|d_A\|_{\ell^1_\nu(\mathbb{Z}^d)} = \|A\|_{\mathcal{C}_\nu}$. Furthermore, as we are in a Hilbert space setting (see also Subsection 2.5 and [50, Theorem 4.16]), $d_A(l)$ is the operator-norm of the n-th side-diagonal $D_A(n)$ of A, given by

$$[D_A(n)]_{k,l} = \begin{cases} A_{k,l} & \text{if } l = k-n \\ 0 & \text{if } l \neq k-n \end{cases}. \tag{41}$$

Proposition 5.1. *Let m be a ν-moderate weight. Then each $A \in \mathcal{C}_\nu$ defines a bounded operator on $\mathcal{B}(\ell^p_m(X;\mathcal{H}))$ for each $1 \leq p \leq \infty$.*

Proof. We follow the proof steps of the exposition [33]. For $g = (g_l)_{l\in\mathbb{Z}^d} \in \ell^p_m(X;\mathcal{H})$ we have $c = (c_l)_{l\in\mathbb{Z}^d} := (\|g_l\|)_{l\in\mathbb{Z}^d} \in \ell^p_m(\mathbb{Z}^d)$. With the notation from (40) we obtain the component-wise estimate

$$\begin{aligned} \|[Ag]_k\| &\leq \sum_{l\in\mathbb{Z}^d} \|A_{k,l}\|\,\|g_l\| \\ &\leq \sum_{l\in\mathbb{Z}^d} d_A(k-l)c_l \\ &= (d_A * c)(k) \qquad (\forall k \in \mathbb{Z}^d). \end{aligned}$$

Therefore, the claim follows from Young's inequality for weighted ℓ^p-spaces (see e.g. [30]), as we have

$$\begin{aligned} \|Ag\|_{\ell^p_m(X;\mathcal{H})} &= \|(\|[Ag]_k\|)_{k\in\mathbb{Z}^d}\|_{\ell^p_m(\mathbb{Z}^d)} \\ &\leq \|d_A * c\|_{\ell^p_m(\mathbb{Z}^d)} \\ &\leq C\|d_A\|_{\ell^1_\nu(\mathbb{Z}^d)}\|c\|_{\ell^p_m(\mathbb{Z}^d)} = C\|A\|_{\mathcal{C}_\nu}\|g\|_{\ell^p_m(X;\mathcal{H})}, \end{aligned}$$

where the constant C stems from the assumption that m is ν-moderate. □

By the above, we may call the Baskakov-Gohberg-Sjöstrand algebra the *algebra of convolution-dominated matrices* [33].

Proposition 5.2. *[44] For any submultiplicative and symmetric weight ν, $\mathcal{C}_\nu$ is a unital Banach *-algebra with respect to matrix multiplication and involution defined as in (12). In particular, $\mathcal{C}_\nu \subseteq \mathcal{B}(\ell^2(X;\mathcal{H}))$.*

Proof. Again we follow the proof steps of the exposition [33]; the verification of $\mathcal{C}_\nu$ being a Banach algebra has already been established in [11, Lemma 2].

We have already observed that $\mathcal{C}_\nu$ is a Banach space. Let $A, B \in \mathcal{C}_\nu$. Then, with the terminology from (40) we have that

$$\begin{aligned} \|[AB]_{k,k-l}\| &\leq \sum_{m\in\mathbb{Z}^d} \|A_{k,m}\|\,\|B_{m,k-l}\| \\ &\leq \sum_{m\in\mathbb{Z}^d} d_A(k-m)\,d_B(l-(k-m)) \\ &= (d_A * d_B)(l), \end{aligned}$$

which yields the pointwise estimate

$$d_{AB}(l) \leq (d_A * d_B)(l) \qquad (\forall l \in \mathbb{Z}^d).$$

Thus, since ν is submultiplicative, Young's inequality yields

$$\begin{aligned}\|AB\|_{\mathcal{C}_\nu} &= \|d_{AB}\|_{\ell^1_\nu(\mathbb{Z}^d)} \\ &\leq \|d_A * d_B\|_{\ell^1_\nu(\mathbb{Z}^d)} \\ &\leq \|d_A\|_{\ell^1_\nu(\mathbb{Z}^d)}\|d_B\|_{\ell^1_\nu(\mathbb{Z}^d)} = \|A\|_{\mathcal{C}_\nu}\|B\|_{\mathcal{C}_\nu},\end{aligned}$$

hence $\mathcal{C}_\nu$ is a Banach algebra. The fact that the involution is an isometry follows from the symmetry of ν. Indeed, since

$$d_{A^*}(l) = \sup_{k\in\mathbb{Z}^d} \|[A^*]_{k,k-l}\| = \sup_{k\in\mathbb{Z}^d} \|(A_{k-l,k})^*\| = \sup_{k\in\mathbb{Z}^d} \|A_{k,k+l}\| = d_A(-l)$$

we see that

$$\begin{aligned}\|A^*\|_{\mathcal{C}_\nu} &= \sum_{l\in\mathbb{Z}^d} d_{A^*}(l)v(l) \\ &= \sum_{l\in\mathbb{Z}^d} d_A(-l)v(l) \\ &= \sum_{l\in\mathbb{Z}^d} d_A(-l)v(-l) = \|A\|_{\mathcal{C}_\nu}.\end{aligned}$$

Since $v(0) = 1$, $\|\mathcal{I}_{\mathcal{B}(\ell^2(X;\mathcal{H}))}\|_{\mathcal{C}_\nu} = 1$, hence $\mathcal{C}_\nu$ has an identity. The fact that $\mathcal{C}_\nu \subseteq \mathcal{B}(\ell^2(X;\mathcal{H}))$ follows from Proposition 5.1. $\square$

As before, verifying that $\mathcal{C}_\nu \subseteq \mathcal{B}(\ell^2(X;\mathcal{H}))$ forms a Wiener pair is a subtle task. However, since this has already been done by Baskakov [11] in 1997, we will only outline his key arguments adapted to our setting (compare with [44] or the proof outline in [33] of the scalar-valued case).

At the heart of Baskakov's proof [11] lies a generalization of the classical Wiener's Lemma [63] to its analogue on absolutely convergent Fourier series with coefficients in a (possibly non-commutative) Banach algebra by Bochner and Phillips [16, Theorem 1].

Given $t \in \mathbb{R}^d$, let $M_t \in \mathcal{B}(\ell^2(X;\mathcal{H}))$ be the *modulation operator* defined as diagonal matrix

$$M_t = \operatorname{diag}[e^{2\pi i k\cdot t}\mathcal{I}_{\mathcal{B}(\mathcal{H})}]_{k\in\mathbb{Z}^d}.$$

Then M_t acts on $(g_l)_{l\in\mathbb{Z}^d} \in \ell^2(X;\mathcal{H})$ via $M_t(g_l)_{l\in\mathbb{Z}^d} = (e^{2\pi i l\cdot t}g_l)_{l\in\mathbb{Z}^d}$ and

$$M : \mathbb{T}^d \longrightarrow \mathcal{B}(\ell^2(X;\mathcal{H})), \quad t \mapsto M_t$$

is a unitary representation of the d-dimensional torus $\mathbb{T}^d$, which is the dual group of the abelian group $\mathbb{Z}^d$ – our underlying index set. Then, to each $A = [A_{k,l}]_{k,l\in\mathbb{Z}^d} \in \mathcal{B}(\ell^2(X;\mathcal{H}))$, we assign the $\mathcal{B}(\ell^2(X;\mathcal{H}))$-valued 1-periodic function

$$f_A(t) := M_t A M_{-t} \qquad (t \in \mathbb{R}^d),$$

i.e. we may identify f_A with a $\mathcal{B}(\ell^2(X;\mathcal{H}))$-valued function on $\mathbb{T}^d$. Note that the canonical matrix representation of each operator $f_A(t)$ is given by

$$\mathbb{M}(f_A(t)) = [A_{k,l}e^{2\pi i(k-l)\cdot t}]_{k,l\in\mathbb{Z}^d}. \tag{42}$$

Now, if A is contained in the unweighted Baskakov-Gohberg-Sjöstrand algebra $\mathcal{C} = \mathcal{C}_1$, then

$$f_A(t) = \sum_{n\in\mathbb{Z}^d} D_A(n)e^{2\pi i n\cdot t} \tag{43}$$

admits a Fourier series of operators, where $D_A(n)$ is the n-th side-diagonal of A is defined in (41), and the Fourier series (43) converges absolutely in the operator norm, since

$$\|f_A(t)\|_{\mathcal{B}(\ell^2(X;\mathcal{H}))} \leq \sum_{n\in\mathbb{Z}^d} \|D_A(n)\|_{\mathcal{B}(\ell^2(X;\mathcal{H}))} = \sum_{n\in\mathbb{Z}^d} d_A(n) = \|A\|_{\mathcal{C}}. \tag{44}$$

Note that the expansion (43) can be deduced directly from the matrix representation (42) of f_A. Now assume that $A \in \mathcal{C}$ is invertible in $\mathcal{B}(\ell^2(X;\mathcal{H}))$. Then each operator $f_A(t)$ is invertible in $\mathcal{B}(\ell^2(X;\mathcal{H}))$ with inverse given by $f_A(t)^{-1} = f_{A^{-1}}(t) = M_t A^{-1} M_{-t}$. Now Bochner's and Phillips' theorem [16, Theorem 1] is applicable and guarantees that $f_{A^{-1}}(t)$ admits an absolutely convergent operator-valued Fourier series for each $t \in \mathbb{T}^d$, whose coefficients are precisely given by the side diagonals $D_{A^{-1}}(n)$ of A^{-1}. Consequently (compare with (44)), $A^{-1} \in \mathcal{C}$ and thus $\mathcal{C}$ is inverse-closed in $\mathcal{B}(\ell^2(X;\mathcal{H}))$.

In the weighted case $\mathcal{C}_\nu$ the proof idea is essentially the same. For the inverse-closedness of $\mathcal{C}_\nu$ in $\mathcal{B}(\ell^2(X;\mathcal{H}))$ to hold, ν needs to satisfy the GRS-condition, since then the Banach algebra $L_\nu(\mathbb{Z}^d)$ as defined in [11] is semi-simple. Instead of the Bochner-Phillips theorem from [16], a more abstractly formulated version, namely [11, Theorem 1], is applied.

We refer the reader to [11] as well as [44] for more details; compare also with [10].

Theorem 5.3. *[11,44] Let ν be a submultiplicative and symmetric weight satisfying the GRS-condition. Then $\mathcal{C}_\nu \subseteq \mathcal{B}(\ell^2(\mathbb{Z}^d;\mathcal{H}))$ forms a Wiener pair. In particular, $\mathcal{C}_\nu$ is a symmetric Banach algebra.*

Remark 5.4. In [13, Theorem 7.2] quantitative estimates for $\|A^{-1}\|_{\mathcal{C}_\nu}$ have been derived for continuous analogs of $\mathcal{C}_\nu$. As described in the last paragraph of the introduction, we believe that a discrete analogue of [13, Theorem 7.2] can be derived via discretization as mentioned in [14, Remark 5.6]. Moreover, we expect that methods of [12] could be used to obtain quantitative (i.e., stronger) statements regarding other Wiener pairs discussed so far. We leave a detailed investigation of that matter to future research.

6. Anisotropic decay conditions

In this section, we will derive further examples of Wiener pairs from a given Wiener pair $\mathcal{A} \subseteq \mathcal{B}(\ell^2(X;\mathcal{H}))$, where we assume $\mathcal{A}$ to be *solid*, meaning that if $A = (A_{k,l})_{k,l\in X} \in \mathcal{A}$ and $B = (B_{k,l})_{k,l\in X}$ is a $\mathcal{B}(\mathcal{H})$-valued matrix such that $\|B_{k,l}\| \leq \|A_{k,l}\|$ for all $k,l \in X$, then $B \in \mathcal{A}$ and $\|B\|_{\mathcal{A}} \leq \|A\|_{\mathcal{A}}$. This will be done by considering derivations on $\mathcal{A}$ (see Section 2.1). The analysis of this subsection is based on the article [37] by Gröchenig and Klotz. Hence, we will often only point out the main ideas and refer the reader to [37] for the details.

Proposition 6.1. *[37, Theorem 3.4] Let $\mathcal{A}$ be a symmetric unital Banach (*-) algebra and $\delta : \mathcal{D}(\delta) \longrightarrow \mathcal{A}$ be a symmetric derivation on $\mathcal{A}$ with $\mathcal{I}_{\mathcal{A}} \in \mathcal{D}(\delta)$. Then $\mathcal{D}(\delta) \subset \mathcal{A}$ is a symmetric unital Banach (*-) algebra and in particular a Wiener pair. Moreover, the quotient rule*

$$\delta(A^{-1}) = -A^{-1}\delta(A)A^{-1}$$

is valid and yields the explicit norm estimate

$$\|A^{-1}\|_{\mathcal{D}(\delta)} \leq \|A^{-1}\|^2_{\mathcal{A}}\|A\|_{\mathcal{D}(\delta)}$$

for all $A \in \mathcal{D}(\delta)$.

The above statement can be generalized by iterative means via *commuting derivations* [37]. More precisely, we iteratively define compositions of derivations via

$$\delta_1 \dots \delta_n : \mathcal{D}(\delta_1 \dots \delta_n) = \mathcal{D}(\delta_1, \mathcal{D}(\delta_2 \dots \delta_n)) \longrightarrow \mathcal{D}(\delta_2 \dots \delta_n) \subseteq \mathcal{A},$$

where we assume that the operators δ_n commute pairwise and that $\mathcal{D}(\delta_1 \dots \delta_n)$ is independent of the order of the operators δ_n $(1 \leq n \leq d)$. Then for any multi-index $\alpha = (\alpha_j)_{j=1}^d \in \mathbb{N}_0^d$, the operator $\delta^\alpha = \prod_{j=1}^d \delta_j^{\alpha_j}$ and its domain $\mathcal{D}(\delta^\alpha)$ are well-defined. We equip $\mathcal{D}(\delta^\alpha)$ with the norm

$$\|A\|_{\mathcal{D}(\delta^\alpha)} = \sum_{\beta \leq \alpha} \|\delta^\beta(A)\|_{\mathcal{A}},$$

which is just the (iteratively defined) graph norm on the corresponding preceding derivation algebra $\mathcal{D}(\delta^\beta)$ $(|\beta| = |\alpha| - 1)$. Furthermore, the Leibniz rule (5) implies the generalized Leibniz rule

$$\delta(AB) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \delta^\beta(A)\delta^{\alpha-\beta}(B) \qquad (\forall A, B \in \mathcal{D}(\delta^\alpha)).$$

This yields

$$\|AB\|_{\mathcal{D}(\delta^\alpha)} \leq C\|A\|_{\mathcal{D}(\delta^\alpha)}\|B\|_{\mathcal{D}(\delta^\alpha)} \qquad (\forall A, B \in \mathcal{D}(\delta^\alpha)),$$

where the positive constant C depends only on α. Hence (see also [37, Lemma 3.6]) $\mathcal{D}(\delta^\alpha)$ is an involutive subalgebra of $\mathcal{A}$. By passing on to the equivalent norm

$$|||A|||_{\mathcal{D}(\delta^\alpha)} := \sup_{\substack{B \in \mathcal{D}(\delta^\alpha) \\ \|B\|_{\mathcal{D}(\delta^\alpha)}=1}} \|A \cdot B\|_{\mathcal{D}(\delta^\alpha)} \qquad (A \in \mathcal{D}(\delta^\alpha)), \tag{45}$$

we obtain that $\mathcal{D}(\delta^\alpha)$ is a Banach *-algebra.

By iterative applications of Proposition 6.1, one can show the following:

Proposition 6.2. *[37, Proposition 3.7]) Let $\mathcal{A}$ be a symmetric Banach algebra and let $\{\delta_1, \dots, \delta_d\}$ be a set of commuting derivations with $\mathcal{I}_\mathcal{A} \in \mathcal{D}(\delta_j)$ for all $1 \leq j \leq d$. Then $\mathcal{D}(\delta^\alpha)$ is inverse-closed in $\mathcal{A}$ for each multi-index $\alpha \in \mathbb{N}_0^d$.*

Following [37], we may apply the above machinery by considering derivations defined by the commutator with respect to a suitable diagonal matrix. Similar ideas have also been pursued in [14, Section 5.5]. Note that the commutator (47) considered in the following is essentially the infinitesimal generator of the group that defines the functions f_A in Section 5 and $\check{\check{T}}(h)$ of [14, Section 5.5], respectively.

Theorem 6.3. *Let $\mathcal{A} \subseteq \mathcal{B}(\ell^2(X;\mathcal{H}))$ be a Wiener pair and assume that $\mathcal{A}$ is solid. Then for every $\alpha = (\alpha_j)_{j=1}^d \in \mathbb{N}_0^d$, the class of $\mathcal{B}(\mathcal{H})$-valued matrices $A = [A_{k,l}]_{k,l \in X}$ satisfying*

$$\left\| \left[\prod_{j=1}^d (1 + |k_j - l_j|)^{\alpha_j} A_{k,l} \right]_{k,l \in X} \right\|_{\mathcal{A}} < \infty \tag{46}$$

*is a solid unital Banach *-algebra with respect to the norm (45), which is inverse-closed in $\mathcal{B}(\ell^2(X;\mathcal{H}))$ and in particular a symmetric Banach algebra.*

Proof. *Step 1.* For $1 \leq j \leq d$, let M_j be the $\mathcal{B}(\mathcal{H})$-valued diagonal matrix given by $M_j = \operatorname{diag}[k_j \cdot \mathcal{I}_{\mathcal{B}(\mathcal{H})}]_{k\in X}$, where k_j is the j-th coordinate of the index $k = (k_j)_{j=1}^d \in X \subseteq \mathbb{R}^d$. Then the formal commutator

$$\delta_j(A) := [M_j, A] = M_j A - AM_j \tag{47}$$

has the canonical matrix representation

$$\mathbb{M}(\delta_j(A)) = [(k_j - l_j)A_{k,l}]_{k,l\in X}.$$

Step 2. Fix some $j \in \{1, \ldots, d\}$. Consider the subspace

$$\mathcal{D}(\delta_j) := \{A \in \mathcal{A} : \delta_j(A) \in \mathcal{A}\}$$

of $\mathcal{A}$. We will show, that

$$\delta_j : \mathcal{D}(\delta_j) \longrightarrow \mathcal{A}$$

is a derivation on $\mathcal{A}$. It is quickly verified that the Leibniz rule (5) is satisfied by δ_j on $\mathcal{D}(\delta_j)$. It is also clear that $\mathcal{D}(\delta_j)$ is invariant under involution and that $\delta_j(A^*) = \delta_j(A)^*$ for all $A \in \mathcal{D}(\delta_j)$. Hence, to show that δ_j is a derivation on $\mathcal{A}$, we have to verify that δ_j is a closed operator. To this end, assume that $\{A^{(n)}\}_{n=1}^\infty = \{[A_{k,l}^{(n)}]_{k,l\in X}\}_{n=1}^\infty$ is a sequence in $\mathcal{D}(\delta_j)$ which converges in $\mathcal{D}(\delta_j)$ to some $\mathcal{B}(\mathcal{H})$-valued matrix $A = [A_{k,l}]_{k,l\in X}$ and that $\{\delta_j(A^{(n)})\}_{n=1}^\infty = \{[(k_j - l_j)A_{k,l}^{(n)}]_{k,l\in X}\}_{n=1}^\infty$ converges in $\mathcal{A}$ to some $\mathcal{B}(\mathcal{H})$-valued matrix $B = [B_{k,l}]_{k,l\in X}$. Then both of these sequences are Cauchy sequences in $\mathcal{A}$ and thus their respective limits A and B are contained in $\mathcal{A}$. Since the norm on $\mathcal{A}$ is solid and therefore only depends on the operator norms of the entries of the corresponding matrices, this implies that for each $k, l \in X$, the sequence $\{A_{k,l}^{(n)}\}_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{B}(\mathcal{H})$ converging to $A_{k,l} \in \mathcal{B}(\mathcal{H})$. Consequently, the Cauchy sequence $\{(k_j - l_j)A_{k,l}^{(n)}\}_{n=1}^\infty$ converges to $(k_j - l_j)A_{k,l} \in \mathcal{B}(\mathcal{H})$ for each $k, l \in X$. On the other hand, by the same argument we have that $\{(k_j - l_j)A_{k,l}^{(n)}\}_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{B}(\mathcal{H})$ converging to $B_{k,l} \in \mathcal{B}(\mathcal{H})$, for each $k, l \in X$. Thus $B_{k,l} = (k_j - l_j)A_{k,l}$ for all $k, l \in X$, which shows that δ_j is a closed operator.

Step 3. For each $j \in \{1, \ldots, d\}$, $\mathcal{I}_{\mathcal{A}} = \operatorname{diag}[\mathcal{I}_{\mathcal{B}(\mathcal{H})}]_{k\in X} \in \mathcal{D}(\delta_j)$, since $\delta_j(\mathcal{I}_{\mathcal{A}}) = 0 \in \mathcal{A}$, and the operators δ_j commute pairwise. Thus, $\{\delta_1, \ldots, \delta_d\}$ is a set of commuting derivations. Hence, for each multi-index $\alpha \in \mathbb{N}_0^d$, δ^α and its domain $\mathcal{D}(\delta^\alpha)$ are well-defined. Furthermore, the associated norm

$$\|A\|_{\mathcal{D}(\delta^\alpha)} = \sum_{\beta\leq\alpha} \|\delta^\beta(A)\|_{\mathcal{A}} = \sum_{\beta\leq\alpha} \left\| \left[\prod_{j=1}^d |k_j - l_j|^{\beta_j} A_{k,l} \right]_{k,l\in X} \right\|_{\mathcal{A}}$$

is solid, since the norm on $\mathcal{A}$ is solid. In particular, since $\mathcal{A}$ is symmetric due to Hulanicki's Lemma, Proposition 6.2 yields that $\mathcal{D}(\delta^\alpha)$ is inverse-closed in $\mathcal{A}$ and consequently also in $\mathcal{B}(\ell^2(X;\mathcal{H}))$. Another application of Hulanicki's Lemma yields that $\mathcal{D}(\delta^\alpha)$ is symmetric.

Step 4. Finally, condition (46) is equivalent to the condition $[A_{k,l}]_{k,l\in X} \in \mathcal{D}(\delta^\alpha)$. In fact, this follows from the solidity of $\mathcal{A}$ and from $\left[\prod_{j=1}^d (k_j - l_j)^{\alpha_j} A_{k,l}\right]_{k,l\in X} = \delta^\alpha(A)$. This completes the proof. □

Note that all of our previously proven examples of inverse-closed sub-algebras $\mathcal{A}$ of $\mathcal{B}(\ell^2(X;\mathcal{H}))$ are solid. Hence, the above theorem applies to each of these cases. We explicitly state the following corollaries of Theorem 6.3 applied to the Wiener pairs $\mathcal{J}_s \subseteq \mathcal{B}(\ell^2(X;\mathcal{H}))$ (see Corollary 4.5), $\mathcal{S}_\nu^1 \subseteq \mathcal{B}(\ell^2(X;\mathcal{H}))$ (see Theorem 3.10) and $\mathcal{C}_\nu \subseteq \mathcal{B}(\ell^2(\mathbb{Z}^d;\mathcal{H}))$ (see Theorem 5.3) respectively.

Corollary 6.4. *For every $s > d$ and $\alpha = (\alpha_j)_{j=1}^d \in \mathbb{N}_0^d$, the class of $\mathcal{B}(\mathcal{H})$-valued matrices $[A_{k,l}]_{k,l\in X}$ satisfying the anisotropic polynomial decay condition*

$$\|A_{k,l}\| \leq C(1+|k-l|)^{-s} \prod_{j=1}^d (1+|k_j - l_j|)^{-\alpha_j} \qquad (\forall k, l \in X)$$

*is a solid unital Banach *-algebra which is inverse-closed in $\mathcal{B}(\ell^2(X;\mathcal{H}))$ and thus symmetric.*

Corollary 6.5. *Let ν be an admissible weight in the sense of Definition 2.2 which satisfies the weak growth condition*

$$\nu(x) \geq C(1+|x|)^\delta \qquad \text{for some } \delta \in (0,1], C > 0.$$

Then for each $\alpha = (\alpha_j)_{j=1}^d \in \mathbb{N}_0^d$, the class of $\mathcal{B}(\mathcal{H})$-valued matrices $[A_{k,l}]_{k,l\in X}$ satisfying the anisotropic Schur-type conditions

$$\sup_{k\in X} \sum_{l\in X} \prod_{j=1}^d (1+|k_j - l_j|)^{\alpha_j} \|A_{k,l}\| \nu(k-l) < \infty,$$
$$\sup_{l\in X} \sum_{k\in X} \prod_{j=1}^d (1+|k_j - l_j|)^{\alpha_j} \|A_{k,l}\| \nu(k-l) < \infty$$

*is a solid unital Banach *-algebra which is inverse-closed in $\mathcal{B}(\ell^2(X;\mathcal{H}))$ and thus symmetric.*

Corollary 6.6. *For every submultiplicative and symmetric weight satisfying the GRS-condition and every $\alpha = (\alpha_j)_{j=1}^d \in \mathbb{N}_0^d$, the class of $\mathcal{B}(\mathcal{H})$-valued matrices $[A_{k,l}]_{k,l\in\mathbb{Z}^d}$ satisfying the condition*

$$\sum_{l\in\mathbb{Z}^d} \prod_{j=1}^d (1+|l_j|)^{\alpha_j} \sup_{k\in\mathbb{Z}^d} \|A_{k,k-l}\| \nu(l) < \infty$$

*is a solid unital Banach *-algebra which is inverse-closed in $\mathcal{B}(\ell^2(\mathbb{Z}^d;\mathcal{H}))$ and thus symmetric.*

Acknowledgments

The authors thank Hans Feichtinger, Karlheinz Gröchenig, Jakob Holböck, Nicki Holighaus, Andreas Klotz, Ilya Krishtal, Arvin Lamando, and Rossen Nenov for their valuable suggestions and related discussions. The authors also thank the anonymous reviewer for the many helpful references, comments and suggestions.

This work is supported by the project 10.55776/P34624 *"Localized, Fusion and Tensors of Frames"* (LoFT) of the Austrian Science Fund (FWF).

References

[1] A. Aldroubi, A. Baskakov, I. Krishtal, Slanted matrices, Banach frames, and sampling, J. Funct. Anal. 255 (7) (2008) 1667–1691.

[2] N. Atreas, J. Benedetto, C. Karanikas, Local sampling for regular wavelet and Gabor expansions, Sampl. Theory Signal Image Process. 2 (01 2003) 2–24.

[3] R. Balan, P. Casazza, C. Heil, Z. Landau, Density, overcompleteness, and localization of frames. I. Theory, J. Fourier Anal. Appl. 12 (04 2006) 105–143.

[4] R. Balan, P.G. Casazza, C. Heil, Z. Landau, Density, overcompleteness, and localization of frames. II. Gabor systems, J. Fourier Anal. Appl. 12 (3) (Jun 2006) 307–344.
[5] R. Balan, J. Christensen, I. Krishtal, K. Okoudjou, J. Romero, Multi-window Gabor frames in amalgam spaces, Math. Res. Lett. 21 (1) (2014) 55–69.
[6] R. Balan, I. Krishtal, An almost periodic noncommutative Wiener's Lemma, J. Math. Anal. Appl. 370 (10 2010) 339–349.
[7] P. Balazs, K. Gröchenig, A guide to localized frames and applications to Galerkin-like representations of operators, in: I. Pesenson, H. Mhaskar, A. Mayeli, Q.T.L. Gia, D.-X. Zhou (Eds.), Frames and Other Bases in Abstract and Function Spaces, Applied and Numerical Harmonic Analysis Series (ANHA), Birkhauser/Springer, 2017.
[8] B. Barnes, Symmetric Banach *-algebras: invariance of spectrum, Stud. Math. 141 (3) (2000) 251–261.
[9] B.A. Barnes, The spectrum of integral operators on Lebesgue spaces, J. Oper. Theory 18 (1) (1987) 115–132.
[10] A.G. Baskakov, Wiener's theorem and the asymptotic estimates of the elements of inverse matrices, Funct. Anal. Appl. 24 (3) (July 1990) 222–224.
[11] A.G. Baskakov, Asymptotic estimates for the entries of the matrix of inverse operators and harmonic analysis, Sib. Math. J. 38 (1) (1997) 10–22.
[12] A.G. Baskakov, Estimates for the entries of inverse matrices and the spectral analysis of linear operators, Izv. Math. 61 (6) (dec 1997) 1113.
[13] A.G. Baskakov, I.A. Krishtal, Memory estimation of inverse operators, J. Funct. Anal. 267 (8) (2014) 2551–2605.
[14] A.G. Baskakov, I.A. Krishtal, Spectral properties of an operator polynomial with coefficients in a Banach algebra, in: Frames and Harmonic Analysis. AMS Special Session on Frames, Wavelets and Gabor Systems and Special Session on Frames, Harmonic Analysis, and Operator Theory, North Dakota State University, Fargo, ND, USA, April 16–17, 2016. Proceedings, American Mathematical Society (AMS), Providence, RI, 2018, pp. 93–114.
[15] A.G. Baskakov, I.A. Krishtal, N.B. Uskova, Method of similar operators in harmonious Banach spaces, 2024.
[16] S. Bochner, R.S. Phillips, Absolutely convergent Fourier expansions for non-commutative normed rings, Ann. Math. 43 (3) (1942) 409–418.
[17] L. Brandenburg, On identifying the maximal ideals in Banach algebras, J. Math. Anal. Appl. 50 (3) (1975) 489–510.
[18] O. Christensen, An Introduction to Frames and Riesz Bases, Birkhäuser, 2016.
[19] O. Christensen, T. Strohmer, The finite section method and problems in frame theory, J. Approx. Theory 133 (2) (2005) 221–237.
[20] E. Cordero, K. Gröchenig, Localization of frames II, Appl. Comput. Harmon. Anal. 17 (2004) 29–47.
[21] P. Dabhi, K. Solanki, Vector valued Beurling algebra analogues of Wiener's theorem, Indian J. Pure Appl. Math. (09 2023) 1–14.
[22] P.A. Dabhi, K.B. Solanki, On Hulanicki and Barnes lemmas for p-Banach algebras, Proc. Indian Acad. Sci. Math. 132 (2022) 73, 13.
[23] M.A. de Gosson, Quantum Harmonic Analysis. An Introduction, De Gruyter, Berlin, Boston, 2021.
[24] R.G. Douglas, Banach Algebra Techniques in Operator Theory, Springer, 1998.
[25] H. Feichtinger, Gewichtsfunktionen auf lokalkompakten Gruppen, Sitz.ber. - Österr. Akad. Wiss. Math.-Nat.wiss. Kl., II 188 (8–10) (01 1979) 451–471.
[26] H.G. Feichtinger, W. Kozek, Quantization of TF Lattice-Invariant Operators on Elementary LCA Groups, Birkhäuser Boston, Boston, MA, 1998, pp. 233–266.
[27] M. Fornasier, K. Gröchenig, Intrinsic localization of frames, Constr. Approx. 22 (3) (2005) 395–415.
[28] I. Gelfand, D. Raikov, G. Shilov, Commutative Normed Rings, Chelsea Publishing Company, Bronx, New York, 1964.
[29] I. Gohberg, M. Kaashoek, H. Woerdeman, The band method for positive and strictly contractive extension problems: an alternative version and new applications, Integral Equ. Oper. Theory 12 (01 1989) 343–382.
[30] K. Gröchenig, Foundations of Time-Frequency Analysis, Birkhäuser, Boston, 2001.
[31] K. Gröchenig, Localization of frames, Banach frames, and the invertibility of the frame operator, J. Fourier Anal. Appl. 10 (2) (2004) 105–132.
[32] K. Gröchenig, Time-frequency analysis of Sjöstrand's class, Rev. Mat. Iberoam. 22 (2) (2006) 703–724.
[33] K. Gröchenig, Wiener's lemma: theme and variations. An introduction to spectral invariance and its applications, chapter 5, in: Applied and Numerical Harmonic Analysis, Birkhäuser, 2010, pp. 175–234.
[34] K. Gröchenig, Z. Rzeszotnik, T. Strohmer, Convergence analysis of the finite section method and Banach algebras of matrices, Integral Equ. Oper. Theory 67 (2) (2010) 183–202.
[35] K. Gröchenig, Localized frames are finite unions of Riesz sequences, Adv. Comput. Math. 18 (02 2003) 149–157.
[36] K. Gröchenig, Weight functions in time-frequency analysis, chapter 15, American Mathematical Society, 11 2007, pp. 343–366.
[37] K. Gröchenig, A. Klotz, Noncommutative approximation: inverse-closed subalgebras and off-diagonal decay of matrices, Constr. Approx. 32 (2010) 429–466.
[38] K. Gröchenig, M. Leinert, Symmetry and inverse-closedness of matrix algebras and functional calculus for infinite matrices, Trans. Am. Math. Soc. 358 (6) (2006) 2695–2711.
[39] K. Gröchenig, J. Toft, The range of localization operators and lifting theorems for modulation and Bargmann-Fock spaces, Trans. Am. Math. Soc. 365 (10 2010).
[40] A. Hulanicki, On the spectrum of convolution operators on groups with polynomial growth, Invent. Math. 17 (1972) 135–142.
[41] T. Hytönen, J. van Neerven, M. Veraar, L. Weis, Analysis in Banach Spaces, Volume i: Martingales and Littlewood-Paley Theory, Springer, 12 2016.
[42] S. Jaffard, Propriétés des matrices "bien localisées" preé de leur diagonale et qualques applications, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 7 (5) (1990) 461–476.
[43] J.W. Jenkins, Symmetry and nonsymmetry in the group algebras of discrete groups, Pac. J. Math. 32 (1970) 131–145.

[44] I. Krishtal, Wiener's lemma: pictures at an exhibition, Rev. Unión Mat. Argent. 52 (2) (01 2011) 61–79.
[45] I.A. Krishtal, Wiener's lemma and memory localization, J. Fourier Anal. Appl. 17 (4) (August 2011) 674–690.
[46] V. Kurbatov, Some algebras of operators majorized by a convolution, Funct. Differ. Equ. 8 (01 2001) 323–333.
[47] L. Köhldorfer, Fusion Frames and Operators, Master's thesis, University of Vienna, 2021.
[48] L. Köhldorfer, P. Balazs, Localization of operator-valued frames, preprint, https://arxiv.org/abs/2503.24170.
[49] L. Köhldorfer, P. Balazs, P. Casazza, S. Heineken, C. Hollomey, P. Morillas, M. Shamsabadhi, A Survey of Fusion Frames in Hilbert Spaces, chapter 21, Springer Nature, Switzerland, 2023.
[50] I.J. Maddox, Infinite Matrices of Operators, Lecture Notes in Mathematics, Springer, Berlin, 1980.
[51] M. Naĭmark, Normed Rings, Woters-Noordhoff, 1970.
[52] M. Naĭmark, Normed Algebras, Wolters-Noordhoff Series of Monographs and Textbooks on Pure and Applied Mathematics, Wolters-Noordhoff, 1972.
[53] C. Shin, Q. Sun, Stability of localized operators, J. Funct. Anal. 256 (11 2008).
[54] J. Sjöstrand, Wiener type algebras of pseudodifferential operators, in: Séminaire Équations aux dérivées partielles (Polytechnique), 1994–1995, pp. 1–19.
[55] E. Skrettingland, On Gabor g-frames and Fourier series of operators, 2020.
[56] T. Strohmer, Rates of convergence for the approximation of dual shift-invariant systems in $\ell^2(\mathbb{Z})$, J. Fourier Anal. Appl. 5 (6) (1999) 599–616.
[57] T. Strohmer, Four short stories about Toeplitz matrix calculations, in: Special Issue on Structured and Infinite Systems of Linear Equations, Linear Algebra Appl. 343–344 (2002) 321–344.
[58] Q. Sun, Wiener's lemma for infinite matrices, Trans. Am. Math. Soc. 359 (7) (2007) 3099–3123.
[59] Q. Sun, Wiener's lemma for infinite matrices II, Constr. Approx. 34 (10 2011) 209–235.
[60] W. Sun, G-frames and g-Riesz bases, J. Math. Anal. Appl. 322 (1) (2006) 437–452.
[61] R. Tessera, The inclusion of the Schur algebra in $b(\ell^2)$ is not inverse-closed, Monatshefte Math. 164 (2009) 115–118.
[62] R. Werner, Quantum harmonic analysis on phase space, J. Math. Phys. 25 (5) (05 1984) 1404–1411.
[63] N. Wiener, Tauberian theorems, Ann. Math. 33 (1) (1932) 1–100.

8. Paper 2

Bibliographic information:
L. Köhldorfer and P. Balazs. Localization of operator-valued frames. *Submitted*. ArXiV: https://arxiv.org/pdf/2503.24170

Contribution:
Lukas Köhldorfer is the main author of the paper and the main contributor to its content.

Localization of operator-valued frames

L. Köhldorfer and P. Balazs

Abstract

We introduce a localization concept for operator-valued frames, where the quality of localization is measured by the associated operator-valued Gram matrix belonging to some suitable Banach algebra. We prove that intrinsic localization of an operator-valued frame is preserved by its canonical dual. Moreover, we show that the series associated to the perfect reconstruction of an operator-valued frame converges not only in the underlying Hilbert space, but also in a whole class of associated (quasi-)Banach spaces. Finally, we apply our results to irregular Gabor g-frames.

1 Introduction

Understanding complicated objects using simpler and easier-to-handle building blocks is an old and established concept in mathematics. In Hilbert space the most common systems of building blocks are orthonormal bases, which allow a unique representation of any vector. However, both in theory and in practice, more flexible systems are often required to analyze vectors. One particularly well-established generalization of orthonormal bases are *frames* [19, 17, 14], which allow redundant and stable representations of vectors in a Hilbert space. The study of frames led to many deep theoretical results, such as versions of the *Balian-Low Theorem* [28, 3], or the *Feichtinger Conjecture*, which was proven by Marcus, Spielmann and Srivastava [41] and is equivalent to many other deep theorems in functional analysis [8]. On the other hand, the abstract properties of frames are also desirable for a variety of applications, such as signal processing [2], compressed sensing [42] and many more, see [14] and the numerous references therein.

