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On the Big Bang formation for cosmological solutions to non-vacuum Einstein equations

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Abstract

Friedman-Lemaître-Robertson-Walker (FLRW) spacetimes arise as natural models for our universe by virtue of the Cosmological Principle. This principle posits that, when viewed at large scales, no point and no direction in our universe has a preferential role. Observations support the Cosmological Principle, i.e., the physical universe is well approximated by FLRW models. However, a full mathematical analysis of these models is necessary to understand to what extent this approximation remains valid in our distant past and future, to which we cannot fully apply observational methods. To this end, this thesis studies the stability of FLRW spacetimes as solutions to the Einstein equations, predominantly towards the Big Bang singularity. This amounts to analysing the blow-up stability of solutions to an elliptic-hyperbolic system of partial differential equations.

The thesis consists of three works, preceded by a brief introduction to cosmological and mathematical aspects of Big Bang formation. In the first work, initial data for the Einstein scalar-field equations close to that of an FLRW spacetime with negative spatial curvature is considered. In the contracting direction, it is shown that the maximal globally hyperbolic development of said initial data is incomplete, forming a quiescent crushing Big Bang singularity. In the expanding direction, their development is complete and asymptotically approaches the vacuum solution, provided that a certain spectral assumption with respect to the spatial reference geometry holds. These results combine into a global stability result for a large class of FLRW spacetimes with negative spatial curvature.

The two subsequent works are concerned with Big Bang formation for the Einstein scalar-field Vlasov system in four and three spacetime dimensions respectively. FLRW solutions with isotropic matter and arbitrary spatial geometry are shown to be past stable, with the spacetime and scalar field behaviour matching that of the first result. Additionally, the Vlasov distribution is shown to be extendible towards the past when viewed on the co-mass shell, while the leading asymptotic order of the components of the Vlasov energy-momentum tensor lightly degenerates. Finally, the $(2+1)$-dimensional stability result is leveraged to control the past asymptotics of $(3+1)$-dimensional vacuum solutions in polarized $U(1)$-symmetry.

Zusammenfassung

Friedman-Lemaître-Robertson-Walker (FLRW) Raumzeiten sind aufgrund des Kosmologischen Prinzips natürliche Modelle unseres Universums. Dieses besagt, dass kein Ort und keine Richtung in unserem Universum eine besondere Rolle hat, wenn man es auf großen Skalen betrachtet. Da das Kosmologische Prinzip von Beobachtungen gestützt ist, kann das physikalische Universum gut von FLRW Raumzeiten modelliert werden. Allerdings ist eine vollständige mathematische Untersuchung dieser Lösungen erforderlich, um zu verstehen, inwieweit diese Approximation in der frühen Vergangenheit oder fernen Zukunft valide bleibt, bei denen Folgerungen aus Messmethoden an ihre Grenzen stoßen. Daher wird in dieser Arbeit die Stabilität von FLRW Modellen als Lösungen der Einstein-Gleichungen untersucht, vornehmlich in Richtung der Urknallsingularität. Dies läuft darauf hinaus, dass man die Blow-up-Stabilität von Lösungen zu einem elliptisch-hyperbolischen System partieller Differentialgleichungen studiert.

Die Dissertation besteht aus drei Arbeiten sowie einer kurzen Einführung in kosmologische und mathematische Aspekte der Urknalltheorie. Im ersten Resultat werden Anfangsdaten für das Einstein-Skalarfeld-System betrachtet, die Daten für eine FLRW Raumzeit mit negativer räumlicher Krümmung naheliegen. In der kontrahierenden Richtung wird gezeigt, dass die maximale global hyperbolische Entwicklung dieser Daten unvollständig ist und zu einer ruhenden, zerdrückenden Urknallsingularität führt. In der expandierenden Richtung ist die Entwicklung vollständig und nähert sich asymptotisch der Vakuumlösung an, sofern die räumliche Referenzgeometrie eine bestimmte Spektralbedingung erfüllt. Diese Resultate lassen sich zu einem globalen Stabilitätsresultat für eine große Klasse an FLRW Raumzeiten mit negativer Raumkrümmung verbinden.

Die zwei folgenden Abschnitte befassen sich mit der Urknallentstehung für das Einstein-Skalarfeld-Vlasov-System in vier beziehungsweise drei Raumzeitdimensionen. Es wird gezeigt, dass FLRW Lösungen mit isotroper Materie und beliebiger räumlicher Geometrie vergangenheitsstabil sind, wobei das asymptotische Verhalten der Raumzeit und des Skalarfelds mit dem im ersten Resultat übereinstimmt. Zusätzlich wird gezeigt, dass die Vlasov-Verteilung sich als Funktion auf der Ko-Massenschale zum Urknall vorsetzen lässt, während die führende Ordnung der Komponenten des Vlasov Energie-Impuls-Tensors leicht degeneriert. Abschließend wird das $(2+1)$-dimensionale Stabilitätsresultat genutzt, um die Vergangenheitsstabilität von $(3+1)$-dimensionalen Vakuumlösungen in polarisierter $U(1)$-Symmetrie zu beweisen.

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Contents

CHAPTER 1

Introduction

The Einstein field equations

$$\operatorname{Ric}[\overline{g}]_{\mu\nu} - \frac{1}{2} R[\overline{g}]_{\mu\nu} + \Lambda \, \overline{g}_{\mu\nu} = 8\pi \, T_{\mu\nu} \tag{EEq}$$

have dominated astrophysics and cosmology since their formulation by Einstein in [Ein16] for $\Lambda = 0$ and in [Ein17] for $\Lambda \in \mathbb{R}$. However, we are still far from a comprehensive understanding of how general solutions to (EEq) behave. In particular, only in the 1980s were mathematicians able to provide statements on the asymptotic behaviour of solutions without restricting the solution space, with pionieering works by Christodoulou and Klainerman [CK93] as well as Friedrich [Fri86]. The goal of this thesis is to contribute to this endeavour by studying the asymptotic properties of a class of cosmological solutions to the Einstein equations.

In this chapter, I will introduce the relevant spacetime and matter models, provide the physical motivations for studying these models and review their historic origins. In addition, I will provide a brief overview of the geometric PDE problems that one encounters naturally when analysing these solutions along with conjectures regarding their asymptotic properties. I will close by outlining the main results that make up this thesis, leaving the discussion of how these fit into more recent developments in the field to later chapters.

1.1. The Cosmological Principle and FLRW spacetimes

When considering measurements, one observes that the universe seems to look the same in all directions. Arguably, this is best seen in the near-isotropic Cosmic Microwave Background (CMB) which serves as a snapshot of an early stage of the universe that was dominated by radiation, see [Hin+13; Col+20] for current data. This goes hand in hand with the more philosophical notion that no point in the universe should, on a large scale, have a privileged position. This reflects the desire that different points in the universe should be subject to the same physical laws as we are on Earth. These observations immediately lead one to the *Cosmological Principle*. It states that the universe should be isotropic and spatially homogeneous at large scales. While we clearly cannot expect this principle to be strictly satisfied in a precise and comprehensive model of our universe, the principle serves as a starting point from which one can attempt to integrate small inhomogeneities and anisotropies to describe our universe more accurately.

Let $(\overline{M}, \overline{g})$ be a $(3+1)$-dimensional spacetime modelling our universe that is subject to the Cosmological Principle in its strictest form, i.e., assume that it is spatially homogeneous and isotropic. If one additionally assumes that our universe admits a global time function $t : \overline{M} \to \mathbb{R}$ such that ∂_t is orthogonal to the level sets $M_s = t^{-1}(\{s\})$, it was shown by Robertson [Rob35] and Walker [Wal37] that $(\overline{M}, \overline{g})$ must be isometric to a spacetime of the form

$$\overline{M} = I \times M, \quad \overline{g}_{FLRW} = -dt^2 + a(t)^2 \gamma \,. \tag{1.1.1}$$

Here, $I \subseteq \mathbb{R}$ is an open interval, a is a smooth positive function on I and (M, γ) is a Riemannian manifold with constant sectional curvature κ. For such a spacetime to provide a solution to (EEq), the scale factor a must satisfy a set of ordinary differential equations called the *Friedman equations* that depend on components of $T_{\mu\nu}$ as well as on κ. In current standard models, including the ΛCDM model which underlies [Hin+13; Col+20], one models the universe by multiple perfect fluids that individually model radiation, matter and dark matter content of the universe. The energy momentum tensor of each fluid is of the form

$$T^{Eu}_{\mu\nu} = (\rho + \mathfrak{p})\, u_\mu\, u_\nu + \mathfrak{p}\, \overline{g}_{\mu\nu},$$

where (u^μ) denotes the relativistic four-velocity of the fluid normalised to $u_\mu\, u^\mu = -1$. The energy density ρ and pressure $\mathfrak{p}$ are related by $\mathfrak{p} = w\,\rho$ with $w = \frac{1}{3}$ for radiation, $w = 0$ for matter and $w = -1$ for dark matter. Denoting the cumulative density and pressure by ρ_{tot} and $\mathfrak{p}_{\text{tot}}$ respectively, the scale factor of the corresponding solution in the ΛCDM-model is then determined by the following version of the Friedman equations.

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi}{3}\,\rho_{\text{tot}} - \frac{\kappa}{a^2} + \frac{\Lambda}{3} \\ \frac{\ddot{a}}{a} &= -\,\frac{4\pi}{3}\,(\rho_{\text{tot}} + 3\mathfrak{p}_{\text{tot}}) + \frac{\Lambda}{3} \end{aligned}$$

Spacetimes of the form (1.1.1) are referred to as Friedman-Lemaître-Robertson-Walker (FLRW) spacetimes. They were discovered independently by Friedman [Fri22; Fri24] and Lemaître [Lem27; Lem31a]; the aforementioned rigidity results were proven independently by Robertson [Rob35] and Walker [Wal37]. However, solutions of this type with positive cosmological constant date back to works of Einstein and de Sitter in 1917, see [Ein17] and [DS17] along with [RP09] for a historical overview. While the Friedman equations first derived by Friedman in [Fri22; Fri24] concern dust solutions, we will derive similar expressions for different matter models and spacetime dimensions and also refer to them as Friedman equations.

1.2. Strong Cosmic Censorship and the $3+1$ decomposition

The idea of the universe emerging from an exceptional state was present early on in mathematical cosmology. For example, in [Lem31b], Lemaître conceived of the universe as emerging from a "primeval atom" where the scale factor is close to zero. Additionally, physical discoveries such as the CMB imply that the universe had an early, dense stage. The modern mathematical interpretation of its origins is strongly influenced by works by Ellis, Hawking and Penrose, see [Haw66a; Haw66b; Haw67; HE68; HP70]. The majority of these results concern globally hyperbolic spacetimes, i.e., spacetimes that contain a "spacelike Cauchy hypersurface" M that is intersected precisely once by every inextendible timelike curve. This hypersurface can be viewed as a snapshot of our universe at a given time. Additionally, assume that the universe is currently expanding, i.e., that the trace of the shape operator on M is, depending on the sign convention used, strictly positive or negative. Under these conditions, the aforementioned works have shown that such spacetimes must have emerged from a singularity given suitable energy conditions for the energy-momentum tensor. It is worth noting that ΛCDM-models that only contain dark matter and the vacuum de Sitter universe violate these energy conditions, but also do not contain a singularity nor have scale factors that approach zero toward the past. The de Sitter universe, in particular, is nevertheless used as a model of the inflationary period of the early universe, as discussed in Section 1.3.1, as the effects of dark matter or a cosmological constant are dominant in that era, but any ΛCDM-model approximating our universe must also contain matter and radiation components, and consequently satisfies the necessary energy conditions, at least when considering the past of a sufficiently early time slice.

However, no claim is made in the aforementioned singularity theorems on the precise nature of this singularity and whether this would match the physical intuition of a "Big Bang" when applied to cosmological solutions. Such incompleteness can either form because the spacetime is indeed inextendible or because there are multiple incompatible ways to develop the spacetime further. The latter implies that full data of the universe at a given time do not fully determine its future or past, and thus that the theory of general relativity is not fully deterministic. Taub-NUT spacetimes fall into this category. These are spatially homogeneous cosmological vacuum spacetimes that admit multiple extensions towards the past, see [Mis67]. However, this behaviour is shown to be unstable in [MT69], in the sense that adding any nearby Einstein solution containing radiation is inextendible. On the other hand, in standard ΛCDM-models as discussed above, the Kretschmann curvature scalar

$$\mathcal{K} = \mathrm{Riem}[\overline{g}]_{\alpha\beta\gamma\delta}\,\mathrm{Riem}[\overline{g}]^{\alpha\beta\gamma\delta}$$

blows up as the scale factor a approaches zero, which means that the spacetime is past C^2-inextendible. Thus, one would informally hope that determinism breaking down is a rare occurance and that "typical" configurations of the universe fully determine its future and allow us to fully extract its past. The *Strong Cosmic Censorship Conjecture* states more formally that the maximal globally hyperbolic development of generic initial data to the Einstein equations should be inextendible in a suitable regularity class. This formulation of the conjecture first appears in Moncrief's work [Mon81]. The conjecture itself is due to Penrose in [Pen79, p. 624f.] where it is stated that any "physically reasonable" spacetime should be globally hyperbolic. Moncrief recast this conjecture in terms of developments of initial data to access this conjecture with analytic methods, as I will discuss momentarily. For a recent review of Cosmic Censorship in light of Penrose having been awarded the 2020 Nobel Prize in Physics, we refer to [Lan21].

While the Strong Cosmic Censorship Conjecture, in its most general form within C^0 solutions, was disproven for the vacuum equations by Dafermos and Luk in [DL17], fully resolving the conjecture remains an active field of study. In light of the behaviour of ΛCDM-models, one commonly analyses the C^2-version of Strong Cosmic Censorship in cosmological settings.[1] Blow-up of the Kretschmann scalar in particular has also been shown to occur for spatially homogeneous solutions and vacuum solutions in presence of symmetry, see [CR95; Rin09b], making it a natural candidate to identify C^2-inextendibility for our purposes.

To study the development of data on a given surface within the Einstein equations, we will study certain foliations $(M_t)_{t\in I}$ of $\overline{M}$ by spacelike Cauchy hypersurfaces. The Einstein equations can then be phrased in terms of the first and second fundamental forms (g_t, k_t) on M_t as well as the lapse function n and shift vector X on $\overline{M}$ which encode the remaining components of the spacetime metric. In the standard four-dimensional setting, this approach is referred to as the $(3+1)$-formalism. It reduces the Einstein equations to a system of ten geometric unknowns, namely to the evolution equation for k as well the constraint equations for g and k, along with equations for matter. This decomposition is occasionally also referred to as the ADM formalism due to a series of works by Arnowitt, Deser and Misner culminating in [ADM62]. Therein, they employ a related decomposition using the conjugate momentum π_t to g_t instead of the second fundamental form k_t to derive a Hamiltonian from the Einstein-Hilbert Lagrangian and to obtain an equivalent system of constrained evolution equations. However, versions of the $(3+1)$-decomposition predate the ADM analysis, including an early version in Gaussian coordinates by Darmois in [Dar27] and an equivalent formulation in general coordinates by Choquet-Bruhat in [FB56].

In practice, the foliation is not available a priori and has to be constructed. Doing so requires a choice of coordinates, which influences the nature of the evolutionary system. For example, the foundational proof by Choquet-Bruhat in [FB52] that data satisfying to the vacuum constraint equations launch a unique Einstein vacuum solution uses harmonic coordinates that cast the equations as a system of nonlinear wave equations for the components of the spacetime metric. Meanwhile, for most of this thesis, we choose coordinates that eliminate the shift, turning the Einstein equations into wave-like equations for the spatial metric g_t coupled to an elliptic equation for the lapse as well as the constraints. This allows us to recast the C^2-Strong Cosmic Censorship conjecture toward the singularity: By considering data close to that of FLRW data, this now becomes a question of blow-up stability for solutions to a system of nonlinear elliptic-hyperbolic differential equations. In Chapter 2, evolution in the direction of expansion is also considered. Here, we verify the lack of a future singularity, i.e., future completeness, by establishing the asymptotic stability of FLRW solutions to this PDE system.

Before providing an overview of these results, we introduce the necessary matter models and review which behaviour one would expect for cosmological solutions near the Big Bang singularity.

[1] There have, however, also been recent advances in C^0-inextendibility of FLRW spacetimes, for example in the works [Sbi22; Sbi23] of Sbierski.

1.3. Matter models

1.3.1. Scalar fields. The scalar field is often introduced as the structurally simplest non-trivial matter model for the Einstein equations. Considering the general case of a scalar field with a potential, it arises from the scalar Lagrangian

$$\mathcal{L} = -\nabla[\overline{g}]^{\alpha}\phi\,\nabla[\overline{g}]_{\alpha}\phi + V(\phi)\,.$$

The associated energy-momentum tensor is given by

$$T^{SF}_{\mu\nu}[\phi,\overline{g}] = \nabla[\overline{g}]_{\mu}\phi\,\nabla[\overline{g}]_{\nu}\phi - \left(\frac{1}{2}\,\nabla[\overline{g}]^{\alpha}\phi\,\nabla[\overline{g}]_{\alpha}\phi + V(\phi)\right)\overline{g}_{\mu\nu}\,.$$

To be compatible with the Einstein tensor, any energy-momentum tensor must be divergence-free. For scalar field matter, this is ensured if the scalar field satisfies the relativistic wave equation

$$\square_{\overline{g}}\,\phi = \overline{g}^{\mu\nu}\,\nabla[\overline{g}]_{\mu}\,\nabla[\overline{g}]_{\nu}\phi = V'(\phi)\,.$$

Below, we will usually restrict ourselves to the fully linear case where $V=0$.

The behaviour of relativistic waves on fixed backgrounds can serve as a toy model for the full Einstein equations given that the Einstein equations in harmonic gauge reduce to a system of nonlinear wave equations. As a matter model in its own right, it is also a subcase of the stiff fluid model, i.e., a perfect fluid with equation of state $\mathfrak{p}=\rho$. For a formal derivation, we refer to [Chr07, Chapter 1]. This can be seen heuristically from considering the density and pressure of this energy-momentum tensor for a spatially homogeneous scalar field. In the nonlinear case, denoting the future directed unit normal by e_0, it is given by

$$\rho = \frac{1}{2}\,(e_0\phi)^2 + V(\phi),\ \ \mathfrak{p} = \frac{1}{2}(e_0\phi)^2 - V(\phi)\,. \tag{1.3.1}$$

Note that the equation of state precisely matches that of a stiff fluid in the linear case. Scalar field and stiff fluid matter both lead to stability mechanisms towards the Big Bang that are discussed in more detail in Section 1.4. For this reason, the inclusion of a scalar field will be essential for all parts of this work.

From a cosmological perspective, however, the nonlinear wave equation plays an increasingly important role in resolving issues with the ΛCDM-model. Approaching the Big Bang in these models, the spacetime becomes increasingly causally disconnected. Hence, there is no structural reason why separate regions in the CMB should have the same temperature, since no information could have passed between them. This is often referred to as the *horizon problem*. Additionally, the standard theory does not explain why the universe seems to be approximately flat – the *flatness problem* – or why it is almost isotropic. The theory of inflation attempts to resolve these issues by introducing a period of accelerated expansion in the early universe that isotropises the matter content and flattens the spatial metric. The easiest way to ensure accelerated expansion in the Friedman equations is with a positive cosmological constant, but this does not allow for the expansion to slow down again and thus might not be compatible with observations. Thus, the next option is to model this via a single or multiple *inflatons*. These are scalar fields with a suitable non-negative potential coupled to the Einstein equations.[2] When the scalar potential is dominant, this acts as dark matter does in the Friedman equations and causes accelerated expansion. While we refer to [Ren06, Chapter 5] for a formal discussion, one can again see this heuristically from (1.3.1) since, for a dominant potential, the equation of state becomes $\rho \approx -\mathfrak{p}$. The precise form of the scalar potential is still up to debate – see, for example the review [Cic+24] discussing inflation driven by various potentials in String Cosmology. Nevertheless, studying the Einstein scalar-field system is a necessary step to rigorously understand one of the most commonly accepted cosmological theories for the Early Universe.

1.3.2. Vlasov matter. The second half of thesis concerns FLRW spacetimes that, on top of a scalar field, contain a collisionless gas described kinetically by the Vlasov equation. We briefly outline the considerations from which this equation arises, following the outline of [SM58; Ehl73].

As a starting point, consider a gas as a collection of point particles. If the movement of such a particle is only affected by gravitational effects, it moves along the worldline $x(s)$ determined by the geodesic equation

$$\frac{dx^\mu}{ds} = p^\mu, \quad \frac{dp^\mu}{ds} = -\Gamma[\overline{g}]^\mu_{\alpha\beta}\, p^\alpha\, p^\beta\,.$$

Each such particle has a fixed relativistic mass $m \geq 0$ given by $m^2 = -p_\mu p^\mu$. For the sake of simplicity, we will consider gases that consist of one particle type, i.e., where all particles have mass m. Thus, $(x,p)(s)$ can be viewed as a curve on the mass shell P which is discussed in more detail in Section 3.2 along with its induced Riemannian structure.

Tracking the behaviour of each individual particle within a gas would be too involved computationally. Instead, the gas is viewed statistically, describing the density of the gas by the particle distribution function $f^\sharp$. If we consider an arbitrary open subset U_t of a spacelike hypersurface $M_t \cong M$ in $\overline{M} \cong I \times M$, the average number of such particles should be preserved by transport along geodesics. More precisely, if Φ_s describes the geodesic flow for some $s > 0$ and dvol_P describes the volume form on constant-in-time leaves $P^\sharp_t$ the mass shell $P^\sharp$, one must have

$$\int_{TU_t} f^\sharp(t,\cdot,\cdot)\,\mathrm{dvol}_{P^\sharp_t} = \int_{\Phi_s(TU_t)} f^\sharp(t+s,\cdot,\cdot)\,\mathrm{dvol}_{P^\sharp_t} = \int_{TU_{t+s}} \Phi_t^* f^\sharp(t+s,\cdot,\cdot)\,\mathrm{dvol}_{P^\sharp_{t+s}}\,.$$

Consequently, $f^\sharp$ must be preserved by the geodesic spray restricted to the mass shell P, given by

$$\mathcal{X} f^\sharp = \left(p^\mu \frac{\partial}{\partial x^\mu} - \Gamma[\overline{g}]^m_{\alpha\beta}\, p^\alpha\, p^\beta \frac{\partial}{\partial p^m}\right) f^\sharp = 0\,. \tag{1.3.2}$$

[2] Note that the vacuum Einstein equations with cosmological constant Λ are equivalent to Einstein scalar-field equations with trivial scalar field and $V(\phi) = \Lambda$.

Similarly, the energy-momentum tensor

$$T^{Vl}_{\mu\nu}[f^\sharp, \overline{g}] = \int_{P^\sharp_{(t,x)}} p_\mu \, p_\nu \, f^\sharp(t,x,p) \, \mathrm{dvol}_P$$

arises naturally from the requirement that the energy-momentum tensor should be preserved when projected along any worldline. The Vlasov equation (1.3.2) also ensures that T^{Vl} is divergence-free.

The Vlasov equation is occasionally also referred to as the Liouville equation or the collisionless Boltzmann equation. The former alternate name arises from the Hamiltonian perspective, where the Vlasov equation arises as an application of Liouville's theorem, i.e.,

$$\partial_t f^\sharp + \left\{ f^\sharp, H \right\} = 0 \,,$$

see [Lio38], to the free particle Hamiltonian $H = \frac{1}{2} \, \overline{g}^{\mu\nu} \, p_\mu \, p_\nu$. Regarding the latter, the full Boltzmann equation

$$\mathcal{X} f^\sharp = Q \left(f^\sharp, f^\sharp \right)$$

describes the behaviour of a particle gas with collisions modelled by the bilinear functional Q on the space of scalar functions on the mass shell. The first extensions of the classical formulation by Boltzmann in [Bol68] to relativistic settings considered collisions in their formulation from the beginning. The first precursor dates back to Jüttner in [Jüt11] who considered the thermodynamical properties of ideal gases in Special Relativity that were modeled with the help of equilibrium solutions to the Boltzmann solution. These ideas were extended to general relativity and expanded upon in the late 1950s and early 1960s, see for example [SM58; TW61; Isr63; Che63; Lin66; Ehl73]. An existence theory for the Einstein-Boltzmann system was first established in [Ban73]. Considering the collisionless case in isolation was popularised by Vlasov in his description of plasma, see [Vla61; Vla68], leading to the (1.3.2) being referred to as the Vlasov equation.

For a general overview regarding the Einstein-Vlasov system, we refer to [And11] as well as [Ren08, Chapter 11]. Cosmologically, Vlasov matter serves as a kinetic descriptor of stellar dynamics within our universe, where each stellar object can be viewed as a "particle" in the matter content of the universe. Since collisions on this scale, i.e., between galaxies and galaxy clusters, are currently very rare, one often considers the Vlasov equation rather than the full Boltzmann equation. For more details on this perspective, see [BT08, Chapters 1.2 and 4]. While one might argue that, toward the past, collisions become more frequent as the universe contracts, we still restrict ourselves to the Vlasov case as understanding the behaviour of the transport component is, at least, a necessary step towards understanding the full system. Additionally, as will be discussed in more detail in Chapter 3, Vlasov matter serves as a simplified model of radiation toward the Big Bang singularity, which is expected to dominate the early universe while also being difficult to control. From this perspective, each Vlasov particle represents a photon within the electromagnetic radiation field.

1.4. The BKL conjecture and quiescent Big Bang formation

The first attempts to understand the behaviour of general cosmological solutions approaching the past singularity date back to the works by Lifshitz, Khalatnikov and Sudakov in [LK61b; LK61a; LSK61], summarized in [LK63]. Therein, the authors considered the vacuum Einstein equations with $\Lambda = 0$. If one upholds the Cosmological Principle to obtain vacuum model solutions, the resulting FLRW solutions are all quotients of Minkowski space and thus do not exhibit a Big Bang. However, if one only assumes spatial homogeneity, this allows for Kasner spacetimes, which are spatially flat spacetimes of the form

$$\overline{M} = I \times \mathbb{T}^3, \quad \overline{g}_{Kas} = -dt^2 + \sum_{i=1}^{3} t^{2q_i} \, (dx^i)^2 \tag{1.4.1}$$

and provide a solution to the Einstein vacuum equations if the following Kasner relations are satisfied.

$$q_1 + q_2 + q_3 = q_1^2 + q_2^2 + q_3^2 = 1 \tag{1.4.2}$$

These spacetimes were first discovered by Kasner in [Kas21]. They also exhibit polynomial Kretschmann scalar blow-up if none of the exponents vanish and can thus serve as useful toy models toward the Big Bang before considering the effects matter models may have on the singularity.

With Kasner spacetimes in mind, Lifshitz and Khalatnikov attempted to analyse the behaviour of general solutions by making the following ansatz.[3]

$$\overline{g} = -dt^2 + \sum_{I=1}^{3} t^{2q_I(x)} \, \omega_I \otimes \omega_I + \dots$$

Here, $\{\omega_I\}_{I=1,2,3}$ is a coframe over M and $\{q_I\}_{I=1,2,3}$ are smooth functions on M that satisfy the Kasner relations (1.4.2) pointwise. Without loss of generality, assume $q_1 \leq q_2 \leq q_3$. The spatial Ricci curvature over constant-time hypersurfaces can then be expanded as

$$\left|\mathrm{Ric}_I{}^J\right| = 2\,\left(\gamma^1_{23}\right)^2\, t^{-2\,(q_2+q_3-q_1)} + \dots = 2\,\left(\gamma^1_{23}\right)^2\, t^{-2-2\,q_1} + \dots\,,$$

where γ^I_{JK} denotes the connection coefficients of the coframe. Since the Kasner relations imply that q_1 is negative except for the extremal case with Kasner exponents $(0,0,1)$, which corresponds to Minkowski spacetime, the leading term diverges at an order higher than t^{-2} if there is a singularity, since the extremal case simply corresponds to Minkowski spacetime. On the other hand, the Einstein equations imply

$$\mathrm{Ric}_I{}^J + \frac{1}{2\sqrt{-\det \overline{g}}}\,\partial_t\left(\sqrt{-\det \overline{g}}\,k_I{}^J\right) = 0\,,$$

where k is the second fundamental form with respect to the constant time foliation. One computes that the latter term is of order t^{-2} or lower. From this, one sees that this ansatz can only be consistent and provide a solution up to the singularity if the leading term in the spatial curvature vanishes, so if one has $\gamma^1_{23} \equiv 0$. Lifshitz and Khalatnikov initially concluded

[3] For a critical view on the validity of this ansatz, see [BT79].

that such past singularities are non-generic phenomena for vacuum solutions to the Einstein equations, as well as solutions in presence of radiation after a similar analysis. In [BKL70; BKL82], a more refined analysis of this ansatz is performed. The authors observe that the leading order terms lead to a "bouncing" phenomenon, where the solution transitions from one Kasner regime to another and that the direction in which the Kasner exponent is negative changes, see [BKL70, p.536f.]. This leads the authors to the *BKL conjecture*: Generically, singularities in cosmology are expected to be oscillatory in the sense that, along each worldline, the spacetime undergoes infinitely many bounces between different Kasner regimes approaching the spacelike singularity. In [LK63], it is theorized that this series of contractions and expansions allows the spacetime to remain causally connected. This is revised in [BK71]. Since then, the BKL conjecture commonly includes that this behaviour is local and modelled by spatially homogeneous dynamics, see [BK71].

In [LK63; BKL70], the authors also observe that the presence of radiation does not affect the heuristics sketched above, and thus conjecture that these behaviours persist for most matter models since one physically expects radiation to the dominant matter type toward the Big Bang. The isotropic solutions containing radiation are determined to be unstable by a perturbative analysis. However, in [BK73], the authors perform similar calculations for the Einstein massless scalar-field system and observe that this ansatz is consistent with the Einstein equations without losing any degrees of freedom.[4] In particular, such solutions should generically exhibit the same asymptotics locally as generalised Kasner solutions which solve the Einstein scalar field system. These are of the form (1.4.1), coupled to an isotropic scalar field given by

$$\partial_t\phi = C\,t^{-1}, \quad \partial_{x^i}\phi = 0$$

for some $C \neq 0$, and satisfy generalised Kasner relations

$$q_1 + q_2 + q_3 = 1, \quad q_1^2 + q_2^2 + q_3^2 + 8\pi\, C^2 = 1\,.$$

The first rigorous proof of this type is due to Andersson and Rendall in [AR01]. Therein, the authors consider so-called *velocity term dominated (VTD) solutions* which are formal solutions of truncated Einstein equations where spatial derivative terms are dropped. These were first introduced in [ELS72] to provide a framework to study the Big Bang in presence of irrotational dust, inspired by [LK63]. In [AR01], the authors show that any analytic VTD solution is associated to a unique analytic Einstein scalar-field solution that asymptotically converges toward the VTD solution approaching the Big Bang. Such solutions are referred to as *quiescent*, a term originating in the work [Bar78] of Barrow.

There are results able to analyse the chaotic, oscillatory BKL regime, but these usually are restricted to spatially homogeneous dynamics or the analysis of the bounces in more rigid symmetry classes, see, for example, [Rin01; BD23] or [CIM90; Rin04] respectively. However,

[4] The authors also note that the heuristic argument in [LK63] fails for fluids with $\mathfrak{p} > \frac{2}{3}\rho$.

when trying to study Big Bang formation and Cosmic Censorship for general solutions, quiescent spacetimes are significantly more tractable leading to a series of recent results, beginning with the past stability of FLRW scalar field spacetimes for flat spatial geometry by Rodnianski and Speck in [RS18b]. The results of this thesis, outlined in the next section, contribute to this effort by studying quiescent Big Bang formation for different FLRW spacetimes and verifying the Strong Cosmic Censorship conjecture near these spacetimes in the process. The state of the art for quiescent Big Bang formation will be discussed in more detail throughout this thesis and is thus omitted here.

From a physical perspective, by quantitatively controlling blow-up quantities such as the Kretschmann scalar towards the singularity, the results contained below verify that small amounts of noise do not lead to significant deviations from the FLRW picture when studying the Big Bang. Such noise on a homogeneous background could be similar to what has been observed in the CMB. In particular, the analysis below strongly indicates that many of the results in this thesis extend to cosmological models involving inflatons, since matter blow-up toward the Big Bang is usually caused by the wave equation itself rather than the nonlinearity. Such an extension is proven, for example, in the recent work [OGPR23] which studies general conditions for initial data to the Einstein non-linear scalar field system to admit quiescent Big Bang formation. However, such a heuristic extension is clearly only valid as long as the nonlinear potential is negligible, i.e., in the pre-inflationary period. Indeed, [OGPR23] requires the rate of expansion on the initial hypersurface to be sufficiently large compared to the initial data size, in part to ensure that the core linear system dominates the asymptotics, and we also require this in Chapter 4, without loss of generality from a mathematical perspective. In light of this, the results below are likely best interpreted physically as steps toward understanding the pre-inflationary period up to and including the Big Bang singularity in the near-isotropic setting. This can hopefully provide the basis to then study general anisotropic data in this period, after which one might be able to rigorously study the origins of our universe using post-inflationary, near-isotropic initial data.

1.5. Overview of results and contributions

To close out this introduction, we provide an overview of the three main works that constitute this thesis as well as my contributions to the respective works.

1.5.1. Chapter 2. This chapter contains the joint work [FU25a] with David Fajman. Therein, we have shown that FLRW spacetimes with negatively curved spatial geometry are globally asympotically stable within the Einstein scalar-field system.

Towards the past, we show that all FLRW spacetimes with scalar field matter and negative spatial curvature exhibit quiescent Big Bang formation, as in the works [RS18b] and [Spe18] that cover flat and spherical spatial geometry respectively. The main challenge lies in controlling the evolution of the spacetime geometry. Both of the aforementioned prior works rely on Lie-transported spatial reference frames adapted to their respective geometries to obtain high order energy estimates. Since the choice of frame is much less clear for hyperbolic spatial geometries, we utilize an alternate approach using Bel-Robinson variables to cast the geometric evolution as a coupled system of Maxwell and transport equations with elliptic constraints. This system can be commuted with the adapted Laplace-Beltrami operator instead of frames, similar to our previous work [FU22].

Towards the future, we first show that vacuum FLRW solutions with hyperbolic spatial geometry (M,γ), also referred to as Milne spacetimes, are stable within the Einstein scalar-field system if the smallest positive eigenvalue of the Laplace-Beltrami operator $-\Delta_\gamma$ satisfies a certain lower bound. This condition is satisfied for hyperbolic three-manifolds of small diameter, including the Weeks manifold, and is likely to hold for a large class of such geometries. The main difficulty is that the canonical scalar-field energy does not decay toward the future. To remedy this, we add an indefinite correction term to obtain decay. The aforementioned spectral condition is needed to ensure that this resulting energy is coercive. The global stability result follows by connecting the past and future asymptotic regimes using a Cauchy stability argument.

1.5.2. Chapter 3. This chapter contains the joint work [FU25b] with David Fajman, which proves that near-FLRW solutions to the Einstein scalar-field Vlasov equations exhibit stable Big Bang formation. This is the first nonlinear stability result containing Vlasov matter coupled to the Einstein equations up to a spacelike singularity without symmetry assumptions. By viewing the Vlasov distribution f as a function on the co-mass shell, we are able to show that $f(t,\cdot,\cdot)$ converges towards a limiting distribution on the Big Bang hypersurface that is close to the FLRW distribution. However, since the metric degenerates towards the past, the components of the Vlasov energy-momentum tensor only remain close to their FLRW counterparts up to a slight asymptotic offset, and the dual distribution $f^\sharp$ on the mass shell becomes highly anisotropic towards the past. Again viewing this through the lense of modelling the pre-inflationary period of our universe, this would imply that radiation is sensitive to any anisotropy that is present, but this may or may not become asymptotically dominant compared to the effect of the inflaton on Big Bang formation.

The analysis for the spacetime geometry and scalar field largely follows that of Chapter 2. In fact, in the course of the analysis, we show that the results of [RS18b; Spe18] can also be proven using the Bel-Robinson approach from Chapter 2. The only additional nuance is due to the coupling of this quiescent mechanism with the Vlasov energy hierarchy: To analyse the Vlasov equation, we split derivatives on the fibers of the co-mass shell into momentum (or vertical) derivatives and horizontal derivatives which are natural extensions of spatial derivatives to the co-mass shell. When commuting the Vlasov equation with derivatives on the co-mass shell, terms containing horizontal derivatives in the commuted Vlasov equation are asymptotically dominated by borderline vertical terms. However, the latter either enter nonlinearly or are linear geometric terms, both of which can be controlled using sharper, low order estimates on the geometric variables for low horizontal orders. In this spirit, one can obtain a closed estimate for an appropriately time-weighted Vlasov energy that can, in turn, be integrated into the energy estimates for the spacetime geometry when introducing appropriate weights. Furthermore, once strong bounds on the quiescent variables are obtained, one can iteratively improve Sobolev bounds on the Vlasov distribution for increasing numbers of horizontal derivatives along this hierarchy.

1.5.3. Chapter 4. This chapter is based on [Urb24] and shows that the results of Chapter 3 extend to (2+1)-dimensional FLRW solutions to the Einstein scalar-field Vlasov system, despite Vlasov and inhomogeneous scalar field terms entering the evolution at higher asymptotic orders. While many of the energy estimates for matter variables can be performed as in the four dimensional setting, the geometric evolution needs to be treated differently. In fact, the Ricci tensor being pure trace in two dimensions leads to cancellations in the energy estimates for the shear that significantly simplify the analysis.

In addition, using a conformal reduction established by Moncrief in [Mon86], I show that this implies that the Strong Cosmic Censorship conjecture holds toward the past for certain four-dimensional polarized $U(1)$-symmetric solutions to the Einstein vacuum equations. This extends similar results for Kasner spacetimes in [FRS23] to Kasner-like vacuum solutions with spatial topology $M \times \mathbb{S}^1$ where M has arbitrary genus.

1.5.4. Contributions by the author. Regarding the joint work [FU25a] in Chapter 2, most of the proof of Big Bang stability, including how the Bel-Robinson formalism is used, is due to myself, with occasional input by my co-author and supervisor, David Fajman. The outline for the proof of future and, subsequently, global stability originates from David Fajman, but I contributed significantly to its execution. With regard to [FU25b] in Chapter 3, its mathematical content, in particular the energy formalism necessary to integrate Vlasov matter into the aforementioned Big Bang stability framework, was developed by me with occasional input by the co-author, David Fajman. The original drafts of both manuscripts were written by me, while David Fajman was heavily involved with revisions and editing subsequent drafts up to and including their current versions. Chapter 4, based on [Urb24], is fully my own work.

CHAPTER 2

Cosmic Censorship near FLRW spacetimes with negative spatial curvature

2.1. Introduction

2.1.1. Setting and main results. We consider the Einstein scalar-field system

$$\mathrm{Ric}[\overline{g}]_{\mu\nu} - \frac{1}{2}R[\overline{g}]\overline{g}_{\mu\nu} = 8\pi T_{\mu\nu}[\overline{g}, \phi] \tag{2.1.1a}$$

$$T_{\mu\nu} = \overline{\nabla}_\mu\phi\overline{\nabla}_\nu\phi - \frac{1}{2}\overline{g}_{\mu\nu}\overline{\nabla}^\alpha\phi\overline{\nabla}_\alpha\phi \tag{2.1.1b}$$

$$\square_{\overline{g}}\phi = 0 \tag{2.1.1c}$$

with initial data $(g_0, k_0, \pi_0, \psi_0)$ on a closed 3-manifold M that admits a negative Einstein metric γ.[1] In this paper, we determine the maximal globally hyperbolic development emanating from such initial data given that it is sufficiently close to the initial data of a homogeneous solution with a non-trivial scalar field.
In the collapsing direction, we prove a stable Big Bang formation and curvature blow-up result, which requires the presence of a non-trivial scalar field. The results complement those in [RS18b; Spe18], which cover flat and spherical spatial geometry. In the expanding direction, we prove a nonlinear future stability result of the corresponding vacuum background solution, which is the Milne model, under a mild condition for the first positive eigenvalue of $-\Delta_\gamma$ (see Definition 2.9.2). As discussed in more detail in Remark 2.9.3, numerical studies (see [CS99; Ino01]) show that this condition holds for an analogue of Weeks space, and suggest that this may hold for all closed hyperbolic 3-manifolds with sectional curvature $-\frac{1}{9}$.

Connecting the two regions, we prove the global stability (i.e., past and future stability) of the spacetime

$$\left([0,\infty) \times M, -dt^2 + a(t)^2\gamma\right), \tag{2.1.2a}$$

given a negative Einstein manifold (M, γ) obeying the aforementioned spectral condition, with

$$a(0) = 0,\ \dot{a} = \sqrt{\frac{1}{9} + \frac{4\pi}{3}C^2a^{-4}} \tag{2.1.2b}$$

for some given constant $C > 0$, and the scalar field given by

$$\partial_t\phi = Ca^{-3},\ \nabla\phi = 0\,. \tag{2.1.2c}$$

[1] Here and throughout, π_0 and ψ_0 prescribe data for $\nabla\phi|_{\Sigma_{t_0}}$ and $\partial_0\phi|_{\Sigma_{t_0}}$ respectively.

The scale factor consequently exhibits the following asymptotic behaviour:

$$a(t) \simeq t^{\frac{1}{3}} \text{ as } t \searrow 0 \text{ and } a(t) \simeq t \text{ as } t \nearrow \infty \tag{2.1.2d}$$

The main result can be split into two parts:

THEOREM 2.1.1 (Big Bang stability – rough version). *Let $(M, g_0, k_0, \pi_0, \psi_0)$ be initial data for the Einstein scalar-field system that is sufficiently close to*

$$(M, a(t_0)^2\gamma, -\dot{a}(t_0)a(t_0)\gamma, 0, Ca(t_0)^{-3}),$$

where $C > 0$ and (M, γ) is a closed Riemannian 3-manifold with $\mathrm{Ric}[\gamma] = -\frac{2}{9}\gamma$ (i.e., a closed negative Einstein manifold with scalar curvature $-\frac{2}{3}$).

Then, the past maximal globally hyperbolic development $((0, t_0] \times M, \overline{g}, \phi)$ of the initial data within the Einstein scalar-field system (2.1.1a)-(2.1.1c) *admits a foliation by CMC hypersurfaces $\Sigma_s = t^{-1}(\{s\})$ with zero shift. This development remains close to the FLRW solution described in* (2.1.2a)-(2.1.2c) *in the past of the initial data slice Σ_{t_0}. In particular, the solution exhibits curvature blow-up of order t^{-4} and every causal geodesic becomes incomplete as t approaches* 0.

THEOREM 2.1.2 (Global stability). *Let $(M, g_0, k_0, \pi_0, \psi_0)$ be initial data as in Theorem 2.1.1. In addition, we suppose that the smallest positive eigenvalue of $-\Delta_\gamma$ acting on scalar functions is strictly greater than $\frac{1}{9}$.*
Then, the initial data admits a maximal globally hyperbolic development $((0, \infty) \times M, \overline{g}, \phi)$ solving the Einstein scalar-field system that, in addition to the results of Theorem 2.1.1, is future (causally) complete. As $t \nearrow \infty$, the solution is attracted by Milne spacetime in the sense that the expansion normalized variables $(\boldsymbol{g}, \boldsymbol{k}, \nabla\phi, \phi')$ (see Definition 2.9.4) converge toward $(\gamma, \frac{1}{3}\gamma, 0, 0)$.

A more detailed statement of Theorem 2.1.1 is provided in Theorem 3.8.1. The additional spectral condition in Theorem 2.1.2 is discussed at the end of Subsection 2.1.3, and the statement itself is proven in Section 2.10 to be an extension of the Milne future stability result in Theorem 2.9.1.

2.1.2. Background material. We now provide context for the previously discussed setting and the results in Theorems 2.1.1 - 2.1.2:

2.1.2.1. *Initial data to the Einstein scalar-field equations.* It is well known that the Einstein equations can, via the 3+1 decomposition, be viewed as an elliptic-hyperbolic system of PDEs (see, for example, [AM03]). This reduces solving the Einstein equations to two problems: finding admissible Einstein initial data in physical space, and then solving the corresponding initial value problem. Regarding the former, initial data to the Einstein scalar-field system takes the form

$$(M, \mathring{g}, \mathring{k}, \mathring{\pi}, \mathring{\psi}),$$

where $\mathring{g}$ and $\mathring{k}$ are symmetric $(0,2)$-tensors on M, $\mathring{\pi}$ is an exact $(0,1)$-tensor (corresponding to $\nabla\phi$) and $\mathring{\psi}$ is a scalar function (corresponding to the future directed normal derivative $\partial_0\phi$ of the scalar field). The initial data must satisfy the Hamiltonian and momentum constraints

$$\mathrm{R}[\mathring{g}] + \left(\mathring{k}^a{}_a\right)^2 - \left(\mathring{k}^a{}_b\mathring{k}^b{}_a\right) = 8\pi\left[|\mathring{\psi}|^2 + |\mathring{\pi}|^2_{\mathring{g}}\right], \tag{2.1.3a}$$

$$\mathrm{div}_{\mathring{g}}\mathring{k} = -8\pi\cdot\mathring{\pi}\cdot\mathring{\psi} \tag{2.1.3b}$$

(see (2.2.16a) and (2.2.16b)), where the indices of $\mathring{k}$ in the first line are raised with respect to $\mathring{g}$.
We note that, in our argument, we will additionally assume that our initial data has constant mean curvature so that our gauges can be satisfied initially – this is enforced on the level of initial data by requiring

$$\mathrm{tr}_{\mathring{g}}\mathring{k} = -3\frac{\dot{a}(t_0)}{a(t_0)}$$

(see (3.2.1)). We will argue in Remark 2.8.1 why the initial data being near-FLRW allows us to assume the initial hypersurface to be CMC without loss of generality.

The results of [FB52; CBG69] show that there exists an embedding[2] $\iota : M \hookrightarrow \iota(M) \subset \overline{M}$ and a maximal solution $(\overline{M}, \overline{g}, \nabla\phi, \partial_0\phi)$ to the Einstein scalar-field equations such that $\iota(M) = \Sigma_{t_0}$ is a Cauchy hypersurface and such that

$$\iota^*\overline{g} = \mathring{g},\ \iota^*k = \mathring{k},\ \iota^*\pi = \mathring{\pi}, \text{ and } \iota^*\partial_0\phi = \mathring{\psi}_0\,.$$

We will perturb around initial data corresponding to data for an FLRW spacetime at time $t = t_0$, i.e.,

$$(M \cong \Sigma_{t_0}, a(t_0)^2\gamma, -\dot{a}(t_0)\,a(t_0)\,\gamma, 0, C\,a(t_0)^{-3})\,.$$

Furthermore, the maximal globally hyperbolic development (MGHD) is unique (up to diffeomorphism), and thus we can assume $(\overline{M}, \overline{g}, \nabla\phi, \partial_0\phi)$ to be globally hyperbolic. However, these statements provide little information on the properties of the MGHD in the future and past of the initial data slice.

2.1.2.2. *Strong Cosmic Censorship.* In their groundbreaking papers on singularity theorems, Hawking [Haw67] and Penrose [Pen65] established very general criteria for the MGHD of spacetimes to become causally geodesically incomplete. Many spacetimes of physical relevance satisfy these criteria, including the spacetimes considered in this article. While giving us more information on the MGHD than the existence and uniqueness results mentioned above, a key issue in the application of this mathematical result to General Relativity is that no statement is made on how precisely the singularity comes about: In particular, such incompleteness (within a given regularity class) could either mean that the geodesic is inextendible – which must be caused by the blow-up of some geometric quantity – or that there exist multiple inequivalent extensions. While the latter behaviour is exhibited even for

[2] We usually ignore the embedding in notation.

some cosmological spacetimes (see, for example, the Taub solutions discussed in [CI93]), such behaviour is usually considered to be unphysical since it would imply a breakdown of determinism. The *Strong Cosmic Censorship Conjecture* (SCCC) posits in its most general form that, for generic solutions to the Einstein equations, this incompleteness instead manifests as inextendibility at a given level of regularity (e.g., $C^0, C^2, C^\infty, \dots$).

In certain frameworks in the homogeneous cosmological setting – i.e. for homogeneous initial data on a closed spatial hypersurface –, it was shown in fundamental works by Chrusciel-Rendall [CR95] and Ringström [Rin09b] that the so called Kretschmann scalar $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ is unbounded where incompleteness manifests. Thus, it is the driving force behind geodesic incompleteness in these cases, forcing C^2-inextendibility of the MGHD. For the purposes of analyzing cosmologically relevant spacetimes, the SCCC is hence often rephrased as follows:

Conjecture 2.1.3 (Cosmological SCCC). *(See e.g. [Rin09b, Chapter 17]) For generic initial data, the Kretschmann scalar is unbounded where causal geodesics become incomplete.*

Theorem 2.1.1, in short, shows that this conjecture is rigorously supported in the case of FLRW spacetimes with negative spatial curvature. More precisely, the past asymptotics of such spacetimes, determined by initial data on Σ_{t_0} as discussed above, are generic in the following sense: There exists an open neighbourhood of said FLRW data within the set of Einstein scalar-field initial data such that the solutions past directed causal geodesics become incomplete, and the incompleteness is driven by blow-up of Kretschmann scalar with the same asymptotics as the FLRW solution. The global result in Theorem 2.1.2 portrays the other side of Cosmic Censorship – as with the past evolution, near-FLRW data fully determines the future of the spacetime in the sense that the MGHD is future complete, again showing that this feature of FLRW spacetimes with negative spatial sectional curvature is generic.

2.1.2.3. *FLRW and generalized Kasner spacetimes with scalar fields.* On a large scale, the universe is often viewed as spatially homogeneous and isotropic, i.e., no point in space and no direction are distinguishable from any other point and direction (referred to as the "Cosmological Principle"). In 1935, it was shown by Robertson and Walker that, under a few very natural additional assumptions, this restricts the class of potential spacetimes to the FLRW class

$$\left(I \times \tilde{M},\, \tilde{g}_{FLRW} = -dt^2 + a(t)^2\tilde{\gamma}\right),$$

where $(\tilde{M}, \tilde{\gamma})$ is a manifold of constant sectional curvature κ and where the scale factor a depends smoothly on t. This holds before taking the Einstein equations into consideration – when doing so, the matter model determines how space expands within the cosmological model via a. We refer to Lemma 3.2.1 for the scalar-field solution for $\kappa = -\frac{1}{9}$, but note that the scale factor behaves like $t^{\frac{1}{3}}$ for scalar-field matter, regardless of spatial geometry, and that the Kretschmann scalar blows up at order $\mathcal{O}\left(t^{-4}\right)$ toward the Big Bang ($t \downarrow 0$).
Spatially flat FLRW spacetimes are a subclass of the closely related *generalized Kasner spacetimes*, which are still spatially homogeneous but anisotropic in general. For scalar field matter,

the spacetime metric is given by

$$\overline{g}_{Kasner} = -dt^2 + \sum_{i=1}^{D} t^{2p_i} dx^i \otimes dx^i, \quad \sum_{i=1}^{D} p_i = 1, \sum_{i=1}^{D} p_i^2 = 1 - 8\pi A^2, \quad \overline{\phi}_{Kasner}(t) = A\log(t)\,.$$

The standard Kasner family is obtained by considering the vacuum case ($A = 0$), and the spatially flat FLRW spacetime by setting $D = 3, p_i = \frac{1}{3}, A = \sqrt{\frac{1}{12\pi}}$. If more than one of the Kasner exponents is non-zero, the generalized Kasner family satisfies the SCCC, also by exhibiting Kretschmann scalar blow-up of order t^{-4}as $t \downarrow 0$ (see [RS18b, (1.8)]).

Kasner spacetimes are of particular relevance to cosmology due to their relationship with the *BKL conjecture*: Heuristically, this conjecture states that the dynamics of cosmological spacetimes near a spacelike singularity generically exhibit chaotic and highly oscillatory behaviour, often referred to as "Mixmaster" behaviour. This behaviour is driven by velocity terms within the Einstein equations and is locally comparable to that of (vacuum) Kasner solutions. However, even if the BKL picture is to be believed in general, scalar-field (or, more generally, stiff-fluid) solutions seem to form an exception to it: They have a dampening effect on said oscillations, thus generating Big Bang stability as shown rigorously in [RS18b; FRS23] for Kasner spacetimes (for more details, see Section 2.1.3). This scenario, often referred to as *quiescent cosmology*, was studied in, for example, [BK73; Bar78; AR01]. With this in mind, both the aforementioned Kasner results and the results within this article, along with the prior FLRW results [RS18b; Spe18], confirm this quiescent effect of scalar fields in cosmology.

We note that one can view this as a scalar field ensuring a specific scenario in the very early universe given a class of initial data, namely matching the asymptotic behaviour of the Big Bang singularity. This fits into the recent use of nonlinear scalar fields in string cosmology, where specific choices of field are made to specific behaviours (e.g., inflation) in the early universe. For a recent review, we refer to [Cic+24].

2.1.3. Relation to previous work. Theorem 2.1.2 is the first theorem about the full global structure of FLRW spacetimes with negatively curved spatial geometry. For such solutions, prior results exclusively concern future stability, which we further discuss below. Besides [Spe18] covering the $\mathbb{S}^3$-case, it is the only open set of initial data for cosmological spacetimes (i.e., without symmetry assumptions) with $\Lambda = 0$ and in absence of accelerated expansion for which the global (future and past) dynamics are now fully understood.[3]

Scalar field matter (and, more generally, matter obeying semilinear wave equations or fluid matter) and their asymptotic behaviour on fixed cosmological backgrounds have been studied extensively, for example in [AR10; AFF19; Bac19; Rin19; BO24c; Rin20; Rin21; Wan21]. While many of the results, in particular [Rin20], manage to analyze very general classes of equations and spacetime geometries, including the wave equation on the FLRW

[3]For a related future stability result in accelerated expansion, see [Rin08] which considers scalar fields with a non-trivial potential.

backgrounds studied in [AFF19; FU22], the methods used are often difficult to apply to the full Einstein scalar-field system. In [FU22], we extended the approach of [AFF19] to be able to deal with various warped product spacetimes, and in particular FLRW spacetimes with negatively curved spatial geometry, by using the spatial Laplace operator to control high order derivatives. The perturbation-adapted analogue of this strategy is at the basis of the energy method in this paper.

We also note that, by the results of [GaNS19], there are non-trivial waves on fixed FLRW backgrounds that converge toward the Big Bang singularity, even if, as demonstrated in [AFF19; FU22], this behaviour is non-generic. Such waves can give rise to convergent asymptotics on cosmological backgrounds as studied in [Rin20]. Thus, it will likely be difficult to replace (2.1.2c) with an arbitrary non-trivial reference wave while keeping past stability intact. However, by restricting to an open neighbourhood near the solution described in (2.1.2a)-(2.1.2c), potential non-generic solutions of this type are excluded. For the more general conditions on initial data that lead to quiescent asymptotics, we refer to [OGPR23], which will be discussed further below.

Theorem 2.1.1 forms the counterpart to the pioneering works by Rodnianski-Speck [RS18a; RS18b] and Speck [Spe18], which cover nonlinear Big Bang stability for FLRW spacetimes with spatial geometry $\mathbb{T}^3$ and $\mathbb{S}^3$ respectively. These results were extended to Kasner spacetimes in [RS22] with $|q_i| < \frac{1}{6}$, and to the full subcritical regime in [FRS23], i.e., (generalized) Kasner spacetimes as discussed in Section 2.1.2.3 with $\max_{i,j,k=1,\dots,D}(p_i + p_j - p_k) < 1$. The former necessitates considering $1 + D$-dimensional Kasner spacetimes with $D \geq 38$, while the latter result also can be satisfied in $D = 3$ for generalized Kasner spacetimes. Recall that this means, in contrast to our setting, that the reference spacetime can be anisotropic, even if the conditions on Kasner exponents rule out extremely anisotropic regimes. As a result, the analysis therein becomes significantly more involved, especially at top order, since approximately monotonic energy identities as used in our work as well as in [RS18b; Spe18] have not been found in these anisotropic settings.

We note that the argument in [FRS23] relies on identifying an almost-diagonal structure for the asymptotics of (combined) connection coefficients for an adapted frame that is carried along by Fermi-Walker transport; this is precisely where subcriticality enters. Given that these no longer can vanish in a reference frame adapted to near-hyperbolic spatial geometry, it is a priori unclear whether this structure is sufficiently maintained.
The impressive recent preprint [OGPR23] by Oude Groeniger, Petersen and Ringström circumvents this issue and uses the equations considered in [FRS23] to establish general conditions for initial data to the Einstein (non-linear) scalar-field equations to give rise to quiescent singularities (see [OGPR23, Theorem 12]). Additionally, they show that a large class of cosmological model solutions to exhibit stable Big Bang formation (see [OGPR23, Theorem 49]). In particular, by only requiring that the mean curvature is sufficiently large compared to the expansion-normalised data, the rescaled connection coefficients can be made to be sufficiently

small even if they are non-trivial in the reference. However, this high level of generality comes at the cost of no longer being able to ensure that the expansion-normalized solution variables themselves, in particular the generalized Kasner exponents, remain close to the reference solution, in contrast to our asymptotic results in Theorem 3.8.1.

Furthermore, Beyer and Oliynyk have recently shown in [BO24a] that, over $\mathbb{T}^3$, the Big Bang formation can be localized in the sense that data given solely on a ball within the initial hypersurface must also cause stable blow-up on a (smaller) ball on the Big Bang hypersurface. While this result further indicates that blow-up behaviour of near-FLRW spacetimes might be, at least, independent of global geometric properties as it seems to be a localizable, we note that proof of localized stability crucially relies on the flatness of the conformal reference spacetime. To be more precise, the proof relies on extending the local initial data to global data for a Fuchsian system of metric and matter quantities as well as, again, connection coefficients for an adapted, Fermi-Walker transported frame. However, the derivation of the system for the former explicitly seems to use flat spatial geometry to obtain the necessary Fuchsian form. This form seems to similarly be broken as soon as the connection coefficients are not perturbed around 0, since this would lead to inhomogeneous error terms of order t^{-1} for the rescaled variables which are stronger than what the method, so far, accounts for.

By contrast, in [RS18b; Spe18], the reference frame itself is used in the commutator method to obtain the necessary energy identities at high orders. In all of these works, it hence is a priori unclear how one could extend these methods to the negative spatial Einstein geometry of (M, γ). We provide an alternative approach that, besides establishing the complementary stability result to [RS18b; Spe18], does not rely on any information on the spatial geometry of the reference manifold in its methodology (although it is of course relevant in determining the FLRW reference solution that we are studying). Instead, we rely on differential operators adapted to the evolved spatial metric. Hence, we believe that our approach may also prove useful for stability problems in spatially inhomogeneous (and hence also anisotropic) settings. In light of [RS22; FRS23] in particular, the main challenge in achieving this would either be to find approximately monotonic energy identities with our Bel-Robinson approach that have not been observed previously, or to also find ways to circumvent the lack thereof.

To obtain Theorem 2.1.1, we use the Laplace-Beltrami-operator (acting, respectively, on scalar functions and tensor fields) with respect to the (rescaled) evolved metric as our commutating operator instead of a fixed reference frame. This, in turn, leads us to replacing the wave-like system for metric and second fundamental form exploited in [RS18b; Spe18] by an evolutionary system in the second fundamental form and Bel-Robinson variables. The latter technique dates back to the fundamental works by Christodoulou-Klainerman [CK90; CK93], where it was used to analyse field equations on Minkowski space and then to show global stability of Minkowski space itself. It has also been applied to the future stability of Milne spacetimes in the vacuum Einstein equations by Anderson-Moncrief in [AM04] and, more recently, within the massive Einstein Klein-Gordon system by Wang in [Wan19]. As far as we are aware, this method has not yet been applied to solutions that are not near-vacuum or in the context of Big Bang singularity formation.

Toward the Big Bang, the solutions exhibit asymptotically velocity dominated (AVTD) behaviour in the sense that they behave, to leading order, like solutions to the Einstein scalar-field equations in CMC gauge with zero shift with all terms involving spatial derivatives set to zero (the "velocity term dominated" (VTD) equations). This behaviour also matches results obtained by studying high regularity solutions (e.g., [AR01]), or related works using Fuchsian methods that prescribe a behaviour at the singularity and then develop it locally, often under additional symmetry assumptions (e.g., [Dam+02; CBIM04; IM02; FL23]). In particular, this asymptotic behaviour leads to the same types of "Kasner footprint states" as in [RS18a; RS18b]: As one approaches the Big Bang, the rescaled variables converge toward tensor fields on the Big Bang hypersurface that precisely solve the truncated VTD equations. Further, the distance between the footprints of the FLRW and the perturbed solution are controlled by the initial data. For example, the rescaled Weingarten map $a^3k^a{}_b$ converges to $(K_{Bang})^a{}_b$ on the Big Bang hypersurface, which is close to $\frac{\sqrt{4\pi}}{3}C\mathbb{I}^a_b$, the rescaled FLRW footprint (see (3.8.2e) and (2.8.5c)).

What remains to be considered to obtain Theorem 2.1.2 is future stability, which we can reduce to future stability of the vacuum solution in the Einstein scalar-field system. This solution, called the Milne spacetime, has been shown to be stable within the set of vacuum solutions – see [AM11] – and a range of other Einstein systems – see, for example , [Wan19; AF20; FW21; FOW24; BF22; BFK19] and related work in lower dimensions, e.g. [AMT97; Mon08; Faj17a; Faj20; Mon23]. As such, our contribution to the study of future stability of Milne spacetimes is that we deal with the massless scalar field matter via corrected energy estimates which are inspired by work of Choquet-Bruhat and Moncrief in [CBM01] for vacuum Einstein equations with $U(1)$-symmetry. Out of the works listed above, only [Wan19; FW21] deal with scalar field matter at all, namely the massive case. These fields exhibit stronger decay toward the future, making the matter components easier to deal with than in our analysis.
The additional spectral condition is needed to ensure coercivity of the corrected scalar field energy. Numerical work, e.g. [CS99; Ino01], does not suggest that this condition is violated by any closed 3-manifold with constant sectional curvature $\kappa = -\frac{1}{9}$, and verifies that is is satisfied, for example, by an analogue of Weeks space in which the metric is appropriately scaled to have the required sectional curvature. The latter is also verified by the recent result [BMP25] that, amongst considering more general related settings, sufficiently constrains the spectrum of the Laplacian on Weeks space. We refer to Remark 2.9.3 where this discussed in more detail.

2.1.4. Challenges in the proof. The contracting and expanding regimes of near-FLRW spacetime are analyzed in two separate and methodologically independent parts. Before providing an overview of both arguments, we summarize the challenges that arise:

2.1.4.1. *Big Bang stability.* The main difficulties in establishing Big Bang stability are three-fold:

Firstly, we have to expect that the solutions are asymptotically velocity term dominated (as argued in Remark 2.8.3, we end up proving that this is the case), and thus that rescaled variables at best exhibit the same asymptotic behaviour as their counterparts in FLRW spacetime, up to a small perturbation in the asymptotic footprint. For example, note that, in the reference FLRW spacetime, one has

$$(k_{FLRW})^i{}_j = -3\frac{\dot{a}}{a}\mathbb{I}^i_j \approx -\frac{1}{t}\mathbb{I}^i_j\,.$$

At best, the shear $\hat{k}^i_j$ of the perturbed solution then behaves like $\frac{\varepsilon}{t}$. In fact, we show that this is the case in (3.4.1b). This implies that the contraction rescaled metric $G_{ij} = a^{-2}g_{ij}$ can only be controlled up to $\mathcal{O}\left(t^{-c\sqrt{\varepsilon}}\right)$ (see (3.4.3c)), since one has $\partial_t g_{ij} \approx -2g_{il}k^l{}_j$ and thus

$$\partial_t G_{ij} \approx G_{il}\hat{k}^l{}_j \approx \frac{\varepsilon}{t} * G\,.$$

However, to be able to use the structure of the evolution equations to cancel terms in our energy arguments, we have to work with adapted quantities. For example, we need to use integration by parts with respect to (Σ_t, G_t) to cancel high order scalar field terms with help of the (rescaled) wave equation that contains Δ_G, or to obtain elliptic estimates from the lapse equation via the operator Δ_G or from the adapted div-curl-system for Σ arising from the constraint equations.

As a result, even the rescaled solution variables will diverge at order $\mathcal{O}\left(t^{-c\sqrt{\varepsilon}}\right)$ toward the singularity, so we need to track and control their rate of divergence within the bootstrap argument. This significantly complicates dealing with nonlinear terms, where the bootstrap assumptions often cannot be inserted naively. This in turn makes coercivity of the energies more involved to establish (see Lemma 2.4.5 and Remark 2.4.6), since this only holds up to curvature errors that also diverge and thus need to be carefully tracked.

Secondly, and in contrast to [RS18b; Spe18], replacing the wave structure of the geometric evolution in the Einstein equations with our less geometry dependent Bel-Robinson framework seems to lose regularity at first glance: The energy estimates for the evolution system for the scalar field energy and the geometric energies can be caricatured as follows:

$$-\frac{d}{dt}\mathfrak{E}^{(L)}(\phi,\cdot) \lesssim \frac{\varepsilon^{\frac{1}{8}}}{t}\left[\mathfrak{E}^{(L)}(\phi,\cdot) + \mathfrak{E}^{(L)}(\Sigma,\cdot)\right] + \dots$$

$$-\frac{d}{dt}\left[\mathfrak{E}^{(L)}(\Sigma,\cdot) + \mathfrak{E}^{(L)}(W,\cdot)\right] + \dots \lesssim \frac{\varepsilon^{\frac{1}{8}}}{t}\left[\mathfrak{E}^{(L)}(\Sigma,\cdot) + \mathfrak{E}^{(L)}(W,\cdot)\right] + \frac{\varepsilon^{-\frac{1}{8}}}{t}\cdot a^4\mathfrak{E}^{(L+1)}(\phi,\cdot) + \dots$$

Herein, the superscript refers to the order of derivatives, while $\mathfrak{E}^{(L)}(\phi,\cdot),\mathfrak{E}^{(L)}(\Sigma,\cdot)$ and $\mathfrak{E}^{(L)}(W,\cdot)$ refer to energies for the scalar field, the rescaled tracefree part Σ of second fundamental form and the Bel-Robinson variables respectively. Thus, it seems that we lose derivatives in the scalar field and are not able to close the argument. This is remedied using the div-curl-system in Σ, see (2.2.36a) and (2.2.36b), which yields a weak estimate of the form

$$a^4\mathfrak{E}^{(L+1)}(\Sigma,\cdot)\lesssim\mathfrak{E}^{(L)}(\phi,\cdot)+\mathfrak{E}^{(L)}(W,\cdot)+\mathfrak{E}^{(L)}(\Sigma,\cdot)+\dots .$$

Combining these estimates to improve the bootstrap assumptions then necessitates an intricately constructed total energy to balance these different types of estimates against one another.

Finally, given (2.1.2c), the rescaled time derivative of the scalar field is not small and does not become so toward the Big Bang. This leads to various terms within the core linearized evolutionary system of both matter and geometry that, if estimated naively, could lead to exponential blow-up toward the singularity. When such terms occur in the scalar field energy evolution, this can be dealt with along similar lines as in [RS18b; Spe18], but we incur additional large terms in our geometric evolution that only cancel using the explicit form of the Friedman equations, which we highlight in Lemma 3.7.1 and its proof.

2.1.4.2. *Future and global stability.* For Milne stability, the canonical Sobolev energies for the scalar field variables, i.e.,

$$\int_M |\phi'|^2_{\boldsymbol{g}}+|\nabla\phi|^2_{\boldsymbol{g}}\,\mathrm{dvol}_{\boldsymbol{g}}$$

and higher order analogues, do not obey useful energy estimates. This can be overcome by adding an indefinite correction term of the type

$$\int_M \phi'(\phi-\overline{\phi})\mathrm{dvol}_{\boldsymbol{g}}$$

to the canonical energy, see Definition 2.9.6. This is similar to what was done in [CBM01] in a $2+1$-dimensional setting, as well as similar to the indefinite terms we introduce in our geometric energy to control the wave system in the metric variables, as in previous work on Milne stability in different matter models, including [AF20; FW21]. That this corrected energy controls Sobolev norms relies on the aforementioned spectral condition. As a result, and unlike for past stability, the specific spatial geometry is crucial in generating decay from energy estimates, even before considering the geometric evolution.

Moreover, we need to transition from the near-FLRW data used to analyze the contracting regime to data in the expanding regime on a distant enough future hypersurface such that it is near-Milne and the future stability result applies. This requires a gauge switch from CMC gauge with zero shift to CMCSH gauge, as well as careful control of the solution variables over a finite time interval using continuous dependence on initial data. For the former, close inspection of [FK20] gives us a diffeomorphism close to the identity that maps the initial data for the metric to new data satisfying the spatially harmonic gauge condition, thus allowing us to switch gauges without losing proximity to the reference solution. This is discussed in detail in Section 2.10.

2.1.5. Proof outline.

2.1.5.1. *Big Bang stability.*

The big picture. The key argument in our Big Bang stability proof is a hierarchized series of energy estimates that establishes the asymptotic behaviour of solution variables toward the singularity. We rely on a bootstrap argument which establishes that energies $\mathfrak{E}^{(L)}$ (see Definition 3.3.3) for the scalar field, the rescaled shear, the Bel-Robinson variables, the lapse and the curvature at worst only diverge slightly. Here, $0 \leq L \leq 18$ denotes the order of derivatives considered. To this end, we make a bootstrap assumption on the solution norm $\mathcal{C}$ (see Definition 3.3.2) which controls the distance of these rescaled variables, as well as the metric itself, to their FLRW counterparts in terms of supremum norms with respect to G, where $G = a^{-2}g$ is the rescaled *adapted* spatial metric (see Definition 3.2.3). We refer to Assumption 3.3.11 and Remark 2.3.19 for the detailed bootstrap assumptions and improvements, as well as to Lemma 2.3.14 for the underpinning local well-posedness result. That this bootstrap argument implies Theorem 2.1.1 follows from a straightforward adaptation of the arguments in [RS18b, Theorem 15.1].

We work with evolution-adapted norms even though $G(t,x)$ degenerates toward the Big Bang singularity. Indeed, since we need to exploit the structure of the evolutionary equations, it is more convenient to have these adapted quantities controlled by the solution norms $\mathcal{H}$ and $\mathcal{C}$ directly instead of having to perform changes of metric at that point. Once the improved energy estimates are shown, a (time-scaled) coercivity notion (see Lemma 2.4.5 and the proof of Corollary 2.7.3) and Sobolev embeddings with respect to the reference metric γ then ensure that these improved estimates translate to $\mathcal{H}$ and $\mathcal{C}$. This then closes the bootstrap. To actually achieve this improved energy behaviour, we derive elliptic energy estimates or integral-type estimates that, once suitably combined and scaled, yield the desired improvements by straightforwardly applying the Gronwall lemma. Additionally, note that we assume that the initial data is close to FLRW data not just in $\mathcal{H}$, which contains precisely the norms needed to control $\mathcal{C}$ by Sobolev embedding, but also scaled smallness assumptions at one order higher, contained in the top order semi-norm $\mathcal{H}_{top}$ (see Assumption 3.3.4). This is needed to ensure that the top order energy is small initially, and thus to close the bootstrap.

Scale factor $a(t)$. The precise structure of the Friedman equations (3.2.3a)-(2.2.4) is crucial not only to control time integral quantities up to the Big Bang hypersurface (see Lemma 3.2.2), but also to ensure that certain terms in the evolution that would otherwise cause large divergences contribute with favourable sign (see the arguments in Lemma 3.5.5

as well as Lemma 3.7.1). It turns out that the sectional curvature entering the Friedman equations actually is not of key importance to large parts of the Big Bang stability analysis: The leading order behaviour of the scale factor toward the Big Bang singularity is determined via the Friedman equation (2.1.2b) by the matter term, not the sectional curvature. This indicates that our method might extend to different settings.

Gauge choice, commutation method and Bel-Robinson variables. We commute the resulting elliptic-hyperbolic Einstein system with the Laplace-Beltrami operator Δ_G with respect to the rescaled evolved spatial metric $G(t,x)$ to obtain higher order energy control. Commuting with this operator has the advantage of leaving many integration-by-parts identities intact. These are needed to provide specific cancellations, e.g., to cancel $\Delta^{\frac{L}{2}+1}\phi$-terms arising from the wave equation when computing $\partial_t\mathfrak{E}^{(L)}(\phi,\cdot)$. We also note that the only feature of the adapted metric we use is that it is close to γ, and do not use any further information on the geometry, e.g., by choosing a specific reference frame in our commutation method. Further, we employ CMC gauge with zero shift to avoid badly behaved shift terms (see Remark 2.1.4).

We still, however, need to deal with the Ricci term in the evolution equation for the second fundamental form. To this end, we consider the Bel-Robinson variables E and B which are Σ_t-tangent symmetric tracefree $(0,2)$-tensors and contain all information of the spacetime Weyl tensor $W[\overline{g}]$ (see Subsection 2.2.4). Suitably projecting the Gauss-Codazzi equations admits additional constraint equations in terms of E and B that allow us to replace the Ricci tensor at the "cost" of introducing Bel-Robinson energies into the formalism, see (2.2.24a) and the rescaled version (3.2.20j). Further, E and B satisfy a Maxwell-type system (see Lemma 2.2.7) that can be exploited to obtain energy estimates and, as with the other evolution equations, is well adapted to commutation with Δ_G.

A priori low order C_G-control. By applying the bootstrap assumptions on $\mathcal{C}$ to the evolution equations, we can immediately deduce improved low order estimates in C^l_G for $l\geq 10$ for the solution variables by inserting them into the respective evolution equations (see Lemma 3.4.2), as well as via the maximum principle for the lapse (see Lemma 2.4.1). These usually still diverge slightly, mostly due to the asymptotic behaviour of G. However and crucially to our argument, at order 0, the renormalized time derivative Ψ of the wave, the rescaled tracefree part Σ of the second fundamental form and the rescaled Bel-Robinson variable $\mathcal{E}$ are in fact $K\varepsilon$-small in C^0_G on the bootstrap interval (see Lemma 2.4.2). If these estimates did not hold, it would lead to terms that diverge at order $\mathcal{O}\left(a^{-3-c\sqrt{\varepsilon}}\right)$ in the differential inequalities, and thus cause exponential energy blow-up of order $\mathcal{O}\left(e^{a^{-c\sqrt{\varepsilon}}}\right)$ that we could no longer control. This behaviour is closely related to the fact that Ψ and Σ converge toward footprint states on the Big Bang hypersurface that remain $K\varepsilon$-small (see (2.8.3c) and (3.8.2e)), and then pass this convergence on to $|\mathcal{E}|_G$ (see (2.8.8a)).

Energy estimates and hierarchy. The main part of the analysis is establishing various energy estimates.

- For the lapse (see Section 2.5), the relevant estimates are direct results of the elliptic lapse equations (3.2.20f)-(3.2.20g). The non-lapse terms on the right hand side of 3.2.20f only diverge slightly toward the Big Bang, in contrast to the divergence at order a^{-4} in (3.2.20f), and thus allows one to show that, at lower derivative order, the lapse converges to 1. However, since the right hand side of (3.2.20g) contains the scalar curvature of G, this estimate loses derivatives. On the other hand, (3.2.20f) does not lose derivatives, and the elliptic nature in fact allows one to estimate lapse energies of order $L+2$ by energies in Σ and the scalar field of order L. This makes it possible to control the higher order lapse term occurring, for example, in (2.2.28c), without losing regularity. Conversely, both of these gains in regularity are at the cost of losing powers of a. In short, (3.2.20g) is needed to establish the asymptotic behaviour of the lapse, and (3.2.20f) to obtain improved energy bounds as a whole.
- The core matter energy estimate (see Lemma 3.5.5) relies on delicate cancellations when computing the time derivative of $\mathfrak{E}^{(L)}(\phi,\cdot)$. While we derive this in a fashion that differs from the energy flux method used in [Spe18], the necessary cancellations to arrive at Lemma 3.5.5 are similar.
- The (rescaled) tracefree component of the second fundamental form Σ (see Lemma 3.5.9) and the (rescaled) Bel-Robinson variables $\mathcal{E}$ and $\mathcal{B}$ (see Lemma 3.5.8) need to be treated simultaneously to deal with the leading curvature term in the evolution of the former by inserting a constraint equation in which $\mathcal{E}$ occurs as the leading term (see (2.2.36d)). However, the matter terms within the evolution of $\mathcal{E}$ and $\mathcal{B}$ contain, firstly, terms where we again need very precise estimates to show that they do not contribute large a^{-3}-divergences, and, secondly, matter terms that lose one order of derivative.
  This order of regularity can be regained using the momentum constraint equation (2.2.36a) and its Bel-Robinson counterpart (2.2.36b) containing $\mathcal{B}$, which leads to a div-curl-system for Σ (see Lemma 3.5.10). This is, again, at the cost of losing powers of a.
- As a result, the core Gronwall argument performed in Proposition 2.7.2 combines energies for the matter variables, Σ and the Bel-Robinson variables, as well as energies for $\mathrm{Ric}[G]$. In particular, the curvature energies are necessary to handle commutation errors within the energy estimates, and improved bounds on them need to be obtained to apply the coercivity results in Lemma 2.4.5 – else, none of energy improvements would extend to improved Sobolev norm bounds and the bootstrap argument would not close.
  As many of the a priori C_G-norm estimates add small additional divergences, it is necessary to perform an induction over derivative orders within this mechanism to deal with lower order error terms. Since Δ_G is elliptic, it is sufficient to perform this for even orders. Along with energies at order $L \in 2\mathbb{N}_0$, the total energy also

includes the energy controlling Σ as well as the scalar field and curvature energies at order $L+1$, appropriately scaled to account for the degenerate elliptic estimate for Σ from Lemma 3.5.10. This remedies the derivative loss in the Bel-Robinson energy and allows one to improve the total energy at each order until reaching $L = 18$, at which point the bootstrap argument can be closed.

- Note that *the metric itself does not enter the core energy mechanism.* In fact, trying to replace control of the Ricci tensor by control of G is likely too imprecise in dealing with high order curvature errors. Instead, control of $G-\gamma$ and $\Gamma[G]-\hat{\Gamma}[\gamma]$ is a consequence of a simple integral energy inequality and the improvements achieved for Σ and matter variables (see Lemma 3.5.11 and Corollary 2.7.3). Since we cannot utilize any additional structure in dealing with the metric, we have to construct our argument carefully to allow for the metric control to be weaker than what one gets for the core variables, while still being sufficiently strong to constitute an improvement and allowing to switch between H_G and H_γ (and, respectively, C_G and C_γ) norms.

We also point to Remark 2.6.1 for a more detailed sketch of how the integral inequalities for the core Gronwall argument are structured and how this leads to the bootstrap improvement for the energies.

2.1.5.2. *Future stability and connecting the regions.* We follow similar lines as in [AF20; FW21] to prove that near-FLRW spacetimes in negative spatial geometry are future stable. Since $\partial_t\phi$ decays like $a^{-3} \simeq t^{-3}$ in the reference spacetime, the sectional curvature becomes dominant in the Friedman equations and the scale factor approaches that of Milne spacetime as t approaches ∞. Hence, if one moves sufficiently far to the future, choosing near-FLRW data with a homogeneous scalar field is equivalent to choosing near-vacuum data. Thus, what we prove first in Section 2.9 is future stability of near-Milne spacetimes under the Einstein scalar-field system. Once this is established, we argue in Section 2.10 how early near-FLRW initial data evolves to data that is sufficiently close to Milne for large enough times, which is essentially a consequence of the scale factor and the (physical) mean curvature approaching that of Milne, up to a multiplicative constant.

In terms of dealing with geometric and elliptic estimates, we can essentially carry over the results of [AF20], as was also done in [FW21], by working in CMCSH gauge and verifying that the matter components are indeed only perturbative terms within the geometric evolution.

This leaves only the scalar field to be examined. Here, we introduce corrective terms to the energies (see Definition 2.9.6) which yield decay estimates for the corrected scalar field energy (see Lemmas 2.9.16 and 2.9.17). That these energies are coercive (see Lemmas 2.9.12 and 2.9.13) requires the aforementioned lower bound for the first positive eigenvalue of $-\Delta_\gamma$.

Remark 2.1.4 (Why not use CMCSH gauge to prove Big Bang stability?). One might consider applying this gauge to Big Bang stability as well since this is precisely the choice of gauge turning the geometric evolution into a wave-like system in (g, k), which seems simpler than our chosen approach in CMC gauge with zero shift. In particular, this would also not rely on any choice of reference frame, and keep the wave structure of the geometric evolution intact, unlike when using Bel-Robinson variables. However, the issue with this approach lies in the shift equation, which would take the following form for the rescaled shift vector $X = a^3\tilde{X}$:

$$
\begin{aligned}
\Delta_G X^l + \mathrm{Ric}[G]^l_m X^m = & -2(N+1)(G^{-1})^{im}(G^{-1})^{jn}\Sigma_{ij}\left(\Gamma^l_{mn} - \hat{\Gamma}^l_{mn}\right) \\
& + 2(G^{-1})^{im}\nabla_i X^n \left(\Gamma^l_{mn} - \hat{\Gamma}^l_{mn}\right) \\
& + \langle \text{error terms in lapse and matter} \rangle
\end{aligned}
\tag{2.1.4}
$$

As a result, the first term has to be expected to diverge at the same rate as the metric, i.e., we expect even low order norms of $\tilde{X}$ to behave like $a^{-3-c\sqrt{\varepsilon}}$ at best up to small prefactors. However, computing the time derivative of an integral over $|G-\gamma|^2_G$ (or derivatives thereof) becomes the integral over the $(\partial_t - \mathcal{L}_{\tilde{X}})$-derivative of this quantity, and hence we get explicit terms of the form $\mathcal{L}_{\tilde{X}}\gamma$ which always exist at highest order and diverge worse than t^{-1}. In short, the fact that the metric cannot be expected to converge to a footprint state leads to leading order terms in the differential energy estimates to carry strongly divergent pre-factors in CMCSH gauge. This obstructs improvements in a tentative bootstrap argument.

2.1.6. Paper outline.

- Sections 3.2-3.8 cover the proof of Big Bang stability:
  - In Section 3.2, we introduce notation and provide the necessary information on the FLRW background solution as well as the equations relevant to the subsequent analysis.
  - Then, in Section 2.3, we discuss the solution norms and energies and state the initial data and bootstrap assumptions.
  - In Section 3.4, improved low order C_G-norm estimates that follow directly from the bootstrap assumptions are established, along with additional formulas and a priori estimates.
  - Section 2.5 concerns the elliptic estimates for the lapse.
  - In Section 2.6, we discuss the energy and Sobolev norm estimates for all other variables, all of which are integral estimates except for the aforementioned elliptic estimate for Σ, as well as a norm bound for $\nabla\phi$ that is not needed for the energy improvement.
  - These are all combined in Section 2.7 to improve the bootstrap assumptions – first for the energies, then for $\mathcal{H}$ and finally $\mathcal{C}$.
  - In Section 3.8, we show how this bootstrap argument implies the main Big Bang stability result (see Theorem 3.8.1, which is the formal version of Theorem 2.1.1).
- Section 2.9 contains the proof of near-Milne future stability.
- In Section 2.10, we show that this is sufficient for future stability of near-FLRW spacetimes, proving Theorem 2.1.2.

- The appendices (Sections 2.A-2.B) collect various basic formulas and commutator expressions as well as error terms and how these can be estimated.

2.2. Big Bang stability: Preliminaries

2.2.1. Notation.

2.2.1.1. *Foliations.* On a spacetime manifold $(\overline{M}, \overline{g})$, we assume the existence of a spacelike Cauchy hypersurface Σ_{t_0} that is diffeomorphic to M. As we argue in Remark 2.8.1, we can assume without loss of generality that it has constant mean curvature. We will ultimately show that there exists a time function t such that the past of $\Sigma_{t_0} = t^{-1}(t_0)$ can be foliated by $\Sigma_s = t^{-1}(s)$ for $s \in (0, t_0)$, and that where the solution exists, this is at least possible up to some $T \in (0, t_0)$. These constant time surfaces are then also spacelike Cauchy hypersurfaces diffeomorphic to M and CMC. We will use this notation throughout with little comment and often simply view Σ_s as $\{s\} \times M$.

2.2.1.2. *Metrics.* The spacetime metric $\overline{g}$ on $\overline{M}$ takes the general form

$$\overline{g} = -n^2 dt^2 + g_{ab}\, dx^a\, dx^b$$

where $n \equiv n(t, x)$ is the lapse function and $g|_{\Sigma_t} \equiv g|_{\Sigma_t}(t, x)$ is a Riemannian metric on Σ_t. We will often simply denote the spatial metric by g. Furthermore, we denote the rescaled spatial metric by $G_{ij} = a^{-2} g_{ij}$ (see Definition 2.2.9) and the tensor-field induced by the matrix inverse of (G_{ij}) by G^{-1}. Similarly, $\det g$ and $\det G$ are also meant as the determinants in the matrix sense. Finally, we define dvol_g and μ_g as the volume form and volume element with regard to g, and the same for γ and G.

2.2.1.3. *Indices and coordinates.* Greek indices $\alpha, \beta, \dots, \mu, \nu, \dots$ run from 0 to 3, lowercase latin indices $a, b, \dots,\ i, j, \dots$ from 1 to 3. The spatial indices on some coordinate neighbourhood $V \subseteq \overline{M}$ are always with regard to the local frame induced by coordinates (x^1, x^2, x^3) on M, applied to each $V \cap \Sigma_t$ by the standard embedding where this intersection is non-empty. The index 0 always denotes components relative to $\partial_0 = n^{-1}\partial_t$, where ∂_t is the derivative associate to the time function t. The Levi-Civita connections associated to $\overline{g}$, respectively g and G, are denoted by $\overline{\nabla}$, respectively ∇.[4]Additionally, for the hyperbolic spatial reference metric γ on M (see Definition 2.2.1), we write the Levi-Civita connection as $\hat{\nabla}$.

Whenever we raise or lower Greek (resp. Latin) indices without additional notation, it is with regard to $\overline{g}$ (resp. g). When we raise indices of a tensor $\mathfrak{T}$ with regard to the rescaled spatial metric G, we flag this by writing $\mathfrak{T}^\sharp$. We never raise or lower with respect to γ. Refer to Section 2.2.1.9 as to how we distinguish taking multiple covariant derivatives from index

[4]Note that g and G have the same Levi-Civita-connection since, on every hypersurface Σ_t, they are related by a scalar multiple.

raising.

2.2.1.4. Σ_t-*tangent tensors.* For any Σ_t-tangent tensor $\xi^{\alpha_1\dots\alpha_r}{}_{\beta_1\dots\beta_s}$, we write $\xi(t)^{a_1\dots a_r}{}_{b_1\dots b_r}$ for the $\overline{g}$-orthogonal projection of ξ onto the hypersurface Σ_t. When clear from context, we will drop the time dependency in notation.

2.2.1.5. *Sign conventions.* Within this paper, the second fundamental form with regard to Σ_t is defined as the $(0,2)$-tensor k given by

$$k(X,Y) = -\overline{g}(\overline{\nabla}_X\partial_0, Y)\,,$$

where X and Y are Σ_t-tangent vectors. The Riemann curvature tensor of $\overline{g}$ is taken to be

$$\overline{\nabla}_\alpha\overline{\nabla}_\beta Z_\gamma - \overline{\nabla}_\beta\overline{\nabla}_\alpha Z_\gamma = \mathrm{Riem}[\overline{g}]_{\alpha\beta\gamma}{}^\delta Z_\delta$$

for the covariant vector field (Z_μ), and the analogous convention holds for all other Riemann curvature tensors that appear.

2.2.1.6. *Constants.* For two nonnegative scalar functions ζ_1, ζ_2, we write $\zeta_1 \lesssim \zeta_2$ if and only if there exists a constant $K > 0$ such that $\zeta_1 \le K\zeta_2$. This implicit constant may depend on information from the FLRW reference solution at the starting point of the evolution (in particular on γ and $a(t_0)$, see Definition 2.2.1) and combinatorial quantities. We extend this notation to a real function ζ_1' by

$$\zeta_1' \lesssim \zeta_2 :\Leftrightarrow \max\left(\zeta_1', 0\right) \lesssim \zeta_2\,.$$

Additionally, we write $\zeta_1 \simeq \zeta_2$ if and only if $\zeta_1 \lesssim \zeta_2 \lesssim \zeta_1$ is satisfied.

2.2.1.7. *Tensor contractions.* We denote by $\boldsymbol{\varepsilon}_{\alpha\beta\gamma\delta}$ the Levi Civita tensor with regard to $\overline{g}$ and define the Levi-Civita tensor on spatial hypersurfaces Σ_t by $\boldsymbol{\varepsilon}[g]_{ijk} = \boldsymbol{\varepsilon}_{0ijk}$. Notice that this corresponds to the Levi-Civita tensor associated to g. Further, $\boldsymbol{\varepsilon}[G]_{ijk} = a^{-3}\boldsymbol{\varepsilon}[g]_{ijk}$ is the Levi-Civita tensor with respect to the rescaled metric G (see (2.2.27a)).

For Σ_t-tangent $(0,2)$-tensors $A, \tilde{A}$ and vector field v, we define the following objects as in [AM04, Section A.2]:

$$\begin{aligned}
A\cdot\tilde{A} &= A_{ab}\tilde{A}^{ab} = \langle A, \tilde{A}\rangle_g\\
(A\odot_g\tilde{A})_{ij} &= A_{ik}\tilde{A}^k{}_j\\
(A\wedge\tilde{A})_i &= \boldsymbol{\varepsilon}_i{}^{jp}A_j{}^q\tilde{A}_{qp}\\
(v\wedge A)_{ab} &= \boldsymbol{\varepsilon}_a{}^{cd}v_cA_{db} + \boldsymbol{\varepsilon}_b{}^{cd}v_cA_{ad}\\
(A\times\tilde{A})_{ij} &= \boldsymbol{\varepsilon}_i{}^{ab}\boldsymbol{\varepsilon}_j{}^{pq}A_{ap}\tilde{A}_{bq} + \frac{1}{3}(A\cdot\tilde{A})g_{ij} - \frac{1}{3}(tr_gA\cdot tr_g\tilde{A})g_{ij}\\
(\mathrm{curl}A)_{ij} = (\mathrm{curl}_gA)_{ij} &= \frac{1}{2}\left[\boldsymbol{\varepsilon}_i{}^{cd}\nabla_dA_{cj} + \boldsymbol{\varepsilon}_j{}^{cd}\nabla_dA_{ci}\right]\\
(\mathrm{div}_gA)_i &= \nabla^bA_{ib}
\end{aligned}$$

The operations $\odot_G$, $\langle\cdot,\cdot\rangle_G$ and div_G are defined analogously, with all indices raised and lowered by G instead of g. Finally, for two $(0,1)$-tensors $\pi,\tilde\pi$, we denote their symmetrized product by

$$(\pi\otimes\tilde\pi)_{ij}=\frac{1}{2}\left(\pi_i\tilde\pi_j+\pi_j\tilde\pi_i\right)\,.$$

For pointwise estimates of these quantities, refer to Lemma 2.A.3.

2.2.1.8. *Schematic term notation.* We will denote as $\mathfrak{T}_1*\cdots*\mathfrak{T}_l$, where $\mathfrak{T}_i$ are Σ_t-tangent tensors, any multiple of $(\mathfrak{T}_i)$, with regard to the rescaled adapted spatial metric G or as standard multiplication if no summation over indices occurs between factors. Constant prefactors and contractions with regard to G are also suppressed in this notation.

When working with terms where such notation is used, we will estimate these inner products by $\lesssim\prod_{i=1}^l|\mathfrak{T}_i|_G$, making any constant in front irrelevant, and further we can view any contraction with regard to G as a product of the non-contracted tensor $\mathfrak{T}$ with G or G^{-1}, and estimate that up to constant by $|G|_G|\mathfrak{T}|_G$, where the first factor is simply $\sqrt{3}$.
For similar products with respect to γ, we denote them by $*_\gamma$.

2.2.1.9. *On multiple derivatives of variables.* For a scalar function ζ, an (r,s)-tensor field $\mathfrak{T}$ and *capitalized* integers $I,J,\dots\in\mathbb{N}_0$, we denote by $\nabla^I\zeta$ and $\nabla^I\mathfrak{T}$ the tensors $\nabla_{l_1}\dots\nabla_{l_I}\zeta$ and
$\nabla_{l_1}\dots\nabla_{l_I}\mathfrak{T}^{i_1\dots i_r}{}_{j_1\dots j_s}$. We extend this notation to other covariant derivatives analogously. To avoid potential ambiguity with an index raised by g, we will apply the following convention:

- If a covariant derivative carries an uppercase letter, a formula with more than one symbol or a positive integer in its superscript, this refers to taking a derivative of that order.
- If a covariant derivative carries a lowercase letter or 0 in its superscript, this refers to an index.

Further, we will only apply this notation where the precise distribution of indices is not important (e.g. in schematic notation, see Section 2.2.1.8).

2.2.2. FLRW spacetimes and the Friedman equations. Herein, we collect the properties of the reference FLRW solution to the Einstein scalar-field system in CMC-transported coordinates. Our main focus will lie on the behaviour of the scale factor as determined by the Friedman equations. Before moving on to that, we collect the information on the spatial geometry we will need:

DEFINITION 2.2.1 (Hyperbolic reference geometry). (M,γ) is a three-dimensional, connected, closed, orientable Riemannian manifold with constant sectional curvature $-\frac{1}{9}$, and hence Ricci tensor $\operatorname{Ric}[\gamma]=-\frac{2}{9}\gamma_{ij}$ and scalar curvature $R[\gamma]=-\frac{2}{3}$.

REMARK 2.2.2 (Orientability is not a restriction). We assume that M is orientable for the sake of simplicity. If M should be non-orientable, we may pass the initial data to the oriented double cover and solve the problem there. Since the result is equivariant with respect to the double covering map, this then solves the original problem.

With this in hand, we can express our classical family of solutions to the Einstein scalar-field system as follows:

LEMMA 2.2.3 (FLRW solutions and Friedman equations). *Consider FLRW spacetimes* $(\overline{M}, \overline{g}_{FLRW})$ *with* $\overline{M} = (0,\infty) \times M$*, where* (M,γ) *is as in Definition 2.2.1 and where*

$$\overline{g}_{FLRW} = -dt^2 + a(t)^2\gamma \tag{2.2.1}$$

holds for some $a \in C^\infty((0,\infty))$*, with the conventions* $a(0) = 0$ *and* $\dot{a}(T) > 0$ *for some arbitrary* $T > 0$*. Further, choose a (smooth) scalar function* ϕ_{FLRW} *such that*

$$\partial_t\phi_{FLRW} = C \cdot a(t)^{-3}, \quad \nabla\phi_{FLRW} = 0, \quad \square_{\overline{g}_{FLRW}}\phi_{FLRW} = 0\,. \tag{2.2.2}$$

Such a pair $(\overline{g}_{FLRW}, \phi_{FLRW})$ *solves the Einstein scalar-field system* (2.1.1a)-(2.1.1b) *on* $\overline{M}$ *if and only if* a *satisfies the Friedman equation*

$$\dot{a} = \sqrt{\frac{1}{9} + \frac{4\pi}{3}C^2a^{-4}}\,. \tag{2.2.3}$$

In particular, one has

$$\ddot{a} = -\frac{8\pi}{3}C^2a^{-5}\,. \tag{2.2.4}$$

PROOF. The first statement follows from explicitly computing $\mathrm{Ric}[\overline{g}]$ as in [O'N83, p.345]. That (2.2.3) implies (2.2.4) follows simply by computing the derivative of $\dot{a}^2$. $\square$

In the subsequent analysis, the following properties of a that follow from (2.2.3) will be crucial for our analysis:

LEMMA 2.2.4 (Scale factor analysis). *Let* a *solve* (2.2.3) *with* $a(0) = 0$*. Then* a *is analytic on* $(0,\infty)$ *and extends to a continuous function on* $[0,\infty)$ *with* $a(t) \simeq t^{\frac{1}{3}}$ *being satisfied near* $t = 0$*. Further, for any* $p > 0$*, there exist constants* $c > 0$ *and* $K_p > 0$*, where* c *is independent of* p *and* K_p *depends analytically on* p*, such that, for any* $t \in (t, t_0]$*, one has*

$$\exp\left(p\int_t^{t_0} a(s)^{-3}\,ds\right) \leq K_p a(t)^{-cp} \tag{2.2.5}$$

and

$$\int_t^{t_0} a(s)^{-3-p}\,ds \lesssim \frac{1}{p}a(t)^{-p}, \quad \int_t^{t_0} a(s)^{-3+p}\,ds \lesssim \frac{1}{p}\,. \tag{2.2.6}$$

Moreover, for any $t \in (0, t_0]$ *and any* $q > 0$*, there exist constants* $c > 0$ *and* $K > 0$ *which both are independent of* q *such that one has*

$$\int_t^{t_0} a(s)^{-3}\,ds \leq \frac{K}{q}a(t)^{-cq}\,. \tag{2.2.7}$$

Finally, (2.2.3) *also implies*

$$\sqrt{\frac{4\pi}{3}}Ca^{-2} \leq \dot{a} \tag{2.2.8}$$

REMARK 2.2.5. We will use the estimates in Lemma 2.2.4 where p is a positive power of ε up to algebraic constants. Then, we can simply replace K_p in (2.2.5) by a uniform constant.

PROOF. For the first statement, we refer to [FU22, Lemma 2.1] with $\gamma = 2$. We also collect from there[5] that, for $t < t_0$,

$$\int_t^{t_0} a(s)^{-3}\,ds \lesssim 1 + \left|\log\left(\frac{t}{t_0}\right)\right|$$

is satisfied. Hence, there exists some $c' > 0$ such that

$$\exp\left(p\int_t^{t_0} a(s)^{-3}\,ds\right) \le K_p \exp(c'\cdot p\log(t_0))\cdot\exp(-c'\cdot p) \le K_p' t^{-c'p}\,.$$

Then (2.2.5) follows by applying $a(t) \simeq t^{\frac{1}{3}}$. Noting that $a^{-3} \simeq \dot{a}/a$ holds, one further has

$$\int_t^{t_0} a(s)^{-3-p}\,ds \lesssim \int_{a(t)}^{a(t_0)} y^{-1-p}dy = \frac{1}{p}\left(a(t)^{-p} - a(t_0)^{-p}\right) \le \frac{1}{p}a(t)^{-p}\,, \tag{2.2.9}$$

and the other inequality in (2.2.6) follows analogously. Finally, (2.2.7) follows directly from (2.2.6) when assuming without loss of generality that $a|_{(t,t_0)}$ only takes values in $(0,1)$.

□

2.2.3. Solutions to the Einstein scalar-field equations in CMC gauge. From here on out, we impose the CMC condition[6]

$$k^l{}_l(t,\cdot) = \tau(t) = -3\frac{\dot{a}(t)}{a(t)}\,. \tag{2.2.10}$$

We use (2.2.3) and (2.2.4) to collect the following formulas for the mean curvature:

$$\partial_t\tau = 12\pi C^2 a^{-6} + \frac{1}{3}a^{-2} \tag{2.2.11}$$

$$\tau^2 = 9\frac{\dot{a}^2}{a^2} = 12\pi C^2 a^{-6} + a^{-2}\,. \tag{2.2.12}$$

We consequently define the trace-free component $\hat{k}$ of k as

$$\hat{k}_{ij} = k_{ij} - \frac{\tau}{3}g_{ij} \tag{2.2.13}$$

and recall that the future directed unit normal to our foliation is written as

$$\partial_0 = n^{-1}\partial_t\,. \tag{2.2.14}$$

With this, we can express the Einstein scalar-field equations in our gauge as follows:

PROPOSITION 2.2.6 (The Einstein scalar-field system in CMC gauge). *A pair $(\overline{g},\phi)$ solves the Einstein scalar-field equations* (2.1.1a)-(2.1.1c) *on $I\times M$ in CMC gauge* (2.2.10) *for some interval $I\subseteq(0,t_0]$, where the scale factor satisfies* (2.2.3)*, if and only if the following equations are satisfied on $I\times M$:*

---

[5]In [FU22, Lemma 2.1], one at first only has $\int_t^{t_0} a(s)^{-3}\,ds \lesssim \log(t_0) - \log(t)$ for t_0 small enough that we can estimate $a(t)$ by $t^{\frac{1}{3}}$ up to constant. However, assuming this inequality were to hold up to $\overline{t} > 0$ and one had $t_0 > \overline{t}$, the contribution $\int_{\overline{t}}^{t_0} a(s)^{-3}\,ds$ only adds a constant that we can absorb into our notation. Similarly, $a(t) \simeq t^{\frac{1}{3}}$ can be assumed to hold on $(0,t_0]$ for any fixed $t_0 > 0$, and we can ignore this technicality in proving the integral formulas in Lemma 2.2.4.

[6]Recall that k is negative in our sign convention, see Section 2.2.1.5.

*The **metric evolution equations***

$$\partial_t g_{ij} = -2nk_{ij} = -2n\hat{k}_{ij} + 2n\frac{\dot a}{a}g_{ij}\,, \tag{2.2.15a}$$

$$\begin{aligned}\partial_t \hat{k}_{ij} = &-\nabla_i\nabla_j n + n\left[\mathrm{Ric}[g]_{ij} - \frac{\dot a}{a}\hat{k}_{ij} - 2\hat{k}_{il}\hat{k}^l_j - 8\pi\nabla_i\phi\nabla_j\phi\right]\\ &+ 4\pi C^2 a^{-6}(n-1)g_{ij} + \frac{1}{9}(3n-1)\,a^{-2}g_{ij}\,,\end{aligned} \tag{2.2.15b}$$

*the **Hamiltonian** and **momentum constraint equations***

$$R[g] + \frac{2}{3}\tau^2 - \langle\hat{k},\hat{k}\rangle_g = 8\pi\left[|\partial_0\phi|^2 + |\nabla\phi|^2_g\right]\,, \tag{2.2.16a}$$

$$\nabla^l\hat{k}_{lj} = -8\pi\nabla_j\phi\cdot\partial_0\phi\,, \tag{2.2.16b}$$

*the **lapse equation***

$$\Delta_g n = -12\pi C^2 a^{-6} - \frac{1}{3}a^{-2} + n\left[\frac{1}{3}a^{-2} + 4\pi C^2 a^{-6} + \langle\hat{k},\hat{k}\rangle_g + 8\pi|\partial_0\phi|^2\right]\,, \tag{2.2.17a}$$

or equivalently by (2.2.16a)

$$\Delta_g n = -12\pi C^2 a^{-6} - \frac{1}{3}a^{-2} + n\left[R[g] - 8\pi|\nabla\phi|^2_g + 12\pi C^2 a^{-6} + a^{-2}\right]\,, \tag{2.2.17b}$$

*and the **wave equation***

$$\Box_{\overline{g}}\phi = -\partial_0^2\phi + n^{-1}g^{ij}\nabla_i n\nabla_j\phi + \Delta_g\phi + \tau\partial_0\phi = 0\,. \tag{2.2.18}$$

Proof. These are standard equations that follow from [Ren08, Chapter 2.3] and applying (2.2.10)-(2.2.12). $\square$

2.2.4. Bel-Robinson variables. In this subsection, we briefly (re-)establish Bel-Robinson variables and how they behave within the Einstein scalar-field system.