Here, we work in a setting which is an extension of the latter. Our work is on operator-valued frames, because, despite its great usefulness in many situations, frame theory also has its limitations. One can think of a frame as a weighted family of rank-one projections. However, in various applications, such as distributed- and parallel processing [12, 9], wireless sensor networks [31], packet encoding [6, 7, 10], visual and hearing systems [43], or geophones in geophysics [16], the linear reconstruction of vectors from *higher-rank measurements* is required. This lead to the study of *fusion frames* [11, 12, 38] and further generalizations of frames such as *operator-valued frames.*

The theory of operator-valued frames (originally called *generalized frames*, or simply *g-frames*) was introduced by W. Sun in [45] and encompasses not only frames and fusion frames, but also other established mathematical notions such as *bounded quasi-projectors* [22, 23], or *time-frequency localization operators* [20]. A countable family $T = (T_k)_{k\in X}$ of bounded operators $T_k \in \mathcal{B}(\mathcal{H})$ on some separable Hilbert space $\mathcal{H}$ is called an operator-valued frame, if there exist positive constants $0 < A \leq B < \infty$, such that

$$A\|f\|^2 \leq \sum_{k\in X} \|T_k f\|^2 \leq B\|f\|^2 \qquad (\forall f \in \mathcal{H}). \tag{1.1}$$

Abstract theory shows [45], that every operator-valued frame $(T_k)_{k\in X}$ allows for a linear, bounded and stable reconstruction of any vector $f \in \mathcal{H}$ from the $\mathcal{H}$-valued sequence $(T_k f)_{k\in X} \in \ell^2(X;\mathcal{H})$. In addition to the possibility of perfect reconstruction, operator-valued frames share also many other parallels to frames, and their properties are well-understood [45, 46, 35].

However, beside the latter mentioned features, operator-valued frames might admit other useful properties, which cannot be captured by the defining inequalities (1.1) alone. For instance, a natural constraint for an operator-valued frame $(T_k)_{k\in X}$ would be the requirement, that it is *well localized* in the sense that different elements T_k essentially act on different parts of the space $\mathcal{H}$ and do not correlate much with each other. For example, we could consider an operator-valued frame $(T_k)_{k\in X}$, which additionally satisfies $\ker(T_l)^\perp \subseteq \ker(T_k)$ whenever $k \neq l$, or, equivalently,

$$\|T_k T_l^*\| = 0 \qquad \text{whenever } k \neq l. \tag{1.2}$$

Although such operator-valued frames do exist (e.g. every orthonormal fusion basis [12, 38] has this property), the latter condition will only be satisfied for a very small class of operator-valued frames. Hence, it seems more reasonable to study a broader class of operator-valued frames satisfying a relaxed version of (1.2), which will be done here.

Our main motivation for writing this article comes from considering *Gabor g-frames* for $L^2(\mathbb{R}^d)$ [44]. These are operator-valued frames $(T_k)_{k\in X}$ consisting of *translations* of a fixed *window operator* in the time-frequency plane, generalizing Gabor frames in $L^2(\mathbb{R}^d)$. We show in Theorem 4.3 that if the window operator is suitably chosen and if the index set $X \subset \mathbb{R}^{2d}$ is sufficiently regular, then there exists some uniform constant $C > 0$ such that

$$\|T_k T_l^*\| \leq C(1 + |k - l|)^{-s} \qquad (\forall k, l \in X), \tag{1.3}$$

where, similar to the standard Gabor setting, the decay parameter $s > 2d$ stems from the operator class to which T belongs. The latter estimate can be interpreted to mean that T_k and T_l correlate very little, whenever k and l are far apart. In particular, since $T_k T_l^*$ is precisely the (k,l)-th entry of the $\mathcal{B}(L^2(\mathbb{R}^d))$-valued Gram matrix G_T associated with the family $(T_k)_{k\in X}$, we may reinterpret (1.3) to saying that the Gram matrix G_T has polynomial off-diagonal decay. In more abstract terms, this means that G_T belongs to the *Jaffard class* of operator-valued matrices, a matrix algebra with manifold convenient properties, that has recently been studied in [37, 36]. This observation suggests to define localization of an operator-valued frame as proposed in [34, Definition 5.7] and analogously done for frames in [24] by declaring an operator-valued frame $T = (T_k)_{k\in X}$ *intrinsically $\mathcal{A}$-localized*, if its associated Gram matrix G_T belongs to a given *solid spectral matrix algebra* $\mathcal{A}$ (see Definition 3.1).

In this article, we provide a systematic study of localized operator-valued frames as defined above. We show that intrinsic localization is preserved under canonical duality. For every mutually localized pair of operator-valued dual frames we define a whole chain of associated co-orbit spaces. We prove that the localization assumption of an operator-valued frame guarantees that all of these spaces are well-defined (quasi-)Banach spaces and show that the series associated to perfect reconstruction of such operator-valued frames not only converge in the underlying Hilbert space, but also in each of these spaces. Finally, we apply our results to (possibly irregular) Gabor g-frames for $L^2(\mathbb{R})$.

2 Preliminaries and Notation

We denote the cardinality of a set X by $|X|$. For $x \in \mathbb{R}^d$, $|x|$ denotes the Euclidean norm on $\mathbb{R}^d$. The symbol $\mathbb{N}$ denotes the set of positive integers $\{1, 2, 3, \dots\}$. The kernel and range of an operator T is denoted by $\mathcal{N}(T)$ and $\mathcal{R}(T)$ respectively, and $\mathcal{I}_X$ denotes the identity element on a given space X. The space of bounded operators between two (quasi-)normed spaces X and Y is denoted by $\mathcal{B}(X,Y)$ and we set $\mathcal{B}(X) := \mathcal{B}(X,X)$. We write $X \simeq Y$ if two (quasi-)normed spaces X and Y are isomorphic and $X \cong Y$ if they are isometrically isomorphic. The topological dual space of a normed space X is denoted by X^*.

Throughout these notes, $\mathcal{H}$ is a separable Hilbert space and $\|\cdot\|$ (without an index) always denotes either the norm in $\mathcal{H}$ or the norm in $\mathcal{B}(\mathcal{H})$. All other occurring norms will be labeled with an index.

2.1 The Moore-Penrose pseudo-inverse

Recall the construction of the (Moore-Penrose) pseudo-inverse of a bounded closed range operator between two Hilbert spaces [14]: For $U \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ having closed range, the restriction $U_0 := U|_{\mathcal{N}(U)^\perp} : \mathcal{N}(U)^\perp \longrightarrow \mathcal{H}_2$, is bounded and injective with $\mathcal{R}(U_0) = \mathcal{R}(U)$. Hence, $U_0^{-1} : \mathcal{R}(U) \longrightarrow \mathcal{N}(U)^\perp$ is well-defined and bounded, and the extension $U^\dagger$ of U_0^{-1}, defined by

$$U^\dagger x = \begin{cases} U_0^{-1}x & \text{if } x \in \mathcal{R}(U) \\ 0 & \text{if } x \in \mathcal{R}(U)^\perp \end{cases}$$

is called the *(Moore-Penrose) pseudo inverse* of U. In case U is surjective we have $U^\dagger = U^*(UU^*)^{-1}$, and if U is even bijective then $U^\dagger = U^{-1}$.

The pseudo-inverse can be characterized as follows.

Lemma 2.1. *[5] Let $U \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ be a bounded operator between Hilbert spaces and assume that U has closed range. The pseudo-inverse of U is the unique operator $U^\dagger : \mathcal{H}_2 \longrightarrow \mathcal{H}_1$, satisfying the three relations*

$$\mathcal{N}(U^\dagger) = \mathcal{R}(U)^\perp, \qquad \mathcal{R}(U^\dagger) = \mathcal{N}(U)^\perp, \qquad UU^\dagger U = U. \tag{2.1}$$

The following formula for the pseudo-inverse is folklore and can − for instance − be deduced from [14, Lemma 2.5.2 (iv)] or [13, Corollary 2.3] together with the relations (2.1).

Lemma 2.2. *Let $U \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ be a bounded operator between Hilbert spaces and assume that U has closed range. Then*

$$U^\dagger = U^*(UU^*)^\dagger.$$

2.2 Weight functions

A *weight function* on $\mathbb{R}^d$, or simply *weight*, is a continuous and positive function $\nu : \mathbb{R}^d \longrightarrow (0, \infty)$. A weight ν is called *submultiplicative*, if

$$\nu(x + x') \le \nu(x)\nu(x') \qquad (\forall x, x' \in \mathbb{R}^d),$$

and *symmetric* if

$$\nu(x) = \nu(-x) \qquad (\forall x \in \mathbb{R}^d).$$

A weight m is called *ν-moderate* if there exists some constant $C > 0$ such that

$$m(x + x') \le Cm(x)\nu(x') \qquad (\forall x, x' \in \mathbb{R}^d).$$

We say that ν satisfies the *GRS-condition* (Gelfand-Raikov-Shilov condition [25]), if

$$\lim_{n\to\infty} (\nu(nz))^{\frac{1}{n}} = 1 \qquad (\forall z \in \mathbb{R}^d).$$

For more background on weight functions see [27].

2.3 Bochner sequence spaces

Occasionally, our analysis will rely on properties on Banach space-valued ℓ^p-spaces, defined as follows. If B is a Banach space, X a countable index set, and $\nu = (\nu_k)_{k\in X}$ a family of positive numbers (e.g. samples of a weight), then for each $p \in (0,\infty]$ the *Bochner sequence space* $\ell^p(X;B)$ [30, Chapter 1] is defined by

$$\ell^p_\nu(X;B) := \{(Y_k)_{k\in X} : Y_k \in B\,(\forall k \in X), (\|Y_k\|_B \cdot \nu_k)_{k\in X} \in \ell^p(X)\}.$$

In case $\nu_k = 1$ for all $k \in X$, we write $\ell^p_\nu(X;B) = \ell^p(X;B)$.

The Bochner sequence spaces $\ell^p_\nu(X;B)$ share many properties with the classical ℓ^p-spaces. In particular (see [30, Chapter 1]), $\ell^p_\nu(X;B)$ equipped with the (quasi-)norm

$$\|(Y_k)_{k\in X}\|_{\ell^p_\nu(X;B)} = \|(\|Y_k\|_B \cdot \nu_k)_{k\in X}\|_{\ell^p(X)},$$

is a Banach space for every $1 \le p \le \infty$ and a quasi-Banach space for $0 < p < 1$. If $1 \le p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the dual space $(\ell^p_\nu(X;B))^*$ of $\ell^p_\nu(X;B)$ is isometrically isomorphic to $\ell^q_{1/\nu}(X;B^*)$. Hence, we identify the latter with $(\ell^p_\nu(X;B))^*$ from now on.

By $\ell^{00}(X;B)$ we denote the space of all finitely supported B-valued sequences indexed by X, and by $\ell^0_\nu(X;B)$ we denote the space of all B-valued sequences $(f_k)_{k\in X}$ such that for all $\varepsilon > 0$ there exists $(g_k)_{k\in X} \in \ell^{00}(X;B)$ satisfying $\sup_{k\in X} \|f_k - g_k\|\nu(l) < \varepsilon$. It is straightforward to see that $(\ell^{00}(X;B), \|\,.\,\|_{\ell^p_\nu(X;B)})$ is a dense subspace of $(\ell^p_\nu(X;B), \|\,.\,\|_{\ell^p_\nu(X;B)})$ for each $p \in [1,\infty)$. By definition, $(\ell^{00}_\nu(X;B), \|\,.\,\|_{\ell^0_\nu(X;B)})$ is also a dense subspace of $(\ell^0_\nu(X;B), \|\,.\,\|_{\ell^0_\nu(X;B)})$, where we set $\|\,.\,\|_{\ell^0_\nu(X;B)} := \|\,.\,\|_{\ell^\infty_\nu(X;B)}$ for notational convenience. Moreover, in [47, Theorem 3.1] it was proven that $(\ell^0_\nu(X;B), \|\,.\,\|_{\ell^0_\nu(X;B)})^*$ is isomorphic to $(\ell^1_{1/\nu}(X;B^*), \|\,.\,\|_{\ell^1_{1/\nu}(X;B^*)})$.

An important property of Bochner sequence spaces is the fact that Riesz-Thorin interpolation holds. For later reference, we state the following special case of [30, Theorem 2.2.1].

Proposition 2.3 (Riesz-Thorin interpolation). *Assume that $A \in \mathcal{B}(\ell^1_\nu(X;B))$ and $A \in \mathcal{B}(\ell^\infty_\nu(X;B))$. Then also $A \in \mathcal{B}(\ell^p_\nu(X;B))$ for every $1 < p < \infty$.*

The following simple lemma will be convenient later on.

Lemma 2.4. *Let $a_{kl} \in \mathbb{C}$ $(k,l \in X)$ be given and $0 < p \le \infty$ be arbitrary. If the $\mathcal{B}(\mathcal{H})$-valued matrix $A = [a_{kl} \cdot \mathcal{I}_{\mathcal{B}(\mathcal{H})}]_{k,l\in X}$ defines an element in $\mathcal{B}(\ell^p_\nu(X;B))$ via matrix multiplication, then the scalar matrix $M = [a_{kl}]_{k,l\in X}$ defines an element in $\mathcal{B}(\ell^p_\nu(X))$ via matrix multiplication.*

Proof. Let $c = (c_l)_{l\in X} \in \ell^p_\nu(X)$ be arbitrary. Then for every unit-norm $h \in B$, $(h_l)_{l\in X} := (c_l h)_{l\in X}$ is contained in $\ell^p_\nu(X;B)$ and $\|(h_l)_{l\in X}\|_{\ell^p_\nu(X;B)} = \|c\|_{\ell^p_\nu(X)}$. Thus

$$\begin{aligned}
\|Mc\|_{\ell^p_\nu(X)} &= \left\| \left(\left| \sum_{l\in X} a_{kl} c_l \right| \|h\| \right)_{k\in X} \right\|_{\ell^p_\nu(X)} \\
&= \left\| \left(\left\| \sum_{l\in X} a_{kl} c_l h \right\| \right)_{k\in X} \right\|_{\ell^p_\nu(X)} \\
&= \|A(h_l)_{l\in X}\|_{\ell^p_\nu(X;B)} \\
&\leq \|A\|_{\mathcal{B}(\ell^p_\nu(X;B))} \|c\|_{\ell^p_\nu(X)},
\end{aligned}$$

as was to be shown. □

We will also need the following special case of the Stein-Weiss interpolation theorem [4, p.115, Theorem 5.4.1].

Proposition 2.5 (Stein-Weiss interpolation)**.** *Let $0 < p \leq \infty$ be arbitrary. If $A \in \mathcal{B}(\ell^p_\nu(X))$ and $A \in \mathcal{B}(\ell^p_{1/\nu}(X))$ then also $A \in \mathcal{B}(\ell^p(X))$.*

Finally, we will often encounter $\mathcal{B}(\mathcal{H})$-valued matrices $A = [A_{k,l}]_{k,l\in X}$, which define bounded operators on (weighted) Bochner sequence spaces with values in $\mathcal{H}$. If $A = [A_{k,l}]_{k,l\in X} \in \mathcal{B}(\ell^p_\nu(X;\mathcal{H}))$ (for some $1 \leq p < \infty$), then its Banach space adjoint A' is an element in $\mathcal{B}(\ell^q_{1/\nu}(X;\mathcal{H}))$ (where $\frac{1}{p} + \frac{1}{q} = 1$) and a direct computation shows that

$$A' = ([A^*_{k,l}]_{k,l\in X})^T \tag{2.2}$$

where the exponent T denotes matrix transposition and $A^*_{k,l}$ is the adjoint of $A_{k,l} \in \mathcal{B}(\mathcal{H})$.

2.4 Operator-valued frames

A countable family $(T_k)_{k\in X}$ of bounded operators $T_k \in \mathcal{B}(\mathcal{H})$ constitutes an *operator-valued frame* (originally called *generalized frame*, or simply *g-frame*) [45] for $\mathcal{H}$, if there exist positive constants $0 < A \leq B < \infty$, such that

$$A\|f\|^2 \leq \sum_{k\in X} \|T_k f\|^2 \leq B\|f\|^2 \qquad (\forall f \in \mathcal{H}). \tag{2.3}$$

From now on, for brevity reason, we call an operator-valued frame a *g-frame*. We will refer to the constants A and B as the *lower* and *upper g-frame bound*, respectively.

We call $(T_k)_{k\in X}$ a *g-Bessel sequence*, whenever the upper (but not necessarily the lower) inequality in (2.3) is satisfied for some prescribed $B > 0$ and all $f \in \mathcal{H}$. Let $T = (T_k)_{k\in X}$ be a g-Bessel sequence in $\mathcal{H}$. Then the following operators are well-defined and bounded [45]:

- The *synthesis operator* $D_T : \ell^2(X;\mathcal{H}) \longrightarrow \mathcal{H}$, defined by
$$D_T(f_k)_{k\in X} = \sum_{k\in X} T_k^* f_k,$$
- The *analysis operator* $C_T : \mathcal{H} \longrightarrow \ell^2(X;\mathcal{H})$, defined by
$$C_T f = (T_k f)_{k\in X},$$
- The *g-frame operator* $S_T := D_T C_T : \mathcal{H} \longrightarrow \mathcal{H}$, given by
$$S_T f = \sum_{k\in X} T_k^* T_k f.$$
- The *g-Gram matrix* $G_T := C_T D_T : \ell^2(X;\mathcal{H}) \longrightarrow \ell^2(X;\mathcal{H})$.

These operators share the same properties with their frame-theoretic pendants [14]. More precisely [45], T being a g-Bessel sequence with bound B implies that D_T and C_T are adjoint to one another and $\|D_T\|_{\mathcal{B}(\ell^2(X;\mathcal{H}),\mathcal{H})} = \|C_T\|_{\mathcal{B}(\mathcal{H},\ell^2(X;\mathcal{H}))} \leq \sqrt{B}$. In particular, $\|T_k\| \leq \sqrt{B}$ for all $k \in X$. Furthermore, both S_T and G_T are self-adjoint and bounded by B in this case. If T is a g-frame, then C_T is bounded, injective and has closed range, D_T is bounded and surjective, and S_T is bounded, self-adjoint, positive and invertible satisfying $A \cdot \mathcal{I}_{\mathcal{H}} \leq S_T \leq B \cdot \mathcal{I}_{\mathcal{H}}$ (in the sense of positive operators). Consequently, S_T may be composed with its inverse (and vice versa) which yields the possibility of *g-frame reconstruction* via

$$f = \sum_{k\in X} T_k^* T_k S_T^{-1} f = \sum_{k\in X} S_T^{-1} T_k^* T_k f \qquad (\forall f \in \mathcal{H}). \tag{2.4}$$

The family $\widetilde{T} = (\widetilde{T}_k)_{k\in X} := (T_k S_T^{-1})_{k\in X}$, which appears in the reconstruction process (2.4), is again a g-frame and called the *canonical dual g-frame*. In particular, if $A \leq B$ are g-frame bounds for T, then $B^{-1} \leq A^{-1}$ are g-frame bounds for $\widetilde{T}$ and $B^{-1} \leq \|S_T^{-1}\| \leq A^{-1}$. More generally [35], if $(T_k^d)_{k\in X}$ is a g-Bessel sequence such that

$$f = \sum_{k\in X} T_k^* T_k^d f = \sum_{k\in X} (T_k^d)^* T_k f \qquad (\forall f \in \mathcal{H}), \tag{2.5}$$

or, equivalently,

$$\mathcal{I}_{\mathcal{B}(\mathcal{H})} = D_T C_{T^d} = D_{T^d} C_T, \tag{2.6}$$

then $(T_k^d)_{k\in X}$ is already a g-frame and called a *dual g-frame* (or simply *dual*) of $(T_k)_{k\in X}$.

Remark 2.6. *Assume that $T = (T_k)_{k\in X}$ is a g-Bessel sequence in $\mathcal{H}$. Then $C_T \in \mathcal{B}(\mathcal{H}, \ell^2(X;\mathcal{H}))$, $D_T \in \mathcal{B}(\ell^2(X;\mathcal{H}), \mathcal{H})$ and $G_T \in \mathcal{B}(\ell^2(X;\mathcal{H}))$ are all bounded. Since we can interpret $\mathcal{H}$ as a direct sum of Hilbert spaces indexed by just a singleton, we see that each of the operators C_T, D_T and G_T is a bounded operator from one direct sum of Hilbert spaces to another direct sum of Hilbert spaces. Hence, we are in the setting of the matrix calculus from [40] (see also [38, Section 3.1] for more details), which guarantees that every bounded operator A between two direct sums of Hilbert spaces can be uniquely represented by a $\mathcal{B}(\mathcal{H})$-valued matrix $\mathbb{M}(A) = [A_{k,l}]$, which acts on elements from a direct sum of Hilbert spaces, viewed as column vectors, via matrix-vector multiplication. Moreover, composition of such operators A and B corresponds to matrix multiplication, i.e. $\mathbb{M}(AB) = \mathbb{M}(A)\cdot\mathbb{M}(B)$, where each entry $[AB]_{k,l} = \sum_n A_{k,n}B_{n,l}$ of $\mathbb{M}(AB)$ converges in the operator norm topology with respect to $\mathcal{B}(\mathcal{H})$ [40, 38]. In particular, we have*

$$\begin{aligned}
\mathbb{M}(D_T) &= \begin{bmatrix} \dots & T_{k-1}^* & T_k^* & T_{k+1}^* & \dots \end{bmatrix}, \\
\mathbb{M}(C_T) &= \begin{bmatrix} \dots & T_{k-1} & T_k & T_{k+1} & \dots \end{bmatrix}^T, \\
\mathbb{M}(G_T) &= \begin{bmatrix} \ddots & \vdots & \vdots & \\ \dots & T_kT_l^* & T_kT_{l+1}^* & \dots \\ \dots & T_{k+1}T_l^* & T_{k+1}T_{l+1}^* & \dots \\ & \vdots & \vdots & \ddots \end{bmatrix}.
\end{aligned} \tag{2.7}$$

where the exponent T denotes matrix transposition. Analogously, if $U = (U_k)_{k\in X}$ is another g-Bessel sequence in $\mathcal{H}$, then the canonical matrix representation of the mixed g-Gram matrix *$G_{T,U} := C_T D_U \in \mathcal{B}(\ell^2(X;\mathcal{H}))$ is the same as in (2.7) after replacing each T_l^* by U_l^*.*

The above mentioned matrix calculus implies that we can identify $\mathcal{B}(\ell^2(X;\mathcal{H}))$ with a Banach *-algebra of $\mathcal{B}(\mathcal{H})$-valued matrices, where the involution * is defined via $([A_{k,l}]_{k,l\in X})^* = [A_{l,k}^*]_{k,l\in X}$ and precisely corresponds to taking adjoints in $\mathcal{B}(\ell^2(X;\mathcal{H}))$, meaning that $\mathbb{M}(A^*) = \mathbb{M}(A)^*$ for all $A \in \mathcal{B}(\ell^2(X;\mathcal{H}))$. This viewpoint is particularly convenient for defining g-frame localization analogously as done in [24] for frames, i.e. by g-Gram matrices belonging to some suitable sub-algebra $\mathcal{A}$ of $\mathcal{B}(\ell^2(X;\mathcal{H}))$ consisting of $\mathcal{B}(\mathcal{H})$-valued matrices, see ahead.

3 Localization of operator-valued frames

Definition 3.1. *Let $\mathcal{A} = \mathcal{A}(X)$ be a unital Banach *-algebra of $\mathcal{B}(\mathcal{H})$-valued matrices $A = [A_{k,l}]_{k,l\in X}$. Then $\mathcal{A}$ is called a* solid spectral matrix algebra*, if*

(1.) $\mathcal{A} \subseteq \mathcal{B}(\ell^2(X;\mathcal{H}))$*, meaning that the inclusion map* $\iota : \mathcal{A} \longrightarrow \mathcal{B}(\ell^2(X;\mathcal{H}))$ *is a unital *-monomorphism (i.e. a faithful representation).*

(2.) $\mathcal{A}$ *is* inverse-closed *in* $\mathcal{B}(\ell^2(X;\mathcal{H}))$*, i.e., if* $A \in \mathcal{A}$ *defines an invertible operator in* $\mathcal{B}(\ell^2(X;\mathcal{H}))$*, then its inverse* $A^{-1} \in \mathcal{B}(\ell^2(X;\mathcal{H}))$ *is contained in* $\mathcal{A}$ *as well,*

(3.) $\mathcal{A}$ *is* solid*: if* $A = [A_{k,l}]_{k,l\in X} \in \mathcal{A}$ *and if* $B = [B_{k,l}]_{k,l\in X}$ *is a* $\mathcal{B}(\mathcal{H})$*-valued matrix such that* $\|B_{k,l}\| \leq \|A_{k,l}\|$ *for all* $k,l \in X$*, then* $B \in \mathcal{A}$ *and* $\|B\|_{\mathcal{A}} \leq \|A\|_{\mathcal{A}}$*.*

For brevity reason, we will call such an algebra $\mathcal{A}$ *simply a* spectral algebra*.*

Spectral algebras are also *pseudo-inverse-closed* in $\mathcal{B}(\ell^2(X;\mathcal{H}))$:

Theorem 3.2. *[24] Let* $\mathcal{K}$ *be a Hilbert space. Suppose that* $\mathcal{A}$ *is a Banach *-algebra which is inverse-closed in* $\mathcal{B}(\mathcal{K})$ *and let* M *be a closed subspace of* $\mathcal{K}$*. If* $A = A^* \in \mathcal{A}$*,* $\mathcal{N}(A) = M^\perp$ *and* $A : M \longrightarrow M$ *is bounded and invertible, then the Moore-Penrose pseudo-inverse* $A^\dagger$ *of* A *is an element in* $\mathcal{A}$*.*

The above result holds also true for possibly non-self-adjoint operators A [24]. Since its proof was omitted by the authors, we present the details for completeness reason.

Corollary 3.3. *[24] Let* $\mathcal{A}$ *be a Banach *-algebra, which is inverse-closed in* $\mathcal{B}(\mathcal{K})$*. Then* $\mathcal{A}$ *is also* pseudo-inverse-closed *in* $\mathcal{B}(\mathcal{K})$*, that is, if* $A \in \mathcal{A}$ *defines a bounded operator in* $\mathcal{B}(\mathcal{K})$ *with closed range, then* $A^\dagger \in \mathcal{A}$*.*

Proof. By Lemma 2.2, the pseudo-inverse of A is given by $A^\dagger = A^*(AA^*)^\dagger$. Since AA^* is self-adjoint and $\mathcal{R}(AA^*) = \mathcal{R}(A)$ is closed, Theorem 3.2 implies that $(AA^*)^\dagger \in \mathcal{A}$. Consequently, $A^\dagger \in \mathcal{A}$ due to the algebra property of $\mathcal{A}$. □

Example 3.4. *Recently, the following examples of spectral algebras of* $\mathcal{B}(\mathcal{H})$*-valued matrices indexed by a relatively separated set* $X \subset \mathbb{R}^d$ *haven been discussed [37]:*
(1.) *For every* $s > d$*, the* Jaffard algebra $\mathcal{J}_s = \mathcal{J}_s(X)$ *of* $\mathcal{B}(\mathcal{H})$*-valued matrices* $A = [A_{k,l}]_{k,l\in X}$ *for which*

$$\sup_{k,l\in X} \|A_{k,l}\|(1 + |k - l|)^s < \infty$$

is a spectral algebra.
(2.) *For every weight* ν *which*

(2a) is of the form $\nu(x) = e^{\rho(\|x\|)}$*, where* $\rho : [0,\infty) \longrightarrow [0,\infty)$ *is a continuous and concave function with* $\rho(0) = 0$*, and* $\|\,.\,\|$ *any norm on* $\mathbb{R}^d$*,*

(2b) satisfies the GRS-condition,

(2c) satisfies the weak growth condition

$$\nu(x) \geq C(1+|x|)^{\delta} \qquad \text{for some } \delta \in (0,1], C > 0,$$

the weighted Schur algebra $\mathcal{S}_\nu^1 = \mathcal{S}_\nu^1(X)$ *of* $\mathcal{B}(\mathcal{H})$*-valued matrices* $A = [A_{k,l}]_{k,l\in X}$ *for which*

$$\max\left\{\sup_{k\in X}\sum_{l\in X}\|A_{k,l}\|\nu(k-l)\,,\,\sup_{l\in X}\sum_{k\in X}\|A_{k,l}\|\nu(k-l)\right\} < \infty$$

is a spectral algebra.
(3.) *For every weight* ν*, which is submultiplicative, symmetric, and satisfies the GRS-condition, the weighted* Baskakov-Gohberg-Sjöstrand algebra $\mathcal{C}_\nu = \mathcal{C}_\nu(\mathbb{Z}^d)$ *of* $\mathcal{B}(\mathcal{H})$*-valued matrices* $A = [A_{k,l}]_{k,l\in\mathbb{Z}^d}$ *for which*

$$\|A\|_{\mathcal{C}_\nu} := \sum_{l\in\mathbb{Z}^d}\sup_{k\in\mathbb{Z}^d}\|A_{k,k-l}\|\nu(l) < \infty$$

is a spectral algebra contained in $\mathcal{B}(\ell^2(\mathbb{Z}^d;\mathcal{H}))$*.*
(4.) *Spectral algebras* $\mathcal{J}_\nu(X)$ *which model more general off-diagonal decay than polynomial decay (and hence generalize* $\mathcal{J}_s$*).*
(5.) *Anisotropic variations of all of the latter mentioned spectral algebras.*

With the above examples of spectral algebras in mind, we now give the main definition of this article, which extends the localization concept from [24] to the g-frame setting. We remark that the special case $\mathcal{A} = \mathcal{C}_\nu(\mathbb{Z}^d)$ of Definition 3.5 is [34, Definition 5.6].

Definition 3.5. *Let* $\mathcal{A} = \mathcal{A}(X)$ *be a spectral algebra and* $T = (T_k)_{k\in X}$ *be a* $\mathcal{B}(\mathcal{H})$*-valued family indexed by* X*. We say that* T *is* intrinsically $\mathcal{A}$-localized*, in short* $T \sim_{\mathcal{A}} T$*, if its associated g-Gram matrix* $G_T = [T_k T_l^*]_{k,l\in X}$ *belongs to* $\mathcal{A}$*. If* $U = (U_k)_{k\in X}$ *is another* $\mathcal{B}(\mathcal{H})$*-valued family indexed by* X*, then we call* U *and* T mutually $\mathcal{A}$-localized*, in short* $U \sim_{\mathcal{A}} T$*, if the mixed g-Gram matrix* $G_{U,T} = [U_k T_l^*]_{k,l\in X}$ *belongs to* $\mathcal{A}$*.*

By the result below, we may always assume that intrinsically localized $\mathcal{B}(\mathcal{H})$-valued sequences are g-Bessel sequences.

Proposition 3.6. *Every intrinsically localized* $\mathcal{B}(\mathcal{H})$*-valued sequence is a g-Bessel sequence.*

Proof. Assume that $T = (T_k)_{k\in X}$ is an intrinsically localized $\mathcal{B}(\mathcal{H})$-valued family indexed by X. Then G_T is contained in some spectral algebra $\mathcal{A} \subseteq \mathcal{B}(\ell^2(X;\mathcal{H}))$ and thus well-defined on $\ell^2(X;\mathcal{H})$, bounded and self-adjoint. This implies that D_T is bounded, since

$$\|D_T(h_l)_{l\in X}\|^2 = \langle G_T(h_l)_{l\in X}, (h_l)_{l\in X}\rangle_{\ell^2(X;\mathcal{H})} \leq \|G_T\|_{\mathcal{B}(\ell^2(X;\mathcal{H}))}\|(h_l)_{l\in X}\|^2_{\ell^2(X;\mathcal{H}))}$$

holds true for all $(h_l)_{l\in X} \in \ell^{00}(X;\mathcal{H})$ and, by density, also for all $(h_l)_{l\in X} \in \ell^2(X;\mathcal{H})$. In particular, $C_T = D_T^*$ is bounded, which implies that T is a g-Bessel sequence. □

3.1 Intrinsic localization and duality

Next, we will show that intrinsic localization of a g-frame is preserved by its canonical dual g-frame. Proving this relies heavily on the properties of a spectral algebra.

The following preparatory result is the g-frame analogue of [29, Lemma 2.14].

Lemma 3.7. *Let $T = (T_k)_{k\in X}$ be a g-frame and $\widetilde{T} = (T_k S_T^{-1})_{k\in X}$ its canonical dual g-frame. Then G_T has closed range and*

$$G_{\widetilde{T}} = G_T^\dagger \qquad \text{and} \qquad G_{T,\widetilde{T}} = G_T G_T^\dagger. \tag{3.1}$$

Proof. By definition of the analysis operator and the g-frame inequalities (2.3), $T = (T_k)_{k\in X}$ is a g-frame if and only if $C_T : \mathcal{H} \longrightarrow \ell^2(X;\mathcal{H})$ is well-defined, bounded and bounded from below. This is equivalent to C_T being a bounded, injective and closed-range operator and $D_T = C_T^*$ being bounded and surjective [15]. In particular, $\mathcal{R}(G_T) = \mathcal{R}(C_T D_T) = \mathcal{R}(C_T)$ is closed and thus $G_T^\dagger$ is well-defined.

In order to show the first relation in (3.1), we verify that $G_{\widetilde{T}}$ satisfies the three characterizing conditions of the pseudo-inverse of G_T from Lemma 2.1. Firstly, since $C_{\widetilde{T}} = C_T S_T^{-1}$, we see that $\mathcal{N}(G_{\widetilde{T}}) = \mathcal{R}(G_{\widetilde{T}})^\perp = \mathcal{R}(C_{\widetilde{T}})^\perp = \mathcal{R}(C_T S_T^{-1})^\perp = \mathcal{R}(C_T)^\perp = \mathcal{R}(G_T)^\perp$, where we also used the fact that g-Gram matrices associated to g-Bessel sequences are self-adjoint. Secondly, we similarly have $\mathcal{R}(G_{\widetilde{T}}) = \mathcal{R}(C_{\widetilde{T}}) = \mathcal{R}(C_T S_T^{-1}) = \mathcal{R}(C_T) = \mathcal{R}(G_T) = \mathcal{N}(G_T)^\perp$. Finally, we have $G_T G_{\widetilde{T}} G_T = C_T D_T C_{\widetilde{T}} D_{\widetilde{T}} C_T D_T = C_T D_T = G_T$ by (2.6).

By using (2.6) again, the second relation in (3.1) follows from the first via $G_T G_T^\dagger = G_T G_{\widetilde{T}} = C_T D_T C_{\widetilde{T}} D_{\widetilde{T}} = C_T D_{\widetilde{T}} = G_{T,\widetilde{T}}$. □

We are now prepared to prove our first main result, which is the g-frame theoretic analogue of [24, Theorem 3.6] and a generalization of [34, Lemma 5.6], where the case $\mathcal{A} = \mathcal{C}_\nu(\mathbb{Z}^d)$ is treated.

Theorem 3.8. *Let $T = (T_k)_{k\in X}$ be an intrinsically $\mathcal{A}$-localized g-frame. Then the following hold:*

(i) The canonical dual g-frame $\widetilde{T} = (T_k S_T^{-1})_{k\in X}$ is intrinsically $\mathcal{A}$-localized.

(ii) T and $\widetilde{T}$ are mutually $\mathcal{A}$-localized.

Proof. By assumption we have $G_T \in \mathcal{A}$. By Lemma 3.7, $G_T^\dagger$ is well-defined, $G_{\widetilde{T}} = G_T^\dagger$ and $G_{T,\widetilde{T}} = G_T G_T^\dagger$. By Corollary 3.3, $\mathcal{A}$ is pseudo-inverse-closed, hence $G_{\widetilde{T}} = G_T^\dagger \in \mathcal{A}$. Since $\mathcal{A}$ is an algebra, we obtain $G_{T,\widetilde{T}} = G_T G_T^\dagger \in \mathcal{A}$. □

From Theorem 3.8, we can derive the following analogue of [29, Corollary 2.16], see also [29, Lemma 2.13].

Theorem 3.9. *The relation $\sim_{\mathcal{A}}$ is symmetric. Furthermore, $\sim_{\mathcal{A}}$ is an equivalence relation on the set of all intrinsically $\mathcal{A}$-localized g-frames indexed by X.*

Proof. If $G_{T,U} \in \mathcal{A}$, then $G_{U,T} = (G_{T,U})^* \in \mathcal{A}$, hence $\sim_{\mathcal{A}}$ is symmetric.

Clearly, $\sim_{\mathcal{A}}$ is a reflexive relation on the set of all intrinsically $\mathcal{A}$-localized g-frames on $\mathcal{H}$. Now, let $R = (R_k)_{k\in X}$, $T = (T_k)_{k\in X}$ and $U = (U_k)_{k\in X}$ be intrinsically $\mathcal{A}$-localized g-frames. To show transitivity, assume that $T \sim_{\mathcal{A}} U$ and $U \sim_{\mathcal{A}} R$. By Theorem 3.8 we have $T \sim_{\mathcal{A}} \widetilde{T}$ and $\widetilde{U} \sim_{\mathcal{A}} \widetilde{U}$, where $\widetilde{T}$ and $\widetilde{U}$ denote the canonical dual g-frames of T and U respectively. Thus, by g-frame reconstruction with respect to the dual frame pairs $(T, \widetilde{T})$ and $(U, \widetilde{U})$ we obtain via (2.6) that

$$G_{T,R} = C_T D_R = C_T D_{\widetilde{T}} C_T D_U C_{\widetilde{U}} D_{\widetilde{U}} C_U D_R = G_{T,\widetilde{T}} G_{T,U} G_{\widetilde{U}} G_{U,R} \in \mathcal{A},$$

i.e. $T \sim_{\mathcal{A}} R$. □

3.2 Operator-valued frames and associated (quasi-) Banach spaces

One of the most remarkable features of localized frames is that their associated frame reconstruction formulae not only hold true in the underlying Hilbert space, but also in a whole class of associated (quasi-)Banach spaces. In this section we show that analogous results hold true in the g-frame setting. In particular, we will consider the associated g-frame analysis and synthesis operators acting on other Banach spaces than $\mathcal{H}$ and $\ell^2(X;\mathcal{H})$ respectively. In order to avoid tedious distinctions of these operators defined on different spaces, we will simply consider them as $\mathcal{B}(\mathcal{H})$-valued matrices, as justified in Remark 2.6.

3.2.1 Admissible weights

One ingredient of the associated (quasi-)Banach spaces we will introduce are a certain class of weight functions, see below.

Definition 3.10. *[24] Let $\mathcal{A}$ be some subfamily of $\mathcal{B}(\ell^2(X;\mathcal{H}))$ (for instance, a spectral algebra) and $0 < p_0 \leq 1$. A weight function ω is called $(\mathcal{A}, p_0)$-*admissible*, if every $A \in \mathcal{A}$ defines a bounded operator $\ell^p_\omega(X;\mathcal{H}) \longrightarrow \ell^p_\omega(X;\mathcal{H})$ for every $p \in (p_0, \infty] \cup \{1\}$.*

Example 3.11. *A weight being $(\mathcal{A}, p_0)$-admissible is a fairly strong assumption. However, recently the following examples of $(\mathcal{A}, p_0)$-admissible weights ω have been shown to exist [37]:*
(1.) If $s > d+r$ for some $r \geq 0$, and ν_r denotes the polynomial weight $\nu_r(x) = (1+|x|)^r$, then every ν_r-moderate weight ω is $(\mathcal{J}_s, \frac{d}{s-r})$-admissible, where $\mathcal{J}_s$ denotes the Jaffard algebra from Example 3.4 (1.).
(2.) If $\mathcal{S}^1_\nu$ denotes the weighted Schur algebra *from Example 3.4 (2.), then every ν-moderate weight ω is $(\mathcal{S}^1_\nu, 1)$-admissible.*
(3.) If $\mathcal{C}_\nu$ denotes the Baskakov-Gohberg-Sjöstrand algebra from Example 3.4 (3.), then every ν-moderate weight ω is $(\mathcal{C}_\nu, 1)$-admissible.