Recall that the Weyl tensor $W \equiv W[\overline{g}]$ is the trace-free component of the spacetime curvature and, in the Einstein scalar-field system, takes the form

$$\begin{aligned}W_{\alpha\beta\gamma\delta} =& \mathrm{Riem}[\overline{g}]_{\alpha\beta\gamma\delta} - P[\overline{g}]_{\alpha\beta\gamma\delta}\,,\\ P[\overline{g}]_{\alpha\beta\gamma\delta} =& \frac{1}{2}\left(\overline{g}_{\alpha\gamma}\mathrm{Ric}[\overline{g}]_{\beta\delta} - \overline{g}_{\beta\gamma}\mathrm{Ric}[\overline{g}]_{\alpha\delta} - \overline{g}_{\alpha\delta}\mathrm{Ric}[\overline{g}]_{\gamma\beta} + \overline{g}_{\beta\delta}\mathrm{Ric}[\overline{g}]_{\alpha\gamma}\right)\\ &- \frac{1}{6}R[\overline{g}]\left(\overline{g}_{\alpha\gamma}\overline{g}_{\beta\delta} - \overline{g}_{\alpha\delta}\overline{g}_{\beta\gamma}\right)\\ =& 4\pi\left(\overline{g}_{\alpha\gamma}\overline{\nabla}_\beta\phi\overline{\nabla}_\delta\phi - \overline{g}_{\beta\gamma}\overline{\nabla}_\alpha\phi\overline{\nabla}_\delta\phi - \overline{g}_{\alpha\delta}\overline{\nabla}_\beta\phi\overline{\nabla}_\gamma\phi + \overline{g}_{\beta\delta}\overline{\nabla}_\alpha\phi\overline{\nabla}_\gamma\phi\right)\\ &- \frac{4\pi}{3}\left(\overline{\nabla}^\rho\phi\overline{\nabla}_\rho\phi\right)\left(\overline{g}_{\alpha\gamma}\overline{g}_{\beta\delta} - \overline{g}_{\alpha\delta}\overline{g}_{\beta\gamma}\right)\,.\end{aligned}$$

We define the dual W^* of the Weyl tensor as

$$W^*_{\alpha\beta\gamma\delta} = \frac{1}{2}\varepsilon_{\alpha\beta\mu\nu}W^{\mu\nu}{}_{\gamma\delta}\,.$$

The electric and magnetic components of the Weyl tensor, referred to as the Bel-Robinson variables from here on, are now defined as

$$E(W)_{\alpha\beta} = W_{\alpha\mu\beta\nu}\partial_0^\mu\partial_0^\nu = W_{\alpha 0\beta 0},\ B(W)_{\alpha\beta} = W^*_{\alpha\mu\beta\nu}\partial_0^\mu\partial_0^\nu = W^*_{\alpha 0\beta 0}\,.$$

We note that, conversely, the Weyl tensor can be fully reconstructed from E and B since the following identities hold:

$$W_{a0c0} = E_{ac}, \quad W_{abc0} = -\varepsilon_{ab}{}^m B_{mc}, \quad W_{abcd} = -\varepsilon_{abi}\varepsilon_{cdj}E^{ij} \tag{2.2.19}$$

By the symmetries of the Weyl tensor as a whole, E and B are symmetric and one has $E_{0\beta} = 0 = B_{0\beta}$. Hence, E and B are symmetric, tracefree Σ_t-tangent $(0,2)$-tensors which we shall simply denote as E_{ij} and B_{ij}.
Further, we define

$$J_{\beta\gamma\delta} := \overline{\nabla}^\alpha W_{\alpha\beta\gamma\delta}, \quad J^*_{\beta\gamma\delta} := \overline{\nabla}^\alpha W^*_{\alpha\beta\gamma\delta}\,. \tag{2.2.20}$$

By applying the Bianchi identity for $\mathrm{Riem}[\overline{g}]$, we gain the explicit expression

$$J_{\beta\gamma\delta} = \frac{1}{2}\left(\overline{\nabla}_\gamma \mathrm{Ric}[\overline{g}]_{\delta\beta} - \overline{\nabla}_\delta \mathrm{Ric}[\overline{g}]_{\gamma\beta}\right) - \frac{1}{12}\left(\overline{g}_{\beta\delta}\overline{\nabla}_\gamma R[\overline{g}] - \overline{g}_{\beta\gamma}\overline{\nabla}_\delta R[\overline{g}]\right)\,. \tag{2.2.21}$$

Using (2.1.1a), we collect:

$$\begin{aligned} J_{i0j} =& 4\pi\Big[\nabla_i(\partial_0\phi)\nabla_j\phi + k^l{}_i\nabla_l\phi\nabla_j\phi - \partial_0\phi\nabla_i\nabla_j\phi \\ &- k_{ij}(\partial_0\phi)^2 - n^{-1}\nabla_i n\cdot\nabla_j\phi\cdot\partial_0\phi\Big] - \frac{2\pi}{3}\Big[\partial_0(\overline{\nabla}^\alpha\phi\overline{\nabla}_\alpha\phi)\Big]g_{ij} \end{aligned} \tag{2.2.22}$$

$$J^*_{i0j} = 4\pi\varepsilon_{lmj}\left(\nabla^l\nabla_i\phi + k^l{}_i\partial_0\phi\right)\nabla^m\phi + \frac{2\pi}{3}\varepsilon_{imj}\nabla^m\left(\overline{\nabla}^\alpha\phi\overline{\nabla}_\alpha\phi\right) \tag{2.2.23}$$

Note that expressions containing $\overline{\nabla}^\alpha\phi\overline{\nabla}_\alpha\phi$ can be ignored throughout our analysis since they are either pure trace or antisymmetric and thus will cancel in inner products with E, B, $\hat{k}$ and their rescaled analogues. The Bel-Robinson variables then behave as follows:

Lemma 2.2.7 (Constraint and evolution equations for Bel-Robinson variables). *If $(\overline{g},\phi)$ is a classical solution to the Einstein scalar-field system* (2.1.1a)-(2.1.1b) *in CMC gauge (see* (2.2.10)*), E and B satisfy the following constraint equations:*

$$E = \mathrm{Ric}[g] + \frac{2}{9}\tau^2 g + \frac{\tau}{3}\hat{k} - \hat{k}\odot_g\hat{k} - 4\pi(\nabla\phi\otimes\nabla\phi) - \left(\frac{8\pi}{3}|\partial_0\phi|^2 + \frac{4\pi}{3}|\nabla\phi|_g^2\right)g \tag{2.2.24a}$$

$$B = -\mathrm{curl}\hat{k} \tag{2.2.24b}$$

Further, they satisfy the following evolution equations:

(2.2.25a)

$$\partial_t E_{ij} = n\mathrm{curl}B_{ij} - (\nabla n\wedge B)_{ij} - \frac{5}{2}n\,(E\times k)_{ij} - \frac{2}{3}n(E\cdot k)g_{ij} - \frac{\tau}{2}n\cdot E_{ij} - \frac{n}{2}\left(J_{i0j}+J_{j0i}\right)$$

(2.2.25b)

$$\partial_t B_{ij} = -n\mathrm{curl}E_{ij} + (\nabla n\wedge E)_{ij} - \frac{5}{2}n\,(B\times k)_{ij} - \frac{2}{3}n(B\cdot k)g_{ij} - \frac{\tau}{2}n\cdot B_{ij} - \frac{n}{2}\left(J^*_{i0j}+J^*_{j0i}\right)$$

PROOF. For (2.2.25a)-(2.2.25b), we refer to [AM04, (3.11a)-(3.11b)].[7] Equations (2.2.24a)-(2.2.24b) follow as in [Wan19, (3.63a)-(3.63b)] from contracting the Gauss-Codazzi constraints. □

REMARK 2.2.8 (Initial data for Bel-Robinson variables). Since the Weyl tensor vanishes over FLRW spacetimes, so do $E(W[\overline{g}_{FLRW}])$ and $B(W[\overline{g}_{FLRW}])$. Furthermore, note that given initial data $(M, \mathring{g}, \mathring{k}, \mathring{\pi}, \mathring{\psi})$ on Σ_{t_0} in the sense discussed in Section 2.1.2.1, and defining $\hat{\mathring{k}} = \mathring{k} - \frac{\tau}{3}\mathring{g}$, we can use (2.2.24a) and (2.2.24b) to define the following $(0,2)$-tensors:

$$\mathring{E} = \mathrm{Ric}[\mathring{g}] + \frac{2}{9}\tau^2\mathring{g} + \frac{\tau}{3}\hat{\mathring{k}} - \hat{\mathring{k}} \odot_{\mathring{g}} \hat{\mathring{k}} - 4\pi\left(\mathring{\pi}\otimes\mathring{\pi}\right) - \left(\frac{8\pi}{3}\left|\mathring{\psi}\right|^2_{\mathring{g}} + \frac{4\pi}{3}\left|\mathring{\pi}\right|^2_{\mathring{g}}\right)\mathring{g} \tag{2.2.26a}$$

$$\mathring{B} = -\,\mathrm{curl}_{\mathring{g}}\hat{\mathring{k}} \tag{2.2.26b}$$

These are easily seen to be symmetric, and the constraints (2.1.3a) and (2.1.3b) on the initial data ensure that they are also tracefree. Hence, any choice of initial data for the Cauchy problem immediately also contains a unique choice of initial data for the Bel-Robinson variables that is consistent with solutions to the Einstein scalar-field equations in CMC gauge.

2.2.5. Rescaled variables and equations. It will be more convenient to work the rescaled and shifted solution variables to measure their distance from the FLRW reference solution. In this subsection, we introduce the renormalized solution variables and restate the Einstein scalar-field system in terms of these variables.

DEFINITION 2.2.9 (Rescaled variables for Big Bang stability). We will consider the rescaled variables

$$G_{ij} = a^{-2}g_{ij}, \quad (G^{-1})^{ij} = a^2 g^{ij}, \quad \Sigma_{ij} = a\hat{k}_{ij} \tag{2.2.27a}$$

$$N = n - 1 \tag{2.2.27b}$$

$$\mathcal{E}_{ij} = a^4 \cdot E_{ij}, \quad \mathcal{B}_{ij} = a^4 \cdot B_{ij} \tag{2.2.27c}$$

$$\Psi = a^3\partial_0\phi - C, \tag{2.2.27d}$$

We note that the scaling of $\mathcal{B}$ in (2.2.27c) is **not** the asymptotic rescaling of B – in fact, we expect B to have (approximate) leading order a^{-2} as one can see in (2.4.4g). However, keeping this scaling parallel to that of $\mathcal{E}$ makes the structurally very similar evolution equations significantly easier to deal with. We also do not rescale N asymptotically, unlike [RS18b; Spe18], but note that N converges to 0 at an order slightly below a^4 at low orders (see (2.3.17h) and (2.8.3a)).

PROPOSITION 2.2.10 (The rescaled Einstein scalar-field system). *The Einstein scalar-field system in CMC gauge as in Proposition 2.2.6 are solved by $(g, \hat{k}, n, \nabla\phi, \partial_0\phi)$ if the rescaled variables $(G, \Sigma, N, \nabla\phi, \Psi, \mathcal{E}, \mathcal{B})$ as in Definition 2.2.9 solve the following set of equations:*[8]

[7] Note that there is a minor error in the statement in [AM04], where the authors forget to symmetrize the J-tensors when applying the symmetrization to (3.14) of that work. This error seems to also occur in [CK93, (7.2.2jk)].

[8] We refer to Lemma 2.A.3 for the scalings that occur when switching between tensor field operations like $\wedge$ and $\wedge_G$.

The ***rescaled metric evolution equations***

$$\partial_t G_{ij} = -2(N+1)a^{-3}\Sigma_{ij} + 2N\frac{\dot a}{a}G_{ij} \tag{2.2.28a}$$

$$\partial_t (G^{-1})^{ij} = 2(N+1)a^{-3}(\Sigma^\sharp)^{ij} - 2N\frac{\dot a}{a}(G^{-1})^{ij} \tag{2.2.28b}$$

$$\begin{aligned}\partial_t\Sigma_{ij} = &-a\nabla_i\nabla_j N + (N+1)\left[a\mathrm{Ric}[G]_{ij} - 2a^{-3}(\Sigma\odot_G\Sigma)_{ij} - 8\pi a\nabla_i\phi\nabla_j\phi\right]\\ &+ 4\pi C^2a^{-3}\cdot NG_{ij} + \frac{1}{9}(3N+2)\,aG_{ij} + N\frac{\dot a}{a}\Sigma_{ij}\end{aligned} \tag{2.2.28c}$$

$$\begin{aligned}\partial_t(\Sigma^\sharp)^a{}_b = &\,\tau N(\Sigma^\sharp)^a{}_b - a\nabla^{\sharp a}\nabla_b N + (N+1)a\left[\left(\mathrm{Ric}[G]^\sharp\right)^a_b + \frac{2}{9}\mathbb{I}^a_b\right]\\ &- 8\pi(N+1)a\nabla^{\sharp a}\phi\nabla_b\phi + N\left(4\pi C^2a^{-3} + \frac{1}{9}a\right)\mathbb{I}^a_b\,,\end{aligned} \tag{2.2.28d}$$

the ***rescaled Hamiltonian and momentum constraints***

$$R[G] + \frac{2}{3} - a^{-4}\langle\Sigma,\Sigma\rangle_G = 8\pi\left[a^{-4}\Psi^2 + 2Ca^{-4}\Psi + |\nabla\phi|^2_G\right] \tag{2.2.29a}$$

$$\nabla^{\sharp m}\Sigma_{ml} = -8\pi\nabla_l\phi(\Psi + C) \tag{2.2.29b}$$

with their Bel-Robinson analogues

$$\begin{aligned}\mathcal{E}_{ij} = &\,a^4\left(\mathrm{Ric}[G]_{ij} + \frac{2}{9}G_{ij}\right) + \frac{\tau}{3}a^3\Sigma_{ij} - (\Sigma\odot_G\Sigma)_{ij} - 4\pi a^4\nabla_i\phi\nabla_j\phi\\ &- \left[\frac{4\pi}{3}a^4|\nabla\phi|^2_G + \frac{8\pi}{3}\Psi^2 + \frac{16\pi}{3}C\Psi\right]G_{ij}\end{aligned} \tag{2.2.29c}$$

$$\mathcal{B}_{ij} = -a^2\mathrm{curl}_G\Sigma_{ij}\,, \tag{2.2.29d}$$

the ***rescaled lapse equation***

$$\Delta N = \left(12\pi C^2a^{-4} + \frac{1}{3}\right)N + (N+1)a^{-4}\left[\langle\Sigma,\Sigma\rangle_G + 8\pi\Psi^2 + 16\pi C\Psi\right] \tag{2.2.30a}$$

or equivalently

$$\Delta N = \left(12\pi C^2a^{-4} + \frac{1}{3}\right)N + (N+1)\left[R[G] + \frac{2}{3} - 8\pi|\nabla\phi|^2_G\right]\,, \tag{2.2.30b}$$

the ***rescaled evolution equations for the Bel-Robinson variables***

(2.2.31a)

$$\begin{aligned}\partial_t\mathcal{E}_{ij} = &\,(3-N)\frac{\dot a}{a}\mathcal{E}_{ij} - a^{-1}(\nabla N\wedge_G\mathcal{B})_{ij} + (N+1)a^{-1}\mathrm{curl}_G\mathcal{B}_{ij}\\ &- (N+1)\left[\frac{5}{2}a^{-3}(\mathcal{E}\times_G\Sigma)_{ij} + \frac{2}{3}a^{-3}\langle\mathcal{E},\Sigma\rangle_G G_{ij}\right]\\ &+ 4\pi(N+1)a^{-3}(\Psi+C)^2\Sigma_{ij} - 4\pi(N+1)\dot a a^3\nabla_i\phi\nabla_j\phi + 4\pi a\nabla_{(i}N\nabla_{j)}\phi(\Psi+C)\\ &- 4\pi a(N+1)\left[\nabla_i\Psi\nabla_j\phi + \nabla_j\Psi\nabla_i\phi + (\Sigma^\sharp)^l_{(i}\nabla_{j)}\phi\nabla_l\phi - (\Psi+C)\nabla_i\nabla_j\phi\right]\\ &+ (N+1)\left[\frac{2\pi}{3}a^6\partial_0\left(a^{-6}(\Psi+C)^2 + a^{-2}|\nabla\phi|^2_G\right) + 4\pi\frac{\dot a}{a}(\Psi+C)^2\right]G_{ij}\end{aligned}$$

(2.2.31b)

$$\begin{aligned}\partial_t\mathcal{B}_{ij} =& \frac{\dot a}{a}(3-N)\,\mathcal{B}_{ij} + a^{-1}(\nabla N \wedge_G \mathcal{E})_{ij} - (N+1)a^{-1}\mathrm{curl}_G\mathcal{E}_{ij}\\ &-(N+1)\left[\frac{5}{2}a^{-3}(\mathcal{B}\times_G\Sigma)_{ij} + \frac{2}{3}a^{-3}\langle\mathcal{B},\Sigma\rangle_G G_{ij}\right]\\ &-4\pi(N+1)\varepsilon[G]_{lmj}\left(a^3\nabla^{\sharp l}\nabla_j\phi\nabla^{\sharp m}\phi + a^{-1}(\Sigma^\sharp)^l{}_i\nabla^{\sharp m}\phi(\Psi+C)\right)\end{aligned}$$

*and the **rescaled wave equation***

$$\partial_t\Psi = a\langle\nabla N,\nabla\phi\rangle_G + a(N+1)\Delta\phi - 3\frac{\dot a}{a}N(\Psi+C) \tag{2.2.32a}$$

along with the evolution equation

$$\partial_t\nabla\phi = a^{-3}(N+1)\nabla\Psi + a^{-3}(\Psi+C)\nabla N\,. \tag{2.2.32b}$$

*Finally, we collect the **rescaled Ricci evolution equation***

$$\begin{aligned}\partial_t\mathrm{Ric}[G]_{ab} =& a^{-3}(N+1)(\Delta_G\Sigma_{ab} - \nabla^{\sharp d}\nabla_a\Sigma_{db} - \nabla^{\sharp d}\nabla_b\Sigma_{da})\\ &+a^{-3}\nabla^{\sharp d}N(2\nabla_d\Sigma_{ab} - \nabla_a\Sigma_{db} - \nabla_b\Sigma_{da})\\ &-a^{-3}\left(\nabla_a N(\mathrm{div}_G\Sigma)_b + \nabla_b(\mathrm{div}_G\Sigma)_a\right) + \Delta_G N(a^{-3}\Sigma_{ab} + \frac{\tau}{3}G_{ab})\\ &-a^{-3}\left(\nabla^{\sharp d}\nabla_a N\cdot\Sigma_{db} + \nabla^{\sharp d}\nabla_b N\cdot\Sigma_{da}\right) + \frac{\tau}{3}\nabla_a\nabla_b N\end{aligned} \tag{2.2.33}$$

as well as (in a coordinate neighbourhood) the Christoffel evolution equation

$$\begin{aligned}\partial_t\Gamma^k_{ij}[G] =& \frac{1}{2}(G^{-1})^{kl}\left(\nabla_i(\partial_t G_{jl}) + \nabla_j(\partial_t G_{il}) - \nabla_l(\partial_t G_{ij})\right)\\ =& -(N+1)a^{-3}\left[\nabla_i(\Sigma^\sharp)^k{}_j + \nabla_j(\Sigma^\sharp)^k{}_i - \nabla^{\sharp k}\Sigma_{ij}\right]\\ &-a^{-3}\left[\nabla_i N(\Sigma^\sharp)^k{}_j + \nabla_j N(\Sigma^\sharp)^k{}_i - \nabla^{\sharp k}N\Sigma_{ij}\right]\\ &+\frac{\dot a}{a}\left[\nabla_i N\cdot\mathbb{I}^k_j + \nabla_j N\cdot\mathbb{I}^k_i - \nabla^{\sharp k}N\cdot G_{ij}\right]\end{aligned} \tag{2.2.34}$$

Proof. For the first identity in (2.2.34), we refer to [CLN06, Lemma 2.27] and insert the evolution equation (2.2.28a). Otherwise, all equations simply follow by computing the effects of rescaling on the equations from Proposition 2.2.6 (respectively the Ricci evolution equation as in [Ren08, Chapter 2.3, (2.32)]) as well as the Bel-Robinson evolution equations (2.2.25a)-(2.2.25b) and constraint equations (2.2.24a)-(2.2.24b). Notice that one already finds a solution to the system in Proposition (2.2.6) with the rescaled variables excluding (2.2.31a), (2.2.31b), (2.2.29c) and (2.2.29d). Conversely, all of the rescaled equations are satisfied by a solution to Proposition 2.2.6 at sufficiently high regularity. Hence, solving the full rescaled system is always sufficient to solve the Einstein system in Proposition 2.2.6 and they are equivalent if the initial data is regular enough to ensure sufficiently high regularity of solutions.

□

2.2.6. Commuted equations. We collect Laplace-commuted versions of the equations for the rescaled variables in Proposition 2.2.10 in this subsection. For the sake of brevity, we will not state all possible commutations for every equation, but restrict ourselves to the ones we actually need within the bootstrap argument. We also refer to Appendix 2.A.2 for expressions for commutators of spatial differential operators with each other and with ∂_t.

The terms written down explicity in Lemma 2.2.11 are ones that dominate the evolution behaviour or that are the largest higher order terms, both of which require careful treatment within the bootstrap argument. The error terms are broadly categorized into three groups:

- "Borderline" terms are terms that critically contribute to the fact that the energies diverge toward the Big Bang singularity. This almost always takes the form of adding energy terms at the same order as the evolved variable scaled by factors of the type εa^{-3} or $\varepsilon a^{-3-c\sqrt{\varepsilon}}$, which causes the energies to slightly diverge since a^{-3} is barely not integrable (see (2.2.6)).
- "Junk" terms are terms that are subcritical in the sense that they lead to integrable error terms, or terms that only contain lower order derivatives of the solution variables.
- "Top" order terms (which only appear in (2.2.38a) and (2.2.38b)) are terms that are junk terms for low order energies, but become borderline terms at top order.

All of these error terms are tracked schematically in Section 2.A.3. Since we will only need L^2_G-bounds on these error terms, which are given in Section 2.A.4, we will treat them as notational "black boxes" outside of the appendix.

Lemma 2.2.11 (Laplace-commuted rescaled equations). *Let $L\in 2\mathbb{N}$, $L\geq 2$. With error terms as defined in Appendix 2.A.3, the system in Proposition 2.2.10 leads to the following Laplace-commuted equations:*

The ***Laplace-commuted rescaled evolution equations for the second fundamental form***

$$\partial_t\Delta^{\frac{L}{2}}\Sigma=-a\nabla^2\Delta^{\frac{L}{2}}N+a(N+1)\Delta^{\frac{L}{2}}\mathrm{Ric}[G]+\mathfrak{S}_{L,Border}+\mathfrak{S}_{L,Junk}\,, \tag{2.2.35}$$

the ***Laplace-commuted rescaled momentum constraint equations***

$$\mathrm{div}_G\Delta^{\frac{L}{2}}\Sigma=-8\pi(\Psi+C)\left[\nabla\Delta^{\frac{L}{2}}\phi+\Delta^{\frac{L}{2}-1}\mathrm{Ric}[G]*\nabla\phi\right]+\nabla\Delta^{\frac{L}{2}-1}\mathrm{Ric}[G]*\Sigma+\mathfrak{M}_{L,Junk} \tag{2.2.36a}$$

and

$$\mathrm{curl}_G\Delta^{\frac{L}{2}}\Sigma=-a^{-2}\Delta^{\frac{L}{2}}\mathcal{B}+\boldsymbol{\varepsilon}[G]*\nabla\Delta^{\frac{L}{2}-1}\mathrm{Ric}[G]*\Sigma+\tilde{\mathfrak{M}}_{L,Junk}\,, \tag{2.2.36b}$$

the ***Laplace-commuted rescaled Hamiltonian constraint equations***

$$\begin{aligned}\Delta^{\frac{L}{2}}R[G]+a^{-4}\sum_{I_1+I_2=L}\nabla^{I_1}\Sigma*\nabla^{I_2}\Sigma\\=16\pi Ca^{-4}\Delta^{\frac{L}{2}}\Psi+a^{-4}\sum_{I_1+I_2=L}\left[\nabla^{I_1}\Psi*\nabla^{I_2}\Psi+\nabla^{I_1+1}\phi*\nabla^{I_2+1}\phi\right]\end{aligned} \tag{2.2.36c}$$

and

$$\Delta^{\frac{L}{2}}\mathrm{Ric}[G]=a^{-4}\Delta^{\frac{L}{2}}\mathcal{E}-\frac{\tau}{3}a^{-1}\Delta^{\frac{L}{2}}\Sigma+\mathfrak{H}_{L,Border}+\mathfrak{H}_{L,Junk}\,, \tag{2.2.36d}$$

the ***Laplace-commuted rescaled lapse equations***

(2.2.37a)
$$\Delta^{\frac{L}{2}+1}N = \left(12\pi C^2 a^{-4} + \frac{1}{3}\right)\Delta^{\frac{L}{2}}N + 16\pi C a^{-4}\cdot\Delta^{\frac{L}{2}}\Psi + \mathfrak{N}_{L,Border} + \mathfrak{N}_{L,Junk}\,,$$

(2.2.37b)
$$\nabla\Delta^{\frac{L}{2}+1}N = \left(12\pi C^2 a^{-4} + \frac{1}{3}\right)\nabla\Delta^{\frac{L}{2}}N + 16\pi C a^{-4}\cdot\nabla\Delta^{\frac{L}{2}}\Psi + \mathfrak{N}_{L+1,Border} + \mathfrak{N}_{L+1,Junk}\,,$$

the ***Laplace-commuted rescaled Bel-Robinson evolution equations***

$$\begin{aligned}\partial_t\Delta^{\frac{L}{2}}\mathcal{E} = {}& \frac{\dot a}{a}\,(3-N)\,\Delta^{\frac{L}{2}}\mathcal{E} + (N+1)a^{-1}\mathrm{curl}_G\Delta^{\frac{L}{2}}\mathcal{B} - a^{-1}\nabla\Delta^{\frac{L}{2}}N\wedge_G\mathcal{B}\\ &+ 4\pi C^2a^{-3}(N+1)\Delta^{\frac{L}{2}}\Sigma + 4\pi a(\Psi+C)\nabla\Delta^{\frac{L}{2}}N\otimes\nabla\phi\\ &+ 4\pi a(\Psi+C)(N+1)\nabla^2\Delta^{\frac{L}{2}}\phi - 8\pi a(N+1)\left(\nabla\phi\otimes\nabla\Delta^{\frac{L}{2}}\Psi\right)\\ &+ \mathfrak{E}_{L,Border} + \mathfrak{E}_{L,top} + \mathfrak{E}_{L,Junk}\end{aligned} \tag{2.2.38a}$$

$$\begin{aligned}\partial_t\Delta^{\frac{L}{2}}\mathcal{B} = {}& \frac{\dot a}{a}(3-N)\Delta^{\frac{L}{2}}\mathcal{B} - (N+1)a^{-1}\mathrm{curl}_G\Delta^{\frac{L}{2}}\mathcal{E} + a^{-1}\nabla\Delta^{\frac{L}{2}}N\wedge_G\mathcal{E}\\ &+ a^3\boldsymbol{\varepsilon}[G]*\nabla^2\Delta^{\frac{L}{2}}\phi*\nabla\phi + \mathfrak{B}_{L,Border} + \mathfrak{B}_{L,top} + \mathfrak{B}_{L,Junk}\,,\end{aligned} \tag{2.2.38b}$$

the ***Laplace-commuted rescaled matter evolution equations***

(2.2.39a)
$$\partial_t\Delta^{\frac{L}{2}}\Psi = a\langle\nabla\Delta^{\frac{L}{2}}N,\nabla\phi\rangle_G + a(N+1)\Delta^{\frac{L}{2}+1}\phi - 3C\frac{\dot a}{a}\Delta^{\frac{L}{2}}N + \mathfrak{P}_{L,Border} + \mathfrak{P}_{L,Junk}$$

(2.2.39b)
$$\partial_t\nabla\Delta^{\frac{L}{2}}\phi = a^{-3}(N+1)\nabla\Delta^{\frac{L}{2}}\Psi + Ca^{-3}\nabla\Delta^{\frac{L}{2}}N + \mathfrak{Q}_{L,Border} + \mathfrak{Q}_{L,Junk}$$

as well as (also allowing $L=0$ *for* (2.2.39d)*)*

$$\begin{aligned}\partial_t\nabla_l\Delta^{\frac{L}{2}}\Psi = {}& a\nabla_l\nabla^{\sharp j}\Delta^{\frac{L}{2}}N\nabla_j\phi + a(N+1)\nabla_l\Delta^{\frac{L}{2}+1}\phi - 3C\frac{\dot a}{a}\nabla_l\Delta^{\frac{L}{2}}N\\ &+ \left(\mathfrak{P}_{L+1,Border}\right)_l + \left(\mathfrak{P}_{L+1,Junk}\right)_l\end{aligned} \tag{2.2.39c}$$

$$\partial_t\Delta^{\frac{L}{2}+1}\phi = a^{-3}(N+1)\Delta^{\frac{L}{2}+1}\Psi + Ca^{-3}\Delta^{\frac{L}{2}+1}N + \mathfrak{Q}_{L+1,Border} + \mathfrak{Q}_{L+1,Junk} \tag{2.2.39d}$$

and the ***Laplace-commuted rescaled Ricci evolution equations***

$$\begin{aligned}\partial_t\Delta^{\frac{L}{2}}\mathrm{Ric}[G]_{ij} = {}& a^{-3}\left(\Delta^{\frac{L}{2}+1}\Sigma_{ij} - 2\nabla^{\sharp m}\nabla_{(i}\Delta^{\frac{L}{2}}\Sigma_{j)m}\right)\\ &- \frac{\dot a}{a}\left(\nabla_i\nabla_j\Delta^{\frac{L}{2}}N + \Delta^{\frac{L}{2}+1}N\cdot G_{ij}\right) + \left(\mathfrak{R}_{L,Border}\right)_{ij} + \left(\mathfrak{R}_{L,Junk}\right)_{ij}\end{aligned} \tag{2.2.40a}$$

(2.2.40b)
$$\begin{aligned}\partial_t\nabla_k\Delta^{\frac{L}{2}}\mathrm{Ric}[G]_{ij} = {}& a^{-3}\left(\nabla_k\Delta^{\frac{L}{2}+1}\Sigma_{ij} - 2\nabla_k\nabla^{\sharp m}\nabla_{(i}\Delta^{\frac{L}{2}}\Sigma_{j)m}\right)\\ &- \frac{\dot a}{a}\left(\nabla_k\nabla_i\nabla_j\Delta^{\frac{L}{2}}N + \nabla_k\Delta^{\frac{L}{2}+1}N\cdot G_{ij}\right)\\ &+ \left(\mathfrak{R}_{L+1,Border}\right)_{ijk} + \left(\mathfrak{R}_{L+1,Junk}\right)_{ijk}\,.\end{aligned}$$

PROOF. The equations (2.2.36c),(2.2.36d) and (2.2.37a) are obtained by simply applying $\Delta^{\frac{L}{2}}$ on both sides of (2.2.29a),(2.2.29c) and (2.2.30a) respectively. For (2.2.36a) and (2.2.36b), we additionally use the commutator formulas (2.A.7e) and (2.A.7f), while for the evolution equations, we apply the respective commutator of ∂_t and spatial derivatives as collected in Lemma 2.A.7 and commute Laplacians past ∇ and curl where needed (see the commutators in Lemma 2.A.5). The commutators with ∂_t only cause additional borderline and junk terms that do not substantially influence the behaviour, while the spatial commutators often lead to high order curvature terms, for example the Ricci terms in (2.2.36a), that need to be more carefully tracked. □

REMARK 2.2.12 (Simplified junk term notation). For junk terms that occur in an inner product with a tracefree symmetric tensor, any terms that are pure trace will immediately cancel and thus do not need to be taken into consideration for the following estimates, even if they have to be written down in the junk terms. Hence, we will denote with a superscript "$\|$" (for example $\mathfrak{H}^{\|}_{L,Junk}$) on a schematic error term the expressions that arise when dropping all terms of the form $\zeta \cdot G$ for some scalar function ζ that occur in this term's definition (see, for example, (2.A.11d)).

2.3. Big Bang stability: Norms, energies and bootstrap assumptions

Herein, we state the norms and energies we use to control the solution variables. These will allow us to state our initial data and bootstrap assumptions, and we then provide which improvement we aim to achieve for the latter. Note that we do not provide the coerciveness of our energies immediately (and actually cannot, at least not in a manner useful to our analysis), but will establish Sobolev norm control in the proof of Corollary 2.7.3, the key ingredient being Lemma 2.4.5. Furthermore, we collect a local well-posedness statement from previous work in Section 2.3.4.

2.3.1. Norms. Recall that γ is the hyperbolic spatial reference metric on M introduced in Definition 2.2.1, which we view as a metric on any foliation hypersurface Σ_t (see Section 2.1.2.1), and G is the rescaled spatial metric arising from the evolution (see Definition 2.2.9).

DEFINITION 2.3.1 (Pointwise norms and volume forms). We denote by $|\cdot|_\gamma$ (resp. $|\cdot|_{G(t,\cdot)}$) the pointwise norm with regard to γ (resp. $G(t,\cdot)$). For the sake of simplicity, we define $|\zeta|_\gamma = |\zeta|_{G(t,\cdot)} = |\zeta(t,\cdot)|$ for any scalar function ζ on Σ_t. The volume forms on Σ_t with respect to γ and $G(t,\cdot)$ are written as dvol_γ and $\mathrm{dvol}_{G(t,\cdot)}$ (or just dvol_G).

DEFINITION 2.3.2 (L^2-norms). Let $\mathfrak{T}$ be a Σ_t-tangent (r,s)-tensor field (for $r,s \geq 0$). Then, we define:

$$\|\mathfrak{T}\|^2_{L^2_\gamma(\Sigma_t)} = \|\mathfrak{T}(t,\cdot)\|^2_{L^2_\gamma(\Sigma_t)} := \int_M |\mathfrak{T}(t,\cdot)|^2_\gamma \,\mathrm{dvol}_\gamma, \tag{2.3.1a}$$

$$\|\mathfrak{T}\|^2_{L^2_G(\Sigma_t)} = \|\mathfrak{T}(t,\cdot)\|^2_{L^2_G(\Sigma_t)} := \int_M |\mathfrak{T}(t,\cdot)|^2_{G(t,\cdot)} \,\mathrm{dvol}_{G(t,\cdot)} \tag{2.3.1b}$$

DEFINITION 2.3.3 (Sobolev norms). Let $\mathfrak{T}$ be as above and $J \in \mathbb{N}_0$. We define:

$$\|\mathfrak{T}\|^2_{\dot{H}^J_\gamma(\Sigma_t)} = \|\mathfrak{T}(t,\cdot)\|^2_{\dot{H}^J_\gamma} = \int_{\Sigma_t} \left|\hat{\nabla}^J \mathfrak{T}(t,\cdot)\right|^2_\gamma \mathrm{dvol}_\gamma \tag{2.3.2a}$$

$$\|\mathfrak{T}\|^2_{\dot{H}^J_G(\Sigma_t)} = \|\mathfrak{T}(t,\cdot)\|^2_{\dot{H}^J_G} = \int_{\Sigma_t} \left|\nabla^J \mathfrak{T}(t,\cdot)\right|^2_{G(t,\cdot)} \mathrm{dvol}_{G(t,\cdot)} \tag{2.3.2b}$$

$$\|\mathfrak{T}\|^2_{H^J_\gamma(\Sigma_t)} = \|\mathfrak{T}(t,\cdot)\|^2_{H^J_\gamma} = \sum_{k=0}^J \|\mathfrak{T}\|^2_{\dot{H}^k_\gamma(\Sigma_t)} \tag{2.3.2c}$$

$$\|\mathfrak{T}\|^2_{H^J_G(\Sigma_t)} = \|\mathfrak{T}(t,\cdot)\|^2_{H^J_G} = \sum_{k=0}^J \|\mathfrak{T}\|^2_{\dot{H}^k_G(\Sigma_t)} \tag{2.3.2d}$$

DEFINITION 2.3.4 (Supremum norms). For $\mathfrak{T}$ as above and $J \in \mathbb{N}_0$, we set:

$$\|\mathfrak{T}\|_{\dot{C}^J_\gamma(\Sigma_t)} = \|\mathfrak{T}(t,\cdot)\|_{\dot{C}^J_\gamma} = \sup_{p\in\Sigma_t} \left|\hat{\nabla}^J \mathfrak{T}(t,\cdot)\right|_\gamma, \qquad \|\mathfrak{T}\|_{C^J_\gamma(\Sigma_t)} = \sum_{k=0}^J \|\mathfrak{T}\|_{\dot{C}^k_\gamma(\Sigma_t)} \tag{2.3.3a}$$

$$\|\mathfrak{T}\|_{\dot{C}^J_G(\Sigma_t)} = \|\mathfrak{T}(t,\cdot)\|_{\dot{C}^J_G} = \sup_{p\in\Sigma_t} \left|\nabla^J \mathfrak{T}(t,\cdot)\right|_{G(t,\cdot)}, \quad \|\mathfrak{T}\|_{C^J_G(\Sigma_t)} = \sum_{k=0}^J \|\mathfrak{T}\|_{\dot{C}^k_G(\Sigma_t)} \tag{2.3.3b}$$

REMARK 2.3.5 (Time dependence is suppressed in notation). When the choice of t and Σ_t is clear from context, we will often drop time dependences of G, $|\cdot|_G$, dvol_G and $\mathfrak{T}$, suppress the hypersurface Σ_t in the Sobolev and supremum norms, and simply write $\int_M$ instead of $\int_{\Sigma_t}$. For example, we write

$$\|\mathfrak{T}\|^2_{L^2_G} = \int_M |\mathfrak{T}|^2_G \, \mathrm{dvol}_G\,.$$

DEFINITION 2.3.6 (Solution norms). We define the following norms to measure the size of near-FLRW solutions:

$$\begin{aligned}\mathcal{H} = {}& \|\Psi\|_{H^{18}_G} + \|\nabla\phi\|_{H^{17}_G} + a^2\|\nabla\phi\|_{\dot{H}^{18}_G} \\ & + \|\Sigma\|_{H^{18}_G} + \|\mathcal{E}\|_{H^{18}_G} + \|\mathcal{B}\|_{H^{18}_G} \\ & + \|G-\gamma\|_{H^{18}_G} + \|\mathrm{Ric}[G] + \frac{2}{9}G\|_{H^{16}_G} + a^{-4}\|N\|_{H^{16}_G} + a^{-2}\|N\|_{\dot{H}^{17}_G} + \|N\|_{\dot{H}^{18}_G}\end{aligned} \tag{2.3.4a}$$

$$\mathcal{H}_{top} = a^2\|\Psi\|_{\dot{H}^{19}_G} + a^4\|\nabla\phi\|_{\dot{H}^{19}_G} + a^2\|\Sigma\|_{\dot{H}^{19}_G} + a^2\|\mathrm{Ric}[G] + \frac{2}{9}G\|_{\dot{H}^{17}_G} \tag{2.3.4b}$$

$$\begin{aligned}\mathcal{C} = {}& \|\Psi\|_{C^{16}_G} + \|\nabla\phi\|_{C^{15}_G} + \|\Sigma\|_{C^{16}_G} + \|\mathcal{E}\|_{C^{16}_G} + \|\mathcal{B}\|_{C^{16}_G} \\ & + \|G-\gamma\|_{C^{16}_G} + \|\mathrm{Ric}[G] + \frac{2}{9}G\|_{C^{14}_G} + a^{-4}\|N\|_{C^{14}_G} + a^{-2}\|N\|_{\dot{C}^{15}_G} + \|N\|_{\dot{C}^{16}_G}\end{aligned} \tag{2.3.4c}$$

$$\begin{aligned}\mathcal{C}_\gamma = {}& \|\Psi\|_{C^{16}_\gamma} + \|\nabla\phi\|_{C^{15}_\gamma} + \|\Sigma\|_{C^{16}_\gamma} + \|\mathcal{E}\|_{C^{16}_\gamma} + \|\mathcal{B}\|_{C^{16}_\gamma} \\ & + \|G-\gamma\|_{C^{16}_\gamma} + \|\mathrm{Ric}[G] + \frac{2}{9}G\|_{C^{14}_\gamma} + a^{-4}\|N\|_{C^{14}_\gamma} + a^{-2}\|N\|_{\dot{C}^{15}_\gamma} + \|N\|_{\dot{C}^{16}_\gamma}\end{aligned} \tag{2.3.4d}$$

REMARK 2.3.7 (Choice of metric and controlling Christoffel symbols). We could equivalently also phrase $\mathcal{H}$ in terms of γ-norms, or predominately use $\mathcal{C}_\gamma$ instead of $\mathcal{C}$, since we include the norms on $G-\gamma$ and $\mathrm{Ric}[G]+\frac{2}{9}G = (\mathrm{Ric}[G]+\frac{2}{9}\gamma)+\frac{2}{9}(G-\gamma)$. We will demonstrate in Lemma 2.7.4 how H_G and C_γ norms can be used to control H_γ and C_G norms. We will also indicate how the initial data and bootstrap assumptions for $\mathcal{C}_\gamma$ and $\mathcal{C}$ are equivalent in

Remarks 2.3.11 and 2.3.18. The main reason for this is that, by successively replacing local coordinates in the expressions of $\Gamma - \hat{\Gamma}$ by $\hat{\nabla}$, one has

$$\|\Gamma - \hat{\Gamma}\|_{C^{l-1}_\gamma(M)} \lesssim P_l(\|G-\gamma\|_{C^l_\gamma(M)}, \|G^{-1}-\gamma^{-1}\|_{C^l_\gamma(M)}) \,. \tag{2.3.5}$$

We choose to work predominately with norms in terms of the rescaled metric since quantities appearing in the Einstein system are naturally contracted by G, not γ, and we commute with differential operators associated with G.

Remark 2.3.8 (Redundancies in the solution norms). The solution norms $\mathcal{H}$, $\mathcal{C}$ and $\mathcal{C}_\gamma$ aren't "optimal" in the sense that controlling the norms of Ψ, $\nabla\phi, \Sigma$ and $G-\gamma$ is entirely sufficient to gain the claimed control (up to constant) on N via the lapse equation, $\mathcal{E}$ and $\mathcal{B}$ via to the constraint equations and $\mathrm{Ric}[G]+\frac{2}{9}G$ via local coordinates. We choose to include all variables in the norms and subsequent assumptions mainly for the sake of convenience.

2.3.2. Energies. The fundamental objects used to control the solution variables are the energies that take the following form:

Definition 2.3.9 (Energies). Let $l \in \mathbb{N}_0$. We define:

$$\begin{aligned}\mathfrak{E}^{(l)}(\phi,t) &= (-1)^l \int_M \Psi\Delta^l\Psi - a^4\phi\Delta^{l+1}\phi \,\mathrm{dvol}_G \\ &= \begin{cases} \int_M |\Delta^{\frac{l}{2}}\Psi|^2 + a^4|\nabla\Delta^{\frac{l}{2}}\phi|^2_G \,\mathrm{dvol}_G & l \text{ even} \\ \int_M |\nabla\Delta^{\frac{l-1}{2}}\Psi|^2_G + a^4|\Delta^{\frac{l+1}{2}}\phi|^2_G \,\mathrm{dvol}_G & l \text{ odd}\end{cases}\end{aligned} \tag{2.3.6a}$$

$$\mathfrak{E}^{(l)}(W,t) = (-1)^l \int_M \langle\mathcal{E},\Delta^l\mathcal{E}\rangle_G + \langle\mathcal{B},\Delta^l\mathcal{B}\rangle_G \,\mathrm{dvol}_G \tag{2.3.6b}$$

$$\mathfrak{E}^{(l)}(\Sigma,t) = (-1)^l \int_M \langle\Sigma,\Delta^l\Sigma\rangle_G \,\mathrm{dvol}_G \tag{2.3.6c}$$

$$\mathfrak{E}^{(l)}(\mathrm{Ric},\cdot) = (-1)^l \int_M \left\langle \mathrm{Ric}[G]+\frac{2}{9}G, \Delta^l\left(\mathrm{Ric}[G]+\frac{2}{9}G\right)\right\rangle_G \mathrm{dvol}_G \tag{2.3.6d}$$

$$\mathfrak{E}^{(l)}(N,\cdot) = (-1)^l \int_M \langle N,\Delta^l N\rangle_G \,\mathrm{dvol}_G \tag{2.3.6e}$$

Usually, we will use integration by parts to distribute derivatives within the energies as in (2.3.6b). Further, we introduce the notation

$$\mathfrak{E}^{(\leq l)} = \sum_{i=0}^{l} \mathfrak{E}^{(i)} \tag{2.3.7}$$

for any of the energies above.

For any $l \in \mathbb{N}_0$ and any smooth functions $f_1, f_2 \in C^\infty(\mathbb{R}_+)$, we have

$$f_1 \cdot f_2 \cdot \mathfrak{E}^{(2l+1)} \leq \frac{f_1^2}{2}\mathfrak{E}^{(2l)} + \frac{f_2^2}{2}\mathfrak{E}^{(2l+2)} \,. \tag{2.3.8}$$

Performing the calculation for Σ as an example, we have:

$$\mathfrak{E}^{(2l+1)}(\Sigma,\cdot) = -\int_M \langle\Delta^l\Sigma,\Delta^{l+1}\Sigma\rangle_G \,\mathrm{dvol}_G \leq \int_M |\Delta^l\Sigma|_G|\Delta^{l+1}\Sigma|_G \,\mathrm{dvol}_G$$

$$\leq \sqrt{\mathfrak{E}^{(2l)}(\Sigma,\cdot)}\sqrt{\mathfrak{E}^{(2l+2)}(\Sigma,\cdot)}$$

Now, (2.3.8) then follows from the Young inequality. As a consequence, we also have

$$\mathfrak{E}^{(\leq 2l)} \lesssim \sum_{m=0}^{l} \mathfrak{E}^{(2m)}\,,\ \mathfrak{E}^{(\leq 2l+1)} \lesssim \sum_{m=0}^{l+1} \mathfrak{E}^{(2m)}\,. \tag{2.3.9}$$

This allows us to largely restrict our analysis to even order energies, outside of how we close the bootstrap argument at top order.

2.3.3. Assumptions on the initial data. With the necessary solution norms and energies now defined, we can now state what we assume near-FLRW initial data to satisfy:

Assumption 2.3.10 (Near-FLRW initial data). *For some small enough $\varepsilon \in (0,1)$ and the solution norms $\mathcal{H}, \mathcal{H}_{top}$ and $\mathcal{C}$ as in Definition 2.3.6, we assume the rescaled initial data to be close to that of the FLRW solution in Lemma 2.2.3 in the following sense:*

$$\mathcal{H}(t_0) + \mathcal{H}_{top}(t_0) + \mathcal{C}(t_0) \lesssim \varepsilon^2 \tag{2.3.10}$$

The assumptions on $\mathcal{H} + \mathcal{H}_{top}$ also imply the following:

$$\begin{aligned} &\mathfrak{E}^{(\leq 18)}(\phi,t_0) + \mathfrak{E}^{(\leq 18)}(\Sigma,t_0) + \mathfrak{E}^{(\leq 18)}(W,t_0) + \mathfrak{E}^{(\leq 16)}(\mathrm{Ric},t_0) \\ &+\|\nabla\phi\|^2_{H^{18}_G} + \mathfrak{E}^{(18)}(N,t_0) + a(t_0)^{-4}\mathfrak{E}^{(17)}(N,t_0) + a(t_0)^{-8}\mathfrak{E}^{(\leq 16)}(N,t_0) \\ &+a(t_0)^4\mathfrak{E}^{(19)}(\phi,t_0) + a(t_0)^4\mathfrak{E}^{(19)}(\Sigma,t_0) + a(t_0)^4\mathfrak{E}^{(17)}(\mathrm{Ric},t_0) \lesssim \varepsilon^4 \end{aligned} \tag{2.3.11}$$

Remark 2.3.11 (Initial data size in $\mathcal{C}_\gamma(t_0)$). Notice that by (2.3.5), (2.3.10) implies that

$$\|\Gamma - \hat{\Gamma}\|_{C^{15}_G(\Sigma_{t_0})} \lesssim \varepsilon^4\,, \tag{2.3.12a}$$

and arguing along similar lines and using $L^2 - L^\infty$-estimates, also

$$\|\Gamma - \hat{\Gamma}\|_{H^{17}_G(\Sigma_{t_0})} \lesssim \varepsilon^4\,. \tag{2.3.12b}$$

In particular, since moving from C^l_γ to C^l_G only requires control on Christoffel symbols to order $l-1$ for general tensors and $l-2$ for scalar functions, as well as zero order control on $G-\gamma$, it follows from (2.3.10) that

$$\mathcal{C}_\gamma(t_0) \lesssim \varepsilon^2 \tag{2.3.13}$$

We refer to the proof of Lemma 2.7.4 for a more detailed term analysis and how a similar argument also applies to the Sobolev norms.

Remark 2.3.12 (Redundancies in the initial data assumptions). Similar to Remark 2.3.8, one could also reduce the initial data assumptions in (2.3.10), especially at top order. In particular, we highlight that the Bel-Robinson energy can be entirely controlled by the other terms that occur due to the additional scaled Σ-energy at order 19, or vice versa we could drop the latter in favour of the former. This will be reflected in Lemma 2.6.10.

Remark 2.3.13 (The volume form). Let μ_G and μ_γ denote the volume elements of G and γ respectively. Since the determinant is a smooth map on invertible matrices, the initial data assumptions on $G-\gamma$ also imply

$$\|\mu_G-\mu_\gamma\|_{C^0_G(\Sigma_{t_0})}=\|\mu_G-\mu_\gamma\|_{C^0_\gamma(\Sigma_{t_0})}\lesssim\varepsilon^2\,. \tag{2.3.14}$$

Consequently, we have

$$\|\mathrm{dvol}_G-\mathrm{dvol}_\gamma\|_{C^0_\gamma(\Sigma_{t_0})}=\mu_\gamma^{-1}\|\mathrm{dvol}_\gamma\|_{C^0_\gamma(\Sigma_{t_0})}\|\mu_{G(t_0,\cdot)}-\mu_\gamma\|_{C^0_\gamma(\Sigma_{t_0})}\lesssim\varepsilon^2$$

and, since $\|G^{-1}-\gamma^{-1}\|_{C^0_\gamma(\Sigma_{t_0})}\lesssim\varepsilon^2$ also follows by a von Neumann series argument from the initial data assumption on $G-\gamma$,

$$\|\mathrm{dvol}_G-\mathrm{dvol}_\gamma\|_{C^0_G(\Sigma_{t_0})}\lesssim\varepsilon^2\,. \tag{2.3.15}$$

2.3.4. Local well-posedness and continuation criteria. For everything that follows, we need to establish that the initial data assumptions above also ensure local well-posedness. For the core system, we translate the local well-posedness result for stiff fluids in [RS18b] to the sub-case of the scalar field system. While statement and proof there are for vanishing spatial sectional curvature and what corresponds to choosing $C=\sqrt{2/3}$, the arguments for our setting are completely analogous.

LEMMA 2.3.14 (Local well-posedness and continuation criteria for the Einstein scalar–field system (Big Bang version), see [RS18b, Theorem 14.1]). *Let $N\geq 4$ be an integer and $(M,\mathring{g},\mathring{k},\mathring{\pi},\mathring{\psi})$ be geometric initial data to the Einstein scalar-field system (see Section 2.1.2.1) and assume that one has*

$$\|\mathring{g}-a(t_0)^2\gamma\|_{H^{N+1}_\gamma(M)}+\|\mathring{k}-\frac{\tau(t_0)}{3}\cdot a(t_0)^2\gamma\|_{H^N_\gamma(M)}+\|\mathring{\pi}\|_{H^N_\gamma(M)}+\|\mathring{\psi}-Ca(t_0)^{-3}\|_{H^N_\gamma(M)}<\infty$$

as well as, for some sufficiently small $\eta'>0$,

$$\|\mathring{\psi}-Ca(t_0)^{-3}\|_{C^0_\gamma(M)}\leq\eta'\,.$$

Then, the CMC-transported Einstein scalar-field system (respectively the rescaled system) is locally well-posed in the following sense: The initial data $(\mathring{g},\mathring{k},\mathring{\pi},\mathring{\psi})$ launches a unique classical solution $(g,k,n,\nabla\phi,\partial_t\phi)$ to (2.2.16a)-(2.2.16b)*,* (2.2.15a)-(2.2.15b)*,* (2.2.18) *and* (2.2.17a) *on $[t_1,t_0]\times M$ for some $t_1\in(0,t_0)$ that satisfies $k^l{}_l=-3\dot{a}a^{-1}$ and $n>0$, launches a solution to the Einstein scalar-field system and such that the variables enjoy the following regularity:*

$$\begin{aligned}
g\in&C^{N-1}_{dt^2+\gamma}([t_1,t_0]\times M)\cap C^0([t_1,t_0],H^{N+1}_\gamma(M))\\
k\in&C^{N-2}_{dt^2+\gamma}([t_1,t_0]\times M)\cap C^0([t_1,t_0],H^{N}_\gamma(M))\\
\nabla\phi\in&C^{N-2}_{dt^2+\gamma}([t_1,t_0]\times M)\cap C^0([t_1,t_0],H^{N}_\gamma(M))\\
\partial_t\phi\in&C^{N-2}_{dt^2+\gamma}([t_1,t_0]\times M)\cap C^0([t_1,t_0],H^{N}_\gamma(M))\\
n\in&C^{N}_{dt^2+\gamma}([t_1,t_0]\times M)\cap C^0([t_1,t_0],H^{N+2}_\gamma(M))
\end{aligned}$$

The rescaled variables $(G,\Sigma,N,\nabla\phi,\Psi)$ enjoy the analogous regularity. If $(\mathfrak{t},t_0]$ is the maximal interval on which the above statements hold, then one either has $\mathfrak{t}=0$ or one of the following blow-up criteria are satisfied:

(1) *The smallest eigenvalue of* $g(t_m,\cdot)$ *converges to 0 for some sequence* $(t_m,x_m)\subseteq(\mathfrak{t},t_0]\times M$ *with* $t_m\downarrow\mathfrak{t}$.
(2) $n(t_m,x_m)$ *converges to* 0 *for some sequence* $(t_m,x_m)\subseteq(\mathfrak{t},t_0]\times M$ *with* $t_m\downarrow\mathfrak{t}$.
(3) $\left(|\partial_0\phi|^2+|\nabla\phi|_g^2\right)(t_m,x_m)$ *converges to* 0 *for some sequence* $(t_m,x_m)\subseteq(\mathfrak{t},t_0]\times M$ *with* $t_m\downarrow\mathfrak{t}$.
(4) $s\in(\mathfrak{t},t_0]\mapsto\|g\|_{C^2_\gamma(\Sigma_s)}+\|k\|_{C^1_\gamma(\Sigma_s)}+\|n\|_{C^2_\gamma(\Sigma_s)}+\|\partial_t\phi\|_{C^1_\gamma(\Sigma_s)}+\|\nabla\phi\|_{C^1_\gamma(\Sigma_s)}$ *is unbounded.*

A NOTE ON THE PROOF. Note that the additional initial data requirement in the stiff-fluid setting that the pressure is strictly positive is covered by the smallness assumption on $\mathring{\psi}-Ca(t_0)^2$, since the pressure corresponds to $|\mathring{\psi}|^2+|\mathring{\pi}|_{\mathring{g}}^2$ and the assumptions on $\partial_t\phi$ and $\nabla\phi$ ensure that (after embedding) this quantity behaves like $C^2a(t_0)^{-6}+\mathcal{O}(\eta')$ at Σ_{t_0}. □

COROLLARY 2.3.15 (Local well-posedness for the Bel-Robinson variables). *Under the assumptions of Lemma 2.3.14, the Bel-Robinson variables E and B corresponding to the Lorentzian metric $\overline{g}=-n^2dt^2+g$ satisfy the equations* (2.2.24a)-(2.2.24b)*, are the unique classical solutions to the evolution equations* (2.2.25a)-(2.2.25b)*, and satisfy*

$$E,B\in C^{N-3}([t_1,t_0]\times M)\cap C([t_1,t_0],H^{N-1}_\gamma(M))$$

PROOF. That E and B satisfy the constraint equations, solve the evolution equations and have the stated regularity on the interval of existence is a direct consequence of Lemma 2.3.14 and the computations in Section 2.2.4. Furthermore, with initial data derived from the constraint equations as in Remark 2.2.8, the hyperbolic system (2.2.24a)-(2.2.24b) launches a unique solution satisfying the regularity above that must then be (E,B). □

For sufficiently regular initial data ($N\geq 21$), it follows that

$$\mathfrak{E}^{(\leq 19)}(\phi,\cdot),\mathfrak{E}^{(\leq 18)}(W,\cdot),\mathfrak{E}^{(\leq 19)}(\Sigma,\cdot),\mathfrak{E}^{(\leq 17)}(\mathrm{Ric},\cdot),\|G-\gamma\|_{H^{18}_G}\in C^1([t_1,t_0])\,,$$

and similarly the square of any supremum norm occurring in $\mathcal{C}$ is continuously differentiable on $[t_1,t_0]$. Strictly speaking, we would need to assume this additional regularity on our initial data for the computations in the following sections (especially Section 2.6) to hold. However, since smooth functions are dense in $H^l(M)$ for any $l\in\mathbb{N}_0$, any bounds on $\mathcal{H}(t)$ and $\mathcal{C}(t)$ that we prove assuming sufficient regularity at Σ_{t_0} then immediately extend to data only satisfying the regularity implied by (2.3.10).
Thus, from here on out, we will assume without loss of generality that all energies and squared norms are continuously differentiable on the domain of existence, and similarly all variables are continuously differentiable for the lower order C_G-norm improvements in Section 2.4.2.

2.3.5. Bootstrap assumption. To keep an overview of the entire bootstrap argument, we state all of the assumptions and comprehensively list how we intend to improve them.

ASSUMPTION 2.3.16 (Bootstrap assumption). *Fix some $t_{Boot}\in[0,t_0)$. Further, let $c_0>0$, let $\sigma\in(\varepsilon^{\frac{1}{8}},1]$ be suitably small such that $c_0\sigma<1$, and $K_0>0$ a suitable constant. For any $t\in(t_{Boot},t_0]$, we assume*

$$\mathcal{C}(t)\leq K_0\varepsilon a(t)^{-c_0\sigma}\,. \tag{2.3.16}$$

REMARK 2.3.17. More explicitly, (2.3.16) means

$$\|\Psi\|_{C_G^{16}} \leq K_0 \varepsilon a^{-c_0\sigma} \tag{2.3.17a}$$

$$\|\nabla\phi\|_{C_G^{15}} \leq K_0 \varepsilon a^{-c_0\sigma} \tag{2.3.17b}$$

$$\|\Sigma\|_{C_G^{16}} \leq K_0 \varepsilon a^{-c_0\sigma} \tag{2.3.17c}$$

$$\|\mathcal{E}\|_{C_G^{16}} \leq K_0 \varepsilon a^{-c_0\sigma} \tag{2.3.17d}$$

$$\|\mathcal{B}\|_{C_G^{16}} \leq K_0 \varepsilon a^{-c_0\sigma} \tag{2.3.17e}$$

$$\|\mathrm{Ric}[G] + \frac{2}{9}G\|_{C_G^{14}} \leq K_0 \varepsilon a^{-c_0\sigma} \tag{2.3.17f}$$

$$\|G-\gamma\|_{C_G^{16}} \leq K_0 \varepsilon a^{-c_0\sigma} \tag{2.3.17g}$$

$$\|N\|_{C_G^{14}} + a^2\|N\|_{\dot{C}_G^{15}} + a^4\|N\|_{\dot{C}_G^{16}} \leq K_0 \varepsilon a^{4-c_0\sigma} \tag{2.3.17h}$$

$$\|\Gamma - \hat{\Gamma}\|_{C_G^{15}} \leq K_0 \varepsilon a^{-c_0\sigma} \tag{2.3.17i}$$

REMARK 2.3.18 (Bootstrap assumptions with respect to γ). Note again that we could equivalently make the above bootstrap assumptions with respect to H_γ- and C_γ-norms: For example, the assumptions (2.3.17i) and (2.3.17g) imply

$$\|\zeta\|_{C_\gamma^l} \lesssim a^{-c\sigma}\|\zeta\|_{C_G^l} + \|\zeta\|_{C_\gamma^{\lceil \frac{l-1}{2}\rceil}}\varepsilon a^{-c\sigma}, \quad \|\mathfrak{T}\|_{C_\gamma^l} \lesssim a^{-c\sigma}\|\mathfrak{T}\|_{C_G^l} + \|\mathfrak{T}\|_{C_\gamma^{\lceil \frac{l}{2}\rceil}}\varepsilon a^{-c\sigma}$$

for any smooth function $\zeta \in C^\infty(\Sigma_t)$, any Σ_t-tangent tensor $\mathfrak{T}$ and a constant $c > 0$. This is essentially a direct consequence of (2.3.5), and we will prove an improved version of this rigorously in Lemma 2.7.4. Applying this to each norm in $\mathcal{C}$, we get

$$\mathcal{C}_\gamma \lesssim \varepsilon a^{-c\sigma} \tag{2.3.18}$$

for some updated constant $c \geq c_0$.

REMARK 2.3.19 (Strategy for the bootstrap improvement). Our goal is to improve the C-norm estimate to

$$\mathcal{C} \leq K_1 \varepsilon^{\frac{9}{8}} a^{-c_1\varepsilon^{\frac{1}{8}}},$$

where $c_1, K_1 > 0$ are positive constants independent of σ and ε. Notice how this is actually an improvement if we choose σ suitably and then choose ε sufficiently small: Any update between K_0 and K_1 can be balanced out since we gain at least the additional prefactor $\varepsilon^{\frac{1}{8}}$ in each estimate, which we can then choose to have been suitably small. Similarly, we improve the power of a if we have $\varepsilon^{\frac{1}{8}} \cdot \sigma^{-1} < \frac{c_0}{c_1}$. If we then retroactively choose σ large enough compared to ε but small overall – for example $\sigma = \varepsilon^{\frac{1}{16}}$ – and then ensure that $\max\{c_0, c_1\}\varepsilon^{\frac{1}{16}} < 1$ as well as $c_1\varepsilon^{\frac{1}{16}} < c_0$ are satisfied by choosing ε to have been small enough, we have strictly improved the bootstrap assumptions.

REMARK 2.3.20 (Conventions within the bootstrap argument). Throughout the rest of the argument, we tacitly assume $t \in (t_{Boot}, t_0]$ if not stated otherwise, and we assume ε and σ to be sufficiently small. In the proof of Theorem 2.8.2, we will choose $\sigma = \varepsilon^{\frac{1}{16}}$, but this explicit choice will not be used or needed up to that point. Finally, we allow $c \geq c_0$ be a constant that we may update from line to line, and will similarly deal with prefactors by

"$\lesssim$"-notation where the constant may change in each line. These updates will always be independent of σ and ε, but may depend on t_0, and the quantities arising from the FLRW reference solution. Hence, we not only assume $c_0\sigma < 1$, but $c\sigma < 1$ throughout the argument.

2.4. Big Bang stability: A priori estimates

In this section, we collect strong low order C_G-norm estimates that follow as an immediate consequence from the bootstrap assumptions, starting with key estimates at the base level and followed by weaker, but still improved estimates at higher levels. Finally, we collect a differentiation formula for integrals with respect to dvol_G as well as a Sobolev estimate that lays the groundwork for energy coercivity. In particular, using the strong C_G-norm estimates, said estimate proves that moving between energies and norms at most incurs an error involving lower order energies of the controlled variable and curvature energies, scaled by $a^{-c\sqrt{\varepsilon}}$.

2.4.1. Strong C^0_G-estimates. First, we establish a pointwise bound on the lapse that actually holds irrespective of the bootstrap assumptions:

Lemma 2.4.1 (Maximum principle for the lapse). *The lapse remains positive and bounded throughout the evolution:*

$$n = N + 1 \in (0, 3] \tag{2.4.1}$$

Proof. Let $t \in \mathbb{R}_+$ be arbitrary and let n_{min} be the minimum of n over Σ_t at (t, x_{min}). Then, $(\Delta_g n)(t, x_{min}) > 0$ holds. If n_{min} were nonpositive, (2.2.17a) would lead to the following contradiction:

$$0 \geq -12\pi C^2 a^{-6} - \frac{1}{3}a^{-2} + n_{min}\left[\frac{1}{3}a^{-2} + 4\pi C^2 a^{-6} + \langle \hat{k}, \hat{k}\rangle_g + 8\pi|\partial_0\phi|^2\right] = \Delta_g n(t, x_{min}) > 0$$

This shows $n > 0$, and the upper bound follows analogously. □

The following estimate will be essential in dealing with borderline terms throughout the bootstrap argument:

Lemma 2.4.2 (Strong C^0_G estimates). *The following estimates hold:*

$$\|\Psi\|_{C^0_G} \lesssim \varepsilon \tag{2.4.2a}$$

$$\|\Sigma\|_{C^0_G} \lesssim \varepsilon \tag{2.4.2b}$$

$$\|\mathcal{E}\|_{C^0_G} \lesssim \varepsilon \tag{2.4.2c}$$

Proof. (2.4.2a): From (2.2.32a), we obtain the following using Lemma 2.4.1 for n, the bootstrap assumptions (2.3.17h) and (2.3.17b) and that $\dot{a} \simeq a^{-2}$ by (2.2.3):

$$|\partial_t\Psi| \lesssim \varepsilon a^{5-c\sigma} + \varepsilon a^{1-c\sigma} + \varepsilon a^{1-c\sigma}|\Psi| + \varepsilon a^{1-c\sigma}$$

After integration, we thus obtain using the initial data assumption (2.3.10):

$$|\Psi(t)| \lesssim |\Psi(t_0)| + \int_t^{t_0} \varepsilon a(s)^{1-c\sigma}\, ds + \int_t^{t_0} \varepsilon a(s)^{1-c\sigma}|\Psi(s)|\, ds$$

$$\lesssim \varepsilon\left(1+\int_t^{t_0} a(s)^{1-c\sigma}\,ds\right)+\int_t^{t_0}\varepsilon a(s)^{1-c\sigma}|\Psi(s)|\,ds$$

By (2.2.6), the integral over $a^{1-c\sigma}$ is bounded since $c\sigma<1$, so the Gronwall lemma now yields (2.4.2a).

(2.4.2b): Notice that

$$\partial_t|\Sigma|_G^2=(\partial_t\Sigma^\sharp)^l{}_m(\Sigma^\sharp)^m{}_l+(\Sigma^\sharp)^l{}_m(\partial_t\Sigma^\sharp)^m{}_l=2(\partial_t\Sigma^\sharp)^l{}_m(\Sigma^\sharp)^m{}_l\le 2|\partial_t\Sigma^\sharp|_G|\Sigma|_G\,. \tag{2.4.3}$$

Now, we consider (2.2.28d) and, using the bootstrap assumptions (2.3.17h), (2.3.17f) and (2.3.17b), get:

$$\begin{aligned}|\partial_t\Sigma^\sharp|_G\lesssim&\,\tau|N||\Sigma^\sharp|_G+|\nabla^\sharp\nabla N|_G a+|N+1|a\left|\mathrm{Ric}[G]^\sharp+\frac{2}{9}G^\sharp\right|_G+|N+1|a|\nabla^\sharp\phi\nabla\phi|_G\\&+\sqrt{3}|N|\cdot\left(4\pi C^2a^{-3}+\frac{1}{9}a\right)\\\lesssim&\,\varepsilon a^{1-c\sigma}|\Sigma|_G+\varepsilon a^{1-c\sigma}\end{aligned}$$

We can now apply Lemma 2.A.2 with $f=|\Sigma|_G^2$, and thus have along with (2.3.10) and (2.4.3):

$$|\Sigma|_G(t)\le|\Sigma|_G(t_0)+\int_t^{t_0}|\partial_t\Sigma^\sharp|_G(s)\,ds\lesssim\varepsilon$$

(2.4.2c): Using the constraint equation (2.2.29c) and that $\langle G,\mathcal{E}\rangle_G=\mathrm{tr}_G\mathcal{E}=0$, one sees

$$|\mathcal{E}|_G^2=\left\langle a^4\left(\mathrm{Ric}[G]+\frac{2}{9}G\right)-\dot a a^2\Sigma-\Sigma\odot_G\Sigma-4\pi a^4\nabla\phi\nabla\phi,\mathcal{E}\right\rangle_G\,.$$

Then, applying the bootstrap assumptions (2.3.17f) and (2.3.17b) shows the Ricci and matter terms are bounded by $\varepsilon a^{4-c\sigma}|\mathcal{E}|_G$, and the a priori estimate (2.4.2b) along with $\dot a a^2\simeq 1$ by (2.2.3) bounds the remaining terms by $\varepsilon|\mathcal{E}|_G$. The statement then follows by dividing by $|\mathcal{E}|_G$ and taking the supremum. □

Note that, in the proof of (2.4.2b), it was essential that we used (2.2.28d) instead of (2.2.28c), since using the latter would incur terms of the type $|\partial_tG|_G|\Sigma|_G^2$and $a^{-3}|\Sigma|_G^3$ when computing the time derivative of $|\Sigma|_G^2$, which, at this point, behave like $\varepsilon a^{-3-c\sigma}|\Sigma|_G^2$, and thus not yield the sharp estimate (or even an improved estimate) that we will need to control borderline terms.

2.4.2. Strong low order C_G-norm estimates. Now, we can prove the main supremum norm estimates in this section:

LEMMA 2.4.3 (Strong low order C_G-norm estimates). *The following estimates hold:*

$$\|\Psi\|_{C_G^{13}}\lesssim\varepsilon a^{-c\sqrt{\varepsilon}} \tag{2.4.4a}$$

$$\|\Sigma\|_{C_G^{12}}\lesssim\varepsilon a^{-c\sqrt{\varepsilon}} \tag{2.4.4b}$$

$$\|G-\gamma\|_{C_G^{12}}\lesssim\sqrt{\varepsilon}a^{-c\sqrt{\varepsilon}} \tag{2.4.4c}$$

$$\|G^{-1}-\gamma^{-1}\|_{C_G^{12}}\lesssim\sqrt{\varepsilon}a^{-c\sqrt{\varepsilon}} \tag{2.4.4d}$$

$$\|\nabla\phi\|_{C^{12}_G}\lesssim\sqrt{\varepsilon}a^{-c\sqrt{\varepsilon}} \tag{2.4.4e}$$

$$\|\mathrm{Ric}[G]+\frac{2}{9}G\|_{C^{10}_G}\lesssim\sqrt{\varepsilon}a^{-c\sqrt{\varepsilon}} \tag{2.4.4f}$$

$$\|\mathcal{B}\|_{C^{11}_G}\lesssim\varepsilon a^{2-c\sqrt{\varepsilon}} \tag{2.4.4g}$$

$$\|\mathcal{E}\|_{C^{12}_G}\lesssim\varepsilon a^{-c\sqrt{\varepsilon}} \tag{2.4.4h}$$

Proof. Before going into the individual estimates, we collect the following commutator term estimates from the expressions in (2.A.10a)-(2.A.10b):

(2.4.5)
$$\|[\partial_t,\nabla^J]\zeta\|_{C^0_G}\lesssim a^{-3}\|N+1\|_{C^{J-1}_G}\|\Sigma\|_{C^{J-1}_G}\|\zeta\|_{C^{J-1}_G}+\frac{\dot a}{a}\|N\|_{C^{J-1}_G}\|\zeta\|_{C^{J-1}_G}$$

(2.4.6)
$$\|[\partial_t,\nabla^J]\mathfrak{T}\|_{C^0_G}\lesssim a^{-3}\|N+1\|_{C^J_G}\left(\|\nabla^J\Sigma\|_{C^0_G}\|\mathfrak{T}\|_{C^0_G}+\|\Sigma\|_{C^{J-1}_G}\|\mathfrak{T}\|_{C^{J-1}_G}\right)+\frac{\dot a}{a}\|N\|_{C^J_G}\|\mathfrak{T}\|_{C^{J-1}_G}$$

With this in hand, we will prove each estimate by iterating over the derivative order as long as the bootstrap assumptions can be applied. In each step, we use the previously obtained estimates at lower order to control the commutator term (with some additional care for $\mathfrak{T}=\Sigma$ which we need to consider first), while we can use similar arguments to those at order 0 to control the "core" of the evolution equations.

To start out, we apply (2.4.2b) on Σ and the bootstrap assumption (2.3.17h) on N to the rescaled evolution equations (2.2.28a)-(2.2.28b) and deduce

$$|\partial_t G^{\pm1}|_G=|\partial_t(G^{\pm1}-\gamma^{\pm1})|_G\lesssim\varepsilon a^{-3}+\varepsilon a^{1-c\sigma}\lesssim\varepsilon a^{-3}\,. \tag{2.4.7}$$

<u>(2.4.4b):</u> We assume

$$\|\Sigma\|_{C^{J-1}_G}\lesssim\varepsilon a^{-c\sqrt{\varepsilon}} \tag{2.4.8}$$

to be satisfied for some $J\in\{1,\dots,12\}$ (For $J=1$, this is true by (2.4.2b)). Observe the following:

$$\partial_t|\nabla^J\Sigma|^2_G=2\langle\partial_t\nabla^J\Sigma,\nabla^J\Sigma\rangle_G+\partial_tG^{-1}*\nabla^J\Sigma*\nabla^J\Sigma$$

Now, we commute (2.2.28d) with ∇^J: As before, $\nabla^J\partial_t\Sigma$ is bounded by $\varepsilon a^{-c\sigma}$ for any admissible J. Hence and using (2.4.7),

$$\partial_t|\nabla^J\Sigma|^2_G\lesssim\varepsilon a^{-3}|\nabla^J\Sigma|^2_G+\left(\varepsilon a^{1-c\sigma}+\|[\partial_t,\nabla^J]\Sigma\|_{C^0_G}\right)|\nabla^J\Sigma|_G$$

is satisfied. Looking at the commutator term using (2.4.6), we have with (2.4.2b) that

$$\|[\partial_t,\nabla^J]\Sigma\|_{C^0_G}\lesssim\varepsilon a^{-3}\|\Sigma\|_{\dot C^J_G}+a^{-3}\cdot\|\Sigma\|^2_{C^{J-1}_G}+\varepsilon a^{1-c\sigma}\|\Sigma\|_{C^{J-1}_G}\,.$$

Altogether, we obtain

$$\partial_t|\nabla^J\Sigma|^2_G\lesssim\left(\varepsilon a^{-3}\|\Sigma\|_{\dot C^J_G}+\varepsilon a^{-c\sigma}+\varepsilon^2a^{-3-c\sqrt{\varepsilon}}\right)|\nabla^J\Sigma|_G\,.$$

With Lemma 2.A.2 as well as the initial data assumption (2.3.10) and the integral formula (2.2.6) with $p=c\sqrt{\varepsilon}$, this implies

$$|\nabla^J\Sigma|_G(t)\lesssim\int_t^{t_0}\varepsilon a^{-3}\|\Sigma\|_{\dot{C}^J_G(\Sigma_s)}\,ds+\varepsilon\left(1+\sqrt{\varepsilon}a^{-c\sqrt{\varepsilon}}\right)$$

and consequently, after taking the supremum on the left and applying the Gronwall lemma,

$$\|\Sigma\|_{\dot{C}^J_G(\Sigma_s)}\lesssim\varepsilon a^{-c\sqrt{\varepsilon}}\,.$$

Combining this with (2.4.8) proves the statement up to order J, and hence shows (2.4.4b) by iterating the argument up to $J=12$.

(2.4.4a): We again assume that

$$\|\Psi\|_{C^{J-1}_G}\lesssim\varepsilon a^{-c\sqrt{\varepsilon}} \tag{2.4.9}$$

holds for $J\in\{1,2,\dots,13\}$. Observe that

$$|\partial_t\nabla^J\Psi|_G\lesssim a\|N+1\|_{C^{J+1}_G}\|\nabla\phi\|_{C^{J+1}_G}+\frac{\dot a}{a}\|\nabla N\|_{C^J_G}(1+\|\Psi\|_{C^J_G})+\|[\partial_t,\nabla^J]\Psi\|_{C^0_G}$$

By (2.3.17h), (2.3.17b) and (2.3.17a), the first two summands can be bounded (up to constant) by $\varepsilon a^{1-c\sigma}$. By (2.4.9), (2.4.4b) and (2.3.17h) and using (2.4.5), the commutator term is bounded (up to constant) by $\varepsilon^2a^{-3-c\sqrt{\varepsilon}}$. Altogether,

$$|\partial_t\nabla^J\Psi|_G\lesssim\varepsilon a^{1-c\sigma}+\varepsilon^2a^{-3-c\sqrt{\varepsilon}}$$

follows. Inserting this and (2.4.7) into

$$|\partial_t\left(|\nabla^J\Psi|_G^2\right)|\leq|\partial_tG^{-1}|_G|\nabla^J\Psi|_G^2+2|\partial_t\nabla^J\Psi|_G\cdot|\nabla^J\Psi|_G$$

implies, with Lemma 2.A.2,

$$\begin{aligned}|\nabla^J\Psi|_G(t)&\leq|\nabla^J\Psi|(t_0)+\int_t^{t_0}\left(\frac12|\partial_tG^{-1}||\nabla^J\Psi|_G+|\partial_t\nabla^J\Psi|_G\right)(s)\,ds\\&\lesssim\varepsilon^2+\int_t^{t_0}\left(\varepsilon a(s)^{-3}|\nabla^J\Psi(s,\cdot)|_G+\varepsilon a(s)^{1-c\sigma}+\varepsilon^2a(s)^{-3-c\sqrt{\varepsilon}}\right)ds\,.\end{aligned}$$

We obtain using (2.2.6):

$$|\nabla^J\Psi|_G(t)\lesssim\varepsilon a(t)^{-c\sqrt{\varepsilon}}+\int_t^{t_0}\varepsilon a(s)^{-3}|\nabla^J\Psi(s,\cdot)|_G\,ds$$

The Gronwall lemma, applying (2.2.5) and taking the supremum over Σ_t then implies $|\nabla^J\Psi|_{\dot{C}^J_G}\lesssim\varepsilon a^{-c\sqrt{\varepsilon}}$. This proves (2.4.2a) by iterating over J and adding up the individual seminorms.

(2.4.4c)-(2.4.4d): Note that (2.4.7) implies (2.4.4c) at order 0 since one has

$$|\partial_t(|G-\gamma|_G)^2|\lesssim|\partial_tG^{-1}|_G|G-\gamma|_G^2+|\partial_t(G-\gamma)|_G|G-\gamma|_G\lesssim\varepsilon a^{-3}(1+|G-\gamma|_G)|G-\gamma|_G$$

which we can apply the Gronwall lemma to after integrating, along with (2.2.7) for the error term, as in the proof of (2.4.4b).

For higher orders, commuting (2.2.28a) with ∇^J and inserting (2.4.4b) and (2.3.17h) implies

$$\|\partial_t\nabla^J(G-\gamma)\|_{C^0_G}\lesssim\varepsilon a^{-3-c\sqrt{\varepsilon}}+\varepsilon a^{1-c\sigma}+\|[\partial_t,\nabla^J](G-\gamma)\|_{C^0_G}$$

with

$$\|[\partial_t,\nabla^J](G-\gamma)\|_{C^0_G}\lesssim\left(\varepsilon a^{-3-c\sqrt{\varepsilon}}+\varepsilon a^{1-c\sigma}\right)\|G-\gamma\|_{C^{J-1}_G}\,.$$

Once again doing the same iterative argument over $J\leq 12$ and assuming the estimate to hold up to $J-1$, this altogether becomes

$$\|\partial_t\nabla^J(G-\gamma)\|_{C^0_G}\lesssim\varepsilon a^{-3-c\sqrt{\varepsilon}},$$

implying with (2.2.6)

$$\|\nabla^J(G-\gamma)\|_{C^0_G}\lesssim\varepsilon^2+\varepsilon\int_t^{t_0}a(s)^{-3-c\sqrt{\varepsilon}}\,ds\lesssim\sqrt{\varepsilon}a^{-c\sqrt{\varepsilon}}\,.$$

The argument for $G^{-1}-\gamma^{-1}$ is completely analogous.

(2.4.4e): We only prove the statement for C^0_G, the full estimate extends from there by the same iterative arguments as above. Considering (2.2.32b), Lemma 2.4.1, (2.4.2a) and (2.2.3), we have

$$|\partial_t\nabla\phi|_G\lesssim a^{-3}\left(|\nabla\Psi|_G+|\nabla N|_G\right)$$

and thus, with (2.4.4a) and the bootstrap assumption (2.3.17h),

$$|\partial_t\nabla\phi|_G\lesssim\varepsilon a^{-3-c\sqrt{\varepsilon}}\,.$$

With (2.4.7), this implies

$$|\partial_t|\nabla\phi|^2_G|\lesssim\varepsilon a^{-3}|\nabla\phi|^2_G+\varepsilon a^{-3-c\sqrt{\varepsilon}}|\nabla\phi|_G$$

and the statement follows as usual by applying Lemma 2.A.2, (2.2.7) and the Gronwall lemma.

(2.4.4f): This follows as in the proof of (2.4.4c) using (2.2.33) and (2.2.28a) and their commuted analogues.

Once again, for C^0_G, we have (2.4.4g): This is obtained immediately from commuting (2.2.29d) with ∇^J and applying (2.4.4b). Notice that the Levi-Civita tensor can be absorbed into the implicit constants since $|\varepsilon[G]|_G=\sqrt{6}$ holds (see (2.A.2c)).

(2.4.4h): This follows like in the proof of (2.4.2c) from applying (2.3.17f), (2.3.17b) and (2.4.4b) to the constraint equation (2.2.29c) commuted with ∇^J.

□

2.4.3. Other useful a priori observations. Before moving on to the energy estimates, we collect a differentiation identity and lay the groundwork for energy coercivity:

LEMMA 2.4.4 (The volume form and differentiation of integrals). *Let $\mu_G=\sqrt{\det G}$ denote the volume element with regard to G. It satisfies*

$$\partial_t\mu_G=\frac{1}{2}\mu_G(G^{-1})^{ij}\partial_t G_{ij}=-N\tau\mu_G\,, \tag{2.4.10}$$

and hence one has

$$\|\mu_G-\mu_\gamma\|_{C^0_G}\lesssim\varepsilon\,. \tag{2.4.11}$$

on $(\Sigma_t)_{t\in(t_{Boot},t_0]}$. *Further, for any differentiable function* ζ, *one has*

$$\partial_t\int_M \zeta \mathrm{dvol}_G=\int_M\partial_t\zeta\mathrm{dvol}_G-\int_M N\tau\cdot\zeta\mathrm{dvol}_G \tag{2.4.12}$$

Proof. From (2.2.28a), we obtain $(G^{-1})^{ij}\partial_t G_{ij}=-2N\tau$, and (2.4.10) follows by

$$\partial_t\mu_G=\frac{1}{2}\sqrt{\det G}(G^{-1})^{ij}\partial_tG_{ij}=-N\tau\mu_G\,.$$

Hence, we have using (2.3.17h) and the initial data estimate (2.3.14) that

$$|\mu_G-\mu_\gamma|(t,\cdot)\lesssim\varepsilon+\int_t^{t_0}\varepsilon a(s)^{1-c\sigma}|\mu_G-\mu_\gamma|(s.\cdot)\,ds$$

holds, and thus (2.4.11) after applying the Gronwall lemma.
Finally, we obtain (2.4.12) by writing $\mathrm{dvol}_G=\frac{\mu_G}{\mu_\gamma}\mathrm{dvol}_\gamma$ and inserting (2.4.10) □

Lemma 2.4.5 (Preliminary Sobolev norm estimates). *Let* ζ *be a scalar function and* $\mathfrak{T}$ *be a symmetric* Σ_t*-tangent* $(0,2)$*-tensor, and let* $l\in\{1,\dots,9\}$. *Then, on* $(t_{Boot},t_0]$, *the following estimates are satisfied: For* $l>5$, *one has:*

$$\|\nabla^2\zeta\|^2_{L^2_G}\lesssim\|\Delta\zeta\|^2_{L^2_G}+a^{-c\sqrt{\varepsilon}}\|\nabla\zeta\|^2_{L^2_G} \tag{2.4.13a}$$

$$\|\zeta\|^2_{H^{2l}_G}\lesssim\|\Delta^l\zeta\|^2_{L^2_G}+a^{-c\sqrt{\varepsilon}}\left(\sum_{m=0}^{l-1}\|\Delta^m\zeta\|^2_{L^2_G}+\|\zeta\|^2_{C^{2l-12}_G}\mathfrak{E}^{(\le 2l-3)}(\mathrm{Ric},\cdot)\right) \tag{2.4.13b}$$

$$\sum_{m=1}^{2l+1}\|\zeta\|^2_{\dot H^m_G}\lesssim\|\nabla\Delta^l\zeta\|^2_{L^2_G}+a^{-c\sqrt{\varepsilon}}\left(\sum_{m=0}^{l-1}\|\nabla\Delta^m\zeta\|^2_{L^2_G}+\|\nabla\zeta\|^2_{C^{2l-12}_G}\mathfrak{E}^{(\le 2l-2)}(\mathrm{Ric},\cdot)\right) \tag{2.4.13c}$$

$$\sum_{m=1}^{2l}\|\nabla\zeta\|^2_{\dot H^m_G}\lesssim\|\nabla\Delta^l\zeta\|^2_{L^2_G}+a^{-c\sqrt{\varepsilon}}\left(\sum_{m=0}^{l-1}\|\nabla\Delta^m\zeta\|^2_{L^2_G}+\|\nabla\zeta\|^2_{C^{2l-11}_G}\mathfrak{E}^{(\le 2l-2)}(\mathrm{Ric},\cdot)\right) \tag{2.4.13d}$$

and

$$\|\mathfrak{T}\|^2_{H^{2l}_G}\lesssim\|\Delta^l\mathfrak{T}\|^2_{L^2_G}+a^{-c\sqrt{\varepsilon}}\left(\sum_{m=0}^{l-1}\|\Delta^m\mathfrak{T}\|^2_{L^2_G}+\|\mathfrak{T}\|^2_{C^{2l-11}_G}\mathfrak{E}^{(\le 2l-2)}(\mathrm{Ric},\cdot)\right) \tag{2.4.14a}$$

$$\sum_{m=1}^{2l+1}\|\mathfrak{T}\|^2_{\dot H^m_G}\lesssim\|\nabla\Delta^l\mathfrak{T}\|^2_{L^2_G}+a^{-c\sqrt{\varepsilon}}\left(\sum_{m=0}^{l-1}\|\nabla\Delta^m\mathfrak{T}\|^2_{L^2_G}+\|\mathfrak{T}\|^2_{C^{2l-10}_G}\mathfrak{E}^{(\le 2l-1)}(\mathrm{Ric},\cdot)\right) \tag{2.4.14b}$$

More precisely, the Ricci energy terms can be dropped in all of the above estimates for $l\le 5$.

Remark 2.4.6. We stress that Lemma 2.4.5 is crucial for everything that follows in multiple ways:

Firstly, the L^2_G-norms containing ζ and $\mathfrak{T}$ on the right hand sides above except (2.4.13d) are in precisely the form the energies in Definition 2.3.9 take. Hence, this is what will actually yield near-coercivity of our energies since the C_G-norms can be controlled by $a^{-c\sqrt{\varepsilon}}$ or better using the a priori estimates from Lemma 2.4.3, as well as (2.3.17h) for the lapse. This will be shown more explicitly as an intermediary step in improving the bootstrap assumptions for $\mathcal{C}$

(see proof of Corollary 2.7.3).

Secondly, a downside of using Δ as the main differential operator to commute with the Einstein scalar-field system is that it creates error terms that we can only bound by Sobolev norms and not directly express as energies. Thus, we need a way to translate this information back to energies to formulate energy inequalities. A lot of this is done "under the hood" in the error term estimates in subsection 2.A.4.

Finally, some top order terms also do not appear in a way that their L^2-norm is directly the square root of an energy (see, for example, the term $a\nabla^2\Delta^{\frac{L}{2}}N$ in (2.2.35)), and some borderline terms would lead to nonintegrable divergences if we were to incur additional divergences in estimation (see, for example, the first term in (2.A.14a)). Lemma 2.4.5 precisely provides a way to relate these terms to energies . Additionally, by applying these estimates for terms of the form $\Delta^{\frac{L}{2}}\zeta$ and $\Delta^{\frac{L}{2}}\mathfrak{T}$, one can avoid high order curvature energies that run the risk of breaking the energy hierarchy.

Proof. Since the arguments for all of the inequalities above are very similar, we only prove (2.4.14a) in full and then briefly address the other estimates.
Letting $\tilde{\mathfrak{T}}_{i_1\dots i_{2l}k_1k_2} = \nabla_{i_1}\dots\nabla_{i_{2l}}\mathfrak{T}_{k_1k_2}$, we compute with the commutator formula (2.A.6c) and strong C_G-norm estimate (2.4.4f):

$$\begin{aligned}\int_M |\nabla^2\tilde{\mathfrak{T}}|_G^2 &= -\int_M \langle\nabla\tilde{\mathfrak{T}},\Delta\nabla\tilde{\mathfrak{T}}\rangle_G\,\mathrm{dvol}_G\\ &= -\int_M\langle\nabla\tilde{\mathfrak{T}},\nabla\Delta\tilde{\mathfrak{T}}\rangle_G\,\mathrm{dvol}_G+\int_M\nabla\tilde{\mathfrak{T}}*\left[\nabla\mathrm{Ric}[G]*\tilde{\mathfrak{T}}+\mathrm{Ric}[G]*\nabla\tilde{\mathfrak{T}}\right]\mathrm{dvol}_G\\ &\lesssim\int_M|\Delta\tilde{\mathfrak{T}}|_G^2\,\mathrm{dvol}_G+\left(1+\left\|\mathrm{Ric}[G]+\frac{2}{9}G\right\|_{C^1_G}\right)\cdot\left[\int_M|\tilde{\mathfrak{T}}|_G^2\,\mathrm{dvol}_G+\int_M|\nabla\tilde{\mathfrak{T}}|_G^2\,\mathrm{dvol}_G\right]\\ &\lesssim\int_M|\Delta\tilde{\mathfrak{T}}|_G^2+a^{-c\sqrt{\varepsilon}}\int_M|\tilde{\mathfrak{T}}|_G^2\,\mathrm{dvol}_G\end{aligned}$$

In the final step, we used integration by parts to obtain

$$a^{-c\sqrt{\varepsilon}}\int_M|\nabla\tilde{\mathfrak{T}}|_G^2\,\mathrm{dvol}_G\le\int_M|\Delta\tilde{\mathfrak{T}}|_G\cdot a^{-c\sqrt{\varepsilon}}|\tilde{\mathfrak{T}}|_G\,\mathrm{dvol}_G\lesssim\int_M\left(|\Delta\tilde{\mathfrak{T}}|_G^2+a^{-2c\sqrt{\varepsilon}}|\tilde{\mathfrak{T}}|_G^2\right)\mathrm{dvol}_G$$

and updated c. This already shows (2.4.14a) for $l=1$. Assume now that (2.4.14a) holds up to some $l\in\mathbb{N}$, $l\le 9$ and *any* symmetric Σ_t-tangent $(0,2)$-tensor field. By applying (2.A.6d), we have

$$\begin{aligned}\Delta\tilde{\mathfrak{T}}&=\nabla^{2l}\Delta\mathfrak{T}+[\Delta,\nabla^2]\nabla^{2l-2}\mathfrak{T}+\dots+\nabla^{2l-2}[\Delta,\nabla^2]\mathfrak{T}\\ &=\nabla^{2l}\Delta\mathfrak{T}+\sum_{I_{\mathrm{Ric}}+I_{\mathfrak{T}}=2l}\nabla^{I_{\mathrm{Ric}}}(\mathrm{Ric}[G]+\frac{2}{9}G)*\nabla^{I_{\mathfrak{T}}}\mathfrak{T}+G*\nabla^{2l}\mathfrak{T}\end{aligned}\tag{2.4.15}$$

Subsequently, we have for $l>5$ using the strong C_G-norm estimate (2.4.4f) for any Ricci term of order 10 or lower that

$$\|\mathfrak{T}\|^2_{\dot{H}^{2(l+1)}_G}=\int_M|\nabla^2\tilde{\mathfrak{T}}|_G^2\,\mathrm{dvol}_G$$

$$\lesssim \|\Delta\mathfrak{T}\|^2_{\dot{H}^{2l}_G} + \left(1+\sqrt{\varepsilon}a^{-c\sqrt{\varepsilon}}\right)\|\mathfrak{T}\|^2_{H^{2l}_G} + \|\mathfrak{T}\|^2_{C^{2l-11}_G}\|\mathrm{Ric}[G]+\frac{2}{9}G\|^2_{H^{2l}_G},$$

and get the same estimate without the final term for $l \leq 5$. By assumption, we can estimate $\|\Delta\mathfrak{T}\|^2_{H^{2l}_G}$, $\|\mathfrak{T}\|^2_{H^{2l}_G}$ and $\|\mathrm{Ric}[G]+\frac{2}{9}G\|^2_{H^{2l}_G}$ as in (2.4.14a), and get the following for $l > 5$:

$$\begin{aligned}
&\int_M |\nabla^{2l+2}\mathfrak{T}|^2_G \mathrm{dvol}_G \\
\lesssim &\left[\|\Delta^l\Delta\mathfrak{T}\|^2_{L^2_G} + a^{-c\sqrt{\varepsilon}}\sum_{m=0}^{l-1}\|\Delta^{m+1}\mathfrak{T}\|^2_{L^2_G} + a^{-c\sqrt{\varepsilon}}\|\Delta\mathfrak{T}\|^2_{C^{2l-12}_G}\mathfrak{E}^{(\leq 2l-2)}(\mathrm{Ric},\cdot)\right] \\
&+\left[a^{-c\sqrt{\varepsilon}}\|\Delta^l\mathfrak{T}\|^2_{L^2_G} + a^{-c\sqrt{\varepsilon}}\sum_{m=0}^{l-1}\|\Delta^m\mathfrak{T}\|^2_{L^2_G} + a^{-c\sqrt{\varepsilon}}\|\mathfrak{T}\|^2_{C^{2l-12}_G}\mathfrak{E}^{(\leq 2l-2)}(\mathrm{Ric},\cdot)\right] \\
&+a^{-c\sqrt{\varepsilon}}\|\mathfrak{T}\|^2_{C^{2l-11}_G}\left[\mathfrak{E}^{(2l)}(\mathrm{Ric},\cdot) + \left(\mathfrak{E}^{(\leq 2l-2)}(\mathrm{Ric},\cdot) + \varepsilon a^{-c\sqrt{\varepsilon}}\mathfrak{E}^{(\leq 2l-2)}(\mathrm{Ric},\cdot)\right)\right] \\
\lesssim &\|\Delta^{l+1}\mathfrak{T}\|^2_{L^2_G} + a^{-c\sqrt{\varepsilon}}\left(\sum_{m=0}^{l}\|\Delta^m\mathfrak{T}\|^2_{L^2_G} + \|\mathfrak{T}\|^2_{C^{2l-10}G}\mathfrak{E}^{(\leq 2l)}(\mathrm{Ric},\cdot)\right)
\end{aligned}$$

For $l = 5$, we get analogous estimates dropping the Ricci energies in the first two lines, and for $l = 4$, the same with all curvature terms dropped.
To prove the statement for $l+1$, it now remains to be shown that $\|\mathfrak{T}\|^2_{\dot{H}^{2l+1}_G}$ can be bounded by the same right hand side (up to constant) as above. By integration by parts, one has

$$\|\nabla^{2l+1}\mathfrak{T}\|^2_{L^2_G} \lesssim \|\nabla^{2l}\mathfrak{T}\|^2_{L^2_G} + \|\Delta\nabla^{2l}\mathfrak{T}\|^2_{L^2_G},$$

where the latter tensor is precisely $\Delta\tilde{\mathfrak{T}}$ which we just treated, and the former is covered by the induction assumption at order $2l$. So, (2.4.14a) now follows for $l+1$, and thus by iteration up to $l = 10$.

The proof of (2.4.14b) is analogous – we note that since we actually only needed a strong estimate on $\|\mathrm{Ric}[G]+2G\|_{C^9_G}$ for the previous inequality, but (2.4.4f) holds at C^{10}_G, this gives enough room to extend the argument in full despite the extra derivative order.
For both, note that we only need to estimate the Ricci terms in the L^2_G-norm if one cannot apply the a priori estimate (2.4.4f) to all $\nabla^{I_{\mathrm{Ric}}}\mathrm{Ric}[G]$ that occur in (2.4.15), and thus we could easily adjust the proof such that the Ricci energy does not occur in any of the proofs as long as $2l-1 \leq 10$ is satisfied, so for $l \leq 5$.

The estimates (2.4.13b)-(2.4.13d) are proved identically, the only difference being that one order of curvature less enters in the commutator terms in (2.4.15), leading to one order less in curvature in total. For (2.4.13a), we note that we can avoid incurring any L^2-norm by carefully repeating the argument we made for $\tilde{\mathfrak{T}}$ using (2.A.6a):

$$\begin{aligned}
\int_M |\nabla^2\zeta|^2_G \,\mathrm{dvol}_G &= \int_M -\langle\nabla\zeta,\nabla\Delta\zeta\rangle_G \,\mathrm{dvol}_G + \int_M \mathrm{Ric}[G]*\nabla\zeta*\nabla\zeta \,\mathrm{dvol}_G \\
&\lesssim \int_M |\Delta\zeta|^2_G \,\mathrm{dvol}_G + a^{-c\sqrt{\varepsilon}}\int_M |\nabla\zeta|^2_G \,\mathrm{dvol}_G
\end{aligned}$$

□

2.5. Big Bang stability: Elliptic lapse estimates

In this section, we study the elliptic structure of the equations (2.2.30a)-(2.2.30b), which admit estimates controlling (time-scaled) lapse energies by other energy quantities. To this end, we recast these equations as follows:

DEFINITION 2.5.1 (Elliptic operators). For any (sufficiently regular) scalar function ζ on Σ_t, we define the differential operators

$$\mathcal{L}\zeta = a^4\Delta\zeta - f\cdot\zeta, \qquad f = \frac{1}{3}a^4 + 12\pi C^2 + \underbrace{\langle\Sigma,\Sigma\rangle_G + 8\pi\Psi^2 + 16\pi C\Psi}_{=:F}, \tag{2.5.1a}$$

$$\tilde{\mathcal{L}}\zeta = a^4\Delta\zeta - \tilde{f}\cdot\zeta, \qquad \tilde{f} = \frac{1}{3}a^4 + 12\pi C^2 + \underbrace{a^4\left[R[G] + \frac{2}{3} - 8\pi|\nabla\phi|^2_G\right]}_{=\tilde{F}}. \tag{2.5.1b}$$

Note that the lapse equations (2.2.30a), respectively (2.2.30b), now read

$$\mathcal{L}N = F, \text{ respectively } \tilde{\mathcal{L}}N = \tilde{F}. \tag{2.5.2}$$

Furthermore, observe that

$$[\mathcal{L},\Delta]\zeta = \Delta f\cdot\zeta + 2\langle\nabla f,\nabla\zeta\rangle_G = \Delta F\cdot\zeta + 2\langle\nabla F,\nabla\zeta\rangle_G, \tag{2.5.3a}$$

$$[\tilde{\mathcal{L}},\Delta]\zeta = \Delta\tilde{F}\cdot\zeta + 2\langle\nabla\tilde{F},\nabla\zeta\rangle_G. \tag{2.5.3b}$$

2.5.1. Elliptic lapse estimates with $\mathcal{L}$. We first study the elliptic operator $\mathcal{L}$, which will admit weak lapse energy estimates in terms of scalar field quantities and Σ, up to curvature errors, that can in particular be utilized at high orders without having to resort to higher derivative levels. Before moving on to the estimates themselves, we collect a couple of inequalities we can deduce from the bootstrap assumptions and strong C_G-norm estimates.

REMARK 2.5.2. There exists a constant $K>0$ such that, for $\varepsilon>0$ small enough, the following estimates hold:

- $F\geq -K\varepsilon$, and equivalently $f\geq 12\pi C^2 - K\varepsilon$. This is ensured by (2.4.2b) and (2.4.2a). In particular, we can assume ε to have been small enough such that $f-6\pi C^2$ can be bounded from below by a positive constant that is independent of ε (for example $3\pi C^2$).
- $|\nabla f|_G = |\nabla F|_G \leq K\varepsilon a^{-c\sqrt{\varepsilon}}$. This is given by (2.4.4b) and (2.4.4a).

LEMMA 2.5.3 (Elliptic estimates with $\mathcal{L}$). *Consider scalar functions ζ, Z on Σ_t that satisfy*

$$\mathcal{L}\zeta = Z. \tag{2.5.4}$$

Then,

$$a^4\|\Delta\zeta\|_{L^2_G} + a^2\|\nabla\zeta\|_{L^2_G} + \|\zeta\|_{L^2_G} \lesssim \|Z\|_{L^2_G} \tag{2.5.5}$$

PROOF. The proof follows along the same lines as that of [Spe18, Lemma 16.5]: First, we obtain the following by multiplying (2.5.4) with $-\zeta$ and integrating: and using that $f-6\pi C^2$

is bounded from below by a positive constant (see the first point in Remark 2.5.2):

$$\int_M (a^4|\nabla\zeta|_G^2 + |\zeta|^2)\mathrm{dvol}_G \lesssim \|Z\|_{L^2_G}^2$$

Next, we multiply (2.5.4) with $a^4\Delta\zeta$ and obtain

$$\begin{aligned}&\int_M \left(a^8|\Delta\zeta|^2 + a^4|\nabla\zeta|_G^2 f\right)\,\mathrm{dvol}_G\\ \leq &\int_M \frac{1}{2}|Z|^2 + \frac{a^8}{2}|\Delta\zeta|^2 + \left(\frac{1}{2}|\zeta|^2 + \frac{a^4}{2}|\nabla\zeta|_G^2\right) a^2\|\nabla f\|_{L^\infty_G}\,\mathrm{dvol}_G\end{aligned}$$

Using the second point in Remark 2.5.2 as well as the previous step, we can now conclude

$$\int_M \left(a^8|\Delta\zeta|^2 + a^4|\nabla\zeta|_G^2\right)\,\mathrm{dvol}_G \lesssim (1+\varepsilon)\|Z\|_{L^2_G}^2$$

and thus the statement after rearranging. □

COROLLARY 2.5.4 (Intermediary elliptic lapse estimate with $\mathcal{L}$). *The following estimates hold for any $l \in \{0,\dots,10\}$:*

$$\begin{aligned}&a^4\|\Delta^{l+1}N\|_{L^2_G} + a^2\|\nabla\Delta^l N\|_{L^2_G} + \|\Delta^l N\|_{L^2_G}\\ \lesssim &\|\Delta^l F\|_{L^2_G} + \underbrace{\varepsilon a^{-c\sqrt{\varepsilon}}\|F\|_{H^{2(l-1)}_G}}_{\text{not present for } l=0} + \underbrace{\varepsilon^2 a^{4-c\sigma}\sqrt{\mathfrak{E}^{(\leq 2l-4)}(\mathrm{Ric},\cdot)}}_{\text{not present for } l\leq 1}\end{aligned}$$

PROOF. We prove the statement by induction over $l \in \mathbb{N}$: For $l = 0$, the estimates immediately follow from (2.5.2) and Lemma 2.5.3. Assume the statement to be satisfied up to $l-1$ for some $l \in \mathbb{N}_0$, $l \leq 11$. We get, applying (2.5.3a) iteratively,

$$\mathcal{L}\Delta^l N = \sum_{I=1}^{2l-1} \nabla^I F * \nabla^{2l-I} N + (N+1)\Delta^l F\,.$$

Applying Lemma 2.5.3 to $\zeta = \Delta^l N$ as well as Lemma 2.4.1 yields

$$a^4\|\Delta^{l+1}N\|_{L^2_G} + a^2\|\nabla\Delta^l N\|_{L^2_G} + \|\Delta^l N\|_{L^2_G} \lesssim \sum_{I=1}^{2l-1}\|\nabla^I F * \nabla^{2l-I}N\|_{L^2_G} + \|\Delta^l F\|_{L^2_G}$$

Hence, using (2.4.13c) (replacing l with $l-1$) and (2.3.17h), (2.4.4a) and (2.4.4b) to estimate low order terms, we get

$$\begin{aligned}&a^4\|\Delta^{l+1}N\|_{L^2_G} + a^2\|\nabla\Delta^l N\|_{L^2_G} + \|\Delta^l N\|_{L^2_G}\\ \lesssim &\|\Delta^l F\|_{L^2_G} + \varepsilon a^{-c\sqrt{\varepsilon}}\left(\sum_{m=0}^{l-1}\|\nabla\Delta^m N\|_{L^2_G} + \varepsilon a^{4-c\sigma}\sqrt{\mathfrak{E}^{(\leq 2l-4)}(\mathrm{Ric},\cdot)}\right)\\ &+ \varepsilon a^{4-c\sigma}\left(\|F\|_{H^{2(l-1)}_G} + \|\nabla\Delta^{l-1}F\|_{L^2_G} + \varepsilon a^{-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq 2l-4)}(\mathrm{Ric},\cdot)}\right)\end{aligned}$$

For the top order lapse term, we can redistribute the divergent prefactor as follows:

$$\varepsilon a^{-c\sqrt{\varepsilon}}\|\nabla\Delta^{l-1}N\|_{L^2_G} \lesssim \varepsilon\|\Delta^l N\|_{L^2_G} + \varepsilon a^{-2c\sqrt{\varepsilon}}\|\Delta^{l-1}N\|_{L^2_G}$$

The lower order lapse terms as well as $\|\nabla\Delta^{l-1}F\|_{L^2_G}$ can be estimated similarly, just without having to redistribute the prefactor. Updating $c>0$ and rearranging then yields the statement at order l for suitably small $\varepsilon>0$, and thus the entire statement after iteration. □

COROLLARY 2.5.5 (Lapse energy estimates with $\mathcal{L}$). *For any $l\in\{0,\dots,9\}$, one has*

$$\begin{aligned}&a^8\mathfrak{E}^{(2(l+1))}(N,\cdot)+a^4\mathfrak{E}^{(2l+1)}(N,\cdot)+\mathfrak{E}^{(2l)}(N,\cdot)\\ \lesssim\,&\varepsilon^2\mathfrak{E}^{(2l)}(\Sigma,\cdot)+\mathfrak{E}^{(2l)}(\phi,\cdot)+\underbrace{\varepsilon^2a^{-c\sqrt{\varepsilon}}\left[\mathfrak{E}^{(\leq 2(l-1))}(\Sigma,\cdot)+\mathfrak{E}^{(\leq 2(l-1))}(\phi,\cdot)\right]}_{\textit{not present for } l=0}\\ &+\underbrace{\left(\varepsilon^4a^{-c\sqrt{\varepsilon}}+\varepsilon^2a^{8-c\sigma}\right)\mathfrak{E}^{(\leq 2l-3)}(\mathrm{Ric},\cdot)}_{\textit{not present for } l\leq 1}\end{aligned}$$

PROOF. Note that, by Corollary 2.5.4, all that needs to be done is to relate all Sobolev norms of F that occur to the respective energies. Schematically, we have

$$\Delta^lF=16\pi(\Delta^l\Psi)(\Psi+C)+2\langle\Delta^l\Sigma,\Sigma\rangle_G+\sum_{I=1}^{2l-1}\left(\nabla^I\Psi*\nabla^{2l-I}\Psi+\nabla^I\Sigma*\nabla^{2l-I}\Sigma\right).$$

For the first two terms, we can use (2.4.2a) and (2.4.2b) to bound $|\Sigma|_G$ and $|\Psi+C|$ by ε and 1 up to constant, respectively. For the remaining terms, we similarly always bound the lower order in L^∞_G with (2.4.4a)-(2.4.4b) and bound the higher order with the energy estimates in Lemma 2.4.5. Further, we can use (2.3.8) to redistribute divergent prefactors onto energies of order $l-2$ and lower. This already incurs the terms on the right hand side of the claimed estimate, and the lower order norms of F only incur at equivalent or weaker error terms. □

2.5.2. Elliptic lapse estimates with $\tilde{\mathcal{L}}$. While the estimates in the previous subsection are useful at high orders, they are not enough to close the bootstrap assumptions for N. This can be achieved by deriving estimates in terms of $\tilde{\mathcal{L}}$ – however, due to the explicit presence of Ricci terms in this version of the lapse equation, we use this to bound N at lower orders. Since the arguments are largely identical to the ones above, we only sketch the proofs.