The following result (whose scalar-analogue was stated in [1]) will be convenient later on.

Lemma 3.12. *Let $\mathcal{A}$ be a spectral algebra and ω be $(\mathcal{A}, p_0)$-admissible. Then $1/\omega$ is $(\mathcal{A}, 1)$-admissible.*

Proof. Choose some arbitrary $A = [A_{k,l}]_{k,l\in X} \in \mathcal{A}$. Then $A^* = [A^*_{l,k}]_{k,l\in X} \in \mathcal{A}$ and in particular $A^* \in \mathcal{B}(\ell^p_\omega(X;\mathcal{H}))$ for $p \in [1,\infty)$ by assumption. Thus, by duality, the Banach space adjoint $(A^*)'$ of A^*, which, by (2.2), is given by $(A^*)' = (A^*)^* = A$, is contained in $\mathcal{B}(\ell^p_{1/\omega}(X;\mathcal{H}))$ for all $1 < p \leq \infty$.

It remains to show that $A \in \mathcal{B}(\ell^1_{1/\omega}(X;\mathcal{H}))$. Since $A^* = [[A^*]_{l,k}]_{k,l\in X} \in \mathcal{A}$, we have by solidity of $\mathcal{A}$ that the matrix $[\|[A^*]_{l,k}\| \cdot \mathcal{I}_{\mathcal{B}(\mathcal{H})}]_{k,l\in X}$ is contained in $\mathcal{A}$ and thus $[\|[A^*]_{l,k}\| \cdot \mathcal{I}_{\mathcal{B}(\mathcal{H})}]_{k,l\in X} \in \mathcal{B}(\ell^\infty_\omega(X;\mathcal{H}))$ by assumption. Lemma 2.4 implies that the scalar matrix $M := [\|[A^*]_{l,k}\|]_{k,l\in X}$ is contained in $\mathcal{B}(\ell^\infty_\omega(X))$.

For $g = (g_l)_{l\in X} \in \ell^1_{1/\omega}(X;\mathcal{H})$ this yields

$$\begin{aligned}
\|Ag\|_{\ell^1_{1/\omega}(X;\mathcal{H})} &\leq \sum_{k\in X}\sum_{l\in X} \|A_{k,l}\|\|g_l\|\frac{1}{\omega(k)} \\
&= \sum_{l\in X}\sum_{k\in X} \|A_{k,l}\|\|g_l\|\frac{1}{\omega(k)} \\
&= \sum_{l\in X} \|g_l\|\frac{1}{\omega(l)} \sum_{k\in X} \|A_{k,l}\|\frac{\omega(l)}{\omega(k)} \\
&\leq \left(\sum_{l\in X} \|g_l\|\frac{1}{\omega(l)}\right) \sup_{m\in X} \sum_{k\in X} \|A_{k,m}\|\frac{\omega(m)}{\omega(k)} \\
&= \|g\|_{\ell^1_{1/\omega}(X;\mathcal{H})} \sup_{m\in X} \left|\sum_{k\in X} \|[A^*]_{m,k}\|\frac{1}{\omega(k)}\right| \omega(m) \\
&= \|g\|_{\ell^1_{1/\omega}(X;\mathcal{H})} \|M(w(k)^{-1})_{k\in X}\|_{\ell^\infty_\omega(X)} \\
&\leq \|M\|_{\mathcal{B}(\ell^\infty_\omega(X))} \|g\|_{\ell^1_{1/\omega}(X;\mathcal{H})},
\end{aligned}$$

where we used that $\|(w(k)^{-1})_{k\in X}\|_{\ell^\infty_\omega(X)} = 1$ in the last step. Note that the interchange of order of summation in the second step is justified by the monotone convergence theorem, since only non-negative terms occur. $\square$

Corollary 3.13. *Let $\mathcal{A}$ be a spectral algebra. Whenever there exists some $(\mathcal{A}, p_0)$-admissible weight ω for some $0 < p_0 \leq 1$, the trivial weight $(1)_{l\in X}$ is $(\mathcal{A}, 1)$-admissible.*

Proof. Let $A = [A_{k,l}]_{k,l\in X} \in \mathcal{A}$ be arbitrary. Then $[\|A_{k,l}\| \cdot \mathcal{I}_{\mathcal{B}(\mathcal{H})}]_{k,l\in X} \in \mathcal{A}$ by solidity of $\mathcal{A}$. By assumption, Lemma 3.12 and Lemma 2.4 this implies that the scalar matrix $C := [\|A_{k,l}\|]_{k,l\in X}$ defines an element in $\mathcal{B}(\ell^p_\omega(X)) \cap \mathcal{B}(\ell^p_{1/\omega}(X))$ for each $1 \leq p \leq \infty$. By Stein-Weiss interpolation (Proposition 2.5) this yields $C \in \mathcal{B}(\ell^p(X))$ for all $1 \leq p \leq \infty$. Now, let $p \in [1, \infty)$ be arbitrary and take any $(f_l)_{l\in X} \in \ell^p(X;\mathcal{H})$. Then $(\|f_l\|)_{l\in X} \in \ell^p(X)$ and thus

$$\begin{aligned}
\sum_{k\in X} \left\|\sum_{l\in X} A_{k,l} f_l\right\|^p &\leq \sum_{k\in X} \left|\sum_{l\in X} \|A_{k,l}\|\|f_l\|\right|^p \\
&\leq \|C\|^p_{\mathcal{B}(\ell^p(X))} \|(f_l)_{l\in X}\|^p_{\ell^p(X;\mathcal{H})},
\end{aligned}$$

which means that $A \in \mathcal{B}(\ell^p(X;\mathcal{H}))$. The modifications of the above for the case $p = \infty$ are obvious. $\square$

3.2.2 Definition of the spaces $\mathcal{H}^p_\omega(T^d, T)$ for $p < \infty$

We are now able to define the spaces $\mathcal{H}^p_\omega(T^d, T)$. Let $\mathcal{A}$ be a spectral algebra and ω be an $(\mathcal{A}, p_0)$-admissible weight. Further, let $T = (T_k)_{k\in X}$ be a g-frame and $T^d = (T^d_k)_{k\in X}$ be any dual g-frame of T such that T^d and T are mutually $\mathcal{A}$-localized (by Theorem 3.8 we may choose the canonical dual g-frame of T in case T is intrinsically $\mathcal{A}$-localized). Consider the space

$$\mathcal{H}^{00}(T) := D_T(\ell^{00}(X;\mathcal{H})) = \left\{ \sum_{k\in K} T_k^* f_k : K \subseteq X, |K| < \infty, f_k \in \mathcal{H}\, (\forall k \in K) \right\}.$$

Since $D_T \in \mathcal{B}(\ell^2(X;\mathcal{H}), \mathcal{H})$ and $D_T(\ell^2(X;\mathcal{H})) = \mathcal{H}$ for every g-frame T, $\mathcal{H}^{00}(T)$ is a dense subspace of $\mathcal{H}$, because $\ell^{00}(X;\mathcal{H})$ is a dense subspace of $\ell^2(X;\mathcal{H})$. For $p \in (p_0, \infty) \cup \{0, 1\}$, we equip $\mathcal{H}^{00}(T)$ with the (quasi-)norm

$$\|f\|_{\mathcal{H}^p_\omega(T^d,T)} := \|C_{T^d} f\|_{\ell^p_\omega(X;\mathcal{H})} := \|(T^d_k f)_{k\in X}\|_{\ell^p_\omega(X;\mathcal{H})}. \tag{3.2}$$

Note that (3.2) indeed defines a norm on $\mathcal{H}^{00}(T)$ for $1 \le p < \infty$ and $p = 0$, and a quasi-norm on $\mathcal{H}^{00}(T)$ in case $p_0 < p < 1$. To see this, observe that for $f \in \mathcal{H}^{00}(T)$ we have $f = D_T(g_k)_{k\in X}$ for some $(g_k)_{k\in X} \in \ell^{00}(X;\mathcal{H})$ and thus

$$\begin{aligned}
\|C_{T^d} f\|_{\ell^p_\omega(X;\mathcal{H})} &= \|C_{T^d} D_T (g_k)_{k\in X}\|_{\ell^p_\omega(X;\mathcal{H})} \\
&= \|G_{T^d,T}(g_k)_{k\in X}\|_{\ell^p_\omega(X;\mathcal{H})} \\
&\le \|G_{T^d,T}\|_{\mathcal{B}(\ell^p_\omega(X;\mathcal{H}))} \|(g_k)_{k\in X}\|_{\ell^p_\omega(X;\mathcal{H})} < \infty,
\end{aligned} \tag{3.3}$$

since $G_{T^d,T} \in \mathcal{A}$ and ω is $(\mathcal{A}, p_0)$-admissible by assumption, whereas the other properties of a (quasi-)norm follow from C_{T^d} being linear and injective as an operator $\mathcal{H} \longrightarrow \ell^2(X;\mathcal{H})$ and $\|\,.\,\|_{\ell^p_\omega(X;\mathcal{H})}$ being a (quasi-)norm.

This justifies the following definition.

Definition 3.14. *Let $\mathcal{A}$ be a spectral algebra, ω be an $(\mathcal{A}, p_0)$-admissible weight, $T = (T_k)_{k\in X}$ be a g-frame and $T^d = (T^d_k)_{k\in X}$ be a dual g-frame of T such that $T^d \sim_{\mathcal{A}} T$. Then for each $p \in (p_0, \infty) \cup \{1\}$ we define $\mathcal{H}^p_\omega(T^d, T)$ as the completion of $\mathcal{H}^{00}(T)$ with respect to the (quasi-)norm $\|\,.\,\|_{\mathcal{H}^p_\omega(T^d,T)}$. Furthermore we analogously define $\mathcal{H}^0_\omega(T^d, T)$ as the completion of $\mathcal{H}^{00}(T)$ with respect to the norm $\|\,.\,\|_{\mathcal{H}^0_\omega(T^d,T)} := \|C_{T^d}\,.\,\|_{\ell^0_\omega(X;\mathcal{H})} = \|C_{T^d}\,.\,\|_{\ell^\infty_\omega(X;\mathcal{H})}$.*

By definition, $\mathcal{H}^p_\omega(T^d, T)$ is a Banach space for each $p \in [1, \infty) \cup \{0\}$, and a quasi-Banach space for $p_0 < p < 1$. Moreover, C_{T^d} extends via density to an isometry $\mathcal{H}^p_\omega(T^d, T) \longrightarrow \ell^p_\omega(X;\mathcal{H})$ for $p \in (p_0, \infty) \cup \{0, 1\}$.

3.2.3 Definition of the space $\mathcal{H}^\infty_\omega(T^d, T)$

The definition of the space $\mathcal{H}^\infty_\omega(T^d, T)$, which we will consider next, is more technical. In order to motivate it, let $T^d = (T^d_k)_{k\in X}$ be a g-frame for $\mathcal{H}$ and consider the dense subspace $\mathcal{H}^{00}(T^d) := D_{T^d}(\ell^{00}(X;\mathcal{H}))$ of $\mathcal{H}$. Then it is easily seen that $(\mathcal{H}, \mathcal{H}^{00}(T^d))$ forms a dual pair with respect to the bilinear map $\mathcal{H} \times \mathcal{H}^{00}(T^d) \longrightarrow \mathbb{C}, (h, g) \mapsto \langle h, g\rangle$. Consequently, the family $\{p_g : g \in \mathcal{H}^{00}(T^d)\}$ of seminorms p_g on $\mathcal{H}$, where $p_g(h) = |\langle g, h\rangle|$, defines a Hausdorff locally convex topology $\sigma(\mathcal{H}, \mathcal{H}^{00}(T^d))$ on $\mathcal{H}$. The topological completion $\overline{(\mathcal{H}, \sigma(\mathcal{H}, \mathcal{H}^{00}(T^d)))}$ consists of all equivalence classes $[\{f_\alpha\}]$ of Cauchy nets $\{f_\alpha\} \subset \mathcal{H}$, where (by definition) two Cauchy nets $\{f_\alpha\}$ and $\{g_\alpha\}$ are equivalent if and only if

$$\forall \varepsilon > 0, \forall g \in \mathcal{H}^{00}(T^d) : \exists \beta : \forall \alpha \geq \beta : |\langle g, f_\alpha - g_\alpha\rangle| < \varepsilon. \tag{3.4}$$

Since any $g \in \mathcal{H}^{00}(T^d)$ has the form $g = \sum_{k\in K}(T^d_k)^* h_k$ with $h_k \in \mathcal{H}$ for each $k \in X$ and $K \subset X$, $|K| < \infty$, (3.4) is equivalent to

$$\forall \varepsilon > 0, \forall k \in X, \forall h \in \mathcal{H} : \exists \beta : \forall \alpha \geq \beta : |\langle h, T^d_k(f_\alpha - g_\alpha)\rangle| < \varepsilon. \tag{3.5}$$

Next, we consider the subspace of all equivalence classes of Cauchy *sequences* $\{f_n\}_{n=1}^\infty \subset \mathcal{H}$ and denote this space by $\mathcal{H}^\infty(T^d, T)$.
Claim. *If $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ are equivalent Cauchy sequences, then the sequence $\{f_n - g_n\}_{n=1}^\infty$ is norm-bounded in $\mathcal{H}$.*

Proof. Recall, that $(\mathcal{H}^{00}(T^d), \|\cdot\|_\mathcal{H})$ is a dense subspace of $(\mathcal{H}, \|\cdot\|_\mathcal{H})$. By the Riesz representation theorem we may identify each $f_n \in \mathcal{H}$ with a linear functional $\ell_{f_n} \in \mathcal{H}^* \subseteq (\mathcal{H}^{00}(T^d))^*$. Hence, by (3.4), two such Cauchy sequences $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ being equivalent implies that the sequence of functionals $\{\ell_{(f_n-g_n)}\}_{n=1}^\infty \subset \mathcal{H}^* \subseteq (\mathcal{H}^{00}(T^d))^*$ is a weak* convergent sequence in $(\mathcal{H}^{00}(T^d))^*$ (with weak*-limit 0). Therefore, by a general principle [15], $\{\ell_{(f_n-g_n)}\}_{n=1}^\infty$ is norm-bounded in $(\mathcal{H}^{00}(T^d))^*$ by some positive constant c'. We will show that $\{\ell_{(f_n-g_n)}\}_{n=1}^\infty$ is also norm-bounded in $\mathcal{H}^*$, which proves the claim. To this end, observe that $(\mathcal{H}^{00}(T^d), \|\cdot\|_\mathcal{H})$ being a dense subspace of $(\mathcal{H}, \|\cdot\|_\mathcal{H})$ implies via the reverse triangle inequality that the unit sphere of $\mathcal{H}^{00}(T^d)$ is a dense subset of the unit sphere of $\mathcal{H}$ (with respect to $\|\cdot\|_\mathcal{H}$). Thus, for every unit vector $h \in \mathcal{H}$ and every $n \in \mathbb{N}$, there exists a unit vector $h_n \in \mathcal{H}^{00}(T^d)$ such that $\|h - h_n\| < \|\ell_{(f_n-g_n)}\|^{-1}_{\mathcal{H}^*}$ (we may assume W.L.O.G. that each $\ell_{(f_n-g_n)}$ is non-zero). This yields for any unit-norm $h \in \mathcal{H}$ that

$$|\ell_{(f_n-g_n)}(h)| \leq |\ell_{(f_n-g_n)}(h - h_n)| + |\ell_{(f_n-g_n)}(h_n)| \leq 1 + c' =: c$$

and the claim follows. $\square$

So, now we look at the subspace $\mathcal{H}^{\infty}(T^d, T)$ of all equivalence classes of Cauchy *sequences* $\{f_n\}_{n=1}^{\infty} \subset \mathcal{H}$ of the Hausdorff locally convex space $\overline{(\mathcal{H}, \sigma(\mathcal{H}, \mathcal{H}^{00}(T^d)))}$. Let B denote the upper g-frame bound of T^d. By the proven claim and (3.5), two such Cauchy sequences being equivalent implies

$$|\langle h, T_k^d(f_n - g_n)\rangle| \leq \|h\|_{\mathcal{H}} \|T_k^d\|_{\mathcal{B}(\mathcal{H})} \|f_n - g_n\|_{\mathcal{H}} \leq \sqrt{B} c \|h\|_{\mathcal{H}} \qquad (\forall k \in X, \forall n \in \mathbb{N}),$$

where c denotes the norm-bound of the sequence $\{\|f_n - g_n\|_{\mathcal{H}}\}_{n=1}^{\infty}$. Thus

$$\sup_{n\in\mathbb{N}} \|C_{T^d}(f_n - g_n)\|_{\ell^{\infty}(X;\mathcal{H})} < \infty. \tag{3.6}$$

In the weighted case we consider a modified version of (3.6) and arrive at the following (compare with [1, Definition 3]).

Definition 3.15. *Let $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ be sequences in $\mathcal{H}$, $T^d = (T_k^d)_{k\in X}$ be a g-frame for $\mathcal{H}$ and ω be a weight. Then we say that $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ are* (T^d, ω)-equivalent*, in short $\{f_n\}_{n=1}^{\infty} \sim_{T^d,\omega} \{g_n\}_{n=1}^{\infty}$, if*

$$\forall k \in X: \lim_{n\to\infty} \|T_k^d(f_n - g_n)\| = 0 \qquad \textit{and} \qquad \sup_{n\in\mathbb{N}} \|C_{T^d}(f_n - g_n)\|_{\ell^{\infty}_{\omega}(X;\mathcal{H})} < \infty.$$

Clearly, the relation $\sim_{T^d,\omega}$ defines an equivalence relation on the set of all sequences in $\mathcal{H}$. We define the space $\mathcal{H}^{\infty}_{\omega}(T^d, T)$ as a certain subspace of the vector space of equivalence classes corresponding to $\sim_{T^d,\omega}$ (compare with [1, Definition 3]).

Definition 3.16. *Let $\mathcal{A}$ be a spectral algebra, ω be an $(\mathcal{A}, p_0)$-admissible weight, $T = (T_k)_{k\in X}$ be a g-frame and $T^d = (T_k^d)_{k\in X}$ be a dual g-frame of T such that $T^d \sim_{\mathcal{A}} T$. Then $\mathcal{H}^{\infty}_{\omega}(T^d, T)$ is defined to be the space of all equivalence classes $f = [\{f_n\}_{n=1}^{\infty}]_{\sim_{T^d,\omega}}$ of sequences $\{f_n\}_{n=1}^{\infty}$ in $\mathcal{H}$ satisfying*

(1.) $\lim_{n\to\infty} T_k^d f_n =: T_k^d f$ *exists in $\mathcal{H}$ for each $k \in X$,*

(2.) $\sup_{n\in\mathbb{N}} \|C_{T^d} f_n\|_{\ell^{\infty}_{\omega}(X;\mathcal{H})} < \infty$.

It is quickly verified that the defining properties (1.) and (2.) do not depend on the choice of the representative sequence $\{f_n\}_{n=1}^{\infty}$. Condition (2.) can be rewritten as

$$\|T_k^d f_n\| \omega(k) \leq C \qquad (\forall k \in X, \forall n \in \mathbb{N}) \tag{3.7}$$

for some positive constant C. Thus $\|T_k^d f\| \omega(k) = \lim_{n\to\infty} \|T_k^d f_n\| \omega(k) \leq C$ for each $k \in X$. In particular

$$\|f\|_{\mathcal{H}^{\infty}_{\omega}(T^d,T)} := \|C_{T^d} f\|_{\ell^{\infty}_{\omega}(X;\mathcal{H})} = \sup_{k\in X} \left\| \lim_{n\to\infty} T_k^d f_n \right\| \omega(k) \tag{3.8}$$

is well-defined. Note that (3.8) certainly defines a semi-norm on $\mathcal{H}^\infty_\omega(T^d,T)$, since limits are linear and $\|\,.\,\|_{\ell^\infty_\omega(X;\mathcal{H})}$ is a norm. If $\|f\|_{\mathcal{H}^\infty_\omega(T^d,T)} = 0$ for some $f = [\{f_n\}_{n=1}^\infty]_{\sim_{T^d,\omega}}$, then $\{f_n\}_{n=1}^\infty \sim_{T^d,\omega} \{0\}_{n=1}^\infty$ and thus $f = 0$ in $\mathcal{H}^\infty_\omega(T^d,T)$. This shows that $\|\,.\,\|_{\mathcal{H}^\infty_\omega(T^d,T)}$ actually defines a norm on $\mathcal{H}^\infty_\omega(T^d,T)$.

We explicitly emphasize that here $T^d_k f \in \mathcal{H}$ is the limit of the sequence $\{T^d_k f_n\}_{n=1}^\infty \subset \mathcal{H}$ (for each $k \in X$). In particular, $C_{T^d} f = (T^d_k f)_{k\in X}$ is understood as a pointwise limit. In this sense, $C_{T^d} : \mathcal{H}^\infty_\omega(T^d,T) \longrightarrow \ell^\infty_\omega(X;\mathcal{H})$ is a well-defined isometry.

3.2.4 Properties of the spaces $\mathcal{H}^p_\omega(T^d,T)$

In the previous subsection, we have shown that $C_{T^d} : \mathcal{H}^\infty_\omega(T^d,T) \longrightarrow \ell^\infty_\omega(X;\mathcal{H})$ is a well-defined isometry. Below we show that its range is closed.

Lemma 3.17. *Under the same assumptions as in Definition 3.16, the map $f = [\{f_n\}_{n=1}^\infty]_{\sim_{T^d,\omega}} \mapsto (\lim_{n\to\infty} T^d_k f_n)_{k\in X} =: C_{T^d} f$ is an isometric isomorphism from $\mathcal{H}^\infty_\omega(T^d,T)$ onto the space $V = \{g \in \ell^\infty_\omega(X;\mathcal{H}) : g = G_{T^d,T}\, g\}$. Furthermore, V is a closed subspace of $\ell^\infty_\omega(X;\mathcal{H})$, and thus $\mathcal{H}^\infty_\omega(T^d,T)$ a Banach space.*

Proof. Let $f = [\{f_n\}_{n=1}^\infty]_{\sim_{T^d,\omega}} \in \mathcal{H}^\infty_\omega(T^d,T)$ be fixed. Then, by definition, we have that $\lim_{n\to\infty} T^d_k f_n =: T^d_k f$ exists in $\mathcal{H}$ for each $k \in X$ and that $C_{T^d} f = (T^d_k f)_{k\in X} \in \ell^\infty_\omega(X;\mathcal{H})$. Since ω is $(\mathcal{A},p_0)$-admissible and $T^d \sim_{\mathcal{A}} T$, we have $G_{T^d,T} \in \mathcal{A} \subset \mathcal{B}(\ell^\infty_\omega(X;\mathcal{H}))$, hence $G_{T^d,T} C_{T^d} f \in \ell^\infty_\omega(X;\mathcal{H})$. We will show that $C_{T^d} f = G_{T^d,T} C_{T^d} f$ as elements in $\ell^\infty_\omega(X;\mathcal{H})$. It suffices to verify the claimed identity component-wise, i.e. to show that

$$\left\| T^d_k f - \sum_{l\in X} [G_{T^d,T}]_{k,l} T^d_l f \right\| = 0 \qquad \text{(for each } k \in X).$$

Since T^d and T is a pair of dual g-frames we have $G_{T^d,T} C_{T^d} f_n = C_{T^d} f_n$ for each $n \in \mathbb{N}$. Therefore, it suffices to show that

$$\lim_{n\to\infty} \left\| \sum_{l\in X} [G_{T^d,T}]_{k,l} T^d_l f_n - \sum_{l\in X} [G_{T^d,T}]_{k,l} T^d_l f \right\| = 0 \qquad \text{(for each } k \in X).$$

The latter follows immediately from an application of the dominated convergence theorem for Bochner spaces [30, Proposition 1.2.5], once we have verified that for each $k \in X$, there exists some family of non-negative numbers $(c^k_l)_{l\in X}$ such that

$$\|[G_{T^d,T}]_{k,l} T^d_l f_n\| \le c^k_l \quad (\forall n \in \mathbb{N}), \qquad \text{and } \sum_{l\in X} c^k_l < \infty.$$

In order to show the latter dominance condition, we fix $k \in X$ and estimate via (3.7)

$$\begin{aligned}\sum_{l\in X} \|[G_{T^d,T}]_{k,l}T_l^d f_n\| &\le \sum_{l\in X} C\|[G_{T^d,T}]_{k,l}\| \cdot w(l)^{-1} \\ &\le Cw(k)^{-1} \sup_{m\in X} \left| \sum_{l\in X} \|[G_{T^d,T}]_{m,l}\| \cdot w(l)^{-1} \right| w(m) =: (*).\end{aligned}$$

Since $G_{T^d,T} = [T_k^d T_l^*]_{k,l\in X}$ is contained in $\mathcal{A}$, so is $[\|T_k^d T_l^*\| \cdot \mathcal{I}_{\mathcal{B}(\mathcal{H})}]_{k,l\in X}$ due to solidity of $\mathcal{A}$. Thus, by Lemma 2.4, the scalar matrix $G := [\|T_k^d T_l^*\|]_{k,l\in X}$ is contained in $\mathcal{B}(\ell^\infty_\omega(X))$ as well. Finally, since $\|(\omega(l)^{-1})_{l\in X}\|_{\ell^\infty_\omega(X)} = 1$, we arrive at

$$(*) = C\omega(k)^{-1}\|G \cdot (\omega(l)^{-1})_{l\in X}\|_{\ell^\infty_\omega(X)} \le C\omega(k)^{-1}\|G\|_{\mathcal{B}(\ell^\infty_\omega(X))} < \infty$$

and the claim is proven. In particular, we have shown that C_{T^d} is an isometry from $\mathcal{H}^\infty_\omega(T^d, T)$ into $V \subseteq \ell^\infty_\omega(X; \mathcal{H})$.

To show that $C_{T^d} : \mathcal{H}^\infty_\omega(T^d, T) \longrightarrow V$ is onto, choose some arbitrary $g = (g_l)_{l\in X} \in V$. Let $F_1 \subseteq F_2 \subseteq \ldots$ be a nested sequence of finite subsets of X such that $\bigcup_{n=1}^\infty F_n = X$ and set $f_n = \sum_{l\in F_n} T_l^* g_l \in \mathcal{H}$. Then

$$\lim_{n\to\infty} T_k^d f_n = \lim_{n\to\infty} \sum_{l\in F_n} T_k^d T_l^* g_l = \sum_{l\in X} [G_{T^d,T}]_{k,l} g_l = g_k \in \mathcal{H}$$

for each $k \in X$. In particular, $\sup_{k\in X} \|\lim_{n\to\infty} T_k^d f_n\|\omega(k) = \|g\|_{\ell^\infty_\omega(X;\mathcal{H})} < \infty$. Consequently, $f := [\{f_n\}_{n=1}^\infty]_{\sim_{T^d,\omega}} \in \mathcal{H}^\infty_\omega(T^d, T)$ and $C_{T^d} f = g$.

Finally, let $\{g_n\}_{n=1}^\infty$ be a sequence in V, which converges in $\ell^\infty_\omega(X; \mathcal{H})$ to some $g \in \ell^\infty_\omega(X; \mathcal{H})$. Then

$$\lim_{n\to\infty} \|g - G_{T^d,T} g_n\|_{\ell^\infty_\omega(X;\mathcal{H})} = \lim_{n\to\infty} \|g - g_n\|_{\ell^\infty_\omega(X;\mathcal{H})} = 0$$

and at the same time

$$\lim_{n\to\infty} \|G_{T^d,T} g - G_{T^d,T} g_n\|_{\ell^\infty_\omega(X;\mathcal{H})} \le \|G_{T^d,T}\|_{\mathcal{B}(\ell^\infty_\omega(X;\mathcal{H}))} \lim_{n\to\infty} \|g - g_n\|_{\ell^\infty_\omega(X;\mathcal{H})} = 0.$$

Since limits in a Banach space are unique we must have $g = G_{T^d,T} g \in V$, hence V is a closed subspace of $\ell^\infty_\omega(X; \mathcal{H})$. □

So far we have shown the following.

Corollary 3.18. *The space $\mathcal{H}^p_\omega(T^d, T)$ is a Banach space for each $p \in [1, \infty] \cup \{0\}$ and a quasi-Banach space for $p_0 < p < 1$. Moreover, C_{T^d} is an isometry $\mathcal{H}^p_\omega(T^d, T) \longrightarrow \ell^p_\omega(X; \mathcal{H})$ for all $p \in (p_0, \infty] \cup \{0, 1\}$.*

Remark 3.19. *Ignoring the weights for the moment, we explicitly emphasize the difference between the spaces $\mathcal{H}^0(T^d,T)$ and $\mathcal{H}^\infty(T^d,T)$. While $\mathcal{H}^0(T^d,T)$ is the completion of $\mathcal{H}^{00}(T)$ with respect to the norm $\|C_{T^d}\cdot\|_{\ell^\infty(X;\mathcal{H})}$, the space $\mathcal{H}^\infty(T^d,T)$ is a complete normed subspace of the topological completion of $\mathcal{H}$ equipped with the $\sigma(\mathcal{H},\mathcal{H}^{00}(T^d))$-topology. In particular, while $\mathcal{H}^0(T^d,T)$ consists of equivalence classes of sequences in $\mathcal{H}^{00}(T)$, with equivalence relation given by*

$$\{f_n\}_{n=1}^\infty \sim \{g_n\}_{n=1}^\infty \Leftrightarrow \lim_{n\to\infty}\sup_{k\in X}\|T_k^d(f_n-g_n)\| = 0,$$

the space $\mathcal{H}^\infty(T^d,T)$ consists of equivalence classes of sequences in $\mathcal{H}$, with equivalence relation given by

$$\begin{aligned}\{f_n\}_{n=1}^\infty \sim_{T^d} \{g_n\}_{n=1}^\infty \Leftrightarrow &\lim_{n\to\infty}\|T_k^d(f_n-g_n)\| = 0 \ (\forall k\in X),\\ &\textit{and } \sup_{n\in\mathbb{N}}\sup_{k\in X}\|T_k^d(f_n-g_n)\| < \infty.\end{aligned}$$

We will prove in Theorem 3.29 that $\mathcal{H}^\infty(T^d,T)$ is isomorphic to the bi-dual $(\mathcal{H}^0(T^d,T))^{**}$ of the non-reflexive space $\mathcal{H}^0(T^d,T)$. In this sense $\mathcal{H}^0(T^d,T)$ can be identified with a closed linear subspace of $\mathcal{H}^\infty(T^d,T)$ [15].

Next, we consider the properties of the synthesis operator in this setting. In order to keep the notation as simple as possible, we use the same symbol for each of the operators

$$D_T : \ell^p_\omega(X;\mathcal{H}) \longrightarrow \mathcal{H}^p_\omega(T^d,T) \qquad (p\in(p_0,\infty)\cup\{0,1\}),$$

densely defined on $\ell^{00}(X;\mathcal{H})$ via

$$D_T(g_l)_{l\in X} = \sum_{l\in X} T_l^* g_l,$$

and

$$D_T : \ell^\infty_\omega(X;\mathcal{H}) \longrightarrow \mathcal{H}^\infty_\omega(T^d,T),$$

defined by the equivalence class

$$D_T(g_l)_{l\in X} = \sum_{l\in X} T_l^* g_l := [\{f_n\}_{n=1}^\infty]_{\sim_{T^d,\omega}}$$

of the partial sums

$$f_n = \sum_{l\in F_n} T_l^* g_l,$$

where $F_1 \subseteq F_2 \subseteq \ldots$ is a nested sequence of finite subsets of X such that $\bigcup_{n=1}^\infty F_n = X$. In fact, the latter definition is independent of the choice of $\{F_n\}_{n=1}^\infty$, as the next result shows.

Proposition 3.20. *Let $\mathcal{A}$ be a spectral algebra, ω be an $(\mathcal{A}, p_0)$-admissible weight, $T = (T_k)_{k\in X}$ be a g-frame and $T^d = (T^d_k)_{k\in X}$ be a dual g-frame of T such that $T^d \sim_{\mathcal{A}} T$. Then, for each $p \in (p_0, \infty] \cup \{0, 1\}$, the synthesis operator*

$$D_T : \ell^p_\omega(X; \mathcal{H}) \longrightarrow \mathcal{H}^p_\omega(T^d, T),$$
$$(g_l)_{l\in X} \mapsto \sum_{l\in X} T^*_l g_l \tag{3.9}$$

is bounded and surjective. For $p < \infty$ the series (3.9) converges unconditionally in $\mathcal{H}^p_\omega(T^d, T)$. In the case $p = \infty$, $D_T g$ is independent of the choice of $\{F_n\}_{n=1}^\infty$ and the series (3.9) converges unconditionally in the $\sigma(\mathcal{H}, \mathcal{H}^{00}(T^d))$-topology. Furthermore, for all $p \in (p_0, \infty] \cup \{0, 1\}$,

$$\|f\|_{\mathcal{H}^p_\omega} \asymp \inf\{\|g\|_{\ell^p_\omega(X;\mathcal{H})} : g \in \ell^p_\omega(X; \mathcal{H}), f = D_T g\}.$$

Proof. First, let $p \in (p_0, \infty) \cup \{0, 1\}$. For $f = D_T(g_l)_{l\in X} \in \mathcal{H}^{00}(T)$ with $g = (g_l)_{l\in X} \in \ell^{00}(X; \mathcal{H})$ we have $C_{T^d} f = G_{T^d,T} g$ and thus

$$\|f\|_{\mathcal{H}^p_\omega(T^d,T)} = \|G_{T^d,T} g\|_{\ell^p_\omega(X;\mathcal{H})} \le \|G_{T^d,T}\|_{\mathcal{B}(\ell^p_\omega(X;\mathcal{H}))} \|g\|_{\ell^p_\omega(X;\mathcal{H})} \tag{3.10}$$

by the $\mathcal{A}$-admissibility of ω and since $G_{T^d,T} \in \mathcal{A}$ by assumption. By density, the inequality (3.10) extends to all $g \in \ell^p_\omega(X; \mathcal{H})$, so D_T is bounded. Now, if $\widetilde{f} = [\{f_n\}_{n=1}^\infty] \in \mathcal{H}^p_\omega(T^d, T)$ for some Cauchy sequence $\{f_n\}_{n=1}^\infty$ in $\mathcal{H}^{00}(T)$, then $\{C_{T^d} f_n\}_{n=1}^\infty$ is a Cauchy sequence in $\ell^p_\omega(X; \mathcal{H})$, which has some limit $g \in \ell^p_\omega(X; \mathcal{H})$. Writing $f := D_T g \in \mathcal{H}^p_\omega(T^d, T)$ we see that

$$\begin{aligned}\lim_{n\to\infty} \|f - f_n\|_{\mathcal{H}^p_\omega(T^d,T)} &= \lim_{n\to\infty} \|C_{T^d}(f - f_n)\|_{\ell^p_\omega(X;\mathcal{H})} \\ &\le \|G_{T^d,T}\|_{\mathcal{B}(\ell^p_\omega(X;\mathcal{H}))} \lim_{n\to\infty} \|g - C_{T^d} f_n\|_{\ell^p_\omega(X;\mathcal{H})} = 0,\end{aligned}$$

where we used that $G_{T^d,T} C_{T^d} = C_{T^d}$ holds true on $\mathcal{H}^{00}(T)$ due to T^d and T being a pair of dual g-frames. Consequently, $\widetilde{f} = f$ in $\mathcal{H}^p_\omega(T^d, T)$ and since $f = D_T g$, we may conclude that $D_T : \ell^p_\omega(X; \mathcal{H}) \longrightarrow \mathcal{H}^p_\omega(T^d, T)$ is onto. The unconditional convergence of the series $D_T g$ in $\mathcal{H}^p_\omega(T^d, T)$ is easily deduced from the latter argument.

Now consider the case $p = \infty$. Pick some arbitrary $g = (g_l)_{l\in X} \in \ell^\infty_\omega(X; \mathcal{H})$ and assume that $F_1 \subseteq F_2 \subseteq \ldots$ is a nested sequence of finite subsets of X such that $\bigcup_{n=1}^\infty F_n = X$. Then $f_n = \sum_{l\in F_n} T^*_l g_l$ is contained in $\mathcal{H}$ for each $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} T^d_k f_n = \lim_{n\to\infty} \sum_{l\in F_n} T^d_k T^*_l g_l = \sum_{l\in X} [G_{T^d,T}]_{k,l} g_l = [G_{T^d,T} g]_k \in \mathcal{H}$$

for each $k \in X$, since $G_{T^d,T} \in \mathcal{A} \subseteq \mathcal{B}(\ell^\infty_\omega(X;\mathcal{H}))$ by assumption. Furthermore, if G denotes the scalar matrix $[\|T_k^d T_l^*\|]_{k,l\in X}$ as in the proof of Lemma 3.17, we have that

$$\begin{aligned}
\sup_{n\in\mathbb{N}}\sup_{k\in X}\|T_k^d f_n\|\omega(k) &\leq \sup_{n\in\mathbb{N}}\sup_{k\in X}\sum_{l\in F_n}\|T_k^d T_l^*\|\|g_l\|\omega(k) \\
&\leq \sup_{k\in X}\left|\sum_{l\in X}\|T_k^d T_l^*\|\|g_l\|\right|\omega(k) \\
&\leq \|G\|_{\mathcal{B}(\ell^\infty_\omega(X))}\|g\|_{\ell^\infty_\omega(X;\mathcal{H})}, \qquad (3.11)
\end{aligned}$$

where we used that $(\|g_l\|)_{l\in X}$ is a scalar sequence in $\ell^\infty_\omega(X)$ with norm equal to $\|g\|_{\ell^\infty_\omega(X;\mathcal{H})}$. Thus, $D_T g = [\{f_n\}_{n=1}^\infty]_{\sim_{T^d,\omega}} \in \mathcal{H}^\infty_\omega(T^d,T)$ with $C_{T^d}(D_T g) = G_{T^d,T}g$. The boundedness of D_T now follows from

$$\|D_T g\|_{\mathcal{H}^\infty_\omega(T^d,T)} = \|G_{T^d,T}g\|_{\ell^\infty_\omega(X;\mathcal{H})} \leq \|G_{T^d,T}\|_{\mathcal{B}(\ell^\infty_\omega(X;\mathcal{H}))}\|g\|_{\ell^\infty_\omega(X;\mathcal{H})}.$$

Next, let $G_1 \subseteq G_2 \subseteq \ldots$ be another nested sequence of finite subsets of X with $\bigcup_{n=1}^\infty G_n = X$. We will show that $\{\sum_{l\in G_n} T_l^* g_l\}_{n=1}^\infty \sim_{T^d,\omega} \{\sum_{l\in F_n} T_l^* g_l\}_{n=1}^\infty$, which, by (3.5), implies that both sequences belong to the same equivalence class in $(\mathcal{H},\sigma(\mathcal{H},\mathcal{H}^{00}(T^d)))$, yielding the unconditional convergence of $D_T g$ in the $\sigma(\mathcal{H},\mathcal{H}^{00}(T^d))$-topology. Setting $H_n = F_n \cup G_n$ ($n \in \mathbb{N}$) gives a nested sequence of finite subsets whose union is X as well. Then, for each $k \in X$,

$$\begin{aligned}
\lim_{n\to\infty}\left\|T_k^d\left(\sum_{l\in H_n}T_l^* g_l - \sum_{l\in F_n}T_l^* g_l\right)\right\| &= \lim_{n\to\infty}\left\|\sum_{l\in H_n\setminus F_n}T_k^d T_l^* g_l\right\| \\
&\leq \lim_{n\to\infty}\sum_{l\in X\setminus F_n}\|T_k^d T_l^* g_l\| \\
&\leq \|g\|_{\ell^\infty_\omega(X;\mathcal{H})}\lim_{n\to\infty}\sum_{l\in X\setminus F_n}\|T_k^d T_l^*\|\omega(l)^{-1} \\
&= 0.
\end{aligned}$$

The last step follows from convergence of $\sum_{l\in X}\|T_k^d T_l^*\|\omega(l)^{-1}$ for each $k \in X$, which we already showed in $(*)$ in the proof of Lemma 3.17. Clearly, it also holds

$$\sup_{n\in\mathbb{N}}\left\|C_{T^d}\left(\sum_{l\in H_n}T_l^* g_l - \sum_{l\in F_n}T_l^* g_l\right)\right\|_{\ell^\infty_\omega(X;\mathcal{H})} < \infty.$$

Thus we have shown that $\{\sum_{l\in H_n} T_l^* g_l\}_{n=1}^\infty \sim_{T^d,\omega} \{\sum_{l\in F_n} T_l^* g_l\}_{n=1}^\infty$. The same argument holds if we replace each F_n by G_n. Therefore, $\{\sum_{l\in G_n} T_l^* g_l\}_{n=1}^\infty \sim_{T^d,\omega}$

$\{\sum_{l\in F_n} T_l^* g_l\}_{n=1}^\infty$, since $\sim_{T^d,\omega}$ is an equivalence relation. Finally, we show that D_T is onto. Choose some arbitrary $f = [\{f_n\}_{n=1}^\infty]_{\sim_{T^d,\omega}} \in \mathcal{H}_\omega^\infty(T^d,T)$. Then $C_{T^d}f \in \ell_\omega^\infty(X;\mathcal{H})$ by Corollary 3.18 and $D_T C_{T^d} f \in \mathcal{H}_\omega^\infty(T^d,T)$ by the above. We claim that $f = D_T C_{T^d} f = [\{\sum_{l\in F_n} T_l^*(T_l^d f)\}_{n=1}^\infty]_{\sim_{T^d,\omega}}$ in $\mathcal{H}_\omega^\infty(T^d,T)$. To prove the claim, it suffices to show that $\{\sum_{l\in F_n} T_l^*(T_l^d f)\}_{n=1}^\infty \sim_{T^d,\omega} \{f_n\}_{n=1}^\infty$. Now, due to Lemma 3.17 we have $C_{T^d}(D_T C_{T^d} f) = G_{T^d,T} C_{T^d} f = C_{T^d} f$, hence

$$\lim_{n\to\infty} \left\| T_k^d \left(\sum_{l\in F_n} T_l^*(T_l^d f) - f_n \right) \right\| = \left\| \sum_{l\in X} T_k^d T_l^*(T_l^d f) - T_k^d f \right\| = 0 \qquad (\forall k \in X).$$

By repeating the routine (3.11), we also see that

$$\sup_{n\in\mathbb{N}} \left\| C_{T^d} \left(\sum_{l\in F_n} T_l^*(T_l^d f) - f_n \right) \right\|_{\ell_\omega^\infty(X;\mathcal{H})} < \infty$$

and the claim is proven.

Finally, showing the corresponding norm equivalences is a standard routine, see e.g. the proof of [24, Proposition 2.4]. □

Theorem 3.21. *Let $\mathcal{A}$ be a spectral algebra, ω be an $(\mathcal{A},p_0)$-admissible weight, $T = (T_k)_{k\in X}$ be a g-frame and $T^d = (T_k^d)_{k\in X}$ be a dual g-frame of T such that $T^d \sim_{\mathcal{A}} T$. Then for each $p \in (p_0,\infty] \cup \{0,1\}$ it holds*

$$f = \sum_{k\in X} T_k^* T_k^d f \qquad (\forall f \in \mathcal{H}_\omega^p(T^d,T)), \tag{3.12}$$

with unconditional convergence in $\mathcal{H}_\omega^p(T^d,T)$ for $p < \infty$ and unconditional convergence in the $\sigma(\mathcal{H},\mathcal{H}^{00}(T^d))$-topology in the case $p = \infty$ respectively.

Proof. The case $p = \infty$ was already treated in the proof of Proposition 3.20. So, let $p \in (p_0,\infty) \cup \{0,1\}$. By Proposition 3.20 and Corollary 3.18 the composition $D_T C_{T^d}$ defines a bounded operator on $\mathcal{H}_\omega^p(T^d,T)$ whose associated series converges unconditionally. Moreover, (3.12) holds true on the subspace $\mathcal{H}^{00}(T)$ of $\mathcal{H}$ since T^d and T is a pair of dual g-frames. Hence, by density of $\mathcal{H}^{00}(T)$ in $\mathcal{H}_\omega^p(T^d,T)$ and boundedness of $D_T C_{T^d}$, this identity extends to all of $\mathcal{H}_\omega^p(T^d,T)$. □

So far, the spaces $\mathcal{H}_\omega^p(T^d,T)$ seem to be heavily dependent on the choice of the dual g-frame pair (T^d,T). However, we will see that $\mathcal{H}_\omega^p(T^d,T)$ is independent of the choice of the dual g-frame pair (T^d,T) as long as we stay in the same localization class $\sim_{\mathcal{A}}$. Before we give a precise formulation of the latter statement, we present an alternative description of the spaces $\mathcal{H}_\omega^p(T^d,T)$ for $p < \infty$.

Proposition 3.22. *Let $\mathcal{A}$ be a spectral algebra, ω be an $(\mathcal{A}, p_0)$-admissible weight, $T = (T_k)_{k\in X}$ be a g-frame and $T^d = (T_k^d)_{k\in X}$ be a dual g-frame of T such that $T^d \sim_{\mathcal{A}} T$. Then for each $p \in (p_0, \infty) \cup \{0, 1\}$, $\mathcal{H}_\omega^p(T^d, T)$ is the completion of the space*

$$V_\omega^p(T^d, T) = \{f \in \mathcal{H} : C_{T^d} f \in \ell_\omega^p(X; \mathcal{H})\}$$

with respect to $\| \,.\, \|_{\mathcal{H}_\omega^p(T^d,T)}$.

Proof. Let $f \in V_\omega^p(T^d, T)$ be arbitrary. Then, by dual g-frame reconstruction, $f = D_T C_{T^d} f = \sum_{k\in X} T_k^* T_k^d f$ with unconditional convergence in $\mathcal{H}$. This series converges unconditionally in $\mathcal{H}_\omega^p(T^d, T)$ as well, since $C_{T^d} f \in \ell_\omega^p(X; \mathcal{H})$ by assumption and thus $f = D_T C_{T^d} f \in \mathcal{H}_\omega^p(T^d, T)$ by Proposition 3.20 and Theorem 3.21. Hence $V_\omega^p(T^d, T) \subseteq \mathcal{H}_\omega^p(T^d, T)$. At the same time, we clearly have that $\mathcal{H}^{00}(T) = D_T(\ell^{00}(X; \mathcal{H})) \subseteq V_\omega^p(T^d, T)$. Since $\mathcal{H}^{00}(T)$ is dense in $\mathcal{H}_\omega^p(T^d, T)$ by definition, the claim follows. □

For exponents $p < \infty$ and weights ω chosen in such a way that $\ell_\omega^p(X; \mathcal{H})$ is continuously embedded in $\ell^2(X; \mathcal{H})$, we obtain the following consequence of the latter, which is the g-frame theoretic version of [24, Proposition 2.3].

Corollary 3.23. *Let $\mathcal{A}$ be a spectral algebra, ω be an $\mathcal{A}$-admissible weight, $T = (T_k)_{k\in X}$ be a g-frame and $T^d = (T_k^d)_{k\in X}$ be a dual g-frame of T such that $T^d \sim_{\mathcal{A}} T$. Let $p < \infty$ and ω be so that $\ell_\omega^p(X; \mathcal{H})$ is continuously embedded in $\ell^2(X; \mathcal{H})$. Then*

$$\mathcal{H}_\omega^p(T^d, T) = V_\omega^p(T^d, T).$$

Proof. By Proposition 3.22, it remains to show that $\mathcal{H}_\omega^p(T^d, T) \subseteq V_\omega^p(T^d, T)$. Let $\widetilde{f} = [\{f_n\}_{n=1}^\infty] \in \mathcal{H}_\omega^p(T^d, T)$ where $\{f_n\}_{n=1}^\infty \subseteq V_\omega^p(T^d, T)$ is a representative Cauchy sequence in the $\| \,.\, \|_{\mathcal{H}_\omega^p(T^d,T)}$-norm. If A denotes the lower g-frame bound of T^d and c the bound coming from the continuous embedding of $\ell_\omega^p(X; \mathcal{H})$ in $\ell^2(X; \mathcal{H})$, then

$$\begin{aligned}\|f_n - f_m\| &\leq A^{-1/2} \|C_{T^d}(f_n - f_m)\|_{\ell^2(X;\mathcal{H})} \\ &\leq A^{-1/2} c \|C_{T^d}(f_n - f_m)\|_{\ell_\omega^p(X;\mathcal{H})} = A^{-1/2} c \|f_n - f_m\|_{\mathcal{H}_\omega^p(T^d,T)}.\end{aligned}$$

Consequently $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{H}$ and thus convergent in $\mathcal{H}$ to some $f \in \mathcal{H}$. Note that this limit f is independent of our choice of the representative sequence of $\widetilde{f}$. Then, $\{C_{T^d} f_n\}_{n=1}^\infty$ is a Cauchy sequence in both $\ell_\omega^p(X; \mathcal{H})$ and $\ell^2(X; \mathcal{H})$. By passing onto their respective $\mathcal{H}$-valued components, we see that these Cauchy sequences must converge to the same limiting $\mathcal{H}$-valued sequence $(h_k)_{k\in X}$. However, by continuity of $C_{T^d} : \mathcal{H} \longrightarrow \ell^2(X; \mathcal{H})$, we must have $(h_k)_{k\in X} = C_{T^d} f$. Thus $C_{T^d} f \in \ell_\omega^p(X; \mathcal{H}) \subseteq \ell^2(X; \mathcal{H})$ and $\widetilde{f} = D_T C_{T^d} f = f \in \mathcal{H}$, as was to be shown. □

Remark 3.24. *The spaces $V^p(T, T^d)$ have also been studied in [39], where the case $1 \leq p \leq 2$ has been mainly considered. Moreover, in [39, Example 3.1], it has been shown that $V^p(T, T^d)$ ($1 \leq p \leq 2$) might be the trivial space $\{0_{\mathcal{H}}\}$ if the given g-frame T is not suitably localized.*

Theorem 3.25. *Let $\mathcal{A}$ be a spectral algebra and ω be an $(\mathcal{A}, p_0)$-admissible weight. Let $T = (T_k)_{k\in X}$ and $U = (U_k)_{k\in X}$ be g-frames and assume that $T^d = (T_k^d)_{k\in X}$ is a dual g-frame of T such that $T^d \sim_{\mathcal{A}} T$, and that $U^d = (U_k^d)_{k\in X}$ is a dual g-frame of U such that $U^d \sim_{\mathcal{A}} U$. If $T^d \sim_{\mathcal{A}} U$ and $U^d \sim_{\mathcal{A}} T$, then, for each $p \in (p_0, \infty] \cup \{0, 1\}$,*

$$\mathcal{H}^p_\omega(T^d, T) = \mathcal{H}^p_\omega(U^d, U)$$

with equivalent norms.

Proof. Let $p \in (p_0, \infty) \cup \{0, 1\}$. By Proposition 3.22, $\mathcal{H}^p_\omega(T^d, T)$ and $\mathcal{H}^p_\omega(U^d, U)$ are the completions of $(V^p_\omega(T^d, T), \|\,.\,\|_{\mathcal{H}^p_\omega(T^d,T)})$ and $(V^p_\omega(U^d, U), \|\,.\,\|_{\mathcal{H}^p_\omega(U^d,U)})$ respectively. Now if $f \in V^p_\omega(T^d, T)$, then $C_{U^d} f = G_{U^d,T} C_{T^d} f$ by g-frame reconstruction in $\mathcal{H}$ and thus

$$\|C_{U^d} f\|_{\ell^p_\omega(X;\mathcal{H})} \leq \|G_{U^d,T}\|_{\mathcal{B}(\ell^p_\omega(X;\mathcal{H}))} \|C_{T^d} f\|_{\ell^p_\omega(X;\mathcal{H})},$$

where we also used the assumption $U^d \sim_{\mathcal{A}} T$ and the $\mathcal{A}$-admissibility of ω. This implies that $V^p_\omega(T^d, T)$ is continuously embedded in $V^p_\omega(U^d, U)$. By switching the roles of (T^d, T) and (U^d, U), we may conclude that $V^p_\omega(U^d, U) = V^p_\omega(T^d, T)$ with equivalent norms. Consequently $\mathcal{H}^p_\omega(T^d, T) = \mathcal{H}^p_\omega(U^d, U)$ with equivalent norms.

It remains to consider the case $p = \infty$.

Step 1. Recall from Lemma 3.17 that the spaces $V(T^d, T) = \{g \in \ell^\infty_\omega(X; \mathcal{H}) : g = G_{T^d,T} g\}$ and $V(U^d, U) = \{g \in \ell^\infty_\omega(X; \mathcal{H}) : g = G_{U^d,U} g\}$ are isometrically isomorphic to $\mathcal{H}^p_\omega(T^d, T)$ and $\mathcal{H}^p_\omega(U^d, U)$ via C_{T^d} and C_{U^d} respectively. Next, observe that the identities $G_{T^d,T} G_{T^d,U} = G_{T^d,U} = G_{T^d,U} G_{U^d,U}$ are valid in $\mathcal{B}(\ell^2(X; \mathcal{H}))$ due to g-frame reconstruction in $\mathcal{H}$ of the respective dual g-frame pairs (T^d, T) and (U^d, U). Since each of these occurring g-Gram matrices is contained in $\mathcal{A}$ by assumption, these identities are also valid in $\mathcal{A}$ and consequently in $\mathcal{B}(\ell^\infty_\omega(X; \mathcal{H}))$ by the $(\mathcal{A}, p_0)$-admissibility of ω. This implies that both $G_{T^d,U} : V(U^d, U) \longrightarrow V(T^d, T)$ and $G_{U^d,T} : V(T^d, T) \longrightarrow V(U^d, U)$ are bounded.

Step 2. Next, note that $C_{U^d} = G_{U^d,T} C_{T^d}$ and $C_{T^d} = G_{T^d,U} C_{U^d}$ as operators $\mathcal{H} \longrightarrow \ell^2(X; \mathcal{H})$ by g-frame reconstruction of the dual g-frame pairs (T^d, T) and (U^d, U). By repeating the proof details of Lemma 3.17 with $G_{U^d,T}$ instead of $G_{T^d,T}$, we also obtain that $G_{U^d,T} C_{T^d} = C_{U^d}$ as an operator $\mathcal{H}^\infty_\omega(U^d, U) \longrightarrow V(U^d, U)$, and similarly, $G_{T^d,U} C_{U^d} = C_{T^d}$ as an operator $\mathcal{H}^\infty_\omega(T^d, T) \longrightarrow V(T^d, T)$.

Step 3. We show that $[\{f_n\}_{n=1}^\infty]_{\sim_{T^d,\omega}} = [\{f_n\}_{n=1}^\infty]_{\sim_{U^d,\omega}}$ as sets for all sequences $\{f_n\}_{n=1}^\infty$ in $\mathcal{H}$. Assume that $\{g_n\}_{n=1}^\infty$ and $\{h_n\}_{n=1}^\infty$ are two sequences in $\mathcal{H}$ such that $\{h_n\}_{n=1}^\infty \sim_{T^d,\omega} \{g_n\}_{n=1}^\infty$. Then, by Step 2,

$$\sup_{n\in\mathbb{N}} \|C_{U^d}(g_n - h_n)\|_{\ell^\infty_\omega(X;\mathcal{H})} \le \|G_{U^d,T}\|_{\mathcal{B}(\ell^\infty_\omega(X;\mathcal{H}))} \sup_{n\in\mathbb{N}} \|C_{T^d}(g_n - h_n)\|_{\ell^\infty_\omega(X;\mathcal{H})} < \infty.$$

Furthermore, since $\{g_n - h_n\}_{n=1}^\infty \sim_{T^d,\omega} \{0\}_{n=1}^\infty$, we have $[\{g_n - h_n\}_{n=1}^\infty]_{\sim_{T^d,\omega}} = [\{0\}_{n=1}^\infty]_{\sim_{T^d,\omega}} = 0 \in \mathcal{H}^\infty_\omega(T^d, T)$, which, again by Step 2, yields

$$\begin{aligned}
\lim_{n\to\infty} \|U_k^d(g_n - h_n)\|\omega(k) &\le \sup_{k\in X} \lim_{n\to\infty} \|U_k^d(g_n - h_n)\|\omega(k) \\
&= \sup_{k\in X} \lim_{n\to\infty} \left\| \sum_{l\in X} U_k^d T_l^* T_l^d (g_n - h_n) \right\| \omega(k) \\
&= \sup_{k\in X} \left\| \sum_{l\in X} U_k^d T_l^* \lim_{n\to\infty} T_l^d (g_n - h_n) \right\| \omega(k) \\
&= \|G_{U^d,T} C_{T^d} 0\|_{\ell^\infty_\omega(X;\mathcal{H})} = 0 \qquad (\text{for each } k \in X),
\end{aligned}$$

where the interchange of the limit and the sum is justified by the dominated convergence theorem as in the proof of Lemma 3.17. Thus

$$\lim_{n\to\infty} \|U_k^d(g_n - h_n)\| = 0 \qquad (\forall k \in X),$$

and consequently, $\{g_n\}_{n=1}^\infty \sim_{U^d,\omega} \{h_n\}_{n=1}^\infty$. We conclude that $[\{f_n\}_{n=1}^\infty]_{\sim_{T^d,\omega}} \subseteq [\{f_n\}_{n=1}^\infty]_{\sim_{U^d,\omega}}$ for all sequences $\{f_n\}_{n=1}^\infty$ in $\mathcal{H}$. The converse inclusion relation is shown analogously by switching the roles of (T^d, T) and (U^d, U).

Step 4. Let $f = [\{f_n\}_{n=1}^\infty]_{\sim_{T^d,\omega}}$ be arbitrary. Then $C_{T^d} f \in V(T^d, T)$ by Lemma 3.17. By Step 3, we have that $[\{f_n\}_{n=1}^\infty]_{\sim_{T^d,\omega}} = [\{f_n\}_{n=1}^\infty]_{\sim_{U^d,\omega}} \in \mathcal{H}^\infty_\omega(T^d, T)$, and by Steps 1 and 2, we have that $C_{U^d} f = (\lim_{n\to\infty} U_k^d f_n)_{k\in X} = G_{U^d,T} C_{T^d} f \in V(U^d, U)$. Thus

$$\|f\|_{\mathcal{H}^\infty_\omega(U^d,U)} = \|C_{U^d} f\|_{\ell^\infty_\omega(X;\mathcal{H})} \le \|G_{U^d,T}\|_{\mathcal{B}(\ell^\infty_\omega(X;\mathcal{H}))} \|C_{T^d} f\|_{\mathcal{H}^\infty_\omega(T^d,T)}.$$

This shows that $\mathcal{H}^\infty_\omega(T^d, T)$ is continuously embedded in $\mathcal{H}^\infty_\omega(U^d, U)$. Analogously, we obtain that $\mathcal{H}^\infty_\omega(U^d, U)$ is continuously embedded in $\mathcal{H}^\infty_\omega(T^d, T)$ after switching the roles of (T^d, T) and (U^d, U). This completes the proof. □

Via the above theorem, we can extend Theorem 3.21 as follows if T and T^d also are intrinsically localized.

Theorem 3.26. *Let $\mathcal{A}$ be a spectral algebra and ω be an $(\mathcal{A}, p_0)$-admissible weight. Let $T = (T_k)_{k\in X}$ be an intrinsically $\mathcal{A}$-localized g-frame and assume that $T^d =$*

$(T_k^d)_{k\in X}$ is an intrinsically $\mathcal{A}$-localized dual g-frame of T such that $T^d \sim_{\mathcal{A}} T$. Then, for each $p \in (p_0, \infty] \cup \{0, 1\}$,

$$\mathcal{H}^p_\omega(T^d, T) = \mathcal{H}^p_\omega(T, T^d)$$

with equivalent norms and

$$f = \sum_{k\in X} T_k^* T_k^d f = \sum_{k\in X} (T_k^d)^* T_k f \qquad (\forall f \in \mathcal{H}^p_\omega(T^d, T))$$

with unconditional convergence in $\mathcal{H}^p_\omega(T^d, T)$ for $p < \infty$ and unconditional convergence in the $\sigma(\mathcal{H}, \mathcal{H}^{00}(T^d))$-topology in the case $p = \infty$ respectively.

Proof. The special case $(U^d, U) = (T, T^d)$ of Theorem 3.25 combined with Theorem 3.21 yields the claim. $\square$

Recall from Theorem 3.8 that if T is an intrinsically $\mathcal{A}$-localized g-frame, then $\widetilde{T} \sim_{\mathcal{A}} T$ and $\widetilde{T} \sim_{\mathcal{A}} \widetilde{T}$, where $\widetilde{T}$ denotes the canonical dual g-frame of T. Therefore, given any $(\mathcal{A}, p_0)$-admissible weight, the spaces $\mathcal{H}^p_\omega(\widetilde{T}, T)$ ($p \in (p_0, \infty] \cup \{0, 1\}$) are intrinsically defined and only depend on T. In particular, we may deduce the following result in this case.

Corollary 3.27. *Let $\mathcal{A}$ be a spectral algebra and ω be an $(\mathcal{A}, p_0)$-admissible weight. If $T = (T_k)_{k\in X}$ is an intrinsically $\mathcal{A}$-localized g-frame, then for each $p \in (p_0, \infty] \cup \{0, 1\}$,*

$$\mathcal{H}^p_\omega(\widetilde{T}, T) = \mathcal{H}^p_\omega(T, \widetilde{T})$$

with equivalent norms and

$$f = \sum_{k\in X} T_k^* T_k S_T^{-1} f = \sum_{k\in X} S_T^{-1} T_k^* T_k f \qquad (\forall f \in \mathcal{H}^p_\omega(\widetilde{T}, T)), \tag{3.13}$$

with unconditional convergence in $\mathcal{H}^p_\omega(\widetilde{T}, T)$ for $p < \infty$ and unconditional convergence in the $\sigma(\mathcal{H}, \mathcal{H}^{00}(\widetilde{T}))$-topology in the case $p = \infty$ respectively.

Remark 3.28. *In the language of [32], Corollary 3.27 says that every intrinsically localized g-frame is a Banach g-frame for $\mathcal{H}^p_\omega(\widetilde{T}, T)$ for each $p \in [1, \infty] \cup \{0\}$.*

We conclude this section with the following result on duality relations. Note that the spaces $\mathcal{H}^q_{1/\omega}(T^d, T)$ that occur below are well-defined, since, by Lemma 3.12, the weight $1/\omega$ is $(\mathcal{A}, 1)$-admissible whenever ω is $(\mathcal{A}, 1)$-admissible.

Theorem 3.29. *Let $\mathcal{A}$ be a spectral algebra, ω be an $(\mathcal{A},1)$-admissible weight, $T=(T_k)_{k\in X}$ be an intrinsically $\mathcal{A}$-localized g-frame and $T^d=(T_k^d)_{k\in X}$ be a dual g-frame of T such that $T^d\sim_{\mathcal{A}} T$. Then*

$$\big(\mathcal{H}^p_\omega(T^d,T)\big)^*\simeq \mathcal{H}^q_{1/\omega}(T,T^d),$$

for all $p\in[1,\infty)$ with $\frac{1}{p}+\frac{1}{q}=1$ and for $(p,q)=(0,1)$. In particular,

$$\mathcal{H}^\infty_\omega(T^d,T)\simeq \big(\mathcal{H}^0_\omega(T^d,T)\big)^{**}.$$

Proof. We separate three cases for p.
Case (a): $1<p<\infty$.

Observe that

$$\beta:\mathcal{H}^p_\omega(T^d,T)\times\mathcal{H}^q_{1/\omega}(T,T^d)\longrightarrow\mathbb{C},\qquad \beta(f,g)=\sum_{k\in X}\langle T_k^d f,T_k g\rangle$$

defines a bounded sesquilinear form. Indeed, by Hölder's inequality,

$$\begin{aligned}|\beta(f,g)|&\le\sum_{k\in X}\|T_k^d f\|\|T_k g\|\\&\le\|C_{T^d}f\|_{\ell^p_\omega(X;\mathcal{H})}\|C_T g\|_{\ell^q_{1/\omega}(X;\mathcal{H})}\\&=\|f\|_{\mathcal{H}^p_\omega(T^d,T)}\|g\|_{\mathcal{H}^q_{1/\omega}(T,T^d)}.\end{aligned}$$

As a consequence, the map

$$\kappa:\mathcal{H}^q_{1/\omega}(T,T^d)\longrightarrow(\mathcal{H}^p_\omega(T^d,T))^*,\quad g\mapsto\kappa(g),$$

where the action of $\kappa(g)$ on any $f\in\mathcal{H}^p_\omega(T^d,T)$ is given by $\kappa(g)(f)=\beta(f,g)$, defines a linear and bounded operator. We will prove that κ is bijective.

In order to prove that κ is surjective, let $\rho\in(\mathcal{H}^p_\omega(T^d,T))^*$ be arbitrary. Then $\rho\circ D_T\in(\ell^p_\omega(X;\mathcal{H}))^*$ due to Proposition 3.20. Since $(\ell^p_\omega(X;\mathcal{H}))^*\cong\ell^q_{1/\omega}(X;\mathcal{H})$, there exists a unique $(g_k)_{k\in X}\in\ell^q_{1/\omega}(X;\mathcal{H})$ with $\rho(D_T(h_k)_{k\in X})=\sum_{k\in X}\langle h_k,g_k\rangle$ for all $(h_k)_{k\in X}\in\ell^p_\omega(X;\mathcal{H})$ [30, Chapter 1]. In particular, $g:=D_{T^d}(g_k)_{k\in X}\in\mathcal{H}^q_{1/\omega}(T,T^d)$ due to Proposition 3.20. We will show that

$$\rho(f)=\sum_{k\in X}\langle T_k^d f,T_k g\rangle\qquad(\forall f\in\mathcal{H}^p_\omega(T^d,T)),\tag{3.14}$$

which readily implies that κ is surjective. By density, it suffices to verify (3.14) for $f\in\mathcal{H}^{00}(T)$. To this end, fix some arbitrary $f\in\mathcal{H}^{00}(T)$ and let $\varepsilon>0$. By density of $\ell^1_{1/\omega}(X;\mathcal{H})$ in $\ell^q_{1/\omega}(X;\mathcal{H})$, there exists some $(h_k^\varepsilon)_{k\in X}\in\ell^1_{1/\omega}(X;\mathcal{H})$, such that $\|(g_k)_{k\in X} - (h_k^\varepsilon)_{k\in X}\|_{\ell^q_{1/\omega}(X;\mathcal{H})} < \varepsilon$. Then, again by Proposition 3.20, $h^\varepsilon := D_{T^d}(h_k^\varepsilon)_{k\in X} \in \mathcal{H}^1_{1/\omega}(T,T^d)$. Using Hölder's inequality, we may start estimating

$$\begin{aligned}
&\left|\rho(f) - \sum_{k\in X}\langle T_k^d f, T_k g\rangle\right| \\
&\leq \left|\rho(f) - \sum_{k\in X}\langle T_k^d f, T_k h^\varepsilon\rangle\right| + \left|\sum_{k\in X}\langle T_k^d f, T_k(g-h^\varepsilon)\rangle\right| \\
&\leq \left|\rho(f) - \sum_{k\in X}\langle T_k^d f, T_k h^\varepsilon\rangle\right| + \sum_{k\in X}\|T_k^d f\|\omega(k)\|T_k(g-h^\varepsilon)\|\omega(k)^{-1} \\
&\leq \left|\rho(f) - \sum_{k\in X}\langle T_k^d f, T_k h^\varepsilon\rangle\right| + \|f\|_{\mathcal{H}^p_\omega(T^d,T)}\|G_{T,T^d}((g_k)_{k\in X} - (h_k^\varepsilon)_{k\in X})\|_{\ell^q_{1/\omega}(X;\mathcal{H})} \\
&< \left|\rho(f) - \sum_{k\in X}\langle T_k^d f, T_k h^\varepsilon\rangle\right| + \varepsilon\|f\|_{\mathcal{H}^p_\omega(T^d,T)}\|G_{T,T^d}\|_{\mathcal{B}(\ell^q_{1/\omega}(X;\mathcal{H}))}. \qquad (3.15)
\end{aligned}$$

Next, let $F_1 \subseteq F_2 \subseteq \ldots$ be some nested sequence of finite subsets of X such that $\bigcup_{n=1}^\infty F_n = X$. Then we have for each $k \in X$ that

$$\begin{aligned}
\frac{1}{\omega(k)}\left\|T_k h^\varepsilon - \sum_{l\in X} T_k(T_l^d)^* h_l^\varepsilon\right\| &= \lim_{n\to\infty}\frac{1}{\omega(k)}\left\|T_k h^\varepsilon - T_k\sum_{l\in F_n}(T_l^d)^* h_l^\varepsilon\right\| \\
&\leq \lim_{n\to\infty}\left\|C_T\left(h^\varepsilon - \sum_{l\in F_n}(T_l^d)^* h_l^\varepsilon\right)\right\|_{\ell^1_{1/\omega}(X;\mathcal{H})} \\
&= \lim_{n\to\infty}\left\|\sum_{l\in X}(T_l^d)^* h_l^\varepsilon - \sum_{l\in F_n}(T_l^d)^* h_l^\varepsilon\right\|_{\mathcal{H}^1_{1/\omega}(T,T^d)} = 0
\end{aligned}$$

by continuity of D_{T^d} and unconditional convergence of $D_{T^d}(h_l^\varepsilon)_{l\in X}$ (see Proposition 3.20). Thus $T_k h^\varepsilon = \sum_{l\in X} T_k(T_l^d)^* h_l^\varepsilon$ in $\mathcal{H}$ for each $k \in X$. Consequently, we may rewrite the first term in (3.15) to

$$\left|\rho(f) - \sum_{k\in X}\langle T_k^d f, T_k h^\varepsilon\rangle\right| = \left|\rho(f) - \sum_{k\in X}\sum_{l\in X}\langle T_k^d f, T_k(T_l^d)^* h_l^\varepsilon\rangle\right|. \qquad (3.16)$$

Now, we interchange the order of summation in (3.16) via Fubini's theorem. This is possible, since $(h_l^\varepsilon)_{l\in X} \in \ell^1_{1/\omega}(X;\mathcal{H})$; indeed,

$$\begin{aligned}
\sum_{l\in X}\sum_{k\in X} |\langle T_k^d f, T_k(T_l^d)^* h_l^\varepsilon\rangle| &\leq \sum_{l\in X}\sum_{k\in X} \|T_k^d f\|\omega(l)\|T_k(T_l^d)^*\|\|h_l^\varepsilon\|\omega(l)^{-1} \\
&\leq \sum_{l\in X} \|h_l^\varepsilon\|\omega(l)^{-1} \sup_{m\in X} \sum_{k\in X} \|T_m^d T_k^*\|\|T_k^d f\|\omega(m) \\
&= \|(h_l^\varepsilon)_{l\in X}\|_{\ell^1_{1/\omega}(X;\mathcal{H})} \|G\cdot(\|T_k^d f\|)_{k\in X}\|_{\ell^\infty_\omega(X)} \\
&\leq \|(h_l^\varepsilon)_{l\in X}\|_{\ell^1_{1/\omega}(X;\mathcal{H})} \|G\|_{\mathcal{B}(\ell^\infty_\omega(X))} \|C_{T^d} f\|_{\ell^\infty_\omega(X)} < \infty,
\end{aligned}$$

where we used that the scalar matrix $G = \left[\|T_k^d T_l^*\|\right]_{k,l\in X}$ is contained in $\mathcal{B}(\ell^\infty_\omega(X))$ due to Lemma 2.4 and since ω is $(\mathcal{A},1)$-admissible. Continuing the calculation (3.16), we see again via Hölder's inequality that

$$\begin{aligned}
\left|\rho(f) - \sum_{k\in X}\langle T_k^d f, T_k h^\varepsilon\rangle\right| &= \left|\rho(f) - \sum_{l\in X}\left\langle \sum_{k\in X} T_k^* T_k^d f, (T_l^d)^* h_l^\varepsilon\right\rangle\right| \\
&= \left|\rho(f) - \sum_{l\in X}\langle f, (T_l^d)^* h_l^\varepsilon\rangle\right| \\
&= \left|\rho(f) - \sum_{l\in X}\langle T_l^d f, h_l^\varepsilon\rangle\right| \\
&\leq \left|\rho(f) - \sum_{l\in X}\langle T_l^d f, g_l\rangle\right| + \left|\sum_{l\in X}\langle T_l^d f, g_l - h_l^\varepsilon\rangle\right| \\
&< |\rho(f) - \rho(D_T C_{T^d} f)| + \varepsilon\|C_{T^d} f\|_{\mathcal{H}^p_\omega(T^d,T)} \\
&= \varepsilon\|f\|_{\mathcal{H}^p_\omega(T^d,T)},
\end{aligned}$$

where we also applied Theorem 3.26 twice. Combining the latter with (3.15) we finally arrive at

$$\left|\rho(f) - \sum_{k\in X}\langle T_k^d f, T_k g\rangle\right| < \varepsilon\|f\|_{\mathcal{H}^p_\omega(T^d,T)}\left(1 + \|G_{T,T^d}\|_{\mathcal{B}(\ell^q_{1/\omega}(X;\mathcal{H}))}\right).$$

Since ε can be arbitrarily small, (3.14) follows.

To show that κ is injective, it suffices to show the existence of a left-inverse of κ. Consider the map $R_{D_T} : (\mathcal{H}^p_\omega(T^d,T))^* \longrightarrow (\ell^p_\omega(X;\mathcal{H}))^*, \rho \mapsto \rho\circ D_T$. Then it is straightforward to verify that R_{D_T} is linear and bounded. Next, let $J : (\ell^p_\omega(X;\mathcal{H}))^* \longrightarrow \ell^q_{1/\omega}(X;\mathcal{H})$ denote the canonical isomorphism satisfying $\mu(\,\cdot\,) = \sum_{k\in X}\langle[\,\cdot\,]_k, [J(\mu)]_k\rangle$ for $\mu \in (\ell^p_\omega(X;\mathcal{H}))^*$ [30, Chapter 1]. Then Proposition 3.20 guarantees that $\Omega := D_{T^d}\circ J\circ R_{D_T} : (\mathcal{H}^p_\omega(T^d,T))^* \longrightarrow \mathcal{H}^q_{1/\omega}(T,T^d)$ is linear and bounded and we claim that Ω is a left-inverse of κ. By boundedness of Ω and κ it suffices to verify the identity $\Omega\kappa(g) = g$ on the dense subspace $\mathcal{H}^{00}(T^d)$ of $\mathcal{H}^q_{1/\omega}(T, T^d)$. To this end, fix some arbitrary $g \in \mathcal{H}^{00}(T^d)$. Then for any $f \in \mathcal{H}^{00}(T)$ we have due to g-frame reconstruction in $\mathcal{H}^{00}(T) \subseteq \mathcal{H}$ that

$$\begin{aligned}
\langle f, g\rangle &= \sum_{k\in X} \langle T^d_k f, T_k g\rangle \\
&= \kappa(g)(f) \\
&= \kappa(g)(D_T C_{T^d} f) \\
&= R_{D_T}\kappa(g)(C_{T^d} f) \\
&= \sum_{k\in X} \langle T^d_k f, [JR_{D_T}\kappa(g)]_k\rangle =: (*).
\end{aligned}$$

On the other hand we also have $\kappa(g) \in \mathcal{H}^*$, which implies $R_{D_T}\kappa(g) \in (\ell^2(X;\mathcal{H}))^*$ and thus $JR_{D_T}\kappa(g) \in \ell^2(X;\mathcal{H})$. Therefore, we may write

$$(*) = \langle C_{T^d} f, JR_{D_T}\kappa(g)\rangle_{\ell^2(X;\mathcal{H})} = \langle f, D_{T^d} JR_{D_T}\kappa(g)\rangle_{\mathcal{H}} = \langle f, \Omega\kappa(g)\rangle_{\mathcal{H}}.$$

In other words, we have shown that $\langle f, \Omega\kappa(g) - g\rangle = 0$ for all $f \in \mathcal{H}^{00}(T)$, which, by density of $\mathcal{H}^{00}(T)$ in $\mathcal{H}$, yields $g = \Omega\kappa g \in \mathcal{H}^{00}(T)$. Thus κ is injective.
Case (b): $p = 0$.

In the case $(p, q) = (0, 1)$ the proof is analogous to case (a); we only point out the following minor differences: The sesquilinear form β on $\mathcal{H}^0_\omega(T^d, T) \times \mathcal{H}^1_{1/\omega}(T, T^d)$ is bounded by $\|f\|_{\mathcal{H}^0_\omega(T^d,T)}\|g\|_{\mathcal{H}^1_{1/\omega}(T,T^d)}$; $\rho \circ D_T \in (\ell^0_\omega(X;\mathcal{H}))^*$ for every $\rho \in \mathcal{H}^0_\omega(T^d, T)$ (again due to Proposition 3.20); and we use that $J : (\ell^0_\omega(X;\mathcal{H}))^* \longrightarrow \ell^1_{1/\omega}(X;\mathcal{H})$ is an isomorphism induced by the sesquilinear form $\sum_{k\in X}\langle f_k, g_k\rangle$ on $\ell^0_\omega(X;\mathcal{H}) \times \ell^1_{1/\omega}(X;\mathcal{H})$ [47, Theorem 3.1]. Note, that the surjectivity of κ can be shown in a more straightforward manner, because in this case $(g_k)_{k\in X}$ is already contained in $\ell^1_{1/\omega}(X;\mathcal{H})$ and no density argument via h^ε is required to justify the change of order of summation.
Case (c): $p = 1$.