REMARK 2.5.6. Note that, when replacing f by $\tilde f$ and F by $\tilde F$ in Remark 2.5.2, the same statements hold for a suitable constant K. In fact, the bootstrap assumptions on $\mathrm{Ric}[G]$ and $\nabla\phi$ even imply $\|\tilde F\|_{C^1_G}\lesssim\varepsilon a^{4-c\sigma}$, noting

$$\left|R[G]+\frac{2}{3}\right|_G\leq|G^{-1}|_G\left|\mathrm{Ric}[G]+\frac{2}{9}G\right|_G\lesssim\left|\mathrm{Ric}[G]+\frac{2}{9}G\right|_G$$

and

$$|\nabla R[G]|_G=\left|\nabla\left(R[G]+\frac{2}{3}\right)\right|_G\lesssim\left|\nabla\left(\mathrm{Ric}[G]+\frac{2}{9}G\right)\right|_G.$$

LEMMA 2.5.7. *Any scalar functions ζ and Z such that*

$$\tilde{\mathcal{L}}\zeta=Z$$

holds satisfy the estimate

$$a^4\|\Delta\zeta\|_{L^2_G}+a^2\|\nabla\zeta\|_{L^2_G}+\|\zeta\|_{L^2_G}\lesssim\|Z\|_{L^2_G}.$$

PROOF. The proof follows identically to Lemma 2.5.3 since all tools relating to f and F used in proving these statements were collected in Remark 2.5.2, and these extend to $\tilde{\mathcal{L}}$ by Remark 2.5.6. $\square$

COROLLARY 2.5.8. *For* $l \in \{0, \dots, 8\}$

$$a^8 \mathfrak{E}^{(2(l+1))}(N,\cdot) + a^4 \mathfrak{E}^{(2l+1)}(N,\cdot) + \mathfrak{E}^{(2l)}(N,\cdot) \lesssim a^8 \mathfrak{E}^{(\leq 2l)}(\mathrm{Ric},\cdot) + \varepsilon a^{8-c\sqrt{\varepsilon}} \|\nabla\phi\|^2_{H^{2l}_G}$$

PROOF. As in the proof of Corollary 2.5.4, this follows by commuting $\tilde{\mathcal{L}}$ with Δ^l iteratively and applying (2.4.4e) and (2.4.4f) to bound lower order terms within the nonlinearities. $\square$

2.6. Big Bang stability: Energy and norm estimates

In this section, we derive energy estimates for matter variables and the geometric quantities as well as Sobolev norm estimates for spatial derivatives of ϕ and for metric quantities. To derive all of the inequalities in this section beside the elliptic inequality in Lemma 2.6.10 and the bound on $\nabla\phi$ in Lemma 2.6.5, we will use the same basic strategy. Hence, we give a brief overview on the form our integral inequalities are going to take and how we intend to obtain improved energy bounds from there:

REMARK 2.6.1 (Integral inequalities and the Gronwall argument). Let $\mathcal{F}_L$ denote an energy or a squared Sobolev(-type) norm at derivative level $L \in 2\mathbb{N}$, for example, $\mathfrak{E}^{(L)}(\phi,\cdot)$. To derive an integral inequality for $\mathcal{F}_L$, we will take its time derivative, apply the respective commuted evolution equations in the integrand, estimate the resulting terms and integrate that inequality. Schematically, the resulting integral inequalities for $\mathcal{F}_L$ then take the following form:

$$\begin{aligned}
&\mathcal{F}_L(t) + \int_t^{t_0} \langle \text{ultimately nonnegative contributions} \rangle \, ds \\
&\lesssim \mathcal{F}_L(t_0) + \int_t^{t_0} \left(\varepsilon^{\frac{1}{8}} a(s)^{-3} + a(s)^{-1-c\sqrt{\varepsilon}} \right) \mathcal{F}_L(s) \, ds \\
&\quad + \int_t^{t_0} a(s)^{-3} \langle \text{other energies/squared Sobolev norms at same derivative level} \rangle \, ds \\
&\quad + \int_t^{t_0} a(s)^{-3-c\sqrt{\varepsilon}} \langle \text{energies/squared Sobolev norms at derivative levels up to } L-2 \rangle \, ds
\end{aligned}$$

For some inequalities, we will not be able to derive any beneficial ε-prefactors in the penultimate line. For example, for $\mathfrak{E}^{(L)}(\Sigma,\cdot)$, linear lapse terms in the evolution of Σ incur a term of the form

$$\int_t^{t_0} a(s)^{-3} \cdot a(s)^4 \|\Delta^{\frac{L}{2}} N\|_{\dot{H}^2_G} \cdot \sqrt{\mathfrak{E}^{(L)}(\Sigma, s)} \, ds$$

on the right hand side, which after applying lapse energy estimates creates $\varepsilon^{-\frac{1}{8}} \mathfrak{E}^{(L)}(\phi,\cdot)$ on the right. However, combining the respective inequalities for the core energy mechanism at each derivative level with appropriate ε-weights, this will then combine to an inequality of the following form for a total energy, which we informally denote by $\mathcal{F}_{\mathrm{total},L}$:

$$\mathcal{F}_{\mathrm{total},L}(t) + \int_t^{t_0} \langle \text{nonnegative quantity} \rangle \, ds \tag{2.6.1}$$

$$\begin{aligned}
&\lesssim \mathcal{F}_{\text{total},L}(t_0) + \int_t^{t_0} \left(\varepsilon^{\frac{1}{8}} a(s)^{-3} + a(s)^{-1-c\sqrt{\varepsilon}}\right) \mathcal{F}_{\text{total},L}(s)\, ds \\
&\quad + \underbrace{\int_t^{t_0} \varepsilon^{\frac{1}{8}} a(s)^{-3-c\varepsilon^{\frac{1}{8}}} \langle\text{already improved terms}\rangle\, ds}_{\text{not present for } L=0} \\
&\quad + \varepsilon^{\frac{1}{4}} \mathcal{F}_{\text{total},L}(t) + \underbrace{\sqrt{\varepsilon}\cdot\langle\text{small lower order terms}\rangle(t)}_{\text{not present for } L=0}
\end{aligned}$$

In the mentioned example, multiplying $\mathfrak{E}^{(L)}(\Sigma,\cdot)$ with the weight $\varepsilon^{\frac{1}{4}}$, in turn, mitigates the otherwise offending term to $\varepsilon^{\frac{1}{8}} a(s)^{-3}\mathfrak{E}^{(L)}(\phi, s)$, which can be absorbed into the first line.

Furthermore, $\varepsilon^{\frac{1}{4}}\mathcal{F}_{\text{total},L}(t)$ in the penultimate line of (6-1) can be absorbed into the left-hand side after updating the implicit constant in "$\lesssim$". Applying the Gronwall lemma (see Lemma 2.A.1) and the initial data assumption (which implies $\mathcal{F}_{\text{total},L}(t_0) \lesssim \varepsilon^4$) then yields

$$\begin{aligned}
\mathcal{F}_{\text{total},L}(t) \lesssim &\left(\varepsilon^4 + \underbrace{\int_t^{t_0} \varepsilon^{\frac{1}{8}} a(s)^{-3-c\varepsilon^{\frac{1}{8}}} \langle\text{improved terms}\rangle\, ds + \sqrt{\varepsilon}\cdot\langle\text{lower order terms}\rangle(t)}_{\text{not present for } L=0}\right) \\
&\cdot \exp\left(K\cdot \int_t^{t_0} \varepsilon^{\frac{1}{8}} a(s)^{-3} + a(s)^{-1-c\sqrt{\varepsilon}}\, ds\right)
\end{aligned}$$

for some constant $K>0$. (2.2.7) and (2.2.6), the exponential factor can be bounded by $a^{-c\varepsilon^{\frac{1}{8}}}$, up to constant and updating $c>0$. Hence, for $L=0$, this implies $\mathcal{F}_{\text{total},0} \lesssim \varepsilon^4 a^{-c\varepsilon^{\frac{1}{8}}}$, and thus leads to improved bounds for base level energy quantities (see Remark 2.3.19 for the precise scaling hierarchy that will achieve). By iterating this argument for $L>0$, the already improved terms will then be bounded (at worst) by $\varepsilon^4 a^{-c\varepsilon^{\frac{1}{8}}}$, and (2.2.6) shows that the first line can be bounded by $\varepsilon^4 a^{-c\varepsilon^{\frac{1}{8}}}$ after updating c. This allows us to bound $\mathcal{F}_{\text{total},L}$ by $\varepsilon^4 a^{-c\varepsilon^{\frac{1}{8}}}$ for any L up to and including top order.

Finally, we mention that, to control energies at order L, we need to consider scaled energies at order $L+1$ within $\mathcal{F}_{total,L}$ – this arises since the scalar field occurs at first order in the evolution equations for $\mathcal{E}$ and $\mathcal{B}$. We avoid losing derivatives by employing the div-curl-estimate in Lemma 2.6.10 at order $L+1$, which allows us to control $a^4\mathfrak{E}^{(L+1)}(\Sigma,\cdot)$ by quantities at order L. This is precisely what generates the non-integral terms in the schematics above. We note that it is crucial that the scalar field occurs at no worse scaling than a^{-1} in (2.2.38a)-(2.2.38b) – else, moving to these time-scaled estimates at order $L+1$ would lose too many powers of a and lead to exponentially divergent terms after applying the Gronwall argument.

Recall that L^2_G-norm estimates for error terms arising in the Laplace-commuted equations in Lemma 2.2.11 are collected in Section 2.A.4. Low order estimates (in particular estimates for $L=2$) could often be improved if needed by more carefully avoiding curvature error terms, but we refrain from doing so where it is not necessary to keep estimates as unified as possible.

2.6.1. Integral and energy estimates for the scalar field.

2.6.1.1. *Scalar field energy estimates.* Over the following two lemmas, we prove the core energy estimates to control the matter variables, which are immediately prepared differently at base, intermediate and top order for the total energy estimates in Section 2.7.

LEMMA 2.6.2. *[Even order scalar field energy estimates] Let $t \in (t_{Boot}, t_0]$. Then, one has*

$$\begin{aligned}\mathfrak{E}^{(0)}(\phi,t)+\int_t^{t_0}\dot a(s)a(s)^3\mathfrak{E}^{(1)}(N,s)+\frac{\dot a(s)}{a(s)}\mathfrak{E}^{(0)}(N,s)\,ds\\ \lesssim \mathfrak{E}^{(0)}(\phi,t_0)+\int_t^{t_0}\varepsilon a(s)^{-3}\mathfrak{E}^{(0)}(\phi,s)+\varepsilon a^{-3}\mathfrak{E}^{(0)}(\Sigma,s)\,ds\,.\end{aligned}\tag{2.6.2}$$

Further, for any $L \in 2\mathbb{N}$, $2 \leq L \leq 18$, the following estimate is satisfied:

$$\begin{aligned}&\mathfrak{E}^{(L)}(\phi,t)+\int_t^{t_0}\dot a(s)a(s)^3\mathfrak{E}^{(L+1)}(N,s)+\frac{\dot a(s)}{a(s)}\mathfrak{E}^{(L)}(N,s)\,ds\\ &\lesssim \mathfrak{E}^{(L)}(\phi,t_0)+\int_t^{t_0}\left(\varepsilon a(s)^{-3}+a(s)^{-1-c\sqrt{\varepsilon}}\right)\mathfrak{E}^{(L)}(\phi,s)\,ds\\ &\quad+\int_t^{t_0}\varepsilon a(s)^{-3}\mathfrak{E}^{(L)}(\Sigma,s)+\varepsilon^{\frac32}a(s)^{-3}\mathfrak{E}^{(L-2)}(\mathrm{Ric},s)\,ds\\ &\quad+\int_t^{t_0}\sqrt{\varepsilon}a(s)^{-3-c\sqrt{\varepsilon}}\mathfrak{E}^{(\leq L-2)}(\phi,s)+\varepsilon a(s)^{-3-c\sqrt{\varepsilon}}\mathfrak{E}^{(\leq L-2)}(\Sigma,s)\,ds\\ &\quad\underbrace{+\int_t^{t_0}\varepsilon^{\frac32}a(s)^{-3-c\sqrt{\varepsilon}}\mathfrak{E}^{(\leq L-4)}(\mathrm{Ric},s)\,ds}_{\text{if } L\geq 4}\end{aligned}\tag{2.6.3}$$

REMARK 2.6.3. This proof relies on two mechanisms: Firstly, we use the structure of the wave equation and integration by parts to cancel the highest order scalar field derivative terms. Getting this cancellation is what necessitates scaling the potential term in the scalar field energy by a^4. Secondly, we deal with the highest order lapse terms using the elliptic structure of the (Laplace-commuted) lapse equation – both in an indirect way by invoking the elliptic energy estimate in Corollary 2.5.5 as well as by directly inserting (2.2.37a) to cancel some ill-behaved terms. While the framework significantly differs from the scalar field energy estimates [Spe18], these two core mechanisms also appear there and play similarly crucial roles.

PROOF. Since the arguments are essentially the same, we will only write down the proof for $L \geq 2$ in full and make short comments throughout the argument which terms do not occur for $L = 0$.
We use the evolution equations (2.2.39a) and (2.2.39b) and Lemma 2.4.4 to compute, for $L \geq 2$,

$$\begin{aligned}-\partial_t\mathfrak{E}^{(L)}(\phi,\cdot)=&\int_M -2\Delta^{\frac L2}\Psi\partial_t\Delta^{\frac L2}\Psi-2a^4\langle\partial_t\nabla\Delta^{\frac L2}\phi,\nabla\Delta^{\frac L2}\phi\rangle_G-a^4(\partial_tG^{-1})^{ij}\nabla_i\Delta^{\frac L2}\phi\nabla_j\Delta^{\frac L2}\phi\\ &\qquad-3N\frac{\dot a}{a}\left[|\Delta^{\frac L2}\Psi|^2+a^4|\nabla\Delta^{\frac L2}\phi|_G^2\right]-4\frac{\dot a}{a}\cdot a^4|\nabla\Delta^{\frac L2}\phi|^2_G\,\mathrm{dvol}_G\end{aligned}$$

$$=\int_M\left(-2a(N+1)\Delta^{\frac L2+1}\phi-2a\langle\nabla\Delta^{\frac L2}N,\nabla\phi\rangle_G+6C\frac{\dot a}{a}\Delta^{\frac L2}N\right)\cdot(\Delta^{\frac L2}\Psi)\tag{2.6.4a}$$

$$- 2a(N+1)\langle\nabla\Delta^{\frac{L}{2}}\Psi, \nabla\Delta^{\frac{L}{2}}\phi\rangle_G - 2Ca\langle\nabla\Delta^{\frac{L}{2}}N, \nabla\Delta^{\frac{L}{2}}\phi\rangle_G \tag{2.6.4b}$$

$$- 2\left(\mathfrak{P}_{L,Border} + \mathfrak{P}_{L,Junk}\right)\cdot\Delta^{\frac{L}{2}}\Psi - 2a^4\langle\mathfrak{Q}_{L,Border} + \mathfrak{Q}_{L,Junk}, \nabla\Delta^{\frac{L}{2}}\phi\rangle_G \tag{2.6.4c}$$

$$+ 2(N+1)a\cdot(\Sigma^\sharp)^{ij}\nabla_i\Delta^{\frac{L}{2}}\phi\nabla_j\Delta^{\frac{L}{2}}\phi - 2N\frac{\dot a}{a}\cdot a^4|\nabla\Delta^{\frac{L}{2}}\phi|_G^2 \tag{2.6.4d}$$

$$- 3N\frac{\dot a}{a}\left[|\Delta^{\frac{L}{2}}\Psi|^2 + a^4|\nabla\Delta^{\frac{L}{2}}\phi|_G^2\right] - 4\frac{\dot a}{a}\cdot a^4|\nabla\Delta^{\frac{L}{2}}\phi|_G^2\,\mathrm{dvol}_G\,. \tag{2.6.4e}$$

Note that, for $L=0$, the equivalent equality holds where the borderline and junk terms are replaced by $-2a^4\Psi\langle\nabla N, \nabla\phi\rangle_G$ (to verify this, insert (2.2.32a) and (2.2.32b) instead of (2.2.39a) and (2.2.39b)). We now go through (2.6.4a)-(2.6.4e) term by term:
After integrating by parts, the first term in (2.6.4a) reads

$$\int_M 2a(N+1)\langle\nabla\Delta^{\frac{L}{2}}\phi, \nabla\Delta^{\frac{L}{2}}\Psi\rangle_G + 2a\langle\nabla N, \nabla\Delta^{\frac{L}{2}}\phi\rangle_G\cdot\Delta^{\frac{L}{2}}\Psi\,\mathrm{dvol}_G\,. \tag{2.6.5}$$

The first term **precisely** cancels the first term in (2.6.4b), while we can use the bootstrap assumption (2.3.17h) to estimate the other term in (2.6.5) up to constant by

$$\varepsilon a^{3-c\sigma}\cdot a^2\|\nabla\Delta^{\frac{L}{2}}\phi\|_{L^2_G}\|\Delta^{\frac{L}{2}}\Psi\|_{L^2_G}\lesssim\varepsilon a^{3-c\sigma}\mathfrak{E}^{(L)}(\phi,\cdot)\,.$$

For the second term in (2.6.4a), we use (2.4.4e) to estimate $\nabla\phi$ and Corollary 2.5.5 at order L to deal with the lapse, getting

$$\begin{aligned}\left|\int_M 2a\langle\nabla\Delta^{\frac{L}{2}}N, \nabla\phi\rangle_G\cdot\Delta^{\frac{L}{2}}\Psi\,\mathrm{dvol}_G\right| &\lesssim\sqrt{\varepsilon}a^{-1-c\sqrt{\varepsilon}}\sqrt{a^4\mathfrak{E}^{(L+1)}(N,\cdot)}\sqrt{\mathfrak{E}^{(L)}(\phi,\cdot)}\\ &\lesssim\varepsilon a^{-1}\cdot a^4\mathfrak{E}^{(L+1)}(N,\cdot) + a^{-1-c\sqrt{\varepsilon}}\mathfrak{E}^{(L)}(\phi,\cdot)\\ &\lesssim\varepsilon^3a^{-1}\mathfrak{E}^{(L)}(\Sigma,\cdot) + a^{-1-c\sqrt{\varepsilon}}\mathfrak{E}^{(L)}(\phi,\cdot)\\ &\quad+\varepsilon^2a^{-1-c\sqrt{\varepsilon}}\mathfrak{E}^{(\leq L-2)}(\Sigma,\cdot) + \varepsilon^2a^{-1-c\sqrt{\varepsilon}}\mathfrak{E}^{(\leq L-2)}(\phi,\cdot)\\ &\quad+\underbrace{\varepsilon^2a^{-1-c\sqrt{\varepsilon}}\mathfrak{E}^{(\leq L-3)}(\mathrm{Ric},\cdot)}_{\text{not present for }L=2}\,.\end{aligned}$$

Repeating this argument for $L=0$, the last two lines do not appear.

To deal with the remaining term in (2.6.4a), we can insert the following zero on the right hand side of the differential equality, where the equality (2.6.6) holds due to (2.2.37a):

$$\begin{aligned}0 &= -\frac{3}{8\pi}\dot a a^3\int_M\mathrm{div}_G\left(\nabla\Delta^{\frac{L}{2}}N\cdot\Delta^{\frac{L}{2}}N\right)\,\mathrm{dvol}_G\\ &= -\frac{3}{8\pi}\dot a a^3\int_M\Delta^{\frac{L}{2}+1}N\cdot\Delta^{\frac{L}{2}}N + |\nabla\Delta^{\frac{L}{2}}N|_G^2\,\mathrm{dvol}_G\\ &= \int_M -\frac{3}{8\pi}\dot a a^3|\nabla\Delta^{\frac{L}{2}}N|_G^2 - \frac{3}{8\pi}\left(\frac{1}{3}\dot a a^3 + 12\pi C^2\frac{\dot a}{a}\right)|\Delta^{\frac{L}{2}}N|^2\\ &\qquad - 6C\frac{\dot a}{a}\Delta^{\frac{L}{2}}N\cdot\Delta^{\frac{L}{2}}\Psi - \frac{3}{8\pi}\dot a a^3\left[\mathfrak{N}_{L,Border} + \mathfrak{N}_{L,Junk}\right]\cdot\Delta^{\frac{L}{2}}N\,\mathrm{dvol}_G\end{aligned} \tag{2.6.6}$$

Note that the first line has a negative sign, so (after absorbing a few terms into it without changing the sign, see namely lapse quantities in (2.6.7) and (2.6.8a)), we pull it to the left hand side of the differential inequality. Further, the first term in the second line of (2.6.6) precisely cancels the third term in (2.6.4a). That leaves the borderline and junk terms in (2.6.6), for which we use (2.A.17b) and (2.A.19d) (along with $\dot{a} \simeq a^{-2}$ due to (2.2.3)) to get, for $L \geq 4$,

$$\begin{aligned}
&\frac{3}{8\pi}\dot{a}a^3 \left| \int_M \left[\mathfrak{N}_{L,Border} + \mathfrak{N}_{L,Junk}\right] \cdot \Delta^{\frac{L}{2}} N \,\mathrm{dvol}_G \right| \\
&\lesssim \varepsilon a^{-3} \left[\mathfrak{E}^{(L)}(\phi,\cdot) + \mathfrak{E}^{(L)}(\Sigma,\cdot) + \mathfrak{E}^{(L)}(N,\cdot)\right] \\
&\quad + \varepsilon a^{-3-c\sqrt{\varepsilon}} \left[\mathfrak{E}^{(\leq L-2)}(\phi,\cdot) + \mathfrak{E}^{(\leq L-2)}(\Sigma,\cdot) + \mathfrak{E}^{(\leq L-2)}(N,\cdot)\right] \\
&\quad + \underbrace{\varepsilon^3 a^{-3} \mathfrak{E}^{(\leq L-2)}(\mathrm{Ric},\cdot) + \varepsilon^3 a^{-3-c\sqrt{\varepsilon}} \mathfrak{E}^{(\leq L-3)}(\mathrm{Ric},\cdot)}_{\text{not present for } L=2}\,.
\end{aligned}$$

Again, the same estimate holds for $L = 0$ with the last two lines dropped.

From (2.6.4a)-(2.6.4b), only the term $-2Ca\langle \nabla\Delta^{\frac{L}{2}} N, \nabla\Delta^{\frac{L}{2}}\phi\rangle_G$ still needs to be handled: Using the inequality (2.2.8) arising from the Friedman equation, we can estimate this by

$$\int_M 2\sqrt{\frac{3}{4\pi}}\dot{a}a^3 |\nabla\Delta^{\frac{L}{2}} N|_G |\nabla\Delta^{\frac{L}{2}}\phi|_G \,\mathrm{dvol}_G \leq \int_M 4\dot{a}a^3 |\nabla\Delta^{\frac{L}{2}}\phi|_G^2 + \frac{3}{16\pi}\dot{a}a^3|\nabla\Delta^{\frac{L}{2}} N|_G^2 \,\mathrm{dvol}_G\,. \tag{2.6.7}$$

Note that the first term precisely cancels the final term in (2.6.4e), while the second term can be absorbed into the first term in (2.6.6) while preserving that term's sign.

To bound the error terms in (2.6.4c), we insert the borderline term estimates (2.A.17d) and (2.A.17e) as well as the junk term estimates (2.A.19f) and (2.A.19h), where (2.3.8) is used to estimate odd order by even order energies where needed. Furthermore, observe that we can estimate the $\mathfrak{Q}_L$-terms as

$$\left(a^2 \|\mathfrak{Q}_L\|_{L^2_G}\right) \cdot \sqrt{\mathfrak{E}^{(L)}(\phi,\cdot)},$$

so all borderline and junk terms arising from it, beside the scalar field energies, are dominated by terms occurring elsewhere.

Finally, all terms that remain, namely (2.6.4d) and the first term in (2.6.4e), can be bounded by $\varepsilon a^{-3}\mathfrak{E}^{(L)}(\phi,\cdot)$ due to the strong base level estimate (2.4.2b) and (2.3.17h). In summary, and always only keeping the worst terms for each energy and squared norm, this yields for $L \geq 4$:

$$- \partial_t \mathfrak{E}^{(L)}(\phi,\cdot) + \dot{a}a^3 \mathfrak{E}^{(L+1)}(N,\cdot) + \frac{\dot{a}}{a}\mathfrak{E}^{(L)}(N,\cdot)$$

$$\lesssim \left(\varepsilon a^{-3} + a^{-1-c\sqrt{\varepsilon}}\right) \mathfrak{E}^{(L)}(\phi,\cdot) + \left(\varepsilon a^{-3} + \sqrt{\varepsilon} a^{-1-c\sqrt{\varepsilon}}\right)\left(a^4 \mathfrak{E}^{(L+1)}(N,\cdot) + \mathfrak{E}^{(L)}(N,\cdot)\right) \tag{2.6.8a}$$

$$+ \varepsilon a^{-3}\mathfrak{E}^{(L)}(\Sigma,\cdot) + \varepsilon^{\frac{3}{2}} a^{-3}\mathfrak{E}^{(L-2)}(\mathrm{Ric},\cdot) + \varepsilon a^{-3-c\sqrt{\varepsilon}}\mathfrak{E}^{(\leq L-2)}(\phi,\cdot) \tag{2.6.8b}$$

$$+ \varepsilon a^{-3-c\sqrt{\varepsilon}}\mathfrak{E}^{(\leq L-2)}(\Sigma,\cdot) + \left[\varepsilon a^{-3-c\sqrt{\varepsilon}} + \sqrt{\varepsilon} a^{-1-c\sqrt{\varepsilon}}\right] \mathfrak{E}^{(\leq L-2)}(N,\cdot) \tag{2.6.8c}$$

$$\text{(2.6.8d)} \quad \underbrace{+\varepsilon^{\frac{3}{2}} a^{-3-c\sqrt{\varepsilon}} \mathfrak{E}^{(\leq L-4)}(\mathrm{Ric}, \cdot)}_{\text{not present for } L=2}$$

The lapse energies in (2.6.8a) can now also be absorbed into those on the left hand side of the inequality by updating the implicit constant in "$\lesssim$". We can treat the lower order lapse energies in (2.6.8c) with Corollary 2.5.5 and see that the resulting terms are all dominated by terms we already have on the right hand side of the inequality above.

Inserting these estimates and integrating over $(t, t_0]$ then yields (2.6.3) for $L \geq 4$, and the statement for $L = 2$ is obtained completely analogously.
As mentioned earlier, (2.6.4c) is replaced by the following term for $L = 0$:

$$\begin{aligned}
&\int_M -2a\Psi \langle \nabla N, \nabla \phi \rangle_G \,\mathrm{dvol}_G \\
&\lesssim \varepsilon a^{-3} \int_M a^2 |\nabla N|_G \cdot a^2 |\nabla \phi|_G \,\mathrm{dvol}_G \lesssim \varepsilon \dot{a} a^3 \mathfrak{E}^{(1)}(N, \cdot) + \varepsilon a^{-3} \mathfrak{E}^{(0)}(\phi, \cdot)\,,
\end{aligned}$$

Here, we applied (2.4.2a) and (2.2.3). Both of these terms can be absorbed into terms that are already present, and (2.6.2) then follows by dealing with terms in $\partial_t \mathfrak{E}^{(0)}(\phi, \cdot)$ as described and integrating. □

To close the argument, we will need a scaled scalar field energy estimate at the odd orders $L + 1$, which is not covered by the previous lemma and we hence establish separately:

Lemma 2.6.4 (Odd order scalar field energy estimate). *For $L \in 2\mathbb{N}$, $2 \leq L \leq 18$, we have:*

$$\begin{aligned}
\text{(2.6.9)} \qquad & a(t)^4 \mathfrak{E}^{(L+1)}(\phi, t) + \int_t^{t_0} \left\{ \dot{a}(s) a(s)^7 \mathfrak{E}^{(L+2)}(N, s) + \dot{a}(s) a(s)^3 \mathfrak{E}^{(L+1)}(N, s) \right\} ds \\
& \lesssim a(t_0)^4 \mathfrak{E}^{(L+1)}(\phi, t_0) + \int_t^{t_0} \left(\varepsilon a(s)^{-3} + a(s)^{-1-c\sqrt{\varepsilon}} \right) \cdot a(s)^4 \mathfrak{E}^{(L+1)}(\phi, s)\, ds \\
& \quad + \int_t^{t_0} \left\{ \varepsilon a(s)^{-3} \cdot a(s)^4 \mathfrak{E}^{(L+1)}(\Sigma, s) + \left(\varepsilon a(s)^{-3} + a(s)^{-1-c\sqrt{\varepsilon}} \right) \mathfrak{E}^{(L)}(\phi, s) \right. \\
& \qquad + \varepsilon a(s)^{-3} \mathfrak{E}^{(L)}(\Sigma, s) + \varepsilon a(s)^{-1-c\sqrt{\varepsilon}} \cdot a(s)^4 \mathfrak{E}^{(L-1)}(\mathrm{Ric}, s) \\
& \qquad + \left(\varepsilon^3 a^{-3} + \varepsilon a^{-1-c\sqrt{\varepsilon}} \right) \mathfrak{E}^{(L-2)}(\mathrm{Ric}, s) \\
& \qquad + \varepsilon a(s)^{-3-c\sqrt{\varepsilon}} \left(\mathfrak{E}^{(\leq L-2)}(\phi, s) + \mathfrak{E}^{(\leq L-2)}(\Sigma, s) \right) \\
& \qquad \left. \underbrace{+ \left(\varepsilon^3 a(s)^{-3-c\sqrt{\varepsilon}} + \varepsilon^2 a(s)^{-1-c\sqrt{\varepsilon}} \right) \mathfrak{E}^{(\leq L-4)}(\mathrm{Ric}, s) \right\}}_{\textit{not present for } L=2} ds
\end{aligned}$$

At order 1, *the analogous estimate holds where the last three lines of* (2.6.9) *are dropped.*

Proof. These estimates follow completely analogously to Lemma 2.6.2, with the exception that high order lapse terms can now be estimated at order $L + 2$ due to the scalar field energy being scaled by a^4. In particular, we note that to deal with the analogous term to (2.6.4a), one now inserts the following zero on the right and applies the commuted lapse equation (2.2.37b):

$$\begin{aligned}
0 =& -\frac{3}{8\pi}\dot{a}a^7\int_M \mathrm{div}_G\left(\nabla\Delta^{\frac{L}{2}}N\cdot\Delta^{\frac{L}{2}+1}N\right)\mathrm{dvol}_G\\
=& \int_M\left\{-\frac{3}{8\pi}\dot{a}a^7|\Delta^{\frac{L}{2}+1}N|^2-\frac{3}{8\pi}\left(\frac{1}{3}\dot{a}a^3+12\pi C^2\frac{\dot{a}}{a}\right)\cdot a^4|\nabla\Delta^{\frac{L}{2}}N|^2_G\right.\\
&\left.-6C\frac{\dot{a}}{a}\cdot a^4\langle\nabla\Delta^{\frac{L}{2}}N,\nabla\Delta^{\frac{L}{2}}\Psi\rangle_G-\frac{3}{8\pi}\dot{a}a^7\langle\mathfrak{N}_{L+1,Border}+\mathfrak{N}_{L+1,Junk},\nabla\Delta^{\frac{L}{2}}N\rangle_G\right\}\mathrm{dvol}_G
\end{aligned}$$

For $L=0$, the argument is again the same as at higher orders with less complicated junk terms. We briefly highlight some specific junk terms: The term analogous to (2.6.4c) is now estimated as follows using (2.4.13a):

$$\begin{aligned}
a^4\cdot\int_M -2a\langle\nabla\Psi\nabla N,\nabla^2\phi\rangle_G\lesssim&\,\varepsilon\int_M a^{\frac{1}{2}}|\nabla N|_G\cdot a^{\frac{1}{2}-c\sqrt{\varepsilon}}\cdot a^4|\nabla\phi|_G\\
\lesssim&\,\varepsilon\dot{a}a^3\mathfrak{E}^{(1)}(N,\cdot)+\varepsilon a^{1-c\sqrt{\varepsilon}}\cdot a^4\mathfrak{E}^{(\leq 1)}(\phi,\cdot)
\end{aligned}$$

Further, note that, by the commutator formula (2.A.8c) and applying (2.4.4e), one has

$$\begin{aligned}
\left|\int_M a^8[\partial_t,\Delta]\phi\cdot\Delta\phi\,\mathrm{dvol}_G\right|\lesssim&\,\varepsilon a^{5-c\sqrt{\varepsilon}}\left(\|\nabla\Sigma\|_{L^2_G}+\|\nabla N\|_{L^2_G}\right)\|\Delta\phi\|_{L^2_G}\\
\lesssim&\,\varepsilon a^{-1-c\sqrt{\varepsilon}}\left(a^4\mathfrak{E}^{(1)}(\phi,\cdot)+a^4\mathfrak{E}^{(1)}(\Sigma,\cdot)\right)+\varepsilon a^{6-c\sigma}\cdot\dot{a}a^3\mathfrak{E}^{(1)}(N,\cdot)\,.
\end{aligned}$$

□

2.6.1.2. *Sobolev norm estimate for $\nabla\phi$.* To improve the bootstrap assumptions on $\nabla\phi$, we will need sharper bounds than those on $a^4\|\nabla\phi\|^2_{H^L}$ that will follow from bounds on $\mathfrak{E}^{(L)}(\phi,\cdot)$:

LEMMA 2.6.5. *Let $l\in(t_{Boot},t_0]$. Then, for $l\in\mathbb{Z}_+$, $l\leq 17$, the following estimate holds:*

$$\|\nabla\phi\|_{H^l_G(\Sigma_t)}\lesssim(1+\varepsilon a(t)^{-c\sqrt{\varepsilon}})\|\Sigma\|_{H^{l+1}_G(\Sigma_t)}+\varepsilon a(t)^{-c\sqrt{\varepsilon}}\|\Psi\|_{H^l_G(\Sigma_t)}$$

PROOF. By (2.4.2a), $\Psi+C>\frac{C}{2}$ holds if ε is chosen small enough. Consequently, we can rearrange (2.2.29b) and apply the product rule to obtain

$$|\nabla^l\nabla\phi|_G=\frac{1}{8\pi}\left|\nabla^l\left(\frac{\mathrm{div}_G\Sigma}{\Psi+C}\right)\right|_G\lesssim\sum_{I_\Sigma+I_\Psi=l}|\nabla^{I_\Sigma+1}\Sigma|_G|\nabla^{I_\Psi}(\Psi+C)|_G\,.$$

The statement then follows by integrating over Σ_t and applying (2.4.2a) and (2.4.2b). □

2.6.2. Energy estimates for the Bel-Robinson variables. In this subsection, we collect the energy estimates for the Bel-Robinson variables:

LEMMA 2.6.6 (Bel-Robinson energy estimates). *Let $t\in(t_{Boot},t_0]$. Then one has*

(2.6.10)

$$\begin{aligned}
&\mathfrak{E}^{(0)}(W,t)+\int_t^{t_0}\int_M\left[8\pi C^2a(s)^{-3}(N+1)\langle\Sigma,\mathcal{E}\rangle_G+6\frac{\dot{a}(s)}{a(s)}(N+1)|\mathcal{E}|^2_G\right]\mathrm{dvol}_G\,ds\\
&\lesssim\mathfrak{E}^{(0)}(W,t_0)+\int_t^{t_0}\left(\varepsilon a(s)^{-3}+a(s)^{-1-c\sqrt{\varepsilon}}\right)\mathfrak{E}^{(0)}(W,s)+\varepsilon^{-\frac{1}{8}}a(s)^{-3}\cdot a(s)^4\mathfrak{E}^{(1)}(\phi,s)\,ds\\
&\qquad+\int_t^{t_0}a(s)^{-1-c\sqrt{\varepsilon}}\mathfrak{E}^{(0)}(\phi,s)+\varepsilon a(s)^{-3}\mathfrak{E}^{(0)}(\Sigma,s)\,ds
\end{aligned}$$

as well as, for $L \in 2\mathbb{N}$, $2 \leq L \leq 18$,

(2.6.11)

$$\begin{aligned}
&\mathfrak{E}^{(L)}(W,t) + \int_t^{t_0}\int_M \left[8\pi C^2 a(s)^{-3}(N+1)\langle \Delta^{\frac{L}{2}}\Sigma, \Delta^{\frac{L}{2}}\mathcal{E}\rangle_G + 6(N+1)\frac{\dot a(s)}{a(s)}|\Delta^{\frac{L}{2}}\mathcal{E}|_G^2\right]\mathrm{dvol}_G\, ds\\
&\lesssim \mathfrak{E}^{(L)}(W,t_0) + \int_t^{t_0}\left(\varepsilon^{\frac18}a(s)^{-3} + a(s)^{-1-c\sqrt{\varepsilon}}\right)\mathfrak{E}^{(L)}(W,s)\,ds\\
&+\int_t^{t_0}\Big\{\varepsilon^{-\frac18}a(s)^{-3}\cdot a(s)^4\mathfrak{E}^{(L+1)}(\phi,s) + \left(\varepsilon^{\frac18}a(s)^{-3}+a(s)^{-1}\right)\mathfrak{E}^{(L)}(\phi,s)\\
&\qquad + \varepsilon a(s)^{-3}\mathfrak{E}^{(L)}(\Sigma,s) + \varepsilon^{\frac78}a(s)^{-3}\cdot a(s)^4\mathfrak{E}^{(L-1)}(\mathrm{Ric},s) + \varepsilon^{\frac{31}{8}}a(s)^{-3}\mathfrak{E}^{(\leq L-2)}(\mathrm{Ric},s)\\
&\qquad + \left(\varepsilon^{\frac{15}{8}}a(s)^{-3-c\sqrt{\varepsilon}} + a(s)^{-1-c\sqrt{\varepsilon}}\right)\mathfrak{E}^{(\leq L-2)}(\phi,s)\\
&\qquad + \varepsilon^{\frac{15}{8}}a(s)^{-3-c\sqrt{\varepsilon}}\left(\mathfrak{E}^{(\leq L-2)}(\Sigma,s) + \mathfrak{E}^{(\leq L-2)}(W,s)\right) + \underbrace{\varepsilon^{\frac{15}{8}}a(s)^{-3-c\sqrt{\varepsilon}}\mathfrak{E}^{(\leq L-4)}(\mathrm{Ric},s)}_{\textit{not present for } L=2}\Big\}ds
\end{aligned}$$

Remark 2.6.7. We preemptively note that the error terms on the left hand side, once combined with the similar terms on the left hand side in Lemma 2.6.8 and given suitable weights, will turn out to have positive sign, even if they do not have definite sign in isolation.

The main idea in deriving this inequality is that we can use the algebraic identity (2.A.3d) and integration by parts to exploit the Maxwell system that lies at the core of the Bel-Robinson evolution equations. As a result, we avoid having higher order energies of the Bel-Robinson variables on the right hand side of the integral energy inequalities (which would break the bootstrap argument), then only having to deal with scalar field and Ricci energies at the next derivative level.

Proof. We first prove (2.6.11), and then explain how the same ideas lead to the simpler estimate (2.6.10). To this end, we start out by taking the time derivative of the energy as usual:

$$\begin{aligned}
-\partial_t\mathfrak{E}^{(L)}(W,\cdot) = \int_M &-3N\frac{\dot a}{a}\left[|\Delta^{\frac{L}{2}}\mathcal{E}|_G^2 + |\Delta^{\frac{L}{2}}\mathcal{B}|_G^2\right] - 2\left(\langle\partial_t\Delta^{\frac{L}{2}}\mathcal{E},\Delta^{\frac{L}{2}}\mathcal{E}\rangle_G + \langle\partial_t\Delta^{\frac{L}{2}}\mathcal{B},\Delta^{\frac{L}{2}}\mathcal{B}\rangle_G\right)\\
&- 2(\partial_t G^{-1})^{i_1j_1}(G^{-1})^{i_2j_2}\left[\Delta^{\frac{L}{2}}\mathcal{E}_{i_1i_2}\Delta^{\frac{L}{2}}\mathcal{E}_{j_1j_2} + \Delta^{\frac{L}{2}}\mathcal{B}_{i_1i_2}\Delta^{\frac{L}{2}}\mathcal{B}_{j_1j_2}\right]\mathrm{dvol}_G
\end{aligned}$$

$\mathcal{E}$ and $\mathcal{B}$ are symmetric and tracefree, thus symmetrizations become redundant, and any scalar product with a tensor that is pure trace or with an antisymmetric tensor can be dropped.[9] With this in hand, we get, inserting (2.2.38a) and (2.2.38b):

(2.6.12a)

$$-\partial_t\mathfrak{E}^{(L)}(W,\cdot) = \int_M\Big\{\frac{\dot a}{a}(-6(N+1)+9N)\left(|\Delta^{\frac{L}{2}}\mathcal{E}|_G^2 + |\Delta^{\frac{L}{2}}\mathcal{B}|_G^2\right)$$

(2.6.12b)

$$+ 2(N+1)a^{-1}\left(\langle\mathrm{curl}_G\Delta^{\frac{L}{2}}\mathcal{E},\Delta^{\frac{L}{2}}\mathcal{B}\rangle_G - \langle\mathrm{curl}_G\Delta^{\frac{L}{2}}\mathcal{B},\Delta^{\frac{L}{2}}\mathcal{E}\rangle_G\right)$$

[9]Recall the superscript "$\|$" notation for error terms, see Remark 2.2.12.

$$+2a^{-1}\left(\langle\nabla\Delta^{\frac{L}{2}}N\wedge_G\mathcal{B},\Delta^{\frac{L}{2}}\mathcal{E}\rangle_G-\langle\nabla\Delta^{\frac{L}{2}}N\wedge_G\mathcal{E},\Delta^{\frac{L}{2}}\mathcal{B}\rangle_G\right)\tag{2.6.12c}$$

$$-8\pi C^2a^{-3}(N+1)\langle\Delta^{\frac{L}{2}}\Sigma,\Delta^{\frac{L}{2}}\mathcal{E}\rangle_G-8\pi a(\Psi+C)\langle\nabla\Delta^{\frac{L}{2}}N\nabla\phi,\Delta^{\frac{L}{2}}\mathcal{E}\rangle_G\tag{2.6.12d}$$

$$-8\pi a(\Psi+C)(N+1)\langle\nabla^2\Delta^{\frac{L}{2}}\phi,\Delta^{\frac{L}{2}}\mathcal{E}\rangle_G\tag{2.6.12e}$$

$$+16\pi a(N+1)\langle\nabla\phi\nabla\Delta^{\frac{L}{2}}\Psi,\Delta^{\frac{L}{2}}\mathcal{E}\rangle_G\tag{2.6.12f}$$

$$+a^3\varepsilon[G]*\nabla\phi*\nabla^2\Delta^{\frac{L}{2}}\phi*\Delta^{\frac{L}{2}}\mathcal{B}\tag{2.6.12g}$$

$$+(N+1)a^{-3}\Sigma*\left(\Delta^{\frac{L}{2}}\mathcal{E}*\Delta^{\frac{L}{2}}\mathcal{E}+\Delta^{\frac{L}{2}}\mathcal{B}*\Delta^{\frac{L}{2}}\mathcal{B}\right)\tag{2.6.12h}$$

$$-2\langle\mathfrak{E}_{L,Border}+\mathfrak{E}_{L,top}+\mathfrak{E}^{\parallel}_{L,Junk},\Delta^{\frac{L}{2}}\mathcal{E}\rangle_G\tag{2.6.12i}$$

$$-2\langle\mathfrak{B}_{L,Border}+\mathfrak{B}_{L,top}+\mathfrak{B}^{\parallel}_{L,Junk},\Delta^{\frac{L}{2}}\mathcal{B}\rangle_G\Bigg\}\,\mathrm{dvol}_G\tag{2.6.12j}$$

For (2.6.12a), we pull $6(N+1)\dot a a^{-1}|\Delta^{\frac{L}{2}}\mathcal{E}|^2_G$ to the left. This leaves

$$\int_M -6\frac{\dot a}{a}|\Delta^{\frac{L}{2}}\mathcal{B}|^2_G+3N\frac{\dot a}{a}|\Delta^{\frac{L}{2}}\mathcal{B}|^2_G+9N\frac{\dot a}{a}|\Delta^{\frac{L}{2}}\mathcal{E}|^2_G\,\mathrm{dvol}_G,$$

where we can estimate the last two terms up to constant by $\varepsilon a^{1-c\sigma}\mathfrak{E}^{(L)}(W,\cdot)$ by (2.3.17h) and can drop the first term since it is nonpositive.

Regarding (2.6.12b), note that we have

$$a^{-1}\left(\langle\mathrm{curl}_G\Delta^{\frac{L}{2}}\mathcal{E},\Delta^{\frac{L}{2}}\mathcal{B}\rangle_G-\ \langle\mathrm{curl}_G\Delta^{\frac{L}{2}}\mathcal{B},\Delta^{\frac{L}{2}}\mathcal{E}\rangle_G\right)=-a^{-1}\mathrm{div}_G\left(\Delta^{\frac{L}{2}}\mathcal{E}\wedge_G\Delta^{\frac{L}{2}}\mathcal{B}\right).$$

Hence, the absolute value of (2.6.12b), using (2.A.4c) for the wedge product and (2.3.17h), can be bounded by:

$$\begin{aligned}\left|\int_M 2a^{-1}(N+1)\mathrm{div}_G\left(\Delta^{\frac{L}{2}}\mathcal{E}\wedge_G\Delta^{\frac{L}{2}}\mathcal{B}\right)\,\mathrm{dvol}_G\right|&=\left|\int_M 2a^{-1}\langle\nabla N,\Delta^{\frac{L}{2}}\mathcal{E}\wedge_G\Delta^{\frac{L}{2}}\mathcal{B}\rangle_G\,\mathrm{dvol}_G\right|\\&\lesssim\int_M a^{-1}|\nabla N|_G|\Delta^{\frac{L}{2}}\mathcal{E}|_G|\Delta^{\frac{L}{2}}\mathcal{B}|_G\,\mathrm{dvol}_G\\&\lesssim\varepsilon a^{3-c\sigma}\mathfrak{E}^{(L)}(W,\cdot)\end{aligned}$$

For (2.6.12c), we use the pointwise wedge product estimate (2.A.4d) and a priori estimates (2.4.2c) and (2.4.4g) to bound it as follows:

$$\begin{aligned}|(2.6.12c)|&\le 2a^{-1}|\nabla\Delta^{\frac{L}{2}}N|_G\left(|\mathcal{B}|_G\cdot|\Delta^{\frac{L}{2}}\mathcal{E}|_G+|\mathcal{E}|_G\cdot|\Delta^{\frac{L}{2}}\mathcal{B}|_G\right)\\&\lesssim\varepsilon a^{-3}\sqrt{a^4\mathfrak{E}^{(L+1)}(N,\cdot)}\sqrt{\mathfrak{E}^{(L)}(W,\cdot)}\\&\lesssim\varepsilon a^{-3}\left(\mathfrak{E}^{(L)}(W,\cdot)+a^4\mathfrak{E}^{(L+1)}(N,\cdot)\right)\end{aligned}$$

We pull the first term of (2.6.12d) to the left as well, and estimate the second using the strong C_G-norm estimates (2.4.2a) and (2.4.4e) by

$$\sqrt{\varepsilon}a^{-1-c\sqrt{\varepsilon}}\sqrt{a^4\mathfrak{E}^{(L+1)}(N,\cdot)}\sqrt{\mathfrak{E}^{(L)}(W,\cdot)} \lesssim a^{-1-c\sqrt{\varepsilon}}\mathfrak{E}^{(L)}(W,\cdot) + \varepsilon a^{-1}\cdot a^4\mathfrak{E}^{(L+1)}(N,\cdot)\,.$$

Moving on to (2.6.12e)-(2.6.12g), we see using (2.4.2a), (2.4.4e), (2.4.13a) with $\zeta = \Delta^{\frac{L}{2}}\phi$ and (2.3.8):

$$\begin{aligned}
|(2.6.12\text{e})| &\lesssim \left(a^{-3}\sqrt{a^4\mathfrak{E}^{(L+1)}(\phi,\cdot)} + a^{-1}\sqrt{\mathfrak{E}^{(L)}(\phi,\cdot)} + a^{-1-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(L-2)}(\phi,\cdot)}\right)\sqrt{\mathfrak{E}^{(L)}(W,\cdot)}\\
&\lesssim \left(\varepsilon^{\frac{1}{8}}a^{-3} + a^{-1-c\sqrt{\varepsilon}}\right)\mathfrak{E}^{(L)}(W,\cdot) + \varepsilon^{-\frac{1}{8}}a^{-3}\cdot a^4\mathfrak{E}^{(L+1)}(\phi,\cdot) + a^{-1}\mathfrak{E}^{(\leq L)}(\phi,\cdot)
\end{aligned}$$

$$\begin{aligned}
|(2.6.12\text{f})| &\lesssim \sqrt{\varepsilon}a^{1-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(L+1)}(\phi,\cdot)}\sqrt{\mathfrak{E}^{(L)}(W,\cdot)}\\
&\lesssim a^{1-c\sqrt{\varepsilon}}\mathfrak{E}^{(L)}(W,\cdot) + \varepsilon a^{1-c\sqrt{\varepsilon}}\mathfrak{E}^{(L+1)}(\phi,\cdot)\,,
\end{aligned}$$

$$\begin{aligned}
|(2.6.12\text{g})| &\lesssim \sqrt{\varepsilon}a^{1-c\sqrt{\varepsilon}}\cdot a^2\|\nabla^2\Delta^{\frac{L}{2}}\phi\|_{L^2_G}\cdot\sqrt{\mathfrak{E}^{(L)}(W,\cdot)}\\
&\lesssim \sqrt{\varepsilon}a^{1-c\sqrt{\varepsilon}}\left(\sqrt{\mathfrak{E}^{(L+1)}(\phi,\cdot)} + a^{-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(L-1)}(\phi,\cdot)}\right)\cdot\sqrt{\mathfrak{E}^{(L)}(W,\cdot)}\\
&\lesssim a^{1-c\sqrt{\varepsilon}}\mathfrak{E}^{(L)}(W,\cdot) + \varepsilon a^{1-c\sqrt{\varepsilon}}\left[\mathfrak{E}^{(L+1)}(\phi,\cdot) + \mathfrak{E}^{(L)}(\phi,\cdot) + \mathfrak{E}^{(\leq L-2)}(\phi,\cdot)\right]
\end{aligned}$$

We can estimate (2.6.12h) by $\varepsilon a^{-3}\mathfrak{E}^{(L)}(W,\cdot)$ as usual, and obtain the following in summary:

$$\begin{aligned}
&-\partial_t\mathfrak{E}^{(L)}(W,\cdot) + 8\pi C^2a^{-3}\int_M(N+1)\langle\Delta^{\frac{L}{2}}\Sigma,\Delta^{\frac{L}{2}}\mathcal{E}\rangle_G\,\mathrm{dvol}_G + 6\frac{\dot a}{a}\int_M(N+1)|\Delta^{\frac{L}{2}}\mathcal{E}|_G^2\,\mathrm{dvol}_G\\
\lesssim&\left(\varepsilon a^{-3} + a^{-1-c\sqrt{\varepsilon}}\right)\mathfrak{E}^{(L)}(W,\cdot) + a^{-1}\mathfrak{E}^{(L+1)}(\phi,\cdot) + a^{-1}\mathfrak{E}^{(L)}(\phi,\cdot)\\
&+\varepsilon a^{-3}\cdot a^4\mathfrak{E}^{(L+1)}(N,\cdot) + a^{-1}\mathfrak{E}^{(\leq L-2)}(\phi,\cdot)\\
&\Big[\|\mathfrak{E}_{L,Border}\|_{L^2_G} + \|\mathfrak{E}_{L,top}\|_{L^2_G} + \|\mathfrak{E}^{\parallel}_{L,Junk}\|_{L^2_G}\\
&+\|\mathfrak{B}_{L,Border}\|_{L^2_G} + \|\mathfrak{B}_{L,top}\|_{L^2_G} + \|\mathfrak{B}^{\parallel}_{L,Junk}\|_{L^2_G}\Big]\sqrt{\mathfrak{E}^{(L)}(W,\cdot)}
\end{aligned}$$

We can now apply Corollary 2.5.5 for $2l = L$ to estimate the lapse energy in the second line (leading to borderline scalar field energy and Σ-energy contributions as well as junk terms), and insert the borderline (see (2.A.17k)), top (see (2.A.18a) and (2.A.18b)) and junk estimates (see (2.A.19n)), dealing with the lapse energies there analogously. In particular, the top order curvature terms arise as follows:

$$\begin{aligned}
\|\mathfrak{E}_{L,top}\|_{L^2_G}\sqrt{\mathfrak{E}^{(L)}(W,\cdot)} &\lesssim \sqrt{\varepsilon}a^{-1-c\sqrt{\varepsilon}}\sqrt{a^4\mathfrak{E}^{(L-1)}(\mathrm{Ric},\cdot)}\sqrt{\mathfrak{E}^{(L)}(W,\cdot)}\\
&\lesssim \varepsilon^{\frac{1}{8}}a^{-1-c\sqrt{\varepsilon}}\mathfrak{E}^{(L)}(W,\cdot) + \varepsilon^{\frac{7}{8}}a^{-1}\cdot a^4\mathfrak{E}^{(L-1)}(\mathrm{Ric},\cdot)\\
\|\mathfrak{B}_{L,top}\|\sqrt{\mathfrak{E}^{(L)}(W,\cdot)} &\lesssim \varepsilon a^{-3}\sqrt{a^4\mathfrak{E}^{(L-1)}(\mathrm{Ric},\cdot)}\sqrt{\mathfrak{E}^{(L)}(W,\cdot)}\\
&\lesssim \varepsilon^{\frac{1}{8}}a^{-3}\mathfrak{E}^{(L)}(W,\cdot) + \varepsilon^{\frac{15}{8}}a^{-3}\cdot a^4\mathfrak{E}^{(L-1)}(\mathrm{Ric},\cdot)
\end{aligned}$$

Hence, both top order curvature terms can be bounded by $\varepsilon^{\frac{7}{8}}a^{-3}\cdot a^4\mathfrak{E}^{(L-1)}(\mathrm{Ric},\cdot)$. Integrating the inequality yields (2.6.11).

For (2.6.10), we get applying (2.2.31a) and (2.2.31b) and again using that $\mathcal{E}$ and $\mathcal{B}$ are symmetric and tracefree:

$$\begin{aligned}
-\partial_t\mathfrak{E}^{(0)}(W,\cdot) = \int_M \Bigg\{ & \frac{\dot a}{a}(-6(N+1)+9N)\left(|\mathcal{E}|_G^2+|\mathcal{B}|_G^2\right)\\
&+2(N+1)\left(\langle\mathrm{curl}\mathcal{E},\mathcal{B}\rangle_G-\langle\mathrm{curl}\mathcal{B},\mathcal{E}\rangle_G\right)\\
&+2\left(\langle\nabla N\wedge\mathcal{B},\mathcal{E}\rangle_G-\langle\nabla N\wedge\mathcal{E},\mathcal{B}\rangle_G\right)\\
&+(N+1)a^{-3}\Sigma*(\mathcal{E}*\mathcal{E}+\mathcal{B}*\mathcal{B})\\
&-8\pi a^{-3}(N+1)(\Psi+C)^2\langle\Sigma,\mathcal{E}\rangle_G\\
&+\left[\dot a a^3\nabla\phi*\nabla\phi+a(\Psi+C)\cdot\nabla N*\nabla\phi\right]*\mathcal{E}\\
&+a(N+1)\left[\nabla\phi*\nabla\Psi+\Sigma*\nabla\phi*\nabla\phi+(\Psi+C)\nabla^2\phi\right]*\mathcal{E}\\
&+(N+1)\varepsilon[G]*\left(a^3\nabla^2\phi*\nabla\phi+a^{-1}(\Psi+C)\Sigma*\nabla\phi\right)*\mathcal{B}\Bigg\}\,\mathrm{dvol}_G
\end{aligned}$$

The first two lines are treated as in the general case. For the third line, we get $\varepsilon a^{3-c\sigma}\mathfrak{E}^{(0)}(W,\cdot)$ with (2.A.4d) and (2.3.17h), while the fourth term is bounded by $\varepsilon a^{-3}\mathfrak{E}^{(0)}(W,\cdot)$ with (2.4.2b). This leaves the surviving matter terms in the final four lines.
We pull $\int_M 8\pi a^{-3}(N+1)C^2\langle\Sigma,\mathcal{E}\rangle_G\mathrm{dvol}_G$ to the left as before. For the remaining terms, we can apply a priori estimates (2.4.2a), (2.4.4a) and (2.4.4e), the bootstrap assumption (2.3.17h) and Lemma 2.4.1 for N, which yields the following bound up to constant remaining terms in the last three lines:

$$\begin{aligned}
\sqrt{\mathfrak{E}^{(0)}(W,\cdot)}\cdot\Bigg[&a^{-1}\cdot a^2\|\nabla^2\phi\|_{L^2_G}+\sqrt{\varepsilon}a^{-1-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(0)}(\phi,\cdot)}\\
&+\left(\varepsilon a^{-3}+\sqrt{\varepsilon}a^{-1-c\sqrt{\varepsilon}}\right)\sqrt{\mathfrak{E}^{(0)}(\Sigma,\cdot)}\Bigg]
\end{aligned}$$

Applying (2.4.13a) to the scalar field norm and then (2.3.8), this leads to (2.6.10) along with the previous observations. □

2.6.3. Energy estimates for the second fundamental form. For the energy estimates for Σ, we again first derive even order integral estimates:

LEMMA 2.6.8 (Energy estimates for the second fundamental form for even orders). *Let $t\in(t_{Boot},t_0]$. Then, one has:*

(2.6.13) $$\mathfrak{E}^{(0)}(\Sigma,t)+2\int_t^{t_0}\int_M\left[a(s)^{-3}(N+1)\langle\mathcal{E},\Sigma\rangle_G+\frac{\dot a(s)}{a(s)}(N+1)|\Delta^{\frac{L}{2}}\Sigma|_G^2\right]\mathrm{dvol}_G\,ds$$

$$\lesssim \mathfrak{E}^{(0)}(\Sigma,t_0)+\int_t^{t_0}\varepsilon^{\frac18}a(s)^{-3}\mathfrak{E}^{(0)}(\Sigma,s)\,ds+\int_t^{t_0}\varepsilon^{-\frac18}a(s)^{-3}\mathfrak{E}^{(0)}(\phi,s)\,ds$$

For $L\in 2\mathbb{N}, L\leq 18$, the following holds:

(2.6.14)

$$\begin{aligned}
&\mathfrak{E}^{(L)}(\Sigma,t)+2\int_t^{t_0}\int_M\left[a(s)^{-3}(N+1)\langle\Delta^{\frac L2}\mathcal{E},\Delta^{\frac L2}\Sigma\rangle_G+\frac{\dot a(s)}{a(s)}(N+1)|\Delta^{\frac L2}\Sigma|_G^2\right]\mathrm{dvol}_G\,ds\\
&\quad\lesssim \mathfrak{E}^{(L)}(\Sigma,t_0)+\int_t^{t_0}\varepsilon^{\frac18}a(s)^{-3}\mathfrak{E}^{(L)}(\Sigma,s)\,ds\\
&\quad+\int_t^{t_0}\Big\{\varepsilon^{-\frac18}a(s)^{-3}\mathfrak{E}^{(L)}(\phi,s)+\varepsilon^{\frac{15}{8}}a(s)^{5-c\sigma}\mathfrak{E}^{(L-1)}(\mathrm{Ric},s)+\varepsilon^2a(s)^{-3}\mathfrak{E}^{(L-2)}(\mathrm{Ric},s)\\
&\qquad+\varepsilon^{\frac{15}{8}}a(s)^{-3-c\sqrt\varepsilon}\mathfrak{E}^{(\leq L-2)}(\Sigma,s)+\left(\varepsilon^{\frac{15}{8}}a(s)^{-3-c\sqrt\varepsilon}+\varepsilon a(s)^{-1-c\sqrt\varepsilon}\right)\mathfrak{E}^{(\leq L-2)}(\phi,s)\\
&\qquad+\underbrace{\varepsilon^{\frac{15}{8}}a(s)^{-3-c\sqrt\varepsilon}\mathfrak{E}^{(\leq L-4)}(\mathrm{Ric},s)}_{\textit{not present for } L=2}\Big\}\,ds
\end{aligned}$$

REMARK 2.6.9. The main hurdle of dealing with the second fundamental form is that a high order curvature term occurs in the evolution equation. It is to precisely this end that the Bel-Robinson variables needed to be introduced, since (2.2.36d) is what facilitates controlling said term without having to use $\mathfrak{E}^{(L)}(\mathrm{Ric},\cdot)$ or similar high order metric energies. Again, the resulting leading terms will turn out to have definite sign when combined with the Bel-Robinson energy estimates above.

PROOF. Here, we omit the proof for the inequality at order zero since is completely analogous in structure to the one for orders 2 and higher and the only differences that arise are that lower order error terms do not occur.
Once again, we start out by differentiating $-\mathfrak{E}^{(L)}(\Sigma,\cdot)$ and insert (2.2.35):

$$\begin{aligned}
-\partial_t\mathfrak{E}^{(L)}(\Sigma,\cdot)&=\int_M -2\left\langle\partial_t\Delta^{\frac L2}\Sigma,\Delta^{\frac L2}\Sigma\right\rangle_G+(\partial_tG^{-1})*\Delta^{\frac L2}\Sigma*\Delta^{\frac L2}\Sigma-3N\frac{\dot a}{a}|\Delta^{\frac L2}\Sigma|_G^2\,\mathrm{dvol}_G\\
&=\int_M\Big\{2a\langle\nabla^2\Delta^{\frac L2}N,\Delta^{\frac L2}\Sigma\rangle_G-2a(N+1)\langle\Delta^{\frac L2}\mathrm{Ric}[G],\Delta^{\frac L2}\Sigma\rangle_G\\
&\qquad+(\partial_tG^{-1})*\Delta^{\frac L2}\Sigma*\Delta^{\frac L2}\Sigma-3N\frac{\dot a}{a}|\Delta^{\frac L2}\Sigma|_G^2\\
&\qquad-2\langle\mathfrak{S}_{L,Border},\Delta^{\frac L2}\Sigma\rangle_G-2\langle\mathfrak{S}^{\parallel}_{L,Junk},\Delta^{\frac L2}\Sigma\rangle_G\Big\}\,\mathrm{dvol}_G
\end{aligned}$$

For the first term, one can use (2.4.13a), Corollary 2.5.5 at order L and (2.3.8) to bound its absolute value by the following:

$$\begin{aligned}
&\lesssim a\|\Delta^{\frac L2}N\|_{\dot H^2_G}\sqrt{\mathfrak{E}^{(L)}(\Sigma,\cdot)}\\
&\lesssim\left[a^{-3}\sqrt{a^8\mathfrak{E}^{(L+2)}(N,\cdot)}+a^{1-c\sqrt\varepsilon}\sqrt{\mathfrak{E}^{(L)}(N,\cdot)}\right]\sqrt{\mathfrak{E}^{(L)}(\Sigma,\cdot)}\\
&\lesssim\Bigg[\varepsilon a^{-3}\sqrt{\mathfrak{E}^{(L)}(\Sigma,\cdot)}+a^{-3}\mathfrak{E}^{(L)}(\phi,\cdot)+\varepsilon a^{-3-c\sqrt\varepsilon}\left(\sqrt{\mathfrak{E}^{(\leq L-2)}(\Sigma,\cdot)}+\sqrt{\mathfrak{E}^{(\leq L-2)}(\phi,\cdot)}\right)
\end{aligned}$$

$$+\underbrace{\left(\varepsilon^2 a^{-3-c\sqrt{\varepsilon}}+\varepsilon a^{1-c\sigma}\right)\sqrt{\mathfrak{E}^{(\leq L-3)}(\mathrm{Ric},\cdot)}}_{\text{not present for }L=2}\Bigg]\sqrt{\mathfrak{E}^{(L)}(\Sigma,\cdot)}$$

$$\lesssim\left(\varepsilon^{\frac18}a^{-3}+a^{1-c\sigma}\right)\mathfrak{E}^{(L)}(\Sigma,\cdot)+\varepsilon^{-\frac18}a^{-3}\mathfrak{E}^{(L)}(\phi,\cdot)+\varepsilon a^{-3-c\sqrt{\varepsilon}}\left[\mathfrak{E}^{(\leq L-2)}(\Sigma,\cdot)+\mathfrak{E}^{(\leq L-2)}(\phi,\cdot)\right]$$
$$+\underbrace{\left(\varepsilon^{\frac{31}{8}}a^{-3}+\varepsilon^2 a^{1-c\sigma}\right)\mathfrak{E}^{(L-2)}(\mathrm{Ric},\cdot)+\left(\varepsilon^{\frac{31}{8}}a^{-3-c\sqrt{\varepsilon}}+\varepsilon^2a^{1-c\sigma}\right)\mathfrak{E}^{(\leq L-4)}(\mathrm{Ric},\cdot)}_{\text{not present for }L=2}$$

Next, we replace the high order curvature term as follows, using the commuted rescaled Hamiltonian constraint equation (2.2.36d) that $\Delta^{\frac{L}{2}}\Sigma$ is tracefree and symmetric:

$$\int_M -2a(N+1)\langle\Delta^{\frac{L}{2}}\mathrm{Ric}[G],\Delta^{\frac{L}{2}}\Sigma\rangle_G\,\mathrm{dvol}_G$$
$$=\int_M -2(N+1)a^{-3}\langle\Delta^{\frac{L}{2}}\mathcal{E},\Delta^{\frac{L}{2}}\Sigma\rangle_G-2(N+1)\frac{\dot a}{a}|\Delta^{\frac{L}{2}}\Sigma|_G^2$$
$$+\langle\mathfrak{H}_{L,Border}+\mathfrak{H}^{\|}_{L,Junk},\Delta^{\frac{L}{2}}\Sigma\rangle_G\,\mathrm{dvol}_G$$

We pull the first two terms to left, only keeping the error terms on the right. After inserting the borderline and junk term estimates for the Hamiltonian constraint equations ((2.A.17a) and (2.A.19c)) and the evolution equation itself ((2.A.17h) and (2.A.19k)), as well as bounding $|\partial_t G^{-1}|\lesssim\varepsilon a^{-3}$ and inserting (2.3.17h) as usual, we obtain (2.6.14) by integrating. □

Additionally, we can exploit the structure of the momentum constraint equations to gain an elliptic estimate for $\mathfrak{E}^{(L+1)}(\Sigma,\cdot)$. Crucially, the upper bound only depends on Σ-, scalar field and Bel-Robinson energies up to order L, and appropriately small and time-scaled curvature contributions up to order $L-1$. This will allow us to close the argument since we do not need to consider the Bel-Robinson energy at order $L+1$ to control Σ at that order, which would require higher order scalar field and curvature energies.

Lemma 2.6.10 (Odd order energy estimate for the second fundamental form). *For any $L\in 2\mathbb{Z}_+$, $2\leq L\leq 18$, we have*

(2.6.15)
$$a^4\mathfrak{E}^{(L+1)}(\Sigma,\cdot)\lesssim\left(a^{4-c\sqrt{\varepsilon}}+\varepsilon a^{2-c\sqrt{\varepsilon}}\right)\mathfrak{E}^{(L)}(\Sigma,\cdot)+\mathfrak{E}^{(L)}(\phi,\cdot)+\mathfrak{E}^{(L)}(W,\cdot)+\varepsilon^2a^4\mathfrak{E}^{(L-1)}(\mathrm{Ric},\cdot)$$
$$+\varepsilon a^{-c\sqrt{\varepsilon}}\mathfrak{E}^{(\leq L-2)}(\phi,\cdot)+a^{2-c\sqrt{\varepsilon}}\mathfrak{E}^{(\leq L-2)}(\Sigma,\cdot)+\varepsilon a^{2-c\sqrt{\varepsilon}}\mathfrak{E}^{(\leq L-2)}(\mathrm{Ric},\cdot)\,.$$

For $L=0$, one analogously has

$$a^4\mathfrak{E}^{(1)}(\Sigma,\cdot)\lesssim\left(a^{4-c\sqrt{\varepsilon}}+\varepsilon a^{2-c\sqrt{\varepsilon}}\right)\mathfrak{E}^{(0)}(\Sigma,\cdot)+\mathfrak{E}^{(0)}(\phi,\cdot)+\mathfrak{E}^{(0)}(W,\cdot)\,.\tag{2.6.16}$$

Proof. We prove the statement for $L\geq 2$, since the proof of (2.6.16) is entirely analogous.

By [AM04, (A.22)], since (Σ_t,g) is a three-dimensional compact Riemannian manifold for any $t\in(t_{Boot},t_0]$, any tracefree $(0,2)$ tensor U_{ij} on (Σ_t,g) satisfies

$$\int_{\Sigma_t}|\nabla U|_g^2+3\mathrm{Ric}[g]\cdot U\cdot U-\frac{R[g]}{2}|U|_g^2\,\mathrm{dvol}_g=\int_{\Sigma_t}|\mathrm{curl}U|_g^2+\frac32|\mathrm{div}_gU|_g^2\,\mathrm{dvol}_g\,.\tag{2.6.17}$$

In particular, for $U=\Delta^{\frac{L}{2}}\Sigma$ and after rescaling, this reads:

$$\begin{aligned}&\int_M \left|\nabla\Delta^{\frac{L}{2}}\Sigma\right|_G^2+3\left(\mathrm{Ric}[G]^\sharp\right)^i{}_j\left(\Delta^{\frac{L}{2}}\Sigma^\sharp\right)^j{}_l\left(\Delta^{\frac{L}{2}}\Sigma^\sharp\right)^l{}_i-\frac{R[G]}{2}|\Delta^{\frac{L}{2}}\Sigma|_G^2\,\mathrm{dvol}_G\\ &=\int_M \frac{3}{2}|\mathrm{div}_G\Delta^{\frac{L}{2}}\Sigma|_G^2+a^2|\mathrm{curl}\Delta^{\frac{L}{2}}\Sigma|_G^2\,\mathrm{dvol}_G\end{aligned}$$

The last two terms on the left hand side can be estimated by $(1+\sqrt{\varepsilon}a^{-c\sqrt{\varepsilon}})\mathfrak{E}^{(L)}(\Sigma,\cdot)$ in absolute value using the strong C_G-norm estimate (2.4.4f). Thus, inserting the Laplace-commuted rescaled momentum constraint equations (2.2.36a) and (2.2.36b), we obtain for a suitable constant $K>0$:

$$\begin{aligned}&\mathfrak{E}^{(L+1)}(\Sigma,\cdot)-K\left(1+\sqrt{\varepsilon}a^{-c\sqrt{\varepsilon}}\right)\mathfrak{E}^{(L)}(\Sigma,\cdot)\\ &\lesssim\int_M\Big\{|\Psi+C|^2\left|\nabla\Delta^{\frac{L}{2}}\phi\right|_G^2+|\nabla\phi|_G^2\left|\Delta^{\frac{L}{2}-1}\mathrm{Ric}[G]\right|_G^2+|\Sigma|_G^2\left|\nabla\Delta^{\frac{L}{2}-1}\mathrm{Ric}[G]\right|_G^2+|\mathfrak{M}_{L,Junk}|_G^2\\ &\quad+a^{-4}|\Delta^{\frac{L}{2}}\mathcal{B}|_G^2+|\Sigma|_G^2\left|\nabla\Delta^{\frac{L}{2}-1}\mathrm{Ric}[G]\right|_G^2+|\nabla\Sigma|_G^2\left|\nabla^2\Delta^{\frac{L}{2}-2}\mathrm{Ric}[G]\right|_G^2+\left|\tilde{\mathfrak{M}}_{L,Junk}\right|_G^2\Big\}\,\mathrm{dvol}_G\end{aligned}$$

After rearranging, using the strong C_G-norm estimates (2.4.2a), (2.4.4e), (2.4.2b) and (2.4.4b) and multiplying by a^4 on both sides, we get

$$\begin{aligned}a^4\mathfrak{E}^{(L+1)}(\Sigma,\cdot)\lesssim&\left(1+\sqrt{\varepsilon}a^{-c\sqrt{\varepsilon}}\right)a^4\mathfrak{E}^{(L)}(\Sigma,\cdot)+\mathfrak{E}^{(L)}(\phi,\cdot)+\mathfrak{E}^{(L)}(W,\cdot)+\varepsilon^2a^4\mathfrak{E}^{(L-1)}(\mathrm{Ric},\cdot)\\ &+\varepsilon^2a^{4-c\sqrt{\varepsilon}}\mathfrak{E}^{(L-2)}(\mathrm{Ric},\cdot)+a^4\|\mathfrak{M}_{L,Junk}\|_{L^2_G}^2+a^4\|\tilde{\mathfrak{M}}_{L,Junk}\|_{L^2_G}^2\,.\end{aligned}$$

The statement follows inserting the estimates (2.A.19a) and (2.A.19b). □

2.6.4. Energy estimates for the curvature. To control commutator errors, we will also need some additional estimates on curvature energies. Unlike the other energies, these inequalities do not rely on any delicate structure within the equations and instead just rely on pointwise estimates, the Young inequality and near-coercivity of energies in the sense of Lemma 2.4.5. For the sake of convenience, we phrase these estimates for $\mathfrak{E}^{(L-2)}(\mathrm{Ric},\cdot)$ since this is the order needed when improving behaviour of the total energy at order L.

LEMMA 2.6.11 (Curvature energy estimates at even orders). *Let $L\in 2\mathbb{Z}$, $4\le L\le 16$ and $t\in(t_{Boot},t_0]$. Then, one has*

$$\begin{aligned}\mathfrak{E}^{(L-2)}(\mathrm{Ric},t)\lesssim&\,\mathfrak{E}^{(L-2)}(\mathrm{Ric},t_0)+\int_t^{t_0}\left(\varepsilon^{\frac{1}{8}}a(s)^{-3}+a(s)^{8-c\sigma}\right)\mathfrak{E}^{(L-2)}(\mathrm{Ric},s)\,ds\\ &+\int_t^{t_0}\Big\{\varepsilon^{-\frac{1}{8}}a(s)^{-3}\left(\mathfrak{E}^{(L)}(\phi,s)+\mathfrak{E}^{(L)}(\Sigma,s)\right)\\ &\quad+\varepsilon^{-\frac{1}{8}}a(s)^{-3-c\sqrt{\varepsilon}}\left(\mathfrak{E}^{(\le L-2)}(\phi,s)+\mathfrak{E}^{(\le L-2)}(\Sigma,s)\right)\\ &\quad+\varepsilon^{\frac{7}{8}}a(s)^{-3-c\sqrt{\varepsilon}}\mathfrak{E}^{(\le L-4)}(\mathrm{Ric},s)\Big\}\,ds\,.\end{aligned}\tag{2.6.18}$$

Additionally,

$$\mathfrak{E}^{(0)}(\mathrm{Ric},t)\lesssim\mathfrak{E}^{(0)}(\mathrm{Ric},t_0)+\int_t^{t_0}\varepsilon^{\frac{1}{8}}a(s)^{-3}\mathfrak{E}^{(0)}(\mathrm{Ric},s)\,ds\tag{2.6.19}$$

$$+\int_t^{t_0}\varepsilon^{-\frac{1}{8}}a(s)^{-3}\left(\mathfrak{E}^{(0)}(\phi,s)+\mathfrak{E}^{(0)}(\Sigma,s)\right)ds\,.$$

PROOF. First, we note that

$$\|\mathrm{div}^\sharp_G\nabla\Delta^{\frac{L}{2}-1}\Sigma\|_{L^2_G}\lesssim\|\nabla^2\Delta^{\frac{L}{2}-1}\Sigma\|_{L^2_G}\lesssim\|\Delta^{\frac{L}{2}}\Sigma\|_{L^2_G}+a^{-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(L-2)}(\Sigma,\cdot)}$$

holds using the low order version of (2.4.14a) with $\mathfrak{T}=\Delta^{\frac{L}{2}-1}\Sigma$ for $l=2$, and similarly

$$\|\nabla^2\Delta^{\frac{L}{2}-1}N\|_{L^2_G}\lesssim\|\Delta^{\frac{L}{2}}N\|_{L^2_G}+a^{-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(L-2)}(N,\cdot)}$$

using (2.4.13b) at order 2. Now, using $\Delta^{\frac{L}{2}-1}G=0$ for $L\geq 4$, we continue as usual by applying (2.2.40a) to the expression below:

$$\begin{aligned}-\partial_t\mathfrak{E}^{(L-2)}(\mathrm{Ric},\cdot)\lesssim\int_M\Big\{&a^{-3}\left(|\Delta^{\frac{L}{2}}\Sigma|_G+|\nabla^2\Delta^{\frac{L}{2}-1}\Sigma|_G\right)|\Delta^{\frac{L}{2}-1}\mathrm{Ric}[G]|_G\\&+\frac{\dot a}{a}\left(|\nabla^2\Delta^{\frac{L}{2}-1}N|_G+|\Delta^{\frac{L}{2}}N|_G\right)|\Delta^{\frac{L}{2}-1}\mathrm{Ric}[G]|_G\\&+(|\mathfrak{R}_{L-2,Border}|_G+|\mathfrak{R}_{L-2,Junk}|_G)\cdot|\Delta^{\frac{L}{2}-1}\mathrm{Ric}[G]|_G\\&+a^{-3}\Sigma*\Delta^{\frac{L}{2}-1}\mathrm{Ric}[G]*\Delta^{\frac{L}{2}-1}\mathrm{Ric}[G]+N\frac{\dot a}{a}|\Delta^{\frac{L}{2}-1}\mathrm{Ric}[G]|^2_G\Big\}\,\mathrm{dvol}_G\end{aligned}$$

Due to the estimates above as well as (2.4.2b) and (2.3.17h), this implies

$$\begin{aligned}-\partial_t\mathfrak{E}^{(L-2)}(\mathrm{Ric},\cdot)\lesssim\,&a^{-3}\left[\sqrt{\mathfrak{E}^{(L)}(\Sigma,\cdot)}+\sqrt{\mathfrak{E}^{(L)}(N,\cdot)}\right]\sqrt{\mathfrak{E}^{(L-2)}(\mathrm{Ric},\cdot)}\\&+a^{-3-c\sqrt{\varepsilon}}\left[\sqrt{\mathfrak{E}^{(\leq L-2)}(\Sigma,\cdot)}+\sqrt{\mathfrak{E}^{(\leq L-2)}(N,\cdot)}\right]\sqrt{\mathfrak{E}^{(L-2)}(\mathrm{Ric},\cdot)}\\&+\left(\|\mathfrak{R}_{L-2,Border}\|_{L^2_G}+\|\mathfrak{R}_{L-2,Junk}\|_{L^2_G}\right)\sqrt{\mathfrak{E}^{(L-2)}(\mathrm{Ric},\cdot)}\\&+\varepsilon a^{-3}\mathfrak{E}^{(L-2)}(\mathrm{Ric},\cdot)\,.\end{aligned}$$

Using Corollary 2.5.5 at order L and distributing terms containing $\mathfrak{E}^{(L-3)}(\mathrm{Ric},\cdot)$ with (2.3.8) as usual, we get

$$\begin{aligned}-\partial_t\mathfrak{E}^{(L-2)}(\mathrm{Ric},\cdot)\lesssim\,&\left[\varepsilon^{\frac{1}{8}}a^{-3}+a^{8-c\sigma}\right]\mathfrak{E}^{(L-2)}(\mathrm{Ric},\cdot)+\varepsilon^{-\frac{1}{8}}a^{-3}\left[\mathfrak{E}^{(L)}(\Sigma,\cdot)+\mathfrak{E}^{(L)}(\phi,\cdot)\right]\\&+\left[\varepsilon^{\frac{31}{8}}a^{-3-c\sqrt{\varepsilon}}+\varepsilon^2a^{8-c\sigma}\right]\mathfrak{E}^{(\leq L-4)}(\mathrm{Ric},\cdot)\\&+\varepsilon^{-\frac{1}{8}}a^{-3-c\sqrt{\varepsilon}}\left[\mathfrak{E}^{(\leq L-2)}(\Sigma,\cdot)+\mathfrak{E}^{(\leq L-2)}(\phi,\cdot)\right]\\&+\left(\|\mathfrak{R}_{L-2,Border}\|_{L^2_G}+\|\mathfrak{R}_{L-2,Junk}\|_{L^2_G}\right)\sqrt{\mathfrak{E}^{(L-2)}(\mathrm{Ric},\cdot)}\,.\end{aligned}$$

Equation (2.6.18) now follows inserting the borderline and junk term estimates (2.A.17i) and (2.A.19l) and applying the lapse energy estimates from Corollary 2.5.5.

Equation (2.6.19) follows almost identically by inserting (2.2.33) instead of (2.2.40a) as well as (2.2.28a) for the additional $\partial_tG*(\mathrm{Ric}[G]+2/9G)$-terms. These can be estimated as

$$\lesssim\int_M a^{-3}\Sigma*\left(\mathrm{Ric}[G]+\frac{2}{9}G\right)+\frac{\dot a}{a}N\cdot G*\left(\mathrm{Ric}[G]+\frac{2}{9}G\right)\mathrm{dvol}_G$$

$$\lesssim a^{-3}\left(\sqrt{\mathfrak{E}^{(0)}(\Sigma,\cdot)}+\sqrt{\mathfrak{E}^{(0)}(N,\cdot)}\right)\sqrt{\mathfrak{E}^{(0)}(\mathrm{Ric},\cdot)}\,,$$

which can be treated as at higher orders. □

Lemma 2.6.12 (Odd order curvature energy estimate). *For $L\in 2\mathbb{N}$, $4\le L\le 18$ and $t\in(t_{Boot},t_0]$,*

(2.6.20)

$$\begin{aligned}
a(t)^4\mathfrak{E}^{(L-1)}(\mathrm{Ric},t)\lesssim\,& a(t_0)^4\mathfrak{E}^{(L-1)}(\mathrm{Ric},t_0)\\
&+\int_t^{t_0}\left(\varepsilon^{\frac18}a(s)^{-3}+a(s)^{-1-c\sqrt{\varepsilon}}\right)\left(a(s)^4\mathfrak{E}^{(L-1)}(\mathrm{Ric},s)\right)ds\\
&+\int_t^{t_0}\varepsilon^{-\frac18}a(s)^{-3}\cdot a(s)^4\mathfrak{E}^{(L+1)}(\Sigma,s)\,ds\\
&+\int_t^{t_0}\Big\{\varepsilon^{-\frac18}a(s)^{-3}\mathfrak{E}^{(L)}(\phi,s)+\left(\varepsilon^{\frac{15}{8}}a(s)^{-3}+a(s)^{-1-c\sqrt{\varepsilon}}\right)\mathfrak{E}^{(L)}(\Sigma,s)\\
&\qquad+\left(\varepsilon^{\frac{15}{8}}a(s)^{-3-c\sqrt{\varepsilon}}+a(s)^{-1-c\sqrt{\varepsilon}}\right)\left(\mathfrak{E}^{(\le L-2)}(\phi,s)+\mathfrak{E}^{(\le L-2)}(\Sigma,s)\right)\\
&\qquad+\varepsilon^{\frac{15}{8}}a(s)^{-3}\mathfrak{E}^{(L-2)}(\mathrm{Ric},\cdot)+\varepsilon^{\frac{15}{8}}a^{-3-c\sqrt{\varepsilon}}\mathfrak{E}^{(\le L-4)}(\mathrm{Ric},s)\Big\}ds
\end{aligned}$$

Proof. The proof is very similar to that of Lemma 2.6.11 since we did not exploit any structure within (2.2.40a) that does not equally occur in (2.2.40b), and thus we omit the details. As in the proof of Lemma 2.6.4, we note that the differences within the estimate come from how top order lapse terms are treated: The scaling of the top order energy allows one to estimate $a^4\mathfrak{E}^{(L+1)}(N,\cdot)$ by scalar field energies and Σ-energies of up to order L and curvature energies up to order $L-3$. □

2.6.5. Sobolev norm estimates for metric objects. To close the bootstrap argument, we need to improve the behaviour of metric quantities in addition to the energy formalism, both to capture the intrinsic behaviour of the metric and to relate energies to supremum norms.

Lemma 2.6.13 (Sobolev norm estimates for Christoffel symbols). *Let U be a coordinate neighbourhood on M, viewed as a coordinate neighbourhood on Σ_t for $t\in(t_{Boot},t_0]$. For any $l\in\mathbb{N}$, $l\le 17$, the following Sobolev estimate then holds:*

$$\|\Gamma-\hat\Gamma\|^2_{H^l_G(U)}\lesssim a^{-c\varepsilon^{\frac18}}\left(\varepsilon^4+\varepsilon^{-\frac14}\sup_{s\in(t,t_0)}\left(\|N\|^2_{H^{l+1}_G(\Sigma_s)}+\|\Sigma\|^2_{H^{l+1}_G(\Sigma_s)}\right)\right)\tag{2.6.21}$$

Proof. Commuting the evolution equation (2.2.34) with ∇^J, we get for $J\in\mathbb{N}$, $J\le 17$:

$$\begin{aligned}
-\partial_t\|\Gamma-\hat\Gamma\|^2_{\dot H^J_G}=\int_U\Big[&(\partial_tG^{-1})*G^{-1}*\dots*G^{-1}*G*\nabla^J(\Gamma-\hat\Gamma)*\nabla^J(\Gamma-\hat\Gamma)\\
&+(G^{-1})*\dots*(G^{-1})*\partial_tG*\nabla^J(\Gamma-\hat\Gamma)*\nabla^J(\Gamma-\hat\Gamma)
\end{aligned}$$

$$+\left(a^{-3}\sum_{I_N+I_\Sigma=J+1}\nabla^{I_N}(N+1)*\nabla^{I_\Sigma}\Sigma+\frac{\dot a}{a}\nabla^{J+1}N\right)*\nabla^J(\Gamma-\hat\Gamma)$$
$$+2\langle[\partial_t,\nabla^J](\Gamma-\hat\Gamma),\nabla^J(\Gamma-\hat\Gamma)\rangle_G-3N\frac{\dot a}{a}\left|\nabla^J(\Gamma-\hat\Gamma)\right|^2_G\Big]\,\mathrm{dvol}_G$$

We recall that (2.4.4c) implies

$$\|\Gamma-\hat\Gamma\|_{C^{11}_G}\lesssim\sqrt{\varepsilon}a^{-c\sqrt{\varepsilon}} \tag{2.6.22}$$

by (2.3.5). It follows from inserting this in (2.A.10b) along with (2.4.2b), (2.4.4b) and (2.3.17h) that

$$\|[\partial_t,\nabla^J](\Gamma-\hat\Gamma)\|_{L^2_G}\lesssim\sqrt{\varepsilon}a^{-3-c\sqrt{\varepsilon}}\|\Sigma\|_{H^J_G}+\sqrt{\varepsilon}a^{-3-c\sqrt{\varepsilon}}\|N\|_{H^J_G}+\varepsilon a^{-3}\|\Gamma-\hat\Gamma\|_{H^{J-1}_G}$$

is satisfied. Consequently and using the same strong C_G-norm bounds along with Lemma 2.4.1, the differential inequality becomes

$$\begin{aligned}-\partial_t\|\Gamma-\hat\Gamma\|^2_{\dot H^J_G}\lesssim&\,\varepsilon^{\frac18}a^{-3}\|\Gamma-\hat\Gamma\|^2_{\dot H^J_G}+\varepsilon^{-\frac18}a^{-3}\left(\|N\|^2_{H^{J+1}_G}+\|\Sigma\|^2_{H^{J+1}_G}\right)\\&+\varepsilon^{\frac78}a^{-3-c\sqrt{\varepsilon}}\|\Sigma\|^2_{H^J_G}+\varepsilon^{\frac78}a^{-3-c\sqrt{\varepsilon}}\|N\|^2_{H^J_G}\\&+\varepsilon^{\frac{15}{8}}a^{-3-c\sqrt{\varepsilon}}\|\Gamma-\hat\Gamma\|^2_{H^{J-1}_G}\,.\end{aligned}$$

Further, we analogously get

$$-\partial_t\|\Gamma-\hat\Gamma\|^2_{L^2_G}\lesssim\varepsilon^{\frac18}a^{-3}\|\Gamma-\hat\Gamma\|^2_{L^2_G}+\varepsilon^{-\frac18}a^{-3}\left(\|\Sigma\|^2_{H^1_G}+\|N\|^2_{H^1_G}\right)$$

and thus, with the Gronwall lemma and (2.2.7),

$$\begin{aligned}\|\Gamma-\hat\Gamma\|^2_{L^2_G(\Sigma_t)}\lesssim&\,a^{-c\varepsilon^{\frac18}}\left(\varepsilon^4+\int_t^{t_0}\varepsilon^{-\frac18}a(s)^{-3}\left(\|\Sigma\|^2_{H^1_G(\Sigma_s)}+\|N\|^2_{H^1_G(\Sigma_s)}\right)ds\right)\\\lesssim&\,a^{-c\varepsilon^{\frac18}}\left(\varepsilon^4+\varepsilon^{-\frac14}\sup_{s\in(t,t_0)}\left(\|\Sigma\|^2_{H^1_G(\Sigma_s)}+\|N\|^2_{H^1_G(\Sigma_s)}\right)\right)\,.\end{aligned}$$

This proves (2.6.21) for $l=0$, and we assume for an iterative argument that the statement has been proved for $l=J-1$. Then, we obtain (estimating the error terms in Σ and N by their supremum immediately):

$$\begin{aligned}-\partial_t\|\Gamma-\hat\Gamma\|^2_{\dot H^J_G}\lesssim&\,\varepsilon^{\frac18}a^{-3}\|\Gamma-\hat\Gamma\|^2_{\dot H^J_G}+\varepsilon^{-\frac18}a^{-3}\left(\|N\|^2_{H^J_G}+\|\Sigma\|^2_{H^J_G}\right)\\&+\varepsilon^{\frac78+4}a^{-3-c\varepsilon^{\frac18}}+\varepsilon^{\frac78}a^{-3-c\varepsilon^{\frac18}}\sup_{s\in(\cdot,t_0)}\left(\|N\|^2_{H^{J-1}(\Sigma_s)}+\|\Sigma\|^2_{H^{J-1}(\Sigma_s)}\right)\,.\end{aligned}$$

After integrating, applying the Gronwall lemma and dealing with the first line as before, we get

$$\begin{aligned}\|\Gamma-\hat\Gamma\|^2_{\dot H^J_G}\lesssim&\,\varepsilon^4a^{-c\varepsilon^{\frac18}}+\varepsilon^{-\frac14}a^{-c\varepsilon^{\frac18}}\sup_{s\in(\cdot,t_0]}\left(\|N\|^2_{H^J_G(\Sigma_s)}+\|\Sigma\|^2_{H^J_G(\Sigma_s)}\right)\\&+\varepsilon^{\frac68+4}a^{-c\varepsilon^{\frac18}}+\varepsilon^{\frac68}a^{-c\varepsilon^{\frac18}}\sup_{s\in(\cdot,t_0)}\left(\|N\|^2_{H^{J-1}(\Sigma_s)}+\|\Sigma\|^2_{H^{J-1}(\Sigma_s)}\right)\,,\end{aligned}$$

where the second line can obviously be absorbed into the first up to constant. Combining this with the assumption yields (2.6.21) for $l=J$ and thus iteratively for all $l\leq 17$. $\square$

LEMMA 2.6.14 (Sobolev norm estimates for the metric). *For any $t\in(t_{Boot},t_0]$ and any $l\in\mathbb{N}$, $l\leq 18$, we have:*

$$\|G-\gamma\|^2_{H^l_G(\Sigma_t)}\lesssim a^{-c\varepsilon^{\frac18}}\left(\varepsilon^4+\varepsilon^{-\frac14}\sup_{s\in(t,t_0)}\left(\|N\|^2_{H^l_G(\Sigma_s)}+\|\Sigma\|^2_{H^l_G(\Sigma_s)}\right)\right) \tag{2.6.23}$$

PROOF. For $l=0$, we compute the following using (2.2.28a) and (2.2.28b):

$$\begin{aligned}-\partial_t\|G-\gamma\|^2_{L^2_G}=&\int_M\Big\{-2(\partial_tG^{-1})^{i_1j_1}(G^{-1})^{i_2j_2}(G-\gamma)_{i_1i_2}(G-\gamma)_{j_1j_2}-2\langle\partial_tG,G-\gamma\rangle_G\\&\qquad-3N\frac{\dot a}{a}|G-\gamma|^2_G\Big\}\mathrm{dvol}_G\\=&\int_M\Big\{(N+1)a^{-3}\left[\Sigma*(G-\gamma)+\Sigma\right]*(G-\gamma)+N\frac{\dot a}{a}|G-\gamma|^2_G\\&\qquad-4N\frac{\dot a}{a}\langle G,G-\gamma\rangle_G\Big\}\mathrm{dvol}_G\end{aligned}$$

We apply (2.4.2b) and (2.3.17h) and get

$$\begin{aligned}-\partial_t\|G-\gamma\|^2_{L^2_G}\lesssim&\,\varepsilon a^{-3}\|G-\gamma\|^2_{L^2_G}+a^{-3}\left(\|\Sigma\|_{L^2_G}+\|N\|_{L^2_G}\right)\|G-\gamma\|_{L^2_G}\\\lesssim&\,\varepsilon^{\frac18}a^{-3}\|G-\gamma\|^2_{L^2_G}+\varepsilon^{-\frac18}a^{-3}\left(\|\Sigma\|^2_{L^2_G}+\|N\|^2_{L^2_G}\right).\end{aligned}$$

After integrating and applying the Gronwall lemma (as well as the initial data assumption), we obtain

$$\|G-\gamma\|^2_{L^2_G(\Sigma_t)}\lesssim a^{-c\varepsilon^{\frac18}}\left(\varepsilon^4+\varepsilon^{-\frac18}\int_t^{t_0}a(s)^{-3}\left(\|\Sigma\|^2_{L^2_G(\Sigma_s)}+\|N\|^2_{L^2_G(\Sigma_s)}\right)ds\right).$$

The statement for $l=0$ now follows taking the supremum over the norms under the integral and applying (2.2.7). This extends to higher orders via the same iteration argument as in Lemma 2.6.13. $\square$

2.7. Big Bang stability: Improving the bootstrap assumptions

In this section, we combine the energy estimates obtained in the last two sections to improve the boostrap assumptions for the energies themselves, and then show how this improves the behaviour of the solution norms. For an outline of the energy improvement arguments that we perform in Section 2.7.1, we refer back to Remark 2.6.1.

Before carrying out the improvements themselves, we quickly collect an estimate that shows that combining Lemmas 2.6.6 and 2.6.8 yields sufficient control on the energies themselves:

LEMMA 2.7.1. *Let $L\in2\mathbb{N}$. Then, the following estimate is satisfied:*

$$16\pi C^2a^{-3}(N+1)\langle\Delta^{\frac L2}\mathcal{E},\Delta^{\frac L2}\Sigma\rangle_G+8\pi C^2\frac{\dot a}{a}(N+1)|\Delta^{\frac L2}\Sigma|^2_G+6\frac{\dot a}{a}(N+1)|\Delta^{\frac L2}\mathcal{E}|^2_G\geq0 \tag{2.7.1}$$

PROOF. First, we recall that $N+1>0$ holds by Lemma 2.4.1. Additionally, we can apply (2.2.8) and the Young inequality and get

$$\begin{aligned}\left|16\pi C^2 a^{-3}(N+1)\langle\Delta^{\frac{L}{2}}\mathcal{E},\Delta^{\frac{L}{2}}\Sigma\rangle_G\right| &\leq 16\pi C^2\cdot\sqrt{\frac{3}{4\pi C^2}}\frac{\dot a}{a}\cdot(N+1)|\Delta^{\frac{L}{2}}\mathcal{E}|_G|\Delta^{\frac{L}{2}}\Sigma|_G\\ &\leq 4(N+1)\frac{\dot a}{a}\cdot\left(\sqrt{3}\cdot|\Delta^{\frac{L}{2}}\mathcal{E}|_G\right)\cdot\left(\sqrt{4\pi C^2}|\Delta^{\frac{L}{2}}\Sigma|_G\right)\\ &\leq 6\frac{\dot a}{a}(N+1)|\Delta^{\frac{L}{2}}\mathcal{E}|_G^2+8\pi C^2\frac{\dot a}{a}(N+1)|\Delta^{\frac{L}{2}}\Sigma|_G^2\,.\end{aligned}$$

□

This shows that $\mathfrak{E}^{(L)}(W,\cdot)+4\pi C^2\mathfrak{E}^{(L)}(\Sigma,\cdot)$ is controlled by the sum of the left hand sides of the inequalities in Lemmas 2.6.6 and 2.6.8 for $L\in 2\mathbb{N}, 0\leq L\leq 18$.

2.7.1. Improving energy bounds.

PROPOSITION 2.7.2 (Improved energy bounds). *Under the bootstrap assumptions (see Assumption 2.3.16) and the initial data assumptions in Assumption* (2.3.10)*, the following improved estimates hold on* $(t_{Boot},t_0]$*:*

$$\mathfrak{E}^{(\leq 18)}(\phi,\cdot)\lesssim\varepsilon^4 a^{-c\varepsilon^{\frac18}} \tag{2.7.2a}$$

$$\mathfrak{E}^{(\leq 18)}(\Sigma,\cdot)\lesssim\varepsilon^{\frac{15}{4}} a^{-c\varepsilon^{\frac18}} \tag{2.7.2b}$$

$$\mathfrak{E}^{(\leq 18)}(W,\cdot)\lesssim\varepsilon^{\frac{15}{4}} a^{-c\varepsilon^{\frac18}} \tag{2.7.2c}$$

$$\mathfrak{E}^{(\leq 16)}(\mathrm{Ric},\cdot)\lesssim\varepsilon^{\frac{7}{2}} a^{-c\varepsilon^{\frac18}} \tag{2.7.2d}$$

$$\mathfrak{E}^{(\leq 16)}(N,\cdot)+a^4\mathfrak{E}^{(17)}(N,\cdot)+a^8\mathfrak{E}^{(18)}(N,\cdot)\lesssim\varepsilon^{\frac{7}{2}} a^{8-c\varepsilon^{\frac18}} \tag{2.7.2e}$$

PROOF. We prove this estimate by performing an induction over even energy orders. Starting at order 0, we first observe that by Lemma 2.7.1, we can bound the (base level) total energy

$$\mathfrak{E}^{(0)}_{total}:=\mathfrak{E}^{(0)}(\phi,\cdot)+\varepsilon^{\frac14}\left(\mathfrak{E}^{(0)}(W,\cdot)+4\pi C^2\mathfrak{E}^{(0)}(\Sigma,\cdot)\right)+a^4\mathfrak{E}^{(1)}(\phi,\cdot)+\varepsilon^{\frac12}\mathfrak{E}^{(1)}(\Sigma,\cdot)$$

by the sum of the left hand side of (2.6.2), the left hand side of (2.6.10) weighted by $\varepsilon^{\frac14}$ and the left hand side of (2.6.13) weighted by $4\pi C^2\cdot\varepsilon^{\frac14}$, and the left hand sides of (2.6.9) and $\varepsilon^{\frac12}\cdot$(2.6.15) extended to $L=0$.[10] Combining said estimates and inserting the initial data assumption from (2.3.11), the following holds in total:

$$\mathfrak{E}^{(0)}_{total}(t)\lesssim\varepsilon^4+\int_t^{t_0}\left(\varepsilon^{\frac18}a(s)^{-3}+a(s)^{-1-c\sqrt{\varepsilon}}\right)\mathfrak{E}^{(0)}_{total}(s)\,ds \tag{2.7.3}$$

[10]We need to weight $\mathfrak{E}^{(0)}(\Sigma,\cdot)$ in the total energy by $K\cdot\varepsilon^{\frac14}$ for some $K>0$ to balance out the $\varepsilon^{-\frac18}$-weight from the scalar field energy on the right hand side of (2.6.13). The weight on the Bel-Robinson energy is then needed to obtain the cancellation in Lemma 2.7.1. The additional weight on $a^4\mathfrak{E}^{(1)}(\Sigma,\cdot)$ is needed so that the div-curl-estimates only generates a term of size $\varepsilon^{\frac14}\mathfrak{E}^{(L)}_{total}$ that can be absorbed later in the argument.

Applying the Gronwall lemma (see Lemma 2.A.1) to (2.7.3), we get for some suitable constant $c' > 0$:

$$\mathfrak{E}^{(0)}_{total}(t) \lesssim \varepsilon^4 \exp\left(c' \int_t^{t_0} \varepsilon^{\frac{1}{8}} a(s)^{-3} + a(s)^{-1-c\sqrt{\varepsilon}}\, ds\right) \lesssim \varepsilon^4 a^{-c'\varepsilon^{\frac{1}{8}}}$$

Now assume that, for $L \in 2\mathbb{N}, 2 \leq L \leq 18$, we have already shown

$$\mathfrak{E}^{(\leq L-2)}(\phi,\cdot) + \varepsilon^{\frac{1}{4}}\left(\mathfrak{E}^{(\leq L-2)}(\Sigma,\cdot) + \mathfrak{E}^{(\leq L-2)}(W,\cdot)\right) \lesssim \varepsilon^4 a^{-c\varepsilon^{\frac{1}{8}}} \tag{2.7.4a}$$

on $(t_{Boot}, t_0]$. Note that (2.7.1) means this holds true for $L = 2$ after updating $c > 0$. Further, if $L \geq 4$ holds, we assume

$$\mathfrak{E}^{(\leq L-4)}(\mathrm{Ric},\cdot) \lesssim \varepsilon^{\frac{7}{2}} a^{-c\varepsilon^{\frac{1}{8}}}\,. \tag{2.7.4b}$$

We will show that these assumptions hold at $L = 4$ after having shown the induction step for $L = 2$. We define, for $2 \leq L \leq 18$,

$$\begin{aligned}\mathfrak{E}^{(L)}_{total} :=\ & \mathfrak{E}^{(L)}(\phi,\cdot) + \varepsilon^{\frac{1}{4}}\left(\mathfrak{E}^{(L)}(W,\cdot) + 4\pi C^2 \mathfrak{E}^{(L)}(\Sigma,\cdot)\right) + a^4 \mathfrak{E}^{(L+1)}(\phi,\cdot) + \varepsilon^{\frac{1}{2}} a^4 \mathfrak{E}^{(L+1)}(\Sigma,\cdot)\\ & + \varepsilon^{\frac{1}{2}} \mathfrak{E}^{(L-2)}(\mathrm{Ric},\cdot) + \varepsilon^{\frac{3}{4}} a^4 \mathfrak{E}^{(L-1)}(\mathrm{Ric},\cdot)\,.\end{aligned}$$

We combine the respective energy estimates with the appropriate scalings[11], namely (in the listed order) (2.6.3), (2.6.11), (2.6.14), (2.6.9), (2.6.15), (2.6.18) and (2.6.20). Observe that the sum of these scaled left hand sides controls $\mathfrak{E}^{(L)}_{total}$ by Lemma 2.7.1. Combining all of these estimates and inserting the initial data assumption (2.3.11), we get the following estimate:

$$\mathfrak{E}^{(L)}_{total}(t) \lesssim \varepsilon^4 + \int_t^{t_0} \left(\varepsilon^{\frac{1}{8}} a(s)^{-3} + a(s)^{-1-c\sqrt{\varepsilon}}\right) \mathfrak{E}^{(L)}_{total}(s)\, ds \tag{2.7.5a}$$

$$+ \int_t^{t_0} \left\{\varepsilon^{\frac{3}{8}} a(s)^{-3-c\sqrt{\varepsilon}} \left[\mathfrak{E}^{(\leq L-2)}(\phi,s) + \mathfrak{E}^{(\leq L-2)}(\Sigma,s)\right]\right. \tag{2.7.5b}$$

$$\left. + \varepsilon^{\frac{17}{8}} a(s)^{-3-c\sqrt{\varepsilon}} \mathfrak{E}^{(\leq L-2)}(W,s) + \underbrace{\varepsilon^{\frac{11}{8}} \mathfrak{E}^{(\leq L-4)}(\mathrm{Ric},s)}_{\text{if } L=4}\right\} ds \tag{2.7.5c}$$

$$+ \varepsilon^{\frac{1}{4}}\left(a(t)^{4-c\sqrt{\varepsilon}} + \varepsilon a(t)^{2-c\sqrt{\varepsilon}}\right) \cdot \varepsilon^{\frac{1}{2}} \mathfrak{E}^{(L)}(\Sigma,t) + \varepsilon^{\frac{1}{2}} \mathfrak{E}^{(L)}(\phi,t) + \varepsilon^{\frac{1}{4}} \cdot \varepsilon^{\frac{1}{4}} \mathfrak{E}^{(L)}(W,t) \tag{2.7.5d}$$

$$+ \varepsilon^{\frac{7}{4}} \cdot \varepsilon^{\frac{3}{4}} a^4 \mathfrak{E}^{(L-1)}(\mathrm{Ric},t) + \varepsilon^{\frac{5}{2}} a^{-c\sqrt{\varepsilon}} \mathfrak{E}^{(\leq L-2)}(\phi,t) + \varepsilon^{\frac{1}{2}} a^{2-c\sqrt{\varepsilon}} \mathfrak{E}^{(\leq L-2)}(\Sigma,t) \tag{2.7.5e}$$

$$+ \varepsilon^{\frac{3}{2}} a^{2-c\sqrt{\varepsilon}} \mathfrak{E}^{(\leq L-2)}(\mathrm{Ric},t) \tag{2.7.5f}$$

We briefly summarize which inequalities contain the listed error term bounds as explicit terms: The first two terms in (2.7.5b) come from (2.6.18) and the latter from (2.6.11), those in (2.7.5c) from (2.6.3) and (2.6.18), and finally the last three lines are precisely the scaled right hand side of (2.6.15). Regarding the curvature energies in the various individual energy estimates, any summand with $\mathfrak{E}^{(L-3)}(\mathrm{Ric},\cdot)$ can be split using (2.3.8), the resulting summands containing $\mathfrak{E}^{(L-2)}(\mathrm{Ric},\cdot)$ can always be absorbed into the total energy term in the

[11]The weights on all terms beside the curvature energies are necessary for the same reasons as at order 0. We need to scale the curvature energy at order L by $\varepsilon^{\frac{1}{2}}$ to account for $\varepsilon^{-\frac{1}{8}} a^{-3} \mathfrak{E}^{(L)}(\Sigma,\cdot)$ in (2.6.18), and the weight on the $L+1$-order curvature energy again needs to be chosen according to that on $\mathfrak{E}^{(L+1)}(\Sigma,\cdot)$.

first line, and anything with $\mathfrak{E}^{(\leq L-4)}(\mathrm{Ric},\cdot)$ is tracked in $\langle\mathrm{Err}\rangle_L$ for $L\geq 4$.

Inserting (2.7.4a)-(2.7.4b), (2.7.5b)-(2.7.5c) can be bounded up to constant by $\varepsilon^{\frac{33}{8}}a^{-3-c\varepsilon^{\frac18}}$. Here, the error term dominating all others arises from

$$\varepsilon^{\frac38}a(s)^{-3-c\sqrt{\varepsilon}}\mathfrak{E}^{(\leq L-2)}(\Sigma,s)\,.$$

Regarding (2.7.5d)-(2.7.5f), notice that the first four summands can be bounded from above by $\varepsilon^{\frac14}\mathfrak{E}^{(L)}_{total}(t)$ up to constant. For the remaining three terms, we can again insert the induction assumptions (2.7.4a)-(2.7.4b), bounding them by $\varepsilon^{\frac{17}{4}}a(t)^{-c\varepsilon^{\frac18}}$.