In the case $p = 1$ (i.e. $q = \infty$), the adaptations of case (a) to this setting are as follows: Recalling that $C_T u = (T_k u)_{k\in X} := (\lim_{n\to\infty} T_k u_n)_{k\in X} \in \ell^\infty_{1/\omega}(X;\mathcal{H})$ whenever $u = [\{u_n\}_{n=1}^\infty]_{\sim_{T,1/\omega}} \in \mathcal{H}^\infty_{1/\omega}(T, T^d)$, we again see via Hölder's inequality that $\beta : \mathcal{H}^1_\omega(T^d, T) \times \mathcal{H}^\infty_{1/\omega}(T, T^d) \longrightarrow \mathbb{C}$, $\beta(f, g) = \sum_{k\in X}\langle T^d_k f, T_k g\rangle$ defines a bounded sesquilinear form. So, κ can be defined again analogously as in case (a).

We show that κ is surjective. As in case (a), for given $\rho \in (\mathcal{H}^1_\omega(T^d, T))^*$, there exists a unique $(g_k)_{k\in X} \in \ell^\infty_{1/\omega}(X;\mathcal{H})$ with $\rho(D_T(h_k)_{k\in X}) = \sum_{k\in X}\langle h_k, g_k\rangle$ for all $(h_k)_{k\in X} \in \ell^1_\omega(X;\mathcal{H})$. We again set $g := D_{T^d}(g_k)_{k\in X} \in \mathcal{H}^\infty_{1/\omega}(T, T^d)$ and calculate via Theorem 3.26

$$
\begin{aligned}
\rho(f) &= \rho(D_T C_{T^d} f) \\
&= \sum_{k\in X} \langle T_k^d f, g_k\rangle \\
&= \sum_{k\in X} \lim_{n\to\infty} \left\langle T_k^d \sum_{l\in F_n} T_l^* T_l^d f, g_k \right\rangle \\
&= \sum_{k\in X} \lim_{n\to\infty} \sum_{l\in F_n} \left\langle T_l^d f, T_l (T_k^d)^* g_k \right\rangle \\
&= \sum_{k\in X} \sum_{l\in X} \left\langle T_l^d f, T_l (T_k^d)^* g_k \right\rangle \\
&= \sum_{l\in X} \sum_{k\in X} \left\langle T_l^d f, T_l (T_k^d)^* g_k \right\rangle \\
&= \sum_{l\in X} \lim_{n\to\infty} \left\langle T_l^d f, T_l \sum_{k\in F_n} (T_k^d)^* g_k. \right\rangle \\
&= \sum_{l\in X} \left\langle T_l^d f, T_l g \right\rangle .
\end{aligned}
$$

This time, the order of summation may be switched due to

$$
\begin{aligned}
\sum_{l\in X}\sum_{k\in X} |\langle T_l^d f, T_l (T_k^d)^* g_k\rangle| &\leq \sum_{l\in X}\sum_{k\in X} \|T_l^d f\| \omega(l) \|T_l (T_k^d)^*\| \|g_k\| \omega(l)^{-1} \\
&\leq \sum_{l\in X} \|T_l^d f\| \omega(l) \sup_{m\in X} \sum_{k\in X} \|T_m (T_k^d)^*\| \|g_k\| \omega(m)^{-1} \\
&= \|f\|_{\mathcal{H}^1_\omega(T^d,T)} \, \|G' \cdot (\|g_l\|)_{l\in X}\|_{\ell^\infty_{1/\omega}(X)} \\
&\leq \|f\|_{\mathcal{H}^1_\omega(T^d,T)} \, \|G'\|_{\mathcal{B}(\ell^\infty_{1/\omega}(X))} \, \|(g_l)_{l\in X}\|_{\ell^\infty_{1/\omega}(X;\mathcal{H})} < \infty,
\end{aligned}
$$

where we analogously used that $G' = \left[\|T_k (T_l^d)^*\|\right]_{k,l\in X} \in \mathcal{B}(\ell^\infty_{1/\omega}(X))$.

Finally, we show the injectivity of κ by proving the existence of a left-inverse Ω of κ. As in case (a), we consider the operator $R_{D_T} \in \mathcal{B}((\mathcal{H}^1_\omega(T^d,T))^*, (\ell^1_\omega(X;\mathcal{H}))^*)$ and the canonical isomorphism $J : (\ell^1_\omega(X;\mathcal{H}))^* \longrightarrow \ell^\infty_{1/\omega}(X;\mathcal{H})$. For arbitrary $f \in \mathcal{H}^{00}(T) \subseteq \mathcal{H}^1_\omega(T^d,T)$ and $g = [\{g_n\}_{n=1}^\infty]_{\sim_{T,1/\omega}} \in \mathcal{H}^\infty_{1/\omega}(T,T^d)$ we see that

$$
\kappa(g)(f) = \sum_{k\in X} \lim_{n\to\infty} \langle T_k^d f, T_k g_n\rangle = \lim_{n\to\infty} \sum_{k\in X} \langle T_k^d f, T_k g_n\rangle = \lim_{n\to\infty} \langle f, g_n\rangle,
$$

where we applied g-frame reconstruction in $\mathcal{H}$ in the third step and the dominated convergence theorem in the second step upon noticing that $C_{T^d} f \in \ell^1_\omega(X;\mathcal{H})$ and

$\|T_k g_n\|\omega(k)^{-1} \leq C$ due to (3.7). At the same time we have as above

$$\begin{aligned}
\kappa(g)(f) &= \kappa(g)(D_T C_{T^d} f)\\
&= R_{D_T}\kappa(g)(C_{T^d} f)\\
&= \sum_{k\in X} \langle T_k^d f, [JR_{D_T}\kappa(g)]_k\rangle\\
&= \lim_{n\to\infty} \left\langle f, \sum_{k\in F_n} (T_k^d)^*[JR_{D_T}\kappa(g)]_k \right\rangle
\end{aligned}$$

with $JR_{D_T}\kappa(g) \in \ell^\infty_{1/\omega}(X;\mathcal{H})$. The composition $\Omega := D_{T^d} \circ J \circ R_{D_T}$ is an element in $\mathcal{B}(\mathcal{H}^1_\omega(T^d,T))^*, \mathcal{H}^\infty_{1/\omega}(T,T^d))$ and we have

$$\begin{aligned}
\Omega\kappa(g) &= \left[\left\{ \sum_{k\in F_n} (T_k^d)^*[JR_{D_T}\kappa(g)]_k \right\}_{n=1}^\infty \right]_{\sim_{T,1/\omega}}\\
&= [\{(\Omega\kappa(g))_n\}_{n=1}^\infty]_{\sim_{T,1/\omega}} \in \mathcal{H}^\infty_{1/\omega}(T,T^d).
\end{aligned}$$

Therefore, we may conclude from the above that

$$\lim_{n\to\infty} \langle f, g_n\rangle = \kappa(g)(f) = \lim_{n\to\infty} \langle f, (\Omega\kappa(g))_n\rangle \qquad (\forall f \in \mathcal{H}^{00}(T)).$$

Specifying to elements of the form $f = T_k^* h \in \mathcal{H}^{00}(T)$ $(k \in X, h \in \mathcal{H})$ implies

$$\lim_{n\to\infty} \langle h, T_k\big(g_n - (\Omega\kappa(g))_n\big)\rangle = 0 \qquad (\forall k \in X, \forall h \in \mathcal{H}). \tag{3.17}$$

However, since $g - \Omega\kappa(g) \in \mathcal{H}^\infty_{1/\omega}(T,T^d)$, we know (see Definition 3.16) that for each $k \in X$ the limit $\lim_{n\to\infty} T_k\big(g_n - (\Omega\kappa(g))_n\big) = T_k(g - \Omega\kappa(g)) \in \mathcal{H}$ exists and has to be zero as a consequence of (3.17). In particular, we have shown that $\lim_{n\to\infty} \|T_k\big(g_n - (\Omega\kappa(g))_n\big)\| = 0$ for each $k \in X$. Since $g - \Omega\kappa(g) \in \mathcal{H}^\infty_{1/\omega}(T,T^d)$ we also have $\sup_{n\in\mathbb{N}} \|C_T\big(g_n - (\Omega\kappa(g))_n\big)\|_{\ell^\infty_{1/\omega}(X;\mathcal{H})} < \infty$. Thus $\{g_n\}_{n=1}^\infty \sim_{T,1/\omega} \{(\Omega\kappa(g))_n\}_{n=1}^\infty$, i.e. $g = \Omega\kappa(g)$ in $\mathcal{H}^\infty_{1/\omega}(T,T^d)$. This completes the proof. □

4 Application to Gabor g-frames

We now turn to the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d)$ and consider Gabor g-frames, which were introduced in [44].

We recall some basic notions from time-frequency analysis [26]. The fundamental operators in time-frequency analysis are the *translation operator* T_x and the *modulation operator* M_ω given by

$$T_x f(t) = f(t-x) \quad \text{and} \quad M_\omega f(t) = e^{2\pi i\omega\cdot t} f(t) \qquad (x,\omega \in \mathbb{R}^d).$$

Their composition $\pi(z) = M_\omega T_x$, where $z = (x, w) \in \mathbb{R}^{2d}$, is called a *time-frequency shift* by z. Each of the operators T_x, M_ω and $\pi(z)$ is unitary on $L^2(\mathbb{R}^d)$. If g is some window function, then the *short-time Fourier transform* (STFT) of a function f with respect to the window g is given by

$$V_g f(x, w) = \int_{\mathbb{R}^d} f(t)\overline{g(t-x)}e^{-2\pi i\omega\cdot t}\, dt \qquad (x, w \in \mathbb{R}^d).$$

Setting $z = (x, w) \in \mathbb{R}^{2d}$, we may rewrite the above to

$$V_g f(z) = \langle f, \pi(z)g\rangle \qquad (z \in \mathbb{R}^{2d})$$

whenever the bracket is well-defined. An important class of function spaces in time-frequency analysis are the *modulation spaces* defined as follows. Let $g_0(t) = 2^{\frac{d}{4}}e^{-\pi t^2}$ denote the normalized Gaussian in $L^2(\mathbb{R}^d)$ (where $t^2 = t\cdot t$) and let $\nu_s(x) = (1 + |x|)^s$ be the standard polynomial weight. Then $M^1_{\nu_s}(\mathbb{R}^d)$ is the Banach space of all $f \in L^2(\mathbb{R}^d)$ such that

$$\|\varphi\|_{M^1_{\nu_s}(\mathbb{R}^d)} = \int_{\mathbb{R}^{2d}} |V_{g_0} f(z)|\nu_s(z)\, dz < \infty. \tag{4.1}$$

The space $S_0 = M^1_{\nu_0}$ is known as *Feichtinger's algebra* [21]. For a ν_s-moderate weight m and $p \in [1, \infty]$ the *modulation space* $M^p_m(\mathbb{R}^d)$ is the space of all elements f in the anti-linear dual space $(M^1_{\nu_s}(\mathbb{R}^d))'$ for which

$$\|\varphi\|_{M^p_m(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^{2d}} |V_{g_0} f(z)|^p m(z)^p\, dz\right)^{\frac{1}{p}} < \infty. \tag{4.2}$$

We collect some important properties of modulation spaces we will need.

Proposition 4.1. *[26] Let m be a ν_s-moderate weight and $p \in [1, \infty]$.*

(a) $M^p_m(\mathbb{R}^d)$ is a Banach space with respect to the norm defined in (4.2).

(b) If $1 \le p_1 \le p_2 \le \infty$ and $m_2(z) \le Cm_1(z)$ for some $C > 0$, then $M^{p_1}_{m_1}(\mathbb{R}^d)$ is continuously and densely embedded into $M^{p_2}_{m_2}(\mathbb{R}^d)$.

(c) $M^2_1(\mathbb{R}^d) = L^2(\mathbb{R}^d)$ with equivalent norms.

Returning to the g-frame setting, we are interested in families consisting of shifts of some *window operator* $T \in \mathcal{B}(L^2(\mathbb{R}^d))$ in the time-frequency plane. More precisely, the *translation of the operator* T by $z \in \mathbb{R}^{2d}$ [33] is given by

$$\alpha_z(T) := (\pi(z) \otimes \pi(z))(T) = \pi(z)T\pi(z)^*.$$

For a countable set $X \subseteq \mathbb{R}^{2d}$ and a window operator $T \in \mathcal{B}(L^2(\mathbb{R}^d))$ we call the family

$$\mathcal{G} = \mathcal{G}(T, X) = (\alpha_k(T))_{k\in X}$$

a *g-Gabor system*. In the terminology of [44], such a g-Gabor system $\mathcal{G}(T, X)$ is called a *Gabor g-frame*, if it is a g-frame for $L^2(\mathbb{R}^d)$, i.e. if there exist positive constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{k\in X} \|\alpha_k(T)f\|^2 \leq B\|f\|^2 \qquad (\forall f \in L^2(\mathbb{R}^d)). \tag{4.3}$$

In [44] *regular* Gabor g-frames (which correspond to $X = \Lambda \subset \mathbb{R}^{2d}$ being a full-rank lattice) were considered. In particular, via Fourier methods for periodic operators, which ultimately rely on the group structure of the lattice k, Skrettingland proved in [44] several results for this class of g-frames, which typically are concluded from suitable localization properties of a given frame in some abstract Hilbert space $\mathcal{H}$ [24]. We will extend some of those results to the case of *irregular* Gabor g-frames (i.e. Gabor g-frames indexed by some arbitrary relatively separated set $X \subset \mathbb{R}^{2d}$) by showing that g-Gabor systems with respect to some suitably nice window operator T are polynomially localized (in the sense of Example 3.4 (1.)).

To this end, we consider the class $\mathcal{B}_{\nu_s\otimes\nu_s}$ of integral operators acting boundedly on $L^2(\mathbb{R}^d)$, whose integral kernel belongs to $M^1_{\nu_s\otimes\nu_s}(\mathbb{R}^{2d})$, where

$$\begin{aligned} M^1_{\nu_s\otimes\nu_s}(\mathbb{R}^{2d}) &\cong M^1_{\nu_s}(\mathbb{R}^d)\hat{\otimes}M^1_{\nu_s}(\mathbb{R}^d) \\ &= \left\{ \sum_{n\in\mathbb{N}} \varphi_n \otimes \psi_n : \sum_{n\in\mathbb{N}} \|\varphi_n\|_{M^1_{\nu_s}(\mathbb{R}^d)} \|\psi_n\|_{M^1_{\nu_s}(\mathbb{R}^d)} < \infty \right\} \end{aligned} \tag{4.4}$$

is the projective tensor product of the Banach space $M^1_{\nu_s}(\mathbb{R}^d)$ with itself. Here $g\otimes f$ denotes the rank-one operator given by $(g\otimes f)(h) = \langle h, f\rangle g$, where $f, g, h \in L^2(\mathbb{R}^d)$. The space $\mathcal{B}_{\nu_s\otimes\nu_s}$ was studied in [44] in detail. We collect the following properties, which we will need.

Proposition 4.2. *[44] Let $T \in \mathcal{B}_{\nu_s\otimes\nu_s}$. Then the following hold:*

(a) There exist sequences $\{\varphi_n\}_{n\in\mathbb{N}}$ and $\{\psi_n\}_{n\in\mathbb{N}}$ in $M^1_{\nu_s}(\mathbb{R}^d)$ with

$$\sum_{n\in\mathbb{N}} \|\varphi_n\|_{M^1_{\nu_s}(\mathbb{R}^d)} \|\psi_n\|_{M^1_{\nu_s}(\mathbb{R}^d)} < \infty,$$

such that T can be written as a sum of rank-one operators

$$T = \sum_{n\in\mathbb{N}} \varphi_n \otimes \psi_n. \tag{4.5}$$

The decomposition (4.5) converges absolutely in $\mathcal{B}_{\nu_s\otimes\nu_s}$, in the space of trace class operators on $L^2(\mathbb{R}^d)$, and in $\mathcal{B}(L^2(\mathbb{R}^d))$.

(b) Let $T^* \in \mathcal{B}(L^2(\mathbb{R}^d))$ denote the adjoint operator of T. Then $T^* \in \mathcal{B}_{\nu_s\otimes\nu_s}$. In particular, if $T = \sum_{n\in\mathbb{N}} \varphi_n \otimes \psi_n$, then $T^* = \sum_{n\in\mathbb{N}} \psi_n \otimes \varphi_n$.

We are now ready to prove the main theorem of this section, which provides a concrete example of an intrinsically localized operator-valued sequence.

Theorem 4.3. *Let $X \subset \mathbb{R}^{2d}$ be relatively separated, $s > 2d$, and $T \in \mathcal{B}_{\nu_s\otimes\nu_s}$. Then the g-Gabor system $\mathcal{G} = \mathcal{G}(T, X)$ is intrinsically $\mathcal{J}_s$-localized. In particular, $\mathcal{G}(T, X)$ is a g-Bessel sequence for $L^2(\mathbb{R}^d)$.*

Proof. If $T \in \mathcal{B}_{\nu_s\otimes\nu_s}$, then, by Proposition 4.2, also $T^* \in \mathcal{B}_{\nu_s\otimes\nu_s}$ and there exist sequences $\{\varphi_n\}_{n\in\mathbb{N}}, \{\psi_n\}_{n\in\mathbb{N}}$ in $M^1_{\nu_s}(\mathbb{R}^d)$ with $\sum_{n\in\mathbb{N}} \|\varphi_n\|_{M^1_{\nu_s}} \|\psi_n\|_{M^1_{\nu_s}} < \infty$ such that $T = \sum_{n\in\mathbb{N}} \varphi_n \otimes \psi_n$ and $T^* = \sum_{n\in\mathbb{N}} \psi_n \otimes \varphi_n$. We first show that there exists some constant $C > 0$ such that

$$\|[G_{\mathcal{G}}]_{k,l}\| \leq C(1 + |k - l|)^{-s} \qquad (\forall k, l \in X).$$

Observe that $[G_{\mathcal{G}}]_{k,l} = \alpha_k(T)\alpha_l(T)^* = \pi(k)T\pi(k)^*\pi(l)T^*\pi(l)^*$. Then for arbitrary $f \in L^2(\mathbb{R}^d)$ we have

$$\begin{aligned}
[G_{\mathcal{G}}]_{k,l} f &= \pi(k)\left(\sum_{m\in\mathbb{N}} \varphi_m \otimes \psi_m\right)\pi(k)^*\pi(l)\left(\sum_{n\in\mathbb{N}} \psi_n \otimes \varphi_n\right)\pi(l)^* f \\
&= \left(\sum_{m\in\mathbb{N}} (\pi(k)\varphi_m) \otimes (\pi(k)\psi_m)\right)\left(\sum_{n\in\mathbb{N}} (\pi(l)\psi_n) \otimes (\pi(l)\varphi_n)\right) f \\
&= \left(\sum_{m\in\mathbb{N}} (\pi(k)\varphi_m) \otimes (\pi(k)\psi_m)\right)\sum_{n\in\mathbb{N}} \langle f, \pi(l)\varphi_n\rangle \pi(l)\psi_n \\
&= \sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{N}} \langle f, \pi(l)\varphi_n\rangle \langle \pi(l)\psi_n, \pi(k)\psi_m\rangle \pi(k)\varphi_m.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|[G_{\mathcal{G}}]_{k,l} f\|_{L^2} &\leq \sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{N}} \|f\|_{L^2}\|\varphi_n\|_{L^2}\|\varphi_m\|_{L^2}|V_{\psi_m}(\pi(l)\psi_n)(k)|, \\
&\leq C_0^2 \sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{N}} \|f\|_{L^2}\|\varphi_n\|_{M^1_{\nu_s}}\|\varphi_m\|_{M^1_{\nu_s}}|V_{\psi_m}\psi_n(k - l)|,
\end{aligned}$$

where, in the latter estimate, we used the covariance property of the STFT [26] and the continuous embedding $M^1_{\nu_s}(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d)$ (see Proposition 4.1). Since $\|g_0\|_{L^2(\mathbb{R}^d)} = 1$, the STFT satisfies the pointwise estimate

$$|V_{\psi_m}\psi_n| \leq \frac{1}{\|g_0\|^2_{L^2(\mathbb{R}^d)}}|V_{g_0}\psi_n| * |V_{\psi_m} g_0| = |V_{g_0}\psi_n| * |V_{\psi_m} g_0|$$

(see [26, Theorem 11.3.7]). Hence, we may apply Young's inequality for weighted L^p-spaces [26, Proposition 11.1.3] and obtain that

$$|V_{\psi_m}\psi_n(k-l)|\nu_s(k-l) \leq \|V_{\psi_m}\psi_n\|_{L^\infty_{\nu_s}} \leq \|V_{g_0}\psi_n\|_{L^\infty_{\nu_s}} \|V_{\psi_m}g_0\|_{L^1_{\nu_s}}.$$

Since ν_s is a symmetric weight it holds $\|V_{\psi_m}g_0\|_{L^1_{\nu_s}} = \|V_{g_0}\psi_m\|_{L^1_{\nu_s}}$. Together with the continuous embedding $M^1_{\nu_s}(\mathbb{R}^d) \hookrightarrow M^\infty_{\nu_s}(\mathbb{R}^d)$ this yields

$$\begin{aligned}
|V_{\psi_m}\psi_n(k-l)|\nu_s(k-l) &\leq \|V_{g_0}\psi_n\|_{L^\infty_{\nu_s}} \|V_{g_0}\psi_m\|_{L^1_{\nu_s}} \\
&= \|\psi_n\|_{M^\infty_{\nu_s}} \|\psi_m\|_{M^1_{\nu_s}} \\
&\leq C_1\|\psi_n\|_{M^1_{\nu_s}} \|\psi_m\|_{M^1_{\nu_s}}.
\end{aligned}$$

Altogether, we see that

$$\begin{aligned}
\|[G_{\mathcal{G}}]_{k,l}\| &\leq C_0^2 \sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{N}} \|\varphi_n\|_{M^1_{\nu_s}}\|\varphi_m\|_{M^1_{\nu_s}}|V_{\psi_m}\psi_n(k-l)| \\
&\leq \frac{C_0^2C_1}{\nu_s(k-l)} \sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{N}} \|\varphi_n\|_{M^1_{\nu_s}}\|\varphi_m\|_{M^1_{\nu_s}}\|\psi_n\|_{M^1_{\nu_s}}\|\psi_m\|_{M^1_{\nu_s}} \\
&= \frac{C_0^2C_1}{\nu_s(k-l)} \left(\sum_{m\in\mathbb{N}} \|\varphi_m\|_{M^1_{\nu_s}}\|\psi_m\|_{M^1_{\nu_s}}\right)\left(\sum_{n\in\mathbb{N}} \|\varphi_n\|_{M^1_{\nu_s}}\|\psi_n\|_{M^1_{\nu_s}}\right) \\
&= C(1+|k-l|)^{-s}.
\end{aligned}$$

Since $k,l \in X$ were arbitrary, the latter implies that $\mathcal{G}(T,X)$ is intrinsically $\mathcal{J}_s$-localized. By Proposition 3.6, $\mathcal{G}(T,X)$ is therefore a g-Bessel sequence. □

In particular, if a Gabor g-system as above is a g-frame for $L^2(\mathbb{R}^d)$, we may apply our machinery established so far and obtain further results. We note that sufficiency conditions for such Gabor g-systems over a lattice being a frame were given in [44].

Theorem 4.4. *Let $X \subset \mathbb{R}^{2d}$ be relatively separated, $s > 2d + r$ for some $r \geq 0$, $T \in \mathcal{B}_{\nu_s\otimes\nu_s}$, and m be a ν_r-moderate weight. If $\mathcal{G} = \mathcal{G}(T,X)$ constitutes a Gabor g-frame for $L^2(\mathbb{R}^d)$, then $\mathcal{G}$ and its canonical dual $\widetilde{\mathcal{G}}$ are intrinsically and mutually $\mathcal{J}_s$-localized. In other words, there exist constants $C_1, C_2, C_3 > 0$ such that*

$$\begin{aligned}
\|T\pi(k)^*\pi(l)T^*\| &\leq C_1(1+|k-l|)^{-s} \qquad (\forall k,l \in X) \\
\|T\pi(k)^*S_{\mathcal{G}}^{-1}\pi(l)T^*\| &\leq C_2(1+|k-l|)^{-s} \qquad (\forall k,l \in X) \\
\|T\pi(k)^*S_{\mathcal{G}}^{-2}\pi(l)T^*\| &\leq C_3(1+|k-l|)^{-s} \qquad (\forall k,l \in X).
\end{aligned} \tag{4.6}$$

Furthermore, for each $p \in (\frac{2d}{s-r}, \infty] \cup \{0\}$, the space $\mathcal{H}^p_m(\widetilde{\mathcal{G}}, \mathcal{G})$ is a well-defined (quasi-) Banach space and it holds

$$f = \sum_{k\in X} \pi(k)T^*T\pi(k)^*S_{\mathcal{G}}^{-1}f = \sum_{k\in X} S_{\mathcal{G}}^{-1}\pi(k)T^*T\pi(k)^*f \qquad (\forall f \in \mathcal{H}^p_m(\widetilde{\mathcal{G}}, \mathcal{G}))$$

with unconditional convergence in $\mathcal{H}^p_m(\widetilde{\mathcal{G}}, \mathcal{G})$ for $p \in (\frac{2d}{s-r}, \infty) \cup \{0\}$ and unconditional convergence in the $\sigma(\mathcal{H}, \mathcal{H}^{00}(\widetilde{\mathcal{G}}))$-topology in the case $p = \infty$ respectively.

Proof. Since $s > 2d$, $\mathcal{J}_s$ is a spectral algebra by Example 3.4 (1.). Furthermore, since $T \in \mathcal{B}_{\nu_s \otimes \nu_s}$, Theorem 4.3 guarantees that $\mathcal{G}$ is intrinsically $\mathcal{J}_s$-localized, which is equivalent to the first inequality in (4.6). If $\widetilde{\mathcal{G}}$ denotes the canonical dual g-frame of $\mathcal{G}$, then Theorem 3.8 implies that $\widetilde{\mathcal{G}} \sim_{\mathcal{J}_s} \mathcal{G}$ and $\widetilde{\mathcal{G}} \sim_{\mathcal{J}_s} \widetilde{\mathcal{G}}$, which is equivalent to the second and third inequality in (4.6) respectively. Moreover, the assumptions on m and Example 3.11 (1.) imply that m is an $(\mathcal{J}_s, \frac{2d}{s-r})$-admissible weight. Hence, an application of Corollary 3.27 yields the remaining part of the theorem. $\square$

Remark 4.5. *We expect that the modulation spaces $M^p_{\nu_s \otimes \nu_s}$ considered in [44] coincide (up to norm-equivalence) with the co-orbit spaces $\mathcal{H}^p_{\nu_s}(\widetilde{\mathcal{G}}, \mathcal{G})$ appearing in the latter theorem. Similarly, we expect that the co-orbit spaces of operators discussed in [18] can be identified with co-orbit spaces associated with certain localized g-frames. We leave a more detailed investigation on this matter to future works.*

Acknowledgments

The authors thank Daniel Freeman, Karlheinz Gröchenig, Jakob Holböck, Ilya Krishtal, Franz Luef and Henry McNulty for their valuable suggestions and related discussions.

This research was funded in whole or in part by the Austrian Science Fund (FWF) 10.55776/P34624. For open access purposes, the author has applied a CC BY public copyright license to any author accepted manuscript version arising from this submission.

References

[1] P. Balazs and K. Gröchenig. A guide to localized frames and applications to Galerkin-like representations of operators. In I. Pesenson, H. Mhaskar, A. Mayeli, Q. T. L. Gia, and D.-X. Zhou, editors, *Frames and Other Bases in Abstract and Function Spaces*, Applied and Numerical Harmonic Analysis series (ANHA). Birkhauser/Springer, 2017.

[2] P. Balazs, N. Holighaus, T. Necciari, and D. Stoeva. Frame Theory for Signal Processing in Psychoacoustics. In R. Balan, J. J. Benedetto, W. Czaja, and K. Okoudjou, editors, *Excursions in Harmonic Analysis Vol. 5,*, pages 225–268. Springer, International Publishing, 2017.

[3] J. Benedetto, C. Heil, and D. Walnut. Differentiation And The Balian-Low Theorem. *Journal of Fourier Analysis and Applications*, 1:355–402, 01 1995.

[4] J. Bergh and J. Löfström. *Interpolation Spaces: An Introduction*. Springer-Verlag, Berlin Heidelberg New York, 1976.

[5] F. J. Beutler and W. L. Root. The operator pseudoinverse in control and systems identification. In M. Z. Nashed, editor, *Generalized Inverses and Applications*, pages 397–494. Academic Press, 1976.

[6] B. Bodmann. Optimal linear transmission by loss-insensitive packet encoding. *Appl. Comp. Har. Anal.*, 22:274–285, 2007.

[7] B. Bodmann, D. Kribs, and V. Paulsen. Decoherence-insensitive quantum communications by optimal c^*-encoding. *IEEE Transactions on Information Theory*, 53(12):4738–4749, 2007.

[8] P. Casazza, O. Christensen, A. Lindner, and R. Vershynin. Frames and the Feichtinger conjecture. *Proceedings of the American Mathematical Society*, 133, 04 2005.

[9] P. Casazza and G. Kutyniok. Robustness of fusion frames under erasures of subspaces and local frame vectors. *Contemporary Math*, 464:149–160, 2008.

[10] P. Casazza, G. Kutyniok, S. Li, and C. Rozell. Modeling sensor networks with fusion frames. *Proceed. of SPIE - Wavelets XII*, 6701:11910–11913, 2007.

[11] P. G. Casazza and G. Kutyniok. Frames of subspaces. *Cont. Math.*, 2004.

[12] P. G. Casazza, G. Kutyniok, and S. Li. Fusion frames and distributed processing. *Appl. Comput. Harmon. Anal.*, 254(1):114–132, 2008.

[13] O. Christensen. Frames and pseudo-inverses. *J. Math. Anal. Appl*, 195(2):401–414, 1995.

[14] O. Christensen. *An Introduction to Frames and Riesz Bases*. Birkhäuser, 2016.

[15] J. B. Conway. *A Course in Functional Analysis*. Graduate Texts in Mathematics. Springer, New York, 2. edition, 1990.

[16] M. S. Craig and R. L. Genter. Geophone array formation and semblance evaluation. *Society of Exploration Geophysicists*, 71, 01 2006.

[17] I. Daubechies, A. Grossmann, and Y. Meyer. Painless non-orthogonal expansions. *J. Math. Phys.*, 27:1271–1283, 1986.

[18] M. Doerfler, F. Luef, H. McNulty, and E. Skrettingland. Time-frequency analysis and coorbit spaces of operators. *Journal of Mathematical Analysis and Applications*, 534:128058, 12 2023.

[19] R. J. Duffin and A. C. Schaeffer. A class of nonharmonic Fourier series. *Trans. Amer. Math. Soc.*, 72:341–366, 1952.

[20] M. Dörfler, H. Feichtinger, and K. Gröchenig. Time-frequency partitions for the Gelfand triple (S_0, L^2, S_0'). *Mathematica Scandinavica*, 98:81–96, 01 2006.

[21] H. G. Feichtinger. On a new segal algebra. *Monatshefte für Mathematik*, 92:269–290, 1981.

[22] M. Fornasier. Decompositions of Hilbert spaces: Local construction of global frames. *Constructive Theory of Functions*, pages 275–281, 2003.

[23] M. Fornasier. Quasi-orthogonal decompositions of structured frames. *Journal of Mathematical Analysis and Applications*, 289(1):180–199, 2004.

[24] M. Fornasier and K. Gröchenig. Intrinsic localization of frames. *Constr. Approx.*, 22(3):395–415, 2005.

[25] I. Gelfand, D. Raikov, and G. Shilov. *Commutative normed rings, by I.Gelfand D.Raikov and G.Shilov*. Chelsea Publishing Company, Bronx, New York, 1964.

[26] K. Gröchenig. *Foundations of Time-Frequency Analysis*. Birkhäuser, Boston, 2001.

[27] K. Gröchenig. *Weight functions in time-frequency analysis*, chapter 15, pages 343–366. American Mathematical Society, 11 2007.

[28] E. Hernández and G. Weiss. *A First Course on Wavelets*. CRC Press, Boca Raton – New York – London – Tokyo, 1996.

[29] J. Holböck. Localized frames and applications. Master's thesis, University of Vienna, 2022.

[30] T. Hytönen, J. van Neerven, M. Veraar, and L. Weis. *Analysis in Banach Spaces, Volume I: Martingales and Littlewood-Paley Theory*. Springer, 12 2016.

[31] S. Iyengar and R. B. Eds. *Distributed Sensor Networks*. Chapman & Hall/CRC CRC Press, Baton Rouge, 2005.

[32] A. Khosravi. Fusion frames and g-frames in banach spaces. *Proc. Indian Acad. Sci. (Math. Sci*, 121:155–164, 04 2011.

[33] W. Kozek. On the generalized weyl correspondence and its application to time-frequency analysis of linear time-varying systems. In *Proceedings of the IEEE-SP International Symposium on Time-Frequency and Time-Scale Analysis*, pages 167–170, 1992.

[34] I. A. Krishtal. Wiener's lemma and memory localization. *Journal of Fourier Analysis and Applications*, 17(4):674–690, August 2011.

[35] G. Kutyniok, V. Paternostro, and F. Philipp. The effect of perturbations of frame sequences and fusion frames on their duals. *Oper. Matrices*, 11:301–336, 2017.

[36] L. Köhldorfer and P. Balazs. On the inverse-closedness of operator-valued matrices with polynomial off-diagonal decay. In *14th International Conference on Sampling Theory and Applications (SampTA 2025), Vienna from July 24 - August 1*, 2025.

[37] L. Köhldorfer and P. Balazs. Wiener pairs of Banach algebras of operator-valued matrices. *Journal of Mathematical Analysis and Applications*, 549(2):129525, 2025.

[38] L. Köhldorfer, P. Balazs, P. Casazza, S. Heineken, C. Hollomey, P. Morillas, and M. Shamsabadhi. *A survey of Fusion Frames in Hilbert Spaces*, chapter 21. Springer Nature Switzerland, 2023.

[39] L. Li and A. Liu. The Banach spaces associated with g-frames. *Banach Journal of Mathematical Analysis*, 16(4), 2022.

[40] I. J. Maddox. *Infinite matrices of operators*. Lecture Notes in Mathematics. Springer, Berlin, 1980.

[41] A. Marcus, D. Spielman, and N. Srivastava. Interlacing Families II: Mixed Characteristic Polynomials and the Kadison-Singer Problem. *Annals of Mathematics*, 182:327–350, 2015.

[42] H. Rauhut, K. Schnass, and P. Vandergheynst. Compressed sensing and redundant dictionaries. *IEEE Trans. Inform. Theory*, 54(5):2210–2219, 2008.

[43] C. Rozell and D. Johnson. Analyzing the robustness of redundant population codes in sensory and feature extraction systems. *Neurocomputing*, 69(10–12):1215–1218, 2006.

[44] E. Skrettingland. On Gabor g-frames and Fourier series of operators. *Studia Mathematica*, 259:25–78, 01 2021.

[45] W. Sun. G-frames and g-Riesz bases. *Journal of Mathematical Analysis and Applications*, 322(1):437 – 452, 2006.

[46] W. Sun. Stability of g-frames. *Journal of Mathematical Analysis and Applications*, 326(2):858–868, 2007.

[47] Y. Yadav and J. Srivastava. Duals of vector valued function spaces $c_0(X, U, M)$, $c(X, U, M)$ and $\ell_\infty(X, U, M)$ defined by Orlicz function. *Acta Mathematica Universitatis Comenianae*, 83(1):29–38, 2014.

9. Paper 3

Bibliographic information:
P. Balazs, L. Köhldorfer and M. Speckbacher. Localized frames without inequalities. *Submitted.* ArXiV: https://arxiv.org/pdf/2506.02862

Contribution:
Peter Balazs, Lukas Köhldorfer and Michael Speckbacher contributed equally to the content of the paper.

LOCALIZED FRAMES WITHOUT INEQUALITIES

P. BALAZS, L. KÖHLDORFER, AND M. SPECKBACHER

Abstract. We consider countable families of vectors in a separable Hilbert space, which are mutually localized with respect to a fixed localized Riesz basis. We prove the equivalence of the frame property and nine conditions that do not involve any inequalities. This is done by studying the properties of their frame-related operators on the co-orbit spaces generated by the reference Riesz basis. We apply our main result to the setting of shift-invariant spaces and obtain new conditions for stable sets of sampling.

1. Introduction

A frame is a natural generalization of the notion of orthonormal bases for a Hilbert space that allows for redundant representations, making them particularly suited for applications, for example, in signal processing. A frame is defined as a countable family of vectors $(\varphi_k)_{k\in X}$ for which the famous chain of inequalities

$$(1) \qquad A\,\|f\|^2 \leq \sum_{k\in X} |\langle f, \varphi_k\rangle|^2 \leq B\,\|f\|^2, \qquad f \in \mathcal{H},$$

is satisfied. In the article [21], Gröchenig gave fourteen equivalent conditions *without inequalities* for a Gabor system over a lattice with non-zero window in the modulation space $M^1(\mathbb{R}^d)$ being a frame. The main result of [21] is that any such Gabor system $\mathcal{G} = \mathcal{G}(g,\Lambda)$ is a frame for $L^2(\mathbb{R}^d)$ if and only if the associated frame related operators satisfy certain mapping properties, for example, if the analysis operator $C_{\mathcal{G}} : M^\infty(\mathbb{R}^d) \to \ell^\infty(\Lambda)$ is injective. The proof of the latter relies on deep results of Gabor systems over lattices, such as the Ron-Shen duality principle [31], Wiener's Lemma for twisted convolutions [23], and a result on linear independence of time-frequency shifts [21].

In many settings, however, frame theoretic results are ultimately not the consequence of an underlying group structure, but follow from localization properties of the frame, see, e.g., kernel theorems (compare [6, 14] with [10]), invertibility results of Gabor frame operators on modulation spaces [19, 23, 25], or a partial generalization [25, Theorem 5.1] of [21, Theorem 3.1]. A frame $(\psi_k)_{k\in X}$ is said to be *localized*, if its Gram matrix is contained in a *spectral matrix algebra* (see [15] or Definition 2). To a localized frame ψ one can then associate a family of Banach spaces, the so-called *co-orbit spaces* $\mathcal{H}^p(\psi)$ $(1 \leq p \leq \infty)$ which are defined in terms of summability conditions on the analysis coefficients $(\langle f, \psi_k\rangle)_{k\in X}$, see Section 2.3 for further details.

Returning to Gabor analysis, we note that the property $g \in M^1(\mathbb{R}^d)$ of the window function can be seen as a localization assumption. Indeed, in the case

2020 *Mathematics Subject Classification.* 42C15, 46B15, 47B37, 42B35, 46B45.

Key words and phrases. localized frames, co-orbit spaces, R-dual sequences, invertibility of frame-related operators.