In summary and after rearranging, (2.7.5a)-(2.7.5f) becomes for some constant $K>0$:

$$\begin{aligned}(1-K\varepsilon^{\frac14})\mathfrak{E}^{(L)}_{total}(t)\lesssim&\,\varepsilon^4+\int_t^{t_0}\left(\varepsilon^{\frac18}a(s)^{-3}+a(s)^{-1-c\sqrt{\varepsilon}}\right)\mathfrak{E}^{(L)}_{total}(s)\,ds+\int_t^{t_0}\varepsilon^{\frac{33}{8}}a(s)^{-3-c\varepsilon^{\frac18}}\,ds\\&+\varepsilon^{\frac{17}{4}}a(t)^{-c\varepsilon^{\frac18}}\\\lesssim&\,\varepsilon^4a(t)^{-c\varepsilon^{\frac18}}+\int_t^{t_0}\left(\varepsilon^{\frac18}a(s)^{-3}+a(s)^{-1-c\sqrt{\varepsilon}}\right)\mathfrak{E}^{(L)}_{total}(s)\,ds\end{aligned}$$

The prefactor on the left hand side is positive for small enough $\varepsilon>0$, and the Gronwall lemma then yields

$$\mathfrak{E}^{(L)}_{total}(t)\lesssim\varepsilon^4a^{-c\varepsilon^{\frac18}}\,. \tag{2.7.6}$$

In particular, this directly implies that the induction assumptions (2.7.4a) and (2.7.4b), using (2.3.9) to cover the skipped odd order, hold at order L, completing the induction step, and clearly also that (2.7.4b) holds for $L-2=2$ using (2.7.6) at order 2. This completes the induction argument, proving (2.7.2a)-(2.7.2d). Finally, applying the obtained improved estimates for $\nabla\phi$ and $\mathrm{Ric}[G]$ to Corollary 2.5.8, we also get (2.7.2e). □

2.7.2. Improving solution norm control. To close the bootstrap argument, it now remains to show that the improved energy bounds also imply improved bounds for $\mathcal{H}$ and $\mathcal{C}$. The former follows almost directly using Lemma 2.4.5:

COROLLARY 2.7.3 (Improved Sobolev norm bounds). *On $(t_{Boot},t_0]$, the following estimates hold:*

$$\mathcal{H}\lesssim\varepsilon^{\frac74}a^{-c\varepsilon^{\frac18}} \tag{2.7.7a}$$

$$\|\Sigma\|^2_{H^{18}_G}\lesssim\varepsilon^{\frac{15}{4}}a^{-c\varepsilon^{\frac18}} \tag{2.7.7b}$$

$$\|N\|^2_{H^{18}_G}\lesssim\varepsilon^4a^{-c\varepsilon^{\frac18}} \tag{2.7.7c}$$

PROOF. First, we apply the improved energy estimates from Proposition 2.7.2 as well as the strong C_G-norm bounds from Lemma 2.4.3 to the near-coercivity estimates in Lemma 2.4.5. With this, we directly obtain the following Sobolev norm estimates (updating c):

$$\|\Psi\|^2_{H^{18}_G}\lesssim\varepsilon^4a^{-c\varepsilon^{\frac18}}+\varepsilon a^{-c\varepsilon^{\frac18}}\cdot\varepsilon^{\frac{15}{4}}a^{-c\varepsilon^{\frac18}}\lesssim\varepsilon^4a^{-c\varepsilon^{\frac18}}$$

$$\begin{aligned}
\|\Sigma\|^2_{H^{18}_G} &\lesssim \varepsilon^{\frac{15}{4}} a^{-c\varepsilon^{\frac{1}{8}}}\\
\|\mathrm{Ric}[G]+\frac{2}{9}G\|^2_{H^{16}_G} &\lesssim \varepsilon^{\frac{7}{2}} a^{-c\varepsilon^{\frac{1}{8}}}\\
\|\mathcal{E}\|^2_{H^{18}_G}+\|\mathcal{B}\|^2_{H^{18}_G} &\lesssim \varepsilon^{\frac{15}{4}} a^{-c\varepsilon^{\frac{1}{8}}}
\end{aligned}$$

By Lemma 2.6.5, we also have

$$\|\nabla\phi\|_{H^{17}_G} \lesssim \left(1+\varepsilon a^{-c\sqrt{\varepsilon}}\right)\|\Sigma\|_{H^{18}_G}+\varepsilon\|\Psi\|_{H^{18}_G} \lesssim \varepsilon^{\frac{15}{4}} a^{-c\varepsilon^{\frac{1}{8}}}\,. \tag{2.7.8}$$

We take particular care in showing that the improved bound holds for $a^2\|\nabla\phi\|_{\dot{H}^{18}_G}$: First, note that (2.7.2d) implies $\mathfrak{E}^{(\leq 17)}(\mathrm{Ric},\cdot)\lesssim \varepsilon^{\frac{7}{2}} a^{-c\varepsilon^{\frac{1}{8}}}$. Applying this along with (2.4.4e) to (2.4.13d), as well as (2.4.13a) in the second line and (2.7.2a) as well as (2.7.8) in the final step, we obtain:

$$\begin{aligned}
a^4\|\nabla\phi\|^2_{H^{18}_G} &\lesssim a^4\|\nabla\Delta^9\phi\|^2_{L^2_G}+a^{4-c\sqrt{\varepsilon}}\sum_{m=0}^{8}\|\nabla\Delta^m\phi\|^2_{L^2_G}+\varepsilon a^{4-c\sqrt{\varepsilon}}\cdot\mathfrak{E}^{(\leq 16)}(\mathrm{Ric},\cdot)\\
&\lesssim a^{4-c\sqrt{\varepsilon}}\left(\|\nabla\Delta^9\phi\|^2_{L^2_G}+\|\nabla\phi\|^2_{H^{17}_G}\right)+\varepsilon^{\frac{9}{2}}a^{-c\varepsilon^{\frac{1}{8}}}\\
&\lesssim a^{-c\sqrt{\varepsilon}}\cdot\mathfrak{E}^{(\leq 18)}(\phi,\cdot)+a^{4-c\sqrt{\varepsilon}}\|\nabla\phi\|^2_{H^{17}_G}+\varepsilon^{\frac{9}{2}}a^{-c\varepsilon^{\frac{1}{8}}}\\
&\lesssim \varepsilon^{\frac{15}{4}}a^{-c\varepsilon^{\frac{1}{8}}}
\end{aligned}$$

Further, inserting (2.7.2a), (2.7.2b) and (2.7.2d) into Corollary 2.5.5 implies

$$a^8\|\Delta^{10}N\|^2_{L^2_G}+a^4\|\nabla\Delta^9 N\|^2_{L^2_G}+\sum_{m=0}^{9}\|\Delta^m N\|^2_{L^2_G}\lesssim \varepsilon^{\frac{11}{4}}a^{-c\varepsilon^{\frac{1}{8}}}$$

and subsequently, applying Lemma 2.4.5 as before,

$$a^8\|N\|^2_{\dot{H}^{20}_G}+a^4\|N\|^2_{\dot{H}^{19}_G}+\|N\|^2_{H^{18}_G}\lesssim \varepsilon^4 a^{-c\varepsilon^{\frac{1}{8}}}\,.$$

Finally, having now shown (2.7.7b) and (2.7.7c), we can apply these to (2.6.23) to get

$$\|G-\gamma\|^2_{H^{18}_G}\lesssim a^{-c\varepsilon^{\frac{1}{8}}}\left(\varepsilon^4+\varepsilon^{-\frac{1}{4}+\frac{15}{4}}+\varepsilon^{-\frac{1}{4}+4}\right)\lesssim \varepsilon^{\frac{7}{2}}a^{-c\varepsilon^{\frac{1}{8}}}\,,$$

proving (2.7.7a). □

Intuitively, the bounds on $\mathcal{C}$ should now follow from $\mathcal{H}$ by the standard Sobolev embedding. However, since both of these norms are with respect to G, the embedding constant may be time dependent. To circumvent this issue, we need to switch between norms with respect to G and γ and then apply the embedding with respect to C_γ and H_γ. The following lemma ensures that we still obtain bootstrap improvements after performing these norm switches:

LEMMA 2.7.4 (Moving between norms). *Let $l\in\mathbb{N}$, $l\leq 18$, ζ be a scalar field, $\mathfrak{T}$ be an arbitrary Σ_t-tangent tensor. Then, for some multivariate polynomial P_l with $P_l(0,0)=0$, we have*

$$\|\zeta\|_{C^l_G(U)}\lesssim a^{-c\sqrt{\varepsilon}}\|\zeta\|_{C^l_\gamma(M)} \tag{2.7.9a}$$

(2.7.9b) $$+ a^{-c\sqrt{\varepsilon}}\|\zeta\|_{C_\gamma^{\max\{0,\lfloor\frac{l-1}{2}\rfloor\}}(M)} P_l\left(\|G-\gamma\|_{C_\gamma^{l-1}(M)}, \|G^{-1}-\gamma^{-1}\|_{C_\gamma^{l-1}(M)}\right)$$

(2.7.9c) $$\|\mathfrak{T}\|_{C_G^l(M)} \lesssim a^{-c\sqrt{\varepsilon}}\|\mathfrak{T}\|_{C_\gamma^l(M)}$$

(2.7.9d) $$+ a^{-c\sqrt{\varepsilon}}\|\mathfrak{T}\|_{C_\gamma^{\max\{0,\lfloor\frac{l-1}{2}\rfloor\}}(M)} P_l\left(\|G-\gamma\|_{C_\gamma^{l}(M)}, \|G^{-1}-\gamma^{-1}\|_{C_\gamma^{l}(M)}\right)$$

as well as the same inequalities with the roles of G and γ reversed. For $l \leq 12$, this reduces to:

(2.7.10a) $$a^{c\sqrt{\varepsilon}}\|\zeta\|_{C_\gamma^l(M)} \lesssim \|\zeta\|_{C_G^l(M)} \lesssim a^{-c\sqrt{\varepsilon}}\|\zeta\|_{C_\gamma^l(M)}$$

(2.7.10b) $$a^{c\sqrt{\varepsilon}}\|\mathfrak{T}\|_{C_\gamma^l(M)} \lesssim \|\mathfrak{T}\|_{C_G^l(M)} \lesssim a^{-c\sqrt{\varepsilon}}\|\mathfrak{T}\|_{C_\gamma^l(M)}$$

Further, one has

(2.7.11a)

$$\begin{aligned}\|\zeta\|^2_{H_\gamma^l(M)} \lesssim\, & a^{-c\sqrt{\varepsilon}}\|\zeta\|^2_{H_G^l(M)}\\ & + a^{-c\varepsilon^{\frac18}}\|\zeta\|^2_{C_G^{\lceil\frac{l-1}{2}\rceil}(M)}\left(\varepsilon^4 + \varepsilon^{-\frac14}\sup_{s\in(\cdot,t_0)}\left(\|N\|^2_{H_G^{l-1}(\Sigma_s)} + \|\Sigma\|^2_{H_G^{l-1}(\Sigma_s)}\right)\right)\end{aligned}$$

(2.7.11b)

$$\begin{aligned}\|\mathfrak{T}\|^2_{H_\gamma^l(M)} \lesssim\, & a^{-c\sqrt{\varepsilon}}\|\mathfrak{T}\|^2_{H_G^l(M)}\\ & + a^{-c\varepsilon^{\frac18}}\|\mathfrak{T}\|^2_{C_G^{\lceil\frac{l-1}{2}\rceil}(M)}\left(\varepsilon^4 + \varepsilon^{-\frac14}\sup_{s\in(\cdot,t_0)}\left(\|N\|^2_{H_G^{l}(\Sigma_s)} + \|\Sigma\|^2_{H_G^{l}(\Sigma_s)}\right)\right)\end{aligned}$$

REMARK 2.7.5. While we only need the tensorial inequalities for gradient vector fields and $(0,2)$-tensors when applied to norms in $\mathcal{H}$ and $\mathcal{C}$, the proof is simpler when considering tensors of arbitrary rank.

PROOF. We restrict ourselves to proving the tensorial statements; the scalar field analogues follow analagously except for the fact that, since $\nabla_i\zeta = \hat{\nabla}_i\zeta = \partial_i\zeta$, error terms caused by Christoffel symbols always enter at one order less. Thus, it remains to show (2.7.9d), (2.7.10b) and (2.7.11b) by iterating over derivative order.
Starting with the base level estimates, we have if $\mathfrak{T}$ is of rank (r,s):

$$\begin{aligned}|\mathfrak{T}|^2_G - |\mathfrak{T}|^2_\gamma = &\Big[G_{i_1j_1}\dots G_{i_rj_r}(G^{-1})^{p_1q_1}\dots(G^{-1})^{p_sq_s} - \gamma_{i_1j_1}\dots\gamma_{i_rj_r}(\gamma^{-1})^{p_1q_1}\dots(\gamma^{-1})^{p_sq_s}\Big]\cdot\\ &\cdot \mathfrak{T}^{i_1\dots i_r}{}_{p_1\dots p_s}\mathfrak{T}^{j_1\dots j_r}{}_{q_1\dots q_s}\end{aligned}$$

We successively replace $G^{\pm1}$ by $(G^{\pm1}-\gamma^{\pm1})+\gamma^{\pm1}$, take the $|\cdot|_\gamma$-norm of each factor and use (2.4.4c)-(2.4.4d). This yields

$$\left||\mathfrak{T}|^2_G - |\mathfrak{T}|^2_\gamma\right| \lesssim \sqrt{\varepsilon}a^{-c\sqrt{\varepsilon}}|\mathfrak{T}|^2_\gamma\,,$$

implying (2.7.9d) (and (2.7.10b)) for $l=0$ after rearranging and taking supremums suitably.
To show (2.7.11b) at base level, consider

$$\int_M|\mathfrak{T}|^2_G\,\mathrm{dvol}_G - \int_M|\mathfrak{T}|^2_\gamma\,\mathrm{dvol}_\gamma$$

$$= \int_M \left(|\mathfrak{T}|_G^2 - |\mathfrak{T}|_\gamma^2\right) \, \mathrm{dvol}_G + \int_M |\mathfrak{T}|_\gamma^2 \frac{\mu_G - \mu_\gamma}{\mu_\gamma} \, \mathrm{dvol}_\gamma \, .$$

We can control the first summand on the right hand side as before, while we have $|\mu_G - \mu_\gamma| \lesssim \varepsilon$ by (2.4.11). Hence,

$$(1 - K\varepsilon)\|\mathfrak{T}\|_{L^2_\gamma}^2 \lesssim (1 + \sqrt{\varepsilon} a^{-c\sqrt{\varepsilon}})\|\mathfrak{T}\|_{L^2_G}^2$$

follows for a suitable constant $K > 0$, implying the statement for small enough $\varepsilon > 0$.

Next, we perform the iteration for (2.7.9d), assuming the statement and the analogue with γ and G reversed to hold up to order $l-1$. As above, note that

$$\left| \left|\nabla^J \mathfrak{T}\right|_G^2 - \left|\hat{\nabla}^J \mathfrak{T}\right|_\gamma^2 \right| \lesssim \sqrt{\varepsilon} a^{-c\sqrt{\varepsilon}} \left|\hat{\nabla}^J \mathfrak{T}\right|_\gamma^2 + (1 + \sqrt{\varepsilon} a^{-c\sqrt{\varepsilon}}) \left| \left|\nabla^J \mathfrak{T}\right|_\gamma^2 - \left|\hat{\nabla}^J \mathfrak{T}\right|_\gamma^2 \right|$$

where we can rewrite the second term as

$$\left| 2\langle \hat{\nabla}^J \mathfrak{T} - \nabla^J \mathfrak{T}, \hat{\nabla} \mathfrak{T}\rangle_\gamma - |\nabla^J \mathfrak{T} - \hat{\nabla}^J \mathfrak{T}|_\gamma^2 \right|$$

and hence obtain (moving between pointwise norms as before)

$$\left| \left|\nabla^J \mathfrak{T}\right|_G^2 - \left|\hat{\nabla}^J \mathfrak{T}\right|_\gamma^2 \right| \lesssim a^{-c\sqrt{\varepsilon}} \left|\hat{\nabla}^J \mathfrak{T}\right|_\gamma^2 + a^{-c\sqrt{\varepsilon}} \left|\nabla^J \mathfrak{T} - \hat{\nabla}^J \mathfrak{T}\right|_\gamma^2 .$$

Regarding $\nabla^J \mathfrak{T} - \hat{\nabla}^J \mathfrak{T}$, we have the following schematic decomposition :

$$\begin{aligned} \nabla^J \mathfrak{T} - \hat{\nabla}^J \mathfrak{T} = & \sum_{I=0}^{J-1} \hat{\nabla}^{J-I-1}(\Gamma - \hat{\Gamma}) *_\gamma \left(\nabla^I \mathfrak{T} + \hat{\nabla}^I \mathfrak{T}\right) \\ & + \langle \text{at least cubic nonlinear terms} \rangle \, , \end{aligned} \tag{2.7.12}$$

Here, $*_\gamma$ encodes the analogous schematic product notation with regard to γ (see subsection 2.2.1.8). Regarding the Christoffel symbols, notice (2.7.9d) with roles of γ and G reversed holding up to $l-1$ implies that, for any $m \in \{0, \dots, l-1\}$ and some multivariate polynomial $\tilde{P}_m$, we have

$$\|\Gamma - \hat{\Gamma}\|_{C^m_\gamma(M)} \lesssim a^{-c\sqrt{\varepsilon}} \tilde{P}_m(\|\Gamma - \hat{\Gamma}\|_{C^m_G(M)}, \|G - \gamma\|_{C^m_G(M)}, \|G^{-1} - \gamma^{-1}\|_{C^m_G(M)}) \, .$$

As explained in Remark 2.3.7, we can bound $\|\Gamma - \hat{\Gamma}\|_{C^m_G(M)}$ by a polynomial in $\|G - \gamma\|_{C^{m+1}_G(M)}$. Hence, we can apply (2.4.4c) to obtain

$$\|\Gamma - \hat{\Gamma}\|_{C^{l-1}_\gamma(M)} \lesssim \sqrt{\varepsilon} a^{-c\sqrt{\varepsilon}} \, . \tag{2.7.13}$$

Moving back to (2.7.12) and just considering the first line for now, this implies

(2.7.14)

$$\begin{aligned} \left| \|\mathfrak{T}\|_{\dot{C}^l_G(M)}^2 - \|\mathfrak{T}\|_{\dot{C}^l_\gamma(M)}^2 \right| \lesssim & \, a^{-c\sqrt{\varepsilon}} \left(\|\mathfrak{T}\|_{\dot{C}^l_\gamma(M)}^2 + \sum_{m=0}^{l-1} \|\nabla^m \mathfrak{T}\|_{C^0_\gamma(M)}^2 \right) \\ & + \left(\|\mathfrak{T}\|_{C_\gamma^{\lceil \frac{l-1}{2} \rceil}(M)}^2 + \sum_{m=0}^{\lceil \frac{l-1}{2} \rceil} \|\nabla^m \mathfrak{T}\|_{C^0_\gamma(M)}^2 \right) \|\Gamma - \hat{\Gamma}\|_{C^{l-1}_\gamma(M)}^2 \\ & + \langle \text{at least cubic nonlinear terms} \rangle \, . \end{aligned}$$

We can rewrite $\nabla^m\mathfrak{T}$-norms in C_γ as ones in C_G up to $a^{-c\sqrt{\varepsilon}}$ as before. Then, we can apply the already obtained estimates up to order $l-1$ show that the first two lines of the right hand side can be estimated by the right hand side of (2.7.9d). The highly nonlinear terms can be dealt with similarly, closing the induction over admissible l. The inequality in (2.7.10b) immediately follows by applying (2.4.4c)-(2.4.4d) and (2.7.13).

Now, assume (2.7.11b) to be proven up to order $J-1$. By analogous arguments as at order zero, we get, after rearranging,

$$\int_M |\hat{\nabla}^J\mathfrak{T}|^2_\gamma \,\mathrm{dvol}_\gamma \lesssim \left|\int_M \left(|\nabla^J\mathfrak{T}|^2_G - |\hat{\nabla}^J\mathfrak{T}|^2_\gamma\right)\,\mathrm{dvol}_G\right| + \int_M |\nabla^J\mathfrak{T}|^2_G \,\mathrm{dvol}_G\,,$$

so we only need to concern ourselves with the first summand. Reversing roles of G and γ compared to the proof of (2.7.9d), we get

$$\left|\,\left|\nabla^J\mathfrak{T}\right|^2_G - \left|\hat{\nabla}^J\mathfrak{T}\right|^2_\gamma\right| \lesssim \sqrt{\varepsilon}a^{-c\sqrt{\varepsilon}}\left|\nabla^J\mathfrak{T}\right|^2_G + a^{-c\sqrt{\varepsilon}}\left|2\langle\nabla^J\mathfrak{T}-\hat{\nabla}^J\mathfrak{T},\nabla\mathfrak{T}\rangle_G - |\nabla^J\mathfrak{T}-\hat{\nabla}^J\mathfrak{T}|^2_G\right|,$$

and have the following, applying Lemma 2.6.13 immediately to estimate $\|\Gamma-\hat{\Gamma}\|_{H^{l-1}_G}$:

$$\begin{aligned}
&\left|\int_V \left\{2\langle\nabla^l\mathfrak{T}-\hat{\nabla}^l\mathfrak{T},\nabla\mathfrak{T}\rangle_G - |\nabla^l\mathfrak{T}-\hat{\nabla}^l\mathfrak{T}|^2_G\right\}\mathrm{dvol}_G\right|\\
&\lesssim a^{-c\sqrt{\varepsilon}}\left(\|\mathfrak{T}\|^2_{H^l_G(M)} + \|\hat{\nabla}^{\leq l-1}\mathfrak{T}\|^2_{H^0_G(M)}\right)\\
&\quad+\left(\|\mathfrak{T}\|^2_{C_G^{\lceil\frac{l-1}{2}\rceil}(M)} + \|\hat{\nabla}^{\leq\lceil\frac{l-1}{2}\rceil}\mathfrak{T}\|^2_{C^0_G(M)}\right)\cdot\\
&\qquad\cdot a^{-c\varepsilon^{\frac18}}\left(\varepsilon^4+\varepsilon^{-\frac14}\sup_{s\in(\cdot,t_0)}\left(\|N\|^2_{H^l_G(\Sigma_s)}+\|\Sigma\|^2_{H^l_G(\Sigma_s)}\right)\right)
\end{aligned}$$

By the same arguments as earlier, we have $\|\hat{\nabla}^{\leq l-1}\mathfrak{T}\|_{H^{l-1}_G(M)} \lesssim a^{-c\sqrt{\varepsilon}}\|\mathfrak{T}\|_{H^{l-1}_\gamma(M)}$ and can then apply the induction hypothesis. This proves (2.7.11b). $\square$

COROLLARY 2.7.6 (Improved C-norm bounds). *On $(t_{Boot},t_0]$, the following estimate is satisfied:*

$$\mathcal{C}+\mathcal{C}_\gamma \lesssim \varepsilon^{\frac74}a^{-c\varepsilon^{\frac18}} \tag{2.7.15}$$

PROOF. We first apply the Sobolev norm estimates in Lemma 2.7.4 to (2.7.7a), to then control $\mathcal{C}_\gamma$ via the standard Sobolev embedding $H^{l+2}_\gamma(M)\hookrightarrow C^l(M)$, and finally control $\mathcal{C}$ with (2.7.9b)-(2.7.9d).
Note that by Lemma 2.4.3, we can control the C_G-norm up to order 10 of every quantity occurring in $\mathcal{H}$ beside the lapse by at worst $\sqrt{\varepsilon}a^{-c\sqrt{\varepsilon}}$, while the bootstrap assumption already implies better behaviour for the lapse. Thus, we can apply (2.7.11a)-(2.7.11b) to every norm appearing in $\mathcal{H}$, and obtain by applying (2.7.7b) and (2.7.7c) in the second line:

$$\begin{aligned}
\mathcal{C}^2_\gamma &\lesssim a^{-c\sqrt{\varepsilon}}\cdot\mathcal{H}^2 + \varepsilon a^{-c\sqrt{\varepsilon}}\cdot a^{-c\varepsilon^{\frac18}}\left(\varepsilon^4+\varepsilon^{-\frac14}\sup_{s\in(t,t_0)}\left(\|N\|^2_{H^{18}_G(\Sigma_s)}+\|\Sigma\|^2_{H^{18}_G(\Sigma_s)}\right)\right)\\
&\lesssim \varepsilon^{\frac72}a^{-c\varepsilon^{\frac18}} + \varepsilon a^{-c\varepsilon^{\frac18}}\left(\varepsilon^4+\varepsilon^{\frac72}\right)
\end{aligned}$$

$$\lesssim \varepsilon^{\frac{7}{2}} \cdot a^{-c\varepsilon^{\frac{1}{8}}}$$

In particular, we can update c such that

$$|P(\|G-\gamma\|_{C^{16}_\gamma(\Sigma_t)}, \|G-\gamma\|_{C^{16}_\gamma(\Sigma_t)})| \lesssim \varepsilon^{\frac{7}{2}} a^{-c\varepsilon^{\frac{1}{8}}}$$

holds for any multivariate polynomial P that appears when applying (2.7.9b)-(2.7.9d). Again using the strong C_G-norm estimates from Lemma 2.4.3, this then implies $\mathcal{C} \lesssim \varepsilon^{\frac{7}{4}} a^{-c\varepsilon^{\frac{1}{8}}}$. □

2.8. Big Bang stability: The main theorem

In this section, we provide the proof of the first main result, Theorem 2.1.1, which we state in more detail in Theorem 2.8.2 below. As in [RS18b; Spe18], most of the work has already been done by establishing the necessary bounds on solution norms.

Remark 2.8.1 (Existence of a CMC hypersurface). As mentioned in Section 2.1.2.1, it may seem that the generality of the results in Theorem 2.8.2 is restricted by taking the initial data on Σ_{t_0} to be CMC. However, as long as one remains close enough to a constant time hypersurface of the FLRW reference metric (which is CMC), one can locally evolve the perturbed data in harmonic gauge to a nearby hypersurface that is CMC and remains close to the FLRW reference solution. To make this a bit more precise, and also since this is a little less involved than the arguments in [RS18b], we will briefly sketch how the arguments from [FK20, Section 2.5] extend to our setting.

First, we once again assume without loss of generality that our initial data is sufficiently regular. Note that we can locally evolve our data within harmonic gauge to get a C^{17}-regular family of metrics with near-FLRW initial data (for well-posedness, consider the analogue of [RS18b, Proposition 14.1]). Consider the Banach manifold $\mathcal{M}^{17}$ formed by the set of C^{17} Lorentz metrics on $I \times \overline{M}$ for an open interval I around t_0 such that the surfaces of constant time are Riemannian, endowed with the norm

$$\|\tilde{g}\| = \|\tilde{n}^2\|_{C^{17}_{dt^2+\gamma}(I\times M)} + \|\tilde{X}\|_{C^{17}_{dt^2+\gamma}(I\times M)} + \|\tilde{g}_t\|_{C^{17}_{dt^2+\gamma}(I\times M)}\,,$$

where $\tilde{g} \in \mathcal{M}^{17}$ has lapse $\tilde{n}$, shift $\tilde{X}$ and spatial metrics $(\tilde{g}_t)_{t\in I}$. Further, for any $f \in C^{17}(M,I)$, we define the embedding $\iota_f : M \hookrightarrow \overline{M}$ by $x \mapsto (f(x), x)$, and subsequently define the smooth map

$$\begin{aligned} H_0 : \mathcal{D} := \{(\tilde{g}, f) \in \mathcal{M}^{17} \times C^{17}(M,I) | \iota_f^*\tilde{g} \text{ is Riemannian}\} &\longrightarrow C^{16}(M) \\ (\tilde{g}, f) &\mapsto \text{mean curvature of } (M, \tilde{g}_t) \text{ embedded along } \iota_f\,. \end{aligned}$$

One easily checks that $(\overline{g}_{FLRW}, t_0)$ is a regular point of H_0. By the implicit function theorem for Banach manifolds, this means there is a (unique) smooth function F that maps an open neighbourhood of $\overline{g}_{FLRW}$ in $\mathcal{M}^{17}$ to an open neighbourbood of the constant function $x \mapsto t_0$ in $C^{17}(M,I)$ such that $H_0(\cdot, F(\cdot)) = \tau(t_0)$ holds in that neighbourhood.

Thus, we can choose a surface Σ' with mean curvature $\tau(t_0)$ near the original Σ_{t_0}. Furthermore, for small enough $\varepsilon > 0$, the initial data on Σ' remains close to the FLRW initial data in the sense of Assumption 2.3.10, using similar arguments to control Sobolev norms. Thus, we can replace Σ_{t_0} by Σ' without loss of generality, proving that the CMC assumption (2.2.10) is not a true restriction.

THEOREM 2.8.2 (Stability of Big Bang formation). *Let $(M, \mathring{g}, \mathring{k}, \mathring{\pi}, \mathring{\psi})$ be initial data to the Einstein scalar-field system as discussed in Section 2.1.2.1. Further, let the data be embedded into a time-oriented 4-manifold such that it induces initial data for the rescaled solution variables (see Definition 2.2.9) at the initial hypersurface Σ_{t_0}. We also assume this rescaled initial data is close to that of the FLRW reference solution (see* (2.2.1) *and* (2.2.2)*) in the sense that*

$$\mathcal{H}(t_0) + \mathcal{H}_{top}(t_0) + \mathcal{C}(t_0) \leq \varepsilon^2 \tag{2.8.1}$$

is satisfied (with $\mathcal{H}$ and $\mathcal{C}$ as in Definition 2.3.6). [12]

Then, the past maximal globally hyperbolic development $((0, t_0] \times M, \overline{g}, \phi)$ of this data within the Einstein scalar-field system (2.1.1a)-(2.1.1c) *in CMC gauge* (2.2.10) *with zero shift is foliated by the CMC hypersurfaces $\Sigma_s = t^{-1}(\{s\})$, and one has*

$$\mathcal{H}(t) + \mathcal{C}(t) + \mathcal{C}_\gamma(t) \lesssim \varepsilon^{\frac{7}{4}} a(t)^{-c\varepsilon^{\frac{1}{8}}} \tag{2.8.2}$$

for some $c > 0$ and any $t \in (0, t_0]$. In particular, this implies the following statements:

Asymptotic behaviour of solution variables: *We denote the solution metric as $\overline{g} = -n^2 dt^2 + g$, the second fundamental form (viewed as a $(1,1)$-tensor) with respect to Σ_t as k and the volume form with regard to g on Σ_t by dvol_g. There exist a smooth function $\Psi_{Bang} \in C^{15}_\gamma(M)$, a $(1,1)$-tensor field $K_{Bang} \in C^{15}_\gamma(M)$ and a volume form $\mathrm{dvol}_{Bang} \in C^{15}_\gamma(M)$ such that the following estimates hold for any $t \in (0, t_0]$:*

$$\|n-1\|_{C^l_\gamma(\Sigma_t)} \lesssim \begin{cases} \varepsilon a(t)^{4-c\varepsilon^{\frac{1}{8}}} & l \leq 14 \\ \varepsilon a(t)^{2-c\varepsilon^{\frac{1}{8}}} & l = 15 \end{cases} \tag{2.8.3a}$$

$$\left\|a^{-3}\mathrm{dvol}_g - \mathrm{dvol}_{Bang}\right\|_{C^l_\gamma(\Sigma_t)} \lesssim \begin{cases} \varepsilon a(t)^{4-c\varepsilon^{\frac{1}{8}}} & l \leq 14 \\ \varepsilon a(t)^{2-c\varepsilon^{\frac{1}{8}}} & l = 15 \end{cases} \tag{2.8.3b}$$

$$\left\|a^3\partial_t\phi - (\Psi_{Bang} + C)\right\|_{C^l_\gamma(\Sigma_t)} \lesssim \begin{cases} \varepsilon a(t)^{4-c\varepsilon^{\frac{1}{8}}} & l \leq 14 \\ \varepsilon a(t)^{2-c\varepsilon^{\frac{1}{8}}} & l = 15 \end{cases} \tag{2.8.3c}$$

$$\left\|\phi(t,\cdot) - \phi(t_0,\cdot) + \int_t^{t_0} a(s)^{-3}\,ds \cdot (\Psi_{Bang} + C)\right\|_{\dot{C}^l_\gamma(\Sigma_t)} \lesssim \begin{cases} \varepsilon a(t)^{4-c\varepsilon^{\frac{1}{8}}} & 1 \leq l \leq 14 \\ \varepsilon a(t)^{2-c\varepsilon^{\frac{1}{8}}} & l = 15 \end{cases} \tag{2.8.3d}$$

$$\left\|a^3 k - K_{Bang}\right\|_{C^l_\gamma(\Sigma_t)} \lesssim \begin{cases} \varepsilon a(t)^{4-c\varepsilon^{\frac{1}{8}}} & l \leq 14 \\ \varepsilon a(t)^{2-c\varepsilon^{\frac{1}{8}}} & l = 15 \end{cases} \tag{2.8.3e}$$

[12]Essentially, this translates to smallness in H^{19}_γ and C^{17}_γ . For $\varepsilon = 0$, the solution is the FLRW reference solution.

Further, these footprint states satisfy the equations[13]

$$(K_{Bang})^a{}_a = -\sqrt{12\pi}C\,, \tag{2.8.4a}$$

$$8\pi(\Psi_{Bang}+C)^2 + (K_{Bang})^a{}_b(K_{Bang})^b{}_a = 12\pi C^2 \tag{2.8.4b}$$

and remain close to the data of the reference solution in the following sense, where $\mathbb{I}$ *denotes the Kronecker symbol:*

$$\|\mathrm{dvol}_\gamma - \mathrm{dvol}_{Bang}\|_{C^{15}_\gamma(M)} \lesssim \varepsilon \tag{2.8.5a}$$

$$\|\Psi_{Bang}\|_{C^{15}_\gamma(M)} \lesssim \varepsilon \tag{2.8.5b}$$

$$\left\| K_{Bang} + \sqrt{\frac{4\pi}{3}}C\mathbb{I} \right\|_{C^{15}_\gamma(M)} \lesssim \varepsilon \tag{2.8.5c}$$

Additionally, there exists a $(0,2)$*-tensor field* $M_{Bang} \in C^{15}_\gamma(M)$ *satisfying*

$$\|M_{Bang} - \gamma\|_{C^{15}_\gamma(M)} \lesssim \varepsilon \tag{2.8.6}$$

and, with $\odot$ *and* exp *meant in the matrix product and exponential sense respectively, one has*

$$\left\| g \odot \exp\left[\left(-2\int_t^{t_0} a(s)^{-3}\,ds\right)\cdot K_{Bang}\right] - M_{Bang}\right\|_{C^l_\gamma(\Sigma_t)} \lesssim \begin{cases} \varepsilon a(t)^{4-c\varepsilon^{\frac18}} & l \leq 14 \\ \varepsilon a(t)^{2-c\varepsilon^{\frac18}} & l = 15\,. \end{cases} \tag{2.8.7}$$

Moreover, the Bel-Robinson variables E *and* B *satisfy the estimates*

$$\|E\|_{C^{16}_\gamma(\Sigma_t)} \lesssim \varepsilon a^{-4-c\varepsilon^{\frac18}} \tag{2.8.8a}$$

$$\|B\|_{C^l_\gamma(\Sigma_t)} \lesssim \begin{cases} \varepsilon a^{-2-c\varepsilon^{\frac18}} & l \leq 15 \\ \varepsilon a^{-4-c\varepsilon^{\frac18}} & l \leq 16\,. \end{cases} \tag{2.8.8b}$$

Causal disconnectedness: *Let* $\boldsymbol{\gamma}$ *be a past directed causal curve on* $((0,t]\times M, \overline{g})$ *for* $t \leq t_0$ *with domain* $[s_1, s_{max})$ *such that* $\boldsymbol{\gamma}(s_1) \in \Sigma_t$ *and* s_{max} *is maximal. Then, there exists a constant* $\mathcal{K} > 0$ *that does not depend on* $\boldsymbol{\gamma}$ *such that one has*

$$L[\boldsymbol{\gamma}] = \int_{s_1}^{s_{max}} \sqrt{(\gamma_{ab})_{\boldsymbol{\gamma}(s)}\dot{\boldsymbol{\gamma}}^a(s)\dot{\boldsymbol{\gamma}}^b(s)}\,ds \leq \mathcal{K}a(t)^{2-c\varepsilon^{\frac18}}\,, \tag{2.8.9}$$

where γ *is the negative Einstein spatial reference metric on* M *(see Definition 2.2.1). Hence, for points* $p, q \in \Sigma_t$ *with* $dist_\gamma(p,q) > 2\mathcal{K}a(t)^{2-c\varepsilon^{\frac18}}$*, the causal pasts of* p *and* q *cannot intersect.*

Geodesic incompleteness: *Let* $\boldsymbol{\gamma}(\mathcal{A})$ *be a past directed, affinely parametrized causal geodesic emanating from* Σ_{t_0}*, where* $\mathcal{A} : (0, t_0] \to [0,\infty)$ *denotes the parameter time that is normalized to* $\mathcal{A}(t_0) = 0$*. Then,*

$$\mathcal{A}(0) \leq \mathcal{K}_1 \cdot |\mathcal{A}'(t_0)| \cdot a(t_0)^{1+K_2\varepsilon}\int_0^{t_0} a(s)^{-1-\mathcal{K}_2\varepsilon}\,ds < \infty\,, \tag{2.8.10}$$

[13] These are precisely the (generalized) Kasner relations, see Section 2.1.2.3.

holds for suitable constants $\mathcal{K}_1, \mathcal{K}_2 > 0$ that are independent of $\boldsymbol{\gamma}$, and thus any such geodesic crashes into the Big Bang hypersurface in finite affine parameter time.

Blow-up: *The norm $|k|_g$ behaves toward the Big Bang hypersurface as follows:*

$$\left\| a^6 |k|_g^2 - (K_{Bang})^i{}_j (K_{Bang})^j{}_i \right\|_{C^0_\gamma(\Sigma_t)} \lesssim \varepsilon a^{4-c\varepsilon^{\frac{1}{8}}} \tag{2.8.11a}$$

Further, with $W[\overline{g}]$ denoting the Weyl curvature and $P[\overline{g}] = \mathrm{Riem}[\overline{g}] - W[\overline{g}]$,

$$\left\| a^{12} P_{\alpha\beta\gamma\delta} P^{\alpha\beta\gamma\delta} - \frac{5}{3} \cdot (8\pi)^2 (\Psi_{Bang} + C)^4 \right\|_{C^0_\gamma(M)} \lesssim \varepsilon a^{4-c\varepsilon^{\frac{1}{8}}} \tag{2.8.11b}$$

is satisfied, whereas there exists a scalar footprint $W_{Bang} \in C^{15}_\gamma(M)$ such that one has

$$\left\| a^{12} W_{\alpha\beta\gamma\delta} W^{\alpha\beta\gamma\delta} - W_{Bang} \right\|_{C^0(M)} \lesssim \varepsilon a^{2-c\varepsilon^{\frac{1}{8}}} . \tag{2.8.11c}$$

Here, W_{Bang} is a fourth order polynomial in $\hat{K}_{Bang} = K_{Bang} + \sqrt{\frac{4\pi}{3}} C \mathbb{I}$ and Ψ_{Bang} and satisfies

$$\|W_{Bang}\|_{C^{15}_\gamma(M)} \lesssim \varepsilon . \tag{2.8.11d}$$

Finally, the scalar curvature $R[\overline{g}]$ and the Ricci curvature invariant $\mathrm{Ric}[\overline{g}]_{\alpha\beta}\mathrm{Ric}[\overline{g}]^{\alpha\beta}$ blow up with the asymptotics

$$\|a^6 R[\overline{g}] - 8\pi(\Psi_{Bang} + C)^2\|_{C^0(M)} \lesssim \varepsilon a^{4-c\varepsilon^{\frac{1}{8}}} , \tag{2.8.11e}$$

$$\|a^{12} \mathrm{Ric}[\overline{g}]_{\alpha\beta} \mathrm{Ric}[\overline{g}]^{\alpha\beta} - (8\pi)^2 (\Psi_{Bang} + C)^4\|_{C^0(M)} \lesssim \varepsilon a^{4-c\varepsilon^{\frac{1}{8}}} , \tag{2.8.11f}$$

and the Kretschmann scalar $\mathcal{K} = \mathrm{Riem}[\overline{g}]_{\alpha\beta\gamma\delta}\mathrm{Riem}[\overline{g}]^{\alpha\beta\gamma\delta}$ exhibits stable blow-up in the following sense:

$$\left\| a^{12} \mathcal{K} - \frac{5}{3} \cdot (8\pi)^2 (\Psi_{Bang} + C)^4 - W_{Bang} \right\|_{C^0(M)} \lesssim \varepsilon a^{2-c\varepsilon^{\frac{1}{8}}} \tag{2.8.11g}$$

REMARK 2.8.3 (The solution variables exhibit AVTD behaviour). The estimates (2.8.3a)-(2.8.3e) and (2.8.7) imply that the solution is asymptotically velocity term dominated (AVTD) in the sense that, toward the Big Bang singularity, they behave at leading order like solutions to the (formal) velocity term dominated equations. These arise by dropping any terms containing spatial derivatives in the decomposed Einstein system, i.e. in (2.2.15a), (2.2.15b), (2.2.17a) and (2.2.18).

PROOF. As argued at the end of Section 2.3.4, we can assume without loss of generality that our initial data is sufficiently regular. Hence, the local existence statement in Lemma 2.3.14 and the initial data requirements (2.8.1) ensure that there exists a local solution to the Einstein scalar-field system on $[t_1, t_0] \times M$ and that the bootstrap assumption (see Assumption 2.3.16) holds on $[t_1, t_0] \times M$ with $t_1 \in (0, t_0)$ and $\sigma = \varepsilon^{\frac{1}{16}}$. Let $\mathfrak{t} \in (0, t_0)$ be such that $(\mathfrak{t}, t_0] \times M$ is the maximal domain on which the solution variables exist and satisfy the bootstrap assumptions. For contradiction, we now assume that $\mathfrak{t} > 0$ were to hold.

Due to Corollary 2.7.6, there exist (summarizing all updates) constants $c_1, K_1 > 0$ such that, for any $t \in (\mathfrak{t}, t_0]$,

$$\mathcal{C}(t) \leq K_1 \varepsilon^{\frac{7}{4}} a(t)^{-c_1 \varepsilon^{\frac{1}{8}}} \tag{2.8.12}$$

If ε is small enough such that $K_1 \varepsilon^{\frac{1}{8}} < K_0$ and $c_1 \varepsilon^{\frac{1}{8}} < c_0 \sigma$ hold, this is a strict improvement of the bootstrap assumption. Furthermore, argued exactly as in the proof of [RS18b, Theorem 15.1], above improvement ensures none of the blow-up criteria of Lemma 2.3.14 are satisfied if $\mathfrak{t} > 0$ were to hold, essentially as a direct consequence of (2.8.12). Hence, the solution could be classically extended to a CMC hypersurface $\Sigma_{\mathfrak{t}}$ diffeomorphic to M while satisfying the improved estimates by continuity, and further to an interval $(\mathfrak{t}', t_0]$ for some $0 < \mathfrak{t}' < \mathfrak{t}$ on which the bootstrap assumptions must then be satisfied, also by continuity. This contradicts the maximality of $(\mathfrak{t}, t_0]$.

Thus, the rescaled solution variables induce a unique solution to the Einstein scalar-field system on $(0, t_0] \times M$ such that (2.8.12) is satisfied for any $t \in (0, t_0]$. The core estimate (2.8.2) follows since Corollaries 2.7.3 and 2.7.6 now hold on $(0, t_0]$.

From (2.8.2), the asymptotic behaviour in (2.8.3a)-(2.8.3e) and (2.8.7) is established as in [RS18b, Theorem 15.1], which we briefly outline: First, we note that (2.8.3a) follows directly from (2.8.2). For the remaining estimates, the arguments are similar, so consider for example $\partial_t \phi$: By the rescaled wave equation (2.2.32a) and (2.8.3a), we have that

$$\|\partial_t \Psi\|_{C^l_\gamma(\Sigma_t)} \lesssim \begin{cases} \varepsilon a^{1-c\varepsilon^{\frac{1}{8}}} & l \leq 14 \\ \varepsilon a^{-1-c\varepsilon^{\frac{1}{8}}} & l = 15 \end{cases}$$

Hence, for an arbitrary decreasing sequence $(t_m)_{m\in\mathbb{N}}$, on $(0, t_0]$ that converges to zero, we have

$$\|\Psi(t_{m_1}, \cdot) - \Psi(t_{m_2}, \cdot)\|_{C^l_\gamma(M)} \lesssim \begin{cases} \varepsilon a(t_{m_1})^{4-c\varepsilon^{\frac{1}{8}}} & l \leq 14 \\ \varepsilon a(t_{m_1})^{2-c\varepsilon^{\frac{1}{8}}} & l = 15 \end{cases}$$

for any $m_1, m_2 \in \mathbb{N}$, $m_1 < m_2$ by (2.2.6). This shows that $\Psi(t_{m_1}, \cdot)$ is a Cauchy sequence in $C^{15}_\gamma(M)$ and hence there exists a limit function $\Psi_{Bang} \in C^{15}_\gamma(M)$ that satisfies

$$\|\Psi(t, \cdot) - \Psi_{Bang}\|_{C^l_\gamma(M)} \lesssim \begin{cases} \varepsilon a(t)^{4-c\varepsilon^{\frac{1}{8}}} & l \leq 14 \\ \varepsilon a(t)^{2-c\varepsilon^{\frac{1}{8}}} & l = 15 \end{cases}$$

for any $t \in (0, t_0]$. Since $\Psi = a^3 n^{-1} \partial_t \phi - C$ holds by definition, (2.8.3c) now follows by examining the Taylor expansion of $n^{-1} - 1$ at 0 using (2.8.3a).

The identity (2.8.4a) follows directly from the CMC condition (2.2.10), the asymptotic behaviour (2.8.3e) of $a^3 k$ and the Friedman equation (2.2.3), while (2.8.4b) follows from the asymptotic limit of the Hamiltonian constraint (2.2.16a) with (2.2.3), (2.8.3a), (2.8.3c) and (2.8.3e) as well as (2.8.2) for lower order terms. The asymptotics in (2.8.7) follows exactly as

in [RS18b, Theorem 15.1], and (2.8.5a)-(2.8.6) are a direct result of the initial data assumptions and applying the respective asymptotic estimates to $t = t_0$.
For the first estimate in (2.8.8b), we apply the momentum constraint (2.2.29d) to get

$$|\nabla^J B|_G = a^{-4}|\nabla^J \mathcal{B}|_G = a^{-2}|\nabla^J \mathrm{curl}_G \Sigma|_G \lesssim a^{-2}|\nabla^{J+1}\Sigma|_G$$

and consequently, with Lemma 2.7.4 as well as (2.4.4g) and (2.8.2),

$$\begin{aligned}\|B\|_{C^{15}_\gamma(\Sigma_t)} &\lesssim a^{-c\sqrt{\varepsilon}}\|B\|_{C^{15}_G(\Sigma_t)} + \varepsilon a^{-2-c\sqrt{\varepsilon}} \cdot P_{15}(\|G-\gamma\|_{C^{15}_\gamma(\Sigma_t)})\\ &\lesssim a^{-2-c\sqrt{\varepsilon}}\|\Sigma\|_{C^{16}_G(\Sigma_t)} + \varepsilon a^{-2-c\sqrt{\varepsilon}} \cdot P_{15}(\|G-\gamma\|_{C^{15}_\gamma(\Sigma_t)})\\ &\lesssim \varepsilon a^{-2-c\varepsilon^{\frac{1}{8}}} .\end{aligned}$$

The remaining estimates in (2.8.8a) and (2.8.8b) are contained in (2.8.2). The results (2.8.9) and (2.8.10) follow as in the proofs of (15.6) and (15.7) in [RS18b, Theorem 15.1] from the asymptotic behaviour of the solution variables in (2.8.3a)-(2.8.3e) and (2.8.7). We briefly sketch the proof of (2.8.10): Consider a geodesic $\boldsymbol{\gamma}$ affinely parametrized by $\mathcal{A}$ as in the statement. The geodesic equations then lead to the following estimate for some suitable $\mathcal{K} > 0$:

$$|\mathcal{A}''| \leq \frac{\dot{a}}{a}|\mathcal{A}'| + \mathcal{K}\left[\frac{\dot{a}}{a}|N| + n^{-1}|\partial_t N| + n^{-1}|\nabla N|_g + n|\hat{k}|_g\right]|\mathcal{A}'| .$$

The leading term is hereby arises from the mean curvature condition. Arguing as with the elliptic estimates in Section 2.5, one can show that $|\partial_t N| \lesssim \varepsilon a^{-1-c\varepsilon^{\frac{1}{8}}}$. Thus, along with the other pointwise bounds on n, g and $\hat{k}$, one obtains

$$|\mathcal{A}''| \leq \frac{\dot{a}}{a}\left(1 + c\varepsilon\right)|\mathcal{A}'|$$

and consequently

$$|\mathcal{A}'(t)| \leq |\mathcal{A}'(t_0)| a(t)^{-1-c\varepsilon}$$

by the Gronwall lemma. (2.8.10) follows by integrating.

Turning to the blow-up behaviour of geometric invariants, observe (2.8.11a) is a direct consequence of (2.8.3e). Regarding (2.8.11c), we first compute using (2.2.19) and standard algebraic manipulations that

$$a^{12}W_{\alpha\beta\gamma\delta}W^{\alpha\beta\gamma\delta} = a^{12}\left(8|E|_g^2 + 8|B|_g^2\right) = 8|\mathcal{E}|_G^2 + 8|\mathcal{B}|_G^2 .$$

By the rescaled constraint equation (2.2.29c), we have

$$\mathcal{E}_{ij} = -\dot{a}a^2\Sigma_{ij} + (\Sigma \odot \Sigma)_{ij} - \left[\frac{8\pi}{3}\Psi^2 + \frac{16\pi}{3}C\Psi\right]G_{ij} + \mathcal{O}\left(\varepsilon a^{4-c\varepsilon^{\frac{1}{8}}}\right) ,$$

for $t \downarrow 0$. Further, by expanding (2.2.3) around $a = 0$, we have $\dot{a}a^2 = \sqrt{\frac{4\pi}{3}}C + \mathcal{O}\left(a^2\right)$. Since $\Sigma^\sharp$ and Ψ converge to footprint states $\hat{K}_{Bang} = K_{Bang} + \sqrt{\frac{4\pi C}{3}}\mathbb{I}$ and Ψ_{Bang} in $C^{15}_\gamma(M)$ respectively, this shows that $8|\mathcal{E}|^2_G$ converges to some $W_{\mathrm{Bang}} \in C^{15}_\gamma(M)$ that can be expressed as a fourth-order polynomial in $\hat{K}_{\text{Bang}}$ and Ψ_{Bang} and satsifies

$$\left\| |\mathcal{E}|_G^2 - \frac{1}{8} W_{Bang} \right\|_{C^0(M)} \lesssim \varepsilon a^{2-c\varepsilon^{\frac{1}{8}}}$$

as well as (2.8.11d). Due to (2.8.8b), the $|\mathcal{B}|_G^2$-term in the Weyl curvature scalar is negligible in comparison, and thus (2.8.11c) immediately follows.
Furthermore, one has

$$P_{\alpha\beta\gamma\delta}P^{\alpha\beta\gamma\delta} = 2\text{Ric}[\overline{g}]_{\alpha\beta}\text{Ric}[\overline{g}]^{\alpha\beta} - \frac{2}{9}R[\overline{g}]^2,$$

and (2.8.11b) is a direct consequence of (2.8.11f) and (2.8.11e), which follow once more with (2.8.3c) and (2.8.3a) as well as (2.8.2) for error terms. Finally, (2.8.11g) is obtained from (2.8.11b)-(2.8.11c). □

2.9. Future stability

The goal of this section is to show the following theorem:

THEOREM 2.9.1 (Future stability of Milne spacetime). *Let the rescaled initial data* $(\boldsymbol{g}, \boldsymbol{k}, \nabla\phi, \phi')$ *on* M *be sufficiently close to* $(\gamma, \frac{1}{3}\gamma, 0, 0)$ *in* $H^5 \times H^4 \times H^4 \times H^4$ *on some initial hypersurface* $\Sigma_{\tau=\tau_0}$ *(see Definition 2.9.4 and Assumption 2.9.7). Then, its maximal globally hyperbolic development* $(\overline{M}, \overline{g}, \phi)$ *within the Einstein scalar-field system in CMCSH gauge is foliated by the CMC Cauchy hypersurfaces* $(\Sigma_\tau)_{\tau\in[\tau_0,0)}$*, is future (causally) complete and exhibits the following asymptotic behaviour:*

$$(\boldsymbol{g}, \boldsymbol{k}, \phi', \nabla\phi)(\tau) \longrightarrow (\gamma, \frac{1}{3}\gamma, 0, 0) \ \textit{as}\ \tau \uparrow 0$$

Since the control of geometric perturbations uses the same arguments as in [AF20], the focus in this section will lie on dealing with the scalar field. The key idea is controlling decay of the scalar field using an indefinite corrective term on top of the canonical energy (see Definition 2.9.6).

2.9.1. Preliminaries.

2.9.1.1. *Notation, gauge and spatial reference geometry.* Within this section, we will decompose the Lorentzian metric as follows:

$$\overline{g} = -n^2dt^2 + g_{ab}(dx^a + X^a)(dx^b + X^bdt) \tag{2.9.1a}$$

We impose CMCSH gauge (see [AM04]) via

$$t = \tau,\ g^{ij}(\Gamma^a_{ij} - \hat{\Gamma}^a_{ij}) = 0\,, \tag{2.9.1b}$$

where $\hat{\Gamma}$ refers to the Christoffel symbols with regards to the spatial reference metric γ.
We extend the notation from the Big Bang stability analysis regarding foliations, derivatives, indices and schematic term notation to this setting (see Section 2.2.1). In particular, Σ_T and Σ_τ will refer to spatial hypersurfaces along which the logarithmic time T (see (2.9.2c)) and the mean curvature τ are constant (see (2.9.2c) on why these are interchangeable), and we will write for example $\Sigma_{T=0}$ when inserting a specific value to avoid potential ambiguity. We

use similar notation for scalar functions and tensors that depend on T or, respectively, τ.

For the extent of the future stability analysis, we have to introduce an additional condition for the spatial geometry beyond Definition 2.2.1:

DEFINITION 2.9.2 (Spectral condition for the Laplacian of the spatial reference manifold). Let $\mu_0(\gamma)$ to be the smallest positive eigenvalue of the Laplace operator $-\Delta_\gamma = -(\gamma^{-1})^{ab}\hat{\nabla}_a\hat{\nabla}_b$ acting on scalar functions, where (M,γ) is as in Definition 2.2.1. (M,γ) additionally is assumed to satisfy

$$\mu_0(\gamma) > \frac{1}{9}\,.$$

REMARK 2.9.3 (Manifolds that satisfy Definition 2.9.2). The available literature on spectra of $-\Delta_\gamma$ usually focuses on hyperbolic manifolds with sectional curvature $\kappa = -1$. Thus, one needs to check that μ_0 is strictly greater than 1 to verify the analogue of Definition 2.9.2 after rescaling.

Numerical works, e.g., [CS99; Ino01], provide evidence for over 250 compact hyperbolic 3-manifolds to satisfy this spectral bound, many of which are closed. In particular, both [CS99][14] and [Ino01] consider the smallest closed orientable hyperbolic 3-manifold, the Weeks space m003(-3,1), and compute that it falls under Definition 2.9.2 with $\mu_0 \approx 27,8$ in [CS99, Table IV] and $26 \lessapprox \mu_0 \lessapprox 27,8$ in [Ino01, Table 2]. Moreover, as demonstrated in [Ino01, Figure 6], many manifolds with small enough diameter d satisfy this condition. In fact, the analytical bound

$$\mu_0 \geq \max\left\{\frac{\pi^2}{2d^2} - \frac{1}{2}, \sqrt{\frac{\pi^4}{d^4} + \frac{1}{4}} - \frac{3}{2}, \frac{\pi^2}{d^2}e^{-d}\right\}$$

(see [CZ95, Theorem 1.1-1.2] with $L = 2$) implies that $\mu_0 > 10$ holds for Weeks space, which has diameter $d \approx 0,843$ (see [CS99, Table V]). Furthermore, [Ino01] finds no closed hyperbolic manifolds that violate this bound. More recently, the Selberg trace formula has been used in [LL22; LL24; BMP25] to compute candidates for eigenvalues of $-\Delta_\gamma$ and related operators, based on an optimization approach originating in [BS07]. In particular, the calculations visualized in [BMP25, Figure 3] demonstrate that one must have $\mu_0 \geq 27,6$ on the Weeks manifold.

We also note that it is conjectured that one at least has $\mu_0 \geq 1$ for any arithmetic hyperbolic 3-manifold (see [Ber03, Conjecture 2.3]). In fact, this is tied to the Ramanujan conjecture for automorphic forms. Finally, one can construct compact manifolds with boundary and with constant sectional curvature -1 where μ_0 becomes arbitrarily small, see [Cal94, Corollary 4.4].

2.9.1.2. *Rescaled variables and Einstein equations.* We will use the standard rescaling of the solution variables by τ:

[14]These results have to be interpreted cautiously since the numerical method cannot detect eigenvalues below 1.

Definition 2.9.4 (Rescaled variables for future stability).

$$\boldsymbol{g}_{ij} = \tau^2 g_{ij}\,, (\boldsymbol{g}^{-1})^{ij} = \tau^{-2} g^{ij}\,,\ \boldsymbol{\Sigma}_{ij} = \tau \hat{k}_{ij} \tag{2.9.2a}$$

$$\boldsymbol{n} = \tau^2 n\,,\ \hat{\boldsymbol{n}} = \frac{\boldsymbol{n}}{3} - 1\,,\ \boldsymbol{X}^a = \tau X^a \tag{2.9.2b}$$

Furthermore, we introduce the logarithmic time

$$T = -\log\left(\frac{\tau}{\tau_0}\right) \Leftrightarrow \tau = \tau_0 e^{-T} \tag{2.9.2c}$$

which satisfies $\partial_T = -\tau\partial_\tau$. Toward the future, τ increases from τ_0 to 0, and thus T increases from 0 to ∞. We additionally introduce:

$$\widetilde{\partial}_0 = \partial_T + \mathcal{L}_{\boldsymbol{X}} = -\tau(\partial_\tau - \mathcal{L}_X) \tag{2.9.2d}$$

$$\phi' = \boldsymbol{n}^{-1}\widetilde{\partial}_0\phi = n^{-1}(-\tau)^{-1}(\partial_\tau - \mathcal{L}_X)\phi \tag{2.9.2e}$$

Moreover, for any scalar function ζ, we denote by $\overline{\zeta}$ the mean integral with respect to $(\Sigma_T, \boldsymbol{g}_T)$.

For symmetric $(0,2)$-tensors h, we define the perturbed Lichnerowicz Laplacian

$$\mathcal{L}_{\boldsymbol{g},\gamma} h_{ab} = -\frac{1}{\mu_{\boldsymbol{g}}}\hat{\nabla}_k\left((\boldsymbol{g}^{-1})^{kl}\mu_{\boldsymbol{g}}\hat{\nabla}_l h_{ab}\right) - 2\mathrm{Riem}[\gamma]_{akbl}(\boldsymbol{g}^{-1})^{kk'}(\boldsymbol{g}^{-1})^{ll'}h_{k'l'}\,. \tag{2.9.3}$$

This operator satisfies

$$(\mathrm{Ric}[\boldsymbol{g}] - \mathrm{Ric}[\gamma])_{ij} = \frac{1}{2}\mathcal{L}_{\boldsymbol{g},\gamma}(\boldsymbol{g}-\gamma)_{ij} + J_{ij}, \quad \|J\|_{H^{l-1}} \lesssim \|\boldsymbol{g}-\gamma\|_{H^l}\,, \tag{2.9.4}$$

see [AM03, Pf. of Theorem 3.1]. Under our conditions for the reference geometry, [Krö15] implies that the smallest positive eigenvalue of $\mathcal{L}_{\gamma,\gamma}$, denoted by λ_0, satisfies $\lambda_0 \geq \frac{1}{9}$, and that $\mathcal{L}_{\gamma,\gamma}$ has trivial kernel. The spectral condition in Definition 2.9.2 is not necessary for this to hold true.

We now collect the $(3+1)$-decomposition of the Einstein scalar-field equations in CMCSH gauge with the help of [AF20, (2.13)-(2.18)]:

Lemma 2.9.5 (Rescaled CMCSH equations). *The rescaled CMCSH Einstein scalar-field equations take the following form: The constraint equations*

$$R[\boldsymbol{g}] - |\boldsymbol{\Sigma}|^2_{\boldsymbol{g}} - \frac{2}{3} = 8\pi\left[|\phi'|^2 + |\nabla\phi|^2_{\boldsymbol{g}}\right],\ \mathrm{div}_{\boldsymbol{g}}\boldsymbol{\Sigma}_b = 8\pi\tau^3\phi'\nabla_b\phi, \tag{2.9.5a}$$

the elliptic lapse and shift equations

$$\left(\Delta_{\boldsymbol{g}} - \frac{1}{3}\right)\boldsymbol{n} = \boldsymbol{n}\left(|\boldsymbol{\Sigma}|^2_{\boldsymbol{g}} + 4\pi\left[|\phi'|^2 + |\nabla\phi|^2_{\boldsymbol{g}}\right]\right) - 1, \tag{2.9.5b}$$

(2.9.5c)

$$\begin{aligned}\Delta_{\boldsymbol{g}}\boldsymbol{X}^a + (\boldsymbol{g}^{-1})^{ab}\mathrm{Ric}[\boldsymbol{g}]_{bm}\boldsymbol{X}^m =& 2(\boldsymbol{g}^{-1})^{am}(\boldsymbol{g}^{-1})^{bn}\nabla_b\boldsymbol{n}\cdot\boldsymbol{\Sigma}_{mn} - (\boldsymbol{g}^{-1})^{ab}\nabla_b\hat{\boldsymbol{n}} + 8\pi\boldsymbol{n}\tau^3\phi'\nabla_b\phi\\ &- 2(\boldsymbol{g}^{-1})^{bk}((\boldsymbol{g}^{-1})^{cl}\boldsymbol{n}\cdot\boldsymbol{\Sigma}_{bc} - \nabla_b\boldsymbol{X}^l)(\Gamma^a_{kl} - \hat{\Gamma}^a_{kl})\,,\end{aligned}$$

the geometric evolution equations

$$\widetilde{\partial}_0\boldsymbol{g}_{ab} = 2\boldsymbol{n}\boldsymbol{\Sigma}_{ab} + 2\hat{\boldsymbol{n}}\boldsymbol{g}_{ab}\ , \tag{2.9.5d}$$

$$\widetilde{\partial}_0(\boldsymbol{g}^{-1})^{ab} = -2\boldsymbol{n}(\boldsymbol{g}^{-1})^{ac}(\boldsymbol{g}^{-1})^{bd}\boldsymbol{\Sigma}_{cd} - 2\hat{\boldsymbol{n}}(\boldsymbol{g}^{-1})^{ab}\,, \tag{2.9.5e}$$

$$\begin{aligned}\widetilde{\partial}_0\boldsymbol{\Sigma}_{ab} = &-2\boldsymbol{\Sigma}_{ab} - \boldsymbol{n}\left(\mathrm{Ric}[\boldsymbol{g}]_{ab} + \frac{2}{9}\boldsymbol{g}_{ab}\right) + \nabla_a\nabla_b\boldsymbol{n} \\ &+ 2\boldsymbol{n}\cdot(\boldsymbol{g}^{-1})^{mn}\boldsymbol{\Sigma}_{am}\boldsymbol{\Sigma}_{bn} - \frac{1}{3}\hat{\boldsymbol{n}}\boldsymbol{g}_{ab} - \hat{\boldsymbol{n}}\Sigma_{ab} - 8\pi\boldsymbol{n}\nabla_a\phi\nabla_b\phi\end{aligned} \tag{2.9.5f}$$

and the wave equation

$$\widetilde{\partial}_0\phi' = \langle\nabla\boldsymbol{n},\nabla\phi\rangle_{\boldsymbol{g}} + \boldsymbol{n}\Delta_{\boldsymbol{g}}\phi + (1-\boldsymbol{n})\phi'\,. \tag{2.9.5g}$$

2.9.1.3. *Energies and data assumptions.* The proof will rely on the following corrected energy quantities:

Definition 2.9.6 (Energies for future stability).

$$\mathbb{E}^{(l)}_{SF} = (-1)^l\int_M\left[\phi'\Delta^l_{\boldsymbol{g}}\phi' - \phi\Delta^{l+1}_{\boldsymbol{g}}\phi\right]\mathrm{dvol}_{\boldsymbol{g}}, \qquad \mathcal{C}^{(l)}_{SF} = (-1)^l\int_M(\phi-\overline{\phi})\Delta^l_{\boldsymbol{g}}\phi'\,\mathrm{dvol}_{\boldsymbol{g}} \tag{2.9.6a}$$

$$E_{SF} = \sum_{m=0}^{4}\left(\mathbb{E}^{(m)}_{SF} + \frac{2}{3}\mathcal{C}^{(m)}_{SF}\right) \tag{2.9.6b}$$

$$\begin{aligned}E_{\mathrm{geom}} = \sum_{m=1}^{5}\Bigg(&\frac{9}{2}\int_M\langle\boldsymbol{g}-\gamma,\mathcal{L}^m_{\boldsymbol{g},\gamma}(\boldsymbol{g}-\gamma)\rangle_{\boldsymbol{g}}\mathrm{dvol}_{\boldsymbol{g}} + \frac{1}{2}\int_M\langle 6\boldsymbol{\Sigma},\mathcal{L}^{m-1}_{\boldsymbol{g},\gamma}(6\boldsymbol{\Sigma})\rangle_{\boldsymbol{g}}\mathrm{dvol}_{\boldsymbol{g}} \\ &+ c_E\int_M\langle 6\boldsymbol{\Sigma},\mathcal{L}^{m-1}_{\boldsymbol{g},\gamma}(\boldsymbol{g}-\gamma)\rangle_{\boldsymbol{g}}\mathrm{dvol}_{\boldsymbol{g}}\Bigg)\end{aligned} \tag{2.9.6c}$$

The constant c_E is given by

$$c_E = \begin{cases}1 & \lambda_0 > \frac{1}{9}\\ 9(\lambda_0-\delta') & \lambda_0 = \frac{1}{9}\,,\end{cases} \tag{2.9.7}$$

where $\delta' > 0$ is chosen to be small enough within the argument.

The Sobolev norms $H^l_{\boldsymbol{g}}$ and $C^l_{\boldsymbol{g}}$ are defined analogously to Definitions 2.3.3 and 2.3.4, with similar conventions on suppressing time dependence in notation whereever possible. Since norms with respect to $\boldsymbol{g}$ and γ are equivalent under the bootstrap assumption (and consequently throughout the entire argument), we will simply denote the norms by H^l and C^l throughout unless the specific metric is crucial.

Assumption 2.9.7 (Initial data assumption). *The initial data on the spatial hypersurface $\Sigma_{T=0}$ is assumed to be small in the following sense:*

$$\begin{aligned}&\|\boldsymbol{g}-\gamma\|_{C^3} + \|\boldsymbol{\Sigma}\|_{C^2} + \|\hat{\boldsymbol{n}}\|_{C^4} + \|\boldsymbol{X}\|_{C^4} + \|\phi'\|_{C^2} + \|\nabla\phi\|_{C^2} \\ &+\|\boldsymbol{g}-\gamma\|_{H^5} + \|\boldsymbol{\Sigma}\|_{H^4} + \|\hat{\boldsymbol{n}}\|_{H^6} + \|\boldsymbol{X}\|_{H^6} + \|\phi'\|_{H^4} + \|\nabla\phi\|_{H^4} \le \delta^2\end{aligned} \tag{2.9.8}$$

Remark 2.9.8 (Local well-posedness toward the future). Under the above initial data assumption, local well-posedness is satisfied by analogizing the arguments for local well-posedness in the vacuum setting (see [AM03, Theorem 3.1]) with the matter coupling added.

Since this only consists of adding another wave equation to the hyperbolic system, the argument is structurally unchanged given appropriate smallness assumptions on ϕ' and $\nabla\phi$ (where ϕ itself does not enter into the Einstein system). As before, we can without loss of generality assume that the initial is sufficiently regular to ensure that E_{geom}, $\mathfrak{E}^{(l)}_{SF}$ and $\mathcal{C}^{(l)}_{SF}$ initially are continuously differentiable (in time) for any $l \leq 4$.

ASSUMPTION 2.9.9 (Bootstrap assumption). *On the bootstrap interval $T \in [0, T_{Boot})$, we assume one has*

$$\begin{aligned}\|\boldsymbol{g}-\gamma\|_{C^3}+\|\boldsymbol{\Sigma}\|_{C^2}+\|\hat{\boldsymbol{n}}\|_{C^4}+\|\boldsymbol{X}\|_{C^4}+\|\phi'\|_{C^2}+\|\nabla\phi\|_{C^2}\\ +\|\boldsymbol{g}-\gamma\|_{H^5}+\|\boldsymbol{\Sigma}\|_{H^4}+\|\hat{\boldsymbol{n}}\|_{H^6}+\|\boldsymbol{X}\|_{H^6}+\|\phi'\|_{H^4}+\|\nabla\phi\|_{H^4}\leq\delta e^{-\frac{T}{2}}\,.\end{aligned}\tag{2.9.9}$$

We only choose not to use "$\lesssim$"-notation in the above assumptions for notational convenience in some technical computations. As before, δ can be chosen to have been sufficiently small for the following estimates to hold and for the decay estimates we derive from the bootstrap assumptions to be strict improvements. Moreover, note that (2.9.9) is satisfied since all of the norms are continuous in time (see Remark 2.9.8)

Before moving on to the energy estimates, we quickly collect the following immediate consequence of the bootstrap assumptions:

LEMMA 2.9.10 (Sobolev estimate for the curvature). *The following estimate holds for any $l \in \mathbb{N}_0$:*

$$\left\|\mathrm{Ric}[\boldsymbol{g}]+\frac{2}{9}\boldsymbol{g}\right\|_{H^l}\lesssim\|\boldsymbol{g}-\gamma\|_{H^{l+2}}+\|\boldsymbol{g}-\gamma\|^2_{H^{l+1}}\tag{2.9.10a}$$

Under the bootstrap assumptions, this implies

$$\left\|\mathrm{Ric}[\boldsymbol{g}]+\frac{2}{9}\boldsymbol{g}\right\|_{C^1}+\left\|\mathrm{Ric}[\boldsymbol{g}]+\frac{2}{9}\boldsymbol{g}\right\|_{H^3}\lesssim\delta e^{-\frac{T}{2}}\tag{2.9.10b}$$

PROOF. By (2.9.4), one has

$$\left\|\mathrm{Ric}[\boldsymbol{g}]+\frac{2}{9}\boldsymbol{g}\right\|_{H^l}\leq\frac{1}{2}\|\mathcal{L}_{\boldsymbol{g},\gamma}(\boldsymbol{g}-\gamma)\|_{H^l}+K\|\boldsymbol{g}-\gamma\|^2_{H^{l+1}}$$

for some suitably large $K > 0$ along with the fact that $\mathcal{L}_{\boldsymbol{g},\gamma}$ is elliptic. This implies the first inequality, while the latter follows from directly from the bootstrap assumption (2.9.9) and by applying the standard Sobolev embedding. □

2.9.2. Elliptic estimates. We briefly collect the elliptic estimates for lapse and shift:

LEMMA 2.9.11 (Elliptic estimates for lapse and shift). *Let $l \in \{3, 4, 5, 6\}$. Then, one has $\boldsymbol{n} \in (0, 3)$ (thus $\hat{\boldsymbol{n}} \in (-1, 0)$) and the following estimates hold:*

$$\|\hat{\boldsymbol{n}}\|_{H^l}\lesssim\delta e^{-\frac{T}{2}}\|\boldsymbol{\Sigma}\|_{H^{l-2}}+\delta^2e^{-T}\|\boldsymbol{g}-\gamma\|_{H^{l-2}}+\delta e^{-\frac{T}{2}}\left[\|\phi'\|_{H^{l-2}}+\|\nabla\phi\|_{H^{l-2}}\right]\tag{2.9.11a}$$

$$\|\boldsymbol{X}\|_{H^l}\lesssim\delta e^{-\frac{T}{2}}\|\boldsymbol{\Sigma}\|_{H^{l-2}}+\delta e^{-\frac{T}{2}}\|\boldsymbol{g}-\gamma\|_{H^{l-1}}+\delta e^{-\frac{T}{2}}\left[\|\phi'\|_{H^{l-2}}+\|\nabla\phi\|_{H^{l-2}}\right]\tag{2.9.11b}$$

PROOF. The pointwise bounds on $\boldsymbol{n}$ follow via (2.9.5b) and the maximum principle as in Lemma 2.4.1. For the remaining estimates, applying elliptic regularity theory to (2.9.5b) and (2.9.5c) implies:

$$\begin{aligned}
\|\hat{\boldsymbol{n}}\|_{H^l} \lesssim{} & \|\boldsymbol{\Sigma}\|_{C^{\lfloor \frac{l-2}{2}\rfloor}}\|\boldsymbol{\Sigma}\|_{H^{l-2}} + \|\nabla\phi\|^2_{C^2}\|\boldsymbol{g}-\gamma\|_{H^{l-2}} \\
&+ \left[\|\nabla\phi\|_{C^2}\left(1+\|\boldsymbol{g}-\gamma\|_{C^2}\right)+\|\phi'\|_{C^2}\right]\left[\|\phi'\|_{H^{l-2}}+\|\nabla\phi\|_{H^{l-2}}\right] \\
\|\boldsymbol{X}\|_{H^l} \lesssim{} & \|\boldsymbol{\Sigma}\|_{C^{\lfloor \frac{l-2}{2}\rfloor}}\|\boldsymbol{\Sigma}\|_{H^{l-2}} + \|\boldsymbol{g}-\gamma\|^2_{H^{l-1}} + \|\nabla\phi\|_{C^1}\|\boldsymbol{g}-\gamma\|_{H^{l-3}} \\
&+ \left[\|\nabla\phi\|^2_{C^2}\left(1+\|\boldsymbol{g}-\gamma\|_{C^2}\right)+\|\phi'\|_{C^2}\right]\left[1+\|\hat{\boldsymbol{n}}\|_{C^2}\right]\left[\|\phi'\|_{H^{l-2}}+\|\nabla\phi\|_{H^{l-2}}\right]
\end{aligned}$$

The statement then follows by inserting (2.9.9). □

2.9.3. Scalar field energy estimates.

2.9.3.1. *Near-coercivity of E_{SF}.* We will be able to prove a decay estimate via a Gronwall argument only for the corrected energy E_{SF} . Hence, we first need to verify that this energy controls the solution norms, for which we first show that it controls the "canonical" scalar field energies:

LEMMA 2.9.12 (Positivity of corrected scalar field energies). *Let*

$$Q = \frac{\sqrt{1+9q}-1}{\sqrt{1+9q}} \quad \textit{with } q = \frac{1}{2}\left(\mu_0(\gamma)-\frac{1}{9}\right).$$

Then, for any $l \in \{0,1,2,3,4\}$ and $\delta > 0$ small enough, one has

$$Q\mathbb{E}^{(l)}_{SF} \le \mathbb{E}^{(l)}_{SF} + \frac{2}{3}\mathcal{C}^{(l)}_{SF}, \quad \textit{hence} \quad Q\sum_{m=0}^{4}\mathbb{E}^{(l)}_{SF} \le E_{SF} \tag{2.9.12}$$

PROOF. We denote the smallest positive eigenvalue of $-\Delta_{\boldsymbol{g}}$ acting on scalar functions on Σ_T by $\mu_0(\boldsymbol{g}_T)$. By the bootstrap assumption (2.9.9) and since μ_0 depends continuously on the metric, we obtain the following for small enough $\delta > 0$:

$$\mu_0(\boldsymbol{g}_T) \ge \mu_0(\gamma) - \frac{1}{2}\left(\mu_0(\gamma)-\frac{1}{9}\right) \ge \frac{1}{9}+q$$

By the Poincaré inequality applied on $(\Sigma_T, \boldsymbol{g}_T)$ (see [CBM01, p.1037]), the above spectral bound implies the following for any $\zeta \in H^1(\Sigma_T)$:

$$\|\zeta-\overline{\zeta}\|^2_{L^2_{\boldsymbol{g}}(\Sigma_T)} \le \mu_0(\boldsymbol{g}_T)^{-1}\|\nabla\zeta\|^2_{L^2_{\boldsymbol{g}}(\Sigma_T)} \le \left(\frac{1}{9}+q\right)^{-1}\|\nabla\zeta\|^2_{L^2_{\boldsymbol{g}}(\Sigma_T)} \tag{2.9.13}$$

For $l=0$, this means

$$\begin{aligned}
\mathbb{E}^{(0)}_{SF} + \frac{2}{3}\mathcal{C}^{(0)}_{SF} &\ge \|\phi'\|^2_{L^2_{\boldsymbol{g}}} + \|\nabla\phi\|^2_{L^2_{\boldsymbol{g}}} - \frac{2}{3}\|\phi-\overline{\phi}\|_{L^2_{\boldsymbol{g}}}\|\phi'\|_{L^2_{\boldsymbol{g}}} \\
&\ge \|\phi'\|^2_{L^2_{\boldsymbol{g}}} + \|\nabla\phi\|^2_{L^2_{\boldsymbol{g}}} - 2\left(1+9q\right)^{-\frac{1}{2}}\|\nabla\phi\|_{L^2_{\boldsymbol{g}}}\|\phi'\|_{L^2_{\boldsymbol{g}}} \\
&\ge \frac{\sqrt{1+9q}-1}{\sqrt{1+9q}}\mathbb{E}^{(0)}_{SF}.
\end{aligned}$$

For $l=1$, notice that we can rewrite $\mathcal{C}^{(1)}_{SF}$ as

$$\mathcal{C}^{(1)}_{SF} = \int_M \langle\nabla\phi,\nabla\phi'\rangle_{\boldsymbol{g}}\,\mathrm{dvol}_{\boldsymbol{g}} = \int_M \left\langle\nabla\phi,\nabla\left(\phi'-\overline{\phi'}\right)\right\rangle_{\boldsymbol{g}}\mathrm{dvol}_{\boldsymbol{g}} = -\int_M \left(\phi'-\overline{\phi'}\right)\Delta_{\boldsymbol{g}}\phi\,\mathrm{dvol}_{\boldsymbol{g}}.$$

Hence, applying (2.9.13) to $\zeta=\phi'$ yields

$$\mathbb{E}^{(1)}_{SF}+\frac{2}{3}\mathcal{C}^{(1)}_{SF}\geq\mathbb{E}^{(1)}_{SF}-2\,(1+9q)^{-\frac{1}{2}}\,\|\nabla\phi'\|_{L^2_{\boldsymbol{g}}}\|\Delta_{\boldsymbol{g}}\phi\|_{L^2_{\boldsymbol{g}}}\geq\frac{\sqrt{1+9q}-1}{\sqrt{1+9q}}\mathbb{E}^{(1)}_{SF}\,.$$

For $l=2,3,4$, notice $\overline{\Delta_{\boldsymbol{g}}\phi}=\overline{\Delta_{\boldsymbol{g}}\phi'}=\overline{\Delta^2_{\boldsymbol{g}}\phi}=0$ holds due to the divergence theorem, hence the argument proceeds as in $l=0,1$. □

LEMMA 2.9.13 (Near-coercivity of corrected scalar field energy). *For any differentiable function ζ and $k\in\{1,2\}$, one has the following under the bootstrap assumptions:*

$$\int_M|\nabla^2\zeta|^2_{\boldsymbol{g}}\,\mathrm{dvol}_{\boldsymbol{g}}\lesssim\int_M|\Delta_{\boldsymbol{g}}\zeta|^2_{\boldsymbol{g}}+|\nabla\zeta|^2_{\boldsymbol{g}}\,\mathrm{dvol}_{\boldsymbol{g}}$$

$$\|\zeta\|^2_{\dot{H}^{2k}}\lesssim\|\Delta^k_{\boldsymbol{g}}\zeta\|^2_{L^2}+\left(\|\zeta\|^2_{\dot{H}^{2k-1}}+\|\zeta\|^2_{\dot{H}^{2k-2}}\right)+\|\nabla\zeta\|^2_{C^1}\left\|\mathrm{Ric}[\boldsymbol{g}]+\frac{2}{9}\boldsymbol{g}\right\|^2_{H^{2k-2}}$$

$$\|\nabla\zeta\|^2_{\dot{H}^{2k}}\lesssim\|\nabla\Delta^k_{\boldsymbol{g}}\zeta\|^2_{L^2}+\left(\|\nabla\zeta\|^2_{\dot{H}^{2k-1}}+\|\nabla\zeta\|^2_{\dot{H}^{2k-2}}\right)+\|\nabla\zeta\|^2_{C^2}\left\|\mathrm{Ric}[\boldsymbol{g}]+\frac{2}{9}\boldsymbol{g}\right\|^2_{H^{2k-2}}$$

Consequently, the following estimate holds:

$$\|\phi'\|^2_{H^4}+\|\nabla\phi\|^2_{H^4}\lesssim E^{(4)}_{SF}+\left(\|\phi'\|^2_{C^2}+\|\nabla\phi\|^2_{C^2}\right)\left\|\mathrm{Ric}[\boldsymbol{g}]+\frac{2}{9}\boldsymbol{g}\right\|^2_{H^2}\tag{2.9.14}$$

PROOF. The inequalities for ζ follows from the same arguments as Lemma 2.4.5, except that we have $\|\mathrm{Ric}[\boldsymbol{g}]\|_{C^1_{\boldsymbol{g}}}\lesssim 1+\delta\lesssim 1$ by Lemma 2.9.10. The final estimate then follows by applying these estimates to $\zeta=\phi'$ and $\zeta=\phi$ and applying Lemma 2.9.12. □

2.9.3.2. *Preparations for energy estimates.* Before proving the energy estimate, we need to establish two technical lemmas: First, we collect a formula to differentiate integrals, and then some estimates needed to deal with the mean value of ϕ in the base level correction term.

LEMMA 2.9.14 (Differentiation of integrals, future stability version). *For any diffentiable function ζ, one has*

$$\partial_T\int_M\zeta\mathrm{dvol}_{\boldsymbol{g}}=\int_M\left(\widetilde{\partial}_0\zeta+3\hat{\boldsymbol{n}}\zeta\right)\,\mathrm{dvol}_{\boldsymbol{g}}\,.\tag{2.9.15}$$

PROOF. As in the proof of (2.4.12), we obtain

$$\begin{aligned}\partial_T\int_M\zeta\mathrm{dvol}_{\boldsymbol{g}}&=\int_M\partial_T\zeta+\frac{\partial_T\mu_{\boldsymbol{g}}}{\mu_{\boldsymbol{g}}}\zeta\,\mathrm{dvol}_{\boldsymbol{g}}\\&=\int_M\partial_T\zeta+3\hat{\boldsymbol{n}}\zeta-\frac{1}{2}(\boldsymbol{g}^{-1})^{ab}\mathcal{L}_{\boldsymbol{X}}\boldsymbol{g}_{ab}\zeta\,\mathrm{dvol}_{\boldsymbol{g}}\\&=\int_M\partial_T\zeta+3\hat{\boldsymbol{n}}\zeta-\mathrm{div}_{\boldsymbol{g}}\boldsymbol{X}\cdot\zeta\,\mathrm{dvol}_{\boldsymbol{g}}\end{aligned}$$

The statement now follows by applying Stokes' theorem to the final term and rearranging. □

LEMMA 2.9.15 (Decay estimate for the integrated time derivative). *For any $T>0$, we have*

$$\int_{\Sigma_T}\phi'\,\mathrm{dvol}_{\boldsymbol{g}}=\left(\int_{\Sigma_{T=0}}\phi'\,\mathrm{dvol}_{\boldsymbol{g}}\right)\cdot e^{-2T}\,.\tag{2.9.16}$$

Consequently, the bootstrap assumptions imply

$$\left|\int_{\Sigma_T} \widetilde{\partial}_0\overline{\phi} \cdot \phi' \,\mathrm{dvol}_{\boldsymbol{g}}\right| \lesssim \delta^3 e^{-\frac{5}{2}T} \tag{2.9.17}$$

for $\delta > 0$ small enough.

Proof. Using that the integral of $\mathrm{div}_{\boldsymbol{g}}(\boldsymbol{n}\nabla\phi)$ vanishes, we compute:

$$\begin{aligned}\partial_T\left(\int_M \phi' \,\mathrm{dvol}_{\boldsymbol{g}}\right) &= \int_M \left(\widetilde{\partial}_0\phi' + 3\hat{\boldsymbol{n}}\phi'\right) \mathrm{dvol}_{\boldsymbol{g}} = \int_M \left[(1-\boldsymbol{n})\phi' + (\boldsymbol{n}-3)\phi'\right] \mathrm{dvol}_{\boldsymbol{g}}\\ &= -2\left(\int_M \phi' \,\mathrm{dvol}_{\boldsymbol{g}}\right)\end{aligned}$$

Hence, (2.9.16) precisely describes the solution to this ODE ($f' = -2f$) with prescribed initial value at $T = 0$, and the initial data assumption (2.9.8) implies

$$\left|\int_M \phi' \,\mathrm{dvol}_{\boldsymbol{g}}\right| \leq \|\phi'\|_{C^0(\Sigma_{T=0})} \mathrm{dvol}_{\boldsymbol{g}}(\Sigma_{T=0}) e^{-2T} \lesssim \delta^2 e^{-2T}.$$

Furthermore, one has by (2.9.15) that

$$\partial_T \mathrm{dvol}_{\boldsymbol{g}}\left(\Sigma_T\right) = \int_{\Sigma_T} 3\hat{\boldsymbol{n}} \mathrm{dvol}_{\boldsymbol{g}} \tag{2.9.18}$$

Consequently, one has

$$\begin{aligned}\widetilde{\partial}_0\overline{\phi} &= \left[-\frac{\partial_T \mathrm{dvol}_{\boldsymbol{g}}\left(\Sigma_T\right)}{\mathrm{dvol}_{\boldsymbol{g}}\left(\Sigma_T\right)} \cdot \overline{\phi} + \frac{1}{\mathrm{dvol}_{\boldsymbol{g}}\left(\Sigma_T\right)} \int_M \left(\widetilde{\partial}_0\phi + 3\hat{\boldsymbol{n}}\phi\right) \mathrm{dvol}_{\boldsymbol{g}}\right]\\ &= \int_M \left(\boldsymbol{n}\phi' + 3\hat{\boldsymbol{n}}(\phi - \overline{\phi})\right) \mathrm{dvol}_{\boldsymbol{g}}.\end{aligned}$$

By applying $|\boldsymbol{n}| < 3$, the adapted Poincare inequality (2.9.13) and the bootstrap assumptions (2.9.9), this implies

$$\left|\widetilde{\partial}_0\overline{\phi}\right| \lesssim \|\phi'\|_{L^2} + \|\nabla\phi\|_{L^2}\|\hat{\boldsymbol{n}}\|_{L^2_{\boldsymbol{g}}} \lesssim \delta e^{-\frac{T}{2}}.$$

The bound (2.9.17) now follows by combining this with (2.9.16). □

2.9.3.3. *Energy estimates.* Now, we can collect the following estimates for the corrected scalar field energies:

Lemma 2.9.16 (Base level estimate for the corrected scalar field energy). *Under the bootstrap assumptions, the following estimate holds for some $K > 0$:*

$$\partial_T E^{(0)}_{SF} \leq -2E^{(0)}_{SF} + K\delta e^{-\frac{T}{2}}\sqrt{E^{(0)}_{SF}}\left(\sqrt{E^{(0)}_{SF}} + \|\boldsymbol{\Sigma}\|_{L^2} + \|\boldsymbol{g} - \gamma\|_{L^2}\right) + K\delta^3 e^{-\frac{5T}{2}} \tag{2.9.19}$$

Proof. We compute, using $[\widetilde{\partial}_0, \nabla]\phi = 0$, $\widetilde{\partial}_0\phi = \boldsymbol{n}\phi'$ and the rescaled wave equation (2.9.5g):

$$\begin{aligned}\partial_T \mathbb{E}^{(0)}_{SF} &= \int_M \left[2\widetilde{\partial}_0\phi' \cdot \phi' + 2\left\langle\nabla\phi, \nabla\widetilde{\partial}_0\phi\right\rangle_{\boldsymbol{g}} + \left(\widetilde{\partial}_0\boldsymbol{g}^{-1}\right)^{ab}\nabla_a\phi\nabla_b\phi + 3\hat{\boldsymbol{n}}\left(|\phi'|^2 + |\nabla\phi|^2_{\boldsymbol{g}}\right)\right] \mathrm{dvol}_{\boldsymbol{g}}\\ &= \int_M \Big[2\left(\langle\nabla\boldsymbol{n}, \nabla\phi\rangle_{\boldsymbol{g}} + \boldsymbol{n}\Delta_{\boldsymbol{g}}\phi + (1-\boldsymbol{n})\phi'\right)\phi' - 2(\boldsymbol{n}\phi') \cdot \Delta_{\boldsymbol{g}}\phi\end{aligned}$$

$$- 2\boldsymbol{n}\langle \boldsymbol{\Sigma}, \nabla\phi\nabla\phi\rangle_{\boldsymbol{g}} + 3\hat{\boldsymbol{n}}|\phi'|^2 + \hat{\boldsymbol{n}}|\nabla\phi|^2_{\boldsymbol{g}}\Big] \,\mathrm{dvol}_{\boldsymbol{g}}$$

With $2(1-\boldsymbol{n}) = -4 - 6\hat{\boldsymbol{n}}$, integration by parts and using the bootstrap assumption (2.9.9) on C-norms, we get for some constant $K>0$ that we update from line to line:

$$\begin{aligned}
\partial_T \mathbb{E}^{(0)}_{SF} \leq& \int_M -4|\phi'|^2_{\boldsymbol{g}} \,\mathrm{dvol}_{\boldsymbol{g}} + K\left[\|\nabla\phi\|_{C^0}\|\hat{\boldsymbol{n}}\|_{H^1}\sqrt{\mathbb{E}^{(0)}_{SF}} + \left(\|\boldsymbol{\Sigma}\|_{C^0} + \|\hat{\boldsymbol{n}}\|_{C^0}\right)\mathbb{E}^{(0)}_{SF}\right] \\
\leq& \int_M -4|\phi'|^2 \,\mathrm{dvol}_{\boldsymbol{g}} + K\delta e^{-\frac{T}{2}}\left(\sqrt{\mathbb{E}^{(0)}_{SF}}\|\hat{\boldsymbol{n}}\|_{H^1} + \mathbb{E}^{(0)}_{SF}\right)
\end{aligned}$$

Similarly and using the same evolution equations, we obtain:

$$\begin{aligned}
\partial_T \mathcal{C}^{(0)}_{SF} =& \int_M \left[\widetilde{\partial}_0\phi\cdot\phi' - \widetilde{\partial}_0\overline{\phi}\cdot\phi' + (\phi-\overline{\phi})\widetilde{\partial}_0\phi' + 3\hat{\boldsymbol{n}}(\phi-\overline{\phi})\phi'\right] \mathrm{dvol}_{\boldsymbol{g}} \\
=& \int_M \left[3|\phi'|^2 + 3\hat{\boldsymbol{n}}|\phi'|^2 + \left(\phi-\overline{\phi}\right)\cdot \mathrm{div}_{\boldsymbol{g}}\left(\boldsymbol{n}\nabla\phi\right) - 2\left(\phi-\overline{\phi}\right)\phi' - \widetilde{\partial}_0\overline{\phi}\cdot\phi'\right] \mathrm{dvol}_{\boldsymbol{g}} \\
\leq& - 2\mathcal{C}^{(0)}_{SF} + \int_M 3\left[|\phi'|^2 - |\nabla\phi|^2_{\boldsymbol{g}}\right] \mathrm{dvol}_{\boldsymbol{g}} + 3\|\hat{\boldsymbol{n}}\|_{C^0}\mathbb{E}^{(0)}_{SF} - \int_M \left(\widetilde{\partial}_0\overline{\phi}\cdot\phi'\right) \mathrm{dvol}_{\boldsymbol{g}}
\end{aligned}$$

Applying Lemma 2.9.15 to the last term, we get:

$$\partial_T \mathcal{C}^{(0)}_{SF} \leq - 2\mathcal{C}^{(0)}_{SF} + K\delta e^{-\frac{T}{2}}\mathbb{E}^{(0)}_{SF} + K\delta^3 e^{-\frac{5}{2}T}$$

Combining these two estimates, inserting (2.9.11a) and (2.9.12) as well as updating K yields:

$$\begin{aligned}
\partial_T E^{(0)}_{SF} =& \partial_T \mathbb{E}^{(0)}_{SF} + \frac{2}{3}\partial_T \mathcal{C}^{(0)}_{SF} \\
=& \int_M \left[\left(-4 + \frac{2}{3}\cdot 3\right)|\phi'|^2 - \frac{2}{3}\cdot 3|\nabla\phi|^2_{\boldsymbol{g}}\right] \mathrm{dvol}_{\boldsymbol{g}} - 2\cdot\frac{2}{3}\mathcal{C}^{(0)}_{SF} \\
&+ K\delta e^{-\frac{T}{2}}\left(\sqrt{\mathbb{E}^{(0)}_{SF}}\|\hat{\boldsymbol{n}}\|_{H^1} + \mathbb{E}^{(0)}_{SF}\right) + K\delta^3 e^{-\frac{5}{2}T} \\
\leq& - 2E^{(0)}_{SF} + K\delta e^{-\frac{T}{2}}\sqrt{E^{(0)}_{SF}}\left(\|\Sigma\|_{L^2} + \|\boldsymbol{g}-\gamma\|_{L^2} + \sqrt{E^{(0)}_{SF}}\right) + K\delta^3 e^{-\frac{5}{2}T}
\end{aligned}$$

□

Lemma 2.9.17 (Higher order estimates for the corrected scalar field energy). *For any $l \in \{1,\dots,4\}$, the following estimate holds:*

$$\begin{aligned}
\partial_T\left(\mathbb{E}^{(l)}_{SF} + \frac{2}{3}\mathcal{C}^{(l)}_{SF}\right) \leq& - 2\left(\mathbb{E}^{(l)}_{SF} + \frac{2}{3}\mathcal{C}^{(l)}_{SF}\right) + K\delta e^{-\frac{T}{2}}\left(\sum_{m=0}^{l}\sqrt{\mathbb{E}^{(m)}_{SF}}\right)\cdot \\
&\cdot\left(\|\phi'\|_{H^l} + \|\nabla\phi\|_{H^l} + \|\Sigma\|_{H^l} + \|\boldsymbol{g}-\gamma\|_{H^l}\right)
\end{aligned}$$

Proof. Starting with $l = 2k$, $k\in\{1,2\}$, one calculates:

$$\partial_T \mathbb{E}^{(2k)}_{SF} = \int_M \Big[2\Delta^k_{\boldsymbol{g}}\widetilde{\partial}_0\phi'\cdot\Delta^k_{\boldsymbol{g}}\phi' + 2\langle\nabla\Delta^k_{\boldsymbol{g}}\phi, \nabla\Delta^k_{\boldsymbol{g}}\widetilde{\partial}_0\phi\rangle_{\boldsymbol{g}} \tag{2.9.20a}$$

$$+ (\widetilde{\partial}_0\boldsymbol{g}^{-1})^{ab}\cdot\nabla_a\Delta^k_{\boldsymbol{g}}\phi\cdot\nabla_b\Delta^k_{\boldsymbol{g}}\phi + 3\hat{\boldsymbol{n}}\left(|\Delta^k\phi'|^2_{\boldsymbol{g}} + |\nabla\Delta^k\phi|^2_{\boldsymbol{g}}\right) \tag{2.9.20b}$$

$$+ 2[\widetilde{\partial}_0, \Delta^k_{\boldsymbol{g}}]\phi'\cdot\Delta^k_{\boldsymbol{g}}\phi' + 2\left\langle[\widetilde{\partial}_0, \nabla\Delta^k_{\boldsymbol{g}}]\phi, \nabla\Delta^k_{\boldsymbol{g}}\phi\right\rangle_{\boldsymbol{g}}\Big] \mathrm{dvol}_{\boldsymbol{g}} \tag{2.9.20c}$$

We insert the rescaled wave equation (2.9.5g) and $\widetilde{\partial}_0\phi = \boldsymbol{n}\phi'$ into the right hand side of (2.9.20a) and obtain for some constant $K>0$:

$$\begin{aligned}
(2.9.20\mathrm{a}) \leq & \int_M \Big[-4|\Delta_{\boldsymbol{g}}^k\phi'|^2 - 6\hat{\boldsymbol{n}}|\Delta_{\boldsymbol{g}}^k\phi'|^2 + \boldsymbol{n}\Delta_{\boldsymbol{g}}^{k+1}\phi\cdot\Delta_{\boldsymbol{g}}^k\phi'\Big]\,\mathrm{dvol}_{\boldsymbol{g}} \\
& + K\|\Delta^k\phi'\|_{L^2}\left(\|\hat{\boldsymbol{n}}\|_{H^{2k+1}}\|\nabla\phi\|_{C^0} + \|\nabla\phi\|_{H^{2k}}\|\hat{\boldsymbol{n}}\|_{C^{2k}}\right) \\
& + \int_M \left[-\boldsymbol{n}\Delta_{\boldsymbol{g}}^k\phi'\cdot\Delta_{\boldsymbol{g}}^{k+1}\nabla\phi - 3\langle\nabla\hat{\boldsymbol{n}},\nabla\Delta_{\boldsymbol{g}}^k\phi\rangle_{\boldsymbol{g}}\cdot\Delta_{\boldsymbol{g}}^k\phi'\right]\mathrm{dvol}_{\boldsymbol{g}} \\
& + K\|\nabla\Delta_{\boldsymbol{g}}^k\phi\|_{L^2}\left(\|\hat{\boldsymbol{n}}\|_{H^{2k+1}}\|\phi'\|_{C^0} + \|\hat{\boldsymbol{n}}\|_{C^{2k}}\|\phi'\|_{H^{2k}}\right) \\
\leq & \int_M -4|\Delta_{\boldsymbol{g}}^k\phi'|^2\,\mathrm{dvol}_{\boldsymbol{g}} \\
& + K\sqrt{\mathbb{E}_{SF}^{(2k)}}\cdot\Big[\left(\|\nabla\phi\|_{C^0}+\|\phi'\|_{C^0}\right)\cdot\|\hat{\boldsymbol{n}}\|_{H^{2k+1}} + \left(\|\nabla\phi\|_{H^{2k}}+\|\phi'\|_{H^{2k}}\right)\cdot\|\hat{\boldsymbol{n}}\|_{C^{2k}}\Big]
\end{aligned}$$

For (2.9.20b), we use (2.9.5e) and the bootstrap assumption (2.9.9) to bound it by $K\delta e^{-\frac{T}{2}}\mathbb{E}_{SF}^{(2k)}$. Regarding (2.9.20c), the commutator formulas (2.B.1a)-(2.B.1b) imply

$$\begin{aligned}
\|[\widetilde{\partial}_0,\Delta_{\boldsymbol{g}}^k]\phi'\|_{L^2} &\lesssim \|\boldsymbol{n}\|_{C^{2k-1}}\left(\|\phi'\|_{C^1}\|\boldsymbol{\Sigma}\|_{\dot{H}^{2k-1}} + \|\boldsymbol{\Sigma}\|_{C^{2k-2}}\|\phi'\|_{H^{2k}}\right) + \|\hat{\boldsymbol{n}}\|_{C^{2k-1}}\|\phi'\|_{H^{2k}} \\
\|[\widetilde{\partial}_0,\nabla\Delta_{\boldsymbol{g}}^k]\phi\|_{L^2} &\lesssim \|\boldsymbol{n}\|_{C^{2k}}\left(\|\nabla\phi\|_{C^1}\|\boldsymbol{\Sigma}\|_{H^{2k}} + \|\boldsymbol{\Sigma}\|_{C^{2k-2}}\|\nabla\phi\|_{H^{2k}}\right) + \|\hat{\boldsymbol{n}}\|_{C^{2k}}\|\nabla\phi\|_{H^{2k}}
\end{aligned}$$

Summarizing, inserting the C-norm bounds from the bootstrap assumption (2.9.9) and updating K, this implies

$$\begin{aligned}
\partial_T\mathbb{E}_{SF}^{(2k)} \leq & \int_M -4|\Delta_{\boldsymbol{g}}^k\phi'|^2\,\mathrm{dvol}_{\boldsymbol{g}} + K\delta e^{-\frac{T}{2}}\mathbb{E}_{SF}^{(2k)} \\
& + K\delta e^{-\frac{T}{2}}\sqrt{\mathbb{E}_{SF}^{(2k)}}\left(\|\phi'\|_{H^{2k}}+\|\nabla\phi\|_{H^{2k}}\right) \\
& + K\delta e^{-\frac{T}{2}}\sqrt{\mathbb{E}_{SF}^{(2k)}}\left(\|\hat{\boldsymbol{n}}\|_{H^{2k+1}}+\|\Sigma\|_{H^{2k}}\right)
\end{aligned}$$

Moving on to the corrective term, we compute:

$$\partial_T\mathcal{C}_{SF}^{(2k)} = \int_M \Big[\Delta_{\boldsymbol{g}}^k\widetilde{\partial}_0\phi\cdot\Delta_{\boldsymbol{g}}^k\phi' + \Delta_{\boldsymbol{g}}^k\phi\cdot\Delta_{\boldsymbol{g}}^k\widetilde{\partial}_0\phi' \tag{2.9.21a}$$

$$+3\hat{\boldsymbol{n}}\cdot\Delta_{\boldsymbol{g}}^k\phi\cdot\Delta_{\boldsymbol{g}}^k\phi' + [\widetilde{\partial}_0,\Delta_{\boldsymbol{g}}^k]\phi\cdot\Delta_{\boldsymbol{g}}\phi' + \Delta_{\boldsymbol{g}}^k\phi\cdot[\widetilde{\partial}_0,\Delta_{\boldsymbol{g}}^k]\phi'\Big]\,\mathrm{dvol}_{\boldsymbol{g}} \tag{2.9.21b}$$

Inserting the evolution equations into the right hand side of (2.9.21a), we can bound that line by

$$\begin{aligned}
\leq & \int_M \left[3|\Delta_{\boldsymbol{g}}^k\phi'|^2 + 3\hat{\boldsymbol{n}}|\Delta_{\boldsymbol{g}}^k\phi'|^2\right]\mathrm{dvol}_{\boldsymbol{g}} + K\|\hat{\boldsymbol{n}}\|_{C^{2k}}\|\phi'\|_{H^{2k-1}}\|\Delta^k\phi'\|_{L^2} \\
& + \int_M \left[-2\Delta_{\boldsymbol{g}}^k\phi\cdot\Delta_{\boldsymbol{g}}^k\phi' + 3\hat{\boldsymbol{n}}\Delta_{\boldsymbol{g}}^k\phi\cdot\Delta_{\boldsymbol{g}}^k\phi' + 3\Delta_{\boldsymbol{g}}^k\phi\cdot\Delta_{\boldsymbol{g}}^{k+1}\phi + 3\hat{\boldsymbol{n}}\Delta_{\boldsymbol{g}}^k\phi\cdot\Delta_{\boldsymbol{g}}^{k+1}\phi\right]\mathrm{dvol}_{\boldsymbol{g}} \\
& + K\left[\|\hat{\boldsymbol{n}}\|_{C^{2k}}\left(\|\nabla\phi\|_{H^{2k}}+\|\phi'\|_{H^{2k-1}}\right) + \left(\|\nabla\phi\|_{C^0}+\|\phi'\|_{C^0}\right)\|\hat{\boldsymbol{n}}\|_{H^{2k+1}}\right]\|\Delta_{\boldsymbol{g}}^k\phi\|_{L^2}
\end{aligned}$$

Note that, after integrating by parts, the last two terms in the second line can be bounded by

$$\int_M -3|\nabla\Delta_{\boldsymbol{g}}\phi|_{\boldsymbol{g}}^2\,\mathrm{dvol}_{\boldsymbol{g}} + \|\hat{\boldsymbol{n}}\|_{C^1}(\|\nabla\Delta^k\phi\|_{L^2}+\|\Delta^k\phi\|_{L^2})\|\nabla\Delta^k\phi\|_{L^2}\,.$$

For the terms in (2.9.21b), notice that the first term can be bounded by $\delta e^{-\frac{T}{2}}\|\nabla\phi\|_{H^{2k-1}}\sqrt{\mathbb{E}^{(2k)}_{SF}}$, while the commutator terms can be estimated as before, with

$$\|[\widetilde{\partial}_0,\Delta^k_{\boldsymbol{g}}]\phi\|_{L^2}\lesssim\|\nabla\phi\|_{C^0}\|\boldsymbol{n}\|_{C^0}\|\boldsymbol{\Sigma}\|_{\dot{H}^{2k-1}}+\|\boldsymbol{n}\|_{C^{2k}}\|\boldsymbol{\Sigma}\|_{C^{2k-2}}\|\nabla\phi\|_{H^{2k-1}}$$

Combining all of the above, we get

$$\begin{aligned}\partial_T\mathcal{C}^{(2k)}_{SF}\leq&-2\mathcal{C}^{(2k)}_{SF}+\int_M\left[3|\Delta^k_{\boldsymbol{g}}\phi'|-3|\nabla\Delta^k_{\boldsymbol{g}}\phi|_{\boldsymbol{g}}\right]\mathrm{dvol}_{\boldsymbol{g}}\\&+K\delta e^{-\frac{T}{2}}\left[\|\phi'\|_{H^{2k}}+\|\nabla\phi\|_{H^{2k}}+\|\hat{\boldsymbol{n}}\|_{H^{2k+1}}+\|\boldsymbol{\Sigma}\|_{H^{2k}}\right]\\&\cdot\left(\sqrt{\mathbb{E}^{(2k)}_{SF}}+\sqrt{\mathbb{E}^{(2k-1)}_{SF}}\right)\end{aligned}$$

Finally, combining both differential estimates yields the statement for $l=2k$. For $l=2k-1$, $k\in\{1,2\}$, the argument is completely analogous and hence omitted. $\square$

2.9.4. Geometric variables. We can take the following results from prior literature, where we additionally apply the elliptic estimates in Lemma 2.9.11:

LEMMA 2.9.18 (Coercivity of geometric energies, [AM11, Lemma 7.4]). *For sufficiently small $\delta>0$, the following estimate holds:*

$$\|\boldsymbol{g}-\gamma\|^2_{H^5}+\|\Sigma\|^2_{H^4}\lesssim E_{geom}\tag{2.9.22}$$

LEMMA 2.9.19 (Geometric energy estimate, [AF20, Lemma 20]). *Let $\delta>0$ be chosen appropriately small, and let*

$$\alpha=\begin{cases}1&\lambda_0>\frac{1}{9}\\1-3\sqrt{\delta'}&\lambda_0=\frac{1}{9}\end{cases},\tag{2.9.23}$$

where $\delta'>0$ is the same as in (2.9.7)*, in particular suitably small. Then, there exists some constant $K>0$ such that the following estimate holds:*

$$\partial_T E_{geom}\leq-2\alpha E_{geom}+KE_{geom}^{\frac{3}{2}}+K\delta e^{-\frac{T}{2}}\sqrt{E_{geom}}\left[\|\phi'\|_{H^4}+\|\nabla\phi\|_{H^4}\right]\tag{2.9.24}$$

2.9.5. Closing the bootstrap. Now, we can collect our estimates to improve the bootstrap assumptions:

PROPOSITION 2.9.20 (Improved bounds for future stability). *Let the bootstrap assumption (see Assumption 2.9.9) be satisfied for $T\in[0,T_{Boot})$ and assume the initial data assumption holds at $T=0$ (see Assumption 2.9.7). For $\delta>0$ sufficiently small and α as in* (2.9.23) *with $\delta'>0$ sufficiently small, the following estimates hold:*

$$\|\phi'\|_{C^2}+\|\nabla\phi\|_{C^2}+\|\phi'\|_{H^4}+\|\nabla\phi\|_{H^4}\lesssim\delta^{\frac{3}{2}}e^{-\alpha T}\tag{2.9.25a}$$

$$\|\boldsymbol{g}-\gamma\|_{C^3}+\|\Sigma\|_{C^2}+\|\boldsymbol{g}-\gamma\|_{H^5}+\|\Sigma\|_{H^4}\lesssim\delta^{\frac{3}{2}}e^{-\alpha T}\tag{2.9.25b}$$

$$\|\hat{\boldsymbol{n}}\|_{C^4}+\|\boldsymbol{X}\|_{C^4}+\|\hat{\boldsymbol{n}}\|_{H^6}+\|\boldsymbol{X}\|_{H^6}\lesssim\delta^3e^{-2\alpha T}\tag{2.9.25c}$$

PROOF. In the following, the positive constant K may be updated from line to line. Combining the estimate from Lemma 2.9.16 as well as those from Lemma 2.9.17 at each level with Lemma 2.9.19 and applying the (near)-coercivity estimates (2.9.14) and (2.9.22) to the right hand sides, we obtain:

$$\begin{aligned}
&\partial_T \left(E^{(4)}_{SF} + E_{geom}\right)\\
\leq &-2E_{SF} + K\delta e^{-\frac{T}{2}}\sqrt{E_{SF}}\left(\sqrt{E_{SF} + \delta^2 e^{-T}\left\|\mathrm{Ric}[\boldsymbol{g}] + \frac{2}{9}\boldsymbol{g}\right\|^2_{H^2}} + \sqrt{E_{geom}}\right) + K\delta^3 e^{-\frac{5}{2}T}\\
&-2\alpha E_{geom} + KE_{geom}^{\frac{3}{2}} + K\delta e^{-\frac{T}{2}}\sqrt{E_{geom}}\sqrt{E_{SF} + \delta^2 e^{-T}\left\|\mathrm{Ric}[\boldsymbol{g}] + \frac{2}{9}\boldsymbol{g}\right\|^2_{H^2}}
\end{aligned}$$

Applying (2.9.10a) to the curvature norms, as well as (2.9.22) to the resulting norms on $\boldsymbol{g}-\gamma$ and (2.9.9) (which implies $\sqrt{E_{geom}} \lesssim \delta e^{-\frac{T}{2}}$), this becomes

$$\partial_T \left(E_{SF} + E_{geom}\right) \leq -2\alpha(E^{(4)}_{SF} + E_{geom}) + K\delta e^{-\frac{T}{2}}\left(E_{SF} + E_{geom}\right) + K\delta^3 e^{-\frac{5}{2}T}\,.$$

and consequently, since $\alpha \leq 1$,

$$\partial_T \left[e^{2\alpha T}\left(E_{SF} + E_{geom}\right)\right] \lesssim \delta e^{-\frac{T}{2}} \cdot e^{2\alpha T}\left(E_{SF} + E_{geom}\right) + \delta^3 e^{-\frac{T}{2}}$$

The Gronwall lemma, along with the initial data assumption (2.9.8), now implies

$$E_{SF} + E_{geom} \lesssim \delta^3 e^{-2\alpha T}\,. \tag{2.9.26}$$

Lemma 2.9.18 and the standard Sobolev embedding then imply (2.9.25b). In particular, this means

$$\left\|\mathrm{Ric}[\boldsymbol{g}] + \frac{2}{9}\boldsymbol{g}\right\|_{H^2} \lesssim \delta^{\frac{3}{2}} e^{-\alpha T} \tag{2.9.27}$$

due to Lemma 2.9.10, and for $\delta' > 0$ small enough, inserting (2.9.26) and (2.9.27) into (2.9.14) shows (2.9.25a). Moreover, (2.9.25c) follows directly from the proof of Lemma 2.9.11 and the already obtained improvements. □

Proof of Theorem 2.9.1. The problem is locally well-posed as outlined in Remark 2.9.8. There then is some maximal interval $[0, T_{Boot})$ for the logarithmic time T – or, equivalently, some maximal time interval $[\tau_0, \tau_{Boot})$ – on which the solution exists and the bootstrap assumptions (see Assumption 2.9.9) are satisfied. By the analogous argument to the proof of Theorem 2.8.2, the decay estimates in Proposition 2.9.20 are strictly stronger than the bootstrap assumptions for small enough $\delta, \delta' > 0$. This implies $T_{Boot} = \infty$ (resp. $\tau_{Boot} = 0$) since we could else extend the solution strictly beyond T_{Boot} while also satisfying the bootstrap assumptions. This proves the convergence statement in Theorem 2.9.1.