$\Lambda = \mathbb{Z}^d$, we note that $\mathcal{G}(g, \mathbb{Z}^d)$ is localized with respect to the unweighted Baskakov-Gohberg-Sjöstrand algebra $\mathcal{C}$ (see Example 4 (3)), and in the case of more irregular sets Λ, similar (more general) properties have been shown [25]. Furthermore, if $\mathcal{G}$ is a frame, the modulation spaces $M^p(\mathbb{R}^d)$ coincide with the co-orbit spaces $\mathcal{H}^p(\mathcal{G})$ $(1 \leq p \leq \infty)$ generated from $\mathcal{G}$ in this case [25, Theorem 3.2]. Naturally, these observations lead to the question, whether abstract localized frames can be characterized in a similar fashion as in [21], namely via properties of their associated frame-related operators on their respective co-orbit spaces.

In this article we give several equivalent conditions for a suitably localized countable family $\psi = (\psi_k)_{k\in X}$ of vectors in some abstract separable Hilbert space $\mathcal{H}$ being a frame. Let C_ψ, D_ψ and S_ψ denote the analysis, synthesis and frame operator respectively, and set $\mathrm{Ran}_p(C_\psi) := C_\psi(\mathcal{H}^p(\varphi))$ and $\mathrm{Ran}_p(D_\psi) := D_\psi(\ell^p(X))$, see Section 2 for the detailed definitions. The formulation of our main result is as follows:

Theorem 1. *Let $\mathcal{A}$ be a spectral algebra and $\varphi = (\varphi_k)_{k\in X}$ be an intrinsically $\mathcal{A}$-localized Riesz basis in $\mathcal{H}$. Let $\psi = (\psi_k)_{k\in X}$ be a family in $\mathcal{H}$, which is mutually $\mathcal{A}$-localized with respect to φ, and let $\omega = (\omega_k)_{k\in X}$ be given by*

$$\omega_k = \sum_{l\in X} \langle \psi_l, \varphi_k\rangle S_\varphi^{-1/2}\varphi_l, \qquad k \in X.$$

Then the following are equivalent.

1. *ψ is a frame for $\mathcal{H}$.*
2. *S_ψ is invertible on $\mathcal{H}^1(\varphi)$.*
3. *S_ψ is invertible on $\mathcal{H}^\infty(\varphi)$.*
4. *$C_\psi : \mathcal{H}^\infty(\varphi) \to \ell^\infty(X)$ is injective and $\mathrm{Ran}_\infty(C_\psi)$ as well as $\mathrm{Ran}_\infty(D_\psi)$ are closed.*
5. *$D_\psi : \ell^1(X) \to \mathcal{H}^1(\varphi)$ is surjective and $\mathrm{Ran}_1(C_\psi)$ is closed.*
6. *$D_\omega : \ell^\infty(X) \to \mathcal{H}^\infty(\varphi)$ is injective and $\mathrm{Ran}_\infty(C_\omega)$ as well as $\mathrm{Ran}_\infty(D_\omega)$ are closed.*
7. *$C_\omega : \mathcal{H}^1(\varphi) \to \ell^1(X)$ is surjective and $\mathrm{Ran}_1(D_w)$ is closed.*
8. *G_ω is invertible on $\ell^1(X)$.*
9. *G_ω is invertible on $\ell^\infty(X)$.*
10. *ω is a Riesz sequence in $\mathcal{H}$.*

In particular, if any of the above conditions is satisfied, then $\mathcal{H}^p(\psi) = \mathcal{H}^p(\varphi)$ with equivalent norms $(1 \leq p \leq \infty)$.

Naturally, the price for working in an abstract Hilbert space is that less structure and less tools are available. This forces us to impose additional assumptions in the statements of Theorem 1, when compared with [21].

First, while the modulation spaces $M^p(\mathbb{R}^d)$ can be intrinsically defined via the *short-time Fourier transform* associated with the omnipresent Gaussian (or any other Schwartz function) [18], the definition and many relevant properties of the co-orbit spaces $\mathcal{H}^p(\varphi)$ depend on the frame property of some explicitly given localized frame φ. This is the reason why we relate the family ψ to an intrinsically localized reference frame φ generating the co-orbit spaces $\mathcal{H}^p(\varphi)$, instead of working with the co-orbit spaces $\mathcal{H}^p(\psi)$ directly.

Second, to our knowledge, there is currently no canonical generalization of the Ron-Shen duality principle to abstract frames in the literature. The Ron-Shen duality principle connects any Gabor frame on a lattice with its (dual) Gabor Riesz sequence generated by the same window on the adjoint lattice. In Theorem 1 we consider the family ω, which, in the language of [33], is a so-called *R-dual of type III* of ψ, and serves as a substitute of the aforementioned Ron-Shen dual. In general, however, an R-dual associated with a Gabor frame on a lattice does not coincide with its Ron-Shen dual, see [11, 33] for a discussion of this matter. In order to guarantee that the R-dual ω is a Riesz sequence if and only if ψ is a frame, we additionally have to assume that the reference system φ is in fact a Riesz basis and not only a frame. Thus, our localization assumption $\psi \sim_{\mathcal{A}} \varphi$ corresponds to the original localization concept from [19] in the case $\mathcal{A} = \mathcal{J}_s$ (see Example 4 (1)), which was followed by several other publications on this topic [1, 3, 4, 5, 15, 20]. At the same time, however, it might be very difficult, if not even impossible for a Gabor frame to be localized with respect to a localized Riesz basis (see also [19, Section 5] for a discussion of this issue).

Finally, several closed range conditions appear in our setting but not in [21, Theorem 3.1]. We show in Example 28 and Remark 29, that each of the closed range conditions appearing in Theorem 1 is in fact necessary in our setting, since otherwise, the statement is simply not true. For Gabor frames, however, these conditions are automatically satisfied, see Example 27.

By the previous observations, Theorem 1 does not directly generalize [21, Theorem 3.1]. However, [21] served as inspiration for this article, and multiple equivalent conditions of [21, Theorem 3.1] can be translated directly into conditions in Theorem 1. We refer to Section 5 for a more detailed comparison.

This article is structured as follows. In Section 2 we fix our notation and summarize some relevant concepts and results from the literature. In Section 3 we prove some auxiliary results, which we utilize in Section 4, where we prove our main result. In Section 5 we compare Theorem 1 to [21, Theorem 3.1], which initiated the research on the content of this article. Finally, Section 6 is devoted to applying our main result in the setting of shift-invariant spaces.

2. Preliminaries and Notation

Throughout this manuscript $\mathcal{H}$ denotes a separable Hilbert space and X a countable (index) set. If not explicitly stated otherwise, $\|\cdot\|$ and the bracket $\langle\cdot,\cdot\rangle$ (without index) denote the norm and the inner product on $\mathcal{H}$, respectively. Whenever we talk about the dual space B' of a Banach space B, we refer to the *anti-linear* (topological) dual space of B. In particular, the identification $(\ell^p(X))' = \ell^q(X)$ (for $1 \leq p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$) is based on the sesquilinear form

$$\big\langle (c_k)_{k\in X}, (d_k)_{k\in X} \big\rangle_{\ell^p\times\ell^q} = \sum_{k\in X} c_k \overline{d_k}$$

in order to be compatible with the Hilbert space duality on $\ell^2(X)$.

2.1. **Basic Frame Theory.** A countable family $\varphi := (\varphi_k)_{k\in X}$ of vectors in $\mathcal{H}$ is called a *frame*, if there exist constants $A, B > 0$ such that

$$A\,\|f\|^2 \leq \sum_{k\in X} |\langle f, \varphi_k\rangle|^2 \leq B\,\|f\|^2, \qquad f \in \mathcal{H}. \tag{2}$$

For a comprehensive introduction to frames in Hilbert space we refer the reader to [12]. If the upper, but not necessarily the lower inequality in (2) is satisfied for some prescribed $B > 0$ and all $f \in \mathcal{H}$, then φ is called a *Bessel sequence*. Whenever φ is a Bessel sequence, both the *analysis operator*, defined as

$$C_\varphi : \mathcal{H} \to \ell^2(X), \qquad C_\varphi f = \big(\langle f, \varphi_k \rangle\big)_{k\in X},$$

and the *synthesis operator*, defined as

$$D_\varphi : \ell^2(X) \to \mathcal{H}, \qquad D_\varphi c = \sum_{k\in X} c_k \varphi_k,$$

are bounded and adjoint to one another, where the latter series converges unconditionally in $\mathcal{H}$. A combination of the analysis and the synthesis operator yields the *frame operator*

$$S_\varphi = D_\varphi C_\varphi : \mathcal{H} \to \mathcal{H}, \qquad S_\varphi f = \sum_{k\in X} \langle f, \varphi_k \rangle \varphi_k.$$

If φ is a frame, then S_φ is bounded, positive, self-adjoint and invertible and the family $\widetilde{\varphi} = \big(\widetilde{\varphi}_k\big)_{k\in X} := \big(S_\varphi^{-1}\varphi_k\big)_{k\in X}$ is another frame, called the *canonical dual frame*, that satisfies

$$f = D_\varphi C_{\widetilde{\varphi}} f = \sum_{k\in X} \langle f, \widetilde{\varphi}_k \rangle \varphi_k = D_{\widetilde{\varphi}} C_\varphi f = \sum_{k\in X} \langle f, \varphi_k \rangle \widetilde{\varphi}_k, \qquad f \in \mathcal{H}. \tag{3}$$

More generally, if $\varphi^d = (\varphi^d)_{k\in X}$ is another family in $\mathcal{H}$ such that $D_\varphi C_{\varphi^d} = D_{\varphi^d} C_\varphi = \mathcal{I}_\mathcal{H}$, then φ^d is called a *dual frame* of φ. If ψ, φ are two countable families of vectors in $\mathcal{H}$ indexed by X, then we define the *cross Gram matrix* as

$$G_{\psi,\varphi} = \big[\langle \varphi_l, \psi_k \rangle\big]_{k,l\in X}. \tag{4}$$

If $\psi = \varphi$ we write $G_\varphi = G_{\varphi,\varphi}$. In general, a (cross) Gram matrix as in (4) might not be bounded on $\ell^2(X)$, but in case that ψ and φ both are Bessel sequences, then $G_{\psi,\varphi}$ defines a bounded operator on $\ell^2(X)$ and

$$G_{\psi,\varphi} = C_\psi D_\varphi$$

and $G^*_{\psi,\varphi} = G_{\varphi,\psi}$. In fact, a countable family φ in $\mathcal{H}$ is a Bessel sequence if and only if its associated Gram matrix G_φ defines a bounded operator on $\ell^2(X)$ [7]. A family $\varphi := (\varphi_k)_{k\in X} \subset \mathcal{H}$ is called a *Riesz sequence* for $\mathcal{H}$, if there exist constants $A, B > 0$ such that

$$A\|c\|^2_{\ell^2(X)} \le \left\| \sum_{k\in X} c_k \varphi_k \right\|^2 \le B\|c\|^2_{\ell^2(X)}, \qquad c \in \ell^2(X).$$

Equivalently, φ is a Riesz sequence if and only if its associated Gram matrix G_φ defines a bounded and invertible operator on $\ell^2(X)$ [7]. A Riesz sequence φ is called a *Riesz basis*, if it is also *complete*, that is, $\overline{\text{span}}(\varphi_k)_{k\in X} = \mathcal{H}$. In particular, a Riesz basis is a frame for which both $G_\varphi \in \mathcal{B}(\ell^2(X))$ and $S_\varphi \in \mathcal{B}(\mathcal{H})$ are invertible. For any Riesz basis φ the family $(S_\varphi^{-1/2}\varphi_k)_{k\in X}$ constitutes an orthonormal basis.

2.2. **The R-dual sequence.** We will consider the following notion of duality for abstract sequences in $\mathcal{H}$. Let $(e_k)_{k\in X}$ and $(h_k)_{k\in X}$ be two orthonormal bases for $\mathcal{H}$, and let $(\varphi_k)_{k\in X}$ be a family of vectors such that $\sum_{n\in X} |\langle \varphi_n, e_k \rangle|^2 < \infty$ for every $k \in X$. Then

$$\omega_k = \sum_{n\in X} \langle \varphi_n, e_k \rangle h_n$$

converges in $\mathcal{H}$ for each $k \in X$ and the family $\omega = (\omega_k)_{k\in X}$ is called an *R-dual* or *Riesz-dual sequence* for φ. It was shown in [11, Propositions 10,11] that φ is a frame for $\mathcal{H}$ if and only if ω is a Riesz sequence.

2.3. **Localized Frames.** As mentioned in the previous section, whenever φ is a Bessel sequence, its associated Gram matrix $G_\varphi = [\langle \varphi_l, \varphi_k\rangle]_{k,l\in X}$ defines a bounded operator on $\ell^2(X)$. Boundedness on $\ell^2(X)$ is equivalent to some form of off-diagonal decay of the matrix entries $\langle \varphi_l, \varphi_k\rangle$, as described, e.g., in [30, Theorem 4.16]. The idea of localization is to impose stronger off-diagonal decay conditions on a given (cross) Gram matrix. Off-diagonal decay of matrices is often described by Banach matrix algebras, which motivated the authors in [15] to consider the following type of matrix algebras.

Definition 2. *A unital Banach *-algebra $\mathcal{A}$ of $\mathbb{C}$-valued matrices $A = [A_{k,l}]_{k,l\in X}$ is called a* solid spectral matrix algebra [15, Section 2.3]*, or simply* spectral algebra*, if*

- *(A0) $\mathcal{A} \subset \bigcap_{1\le p\le\infty} \mathcal{B}(\ell^p(X))$, i.e., each $A \in \mathcal{A}$ defines a bounded operator on $\ell^p(X)$ for each $1 \le p \le \infty$.*
- *(A1) $\mathcal{A} \subset \mathcal{B}(\ell^2(X))$, i.e., each $A \in \mathcal{A}$ defines a bounded operator on $\ell^2(X)$.*
- *(A2) $\mathcal{A}$ is* inverse-closed *in $\mathcal{B}(\ell^2(X))$, i.e., if $A \in \mathcal{A}$ is invertible in $\mathcal{B}(\ell^2(X))$, then its bounded inverse $A^{-1} \in \mathcal{B}(\ell^2(X))$ is contained in $\mathcal{A}$ as well.*
- *(A3) $\mathcal{A}$ is* solid*, i.e., if $A = [A_{k,l}]_{k,l\in X} \in \mathcal{A}$ and $|B_{k,l}| \le |A_{k,l}|$ for all $k, l \in X$, then $B = [B_{k,l}]_{k,l\in X} \in \mathcal{A}$ and $\|B\|_\mathcal{A} \le \|A\|_\mathcal{A}$.*

Remark 3. *Condition (A2) is equivalent to* spectral invariance *of $\mathcal{A}$ in $\mathcal{B}(\ell^2(X))$, that is, $\sigma_\mathcal{A}(A) = \sigma_{\mathcal{B}(\ell^2(X))}(A)$ for all $A \in \mathcal{A}$* [22, Lemma 2.4]*, where $\sigma_\mathcal{A}(A)$ and $\sigma_{\mathcal{B}(\ell^2(X))}(A)$ denote the spectrum of A in the Banach algebras $\mathcal{A}$ and $\mathcal{B}(\ell^2(X))$ respectively. As a consequence, the holomorphic (Riesz) functional calculi (see e.g.* [13, Chap. VII]*) of $\mathcal{A}$ and $\mathcal{B}(\ell^2(X))$ coincide. More precisely, if $A \in \mathcal{A}$, $U \subseteq \mathbb{C}$ is an open neighborhood of (a connected component of) $\sigma_\mathcal{A}(A) = \sigma_{\mathcal{B}(\ell^2(X))}(A)$, $\Gamma \subseteq U$ is a contour of $\sigma_\mathcal{A}(A) = \sigma_{\mathcal{B}(\ell^2(X))}(A)$, and if $z \mapsto f(z)$ is holomorphic on U, then*

$$f(A) = \int_\Gamma f(z)(z - A)^{-1}\, dz$$

defines the same element in $\mathcal{A}$ and $\mathcal{B}(\ell^2(X))$.

Strictly speaking, the authors in [15] considered algebras satisfying properties (A1−A3) without including property (A0) in their definition. In the language of [15, Definition 2], property (A0) can be rephrased to saying that the constant weight 1 is *$\mathcal{A}$-admissible with critical index* $p_0 = 1$. Here we assume for simplicity, that a spectral algebra automatically satisfies (A0) as well, since many of the classical examples of spectral algebras do so anyway, see below.

Example 4. *Let $X \subset \mathbb{R}^d$ be a relatively separated set. Then the following classes of matrices are examples of spectral algebras in the sense of Definition 2, see also* [29] *for their $\mathcal{B}(\mathcal{H})$-valued analogs.*
(1) *For every $s > d$, the* Jaffard algebra *$\mathcal{J}_s = \mathcal{J}_s(X)$ of matrices $A = [A_{k,l}]_{k,l\in X}$ for which*

$$\sup_{k,l\in X} |A_{k,l}|(1 + |k - l|)^s < \infty$$

is a spectral algebra [27].

(2) *For every continuous function* $\nu : \mathbb{R}^d \to [0, \infty)$ *which*

(2a) is of the form $\nu(x) = e^{\rho(\|x\|)}$*, where* $\rho : [0, \infty) \longrightarrow [0, \infty)$ *is a continuous and concave function with* $\rho(0) = 0$*, and* $\|\cdot\|$ *any norm on* $\mathbb{R}^d$*,*

(2b) satisfies the GRS-condition (Gelfand-Raikov-Shilov condition [16]*)*

$$\lim_{n\to\infty} (\nu(nz))^{\frac{1}{n}} = 1, \qquad z \in \mathbb{R}^d,$$

(2c) satisfies the weak growth condition

$$\nu(x) \geq C(1 + |x|)^\delta \qquad \text{for some } \delta \in (0, 1], C > 0,$$

the weighted Schur algebra $\mathcal{S}^1_\nu = \mathcal{S}^1_\nu(X)$ *of matrices* $A = [A_{k,l}]_{k,l\in X}$ *for which*

$$\max \left\{ \sup_{k\in X} \sum_{l\in X} |A_{k,l}|\nu(k - l)\, , \sup_{l\in X} \sum_{k\in X} |A_{k,l}|\nu(k - l) \right\} < \infty$$

is a spectral algebra [24, 34]*.*

(3) *For every continuous function* $\nu : \mathbb{R}^d \to [0, \infty)$ *which*

(3a) is submultiplicative*, i.e.,* $\nu(x + y) \leq \nu(x)\nu(y)$ *for every* $x, y \in \mathbb{R}^d$*,*

(3b) is symmetric*, i.e.,* $\nu(-x) = \nu(x)$ *for every* $x \in \mathbb{R}^d$*,*

(3c) satisfies the GRS-condition,

the weighted Baskakov-Gohberg-Sjöstrand algebra $\mathcal{C}_\nu = \mathcal{C}_\nu(\mathbb{Z}^d)$ *of matrices* $A = [A_{k,l}]_{k,l\in\mathbb{Z}^d}$ *for which*

$$\sum_{l\in\mathbb{Z}^d} \sup_{k\in\mathbb{Z}^d} |A_{k,k-l}|\nu(l) < \infty$$

is a spectral algebra [8, 17, 32]*.*

(4) *Anisotropic versions of each of the aforementioned algebras are further examples of spectral algebras* [29, Section 6]*.*

For a scalar matrix $A = [A_{k,l}]_{k,l\in X}$, we define the complex conjugate $\overline{A}$ of A and the absolute value $|A|$ of A pointwise, i.e., by $\overline{A} := \left[\,\overline{A_{k,l}}\,\right]_{k,l\in X}$ and $|A| := \left[\,|A_{k,l}|\,\right]_{k,l\in X}$. We explicitly note that by the solidity condition (A3), it holds

$$A \in \mathcal{A} \qquad \Longleftrightarrow \qquad \overline{A} \in \mathcal{A} \qquad \Longleftrightarrow \qquad |A| \in \mathcal{A}.$$

We will use this fact several times without further reference.

Definition 5. [15, Definition 3] *Let* $\mathcal{A}$ *be a spectral algebra. Two countable families* $\psi = (\psi_k)_{k\in X}$ *and* $\varphi = (\varphi_k)_{k\in X}$ *are called* mutually$\mathcal{A}$-localized*, in short* $\psi \sim_\mathcal{A} \varphi$*, if* $G_{\psi,\varphi} \in \mathcal{A}$*. Similarly,* ψ *is called (intrinsically)* $\mathcal{A}$*-localized if* $\psi \sim_\mathcal{A} \psi$*, i.e.,* $G_\psi \in \mathcal{A}$*.*

Note that by involution in $\mathcal{A}$ it holds that $\psi \sim_\mathcal{A} \varphi$ if and only if $\varphi \sim_\mathcal{A} \psi$. We will use this fact without further reference.

One of the main results in [15] states that localization of a frame is preserved under canonical duality. More precisely:

Theorem 6. [15, Theorem 3.6] *Let* φ *be a frame in* $\mathcal{H}$*. If* $\varphi \sim_\mathcal{A} \varphi$ *then also* $\widetilde{\varphi} \sim_\mathcal{A} \widetilde{\varphi}$ *and* $\varphi \sim_\mathcal{A} \widetilde{\varphi}$*.*

The latter theorem immediately implies the following simple lemma, which we will also implicitly apply several times without further notice. For completeness we present the simple proof.

Lemma 7. *Let $\varphi = (\varphi_k)_{k\in X}$ an $\mathcal{A}$-localized frame for $\mathcal{H}$ and $\psi = (\psi_k)_{k\in X}$ be any countable family in $\mathcal{H}$ such that $\psi \sim_{\mathcal{A}} \varphi$. Then it holds $\psi \sim_{\mathcal{A}} \widetilde{\varphi}$ and $\psi \sim_{\mathcal{A}} \psi$.*

Proof. By assumption, $G_{\psi,\varphi} \in \mathcal{A}$ and by Theorem 6, $G_{\widetilde{\varphi}} \in \mathcal{A}$. Thus, by frame reconstruction and the algebra property of $\mathcal{A}$, $G_{\psi,\widetilde{\varphi}} = G_{\psi,\varphi}G_{\widetilde{\varphi}} \in \mathcal{A}$. Similarly, the latter implies that $G_\psi = G_{\psi,\varphi}G_{\widetilde{\varphi},\psi} \in \mathcal{A}$. □

A remarkable feature of a localized frame φ in $\mathcal{H}$ is, that the reconstruction formula (3) does not only converge in the underlying Hilbert space $\mathcal{H}$, but also in a whole chain of associated Banach spaces defined below.

Definition 8. [15, Definition 5] *Let $\mathcal{A}$ be a spectral algebra, φ be a frame for $\mathcal{H}$, and φ^d be some dual frame of φ such that $\varphi \sim_{\mathcal{A}} \varphi^d$. Let*

$$\mathcal{H}^{00}(\varphi) := D_\varphi(\ell^{00}(X)) = \left\{ \sum_{k\in X} c_k\varphi_k : \ c \in \ell^{00}(X) \right\} \tag{5}$$

be the space of finite linear combinations of elements from φ. For $1 \le p < \infty$, the co-orbit space *$\mathcal{H}^p(\varphi^d, \varphi)$ is defined as the norm completion of $\mathcal{H}^{00}(\varphi)$ with respect to the norm $\|f\|_{\mathcal{H}^p(\varphi^d,\varphi)} := \left\|C_{\varphi^d} f\right\|_{\ell^p(X)}$.*

Under the above assumptions, the spaces are well-defined non-trivial Banach spaces with $\mathcal{H}^p(\varphi^d, \varphi) \subseteq \mathcal{H}^q(\varphi^d, \varphi)$ for all $1 \le p, q < \infty$ and $\mathcal{H}^2(\varphi^d, \varphi) = \mathcal{H}$. Furthermore, the analysis operator $C_{\widetilde{\varphi}} : \mathcal{H}^{00}(\varphi) \to \ell^{00}(X)$ extends via density to an isometry $C_{\widetilde{\varphi}} : \mathcal{H}^p(\varphi) \to \ell^p(X)$ $(1 \le p < \infty)$.

Remark 9. *The definition of $\mathcal{H}^p(\varphi^d, \varphi)$ seems to depend heavily on the dual frame pair (φ, φ^d). However, if (ψ, ψ^d) is another pair of dual frames satisfying $\psi \sim_{\mathcal{A}} \psi^d$, and if $\psi^d \sim_{\mathcal{A}} \varphi$ and $\varphi^d \sim_{\mathcal{A}} \psi$, then $\mathcal{H}^p(\varphi^d, \varphi) = \mathcal{H}^p(\psi^d, \psi)$ with equivalent norms* [28, Theorem 3.25].

If φ is an intrinsically $\mathcal{A}$-localized frame, then $\varphi \sim_{\mathcal{A}} \widetilde{\varphi}$ and $\widetilde{\varphi} \sim_{\mathcal{A}} \widetilde{\varphi}$ by Theorem 6. In particular, due to the above remark, it holds $\mathcal{H}^p(\widetilde{\varphi}, \varphi) = \mathcal{H}^p(\varphi, \widetilde{\varphi})$ with equivalent norms and we write $\mathcal{H}^p(\varphi) := \mathcal{H}^p(\widetilde{\varphi}, \varphi)$ in this case $(1 \le p < \infty)$.

The definition of the space $\mathcal{H}^\infty(\varphi)$ associated to an $\mathcal{A}$-localized frame φ in $\mathcal{H}$ is more subtle. One could define it as the anti-linear topological dual space of $\mathcal{H}^1(\varphi)$ (see Theorem 15), or as the completion of $\mathcal{H}$ with respect to the Hausdorff locally convex $\sigma(\mathcal{H}, \mathcal{H}^{00}(\widetilde{\varphi}))$-topology as done in [15, 19]. In this article, we work with the definition from [5] (see also [28]) given below.

Definition 10. [5, 28] *Let $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ be sequences in $\mathcal{H}$ and φ be a frame for $\mathcal{H}$ with canonical dual frame $\widetilde{\varphi}$. Then we call $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ equivalent, in short $(f_n)_{n\in\mathbb{N}} \sim_\varphi (g_n)_{n\in\mathbb{N}}$, if*

$$\lim_{n\to\infty} \langle f_n - g_n, \widetilde{\varphi}_k\rangle = 0 \qquad (\forall k \in X) \qquad \textit{and} \qquad \sup_{n\in\mathbb{N}} \|C_{\widetilde{\varphi}}(f_n - g_n)\|_{\ell^\infty(X)} < \infty.$$

Clearly the relation $\sim_\varphi$ defines an equivalence relation on the space of all sequences in $\mathcal{H}$.

Definition 11. [5, 28] *Let $\mathcal{A}$ be a spectral algebra, and φ be an $\mathcal{A}$-localized frame for $\mathcal{H}$ with canonical dual frame $\widetilde{\varphi}$. Then $\mathcal{H}^\infty(\varphi)$ is defined to be the space of all equivalence classes $f = [(f_n)_{n\in\mathbb{N}}]_{\sim_\varphi}$ of sequences $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{H}$ satisfying*

(i) $\lim_{n\to\infty}\langle f_n, \widetilde{\varphi}_k\rangle =: \langle f, \widetilde{\varphi}_k\rangle$ *exists for each $k \in X$,*
(ii) $\sup_{n\in\mathbb{N}} \|C_{\widetilde{\varphi}} f_n\|_{\ell^\infty(X)} < \infty$.

It is quickly verified that the defining properties (i) and (ii) do not depend on the choice of the representative sequence $(f_n)_{n\in\mathbb{N}}$. Furthermore, condition (ii) implies that there is a positive constant C such that

$$|\langle f_n, \widetilde{\varphi}_k\rangle| \leq C \qquad \text{for each } k \in X \text{ and each } n \in \mathbb{N}. \tag{6}$$

This implies that $|\langle f, \widetilde{\varphi}_k\rangle| \leq C$ for all $k \in X$. In particular,

$$\|f\|_{\mathcal{H}^\infty(\varphi)} := \|C_{\widetilde{\varphi}} f\|_{\ell^\infty(X)} = \sup_{k\in X} \left| \lim_{n\to\infty} \langle f_n, \widetilde{\varphi}_k\rangle \right| \tag{7}$$

is well-defined and, in fact, defines a norm on $\mathcal{H}^\infty(\varphi)$ with respect to which $\mathcal{H}^\infty(\varphi)$ is complete. We explicitly emphasize that here $\langle f, \widetilde{\varphi}_k\rangle$ is not an inner product in $\mathcal{H}$ but (by abusing notation) the limit of the sequence $\big(\langle f_n, \widetilde{\varphi}_k\rangle\big)_{n\in\mathbb{N}}$ (for each $k \in X$). Analogously, $C_{\widetilde{\varphi}} f = (\langle f, \widetilde{\varphi}_k\rangle)_{k\in X}$ is understood as a pointwise limit. In this sense, $C_{\widetilde{\varphi}} : \mathcal{H}^\infty(\varphi) \to \ell^\infty(X)$ defines an isometry.

Proposition 12. [15, Theorem 3.8], [28, Corollary 3.27] *Let $\mathcal{A}$ be a spectral algebra, and $\varphi = (\varphi_k)_{k\in X}$ be an $\mathcal{A}$-localized frame for $\mathcal{H}$ with canonical dual frame $\widetilde{\varphi}$. Then $\mathcal{H}^\infty(\varphi)$ is a Banach space with respect to the norm $\|C_{\widetilde{\varphi}} \cdot \|_{\ell^\infty(X)}$, and $\|C_\varphi \cdot \|_{\ell^\infty(X)}$ is an equivalent norm.*

The properties of the operators D_φ and S_φ on their respective domains are summarized below.

Proposition 13. [15, Proposition 2.4],[28, Proposition 3.20] *Let $\mathcal{A}$ be a spectral algebra, and $\varphi = (\varphi_k)_{k\in X}$ be an $\mathcal{A}$-localized frame for $\mathcal{H}$ with canonical dual frame $\widetilde{\varphi}$. Then for each $1 \leq p < \infty$, the synthesis operator*

$$\begin{aligned} D_\varphi : \ell^{00}(X) &\longrightarrow \mathcal{H}^{00}(\varphi), \\ (c_l)_{l\in X} &\mapsto \sum_{l\in X} c_l \varphi_l \end{aligned} \tag{8}$$

continuously extends to a bounded and surjective operator $D_\varphi : \ell^p(X) \to \mathcal{H}^p(\varphi)$ and the associated series (8) converges unconditionally in $\mathcal{H}^p(\varphi)$.

Similarly, if $F_1 \subseteq F_2 \subseteq \dots$ is a nested sequence of finite subsets of X such that $\bigcup_{n=1}^\infty F_n = X$, then $f_n := \sum_{l\in F_n} c_l \psi_l$ is contained in $\mathcal{H}$ for each $n \in \mathbb{N}$ and

$$c = (c_l)_{l\in X} \mapsto D_\varphi c := \big[(f_n)_{n\in\mathbb{N}}\big]_{\sim_\varphi} = \left[\left(\sum_{l\in F_n} c_l \psi_l \right)_{n\in\mathbb{N}} \right]_{\sim_\varphi} \tag{9}$$

defines a bounded and surjective operator $D_\varphi : \ell^\infty(X) \to \mathcal{H}^\infty(\varphi)$ and $D_\varphi c$ is independent of the choice of $\{F_n\}_{n=1}^\infty$, i.e. (9) converges unconditionally in the $\sigma(\mathcal{H}, \mathcal{H}^{00}(\widetilde{\varphi}))$-topology.

Theorem 14. [15, Theorem 3.8],[28, Corollary 3.27] *Let $\mathcal{A}$ be a spectral algebra, and $\varphi = (\varphi_k)_{k\in X}$ be an $\mathcal{A}$-localized frame for $\mathcal{H}$ with canonical dual frame $\widetilde{\varphi}$. Then for each $1 \leq p \leq \infty$, the frame operator $S_\varphi = D_\varphi C_\varphi : \mathcal{H}^p(\varphi) \to \mathcal{H}^p(\varphi)$ is bounded and invertible.*

We conclude this section with the following result on the anti-linear topological dual space of $\mathcal{H}^p(\varphi)$.

Theorem 15. [5, Proposition 2] *Let $\mathcal{A}$ be a spectral algebra, and $\varphi = (\varphi_k)_{k\in X}$ be an $\mathcal{A}$-localized frame for $\mathcal{H}$ with canonical dual frame $\widetilde{\varphi}$. Then*

$$(\mathcal{H}^p(\varphi))' = \mathcal{H}^q(\varphi)$$

with equivalent norms for all $1 \le p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, where the duality relation is given by

$$\langle f, g\rangle_{\mathcal{H}^p\times\mathcal{H}^q} = \langle C_{\widetilde{\varphi}} f, C_\varphi g\rangle_{\ell^p\times\ell^q} = \sum_{k\in X} [C_{\widetilde{\varphi}} f]_k \overline{[C_\varphi g]_k}.$$

3. Auxiliary Results

In this section we prepare the tools that will be important for the proof of our main theorem.

3.1. **Frame-related Operators.** First, we collect some preliminary results on properties of the frame-related operators associated with localized sequences.

Lemma 16. *Let φ be a Riesz sequence in $\mathcal{H}$ that satisfies $\varphi \sim_{\mathcal{A}} \varphi$. Then $G_\varphi : \ell^p(X) \to \ell^p(X)$ is bounded and invertible for every $1 \le p \le \infty$.*

Proof. By the localization assumption we have that $G_\varphi \in \mathcal{A}$. The Riesz sequence property implies that G_φ is invertible on $\mathcal{B}(\ell^2(X))$, which implies by inverse closedness of $\mathcal{A}$ in $\mathcal{B}(\ell^2(X))$ that $(G_\varphi)^{-1} \in \mathcal{A}$. Consequently, G_φ is invertible on all $\ell^p(X)$-spaces ($1 \le p \le \infty$). □

Corollary 17. *Let φ be a Riesz basis for $\mathcal{H}$ that satisfies $\varphi \sim_{\mathcal{A}} \varphi$. Then $D_\varphi : \ell^p(X) \to \mathcal{H}^p(\varphi)$ is injective for every $1 \le p \le \infty$.*

Proof. We may write $G_\varphi : \ell^p(X) \to \ell^p(X)$ as $G_\varphi = C_\varphi D_\varphi$ with $D_\varphi : \ell^p(X) \to \mathcal{H}^p(\varphi)$ being bounded (Proposition 13). The result then follows immediately from Lemma 16. □

The next two lemmata generalize [19, Proposition 8], where the case of the Jaffard algebra $\mathcal{A} = \mathcal{J}_s$ is treated. They are also similar to [5, Lemma 6 and 7], except for the frame assumption on ψ. In fact, our conclusion is also weaker due to the missing frame assumption, i.e., we cannot conclude here that C_ψ has closed range, as explicitly demonstrated in Example 28 (iv).

Lemma 18. *Let $\varphi = (\varphi_k)_{k\in X}$ be a frame in $\mathcal{H}$ and $\psi = (\psi_k)_{k\in X}$ be a family in $\mathcal{H}$ that satisfies $\psi \sim_{\mathcal{A}} \varphi$. Then $C_\psi : \mathcal{H}^p(\varphi) \to \ell^p(X)$ is well-defined and bounded for each $1 \le p \le \infty$. In particular, ψ is a Bessel sequence in $\mathcal{H}$.*

Proof. First, let $1 \le p < \infty$. Let $f \in \mathcal{H}^{00}(\varphi)$ be arbitrary. Then $C_{\widetilde{\varphi}} f \in \ell^p(X)$ and

$$\|C_\psi f\|_{\ell^p(X)} = \|G_{\psi,\varphi} C_{\widetilde{\varphi}} f\|_{\ell^p(X)} \le \|G_{\psi,\varphi}\|_{\mathcal{B}(\ell^p(X))} \|f\|_{\mathcal{H}^p(\varphi)},$$

where we used frame reconstruction in $\mathcal{H}^{00}(\varphi) \subseteq \mathcal{H}$ with respect to φ and $\widetilde{\varphi}$, as well as property (A0) from Definition 2. Thus, by density, C_ψ extends to a bounded operator $\mathcal{H}^p(\varphi) \to \ell^p(X)$ which we again denote by C_ψ.

Now let $p = \infty$. Let $f = \big[(f_n)_{n\in\mathbb{N}}\big]_{\sim_\varphi} \in \mathcal{H}^\infty(\varphi)$ be arbitrary. Then, by definition, $\lim_{n\to\infty} \langle f_n, \widetilde{\varphi}_k\rangle =: \langle f, \widetilde{\varphi}_k\rangle$ converges for every $k \in X$ and $\sup_{n\in\mathbb{N}} \|C_{\widetilde{\varphi}} f_n\|_{\ell^\infty(X)} \le C$ for some $C > 0$, as well as $C_{\widetilde{\varphi}} f := (\langle f, \widetilde{\varphi}_k\rangle)_{k\in X} \in \ell^\infty(X)$. This implies that

$$\sum_{l\in X} |\langle f_n, \widetilde{\varphi}_l\rangle||\langle \varphi_l, \psi_k\rangle| \le C \sup_{k\in X} \sum_{l\in X} |\langle \varphi_l, \psi_k\rangle| = C\|G_{\psi,\varphi}\|_{\mathcal{B}(\ell^\infty(X))} < \infty \tag{10}$$

for each $k \in X$ and each $n \in \mathbb{N}$. Hence we may apply the dominated convergence theorem and obtain via frame reconstruction

$$\lim_{n\to\infty}\langle f_n,\psi_k\rangle = \lim_{n\to\infty}\sum_{l\in X}\langle f_n,\widetilde{\varphi}_l\rangle\langle\varphi_l,\psi_k\rangle = \sum_{l\in X}\langle f,\widetilde{\varphi}_l\rangle\langle\varphi_l,\psi_k\rangle = [G_{\psi,\varphi}C_{\widetilde{\varphi}}f]_k$$

for each $k \in X$. Consequently,

$$\begin{aligned}\|C_\psi f\|_{\ell^\infty(X)} &= \left\|\left(\lim_{n\to\infty}\langle f_n,\psi_k\rangle\right)_{k\in X}\right\|_{\ell^\infty(X)}\\ &= \|G_{\psi,\varphi}C_{\widetilde{\varphi}}f\|_{\ell^\infty(X)}\\ &\le \|G_{\psi,\varphi}\|_{\mathcal{B}(\ell^\infty(X))}\|C_{\widetilde{\varphi}}f\|_{\ell^\infty(X)}\\ &= \|G_{\psi,\varphi}\|_{\mathcal{B}(\ell^\infty(X))}\|f\|_{\mathcal{H}^\infty(\varphi)},\end{aligned}$$

as was to be shown.