Finally, the decay estimates imply that $|\nabla n|_g$, respectively $|k|_g$, are bounded by $\tau^{\alpha-1}$, respectively $\tau^{\alpha+1}$, up to constant on $[\tau_0, \tau)$. Since α is at worst slightly smaller than 1, both functions are integrable on $[\tau_0, 0)$ for suitably small $\delta' > 0$. By [CBC02], this means the spacetime is future complete. □

2.10. Global stability

To prove Theorem 2.1.2, what still needs to be shown is that initial data as in Theorem 2.1.1 develops from Σ_{t_0} to some hypersurface $\Sigma_{t_1} \equiv \Sigma_{\tau(t_1)}$ in its future such that the data in Σ_{t_1} is near-Milne in the sense of Assumption 2.9.7 and in CMCSH gauge. From there, near-Milne stability yields the behaviour in the future of $\Sigma_{\tau(t_1)}$, and hence future stability of near-FLRW spacetimes as in Theorem 2.1.2.

PROOF OF THEOREM 2.1.2. Within this proof, t will denote the "physical" time coordinate used throughout the Big Bang stability analysis, while τ denotes the mean curvature time used within CMCSH gauge.

Consider initial data $(g, k, \nabla\phi, \partial_0\phi)$ induced on the CMC hypersurface Σ_{t_0} within $\overline{M}$ such that the rescaled variables are close to FLRW reference data in the sense of Theorem 2.8.2. Moreover, let $(\mathring{g}, \mathring{k}, \mathring{\pi}, \mathring{\psi})$ be the geometric initial data on M that induce it via the embedding $\iota : M \hookrightarrow \overline{M}$.

Notice that

$$P : H^{20}_\gamma(M) \to H^{18}_\gamma(M),\, Y^i \mapsto \Delta_\gamma Y^i + (\gamma^{-1})^{il}\mathrm{Ric}[\gamma]_{lj}Y^j = \Delta_\gamma Y^i - \frac{2}{9}Y^i$$

is an isomorphism since Δ_γ has no positive eigenvalues. Hence, using [FK20, Theorem 2.5, Remark 2.6], there is a metric $\mathring{g}'$ isometric to $\mathring{g}$ that remains close in $H^{18}_\gamma(M)$ to γ and satisfies

$$((\mathring{g}')^{-1})^{ij}\left(\Gamma[\mathring{g}']^k_{ij} - \hat{\Gamma}[\gamma]^k_{ij}\right) = 0.$$

Let $\theta \in \mathrm{Diff}(M)$ be the diffeomorphism such that $\theta^*\mathring{g} = \mathring{g}'$, then the proof of [FK20, Theorem 2.5] implies that θ can be chosen close to the identity map within $H^{18}(\mathrm{Diff}(M))$, and consequently that $\theta^*\mathring{k} = \mathring{k}'$, $\theta^*\mathring{\pi} = \mathring{\pi}'$ and $\theta^*\mathring{\psi} = \mathring{\psi}'$ remain close to $-\dot{a}(t_0)a(t_0)\gamma$, 0 and $Ca(t_0)^{-3}$ in $H^{18}_\gamma(M)$. By the same argument as in Remark 2.8.1, we can now evolve this data locally and obtain a new initial hypersurface Σ' close to Σ_{t_0} that is in CMCSH gauge and that $(g, k, \nabla\phi, \partial_0\phi)$ is close to the reference data in the sense of Assumption 2.3.10, exchanging the initial time t_0 by some close time t_0'.

Since τ is strictly increasing, $t \equiv t(\tau)$ exists and we can interchangeably view a as a function in t or τ with some abuse of notation. The Friedman equation (2.2.3) implies $\partial_t a \geq \frac{1}{9}$ and thus $a(t) \geq \frac{1}{9}t$ on $(0,\infty)$, as well as

$$-\tau = 3\frac{\dot{a}}{a} = \frac{1}{a} + \langle\text{lower order terms}\rangle \text{ as } t \to \infty \text{ (resp. } \tau \to 0) .$$

We choose $t_1 > \max\{1, t_0'\}$ large enough (resp. $\tau(t_1) \equiv \tau_0$ small enough) that the following estimates hold for some small $\chi \in (0, \frac{1}{2})$ that depends only on δ:

$$Ca(t_1)^{-3}\tau(t_1)^{-1} \leq \chi \tag{2.10.1}$$

$$-\tau(t_1) \cdot a(t_1) \in [1-\chi, 1+\chi] \tag{2.10.2}$$

As the solution is Cauchy stable, i.e., it and its maximal time of existence depend continuously upon the initial data,[15] one can choose $\varepsilon > 0$ in the analogue of Assumption 2.3.10 small enough to ensure the following: The solution exists until $t_1 > t_0'$ and $(a^{-2}g, a\hat{k}, \nabla\phi, a^3\widetilde{\partial_0}\phi)$ remain $K\varepsilon$-close to $(\gamma, 0, 0, C)$ in $H^6_\gamma \times H^5_\gamma \times H^5_\gamma \times H^5_\gamma$ for some suitable $K > 0$ along the slab $\cup_{s\in[t_0', t_1]}\Sigma_s$. What now remains to be shown is that this implies Assumption 2.9.7 in the sense that, if ε is small enough, δ can be made as small as necessary for Theorem 2.9.1 to apply.

Note that the scalings in Definition 2.9.4 can be rewritten as

$$\boldsymbol{g} - \gamma = (\tau \cdot a)^2 \cdot (a^{-2}g - \gamma) + (\tau^2 \cdot a^2 - 1)\gamma, \quad \boldsymbol{\Sigma} = \frac{\tau}{a}(a\hat{k}),$$
$$\phi' = C\left(-\tau^{-1} \cdot a^{-3}\right) + \left(-\tau^{-1} \cdot a^{-3}\right) \cdot (a^3 n^{-1}(\partial_\tau - \mathcal{L}_X)\phi - C).$$

Since (2.10.2) implies $\tau \cdot a$ is close to -1 at t_1, $\|(\tau \cdot a)^2(a^{-2}g - \gamma)\|_{H^6}$ can be bounded by $\frac{\delta^3}{2}$ for small enough ε. Choosing $\chi < \frac{\delta^3}{2}$ then implies $\|\boldsymbol{g} - \gamma\|_{H^6(\Sigma_{\tau_0})} < \delta^3$. That $\|\boldsymbol{\Sigma}\|_{H^5}$ can be made smaller than δ^3 for small enough $\varepsilon > 0$ follows since $\frac{\tau}{a}$ behaves like $\frac{1}{a^2}$ up to a constant by (2.10.2).
For the normal derivative of the wave, notice that $|C\left(-\tau^{-1} \cdot a^{-3}\right)|$ is bounded by χ due to (2.10.1), and that $-\tau^{-1}a^{-3}$ is equivalent to a^{-2} by (2.10.2). Hence, we can similarly ensure that ϕ' is bounded in H^5 by δ^3. Since $\nabla\phi$ is not changed in either rescaling, and bounds on lapse and shift (up to constant) follow from the elliptic estimates in Lemma 2.9.11, it follows each individual norm in Assumption 2.9.7 can be bounded by δ^3 up to constants that depend only on γ, and hence the initial data assumption itself can be satisfied for suitably small $\delta > 0$.
This proves that we can develop from initial data for the Big Bang stability proof to near-Milne initial data within a CMCSH foliation, and thus we obtain Theorem 2.1.2 from Theorems 2.1.1 and 2.9.1. □

[15] For the argument for Einstein vacuum in CMCSH gauge, see [AM04, Theorem 3.1]. As with local existence, the argument in the Einstein scalar-field system is largely identical since the only difference amounts to coupling the hyperbolic parts of the system with a further hyperbolic one.

2.A. Appendix – Big Bang Stability

2.A.1. Basic formulas and estimates.

2.A.1.1. *Tools from elementary calculus.*

LEMMA 2.A.1 (A Gronwall lemma). *Let $f, \chi, \xi : [a,b] \longrightarrow \mathbb{R}$ be continuous functions such that $\chi \geq 0$, ξ is decreasing and, for any $s \in [a,b]$,*

$$f(s) \leq \int_s^b \chi(r) f(r)\, dr + \xi(s)$$

is satisfied. Then, for any $t \in [a,b]$, we have

$$f(t) \leq \xi(t) \exp\left(\int_t^b \chi(r)\, dr\right).$$

PROOF. This follows by standard arguments as in [Dra03, Corollary 2-3]. □

LEMMA 2.A.2 (A weak fundamental theorem of calculus for square roots). *Let $f : (0, t_0] \longrightarrow \mathbb{R}_0^+$ be a C^1-function. Then, we have for any $t \in (0, t_0]$:*

$$\sqrt{f(t)} \leq \sqrt{f(t_0)} + \int_t^{t_0} \frac{|f'(s)|}{2\sqrt{f(s)}}\, ds \tag{2.A.1}$$

PROOF. This follows from a straightforward application of the monotone convergence theorem to $g_n = \sqrt{f + \frac{1}{n}}$. □

2.A.1.2. *Levi-Civita tensor identities.* Herein, we collect some basic identities for the Levi-Civita tensor $\boldsymbol{\varepsilon}[g]$: Firstly, it satisfies the contraction identities, where $\mathbb{I}^a_b$ denotes the Kronecker-symbol:

$$\boldsymbol{\varepsilon}^{ai_1i_2}\boldsymbol{\varepsilon}_{aj_1j_2} = \mathbb{I}^{i_1}_{j_1}\mathbb{I}^{i_2}_{j_2} - \mathbb{I}^{i_1}_{j_2}\mathbb{I}^{i_2}_{j_1} \tag{2.A.2a}$$

$$\boldsymbol{\varepsilon}^{abi}\boldsymbol{\varepsilon}_{abj_2} = 2\mathbb{I}^i_j \tag{2.A.2b}$$

$$\boldsymbol{\varepsilon}^{abc}\boldsymbol{\varepsilon}_{abc} = 6 \tag{2.A.2c}$$

$$\nabla\boldsymbol{\varepsilon} = 0 \tag{2.A.2d}$$

The analogous formulas hold for $\boldsymbol{\varepsilon}[G]$ when raising indices with regard to G instead of g. For a tracefree and symmetric Σ_t-tangent $(0,2)$-tensor $\mathfrak{T}$ and a Σ_t-tangent $(0,2)$-tensor $\mathfrak{A}$, the following simplified identities hold:

$$(\mathfrak{T} \times \mathfrak{A})_{ij} = \boldsymbol{\varepsilon}_i{}^{ab}\boldsymbol{\varepsilon}_j{}^{pq}\mathfrak{T}_{ap}\mathfrak{A}_{bq} + \frac{1}{3}(\mathfrak{T}\cdot\mathfrak{A})g_{ij} \tag{2.A.3a}$$

$$(\mathfrak{T} \times g)_{ij} = -\mathfrak{T}_{ij} \tag{2.A.3b}$$

$$(\mathfrak{T} \times k)_{ij} = -\frac{\tau}{3}\mathfrak{T}_{ij} + (\mathfrak{T} \times \hat{k})_{ij} \tag{2.A.3c}$$

Further, note the following formulas (for $\tilde{\mathfrak{T}}$ as $\mathfrak{T}$, $\tilde{\mathfrak{A}}$ as $\mathfrak{A}$ and any Σ_t-tangent $(0,1)$-tensor ξ) (see [AM04, p.30]):

$$\mathrm{div}_g(\mathfrak{A} \wedge \tilde{\mathfrak{A}}) = -\mathrm{curl}_g\mathfrak{A}\cdot\tilde{\mathfrak{A}} + \mathfrak{A}\cdot\mathrm{curl}_g\tilde{\mathfrak{A}} \tag{2.A.3d}$$

$$\mathfrak{A}\cdot(\xi \wedge \tilde{\mathfrak{A}}) = -2\xi\cdot(\mathfrak{A}\wedge\tilde{\mathfrak{A}}) \tag{2.A.3e}$$

$$\mathfrak{T} \cdot (\mathfrak{A} \times \tilde{\mathfrak{T}}) = (\mathfrak{T} \times \mathfrak{A}) \cdot \tilde{\mathfrak{T}} \tag{2.A.3f}$$

2.A.1.3. *Estimates on contracted tensors.*

LEMMA 2.A.3. *Let $\mathfrak{S}, \mathfrak{T}$ be traceless and symmetric Σ_t-tangent $(0,2)$-tensors, $\mathfrak{M}, \mathfrak{N}$ symmetric Σ_t-tangent $(0,2)$-tensors and ξ a Σ_t-tangent $(0,1)$-tensor. We define G, G^{-1} and $|\cdot|_G$ via* (2.2.27a)*). Then:*

$$|\mathfrak{M} \odot_G \mathfrak{N}|_G \leq |\mathfrak{M}|_G |\mathfrak{N}|_G, \quad \mathfrak{M} \odot_g \mathfrak{N} = a^{-2} \mathfrak{M} \odot_G \mathfrak{N} \tag{2.A.4a}$$

$$|\mathfrak{S} \times_G \mathfrak{T}|_G \lesssim |\mathfrak{S}|_G |\mathfrak{T}|_G, \quad (\mathfrak{S} \times \mathfrak{T})_{ij} = a^{-3} (\mathfrak{S} \times_G \mathfrak{T})_{ij} \tag{2.A.4b}$$

$$|\mathfrak{S} \wedge_G \mathfrak{T}|_G \leq |\mathfrak{S}|_G |\mathfrak{T}|_G, \quad (\mathfrak{S} \wedge \mathfrak{T})_l = a^{-3} (\mathfrak{S} \wedge_G \mathfrak{T}) \tag{2.A.4c}$$

$$|\xi \wedge_G \mathfrak{T}|_G \leq |\xi|_G |\mathfrak{T}|_G, \quad (\xi \wedge \mathfrak{T})_{ij} = a^{-1} (\xi \wedge_G \mathfrak{T})_{ij} \tag{2.A.4d}$$

$$|\mathrm{curl}_G \mathfrak{M}|_G \lesssim |\nabla \mathfrak{M}|_G, \quad \mathrm{curl}_g \mathfrak{M}_{ij} = a^{-1} \mathrm{curl}_G \mathfrak{M}_{ij} \tag{2.A.4e}$$

PROOF. The estimates with respect to the unrescaled metric are direct consequences of the contraction identities (2.A.2a)-(2.A.2c) replacing g with G, and the scalings follow simply by tracking the effects of the rescaling in Definition 2.2.9. In particular,

$$\boldsymbol{\varepsilon}[g]_i{}^{cd} = g^{cj} g^{dk} \boldsymbol{\varepsilon}[g]_{ijk} = \left(a^{-2} (G^{-1})^{cj}\right) \left(a^{-2} (G^{-1})^{dk}\right) a^3 \boldsymbol{\varepsilon}[G]_{ijk} = a^{-1} \varepsilon[G]_i{}^{\sharp cd} \tag{2.A.5}$$

determines the scaling of the Levi-Civita-Symbol. $\square$

2.A.2. Commutators. Herein, we collect a variety of commutators of spatial derivative operators with each other as well as with time derivatives. While these mostly follow by standard computations, we use the fact that our spatial hypersurfaces are three-dimensional to significantly simplify the spatial commutator formulas, and need to apply the rescaled equations from Proposition 2.2.10 for the time derivative formulas.

For higher order commutators, we denote by $\mathfrak{J}$ terms within the commutator formula that contribute junk terms at any point where this commutator formula is used. Furthermore, in the following, ζ denotes a scalar function on $\overline{M}$ and $\mathfrak{T}$ denotes a Σ_t-tangent, symmetric $(0,2)$-tensor, always with sufficient regularity for the equations to make sense. Moreover, recall the schematic $*$-notation as introduced in subsection 2.2.1.8.

COROLLARY 2.A.4 (Schematic first order spatial commutators). *For ζ and $\mathfrak{T}$ as above, the following identities hold:*

$$[\Delta, \nabla]\zeta = \mathrm{Ric}[G] * \nabla\zeta \tag{2.A.6a}$$

$$[\Delta, \nabla^2]\zeta = \mathrm{Ric}[G] * \nabla^2\zeta + \nabla\mathrm{Ric}[G] * \nabla\zeta \tag{2.A.6b}$$

$$[\Delta, \nabla]\mathfrak{T} = \mathrm{Ric}[G] * \nabla\mathfrak{T} + \nabla\mathrm{Ric}[G] * \mathfrak{T} \tag{2.A.6c}$$

$$[\Delta, \nabla^2]\mathfrak{T} = \mathrm{Ric}[G] * \nabla^2\mathfrak{T} + \nabla\mathrm{Ric}[G] * \nabla\mathfrak{T} + \nabla^2\mathrm{Ric}[G] * \mathfrak{T} \tag{2.A.6d}$$

$$[\Delta, \mathrm{div}_G]\mathfrak{T} = \mathrm{Ric}[G] * \nabla\mathfrak{T} + \nabla\mathrm{Ric}[G] * \mathfrak{T} \tag{2.A.6e}$$

$$[\Delta, \mathrm{curl}_G] = \boldsymbol{\varepsilon}[G] * (\mathrm{Ric}[G] * \nabla\mathfrak{T} + \nabla\mathrm{Ric}[G] * \mathfrak{T}) \tag{2.A.6f}$$

PROOF. Since we are working in three spatial dimensions, the following identity holds:

$$\mathrm{Riem}[G]_{ijkl} = G_{ik}\mathrm{Ric}[G]_{jl} - G_{il}\mathrm{Ric}[G]_{jk} + G_{jl}\mathrm{Ric}[G]_{ik} - G_{jk}\mathrm{Ric}[G]_{il}$$

$$-\frac{1}{2}(G^{-1})^{mn}\mathrm{Ric}[G]_{mn}(G_{ik}G_{jl}-G_{il}G_{jk})$$

Hence, for any $I \in \mathbb{N}_0$, any $\nabla^I\mathrm{Riem}[G]$-term reduces to a sum of products and contractions of $\nabla^I\mathrm{Ric}[G]$ with various metric tensors that are all suppressed in schematic notation. With this in mind, the above statements are simply direct consequences of standard commutation cormulas and (2.A.5). □

LEMMA 2.A.5 (Higher order spatial commutators). *For $l \in \mathbb{N}$, $l \geq 2$, the following formulas hold (and extend to $l = 1$ when dropping any term involving Δ^{l-2}):*

$$[\Delta^l,\nabla]\zeta = \Delta^{l-1}\mathrm{Ric}[G] * \nabla\zeta + \nabla\Delta^{l-2}\mathrm{Ric}[G] * \nabla^2\zeta + \mathfrak{J}([\Delta^l,\nabla]\zeta) \tag{2.A.7a}$$

$$[\Delta^l,\nabla^2]\zeta = \nabla\Delta^{l-1}\mathrm{Ric}[G] * \nabla\zeta + \nabla^2\Delta^{l-2}\mathrm{Ric}[G] * \nabla^2\zeta + \mathfrak{J}([\Delta^l,\nabla^2]\zeta) \tag{2.A.7b}$$

$$[\Delta^l,\nabla]\mathfrak{T} = \nabla\Delta^{l-1}\mathrm{Ric}[G] * \mathfrak{T} + \nabla^2\Delta^{l-2}\mathrm{Ric}[G] * \nabla\mathfrak{T} + \mathfrak{J}([\Delta^l,\nabla]\mathfrak{T})\,, \tag{2.A.7c}$$

$$[\Delta^l,\nabla^2]\mathfrak{T} = \nabla^2\Delta^{l-1}\mathrm{Ric}[G] * \mathfrak{T} + \nabla^3\Delta^{l-2}\mathrm{Ric}[G] * \nabla\mathfrak{T} + \mathfrak{J}([\Delta^l,\nabla^2]\mathfrak{T})\,, \tag{2.A.7d}$$

$$[\Delta^l,\mathrm{div}_G]\mathfrak{T} = \nabla\Delta^{l-1}\mathrm{Ric}[G] * \mathfrak{T} + \nabla^2\Delta^{l-2}\mathrm{Ric}[G] * \nabla\mathfrak{T} + \mathfrak{J}([\Delta^l,\mathrm{div}_G]\mathfrak{T})\,, \tag{2.A.7e}$$

$$[\Delta^l,\mathrm{curl}_G]\mathfrak{T} = \boldsymbol{\varepsilon}[G] * \left(\nabla\Delta^{l-1}\mathrm{Ric}[G] * \mathfrak{T} + \nabla^2\Delta^{l-2}\mathrm{Ric}[G] * \nabla\mathfrak{T}\right) + \mathfrak{J}([\Delta^l,\mathrm{curl}_G]\mathfrak{T}) \tag{2.A.7f}$$

with junk terms, where $\mathcal{I} = I_1 + \cdots + I_{l-m}$,

$$\mathfrak{J}([\Delta^l,\nabla]\zeta) = \sum_{\substack{I_1+I_\zeta=2(l-1),\\ I_\zeta\geq 2}} \nabla^{I_1}\mathrm{Ric}[G] * \nabla^{I_\zeta+1}\zeta + \sum_{m=0}^{l-2}\sum_{\mathcal{I}+I_\zeta=2m} \nabla^{I_1}\mathrm{Ric}[G] * \cdots * \nabla^{I_{l-m}}\mathrm{Ric}[G] * \nabla^{I_\zeta+1}\zeta$$

$$\mathfrak{J}([\Delta^l,\nabla^2]\zeta) = \sum_{\substack{I_1+I_\zeta=2(l-1)+1,\\ I_1,I_\zeta\geq 2}} \nabla^{I_1}\mathrm{Ric}[G] * \nabla^{I_\zeta+1}\zeta + \sum_{m=0}^{l-2}\sum_{\mathcal{I}+I_\zeta=2m+1} \nabla^{I_1}\mathrm{Ric}[G] * \cdots * \nabla^{I_{l-m}}\mathrm{Ric}[G] * \nabla^{I_\zeta+1}\zeta$$

$$\mathfrak{J}([\Delta^l,\nabla]\mathfrak{T}) = \sum_{\substack{I_1+I_\mathfrak{T}=2(l-1)+1,\\ I_\mathfrak{T}\geq 2}} \nabla^{I_1}\mathrm{Ric}[G] * \nabla^{I_\mathfrak{T}}\mathfrak{T} + \sum_{m=0}^{l-2}\sum_{\mathcal{I}+I_\mathfrak{T}=2m+1} \nabla^{I_1}\mathrm{Ric}[G] * \cdots * \nabla^{I_{l-m}}\mathrm{Ric}[G] * \nabla^{I_\mathfrak{T}}\mathfrak{T}$$

$$\mathfrak{J}([\Delta^l,\nabla^2]\mathfrak{T}) = \sum_{\substack{I_1+I_\mathfrak{T}=2l,\\ I_\mathfrak{T}\geq 2}} \nabla^{I_1}\mathrm{Ric}[G] * \nabla^{I_\mathfrak{T}}\mathfrak{T} + \sum_{m=0}^{l-2}\sum_{\mathcal{I}+I_\mathfrak{T}=2m+2} \nabla^{I_1}\mathrm{Ric}[G] * \cdots * \nabla^{I_{l-m}}\mathrm{Ric}[G] * \nabla^{I_\mathfrak{T}}T$$

$$\mathfrak{J}([\Delta^l,\mathrm{div}_G]\mathfrak{T}) = \sum_{\substack{I_1+I_\mathfrak{T}=2(l-1)+1,\\ I_\mathfrak{T}\geq 2}} \nabla^{I_1}\mathrm{Ric}[G] * \nabla^{I_\mathfrak{T}}\mathfrak{T} + \sum_{m=0}^{l-2}\sum_{\mathcal{I}+I_\mathfrak{T}=2m+1} \nabla^{I_1}\mathrm{Ric}[G] * \cdots * \nabla^{I_{l-m}}\mathrm{Ric}[G] * \nabla^{I_\mathfrak{T}}\mathfrak{T}$$

$$\mathfrak{J}([\Delta^l,\mathrm{curl}_G]\mathfrak{T}) = \boldsymbol{\varepsilon}[G] * \left[\sum_{\substack{I_1+I_\mathfrak{T}=2(l-1)+1,\\ I_\mathfrak{T}\geq 2}} \nabla^{I_1}\mathrm{Ric}[G] * \nabla^{I_\mathfrak{T}}\mathfrak{T} + \sum_{m=0}^{l-2}\sum_{\substack{\mathcal{I}+\\ +I_\mathfrak{T}=2m+1}} \nabla^{I_1}\mathrm{Ric}[G] * \cdots * \nabla^{I_{l-m}}\mathrm{Ric}[G] * \nabla^{I_\mathfrak{T}}\mathfrak{T}\right].$$

PROOF. The formulas follow by applying the formulas from Lemma 2.A.4 inductively. □

LEMMA 2.A.6 (Time derivative commutators). *With respect to a solution to the Einstein scalar-field system as in Proposition 2.2.6, the following commutator formulas hold:*

$$[\partial_t,\nabla_i]\zeta = 0 \tag{2.A.8a}$$

$$[\partial_t,\nabla^{\sharp i}]\zeta = 2(N+1)a^{-3}\Sigma^{\sharp ij}\nabla_j\zeta - 2N\frac{\dot{a}}{a}\nabla^{\sharp i}\zeta \tag{2.A.8b}$$

$$[\partial_t,\Delta]\zeta = 2(N+1)a^{-3}\langle\Sigma,\nabla^2\zeta\rangle_G - 2N\frac{\dot{a}}{a}\Delta\zeta \tag{2.A.8c}$$

$$- 2(N+1)a^{-3}\langle \mathrm{div}_G\Sigma, \nabla\zeta\rangle_G - 2a^{-3}\langle\Sigma, \nabla N\nabla\zeta\rangle_G + \frac{\dot a}{a}\langle\nabla N, \nabla\zeta\rangle_G$$

$$[\partial_t, \nabla]\mathfrak{T} = a^{-3}\left((N+1)\,\nabla\Sigma + \Sigma * \nabla N\right) * \mathfrak{T} + \frac{\dot a}{a}\,\nabla N * \mathfrak{T} \tag{2.A.8d}$$

$$\begin{aligned}[\partial_t, \Delta]\mathfrak{T} = {}& a^{-3}(N+1)\Sigma * \nabla^2\mathfrak{T} + \frac{\dot a}{a}N\Delta\mathfrak{T} + a^{-3}\nabla((N+1)\Sigma) * \nabla\mathfrak{T} + \frac{\dot a}{a}\nabla N * \nabla\mathfrak{T}\\ &+ a^{-3}\nabla^2((N+1)\Sigma) * \mathfrak{T} - \frac{\dot a}{a}\nabla^2 N * \mathfrak{T}\end{aligned} \tag{2.A.8e}$$

Proof. Equation (2.A.8a) is simply that coordinate derivatives commute, and (2.A.8b) follows by applying (2.2.28b) and the product rule.
For the commutators (2.A.8c), (2.A.8d) and (2.A.8e), we write out the covariant derivatives in local coordinates, apply the product rule, and then the evolution equations (2.2.28b) and (2.2.34) for the inverse metric and Christoffel symbols. ☐

Lemma 2.A.7 (High order time derivative commutators). *For $l \in \mathbb{N}$, $l \geq 2$, the time derivative commutators take the form*

(2.A.9a)
$$\begin{aligned}[\partial_t, \Delta^l]\zeta = {}& 2a^{-3}(N+1)\langle\Sigma, \nabla^2\Delta^{l-1}\zeta\rangle_G + a^{-3}\nabla\Sigma * \nabla^3\Delta^{l-2}\zeta\\ &- 2(N+1)a^{-3}\langle\mathrm{div}_G\Delta^{l-1}\Sigma, \nabla\zeta\rangle_G + (N+1)a^{-3}\nabla^{2l-3}\mathrm{Ric} * \Sigma * \nabla\zeta + \mathfrak{J}([\partial_t, \Delta^l]\zeta),\end{aligned}$$

(2.A.9b)
$$\begin{aligned}[\partial_t, \nabla\Delta^l]\zeta = {}& 2a^{-3}(N+1)\langle\Sigma, \nabla^3\Delta^{l-1}\zeta\rangle_G + a^{-3}(N+1)\nabla\Sigma * \nabla^{2l}\zeta\\ &- 2(N+1)a^{-3}\langle\nabla\mathrm{div}_G\Delta^{l-1}\Sigma, \nabla\zeta\rangle_G\\ &+ \frac{\dot a}{a}\langle\nabla^2\Delta^{l-1}N, \nabla\zeta\rangle_G + (N+1)a^{-3}\nabla^{2l-2}\mathrm{Ric}[G] * \Sigma * \nabla\zeta\\ &+ \mathfrak{J}([\partial_t, \nabla\Delta^l]\zeta)\end{aligned}$$

(2.A.9c)
$$\begin{aligned}[\partial_t, \Delta^l]\mathfrak{T} = {}& a^{-3}\left(\Sigma * \nabla^2\Delta^{l-1}\mathfrak{T} + \nabla\Sigma * \nabla^3\Delta^{l-2}\mathfrak{T} + \nabla\mathfrak{T} * \nabla\Delta^{l-1}\Sigma + \mathfrak{T} * \Delta^l\Sigma\right)\\ &+ a^{-3}\left((N+1)\Sigma * \mathfrak{T} * \nabla^2\Delta^{l-2}\mathrm{Ric}[G] + \nabla((N+1)\Sigma * \mathfrak{T}) * \nabla^{2l-3}\mathrm{Ric}[G]\right)\\ &+ \frac{\dot a}{a}\Delta^l N \cdot \mathfrak{T} + \frac{\dot a}{a}\nabla\Delta^{l-1}N * \nabla\mathfrak{T} + \mathfrak{J}([\partial_t, \Delta^l])\mathfrak{T}\,,\end{aligned}$$

(2.A.9d)
$$\begin{aligned}[\partial_t, \nabla\Delta^l]\mathfrak{T} = {}& a^{-3}\nabla\Sigma * \Delta^l\mathfrak{T} + a^{-3}(N+1)\Sigma * \nabla^3\Delta^{l-1}\mathfrak{T} + a^{-3}(N+1)\mathfrak{T} * \nabla\Delta^l\Sigma\\ &+ \frac{\dot a}{a}\nabla\Delta^l N * \mathfrak{T} + \frac{\dot a}{a}\nabla^2\Delta^{l-1}N * \nabla\mathfrak{T} + a^{-3}(N+1)\Sigma * \nabla^3\Delta^{l-2}\mathrm{Ric}[G] * \mathfrak{T}\\ &+ \mathfrak{J}([\partial_t, \nabla\Delta^l]\mathfrak{T})\,,\end{aligned}$$

where the junk terms are, where $\mathcal{I} = \sum_{i=1}^{l-m-1} I_i$,

(2.A.9e)
$$\mathfrak{J}([\partial_t, \Delta^l]\zeta) = a^{-3}\sum_{\substack{I_N + I_\Sigma + I_\zeta = 2(l-1)\\ I_\zeta \leq 2(l-2)}} \nabla^{I_N}(N+1) * \nabla^{I_\Sigma}\Sigma * \nabla^{I_\zeta+2}\zeta$$

$$+\frac{\dot a}{a}\sum_{\substack{I_N+I_\zeta=2l\\ I_\zeta\geq 2}}\nabla^{I_N}N*\nabla^{I_\zeta}\zeta$$

$$+a^{-3}\sum_{m=0}^{l-2}\sum_{I_N+I_\Sigma+I_\zeta+\mathcal{I}=2m}\nabla^{I_N}(N+1)*\nabla^{I_\Sigma}\Sigma*\nabla^{I_1}\mathrm{Ric}[G]*\dots*\nabla^{I_{l-m-1}}\mathrm{Ric}[G]*\nabla^{I_\zeta+2}\zeta$$

$$+a^{-3}\sum_{m=0}^{l-2}\sum_{\substack{I_N+I_\Sigma+I_\zeta+\mathcal{I}=2m\\ I_1\neq 2l-4}}\nabla^{I_N}(N+1)*\nabla^{I_\Sigma}\Sigma*\nabla^{I_1+1}\mathrm{Ric}[G]*\dots*\nabla^{I_{l-m-1}}\mathrm{Ric}[G]*\nabla^{I_\zeta+1}\zeta$$

$$+\frac{\dot a}{a}\sum_{m=0}^{l-1}\sum_{\substack{I_N+\mathcal{I}+I_\zeta=2m-1\\ I_\zeta\neq 2(l-1)}}\nabla^{I_N}N*\nabla^{I_1}\mathrm{Ric}[G]*\dots*\nabla^{I_{l-m-1}}\mathrm{Ric}[G]*\nabla^{I_\zeta+1}\zeta$$

(2.A.9f)

$$\mathfrak{I}([\partial_t,\nabla\Delta^l]\zeta)=\frac{\dot a}{a}\sum_{\substack{I_N+I_\zeta=2l\\ I_\zeta\neq 0}}\nabla^{I_N}N*\nabla^{I_\zeta+1}\zeta+a^{-3}\sum_{\substack{I_N+I_\Sigma+I_\zeta=2(l-1)+1\\ (I_\Sigma,I_\zeta)\neq(0,2(l-1)+1),(1,2(l-1))}}\nabla^{I_N}(N+1)*\nabla^{I_\Sigma}\Sigma*\nabla^{I_\zeta+1}\zeta$$

$$+a^{-3}\sum_{m=0}^{l-2}\sum_{I_N+I_\Sigma+I_\zeta+\mathcal{I}=2m+1}\nabla^{I_N}(N+1)*\nabla^{I_\Sigma}\Sigma*\nabla^{I_1}\mathrm{Ric}[G]*\dots*\nabla^{I_{l-m-1}}\mathrm{Ric}[G]*\nabla^{I_\zeta+2}\zeta$$

$$+a^{-3}\sum_{m=0}^{l-2}\sum_{\substack{I_N+I_\Sigma+I_\zeta+\mathcal{I}=2m+1\\ I_1\neq 2l-3}}\nabla^{I_N}(N+1)*\nabla^{I_\Sigma}\Sigma*\nabla^{I_1+1}\mathrm{Ric}[G]*\dots*\nabla^{I_{l-m-1}}\mathrm{Ric}[G]*\nabla^{I_\zeta+1}\zeta$$

$$+\frac{\dot a}{a}\sum_{m=0}^{l-1}\sum_{\substack{I_N+\mathcal{I}+I_\zeta=2m\\ I_\zeta\neq 2(l-1)}}\nabla^{I_N}N*\nabla^{I_1}\mathrm{Ric}[G]*\dots*\nabla^{I_{l-m-1}}\mathrm{Ric}[G]*\nabla^{I_\zeta+1}\zeta$$

(2.A.9g)

$$\mathfrak{I}([\partial_t,\Delta^l])\mathfrak{T}=a^{-3}\sum_{I_N+I_\Sigma+I_\mathfrak{T}=2l}\nabla^{I_N}N*\nabla^{I_\Sigma}\Sigma*\nabla^{I_\mathfrak{T}}\mathfrak{T}+a^{-3}\sum_{\substack{I_\Sigma+I_\mathfrak{T}=2l\\ I_\Sigma,I_\mathfrak{T}\geq 2}}\nabla^{I_\Sigma}\Sigma*\nabla^{I_\mathfrak{T}}\mathfrak{T}$$

$$+a^{-3}\sum_{m=0}^{l-1}\sum_{\substack{I_N+I_\Sigma+I_\mathfrak{T}+\mathcal{I}=2m\\ I_1<2l-3}}\nabla^{I_N}(N+1)*\nabla^{I_\Sigma}\Sigma*\nabla^{I_1}\mathrm{Ric}[G]*\dots*\nabla^{I_{l-m-1}}\mathrm{Ric}[G]*\nabla^{I_\mathfrak{T}}\mathfrak{T}$$

$$+\frac{\dot a}{a}\sum_{I_N+I_\mathfrak{T}=2l,\,I_\mathfrak{T}\geq 2}\nabla^{I_N}N*\nabla^{I_\mathfrak{T}}\mathfrak{T}$$

(2.A.9h)

$$\mathfrak{I}([\partial_t,\nabla\Delta^l]\mathfrak{T})=a^{-3}\Sigma*\nabla N*\Delta^l\mathrm{Ric}[G]+a^{-3}N*\nabla\Sigma*\Delta^l\mathfrak{T}+\frac{\dot a}{a}\nabla N*\Delta^l\mathfrak{T}+\nabla\mathfrak{I}([\partial_t,\Delta^l]\mathfrak{T})$$

We can extend the formulas to $l=1$ by dropping any term which would contain negative powers of Δ or a multiindex of negative order.

Proof. This follows by iteratively applying the commutators in Lemma 2.A.6. □

While all of the above commutators will be essential for the mainline argument, the a priori estimates require the following commutators:

Lemma 2.A.8 (Auxiliary commutators). *Let $J\in\mathbb{N}$. Then, we have:*

$$[\partial_t,\nabla^J]\zeta=a^{-3}\sum_{I_N+I_\Sigma+I_\zeta=l-1,\,I_\zeta<J-1}\nabla^{I_N}(N+1)*\nabla^{I_\Sigma}\Sigma*\nabla^{I_\zeta+1}\zeta \tag{2.A.10a}$$

$$+\frac{\dot a}{a}\sum_{I_N+I_\zeta=J-1,I_N>0}\nabla^{I_N}N*\nabla^{I_\zeta+1}\zeta$$

$$[\partial_t,\nabla^J]\mathfrak{T}=a^{-3}\sum_{I_N+I_\Sigma+I_{\mathfrak{T}}=J,\,I_{\mathfrak{T}}<J}\nabla^{I_N}(N+1)*\nabla^{I_\Sigma}\Sigma*\nabla^{I_{\mathfrak{T}}}\mathfrak{T}+\frac{\dot a}{a}\sum_{I_N+I_{\mathfrak{T}}=J,I_N>0}\nabla^{I_N}N*\nabla^{I_{\mathfrak{T}}}\mathfrak{T}\tag{2.A.10b}$$

Proof. For $J=1$, this has already been shown in (2.A.8a) and (2.A.8d). For higher orders, the formulas follow from a straightforward induction argument using that, in local coordinates, we schematically have

$$[\partial_t,\nabla^J]\zeta=[\partial_t,\nabla]\nabla^{J-1}\zeta+\nabla[\partial_t,\nabla^{J-1}]\zeta=(\partial_t\Gamma[G])*\nabla^{J-1}\zeta+\nabla[\partial_t,\nabla^{J-1}]\zeta$$

and analogously replacing ζ with $\mathfrak{T}$. □

2.A.3. Borderline and junk terms.

Definition 2.A.9 (Error terms). Let $L\in 2\mathbb{N}$, $L\geq 2$. Then, the error terms in the Laplace-commuted equations stated in Lemma 2.2.11 take the following form:

For the constraint equations, we have

$$\begin{aligned}\mathfrak{M}_{L,Junk}=&-8\pi(\Psi+C)\nabla\Delta^{\frac{L}{2}-2}\mathrm{Ric}[G]*\nabla^2\phi+\nabla^{L-2}\mathrm{Ric}[G]*\nabla\Sigma+\underbrace{\nabla^{L-3}\mathrm{Ric}[G]*\nabla^2\Sigma}_{\text{if }L\neq 2}\\&+\sum_{I_\Psi+I_\phi=L,\,I_\Psi\neq 0}\nabla^{I_\Psi}\Psi*\nabla^{I_\phi+1}\phi+8\pi(\Psi+C)\mathfrak{I}([\Delta^{\frac{L}{2}},\nabla]\phi)-\mathfrak{I}([\Delta^{\frac{L}{2}},\mathrm{div}_G]\Sigma)\end{aligned}\tag{2.A.11a}$$

$$\tilde{\mathfrak{M}}_{L,Junk}=\underbrace{-\boldsymbol{\varepsilon}[G]*\nabla^{L-3}\mathrm{Ric}[G]*\nabla\Sigma}_{\text{if }L\neq 2}-\mathfrak{I}([\Delta^{\frac{L}{2}},\mathrm{curl}_G]\Sigma)\tag{2.A.11b}$$

$$\mathfrak{H}_{L,Border}=a^{-4}\left[\Sigma*\Delta^{\frac{L}{2}}\Sigma+\nabla\Sigma*\nabla^{L-1}\Sigma\right]\tag{2.A.11c}$$

$$\begin{aligned}\mathfrak{H}_{L,Junk}=&\sum_{I_1+I_2=L}\nabla^{I_1+1}\phi*\nabla^{I_2+1}\phi+a^{-4}\sum_{I_1+I_2=L,I_i\geq 2}\nabla^{I_1}\Sigma*\nabla^{I_2}\Sigma\\&+\Delta^{\frac{L}{2}}\left[\frac{4\pi}{3}|\nabla\phi|^2_G+\frac{8\pi}{3}a^{-4}\Psi^2+\frac{16\pi}{3}Ca^{-4}\Psi\right]\cdot G\end{aligned}\tag{2.A.11d}$$

The lapse equation error terms are

$$\begin{aligned}\mathfrak{N}_{L,Border}=&a^{-4}(N+1)\left(\Sigma*\Delta^{\frac{L}{2}}\Sigma+\nabla\Sigma*\nabla^{L-1}\Sigma+\Psi*\Delta^{\frac{L}{2}}\Psi+\nabla\Psi*\nabla^{L-1}\Psi\right)\\&+a^{-4}\left[|\Sigma|^2_G+\Psi^2+\Psi\right]*\Delta^{\frac{L}{2}}N+a^{-4}\nabla\left[|\Sigma|^2_G+\Psi^2+\Psi\right]*\nabla^{L-1}N\end{aligned}\tag{2.A.12a}$$

$$\begin{aligned}\mathfrak{N}_{L,Junk}=&a^{-4}\sum_{\substack{I_N+I_1+I_2=L;\\ I_N\leq L-2;\,I_N>0\text{ or }I_1\leq I_2\leq L-2}}\nabla^{I_N}(N+1)*\left(\nabla^{I_1}\Sigma*\nabla^{I_2}\Sigma+\nabla^{I_1}\Psi*\nabla^{I_2}\Psi\right)\\&+a^{-4}N*\Delta^{\frac{L}{2}}\Psi+a^{-4}\sum_{I_N+I_\Psi=L;\,I_\Psi\geq 2,\,I_N\geq 1}\nabla^{I_N}N*\nabla^{I_\Psi}\Psi\,,\end{aligned}\tag{2.A.12b}$$

as well as

$$\begin{aligned}\mathfrak{N}_{L+1,Border}=&a^{-4}(N+1)\Big(\Sigma*\nabla\Delta^{\frac{L}{2}}\Sigma+\nabla\Sigma*\nabla^L\Sigma+\nabla^2\Sigma*\nabla^{L-1}\Sigma+\Psi*\nabla\Delta^{\frac{L}{2}}\Psi\\&+\nabla\Psi*\nabla^L\Psi+\nabla^2\Psi*\nabla^{L-1}\Psi\Big)+a^{-4}\left[|\Sigma|^2_G+\Psi^2+\Psi\right]*\nabla\Delta^{\frac{L}{2}}N\\&+\nabla\Psi*\nabla^LN+\nabla^2\Psi*\nabla^{L-1}\Psi\end{aligned}\tag{2.A.12c}$$

$$\begin{aligned}\mathfrak{N}_{L+1,Junk}=&a^{-4}\sum_{\substack{I_N+I_1+I_2=L;\\ I_N<L+1;\,I_N>0\text{ or }I_1\geq I_2>2}}\nabla^{I_N}(N+1)*\left(\nabla^{I_1}\Sigma*\nabla^{I_2}\Sigma+\nabla^{I_1}\Psi*\nabla^{I_2}\Psi\right)\\&+a^{-4}N*\nabla\Delta^{\frac{L}{2}}\Psi+a^{-4}\sum_{I_N+I_\Psi=L+1;\,I_N,I_\Psi>2}\nabla^{I_N}N*\nabla^{I_\Psi}\Psi\end{aligned}\tag{2.A.12d}$$

whereas the scalar field error terms read

(2.A.13a)
$$\mathfrak{P}_{L,Border} = -3\Psi\frac{\dot a}{a}\Delta^{\frac{L}{2}}N + \frac{\dot a}{a}\nabla\Psi * \nabla^{L-1}N + 2a^{-3}(N+1)\langle\Sigma, \nabla^2\Delta^{\frac{L}{2}-1}\Psi\rangle_G + 2a^{-3}(N+1)\nabla^{L-3}\mathrm{Ric} * \Sigma * \nabla\Psi$$
$$- 2a^{-3}(N+1)\langle\mathrm{div}_G\Delta^{\frac{L}{2}-1}\Sigma, \nabla\Psi\rangle_G + a^{-3}(N+1)\nabla\Sigma * \nabla^3\Delta^{\frac{L}{2}-2}\Psi$$

(2.A.13b)
$$\mathfrak{P}_{L,Junk} = \frac{\dot a}{a}\sum_{I_N+I_\Psi=L,\, I_\Psi\geq 2}\nabla^{I_N}N * \nabla^{I_\Psi}\Psi + a\sum_{I_N+I_\phi=L+1,\, I_N,I_\phi\neq 0}\nabla^{I_N}N * \nabla^{I_\phi+1}\phi + \mathfrak{J}([\partial_t, \Delta^{\frac{L}{2}}]\Psi)$$

(2.A.13c)
$$\mathfrak{Q}_{L,Border} = a^{-3}\Psi\nabla\Delta^{\frac{L}{2}}N + a^{-3}(N+1)\Sigma * \nabla^3\Delta^{\frac{L}{2}-1}\phi + a^{-3}(N+1)\nabla^L\phi * \nabla\Sigma$$

(2.A.13d)
$$\mathfrak{Q}_{L,Junk} = a^{-3}\sum_{I_N+I_\Psi=L+1,\, I_N,I_\Psi\neq 0}\nabla^{I_N}N * \nabla^{I_\Psi}\Psi + a^{-3}\nabla\Delta^{\frac{L}{2}-1}N * \nabla^2\phi * \Sigma$$
$$+ a^{-3}(N+1)\nabla^2\Delta^{\frac{L}{2}-1}\Sigma * \nabla\phi + (N+1)a^{-3}\nabla\Delta^{\frac{L}{2}-1}\mathrm{Ric}[G] * \Sigma * \nabla\phi$$
$$+ a^{-3}\nabla^{L-2}\mathrm{Ric}[G] * ((N+1) * \Sigma * \nabla\phi) + \frac{\dot a}{a}\langle\nabla^2\Delta^{\frac{L}{2}-1}N, \nabla\phi\rangle_G + \mathfrak{J}([\partial_t, \nabla\Delta^{\frac{L}{2}}]\phi)$$

and

$$\mathfrak{P}_{L+1,Border} = -3\Psi\frac{\dot a}{a}\nabla\Delta^{\frac{L}{2}}N + \frac{\dot a}{a}\nabla\Psi * \nabla^2\Delta^{\frac{L}{2}-1}N + 2a^{-3}\langle\Sigma, \nabla^3\Delta^{\frac{L}{2}-1}\Psi\rangle_G$$
$$+ a^{-3}(N+1)\nabla\Sigma * \nabla^L\Psi + 2a^{-3}\nabla^{L-2}\mathrm{Ric} * \Sigma * \nabla\Psi$$
$$+ a^{-3}(N+1)\nabla^2\Delta^{\frac{L}{2}-1}\Sigma * \nabla\Psi \tag{2.A.13e}$$

$$\mathfrak{P}_{L+1,Junk} = \frac{\dot a}{a}\sum_{I_N+I_\Psi=L+1,\, I_\Psi\geq 2}\nabla^{I_N}N * \nabla^{I_\Psi}\Psi + a\sum_{I_N+I_\phi=L+2,\, I_N,I_\phi\neq 0}\nabla^{I_N}N * \nabla^{I_\phi+1}\phi$$
$$+ \mathfrak{J}([\partial_t, \nabla\Delta^{\frac{L}{2}}]\Psi) \tag{2.A.13f}$$

$$\mathfrak{Q}_{L+1,Border} = a^{-3}\Psi\Delta^{\frac{L}{2}+1}N + a^{-3}\nabla\Psi * \nabla\Delta^{\frac{L}{2}}N + a^{-3}(N+1)\Sigma * \nabla^2\Delta^{\frac{L}{2}}\phi + \frac{\dot a}{a}\nabla\Delta^{\frac{L}{2}}N * \nabla\phi$$
$$+ a^{-3}(N+1)\nabla\Sigma * \nabla^2\Delta^{\frac{L}{2}-1}\phi \tag{2.A.13g}$$

$$\mathfrak{Q}_{L+1,Junk} = a^{-3}\sum_{I_N+I_\Psi=L+2,\, 2\leq I_\Psi\leq L+1}\nabla^{I_N}N * \nabla^{I_\Psi}\Psi + a^{-3}(N+1)\nabla^{L-2}\mathrm{Ric}[G] * \Sigma * \nabla\phi$$
$$+ a^{-3}(N+1)\nabla^2\Delta^{\frac{L}{2}-1}\Sigma * \nabla\phi + \mathfrak{J}([\partial_t, \Delta^{\frac{L}{2}+1}]\phi) \tag{2.A.13h}$$

as well as

$$\mathfrak{Q}_{1,Border} = a^{-3}\Psi\Delta N + a^{-3}(N+1)\Sigma * \nabla^2\phi \tag{2.A.13i}$$
$$\mathfrak{Q}_{1,Junk} = a^{-3}\nabla\Psi * \nabla N + a^{-3}(N+1)\nabla\Sigma * \nabla\phi + \mathfrak{J}([\partial_t, \Delta]\phi)\,. \tag{2.A.13j}$$

The commuted rescaled evolution equation for Σ has the error terms

(2.A.14a)
$$\mathfrak{S}_{L,Border} = a^{-3}(N+1)\left(\Sigma * \nabla^2\Delta^{\frac{L}{2}-1}\Sigma + \nabla\Sigma * \nabla^3\Delta^{\frac{L}{2}-2}\Sigma\right) + a^{-3}\left(\Delta^{\frac{L}{2}}N\cdot(\Sigma * \Sigma) + \nabla\Delta^{\frac{L}{2}-1}N * \nabla\Sigma * \Sigma\right)$$
$$+ a^{-3}(N+1)\Sigma * \Sigma * \nabla^2\Delta^{\frac{L}{2}-2}\mathrm{Ric}[G] + \frac{\dot a}{a}\Delta^{\frac{L}{2}}N * \Sigma + \frac{\dot a}{a}\nabla\Delta^{\frac{L}{2}-1}N * \nabla\Sigma$$
$$+ \underbrace{a^{-3}[(N+1)\nabla\Sigma * \Sigma + \nabla N * \Sigma * \Sigma] * \nabla^{L-3}\mathrm{Ric}[G]}_{\text{not present for } L=2}$$

(2.A.14b)
$$\mathfrak{S}_{L,Junk} = -a[\Delta^{\frac{L}{2}}, \nabla^2]N + a\sum_{I_N+I_{\mathrm{Ric}}=L, I_N\neq 0}\nabla^{I_N}N * \nabla^{I_{\mathrm{Ric}}}\mathrm{Ric}[G] + \frac{\dot a}{a}\sum_{I_N+I_\Sigma=L}\nabla^{I_N}N * \nabla^{I_\Sigma}\Sigma$$
$$+ a^{-3}\sum_{I_1+I_2=L,\, I_i>0}\nabla^{I_1}\Sigma * \nabla^{I_2}\Sigma + a^{-3}\sum_{I_N+I_1+I_2=L,\, I_N<L}\nabla^{I_N}N * \nabla^{I_1}\Sigma * \nabla^{I_2}\Sigma$$

$$+ a \sum_{I_N+I_1+I_2=L} \nabla^{I_N}(N+1) * \nabla^{I_1+1}\phi * \nabla^{I_2+1}\phi + \left(4\pi C^2 a^{-3} + \frac{1}{3}a\right)\Delta^{\frac{L}{2}} N \cdot G + \mathfrak{J}([\partial_t, \Delta^{\frac{L}{2}}]\Sigma)$$

while the commuted Ricci tensor evolution equations have error terms, where $\mathcal{I} = \sum_{i=1}^{L/2-m+1} I_i$,

(2.A.15a) $$\mathfrak{R}_{L,Border} = a^{-3}\left[\nabla^{L+2}N \cdot \Sigma + \nabla^{L+1}N * \nabla\Sigma + \Sigma * \nabla^2\Delta^{\frac{L}{2}-1}\mathrm{Ric}[G] + \nabla\Sigma * \nabla^{L-1}\mathrm{Ric}[G]\right]$$

(2.A.15b) $$\mathfrak{R}_{L+1,Border} = a^{-3}\left[\nabla^{L+3}N \cdot \Sigma + \nabla^{L+2}N * \nabla\Sigma + \Sigma * \nabla^3\Delta^{\frac{L}{2}-1}\mathrm{Ric}[G] + \nabla\Sigma * \nabla^{L}\mathrm{Ric}[G]\right]$$

(2.A.15c) $$\begin{aligned}\mathfrak{R}_{L,Junk} &= a^{-3} \sum_{I_N+I_\Sigma=L+2,\, I_\Sigma\geq 2} \nabla^{I_N}N * \nabla^{I_\Sigma}\Sigma \\ &\quad + a^{-3} \sum_{\substack{I_N+I_\Sigma+I_{\mathrm{Ric}}=L \\ (I_\Sigma,I_{\mathrm{Ric}})\neq(0,L),(1,L-1)}} \nabla^{I_N}(N+1) * \nabla^{I_\Sigma}\Sigma * \nabla^{I_{\mathrm{Ric}}}\mathrm{Ric}[G] \\ &\quad + a^{-3}\sum_{m=0}^{\frac{L}{2}-1} \sum_{I_N+I_\Sigma+\mathcal{I}=2m} \nabla^{I_N}(N+1) * \nabla^{I_\Sigma}\Sigma * \nabla^{I_1}\mathrm{Ric}[G] * \dots * \nabla^{I_{\frac{L}{2}-m+1}}\mathrm{Ric}[G] \\ &\quad + \frac{\dot a}{a}\left([\Delta^{\frac{L}{2}}, \nabla^2]N + \Delta^{\frac{L}{2}}N * \mathrm{Ric}[G] + \nabla^{L-1}N * \nabla\mathrm{Ric}[G]\right) + \mathfrak{J}([\partial_t, \Delta^{\frac{L}{2}}]\mathrm{Ric}[G])\end{aligned}$$

(2.A.15d) $$\begin{aligned}\mathfrak{R}_{L+1,Junk} &= a^{-3} \sum_{I_N+I_\Sigma=L+3,\, I_\Sigma\geq 2} \nabla^{I_N}N * \nabla^{I_\Sigma}\Sigma \\ &\quad + a^{-3} \sum_{\substack{I_N+I_\Sigma+I_{\mathrm{Ric}}=L \\ (I_\Sigma,I_{\mathrm{Ric}})\neq(0,L+1),(1,L)}} \nabla^{I_N}(N+1) * \nabla^{I_\Sigma}\Sigma * \nabla^{I_{\mathrm{Ric}}}\mathrm{Ric}[G] \\ &\quad + a^{-3}\sum_{m=0}^{\frac{L}{2}-1} \sum_{I_N+I_\Sigma+\mathcal{I}=2m+1} \nabla^{I_N}(N+1) * \nabla^{I_\Sigma}\Sigma * \nabla^{I_1}\mathrm{Ric}[G] * \dots \nabla^{I_{\frac{L}{2}-m+1}}\mathrm{Ric}[G] \\ &\quad + \frac{\dot a}{a}\left(\nabla[\Delta^{\frac{L}{2}}, \nabla^2]N + \nabla\Delta^{\frac{L}{2}}N * \mathrm{Ric}[G] + \nabla^2\Delta^{\frac{L}{2}-1}N * \nabla\mathrm{Ric}[G]\right) + \mathfrak{J}([\partial_t, \nabla\Delta^{\frac{L}{2}}]\mathrm{Ric}[G])\end{aligned}$$

Finally, the Bel-Robinson evolution error terms are

(2.A.16a)

$$\begin{aligned}\mathfrak{E}_{L,Border} &= \frac{\tau}{3}\left(\Delta^{\frac{L}{2}}N \cdot \mathcal{E} + \nabla^{L-1}N * \nabla\mathcal{E}\right) - a^{-1}\left(\Delta^{\frac{L}{2}}\mathcal{E} \times \Sigma + \mathcal{E} \times \Delta^{\frac{L}{2}}\Sigma\right) \\ &\quad + a^{-3}\boldsymbol{\varepsilon}[G] * \boldsymbol{\varepsilon}[G] * \left(\nabla^{L-1}\mathcal{E} * \nabla\Sigma + \nabla\mathcal{E} * \nabla^{L-1}\Sigma\right) \\ &\quad + a^{-3}\Delta^{\frac{L}{2}}N \cdot (\mathcal{E} * \Sigma) + a^{-3}\nabla\Delta^{\frac{L}{2}-1}N * [\nabla\mathcal{E} * \Sigma + \mathcal{E} * \nabla\Sigma] \\ &\quad + a^{-3}\left(\Sigma * \nabla^2\Delta^{\frac{L}{2}-1}\mathcal{E} + \nabla\Sigma * \nabla^3\Delta^{\frac{L}{2}-2}\mathcal{E} + \nabla\mathcal{E} * \nabla\Delta^{\frac{L}{2}-1}\Sigma + \mathcal{E} * \Delta^{\frac{L}{2}}\Sigma\right) \\ &\quad + a^{-3}\left[(N+1)\Sigma * \mathcal{E} * \nabla^2\Delta^{\frac{L}{2}-2}\mathrm{Ric}[G] + \underbrace{\nabla\left((N+1) * \Sigma * \mathcal{E}\right) * \nabla^{L-3}\mathrm{Ric}[G]}_{\text{if } L\neq 2}\right] \\ &\quad + 4\pi a^{-3}(\Psi+C)^2\Delta^{\frac{L}{2}}N \cdot \Sigma + 4\pi a^{-3}\nabla^{L-1}N * \left[(\Psi+C)^2\nabla\Sigma + 2(\Psi+C) * \nabla\Psi * \Sigma\right] \\ &\quad + 4\pi a^{-3}(N+1)\left[(\Psi^2 + 2C\Psi)\Delta^{\frac{L}{2}}\Sigma + 2(\Psi+C)\Delta^{\frac{L}{2}}\Psi \cdot \Sigma\right] \\ &\quad + 4\pi a^{-3}\nabla^{L-1}\Sigma * \left[(\Psi+C)^2\nabla N + 2(N+1)(\Psi+C)\nabla\Psi\right] \\ &\quad + 4\pi a^{-3}(\Psi+C)\nabla^{L-1}\Psi * [(N+1)\nabla\Sigma + \nabla N * \Sigma]\end{aligned}$$

(2.A.16b)

$$\mathfrak{E}_{L,top} = a^{-1}(N+1)\boldsymbol{\varepsilon}[G] * \mathcal{B} * \nabla\Delta^{\frac{L}{2}-1}\mathrm{Ric}[G] + a(N+1)(\Psi+C)\nabla\Delta^{\frac{L}{2}-1}\mathrm{Ric}[G] * \nabla\phi$$

(2.A.16c)

$$\mathfrak{E}_{L,Junk} = \frac{\dot a}{a}\sum_{I_N+I_{\mathcal{E}}=L,\, I_N\leq L-2} \nabla^{I_N}N * \nabla^{I_{\mathcal{E}}}\mathcal{E}$$

$$
\begin{aligned}
&+ a^{-1}\boldsymbol{\varepsilon}[G] * \left[\sum_{I_N+I_{\mathcal{B}}=L+1,\, I_N, I_{\mathcal{B}}\leq L} \nabla^{I_N} N * \nabla^{I_{\mathcal{B}}}\mathcal{B} + (N+1)\nabla^2\Delta^{\frac{L}{2}-2}\mathrm{Ric}[G] * \nabla\mathcal{B}\right]\\
&+ a^{-3}\boldsymbol{\varepsilon}[G] * \boldsymbol{\varepsilon}[G] * \sum_{\substack{I_N+I_{\mathcal{E}}+I_\Sigma=L,\\ I_N\leq L-2;\, I_N>0 \text{ or } I_{\mathcal{E}}, I_\Sigma\geq 2}} \nabla^{I_N} N * \nabla^{I_{\mathcal{E}}}\mathcal{E} * \nabla^{I_\Sigma}\Sigma\\
&+ a \sum_{\substack{I_N+I_\Psi+I_\phi=L+1\\ I_N, I_\Psi, I_\phi\neq L+1}} \nabla^{I_N}(N+1) * \nabla^{I_\Psi}(\Psi+C) * \nabla^{I_\phi+1}\phi\\
&+ a^{-3} \sum_{\substack{I_N+I_\Sigma+I_1+I_2=L\\ I_N, I_\Sigma, I_i\leq L-2}} \nabla^{I_N}(N+1) * \nabla^{I_\Sigma}\Sigma * \nabla^{I_1}(\Psi+C) * \nabla^{I_2}(\Psi+C)\\
&+ \dot{a}a^3 \sum_{I_N+I_1+I_2=L} \nabla^{I_N}(N+1) * \nabla^{I_1+1}\phi * \nabla^{I_2+1}\phi\\
&+ a \sum_{I_N+I_\Sigma+I_1+I_2=L} \nabla^{I_N}(N+1) * \nabla^{I_\Sigma}\Sigma * \nabla^{I_1+1}\phi * \nabla^{I_2+1}\phi\\
&+ a^{-1}\varepsilon[G] * \mathcal{B} * [\Delta^{\frac{L}{2}}, \nabla]N + 4\pi a(N+1)(\Psi+C)\left[\nabla^{L-2}\mathrm{Ric}[G] * \nabla^2\phi + \mathfrak{J}([\Delta^{\frac{L}{2}}, \nabla^2]\phi)\right]\\
&+ a\left\{(\Psi+C)[\Delta^{\frac{L}{2}}, \nabla]N + (N+1)[\Delta^{\frac{L}{2}}, \nabla]\Psi\right\} * \nabla\phi + (N+1)a^{-1}\mathfrak{J}([\Delta^{\frac{L}{2}}, \mathrm{curl}_G]\mathcal{B}) + \mathfrak{J}([\partial_t, \Delta^{\frac{L}{2}}]\mathcal{E})\\
&+ \Delta^{\frac{L}{2}}\left[a^{-3}(N+1)\mathcal{E} * \Sigma + \frac{2\pi}{3}a^6(N+1)\left(\partial_0\left(a^{-6}(\Psi+C)^2 + a^{-2}|\nabla\phi|_G^2\right) + 4\pi\frac{\dot{a}}{a}(\Psi+C)^2\right)\right]\cdot G
\end{aligned}
$$

(2.A.16d)

$$
\begin{aligned}
\mathfrak{B}_{L,Border} =& \frac{\tau}{3}\left(\Delta^{\frac{L}{2}}N\cdot\mathcal{B} + \nabla^{L-1}N * \nabla\mathcal{B}\right) - a^{-1}\left(\Delta^{\frac{L}{2}}\mathcal{B}\times\Sigma + \mathcal{B} * \Delta^{\frac{L}{2}}\Sigma\right)\\
&+ a^{-3}\boldsymbol{\varepsilon}[G] * \boldsymbol{\varepsilon}[G] * \left(\nabla^{L-1}\mathcal{B} * \nabla\Sigma + \nabla\mathcal{B} * \nabla^{L-1}\Sigma\right)\\
&+ a^{-3}\Delta^{\frac{L}{2}}N\cdot(\mathcal{B} * \Sigma) + a^{-3}\nabla\Delta^{\frac{L}{2}-1}N\cdot[\nabla\mathcal{B} * \Sigma + \mathcal{B} * \nabla\Sigma]\\
&+ a^{-3}\left(\Sigma * \nabla^2\Delta^{\frac{L}{2}-1}\mathcal{B} + \nabla\Sigma * \nabla^3\Delta^{\frac{L}{2}-2}\mathcal{B} + \nabla\mathcal{B} * \nabla\Delta^{\frac{L}{2}-1}\Sigma + \mathcal{B} * \Delta^{\frac{L}{2}}\Sigma\right)\\
&+ a^{-3}\left[(N+1)\Sigma * \mathcal{B} * \nabla^{L-2}\mathrm{Ric}[G] + \underbrace{\nabla\left((N+1) * \Sigma * \mathcal{B}\right) * \nabla^{L-3}\mathrm{Ric}[G]}_{\text{if } L\neq 2}\right]\\
&+ a^{-1}(N+1)(\Psi+C)\cdot\boldsymbol{\varepsilon}[G] * \nabla\Delta^{\frac{L}{2}}\phi * \Sigma + a^{-1}\boldsymbol{\varepsilon}[G] * \nabla^2\nabla^L\phi * \nabla\left((N+1)(\Psi+C)\Sigma\right)
\end{aligned}
$$

(2.A.16e)

$$
\mathfrak{B}_{L,top} = a^3(N+1)\boldsymbol{\varepsilon}[G] * \nabla\Delta^{\frac{L}{2}-1}\mathrm{Ric}[G] * \nabla\phi * \nabla\phi + a^{-1}\boldsymbol{\varepsilon}[G] * \mathcal{E} * \nabla\Delta^{\frac{L}{2}-1}\mathrm{Ric}[G]
$$

(2.A.16f)

$$
\begin{aligned}
\mathfrak{B}_{L,Junk} =& \frac{\dot{a}}{a}\sum_{I_N+I_{\mathcal{B}}=L, I_N\leq L-2} \nabla^{I_N}N * \nabla^{I_{\mathcal{B}}}\mathcal{B}\\
&+ a^{-1}\boldsymbol{\varepsilon}[G] * \left[\sum_{I_N+I_{\mathcal{E}}=L+1, I_N, I_{\mathcal{E}}\leq L} \nabla^{I_N}N * \nabla^{I_{\mathcal{E}}}\mathcal{E} + \nabla^2\Delta^{\frac{L}{2}-2}\mathrm{Ric}[G] * \nabla\mathcal{E}\right]\\
&+ a^{-3}\boldsymbol{\varepsilon}[G] * \boldsymbol{\varepsilon}[G] * \sum_{\substack{I_N+I_{\mathcal{E}}+I_\Sigma=L,\\ I_N\leq L-2;\, I_N>0 \text{ or } I_{\mathcal{B}}, I_\Sigma\geq 2}} \nabla^{I_N}(N+1) * \nabla^{I_{\mathcal{B}}}\mathcal{B} * \nabla^{I_\Sigma}\Sigma\\
&+ a^3\boldsymbol{\varepsilon}[G] * \sum_{\substack{I_N+I_1+I_2=L,\\ I_N>0 \text{ or } I_2<L}} \nabla^{I_N}(N+1) * \nabla^{I_1+1}\phi * \nabla^{I_2+2}\phi\\
&+ a^{-1}\boldsymbol{\varepsilon}[G] * \sum_{\substack{I_N+I_\Psi+I_\phi+I_\Sigma=L\\ I_\phi\leq L-2}} \nabla^{I_N}(N+1) * \nabla^{I_\Psi}(\Psi+C) * \nabla^{I_\phi+1}\phi * \nabla^{I_\Sigma}\Sigma\\
&+ a^{-1}\varepsilon[G] * \mathcal{E} * [\Delta^{\frac{L}{2}}, \nabla]N\\
&+ a^{-1}(N+1)(\Psi+C)\cdot\boldsymbol{\varepsilon}[G] * \Sigma * \left[\Delta^{\frac{L}{2}}, \nabla\right]\phi
\end{aligned}
$$

$$+ a^3(N+1)\varepsilon[G] * \nabla\phi * \left(\nabla^2\Delta^{\frac{L}{2}-2}\mathrm{Ric}[G] * \nabla^2\phi + \mathfrak{I}([\Delta^{\frac{L}{2}}, \nabla]\phi)\right)$$
$$- a^{-1}(N+1)\mathfrak{I}([\Delta^{\frac{L}{2}}, \mathrm{curl}_G]\mathcal{E}) + \mathfrak{I}([\partial_t, \Delta^{\frac{L}{2}}]\mathcal{B}) + \Delta^{\frac{L}{2}}\left[a^{-3}(N+1)\mathcal{B} * \Sigma\right] \cdot G$$
$$+ \Delta^{\frac{L}{2}}\left[4\pi a^2\nabla^{\sharp m}\phi(\Psi + C) + \frac{2\pi}{3}a^5\Delta^{\frac{L}{2}}\nabla^{\sharp m}\left(a^{-6}(\Psi+C)^2 + a^{-2}|\nabla\phi|^2_G\right)\right]\varepsilon[G]_{(\cdot)m(\cdot)}$$

2.A.4. L^2_G error term estimates. In this subsection, we collect how the error terms can be controlled in terms of energies as well as homogeneous Sobolev norms of ϕ. We do not claim that these estimates are optimal – in particular, we note that at low order (like $L = 2$), many of the curvature errors that appear in the estimates below could be avoided entirely: These arise as a result of applying the general estimates in Lemma 2.4.5 where the Ricci tensor does not naturally occur in the respective equations, and can be avoided at low orders by applying (2.4.4f) on all curvature terms that occur.
Instead of optimality, we try to keep both notation and form of the error term estimates as simple as possible and the energy estimates between base and top level as unified as possible. In particular, we track the "worst" curvature energy occurring at high orders for all estimates below, even if these terms are added in artificially for low orders.

LEMMA 2.A.10 (Estimates for borderline error terms). *Let $L \in 2\mathbb{Z}_+$, $L \leq 20$. Then, the following estimates hold:*

(2.A.17a)
$$\|\mathfrak{H}_{L,Border}\|_{L^2_G} \lesssim \varepsilon a^{-4}\sqrt{\mathfrak{E}^{(L)}(\Sigma,\cdot)} + \varepsilon a^{-4-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-2)}(\Sigma,\cdot)} + \varepsilon^2 a^{-4-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-2)}(\mathrm{Ric},\cdot)}$$
(2.A.17b)
$$\begin{aligned}\|\mathfrak{N}_{L,Border}\|_{L^2_G} \lesssim\, &\varepsilon a^{-4}\left[\sqrt{\mathfrak{E}^{(L)}(\phi,\cdot)} + \sqrt{\mathfrak{E}^{(L)}(\Sigma,\cdot)}\right] + \varepsilon a^{-4}\sqrt{\mathfrak{E}^{(L)}(N,\cdot)}\\ &+ \varepsilon a^{-4-c\sqrt{\varepsilon}}\left[\sqrt{\mathfrak{E}^{(\leq L-2)}(\phi,\cdot)} + \sqrt{\mathfrak{E}^{(\leq L-2)}(\Sigma,\cdot)} + \sqrt{\mathfrak{E}^{(\leq L-2)}(N,\cdot)}\right]\\ &+ \underbrace{\varepsilon^2 a^{-4}\sqrt{\mathfrak{E}^{(\leq L-2)}(\mathrm{Ric},\cdot)} + \varepsilon a^{-4-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-4)}(\mathrm{Ric},\cdot)}}_{\text{not present for } L=2}\end{aligned}$$

(2.A.17c)
$$\begin{aligned}\|\mathfrak{N}_{L+1,Border}\|_{L^2_G} \lesssim\, &\varepsilon a^{-6}\left[\sqrt{a^4\mathfrak{E}^{(L+1)}(\phi,\cdot)} + \sqrt{a^4\mathfrak{E}^{(L+1)}(\Sigma,\cdot)}\right] + \varepsilon a^{-6}\sqrt{a^4\mathfrak{E}^{(L+1)}(N,\cdot)}\\ &+ \varepsilon a^{-4}\left[\sqrt{\mathfrak{E}^{(L)}(\phi,\cdot)} + \sqrt{\mathfrak{E}^{(L)}(\Sigma,\cdot)} + \sqrt{\mathfrak{E}^{(L)}(N,\cdot)}\right]\\ &+ \varepsilon a^{-4-c\sqrt{\varepsilon}}\left[\sqrt{\mathfrak{E}^{(\leq L-2)}(\phi,\cdot)} + \sqrt{\mathfrak{E}^{(\leq L-2)}(\Sigma,\cdot)} + \sqrt{\mathfrak{E}^{(\leq L-2)}(N,\cdot)}\right]\\ &+ \varepsilon^2 a^{-4-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-2)}(\mathrm{Ric},\cdot)}\end{aligned}$$
(2.A.17d)
$$\begin{aligned}\|\mathfrak{P}_{L,Border}\|_{L^2_G} \lesssim\, &\varepsilon a^{-3}\sqrt{\mathfrak{E}^{(L)}(\phi,\cdot)} + \varepsilon a^{-3}\sqrt{\mathfrak{E}^{(L)}(N,\cdot)} + \varepsilon a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-2)}(N,\cdot)}\\ &+ \varepsilon a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-2)}(\phi,\cdot)} + \varepsilon a^{-3}\sqrt{\mathfrak{E}^{(L)}(\Sigma,\cdot)} + \varepsilon a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(L-2)}(\Sigma,\cdot)}\end{aligned}$$

$$+\underbrace{\varepsilon^2 a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-3)}(\mathrm{Ric},\cdot)}}_{\textit{not present for } L=2}$$

(2.A.17e)

$$\|\mathfrak{Q}_{L,Border}\|_{L^2_G} \lesssim \varepsilon a^{-3}\sqrt{\mathfrak{E}^{(L+1)}(N,\cdot)} + \varepsilon a^{-3}\sqrt{a^{-4}\mathfrak{E}^{(L)}(\phi,\cdot)} + \varepsilon a^{-3-c\sqrt{\varepsilon}}\sqrt{a^{-4}\mathfrak{E}^{(\leq L-2)}(\phi,\cdot)}$$

(2.A.17f)

$$\begin{aligned}\|\mathfrak{P}_{L+1,Border}\|_{L^2_G} \lesssim{}& \varepsilon a^{-3}\sqrt{\mathfrak{E}^{(L+1)}(\phi,\cdot)} + \varepsilon a^{-3}\sqrt{\mathfrak{E}^{(L+1)}(N,\cdot)} + \varepsilon a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-1)}(N,\cdot)}\\ &+ \varepsilon a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-1)}(\phi,\cdot)} + \varepsilon a^{-3}\sqrt{\mathfrak{E}^{(L+1)}(\Sigma,\cdot)}\\ &+ \varepsilon a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(L-1)}(\Sigma,\cdot)} + \varepsilon^2 a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-2)}(\mathrm{Ric},\cdot)}\end{aligned}$$

(2.A.17g)

$$\begin{aligned}\|\mathfrak{Q}_{L+1,Border}\|_{L^2_G} \lesssim{}& \varepsilon a^{-3}\sqrt{\mathfrak{E}^{(L+2)}(N,\cdot)} + \varepsilon a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(L+1)}(N,\cdot)}\\ &+ \varepsilon a^{-3}\sqrt{a^{-4}\mathfrak{E}^{(L+1)}(\phi,\cdot)} + \varepsilon a^{-3-c\sqrt{\varepsilon}}\sqrt{a^{-4}\mathfrak{E}^{(\leq L-1)}(\phi,\cdot)}\end{aligned}$$

(2.A.17h)

$$\begin{aligned}\|\mathfrak{S}_{L,Border}\|_{L^2_G} \lesssim{}& \varepsilon a^{-3}\sqrt{\mathfrak{E}^{(L)}(\Sigma,\cdot)} + \varepsilon a^{-3}\sqrt{\mathfrak{E}^{(L)}(N,\cdot)} + \varepsilon^2 a^{-3}\sqrt{\mathfrak{E}^{(L-2)}(\mathrm{Ric},\cdot)}\\ &+ \varepsilon a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-2)}(\Sigma,\cdot)} + \varepsilon a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-2)}(N,\cdot)}\\ &+ \underbrace{\varepsilon^2 a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-4)}(\mathrm{Ric},\cdot)}}_{\textit{not present for } L=2}\end{aligned}$$

(2.A.17i)

$$\begin{aligned}\|\mathfrak{R}_{L,Border}\|_{L^2_G} \lesssim{}& \varepsilon a^{-3}\sqrt{\mathfrak{E}^{(L+2)}(N,\cdot)} + \varepsilon a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L)}(N,\cdot)}\\ &+ \varepsilon a^{-3}\sqrt{\mathfrak{E}^{(L)}(\mathrm{Ric},\cdot)} + \varepsilon a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-2)}(\mathrm{Ric},\cdot)}\end{aligned}$$

(2.A.17j)

$$\begin{aligned}\|\mathfrak{R}_{L+1,Border}\|_{L^2_G} \lesssim{}& \varepsilon a^{-3}\sqrt{\mathfrak{E}^{(L+3)}(N,\cdot)} + \varepsilon a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L+1)}(N,\cdot)}\\ &+ \varepsilon a^{-3}\sqrt{\mathfrak{E}^{(L+1)}(\mathrm{Ric},\cdot)} + \varepsilon a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-1)}(\mathrm{Ric},\cdot)}\end{aligned}$$

(2.A.17k)

$$\begin{aligned}&\|\mathfrak{E}_{L,Border}\|_{L^2_G} + \|\mathfrak{B}_{L,Border}\|_{L^2_G}\\ &\lesssim \varepsilon a^{-3}\sqrt{\mathfrak{E}^{(L)}(\phi,\cdot)} + \varepsilon a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-2)}(\phi,\cdot)}\\ &\quad+ \varepsilon a^{-3}\left(\sqrt{\mathfrak{E}^{(L)}(N,\cdot)} + \sqrt{\mathfrak{E}^{(L)}(\Sigma,\cdot)}\right) + \varepsilon a^{-3}\sqrt{\mathfrak{E}^{(L)}(W,\cdot)}\\ &\quad+ \varepsilon a^{-3-c\sqrt{\varepsilon}}\left(\sqrt{\mathfrak{E}^{(\leq L-2)}(N,\cdot)} + \sqrt{\mathfrak{E}^{(\leq L-2)}(\Sigma,\cdot)} + \sqrt{\mathfrak{E}^{(\leq L-2)}(W,\cdot)}\right)\\ &\quad+ \varepsilon^2 a^{-3}\sqrt{\mathfrak{E}^{(L-2)}(\mathrm{Ric},\cdot)} + \underbrace{\varepsilon^2 a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-4)}(\mathrm{Ric},\cdot)}}_{\textit{not present for } L=2}\end{aligned}$$

Proof. All of these estimates follow from applying L^2_G-L^∞_G-type Hölder estimates to the individual nonlinear terms. The lower order terms are either controlled by the zero order estimates in subsection 2.4.1 or the a priori estimates in Lemma 2.4.3. Furthermore, we apply Lemma 2.4.5, along with again Lemma 2.4.3, to translate L^2_G-norms into energies up to additional curvature energy terms. For the sake of simplicity, we always estimate $\frac{\dot{a}}{a}$ by a^{-3} up to constant (see (2.2.3)), and liberally apply (2.3.8) to deal with odd order energies and to distribute $a^{-c\sqrt{\varepsilon}}$ factors to lower orders while updating $c > 0$ wherever this is convenient. $\square$

Lemma 2.A.11 (Estimates for top order error terms).

$$\|\mathfrak{E}_{L,top}\|_{L^2_G} \lesssim \sqrt{\varepsilon} a^{1-c\sqrt{\varepsilon}} \sqrt{\mathfrak{E}^{(L-1)}(\mathrm{Ric},\cdot)} = \sqrt{\varepsilon} a^{-1-c\sqrt{\varepsilon}} \sqrt{a^4 \mathfrak{E}^{(L-1)}(\mathrm{Ric},\cdot)} \tag{2.A.18a}$$

$$\|\mathfrak{B}_{L,top}\|_{L^2_G} \lesssim \varepsilon a^{-1} \sqrt{\mathfrak{E}^{(L-1)}(\mathrm{Ric},\cdot)} = \varepsilon a^{-3} \sqrt{a^4 \mathfrak{E}^{(L-1)}(\mathrm{Ric},\cdot)} \tag{2.A.18b}$$

Proof. This follows directly using (2.4.2c) and (2.4.4g) for the Bel-Robinson terms as well as (2.4.4e). $\square$

Lemma 2.A.12 (Junk terms). *Recalling $\|$-notation from Remark 2.2.12, the following hold:*

(2.A.19a)

$$\begin{aligned}\|\mathfrak{M}_{L,Junk}\|_{L^2_G} \lesssim& \varepsilon a^{-2-c\sqrt{\varepsilon}} \sqrt{\mathfrak{E}^{(\leq L-1)}(\phi,\cdot)} + a^{-2-c\sqrt{\varepsilon}} \sqrt{\mathfrak{E}^{(\leq L-2)}(\phi,\cdot)}\\ &+ a^{-c\sqrt{\varepsilon}} \sqrt{\mathfrak{E}^{(\leq L-1)}(\Sigma,\cdot)} + \sqrt{\varepsilon} a^{-c\sqrt{\varepsilon}} \sqrt{\mathfrak{E}^{(\leq L-2)}(\mathrm{Ric},\cdot)}\end{aligned}$$

(2.A.19b)

$$\|\tilde{\mathfrak{M}}_{L,Junk}\|_{L^2_G} \lesssim \varepsilon a^{-1-c\sqrt{\varepsilon}} \sqrt{\mathfrak{E}^{(\leq L-2)}(\mathrm{Ric},\cdot)} + a^{-1-c\sqrt{\varepsilon}} \sqrt{\mathfrak{E}^{(\leq L-1)}(\Sigma,\cdot)}$$

(2.A.19c)

$$\begin{aligned}\|\mathfrak{H}^{\|}_{L,Junk}\|_{L^2_G} \lesssim& \varepsilon a^{-4-c\sqrt{\varepsilon}} \sqrt{\mathfrak{E}^{(\leq L-2)}(\Sigma,\cdot)} + \sqrt{\varepsilon} a^{-2-c\sqrt{\varepsilon}} \sqrt{\mathfrak{E}^{(\leq L)}(\phi,\cdot)}\\ &+ \varepsilon a^{-c\sqrt{\varepsilon}} \sqrt{\mathfrak{E}^{(\leq L-2)}(\mathrm{Ric},\cdot)}\end{aligned}$$

(2.A.19d)

$$\begin{aligned}\|\mathfrak{N}_{L,Junk}\|_{L^2_G} \lesssim& \varepsilon a^{-4-c\sqrt{\varepsilon}} \sqrt{\mathfrak{E}^{(\leq L-2)}(N,\cdot)} + \varepsilon a^{-c\sigma} \sqrt{\mathfrak{E}^{(L)}(\phi,\cdot)}\\ &+ \varepsilon a^{-4-c\sqrt{\varepsilon}} \left[\sqrt{\mathfrak{E}^{(\leq L-2)}(\phi,\cdot)} + \sqrt{\mathfrak{E}^{(\leq L-2)}(\Sigma,\cdot)}\right]\\ &+ \underbrace{\varepsilon a^{-4} \sqrt{\mathfrak{E}^{(\leq L-2)}(\mathrm{Ric},\cdot)} + \varepsilon a^{-4-c\sqrt{\varepsilon}} \sqrt{\mathfrak{E}^{(\leq L-4)}(\mathrm{Ric},\cdot)}}_{\textit{not present for } L=2}\end{aligned}$$

(2.A.19e)

$$\begin{aligned}\|\mathfrak{N}_{L+1,Junk}\|_{L^2_G} \lesssim& \varepsilon a^{-4-c\sqrt{\varepsilon}} \sqrt{\mathfrak{E}^{(\leq L-1)}(N,\cdot)} + \varepsilon a^{-c\sigma} \left(\sqrt{\mathfrak{E}^{(L+1)}(\phi,\cdot)} + \sqrt{\mathfrak{E}^{(L+1)}(\Sigma,\cdot)}\right)\\ &+ \varepsilon a^{-4-c\sqrt{\varepsilon}} \left[\sqrt{\mathfrak{E}^{(\leq L-1)}(\phi,\cdot)} + \sqrt{\mathfrak{E}^{(\leq L-1)}(\Sigma,\cdot)}\right]\end{aligned}$$

$$+\varepsilon^2 a^{-4}\sqrt{\mathfrak{E}^{(\leq L-1)}(\mathrm{Ric},\cdot)}+\underbrace{\varepsilon^2 a^{-4-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-3)}(\mathrm{Ric},\cdot)}}_{\textit{not present for } L=2}$$

(2.A.19f)

$$\begin{aligned}\|\mathfrak{P}_{L,Junk}\|_{L^2_G}\lesssim{}&\varepsilon a^{1-c\sigma}\sqrt{\mathfrak{E}^{(L)}(\phi,\cdot)}+\varepsilon a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-2)}(\phi,\cdot)}\\&+\varepsilon a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-2)}(\Sigma,\cdot)}+\sqrt{\varepsilon}a^{1-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(L)}(N,\cdot)}\\&+\left[\varepsilon a^{-3-c\sqrt{\varepsilon}}+\sqrt{\varepsilon}a^{-1-c\sqrt{\varepsilon}}\right]\sqrt{\mathfrak{E}^{(\leq L-2)}(N,\cdot)}\\&+\underbrace{\varepsilon^2 a^{1-c\sigma}\sqrt{\mathfrak{E}^{(\leq L-2)}(\mathrm{Ric},\cdot)}+\varepsilon^2 a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-3)}(\mathrm{Ric},\cdot)}}_{\textit{not present for } L=2}\end{aligned}$$

(2.A.19g)

$$\begin{aligned}\|\mathfrak{P}_{L+1,Junk}\|_{L^2_G}\lesssim{}&\varepsilon a^{1-c\sigma}\sqrt{\mathfrak{E}^{(L+1)}(\phi,\cdot)}+\varepsilon a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-1)}(\phi,\cdot)}\\&+\varepsilon a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-1)}(\Sigma,\cdot)}+\sqrt{\varepsilon}a^{1-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(L+1)}(N,\cdot)}\\&+\left[\varepsilon a^{-3-c\sqrt{\varepsilon}}+\sqrt{\varepsilon}a^{-1-c\sqrt{\varepsilon}}\right]\sqrt{\mathfrak{E}^{(\leq L-1)}(N,\cdot)}\\&+\varepsilon^2 a^{-1-c\sigma}\sqrt{a^4\mathfrak{E}^{(L-1)}(\mathrm{Ric},\cdot)}\\&+\left(\varepsilon^2 a^{-1-c\sigma}+\varepsilon^2 a^{-3-c\sqrt{\varepsilon}}\right)\sqrt{\mathfrak{E}^{(\leq L-2)}(\mathrm{Ric},\cdot)}\end{aligned}$$

(2.A.19h)

$$\begin{aligned}\|\mathfrak{Q}_{L,Junk}\|_{L^2_G}\lesssim{}&\varepsilon a^{-1-c\sqrt{\varepsilon}}\sqrt{a^{-4}\mathfrak{E}^{(L)}(\phi,\cdot)}+\varepsilon a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-2)}(\phi,\cdot)}\\&+\sqrt{\varepsilon}a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L)}(\Sigma,\cdot)}+\sqrt{\varepsilon}a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L)}(N,\cdot)}\\&+\underbrace{\varepsilon a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-2)}(\mathrm{Ric},\cdot)}}_{\textit{not present for } L=2}\end{aligned}$$

(2.A.19i)

$$\|\mathfrak{Q}_{1,Junk}\|\lesssim\varepsilon a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(1)}(N,\cdot)}+\varepsilon^{\frac{3}{2}}a^{-3-c\sqrt{\varepsilon}}\|\nabla\phi\|_{L^2_G}+\sqrt{\varepsilon}a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq 1)}(\Sigma,\cdot)}$$

(2.A.19j)

$$\begin{aligned}\|\mathfrak{Q}_{L+1,Junk}\|_{L^2_G}\lesssim{}&\varepsilon a^{-1-c\sqrt{\varepsilon}}\sqrt{a^{-4}\mathfrak{E}^{(L+1)}(\phi,\cdot)}+\varepsilon a^{-3-c\sqrt{\varepsilon}}\mathfrak{E}^{(\leq L-1)}(\phi,\cdot)\\&+\sqrt{\varepsilon}a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L+1)}(\Sigma,\cdot)}+\sqrt{\varepsilon}a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L+1)}(N,\cdot)}\\&+\varepsilon a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-1)}(\mathrm{Ric},\cdot)}\end{aligned}$$

(2.A.19k)

$$\begin{aligned}\|\mathfrak{S}^{\|}_{L,Junk}\|_{L^2_G}\lesssim{}&\varepsilon a^{1-c\sigma}\sqrt{\mathfrak{E}^{(L)}(\Sigma,\cdot)}+\varepsilon a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-2)}(\Sigma,\cdot)}+\sqrt{\varepsilon}a^{-1-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(L)}(\phi,\cdot)}\\&+\left(\varepsilon a^{-3}+a^{1-c\sqrt{\varepsilon}}\right)\sqrt{\mathfrak{E}^{(\leq L)}(N,\cdot)}+\varepsilon a^{5-c\sigma}\sqrt{\mathfrak{E}^{(\leq L-1)}(\mathrm{Ric},\cdot)}\end{aligned}$$

$$+\underbrace{\varepsilon a^{-3}\sqrt{\mathfrak{E}^{(L-2)}(\mathrm{Ric},\cdot)}+\varepsilon a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-4)}(\mathrm{Ric},\cdot)}}_{\textit{not present for } L=2}$$

(2.A.19l)

$$\begin{aligned}\|\mathfrak{R}_{L,Junk}\|_{L^2_G}\lesssim\,&\varepsilon^2a^{1-c\sigma}\sqrt{\mathfrak{E}^{(\leq L-1)}(\mathrm{Ric},\cdot)}+\varepsilon a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-2)}(\mathrm{Ric},\cdot)}\\&+\varepsilon a^{1-c\sigma}\sqrt{\mathfrak{E}^{(\leq L+2)}(\Sigma,\cdot)}+a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L)}(\Sigma,\cdot)}\\&+a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L)}(N,\cdot)}\end{aligned}$$

(2.A.19m)

$$\begin{aligned}\|\mathfrak{R}_{L+1,Junk}\|_{L^2_G}\lesssim\,&\varepsilon^2a^{1-c\sigma}\sqrt{\mathfrak{E}^{(\leq L)}(\mathrm{Ric},\cdot)}+\varepsilon a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-1)}(\mathrm{Ric},\cdot)}\\&+\varepsilon a^{1-c\sigma}\sqrt{\mathfrak{E}^{(\leq L+3)}(\Sigma,\cdot)}+a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L+1)}(\Sigma,\cdot)}\\&+a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L+1)}(N,\cdot)}\end{aligned}$$

(2.A.19n)

$$\begin{aligned}&\|\mathfrak{E}^{||}_{L,Junk}\|_{L^2_G}+\|\mathfrak{B}^{||}_{L,Junk}\|_{L^2_G}\\&\quad\lesssim\varepsilon a^{-1-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L)}(W,\cdot)}+\varepsilon a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-2)}(W,\cdot)}\\&\qquad+\varepsilon a^{-1-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(L)}(\phi,\cdot)}+\left(\varepsilon a^{-3-c\sqrt{\varepsilon}}+a^{-1-c\sqrt{\varepsilon}}\right)\sqrt{\mathfrak{E}^{(\leq L-2)}(\phi,\cdot)}\\&\qquad+\sqrt{\varepsilon}a^{1-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L)}(N,\cdot)}+\varepsilon a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-2)}(N,\cdot)}\\&\qquad+\varepsilon a^{-1-c\sigma}\sqrt{\mathfrak{E}^{(L)}(\Sigma,\cdot)}+\varepsilon a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-2)}(\Sigma,\cdot)}\\&\qquad+\varepsilon a^{-3}\sqrt{\mathfrak{E}^{(L-2)}(\mathrm{Ric},\cdot)}+\varepsilon a^{-3-c\sqrt{\varepsilon}}\underbrace{\sqrt{\mathfrak{E}^{(\leq L-4)}(\mathrm{Ric},\cdot)}}_{\textit{not present for } L=2}\end{aligned}$$

Proof. Once again, this follows by applying the a priori estimates from subsection 2.4.1 and Lemma 2.4.3, as well as the bootstrap assumption (2.3.17h) for the lapse, to deal with the lower order terms in the nonlinearities, and then applying Lemma 2.4.5 as well as (2.3.8) wherever this is necessary. Further, especially in (2.A.19n), it is often more convenient to use the bootstrap assumption for $\|\nabla\phi\|_{C_G}$ instead of the a priori estimate (2.4.4e) to gain higher powers of ε in prefactors.
Recognizing that every low order curvature term can be estimated up to constant by $a^{-c\sqrt{\varepsilon}}$ at worst (see (2.4.4f)), we also note that any of the highly nonlinear curvature terms in $\mathfrak{J}$-expressions turn out to be negligible after updating c compared to Ricci energies arising from applying Lemma 2.4.5 or compared to junk terms in which $\mathrm{Ric}[G]$ is tracked explicitly. □

2.B. Appendix – Future stability

Here, we collect the commutators in CMCSH gauge necessary to study the commuted scalar-field equations:

LEMMA 2.B.1 (Commutator formulas for future stabilty). *Let ζ be a scalar function on Σ_T. Then, the following formulas hold:*

$$
\begin{aligned}
[\widetilde{\partial}_0, \nabla]\zeta &= 0 \\
[\widetilde{\partial}_0, \Delta_{\boldsymbol{g}}]\zeta &= (\widetilde{\partial}_0(\boldsymbol{g}^{-1})^{ab})\nabla_a\nabla_b\zeta - 2(\boldsymbol{g}^{-1})^{ab}\left(\mathrm{div}_{\boldsymbol{g}}(\boldsymbol{n}\boldsymbol{\Sigma})_a - 2\nabla_a\boldsymbol{n}\right)\nabla_b\zeta
\end{aligned}
$$

Schematically, for $k \in \mathbb{N}$, this implies

(2.B.1a)
$$[\widetilde{\partial}_0, \Delta^k_{\boldsymbol{g}}]\zeta = \sum_{I_{\boldsymbol{n}}+I_{\boldsymbol{\Sigma}}+I_\zeta=2k-1} \nabla^{I_{\boldsymbol{n}}}\boldsymbol{n} *_{\boldsymbol{g}} \nabla^{I_{\boldsymbol{\Sigma}}}\boldsymbol{\Sigma} *_{\boldsymbol{g}} \nabla^{I_\zeta+1}\zeta + \sum_{I_{\hat{\boldsymbol{n}}}+I_\zeta=2k-1} \nabla^{I_{\hat{\boldsymbol{n}}}}\hat{\boldsymbol{n}} *_{\boldsymbol{g}} \nabla^{I_\zeta+1}\zeta$$

(2.B.1b)
$$[\widetilde{\partial}_0, \nabla\Delta^k_{\boldsymbol{g}}]\zeta = \sum_{I_{\boldsymbol{n}}+I_{\boldsymbol{\Sigma}}+I_\zeta=2k} \nabla^{I_{\boldsymbol{n}}}\boldsymbol{n} *_{\boldsymbol{g}} \nabla^{I_{\boldsymbol{\Sigma}}}\boldsymbol{\Sigma} *_{\boldsymbol{g}} \nabla^{I_\zeta+1}\zeta + \sum_{I_{\hat{\boldsymbol{n}}}+I_\zeta=2k} \nabla^{I_{\hat{\boldsymbol{n}}}}\hat{\boldsymbol{n}} *_{\boldsymbol{g}} \nabla^{I_\zeta+1}\zeta$$

PROOF. This follows from straightfoward computations, similar to Lemma 2.A.6 for the low order commutators and to Lemma 2.A.7 for higher orders. $\square$

CHAPTER 3

On the past maximal development of near-FLRW data for the Einstein scalar-field Vlasov system

3.1. Introduction

3.1.1. Setting and Results. The goal of this paper is to study the past behaviour of the Einstein equations in presence of both scalar field and massive ($m = 1$) or massless ($m = 0$) Vlasov matter near Friedman-Lemaître-Robertson-Walker (FLRW) solutions. On a $(3+1)$–dimensional spacetime $(\overline{M}, \overline{g})$, these equations, which we refer to as the Einstein scalar-field Vlasov (ESFV) system, are given as follows:

$$\mathrm{Ric}[\overline{g}]_{\mu\nu} - \frac{1}{2}R[\overline{g}]\overline{g}_{\mu\nu} = 8\pi\left(T^{SF}_{\mu\nu}[\overline{g},\phi] + T^{Vl}_{\mu\nu}[\overline{g},f]\right) \tag{3.1.1a}$$

$$T^{SF}_{\mu\nu}[\overline{g},\phi] = \overline{\nabla}_\mu\phi\overline{\nabla}_\nu\phi - \frac{1}{2}\overline{g}_{\mu\nu}\overline{\nabla}^\alpha\phi\overline{\nabla}_\alpha\phi \tag{3.1.1b}$$

$$T^{Vl}_{\mu\nu}[\overline{g},f] = \int_{P_{(t,x)}} p_\mu p_\nu f(t,x,p)\mathrm{dvol}_{P_{(t,x)}} \tag{3.1.1c}$$

$$\square_{\overline{g}}\phi = 0 \tag{3.1.1d}$$

$$\mathcal{X}f = 0 \tag{3.1.1e}$$

Here, $\mathcal{X}$ refers to the geodesic spray with respect to $\overline{g}$ on the co-mass shell

$$P = \bigsqcup_{(t,x)\in\overline{M}} P_{(t,x)}, \text{ where } P_{(t,x)} = \left\{p \in T^*_{(t,x)}\overline{M} \,|\, p_\mu p^\mu = -m^2,\ p^0 > 0\right\}.$$

FLRW solutions take the form

$$(I \times M, -dt^2 + a(t)^2\gamma)\,, \tag{3.1.2a}$$

where (M,γ) is a closed 3-manifold of constant sectional curvature $\kappa \in \mathbb{R}$ and the scale factor $a \in C^\infty(I)$ satisfies the Friedman equation (3.2.3a) with $a(0) = 0$. For $C \in \mathbb{R}^+$ and $\mathcal{F} \in C^\infty_c(\mathbb{R}^+_0, \mathbb{R}^+_0)$, the scalar field $\phi_{FLRW} \in C^\infty(\overline{M})$ and the Vlasov distribution function $f_{FLRW} \in C^\infty(P)$ are given by

$$\partial_t\phi_{FLRW} = Ca^{-3},\ \nabla\phi_{FLRW} = 0 \text{ and} \tag{3.1.2b}$$

$$f_{FLRW}(t,x,p) = \mathcal{F}((\gamma^{-1})^{ij}p_ip_j) = \mathcal{F}(a(t)^2|p|^2_{g_{FLRW}})\,. \tag{3.1.2c}$$

We remark that the scale factor behaves like $t^{\frac{1}{3}}$ toward the past singularity at $t = 0$ (see Lemma 3.2.2). The main result, stated in full in Theorem 3.8.1, can be summarized as follows.

THEOREM 3.1.1 (Past stability of FLRW solutions to the ESFV system, short version). *Let M be a closed Riemannian 3-manifold that admits a metric γ of constant sectional curvature. Let $\left(M, \mathring{g}, \mathring{k}, \mathring{\pi}, \mathring{\psi}, \mathring{f}\right)$ be CMC initial data in the sense discussed in Section 3.2.3 to the ESFV system* (3.1.1) *that is sufficiently close to FLRW initial data with isotropic matter given by a non-trivial scalar field and Vlasov initial data $\mathring{f}_{FLRW}$ with compact momentum support. In the case of massless particles, additionally assume that $\mathring{f}_{FLRW}$ and $\mathring{f}$ vanish in an open neighbourhood of the zero section of P. Then, the following statements hold.*

(1) *The past maximal globally hyperbolic development $(\overline{M}, \overline{g}, \phi, f)$ within the Einstein scalar-field Vlasov system is foliated by CMC hypersurfaces $M_s = t^{-1}(s),\ s > 0$, and exhibits stable Kretschmann scalar blow-up of order t^{-4} toward the Big Bang singularity at $t = 0$.*

(2) *The solution variables (g, k, ϕ, f) exhibit asymptotically velocity term dominated behaviour towards the Big Bang singularity. In particular, in an appropriate function space, $f(t, \cdot, \cdot)$ converges to a limiting distribution f_{Bang} on the cotangent bundle of the Big Bang hypersurface. This distribution is close to the initial distribution and vanishes for sufficiently large and, in the massless case, sufficiently small momenta.*

If one additionally assumes that the momentum support of $\mathring{f}$ is close to that of the FLRW solution, then so is that of f_{Bang}, see Corollary 3.8.3. We note that, when viewing the Vlasov distribution on the mass shell rather than the co-mass shell, the particle momenta concentrate in directions determined by the leading order asymptotics of the shear, as is discussed in Remark 3.8.5.

Additionally, for $\kappa > 0$, Theorem 3.1.1 also implies stable Big Crunch formation toward the future: The respective FLRW solutions exist on $\overline{M} = (0, T) \times M$ for some $T \in \mathbb{R}^+$ and are preserved under the time reflection $t \in (0, T) \rightsquigarrow T - t$ (see Lemma 3.2.1 and 3.2.2). Thus, the entire analysis toward the Big Bang singularity can be re-applied approaching the Big Crunch singularity in the case of positive spatial curvature.

This is, to our knowledge, the first asymptotic stability result in the contracting direction involving Vlasov matter that does not require any symmetry assumptions. Furthermore, since Vlasov matter asymptotically matches the equation of state of radiation (see Section 3.1.2.2), this extends the regime in which the presence of a scalar field stabilizes Big Bang formation compared to the recent result [BO24b] (see the discussion in Section 3.1.3.2).

3.1.2. Background.

3.1.2.1. *Singularity formation and quiescent cosmology.* Since the singularity theorems by Penrose [Pen65] and Hawking [Haw67], understanding the nature of singularity formation has been one of the core pursuits of mathematical relativity. While these theorems describe under which conditions spacetimes become causally incomplete, they provide no information about the type of singularity. In cosmological settings, the two following conjectures concerning past singularity formation have garnered particular attention:

Firstly, the *Strong Cosmic Censorship (SCC) conjecture* states that the maximal development of generic initial data becomes inextendible when it becomes geodesically incomplete. In view of results in the homogeneous setting by Chruściel and Rendall [CR95] and Ringström [Rin00; Rin01][1] as well in Gowdy symmetry by Ringström [Rin09a], the cosmological version of this conjecture usually specifies this to be C^2-inextendibility caused by blow-up of the Kretschmann scalar. Since directly computing the asymptotic behaviour of the Kretschmann scalar given general initial data is often difficult if not impossible, evidence toward this conjecture is commonly provided by showing that the Kretschmann scalar blow-up of a particular explicit solution is stable within solutions to the Einstein equations. We will discuss the available results in this direction for FLRW and Kasner spacetimes in Section 3.1.3.1.