Finally, boundedness of $C_\psi : \mathcal{H}^2(\varphi) \to \ell^2(X)$ is equivalent to ψ being a Bessel sequence, since $\mathcal{H}^2(\varphi) = \mathcal{H}$. □

Lemma 19. *Let $\varphi = (\varphi_k)_{k\in X}$ be a frame in $\mathcal{H}$ and $\psi = (\psi_k)_{k\in X}$ be a family in $\mathcal{H}$ that satisfies $\psi \sim_{\mathcal{A}} \widetilde{\varphi}$. Then $D_\psi : \ell^p(X) \to \mathcal{H}^p(\varphi)$, defined analogously as in (8) and (9), is well-defined and bounded for each $1 \le p \le \infty$ and the associated series converges unconditionally (resp. unconditionally in the $\sigma(\mathcal{H},\mathcal{H}^{00}(\widetilde{\varphi}))$-topology in the case $p = \infty$).*

Proof. Assume that $1 \le p < \infty$. Then, by [28, Proposition 3.22], $V^p(\varphi) = \{f \in \mathcal{H} : C_{\widetilde{\varphi}}f \in \ell^p(X)\}$ is a norm-dense subspace of $\mathcal{H}^p(\varphi)$. For $c = (c_l)_{l\in X} \in \ell^{00}(X)$ we have that $D_\psi c \in \mathcal{H}$ as it is a finite linear combination of elements from ψ. Since

$$\|D_\psi c\|_{\mathcal{H}^p(\varphi)} = \|C_{\widetilde{\varphi}}D_\psi c\|_{\ell^p(X)} = \|G_{\widetilde{\varphi},\psi}c\|_{\ell^p(X)} \le \|G_{\widetilde{\varphi},\psi}\|_{\mathcal{B}(\ell^p(X))}\|c\|_{\ell^p(X)},$$

we see that D_ψ maps $\ell^{00}(X)$ boundedly into $V^p(\varphi) \subseteq \mathcal{H}^p(\varphi)$. Hence, the claim follows via density.

Now consider the case $p = \infty$. Pick some arbitrary $c = (c_l)_{l\in X} \in \ell^\infty(X)$ and assume that $F_1 \subseteq F_2 \subseteq \dots$ is a nested sequence of finite subsets of X such that $\bigcup_{n=1}^\infty F_n = X$. Then $f_n := \sum_{l\in F_n} c_l\psi_l$ is contained in $\mathcal{H}$ for each $n \in \mathbb{N}$ and

$$\lim_{n\to\infty}\langle f_n,\widetilde{\varphi}_k\rangle = \lim_{n\to\infty}\sum_{l\in F_n}\langle\psi_l,\widetilde{\varphi}_k\rangle c_l = \sum_{l\in X}[G_{\widetilde{\varphi},\psi}]_{k,l}c_l = [G_{\widetilde{\varphi},\psi}c]_k \in \mathbb{C} \tag{11}$$

for each $k \in X$, since $G_{\widetilde{\varphi},\psi} \in \mathcal{A} \subseteq \mathcal{B}(\ell^\infty(X))$ by assumption. Furthermore,

$$\begin{aligned}|\langle f_n,\widetilde{\varphi}_k\rangle| &\le \sum_{l\in F_n}\left|[G_{\widetilde{\varphi},\psi}]_{k,l}c_l\right|\\ &\le \sup_{k\in X}\sum_{l\in X}\left|[G_{\widetilde{\varphi},\psi}]_{k,l}\right||c_l|\\ &\le \left\|\left|G_{\widetilde{\varphi},\psi}\right|\right\|_{\mathcal{B}(\ell^\infty(X))}\|c\|_{\ell^\infty(X)}\end{aligned}$$

for all $k \in X$ and all $n \in \mathbb{N}$, where we used that $\left|G_{\widetilde{\varphi},\psi}\right| \in \mathcal{A} \subseteq \mathcal{B}(\ell^\infty(X))$ due to solidity of $\mathcal{A}$. This means that $D_\psi c = \left[(f_n)_{n\in\mathbb{N}}\right]_{\sim_\varphi} \in \mathcal{H}^\infty(\varphi)$. The boundedness of D_ψ follows immediately via (11), as we have

$$\|D_\psi c\|_{\mathcal{H}^\infty(\varphi)} = \sup_{k\in X}\left|[G_{\widetilde{\varphi},\psi}c]_k\right| \le \|G_{\widetilde{\varphi},\psi}\|_{\mathcal{B}(\ell^\infty(X))}\|c\|_{\ell^\infty(X)}.$$

To show the unconditional convergence of $D_\psi c$ with respect to the $\sigma(\mathcal{H},\mathcal{H}^{00}(\widetilde{\varphi}))$-topology, it suffices to show that $D_\psi c$ is independent of the choice of $\{F_n\}_{n=1}^\infty$ [28, Sec. 3.2.4]. So, let $J_1 \subseteq J_2 \subseteq \dots$ be another nested sequence of finite subsets of X with $\bigcup_{n=1}^\infty J_n = X$. We will show that $\{\sum_{l\in J_n} c_l\psi_l\}_{n=1}^\infty \sim_\varphi \{\sum_{l\in F_n} c_l\psi_l\}_{n=1}^\infty$. Setting $H_n = F_n \cup J_n$ ($n \in \mathbb{N}$) gives a nested sequence of finite subsets whose union is X as well. Then, for each $k \in X$,

$$\begin{aligned}\lim_{n\to\infty}\left|\left\langle \sum_{l\in H_n} c_l\psi_l - \sum_{l\in F_n} c_l\psi_l, \widetilde{\varphi}_k\right\rangle\right| &= \lim_{n\to\infty}\left|\left\langle \sum_{l\in H_n\setminus F_n} c_l\psi_l, \widetilde{\varphi}_k\right\rangle\right| \\ &\le \|c\|_{\ell^\infty_\omega(X)} \lim_{n\to\infty}\sum_{l\in X\setminus F_n} |\langle \psi_l, \widetilde{\varphi}_k\rangle| = 0,\end{aligned}$$

since

$$\sup_{k\in X}\sum_{l\in X} |\langle \psi_l, \widetilde{\varphi}_k\rangle| = \|G_{\widetilde{\varphi},\psi}\|_{\mathcal{B}(\ell^\infty(X))} < \infty.$$

Clearly, it also holds

$$\sup_{n\in\mathbb{N}}\left\| C_{\widetilde{\varphi}}\left(\sum_{l\in H_n} c_l\psi_l - \sum_{l\in F_n} c_l\psi_l\right)\right\| < \infty.$$

Thus we have shown that $\{\sum_{l\in H_n} c_l\psi_l\}_{n=1}^\infty \sim_\varphi \{\sum_{l\in F_n} c_l\psi_l\}_{n=1}^\infty$. The same argument holds if we replace F_n by J_n. Thus, $\{\sum_{l\in J_n} c_l\psi_l\}_{n=1}^\infty \sim_\varphi \{\sum_{l\in F_n} c_l\psi_l\}_{n=1}^\infty$, since $\sim_{\widetilde{\varphi}}$ is an equivalence relation. □

When we consider the operators $C_\psi : \mathcal{H}^p(\varphi) \to \ell^p(X)$ and $D_\psi : \ell^p(X) \to \mathcal{H}^p(\varphi)$, $1 \le p \le \infty$, we write $\mathrm{Ran}_p(C_\psi), \mathrm{Ran}_p(D_\psi)$ and $\mathrm{Ker}_p(C_\psi), \mathrm{Ker}_p(D_\psi)$ for their respective ranges and nullspaces.

The next two lemmata are mild generalizations of parts of [5, Lemma 7].

Lemma 20. *Let $\varphi = (\varphi_k)_{k\in X}$ be a frame in $\mathcal{H}$ satisfying $\varphi \sim_{\mathcal{A}} \varphi$ and $\psi = (\psi_k)_{k\in X}$ be a family in $\mathcal{H}$ that satisfies $\psi \sim_{\mathcal{A}} \varphi$. Let $1 \le p < \infty$ be arbitrary and $\frac{1}{p} + \frac{1}{q} = 1$. Then the adjoint $D'_\psi : \mathcal{H}^q(\varphi) \to \ell^q(X)$ of $D_\psi : \ell^p(X) \to \mathcal{H}^p(\varphi)$ is given by C_ψ.*

Proof. We first observe that for each $1 \le p < \infty$, we have $\psi_l \in \mathcal{H}^p(\varphi)$ for each $l \in X$ due to

$$\|\psi_l\|_{\mathcal{H}^p(\varphi)} = \|C_{\widetilde{\varphi}}\psi_l\|_{\ell^p(X)} \le \|C_{\widetilde{\varphi}}\psi_l\|_{\ell^1(X)} \le \sup_{l\in X}\sum_{k\in X} |\langle \psi_l, \widetilde{\varphi}_k\rangle| = \|G_{\widetilde{\varphi},\psi}\|_{\mathcal{B}(\ell^1(X))}.$$

We consider the case $(p,q) = (1,\infty)$. Let $f = \left[(f_n)_{n\in\mathbb{N}}\right]_{\sim_\varphi} \in \mathcal{H}^\infty(\varphi)$ and $c = (c_k)_{k\in X} \in \ell^1(X)$ be arbitrary. Then

$$\langle c, D'_\psi f\rangle_{\ell^1\times\ell^\infty} = \langle D_\psi c, f\rangle_{\mathcal{H}^1\times\mathcal{H}^\infty} = \sum_{k\in X} c_k\langle \psi_k, f\rangle_{\mathcal{H}^1\times\mathcal{H}^\infty}.$$

According to Theorem 15, we have that

$$\begin{aligned}\langle \psi_k, f\rangle_{\mathcal{H}^1\times\mathcal{H}^\infty} &= \sum_{l\in X}\langle \psi_k, \widetilde{\varphi}_l\rangle \lim_{n\to\infty}\langle \varphi_l, f_n\rangle \\ &= \lim_{n\to\infty}\sum_{l\in X}\langle \psi_k, \widetilde{\varphi}_l\rangle\langle \varphi_l, f_n\rangle \\ &= \lim_{n\to\infty}\langle \psi_k, f_n\rangle = \overline{[C_\psi f]_k},\end{aligned}$$

where we applied the dominated convergence theorem (justified analogously as in (10) upon noticing that $\|C_\varphi \cdot \|_{\ell^\infty(X)} \asymp \|C_{\widetilde{\varphi}} \cdot \|_{\ell^\infty(X)}$ (c.f. Proposition 12)) in the second step and frame reconstruction in the third step. On the other hand,

$$\langle c, C_\psi f\rangle_{\ell^1\times\ell^\infty} = \sum_{k\in X} c_k \overline{[C_\psi f]_k},$$

which yields the claim. For the other values of p and q we proceed analogously by showing the identity for $f \in \mathcal{H}^{00}(\varphi)$ and then argue via density. $\square$

Lemma 21. *Let $\varphi = (\varphi_k)_{k\in X}$ be a frame in $\mathcal{H}$ satisfying $\varphi \sim_{\mathcal{A}} \varphi$ and $\psi = (\psi_k)_{k\in X}$ be a family in $\mathcal{H}$ that satisfies $\psi \sim_{\mathcal{A}} \varphi$. Let $1 \le p < \infty$ be arbitrary and $\frac{1}{p} + \frac{1}{q} = 1$. Then the Banach space adjoint $C'_\psi : \ell^q(X) \to \mathcal{H}^q(\varphi)$ of $C_\psi : \mathcal{H}^p(\varphi) \to \ell^p(X)$ is given by D_ψ.*

Proof. We first consider the case $(p, q) = (1, \infty)$. Let $f \in \mathcal{H}^{00}(\varphi) \subseteq \mathcal{H}^1(\varphi)$ and $c = (c_l)_{l\in X} \in \ell^\infty$ be arbitrary. According to Lemma 19, $D_\psi c = \big[(f_n)_{n\in\mathbb{N}}\big]_{\sim_\varphi} \in \mathcal{H}^\infty(\varphi)$, where $f_n = \sum_{l\in F_n} c_l\psi_l$ and $F_1 \subseteq F_2 \subseteq \dots$ is a nested sequence of finite subsets of X such that $\bigcup_{n=1}^\infty F_n = X$. Then we see that

$$\begin{aligned}
\langle f, C'_\psi c\rangle_{\mathcal{H}^1\times\mathcal{H}^\infty} &= \langle C_\psi f, c\rangle_{\ell^1\times\ell^\infty}\\
&= \sum_{l\in X} \langle f, \psi_l\rangle \overline{c_l}\\
&= \lim_{n\to\infty} \left\langle f, \sum_{l\in F_n} c_l\psi_l \right\rangle\\
&= \lim_{n\to\infty} \left\langle \sum_{k\in X} \langle f, \widetilde{\varphi}_k\rangle \varphi_k, \sum_{l\in F_n} c_l\psi_l \right\rangle\\
&= \sum_{k\in X} \langle f, \widetilde{\varphi}_k\rangle \lim_{n\to\infty} \left\langle \varphi_k, \sum_{l\in F_n} c_l\psi_l \right\rangle\\
&= \langle f, D_\psi c\rangle_{\mathcal{H}^1\times\mathcal{H}^\infty},
\end{aligned}$$

where we used frame reconstruction in the fourth step and the dominated convergence theorem (justified analogously as in (10) upon noticing that $\|C_\varphi \cdot \|_{\ell^\infty(X)} \asymp \|C_{\widetilde{\varphi}} \cdot \|_{\ell^\infty(X)}$ (c.f. Proposition 12)) in the fifth step. Via density we conclude that $\langle f, C'_\psi c\rangle_{\mathcal{H}^1\times\mathcal{H}^\infty} = \langle f, D_\psi c\rangle_{\mathcal{H}^1\times\mathcal{H}^\infty}$ for all $f \in \mathcal{H}^1(\varphi)$ and the claim is proven. For other values of p and q the proof is similar. $\square$

As a consequence we immediately obtain the following.

Corollary 22. *Let $\varphi = (\varphi_k)_{k\in X}$ be a frame in $\mathcal{H}$ satisfying $\varphi \sim_{\mathcal{A}} \varphi$ and $\psi = (\psi_k)_{k\in X}$ be a family in $\mathcal{H}$ that satisfies $\psi \sim_{\mathcal{A}} \varphi$. For $1 \le p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ the following hold.*

(i) $S_\psi \in \mathcal{B}(\mathcal{H}^p(\varphi))$ and $S'_\psi = S_\psi \in \mathcal{B}(\mathcal{H}^q(\varphi))$.
(ii) $G_\psi \in \mathcal{B}(\ell^p(X))$ and $G'_\psi = G_\psi \in \mathcal{B}(\ell^q(X))$.

We conclude this subsection with the following simple consequence of Remark 3.

Lemma 23. *Let φ be an intrinsically $\mathcal{A}$-localized Riesz basis in $\mathcal{H}$. Then $G_\varphi^\alpha \in \mathcal{A}$ for every $\alpha \in \mathbb{R}$.*

Proof. Since $\mathcal{A}$ is inverse-closed in $\mathcal{B}(\ell^2(X))$, the spectral identity $\sigma_{\mathcal{B}(\ell^2(X))}(A) = \sigma_{\mathcal{A}}(A)$ holds true for all $A \in \mathcal{A}$, and the holomorphic functional calculi associated with $\mathcal{A}$ and $\mathcal{B}(\ell^2(X))$ coincide (see Remark 3). In particular, if $A \leq B$ denote the Riesz basis constants of φ, then $\sigma_{\mathcal{A}}(G_\varphi) = \sigma_{\mathcal{B}(\ell^2(X))}(G_\varphi) \subseteq [A, B]$. Thus, there exists a closed path Γ contained in the right half plane $\mathrm{Re}(z) > 0$ with $\sigma_{\mathcal{B}(\ell^2(X))}(G_\varphi) = \sigma_{\mathcal{A}}(G_\varphi) \subset \mathrm{int}(\Gamma)$. Since $f(z) = z^\alpha$ is holomorphic on $\mathrm{Re}(z) > 0$, the holomorphic functional calculus yields $G_\varphi^\alpha \in \mathcal{A}$. □

Corollary 24. *Let φ be an intrinsically $\mathcal{A}$-localized Riesz basis in $\mathcal{H}$. Then both $G_{S_\varphi^{-1/4}\varphi}$ and $G_{S_\varphi^{-3/4}\varphi}$ are contained in $\mathcal{A}$.*

Proof. Since φ is a Riesz basis, $(S_\varphi^{-1/2}\varphi_k)_{k\in X}$ is an orthonormal basis. By a simple calculation we see that $\left(G_{S_\varphi^{-1/4}\varphi}\right)^2 = G_\varphi$. Thus, by Lemma 23, $G_{S_\varphi^{-1/4}\varphi} = G_\varphi^{1/2} \in \mathcal{A}$. Similarly, we have $(G_{S_\varphi^{-3/4}\varphi})^2 = G_{\widetilde{\varphi}} \in \mathcal{A}$ due to Theorem 6, which, by Lemma 23, implies $G_{S_\varphi^{-3/4}\varphi} \in \mathcal{A}$. □

3.2. **Localization of the R-dual sequence.** One main ingredient in Theorem 1 is the family ω, which, in the language of [33], is a so-called *R-dual of type III* of ψ.

Let $(\psi_k)_{k\in X}$ be a Bessel sequence in $\mathcal{H}$ and let $\varphi = (\varphi_k)_{k\in X}$ be a Riesz basis in $\mathcal{H}$. Then $(S_\varphi^{-1/2}\varphi_k)_{k\in X}$ is an orthonormal basis in $\mathcal{H}$ [12]. Consequently,

$$\omega_k = \sum_{l\in X} \langle \psi_l, \varphi_k\rangle S_\varphi^{-1/2}\varphi_l$$

converges in $\mathcal{H}$ for each $k \in X$, defining a family $\omega = (\omega_k)_{k\in X}$ of vectors in $\mathcal{H}$. One important observation for proving proving Theorem 1 is, that $\omega \sim_{\mathcal{A}} \varphi$ whenever $\psi \sim_{\mathcal{A}} \varphi$, see below.

Lemma 25. *Let $\mathcal{A}$ be a spectral algebra and $\varphi = (\varphi_k)_{k\in X}$ be a Riesz basis in $\mathcal{H}$ that satisfies $\varphi \sim_{\mathcal{A}} \varphi$. Let $\psi = (\psi_k)_{k\in X}$ be a sequence in $\mathcal{H}$ satisfying $\psi \sim_{\mathcal{A}} \varphi$ and let $\omega = (\omega_k)_{k\in X}$ be given by*

$$\omega_k = \sum_{l\in X} \langle \psi_l, \varphi_k\rangle S_\varphi^{-1/2}\varphi_l \qquad (k \in X).$$

Then $\omega \sim_{\mathcal{A}} \varphi$, $\omega \sim_{\mathcal{A}} \widetilde{\varphi}$ and $\omega \sim_{\mathcal{A}} \omega$. In particular, ω is a Bessel sequence in $\mathcal{H}$.

Proof. First, note that ψ is a Bessel sequence, since $G_\psi \in \mathcal{A} \subset \mathcal{B}(\ell^2(X))$ according to Lemma 7, whence ω is a well-defined sequence in $\mathcal{H}$. Next, recall from Corollary 24 that $G_{S_\varphi^{-1/4}\varphi} \in \mathcal{A}$. Since

$$\langle \varphi_n, \omega_k\rangle = \sum_{l\in X} \langle \varphi_k, \psi_l\rangle \left\langle S_\varphi^{-1/4}\varphi_n, S_\varphi^{-1/4}\varphi_l\right\rangle = \sum_{l\in X} [G_{\psi,\varphi}^*]_{k,l} \left[G_{S_\varphi^{-1/4}\varphi}\right]_{l,n}$$

for all $k, n \in X$, we see that $G_{\omega,\varphi} = G_{\psi,\varphi}^* G_{S_\varphi^{-1/4}\varphi} \in \mathcal{A}$.

Similarly, since $G_{S_\varphi^{-3/4}\varphi} \in \mathcal{A}$ and

$$\langle \widetilde{\varphi}_n, \omega_k\rangle = \sum_{l\in X} \langle \varphi_k, \psi_l\rangle \left\langle S_\varphi^{-3/4}\varphi_n, S_\varphi^{-3/4}\varphi_l\right\rangle = \sum_{l\in X} [G_{\psi,\varphi}^*]_{k,l} \left[G_{S_\varphi^{-3/4}\varphi}\right]_{l,n}$$

for all $k, n \in X$, we obtain $G_{\omega,\widetilde{\varphi}} = G_{\psi,\varphi}^* G_{S_\varphi^{-3/4}\varphi} \in \mathcal{A}$.

Finally, $G_\omega = G_{\omega,\widetilde{\varphi}} G_{\varphi,\omega} \in \mathcal{A}$, hence $G_\omega \in \mathcal{B}(\ell^2(X))$, which implies that ω is a Bessel sequence in $\mathcal{H}$. □

As a consequence, all results appearing in the previous subsection apply to ω, which will be crucial for proving Theorem 1.

3.3. **Complex Interpolation.** Computing the complex interpolation spaces for a couple of closed subspaces $(X_0, X_1) \subset (L^{p_0}(\Omega), L^{p_1}(\Omega))$ is nontrivial and solutions for certain special cases rely, for example, on the complemented subspace property [35, Theorem 1.17.1]. For our purposes it will however be sufficient to observe that the complex interpolation spaces $(X_0, X_1)_\theta$, $\theta \in (0,1)$ are continuously embedded in $L^{p_\theta}(\Omega)$, $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. The proof is essentially a repetition of one direction of the proof of the standard complex interpolation result for $L^p(\Omega)$ spaces. For the convenience of the reader we repeat it here and refer to [9] for a detailed exposition on interpolation spaces.

Lemma 26. *Let (Ω, μ) be a measure space with nonnegative measure μ, $L^p(\Omega) = L^p(\Omega, \mu)$, and $X_0 \subset L^{p_0}(\Omega)$ and $X_1 \subset L^{p_1}(\Omega)$, $1 \leq p_0, p_1 \leq \infty$, be closed subspaces that form a compatible couple (X_0, X_1) with $X_0 \subset X_1$. Moreover, let $\theta \in (0,1)$ and $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Then $L^{p_\theta}(\Omega) \cap X_0 \hookrightarrow (X_0, X_1)_\theta$, that is*

$$\|f\|_\theta \leq \|f\|_{L^{p_\theta}(\Omega)}, \qquad f \in X_0.$$

Proof. First, note that $X_0 \cap X_1 = X_0$ and $X_0 + X_1 = X_1$. Define $\mathcal{F}(X_0, X_1)$ as the space of X_1-valued functions that are holomorphic on the strip $S = \{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$, continuous on its closure and satisfy $\{F(it) : \ t \in \mathbb{R}\} \subset X_0$, as well as $\{F(1+it) : \ t \in \mathbb{R}\} \subset X_1$. When equipped with the norm

$$\|F\|_{\mathcal{F}(X_0,X_1)} := \max\Big\{ \sup_{t\in\mathbb{R}} \|F(it)\|_{X_0}, \sup_{t\in\mathbb{R}} \|F(1+it)\|_{X_1} \Big\},$$

$\mathcal{F}(X_0, X_1)$ is a Banach space. The complex interpolation space $(X_0, X_1)_\theta$, $\theta \in (0,1)$, is then given by

$$(X_0, X_1)_\theta = \big\{ f \in X_1 : \ f = F(\theta), \ F \in \mathcal{F}(X_0, X_1) \big\}$$

equipped with the norm

$$\|f\|_\theta = \inf \big\{ \|F\|_{\mathcal{F}(X_0,X_1)} : f = F(\theta), \ F \in \mathcal{F}(X_0, X_1) \big\}.$$

Let $f \in X_0$. Since $X_0 \subset L^{p_0}(\Omega) \cap L^{p_1}(\Omega)$ we may without loss of generality assume that $\|f\|_{L^{p_\theta}(\Omega)} = 1$ and choose $F^*(z) = |f|^{p_\theta/p(z)-1} f$, where $\frac{1}{p(z)} = \frac{1-z}{p_0} + \frac{z}{p_1}$. Clearly $F^* \in \mathcal{F}(X_0, X_1)$, $F^*(\theta) = f$, and

$$\begin{aligned}
\|f\|_\theta &\leq \|F^*\|_{\mathcal{F}(X_0,X_1)} \\
&= \max\Big\{ \sup_{t\in\mathbb{R}} \big\| |f|^{p_\theta(\frac{1-it}{p_0} + \frac{it}{p_1})-1} f \big\|_{L^{p_0}(\Omega)}, \sup_{t\in\mathbb{R}} \big\| |f|^{p_\theta(-\frac{it}{p_0} + \frac{1+it}{p_1})-1} f \big\|_{L^{p_1}(\Omega)} \Big\} \\
&= \max\Big\{ \|f\|^{1/p_0}_{L^{p_\theta}(\Omega)}, \|f\|^{1/p_1}_{L^{p_\theta}(\Omega)} \Big\} = 1 = \|f\|_{L^{p_\theta}(\Omega)}.
\end{aligned}$$

□

4. Proof of Theorem 1

Proof. We show the equivalences of Theorem 1 via a chain of implications that is depicted in Figure 1. Before we start, observe that by Lemma 25 as well as Lemma 18 and Lemma 19, all operators occurring in the statements **(2)**–**(9)** are bounded on their respective domains.

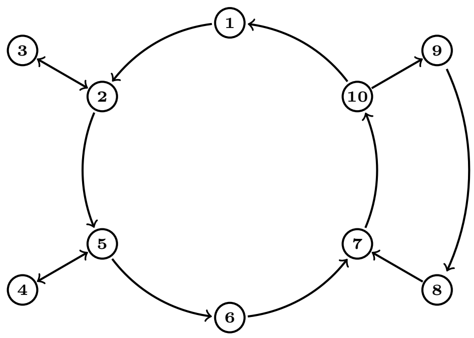

FIGURE 1. Graph showing which implications of the equivalent statements in Theorem 1 are shown.

(1)⇒(2): The assumptions imply that ψ is an intrinsically $\mathcal{A}$-localized frame, since $G_\psi = G_{\psi,\varphi} G_{\widetilde{\varphi}} G_{\varphi,\psi} \in \mathcal{A}$. Consequently, by Theorem 14, S_ψ is bounded and invertible on $\mathcal{H}^1(\psi)$. By Remark 9 it holds that $\mathcal{H}^1(\psi) = \mathcal{H}^1(\varphi)$ with equivalent norms. Hence (2) follows.

(2)⇔(3): This follows from duality via Corollary 22 (i).

(2)⇒(5): Since $S_\psi = D_\psi C_\psi : \mathcal{H}^1(\varphi) \to \mathcal{H}^1(\varphi)$ is bijective, $D_\psi : \ell^1(X) \to \mathcal{H}^1(\varphi)$ has to be surjective and $C_\psi : \mathcal{H}^1(\varphi) \to \ell^1(X)$ injective.

To prove the remaining part of the statement, let us assume to the contrary that $\mathrm{Ran}_1(C_\psi)$ is not closed in $\ell^1(X)$. Then $D_\psi : \overline{\mathrm{Ran}_1(C_\psi)} \to \mathcal{H}^1(\varphi)$ cannot be injective since $D_\psi : \mathrm{Ran}_1(C_\psi) \to \mathcal{H}^1(\varphi)$ is necessarily bijective. Therefore, we may assume that there exists some $c \in \overline{\mathrm{Ran}_1(C_\psi)} \setminus \{0\}$ such that $D_\psi c = 0$ and $\|c\|_{\ell^1(X)} = 1$. Now let $(c^{(n)})_{n\in\mathbb{N}} \subset \mathrm{Ran}_1(C_\psi)$ be a sequence with $\|c^{(n)}\|_{\ell^1(X)} = 1$ for each $n \in \mathbb{N}$, which converges in $\ell^1(X)$ to c. By continuity of D_ψ (Lemma 19) we obtain that $D_\psi c^{(n)} \to 0$ in $\mathcal{H}^1(\varphi)$ as $n \to \infty$. If $(f_n)_{n\in\mathbb{N}} \subset \ell^1(X)$ is the sequence with $C_\psi f_n = c^{(n)}$ (for each $n \in \mathbb{N}$), then we see that

$$\lim_{n\to\infty} \|f_n\|_{\mathcal{H}^1(\varphi)} \lesssim \lim_{n\to\infty} \|S_\psi f_n\|_{\mathcal{H}^1(\varphi)} = \lim_{n\to\infty} \|D_\psi c^{(n)}\|_{\mathcal{H}^1(\varphi)} = 0.$$

Since $G_{\psi,\varphi} \in \mathcal{A} \subset \mathcal{B}(\ell^1(X))$, this finally implies that

$$\lim_{n\to\infty} \|c^{(n)}\|_{\ell^1(X)} = \lim_{n\to\infty} \|G_{\psi,\varphi} C_{\widetilde{\varphi}} f_n\|_{\ell^1(X)} \lesssim \lim_{n\to\infty} \|f_n\|_{\mathcal{H}^1(\varphi)} = 0,$$

a contradiction. Consequently, $\mathrm{Ran}_1(C_\psi)$ must be closed in $\ell^1(X)$.

(4)⇔(5): By duality, Lemma 20 and the closed range theorem, $C_\psi : \mathcal{H}^\infty(\varphi) \to \ell^\infty(X)$ is injective and has closed range if and only if $D_\psi : \ell^1(X) \to \mathcal{H}^1(\varphi)$ is surjective. By analogous reasoning via Lemma 21, $\mathrm{Ran}_1(C_\psi)$ is closed if and only $\mathrm{Ran}_\infty(D_\psi)$ is closed.

(5)⇒(6): First, we show that $D_\omega : \ell^\infty \to \mathcal{H}^\infty(\varphi)$ is injective. By Theorem 14, $S_\varphi \in \mathcal{B}(\mathcal{H}^\infty(\varphi))$ is bounded and invertible, i.e., 0 is contained in the resolvent set of S_φ with respect to $\mathcal{B}(\mathcal{H}^\infty(\varphi))$. Since $z \mapsto z^{1/2}$ is holomorphic on any open subset $U \subset \mathbb{C}$ that does not contain the origin, we may deduce from the holomorphic functional calculus that $S_\varphi^{1/2} \in \mathcal{B}(\mathcal{H}^\infty(\varphi))$ is bounded and invertible as well. Now let $c \in \ell^\infty(X)$ be such that $0 = D_\omega c$ in $\mathcal{H}^\infty(\varphi)$. By definition, this means that

$\lim_{n\to\infty}\langle [D_\omega c]_n, \widetilde{\varphi}_k\rangle = 0$ for each $k \in X$. Letting $F_1 \subseteq F_2 \subseteq \dots$ be a nested sequence of finite subsets of X with $\bigcup_{n=1}^\infty F_n = X$, and recalling from Corollary 24 that $G_{S_\varphi^{-3/4}\varphi} \in \mathcal{A} \subseteq \mathcal{B}(\ell^\infty(X))$, we see that

$$
\begin{aligned}
0 &= \lim_{n\to\infty} \left\langle \sum_{l\in F_n} c_l \omega_l, \widetilde{\varphi}_k \right\rangle \\
&= \sum_{l\in X} c_l \left\langle \sum_{m\in X} \langle \psi_m, \varphi_l\rangle S_\varphi^{-1/2}\varphi_m, \widetilde{\varphi}_k \right\rangle \\
&= \sum_{m\in X} \langle S_\varphi^{-1/2}\varphi_m, \widetilde{\varphi}_k\rangle \sum_{l\in X} c_l \langle \psi_m, \varphi_l\rangle \\
&= \sum_{m\in X} \left[G_{S_\varphi^{-3/4}\varphi}\right]_{k,m} \left[\overline{C_\psi} D_\varphi \overline{c}\right]_m \qquad \text{(for each } k \in X\text{)},
\end{aligned} \tag{12}
$$

where $\overline{C_\psi} : \mathcal{H}^\infty(\varphi) \to \ell^\infty(X)$ is defined by $\overline{C_\psi} f = \left(\overline{[C_\psi f]_k}\right)_{k\in X}$. Note that we are allowed to interchange the order of summation since

$$
\sup_{k\in X} \sum_{m\in X} \left|G_{S_\varphi^{-3/4}\varphi}\right|_{k,m} \cdot \sup_{s\in X} \sum_{l\in X} \left|G_{\varphi,\psi}\right|_{s,l} < \infty,
$$

which follows from $G_{S_\varphi^{-3/4}\varphi}, G_{\varphi,\psi} \in \mathcal{A} \subseteq \mathcal{B}(\ell^\infty(X))$. From (12) we know that $G_{S_\varphi^{-3/4}\varphi}\overline{C_\psi} D_\varphi \overline{c} = 0$ in $\ell^\infty(X)$. By Lemma 16, $G_{\widetilde{\varphi}} \in \mathcal{A} \subset \mathcal{B}(\ell^\infty(X))$ is invertible. Arguing as in the proof of Lemma 25, we apply the holomorphic functional calculus to show that $G_{S_\varphi^{-3/4}\varphi} = (G_{\widetilde{\varphi}})^{1/2} \in \mathcal{A} \subset \mathcal{B}(\ell^\infty(X))$ is invertible as well. Consequently, $\overline{C_\psi} D_\varphi \overline{c} = 0$. By **(4)⇔(5)**, $C_\psi : \mathcal{H}^\infty(\varphi) \to \ell^\infty(X)$ is injective, which is equivalent to $\overline{C_\psi} : \mathcal{H}^\infty(\varphi) \to \ell^\infty(X)$ being injective. Consequently, $D_\varphi \overline{c} = 0$. Finally, an application of Corollary 17 yields $c = 0$.

Next, we show that $\mathrm{Ran}_\infty(D_\omega)$ is closed. Let $(D_\omega c^{(m)})_{m\in\mathbb{N}} \subset \mathcal{H}^\infty(\varphi)$ be a sequence in $\mathrm{Ran}_\infty(D_\omega)$ which converges in $\mathcal{H}^\infty(\varphi)$ to some $f = \left[(f_n)_{n\in\mathbb{N}}\right]_{\sim_\varphi} \in \mathcal{H}^\infty(\varphi)$. In particular, for each $m \in \mathbb{N}$, we have by definition that $D_\omega c^{(m)} = \left[\left(\sum_{l\in F_n^{(m)}} c_l^{(m)} \omega_l\right)_{n\in\mathbb{N}}\right]_{\sim_\varphi}$, where $F_1^{(m)} \subseteq F_2^{(m)} \subseteq \dots$ is a nested sequence of finite subsets of X so that $X = \bigcup_{n\in F_n^{(m)}}$. By Lemma 19, we may assume without loss of generality that for each $n \in \mathbb{N}$, the sets $F_n^{(m)}$ $(m \in \mathbb{N})$ are equal. For simplicity, we write $F_n = F_n^{(m)}$. Thus we have

$$
\begin{aligned}
0 &= \lim_{m\to\infty} \left\| D_\omega c^{(m)} - f \right\|_{\mathcal{H}^\infty(\varphi)} \\
&= \lim_{m\to\infty} \sup_{k\in X} \lim_{n\to\infty} \left| \left\langle \sum_{l\in F_n} c_l^{(m)} \omega_l - f_n, \widetilde{\varphi}_k \right\rangle \right| \\
&= \lim_{m\to\infty} \sup_{k\in X} \lim_{n\to\infty} \left| \left\langle \sum_{l\in F_n} c_l^{(m)} \sum_{i\in X} \langle \psi_i, \varphi_l\rangle S_\varphi^{-1/2}\varphi_i - f_n, \widetilde{\varphi}_k \right\rangle \right| \\
&= \lim_{m\to\infty} \sup_{k\in X} \lim_{n\to\infty} \left| \sum_{i\in X} \sum_{l\in F_n} \langle \psi_i, \varphi_l\rangle \langle S_\varphi^{-1/2}\varphi_i, \widetilde{\varphi}_k\rangle c_l^{(m)} - [C_{\widetilde{\varphi}} f]_k \right|
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{m\to\infty} \sup_{k\in X} \left| \sum_{i\in X} \langle S_\varphi^{-1/2}\varphi_i, \widetilde{\varphi}_k\rangle \lim_{n\to\infty} \sum_{l\in F_n} \langle \psi_i, \varphi_l\rangle c_l^{(m)} - [C_{\widetilde{\varphi}} f]_k \right| \\
&= \lim_{m\to\infty} \sup_{k\in X} \left| \sum_{i\in X} \langle S_\varphi^{-1/2}\varphi_i, \widetilde{\varphi}_k\rangle \lim_{n\to\infty} \left\langle \psi_i, \left[D_\varphi \overline{c^{(m)}} \right]_n \right\rangle - [C_{\widetilde{\varphi}} f]_k \right| \\
&= \lim_{m\to\infty} \sup_{k\in X} \left| \sum_{i\in X} \langle S_\varphi^{-1/2}\varphi_i, \widetilde{\varphi}_k\rangle \left[\overline{C_\psi} D_\varphi \overline{c^{(m)}} \right]_i - [C_{\widetilde{\varphi}} f]_k \right| \\
&= \lim_{m\to\infty} \left\| C_{\widetilde{\varphi}} S_\varphi^{-1/2} D_\varphi \overline{C_\psi} D_\varphi \overline{c^{(m)}} - C_{\widetilde{\varphi}} f \right\|_{\ell^\infty(X)},
\end{aligned}
$$

where we applied the dominated convergence theorem in the fifth step. Since $C_{\widetilde{\varphi}} S_\varphi^{1/2} D_\varphi \in \mathcal{B}(\ell^\infty(X))$, the latter implies that

$$
0 = \lim_{m\to\infty} \left\| \overline{C_\psi} D_\varphi \overline{c^{(m)}} - C_{\widetilde{\varphi}} S_\varphi^{1/2} f \right\|_{\ell^\infty(X)}.
$$

This means that $\left(\overline{C_\psi} D_\varphi \overline{c^{(m)}}\right)_{m\in\mathbb{N}}$ is a sequence in $\mathrm{Ran}_\infty\left(\overline{C_\psi}\right)$ which converges in $\ell^\infty(X)$ to $C_{\widetilde{\varphi}} S_\varphi^{1/2} f \in \ell^\infty(X)$. Since $\overline{C_\psi}$ has closed range due to C_ψ having closed range by **(4)**$\Leftrightarrow$**(5)**, we must have $C_{\widetilde{\varphi}} S_\varphi^{1/2} f = \overline{C_\psi} h$ for some $h \in \mathcal{H}^\infty(\varphi)$. Since $D_\varphi : \ell^\infty(X) \to \mathcal{H}^\infty(\varphi)$ is surjective by Proposition 13, there exists some $\overline{c} \in \ell^\infty(X)$ with $h = D_\varphi \overline{c}$. Altogether, we obtain

$$
0 = \lim_{m\to\infty} \left\| \overline{C_\psi} D_\varphi \overline{c^{(m)}} - \overline{C_\psi} D_\varphi \overline{c} \right\|_{\ell^\infty(X)}.
$$

Since $C_{\widetilde{\varphi}} S_\varphi^{-1/2} D_\varphi \in \mathcal{B}(\ell^\infty(X))$, the latter yields

$$
0 = \lim_{m\to\infty} \left\| C_{\widetilde{\varphi}} S_\varphi^{-1/2}\ D_\varphi \overline{C_\psi} D_\varphi (\overline{c^{(m)}} - \overline{c}) \right\|_{\ell^\infty(X)}.
$$

By going through the calculations from above in reverse order, we arrive at

$$
0 = \lim_{m\to\infty} \left\| D_\omega c^{(m)} - D_\omega c \right\|_{\mathcal{H}^\infty(\varphi)},
$$

which yields the claim.