Secondly, the *BKL conjecture* (originally formulated in the work of Belinskiĭ, Khalatnikov and Lifshitz [BKL70]) states that the asymptotics of vacuum cosmological solutions to the Einstein equations toward the Big Bang are driven by spatial inhomogeneities via spatial Ricci curvature terms. This should then cause the metric evolution to become highly oscillatory and chaotic, often referred to as "Mixmaster" behaviour (see [Mis69]). Originally, it was conjectured in [BKL70] that introducing matter to the Einstein equations does not change the leading order asymptotics compared the vacuum BKL picture. However, Belinskiĭ and Khalatnikov later observed in [BK73] that scalar field and stiff fluid matter appear to have an exceptional role: These matter models exhibit *quiescent Big Bang formation*, i.e., this choice of matter dampens any potential oscillations. This generates stable past asymptotics with asymptotically velocity term dominated (AVTD) asymptotics, i.e., the solution agrees at leading order with the asymptotics of the formal system obtained by dropping all spatial derivative terms from the equations. In particular, this would mean that the SCC conjecture holds for such solutions, since the Kretschmann scalar blows up for FLRW solutions with non-trivial scalar field. For a recent comprehensive result on which types of initial data to the Einstein nonlinear scalar field system generate quiescent singularities, see [OGPR23].

3.1.2.2. *Vlasov matter.* The universe in its current state can, at large scales, reasonably be modelled as a collisionless self-gravitating gas, where stars are viewed as particles with rest mass m and are distributed according to a distribution function f (see [BT08, Chapters 1.2 and 4.1]). Since one needs to require that all particles of the gas travel along timelike geodesics, f must satisfy the Vlasov equation (3.1.1e). The asymptotic behaviour of massive Vlasov matter on FLRW spacetimes in particular corresponds very well to the dominant matter components expected in standard cosmological models in the early and late universe (see, for example, [Ryd16, Sections 5.5 and 6.4]): Toward the past, where the universe is believed to have been radiation dominated, the density of massive and massless Vlasov matter

[1] see also [Rin09b, Chapter 22]

behaves like a^{-4} and the equation of state approaches that of a radiation fluid ($\mathfrak{p}^{Vl} \approx \frac{1}{3}\rho^{Vl}$; this holds with equality in the massless case). Toward the future, where one expects matter domination, the pressure converges to 0, approximating the equation of state of dust, and the Vlasov density is of order a^{-3}. Thus, even though a fully physical model of gas toward the Big Bang may need to include collisions, Vlasov matter is already a realistic matter type to consider from the perspective of FLRW cosmology.

While we will discuss some relevant results regarding past existence and asymptotic behaviour in Section 3.1.3, we also refer to [And11] for a more general overview of results regarding the Einstein-Vlasov system.

3.1.3. Relationship with previous work. This paper both expands the known regime of quiescent Big Bang formation caused by scalar fields and, to our knowledge, is the first result on past stability in presence of Vlasov matter without symmetry. Thus, we will proceed as follows: First, we compare our approach with the existing literature on Big Bang formation for the Einstein scalar-field system. Then, we contrast them with results for Einstein scalar-field fluid systems, since modelling the universe as a fluid is a common approach that is closely related to the Vlasov approach (see Section 3.1.2.2). Finally, we compare our results to those for Einstein (scalar-field) Vlasov matter.

3.1.3.1. *Quiescent singularity formation for scalar field matter.* Rodnianski and Speck proved the first result [RS18a] regarding past stability of the Einstein scalar-field system without symmetries, showing that Big Bang formation in FLRW spacetimes with spatial geometry $\mathbb{T}^3$ is linearly stable and asymptotically velocity term dominated, i.e., the spacetime and matter variables asymptotically behave like the solution to the velocity dominated ODE. This was extended to nonlinear stability in [RS18b], and then in turn to spherical and closed hyperbolic spatial hypersurfaces in [Spe18], respectively [FU25a], see Chapter 2. In the case of flat spatial geometry, the regime of past asymptotic stability has been extended to moderately anisotropic generalized Kasner spacetimes[2] in [RS22] and then to the all subcritical[3] ones in [FRS23]. That these results can be localized was first shown for spatially flat FLRW spacetimes in [BO24a], and in a recent preprint also for the entire subcritical Kasner regime in [BOZ25]. In all of these cases, the Kretschmann scalar exhibits stable blow-up at order t^{-4} approaching the Big Bang ($t \downarrow 0$); the solutions are AVTD. Furthermore, Big Bang stability near FLRW spacetimes can be localized, see [BO24a]. These recent developments indicate that spatial geometry has little bearing on the past asymptotic behaviour of non-vacuum solutions to the Einstein scalar-field equations. In fact, in their recent work, Oude Groeniger, Petersen and Ringström [OGPR23] were able to establish a very general criterion

[2] *Generalized Kasner spacetimes* are solutions to the Einstein scalar-field system of the form

$$\overline{g}_{Kas} = -dt^2 + \sum_{i=1}^{d} t^{2p_i} dx^i \otimes dx^i, \quad \phi_{Kas} = A\log(t), \quad \text{where } \sum_{i=1}^{d} p_i = 1, \quad \sum_{i=1}^{d} p_i^2 = 1 - 8\pi A^2, \quad A \in [0,1].$$

Note that, except for the FLRW case $p_1 = \cdots = p_d$, these are spatially homogeneous but not isotropic. Solutions in the vacuum case ($A = 0$) are simply referred to as Kasner spacetimes.

[3] A generalized Kasner spacetime is called subcritical if the Kasner exponents $(p_i)_{i=1,\dots,d}$ satisfy $p_i + p_j - p_k < 1$ for all $i, j, k = 1, \dots, d$.

for quiescent Big Bang formation in the Einstein non-linear scalar-field system, essentially only requiring the mean curvature to be large enough compared to the initial data and the eigenvalues of the initial Weingarten map are to be sufficiently far from one another (see [OGPR23, Theorem 12]). In terms of Big Bang stability, the latter may be dropped near a wide class of cosmological model solutions, including FLRW spacetimes (see [OGPR23, Theorem 49]).

This independence of spatial geometry is reflected in our result insofar as that we allow space in the reference FLRW spacetime to be of arbitrary constant sectional curvature. To be more precise, the energy formalism we use for the scalar field and spacetime variables extends the approach in Chapter 2 to FLRW spacetimes with spatial geometry $\mathbb{T}^3$ and $\mathbb{S}^3$, and then couples this with an energy mechanism for Vlasov matter. However, restricting ourselves to isotropic reference spacetimes is essential when Vlasov matter is present. Else, both the tracefree part of T^{Vl}_{ij} and the energy flux no longer vanish in the reference spacetime, and one would likely lose control of the shear and related constrained quantities. Thus, while it might be possible to integrate Vlasov stability analysis into the frameworks of [FRS23] or [OGPR23], which both included anisotropic spacetimes, there seems to be no loss of generality in restricting ourselves to a methodology that only considers the case of an isotropic reference. In particular, we note that one likely cannot use the same approach as in [BO24a] to establish localized stability in the scalar-field Vlasov setting, as this seems bound to run into similar issues as radiation fluids do in [BO24b] (see Section 3.1.3.2).

3.1.3.2. *Past stability in Einstein (scalar-field) Euler models.* Given how robust scalar field matter appears to be when considering past asymptotics of cosmological spacetimes, it is a natural next step to analyze whether it can dampen the oscillatory behaviour one would otherwise expect from matter models with AVTD behaviour, as discussed in Section 3.1.2.1.

For example, the dynamics of spatially homogeneous solutions to the Einstein-Euler equations with $c_s^2 \in (0,1)$ are well understood to asymptote toward vacuum solutions, often of Kasner type, that exhibit anisotropic curvature blow-up (see, for example, [WE97; Rin01; CH03]). Additionally, Beyer and Le Floch analysed homogeneous fluids on fixed $(1+d)-$dimensional Kasner backgrounds with linear equation of state $\mathfrak{p} = c_s^2\rho$ in [BL17]. In this setting, the fluid velocity diverges toward the Big Bang in the supercritical case $c_s^2 < \frac{1}{d}$, is time-independent in the critical case $c_s^2 = \frac{1}{d}$, and vanishes toward the Big Bang in the subcritical case $1 \geq c_s^2 > \frac{1}{d}$ (see [BL17, Theorem 3.3]). In [BL17, Theorems 4.2-4.3], they showed that subcritical and certain critical solutions to the Einstein-Euler system in Gowdy symmetry are velocity dominated, while this can even break down on the Kasner background in the supercritical case (see [BL17, Section 4.3]).

Given the AVTD nature of Big Bang formation in the Einstein scalar-field system, one could thus hope that coupling with subcritical fluids preserves quiescence. Indeed, Beyer and Oliynyk were able to prove in [BO24b] that subcritical FLRW solutions to the Einstein scalar-field Euler system are past stable (see [BO24b, Theorems 3.1 and 3.6]). However, the critical case cannot be controlled by the Fuchsian approach in [BO24b] since leading terms in the evolution of the fluid velocity become indefinite (see [BO24b, Remark 9.2]), which may cause the fluid to degenerate. However, the results of [BL17] indicate that the critical regime may still exhibit stability under stronger assumptions.

Since the critical case corresponds to radiation fluids in spacetime dimension $1 + d = 4$, and thus to the asymptotic behaviour of Vlasov matter, our results essentially extend the stability regime of [BO24b] to the critical regime for $m = 0$ and slightly beyond for $m = 1$. That FLRW solutions to (3.1.1) fall into a critical regime is reflected in the fact that only the Vlasov distribution itself remains close to the leading term of its velocity term dominated counterpart, while the natural spatial metric on slices of the CMC foliation cannot be controlled sharply. This causes components of the Vlasov energy momentum tensor to diverge, including the macroscopic pressure and density that one would like to compare to the fluid model. Nevertheless, the solution variables that determine the leading order asymptotics of the spacetime fully retain their AVTD behaviour (see Remark 3.8.2). Further, that spatial velocity directions cannot be ignored toward the Big Bang is also strongly indicated by even VTD radiation fluids on Kasner backgrounds exhibiting this feature, and that stability of their initial profile can only be ensured under additional symmetry assumptions on the initial data (see [BL17, Theorem 4.3]).

3.1.3.3. *Vlasov matter in cosmological spacetimes.* The Einstein-Vlasov system has been studied extensively, especially in the context of future nonlinear stability, while little is known in the contracting regime.

In $2+1$ gravity, the first author has analyzed the past and future maximal development of cosmological solutions to the Einstein-Vlasov system (see [Faj16; Faj17a; Faj17b; Faj18]). In $3+1$ dimensions, the system is known to be nonlinearly stable near Minkowski space (see [Tay17; LT20; FJS21; Big+21]) and near Milne spacetimes (see [AF20; BF22]). Moreover, future stability is known for the Einstein nonlinear scalar-field massive-Vlasov system with accelerated expansion (see [Rin13]). Most of the tools used therein were extended to the massless setting in [Sve12, Paper B]. The works [Rin13] by Ringström and [Sve12, Paper B] by Svedberg are of particular relevance to this paper, since the local well-posedness results therein can be carried over to our setting. We discuss this in Section 3.3.3.

Regarding the past direction, it was shown in [DR16] that Kretschmann blow-up occurs for any surface-symmetric non-vacuum cosmological solution to the Einstein-Vlasov system, but no statement on the nature of blow-up is made. The first stability result for cosmological solutions to the Einstein-Vlasov system was due to Rein (see [Rei96]) in the same setting; the approach was extended to the scalar-field Vlasov setting by Tegankong, Noutchegueme and Rendall in [TNR04; TR06], as well as to the case of $\Lambda \neq 0$ in [Tch04]. In these works, the authors studied solutions of the form

$$-e^{-2\mu(a,r)}da^2 + e^{2\lambda(a,r)}dr^2 + a^2\sigma_\varepsilon$$

and obtained a lower bound on the Kretschmann scalar by a^{-6}, ensuring that the Strong Cosmic Censorship conjecture holds in this symmetry class. However, when comparing our results to [Rei96], it should be noted that near-FLRW spacetimes are not covered by the stability result [Rei96, Theorem 5.3], as discussed in the subsequent Remark. Indeed, in these coordinates, FLRW spacetimes correspond to the choices

$$\lambda(a,r) = \ln(a), \quad e^{-2\mu(a)} = \frac{8\pi}{3}a^2\rho(a)\,.$$

Renormalised Vlasov solutions $f^\sharp$ on the mass shell depend on the expansion rescaled velocity variables w and F. In [Rei96], the maximum of these quantities on the support of $f^\sharp$ at the initial hypersurface $a = 1$ is denoted by w_0 and F_0, respectively. This notation in hand, one computes that FLRW spacetimes satisfy

$$\begin{aligned}
&10\pi^2 w_0 F_0\sqrt{1+w_0^2+F_0}\left\|e^{2\mu(1)}\right\|_{C^0}\left\|f^\sharp_{FLRW}(1)\right\|_{C^0}\\
\geq\,&5\pi\int_{-w_0}^{w_0}\int_0^{F_0}\sqrt{1+w^2+F}\mathcal{F}(w^2+F)\,dF dw = 5\pi e^{2\mu(1)}\rho(1) = \frac{15}{8}\,.
\end{aligned}$$

The requirement in [Rei96, Theorems 4.1 and 5.3] that the quantity on the left-hand side is bounded from above by $\frac{1}{2}$ thus excludes near-FLRW spacetimes from the analysis. There are no such restrictions in the scalar-field Vlasov case studied by Tegankong and Rendall in [TR06] so that the aforementioned lower bound on the Kretschmann scalar still applies to FLRW solutions. This bound is sufficient to show that the Strong Cosmic Censorship conjecture is satisfied for surface symmetric cosmological solutions to (3.1.1). However, it does not necessarily reflect the leading order asymptotic behaviour of the Kretschmann scalar. Indeed, in presence of a scalar field, the Kretschmann scalar of the FLRW solution blows up at order a^{-12} and for the pure Vlasov case at order a^{-8}. In (3.8.5), we show that this blow-up is stable, i.e., near-FLRW initial data also leads to blow-up at order a^{-12} and the rescaled Kretschmann scalar remains close to its FLRW counterpart. The gap between our work and the lower bounds in [Rei96; TR06] appears since non-negative terms which are dominant near FLRW solutions are dropped in their computations. Moreover, in [TR06, Theorem 3.5], it is shown that the expansion normalized eigenvalues of the second fundamental form converge under rather restrictive assumptions on the behaviour on $f^\sharp$ along the entire evolution.[4] In (3.8.2e), we show that this is possible for generic near-FLRW data. Thus, our results are not only the first to establish stable past asymptotics without symmetry assumptions, but also refine previous results [Rei96; TR06] in surface symmetry when considering near FLRW spacetimes. It is also worth noting that [DR16] does not make a statement on the asymptotics

[4]In [TR06], the proof of Theorem 3.5 uses inequality (3.3) from Proposition 3.2. This proposition assumes that the support of the expansion normalised Vlasov distribution function converges in the p^1-direction at the rate t^α for some $\alpha > 0$. As the authors remark after the proof of Proposition 3.2, it is not clear whether such an estimate holds for the past maximal development if $f \neq 0$ or how the result could be utilized for a potential bootstrap argument.

of the Vlasov distribution itself. In [Rei96; TR06], the velocity support of the Vlasov distribution is contained in arbitrarily small, uniformly shrinking balls around the zero section. By contrast, we show in (3.8.4a) that, in a weighted C^{10}-space, the Vlasov distribution converges as a function on the co-mass shell to a footprint that is close to the initial distribution. However, one must exhibit some caution in comparing these results, as all of the analysis mentioned above is performed while considering the Vlasov distribution function on the mass shell, while it only converges toward the Big Bang in this work as a function on the co-mass shell. Indeed, when working on the mass shell, particle velocities concentrate in negative eigendirections of the expansion normalised shear $K^{\|}_{Bang}$, as is discussed in Remark 3.8.5. We leave it open to future work whether any $K^{\|}_{Bang}$ close to 0 can be realised given suitable near-FLRW initial data, but draw attention to some works closely related to this question: Since the evolution of the shear is dominated by the scalar field in the ESFV system, it is reasonable to use results in the Einstein scalar-field system for guidance. since the AVTD asymptotics in these settings are also dominated by the linearized system, we may also consider the linearized Einstein scalar-field equations for (relative) simplicity. In his recent result [Li24b], Li established an isomorphism between near-Kasner initial data and data on the Big Bang hypersurface for the linear Einstein scalar-field system. While no nonlinear analogue to this result is currently available, in the preprint [FG25], Franco-Grisales shows that quiescent initial data on the Big Bang hypersurface for the Einstein non-linear scalar field system that admits locally Gaussian developments, akin to the analytic setting studies by Andersson and Rendall in [AR01]. This notion of geometric initial data at the singularity differs from that the scattering data of [Li24b], instead having been established by Ringström in [Rin25]. Nevertheless, the underlying singular initial data bear a strong resemblance to the asymptotic data obtained in [OGPR23], making [FG25] a promising step in working toward a full nonlinear scattering result even if, therein, the author constructs locally Gaussian developments near the singularity that do not match with the CMC gauge used in [OGPR23] to approach the Big Bang. In summary, these results all indicate that some form of scattering result is likely to extend the full Einstein scalar-field system, and then the ESFV system. Thus, we conjecture that this concentration of velocities is generic since, if no negative eigendirection exists, $K^{\|}_{Bang}$ must vanish identically. Finally, we note that the co-mass shell perspective has proven useful in tackling past global existence in spatial homogeneity, see [Tch04], and T^2-symmetry, see [Wea04].

3.1.4. Proof outline. We prove stability via a bootstrap argument on supremum norms of the spacetime metric components, the scalar field and components of the Vlasov energy-momentum-tensor as well as on low order supremum norms of the Vlasov distribution function (see (3.3.13)). The core argument to improve these assumptions relies on a series of energy estimates (see Definition 3.3.3), where the first two energy hierarchies adapt the arguments of Chapter 2 to the current setting, and the third hierarchy provides sufficiently strong bounds on Vlasov energies.

We will first discuss the structural features of the Vlasov equation. Then, we discuss how the argument of Chapter 2 is adapted to include Vlasov matter. We refer to Section 2.1.4.1 for a more detailed discussion of the bootstrap mechanism and the energy hierarchy for the spacetime and the scalar field variables.

3.1.4.1. *Features of Vlasov matter toward the Big Bang.*

By viewing the Vlasov distribution as a function on the co-mass shell, the Vlasov equation admits a particularly simple form in CMCTC gauge (see (3.2.19a)). In particular, momentum characteristics remain uniformly bounded throughout the analysis if the momentum support is bounded initially. However, this structure becomes significantly more complicated once one commutes the equation with covariant derivatives adapted to this foliation. Firstly, one generates terms of order t^{-1} containing the shear which, a priori, prevent us from obtaining integrable estimates. This essentially means that shear terms reappear that one would also have at order zero in the mass shell formulation, see (3.3.12). Secondly, even if one were to use a different commutation method, (3.2.19a) still contains high order lapse terms that can only be estimated at the cost of scaling and are thus effectively borderline on an energy level. To obtain closed estimates, we leverage the internal structure of the commuted Vlasov equation in, to our knowledge, a novel manner: Toward the past, terms containing momentum derivatives ("vertical derivatives") of f may diverge faster than those with derivatives orthogonal to the momentum ("horizontal derivatives"). However, such terms either occur nonlinearly and thus inherit smallness from lower order a priori estimates (see Lemma 3.4.2), or are linear high order shear and lapse terms arising due to the reference distribution. In particular, when commuting the Vlasov equation with, for example, K horizontal and $L-K$ vertical derivatives, those terms with more than K horizontal derivatives remain integrable in time toward the Big Bang even when the Vlasov distribution function itself can only be controlled imprecisely (e.g., using the bootstrap assumption). This allows us to obtain improved estimates by iterating over increasing K, given one takes care regarding the coupling with lapse and shear terms as discussed below.

We remark that the horizontal derivative as it occurs in the Vlasov equation involves Christoffel symbols of the evolved metric and is thus not necessarily globally defined. While low order supremum bounds can essentially be proven locally, energy estimates can only be obtained globally. However, using the blockdiagonal structure of suitably rescaled Sasaki metrics on the cotangent bundle, along with the associated covariant derivatives, one can still contract over the subspace of vertical derivatives and its complement (see Remark 3.2.4), which allows one to globally define energies that only measure a fixed number of horizontal derivatives of f. This is crucial as it allows us to exploit this scaling hierarchy within our global energy formalism.

3.1.4.2. *Integrating Vlasov matter into the scalar field stability mechanism.*

Initial data and the bootstrap argument. We may take initial data to be CMC without loss of generality, and are thus able to work in CMC gauge with zero shift. Furthermore, we assume that the solution variables are ε^2-close to their FLRW counterparts in Sobolev spaces H^ℓ with $\ell \leq 19$, with their norms collected in $\mathcal{H}$ and $\mathcal{H}_{top}$, as well as in C^m with $m \leq 16$, collected in $\mathcal{C}$, see (3.2.19a). These spaces are defined with respect to the contraction normalized spatial metric G (see (3.2.9a)) or, for the Vlasov distribution function, the rescaled Sasaki metric $\underline{G}_0$ (see (3.2.18)).

On the bootstrap interval, we first assume

$$\mathcal{C} \lesssim \varepsilon a^{-c\sigma}$$

for $c > 0$ and some sufficiently small $\sigma \gg \varepsilon^{\frac{1}{8}}$. This small divergent factor is necessary due to the spatial metric in particular degenerating toward the Big Bang singularity. Our goal is to improve this bound to

$$\mathcal{C} \lesssim \varepsilon^{\frac{7}{4}} a^{-c'\varepsilon^{\frac{1}{8}}}$$

for some $c' > 0$ by establishing improved energy bounds. We then extend these to $\mathcal{H}$ with conditional coercivity estimates and then to $\mathcal{C}$ using Sobolev embeddings with respect to γ. Regarding Vlasov matter, Lemma 3.4.14 provides control on the energy-momentum components based on specific Vlasov energies, which allows us to control the Vlasov matter quantities in $\mathcal{C}$ directly upon Sobolev embedding with respect to γ. While we may still obtain strong Sobolev bounds for the Vlasov distribution function using similar conditional coercivity bounds with respect to the Sasaki metric, this is not required for the bootstrap argument. Additionally, we assume a weaker bound on low order derivatives of $f - f_{FLRW}$ that are not exclusively vertical, see (3.3.13b). This is only needed to obtain a priori bounds on the Vlasov distribution function which then also improve upon this assumption.

That the bootstrap assumptions hold on an initial interval in the chosen gauge is ensured by the local well-posedness result in Lemma 3.3.9. To establish this, we take the local existence, continuation criteria and Cauchy stability results in harmonic gauge from [Rin13] in the massive and [Sve12, Paper B] in the massless case. From there, we argue as in [Sha11; RS18b] to extend the result from harmonic gauge to CMC gauge, where the continuation criteria need to be adapted accordingly.

A priori bounds. To deal with nonlinear terms in the evolution equations, we need improved bounds on the solution variables at low order. In the FLRW picture, the Vlasov matter quantities are dominated by the scalar field. More precisely, the energy density and pressure of Vlasov matter behave like a^{-4} while their scalar field counterparts behave like a^{-6}. This is preserved by the bootstrap assumptions up to a factor of $a^{\pm\sigma}$. Meanwhile, the respective energy fluxes T^{SF}_{0i} and T^{Vl}_{0i} vanish for the FLRW spacetime, and their perturbed counterparts can both only be bounded by $\varepsilon a^{-3-c\sigma}$. Since, however, the energy flux only occurs in the momentum constraint equation, which is not used to establish a priori bounds, low order improvements for shear, Bel-Robinson, metric and scalar field variables are then obtained as in Section 2.4, where the Vlasov terms are always asymptotically negligible, see Lemma 3.4.1f..

As for the Vlasov distribution itself, the distribution function at order zero and the evolution of its support can be dealt with by applying the method of characteristics to the rescaled Vlasov equation (3.2.19a). The commuted equations at order L cause borderline error terms of the same order. As outlined in Section 3.1.4.1, these borderline terms can only add vertical derivatives and are nonlinear, so that the other low order bounds can be applied. Thus, one still obtains a priori estimates for L vertical derivatives of $f - f_{FLRW}$ along similar lines as at order zero, using the low order Vlasov bootstrap assumption on the horizontal derivatives. This can be repeated over increasing order of horizontal derivatives, where all borderline terms contain fewer horizontal derivatives and hence are already improved. In the case where all derivatives are horizontal, the bootstrap assumption is not required, yielding a stronger improvement than previously obtained for the mixed terms. Successively replacing the bootstrap assumption with this improved bound then yields sufficiently strong a priori bounds on the distribution function that, in particular, improve (3.3.13b).

Energy estimates. Regarding Vlasov energies, the transport equation for the distribution function f yields a mechanism for energy near-conservation using integration by parts (see Lemma 3.6.2). We then use Sasaki-commuted Vlasov equations to obtain high order energy estimates. At these higher orders, we again need to deal with borderline error terms: Considering Vlasov terms with K horizontal and $L - K$ vertical derivatives, differentiating in time and applying the Vlasov equation, commuting ∂_t with horizontal Sasaki derivatives leads to error terms of the form

$$a^{-3} * \nabla\Sigma * |\underline{\nabla}_{vert}^{L-K} \underline{\nabla}_{hor}^{K-1}(f - f_{FLRW})|^2_{\underline{G}_0}\,. \tag{3.1.3}$$

At first glance, these diverge too strongly for the argument to close, since $\nabla\Sigma$ can only be bounded by $\varepsilon a^{-c\sqrt{\varepsilon}}$ using the a priori bound (3.4.3b). However, as for the a priori estimates, such terms always contain less horizontal derivatives than the terms they originate from. Moreover, they do not occur in the purely vertical case. Thus, we can obtain a suitable energy estimate for the case of only having vertical derivatives, up to a term of the form

$$\int_t^{t_0} a(s)^{-1-c\sigma-\omega} \cdot a(s)^{\omega} \mathfrak{E}^{(L)}_{1,1}(f, s)\, ds$$

for some fixed $\omega > 0$. Similarly, we obtain a series of hierarchized energy estimates for mixed and purely horizontal Vlasov energies, where energies with K horizontal derivatives are scaled by $a^{(K+1)\omega}$. Said scaling means that borderline terms such as (3.1.3) are scaled more strongly than the energies they are controlled by, and thus do not enter the energy estimates as borderline terms. This can be combined to a total Vlasov energy estimate on the sum of all Vlasov energies at order L with their respective scaling, see Lemma 3.7.3.

Additionally, we note that the Sasaki metric contains first order derivatives of the spatial metric in its components, and the Vlasov equation also contains first order derivatives. Hence, it is not obvious that our commutation method does not lose regularity. In fact, to ensure that high order metric errors arising from commutator errors are kept under control, we use scaled curvature energy estimates, see (3.5.19d) and (3.5.19g).

To ultimately improve the bootstrap assumptions on Vlasov matter quantities, these scaled energy estimates are insufficient. To complete the argument, we derive a complementary, unscaled series of estimates with the same hierarchical approach. Regarding terms of the type described in (3.1.3), the fact that $\nabla\Sigma$ is small, albeit divergent, ensures that no loss of ε-control occurs as soon as we allow for horizontal derivatives. Otherwise, these energy estimates are proven in much the same way as their scaled counterparts. Since they are slightly more delicate, we always focus on proving the latter type in Section 3.6 and sketch the relevant adaptations for the former scaled estimates.

Energy estimates for spacetime quantities and scalar field matter follow along similar lines as in Chapter 2, since Vlasov energies enter the respective estimates with time-integrable weights, as we discuss in Section 3.5.2. In particular, since the sectional curvature is negligible compared to both matter contributions in the Friedman equation (3.2.3a) and we do not make use of the specific value of sectional curvature otherwise, extending the argument of Chapter 2 to $\kappa \in \mathbb{R}$ also only leads to error terms of low order (see, in particular, Lemma 3.7.1). When considering how Vlasov matter couples into the remaining field equations, the rescaled momentum constraint (3.2.20i) is the only term that is not evidently asymptotically weaker than scalar field matter. Here, the leading scalar field term is $Ca^{-3}\nabla\phi$. Note that scalar field energies scale $\nabla\phi$ by a^4. Accounting for this, Vlasov energies always enter spacetime and scalar field energy estimates with time integrable weights that are strictly smaller than those for scalar field energies. In anticipation of the total energy discussed next paragraph, we also derive energy estimates scaled by powers of $a^{\frac{\omega}{4}}$ for the shear, for Bel-Robinson variables and curvature terms, which follow by similar arguments as their unscaled counterparts.

Bootstrap improvement and the main theorem. Structurally, our strategy to obtain strong energy bounds is to construct estimates of the following schematic form

(3.1.4)
$$\mathcal{F}(t) \lesssim \mathcal{F}(t_0) + \int_t^{t_0} \left(\varepsilon^{\frac{1}{8}} s^{-1} + \langle\text{time-integrable terms}\rangle\right) \mathcal{F}(s)\, ds + \langle\text{already improved terms}\rangle$$

As the first step in this procedure, we slightly oversimplify the total energy, see Definition 3.7.4, into the caricature

$$\mathfrak{E}_{total} = \mathfrak{E}_{total,SF} + \left(\varepsilon^{\frac{1}{4}} + a^{\frac{\omega}{4}}\right) \mathfrak{E}_{total,metric} + \mathfrak{E}_{total,Vl}\,.$$

Here, the contribution

$$\mathfrak{E}_{quiesc} = \mathfrak{E}_{total,SF} + \varepsilon^{\frac{1}{4}} \mathfrak{E}_{total,metric}$$

corresponds to the main terms of the total energy for the Einstein scalar-field system as in Chapter 2, and $\mathfrak{E}_{total,Vl}$ is a hierarchical combination of Vlasov energies as outlined above, where any individual term is scaled by at least a^{ω}. This leads to the leading order scalar field and metric terms entering from Vlasov energy estimates to be bounded by

$$\int_t^{t_0} a(s)^{-3-c\sqrt{\varepsilon}+\frac{\omega}{8}} \cdot \left(\mathfrak{E}_{total,SF}(s) + a(s)^{\frac{\omega}{4}}\mathfrak{E}_{total,metric}(s)\right)\, ds \tag{3.1.5}$$

up to multiplicative constant. In other words, if we choose $\omega \gg \sigma > 0$, this is compatible with (3.1.4) applied to $\mathfrak{E}_{total}$ since the prefactor in (3.1.5) is integrable in time, see Lemma 3.2.2. On the other hand, Vlasov energies at worst couple into the estimates of the remaing energies as

$$\int_t^{t_0} a(s)^{-2-c\sqrt{\varepsilon}-20\,\omega}\mathfrak{E}_{total,Vl}(s)\, ds\,. \tag{3.1.6}$$

Consequently, this prefactor can be kept time-integrable if one chooses ω to be sufficiently small, while still significantly larger than σ. In summary, this yields an estimate of the form (3.1.4) for $\mathfrak{E}_{total}$ and consequently to the strong bound $\mathfrak{E}_{total} \lesssim \varepsilon^4 a^{-c\varepsilon^{\frac{1}{8}}}$, see Proposition 3.7.6.

While this bound is sufficient to improve solely the ε-prefactor in $\mathcal{C}$, it would only show that geometric and Vlasov matter energies diverge at rates no worse than $a^{-c\omega}$ for some $c > 0$, which is insufficient to improve the boostrap assumption since ω must be sufficiently larger than σ. However, this bound is still strong enough to ensure that (3.1.6) can be bounded by ε^4 up to multiplicative constant. Thus, this term is already sufficiently improved, and we can essentially reapply the same schematic argument to $\mathfrak{E}_{quiesc}$ as to the total energy in Chapter 2 to obtain an estimate of the form (3.1.4). This leads to strong bounds on all components of the spacetime metric, see Proposition 3.7.7 and Corollary 3.7.8. To obtain strong Vlasov energy bounds (see Proposition 3.7.9), we then iterate the unscaled Vlasov energy estimates over horizontal and total derivative orders, using that the other energy quantities are already improved to obtain a series of inequalities of the schematic form (3.1.4).

The energy bounds then yield an improvement of the bootstrap assumption (3.3.13a) as outlined at the start of this subsection, closing the bootstrap argument. Most parts of the main theorem, including geodesic incompleteness and curvature blow-up stability, now follow from these improved bounds just as in [RS18b, Theorem 15.1] or Theorem 2.8.2. In particular, one can perform a similar argument for the Vlasov distribution itself, though one loses one derivative order in doing so since one needs to directly estimate spatial and momentum derivatives of the distribution that appear in the Vlasov equation.

3.1.5. Structure of the chapter. In Section 3.2, we introduce notation and the decomposed Einstein equations as well as some useful formulas. Section 3.3 collects the necessary solution norms to phrase the initial data and bootstrap assumptions, and we establish a local well-posedness result therein along with continuaton criteria for solutions to the ESFV system. In Section 3.4, we prove a priori bounds and estimates for the solution variables based on the bootstrap assumptions, including low order supremum norm bounds and some near-norm equivalencies. Energy estimates for spacetime and scalar field variables are collected in Section 3.5, and the energy estimates for Vlasov matter in Section 3.6. All of these energy estimates are combined in Section 3.7, which is then used to prove Big Bang stability in Section 3.8.

3.2. Preliminaries

3.2.1. Notation.

3.2.1.1. *Foliation, CMC gauge and spacetime objects.* In this paper, we study the Einstein equations in CMC gauge with zero shift unless stated otherwise. Our $(3+1)$-dimensional spacetime $(\overline{M},\overline{g})$ is thus foliated by constant mean curvature Cauchy hypersurfaces $M_s = t^{-1}(\{s\})$ for a time function t. These hypersurfaces are diffeomorphic to the closed orientable 3-manifold M, and thus we will often replace M_s by $\{s\}\times M$ or simply M without further comment. With respect to this foliation, the spacetime metric is written as

$$\overline{g} = -n^2dt^2 + g_{ab}dx^a dx^b\,.$$

We write $\xi(t)^{a_1\dots a_r}{}_{b_1\dots b_r}$ for the $\overline{g}$-orthogonal projection of any M_t-tangent tensor $\xi^{\alpha_1\dots\alpha_r}{}_{\beta_1\dots\beta_s}$ onto the hypersurface M_t. We usually suppress the time dependency in this notation.

The $(0,2)$-tensor

$$k(X,Y) = -\overline{g}(\nabla_X\partial_0, Y)$$

denotes the second fundamental form with respect to this foliation, where $\partial_0 = n^{-1}\partial_t$ is the future directed timelike unit normal. For the smooth scale factor a (see Lemma 3.2.1) and when working in CMC gauge, the mean curvature of M_t is given by

$$\tau(t) := -3\frac{\dot{a}(t)}{a(t)}\,. \tag{3.2.1}$$

Defining the dual of the Weyl tensor $W\equiv W[\overline{g}]$ by

$$W^*_{\alpha\beta\gamma\delta} = \frac{1}{2}\varepsilon[\overline{g}]_{\alpha\beta\mu\nu}W^{\mu\nu}{}_{\gamma\delta},$$

the Bel-Robinson variables are M_t-tangent (0,2)-tensors defined by

$$E_{ac} := W_{a0c0},\, B_{ac} := W^*_{a0c0}\,.$$

Finally, we write

$$J_{\beta\gamma\delta} := \overline{\nabla}^\alpha W_{\alpha\beta\gamma\delta},\;\; J^*_{\beta\gamma\delta} = \overline{\nabla}^\alpha W^*_{\alpha\beta\gamma\delta}\,.$$

Applying the Bianchi identity to $\mathrm{Riem}[\overline{g}]$ allows one to reexpress these as follows

$$\begin{aligned}
J_{\beta\gamma\delta} &= \frac{1}{2}\left(\overline{\nabla}_\gamma\mathrm{Ric}[\overline{g}]_{\beta\delta} - \overline{\nabla}_\delta\mathrm{Ric}[\overline{g}]_{\beta\gamma}\right) - \frac{1}{12}\left(\overline{g}_{\beta\delta}\overline{\nabla}_\gamma\mathrm{Ric}[\overline{g}] - \overline{g}_{\beta\gamma}\overline{\nabla}_\delta R[\overline{g}]\right)\,,\\
J^*_{\beta\gamma\delta} &= \frac{1}{2}\varepsilon[\overline{g}]^{\mu\nu}{}_{\gamma\delta}\overline{\nabla}_\mu\mathrm{Ric}[\overline{g}]_{\nu\beta} + \frac{1}{12}\varepsilon[\overline{g}]_{\beta\nu\gamma\delta}\overline{\nabla}^\nu R[\overline{g}]\,.
\end{aligned}$$

The Einstein equations and CMCTC gauge imply the following explicit formulas.

$$\begin{aligned}\frac{1}{8\pi}J_{i0j} &= \frac{1}{2}\left(\overline{\nabla}_0 T_{ij} - \overline{\nabla}_j T_{i0}\right) - \frac{1}{12}g_{ij}\overline{\nabla}_0 T\,,\\ \frac{1}{8\pi}J^*_{i0j} &= \frac{1}{2}\varepsilon[g]^{kl}{}_i\overline{\nabla}_k T_{jl} - \frac{1}{12}\varepsilon[g]^k{}_{ij}\overline{\nabla}_k T\,.\end{aligned}$$

Since one has $T = T^{SF} + T^{Vl}$, we can analogously decompose these expressions into scalar field and Vlasov components, which we denote by $(J^{SF})^{(*)}_{i0j}$ and $(J^{Vl})^{(*)}_{i0j}$ respectively.

3.2.1.2. *Indices.* We apply the following index conventions:

- Lowercase Latin indices $a, b, \dots, i, j, \dots$ run from 1 to 3. In the context of spatial derivatives or components of spatial tensors, these are always with respect to an arbitrary local coordinate system $(x^i)_{i=1,2,3}$ on M.
- Lowercase Greek indices $\alpha, \beta, \dots, \mu, \nu, \dots$ are spacetime indices and run from 0 to 3. Unless stated otherwise, the index 0 denotes components relative to ∂_0, and indices $1, 2, 3$ denote spatial indices as discussed above.
- Uppercase Latin letters $I, J, \dots$ index objects on the cotangent bundle T^*M (or, respectively, the mass shell P, see Section 3.2.5) and run from 1 to 6. Here, indices 1 to 3 refer to spatial coordinates and horizontal directions, while indices 4 to 6 refer to momentum coordinates and vertical directions (also see Section 3.2.5). We note that we will always use coordinates $(p_i)_{i=1,2,3}$ on the cotangent planes canonically induced from a local coordinate system $(x^i)_{i=1,2,3}$, see also Remark 3.2.5.

Indices are raised and lowered with respect to $\overline{g}$ and g. When raising, respectively lowering, with respect to the rescaled metric G as in Definition 3.2.3, we denote this by $\mathfrak{T}^\sharp$, respectively $\mathfrak{T}^\flat$. We will not raise or lower indices with respect to cotangent bundle metrics to avoid potential confusion with schematic notation (see Section 3.2.1.3).

3.2.1.3. *Metric conventions and notation.* For any Semi-Riemannian metric $\mathfrak{g}$, its Levi-Civita connection is denoted by $\nabla[\mathfrak{g}]$, and the Riemannian curvature tensor is defined by

$$\nabla[\mathfrak{g}]_\alpha \nabla[\mathfrak{g}]_\beta Z_\gamma - \nabla[\mathfrak{g}]_\beta \nabla[\mathfrak{g}]_\alpha Z_\gamma = \mathrm{Riem}[\mathfrak{g}]_{\alpha\beta\gamma}{}^\delta Z_\delta\,.$$

The Levi-Civita tensor is written as $\varepsilon[\mathfrak{g}]$. If $\mathfrak{g}$ is Riemannian, $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ and $|\cdot|_{\mathfrak{g}}$ denote the inner product and norm respectively. Further, the respective Japanese bracket is defined by $\langle\cdot\rangle_{\mathfrak{g}} = \sqrt{1+|\cdot|^2_{\mathfrak{g}}}$. The volume form and element of $\mathfrak{g}$ are written as $\mathrm{dvol}_{\mathfrak{g}}$ and $\mu_{\mathfrak{g}} = \sqrt{\det\mathfrak{g}}$. Next, we define the following operations on $(0,2)$-tensors $A, \tilde{A}$ and vector fields v over M, where all indices are raised and lowered with respect to $\mathfrak{g}$:

$$\begin{aligned}(A \odot_{\mathfrak{g}} \tilde{A})_{ij} &= A_{ik}\tilde{A}^k{}_j\\ (A \wedge_{\mathfrak{g}} \tilde{A})_i &= \varepsilon[\mathfrak{g}]_i{}^{jp}A_j{}^q\tilde{A}_{qp}\\ (v \wedge_{\mathfrak{g}} A)_{ab} &= \varepsilon[\mathfrak{g}]_a{}^{cd}v_c A_{db} + \varepsilon[\mathfrak{g}]_b{}^{cd}v_c A_{ad}\\ (A \times_{\mathfrak{g}} \tilde{A})_{ij} &= \varepsilon[\mathfrak{g}]_i{}^{ab}\varepsilon[\mathfrak{g}]_j{}^{pq}A_{ap}\tilde{A}_{bq} + \frac{1}{3}\langle A, \tilde{A}\rangle_{\mathfrak{g}}\mathfrak{g}_{ij} - \frac{1}{3}(\mathrm{tr}_{\mathfrak{g}}A)(\mathfrak{tr}_{\mathfrak{g}}\tilde{A})g_{ij}\end{aligned}$$

$$(\mathrm{curl}_{\mathfrak{g}}A)_{ij} = \frac{1}{2}\left[\varepsilon[\mathfrak{g}]_i{}^{cd}\nabla[\mathfrak{g}]_d A_{cj} + \varepsilon[\mathfrak{g}]_j{}^{cd}\nabla[\mathfrak{g}]_d A_{ci}\right]$$
$$(\mathrm{div}_{\mathfrak{g}}A)_i = \nabla[\mathfrak{g}]^b A_{ib}$$

We extend this notation to M_t-tangent tensor field and time-dependent Riemannian metrics $\mathfrak{g}_t$ on M_t analogously. To simplify notation, we will write $*_{\mathfrak{g}}$ to denote any contraction between two tensors with respect to $\mathfrak{g}$, up to an implicit multiplicative constant. For $m \in \mathbb{N}$, we write

$$\mathfrak{T}^{*m} = \underbrace{\mathfrak{T} * \cdots * \mathfrak{T}}_{m-\text{times}}.$$

If no metric is specified, $*$ contracts indices using the expansion normalized spatial metric G, see Definition 3.2.3. Multiple covariant derivatives of a tensor $\mathfrak{T}$ with respect to (projected components of a) Levi-Civita connection $\nabla[\mathfrak{g}]$ are denoted by $\nabla[\mathfrak{g}]^I\mathfrak{T}$. These superscripts indicate the number of derivatives instead of an index if and only if they are uppercase Latin letters or formulas.

We note that the Weyl tensor vanishes in three dimensions, so that

$$\mathrm{Riem}[\mathfrak{g}]_{ijkl} = \mathrm{Ric}[\mathfrak{g}]_{ik}\mathfrak{g}_{jl} - \mathrm{Ric}[\mathfrak{g}]_{il}\mathfrak{g}_{jk} + \mathrm{Ric}[\mathfrak{g}]_{jl}\mathfrak{g}_{ik} - \mathrm{Ric}[\mathfrak{g}]_{jk}\mathfrak{g}_{il} - \frac{1}{2}R[\mathfrak{g}]\left(\mathfrak{g}_{ik}\mathfrak{g}_{jl} - \mathfrak{g}_{il}\mathfrak{g}_{jk}\right).$$

For three-dimensional Riemannian metrics $\mathfrak{g}$, $\mathrm{Riem}[\mathfrak{g}]$ and $\mathrm{Ric}[\mathfrak{g}]$ are equivalent in our schematic notation and will be treated as such throughout.

3.2.1.4. *Constants.* For scalar functions ζ_1, ζ_2 with $\zeta_2 \geq 0$, we write $\zeta_1 \lesssim \zeta_2$ if and only if there exists a constant $K > 0$ such that $\max(\zeta_1, 0) \leq K\zeta_2$, and $\zeta_1 \simeq \zeta_2$ if $\zeta_1, \zeta_2 \geq 0$, $\zeta_1 \lesssim \zeta_2$ and $\zeta_2 \lesssim \zeta_1$. Implicit constants are allowed to depend on data from the FLRW reference solution as well as the initial time t_0.

3.2.2. The background solution and the Friedman equation. We collect properties of the FLRW reference solution:

Lemma 3.2.1 (FLRW solutions to the ESFV system). *Consider an FLRW spacetime $(\overline{M}, \overline{g}_{FLRW})$ of the form*

$$(\overline{M} = (0,T) \times M,\ \overline{g}_{FLRW} = -dt^2 + a(t)^2\gamma)$$

for $T \in \mathbb{R}^+ \cup \{\infty\}$, a closed orientable 3-manifold (M, γ) of constant sectional curvature $\kappa \in \mathbb{R}$ and $a \in C^\infty(0,\infty)$. Further, consider isotropic scalar field matter and Vlasov matter, i.e.,

$$\partial_t\phi_{FLRW} = Ca(t)^{-3}, \quad \nabla\phi_{FLRW} = 0, \tag{3.2.2a}$$

$$f_{FLRW}(t,x,p_i) = \mathcal{F}\left((\gamma^{-1})^{ij}p_ip_j\right), \quad \mathcal{F} \in C_c^\infty(\mathbb{R}_0^+, \mathbb{R}_0^+) \tag{3.2.2b}$$

and

$$\rho^{Vl}_{FLRW}(t) = 4\pi\int_0^\infty R^2\sqrt{m^2a(t)^2 + R^2}\mathcal{F}(R^2)dR, \tag{3.2.2c}$$

$$\mathfrak{p}^{Vl}_{FLRW}(t) = \frac{4\pi}{3}\int_0^\infty \frac{R^4}{\sqrt{m^2a(t)^2+R^2}}\mathcal{F}(R^2)dR\,. \tag{3.2.2d}$$

Then $(\overline{M}, \overline{g}_{FLRW}, \phi_{FLRW}, f_{FLRW})$ *is a solution to the ESFV system* (3.1.1) *if and only if*

$$\dot{a}^2 = \frac{4\pi}{3}C^2a^{-4} + \frac{8\pi}{3}a^{-2}\rho^{Vl}_{FLRW} - \kappa\,. \tag{3.2.3a}$$

Further, one has

$$\ddot{a} = -\frac{8\pi}{3}C^2a^{-5} - \frac{4\pi}{3}a^{-3}\left(\rho^{Vl}_{FLRW} + 3\mathfrak{p}^{Vl}_{FLRW}\right) \tag{3.2.3b}$$

Proof. This is a standard computation, see [O'N83, p. 345] . □

Given the Friedman equations (3.2.3), the mean curvature (see (3.2.1)) satisfies the following identities

$$\tau^2 = 12\pi C^2a^{-6} + 24\pi a^{-4}\rho^{Vl}_{FLRW} - 9\kappa a^{-2} \tag{3.2.4a}$$

$$\partial_t\tau = 12\pi C^2a^{-6} + 12\pi a^{-4}\left(\rho^{Vl}_{FLRW} + \mathfrak{p}^{Vl}_{FLRW}\right) - 3\kappa a^{-2} \tag{3.2.4b}$$

Since the scalar field remains the leading term in (3.2.3a) approaching $a=0$, the asymptotic behaviour of the scale factor toward the Big Bang singularity is not affected by Vlasov matter or the sectional curvature. More precisely, it satisfies the following properties.

Lemma 3.2.2 (Scale factor analysis). *Let a be a solution to* (3.2.3a) *for $C>0$ with $a(0)=0$ that is positive somewhere. Then, there exists some maximal time of existence $0<T\leq\infty$ with $a\in C([0,T),\mathbb{R}^+_0)\cap C^\omega((0,T),\mathbb{R}^+)$. For $\kappa\leq 0$, a is strictly increasing and one has $T=\infty$. Else, $T>0$ is the unique time such that $\dot{a}(\frac{T}{2})=0$. Then, a is strictly increasing on $[0,\frac{T}{2}]$ and $t\in(-\frac{T}{2},\frac{T}{2})\mapsto a\left(t+\frac{T}{2}\right)$ is even.*

As $t\downarrow 0$, one has $a(t)\simeq t^{\frac{1}{3}}$. Further, letting $t_0\in\mathbb{R}^+$, with $t_0\leq\frac{T}{2}$ if $\kappa>0$, the following bounds hold for any $t\in(0,t_0)$, any $q>0$ and constants $c,K>0$ that are independent of t,t_0,p and q.

$$\sqrt{\frac{4\pi}{3}}Ca(t)^{-2} \leq \dot{a}(t)+\kappa \tag{3.2.5}$$

$$\int_t^{t_0} a(s)^{-3+p}\,ds \leq \begin{cases} \frac{1}{p}a(t_0)^p & p>0 \\ \frac{1}{|p|}a(t)^{-|p|} & p<0 \end{cases} \tag{3.2.6}$$

$$\int_t^{t_0} a(s)^{-3}\,ds \leq \frac{K}{q}a(t)^{-cq} \tag{3.2.7}$$

Proof. For the first statement, we note that the functions

$$a\mapsto\int_0^\infty R^2\sqrt{m^2a^2+R^2}\mathcal{F}(R^2)dR,\ a\mapsto\int_0^\infty\frac{R^4}{\sqrt{m^2a^2+R^2}}\mathcal{F}(R^2)dR$$

are analytic on $(0,\infty)$ since $\mathcal{F}$ is compactly supported. Thus, (3.2.3a) is of the form $\dot{a}=G(a)$, where G is analytic on $(0,\infty)$. By the same argument as in the proof of [FU22, Lemma 2.2], the first statement holds for $0<T\leq\infty$ with $T=\infty$ for $\kappa\leq 0$. The case of $\kappa>0$ follows as in the proof of [Spe18, Lemma 3.5]. (3.2.6)-(3.2.7) are proven as in Lemma 2.2.4 for $\kappa \le 0$ and [Spe18, Lemma 3.7] for $\kappa > 0$. (3.2.5) follows immediately from (3.2.3a). $\square$

Below, in the case where $\kappa > 0$, we will always tacitly assume $t_0 \le \frac{T}{2}$, i.e., we take our initial data during the era in which the universe does not contract. By Lemma 3.3.6, we may do so without loss of generality. By reflection at $\frac{T}{2}$, all of our statements regarding Big Bang stability in this case also apply identically to stability of the Big Crunch as $t \uparrow T$.

3.2.3. Initial data. Herein, we briefly describe the form initial data for the ESFV system (3.1.1) takes within this paper: Consider data

$$(M, \mathring{g}, \mathring{k}, \mathring{\pi}, \mathring{\psi}, \mathring{f}),$$

where $\mathring{g}$ is a positive definite symmetric $(0,2)$-tensor on M, $\mathring{k}$ a symmetric $(0,2)$-tensor, $\mathring{\pi}$ is an exact $(0,1)$-tensor, and $\mathring{\psi}$ and $\mathring{f}$ are scalar functions on M and T^*M respectively. Moreover, since $\mathring{f}$ represents a particle distribution function, we will assume $\mathring{f} \ge 0$.
For constant mean curvature (CMC) initial data, we additionally require that

$$\mathrm{tr}_{\mathring{g}}\mathring{k} = -3\frac{\dot{a}(t_0)}{a(t_0)}\,.$$

Additionally, the following constraint equations must hold: For $x \in M$ and understanding indices to be lifted with respect to $\mathring{g}$, we require,

(3.2.8a)
$$\left\{\mathrm{R}[\mathring{g}] + 1 - \left(\mathring{k}^a{}_b\mathring{k}^b{}_a\right)\right\}(x) = 8\pi\left[|\mathring{\psi}(x)|^2 + |\mathring{\pi}|^2_{\mathring{g}_x}\right] + 16\pi\int_{T^*_xM}\sqrt{m^2 + |p|^2_{\mathring{g}_x}}\,\mathring{f}(x,p)\,\mu_{\mathring{g}_x}^{-1}\,dp \quad \text{and}$$

(3.2.8b)
$$\{\mathrm{div}_{\mathring{g}}\mathring{k}\}_i(x) = -\,8\pi\,\mathring{\psi}(x)\,\mathring{\pi}_i(x) - 8\pi\int_{T^*_xM} p_i\mathring{f}(x,p)\mu_{\mathring{g}_x}^{-1}\,dp\,.$$

For the Vlasov quantities in these equations to be well-defined, we will assume $\mathring{f}$ to have compact momentum support, i.e., $\{p\,|\,(x,p) \in \mathrm{supp}\mathring{f}\}$ is a compact set. In particular, this implies that $(x,p) \mapsto |p|_{\gamma_x}$ is uniformly bounded on $\mathrm{supp}\mathring{f}$.

In the following, we understand initial data to be embedded into a spacetime $(\overline{M}, \overline{g})$ along ι such that $(\overline{M}, \overline{g}, \phi, f)$ solves the ESFV system with

$$\iota(M) = M_{t_0},\ \iota^*\overline{g} = \mathring{g},\ \iota^*k = \mathring{k},\ \iota^*(\nabla\phi) = \mathring{\pi},\ \iota^*\partial_0\phi = \mathring{\psi}_0 \text{ and } (T^*\iota)^*f(t_0,\cdot,\cdot) = \mathring{f},$$

where $T^*\iota$ denotes the cotangent map of ι.[5] The maximal globally hyperbolic development of such data is unique, as shown in [FB52; CBG69]. We thus assume said solution to be globally hyperbolic throughout. FLRW initial data is, surpressing the embedding, given by

$$\left(M, a(t_0)^2\gamma, -\dot{a}(t_0)a(t_0)\gamma, 0, Ca(t_0)^{-3}, (x,p) \mapsto \mathcal{F}(|p|^2_\gamma)\right)\,.$$

3.2.4. Expansion normalised variables.

[5]We note that, for the Vlasov distribution, we view the co-mass shell P as foliated by $(P_{(t,\cdot)})_{t\in I}$ and identify each leaf of this foliation with T^*M_t here and throughout.

DEFINITION 3.2.3 (Expansion normalised variables for Big Bang stability). We will use the following expansion normalised variables:

$$G_{ij} = a^{-2}\, g_{ij}, \quad (G^{-1})^{ij} = a^2\, g^{ij}, \quad \Sigma_{ij} = a\,\hat{k}_{ij} = a\left(k_{ij} - \frac{\tau}{3}\, g_{ij}\right) \tag{3.2.9a}$$

$$N = n - 1 \tag{3.2.9b}$$

$$\mathcal{E}_{ij} = a^4\, E_{ij}, \quad \mathcal{B}_{ij} = a^4\, B_{ij} \tag{3.2.9c}$$

$$\Psi = a^3 \partial_0 \phi - C \tag{3.2.9d}$$

Further, on the co-mass shell

$$P = \bigsqcup_{(t,x)\in\overline{M}} P_{(t,x)}, \text{ where } P_{(t,x)} = \{p \in T^*_{(t,x)}\overline{M} \,|\, p_\mu p^\mu = -m^2,\ p_0 < 0\} \tag{3.2.10}$$

with $m = 0$ in the massless and $m = 1$ in the massive case, we define the rescaled momentum variables

$$v_i = p_i, \quad v^0 = ap^0 = \sqrt{m^2a^2 + |p|^2_G}, \quad v^0_\gamma = \sqrt{m^2a^2 + |v|^2_\gamma}\,, \tag{3.2.11}$$

We refer to Remark 3.2.5 regarding the pointwise norms $|v|_G$ and $|v|_\gamma$. For any $t \in (0, t_0]$, we define the following momentum support bounds:

$$\mathcal{P}(t) := \sup_{x\in M_t} \sup_{v\in \mathrm{supp} f(t,x,\cdot)} |v|_G\,, \tag{3.2.12a}$$

$$\mathcal{P}^0(t) := \sup_{x\in M_t} \sup_{v\in \mathrm{supp} f(t,x,\cdot)} v^0\,. \tag{3.2.12b}$$

$\mathcal{P}_\gamma$, respectively $\mathcal{P}^0_\gamma$, are defined replacing $|v|_G$ with $|v|_\gamma$, respectively v^0 with v^0_γ. Finally, we write the rescaled components of the Vlasov energy-momentum tensor as follows:

$$\begin{aligned}
\rho^{Vl} &= a^4 T^{Vl}_{00}[\overline{g}, f] = \int_{T^*_\cdot M} v^0 f(t,x,v)\mu_G^{-1} dv \\
(S^{Vl})^i{}_j &= a^4 (T^{Vl})^i{}_j[\overline{g}, f] = \int_{T^*_\cdot M} \frac{v_j (G^{-1})^{il} v_l}{v^0} f(t,x,v)\mu_G^{-1} dv \\
\mathfrak{p}^{Vl} &= \frac{1}{3}\mathrm{tr}_G S^{Vl} = \frac{1}{3}\int_{T^*_\cdot M} \frac{|v|^2_G}{v^0} f(t,x,v)\mu_G^{-1} dv \\
(S^{Vl,\parallel})^i{}_j &= (S^{Vl})^i{}_j - \mathfrak{p}^{Vl}\mathbb{I}^i_j \\
\jmath^{Vl}_l &= a^3 T^{Vl}_{0l}[\overline{g}, f] = \int_{T^*_\cdot M} v_l f(t,x,v)\mu_G^{-1} dv
\end{aligned}$$

Note that, in the massless case $m = 0$, one has $\rho^{Vl} = \frac{1}{3}\mathfrak{p}^{Vl}$. Furthermore, since f_{FLRW} is isotropic, one has $\jmath^{Vl}_{FLRW} = 0$ and also

$$\int_{T^*_\cdot M} v\, f_{FLRW}(t,\cdot,v)\mu_G^{-1}\, dv = \frac{\mu_\gamma}{\mu_G}\, \jmath^{Vl}_{FLRW} = 0\,. \tag{3.2.13}$$

This implies

$$\jmath^{Vl} = a^{-3}\int_{T^*_\cdot M} v(f - f_{FLRW})(t,x,v)\, \mu_G^{-1}\, dv. \tag{3.2.14}$$

3.2.5. Sasaki metrics. In this section, we review properties of the Sasaki metric and the resulting geometry on the cotangent bundle that will underpin our analysis of the Vlasov equation. For a more detailed overview, we refer to [CGS22, Section 2.4].

The rescaled metric G on M_t induces the Sasaki metric on T^*M_t by

$$\underline{G} = G_{ij}dx^i dx^j + (G^{-1})^{ij} Dv_i Dv_j \, ,$$

where $Dv_i = dv_i - \Gamma[G]^j_{ik} v_j dx^k$. In the following, $\underline{\nabla}$ denotes the covariant derivative with respect to $\underline{G}$. We will express this connection using the local frame

$$\mathbf{A}_i = \partial_{x^i} + v_j \Gamma[G]^j_{ik} \partial_{v_k}, \quad \mathbf{B}^i = \partial_{v_i} \, . \tag{3.2.15}$$

Note that one has $Dv_i(\mathbf{A}_j) = 0$ for $i, j = 1, 2, 3$. We will denote this frame by $\{S_{\underline{I}}\}_{\underline{I}=1,\dots,6}$ with, for $i = 1, 2, 3$, $S^{\underline{I}} = \mathbf{A}_i$ for $\underline{I} = i$ and $S_{\underline{I}} = \mathbf{B}^i$ for $\underline{I} = i + 3$. Coefficients with respect to this frame will consequently be labeled by underlined, capitalized Latin indices. For reasons explained below, we will refer to $\{\mathbf{A}_i\}_{i=1,2,3}$ as the horizontal frame components, and $\{\mathbf{B}^i\}_{i=1,2,3}$ as the vertical frame components. For the sake of notational convenience, we will simply write $\underline{i}$ and $\underline{i+3}$ to distinguish between horizontal and vertical elements of the frame $S_{\underline{I}}$. For any tensor $\mathcal{T}$ on T^*M_t, we further simplify notation as follows:

$$\underline{\nabla}_i \mathcal{T} = \underline{\nabla}_{\underline{i}} \mathcal{T} = \underline{\nabla}_{\mathbf{A}_i} \mathcal{T}, \quad \underline{\nabla}^{i+3} \mathcal{T} = \underline{\nabla}_{\underline{i+3}} \mathcal{T} = \underline{\nabla}_{\mathbf{B}^i} \mathcal{T} \, , \tag{3.2.16}$$

where $i = 1, 2, 3$. With respect to the frame $\{S_{\underline{i}}\}$, the connection coeffients of $\underline{G}$ are as follows (see [AF20, Appendix D] for the tangent bundle analogue).

$$\underline{\Gamma}^{\underline{k}}{}_{\underline{i}\,\underline{j}} = \Gamma[G]^k_{ij} \qquad \underline{\Gamma}^{\underline{k+3}}{}_{\underline{i}\,\underline{j}} = \frac{1}{2} v_l \mathrm{Riem}[G]^l{}_{kij} \tag{3.2.17a}$$

$$\underline{\Gamma}^{\underline{k}}{}_{\underline{(i+3)}\,\underline{j}} = \frac{1}{2} v_l \mathrm{Riem}[G]^{\sharp lik}{}_j \qquad \underline{\Gamma}^{\underline{k}}{}_{\underline{i}\,\underline{(j+3)}} = \frac{1}{2} v_l \mathrm{Riem}[G]^{\sharp ljk}{}_i \tag{3.2.17b}$$

$$\underline{\Gamma}^{\underline{k+3}}{}_{\underline{(i+3)}\,\underline{j}} = -\Gamma[G]^i_{jk} \qquad \underline{\Gamma}^{\underline{k+3}}{}_{\underline{i}\,\underline{(j+3)}} = \underline{\Gamma}^{k}{}_{\underline{(i+3)}\,\underline{j+3}} = \underline{\Gamma}^{\underline{k+3}}{}_{\underline{(i+3)}\,\underline{(j+3)}} = 0 \tag{3.2.17c}$$

We also introduce the rescaled Sasaki metric as follows

$$\underline{G}_0(\mathbf{A}_i, \mathbf{A}_j) = G_{ij}, \quad \underline{G}_0(\mathbf{A}_i, \mathbf{B}^j) = 0, \quad \underline{G}_0(\mathbf{B}^i, \mathbf{B}^j) = (v^0)^{-2} (G^{-1})^{ij} \tag{3.2.18}$$

as well as the following analogues with respect to γ.

$$\begin{aligned} \hat{D}v_i &= dv_i - \hat{\Gamma}^j_{ik} v_j dx^k \\ \underline{\gamma} &= \gamma_{ij} dx^i dx^j + (\gamma^{-1})^{ij} \hat{D}v_i \hat{D}v_j \\ \underline{\gamma}_0 &= \gamma_{ij} dx^i dx^j + (m^2 + |v|^2_\gamma)^{-1} (\gamma^{-1})^{ij} \hat{D}v_i \hat{D}v_j \end{aligned}$$

Remark 3.2.4 (Notation for horizontal and vertical covariant derivatives). A technical difficulty arises from the fact that the horizontal frame $\{\mathbf{A}_i\}_{i=1,2,3}$ is not globally defined. On the other hand, it naturally arises in the Vlasov equation (3.2.19a), and our hierarchy will require us to treat horizontal and vertical covariant derivatives differently. One can, however, split vector fields on T^*M_t into their horizontal and vertical components globally.

Let $\pi_{T^*M} : T^*M \to M$ denote the canonical projection and $T\pi_{T^*M} : TT^*M \to T^*M$ its tangent map. Recall that $\ker T\pi_{T^*M}$ defines the vertical cotangent bundle. Let $\mathfrak{V}(T^*M)$ be the space of sections of TT^*M over T^*M which are valued in $\ker T\pi_{T^*M}$. Then, $\mathfrak{V}(T^*M)$ is spanned by $\{\mathbf{B}^1, \mathbf{B}^2, \mathbf{B}^3\}$. Its orthogonal complement $\mathfrak{V}(T^*M)^\perp$ admits the local basis $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3\}$. We now split vector fields using the $\underline{G}$-orthogonal projection

$$\mathbb{P} : \mathfrak{X}(TT^*M) \longrightarrow \mathfrak{V}(T^*M),\ \mathbb{P}W = \sum_{i=1,3} \underline{G}(\mathbf{B}^i, W)\mathbf{B}^i\,,$$

onto vertical vector fields. Covariant derivatives can now also be split into horizontal and vertical components by taking

$$\underline{\nabla}_{vert} : \mathfrak{X}(T^*M) \times \mathfrak{X}(TT^*M) \longrightarrow \mathfrak{X}(TT^*M),\ (\underline{\nabla}_{vert})_W Z = \underline{\nabla}_{\mathbb{P}(W)} Z$$

and analogously for $\underline{\nabla}_{hor}$ replacing $\mathbb{P}$ with $\mathbb{P}^\perp = 1 - \mathbb{P}$. In particular, extending this to tensors, contractions along these projected covariant derivatives reduce to expressions of the following type for scalar functions ξ_1, ξ_2 on T^*M:

$$\begin{aligned}&\langle \underline{\nabla}_{vert}^{L-K}\underline{\nabla}_{hor}^{K}\xi_1, \underline{\nabla}_{vert}^{L-K}\underline{\nabla}_{hor}^{K}\xi_1\rangle_{\underline{G}_0}\\ =& \Big[(v^0)^{2(L-K)}G_{i_1j_1}\dots G_{i_{L-K}j_{L-K}}(G^{-1})^{k_1l_1}\dots(G^{-1})^{k_Ml_M}\cdot\\ &\cdot\underline{\nabla}^{i_1+3}\dots\underline{\nabla}^{i_{L-K}+3}\underline{\nabla}_{k_1}\dots\underline{\nabla}_{k_M}\xi_1\,\underline{\nabla}_{j_1+3}\dots\underline{\nabla}_{j_{L-K}+3}\underline{\nabla}_{l_1}\dots\underline{\nabla}_{l_M}\xi_2\Big]\end{aligned}$$

The term $\langle \boldsymbol{X}\underline{\nabla}_{vert}^{L-M}\underline{\nabla}_{hor}\xi_1, \underline{\nabla}_{vert}^{L-M}\underline{\nabla}_{hor}\xi_2\rangle_{\underline{G}_0}$ (see (3.2.19b)) decomposes similarly. By construction, such expressions are defined globally on T^*M.For the sake of simplicity, we will often view $\underline{\nabla}_{vert}$ as an operator defined on $\mathfrak{V}(T^*M)\times\mathfrak{X}(TT^*M)$, i.e., as only applying vertical covariant derivatives, and analogously view $\underline{\nabla}_{hor}$.

Remark 3.2.5 (Lifting to the mass co-shell). In the following, we will need $\underline{\nabla}$ (and the derived connections $\underline{\nabla}_{vert}$ and $\underline{\nabla}_{hor}$ as well as $\hat{\underline{\nabla}}$) to act on quantities that are not a priori defined on the co-mass shell and thus, technically, need to be lifted to it. We will suppress such lifts throughout the paper: For example, when considering any M_t-tangent tensor $\mathfrak{T}$, $\underline{\nabla}\mathfrak{T}$ will refer to the covariant Sasaki derivative on the pullback of $\mathfrak{T}$ along the canonical projection $\pi_t : P_t \simeq T^*M_t \to M_t$. In particular, one has

$$\underline{\nabla}_i\mathfrak{T} = \nabla_i\mathfrak{T}, \quad \underline{\nabla}^{i+3}\mathfrak{T} = 0\,.$$

Conversely, we lift the momentum variables v to the covector field $\tilde{v}$ on the co-mass shell at time t by

$$\tilde{v}_i = 0, \quad \tilde{v}_{i+3} = v_i\,,$$

and simply write $\underline{\nabla}v_i := \underline{\nabla}\tilde{v}_{i+3}$. We collect various derivatives of momentum variables in Lemma 3.2.7. We write $v^{\sharp i} = (\underline{G}|_{vert}^{-1})^{ij}v_j = (G^{-1})^{ij}v_j$ and define $\underline{\nabla}v^\sharp$ analogously, as well as $|v|_G$ and $|v|_\gamma$.

3.2.6. $3+1$-decomposition of the Einstein equations.

3.2.6.1. *The rescaled Vlasov equation.* The Vlasov equation (3.1.1e) reads

$$\mathcal{X}f = \overline{g}^{\alpha\beta}p_\alpha\partial_\beta f + \overline{g}^{\lambda\mu}\overline{\Gamma}^\nu_{i\lambda}p_\mu p_\nu\partial_{p_i}f$$

$$= n^{-1}p^0\left(\partial_t f + \frac{g^{ij}p_i}{p^0}n\mathbf{A}_j f - p^0\nabla_i n\mathbf{B}^i f\right) = 0\,.$$

Above, we used that connection coefficient terms containing precisely one zero index precisely cancel in these coordinates since $p^0 = -p_0$ and consequently

$$p_0 p_j g^{jl}\overline{\Gamma}^0_{il} + p^0 p_j \overline{\Gamma}^j_{i0} = p_j\left(-p_0 g^{jl}k_{il} - p^0 k^j{}_i\right) = 0.$$

Rearranging and using the rescaled variables from Definition 3.2.3, the Vlasov equation becomes

$$\partial_t f = -a^{-1}(N+1)\frac{v^{\sharp j}}{v^0}\mathbf{A}_j f + a^{-1}v^0\nabla_j N\mathbf{B}^j f =: \boldsymbol{X}f \tag{3.2.19a}$$

We extend $\boldsymbol{X}$ to T^*M_t-tangent tensors $\mathcal{T}$ on the co-mass shell as follows:

$$\boldsymbol{X}\mathcal{T} = -a^{-1}(N+1)\frac{v^{\sharp j}}{v^0}\underline{\nabla}_j\mathcal{T} + a^{-1}v^0\nabla_j N\mathbf{B}^j\mathcal{T} \tag{3.2.19b}$$

3.2.6.2. *Constraint equations and equations for the rescaled spacetime and scalar field variables.* We collect the evolution equations for the shear, the metric and derived variables:

$$\partial_t G_{ij} = -\,2(N+1)a^{-3}\Sigma_{ij} + 2N\frac{\dot{a}}{a}G_{ij} \tag{3.2.20a}$$

$$\partial_t (G^{-1})^{ij} = 2(N+1)a^{-3}(\Sigma^\sharp)^{ij} - 2N\frac{\dot{a}}{a}(G^{-1})^{ij} \tag{3.2.20b}$$

(3.2.20c)
$$\begin{aligned}\partial_t(\Sigma^\sharp)^i{}_j =& \tau N(\Sigma^\sharp)^i{}_j - a\nabla^{\sharp i}\nabla_j N + (N+1)a\left[\left(\mathrm{Ric}[G]^\sharp\right)^i{}_j - 2\kappa\mathbb{I}^i_j\right]\\ &- 8\pi(N+1)a\nabla^{\sharp i}\phi\nabla_j\phi + N\left[4\pi C^2 a^{-3} + 4\pi a^{-1}\left(\rho^{Vl}_{FLRW} + \mathfrak{p}^{Vl}_{FLRW}\right) - \kappa a\right]\mathbb{I}^i_j\\ &- 8\pi a^{-1}((S^{Vl})^\parallel)^i{}_j - 4\pi a^{-1}\left[\left(\rho^{Vl} - \rho^{Vl}_{FLRW}\right) - \left(\mathfrak{p}^{Vl} - \mathfrak{p}^{Vl}_{FLRW}\right)\right]\mathbb{I}^i_j\end{aligned}$$

(3.2.20d)
$$\begin{aligned}\partial_t\Gamma^k_{ij}[G] =& -(N+1)a^{-3}\left[\nabla_i(\Sigma^\sharp)^k{}_j + \nabla_j(\Sigma^\sharp)^k{}_i - \nabla^{\sharp k}\Sigma_{ij}\right]\\ &- a^{-3}\left[\nabla_i N(\Sigma^\sharp)^k{}_j + \nabla_j N(\Sigma^\sharp)^k{}_i - \Sigma_{ij}\nabla^{\sharp k}N\right]\\ &+ \frac{\dot{a}}{a}\left[\nabla_i N\cdot\mathbb{I}^k_j + \nabla_j N\cdot\mathbb{I}^k_i - \nabla^{\sharp k}N\cdot G_{ij}\right]\end{aligned}$$

(3.2.20e)
$$\begin{aligned}\partial_t\mathrm{Ric}[G]_{ab} =& a^{-3}(N+1)(\Delta_G\Sigma_{ab} - \nabla^{\sharp d}\nabla_a\Sigma_{db} - \nabla^{\sharp d}\nabla_b\Sigma_{da})\\ &+ a^{-3}\nabla^{\sharp d}N(2\nabla_d\Sigma_{ab} - \nabla_a\Sigma_{db} - \nabla_b\Sigma_{da})\\ &- a^{-3}\left(\nabla_a N(\mathrm{div}_G\Sigma)_b + \nabla_b(\mathrm{div}_G\Sigma)_a\right) + \Delta_G N\left(a^{-3}\Sigma_{ab} + \frac{\tau}{3}G_{ab}\right)\\ &- a^{-3}\left(\nabla^{\sharp d}\nabla_a N\cdot\Sigma_{db} + \nabla^{\sharp d}\nabla_b N\cdot\Sigma_{da}\right) + \frac{\tau}{3}\nabla_a\nabla_b N\end{aligned}$$

The rescaled lapse equations[6] read

$$\text{(3.2.20f)}\quad \Delta_G N = \left[12\pi C^2 a^{-4} - 3\kappa + 12\pi a^{-2}\left(\rho^{Vl}_{FLRW} + \mathfrak{p}^{Vl}_{FLRW}\right)\right] N + (N+1)a^{-4}\left(|\Sigma|^2_G + 16\pi C\Psi + 8\pi\Psi^2\right) + 4\pi a^{-2}(N+1)\left[\left(\rho^{Vl} - \rho^{Vl}_{FLRW}\right) + \left(\mathfrak{p}^{Vl} - \mathfrak{p}^{Vl}_{FLRW}\right)\right],$$

(3.2.20g)

$$\Delta_G N = \left[12\pi C^2 a^{-4} - 3\kappa + 12\pi a^{-2}\left(\rho^{Vl}_{FLRW} + \mathfrak{p}^{Vl}_{FLRW}\right)\right] N + (N+1)(R[G] - 6\kappa) + 12\pi a^{-2}(N+1)\left[\left(\mathfrak{p}^{Vl} - \mathfrak{p}^{Vl}_{FLRW}\right) - \left(\rho^{Vl} - \rho^{Vl}_{FLRW}\right)\right].$$

We arranged the Vlasov terms so that the evolved density and pressure appear as in the solution norms used later on (see (3.3.2c)), so that we can expect these terms to be small, at least initially.
The constraint equations read

(3.2.20h)

$$R[G] - 6\kappa - a^{-4}\langle\Sigma,\Sigma\rangle_G = 8\pi\left[a^{-4}\Psi^2 + 2Ca^{-4}\Psi + |\nabla\phi|^2_G\right] + 16\pi a^{-2}\left[\rho^{Vl} - \rho^{Vl}_{FLRW}\right],$$

$$\text{(3.2.20i)}\quad (\mathrm{div}_G\Sigma)_l = -8\pi\nabla_l\phi(\Psi + C) + 8\pi j^{Vl},$$

with the respective Bel-Robinson analogues

(3.2.20j)

$$\begin{aligned}\mathcal{E}_{ij} &= a^4\mathrm{Ric}[G]_{ij} + \frac{2}{9}\tau^2 a^6 G_{ij} + \frac{\tau}{3}a^3\Sigma_{ij} - (\Sigma\odot_G\Sigma)_{ij} - 4\pi a^4 T_{ij} - \frac{4\pi}{3}(3T_{00} - \mathrm{tr}_{\overline{g}}T)a^6 G_{ij}\\ &= a^4\left(\mathrm{Ric}[G]_{ij} - 2\kappa G_{ij}\right) + \frac{\tau}{3}a^3\Sigma_{ij} - (\Sigma\odot_G\Sigma)_{ij} - 4\pi a^4\nabla_i\phi\nabla_j\phi - 4\pi a^2 (S^{Vl})^{\parallel,\flat}_{ij}\\ &\quad - \left[\frac{4\pi}{3}a^4|\nabla\phi|^2_G + \frac{8\pi}{3}\Psi^2 + \frac{16\pi}{3}C\Psi\right]G_{ij} - \frac{16\pi}{3}\left(\rho^{Vl} - \rho^{Vl}_{FLRW}\right)\cdot a^2 G_{ij},\end{aligned}$$

(3.2.20k)

$$\mathcal{B}_{ij} = -a^3\mathrm{curl}_g\Sigma_{ij}.$$

The evolution equations for the Bel-Robinson variables are as follows:

(3.2.20l)

$$\begin{aligned}\partial_t\mathcal{E}_{ij} = &(3-N)\frac{\dot a}{a}\mathcal{E}_{ij} - a^{-1}(\nabla N\wedge_G\mathcal{B})_{ij} + (N+1)a^{-1}\mathrm{curl}_G\mathcal{B}_{ij}\\ &- (N+1)\left[\frac{5}{2}a^{-3}(\mathcal{E}\times_G\Sigma)_{ij} + \frac{2}{3}a^{-3}\langle\mathcal{E},\Sigma\rangle_G G_{ij}\right]\\ &+ 4\pi(N+1)a^{-3}(\Psi + C)^2\Sigma_{ij} - 4\pi(N+1)\dot a a^3\nabla_i\phi\nabla_j\phi + 4\pi a\nabla_{(i}N\nabla_{j)}\phi(\Psi + C)\end{aligned}$$

[6] Recall that, for $\kappa > 0$, we have assumed to be in the contracting regime without loss of generality (see Lemma 3.2.2 and the subsequent remark). Thus, recalling (3.2.4b)

$$\partial_t\tau = 12\pi C^2 a^{-4} - 3\kappa + 12\pi a^{-2}\left(\rho^{Vl}_{FLRW} + \mathfrak{p}^{Vl}_{FLRW}\right) \geq 0$$

holds and the lapse equations (3.2.20f) and (3.2.20g) are always solvable.

$$
\begin{aligned}
&- 4\pi a(N+1)\left[\nabla_i\Psi\nabla_j\phi + \nabla_j\Psi\nabla_i\phi + (\Sigma^\sharp)^l_{(i}\nabla_{j)}\phi\nabla_l\phi - (\Psi+C)\nabla_i\nabla_j\phi\right]\\
&+ (N+1)\left[\frac{2\pi}{3}a^6\partial_0\left(a^{-6}(\Psi+C)^2 + a^{-2}|\nabla\phi|^2_G\right) + 4\pi\frac{\dot a}{a}(\Psi+C)^2\right]G_{ij}\\
&+ a^4\frac{N+1}{2}\left[J^{Vl}_{i0j} + J^{Vl}_{i0j}\right]
\end{aligned}
$$

(3.2.20m)

$$
\begin{aligned}
\partial_t\mathcal{B}_{ij} =& \frac{\dot a}{a}(3-N)\mathcal{B}_{ij} + a^{-1}(\nabla N\wedge_G\mathcal{E})_{ij} - a^{-1}(N+1)\mathrm{curl}_G\mathcal{E}_{ij}\\
&- (N+1)\left[\frac{5}{2}a^{-3}(\mathcal{B}\times_G\Sigma)_{ij} + \frac{2}{3}a^{-3}\langle\mathcal{B},\Sigma\rangle_G G_{ij}\right]\\
&- 4\pi(N+1)\varepsilon[G]_{lmj}\left(a^3\nabla^{\sharp l}\nabla_j\phi\nabla^{\sharp m}\phi + a^{-1}(\Sigma^\sharp)^l{}_i\nabla^{\sharp m}\phi(\Psi+C)\right)\\
&+ a^4\frac{N+1}{2}\left[(J^{Vl})^*_{i0j} + (J^{Vl})^*_{j0i}\right]
\end{aligned}
$$

To compute the new matter terms in the Bel Robinson evolution equations, we first insert (3.2.19a) and then integrate by parts for any vertical derivatives that occur to obtain

(3.2.20n)

$$
\begin{aligned}
\frac{N+1}{2\cdot 8\pi}\left(J^{Vl}_{i0j} + J^{Vl}_{j0i}\right) =& \frac{N+1}{2}\left(\overline{\nabla}_0 T^{Vl}_{ij} - \overline{\nabla}_{(i}T^{Vl}_{j)0}\right) - \frac{N+1}{12}g_{ij}\overline{\nabla}_0 T^{Vl}\\
=& -(N+1)\frac{\dot a}{a}\cdot a^{-2}(S^{Vl})^{\parallel,\flat}_{ij} - \frac{1}{2}(N+1)a^{-3}\nabla_{(i}\jmath^{Vl}_{j)}\\
&+ \frac{1}{2}a^{-3}\int_{T^*M}\left[\frac{v_iv_jv^{\sharp k}}{(v^0)^2}(N+1)\underline{\nabla}_k f + \mathbf{B}^k(v_iv_j)\nabla_k N f\right]\mu_G^{-1}dv\\
&- (N+1)a^{-5}\int_{T^*M}\Sigma_{kl}\frac{v_iv_jv^{\sharp k}v^{\sharp l}}{4(v^0)^3}f\,\mu_G^{-1}dv\\
&- \frac{1}{2}\frac{\dot a}{a}\cdot a^{-2}(N+1)\int_{T^*M}\frac{v_iv_j}{v^0}\cdot\frac{m^2a^2}{(v^0)^2}f\,\mu_G^{-1}dv + \langle\text{pure trace terms}\rangle
\end{aligned}
$$

Furthermore, we have

(3.2.20o)

$$
\begin{aligned}
&\frac{N+1}{2\cdot 8\pi}\left((J^{Vl})^*_{i0j} + (J^{Vl})^*_{j0i}\right) = \frac{1}{2}a^{-1}\varepsilon[G]^{\sharp kl}{}_{(i}\overline{\nabla}_k T^{Vl}_{j)l}\\
&= \frac{1}{2}a^{-4}\varepsilon[G]^{\sharp kl}{}_{(i}\int_{T^*M}\frac{v_{j)}v_l}{v^0}\underline{\nabla}_k f\,\mu_G^{-1}dv - \frac{1}{2}a^{-5}\varepsilon[G]^{\sharp kl}{}_{(i}\Sigma_{j)k}\int_{T^*M}v_l f\,\mu_G^{-1}dv\,.
\end{aligned}
$$

Finally, the rescaled wave equation reads

$$
\partial_t\Psi = a(N+1)\Delta\phi + a\langle\nabla N,\nabla\phi\rangle_G - 3\frac{\dot a}{a}N(\Psi+C)\,. \tag{3.2.20p}
$$

3.2.7. Miscellaneous formulas. To finish this preliminary section, we collect some useful commutator and derivative formulas which can all be proven by straightforward computation:

LEMMA 3.2.6 (Commuting derivatives and integrals). *Let $\zeta:\overline{M}\to\mathbb{R}$ and $\xi:P\to\mathbb{R}$ be sufficiently regular. Then, the following statements hold:*

$$\partial_t \int_M \zeta \mathrm{dvol}_G = \int_M (\partial_t \zeta - N\tau\zeta)\,\mathrm{dvol}_G \tag{3.2.21}$$

$$\partial_t \int_M \int_{T^*_\cdot M} \xi \mathrm{dvol}_{\underline{G}} = \int_M \int_{T^*_\cdot M} \partial_t \xi \mathrm{dvol}_{\underline{G}} \tag{3.2.22}$$

$$\left[\nabla_{i_1} \dots \nabla_{i_L} \int_{T^*_\cdot M} \xi(t,\cdot,v) \mathrm{dvol}_{\underline{G}|_{T^*_\cdot M}}\right](x) = \int_{T^*_\cdot M} \underline{\nabla}_{i_1} \dots \underline{\nabla}_{i_L} \xi(t,x,v) \mathrm{dvol}_{\underline{G}|_{T^*_x M}} \tag{3.2.23}$$

Lemma 3.2.7 (Covariant derivatives of momentum functions). *Recalling the notation from Remark 3.2.5, the following formulas hold:*

$$\underline{\nabla}_j v_k = \underline{\nabla}_j v^0 = \underline{\nabla}_j \langle v \rangle_G = \underline{\nabla}_j \left(\frac{v_k}{v^0}\right) = 0 \tag{3.2.24a}$$

$$\underline{\nabla}_j v^0_\gamma = \left[\Gamma^l_{jk} - \hat{\Gamma}^l_{jk}\right] (\gamma^{-1})^{kr} \frac{v_r v_l}{v^0_\gamma} \tag{3.2.24b}$$

$$\underline{\nabla}^{j+3} v^0 = \frac{v^{\sharp j}}{v^0}, \quad \underline{\nabla}^{j+3} \langle v \rangle_G = \frac{v^{\sharp j}}{\langle v \rangle_G}, \quad \underline{\nabla}^{j+3} \left(\frac{v_k}{v^0}\right) = \frac{1}{v^0}\mathbb{I}^j_k - \frac{v_k v^{\sharp j}}{(v^0)^3} \tag{3.2.24c}$$

Lemma 3.2.8 (Frame commutators). *Let $\xi : P \to \mathbb{R}$ be sufficiently regular. Then, one has:*

$$[\mathbf{A}_i, \mathbf{A}_j]\xi = v_k \mathrm{Riem}[G]^k{}_{lij} \mathbf{B}^l \xi \tag{3.2.25a}$$

$$[\mathbf{A}_i, \mathbf{B}^j]\xi = -\Gamma[G]^j_{il} \mathbf{B}^l \xi \tag{3.2.25b}$$

$$[\mathbf{A}_i, v_j \mathbf{B}^j]\xi = 0 \tag{3.2.25c}$$

$$[\partial_t, \underline{\nabla}_i]\xi = v_k \left(\partial_t \Gamma[G]^k_{ij}\right) \mathbf{B}^j \xi \tag{3.2.25d}$$

$$[\partial_t, \underline{\nabla}^{j+3}]\xi = 0 \tag{3.2.25e}$$

$$\begin{aligned} [\boldsymbol{X}, \mathbf{A}_i]\xi = & -a^{-1}(N+1)\left[\frac{v^{\sharp j} v_l}{v^0} \mathrm{Riem}[G]^l{}_{mji} \mathbf{B}^m \xi - \frac{v_j}{v^0} \Gamma[G]^j_{ik} \mathbf{B}^k \xi\right] \\ & + a^{-1} \frac{v_j}{v^0} \nabla_i N \mathbf{A}_j \xi - a^{-1} v^0 (\nabla_i \nabla_j N) \mathbf{B}^j \xi \end{aligned} \tag{3.2.25f}$$

$$\begin{aligned} [\boldsymbol{X}, \mathbf{B}^i]\xi = & \, a^{-1} \left(\frac{1}{v^0}(G^{-1})^{ij} - \frac{v^{\sharp i} v^{\sharp j}}{(v^0)^3}\right)(N+1)\mathbf{A}_j \xi \\ & + a^{-1} \frac{v^{\sharp j}}{v^0} \Gamma[G]^i_{jl} \mathbf{B}^l \xi - a^{-1} \frac{v^{\sharp i}}{v^0} (\nabla_j N) \mathbf{B}^j \xi \end{aligned} \tag{3.2.25g}$$

Proof. We take (3.2.25a)-(3.2.25b) from [CGS22, (38)]. (3.2.25c)-(3.2.25e) are immediate. (3.2.25f) and (3.2.25g) follow with (3.2.25a)-(3.2.25c) and the product rule. In particular, for (3.2.25f), one computes

$$\begin{aligned} & v^0 (\nabla_j N) \mathbf{B}^j \mathbf{A}_i \xi - \mathbf{A}_i \left(v^0 \nabla_j N \mathbf{B}^j \xi\right) \\ = & \, v^0 \left[(\nabla_j N) \cdot \Gamma[G]^j_{ik} \mathbf{B}^k \xi - (\partial_i \nabla_j N) \mathbf{B}^j \xi\right] \\ = & -v^0 (\nabla_i \nabla_j N) \mathbf{B}^j \xi\,. \end{aligned}$$

□

3.3. Norm setup, local well-posedness and bootstrap assumptions

3.3.1. Norms and energies. In this section, we introduce the necessary norms to state our initial data and bootstrap assumptions, as well the energies we will utilise in our bootstrap improvement mechanism.

DEFINITION 3.3.1 (Norms and function spaces). Let $\mathfrak{T}$ be a M_t-tangent tensor, let ξ be a function on P, let $z \in M$ and let $\mu \in \mathbb{R}_0^+$ as well as $K \in \mathbb{N}$. We define the following semi-norms:

$$\|\mathfrak{T}\|_{\dot{C}^K_G(M_t)} := \sup_{x \in M_t} \left(|\nabla^K \mathfrak{T}|_G\right)_x \tag{3.3.1a}$$

$$\|\xi\|_{\dot{C}^K_{\mu,\underline{G}_0}(T^*M_t)} := \sup_{(x,v)\in T^*M_t} \langle v\rangle_G^\mu \left|\underline{\nabla}^K \xi(t,x,v)\right|_{\underline{G}_0} \tag{3.3.1b}$$

$$\|\mathfrak{T}\|_{\dot{H}^K_G(M_t)} := \left(\int_{M_t} |\nabla^K \mathfrak{T}|^2_G \,\mathrm{dvol}_G\right)^{\frac{1}{2}} \tag{3.3.1c}$$

$$\|\xi\|_{\dot{H}^K_{\mu,\underline{G}_0,0}(T^*M_t)} := \left(\int_{T^*M_t} \langle v\rangle_G^{2\mu} \frac{1+|v|^2_G}{|v|^2_G} |\underline{\nabla}^K \xi(t,x,v)|^2_{\underline{G}_0} \,\mathrm{dvol}_{\underline{G}}\right)^{\frac{1}{2}} \tag{3.3.1d}$$

$$\|\xi\|_{\dot{H}^K_{\mu,\underline{G}_0,1}(T^*M_t)} := \left(\int_{T^*M_t} \langle v\rangle_G^{2\mu} |\underline{\nabla}^K \xi(t,x,v)|^2_{\underline{G}_0} \mathrm{dvol}_{\underline{G}}\right)^{\frac{1}{2}} \tag{3.3.1e}$$

(3.3.1f)

$$\|\xi(t,z,\cdot)\|_{\dot{H}^K_{\mu,\underline{G}_0|_{vert},0}(T^*_x M_t)} := \left(\int_{T^*_x M_t} \langle v\rangle_G^{2\mu} \frac{1+|v|^2_G}{|v|^2_G} |\underline{\nabla}^K_{vert} \xi(t,z,v)|^2_{\underline{G}_0|_{vert}} \,\mathrm{dvol}_{\underline{G}|_{vert}}\right)^{\frac{1}{2}}$$

(3.3.1g)

$$\|\xi(t,z,\cdot)\|_{\dot{H}^K_{\mu,\underline{G}_0|_{vert},1}(T^*_x M_t)} := \left(\int_{T^*_x M_t} \langle v\rangle_G^{2\mu} |\underline{\nabla}^K_{vert} \xi(t,z,v)|^2_{\underline{G}_0|_{vert}} \mathrm{dvol}_{\underline{G}|_{vert}}\right)^{\frac{1}{2}}$$

The corresponding norms when replacing G with γ or $\underline{G}_{(0)}$ with $\underline{\gamma}_{(0)}$ are defined analogously, as well as the respective H^K- and C^K-norms. The corresponding function spaces are defined by completion with the respective norm. Finally, we write $\|\cdot\|_{\dot{H}^K_{\mu,\underline{G}_0}} := \|\xi\|_{\dot{H}^K_{\mu,\underline{G}_0,1}}$ (and similar for the other Sasaki Sobolev norms) and define

$$\|\xi\|_{L^\infty_x H^K_{\mu,\underline{G}_0|_{vert},m}(T^*M_t)} = \sup_{x\in M} \|\xi(t,x,\cdot)\|_{\dot{H}^K_{\mu,\underline{G}_0|_{vert},m}(T^*_x M_t)} \,. \tag{3.3.1h}$$

The following combined solution norms will be used to efficiently encode the initial data and bootstrap assumptions:

DEFINITION 3.3.2 (Solution norms). We define the following norms to measure the size of near-FLRW solutions:

(3.3.2a)

$$\begin{aligned}\mathcal{H}(t) = {} & \|\Psi\|_{H^{18}_G(M_t)} + \|\nabla\phi\|_{H^{17}_G(M_t)} + a^2\|\nabla\phi\|_{\dot{H}^{18}_G(M_t)} \\ & + \|\Sigma\|_{H^{18}_G(M_t)} + \|\mathcal{E}\|_{H^{18}_G(M_t)} + \|\mathcal{B}\|_{H^{18}_G(M_t)} \\ & + \|G-\gamma\|_{H^{18}_G(M_t)} + \|\mathrm{Ric}[G] - 2\kappa G\|_{H^{16}_G(M_t)} + a^{-2}\|N\|_{H^{16}_G(M_t)}\end{aligned}$$

$$+ \|f - f_{FLRW}\|_{H^{18}_{1,\underline{G}_0}(T^*M_t)}$$

(3.3.2b)

$$\mathcal{H}_{top}(t) = a^2\|\Psi\|_{\dot{H}^{19}_G(M_t)} + a^4\|\nabla\phi\|_{\dot{H}^{19}_G(M_t)} + a^2\|\Sigma\|_{\dot{H}^{19}_G(M_t)} + a^2\|\mathrm{Ric}[G] - 2\kappa G\|_{\dot{H}^{17}_G(M_t)} + a^2\|f - f_{FLRW}\|_{\dot{H}^{19}_{1,\underline{G}_0}(T^*M_t)}$$

(3.3.2c)

$$\mathcal{C}(t) = \|\Psi\|_{C^{16}_G(M_t)} + \|\nabla\phi\|_{C^{15}_G(M_t)} + \|\Sigma\|_{C^{16}_G(M_t)} + \|\mathcal{E}\|_{C^{16}_G(M_t)} + \|\mathcal{B}\|_{C^{16}_G(M_t)} + \|G - \gamma\|_{C^{16}_G(M_t)} + \|\mathrm{Ric}[G] - 2\kappa G\|_{C^{14}_G(M_t)} + a^{-2}\|N\|_{C^{14}_G(M_t)} + \|\rho^{Vl} - \rho^{Vl}_{FLRW}\|_{C^{16}_G(M_t)} + \|\mathfrak{p}^{Vl} - \mathfrak{p}^{Vl}_{FLRW}\|_{C^{16}_G(M_t)} + \|\jmath^{Vl}\|_{C^{16}_G(M_t)} + \|S^{Vl,\|}\|_{C^{16}_G(M_t)}$$

We note that we do not include the highest order lapse norm that we could control in $\mathcal{H}$ and $\mathcal{C}$: As one can see from Section 3.5.1, we can in fact control lapse energies up to order 21, but at high orders, these scaled energy estimates become progressively weaker. Conversely, for the bootstrap assumption, we only need to assume low order lapse bounds to control nonlinear error terms, and in particular will need to use that the lapse converges to 1 near the Big Bang hypersurface. Thus, there is no benefit in including higher order lapse norms in $\mathcal{C}$ for the bootstrap assumption. In fact, including the lapse in $\mathcal{H}$ is redundant due to the elliptic estimates in Section 3.5.1, and we only include it to for the sake of convenience. To improve the bootstrap assumptions, we will make use of the following energies:

Definition 3.3.3 (Energies). Let $K, L \in \mathbb{N}$, and, where K occurs, and $0 < K \leq L$. We define:

(3.3.3a) $$\mathfrak{E}^{(L)}(\phi, \cdot) = (-1)^L \int_M \Psi\Delta^L\Psi - a^4\phi\Delta^{L+1}\phi \,\mathrm{dvol}_G$$

(3.3.3b) $$\mathfrak{E}^{(L)}(W, \cdot) = (-1)^L \int_M \langle\mathcal{E}, \Delta^L\mathcal{E}\rangle_G + \langle\mathcal{B}, \Delta^L\mathcal{B}\rangle_G \,\mathrm{dvol}_G$$

(3.3.3c) $$\mathfrak{E}^{(L)}(\Sigma, \cdot) = (-1)^L \int_M \langle\Sigma, \Delta^L\Sigma\rangle_G \,\mathrm{dvol}_G$$

(3.3.3d) $$\mathfrak{E}^{(L)}(\mathrm{Ric}, \cdot) = (-1)^L \int_M \left\langle \mathrm{Ric}[G] - 2\kappa G, \Delta^L\left(\mathrm{Ric}[G] - 2\kappa G\right)\right\rangle_G \mathrm{dvol}_G$$

(3.3.3e) $$\mathfrak{E}^{(L)}(N, \cdot) = (-1)^L \int_M \langle N, \Delta^L N\rangle_G \,\mathrm{dvol}_G$$

(3.3.3f) $$\mathfrak{E}^{(L)}_{\mu,0}(f, \cdot) = \int_{T^*M} \langle v\rangle_G^{2\mu} |\underline{\nabla}^L_{vert}(f - f_{FLRW})|^2_{\underline{G}_0} \,\mathrm{dvol}_{\underline{G}}$$

(3.3.3g) $$\mathfrak{E}^{(L)}_{\mu,K}(f, \cdot) = \int_{T^*M} \langle v\rangle_G^{2\mu} |\underline{\nabla}^{L-K}_{vert}\,\underline{\nabla}^K_{hor} f|^2_{\underline{G}_0} \,\mathrm{dvol}_{\underline{G}}$$

(3.3.3h) $$\mathfrak{E}^{(\leq L)}_{\mu,\leq K}(f, \cdot) = \sum_{R=1}^{L} \sum_{S=0}^{\min\{K,L\}} \mathfrak{E}^{(R)}_{\mu,S}(f, \cdot)$$

3.3.2. Initial data assumptions. Using the solution norms introduced above, we can compactly state our initial data assumption as follows:

Assumption 3.3.4 (Initial data assumptions). *Let $(M, \mathring{g}, \mathring{k}, \mathring{\pi}, \mathring{\psi}, \mathring{f})$ be CMC initial data to the ESFV system as introduced in Section 3.2.3. In particular, $\mathring{f} \geq 0$ is assumed to*

have compact momentum support, and in the case of massless Vlasov matter, we additionally assume that f vanishes on an open neighbourhood of the zero section of T^*M.
For some sufficiently small $\varepsilon > 0$ and some $t_0 > 0$, we assume

$$\mathcal{H}(t_0) + \mathcal{H}_{top}(t_0) + \mathcal{C}(t_0) \leq \varepsilon^2 \tag{3.3.4}$$

as well as

$$\|f - f_{FLRW}\|_{C^{11}_{1,\underline{G}_0}(T^*M_{t_0})} \leq \varepsilon^2 \,. \tag{3.3.5}$$

3.3.3. Local well-posedness. For all that follows, we need to establish that solutions to the ESFV system (3.1.1) locally exist in CMC gauge with zero shift, that all solution norms and energies are sufficiently regular in time and that we can extend solutions as long as we can control the size of our variables sufficiently well. To this end, we first collect local well-posedness results in harmonic gauge from [Rin13; Sve12], and then sketch how their proofs can be combined with results for the Einstein stiff-fluid system in CMC gauge to obtain the result for the ESFV system. In particular, we also collect a Cauchy stability result in harmonic gauge to argue that restricting the analysis to CMC data can be done without loss of generality (see Remark 3.3.7).

Lemma 3.3.5 (Local well-posedness for the ESFV system in harmonic gauge). *Let $l \in \mathbb{N}$, $l \geq 3$ and $\mu \in \mathbb{R}$, $\mu \geq 3$. Further, let $m = 0, 1$ denote the Vlasov mass, and let $(\mathring{g}, \mathring{k}, \mathring{\pi}, \mathring{\psi}, \mathring{f})$ be initial data to the ESFV system* (3.1.1) *as in Section 3.2.3 with*

$$\|\mathring{g}\|_{H^{l+1}_\gamma(M)} + \|\mathring{k}\|_{H^l_\gamma(M)} + \|\mathring{\pi}\|_{H^l_\gamma(M)} + \|\mathring{\psi}\|_{H^l_\gamma(M)} < \infty \tag{3.3.6a}$$

as well as

$$\mathring{f} \in H^l_{\mu,\underline{\gamma}_0,m}(T^*M_{t_0}), \tag{3.3.6b}$$

Then, a local solution $(\overline{M}, \overline{g}, \nabla\phi, \partial_0\phi, f)$ in harmonic gauge with Vlasov-mass m exists in the following sense: For some $h > 0$ and writing $J = (t_0 - h, t_0 + h)$, a spacetime manifold $\overline{M}$ can be foliated by spacelike Cauchy hypersurfaces $(M_t)_{t\in J}$, each of which is endowed with a collection of coordinates $(t \equiv x^0_U, x^1_U, x^2_U, x^3_U)$, where x^i_U are spatial coordinates on an open subset $U \subseteq M$ and M is covered by finitely many such coordinate neighbourhoods. In these coordinates, the harmonic gauge constraint

$$\overline{g}^{\rho\sigma}\left(\Gamma[\overline{g}]^\nu_{\rho\sigma} - \Gamma[\overline{g}_{FLRW}]^\nu_{\rho\sigma}\right) = 0$$

is satisfied, one has

$$(\overline{g}_{FLRW})_{ij} = a(t)^2\gamma_{ij}\,,$$

and $(\overline{M}, \overline{g}, \nabla\phi, \partial_0\phi, f)$ solves (3.1.1). *Furthermore, the spacetime metric components and matter variables enjoy the following regularity in these coordinates:*

$$\begin{aligned}
\overline{g}_{\rho\sigma} &\in C^{l-1}(J \times M) \cap C^0(J, H^{l+1}_\gamma(M))\\
\partial_t\overline{g}_{\rho\sigma} &\in C^{l-2}(J \times M) \cap C^0(J, H^l_\gamma(M))\\
\partial_\rho\phi &\in C^{l-2}(J \times M) \cap C^0(J, H^l_\gamma(M))
\end{aligned}$$

$$\left((t,x)\mapsto \|f(t,x,\cdot)\|^2_{H^1_{\mu,\underline{\gamma}_0|_{vert},m}(T^*_xM)}\right)\in C^{l-1}_{dt^2+\gamma}(J\times M)$$

For $l\geq 4$, one additionally has

$$f\in C^{l-4}_{dt^2+\underline{\gamma}}(J\times T^*M)\,. \tag{3.3.7}$$

If $(\mathfrak{t},t_0]$ is the past maximal interval of existence, then one has

$$\lim_{t\downarrow\mathfrak{t}}\left(\|\overline{g}_{\rho\sigma}\|_{C^2_\gamma(M_t)}+\|\partial_t\overline{g}_{\rho\sigma}\|_{C^1_\gamma(M_t)}+\|\partial_\mu\phi\|_{C^1(M_t)}\right)=\infty\,, \tag{3.3.8a}$$

$$\lim_{t\downarrow\mathfrak{t}}\sup_{x\in M}\left[\int_{T^*_xM}\langle p\rangle_\gamma^{2\mu}\cdot\left(\frac{\langle p\rangle_\gamma}{p^0_\gamma}\right)^{2l}\left(|f(t,x,p)|^2+(p^0_\gamma)^2\,|\mathbf{B}f(t,x,p)|^2_\gamma\right)\,\mu_\gamma^{-1}dp\right]=\infty\,. \tag{3.3.8b}$$

or $\mathfrak{t}=-\infty$.