Finally, we show that $\mathrm{Ran}_\infty(C_\omega)$ is closed by applying a similar argument. By Lemma 20 and the closed range theorem, it suffices to prove that $D_\omega : \ell^1(X) \to \mathcal{H}^1(\varphi)$ has closed range. To this end, let $(D_\omega c^{(m)})_{m\in\mathbb{N}} \subset \mathcal{H}^1(\varphi)$ be a sequence in $\mathrm{Ran}_1(D_\omega)$ which converges in $\mathcal{H}^1(\varphi)$ to some $f \in \mathcal{H}^1(\varphi)$. Next, employing the holomorphic functional calculus with respect to $\mathcal{B}(\mathcal{H}^1(\varphi))$, we see that $S_\varphi^{1/2} \in \mathcal{B}(\mathcal{H}^1(\varphi))$ is bounded and invertible. Analogously to the previous part of the proof, we calculate

$$\begin{aligned}
0 &= \lim_{m\to\infty} \left\| D_\omega c^{(m)} - f \right\|_{\mathcal{H}^1(\varphi)} \\
&= \lim_{m\to\infty} \sum_{k\in X} \left| \left\langle \sum_{l\in X} c_l^{(m)} \omega_l, \widetilde{\varphi}_k \right\rangle - [C_{\widetilde{\varphi}} f]_k \right| \\
&= \lim_{m\to\infty} \sum_{k\in X} \left| \left\langle \sum_{l\in X} c_l^{(m)} \sum_{i\in X} \langle \psi_i, \varphi_l \rangle S_\varphi^{-1/2} \varphi_i, \widetilde{\varphi}_k \right\rangle - [C_{\widetilde{\varphi}} f]_k \right| \\
&= \lim_{m\to\infty} \sum_{k\in X} \left| \sum_{i\in X} \langle S_\varphi^{-1/2} \varphi_i, \widetilde{\varphi}_k \rangle \sum_{l\in X} \langle \psi_i, \varphi_l \rangle c_l^{(m)} - [C_{\widetilde{\varphi}} f]_k \right| \\
&= \lim_{m\to\infty} \left\| C_{\widetilde{\varphi}} S_\varphi^{-1/2} D_\varphi \overline{C_\psi} D_\varphi \overline{c^{(m)}} - C_{\widetilde{\varphi}} f \right\|_{\ell^1(X)} .
\end{aligned}$$

Continuity of $C_{\widetilde{\varphi}} S_\varphi^{1/2} D_\varphi \in \mathcal{B}(\ell^1(X))$ implies

$$0 = \lim_{m\to\infty} \left\| \overline{C_\psi} D_\varphi \overline{c^{(m)}} - C_{\widetilde{\varphi}} S_\varphi^{1/2} f \right\|_{\ell^1(X)} .$$

This means that $\left(\overline{C_\psi} D_\varphi \overline{c^{(m)}}\right)_{m\in\mathbb{N}}$ is a sequence in $\mathrm{Ran}_1\left(\overline{C_\psi}\right)$ which converges in $\ell^1(X)$ to $C_{\widetilde{\varphi}} S_\varphi^{1/2} f \in \ell^1(X)$. Since $\mathrm{Ran}_1\left(\overline{C_\psi}\right)$ is closed due to $\mathrm{Ran}_1\left(C_\psi\right)$ being closed, we must have $C_{\widetilde{\varphi}} S_\varphi^{1/2} f = \overline{C_\psi} h$ for some $h \in \mathcal{H}^1(\varphi)$. Since $D_\varphi : \ell^1(X) \to \mathcal{H}^1(\varphi)$ is surjective due to Proposition 13, there exists some $\overline{c} \in \ell^1(X)$ with $h = D_\varphi \overline{c}$. Altogether, we obtain

$$0 = \lim_{m\to\infty} \left\| \overline{C_\psi} D_\varphi \overline{c^{(m)}} - \overline{C_\psi} D_\varphi \overline{c} \right\|_{\ell^1(X)} .$$

Since $C_{\widetilde{\varphi}} S_\varphi^{-1/2} D_\varphi \in \mathcal{B}(\ell^1(X))$, the latter yields

$$0 = \lim_{m\to\infty} \left\| C_{\widetilde{\varphi}} S_\varphi^{-1/2}\ D_\varphi \overline{C_\psi} D_\varphi (\overline{c^{(m)}} - \overline{c}) \right\|_{\ell^1(X)} .$$

Finally, by going through the calculations from above in reverse order, we finally arrive at

$$0 = \lim_{m\to\infty} \left\| D_\omega c^{(m)} - D_\omega c \right\|_{\mathcal{H}^1(\varphi)} ,$$

as was to be shown.

(6)⇔(7): Lemma 25 ensures that we may apply Lemma 21 to deduce that the Banach space adjoint of $C_\omega : \mathcal{H}^1(\varphi) \to \ell^1(X)$ is $D_\omega : \ell^\infty(X) \to \mathcal{H}^\infty(\varphi)$. Therefore, $\mathrm{Ker}_\infty(D_\omega) = \{0\}$ if and only if $\mathrm{Ran}_1(C_\omega)$ is dense in $\ell^1(X)$. The equivalence of the conditions on the closedness of $\mathrm{Ran}_\infty(C_\omega)$ and $\mathrm{Ran}_1(D_\omega)$ as well as $\mathrm{Ran}_\infty(D_\omega)$ and $\mathrm{Ran}_1(C_\omega)$ is due to the closed range theorem.

(7)⇒(10): Since $\mathrm{Ran}_1(C_\omega) = \ell^1(X)$, it follows that $\mathrm{Ran}_1(C_\omega)$ is dense in $\ell^2(X)$ and so is $\mathrm{Ran}_2(C_\omega)$. We thus deduce that $\mathrm{Ker}_1(D_\omega) \subset \mathrm{Ker}_2(D_\omega) = \{0\}$, that is, $D_\omega : \ell^1(X) \to \mathrm{Ran}_1(D_\omega)$ is bijective. Since $\mathrm{Ran}_1(D_\omega)$ is closed, it follows from the bounded inverse theorem that there exist constants $A_1, B_1 > 0$ such that

$$A_1 \|c\|_{\ell^1(X)} \le \|D_\omega c\|_{\mathcal{H}^1(\varphi)} \le B_1 \|c\|_{\ell^1(X)}, \qquad c \in \ell^1(X). \tag{13}$$

Next, observe that by **(6)⇔(7)**, $D_\omega : \ell^\infty(X) \to \mathcal{H}^\infty(\varphi)$ is injective and has closed range. We may therefore apply the bounded inverse theorem again to obtain constants $A_\infty, B_\infty > 0$ such that

$$A_\infty \|c\|_{\ell^\infty(X)} \leq \|D_\omega c\|_{\mathcal{H}^\infty(\varphi)} \leq B_\infty \|c\|_{\ell^\infty(X)}, \qquad c \in \ell^\infty(X). \tag{14}$$

Since $C_\omega : \mathcal{H}^1(\varphi) \to \ell^1(X)$ is surjective, there exists $(\gamma_k)_{k\in X} \subset \mathcal{H}^1(\varphi)$ such that

$$\langle \gamma_k, \omega_l \rangle = \delta_{k,l}, \qquad k, l \in X.$$

Applying (13) and (14) respectively allows us to infer that for $p \in \{1, \infty\}$

$$B_p^{-1} \|D_\omega c\|_{\mathcal{H}^p(\varphi)} \leq \|c\|_{\ell^p(X)} = \|C_\gamma D_\omega c\|_{\ell^p(X)} = \|c\|_{\ell^p(X)} \leq A_p^{-1} \|D_\omega c\|_{\mathcal{H}^p(\varphi)}.$$

In particular,

$$\|C_\gamma f\|_{\ell^p(X)} \leq A_p^{-1} \|f\|_{\mathcal{H}^p(\varphi)}, \qquad \text{for every } f \in \mathrm{Ran}_p(D_\omega), \text{ and } p \in \{1, \infty\}.$$

Note that $\big(\mathrm{Ran}_1(D_\omega), \mathrm{Ran}_\infty(D_\omega)\big)$ forms a compatible pair of Banach spaces with $\mathrm{Ran}_1(D_\omega) \subset \mathrm{Ran}_\infty(D_\omega)$. It follows from complex interpolation that

$$\|C_\gamma f\|_{\ell^{p_\theta}(X)} \leq A_1^{\theta-1} A_\infty^{-\theta} \|f\|_\theta, \qquad \text{for every } f \in \mathrm{Ran}_1(D_\omega), \text{ and } 0 \leq \theta \leq 1.$$

Since $\mathrm{Ran}_p(D_\omega)$ can be identified with a closed subspace of $\ell^p(X)$ via $C_{\widetilde{\varphi}}$, we may apply Lemma 26 to deduce that

$$\|f\|_\theta = \|C_{\widetilde{\varphi}} f\|_\theta \leq \|C_{\widetilde{\varphi}} f\|_{\ell^{p_\theta}(X)} = \|f\|_{\mathcal{H}^{p_\theta}(\varphi)}.$$

Specifying $\theta = 1/2$ we have $p_{1/2} = 2$ and applying a density argument then yields

$$\|C_\gamma f\|_{\ell^2(X)} \lesssim \|f\|_{\mathcal{H}}, \qquad f \in \mathrm{Ran}_2(D_\omega).$$

Finally,

$$\|c\|_{\ell^2(X)} = \|C_\gamma D_\omega c\|_{\ell^2(X)} \lesssim \|D_\omega c\|_{\mathcal{H}} \lesssim \|c\|_{\ell^2(X)}, \qquad c \in \ell^2(X),$$

where the inequality on the right hand side is a consequence of Lemma 25. In other words, ω is a Riesz sequence in $\mathcal{H}$.

(10)⇔(1): Note that $\big(S_\varphi^{1/2}\psi_k\big)_{k\in X}$ is a Bessel sequence because $(\psi_k)_{k\in X}$ is a Bessel sequence by Lemma 18 and $S_\varphi^{1/2} \in \mathcal{B}(\mathcal{H})$. Since $S_\varphi^{1/2}$ is invertible (see, e.g., [12]), we may rewrite ω_k as

$$\omega_k = \sum_{l\in X} \langle S_\varphi^{1/2}\psi_l, S_\varphi^{-1/2}\varphi_k \rangle S_\varphi^{-1/2}\varphi_l,$$

which means that $(\omega_k)_{k\in X}$ is an *R-dual* sequence of $\big(S_\varphi^{1/2}\psi_k\big)_{k\in X}$ in the sense of [11]. By [11, Theorem 2] we know that $(\omega_k)_{k\in X}$ is a Riesz sequence if and only if $\big(S_\varphi^{1/2}\psi_k\big)_{k\in X}$ is a frame, which in turn is equivalent to $(\psi_k)_{k\in X}$ being a frame, since $S_\varphi^{1/2} \in \mathcal{B}(\mathcal{H})$ is invertible.

(10)⇒(9): This implication follows directly from Lemma 16, since we have that $\omega \sim_{\mathcal{A}} \omega$ by Lemma 25.

(9)⇔(8): This follows from duality via Corollary 22 (ii).

(8)⇒(7): Since $G_\omega = C_\omega D_\omega : \ell^1(X) \to \ell^1(X)$ is invertible, it follows that $C_\omega : \mathcal{H}^1(\varphi) \to \ell^1(X)$ is surjective.

To prove the second statement, let us assume to the contrary that $\mathrm{Ran}_1(D_\omega)$ is not closed in $\mathcal{H}^1(\varphi)$. Then $C_\omega : \overline{\mathrm{Ran}_1(D_\omega)} \to \ell^1(X)$ cannot be injective. Indeed, $C_\omega : \mathrm{Ran}_1(D_\omega) \to \ell^1(X)$ is necessarily bijective as G_ω is invertible. We may therefore assume that there exists some $f \in \overline{\mathrm{Ran}_1(D_\omega)} \setminus \{0\}$ such that $C_\omega f = 0$ and $\|f\|_{\mathcal{H}^1(\varphi)} = 1$. Now let $\{f_n\}_{n=1}^\infty \subset \mathrm{Ran}_1(D_\omega)$ be a unit norm sequence that converges to f in $\mathcal{H}^1(\varphi)$. From the continuity of C_ω (Lemma 18) we deduce that $C_\omega f_n \to 0$ as $n \to \infty$. Let $(c^{(n)})_{n\in\mathbb{N}} \subset \ell^1(X)$ be a sequence which satisfies $D_\omega c^{(n)} = f_n$ for each $n \in \mathbb{N}$. Then

$$\|c^{(n)}\|_{\ell^1(X)} \lesssim \|G_\omega c^{(n)}\|_{\ell^1(X)} = \|C_\omega f_n\|_{\ell^1(X)} \to 0 \qquad (\text{as } n \to \infty).$$

This implies that

$$\|f_n\|_{\mathcal{H}^1(\varphi)} = \|D_\omega c^{(n)}\|_{\mathcal{H}^1(\varphi)} \lesssim \|c^{(n)}\|_{\ell^1(X)} \to 0 \qquad (\text{as } n \to \infty),$$

which contradicts $\|f_n\|_{\mathcal{H}^1(\varphi)} = 1$ for all $n \in \mathbb{N}$.

Finally, the last claim of Theorem 1 follows from Remark 9. □

5. A Comparison to "Gabor Frames Without Inequalities"

As already mentioned in the introduction, the main insight of [21] (explicitly stated as a corollary therein) was that, for a window function g in the modulation space $M^1(\mathbb{R}^d)$, the Gabor system $\mathcal{G} = \mathcal{G}(g, \Lambda)$ over a full-rank lattice $\Lambda \subset \mathbb{R}^{2d}$ being a frame for $L^2(\mathbb{R}^d)$ is equivalent to the injectivity of the analysis operator $C_\mathcal{G} : M^\infty(\mathbb{R}^d) \to \ell^\infty(\Lambda)$. In contrast to Theorem 1, no closed range condition is listed in this or any other of the conditions from [21, Theorem 3.1]. This is due to the fact that they are automatically fulfilled in this setting, as the next example illustrates.

Example 27. *For $g \in M^1(\mathbb{R}^d)$ and $\Lambda \subset \mathbb{R}^{2d}$ a full-rank lattice, the following are equivalent:*

(1) $\mathcal{G} = \mathcal{G}(g, \Lambda)$ is a Gabor frame for $L^2(\mathbb{R}^d)$.
(2) $C_\mathcal{G} : M^\infty(\mathbb{R}^d) \to \ell^\infty(\Lambda)$ is injective and has closed range.
(3) $C_\mathcal{G} : M^\infty(\mathbb{R}^d) \to \ell^\infty(\Lambda)$ is injective.

Indeed, (2)⇒(3) is trivial, and (3)⇒(1) was proven in [21]. *To show (1)⇒(2), we note that due to $g \in M^1(\mathbb{R}^d)$, $S_\mathcal{G} = D_\mathcal{G} C_\mathcal{G}$ is bounded and invertible on $M^p(\mathbb{R}^d)$ for all $1 \le p \le \infty$* [21, Theorem 3.1]. *From the latter, we may deduce the desired closed range condition analogously as in the proof of Theorem 1.*

In a similar manner, other closed range conditions from [21, Theorem 3.1] *can be derived.*

At the same time, however, all closed range conditions appearing in Theorem 1 are in fact necessary. To demonstrate this, we "naively" transfer the items (iii)-(vi), (ix), (x) and (xiii) from [21, Theorem 3.1] to our setting, obtaining the following list of conditions:

(a) $S_\psi : \mathcal{H}^\infty(\varphi) \to \mathcal{H}^\infty(\varphi)$ is injective,
(b) $C_\psi : \mathcal{H}^\infty(\varphi) \to \ell^\infty(X)$ is injective,
(c) $D_\psi : \ell^1(X) \to \mathcal{H}^1(\varphi)$ has dense range,
(d) $D_\omega : \ell^\infty(X) \to \mathcal{H}^\infty(\varphi)$ is injective,
(e) $C_\omega : \mathcal{H}^1(\varphi) \to \ell^1(X)$ has dense range,
(f) $G_\omega : \ell^\infty(X) \to \ell^\infty(X)$ is injective.

Below, we provide an elementary example of families ψ and ω in $\mathcal{H}$, that are localized with respect to a self-localized Riesz basis φ, satisfy all the conditions (a)−(f), but are not a frame, respectively not a Riesz sequence.

Example 28. *Let $X = \mathbb{N}$, $\mathcal{A}$ be any solid spectral algebra, φ be an $\mathcal{A}$-localized Riesz basis (for instance, an orthonormal basis) and define $\psi = (\psi_k)_{k\in\mathbb{N}} := \big(\frac{1}{k}\widetilde{\varphi}_k\big)_{k\in\mathbb{N}}$. Then $\omega_k = \frac{1}{k}S_\varphi^{-1/2}\varphi_k$ for each $k \in \mathbb{N}$, and the following hold:*

- *(i) One has $\psi \sim_{\mathcal{A}} \varphi$ and, consequently, $\omega \sim_{\mathcal{A}} \varphi$;*
- *(ii) ψ is not a frame and, consequently, ω not a Riesz sequence in $\mathcal{H}$;*
- *(iii) Items (a)-(f) from above are satisfied;*
- *(iv) The range of $C_\omega : \mathcal{H}^1(\varphi) \to \ell^1(\mathbb{N})$ is not closed in $\ell^1(\mathbb{N})$.*

Proof. (*i*) It holds $\psi \sim_{\mathcal{A}} \varphi$, because $G_{\psi,\varphi} = \big[\frac{1}{k}\delta_{k,l}\big]_{k,l\in\mathbb{N}}$ is contained in $\mathcal{A}$ due to solidity of $\mathcal{A}$, since $\mathcal{I}_{\mathcal{A}} = [\delta_{k,l}]_{k,l\in\mathbb{N}} \in \mathcal{A}$. The fact that $\omega \sim_{\mathcal{A}} \varphi$ is shown analogously or via Lemma 25.

(*ii*) It is easy to see, that ψ is *not* a frame, hence ω cannot be a Riesz sequence according to Theorem 1.

(*iii*) The Gram matrix $G_\omega = \big[\frac{1}{k^2}\delta_{k,l}\big]_{k,l\in\mathbb{N}} : \ell^\infty(\mathbb{N}) \to \ell^\infty(\mathbb{N})$ is obviously injective, that is, (f) is satisfied. Consequently, also (d) has to be satisfied, and (e) follows from (d) via duality upon an application of Lemma 21.

Next, we show that (a) is satisfied. Let $f = [(f_n)_{n\in\mathbb{N}}]_{\sim_\varphi}, g = [(g_n)_{n\in\mathbb{N}}]_{\sim_\varphi} \in \mathcal{H}^\infty(\varphi)$ and assume that $S_\psi(f-g) = \sum_{l\in\mathbb{N}} \frac{1}{l^2}\langle f-g, \widetilde{\varphi}_l\rangle\widetilde{\varphi}_l = 0$. Then $0 = C_\varphi S_\psi(f-g) = \big(\frac{1}{k^2}\langle f-g, \widetilde{\varphi}_k\rangle\big)_{k\in\mathbb{N}}$. Thus $0 = \langle f-g, \widetilde{\varphi}_k\rangle = \lim_{n\to\infty}\langle f_n - g_n, \widetilde{\varphi}_k\rangle$ for each $k \in \mathbb{N}$ and in particular $(f_n)_{n\in\mathbb{N}} \sim_\varphi (g_n)_{n\in\mathbb{N}}$. This means that $f - g = 0 \in \mathcal{H}^\infty(\varphi)$ and (a) is shown. Statement (b) follows immediately from (a), and (c) follows from (b) via duality and Lemma 21.

(*iv*) Since we have already shown that $C_\omega : \mathcal{H}^1(\varphi) \to \ell^1(\mathbb{N})$ has dense range, it suffices to show that $C_\omega : \mathcal{H}^1(\varphi) \to \ell^1(\mathbb{N})$ is not surjective. But the latter is clear, since $C_\omega = \big(\bigoplus_{k\in\mathbb{N}}\frac{1}{k}\big)\, C_\varphi S_\varphi^{-1/2}$, where $C_\varphi S_\varphi^{-1/2} : \mathcal{H}^1(\varphi) \to \ell^1(\mathbb{N})$ is bijective (this follows from Corollary 17), hence $\mathrm{Ran}_1(C_\omega) = \{(c_k)_{k\in\mathbb{N}} : c_k = a_k/k, (a_k)_{k\in\mathbb{N}} \in \ell^1(X)\}$ which is not equal to $\ell^1(X)$. $\square$

Remark 29. *Via duality (Lemma 21) and the closed range theorem, we may conclude from Example 28 (iv), that $D_\omega : \ell^\infty(\mathbb{N}) \to \mathcal{H}^\infty(\varphi)$ does not have closed range as well. Furthermore, arguing as in the proof of* ***(5)⇔(6)****, one may show that $D_\omega : \ell^\infty(\mathbb{N}) \to \mathcal{H}^\infty(\varphi)$ having closed range is actually equivalent to $C_\psi : \mathcal{H}^\infty(\varphi) \to \ell^\infty(\mathbb{N})$ having closed range. Thus $C_\psi : \mathcal{H}^\infty(\varphi) \to \ell^\infty(\mathbb{N})$ does not have closed range in the above example.*

Similarly, by switching the roles of ψ and ω in Example 28, one readily sees that it is possible to find ψ and ω which are not a frame (respectively a Riesz sequence), violate the remaining closed range conditions appearing in Theorem 1, while meeting all other conditions appearing in the same result.

Thus, each of the closed range conditions appearing in Theorem 1 is necessary for the statement to remain true.

6. Shift-Invariant Spaces

In this section, we apply Theorem 1 to the setting of shift-invariant systems. For a comprehensive overview to shift-invariant spaces, we refer the reader to [2].

Let T_k denote the translation operator on $L^p(\mathbb{R}^d)$ ($k \in \mathbb{R}^d$), given by $T_k f(t) = f(t-k)$. For a generator $\varphi \in L^2(\mathbb{R}^d)$, let $V^p(\varphi)$ be the *shift-invariant space* (formally) defined by

$$V^p(\varphi) = \left\{ \sum_{k\in\mathbb{Z}^d} c_k T_k \varphi : \; c \in \ell^p(\mathbb{Z}^d) \right\} \qquad (1 \leq p \leq \infty).$$

We assume that

(i) φ is continuous;
(ii) $|\varphi(x)| \lesssim (1+|x|)^{-s}$ for some $s > d$; and
(iii) $(T_k\varphi)_{k\in\mathbb{Z}^d}$ forms a Riesz basis for $V^2(\varphi)$.

Then the assumptions (i)−(iii) ensure that $V^p(\varphi)$ is a closed subspace of $L^p(\mathbb{R}^d)$ $(1 \leq p \leq \infty)$, and that $V^2(\varphi)$ is a reproducing kernel Hilbert space with kernel K_x^φ, i.e.

$$f(x) = \langle f, K_x^\varphi \rangle, \qquad f \in V^2(\varphi), x \in \mathbb{R}^d. \tag{15}$$

It was shown in [19] that

$$|\langle T_k\varphi, T_l\varphi\rangle| \lesssim (1+|k-l|)^{-s}, \qquad k,l \in \mathbb{Z}^d,$$

i.e., the Riesz basis $(T_k\varphi)_{k\in\mathbb{Z}^d}$ is intrinsically $\mathcal{J}_s(\mathbb{Z}^d)$-localized, where $\mathcal{J}_s(\mathbb{Z}^d)$ denotes the Jaffard algebra from Example 4 (1). Furthermore, from [19] we also know that the associated co-orbit spaces $\mathcal{H}^p\big((T_k\varphi)_{k\in\mathbb{Z}^d}\big)$ coincide with the shift-invariant spaces $V^p(\varphi)$ $(1 \leq p \leq \infty)$ and

$$|\langle K_x^\varphi, T_l\varphi\rangle| = |\varphi(x-l)| \lesssim (1+|x-l|)^{-s}, \qquad x \in \mathbb{R}^d, l \in \mathbb{Z}^d. \tag{16}$$

Let $X = (x_k)_{k\in\mathbb{Z}^d} \subset \mathbb{R}^d$ be discrete. If $\psi = (\psi_k)_{k\in\mathbb{Z}^d} := (K_{x_k}^\varphi)_{k\in\mathbb{Z}^d}$ forms a frame for $V^2(\varphi)$ with frame bounds $0 < A \leq B$, then X is called a *stable set of sampling* for $V^2(\varphi)$, since, due to (15), the frame inequality (1) takes the form

$$A\|f\|^2_{L^2(\mathbb{R}^d)} \leq \sum_{k\in\mathbb{Z}^d} |f(x_k)|^2 \leq B\|f\|^2_{L^2(\mathbb{R}^d)}, \qquad f \in V^2(\varphi),$$

in this case. Similarly, X is called a *stable set of sampling* for $V^p(\varphi)$ $(1 \leq p \leq \infty)$, if there exist constants $A_p, B_p > 0$ such that

$$A_p\|f\|_{L^p(\mathbb{R}^d)} \leq \|(f(x_k))_{k\in\mathbb{Z}^d}\|_{\ell^p(\mathbb{Z}^d)} \leq B_p\|f\|_{L^p(\mathbb{R}^d)}, \qquad f \in V^p(\varphi).$$

Now, assume that $X = (x_k)_{k\in\mathbb{Z}^d} \subset \mathbb{R}^d$ is relatively separated and satisfies the mild perturbation condition

$$|x_k - k| \leq C, \qquad \text{for some } C > 0 \text{ and every } k \in \mathbb{Z}^d. \tag{17}$$

Then, since

$$(1+|k-l|)^s \leq (1+|k-x_k|)^s(1+|x_k-l|)^s, \qquad k,l \in \mathbb{Z}^d,$$

the estimate (16) implies

$$|\langle \psi_k, T_l\varphi\rangle| \lesssim (1+|k-x_k|)^s(1+|k-l|)^{-s} \lesssim (1+|k-l|)^{-s}, \qquad k,l \in \mathbb{Z}^d,$$

which means that ψ and $(T_k\varphi)_{k\in\mathbb{Z}^d}$ are mutually $\mathcal{J}_s(\mathbb{Z}^d)$-localized. Thus all assumptions of Theorem 1 are met in this setting and all of the conditions **(1)**−**(10)** provide necessary and sufficient conditions for such a set X being a stable set of sampling. Instead of repeating all these conditions, we will only reformulate conditions **(1)**, **(4)** and **(8)**−**(10)**, since they seem more insightful to us in this case than the remaining ones.

Before doing so, we compute the R-dual ω of ψ. Since the frame operator S_φ commutes with translations by $k \in \mathbb{Z}^d$, so does $S_\varphi^{-1/2}$, and we see that $S_\varphi^{-1/2} T_k \varphi = T_k S_\varphi^{-1/2} \varphi = T_k \gamma$ for $\gamma := S_\varphi^{-1/2} \varphi$ and $k \in \mathbb{Z}^d$. Consequently,

$$\omega_k = \sum_{l \in \mathbb{Z}^d} \langle \psi_l, T_k \varphi \rangle S_\varphi^{-1/2} T_l \varphi = \sum_{l \in \mathbb{Z}^d} \overline{\varphi(x_l - k)} T_l \gamma, \qquad k \in \mathbb{Z}^d. \tag{18}$$

Note, that in case $X = (x_k)_{k \in \mathbb{Z}^d} = \mathbb{Z}^d$, the R-dual ω of ψ is again a family of shifts of a single generator η as

$$\omega_k = T_k \sum_{l \in \mathbb{Z}^d} \overline{\varphi(l - k)} T_{l-k} \gamma = T_k \sum_{l \in \mathbb{Z}^d} \overline{\varphi(l)} T_l \gamma = T_k \eta, \qquad k \in \mathbb{Z}^d.$$

Computing the entries of the associated Gram matrix G_ω, we see that

$$\begin{aligned} [G_\omega]_{k,n} = \langle \omega_n, \omega_k \rangle &= \left\langle \sum_{l \in \mathbb{Z}^d} \overline{\varphi(x_l - n)} S_\varphi^{-1/2} T_l \varphi, \sum_{m \in \mathbb{Z}^d} \overline{\varphi(x_m - k)} S_\varphi^{-1/2} T_m \varphi \right\rangle \\ &= \sum_{l \in \mathbb{Z}^d} \varphi(x_l - k) \overline{\varphi(x_l - n)}, \end{aligned} \tag{19}$$

since $(S_\varphi^{-1/2} T_l \varphi)_{l \in \mathbb{Z}^d}$ is an orthonormal basis.

Theorem 30. *Assume that $\varphi \in L^2(\mathbb{R}^d)$ satisfies the conditions (i)–(iii), and let $(x_k)_{k \in \mathbb{Z}^d} \subset \mathbb{R}^d$ be relatively separated. Let G_ω be the autocorrelation matrix with entries given by (19) and assume that condition (17) holds. Then the following are equivalent.*

- *(a) $(x_k)_{k \in \mathbb{Z}^d}$ is a stable set of sampling for $V^2(\varphi)$.*
- *(b) $(x_k)_{k \in \mathbb{Z}^d}$ is a stable set of sampling for $V^\infty(\varphi)$ and $D_\psi : \ell^\infty \to V^\infty(\varphi)$ has closed range.*
- *(c) G_ω is invertible on $\ell^1(\mathbb{Z}^d)$.*
- *(d) G_ω is invertible on $\ell^\infty(\mathbb{Z}^d)$.*
- *(e) G_ω is invertible on $\ell^2(\mathbb{Z}^d)$.*

Remark 31. *In the language of* [26]*, the assumptions (i)–(iii) guarantee that φ is contained in the Wiener amalgam space $W_0 = W(C, \ell^1)$ and has* stable integer shifts*, which, by* [26, Theorem 3.1]*, yields several equivalent conditions for a relatively separated set $(x_k)_{k \in \mathbb{Z}} \subset \mathbb{R}$ being a stable set of sampling for $V^2(\varphi)$. In particular, using a non-commutative version of Wiener's lemma (*[25, Proposition 8.1]*), the authors of* [26] *showed that $(x_k)_{k \in \mathbb{Z}^d}$ being a stable set of sampling for $V^p(\varphi)$ for* some $1 \le p \le \infty$ *is equivalent to $(x_k)_{k \in \mathbb{Z}^d}$ being a stable set of sampling for $V^p(\varphi)$ for* all $1 \le p \le \infty$ *(hence the closed range condition of $D_\psi : \ell^\infty \to V^\infty(\varphi)$ from (b) can actually be omitted). Connected to that, we assume condition (17) in order to guarantee localization with respect to the Jaffard algebra $\mathcal{J}_s(\mathbb{Z}^d)$ and utilize inverse-closedness in $\mathcal{B}(\ell^2(\mathbb{Z}^d))$, which can be viewed as another non-commutative version of Wiener's lemma.*

We also note that the autocorrelation matrix G_ω is given by the product $G_\omega = P_X(\varphi)^* P_X(\varphi)$, where $P_X(\varphi) = (\varphi(x_l - k))_{l,k \in \mathbb{Z}}$ is the matrix appearing in [26, Theorem 3.1]*. According to* [26, Theorem 3.1] *(in the case $d = 1$), stable sampling is equivalent to $P_\Gamma(\varphi) : \ell^\infty(\mathbb{Z}^d) \to \ell^\infty(\Gamma)$ being bounded and injective for* all *weak limits Γ of integer translates of $(x_k)_{k \in \mathbb{Z}}$. In contrast, conditions (c)–(e) from above only concern* one *specific operator G_ω. To the best of our knowledge, these invertibility conditions on the autocorrelation matrix G_ω are new.*

Acknowledgements

The authors would like to thank K. Gröchenig for pointing out the connection between Theorem 30 and [26].

This research was funded by the Austrian Science Fund (FWF) 10.55776/P34624 (P.B. and L.K.) and 10.55776/PAT1384824 (M.S.). For open access purposes, the authors have applied a CC BY public copyright license to any author-accepted manuscript version arising from this submission.

References

[1] A. Aldroubi, A. Baskakov, and I. Krishtal. Slanted matrices, banach frames, and sampling. *Journal of Functional Analysis*, 255(7):1667 – 1691, 2008.

[2] A. Aldroubi and K. Gröchenig. Nonuniform sampling and reconstruction in shift-invariant spaces. *SIAM Rev.*, 43(4):585–620, 2001.

[3] R. Balan, P. Casazza, C. Heil, and Z. Landau. Density, Overcompleteness, and Localization of Frames. I. Theory. *Journal of Fourier Analysis and Applications*, 12:105–143, 04 2006.

[4] R. Balan, P. G. Casazza, C. Heil, and Z. Landau. Density, overcompleteness, and localization of frames. II. Gabor systems. *Journal of Fourier Analysis and Applications*, 12(3):307–344, Jun 2006.

[5] P. Balazs and K. Gröchenig. A guide to localized frames and applications to Galerkin-like representations of operators. In I. Pesenson, H. Mhaskar, A. Mayeli, Q. T. L. Gia, and D.-X. Zhou, editors, *Frames and Other Bases in Abstract and Function Spaces*, Applied and Numerical Harmonic Analysis series (ANHA). Birkhauser/Springer, 2017.

[6] P. Balazs, K. Gröchenig, and M. Speckbacher. Kernel theorems in coorbit theory. *Trans. Am. Math. Soc. Ser. B*, 6:346–364, 2019.

[7] P. Balazs, D. Stoeva, and J.-P. Antoine. Classification of General Sequences by Frame-Related Operators. *Sampl. Theory Signal Image Process.*, 10(2):151–170, 2011.

[8] A. G. Baskakov. Wiener's theorem and the asymptotic estimates of the elements of inverse matrices. *Funct. Anal. Appl.*, 24(3):222–224, 1990.

[9] J. Bergh and J. Löfström. *Interpolation Spaces. An Introduction*, volume 223 of *Grundlehren der Mathematischen Wissenschaften*. Springer, Berlin - New York, 1976.

[10] D. Bytchenkoff, M. Speckbacher, and P. Balazs. Kernel theorems for operators on co-orbit spaces associated with localised frames. *J. Math. Anal. Appl.*, 551(1), 2025.

[11] P. G. Casazza, G. Kutyniok, and M. C. Lammers. Duality principles in frame theory. *J. Fourier Anal. Appl.*, 10(4):383–408, 2004.

[12] O. Christensen. *An Introduction to Frames and Riesz Bases*. Birkhäuser, 2016.

[13] J. B. Conway. *A Course in Functional Analysis*. Graduate Texts in Mathematics. Springer, New York, 2. edition, 1990.

[14] H. G. Feichtinger. Un espace de Banach de distributions tempérées sur les groupes localement compacts abéliens. *C. R. Acad. Sci. Paris Sér. A-B*, 290(17):A791–A794, 1980.

[15] M. Fornasier and K. Gröchenig. Intrinsic localization of frames. *Constr. Approx.*, 22(3):395–415, 2005.

[16] I. Gelfand, D. Raikov, and G. Shilov. *Commutative Normed Rings*. Chelsea Publishing Company, Bronx, New York, 1964.

[17] I. Gohberg, M. Kaashoek, and H. Woerdeman. The band method for positive and strictly contractive extension problems: An alternative version and new applications. *Integral Equations Operator Theory*, 12:343–382, 01 1989.

[18] K. Gröchenig. *Foundations of Time-Frequency Analysis*. Birkhäuser, Boston, 2001.

[19] K. Gröchenig. Localization of frames, Banach frames and the invertibility of the frame operator. *J. Fourier Anal. Appl.*, 10:105–132, 2004.

[20] K. Gröchenig. Localization of frames, Banach frames, and the invertibility of the frame operator. *J. Fourier Anal. Appl.*, 10(2):105–132, 2004.

[21] K. Gröchenig. Gabor frames without inequalities. *Int. Math. Res. Not. IMRN*, 2007(23):ID rnm111, 21, 2007.

[22] K. Gröchenig. *Wiener's lemma: Theme and variations. An introduction to spectral invariance and its applications.*, chapter 5, pages 175 – 234. Applied and Numerical Harmonic Analysis. Birkhäuser, 2010.

[23] K. Gröchenig and M. Leinert. Wiener's lemma for twisted convolution and Gabor frames. *J. Amer. Math. Soc.*, 17(1):1–18, 2004.

[24] K. Gröchenig and M. Leinert. Symmetry and inverse-closedness of matrix algebras and functional calculus for infinite matrices. *Trans. Amer. Math. Soc.*, 358(6):2695–2711, 2006.

[25] K. Gröchenig, J. Ortega-Cerdà, and J. L. Romero. Deformation of Gabor systems. *Adv. Math.*, 277:388–425, 2015.

[26] K. Gröchenig, J.-L. Romero, and J. Stöckler. Sampling theorems for shift-invariant spaces, Gabor frames, and totally positive functions. *Invent. Math.*, 211:1119–1148, 2018.

[27] S. Jaffard. Propriétés des matrices "bien localisées" preé de leur diagonale et qualques applications. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 7(5):461–476, 1990.

[28] L. Köhldorfer and P. Balazs. Localization of operator-valued frames. *arXiv:2503.24170*, 2025.

[29] L. Köhldorfer and P. Balazs. Wiener pairs of Banach algebras of operator-valued matrices. *Journal of Mathematical Analysis and Applications*, 549(2):129525, 2025.

[30] I. J. Maddox. *Infinite Matrices of Operators.* Lecture Notes in Mathematics. Springer, Berlin, 1980.

[31] A. Ron and Z. Shen. Weyl-Heisenberg frames and Riesz bases in $L^2(\mathbb{R}^d)$. *Duke Math. J.*, 89(2):237–282, 1997.

[32] J. Sjöstrand. Wiener type algebras of pseudodifferential operators. *Séminaire Équations aux dérivées partielles (Polytechnique)*, pages 1–19, 1994-1995.

[33] D. T. Stoeva and O. Christensen. On R-duals and the duality principle in Gabor analysis. *J. Fourier Anal. Appl.*, 21:383–400, 2015.

[34] Q. Sun. Wiener's lemma for infinite matrices. *Trans. Amer. Math. Soc.*, 359(7):3099–3123, 2007.

[35] H. Triebel. *Interpolation Theory, Function Spaces, Differential Operators*, volume 18. North Holland Publishing Company, 1978.

(P. B.) Acoustics Research Institute, Austrian Academy of Sciences, Dominikanerbastei 16, 1010 Vienna

Austria and Acoustics, Analysis and AI, Interdiscliplinary Transformation University Austria, Science Park 4, 4040 Linz, Austria
Email address: peter.balazs@oeaw.ac.at

(L. K.) Acoustics Research Institute, Austrian Academy of Sciences, Dominikanerbastei 16, 1010 Vienna, Austria
Email address: lukas.koehldorfer@oeaw.ac.at

(M. S.) Acoustics Research Institute, Austrian Academy of Sciences, Dominikanerbastei 16, 1010 Vienna, Austria
Email address: michael.speckbacher@oeaw.ac.at