Proof-Outline. As argued, for example, in [Rin13, Section 22.2.3], the Einstein equations and the wave equation for the scalar field can be modified in harmonic gauge such that one can equivalently solve a quasilinear system in $\overline{g}_{\mu\nu}$ and ϕ, coupled with the transport equation for f. For such systems, [Rin13, Proposition 19.76 and Lemma 19.83] lead to the stated result for $m=1$, and respectively [Sve12, Paper B, Proposition 8.48 and Lemma 9.1] for $m=0$, since one easily checks that all operators and nonlinearities remain admissible when no scalar field potential is present (see [Rin13, Section 19.1], respectively [Sve12, Paper B, Section 8.1]). We also note that, while the results in [Rin13; Sve12] are proven while viewing the Vlasov distribution as a function on the mass shell, rather than the co-mass shell, we can induce initial data $\mathring{f}^\sharp:TM\to\mathbb{R}$ on the mass shell by

$$\mathring{f}^\sharp(x,q)=\mathring{f}\,(x,p(q,\mathring{g}))\,,\quad p_i(q,\mathring{g})=\mathring{g}_{ij}\,q^j\,,$$

which, due to the regularity requirements for $\mathring{g}$, enjoys the same Sobolev regularity as $\mathring{f}$, but now with respect to the corresponding weighted Sasaki metric induced on the tangent bundle.[7] Then working with canonical coordinates q^μ on the mass shell, one can apply the aforementioned results and transform the corresponding solution $f^\sharp$ back via

$$f(t,x,p)=f^\sharp(t,x,q(\overline{g},p)),\quad q^i(\overline{g},p)=\overline{g}^{i\mu}p_\mu\,.$$

Again, the regularity of the metric components that the regularity of $f^\sharp$ transfers to f in the corresponding function spaces with respect to the cotangent bundle and the co-mass shell. □

We note that the non-negativity of the Vlasov distribution function is ensured by the Vlasov equation, and can be argued as in Lemma 3.4.5. Furthermore, uniqueness follows from uniqueness of the maximal globally hyperbolic development (see Section 3.2.3).
To prove that we can take initial data to be CMC without loss of generality, we need the following result:

[7] Note that the gauge choice in [Rin13; Sve12] is precisely such that one has

$$\overline{g}|_{\{t=t_0\}}=-dt^2+\mathring{g}\,.$$

LEMMA 3.3.6 (Cauchy stability of near-FLRW solutions to the ESFV system). *Let $l \in \mathbb{N}$, $l \geq 3$ and $\mu \in \mathbb{R}$, $\mu \geq 3$, and let $\delta > 0$ be sufficiently small. Further, consider initial data $(M, \mathring{g}, \mathring{k}, \mathring{\pi}, \mathring{\psi}, \mathring{f})$ to the ESFV system with Vlasov mass $m = 0, 1$ (again as in Section 3.2.3) such that one has*

(3.3.9a)

$$\|\mathring{g} - a(t_0)^2\gamma\|_{H^{l+1}_\gamma(M)} + \|\mathring{k} - \frac{\tau(t_0)}{3} \cdot a(t_0)^2\gamma\|_{H^l_\gamma(M)} + \|\mathring{\pi}\|_{H^l_\gamma(M)} + \|\mathring{\psi} - Ca(t_0)^{-3}\|_{H^l_\gamma(M)} < \delta$$

and

$$\|(\mathring{f} - f_{FLRW})(t_0, \cdot, \cdot)\|_{H^l_{\mu,\gamma_0,m}(T^*M)} < \delta \,. \tag{3.3.9b}$$

hold. In the massless Vlasov case, we additionally assume that $\mathring{f}$ vanishes in an open neighbourhood of $\{(x,p) \in T^*M \,|\, p = 0\}$. Then the solution in the sense of Lemma 3.3.5 satisfies the following bound for some $K > 0$ that is independent of $\delta > 0$ for any $t \in [t_1, t_0], t_1 > \mathfrak{t}$:

$$\begin{aligned} &\|g - a^2\gamma\|_{H^{l+1}_\gamma(M_t)} + \|k - \frac{\tau}{3}a^2\gamma\|_{H^l_\gamma(M_t)} + \|\nabla\phi\|_{H^l_\gamma(M_t)} \\ &+\|\partial_0\phi - Ca^{-3}\|_{H^l_\gamma(M_t)} + \|f - f_{FLRW}\|_{H^l_{\mu,\gamma_0,m}(T^*M_t)} \leq K\delta \end{aligned} \tag{3.3.10}$$

PROOF-OUTLINE. For the massive case, this can be proven as in [Rin13, Corollary 24.10], which in turn follows along similar lines as [Rin09b, Theorem 15.10]. Essentially, the argument therein reduces to using the respective result for the metric components and matter variables in harmonic gauge from [Rin13, Corollary 20.7] and patching these local results to a spatially global result. The aforementioned statements are, at first glance, only applicable for smooth initial data, but this clearly extends to non-smooth data by a standard approximation argument.
This mostly applies analogously to $m = 0$ – the only issue that arises is that (3.3.8b) is regularity dependent, thus solutions at high regularity might only be extendible in lower regularity. However, if the distribution function vanishes on an open neighbourhood of the zero section throughout the evolution, any choice of $l \geq 3$ in (3.3.8b) is equivalent to any other (see the proof of [Sve12, Paper B, Lemma 9.2]). That this is the case can be shown as in [Sve12, Paper B, Lemma 8.25], analogous to the proof of (3.4.7). □

REMARK 3.3.7 (Existence of a CMC hypersurface). Before moving on to our main well-posedness result in CMC gauge with zero shift, we illustrate why the above results imply that, as in [RS18b; Spe18] and Chapter 2, it is is not a true restriction to take initial data to be CMC :

If the initial data is close to FLRW data as in (3.3.4), but not CMC, Lemma 3.3.5 yields a local-in-time solution by evolving in harmonic gauge, and by Lemma 3.3.6, we can restrict ourselves to a time interval such that we remain close to the FLRW data throughout the local time evolution. An implicit function theorem argument as in [FK20, Section 2.5] now implies that, within this local solution, we can find a Cauchy hypersurface Σ' close to the initial hypersurface M_{t_0} that is CMC. The adaptation of the argument in [FK20, Section 2.5] can be done as sketched in Remark 2.8.1: The only noteworthy change compared to the latter is in computing that $(\overline{g}_{FLRW}, t_0)$ remains a regular point of the mean curvature map H under the addition of Vlasov matter and regardless of sectional curvature. One computes using (3.2.3a):

$$a(t_0)^2 dH_{(\overline{g}_{FLRW},t_0)}(0,w) = \left[\frac{1}{3}\Delta_\gamma + \kappa - 4\pi C^2 a(t_0)^{-4} - 4\pi a(t_0)^{-2}\left(\rho^{Vl}_{FLRW} + \mathfrak{p}^{Vl}_{FLRW}\right)\right] w$$

Since we assume $f_{FLRW} \geq 0$ and $C > 0$, and since Δ_γ has no positive eigenvalues, this clearly is a regular point for $\kappa \leq 0$. For $\kappa > 0$, $\kappa < 4\pi C^2 a(t_0)^{-4}$ can be ensured for small enough $t_0 > 0$, which Lemma 3.3.6 allows us to take without loss of generality.

Having obtained a CMC hypersurface, we need to ensure that Lemma 3.3.6 is sufficient to ensure Assumption 3.3.4 for suitably small $\delta > 0$. This follows from Sobolev embedding for all terms except those associated with Vlasov matter: The Vlasov equation ensures that f is still nonnegative (as in Lemma 3.4.5), and that it has compact momentum support as well as, in the massless case, support bounded away from the zero section (as in Lemma 3.4.6). In particular, for $\delta < 1$, any momentum weights can be dropped or added up to a constant depending on f_{FLRW}, and thus $\|f - f_{FLRW}\|_{H^{18}_{1,\gamma_0}(T^*M_t)} \lesssim \delta$ also holds. Using a co-mass shell analogue of [Rin13, Lemma 15.29] or estimates similar to the energy bounds in Lemma 3.4.14, this ensures the equivalent bound on all Vlasov matter terms in $\mathcal{C}$ after potentially updating constants. Finally, (3.3.5) can be ensured since [Rin13, Lemma 15.29] applied analogously to the co-mass shell and the standard Sobolev embedding on $\mathbb{R}^3$ imply

$$\|f - f_{FLRW}\|_{C^{11}_{1,\gamma_0}(T^*M_t)} \lesssim \sup_{x\in M} \|f - f_{FLRW}\|_{L^\infty_x H^{13}_{1,\gamma_0|vert}(T^*_x M_t)} \lesssim \|f - f_{FLRW}\|_{H^{13}_{1,\gamma_0}(T^*M_t)} .$$

Again, we note that, on the momentum support of $f - f_{FLRW}$ on $[t_1, t_0] \times M$, $\langle v\rangle_\gamma \simeq 1$ holds.

Remark 3.3.8 (Smallness of t_0). To establish a strong total energy bound in Proposition 3.7.6, we will need to assume that $a(t)$ is sufficiently small for $t \in (t_{Boot}, t_0]$, which simply means that t_0 needs to be sufficiently small since a is strictly increasing on $(0, T/2)$, see Lemma 3.2.2. Lemma 3.3.6 ensures that, if we choose the initial perturbation to be small enough, we can use Cauchy stability to ensure that the solution remains sufficiently close to the FLRW solution in any fixed time interval $[t_1, t_0]$. We can then run the argument in Remark 3.3.7 to find a nearby CMC hypersurface, and can thus overall assume $t_0 > 0$ to be sufficiently small without loss of generality.

Having now prepared initial data to be CMC as well as sufficiently close to the Big Bang if needed, we can state our main local well-posedness result:

Lemma 3.3.9 (Local well-posedness in CMC gauge with zero shift). *Let* $(M, \mathring{g}, \mathring{k}, \mathring{\pi}, \mathring{\psi}, \mathring{f})$ *be CMC initial data as in Lemma 3.3.5 and let* $\mu \geq 3$. *Additionally, assume that*

$$\inf_{x\in M} \mathring{\psi}(x)^2 - |\mathring{\pi}|^2_{\mathring{g}_x} > 0 \ .$$

Then, this data launches a unique solution $(\overline{M}, g, k, \nabla\phi, \partial_0\phi, \overline{f})$ *of the CMC-transported ESFV-system (see* (3.1.1) *with CMC condition* (3.2.1)*) toward the past, foliated by CMC* *hypersurfaces* $(M_t)_{t\in J}$ *with* $J=(t_0-h,t_0]$ *for some* $h>0$. *One has:*

$$\begin{aligned}
g &\in C^{l-1}_{dt^2+\gamma}(J\times M)\cap C^0(J,H^{l+1}_\gamma(M))\\
k &\in C^{l-2}_{dt^2+\gamma}(J\times M)\cap C^0(J,H^{l}_\gamma(M))\\
\nabla\phi &\in C^{l-2}_{dt^2+\gamma}(J\times M)\cap C^0(J,H^{l}_\gamma(M))\\
\partial_t\phi &\in C^{l-2}_{dt^2+\gamma}(J\times M)\cap C^0(J,H^{l}_\gamma(M))\\
n &\in C^{l}_{dt^2+\gamma}(J\times M)\cap C^0(J,H^{l+2}_\gamma(M))\\
f &\in C^{l-4}_{dt^2+\underline{\gamma}}(J\times TM)\cap C^{l-1}(J,L^\infty_x H^1_{\mu,\underline{\gamma}_0|_{vert},m}(T^*_{(\cdot)}M))
\end{aligned}$$

For the past-maximal interval of existence $(\mathfrak{t},t_0]$, *one has* $\mathfrak{t}=0$ *or one of the following blow-up criteria are satisfied:*
There exists a sequence (t_m,x_m) *with* $t_m\downarrow\mathfrak{t}$ *such that*

(1) *such that the smallest eigenvalue of* $g(t_m,x_m)$ *converges to* 0,
(2) *or such that* $n(t_m,x_m)$ *converges to* 0,
(3) *or such that* $\left(|\partial_0\phi|^2+|\nabla\phi|^2_g\right)(t_m,x_m)$ *converges to* 0,
(4) *or one of the following maps is unbounded:*

(3.3.11a)

$$s\in(\mathfrak{t},t_0]\mapsto\|g\|_{C^2_\gamma(M_s)}+\|k\|_{C^1_\gamma(M_s)}+\|n\|_{C^2_\gamma(M_s)}+\|\partial_t\phi\|_{C^1_\gamma(M_s)}+\|\nabla\phi\|_{C^1_\gamma(M_s)}\,,$$

(3.3.11b)

$$s\in(\mathfrak{t},t_0]\mapsto\|\rho^{Vl}\|_{C^1_\gamma(M_s)}+\|\jmath^{Vl}\|_{C^1_\gamma(M_s)}+\|S^{Vl}\|_{C^1_\gamma(M_s)}\,,$$

(3.3.11c)

$$s\in(\mathfrak{t},t_0]\mapsto\sup_{x\in M_s}\left[\int_{T^*_xM_s}\langle p\rangle_\gamma^{2\mu}\cdot\left(\frac{\langle p\rangle_\gamma}{p^0_\gamma}\right)^{2l}\left(|f(s,x,p)|^2+(p^0_\gamma)^2\,|\mathbf{B}f(s,x,p)|^2_\gamma\right)\mu_\gamma^{-1}dp\right]$$

Proof-Outline. In short, this can be proven along the same lines as in the proof of [RS18b, Theorem 14.1] for the stiff fluid, with the analysis of Vlasov matter following from [Rin13; Sve12] as above. The continuation criterion (3.3.11a) then extends from the scalar field case, (3.3.11b) is the direct analogue of (3.3.11a) for the Vlasov matter quantities that occur in the elliptic-hyperbolic system for the metric variables, and (3.3.11c) is needed to ensure boundedness of coefficients arising from Vlasov energy estimates as in the proof in harmonic gauge.

To be more precise, the proof of [RS18b, Theorem 14.1] relies on showing that a modified system of equations leads to an elliptic-hyperbolic system in which the second fundamental form satisfies a wave equation, where its evolution equation is considered as an additional constraint quantity. The only properties of stiff fluid matter that are used in the proof are that the stiff fluid equation implies that the associated energy-momentum tensor being divergence-free, that it satisfies the strong energy condition and that it generates standard energy estimates. The former two are also satisfied for our matter model (where we again note that one has $f\geq 0$, and thus the strong energy condition is ensured). Since scalar field matter constitutes a subcase of stiff fluid matter, the energy estimates for stiff fluids apply to the scalar field as well. Thus, energy estimates for the geometry and scalar field can be straightforwardly extended from the stiff fluid setting.

The necessary energy estimates for the Vlasov distribution function can be established as in [Rin13; Sve12], where they are proven for a wide class of relativistic transport operators of the form

$$L\xi = \partial_t \xi + \frac{z^i}{q^0}\partial_{x^i}\xi + \frac{G^i}{q^0}\partial_{q^i}\xi\,,$$

where $z^i \equiv z^i[u]$ and $G^i \equiv G^i[u]$ are sufficiently regular coefficient functions depending on the quantities u that solve the coupled system of wave equations, which correspond to the spacetime metric components. Again, note that these statements are proven for the mass shell, so we need to consider the Vlasov equation on the mass shell, i.e., apply this to the equation

$$L f^\sharp = \partial_t \xi + \frac{q^i}{q^0}\, n\, \partial_{x^i} f^\sharp + \left(-\frac{q^i q^l}{q^0}\, n\, \Gamma[g]^j_{il} - q^0\, g^{ij}\, \nabla_i n + 2n\, k^j{}_i\, q^i\right)\partial_{q^j}\xi = 0\,. \tag{3.3.12}$$

Nevertheless, we can establish analogous estimates for these components using the elliptic-hyperbolic system from [RS18b, Theorem 14.1], and the transport operator for $f^\sharp$ is actually simpler than the one that needs to be considered in harmonic gauge, energy estimates for the Vlasov distribution also extend. Finally, the components of the Vlasov energy momentum tensor can be bounded up to constant by the energies for the distribution function as in Lemma 3.4.14.

Thus, we can prove local well-posedness of the modified elliptic-hyperbolic system metric and scalar field coupled to the Vlasov equation by a constructing a similar convergent sequence as in [Rin13, Proposition 19.76] and [Sve12, Paper B, Proposition 8.48]. That this then also solves the Einstein equations – i.e., that the constraints are propagated – can again be proven as in the stiff fluid setting since we only use properties of the energy momentum tensor shared by both matter models.

The blow-up criteria follow by combining what arises from the elliptic-hyperbolic system (extending from [RS18b, Theorem 14.1]) with those arising from the transport equation from Lemma 3.3.5. For the latter, (3.3.8a) is covered by the requirements (3.3.11a) from the elliptic-hyperbolic system, and thus only (3.3.8b) needs to be added. □

REMARK 3.3.10 (On regularity of norms and energies). In addition to the initial data assumptions (see Assumption 3.3.4), we can assume, without loss of generality by a standard approximation argument, that the initial data is sufficiently regular. More precisely, using Lemma 3.3.9, we will assume tacitly $l \in \mathbb{N}$ to be sufficiently large such that all norms and energies are continuously differentiable in time. We note that, in the massless case, the blow-up criterion (3.3.8b) is dependent on how regular we choose our initial data to be a priori.

However, due to control on the support of f (see Lemma 3.4.6), it will turn out that this additional weight can be ignored when checking this criterion (see Proof of Theorem 3.8.1).

3.3.4. Bootstrap assumptions. By Lemma 3.3.9, initial data as in Assumption 3.3.4 generates a local-in-time solution toward the past such that $\mathcal{C}(t)$ and $\|f-f_{FLRW}\|_{C^{11}_{1,\underline{G}_0}(T^*M_t)}$ are continuous in t. Thus, the following bootstrap assumption can be satisfied:

ASSUMPTION 3.3.11 (Bootstrap assumptions). *Let $\sigma=\varepsilon^{\frac{1}{16}}$, fix $K_0>0$ and $c_0>0$ such that $c_0\sigma<1$. We assume that, for the bootstrap time $t_{Boot}\in[0,t_0)$, the following holds for any $t\in(t_{Boot},t_0]$:*

$$\mathcal{C}(t)\leq K_0\varepsilon a^{-c_0\sigma}\,. \tag{3.3.13a}$$

Additionally, we assume

$$\|\underline{\nabla}_{hor}(f-f_{FLRW})\|_{C^{10}_{1,\underline{G}_0}(T^*M_t)}\leq K_0\varepsilon^{\frac14}a^{-c\sigma} \tag{3.3.13b}$$

Our goal is to show that the bootstrap assumptions imply

$$\mathcal{C}\leq K_1\varepsilon^{\frac74}a^{-c_1\varepsilon^{\frac18}},\quad \|\underline{\nabla}_{hor}(f-f_{FLRW})\|_{C^{10}_{1,\underline{G}_0}(T^*M_t)}\leq K_1\sqrt{\varepsilon}a^{-c\sqrt{\varepsilon}}$$

for suitable constants $c_1,K_1>0$ and any $t\in(t_{Boot},t_0]$, which is a strict improvement if $\varepsilon>0$ is chosen sufficiently small.

REMARK 3.3.12 (Lapse bounds). We note that, in particular, (3.3.13a) implies

$$\|N\|_{C^{14}_G(M_t)}\lesssim\varepsilon a(t)^{2-c\sigma}\,, \tag{3.3.14}$$

and we will in fact only be able to improve the convergence rate to $\varepsilon a^{2-c\varepsilon^{\frac18}}$ by Theorem 3.8.1. Thus, the lapse converges less strongly than in the pure scalar field case, and the bootstrap assumption is also weaker (see (2.3.17h)). This is due to Vlasov matter being the asymptotically strongest term in (3.2.20g). However, it does not lead to substantial changes in the argument since the fact that the lapse converges at a rate stronger than, for example, $a(t)$ for sufficiently small $\sigma>0$ is enough to make otherwise borderline terms containing low orders of N converge.

3.4. A priori estimates

3.4.1. Strong low order bounds on spacetime metric and scalar field variables. In this section, we collect the necessary improved low order estimates for all variables that are not the Vlasov distribution function or associated matter quantities. Since Vlasov matter is asympotically negligible in the evolution of metric variables and the shear as well as in the Hamiltonian and Bel-Robinson constraint equations, and does not occur in the wave equation at all, all of these bounds follow almost directly from Sections 2.4.1 and 2.4.2, and we only sketch how one controls the occuring Vlasov terms with the bootstrap assumption.

LEMMA 3.4.1 (Strong C^0_G bounds). *For $t\in(t_{Boot},t_0]$, the following estimates hold:*

$$\|\Psi\|_{C^0_G(M_t)}\lesssim\varepsilon \tag{3.4.1a}$$

$$\|\Sigma\|_{C^0_G(M_t)} + \|\Sigma^\sharp\|_{C^0_\gamma(M_t)} \lesssim \varepsilon \tag{3.4.1b}$$

$$\|\mathcal{E}\|_{C^0_G(M_t)} \lesssim \varepsilon \tag{3.4.1c}$$

PROOF. (3.4.1a) is proven identically to the scalar field case (see (2.4.4a)), replacing the lapse bootstrap assumption therein with (3.3.14).
Regarding (3.4.1b), one has

$$-\partial_t|\Sigma|^2_G = 2\left(\partial_t\Sigma^\sharp\right)^i{}_j\left(\Sigma^\sharp\right)^j{}_i \lesssim \left|(\partial_t\Sigma^\sharp)^\parallel\right|_G |\Sigma|_G\,, \tag{3.4.2}$$

where we recall that $(\partial_t\Sigma^\sharp)^\parallel$ is the tracefree, symmetric part of $\partial_t\Sigma^\sharp$. Estimating all non-Vlasov terms as in the proof of (2.4.2b) using (3.3.13a), we have

$$\left|\partial_t\Sigma^\sharp\right|_G \lesssim \varepsilon a^{-1-c\sigma}(1+|\Sigma|_G) + a^{-1}\left|S^{Vl,\parallel}\right|_G\,.$$

That this bound is weaker than its analogue in the pure scalar field case is, again, down to (3.3.14). The Vlasov term can also be bounded by $\lesssim \varepsilon a^{-1-c\sigma}$ using the bootstrap assumption (3.3.13a). Integrating (3.4.2) and inserting this bound along with (3.3.4) and (3.2.6) then implies

$$|\Sigma|^2_G(t) \lesssim |\Sigma|^2_G(t_0) + \int_t^{t_0} \varepsilon a(s)^{-1-c\sigma}\,ds \lesssim \varepsilon.$$

The bound for $|\Sigma^\sharp|_{C^0_\gamma}$ is proven identically since the bootstrap assumption on $G-\gamma$ ensures that all quantities in $\mathcal{C}$ satisfy the bootstrap assumptions with C_G replaced by C_γ, up to updating constants.
(3.4.1c) then follows directly from (3.2.20j): Due to $\langle G,\mathcal{E}\rangle_G = 0$, the second line in (3.2.20j) can be ignored, and using (3.4.1b) for all shear terms and (3.3.13a) for all other terms, including Vlasov matter, one has $\langle\mathcal{E},\mathcal{E}\rangle_G \lesssim \varepsilon|\mathcal{E}|_G$ and thus the statement. □

LEMMA 3.4.2 (Strong low-order C_G-bounds for spacetime and scalar field variables). *For $t\in(t_{Boot},t_0]$, one has:*

$$\|\Psi\|_{C^{13}_G(M_t)} \lesssim \varepsilon a(t)^{-c\sqrt{\varepsilon}} \tag{3.4.3a}$$

$$\|\Sigma\|_{C^{12}_G(M_t)} \lesssim \varepsilon a(t)^{-c\sqrt{\varepsilon}} \tag{3.4.3b}$$

$$\|G^{\pm1}-\gamma^{\pm1}\|_{C^{12}_G(M_t)} \lesssim \sqrt{\varepsilon} a(t)^{-c\sqrt{\varepsilon}} \tag{3.4.3c}$$

$$\|\nabla\phi\|_{C^{12}_G(M_t)} \lesssim \sqrt{\varepsilon} a(t)^{-c\sqrt{\varepsilon}} \tag{3.4.3d}$$

$$\|\mathrm{Ric}[G]-2\kappa\, G\|_{C^{10}_G(M_t)} \lesssim \sqrt{\varepsilon} a(t)^{-c\sqrt{\varepsilon}} \tag{3.4.3e}$$

$$\|\mathcal{E}\|_{C^{12}_G(M_t)} \lesssim \varepsilon a(t)^{-c\sqrt{\varepsilon}} \tag{3.4.3f}$$

$$\|\mathcal{B}\|_{C^{11}_G(M_t)} \lesssim \varepsilon a(t)^{2-c\sqrt{\varepsilon}} \tag{3.4.3g}$$

PROOF. Regarding (3.4.3b), (3.3.13a) implies

$$\left|\nabla^J\partial_t\Sigma^\sharp\right|_G \lesssim \varepsilon a^{-1-c\sigma},$$

where the Vlasov matter terms and the lapse bound (3.3.14) lead to the leading terms. Since the integral of the right hand side over $[0,t_0]$ is still bounded by ε up to constant due to (3.2.6), (3.4.3b) is then proven as in Lemma 2.4.3.

(3.4.3a), (3.4.3c)-(3.4.3e) and (3.4.3g) are also proven almost entirely identically to their analogues in Chapter 2, see Lemma 2.4.3, the only differences arising again from the weaker bootstrap bound (3.3.14) on the lapse that is, as with the shear, sufficient to prove that terms containing N are asymptotically negligible. Finally, (3.4.3g) is proven by applying ∇^J to (3.2.20j) and then arguing as in the C^0_G-case, using (3.4.3b) for the shear and (3.3.13a) for the remaining terms. $\square$

Finally, we collect that the volume element μ_G is uniformly bounded:

LEMMA 3.4.3. *For $\mu_G = \sqrt{\det G}$ and $t \in (t_{Boot}, t_0]$, one has*

$$\|\mu_G - \mu_\gamma\|_{C^0(M_t)} \lesssim \varepsilon\,. \tag{3.4.4}$$

PROOF. This follows immediately from $\partial_t \mu_G = -N\tau\mu_G$, (3.3.14), the initial data assumption and the Gronwall lemma. $\square$

3.4.2. A priori observations for Vlasov matter.

REMARK 3.4.4 (On the charateristic method). The following a priori estimates involving the Vlasov distribution function will rely on the fact that initial value problems of the form

$$\partial_t\xi + b(t,x,v)^i\partial_{x^i}\xi(t,x,v) + B_i(t,x,v)_i\partial_{v_i}\xi(t,x,v) + c(t,x,v,\xi) = 0,\ \ \xi(0,x,v) = \xi_0(x,v)$$

can be locally solved using the method of characteristics, i.e.: There exist functions $X : I \to U \subseteq \mathbb{R}^3$, $V : I \to \mathbb{R}^3$, $Z : I \to \mathbb{R}$ with

$$Z(s) = \xi(s, X(s), V(s))$$

that satisfy the following equations:

$$\begin{aligned}\dot{X}^i(s) &= b^i(s, X(s), V(s))\\ \dot{V}_i(s) &= B_i(s, X(s), V(s))\\ \dot{Z}(s) &= -c(s, X(s), V(s), Z(s))\end{aligned}$$

This is a consequence of [Eva98, p. 107, Theorem 2]. In particular, we will apply this to equations that are shifts or derivatives of the rescaled Vlasov equation (3.2.19a). Since, on the bootstrap interval, the solution to the Vlasov equation exists and is unique, it can be expressed in this fashion in the entire bootstrap interval after choosing inital data on some hypersurface M_t for $t \in (t_{Boot}, t_0]$. In particular, solutions to (3.2.19a) remain constant along integral curves (X, V) of the following characteristic system:

$$\frac{dX^i}{dt} = a^{-1}(N+1)\frac{(G^{-1})^{ij}V_j}{\sqrt{m^2a^2 + |V|_G^2}} \tag{3.4.5a}$$

$$\frac{dV_i}{dt} = a^{-1}(N+1)(G^{-1})^{jk}\Gamma[G]^l_{ik}\frac{V_jV_l}{\sqrt{m^2a^2+|V|^2_G}} - a^{-1}\sqrt{m^2a^2+|V|^2_G}\nabla_i N \tag{3.4.5b}$$

We introduce the following notation:

$$\mathcal{V} = |V|_G, \quad V^0 = \sqrt{m^2a^2 + \mathcal{V}^2}, \quad V^0_\gamma = \sqrt{m^2a^2 + |V|^2_\gamma}$$

LEMMA 3.4.5 (Non-negativity of the Vlasov distribution function). *For any $t \in (t_{Boot}, t_0]$, $f(t,\cdot,\cdot)$ is nonnegative, as well as $\rho^{Vl}(t,\cdot)$ and $\mathfrak{p}^{Vl}(t,\cdot)$.*

PROOF. Recall that $s \mapsto f(s, X(s), V(s))$ remains constant for characteristics (X, V) satisfying (3.4.5) and that $\mathring{f}$ is nonnegative. Thus, for any $(x, v) \in T^*M_t$, there exist $(x_{init}, v_{init}) \in T^*M$ such that one has

$$0 \leq \mathring{f}(x_{init}, v_{init}) = f(t_0, x_{init}, v_{init}) = f(t, x, v)\,.$$

Since v^0 is also non-negative, so are then $\rho^{Vl}(t,\cdot)$ and $\mathfrak{p}^{Vl}(t,\cdot)$. $\square$

LEMMA 3.4.6 (Bounds for the momentum support). *There exists a constant $K > 0$ that is independent of ε such that the following bounds hold:*

$$\mathcal{P}(t) \leq \mathcal{P}^0(t) \leq K a(t)^{-c\sqrt{\varepsilon}} \tag{3.4.6a}$$

$$\mathcal{P}_\gamma(t) \leq \mathcal{P}^0_\gamma(t) \leq K \tag{3.4.6b}$$

Furthermore, for $m = 0$, one additionally has that the momentum support of f is bounded away from zero as follows:

$$\sup\left\{|v|_G^{-2} \mid x \in M_t,\ v \in \mathrm{supp} f(t, x, \cdot)\right\} \leq K a(t)^{-c\sqrt{\varepsilon}} \tag{3.4.7}$$

$$\sup\left\{|v|_\gamma^{-2} \mid x \in M_t,\ v \in \mathrm{supp} f(t, x, \cdot)\right\} \leq K \tag{3.4.8}$$

PROOF. Noting $\mathcal{V} \leq V^0$, we get the following from the characteristic equation (3.4.5b):

(3.4.9)

$$\begin{aligned}
-\frac{d(V^0)^2}{dt} &\lesssim -2\frac{\dot{a}}{a}\cdot m^2 a^2 + |\partial_t G^{-1}|_G \mathcal{V}^2 + a^{-1}|N+1||\Gamma[G]|_G \frac{\mathcal{V}^3}{V^0} + a^{-1} V^0 \mathcal{V}^2 |\nabla N|_G \\
&\lesssim \left(a^{-3}\|\Sigma\|_{C^0_G} + a^{-3}\|N\|_{C^0_G} + a^{-1}(\|G-\gamma\|_{C^1_G} + \|\gamma\|_{C^1_G}) + a^{-1}\|N\|_{\dot{C}^1_G}\right)(V^0)^2
\end{aligned}$$

Using (3.3.13a) for the metric and lapse as well as (3.4.1b), this can be estimated as follows:

$$-\frac{d(V^0)^2}{dt} \lesssim \left(\varepsilon a^{-3} + a^{-1-c\sigma}\right)(V^0)^2$$

Updating $c > 0$, the Gronwall lemma then implies

$$(V^0(t))^2 \leq (V^0(t_0))^2 \cdot a(t)^{-2c\sqrt{\varepsilon}}\,. \tag{3.4.10}$$

For any $x \in M_t$ and any $v \in \mathrm{supp} f(t, x, \cdot)$, (x, v) must lie on a unique characteristic (X, V) emanating from $(X(t_0), V(t_0)) \in \mathrm{supp} f(t_0, \cdot, \cdot)$. Since $\mathring{f}$ has compact momentum support, (3.4.10) implies

$$V^0(t) \lesssim \mathcal{P}(t_0) a(t)^{-c\sqrt{\varepsilon}}$$

for $V(t) = v \in \mathrm{supp} f(t, x, \cdot)$, and thus (3.4.6a) after taking the supremum in v. (3.4.6b) is proven identically, noting that since $\partial_t \gamma = 0$, one has

$$-\frac{d(V^0_\gamma)^2}{dt} \lesssim a^{-1-c\sigma}(V^0_\gamma)^2$$

and thus V^0_γ remains uniformly bounded. (3.4.7)-(3.4.8) also follow similarly, using that $\mathring{f}$ vanishes on an open neighbourhood of the zero section.

$\square$

To obtain strong low order bounds on the Vlasov distribution function, we will need to compare it to the reference distribution function f_{FLRW} while using the Sasaki connection with respect to G. To control the resulting error terms, we first collect the following bounds on the reference distribution:

LEMMA 3.4.7 (Derivatives of the reference distribution function). *For $0 < K < L \leq 19$, one has*

(3.4.11a)
$$\|\underline{\nabla}_{vert}^{L-K}\underline{\nabla}_{hor}^{K} f_{FLRW}\|_{L^2_{1,\underline{G}_0}(T^*M_t)} \lesssim a(t)^{-c\sqrt{\varepsilon}}\Big(\underbrace{\sqrt{\varepsilon}\|\mathrm{Ric}[G]-2\kappa G\|_{H^{K-1}_G(M_t)}}_{if\ K>1} + \|\Gamma-\hat{\Gamma}\|_{H^{K-1}_G(M_t)} + \|G^{-1}-\gamma^{-1}\|_{H^{K-1}_G(M_t)}\Big)$$

Additionally, for $L > 0$, the following holds:

(3.4.11b)
$$\|\underline{\nabla}_{hor}^{L} f_{FLRW}\|_{L^2_{1,\underline{G}_0}(T^*M_t)} \lesssim a(t)^{-c\sqrt{\varepsilon}}\Big(\underbrace{\sqrt{\varepsilon}\|\mathrm{Ric}[G]-2\kappa G\|_{H^{L-2}_G(M_t)}}_{if\ L>2} + \|\Gamma-\hat{\Gamma}\|_{H^{L-1}_G(M_t)} + \|G^{-1}-\gamma^{-1}\|_{H^{L-1}_G(M_t)}\Big)$$

Finally, let $0 < K < L \leq 12$. Then, one has

$$\|\underline{\nabla}_{vert}^{L-K}\underline{\nabla}_{hor}^{K} f_{FLRW}\|_{C^0_{1,\underline{G}_0}(T^*M_t)} \lesssim \sqrt{\varepsilon}a(t)^{-c\sqrt{\varepsilon}}\,, \tag{3.4.11c}$$

$$\|\underline{\nabla}_{vert}^{L-(K-1)}\underline{\nabla}_{hor}^{K-1}\boldsymbol{X} f_{FLRW}\|_{C^0_{1,\underline{G}_0}(T^*M_t)} \lesssim \varepsilon a(t)^{-1-c\sigma}\,. \tag{3.4.11d}$$

Recall that, by Remark 3.2.4, all of the objects above are well-defined.

PROOF. Before going into the proof, we note that vertical derivatives of f_{FLRW} are a linear combination of terms of the form

$$v^{*_\gamma m_1}\cdot\mathcal{F}^{(m_2)}(|v|^2_\gamma)$$

with $m_1, m_2 \in \mathbb{N}$. Since we have $\mathcal{F} \in C_0^\infty(\mathbb{R}_0^+)$, the distribution term is uniformly bounded. Further, on the support of f_{FLRW}, v is uniformly bounded with respect to $\underline{\gamma}|_{vert}$. Thus,

$$(v^0)^L|\underline{\nabla}_{vert}^L f_{FLRW}(t,x,v)|_{\underline{G}} = |\underline{\nabla}_{vert}^L f_{FLRW}(t,x,v)|_{\underline{G}_0} \lesssim a(t)^{-c\sqrt{\varepsilon}}$$

holds by (3.4.6a) and (3.4.3c) for any $L > 0$, as well as

$$\|\underline{\nabla}_{vert}^L f_{FLRW}\|_{L^2_{1,\underline{G}_0}} \lesssim a^{-c\sqrt{\varepsilon}}$$

(also using (3.4.7) for $m = 0$).
We compute:

$$\begin{aligned}\mathbf{A}_i f_{FLRW}(\cdot,\cdot,v) &= \mathcal{F}'(|v|^2_\gamma)\Big[\partial_i(\gamma^{-1})^{jk}v_jv_k + 2\Gamma^r_{ij}v_kv_r(\gamma^{-1})^{jk}\Big]\\ &= \mathcal{F}'(|v|^2_\gamma)\Big[-\hat{\Gamma}^j_{il}(\gamma^{-1})^{lk}v_jv_k - \hat{\Gamma}^k_{il}(\gamma^{-1})^{jl}v_jv_k + 2\Gamma^r_{ij}v_kv_r\Big]\\ &= 2\mathcal{F}'(|v|^2_\gamma)\,(\gamma^{-1})^{jk}\,\Big[\Gamma^l_{ij}-\hat{\Gamma}^l_{ij}\Big]\,v_kv_l\end{aligned} \tag{3.4.12}$$

$$= 2\mathcal{F}'(|v|^2_\gamma)\,\left[(G^{-1})^{jk}-(G^{-1}-\gamma^{-1})^{jk}\right]\,\left[\Gamma^l_{ij}-\hat{\Gamma}^l_{ij}\right]\,v_kv_l$$

Thus, we can estimate

$$|\underline{\nabla}_{hor}f_{FLRW}(\cdot,\cdot,v)|_{\underline{G}_0}\lesssim|\Gamma-\hat{\Gamma}|_G\left(1+|G^{-1}-\gamma^{-1}|_G\right)|v|^2_G\chi_{\mathrm{supp}\mathcal{F}}(|v|^2_\gamma)\,.$$

This proves (3.4.11c) at order 1 follows by integrating over $(T^*M,\underline{G}_0)$ and using that $\mathcal{F}$ has compact support. (3.4.11b) for $L=1$ follows from the same estimates by inserting (3.4.3c). For $L=2$, notice that

$$\underline{\nabla}_i\underline{\nabla}_jf_{FLRW}=\mathbf{A}_i\mathbf{A}_jf_{FLRW}-\Gamma^k_{ij}\mathbf{A}_kf_{FLRW}-\frac{1}{2}v_l\mathrm{Riem}[G]^l{}_{kij}\mathbf{B}^kf_{FLRW} \tag{3.4.13}$$

The first two terms in (3.4.13) can be dealt with as before, while for the final term, we compute

$$\begin{aligned}-\frac{1}{2}v_l\mathrm{Riem}[G]^l{}_{kij}\,\mathbf{B}^kf_{FLRW}(\cdot,\cdot,v)&=-v_l\mathrm{Riem}[G]^l{}_{kij}(\gamma^{-1})^{mk}v_m\cdot\mathcal{F}'(|v|^2_\gamma)\\&=\left[\mathrm{Riem}[G]^l{}_{kij}(G^{-1}-\gamma^{-1})^{mk}v_lv_m-\underbrace{v_kv_l\mathrm{Riem}[G]^{\sharp lk}{}_{ij}}_{=0}\right]\cdot\mathcal{F}'(|v|^2_\gamma)\,.\end{aligned}$$

Thus, this can be estimated in $L^2_{1,\underline{G}_0}$ by $a^{-c\sqrt{\varepsilon}}\|G^{-1}-\gamma^{-1}\|^2_{L^2_G}$ using that $\mathcal{F}$ is compactly supported as well as (3.4.3c) and (3.4.3e). This yields (3.4.11b) for $L=2$. For $L\geq3$, we repeat this procedure using the product rule, incurring curvature error terms. (3.4.11a) is proven similarly, and so is (3.4.11c) using the a priori estimates (3.4.3e), (3.4.3c) and (3.4.6a).

Regarding (3.4.11d), we split up $\boldsymbol{X}f_{FLRW}$ into the horizontal and vertical derivative terms and prove that both parts satisfy the claimed bound: The terms arising from the horizontal components of $\boldsymbol{X}f$ can be bounded directly using (3.4.11c) since $\mathbf{A}_i\xi=\underline{\nabla}_i\xi$. For the remaining vertical components, we may encounter terms where only vertical derivatives fall on f_{FLRW}. However, since we are only interested in a uniform pointwise bound, we don't need to ensure that all Christoffel terms that arise from the Sasaki connection coefficients reduce to difference tensor terms, and note that (3.4.3c) also implies the crude bound

$$|\partial^{\leq11}\Gamma|_G\lesssim a^{-c\sqrt{\varepsilon}}$$

in a coordinate neighbourhood on which $\hat{\Gamma}$ remains bounded. The bound then follows for the vertical parts of $\boldsymbol{X}f$ using (3.3.13a) and that M is compact. □

Lemma 3.4.8 ($C^0_{1,\cdot}$-estimates for the Vlasov distribution function). *For $t\in(t_{Boot},t_0]$, one has:*

$$\|f-f_{FLRW}\|_{C^0_{1,\underline{\gamma}_0}(T^*M_t)}\lesssim\varepsilon \tag{3.4.14a}$$

$$\|f-f_{FLRW}\|_{C^0_{1,\underline{G}_0}(T^*M_t)}\lesssim\varepsilon a(t)^{-c\sqrt{\varepsilon}} \tag{3.4.14b}$$

Proof. We start out by rewriting the rescaled Vlasov equation (3.2.19a) as an inhomogeneous transport equation for $f-f_{FLRW}$:

$$\partial_t\left(f-f_{FLRW}\right)=\boldsymbol{X}(f-f_{FLRW})-a^{-1}\frac{v^{\sharp j}}{v^0}(N+1)\mathbf{A}_jf_{FLRW}+a^{-1}v^0\,\nabla_jN\,\mathbf{B}^jf_{FLRW}$$

Let (X, V, Z) be the corresponding characteristics satisfying (3.4.5) emanating from $(x, v) \in T^*M_t$ with $Z(t) = (f - f_{FLRW})(t, x, v)$. Then, we have

$$\frac{dZ}{dt} = a^{-1}\frac{V_l}{V^0}(G^{-1})^{lj}(N+1)\mathbf{A}_j f_{FLRW} - a^{-1}V^0\,\nabla_j N\,\mathbf{B}^j f_{FLRW}$$

Note that, on the support of $(f - f_{FLRW})(t, x, \cdot)$, V^0 can be controlled by $a^{-c\sqrt{\varepsilon}}$ using (3.4.3c) and (3.4.6a). We obtain in total:

$$\left|\frac{dZ}{dt}\right| \lesssim \varepsilon a^{-1-c\sigma} \tag{3.4.15}$$

Using the initial data assumption (3.3.5) and again (3.4.6a) and (3.4.3c) to control v^0 on the support of $f - f_{FLRW}$, this implies

$$\begin{aligned}
&|\langle v\rangle_\gamma (f - f_{FLRW})(t, x, v)|\\
\leq\,& \langle v\rangle_\gamma |(f - f_{FLRW})(t, X(t), V(t)) - (f - f_{FLRW})(t_0, X(t_0), V(t_0))|\\
&+ \langle v\rangle_\gamma |(f - f_{FLRW})(t_0, X(t_0), V(t_0))|\\
\leq\,& \langle v\rangle_\gamma \left(\chi_{\mathrm{supp}(f - f_{FLRW})(t,x,\cdot)}(v)\right) |Z(t) - Z(t_0)| + \|f - f_{FLRW}\|_{C^0_{1,\underline{\gamma}_0}(T^*M_{t_0})}\\
\lesssim\,& \left[\int_t^{t_0} \varepsilon a(s)^{-1-c\sigma}\,ds + \varepsilon^2\right]
\end{aligned}$$

(3.4.14a) now follows by (3.2.6). (3.4.14b) can be obtained by the same argument or from (3.4.14a) using (3.4.3c). □

Lemma 3.4.9 (Strong $C^l_{1,\underline{G}_0}$-norm estimates for the Vlasov distribution function). *Let $t \in (t_{Boot}, t_0]$. Then, the following holds:*

$$\|f - f_{FLRW}\|_{C^{11}_{1,\underline{G}_0}(T^*M_t)} \lesssim \sqrt{\varepsilon}a(t)^{-c\sqrt{\varepsilon}} \tag{3.4.16}$$

Note that, for $\varepsilon > 0$ suitably small, (3.4.16) is a bootstrap improvement for (3.3.13b).

Proof. The following proof will require us to choose a spatial coordinate frame to express horizontal frame derivatives, so we must first establish suitable coordinate systems to cover the entire evolution: Since M is compact, there exists some sufficiently small $r > 0$ such that an open ball $B_r(x)$ of radius $r > 0$ around any $x \in M$ is a coordinate neighbourhood, and there exist $x_1, \dots, x_m \in M$ such that $\{B_{\frac{r}{2}}(x_i)\}_{i=1,\dots,m}$ covers M. Furthermore, for (3.4.5a), $|\frac{d}{dt}X|_G \lesssim a^{-1}$ holds, which is integrable on $[0, t_0]$ (see (3.2.6)), since $|N+1| \lesssim 1$ holds by the bootstrap assumption (3.3.14). (3.4.5a) will describe the spatial characteristics for all systems considered in this proof. In particular, there exists a finite partition of $(t_{Boot}, t_0]$ by intervals $(t_j, t_{j-1}]$ such that $\int_{t_{j+1}}^{t_j} a(s)^{-1}\,ds < \frac{r}{2}$ holds. Thus, for any $y \in B_{\frac{r}{2}}(x_i)$, the characteristic X emanating from y at time t_j is contained in the coordinate neighbourhood $B_r(x_i)$ on $(t_j, t_{j-1}]$. Thus, we can prove (3.4.16) for any $t \in (t_1, t_0]$ using local coordinates by having found a choice of coordinates that remains valid throughout the evolution, and since we can cover M in a finite amount of such coordinate neighbourhoods. Finally, the bounds then extend to t_1 by continuity, and thus we can prove the statement iteratively on the entire bootstrap interval by proving the bound on $(t_j, t_{j-1}]$ from data emanating from $M_{t_{j-1}}$.

Having now justified that we can essentially perform this proof in local coordinates, we move on to the proof itself: First, we commute (3.2.19a) with $\mathbf{B}^i$ to obtain

$$\begin{aligned}\partial_t\left(v_\gamma^0\mathbf{B}^i(f-f_{FLRW})\right)=&v_\gamma^0\mathbf{B}^i\partial_t f+\frac{\partial_t v_\gamma^0}{v_\gamma^0}v_\gamma^0\mathbf{B}^i(f-f_{FLRW})\\=&\boldsymbol{X}\left(v_\gamma^0\mathbf{B}^i(f-f_{FLRW})\right)+m^2\frac{\dot{a}a}{(v_\gamma^0)^2}v_\gamma^0\mathbf{B}^i(f-f_{FLRW})\\&+v_\gamma^0\mathbf{B}^i\boldsymbol{X}f_{FLRW}+(\mathcal{K}_{1,0})^i\end{aligned}$$

with

$$(\mathcal{K}_{1,0})^i(\cdot,\cdot,v)=v_\gamma^0\,[\mathbf{B}^i,\boldsymbol{X}](f-f_{FLRW})(\cdot,\cdot,v)+\frac{\boldsymbol{X}v_\gamma^0}{v_\gamma^0}\,\mathbf{B}^i(f-f_{FLRW})$$

(see (3.2.25g)). Moving to the corresponding characteristic system (X,V,Z^i), X and V again satisfy (3.4.5) emanating from $(x,v)\in T^*M_t$, while $Z^i(t)=v_\gamma^0\mathbf{B}^i(f-f_{FLRW})(t,X(t),V(t))$ satisfies

$$\frac{dZ^i}{dt}=m^2\frac{\dot{a}a}{(V_\gamma^0(t))^2}\,Z^i+(\mathcal{K}_{1,0})^i(t,X(t),V(t))+V_\gamma^0(t)\,\mathbf{B}^i\boldsymbol{X}f_{FLRW}(t,X(t),V(t))\,.$$

The horizontal term derivative term in (3.2.25g) can be bounded with respect to γ by $\varepsilon^{\frac{1}{4}}a^{-c\sigma}$ using (3.3.13b) and (3.3.13a), while the remaining terms can be bounded by $a^{-1-c\sigma}|Z|_\gamma$ using (3.3.13a). Further and using (3.4.3c) to switch metrics where needed, one computes

$$|\boldsymbol{X}V_\gamma^0|\lesssim a^{-1-c\sqrt{\varepsilon}}(|\Gamma-\hat{\Gamma}|_G+|\nabla N|_G)V_\gamma^0\,,$$

so, with (3.3.13a), the remaining terms can be bounded by $\varepsilon a^{-1-c\sigma}|Z|_\gamma$. In summary, we obtain

$$|\mathcal{K}_{1,0}(\cdot,X,V)|_\gamma\lesssim\varepsilon^{\frac{1}{4}}a^{-1-c\sigma}+a^{-1-c\sigma}|Z|_\gamma\,.$$

Due to (3.4.11d) and (3.4.3c) and since f_{FLRW} has compact momentum support with respect to γ, we also have

$$V_\gamma^0|\mathbf{B}\boldsymbol{X}f_{FLRW}|_\gamma\lesssim\varepsilon a^{-1-c\sigma}\,.$$

Thus, dropping the remaining term since it has favourable sign, one altogether has

$$-\frac{1}{2|Z|_\gamma}\frac{d}{dt}\left[|Z|_\gamma^2\right]\lesssim a^{-1-c\sigma}|Z|_\gamma+\varepsilon a^{-1-c\sigma}+\varepsilon^{\frac{1}{4}}a^{-1-c\sigma}\,.$$

After integrating and applying the Gronwall lemma as well as the scale factor integral bounds from Lemma 3.2.2 and the initial data assumption, yields

$$|\underline{\nabla}_{vert}(f-f_{FLRW})(t,X(t),V(t))|_{\underline{\gamma}_0}\lesssim\varepsilon^{\frac{1}{4}}\,. \tag{3.4.17}$$

Repeating the same procedure for $\mathbf{A}_i$, we have

$$\partial_t\mathbf{A}_i(f-f_{FLRW})=\boldsymbol{X}\mathbf{A}_i(f-f_{FLRW})+\mathbf{A}_i\boldsymbol{X}f_{FLRW}+(\mathcal{K}_{1,1})_i$$

with

$$(\mathcal{K}_{1,1})_i=[\mathbf{A}_i,\boldsymbol{X}](f-f_{FLRW})+[\partial_t,\mathbf{A}_i](f-f_{FLRW})\,,$$

where (3.2.25f) and (3.2.25d) imply, using (3.3.13a) and (3.4.1b):

$$|[\boldsymbol{X},\mathbf{A}_{(\cdot)}](f-f_{FLRW})|_\gamma \lesssim a^{-1-c\sigma}|\mathbf{A}_{(\cdot)}(f-f_{FLRW})|_\gamma+\varepsilon a^{1-c\sigma}|\underline{\nabla}_{vert}(f-f_{FLRW})|_{\underline{\gamma}_0}$$
$$|[\partial_t,\mathbf{A}_{(\cdot)}](f-f_{FLRW})|_\gamma \lesssim \varepsilon a^{-3-c\sqrt{\varepsilon}}|\underline{\nabla}_{vert}(f-f_{FLRW})|_{\underline{\gamma}_0}$$

Using (3.4.17), it follows that

$$|\mathcal{K}_{1,1}|_\gamma \lesssim a^{-1-c\sigma}|\mathbf{A}_{(\cdot)}(f-f_{FLRW})|_\gamma+\varepsilon^{\frac{5}{4}}a^{-3-c\sqrt{\varepsilon}}$$

With (3.4.11d), one also computes

$$|\mathbf{A}_{(\cdot)}\boldsymbol{X}f_{FLRW}|_\gamma \lesssim \varepsilon a^{-1-c\sigma}+\varepsilon a^{-3-c\sqrt{\varepsilon}},$$

and now obtains that

$$|\mathbf{A}_{(\cdot)}(f-f_{FLRW})(t,X(t),V(t))|_\gamma \lesssim \sqrt{\varepsilon}a^{-c\sqrt{\varepsilon}} \tag{3.4.18}$$

holds. Finally, we can redo the proof of (3.4.17), replacing (3.3.13b) with the improved bound (3.4.18), which improves the bound in (3.4.17) to $\sqrt{\varepsilon}a^{-c\sqrt{\varepsilon}}$ after updating constants. Altogether, one now concludes as in the proof of Lemma 3.4.8 that

$$\|\underline{\nabla}(f-f_{FLRW})\|_{C^0_{1,\underline{\gamma}_0}(T^*U_t)} \lesssim \sqrt{\varepsilon}a(t)^{-c\sqrt{\varepsilon}}$$

holds for suitably chosen coordinate neighbourhoods as outlined at the start of the proof, and thus the same bounds on T^*M_t. With (3.4.3c), this also implies

$$\|f-f_{FLRW}\|_{\dot{C}^1_{1,\underline{G}_0}(T^*M_t)} \lesssim \sqrt{\varepsilon}a(t)^{-c\sqrt{\varepsilon}}.$$

We now sketch how to extend this procedure to higher orders inductively, since the line of argumentation is almost identical: Assume the improved bound has been proven up to $C^{L-1}_{1,\underline{G}_0}$ for $L\le 11$. Commuting (3.2.19a) with $(v^0_\gamma)^L\mathbf{B}^{i_1}\dots\mathbf{B}^{i_L}$ implies the following transport system:

$$(\partial_t-\boldsymbol{X})\mathbf{B}^{i_1}\dots\mathbf{B}^{i^L}(f-f_{FLRW})=(\mathcal{K}_{L,0})^{i_1\dots i_L}+(v^0_\gamma)^L\mathbf{B}^{i_1}\dots\mathbf{B}^{i_L}\boldsymbol{X}f_{FLRW}$$

with

$$\begin{aligned}(\mathcal{K}_{L,0})^{i_1\dots i_L}&=[(v^0_\gamma)^L\mathbf{B}^{i_1}\dots\mathbf{B}^{i^L},\boldsymbol{X}](f-f_{FLRW})\\&=(v^0_\gamma)^L\left[\mathbf{B}^{i_1}\dots\mathbf{B}^{i_{L-1}}[\mathbf{B}^{i_L},\boldsymbol{X}](f-f_{FLRW})+\dots+[\mathbf{B}^{i_1},\boldsymbol{X}]\mathbf{B}^{i_2}\dots\mathbf{B}^{i^L}(f-f_{FLRW})\right]\\&\quad+(\boldsymbol{X}(v^0_\gamma)^L)\mathbf{B}^{i_1}\dots\mathbf{B}^{i_L}(f-f_{FLRW})\end{aligned}$$

Once again, the characteristics (X,V) are as in (3.4.5), while one has

$$\frac{dZ^{i_1\dots i_L}}{dt}=(\mathcal{K}_{L,0})^{i_1\dots i_L}+\mathbf{B}^{i_1}\dots\mathbf{B}^{i_L}\boldsymbol{X}f_{FLRW}$$

with

$$(V^0_\gamma)^L|\mathcal{K}_{L,0}|_\gamma \lesssim a^{-1-c\sigma}(V^0_\gamma)^L|Z|_\gamma+\varepsilon^{\frac{1}{4}}a^{-1-c\sigma}+\varepsilon a^{-1-c\sigma}\|f-f_{FLRW}\|_{C^{L-1}_{1,\underline{G}_0}}$$

and

$$(V^0_\gamma)^L\left|\mathbf{B}^L\boldsymbol{X}f_{FLRW}\right|_\gamma \lesssim \varepsilon a^{-1-c\sigma}.$$

This yields

$$\left(V^0_\gamma(t)\right)^L \left|\mathbf{B}^L(f-f_{FLRW})(t,X(t),V(t))\right|_\gamma \lesssim \varepsilon^{\frac14} a(t)^{-c\sqrt{\varepsilon}}$$

as with $L=0$.

From here, we can repeat this procedure for $(v^0_\gamma)^{L-R}\mathbf{B}^{L-R}\mathbf{A}^R$ by iterating over increasing R for $R=1,\dots,L$, where we will have proven

$$\left(V^0_\gamma(t)\right)^{L-\tilde R} \left|\mathbf{B}^{L-\tilde R}\mathbf{A}^{\tilde R}(f-f_{FLRW})(t,X(t),V(t))\right|_\gamma \lesssim \varepsilon^{\frac14} a(t)^{-c\sqrt{\varepsilon}}$$

for $0\le \tilde R<R$ in the previous steps. Additionally, the induction hypothesis allows us to use (3.4.16) up to order $L-1$. We use all of these bounds to control Vlasov terms of lower order and of order L with less horizontal derivaitves in $\mathcal{K}_{L,R}$, along with(3.4.1b),(3.4.3b) and (3.3.13a) to control shear, metric and lapse as well as (3.4.6a) and (3.4.3c) to control superfluous momentum terms. Additionally, for $R<L$, we use the bootstrap assumption (3.3.13b) to bound the term

(3.4.19)
$$-a^{-1}(N+1)\left(\frac{1}{v^0}\mathbb{I}^j_{i_K}-\frac{v^\flat_{i_K}v^j}{(v^0)^3}\right)\mathbf{B}^{i_1}\dots\mathbf{B}^{i_{K-1}}\mathbf{B}^{i_{K+1}}\dots\mathbf{B}^{i_{L-R}}\mathbf{A}_{i_{L-R+1}}\dots\mathbf{A}_{i_L}\mathbf{A}_j(f-f_{FLRW})\,,$$

which occurs when applying the product rule to the horizontal derivative term in $\boldsymbol{X}$, by $\varepsilon^{\frac14}a^{-1-c\sigma}$. In total and now considering $Z^{i_1\dots i_{L-R}}{}_{i_{L-R+1}\dots i_L}$ in analogy to above, this then implies

$$|\mathcal{K}_{L-R,R}|_\gamma \lesssim \begin{cases} a^{-1-c\sigma}|Z|_\gamma+\varepsilon^{\frac14}a^{-1-c\sigma}+\varepsilon^{\frac54}a^{-3-c\sqrt{\varepsilon}} & R<L\\ a^{-1-c\sigma}|Z|_\gamma+\varepsilon^{\frac54}a^{-3-c\sqrt{\varepsilon}} & R=L\end{cases}. \tag{3.4.20}$$

We again use Lemma 3.4.7 to control the reference terms. In total, at order L, we use strong low order bounds on

$$\|\mathrm{Ric}[G]-2\kappa G\|_{C^{L-1}_G(M_t)},\ \|\Gamma-\hat\Gamma\|_{C^{L-1}_G(M_t)},\ \text{and } \|\Sigma\|_{C^L_G(M_t)}$$

(along with using the bootstrap assumption (3.3.14) for $\|N\|_{C^{L+1}_G(M_t)}$). Thus, this procedure can only be performed until $L=11$. Altogether, one obtains

$$\sum_{R=0}^L (V^0_\gamma(t))^{L-R}\left|\mathbf{B}^{L-R}\mathbf{A}^R(f-f_{FLRW})(t,X(t),V(t))\right|_\gamma \lesssim \varepsilon^{\frac14}a(t)^{-c\sqrt{\varepsilon}}\,,$$

and, since (3.4.19) does not occur for $R=L$, also

$$\left|\mathbf{A}^L(f-f_{FLRW})(t,X(t),V(t))\right|_\gamma \lesssim \sqrt{\varepsilon}a(t)^{-c\sqrt{\varepsilon}}$$

just like for $L=R=1$. Now, we iterate over decreasing R: Assume that one has proven, for any $\tilde R>R\ge 0$,

$$\left(V^0_\gamma(t)\right)^{L-\tilde R}\left|\mathbf{B}^{L-\tilde R}\mathbf{A}^{\tilde R}(f-f_{FLRW})(t,X(t),V(t))\right|_\gamma \lesssim \sqrt{\varepsilon}a(t)^{-c\sqrt{\varepsilon}}\,.$$

Then, one can bound (3.4.19) by $\sqrt{\varepsilon}a^{-1-c\sqrt{\varepsilon}}$ instead of applying (3.3.13b), improving the error term bound (3.4.20) as before and thus yielding the same bound as above for $\tilde R=R$.

In total, we obtain

$$\sum_{r=0}^{L}(V_\gamma^0(t))^{L-R}|\mathbf{B}^{L-R}\mathbf{A}^R(f-f_{FLRW})(t,X(t),V(t))|_\gamma \lesssim \sqrt{\varepsilon}a(t)^{-c\sqrt{\varepsilon}}\,.$$

Finally, note that (3.4.3e) and (3.4.3c) imply

$$\begin{aligned}|\underline{\nabla}_{vert}^{L-R}\underline{\nabla}_{hor}^{R}(f-f_{FLRW})|_{\underline{G}_0} &\lesssim |\mathbf{B}^{L-R}\mathbf{A}^R(f-f_{FLRW})|_{\underline{G}_0}+a^{-c\sqrt{\varepsilon}}|\underline{\nabla}^{L-1}(f-f_{FLRW})|_{\underline{G}_0}\\ &\lesssim \sqrt{\varepsilon}a^{-c\sqrt{\varepsilon}},\end{aligned}$$

and subsequently (3.4.16) at order L. This closes the induction over $L\leq 11$, proving (3.4.16).
□

Additionally, the bounds on the Vlasov distribution function admit bounds on its energy-momentum tensor:

Corollary 3.4.10 (Strong low order bounds on Vlasov matter quantities). *The following bounds hold:*

(3.4.21)

$$\|\rho^{Vl}-\rho^{Vl}_{FLRW}\|_{C^{11}_G(M_t)}+\|\mathfrak{p}^{Vl}-\mathfrak{p}^{Vl}_{FLRW}\|_{C^{11}_G(M_t)}+\|\jmath^{Vl}\|_{C^{11}_G(M_t)}+\|S^{Vl,\parallel}\|_{C^{11}_G(M_t)}\lesssim\sqrt{\varepsilon}a(t)^{-c\sqrt{\varepsilon}}$$

Proof. Starting with $\rho^{Vl}-\rho^{Vl}_{FLRW}$, we compute the following:

$$\left(\rho^{Vl}-\rho^{Vl}_{FLRW}\right)(t,x)=\int_{\mathbb{R}^3}\left[f(t,x,v)-\frac{v^0_\gamma}{v^0}\frac{\mu_G}{\mu_\gamma}f_{FLRW}(t,x,v)\right]v^0\mu_G^{-1}dv$$

Note that one has

$$\left|v^0-v^0_\gamma\right|=\frac{(v^0)^2-(v^0_\gamma)^2}{v^0+v^0_\gamma}\leq\frac{\left|(G^{-1}-\gamma^{-1})^{ij}v_iv_j\right|}{v^0+v^0_\gamma}\lesssim|G^{-1}-\gamma^{-1}|_G\cdot v^0.$$

Furthermore, by expanding the determinant and using (3.4.4) and $|\mu_\gamma^{-1}|\lesssim 1$, one obtains with (3.4.3c) that

$$|\mu_G^{-1}-\mu_\gamma^{-1}|\lesssim|\det G^{-1}-\det\gamma^{-1}|\lesssim a^{-c\sqrt{\varepsilon}}|G^{-1}-\gamma^{-1}|_G\,.$$

Since f_{FLRW} is also uniformly bounded, one now obtains:

(3.4.22)

$$\left|\rho^{Vl}-\rho^{Vl}_{FLRW}\right|\lesssim\int_{\mathrm{supp}(f-f_{FLRW})}\left[|(f-f_{FLRW})(\cdot,\cdot,v)|+a^{-c\sqrt{\varepsilon}}|G^{-1}-\gamma^{-1}|_G\right]v^0\,\mu_G^{-1}dv$$

By (3.4.6a), the integral over any polynomial of v^0 is bounded by $a^{-c\sqrt{\varepsilon}}$, and thus the bound follows at order 0 by applying the uniform estimates (3.4.14b) and (3.4.3c).
Regarding $S^{Vl,\parallel}$, we compute

$$\begin{aligned}(S^{Vl,\parallel})^i{}_j=&\int_{T^*M}\frac{(G^{-1})^{il}v_lv_j-\frac{1}{3}|v|^2_G\mathbb{I}^i_j}{v^0}(f-f_{FLRW})(\cdot,\cdot,v)\mu_G^{-1}\,dv\\ &+\int_{T^*M}\left[\left(\frac{(G^{-1})^{il}}{v^0}-\frac{(\gamma^{-1})^{il}}{v^0_\gamma}\right)v_lv_j-\frac{1}{3}\left(\frac{|v|^2_G}{v^0}-\frac{|v|^2_\gamma}{v^0_\gamma}\right)\mathbb{I}^i_j\right]\mathcal{F}(|v|^2_\gamma)\mu_G^{-1}\,dv\end{aligned}$$

$$+\frac{\mu_\gamma}{\mu_G}\left((S^{Vl}_{FLRW})^i{}_j-\mathfrak{p}^{Vl}_{FLRW}\mathbb{I}^i_j\right).$$

Note that the final line vanishes since f_{FLRW} is isotropic. Regarding the second line, note one has, with (3.4.3c),

$$\left|\frac{1}{v^0}-\frac{1}{v^0_\gamma}\right|=\frac{\left|(v^0)^2-(v^0_\gamma)^2\right|}{v^0\,v^0_\gamma\,\left(v^0+v^0_\gamma\right)}\leq\frac{|G^{-1}-\gamma^{-1}|_\gamma}{v^0}\lesssim a^{-c\sqrt{\varepsilon}}\,\frac{|G^{-1}-\gamma^{-1}|_G}{v^0}.$$

Consequently, the pointwise $|\cdot|_G$-norm of the second line is bounded by

$$\lesssim a^{-c\sqrt{\varepsilon}}\,|G^{-1}-\gamma^{-1}|_G\int_{T^*M}v^0_\gamma\,\mathcal{F}(|v|^2_\gamma)\,\mu_\gamma^{-1}\,dv\lesssim\sqrt{\varepsilon}a^{-c\sqrt{\varepsilon}}$$

using (3.4.3c). From here, the argument follows as for the density. Bounds for the pressure and flux are proven similarly at order zero.
At order $L\in\{1,\dots,11\}$, note that one has $\nabla^L\rho^{Vl}_{FLRW}=\nabla^L\mathfrak{p}^{Vl}_{FLRW}=0$ as well as

$$\|\underline{\nabla}^L_{hor}f\|_{C^0_{1,\underline{G}_0}(T^*M)}\lesssim\sqrt{\varepsilon}a^{-c\sqrt{\varepsilon}}$$

using (3.4.16) and (3.4.11c). The higher order bounds now follow in a similar, but simpler fashion to the arguments at order zero using (3.4.6a). □

Finally, we collect the following lemma that will be needed to bound the evolution of weights in our energy approach:

LEMMA 3.4.11 (Evolution bound on momentum variables).

$$-\frac{\partial_t v^0}{v^0}\lesssim\varepsilon a^{-3},\quad -\frac{\partial_t\langle v\rangle_G}{\langle v\rangle_G}\lesssim\varepsilon a^{-3}\tag{3.4.23}$$

PROOF. One computes the following:

$$\partial_t v^0=\frac{1}{2v^0}\left[2m^2a\dot a+(\partial_tG^{-1})^{ij}v_iv_j\right]=(N+1)a^{-3}\Sigma^{\sharp ij}\frac{v_iv_j}{v^0}-N\frac{\dot a}{a}\frac{|v|^2_G}{v^0}+\frac{\dot a}{a}\cdot\frac{m^2a^2}{v^0}$$

The statement then follows from (3.4.1b) and (3.3.13a), where the final summand has favourable sign and is thus dropped. The estimate for $\langle v\rangle_G$ is obtained identically. □

3.4.3. Semi-norm equivalences and commutators for the Vlasov distribution. In this section, we collect estimates on covariant commutator errors for functions on the mass shell, that in particular will imply that the Vlasov energies (3.3.3f)-(3.3.3g) are coercive up to error terms dependent on lower order Vlasov energies as well as metric norms and energies. This will be sufficient to deduce control of the Vlasov matter quantities in $\mathcal{H}$ and $\mathcal{C}$ from improved energy bounds in Section 3.7.

LEMMA 3.4.12 (Commutator formulas for the Sasaki metric). *Let ξ be a sufficiently regular function on $\cup_{t\in(t_{Boot},t_0],x\in M}P_{(t,x)}$ and let $M_1,M_2\in\mathbb{N}$. Then, understanding that terms do not occur if they would involve negative orders of derivatives, the following schematic expressions hold:*

$$\begin{aligned}[\underline{\nabla}^{M_1}_{vert}\underline{\nabla}^{M_2}_{hor},\underline{\nabla}_{hor}]\xi=&\,v*\mathrm{Ric}[G]*\underline{\nabla}^{M_1-1}_{vert}\underline{\nabla}^{M_2}_{hor}\xi+v*\mathrm{Ric}[G]*\underline{\nabla}^{M_1+1}_{vert}\underline{\nabla}^{M_2-1}_{hor}\xi\\&+\underbrace{v*\nabla^{M_2}(\mathrm{Ric}[G]-2\kappa G)*\underline{\nabla}^{M_1}_{vert}\xi}_{if\ M_2>0}+\Xi_{M_1,M_2,hor}\xi\end{aligned}\tag{3.4.24a}$$

$$[\partial_t, \underline{\nabla}^{M_1}_{vert}\underline{\nabla}^{M_2}_{hor}]\xi = v * \nabla^{M_2-2}\partial_t \mathrm{Ric}[G] * \underline{\nabla}^{M_1+1}_{vert}\xi + v * \partial_t \Gamma[G] * \underline{\nabla}^{M_1+1}_{vert}\underline{\nabla}^{M_2-1}_{hor}\xi + \Xi_{M_1,M_2,t}\xi \tag{3.4.24b}$$

If, on the momentum support of $\xi(t,\cdot,\cdot)$, v^0 *is uniformly bounded by* $a(t)^{-c\sqrt{\varepsilon}}$ *up to constant, the lower order error terms can be bounded as follows:*

$$\|\langle v\rangle_G (v^0)^{M_1}\Xi_{M_1,M_2,hor}\xi\|_{L^2_{1,\underline{G}}(T^*M_t)} \lesssim a(t)^{-c\sqrt{\varepsilon}}\|\xi\|_{H^{M_1+M_2-1}_{1,\underline{G}_0}(T^*M_t)} + a(t)^{-c\sqrt{\varepsilon}}\|\mathrm{Ric}[G]-2\kappa G\|_{H^{M_2-1}_G(M_t)}\|\xi\|_{C^{M_1+1}_{1,\underline{G}_0}(T^*M_t)} \tag{3.4.25a}$$

$$\|\langle v\rangle_G (v^0)^{M_1}\Xi_{M_1,M_2,t}\xi\|_{L^2_{1,\underline{G}}(T^*M_t)} \lesssim \varepsilon a(t)^{-3-c\sqrt{\varepsilon}}\|\xi\|_{H^{M_1+M_2-1}_{1,\underline{G}_0}(T^*M_t)} + a(t)^{-c\sqrt{\varepsilon}}\left(\|\partial_t\mathrm{Ric}[G]\|_{H^{M_2-2}_G(M_t)} + \|\partial_t\Gamma[G]\|_{H^{M_2-2}_G(M_t)}\right)\|\xi\|_{C^{M_1+1}_{1,\underline{G}_0}(T^*M_t)} \tag{3.4.25b}$$

By Lemma 3.4.6, (3.4.25) applies for $\xi = f$.

Proof. These formulas follow from straightforward computations using the commutator formulas in Lemma 3.2.8 and the connection coefficient expressions from (3.2.17). For the error bounds, we recall the bounds (3.4.6a) on the momentum support, and estimate lower order curvature and metric errors with (3.4.3e) and (3.4.3c) to obtain (3.4.25a). For (3.4.24b), recall (3.2.20e) and (3.2.20d) and use (3.4.3b) as well as the bootstrap assumption to bound lower order terms in the rescaled shear and lapse respectively. $\square$

Lemma 3.4.13 (Rearrangement of derivatives in intermediate Vlasov energies). *Let* $0 < K < L$ *and let* $\mathcal{D}_{L,K}$ *denote a combination of* $L-K$ *vertical and* K *horizontal covariant derivatives with respect to the Sasaki metric* $\underline{G}$. *Then, the following holds:*

$$\left|\|\mathcal{D}_{L,K}(f-f_{FLRW})\|^2_{L^2_{1,\underline{G}_0}(T^*M_t)} - \|\underline{\nabla}^{L-K}_{vert}\underline{\nabla}^K_{hor}(f-f_{FLRW})\|^2_{L^2_{1,\underline{G}_0}(T^*M_t)}\right| \lesssim a(t)^{-c\sqrt{\varepsilon}}\left(\mathfrak{E}^{(\leq L-1)}_{1,\leq K}(f,t) + \varepsilon\mathfrak{E}^{(\leq K-1)}(\mathrm{Ric},t) + \|\Gamma-\hat{\Gamma}\|^2_{H^{K-1}_G(M_t)} + \|G^{\pm1}-\gamma^{\pm1}\|^2_{H^{K-1}_G(M_t)}\right) \tag{3.4.26a}$$

Similarly, one has for $1 \leq K < L$,

$$\left|\|\mathcal{D}_{L,K}f\|^2_{L^2_{1,\underline{G}_0}(T^*M_t)} - \mathfrak{E}^{(L)}_{L,K}(f,t)\right| \lesssim a(t)^{-c\sqrt{\varepsilon}}\mathfrak{E}^{(\leq L-1)}_{1,\leq K}(f,t) + \varepsilon a(t)^{-c\sqrt{\varepsilon}}\mathfrak{E}^{(\leq K-1)}(\mathrm{Ric},t) + a(t)^{-c\sqrt{\varepsilon}}\left(\|\Gamma-\hat{\Gamma}\|^2_{L^2_G(M_t)} + \|G^{\pm1}-\gamma^{\pm1}\|^2_{H^1_G(M_t)}\right). \tag{3.4.26b}$$

If one additionally takes $K \leq 10$ *in any of the above bounds, all terms but the Vlasov energy quantities can be dropped.*

Proof. Reviewing the commutator formulas from Lemma 3.2.8, one schematically has for $K > 0$:

$$\mathcal{D}_{L,K}\xi - \underline{\nabla}^{L-K}_{vert}\underline{\nabla}^K_{hor}\xi = \sum_{I=0}^{K-1}\left[v * \nabla^I \mathrm{Riem}[G] * \underline{\nabla}^{L-I-1}\xi + \nabla^I\mathrm{Riem}[G] * \underline{\nabla}^{L-I}\xi\right] + \langle \text{lower order nonlinear terms containing } v, \mathrm{Riem}[G], \xi\rangle$$

where we recall $\mathrm{Riem}[G] = \mathrm{Ric}[G] * G$. Thus, the product rule and the a priori estimates (3.4.3e), (3.4.6a) and (3.4.16) yield the following:

$$\begin{aligned}&\left| \|\mathcal{D}_{L,K}(f - f_{FLRW})\|^2_{L^2_{1,\underline{G}_0}(T^*M)} - \|\underline{\nabla}^{L-K}_{vert}\underline{\nabla}^K_{hor}(f - f_{FLRW})\|^2_{L^2_{1,\underline{G}_0}(T^*M)} \right| \\ &\lesssim a^{-c\sqrt{\varepsilon}} \|f - f_{FLRW}\|^2_{H^{L-1}_{1,\underline{G}_0}(T^*M)} + \varepsilon a^{-c\sqrt{\varepsilon}} \mathfrak{E}^{(\leq K-1)}(\mathrm{Ric}, \cdot)\end{aligned}$$

Furthermore, after applying the triangle inequality to $\|\underline{\nabla}^{L-K}_{vert}\underline{\nabla}^K_{hor}(f - f_{FLRW})\|_{L^2_{1,\underline{G}_0}(T^*M)}$, we can estimate the reference terms with (3.4.11a). (3.4.26a) then follows by iterating over iteratively replacing mixed terms on the right hand side until only Vlasov energy terms remain.

For (3.4.26b), we largely proceed as above replacing $f - f_{FLRW}$ with f. However, whenever we encounter a term with only one horizontal derivative by commuting, we replace f with $(f - f_{FLRW}) + f_{FLRW}$. For the former, we simply use (3.4.26a). Regarding the latter, the analogous bounds hold as for $\mathcal{D}_{L,1} f_{FLRW}$ as for $\underline{\nabla}^{L-1}_{vert}\underline{\nabla}_{hor} f_{FLRW}$ in (3.4.11a), except for incurring metric terms up to second order since taking vertical derivatives causes metric terms to occur at order zero that need to be differentiated: For example, one has

$$\begin{aligned}\underline{\nabla}_{hor}\underline{\nabla}_{vert} f_{FLRW} &= \underline{\nabla}_{hor}(\mathcal{F}'(|v|^2_\gamma) * v * \gamma) \\ &= v * \left(\underline{\nabla}_{hor}\mathcal{F}'(|v|^2_\gamma) * \gamma^{-1} + \mathcal{F}'(|v|^2_\gamma) * \nabla(G^{-1} - \gamma^{-1})\right) .\end{aligned}$$

Note that, since we only need to apply (3.4.11a) when only one horizontal derivative occurs, we do not incur higher order metric errors in (3.4.26b). □

Since Lemma 3.4.13 allows us to exchange arrange covariant derivatives in arbitrary order up to curvature errors, this will imply that Vlasov energies are coercive up to lower order metric errors, along with bounds on the reference distribution function to control the purely vertical case. We will use this in Corollary 3.7.11 to improve bounds on Sobolev norms containing $f - f_{FLRW}$. To then improve the bootstrap assumptions, we need to show that these sufficiently control the energy momentum tensor, which is ensured by the following lemma:

LEMMA 3.4.14 (Energy bounds for Vlasov energy-momentum tensor components). *For $L \in \mathbb{Z}_+$ and $t \in (t_{Boot}, t_0]$, the following bounds hold:*

$$\|\rho^{Vl} - \rho^{Vl}_{FLRW}\|_{L^2_G(M_t)} + \|\mathfrak{p}^{Vl} - \mathfrak{p}^{Vl}_{FLRW}\|_{L^2_G(M_t)} \lesssim a(t)^{-c\sqrt{\varepsilon}} \left(\sqrt{\mathfrak{E}^{(0)}_{1,0}(f,t)} + \|G^{-1} - \gamma^{-1}\|_{L^2_G(M_t)}\right) \tag{3.4.27a}$$

$$\|\rho^{Vl} - \rho^{Vl}_{FLRW}\|_{\dot{H}^L_G(M_t)} + \|\mathfrak{p}^{Vl} - \mathfrak{p}^{Vl}_{FLRW}\|_{\dot{H}^L_G(M_t)} \lesssim a(t)^{-c\sqrt{\varepsilon}} \sqrt{\mathfrak{E}^{(L)}_{1,L}(f,t)} \tag{3.4.27b}$$

$$\|\jmath^{Vl}\|_{\dot{H}^L_G(M_t)} \lesssim a(t)^{-c\sqrt{\varepsilon}} \sqrt{\mathfrak{E}^{(L)}_{1,L}(f,t)} \tag{3.4.27c}$$

$$\|(S^{Vl})^{\|}\|_{\dot{H}^L_G(M_t)} \lesssim a(t)^{-c\sqrt{\varepsilon}} \left(\sqrt{\mathfrak{E}^{(L)}_{1,L}(f,t)} + \underbrace{\|G^{-1}-\gamma^{-1}\|_{L^2_G(M_t)}}_{\text{if } L=0}\right) \tag{3.4.27d}$$

PROOF. Note that we can apply Jensen's inequality on continuous functions ξ on T^*M that have compact support after renormalising, i.e., for $V(\mathrm{supp}\xi) = \int_{\mathrm{supp}\xi} \mu_G^{-1} dv$,

$$\begin{aligned}\int_M \left(\int_{T^*M} v^0 \xi \mu_G^{-1} dv\right)^2 \mathrm{dvol}_G &= \int_M V(\mathrm{supp}\xi)^2 \left(V(\mathrm{supp}\xi)^{-1} \int_{\mathrm{supp}\xi} v^0 \xi\, \mu_G^{-1}\, dv\right)^2 \mathrm{dvol}_G \\ &\leq \int_{T^*M} V(\mathrm{supp}\xi)(v^0)^2 \xi^2\, \mathrm{dvol}_{\underline{G}}\,,\end{aligned}$$

and that

$$V(\mathrm{supp}\xi) \lesssim \mathcal{P}^3 \lesssim a^{-c\sqrt{\varepsilon}}$$

using (3.4.6a). Thus, recalling (3.4.22), we compute for the energy density:

$$\begin{aligned}&\|\rho^{Vl} - \rho^{Vl}_{FLRW}\|^2_{L^2_G(M_t)} \\ &\lesssim \int_{M_t} \left(\int_{\mathrm{supp}(f-f_{FLRW})(t,x,\cdot)} \left[|f-f_{FLRW}| + |G^{-1}-\gamma^{-1}|_G\right] v^0\, \mu_G^{-1} dv\right)^2 \mathrm{dvol}_G \\ &\lesssim a(t)^{-c\sqrt{\varepsilon}} \int_{T^*M_t} (v^0)^2 \left[|f-f_{FLRW}|^2 + |G^{-1}-\gamma^{-1}|^2_G\right] \mathrm{dvol}_{\underline{G}} \\ &\lesssim a(t)^{-c\sqrt{\varepsilon}} \left(\mathfrak{E}^{(0)}_{1,0}(f,t) + a(t)^{-c\sqrt{\varepsilon}} \|G^{-1}-\gamma^{-1}\|^2_{L^2_G(M_t)}\right)\end{aligned}$$

The computations for the pressure are fully analogous, proving (3.4.27a) after updating constants. By (3.2.14), we can express $\jmath^{Vl}$ as a weighted integral over $f - f_{FLRW}$, thus avoiding the metric error term. Thus, (3.4.27c) for $L = 0$ follows directly by applying the Jensen inequality as above. Regarding $S^{Vl,\|}$, one can expand the term as in the proof of Corollary 3.4.10 to prove (3.4.27d) for $L = 0$. Higher orders are proven fully analogously, observing as in the proof of Lemma 3.4.10 that $\nabla \rho^{Vl}_{FLRW} = \nabla \mathfrak{p}^{Vl}_{FLRW} = 0$, as well as the Vlasov energy definition (3.3.3g) – this allows us directly apply the Jensen inequality to all matter variables without having to expand the integrand. $\square$

3.5. Estimates for spacetime and scalar field variables

All of the estimates in this and the following section hold for $t \in (t_{Boot}, t_0]$. Furthermore, we let $\omega \gg \sigma > 0$ be a constant that is suitably small: For example, if σ is chosen small enough that $c\varepsilon^{\frac{1}{8}} < c\sigma < \frac{1}{1000}$ holds throughout the proof, $\omega = \frac{1}{100}$ is an admissible choice, but this choice is far from optimal. In general, ω only needs to be small compared to the order of derivatives used in our estimates, but large compared to σ.

3.5.1. Lapse energy estimates. In this section, we collect estimates for the lapse that follow from (3.2.20f)-(3.2.20g). In particular, in contrast to the estimates in Section 2.5, we will also prove estimates for odd order lapse energies, since these will be necessary to control high order lapse terms.

Definition 3.5.1 (Elliptic operators).

$$\mathcal{L}\zeta = a^4\Delta\zeta - h\cdot\zeta, \qquad h = -3\kappa a^4 + 12\pi C^2 + 12\pi a^2\left(\rho^{Vl}_{FLRW} + \mathfrak{p}^{Vl}_{FLRW}\right) + H \tag{3.5.1a}$$

$$H = \langle\Sigma,\Sigma\rangle_G + 8\pi\Psi^2 + 16\pi C\Psi + 4\pi a^2\left(\rho^{Vl} - \rho^{Vl}_{FLRW} + 3\mathfrak{p}^{Vl} - 3\mathfrak{p}^{Vl}_{FLRW}\right)$$

$$\tilde{\mathcal{L}}\zeta = a^4\Delta\zeta - \tilde{h}\cdot\zeta, \qquad \tilde{h} = -3\kappa a^4 + 12\pi C^2 + 12\pi a^2(\mathfrak{p}^{Vl}_{FLRW} + \rho^{Vl}_{FLRW}) + \tilde{H} \tag{3.5.1b}$$

$$\tilde{H} = a^4\left[R[G] + \frac{2}{3} - 8\pi|\nabla\phi|^2_G\right] + 12\pi a^2\left[(\mathfrak{p}^{Vl} - \mathfrak{p}^{Vl}_{FLRW}) - (\rho^{Vl} - \rho^{Vl}_{FLRW})\right]$$

With these operators, the rescaled lapse equations (3.2.20f)-(3.2.20g) then read

$$\mathcal{L}N = H \text{ and } \tilde{\mathcal{L}}N = \tilde{H}\,, \tag{3.5.2}$$

and admit the following estimates:

Lemma 3.5.2 (Elliptic estimates for $\mathcal{L}$ and $\tilde{\mathcal{L}}$). *For scalar functions ζ, Z on M_t with $\mathcal{L}\zeta = Z$ or $\tilde{\mathcal{L}}\zeta = Z$, one has*

$$a^4\|\Delta\zeta\|_{L^2_G(M)} + a^2\|\nabla\zeta\|_{L^2_G(M)} + \|\zeta\|_{L^2_G(M)} \lesssim \|Z\|_{L^2_G(M)} \tag{3.5.3}$$

Proof. The only difference in the proof in the pure scalar-field case is that Vlasov terms occur in h and $\tilde{h}$. However, we observe that $\rho^{Vl}_{FLRW}+\mathfrak{p}^{Vl}_{FLRW}$ is nonnegative and only depends on t, while one has $|H| \lesssim \varepsilon$ by the (3.4.1b), (3.4.1a) and (3.3.13a), as well as $|\tilde{H}| \lesssim \varepsilon a^{2-c\sigma}$ by (3.3.13a). Hence, one has

$$h, \tilde{h} \geq 6\pi C^2 \text{ and } |\nabla h|_G, |\nabla\tilde{h}|_G \lesssim \varepsilon \tag{3.5.4}$$

for small enough $\varepsilon > 0$, which were the only properties needed to prove the above lemma in the scalar field case. □

Lemma 3.5.3 (Odd order elliptic estimates for $\mathcal{L}$ and $\tilde{\mathcal{L}}$). *Consider scalar functions ζ, Z with $\mathcal{L}\zeta = Z$ or $\tilde{\mathcal{L}}\zeta = Z$. Then, one has*

$$a^4\|\nabla\Delta\zeta\|_{L^2_G(M)} + a^2\|\Delta\zeta\|_{L^2_G(M)} + \|\nabla\zeta\|_{L^2_G(M)} \lesssim \|Z\|_{\dot{H}^1_G(M)} + \sqrt{\varepsilon}a^{-c\sqrt{\varepsilon}}\|Z\|_{L^2_G(M)} \tag{3.5.5}$$

Proof. We only prove the statement for $\mathcal{L}$, the proof is identical for $\tilde{\mathcal{L}}$. First, note that

$$\nabla Z = \nabla(\mathcal{L}\zeta) = a^4\nabla\Delta\zeta - h\nabla\zeta + \zeta\nabla h.$$

Contracting $\nabla_i\mathcal{L}\zeta$ with $-\nabla^{\sharp i}\zeta$ and integrating, we thus obtain

$$\int_M a^4|\Delta\zeta|^2_G + h|\nabla\zeta|^2 - \zeta\langle\nabla h,\nabla\zeta\rangle_G \,\mathrm{dvol}_G = -\int_M \langle\nabla Z,\nabla\zeta\rangle_G \,\mathrm{dvol}_G\,.$$

Using integration by parts to rewrite the third term on the left, applying (3.5.4) and rearranging then yields

$$\int_M a^4|\nabla^2\zeta|^2_G + |\nabla\zeta|^2_G \,\mathrm{dvol}_G \lesssim \int_M |\nabla Z|^2_G + \|\Delta h\|_{L^\infty_G}|\zeta|^2_G \,\mathrm{dvol}_G\,.$$

Using the a priori estimates (3.4.3b) and (3.4.3a) as well as (3.3.13a) for the Vlasov terms, along with Lemma 3.5.2, this then becomes

$$\int_M a^4|\nabla^2\zeta|_G + |\nabla\zeta|^2_G \,\mathrm{dvol}_G \lesssim \|Z\|^2_{\dot{H}^1} + \sqrt{\varepsilon}a^{-c\sqrt{\varepsilon}}\|Z\|^2_{L^2_G}\,.$$

Now contracting $\nabla_i\mathcal{L}\zeta$ with $a^4\nabla^{\sharp i}\Delta\zeta$ and integrating and using integration by parts, one has

$$\begin{aligned}
&\int_M a^8|\nabla\Delta\zeta|^2_G + ha^4|\Delta\zeta|^2 + a^4\langle\nabla h,\nabla\zeta\rangle\Delta\zeta + a^4\zeta\langle\nabla h,\nabla\Delta\zeta\rangle_G\,\mathrm{dvol}_G\\
=&\int_M a^8|\nabla\Delta\zeta|^2_G + ha^4|\Delta\zeta|^2 - a^4\Delta h\cdot\zeta\Delta\zeta\,\mathrm{dvol}_G\\
=&\int_M a^4\langle\nabla Z,\nabla\Delta\zeta\rangle_G\,\mathrm{dvol}_G\,.
\end{aligned}$$

Thus, by (3.5.4), the following holds for some $K>0$:

$$\begin{aligned}
&\int_M a^8|\nabla\Delta\zeta|^2_G + 6\pi C^2a^4|\Delta\zeta|^2\,\mathrm{dvol}_G\\
\leq&\frac{1}{2}\|Z\|^2_{\dot{H}^1_G(M)} + \int_M \frac{1}{2}a^8|\nabla\Delta\zeta|^2_G + K\varepsilon\left(|\zeta|^2 + a^4|\Delta\zeta|^2\right)\,\mathrm{dvol}_G
\end{aligned}$$

The statement then follows for small enough ε by rearranging and combining both bounds. □

Lemma 3.5.4 (Lapse energy estimate). *The following estimates hold:*

$$\begin{aligned}
&a^8\mathfrak{E}^{(2)}(N,\cdot) + a^4\mathfrak{E}^{(1)}(N,\cdot) + \mathfrak{E}^{(0)}(N,\cdot)\\
&\lesssim \varepsilon^2\mathfrak{E}^{(0)}(\Sigma,\cdot) + \mathfrak{E}^{(0)}(\phi,\cdot) + a^4\mathfrak{E}^{(0)}_{1,0}(f,\cdot) + a^{4-c\sqrt{\varepsilon}}\|G^{-1}-\gamma^{-1}\|^2_{L^2_G(M)}\,,
\end{aligned}\tag{3.5.6a}$$

$$\begin{aligned}
&a^8\mathfrak{E}^{(2)}(N,\cdot) + a^4\mathfrak{E}^{(1)}(N,\cdot) + \mathfrak{E}^{(0)}(N,\cdot)\\
&\lesssim a^8\mathfrak{E}^{(0)}(\mathrm{Ric},\cdot) + \varepsilon a^{8-c\sqrt{\varepsilon}}\|\nabla\phi\|^2_{L^2_G(M)} + a^{4-c\sqrt{\varepsilon}}\mathfrak{E}^{(0)}_{1,0}(f,\cdot) + a^{4-c\sigma}\|G^{-1}-\gamma^{-1}\|^2_{L^2_G(M)}
\end{aligned}\tag{3.5.6b}$$

For any $L\in\{1,\dots,18\}$, one additionally has:

$$\begin{aligned}
&a^8\mathfrak{E}^{(L+2)}(N,\cdot) + a^4\mathfrak{E}^{(L+1)}(N,\cdot) + \mathfrak{E}^{(L)}(N,\cdot)\\
&\lesssim \varepsilon^2\mathfrak{E}^{(L)}(\Sigma,\cdot) + \mathfrak{E}^{(L)}(\phi,\cdot) + \varepsilon^2a^{-c\sqrt{\varepsilon}}\left[\mathfrak{E}^{(\leq\max\{0,L-2\})}(\Sigma,\cdot) + \mathfrak{E}^{(\leq\max\{0,L-2\})}(\phi,\cdot)\right]\\
&\quad+\underbrace{\left(\varepsilon^4a^{-c\sqrt{\varepsilon}} + \varepsilon^2a^{2-c\sigma}\right)\mathfrak{E}^{(\leq L-3)}(\mathrm{Ric},\cdot)}_{\text{not present for } L\leq 2} + a^{4-c\sqrt{\varepsilon}}\mathfrak{E}^{(L)}_{1,L}(f,\cdot)
\end{aligned}\tag{3.5.6c}$$

$$\begin{aligned}
&a^8\mathfrak{E}^{(L+2)}(N,\cdot) + a^4\mathfrak{E}^{(L+1)}(N,\cdot) + \mathfrak{E}^{(L)}(N,\cdot)\\
&\lesssim a^8\mathfrak{E}^{(\leq L)}(\mathrm{Ric},\cdot) + \varepsilon a^{8-c\sqrt{\varepsilon}}\|\nabla\phi\|^2_{H^L_G(M)} + a^{4-c\sqrt{\varepsilon}}\mathfrak{E}^{(L)}_{1,L}(f,\cdot)
\end{aligned}\tag{3.5.6d}$$

Proof. This proof now proceeds as in the pure scalar field setting (see Corollaries 2.5.5 and 2.5.8): We commute (3.5.2) with Δ^l and $\nabla\Delta^l$ for $l=\lfloor\frac{L}{2}\rfloor=0,\dots,9$ and apply the elliptic estimates from Lemma 3.5.2 and 3.5.3 respectively. For the estimates with respect to $\mathcal{L}$, the right hand sides at order L are of the form

$$\sum_{I=1}^{L-1}\nabla^IH*\nabla^{L-I}N + (N+1)\begin{cases}\Delta^{\frac{L}{2}}H & L\text{ even}\\ \nabla\Delta^{\lfloor\frac{L}{2}\rfloor}H & L\text{ odd}\end{cases}.$$

Again, the only changes occur when considering Vlasov terms: At high order, we apply Lemma 3.4.14 to estimate the density and pressure terms by Vlasov energies up to curvature errors. Conversely, low order Vlasov terms are bounded using (3.4.21). Regarding the remaining quantities, we note that Lemma 2.4.5 fully extends to our setting, allowing us to move between Sobolev norms and energies up to curvature errors at high orders. □

3.5.2. Energy and norm estimates for the scalar field and remaining spacetime variables. In this section, we collect individual energy and Sobolev norm estimates for the shear as well as scalar field and metric variables. Compared to the estimates in Section 2.6 and their respective proofs, there are two differences we need to account for: Firstly, we now consider arbitrary (constant) spatial sectional curvature instead of $\kappa = -\frac{1}{9}$, which leads to a slightly more refined argument being necessary to prove Lemma 3.5.5. Secondly, Vlasov matter now enters the respective equations and the above lapse estimates. Beyond needing to estimate Vlasov matter terms where they occur, the fact that Vlasov matter enters into the elliptic lapse estimate (3.5.6c) leads to a linear coupling between scalar field and Vlasov matter. To deal with this, we need to introduce an additional time-scaling hierarchy in the total energy along with a hierarchy based on ε-scaling, see Definition 3.7.4. To this end, we additionally derive time-scaled estimates along similar lines as their unscaled analogues, and often distribute both types of weights in anticipation of the total energies in Definition 3.7.4. The proofs of the latter otherwise will follow by the same arguments as in the pure scalar field case, so we will often only focus on how these changes are accounted for in the respective proofs.

LEMMA 3.5.5 (Energy estimates for the scalar field). *For $L \in 2\mathbb{Z}_+$, $L \leq 18$, the following estimate holds:*

$$\begin{aligned}
\text{(3.5.7a)}\qquad & \mathfrak{E}^{(L)}(\phi,t) + \int_t^{t_0} \dot{a}(s)a(s)^3\mathfrak{E}^{(L+1)}(N,s) + \frac{\dot{a}(s)}{a(s)}\mathfrak{E}^{(L)}(N,s)\,ds \\
&\lesssim \mathfrak{E}^{(L)}(\phi,t_0) + \int_t^{t_0} \left(\varepsilon a(s)^{-3} + a(s)^{-1-c\sqrt{\varepsilon}}\right)\mathfrak{E}^{(L)}(\phi,s)\,ds \\
&\quad + \int_t^{t_0} \Big\{ \varepsilon^{\frac{3}{4}} a(s)^{-3}\cdot\varepsilon^{\frac{1}{4}}\mathfrak{E}^{(L)}(\Sigma,s) + \varepsilon a(s)^{-3}\cdot\varepsilon^{\frac{1}{2}}\mathfrak{E}^{(L-2)}(\mathrm{Ric},s) \\
&\qquad + a(s)^{-2-c\sqrt{\varepsilon}-(L+1)\omega}\cdot a(s)^{(L+1)\omega}\mathfrak{E}^{(\leq L)}_{1,\leq L}(f,s) \\
&\qquad + a(s)^{-2-c\sqrt{\varepsilon}-\frac{\omega}{2}}\cdot a(s)^{\frac{\omega}{2}}\|G^{-1}-\gamma^{-1}\|^2_{L^2_G(M_s)} \\
&\qquad + \sqrt{\varepsilon}a(s)^{-3-c\sqrt{\varepsilon}}\mathfrak{E}^{(\leq L-2)}(\phi,s) + \varepsilon^{\frac{3}{4}}a(s)^{-3-c\sqrt{\varepsilon}}\cdot\varepsilon^{\frac{1}{4}}\mathfrak{E}^{(\leq L-2)}(\Sigma,s)\Big\}\,ds \\
&\quad \underbrace{+\int_t^{t_0} \varepsilon a(s)^{-3-c\sqrt{\varepsilon}}\cdot\varepsilon^{\frac{1}{2}}\mathfrak{E}^{(\leq L-4)}(\mathrm{Ric},s)\,ds}_{\text{if } L\geq 4}
\end{aligned}$$

Similarly, one has

$$\text{(3.5.7b)}\qquad a(t)^4\mathfrak{E}^{(L+1)}(\phi,t) + \int_t^{t_0} \left\{\dot{a}(s)a(s)^7\mathfrak{E}^{(L+2)}(N,s) + \dot{a}(s)a(s)^3\mathfrak{E}^{(L+1)}(N,s)\right\} ds$$

$$
\begin{aligned}
&\lesssim a(t_0)^4\mathfrak{E}^{(L+1)}(\phi,t_0)+\int_t^{t_0}\left(\varepsilon a(s)^{-3}+a(s)^{-1-c\sqrt{\varepsilon}}\right)\cdot a(s)^4\mathfrak{E}^{(L+1)}(\phi,s)\,ds\\
&\quad+\int_t^{t_0}\Big\{\varepsilon^{\frac12}a(s)^{-3}\cdot\varepsilon^{\frac12}a(s)^4\mathfrak{E}^{(L+1)}(\Sigma,s)+\left(\varepsilon a(s)^{-3}+a(s)^{-1-c\sqrt{\varepsilon}}\right)\mathfrak{E}^{(L)}(\phi,s)\\
&\qquad+\varepsilon^{\frac34}a(s)^{-3}\cdot\varepsilon^{\frac12}\mathfrak{E}^{(L)}(\Sigma,s)+\varepsilon^{\frac14}a(s)^{-1-c\sqrt{\varepsilon}}\cdot\varepsilon^{\frac34}a(s)^4\mathfrak{E}^{(L-1)}(\mathrm{Ric},s)\\
&\qquad+\left(\varepsilon^{\frac52}a(s)^{-3}+\varepsilon^{\frac12}a(s)^{-1-c\sqrt{\varepsilon}}\right)\cdot\varepsilon^{\frac12}\mathfrak{E}^{(L-2)}(\mathrm{Ric},s)\\
&\qquad+a(s)^{-2-c\sqrt{\varepsilon}-(L+2)\omega}\cdot a(s)^{4+(L+2)\omega}\mathfrak{E}^{(L+1)}_{1,L+1}(f,s)\\
&\qquad+a(s)^{2-c\sqrt{\varepsilon}-(L+1)\omega}\cdot a(s)^{(L+1)\omega}\mathfrak{E}^{(\le L)}_{1,\le L}(f,s)\\
&\qquad+a(s)^{2-c\sqrt{\varepsilon}-\frac{\omega}{2}}\cdot a(s)^{\frac{\omega}{2}}\|G^{-1}-\gamma^{-1}\|^2_{L^2_G(M_s)}\Big\}\,ds
\end{aligned}
$$

For $L=0$, analogous estimates hold with all energies of order $L-1$ or lower dropped from the respective right hand sides.

Proof. We only sketch how to adapt the proof of (3.5.7a) from that of Lemma 2.6.2 – analogous adaptations can then be applied to the proof of Lemma 2.6.4 to obtain (3.5.7b) as well as the low order analogues.
After taking the time derivative, inserting (3.2.20p) and integrating by parts, the following schematic equality holds:

(3.5.8a)
$$-\partial_t\mathfrak{E}^{(L)}(\phi,\cdot)\lesssim\int_M\Big\{6C\frac{\dot a}{a}\Delta^{\frac L2}N\cdot\Delta^{\frac L2}\Psi+3a\langle\nabla\Delta^{\frac L2}N,\nabla\phi\rangle_G\cdot\Delta^{\frac L2}\Psi$$

(3.5.8b)
$$-2Ca\langle\nabla\Delta^{\frac L2}N,\nabla\Delta^{\frac L2}\phi\rangle_G\Big\}-4\dot a a^3|\nabla\Delta^{\frac L2}\phi|^2_G\,\mathrm{dvol}_G+\langle\text{error terms}\rangle$$

Regarding the first term in (3.5.8a), we insert the following zero using (3.2.20f) and (3.2.3a):

$$
\begin{aligned}
0&=-\frac{3}{8\pi}\int_M\mathrm{div}_G\left(\nabla\Delta^{\frac L2}N\cdot\Delta^{\frac L2}N\right)\mathrm{dvol}_G\\
&=\int_M-\frac{3}{8\pi}\dot a a^3|\nabla\Delta^{\frac L2}N|^2_G-\frac{3}{8\pi}\left(-\frac{\kappa}{8\pi}\dot a a^3+\frac92C^2\frac{\dot a}{a}+\frac92a^{-2}\left(\rho^{Vl}_{FLRW}-\mathfrak{p}^{Vl}_{FLRW}\right)\right)|\Delta^{\frac L2}N|^2\\
&\quad-6C\frac{\dot a}{a}\Delta^{\frac L2}N\cdot\Delta^{\frac L2}\Psi+\frac92a^{-2}\Delta^{\frac L2}\left[\left(\rho^{Vl}-\rho^{Vl}_{FLRW}\right)-\left(\mathfrak{p}^{Vl}-\mathfrak{p}^{Vl}_{FLRW}\right)\right]\Delta^{\frac L2}N\\
&\quad+\langle\text{borderline and junk terms}\rangle\,\mathrm{dvol}_G
\end{aligned}
$$

Regarding the first line, the first and third term can be pulled to the left hand side, while the second and fourth can be bounded by $\lesssim a^{-2}\mathfrak{E}^{(L)}(N,\cdot)$. The first term in the second line precisely cancels the first term in (3.5.8a), and the term containing Vlasov matter can be bounded by

(3.5.9)
$$a^{-2-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\le L)}_{1,\le L}(f,\cdot)}\sqrt{\mathfrak{E}^{(L)}(N,\cdot)}$$

by Lemma 3.4.14. Note that, for the same argument at $L=0$, this also incurs a metric term. The second term in (3.5.8a) and the explicit term in (3.5.8b) can be treated similarly, so we focus on the latter, denoting it by $(*)$: Applying (3.2.5), we obtain the following:

$$\begin{aligned}|(*)| &\leq \int_M 2\sqrt{\frac{3}{4\pi}}\left(\dot{a}a^3+\kappa a^3\right)|\nabla\Delta^{\frac{L}{2}}N|_G|\nabla\Delta^{\frac{L}{2}}\phi|_G\,\mathrm{dvol}_G\\ &\leq \int_M\left(4\dot{a}a^3|\nabla\Delta^{\frac{L}{2}}\phi|_G^2+\frac{3}{16\pi}\dot{a}a^3|\nabla\Delta^{\frac{L}{2}}N|_G^2\right)\mathrm{dvol}_G\\ &\quad+|\kappa|a^{-1}\left(\mathfrak{E}^{(L)}(\phi,\cdot)+a^4\mathfrak{E}^{(L+1)}(N,\cdot)\right)\end{aligned}$$

The first term cancels with the second term in (3.5.8b), the second can be pulled to the left and absorbed into already present lapse energy terms after adapting constants.
Combining these observations and dealing with borderline and junk terms as in the pure scalar field setting, one obtains the following differential estimate:

$$-\partial_t\mathfrak{E}^{(L)}(\phi,\cdot)+\dot{a}a^3\mathfrak{E}^{(L+1)}(N,\cdot)+\frac{\dot{a}}{a}\mathfrak{E}^{(L)}(N,\cdot)$$

$$\lesssim\left(\varepsilon a^{-3}+a^{-1-c\sigma}\right)\mathfrak{E}^{(L)}(\phi,\cdot)+\left(\varepsilon a^{-3}+a^{-2-c\sqrt{\varepsilon}}\right)\left(a^4\mathfrak{E}^{(L+1)}(N,\cdot)+\mathfrak{E}^{(L)}(N,\cdot)\right)\tag{3.5.10a}$$

$$+\varepsilon a^{-3}\mathfrak{E}^{(L)}(\Sigma,\cdot)+\varepsilon^{\frac{3}{2}}a^{-3}\mathfrak{E}^{(L-2)}(\mathrm{Ric},\cdot)+\varepsilon a^{-3-c\sqrt{\varepsilon}}\mathfrak{E}^{(\leq L-2)}(\phi,\cdot)\tag{3.5.10b}$$

$$+\varepsilon a^{-3-c\sqrt{\varepsilon}}\mathfrak{E}^{(\leq L-2)}(\Sigma,\cdot)+\left[\varepsilon a^{-3-c\sqrt{\varepsilon}}+\sqrt{\varepsilon}a^{-1-c\sqrt{\varepsilon}}\right]\mathfrak{E}^{(\leq L-2)}(N,\cdot)\tag{3.5.10c}$$

$$+a^{-2-c\sqrt{\varepsilon}}\mathfrak{E}^{(\leq L)}_{1,\leq L}(f,\cdot)+\underbrace{\varepsilon^{\frac{3}{2}}a^{-3-c\sqrt{\varepsilon}}\mathfrak{E}^{(\leq L-4)}(\mathrm{Ric},\cdot)}_{\text{not present for }L=2}\tag{3.5.10d}$$

The statement then follows by applying Lemma 3.5.4 to the lapse energies in (3.5.10a) and (3.5.10c) and integrating. We note that, for $L=0$, one not only incurs metric norms by applying elliptic lapse estimates at low orders, but also when estimating (3.5.9). ☐

We will need an additional norm bound to obtain optimal control for the scalar field:

LEMMA 3.5.6 (Norm bound for $\nabla\phi$).

$$\|\nabla\phi\|_{H^l_G(M)}\lesssim(1+\varepsilon a^{-c\sqrt{\varepsilon}})\|\Sigma\|_{H^{l+1}_G(M)}+\sqrt{\varepsilon}a^{-c\sqrt{\varepsilon}}\|\Psi\|_{H^l_G(M)}+(1+\varepsilon a^{-c\sqrt{\varepsilon}})\|\jmath^{Vl}\|_{H^l_G(M)}$$

PROOF. The momentum constraint (3.2.20i) leads to the following expression after rearranging and using (3.4.1a) to ensure $\Psi+C>\frac{C}{2}$:

$$\begin{aligned}|\nabla^l\nabla\phi|_G&=\left|\nabla^l\left(\frac{\frac{1}{8\pi}\mathrm{div}_G\Sigma-\jmath^{Vl}}{\Psi+C}\right)\right|_G\\ &\lesssim|\nabla^{l+1}\Sigma|_G+|\nabla^l\jmath^{Vl}|_G+\sum_{I_1+I_2=l-1}\left(|\nabla^{I_1+1}\Sigma|_G+|\nabla^{I_1}\jmath^{Vl}|_G\right)|\nabla^{I_2+1}\Psi|_G\end{aligned}$$

The statement then follows by applying the Young inequality with (3.4.3b), (3.4.21) and (3.4.1a). ☐

Before moving on the energy estimates for Bel-Robinson variables, we prepare the necessary bounds on the Vlasov matter components that occur in the Bel-Robinson evolution equations (3.2.20l) and (3.2.20m):

LEMMA 3.5.7 (Bounds for Bel-Robinson Vlasov matter components). *Recalling* (3.2.20n) *and* (3.2.20o), *writing*

$$(J^{Vl})^{(*)}_{i0j,symm}=\frac{1}{2}\left[(J^{Vl})^{(*)}_{i0j}+(J^{Vl})^{(*)}_{j0i}\right]$$

and using $J^{Vl,\parallel}_{(\cdot)0(\cdot),symm}$ *to denote the same expression as in* (3.2.20n) *where pure trace terms are dropped, one has*

(3.5.11a)

$$\begin{aligned}&\|a^4(N+1)J^{Vl,\parallel}_{(\cdot)0(\cdot),symm}\|_{L^2_G(M)}\\&\lesssim a^{-1-c\sqrt{\varepsilon}-\omega}\sqrt{a^{4+2\omega}\mathfrak{E}^{(1)}_{1,\leq 1}(f,\cdot)}+a^{-1-c\sqrt{\varepsilon}-\frac{\omega}{2}}\left(\sqrt{a^{\omega}\mathfrak{E}^{(0)}_{1,0}(f,\cdot)}+a^{\frac{\omega}{4}}\|G^{\pm1}-\gamma^{\pm1}\|_{L^2_G(M)}\right)\\&\quad+a^{-1-c\sqrt{\varepsilon}}\sqrt{a^4\mathfrak{E}^{(1)}(N,\cdot)}+a^{1-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(0)}(N,\cdot)}+a^{-1-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(0)}(\Sigma,\cdot)}\end{aligned}$$

as well as, for $L\in 2\mathbb{Z}_+, L\leq 18$,

$$\begin{aligned}&\|a^4\Delta^{\frac{L}{2}}\left((N+1)J^{Vl,\parallel}_{(\cdot)0(\cdot),symm}\right)\|_{L^2_G(M)}\\&\lesssim a^{-1-c\sqrt{\varepsilon}-\frac{L+2}{2}\omega}\sqrt{a^{4+(L+2)\omega}\mathfrak{E}^{(L+1)}_{1,L+1}(f,\cdot)}+a^{-1-c\sqrt{\varepsilon}-\frac{L+1}{2}\omega}\sqrt{a^{(L+1)\omega}\mathfrak{E}^{(\leq L)}_{1,\leq L}(f,\cdot)}\\&\quad+a^{-1-c\sqrt{\varepsilon}}\sqrt{a^4\mathfrak{E}^{(L+1)}(N,\cdot)}+a^{1-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(L)}(N,\cdot)}\\&\quad+\sqrt{\varepsilon}a^{1-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-2)}(N,\cdot)}+a^{-1-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L)}(\Sigma,\cdot)}\\&\quad+\sqrt{\varepsilon}a^{-1-c\sigma}\sqrt{a^4\mathfrak{E}^{(L-1)}(\mathrm{Ric},\cdot)}+\sqrt{\varepsilon}a^{-1-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-2)}(\mathrm{Ric},\cdot)}\end{aligned}\tag{3.5.11b}$$

and, for $L\in 2\mathbb{N}, L\leq 18$,

$$\begin{aligned}&\|a^4\Delta^{\frac{L}{2}}\left((N+1)J^{Vl,*}_{(\cdot)0(\cdot),symm}\right)\|_{L^2_G}\\&\lesssim a^{-2-c\sqrt{\varepsilon}-\frac{(L+2)}{2}\omega}\sqrt{a^{4+(L+2)\omega}\mathfrak{E}^{(L+1)}_{1,L+1}(f,\cdot)}+\varepsilon a^{-1-c\sqrt{\varepsilon}-\frac{L+1}{2}\omega}\sqrt{a^{(L+1)\omega}\mathfrak{E}^{(\leq L)}_{1,L}(f,\cdot)}\\&\quad+a^{-1-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L)}(\Sigma,\cdot)}+\underbrace{\varepsilon a^{-1-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-2)}(\mathrm{Ric},\cdot)}}_{\text{if } L\neq 0}\end{aligned}$$

PROOF. We can directly use Lemma 3.4.14 to control $(S^{Vl})^{\parallel}$ and ∇J^{Vl}. Additionally, we note that the final term in (3.2.20n) can be estimated similarly to $(S^{Vl})^{\parallel}$ in Lemma 4.14 for $L\geq 2$ after observing that $m^2a^2\leq (v^0)^2$. For the zero order estimate, we need to expand the term as follows, using that f_{FLRW} is isotropic.

(3.5.12)

$$\begin{aligned}\int_{T^*_\cdot M}\frac{v_iv_j}{v^0}\frac{m^2a^2}{(v^0)^2}f(\cdot,\cdot,v)\,\mu_G^{-1}\,dv=&\int_{T^*_\cdot M}\frac{v_iv_j}{v^0}\frac{m^2a^2}{(v^0)^2}(f-f_{FLRW})(\cdot,\cdot,v)\,\mu_G^{-1}\,dv\\&+\int_{T^*_\cdot M}v_iv_j\left(\frac{m^2a^2}{(v^0)^3}-\frac{m^2a^2}{(v^0_\gamma)^3}\right)f_{FLRW}(\cdot,v)\,\mu_G^{-1}\,dv\\&+(\gamma_{ij}-G_{ij})\int_{T^*_\cdot M}\frac{1}{3}\frac{|v|^2_\gamma}{v^0_\gamma}\frac{m^2a^2}{(v^0_\gamma)^2}f_{FLRW}(\cdot,v)\,\mu_G^{-1}\,dv+\langle\text{pure trace}\rangle\end{aligned}$$

For the second line, note that one has, using (3.4.3c) in the second estimate,

$$\left|\frac{m^2a^2}{(v^0)^3}-\frac{m^2a^2}{(v^0_\gamma)^3}\right| \leq \left|\frac{(m^2a^2+|v|^2_\gamma)^3-(m^2a^2+|v|^2_G)^3}{(v^0)^3\left((v^0)^3+(v^0_\gamma)^3\right)}\frac{m^2a^2}{(v^0_\gamma)^3}\right| \lesssim a^{-c\sqrt{\varepsilon}}\frac{|G^{-1}-\gamma^{-1}|_G}{v^0_\gamma}$$

Dropping the pure trace term, the right hand side of (3.5.12) can thus be bounded in absolute value by the following, using (3.4.6a) and (3.4.6b).

$$\lesssim \int_{T^*M} v^0|(f-f_{FLRW})(\cdot,\cdot,v)|\,\mu_G^{-1}\,dv + a^{-c\sqrt{\varepsilon}}\left|G^{\pm1}-\gamma^{\pm1}\right|_G$$

From here, one can proceed as in the proof of Corollary 3.4.10 and Lemma 3.4.14, and checks that the last line of (3.2.20n), after dropping pure trace terms, can be bounded by the second term on the right hand side of (3.5.11a).

For the remaining terms, recall that we can apply (3.2.23) to pull the covariant derivative past the respective momentum integrals. By Lemma 3.2.7, any of the prefactors depending only on the momentum are uniformly bounded by v^0 up to constant and vanish when hit by a horizontal derivative. Keeping this in mind, we apply the product rule: When estimating high order Vlasov terms, we apply the Jensen inequality as in the proof of Lemma 3.4.14 to connect the respectice quantities to our Vlasov energies, and apply the low order bounds (3.4.3b) and (3.3.13a) for shear and lapse respectively. On the other hand, if most derivatives hit lapse or shear terms, we estimate the respective distribution function terms with (3.4.16) by $a^{-c\sqrt{\varepsilon}}$, and simply apply (3.4.6a) to deal with the remaining integrals over functions of v. Finally, recall Lemma 2.4.5 to bound Sobolev norms in Σ and N to their respective energy terms up to curvature errors. □

COROLLARY 3.5.8 (Energy estimates for Bel-Robinson variables). *The following holds for $L\in 2\mathbb{N}$, $2\leq L\leq 18$:*

(3.5.13a)

$$\begin{aligned}
&a(t)^{\frac{\omega}{4}}\mathfrak{E}^{(L)}(W,t)\\
&+\int_t^{t_0}\int_M a(s)^{\frac{\omega}{4}}\left[8\pi C^2 a(s)^{-3}(N+1)\langle\Delta^{\frac{L}{2}}\Sigma,\Delta^{\frac{L}{2}}\mathcal{E}\rangle_G+6\frac{\dot a(s)}{a(s)}|\Delta^{\frac{L}{2}}\mathcal{E}|^2_G\right]\mathrm{dvol}_G\,ds\\
&\lesssim a(t_0)^{\frac{\omega}{4}}\mathfrak{E}^{(L)}(W,t_0)+\int_t^{t_0}\left(\varepsilon a(s)^{-3}+a(s)^{-3+\frac{\omega}{8}}+a(s)^{-2-c\sqrt{\varepsilon}}\right)a(s)^{\frac{\omega}{4}}\mathfrak{E}^{(L)}(W,s)\,ds\\
&+\int_t^{t_0}\left\{\left(a(s)^{-3+\frac{\omega}{8}}+a(s)^{-1-c\sqrt{\varepsilon}}\right)\left(a(s)^4\mathfrak{E}^{(L+1)}(\phi,s)+\mathfrak{E}^{(L)}(\phi,s)\right)\right.\\
&\qquad+\left(\varepsilon a(s)^{-3}+a(s)^{-1-c\sqrt{\varepsilon}}\right)a(s)^{\frac{\omega}{4}}\mathfrak{E}^{(L)}(\Sigma,s)+\varepsilon^{\frac14}a(s)^{-3+\frac{\omega}{8}}\cdot\varepsilon^{\frac34}a(s)^4\mathfrak{E}^{(L-1)}(\mathrm{Ric},s)\\
&\qquad+\left(\varepsilon^{\frac72}a(s)^{-3+\frac{\omega}{8}}+\varepsilon^{\frac12}a(s)^{-1-c\sqrt{\varepsilon}+\frac{\omega}{8}}\right)\cdot\varepsilon^{\frac12}\mathfrak{E}^{(\leq L-2)}(\mathrm{Ric},s)\\
&\qquad+a(s)^{-2-c\sqrt{\varepsilon}-(L+2)\omega}\cdot a(s)^{4+(L+2)\omega}\mathfrak{E}^{(L+1)}_{1,L+1}(f,s)\\
&\qquad+a(s)^{-1-c\sqrt{\varepsilon}-(L+1)\omega}\cdot a(s)^{(L+1)\omega}\mathfrak{E}^{(\leq L)}_{1,\leq L}(f,s)+a(s)^{-1-c\sqrt{\varepsilon}-\frac{\omega}{4}}\cdot a(s)^{\frac{\omega}{2}}\|G^{\pm1}-\gamma^{\pm1}\|^2_{L^2_G(M_s)}
\end{aligned}$$

$$
\begin{aligned}
&+\left(\varepsilon^2 a(s)^{-3-c\sqrt{\varepsilon}+\frac{\omega}{8}}+a(s)^{-1-c\sqrt{\varepsilon}+\frac{\omega}{8}}\right)\mathfrak{E}^{(\leq L-2)}(\phi,s)\\
&+\varepsilon a(s)^{-3-c\sqrt{\varepsilon}}\cdot a(s)^{\frac{\omega}{4}}\left(\mathfrak{E}^{(\leq L-2)}(\Sigma,s)+\mathfrak{E}^{(\leq L-2)}(W,s)\right)\\
&+\underbrace{\varepsilon^{\frac12}a(s)^{-3-c\sqrt{\varepsilon}+\frac{\omega}{8}}\cdot\varepsilon^{\frac12}\mathfrak{E}^{(\leq L-4)}(\mathrm{Ric},s)}_{\textit{not present for } L=2}\Bigg\}\,ds\,.
\end{aligned}
$$

(3.5.13b)

$$
\begin{aligned}
&\mathfrak{E}^{(L)}(W,t)+\int_t^{t_0}\int_M\left[8\pi C^2 a(s)^{-3}(N+1)\langle\Delta^{\frac{L}{2}}\Sigma,\Delta^{\frac{L}{2}}\mathcal{E}\rangle_G+6\frac{\dot a(s)}{a(s)}|\Delta^{\frac{L}{2}}\mathcal{E}|_G^2\right]\mathrm{dvol}_G\,ds\\
&\lesssim\mathfrak{E}^{(L)}(W,t_0)+\int_t^{t_0}\left(\varepsilon^{\frac18}a(s)^{-3}+a(s)^{-2-c\sqrt{\varepsilon}}\right)\mathfrak{E}^{(L)}(W,s)\,ds\\
&+\int_t^{t_0}\Bigg\{\varepsilon^{-\frac18}a(s)^{-3}\cdot a(s)^4\mathfrak{E}^{(L+1)}(\phi,s)+\left(\varepsilon^{\frac18}a(s)^{-3}+a(s)^{-1}\right)\mathfrak{E}^{(L)}(\phi,s)\\
&\quad+\left(\varepsilon a(s)^{-3}+a(s)^{-1-c\sqrt{\varepsilon}}\right)\mathfrak{E}^{(L)}(\Sigma,s)\\
&\quad+\varepsilon^{\frac18}a(s)^{-3}\cdot\varepsilon^{\frac34}a(s)^4\mathfrak{E}^{(L-1)}(\mathrm{Ric},s)+\left(\varepsilon^{\frac{27}{8}}a(s)^{-3}+\varepsilon^{\frac12}a(s)^{-1-c\sqrt{\varepsilon}}\right)\cdot\varepsilon^{\frac12}\mathfrak{E}^{(\leq L-2)}(\mathrm{Ric},s)\\
&\quad+a(s)^{-2-c\sqrt{\varepsilon}-(L+2)\omega}\cdot a(s)^{4+(L+2)\omega}\mathfrak{E}^{(L+1)}_{1,L+1}(f,s)\\
&\quad+a(s)^{-1-c\sqrt{\varepsilon}-(L+1)\omega}\cdot a(s)^{(L+1)\omega}\mathfrak{E}^{(\leq L)}_{1,\leq L}(f,s)+a(s)^{-1-c\sqrt{\varepsilon}-\frac{\omega}{4}}\cdot a(s)^{\frac{\omega}{2}}\|G^{\pm1}-\gamma^{\pm1}\|^2_{L^2_G(M_s)}\\
&\quad+\left(\varepsilon^{\frac{15}{8}}a(s)^{-3-c\sqrt{\varepsilon}}+a(s)^{-1-c\sqrt{\varepsilon}}\right)\mathfrak{E}^{(\leq L-2)}(\phi,s)\\
&\quad+\varepsilon a(s)^{-3-c\sqrt{\varepsilon}}\left(\mathfrak{E}^{(\leq L-2)}(\Sigma,s)+\mathfrak{E}^{(\leq L-2)}(W,s)\right)+\underbrace{\varepsilon^{\frac12}a(s)^{-3-c\sqrt{\varepsilon}}\cdot\varepsilon^{\frac12}\mathfrak{E}^{(\leq L-4)}(\mathrm{Ric},s)}_{\textit{not present for } L=2}\Bigg\}\,ds\,.
\end{aligned}
$$

For $L=0$, the analogues of (3.5.13) *hold where all energies of order below L are dropped from the right hand sides.*

PROOF. The proof proceeds exactly as in the pure scalar field case (see Lemma 2.6.6), where we use Lemma 3.5.7 to bound the Vlasov matter terms, and apply Lemma 3.5.4 to bound lapse energies. □

LEMMA 3.5.9 (Integral energy estimates for the second fundamental form). *For $L\in 2\mathbb{Z}_+, L\leq 18$, the following holds:*

(3.5.14a)

$$
\begin{aligned}
&a(t)^{\frac{\omega}{4}}\mathfrak{E}^{(L)}(\Sigma,t)\\
&+2\int_t^{t_0}\int_M a(s)^{\frac{\omega}{4}}\left[a(s)^{-3}(N+1)\langle\Delta^{\frac{L}{2}}\mathcal{E},\Delta^{\frac{L}{2}}\Sigma\rangle_G+\frac{\dot a(s)}{a(s)}(N+1)|\Delta^{\frac{L}{2}}\Sigma|_G^2\right]\mathrm{dvol}_G\,ds\\
&\lesssim a(t_0)^{\frac{\omega}{4}}\mathfrak{E}^{(L)}(\Sigma,t_0)+\int_t^{t_0}\left(\varepsilon a(s)^{-3}+a(s)^{-3+\frac{\omega}{8}}+a(s)^{-1-c\sqrt{\varepsilon}}\right)a(s)^{\frac{\omega}{4}}\mathfrak{E}^{(L)}(\Sigma,s)\,ds
\end{aligned}
$$

$$+\int_t^{t_0}\Big\{a(s)^{-3+\frac{\omega}{8}}\mathfrak{E}^{(L)}(\phi,s)+\varepsilon^{\frac32}a(s)^{3-c\sigma+\frac{\omega}{4}}\cdot\varepsilon^{\frac12}\mathfrak{E}^{(L-1)}(\mathrm{Ric},s)$$

$$+\varepsilon^{\frac32}a(s)^{-3+\frac{\omega}{8}}\cdot\varepsilon^{\frac12}\mathfrak{E}^{(L-2)}(\mathrm{Ric},s)+\varepsilon^2a(s)^{-3-c\sqrt{\varepsilon}}\cdot a(s)^{\frac{\omega}{4}}\mathfrak{E}^{(\leq L-2)}(\Sigma,s)$$

$$+\varepsilon^2a(s)^{-3-c\sqrt{\varepsilon}+\frac{\omega}{8}}\mathfrak{E}^{(\leq L-2)}(\phi,s)+\underbrace{\varepsilon^{\frac32}a(s)^{-3-c\sqrt{\varepsilon}+\frac{\omega}{8}}\cdot\varepsilon^{\frac12}\mathfrak{E}^{(\leq L-4)}(\mathrm{Ric},s)}_{\textit{not present for }L=2}$$

$$+a(s)^{-1-c\sqrt{\varepsilon}-(L+1)\omega}\cdot\left(a(s)^{(L+1)\omega}\mathfrak{E}^{(\leq L)}_{1,\leq L}(f,s)+a(s)^{\frac{\omega}{2}}\|G^{-1}-\gamma^{-1}\|^2_{L^2_G(M_s)}\right)\Big\}\,ds$$

(3.5.14b)

$$\mathfrak{E}^{(L)}(\Sigma,t)+2\int_t^{t_0}\int_M\left[a(s)^{-3}(N+1)\langle\Delta^{\frac L2}\mathcal{E},\Delta^{\frac L2}\Sigma\rangle_G+\frac{\dot a(s)}{a(s)}(N+1)|\Delta^{\frac L2}\Sigma|^2_G\right]\mathrm{dvol}_G\,ds$$

$$\lesssim\mathfrak{E}^{(L)}(\Sigma,t_0)+\int_t^{t_0}\left(\varepsilon^{\frac18}a(s)^{-3}+a(s)^{-1-c\sqrt{\varepsilon}}\right)\mathfrak{E}^{(L)}(\Sigma,s)\,ds$$

$$+\int_t^{t_0}\Big\{\varepsilon^{-\frac18}a(s)^{-3}\mathfrak{E}^{(L)}(\phi,s)+\varepsilon^{\frac54}a(s)^{3-c\sigma}\cdot\varepsilon^{\frac34}\mathfrak{E}^{(L-1)}(\mathrm{Ric},s)+\varepsilon^2a(s)^{-3}\mathfrak{E}^{(L-2)}(\mathrm{Ric},s)$$

$$+\varepsilon^2a(s)^{-3-c\sqrt{\varepsilon}}\mathfrak{E}^{(\leq L-2)}(\Sigma,s)+\left(\varepsilon^{\frac{15}{8}}a(s)^{-3-c\sqrt{\varepsilon}}+\varepsilon a(s)^{-1-c\sqrt{\varepsilon}}\right)\mathfrak{E}^{(\leq L-2)}(\phi,s)$$

$$+a(s)^{-1-c\sqrt{\varepsilon}-(L+1)\omega}\cdot\left(a(s)^{(L+1)\omega}\mathfrak{E}^{(\leq L)}_{1,\leq L}(f,s)+a(s)^{(L+1)\omega}\|G^{-1}-\gamma^{-1}\|^2_{L^2_G(M_s)}\right)$$

$$+\underbrace{\varepsilon^{\frac32}a(s)^{-3-c\sqrt{\varepsilon}}\cdot\varepsilon^{\frac12}\mathfrak{E}^{(\leq L-4)}(\mathrm{Ric},s)}_{\textit{not present for }L=2}\Big\}\,ds$$

For $L=0$, the same estimate holds with all energies of order $L-1$ or lower dropped from the upper bound.

Proof. Note that the only term that is not pure trace that the presence of Vlasov matter adds to the evolution equation is $S^{Vl,\parallel}$. Hence, commuting (3.2.20c) with $\Delta^{\frac L2}$ and carrying over error term notation from (2.A.14), we have the following:

(3.5.15a) $$-\partial_t\mathfrak{E}^{(L)}(\Sigma,\cdot)=\int_M\Big\{2a\langle\nabla^2\Delta^{\frac L2}N,\Delta^{\frac L2}\Sigma\rangle_G-2a(N+1)\langle\Delta^{\frac L2}\mathrm{Ric}[G],\Delta^{\frac L2}\Sigma\rangle_G$$

(3.5.15b) $$+16\pi a^{-1}\langle\Delta^{\frac L2}S^{Vl,\parallel},\Delta^{\frac L2}\Sigma\rangle_G$$

(3.5.15c) $$+(\partial_tG^{-1})*G^{-1}*\Delta^{\frac L2}\Sigma*\Delta^{\frac L2}\Sigma-3N\frac{\dot a}{a}|\Delta^{\frac L2}\Sigma|^2_G$$

(3.5.15d) $$-2\langle\mathfrak{S}_{L,Border},\Delta^{\frac L2}\Sigma\rangle_G-2\langle\mathfrak{S}^{\parallel}_{L,Junk},\Delta^{\frac L2}\Sigma\rangle_G\Big\}\,\mathrm{dvol}_G$$

Regarding (3.5.15a), we apply the lapse energy estimate (3.5.6c) (resp., for $L=0$, (3.5.6a)) for the first term and have the following:

$$a\|\nabla^2\Delta^{\frac L2}N\|_{L^2_G}\sqrt{\mathfrak{E}^{(L)}(\Sigma,\cdot)}\lesssim a^{-3}\sqrt{a^8\mathfrak{E}^{(L+2)}(N,\cdot)+\mathfrak{E}^{(L)}(N,\cdot)}\sqrt{\mathfrak{E}^{(L)}(\Sigma,\cdot)}$$

$$\lesssim\left(\varepsilon^{\frac18}a^{-3}+a^{-1-c\sqrt{\varepsilon}}\right)\mathfrak{E}^{(L)}(\Sigma,\cdot)+\varepsilon^{-\frac18}a^{-3}\mathfrak{E}^{(L)}(\phi,\cdot)+a^{-1-c\sqrt{\varepsilon}-(L+1)\omega}\cdot a^{(L+1)\omega}\mathfrak{E}^{(L)}_{1,L}(f,\cdot)$$

$$+\begin{cases} a^{-1-c\sqrt{\varepsilon}-\frac{\omega}{2}}\cdot a^{\frac{\omega}{2}}\|G^{-1}-\gamma^{-1}\|^2_{L^2_G(M)} & L=0\\ \varepsilon^{\frac{15}{8}}a^{-3-c\sqrt{\varepsilon}}\left(\mathfrak{E}^{(\leq L-2)}(\phi,\cdot)+\mathfrak{E}^{(\leq L-2)}(\Sigma,\cdot)\right)+\varepsilon^{\frac{15}{8}}a^{-3-c\sqrt{\varepsilon}}\mathfrak{E}^{(\leq L-3)}(\mathrm{Ric},\cdot) & L\geq 2\end{cases}$$

For the second term in (3.5.15a), we insert the Bel-Robinson constraint (3.2.20j). High order lapse terms are treated similarly as above, and the only Vlasov term that is tracefree and thus isn't cancelled by the inner product is

$$-2(N+1)a^{-3}\cdot 4\pi a^2\langle\Delta^{\frac{L}{2}}S^{Vl,\parallel},\Delta^{\frac{L}{2}}\Sigma\rangle_G\,.$$

This term is bounded in absolute value by

$$a^{-1-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(L)}(\Sigma,\cdot)}\left(\sqrt{\mathfrak{E}^{(L)}_{1,L}(f,\cdot)}+\underbrace{\|G^{-1}-\gamma^{-1}\|_{L^2_G}}_{\text{if L=0}}\right)$$

up to constant with Lemma 3.4.14, as is the term in (3.5.15b). (3.5.15c)-(3.5.15d) only contribute error terms that can be dealt with as in the scalar field setting, accounting for (3.3.14). □

LEMMA 3.5.10 (Elliptic energy estimate for the second fundamental form). *For any $L\in 2\mathbb{Z}_+$, $2\leq L\leq 18$, we have*

(3.5.16a)

$$\begin{aligned}a^4\mathfrak{E}^{(L+1)}(\Sigma,\cdot)\lesssim&\left(a^{4-c\sqrt{\varepsilon}}+\varepsilon a^{2-c\sqrt{\varepsilon}}\right)\mathfrak{E}^{(L)}(\Sigma,\cdot)+\mathfrak{E}^{(L)}(\phi,\cdot)+\mathfrak{E}^{(L)}(W,\cdot)\\&+\varepsilon^{\frac{5}{4}}\cdot\varepsilon^{\frac{3}{4}}a^4\mathfrak{E}^{(L-1)}(\mathrm{Ric},\cdot)+a^{4-c\sqrt{\varepsilon}-(L+1)\omega}\cdot a^{(L+1)\omega}\mathfrak{E}^{(L)}_{1,L}(f,\cdot)\\&+\varepsilon a^{2-c\sqrt{\varepsilon}}\mathfrak{E}^{(\leq L-2)}(\phi,\cdot)+a^{2-c\sqrt{\varepsilon}}\mathfrak{E}^{(\leq L-2)}(\Sigma,\cdot)+\varepsilon^{\frac{1}{2}}a^{2-c\sqrt{\varepsilon}}\cdot\varepsilon^{\frac{1}{2}}\mathfrak{E}^{(\leq L-2)}(\mathrm{Ric},\cdot)\,.\end{aligned}$$

For $L=0$, one analogously has

(3.5.16b) $$a^4\mathfrak{E}^{(1)}(\Sigma,\cdot)\lesssim\left(a^{4-c\sqrt{\varepsilon}}+\varepsilon a^{2-c\sqrt{\varepsilon}}\right)\mathfrak{E}^{(0)}(\Sigma,\cdot)+\mathfrak{E}^{(0)}(\phi,\cdot)+\mathfrak{E}^{(0)}(W,\cdot)+a^4\mathfrak{E}^{(0)}_{1,0}(f,\cdot)\,.$$

PROOF. Recall from the proof of Lemma 2.6.10 that, since $\Delta^{\frac{L}{2}}\Sigma$ is tracefree, one has

$$\mathfrak{E}^{(L+1)}(\Sigma,\cdot)\lesssim a^{-c\sqrt{\varepsilon}}\mathfrak{E}^{(L)}(\Sigma,\cdot)+\int_M|\mathrm{div}_G\Delta^{\frac{L}{2}}\Sigma|^2_G+|\mathrm{curl}_G\Delta^{\frac{L}{2}}\Sigma|^2_G\,\mathrm{dvol}_G$$

Inserting commuted analogues of (3.2.20i) and (3.2.20k), we can argue as in the scalar field setting, where the commuted analogue of (3.2.20i) now additionally contains the term

$$8\pi\Delta^{\frac{L}{2}}\jmath^{Vl},$$

where we again apply Lemma 3.4.14, leading to the respective Vlasov energy terms. □

We close out this section by collecting all necessary bounds for metric variables:

LEMMA 3.5.11 (Sobolev norm estimates for metric variables). *For any $l\in\mathbb{N}$, $l\leq 18$, we have the following:*

(3.5.17a)

$$a(t)^{\frac{\omega}{2}}\|G^{\pm1}-\gamma^{\pm1}\|^2_{H^l_G(M_t)} \lesssim \varepsilon^4 + \int_t^{t_0} \left(a(s)^{-3+\frac{\omega}{8}} + a(s)^{-1-c\sigma}\right)\cdot a(s)^{\frac{\omega}{2}}\|G^{\pm1}-\gamma^{\pm1}\|^2_{H^l_G(M_s)}\,ds$$
$$+\int_t^{t_0} a(s)^{-3+\frac{\omega}{8}-c\sqrt{\varepsilon}}\left(a(s)^{\frac{\omega}{4}}\mathfrak{E}^{(\leq l)}(\Sigma,s)+\mathfrak{E}^{(\leq l)}(\phi,s)+\underbrace{\varepsilon\mathfrak{E}^{(\leq l-2)}(\mathrm{Ric},s)}_{if\, l\geq 3}\right)ds$$
$$+\int_t^{t_0} a(s)^{-1-c\sqrt{\varepsilon}-(l+1)\omega}\cdot a(s)^{(l+1)\omega}\mathfrak{E}^{(\leq l)}_{1,\leq l}(f,s)\,ds$$

(3.5.17b)

$$\|G^{\pm1}-\gamma^{\pm1}\|^2_{H^l_G(M_t)} \lesssim a^{-c\varepsilon^{\frac{1}{8}}}\left(\varepsilon^4+\varepsilon^{-\frac{1}{4}}\sup_{s\in(t,t_0)}\left(\|N\|^2_{H^l_G(M_s)}+\|\Sigma\|^2_{H^l_G(M_s)}\right)\right)$$

For $l>0$, one additionally has

(3.5.18a) $$a(t)^{\beta+\frac{\omega}{2}}\|\Gamma-\hat{\Gamma}\|^2_{H^{l-1}_G(M_t)}$$
$$\lesssim \varepsilon^4+\int_t^{t_0}\left(a(s)^{-3+\frac{\omega}{8}-c\sqrt{\varepsilon}}+a(s)^{-1-c\sigma}\right)a(s)^{\beta+\frac{\omega}{2}}\|\Gamma-\hat{\Gamma}\|^2_{H^{l-1}_G(M_s)}\,ds$$
$$+\int_t^{t_0} a(s)^{-3+\frac{\omega}{8}}\cdot a(s)^{\beta}\left(a(s)^{\frac{\omega}{4}}\mathfrak{E}^{(\leq l)}(\Sigma,s)+\mathfrak{E}^{(\leq l)}(\phi,s)+\underbrace{\varepsilon\mathfrak{E}^{(\leq l-2)}(\mathrm{Ric},s)}_{if\, l\geq 3}\right)ds$$
$$+\int_t^{t_0} a(s)^{-1-c\sqrt{\varepsilon}-(l+1)\omega}\cdot a(s)^{\beta+(l+1)\omega}\mathfrak{E}^{(\leq l)}_{1,\leq l}(f,s)\,ds \quad and$$

(3.5.18b) $$\|\Gamma-\hat{\Gamma}\|^2_{H^{l-1}_G(M_t)}\lesssim a(t)^{-c\varepsilon^{\frac{1}{8}}}\left(\varepsilon^4+\varepsilon^{-\frac{1}{4}}\sup_{s\in(t,t_0)}\left(\|N\|^2_{H^l_G(M_s)}+\|\Sigma\|^2_{H^l_G(M_s)}\right)\right)$$

as well as the following:

(3.5.19a)

$$a(t)^{\frac{\omega}{2}}\,\mathfrak{E}^{(L-2)}(\mathrm{Ric},t)\lesssim a(t_0)^{\frac{\omega}{2}}\mathfrak{E}^{(L-2)}(\mathrm{Ric},t_0)$$
$$+\int_t^{t_0}\left(\varepsilon^{\frac{1}{8}}a(s)^{-3}+a(s)^{-3+\frac{\omega}{8}}+a(s)^{-1-c\sqrt{\varepsilon}}\right)a(s)^{\frac{\omega}{2}}\mathfrak{E}^{(L-2)}(\mathrm{Ric},s)\,ds$$
$$+\int_t^{t_0}\Big\{a(s)^{-3+\frac{\omega}{8}}\left(\mathfrak{E}^{(L)}(\phi,s)+a(s)^{\frac{\omega}{4}}\mathfrak{E}^{(L)}(\Sigma,s)\right)$$
$$+a(s)^{-1-c\sqrt{\varepsilon}-(L+1)\omega}\cdot a(s)^{(L+1)\omega}\mathfrak{E}^{(\leq L)}_{1,\leq L}(f,s)$$
$$+a(s)^{-3+\frac{\omega}{8}-c\sqrt{\varepsilon}}\left(\mathfrak{E}^{(\leq L-2)}(\phi,s)+a(s)^{\frac{\omega}{4}}\mathfrak{E}^{(\leq L-2)}(\Sigma,s)\right)$$
$$+\varepsilon^{\frac{7}{8}}a(s)^{-3-c\sqrt{\varepsilon}}\cdot a(s)^{\frac{\omega}{2}}\mathfrak{E}^{(\leq L-4)}(\mathrm{Ric},s)\Big\}\,ds\,,$$

(3.5.19b)

$$a(t)^{4+\omega}\,\mathfrak{E}^{(L-1)}(\mathrm{Ric},t)\lesssim\varepsilon^4+\int_t^{t_0}\left(\varepsilon^{\frac{1}{8}}a(s)^{-3}+a(s)^{-3+\frac{\omega}{4}}\right)\cdot a(s)^{4+\omega}\mathfrak{E}^{(L-1)}(\mathrm{Ric},s)\,ds$$

$$+\int_t^{t_0}\Big\{ a(s)^{-3+\frac{\omega}{4}}\cdot a(s)^{4+\frac{\omega}{2}}\mathfrak{E}^{(L+1)}(\Sigma,s)+\left(a(s)^{-1-c\sqrt{\varepsilon}}+\varepsilon^2 a(s)^{-3+\frac{\omega}{2}}\right)a(s)^{\frac{\omega}{4}}\mathfrak{E}^{(L)}(\Sigma,s)$$
$$+\left(a(s)^{-3+\frac{3\omega}{4}}+a(s)^{-1-c\sqrt{\varepsilon}}\right)\mathfrak{E}^{(L)}(\phi,s)$$
$$+\left(a(s)^{-1-c\sqrt{\varepsilon}}+\varepsilon^2 a(s)^{-3-c\sqrt{\varepsilon}+\frac{\omega}{2}}\right)\left(\mathfrak{E}^{(\leq L-2)}(\phi,s)+a(s)^{\frac{\omega}{4}}\mathfrak{E}^{(\leq L-2)}(\Sigma,s)\right)$$
$$+a(s)^{-1-c\sigma-(L+1)\omega}\cdot a(s)^{(L+1)\omega}\mathfrak{E}^{(\leq L)}_{1,\leq L}(f,s)$$
$$+\varepsilon a(s)^{-3+\frac{\omega}{4}}\cdot\, a(s)^{\frac{\omega}{2}}\mathfrak{E}^{(L-2)}(\mathrm{Ric},s)+\varepsilon a(s)^{-3-c\sqrt{\varepsilon}+\frac{\omega}{4}}\cdot a(s)^{\frac{\omega}{2}}\mathfrak{E}^{(\leq L-4)}(\mathrm{Ric},s)\Big\}\,ds$$

$$\mathfrak{E}^{(L-2)}(\mathrm{Ric},t)\lesssim\mathfrak{E}^{(L-2)}(\mathrm{Ric},t_0)+\int_t^{t_0}\left(\varepsilon^{\frac18}a(s)^{-3}+a(s)^{-1-c\sqrt{\varepsilon}}\right)\mathfrak{E}^{(L-2)}(\mathrm{Ric},s)\,ds \tag{3.5.19c}$$
$$+\int_t^{t_0}\Big\{\varepsilon^{-\frac18}a(s)^{-3}\left(\mathfrak{E}^{(L)}(\phi,s)+\mathfrak{E}^{(L)}(\Sigma,s)\right)$$
$$+a(s)^{-1-c\sqrt{\varepsilon}-(L+1)\omega}\cdot a(s)^{(L+1)\omega}\mathfrak{E}^{(\leq L)}_{1,\leq L}(f,s)$$
$$+\varepsilon^{-\frac18}a(s)^{-3-c\sqrt{\varepsilon}}\left(\mathfrak{E}^{(\leq L-2)}(\phi,s)+\mathfrak{E}^{(\leq L-2)}(\Sigma,s)\right)$$
$$+\varepsilon^{\frac78}a(s)^{-3-c\sqrt{\varepsilon}}\mathfrak{E}^{(\leq L-4)}(\mathrm{Ric},s)\Big\}\,ds$$

$$a(t)^4\,\mathfrak{E}^{(L-1)}(\mathrm{Ric},t)\lesssim\varepsilon^4+\int_t^{t_0}\left(\varepsilon^{\frac18}a(s)^{-3}+a(s)^{-1-c\sqrt{\varepsilon}}\right)\cdot a(s)^4\mathfrak{E}^{(L-1)}(\mathrm{Ric},s)\,ds \tag{3.5.19d}$$
$$+\int_t^{t_0}\Big\{\varepsilon^{-\frac18}a(s)^{-3}\cdot a(s)^4\mathfrak{E}^{(L+1)}(\Sigma,s)+\left(\varepsilon^{\frac{15}{8}}a(s)^{-3}+a(s)^{-1-c\sqrt{\varepsilon}}\right)\mathfrak{E}^{(L)}(\Sigma,s)$$
$$+\left(\varepsilon^{-\frac18}a(s)^{-3}+a(s)^{-1-c\sqrt{\varepsilon}}\right)\mathfrak{E}^{(L)}(\phi,s)$$
$$+\left(\varepsilon^{\frac{15}{8}}a(s)^{-3-c\sqrt{\varepsilon}}+a(s)^{-1-c\sqrt{\varepsilon}}\right)\left(\mathfrak{E}^{(\leq L-2)}(\phi,s)+\mathfrak{E}^{(\leq L-2)}(\Sigma,s)\right)$$
$$+a(s)^{-1-c\sigma-(L+1)\omega}\cdot a(s)^{(L+1)\omega}\mathfrak{E}^{(\leq L)}_{1,\leq L}(f,s)$$
$$+\varepsilon a(s)^{-3}\mathfrak{E}^{(L-2)}(\mathrm{Ric},s)+\varepsilon a(s)^{-3-c\sqrt{\varepsilon}}\mathfrak{E}^{(\leq L-4)}(\mathrm{Ric},s)\Big\}\,ds$$

For $L=2$, the previous two estimates extend if all energies with negative order are dropped, while one additionally has the following:

$$a(t)^{\frac{\omega}{2}}\mathfrak{E}^{(0)}(\mathrm{Ric},t)\lesssim a(t_0)^{\frac{\omega}{2}}\mathfrak{E}^{(0)}(\mathrm{Ric},t_0)+\int_t^{t_0}\left(a(s)^{-3+\frac{\omega}{8}}+a(s)^{-1-c\sqrt{\varepsilon}}\right)\mathfrak{E}^{(0)}(\mathrm{Ric},s)\,ds \tag{3.5.19e}$$
$$+\int_t^{t_0}a(s)^{-3+\frac{\omega}{8}}\left(\mathfrak{E}^{(0)}(\phi,s)+a(s)^{\frac{\omega}{4}}\mathfrak{E}^{(0)}(\Sigma,s)\right)$$
$$+a(s)^{-1-c\sqrt{\varepsilon}-\omega}\left(a(s)^{\omega}\mathfrak{E}^{(0)}_{1,0}(f,s)+a(s)^{\frac{\omega}{2}}\|G^{-1}-\gamma^{-1}\|^2_{L^2_G(M_s)}\right)\,ds$$

$$\mathfrak{E}^{(0)}(\mathrm{Ric},t)\lesssim\mathfrak{E}^{(0)}(\mathrm{Ric},t_0)+\int_t^{t_0}\left(\varepsilon^{\frac18}a(s)^{-3}+a(s)^{-1-c\sqrt{\varepsilon}}\right)\mathfrak{E}^{(0)}(\mathrm{Ric},s)\,ds \tag{3.5.19f}$$
$$+\int_t^{t_0}\varepsilon^{-\frac18}a(s)^{-3}\left(\mathfrak{E}^{(0)}(\phi,s)+\mathfrak{E}^{(0)}(\Sigma,s)\right)\,ds$$

$$+\int_t^{t_0} a(s)^{-1-c\sqrt{\varepsilon}-\omega}\left(a(s)^{\omega}\mathfrak{E}^{(0)}_{1,0}(f,s)+a(s)^{\frac{\omega}{2}}\|G^{-1}-\gamma^{-1}\|^2_{L^2_G(M_s)}\right)ds$$

Finally, one has the following bound:

(3.5.19g)

$$a^8\mathfrak{E}^{(l)}(\mathrm{Ric},\cdot)\lesssim\mathfrak{E}^{(l)}(W,\cdot)+\mathfrak{E}^{(l)}(\Sigma,\cdot)+\varepsilon a^{-c\sqrt{\varepsilon}}\mathfrak{E}^{(\leq l-2)}(\Sigma,\cdot)+\mathfrak{E}^{(l)}(\phi,\cdot)+\varepsilon a^{-c\sqrt{\varepsilon}}\mathfrak{E}^{(\leq l-2)}(\phi,\cdot)$$
$$+a^{4-c\sqrt{\varepsilon}}\mathfrak{E}^{(\leq l)}_{1,\leq l}(f,\cdot)\underbrace{+a^{4-c\sqrt{\varepsilon}}\|G^{-1}-\gamma^{-1}\|^2_{L^2_G}}_{\textit{if } l=0}\underbrace{+\varepsilon\mathfrak{E}^{(l-2)}(\mathrm{Ric},\cdot)+\varepsilon a^{-c\sqrt{\varepsilon}}\mathfrak{E}^{(\leq l-4)}(\mathrm{Ric},\cdot)}_{\textit{if } l\geq 4}$$

Proof. Since (3.2.20a), (3.2.20d) and (3.2.20e) do not contain Vlasov matter variables, the proofs for (3.5.17b), (3.5.18b) and (3.5.19c)-(3.5.19f) are largely unchanged to those of Lemmas 2.6.11-2.6.14, where we apply Lemma 3.5.4 to estimate lapse energy terms. Regarding the scaled estimates, we consider (3.5.17a) for even $l\geq 2$; the other cases as well as (3.5.18a) are proven similarly: Commuting (3.2.20a) with $\Delta^{\frac{l}{2}}$, we have the following:

$$-\partial_t\left(a(t)^{\frac{\omega}{2}}\|G^{-1}-\gamma^{-1}\|^2_{H^l_G}\right)$$
$$\lesssim a^{-3-c\sqrt{\varepsilon}+\frac{\omega}{2}}\left(\sqrt{\mathfrak{E}^{(\leq l)}(\Sigma,\cdot)}+\sqrt{\mathfrak{E}^{(\leq l)}(N,\cdot)}+\sqrt{\mathfrak{E}^{(\leq l-2)}(\mathrm{Ric},\cdot)}\right)\|G^{-1}-\gamma^{-1}\|_{H^l_G}$$

We apply the Young inequality to, for example, the shear term as follows:

$$a^{-3-c\sqrt{\varepsilon}+\frac{\omega}{2}}\sqrt{\mathfrak{E}^{(\leq l)}(\Sigma,\cdot)}\|G^{-1}-\gamma^{-1}\|_{H^l_G}$$
$$\lesssim a^{-3+\frac{\omega}{8}}\cdot a^{\frac{\omega}{2}}\|G^{-1}-\gamma^{-1}\|^2_{H^l_G}+a^{-3+\frac{\omega}{8}-c\sqrt{\varepsilon}}\cdot a^{\frac{\omega}{4}}\mathfrak{E}^{(\leq l)}(\Sigma,\cdot)$$

The statement then follows by inserting Lemma 3.5.4, applying the Young inequality similarly to the other terms and integrating. From comparing (3.2.20b) to (3.2.20a), one quickly observes that the analogous bound for the inverse metric follows identically.
Finally, (3.5.19g) is a direct consequence of applying $\Delta^{\frac{l}{2}}$ (resp. $\nabla\Delta^{\frac{l-1}{2}}$) to the constraint equation (3.2.20j), rearranging and applying the a priori estimates (3.4.1b), (3.4.3b), (3.4.3d), (3.4.1a) and (3.4.3a). In particular, regarding the Vlasov terms, one uses (3.4.27d) and (3.4.27a) to connect their norms to our energy quantities. For the remaining terms, we use estimates as in Lemma 2.4.5, along with the previously mentioned a priori estimates, to convert Sobolev norms into energies up to curvature errors. □

3.6. Vlasov energy estimates

In this section, we collect the necessary energy estimates to obtain the bootstrap improvements in the following section. In particular, for each derivative order and each number of horizontal derivatives in the energy, we prove both a time-scaled estimate, which are all combined in Lemma 3.7.3 to obtain bootstrap improvements for the spacetime and scalar field variables, as well as non-scaled estimates that lose smallness, which we use to improve the bootstrap assumptions on Vlasov matter. Regarding the time-scaled estimates, we scale by a^{ω} and powers there of to indicate scaling necessary within the hierarchy of vertical and horizontal derivatives at a given order (foreshadowing (3.7.2)), and by a^β to indicate additional scaling necessary for the total energy at odd derivative orders (foreshadowing the final line of (3.7.5c)).

3.6.1. Preliminaries. First, we collect some tools that will be used throughout:

REMARK 3.6.1 (Integrating M_t-tangent tensors over the cotangent space). We recall that by Lemma 3.4.6, positive powers of $|v|_G$ can be bounded by $a^{-c\sqrt{\varepsilon}}$, and thus integrals over powers of $|v|_G$ and v^0 on the support of f, f_{FLRW} or $f-f_{FLRW}$ are bounded by $a^{-c\sqrt{\varepsilon}}$. In particular, for any M_t-tangent tensor $\mathfrak{T}$ and any $\alpha>-3$, this implies

$$\int_{T^*M_t}|\mathfrak{T}|_G^2\cdot\langle v\rangle_G^{2\alpha}\,\mathrm{dvol}_{\underline{G}}=\int_{M_t}|\mathfrak{T}|_G^2\int_{T^*_xM_t}\left(\langle v\rangle_G^{2\alpha}\mu_G^{-1}dv\right)\mathrm{dvol}_G\lesssim\|\mathfrak{T}\|^2_{L^2_G(M_t)}\cdot a^{-c\sqrt{\varepsilon}}\,.$$

We use this in the sequel without further comment, usually when needing to estimate high order terms of $G-\gamma$, Σ, N or scalar field quantities.

Next, we collect the core estumate that ensures the transport equation almost conserves Vlasov energies:

LEMMA 3.6.2 (Energy mechanism for the Vlasov equation). *Let $\mu\geq 0$ and $0\leq L\leq 19$. Then, the following holds:*

$$\int_{T^*M_t}\langle v\rangle^{2\mu}\langle \boldsymbol{X}\underline{\nabla}_{vert}^L(f-f_{FLRW}),\underline{\nabla}_{vert}^L(f-f_{FLRW})\rangle_{\underline{G}_0}\mathrm{dvol}_{\underline{G}}\lesssim\varepsilon a^{1-c\sigma}\mathfrak{E}^{(L)}_{\mu,0}(f,\cdot)$$

Similarly, if one additionally has $1\leq K\leq L$, we have

$$\int_{T^*M_t}\langle v\rangle^{2\mu}\langle \boldsymbol{X}\underline{\nabla}_{vert}^{L-K}\underline{\nabla}_{hor}^K f,\underline{\nabla}_{vert}^{L-K}\underline{\nabla}_{hor}^K f\rangle_{\underline{G}_0}\mathrm{dvol}_{\underline{G}}\lesssim\varepsilon a^{1-c\sigma}\mathfrak{E}^{(L)}_{\mu,K}(f,\cdot)\,.$$

PROOF. Both estimates are proved identically, so we only write out the proof for the second. Writing out $\boldsymbol{X}$ (see (3.2.19b)), the left hand side takes the following form:

$$\begin{aligned}&\int_{T^*M_t}\langle v\rangle_G^{2\mu}(v^0)^{2(L-K)}\cdot\\&\qquad\cdot\left\langle\left[-a^{-1}\frac{v^{\sharp j}}{v^0}(N+1)\underline{\nabla}_j+a^{-1}v^0\nabla_jN\,\mathbf{B}^j\right]\underline{\nabla}_{vert}^{L-K}\underline{\nabla}_{hor}^K f,\underline{\nabla}_{vert}^{L-K}\underline{\nabla}_{hor}^K f\right\rangle_{\underline{G}}\mathrm{dvol}_{\underline{G}}\end{aligned}$$

Regarding the first term, we integrate by parts with respect to $\underline{\nabla}_j$, noting that the horizontal covariant derivative of any momentum weight vanishes. For the remaining term, we use Fubini's theorem to replace the integral by one over $\int_M\int_{T^*_xM_t}$ and then integrate by parts in $\mathbf{B}$. This leads to the following after rearranging and applying (3.2.24c):

$$\begin{aligned}&\int_{T^*M_t}\langle v\rangle_G^{2\mu}\langle \boldsymbol{X}\underline{\nabla}_{vert}^L(f-f_{FLRW}),\underline{\nabla}_{vert}^L(f-f_{FLRW})\rangle_{\underline{G}_0}\mathrm{dvol}_{\underline{G}}\\&=\int_{T^*M_t}a^{-1}\langle v\rangle_G^{2\mu}\left(v^{\sharp j}\nabla_jN\right)v^0\left(2\mu\langle v\rangle_G^{-2}+2(L-K)(v^0)^{-2}\right)|\underline{\nabla}_{vert}^{L-K}\underline{\nabla}_{hor}^K f|^2_{\underline{G}_0}\,\mathrm{dvol}_{\underline{G}}\end{aligned}$$

The statement then follows using (3.3.14). □

To be able to apply the above identity, we need to commute ∂_t past the respective covariant Sasaki derivatives, apply (3.2.19a), and then commute $\boldsymbol{X}$ back past them. We have already collected error term formulas for the former in Lemma 3.4.12, and now collect schematic formulas for the latter:

LEMMA 3.6.3 (The commuted rescaled Vlasov equation). *The following commuted analogues of* (3.2.19a) *hold: For* $1 \leq L \leq 19$, *one has*

(3.6.1a)
$$\begin{aligned}\underline{\nabla}_{vert}^L \partial_t f = &\boldsymbol{X}\underline{\nabla}_{vert}^L (f - f_{FLRW}) + a^{-1}(N+1)\frac{(v^0)^2 + v * v}{(v^0)^3} * \underline{\nabla}_{vert}^{L-1}\underline{\nabla}_{hor}(f - f_{FLRW})\\ &+ a^{-1}(N+1)\frac{v*v}{v^0} * \mathrm{Ric}[G] * \underline{\nabla}_{vert}^{L-1}\underline{\nabla}_{hor}(f - f_{FLRW})\\ &+ a^{-1}\frac{v}{v^0} * \nabla N * \underline{\nabla}_{vert}^L (f - f_{FLRW})\\ &+ \mathfrak{F}_{L,0} + \underline{\nabla}_{vert}^L \boldsymbol{X} f_{FLRW}\end{aligned}$$

where one has, for $L \geq 2$,

(3.6.1b)
$$\|(v^0)^L \mathfrak{F}_{L,0}\|_{L^2_{1,\underline{G}}(T^*M)} \lesssim a^{-1-c\sigma}\left(\sqrt{\mathfrak{E}^{(\leq L-1)}_{1,\leq 1}(f,\cdot)} + \|\Gamma - \hat{\Gamma}\|_{L^2_G(M)} + \|G^{-1} - \gamma^{-1}\|_{H^1_G(M)}\right)$$

and said error term vanishes for $L = 1$, *as well as*

$$\begin{aligned}\|(v^0)^L \underline{\nabla}_{vert}^L \boldsymbol{X} f_{FLRW}\|_{L^2_{1,\underline{G}}(T^*M)} \lesssim &\, a^{-3-c\sqrt{\varepsilon}}\left(\sqrt{a^4\mathfrak{E}^{(1)}(N,\cdot)} + \sqrt{\mathfrak{E}^{(0)}(N,\cdot)}\right)\\ &+ a^{-1-c\sqrt{\varepsilon}}\left(\|\Gamma - \hat{\Gamma}\|_{L^2_G(M)} + \|G^{-1} - \gamma^{-1}\|_{L^2_G(M)}\right).\end{aligned} \tag{3.6.2}$$

The latter bound also extends to $L = 0$.
Furthermore, for $0 < K < L$, *we have*

$$\begin{aligned}\underline{\nabla}_{vert}^{L-K}\underline{\nabla}_{hor}^K \partial_t f = &\boldsymbol{X}\underline{\nabla}_{vert}^{L-K}\underline{\nabla}_{hor}^K f + a^{-1}(N+1)\nabla^{K-1}\mathrm{Ric}[G] * \underline{\nabla}_{vert}^{L-K+1}\left(\frac{v*v}{v^0}f\right)\\ &+ a^{-1}(N+1)\left(\frac{1}{v^0} + \frac{v*v}{(v^0)^3}\right) * \underline{\nabla}_{vert}^{L-K-1}\underline{\nabla}_{hor}^{K+1} f\\ &+ a^{-1}\frac{v}{v^0} * \nabla N * \mathbf{B}\underline{\nabla}_{vert}^{L-K-1}\underline{\nabla}_{hor}^K f\\ &+ \mathfrak{F}_{L,K}\end{aligned} \tag{3.6.3a}$$

with

(3.6.3b)
$$\begin{aligned}\|(v^0)^{L-K}\mathfrak{F}_{L,K}\|_{L^2_{1,\underline{G}}(T^*M)} \lesssim &\, a^{-1-c\sigma}\sqrt{\mathfrak{E}^{(\leq L-1)}_{1,\leq K}(f,\cdot)} + a^{-1-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq K+1)}(N,\cdot)}\\ &\underbrace{+\sqrt{\varepsilon}a^{-1-c\sqrt{\varepsilon}}\left(\sqrt{\mathfrak{E}^{(\leq K-1)}(\mathrm{Ric},\cdot)} + \|\Gamma - \hat{\Gamma}\|_{H^1_G(M)} + \|G^{-1} - \gamma^{-1}\|_{H^1_G(M)}\right)}_{\text{if } K\geq 2}\end{aligned}$$

as well as

(3.6.4a)
$$\underline{\nabla}_{hor}^L \partial_t f = \boldsymbol{X}\underline{\nabla}_{hor}^L f + a^{-1}(N+1)\nabla^{L-1}\mathrm{Ric}[G] * \underline{\nabla}_{vert}\left(\frac{v*v}{v^0}f\right) + a^{-1}\frac{v}{v^0}\nabla N * \underline{\nabla}_{hor}^L f + \mathfrak{F}_{L,L}$$

with

$$\begin{aligned}\|\mathfrak{F}_{L,L}\|_{L^2_{1,\underline{G}}(T^*M)} \lesssim\, &\varepsilon a^{-1-c\sigma}\sqrt{\mathfrak{E}^{(L)}_{1,\leq L-1}(f,\cdot)} + a^{-1-c\sigma}\sqrt{\mathfrak{E}^{(\leq L-1)}_{1,\leq L-1}(f,\cdot)}\\ &+ a^{-3-c\sqrt{\varepsilon}}\sqrt{a^4\mathfrak{E}^{(L+1)}(N,\cdot)} + a^{-1-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L)}(N,\cdot)}\\ &+ \underbrace{\sqrt{\varepsilon}a^{-1-c\sqrt{\varepsilon}}\left(\sqrt{\mathfrak{E}^{(\leq L-2)}(\mathrm{Ric},\cdot)} + \|\Gamma-\hat{\Gamma}\|_{H^1_G(M)} + \|G^{-1}-\gamma^{-1}\|_{H^1_G(M)}\right)}_{\text{if } L\geq 2}.\end{aligned} \tag{3.6.4b}$$

Proof. We first consider the purely vertical case (3.6.1a), then the purely horizontal case (3.6.4a), and then briefly (3.6.3a) as a combination of both cases before discussing the respective error bounds.

For (3.6.1a), we first replace f with $(f-f_{FLRW})+f_{FLRW}$, which allows us to collect all reference error terms in $\underline{\nabla}^L_{vert}\boldsymbol{X}f_{FLRW}$. Since $\mathbf{B}_i$ and $\underline{\nabla}_{i+3}$ coincide when acting on purely vertical tensors (see (3.2.17)) and vertical frame derivatives commute, the only error terms generated from the vertical term in computing (3.6.1a) are down to applying the product rule to momentum prefactors. Regarding the horizontal term in $\boldsymbol{X}$, we additionally need to apply (3.4.24a) for $M_2=0$, which yields the term in the second line of (3.6.1a) as the leading error term, while all others are absorbed into the error terms. In particular, we note that this commutation only generates terms of the type in the second line of (3.6.1a), where we can always pull the horizontal derivative back to the front up to error terms as in Lemma 3.4.13 that are absorbed into $\mathfrak{F}_{L,0}$. We also note that, whenever vertical derivatives hit v-prefactors, both a power of v and a derivative are lost, and thus the momentum scaling of lower order error terms always precisely matches that of the respective $\underline{G}_0$-norm, up to estimating v by v^0 in $|\cdot|_G$.

For (3.6.4a), we need to ensure that, after taking horizontal derivatives of $\boldsymbol{X}f$ and applying the product rule, we can still express all terms covariantly despite Christoffel symbols occuring explicitly in the commutator $[\mathbf{A}_i,\mathbf{B}^j]f=-\Gamma[G]^j_{il}\mathbf{B}^lf$, see (3.2.25b). To this end, we commute horizontal derivatives past the vertical part of the operator $\boldsymbol{X}$ as follows.

$$\begin{aligned}\underline{\nabla}_i\left[v^0\nabla_jN\mathbf{B}^jf\right] &= v^0\mathbf{A}_j\left[\nabla_jN\mathbf{B}^jf\right] = v^0\left[\partial_i\nabla_jN\,\mathbf{B}^jf+\nabla_jN\,\mathbf{A}_i\mathbf{B}^jf\right]\\ &= v^0\left[\partial_i\nabla_jN\,\mathbf{B}^jf+\nabla_jN\,\left(\mathbf{B}^j\mathbf{A}_if-\Gamma[G]^j_{il}\mathbf{B}^lf\right)\right]\\ &= v^0\left[\nabla_i\nabla_jN\,\mathbf{B}^jf+\nabla_jN\mathbf{B}^j\underline{\nabla}_if\right]\end{aligned}$$

At higher orders, one additionally incurs terms of the form $v*\mathrm{Riem}[G]$ from connection coefficients with two horizontal indices. Thus, repeating this argument, we obtain the following schematic representation:

$$\begin{aligned}&[\underline{\nabla}^L_{hor},v^0\nabla_jN\,\mathbf{B}^jf]f\\ =&\sum_{K_N+K_f=L,\,K_f\neq L} v*\nabla^{K_N+1}N*\mathbf{B}\underline{\nabla}^{K_f}_{hor}f\end{aligned}$$

$$+ \sum_{K_{\mathrm{Ric}}+K_N+K_f=L-2} v * \nabla^{K_{\mathrm{Ric}}}\mathrm{Ric}[G] * \nabla^{K_N+1}N * \mathbf{B}\underline{\nabla}_{hor}^{K_f} f$$
$$+ \langle \text{low order nonlinear error terms with multiple curvature terms} \rangle$$

Regarding the horizontal term in $\boldsymbol{X}f$, we apply (3.4.24a) with $M_1 = 0, M_2 = L$, again noting that we can rearrange the covariant derivatives up to error terms into the order presented, and can deal with the error terms as in Lemma 3.4.13. Finally, (3.6.3a) follows from combining the ideas from both of these proofs, commuting with horizontal derivative terms as in the purely vertical case and noting that $\mathbf{B}$ and $\underline{\nabla}_{vert}^{L-K}$ commute.

The error bounds (3.6.4b), (3.6.3b) and (3.6.1b) simply follow $L^2 - L^\infty-$estimating the remaining terms and using (3.4.16) for low order Vlasov terms, (3.4.3e) for the curvature terms and (3.4.6a) to control momentum terms as well as the lapse bootstrap assumption from (3.3.13a). In particular, we note that $\mathbf{B}f$ and higher order vertical derivatives of it are only bounded by $a^{-c\sqrt{\varepsilon}}$, leading to the lapse energies in the second line of (3.6.4b). Furthermore, we apply (3.4.26b) to order the derivatives in the Vlasov error terms as needed, and note that the leading curvature terms arise from this procedure.

Finally, for (3.6.2), we recall that v^0, $\langle v\rangle_G$ and $\mathbf{B}f_{FLRW}$ are bounded by $a^{-c\sqrt{\varepsilon}}$ on the support of f_{FLRW} since the latter has bounded support with respect to γ and one has (3.4.3c). Using (3.4.12) for the horizontal term in $\boldsymbol{X}$ then directly leads to the stated bound. □

3.6.2. Vlasov energy estimates. Recalling our conventions on t and ω from the start of this section, we now start by proving the Vlasov integral energy estimate at order 0:

LEMMA 3.6.4 (Zero order Vlasov energy estimates). *The following estimate holds:*

(3.6.5a)
$$\begin{aligned}
a(t)^\omega \mathfrak{E}^{(0)}_{1,0}(f,t) \lesssim{}& \varepsilon^4 + \int_t^{t_0} \left(a(s)^{-3+\frac{\omega}{4}} + \varepsilon\, a^{-3}\right) \cdot a(s)^\omega \mathfrak{E}^{(0)}_{1,0}(f,s)\, ds \\
&+ \int_t^{t_0} a(s)^{-3-c\sqrt{\varepsilon}+\frac{\omega}{4}} \left(\varepsilon^2 \mathfrak{E}^{(0)}(\Sigma,s) + \mathfrak{E}^{(0)}(\phi,s) + a(s)^{4+\frac{\omega}{2}} \|\Gamma - \hat{\Gamma}\|^2_{L^2_G(M_s)}\right) ds \\
&+ \int_t^{t_0} a(s)^{-1-c\sqrt{\varepsilon}+\frac{\omega}{4}} \cdot a(s)^{\frac{\omega}{2}} \|G^{-1} - \gamma^{-1}\|^2_{L^2_G(M_s)}\, ds
\end{aligned}$$

(3.6.5b)
$$\begin{aligned}
\mathfrak{E}^{(0)}_{1,0}(f,t) \lesssim{}& \varepsilon^4 + \int_t^{t_0} \left(a(s)^{-1-c\sigma} + \varepsilon^{\frac{1}{8}} a(s)^{-3}\right) \mathfrak{E}^{(0)}_{1,0}(f,s)\, ds \\
&+ \int_t^{t_0} \varepsilon^{-\frac{1}{8}} a(s)^{-3-c\sqrt{\varepsilon}} \left(\varepsilon^2 \mathfrak{E}^{(0)}(\Sigma,s) + \mathfrak{E}^{(0)}(\phi,s)\right) ds \\
&+ \int_t^{t_0} a(s)^{-1-c\sqrt{\varepsilon}} \left(\|\Gamma - \hat{\Gamma}\|^2_{L^2_G(M_s)} + \|G^{-1} - \gamma^{-1}\|^2_{L^2_G(M_s)}\right) ds
\end{aligned}$$

PROOF. We first prove the unscaled estimate (3.6.5b) before sketching the necessary adaptations to arrive at (3.6.5a):

$$-\partial_t \mathfrak{E}^{(0)}_{1,0}(f,\cdot) = -\int_{T^*M} 2\frac{\partial_t \langle v\rangle_G}{\langle v\rangle_G}\langle v\rangle_G^2 |f-f_{FLRW}|^2\,\mathrm{dvol}_{\underline{G}} \\ -2\int_{T^*M}\langle v\rangle_G^2(f-f_{FLRW})\cdot\partial_t f\,\mathrm{dvol}_{\underline{G}} \tag{3.6.6}$$

The first line can be bounded by $\lesssim \varepsilon a^{-3}\mathfrak{E}^{(0)}_{1,0}(f,\cdot)$ with (3.4.23) and (3.3.13a). We insert the rescaled Vlasov equation (3.2.19a) in the second line and expand as follows:

$$-2\int_{T^*M}\langle v\rangle_G^2(f-f_{FLRW})\cdot\partial_t f\,\mathrm{dvol}_{\underline{G}}$$

$$= -2\int_{T^*M}\langle v\rangle_G^2(f-f_{FLRW})\cdot \boldsymbol{X}(f-f_{FLRW})\,\mathrm{dvol}_{\underline{G}} \tag{3.6.7a}$$

$$-2\int_{T^*M}\langle v\rangle_G^2(f-f_{FLRW})\boldsymbol{X}f_{FLRW} \tag{3.6.7b}$$

(3.6.7a) is bounded by $\varepsilon a^{-3}\mathfrak{E}^{(0)}_{1,0}(f,\cdot)$ due to core energy mechanism from Lemma 3.6.2. Regarding (3.6.7b), we apply Lemma 3.5.4 for $L=0$ to (3.6.2) and collect:

$$\|\boldsymbol{X}f_{FLRW}\|_{L^2_{1,\underline{G}}(T^*M)} \lesssim a^{-1-c\sqrt{\varepsilon}}\left[\|G^{-1}-\gamma^{-1}\|_{L^2_G(M)}+\sqrt{\mathfrak{E}^{(0)}_{1,0}(f,\cdot)}\right] \\ + a^{-3-c\sqrt{\varepsilon}}\left[\sqrt{\varepsilon^2\mathfrak{E}^{(0)}(\Sigma,\cdot)}+\sqrt{\mathfrak{E}^{(0)}(\phi,\cdot)}+a^4\|\Gamma-\hat{\Gamma}\|_{L^2_G(M)}\right]$$

Putting all of the above together and applying the Young inequality on each summand, we obtain

$$-\partial_t\mathfrak{E}^{(0)}_{1,0}(f,\cdot)\lesssim \left(a^{-1-c\sigma}+\varepsilon^{\frac18}a^{-3}\right)\mathfrak{E}^{(0)}_{1,0}(f,\cdot) \\ +\varepsilon^{-\frac18}a^{-3-c\sqrt{\varepsilon}}\left(\varepsilon^2\mathfrak{E}^{(0)}(\Sigma,\cdot)+\mathfrak{E}^{(0)}(\phi,\cdot)+a^4\|\Gamma-\hat{\Gamma}\|^2_{L^2_G}\right) \\ + a^{-1-c\sqrt{\varepsilon}}\|G^{-1}-\gamma^{-1}\|^2_{L^2_G}$$

and thus the result after applying the Gronwall lemma.

For (3.6.5a), we argue along the same lines, with the estimates for (3.6.6) and (3.6.7a) treated identically and where $(-\partial_t a^\omega)\mathfrak{E}^{(0)}(f,\cdot)$ can be dropped since it is nonpositive. Regarding the terms in (3.6.7b), we apply the same estimates, but now distribute the additional Vlasov energy scaling to mitigate the divergence of the other energies:

$$a^{-3}\cdot\left(a^{\frac{\omega}{4}}\|\boldsymbol{X}f_{FLRW}\|_{L^2_{\underline{G}}}\right)\cdot\left(a^{\frac{3\omega}{4}}\sqrt{\mathfrak{E}^{(0)}_{1,0}(f,\cdot)}\right) \\ \lesssim a^{-3+\frac{\omega}{4}}\left(a^\omega\mathfrak{E}^{(0)}_{1,0}(f,\cdot)\right)+a^{-3-c\sqrt{\varepsilon}+\frac{\omega}{4}}\left(\varepsilon^2\mathfrak{E}^{(0)}(\Sigma,\cdot)+\mathfrak{E}^{(0)}(\phi,\cdot)+a^{4+\frac{\omega}{2}}\|\Gamma-\hat{\Gamma}\|^2_{L^2_G}\right) \\ + a^{-1-c\sqrt{\varepsilon}+\frac{\omega}{4}}\cdot a^{\frac{\omega}{2}}\|G^{-1}-\gamma^{-1}\|^2_{L^2_G}\,.$$

□

LEMMA 3.6.5 (Vertical Vlasov energy estimates). *Let $\beta \in \{0,4\}$. Then:*

(3.6.8)

$$\begin{aligned} a(t)^{\beta+\omega}\,\mathfrak{E}^{(L)}_{1,0}(f,t) \lesssim \varepsilon^4 &+ \int_t^{t_0} \left(a(s)^{-1-c\sigma} + a(s)^{-3+\frac{\omega}{4}} + \varepsilon a(s)^{-3}\right) a(s)^{\beta+\omega}\mathfrak{E}^{(L)}_{1,0}(f,s)\,ds \\ &+ \int_t^{t_0} a(s)^{-1-c\sigma-\omega}\cdot a(s)^{\beta+2\omega}\mathfrak{E}^{(\le L)}_{1,\le 1}(f,s)\,ds \\ &+ \int_t^{t_0} a(s)^{-3-c\sqrt{\varepsilon}+\beta+\frac{\omega}{2}}\left(\mathfrak{E}^{(0)}(\phi,s) + \varepsilon^2 a(s)^{\frac{\omega}{4}}\mathfrak{E}^{(0)}(\Sigma,s)\right)\,ds \\ &+ \int_t^{t_0} a(s)^{-1-c\sqrt{\varepsilon}+\beta+\frac{\omega}{4}}\left(a(s)^{\frac{\omega}{2}}\|\Gamma-\hat{\Gamma}\|^2_{L^2_G(M_s)} + a(s)^{\frac{\omega}{2}}\|G^{-1}-\gamma^{-1}\|^2_{L^2_G(M_s)}\right)\,ds \end{aligned}$$

$$\begin{aligned} \mathfrak{E}^{(L)}_{1,0}(f,t) \lesssim \varepsilon^4 &+ \int_t^{t_0}\left(a(s)^{-1-c\sigma} + \varepsilon^{\frac{1}{8}}a(s)^{-3}\right)\mathfrak{E}^{(L)}_{1,0}(f,s)\,ds \\ &+ \int_t^{t_0} a(s)^{-1-c\sigma-2\omega}\cdot a(s)^{2\omega}\mathfrak{E}^{(\le L)}_{1,\le 1}(f,s)\,ds \\ &+ \int_t^{t_0}\varepsilon^{-\frac{1}{8}}a(s)^{-3-c\sqrt{\varepsilon}+\beta}\left(\mathfrak{E}^{(0)}(\phi,s) + \varepsilon^2\mathfrak{E}^{(0)}(\Sigma,s)\right)\,ds \\ &+ \int_t^{t_0} a(s)^{-1-c\sqrt{\varepsilon}}\left(\|\Gamma-\hat{\Gamma}\|^2_{L^2_G(M_s)} + \|G^{-1}-\gamma^{-1}\|^2_{L^2_G(M_s)}\right)\,ds \end{aligned} \tag{3.6.9}$$

PROOF. We only prove (3.6.9) in full since the other cases are proven similarly, keeping in mind that $-\partial_t(a^{\beta+\omega})\mathfrak{E}^{(L)}_{1,0} \le 0$.
Since ∂_t and $\mathbf{B}_i$ commute and since $\underline{\nabla}^L_{vert}\xi = \mathbf{B}^L\xi$ holds, we get the following:

$$\begin{aligned} -\partial_t\mathfrak{E}^{(L)}_{1,0}(f,\cdot) = &-\int_{T^*M}\langle v\rangle^2_G\left(2\frac{\partial_t\langle v\rangle_G}{\langle v\rangle_G} + 2L\frac{\partial_t v^0}{v^0}\right)|\underline{\nabla}^L_{vert}(f-f_{FLRW})|^2_{\underline{G}_0}\mathrm{dvol}_{\underline{G}} \\ &-\int_{T^*M}\langle v\rangle^2_G(v^0)^{2L}\partial_t G *_{\underline{G}} \underline{\nabla}^L_{vert}(f-f_{FLRW}) *^L_{\underline{G}} \underline{\nabla}_{vert}(f-f_{FLRW})\,\mathrm{dvol}_{\underline{G}} \\ &-2\int_{T^*M}\langle v\rangle^2_G\langle\underline{\nabla}_{vert}(f-f_{FLRW}),\underline{\nabla}_{vert}\partial_t f\rangle_{\underline{G}_0}\,\mathrm{dvol}_{\underline{G}} \end{aligned}$$

The first two lines are bounded by $\varepsilon a^{-3}\mathfrak{E}^{(1)}_{1,0}(f,\cdot)$ up to constant, with the first line due to (3.4.23) and (3.3.14), and with the second since $|\partial_t G|_G \lesssim \varepsilon a^{-3}$ holds by (3.4.1b) and (3.3.14) applied to (3.2.20a). Regarding the final line, we insert (3.6.1a):

$$-2\int_{T^*M}\langle v\rangle^2_G\langle\underline{\nabla}^L_{vert}(f-f_{FLRW}),\underline{\nabla}^L_{vert}\partial_t f\rangle_{\underline{G}_0}\,\mathrm{dvol}_{\underline{G}}$$

$$\lesssim -2\int_{T^*M}\langle v\rangle^2_G(v^0)^{2L}\langle\underline{\nabla}^L_{vert}(f-f_{FLRW}),\underline{\nabla}^L_{vert}\boldsymbol{X}f_{FLRW})\rangle_{\underline{G}}\mathrm{dvol}_{\underline{G}} \tag{3.6.10a}$$

$$+\int_{T^*M}\langle v\rangle^2_G\Big[a^{-1}|N+1|\left(1+|v|^2_G|\mathrm{Ric}[G]|_G\right)|\underline{\nabla}^{L-1}_{vert}\underline{\nabla}_{hor}(f-f_{FLRW})|_{\underline{G}_0} \tag{3.6.10b}$$

$$+a^{-1}|\nabla N|_G|\underline{\nabla}^L_{vert}(f-f_{FLRW})|_{\underline{G}_0}\Big]|\underline{\nabla}^L_{vert}(f-f_{FLRW})|_{\underline{G}_0}\,\mathrm{dvol}_{\underline{G}} \tag{3.6.10c}$$

$$+\left(\|\mathfrak{F}_{L,0}\|_{L^2_{1,\underline{G}}} + \|\underline{\nabla}^L_{vert}\boldsymbol{X}f_{FLRW}\|_{L^2_{1,\underline{G}}}\right)\sqrt{\mathfrak{E}^{(L)}_{1,0}(f,\cdot)} \tag{3.6.10d}$$

Going through the right hand side line by line, (3.6.10a) can be bounded by $\varepsilon a^{-3}\mathfrak{E}^{(L)}_{1,0}(f,\cdot)$ by Lemma 3.6.2. For (3.6.10b), we apply (3.4.3e) and (3.4.6a) on the prefactor to estimate it as follows:

$$|(3.6.10b)| \lesssim a^{-1-c\sqrt{\varepsilon}}\left(\sqrt{\mathfrak{E}^{(L)}_{1,1}(f,\cdot)}+\|\underline{\nabla}^{L-1}_{vert}\underline{\nabla}_{hor}f_{FLRW}\|_{L^2_{1,\underline{G}_0}}\right)\sqrt{\mathfrak{E}^{(L)}_{1,0}(f,\cdot)}$$

For the reference term, we apply (3.4.11a) for $K=1$, and obtain the bound

$$\begin{aligned}|(3.6.10b)| \lesssim & a^{-1-c\sqrt{\varepsilon}}\mathfrak{E}^{(L)}_{1,1}(f,\cdot)+a^{-1-c\sqrt{\varepsilon}}\left(\|\Gamma-\hat{\Gamma}\|^2_{L^2_G(M_t)}+\|G^{-1}-\gamma^{-1}\|^2_{L^2_G(M_t)}\right)\\ &+a^{-1-c\sqrt{\varepsilon}}\mathfrak{E}^{(L)}_{1,0}(f,\cdot)\,.\end{aligned}$$

For (3.6.10c), we apply (3.4.6a) and (3.3.14) to bound the terms by $\varepsilon a^{1-c\sigma}\mathfrak{E}^{(L)}_{1,0}$. Finally, we apply (3.6.1b) and (3.6.2) to bound (3.6.10d), where we insert Lemma 3.5.4 at order $l=0$ for the occuring lapse energy terms and then apply the Young inequality. The integral estimate (3.6.8) then follows by integrating.
For $\omega'=\omega$, we again use the additional scaling to prevent otherwise strongly divergent energy terms to diverge more than a^{-3} as in the zero order case. For the Vlasov energy gaining a horizontal derivative, we distribute the scaling as follows:

$$a^\omega\sqrt{\mathfrak{E}^{(L)}_{1,0}(f,\cdot)}\sqrt{\mathfrak{E}^{(L)}_{1,1}(f,\cdot)}=\sqrt{a^\omega\mathfrak{E}^{(L)}_{1,0}(f,\cdot)}\cdot\sqrt{a^{-\omega}\cdot a^{2\omega}\mathfrak{E}^{(L)}_{1,1}(f,\cdot)}$$

The respective proofs with $\beta=4$ instead of $\beta=0$ are identical. □

LEMMA 3.6.6 (Mixed Vlasov energy estimates). *Let $0<K<L$ and $\beta\in\{0,4\}$. Then, the following estimates hold:*

(3.6.11)

$$\begin{aligned}a(t)^{\beta+(K+1)\omega}\,\mathfrak{E}^{(L)}_{1,K}(f,t)\lesssim&\ \varepsilon^4+\int_t^{t_0}\left(a(s)^{-3+\frac{\omega}{4}}+\varepsilon^{\frac18}a(s)^{-3}\right)a(s)^{\beta+(K+1)\omega}\mathfrak{E}^{(L)}_{1,K}(f,s)\,ds\\ &+\int_t^{t_0}a(s)^{-1-\omega}\cdot a(s)^{\beta+(K+2)\omega}\mathfrak{E}^{(L)}_{1,K+1}(f,s)\,ds\\ &+\int_t^{t_0}\left(a(s)^{-1-c\sqrt{\varepsilon}+\frac{\omega}{2}}+\varepsilon^2a(s)^{-3-c\sqrt{\varepsilon}+\frac{\omega}{2}}\right)\cdot a(s)^{\beta+((K-1)+1)\omega}\mathfrak{E}^{(\leq L)}_{1,\leq K-1}(f,s)\,ds\\ &+\int_t^{t_0}a(s)^{-3-c\sqrt{\varepsilon}+\frac{\omega}{2}}a(s)^\beta\left(a(s)^{\frac{\omega}{2}}\mathfrak{E}^{(\leq K)}(\Sigma,s)+\mathfrak{E}^{(\leq K)}(\phi,s)\right)\,ds\\ &+\int_t^{t_0}\left(a(s)^{-1-c\sqrt{\varepsilon}+K\omega}+\varepsilon a(s)^{-3-c\sqrt{\varepsilon}+K\omega}\right)\cdot a(s)^{\beta+\frac{\omega}{2}}\mathfrak{E}^{(\leq K-1)}(\mathrm{Ric},s)\,ds\\ &+\int_t^{t_0}\left(a(s)^{-1-c\sqrt{\varepsilon}+\frac{(2K+1)\omega}{2}}+\varepsilon a(s)^{-3-c\sqrt{\varepsilon}+K\omega}\right)\cdot\\ &\quad\cdot\left(a(s)^{\beta+\frac{\omega}{2}}\|\Gamma-\hat{\Gamma}\|^2_{H^1_G(M_s)}+a(s)^{\beta+\frac{\omega}{2}}\|G^{-1}-\gamma^{-1}\|^2_{L^2_G(M_s)}\right)\,ds\end{aligned}$$

(3.6.12)

$$\mathfrak{E}^{(L)}_{1,K}(f,t)\lesssim\varepsilon^4+\int_t^{t_0}\left(a(s)^{-1-c\sigma}+\varepsilon^{\frac18}a(s)^{-3}\right)\mathfrak{E}^{(L)}_{1,K}(f,s)+a(s)^{-1}\mathfrak{E}^{(L)}_{1,K+1}(f,s)\,ds$$

$$+\int_t^{t_0}\left(\varepsilon^{\frac{7}{8}}a(s)^{-3-c\sqrt{\varepsilon}}+a(s)^{-1-c\sqrt{\varepsilon}}\right)\left(\mathfrak{E}^{(L)}_{1,K-1}(f,s)+\mathfrak{E}^{(\leq L-1)}_{1,\leq K-1}(f,s)\right)ds$$
$$+\int_t^{t_0}\varepsilon^{-\frac{1}{8}}a(s)^{-3-c\sqrt{\varepsilon}}\left(\mathfrak{E}^{(\leq K)}(\Sigma,s)+\mathfrak{E}^{(\leq K)}(\phi,s)\right)ds$$
$$+\int_t^{t_0}\left(a(s)^{-1-c\sqrt{\varepsilon}}+\varepsilon^{\frac{15}{8}}a(s)^{-3-c\sqrt{\varepsilon}}\right)\mathfrak{E}^{(\leq K-1)}(\mathrm{Ric},s)\,ds$$
$$+\int_t^{t_0}\left(a(s)^{-1-c\sqrt{\varepsilon}}+\varepsilon^{\frac{15}{8}}a(s)^{-3-c\sqrt{\varepsilon}}\right)\left(\|\Gamma-\hat{\Gamma}\|^2_{H^1_G(M_s)}+\|G^{-1}-\gamma^{-1}\|^2_{L^2_G(M_s)}\right)ds$$

For $K<10$, the curvature energies can be dropped.

Proof. We again focus on proving the unscaled energy inequality (3.6.12), and touch on how the scale factors are distributed for (3.6.11) at the end. One computes:

$$-\partial_t\int_{T^*M}|v|^2_G|\underline{\nabla}^{L-K}_{vert}\underline{\nabla}^K_{hor}f|^2_{\underline{G}_0}\,\mathrm{dvol}_{\underline{G}}$$
$$=-\int_{T^*M}\langle v\rangle^2_G\left(2\frac{\partial_t\langle v\rangle_G}{\langle v\rangle_G}+2(L-K)\frac{\partial_t v^0}{v^0}\right)|\underline{\nabla}^{L-K}_{vert}\underline{\nabla}^K_{hor}f|^2_{\underline{G}_0}\mathrm{dvol}_{\underline{G}} \tag{3.6.13a}$$
$$-\int_{T^*M}\langle v\rangle^2_G(v^0)^{2(L-K)}(\partial_t\underline{G}^{-1})*_{\underline{G}}\underline{\nabla}^{L-K}_{vert}\underline{\nabla}^K_{hor}f*_{\underline{G}}\underline{\nabla}^{L-K}_{vert}\underline{\nabla}^K_{hor}f\,\mathrm{dvol}_{\underline{G}} \tag{3.6.13b}$$
$$-2\int_{T^*M}\langle v\rangle^2_G\langle[\partial_t,\underline{\nabla}^{L-K}_{vert}\underline{\nabla}^K_{hor}]f,\underline{\nabla}^{L-K}_{vert}\underline{\nabla}^K_{hor}f\rangle_{\underline{G}_0}\,\mathrm{dvol}_{\underline{G}} \tag{3.6.13c}$$
$$-2\int_{T^*M}\langle v\rangle^2_G\langle\underline{\nabla}^{L-K}_{vert}\underline{\nabla}^K_{hor}\partial_t f,\underline{\nabla}^{L-K}_{vert}\underline{\nabla}^K_{hor}f\rangle_{\underline{G}_0}\,\mathrm{dvol}_{\underline{G}} \tag{3.6.13d}$$

(3.6.13a)-(3.6.13b) can be bounded by $\varepsilon a^{-3}\mathfrak{E}^{(L)}_{1,K}(f,\cdot)$ up to constant as in the purely vertical case. Regarding (3.6.13c), (3.4.24b) implies the following:

$$\begin{aligned}&\left|\langle[\partial_t,\underline{\nabla}^{L-K}_{vert}\underline{\nabla}^K_{hor}]f,\underline{\nabla}^{L-K}_{vert}\underline{\nabla}^K_{hor}f\rangle_{\underline{G}_0}\right|\\ \lesssim\,&|\underline{\nabla}^{L-K}_{vert}\underline{\nabla}^K_{hor}f|_{\underline{G}_0}\cdot\left[\frac{|v|_G}{v^0}|\underline{\nabla}^{L-K+1}_{vert}f|_{\underline{G}_0}|\nabla^{K-2}\partial_t\mathrm{Ric}[G]|_G\right.\\ &\left.+\frac{|v|_G}{v^0}|\partial_t\Gamma[G]|_G|\underline{\nabla}^{L-K+1}_{vert}\underline{\nabla}^{K-1}_{hor}f|_{\underline{G}_0}+(v^0)^{L-K}|\Xi_{L-K,K,t}f|_{\underline{G}}\right]\end{aligned}$$

Recall that, by integration by parts and the Young inequality, one has

$$a^{-3-c\sqrt{\varepsilon}}\mathfrak{E}^{(l-1)}(\Sigma,\cdot)\lesssim a^{-3}\mathfrak{E}^{(l)}(\Sigma,\cdot)+a^{-3-2c\sqrt{\varepsilon}}\mathfrak{E}^{(l-2)}(\Sigma,\cdot)$$

and the analogous estimates for $\mathfrak{E}^{(l-1)}(N,\cdot)$ and $\mathfrak{E}^{(l-1)}(\mathrm{Ric},\cdot)$. Applying (3.4.3b) and (3.3.14) to (3.2.20e) and (3.2.20d), as well as updating $c>0$, one then gets

$$\begin{aligned}&\|\partial_t\mathrm{Ric}[G]\|_{H^{l-2}_G}+\|\partial_t\Gamma[G]\|_{H^{l-1}_G}\\ \lesssim\,&a^{-3}\left(\sqrt{\mathfrak{E}^{(l)}(\Sigma,\cdot)}+\sqrt{\mathfrak{E}^{(l)}(N,\cdot)}\right)+a^{-3-c\sqrt{\varepsilon}}\left(\sqrt{\mathfrak{E}^{(\leq l-2)}(\Sigma,\cdot)}+\sqrt{\mathfrak{E}^{(\leq l-2)}(N,\cdot)}\right)\\ &+\varepsilon a^{-3-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq l-2)}(\mathrm{Ric},\cdot)}\end{aligned}$$

as well as

$$\|\partial_t \mathrm{Ric}[G]\|_{C^{10}_G(M)} + \|\partial_t \Gamma[G]\|_{C^{11}_G(M)} \lesssim \varepsilon a^{-3-c\sqrt{\varepsilon}} .$$

Further, the supremum norm over $\xi = f$ in (3.4.25b) can be bounded by $a^{-c\sqrt{\varepsilon}}$ due to (3.4.16) whenever it occurs. Altogether, this implies

$$\begin{aligned}|(3.6.13c)| \lesssim \varepsilon a^{-3-c\sqrt{\varepsilon}} &\left(\sqrt{\mathfrak{E}^{(L)}_{1,K-1}(f,\cdot)} + \sqrt{\mathfrak{E}^{(\leq L-1)}_{1,\leq K-1}(f,\cdot)}\right)\\ &+ a^{-3-c\sqrt{\varepsilon}}\left(\sqrt{\mathfrak{E}^{(\leq K)}(\Sigma,\cdot)} + \sqrt{\mathfrak{E}^{(\leq K)}(N,\cdot)} + \varepsilon\sqrt{\mathfrak{E}^{(\leq K-2)}(\mathrm{Ric},\cdot)}\right) .\end{aligned}$$

Considering (3.6.13d), we can apply (3.6.3a) and using $|v|_G \leq v^0$ as well as $|N+1| \lesssim 1$ to obtain the following:

$$(3.6.13d) + 2\int_{T^*M} \langle v\rangle_G^2 \langle \boldsymbol{X}\underline{\nabla}^{L-K}_{vert}\underline{\nabla}^K_{hor} f, \underline{\nabla}^{L-K}_{vert}\underline{\nabla}^K_{hor} f\rangle \mathrm{dvol}_{\underline{G}}$$

(3.6.14a)

$$\lesssim \int_{T^*M} \langle v\rangle_G^2 \Big[a^{-1}|\nabla^{K-1}\mathrm{Ric}[G]|_G \cdot (v^0)^{(L-K)} \left|\underline{\nabla}^{L-K+1}_{vert}\left(\frac{v*v}{v^0}f\right)\right|_{\underline{G}} + a^{-1}|\underline{\nabla}^{L-K-1}_{vert}\underline{\nabla}^{K+1}_{hor} f|_{\underline{G}_0}$$

(3.6.14b)

$$+ a^{-1}(v^0)^{L-K}|\nabla N|_G |\mathbf{B}\underline{\nabla}^{L-K-1}_{vert}\underline{\nabla}^K_{hor} f|_{\underline{G}} + (v^0)^{L-K}|\mathfrak{F}_{L,K}|_{\underline{G}}\Big] \cdot |\underline{\nabla}^{L-K}_{vert}\underline{\nabla}^K_{hor} f|_{\underline{G}_0}\,\mathrm{dvol}_{\underline{G}}$$

The extra term on the left hand side is dealt with using Lemma 3.6.2. Using (3.4.6a) and (3.4.16), the vertical derivative factor in the first term of (3.6.14a) can be estimated by $a^{-c\sqrt{\varepsilon}}$. Regarding the first term in (3.6.14b), note that one schematically has

$$\mathbf{B}\underline{\nabla}^{L-K-1}_{vert}\underline{\nabla}^K_{hor} f = \underline{\nabla}^{L-K}_{vert}\underline{\nabla}^K_{hor} f + \Gamma[G] * \underline{\nabla}^{L-1} f,$$

where the second term always contains horizontal derivatives except for the case $K = 1$. Then, one expands the term as follows:

$$|(v^0)^{L-1}\Gamma[G] * \underline{\nabla}^{L-1}_{vert} f|_{\underline{G}} \lesssim a^{-c\sqrt{\varepsilon}}\left(|\underline{\nabla}^{L-1}_{vert}(f - f_{FLRW})|_{\underline{G}_0} + |\underline{\nabla}^{L-1}_{vert} f_{FLRW}|_{\underline{G}_0}\right) .$$

The former can be related back to $\mathfrak{E}^{(L-1)}_{1,0}(f,\cdot)$ in estimation and the latter is simply bounded uniformly by $a^{-c\sqrt{\varepsilon}}$. Else, this lower order term can be bounded by $\sqrt{\mathfrak{E}^{(L-1)}_{1,\leq K-1}(f,\cdot)}$ in estimation. We can treat $\mathbf{B}\underline{\nabla}^{L-K}_{vert}\underline{\nabla}^{K-1}_{hor} f$ identically, except that the case of only vertical derivatives appearing in error terms now happens for $K \leq 2$. Thus and applying (3.4.3e) and (3.4.16) as well as (3.3.13a) for the lapse, we obtain

$$\begin{aligned}(3.6.13d) \lesssim &\left(a^{-1-c\sigma} + \varepsilon^{\frac{1}{8}} a^{-3}\right)\mathfrak{E}^{(L)}_{1,K}(f,\cdot) + a^{-1}\mathfrak{E}^{(L)}_{1,K+1}(f,\cdot) + \varepsilon^{\frac{15}{8}} a^{-3-c\sqrt{\varepsilon}}\mathfrak{E}^{(L)}_{1,K-1}(f,\cdot)\\ &+ \varepsilon a^{-3-c\sqrt{\varepsilon}}\mathfrak{E}^{(\leq L-1)}_{1,\leq K-1}(f,\cdot) + a^{-1-c\sqrt{\varepsilon}}\mathfrak{E}^{(K-1)}(\mathrm{Ric},\cdot) + a^{-3-c\sqrt{\varepsilon}} a^4 \mathfrak{E}^{(1)}(N,\cdot)\\ &+ \|(v^0)^{(L-K)}\mathfrak{F}_{L,K}\|_{L^2_{1,\underline{G}}}\sqrt{\mathfrak{E}^{(L)}_{1,K}(f,\cdot)}\end{aligned}$$

Note that the last term in the second line comes from the discussed error terms arising from replacing $\mathbf{B}$ with $\underline{\nabla}_{vert}$ at low horizontal orders.

The statement now follows by inserting (3.6.3b), combining the bounds for (3.6.13a)-(3.6.13d), inserting Lemma 3.5.4 and integrating.

For the scaled version, the main point is that one needs to be careful to ensure that $\mathfrak{E}^{(L)}_{1,K-1}(f,\cdot)$ terms do not enter at a power worse than a^{-3}, which is the case in the unscaled version. To this end, we distribute the scaling as follows:

$$a^{-3-c\sqrt{\varepsilon}+(K+1)\omega}\sqrt{\mathfrak{E}^{(L)}_{1,K}(f,\cdot)}\sqrt{\mathfrak{E}^{(L)}_{1,K-1}(f,\cdot)} = a^{-3+\frac{\omega}{2}}\sqrt{a^{(K+1)\omega}\mathfrak{E}^{(L)}_{1,K}(f,\cdot)}\sqrt{a^{K\omega-2c\sqrt{\varepsilon}}\mathfrak{E}^{(L)}_{1,K}(f,\cdot)}$$

Then, we apply the Young inequality and update c. We note that one never encounters more than K horizontal derivatives in the error terms and thus can never lose scaling. For the non-Vlasov energies, we distribute scaling as in the previous arguments. Again, replacing $\beta=0$ with $\beta=4$ goes through identically. $\square$

LEMMA 3.6.7 (Horizontal Vlasov energy estimates). *Let $\beta\in\{0,4\}$ and $L\geq 2$. Then:*

(3.6.15)

$$\begin{aligned}
a(t)^{\beta+(L+1)\omega}\mathfrak{E}^{(L)}_{1,L}(f,t) \lesssim{}& \varepsilon^4 + \int_t^{t_0}\left(a(s)^{-3+\frac{\omega}{2}}+\varepsilon^{\frac18}a(s)^{-3}\right)a(s)^{\beta+(L+1)\omega}\mathfrak{E}^{(L)}_{1,L}(f,s)\,ds\\
&+\int_t^{t_0}\varepsilon a(s)^{-3-c\sqrt{\varepsilon}+\frac{\omega}{2}}\cdot a(s)^{\beta+L\omega}\mathfrak{E}^{(\leq L)}_{1,\leq L-1}(f,s)\,ds\\
&+\int_t^{t_0}a(s)^{-3-c\sqrt{\varepsilon}+\omega}\cdot a(s)^{\beta}\left(a(s)^{\frac{\omega}{4}}\mathfrak{E}^{(\leq L)}(\Sigma,s)+\mathfrak{E}^{(\leq L)}(\phi,s)\right)ds\\
&+\int_t^{t_0}a(s)^{-3-c\sqrt{\varepsilon}+\frac{(2L-1)\omega}{2}}\cdot a(s)^{4+\beta+\omega}\mathfrak{E}^{(L-1)}(\mathrm{Ric},s)\,ds\\
&+\int_t^{t_0}a(s)^{-3-c\sqrt{\varepsilon}+L\omega}\,a(s)^{\beta+\frac{\omega}{2}}\mathfrak{E}^{(L-2)}(\mathrm{Ric},s)\,ds\\
&+\int_t^{t_0}\left(\varepsilon a(s)^{-3-c\sqrt{\varepsilon}+\omega}+a(s)^{-1-c\sqrt{\varepsilon}}\right)a(s)^{\beta+\frac{\omega}{2}}\mathfrak{E}^{(\leq L-4)}(\mathrm{Ric},s)\,ds\\
&+\int_t^{t_0}\left(a(s)^{-1-c\sqrt{\varepsilon}+\frac{(2L+1)\omega}{2}}+\varepsilon a(s)^{-3-c\sqrt{\varepsilon}+L\omega}\right)\cdot\\
&\qquad\cdot\left(a(s)^{\beta+\frac{\omega}{2}}\|\Gamma-\hat{\Gamma}\|^2_{H^1_G(M_s)}+a(s)^{\beta+\frac{\omega}{2}}\|G^{-1}-\gamma^{-1}\|^2_{H^2_G(M_s)}\right)ds
\end{aligned}$$

(3.6.16)

$$\begin{aligned}
\mathfrak{E}^{(L)}_{1,L}(f,t) \lesssim{}& \varepsilon^4 + \int_t^{t_0}\left(a(s)^{-1-c\sigma}+\varepsilon^{\frac18}a(s)^{-3}\right)\mathfrak{E}^{(L)}_{1,L}(f,s)+\varepsilon^{\frac{15}{8}}a(s)^{-3-c\sqrt{\varepsilon}}\mathfrak{E}^{(L)}_{1,L-1}(f,s)\,ds\\
&+\int_t^{t_0}\varepsilon^{-\frac18}a(s)^{-3-c\sqrt{\varepsilon}}\left(\mathfrak{E}^{(\leq L)}(\Sigma,s)+\mathfrak{E}^{(\leq L)}(\phi,s)\right)ds\\
&+\int_t^{t_0}\varepsilon^{\frac78}a(s)^{-3-c\sqrt{\varepsilon}}\cdot\left(a(s)^4\mathfrak{E}^{(L-1)}(\mathrm{Ric},s)+\mathfrak{E}^{(\leq L-2)}(\mathrm{Ric},s)\right)ds
\end{aligned}$$

For $L\leq 10$, the curvature energies can be dropped. For $L=1$, one has

(3.6.17)

$$
\begin{aligned}
a(t)^{4+2\omega}\mathfrak{E}^{(1)}_{1,1}(f,t) \lesssim\, & \varepsilon^4+\int_t^{t_0}\left(a(s)^{-1-c\sqrt{\varepsilon}}+a(s)^{-3+\frac{\omega}{2}}+\varepsilon^{\frac18}a(s)^{-3}\right)\cdot a(s)^{4+2\omega}\mathfrak{E}^{(1)}_{1,1}(f,s)\,ds\\
&+\int_t^{t_0}\varepsilon a(s)^{-3-c\sqrt{\varepsilon}+\frac{\omega}{2}}\cdot a(s)^{4+\omega}\mathfrak{E}^{(1)}_{1,0}(f,s)\,ds\\
&+\int_t^{t_0}a(s)^{-3-c\sqrt{\varepsilon}+\frac54\omega}\cdot a(s)^4\left(a(s)^{\frac{\omega}{4}}\mathfrak{E}^{(1)}(\Sigma,s)+\mathfrak{E}^{(1)}(\phi,s)\right)\,ds\\
&+\int_t^{t_0}a(s)^{-1-c\sqrt{\varepsilon}+\frac32\omega}\cdot a(s)^{4+\frac{\omega}{2}}\|\Gamma-\hat{\Gamma}\|^2_{L^2_G(M_s)}\,ds\\
&+\int_t^{t_0}a(s)^{-1-c\sqrt{\varepsilon}+\frac32\omega}\cdot a(s)^{\frac{\omega}{2}}\|G^{-1}-\gamma^{-1}\|^2_{L^2_G(M_s)}\,ds
\end{aligned}
$$

(3.6.18)

$$
\begin{aligned}
\mathfrak{E}^{(1)}_{1,1}(f,t)\lesssim\, & \varepsilon^4+\int_t^{t_0}\left(a(s)^{-1-c\sigma}+\varepsilon^{\frac18}a(s)^{-3}\right)\mathfrak{E}^{(1)}_{1,1}(f,s)+\varepsilon^{\frac{15}{8}}a(s)^{-3-c\sqrt{\varepsilon}+\omega}\mathfrak{E}^{(1)}_{1,0}(f,s)\,ds\\
&+\int_t^{t_0}\varepsilon^{-\frac18}a(s)^{-3-c\sqrt{\varepsilon}}\left(\mathfrak{E}^{(\leq 1)}(\Sigma,s)+\mathfrak{E}^{(\leq 1)}(\phi,s)\right)\,ds\\
&+\int_t^{t_0}a(s)^{-1-c\sqrt{\varepsilon}}\left(\|\Gamma-\hat{\Gamma}\|^2_{L^2_G(M_s)}+\|G^{-1}-\gamma^{-1}\|^2_{L^2_G(M_s)}\right)\,ds
\end{aligned}
$$

Proof. The proof of these estimates is analogous to that of Lemma 3.6.6 since (3.6.4a) is just a simpler version of (3.6.3a), and the commutator formula (3.4.24b) is also used for $M_1=0$ in a similar fashion to the previous proof. As a difference in estimation, we note that the high order curvature term present in (3.6.4a) is now of order $L-1$, and thus needs to be thought of as being of order $L+1$ in the metric, i.e., of higher order than the Vlasov energy itself. Consequently, the resulting energy controlling this term needs to be additionally scaled to allow the argument to close, which makes the resulting energy term borderline. □

3.7. Improving the bootstrap assumptions

In this section, we combine the energy estimates from Sections 3.5, and 3.6 to improve the bootstrap assumption (3.3.13a) – recall that (3.3.13b) is already improved due to (3.4.16). To this end, we first derive improved energy bounds for the spacetime and scalar field variables and then use these to derive bounds for Vlasov energies. We then translate these bounds to Sobolev norm bounds, and finally to bounds on $\mathcal{C}$.
In the following, we again assume $t\in(t_{Boot},t_0]$ and that $\omega\gg\sigma$ is a suitably small constant, see also the start of Section 5.

To deal with linear borderline terms in the energy estimates for the shear (see Lemma 3.5.9) and Bel-Robinson variables (see Lemma 3.5.8), we need to extend Lemma 2.7.1 to ensure that the error terms essentially have favourable sign even for flat and spherical spatial geometry:

LEMMA 3.7.1. *Let $L \in 2\mathbb{N}$. Then, the following estimate is satisfied:*

(3.7.1)

$$\int_M \left[16\pi C^2 a^{-3}(N+1)\langle\Delta^{\frac{L}{2}}\mathcal{E},\Delta^{\frac{L}{2}}\Sigma\rangle_G + 8\pi C^2\frac{\dot a}{a}(N+1)|\Delta^{\frac{L}{2}}\Sigma|_G^2 + 6\frac{\dot a}{a}(N+1)|\Delta^{\frac{L}{2}}\mathcal{E}|_G^2\right]\mathrm{dvol}_G$$
$$\begin{cases}\geq 0 & \kappa\leq 0\\ \gtrsim -a^{-1}\left[4\pi C^2\mathfrak{E}^{(L)}(\Sigma,\cdot)+\mathfrak{E}^{(L)}(W,\cdot)\right] & \kappa>0\end{cases}$$

PROOF. Firstly, the bootstrap assumption (3.3.14) implies $N+1>0$ for $\varepsilon>0$ small enough. For $\kappa\leq 0$, we have $a^{-3}\leq\frac{3}{4\pi C^2}\frac{\dot a}{a}$ by (3.2.3a) since the Vlasov density is nonnegative. Then, the statement follows directly from this and the Young inequality as in Lemma 2.7.1. Else, one has

$$a^{-3}\leq\frac{3}{4\pi C^2}\frac{\sqrt{\dot a^2+\kappa}}{a}\leq\frac{3}{4\pi C^2}\left(\frac{\dot a}{a}+\sqrt{\kappa}a^{-1}\right)$$

and thus the integrand on the left hand side in (3.7.1) is bounded from below by

$$\gtrsim -a^{-1}|\Delta^{\frac{L}{2}}\Sigma|_G|\Delta^{\frac{L}{2}}\mathcal{E}|_G\,,$$

implying the statement. □

To obtain bounds on the spacetime and scalar field energies, we need to combine Vlasov energies at order L with appropriate scaling as follows:

DEFINITION 3.7.2 (Total scaled Vlasov energies). For $0\leq L\leq 19$, we define

$$\mathfrak{E}^{(L)}_{total,Vl}=\sum_{K=0}^{L}a^{(K+1)\omega}\mathfrak{E}^{(L)}_{1,K}(f,\cdot)\,. \tag{3.7.2}$$

With this in hand, we can combine the estimates from Section 3.6 as follows:

LEMMA 3.7.3 (Total scaled Vlasov energy estimate). *For $L=1$, one has*

(3.7.3)

$$\begin{aligned}a(t)^4\mathfrak{E}^{(1)}_{total,Vl}(t)\lesssim\varepsilon^4&+\int_t^{t_0}\left(a(s)^{-1-\omega-c\sigma}+a(s)^{-3+\frac{\omega}{4}-c\sqrt{\varepsilon}}+\varepsilon^{\frac18}a(s)^{-3}\right)a(s)^4\mathfrak{E}^{(1)}_{total,Vl}(s)\,ds\\&+\int_t^{t_0}a(s)^{-3-c\sqrt{\varepsilon}+\frac{\omega}{2}}\cdot a(s)^4\left(\mathfrak{E}^{(\leq1)}(\phi,s)+a(s)^{\frac{\omega}{4}}\mathfrak{E}^{(\leq1)}(\Sigma,s)+\varepsilon^2\mathfrak{E}^{(0)}(\Sigma,s)\right)ds\\&+\int_t^{t_0}a(s)^{-1-c\sqrt{\varepsilon}}\left(a(s)^{4+\frac{\omega}{2}}\|\Gamma-\hat\Gamma\|^2_{L^2_G(M_s)}+a(s)^{4+\frac{\omega}{2}}\|G^{-1}-\gamma^{-1}\|^2_{L^2_G(M_s)}\right)ds\end{aligned}$$

Let $L\geq 2$. Then, the following holds:

(3.7.4)

$$\begin{aligned}a(t)^\beta\mathfrak{E}^{(L)}_{total,Vl}(t)\lesssim&\,\varepsilon^4+\int_t^{t_0}\left(a(s)^{-1-\omega-c\sigma}+a(s)^{-3+\frac{\omega}{4}}+\varepsilon^{\frac18}a(s)^{-3}\right)a(s)^\beta\mathfrak{E}^{(L)}_{total,Vl}(s)\,ds\\&+\int_t^{t_0}a(s)^{-3-c\sqrt{\varepsilon}+\frac{\omega}{4}}\cdot a(s)^\beta\left(\mathfrak{E}^{(L)}(\phi,s)+a(s)^{\frac{\omega}{4}}\mathfrak{E}^{(L)}(\Sigma,s)\right)ds\end{aligned}$$

$$
\begin{aligned}
&+\int_t^{t_0} \varepsilon a(s)^{-3-c\sqrt{\varepsilon}+(L-1)\omega}\cdot a(s)^{4+\beta}\mathfrak{E}^{(L-1)}(\mathrm{Ric},s)\,ds\\
&+\int_t^{t_0} a(s)^{-3-c\sqrt{\varepsilon}+L\omega}\cdot a(s)^{\beta}\mathfrak{E}^{(L-2)}(\mathrm{Ric},s)\,ds\\
&+\int_t^{t_0} a(s)^{-3-c\sqrt{\varepsilon}+\frac{\omega}{4}}\cdot a(s)^{\beta}\left(\mathfrak{E}^{(\leq L-2)}(\phi,s)+a(s)^{\frac{\omega}{4}}\mathfrak{E}^{(\leq L-2)}(\Sigma,s)\right)\,ds\\
&+\int_t^{t_0}\left(a(s)^{-1-\omega-c\sigma}+a(s)^{-3-c\sqrt{\varepsilon}+\frac{\omega}{2}}\right)a(s)^{\beta+(L-1)\omega}\mathfrak{E}^{(\leq L-2)}_{1,\leq L-2}(f,s)\,ds\\
&+\int_t^{t_0}\left(a(s)^{-1-c\sqrt{\varepsilon}+\omega}+\varepsilon a(s)^{-3-c\sqrt{\varepsilon}+\omega}\right)\mathfrak{E}^{(\leq L-4)}(\mathrm{Ric},s)\,ds\\
&+\int_t^{t_0}\left(a(s)^{-1-c\sqrt{\varepsilon}+\beta}+a(s)^{-3-c\sqrt{\varepsilon}+\beta+\omega}\right)\cdot\\
&\qquad\cdot\left(a(s)^{\frac{\omega}{2}}\|\Gamma-\hat{\Gamma}\|^2_{H^1_G(M_s)}+a(s)^{\frac{\omega}{2}}\|G^{-1}-\gamma^{-1}\|^2_{H^2_G(M_s)}\right)\,ds
\end{aligned}
$$

Proof. This is a combination of Lemma 3.6.5, 3.6.6 and 3.6.7. Observe that the final line arises from Lemma 3.6.5, while the remaining lines arise from Lemma 3.6.6 for $K=L-1$ and Lemma 3.6.7. Furthermore, we immediately rewrote $\mathfrak{E}^{(L-1)}_{1,\leq K}(f,\cdot)$ as follows: First, let $\mathcal{K}_{L,K}>0$ be a suitable constant depending on L and K and $\underline{\nabla}_v=v_l\underline{\nabla}_{\mathbf{B}^l}$. For $L-1>K>0$, we integrate by parts:

$$
\begin{aligned}
a^{-c\sqrt{\varepsilon}}\mathfrak{E}^{(L-1)}_{1,K}(f,\cdot)=&-a^{-c\sqrt{\varepsilon}}\int_{T^*M}\langle v\rangle_G^2(v^0)^{2(L-K-1)}\langle\underline{\nabla}^{L-K-2}_{vert}\underline{\nabla}^K_{hor}f,\underline{\nabla}^{L-K}_{vert}\underline{\nabla}^K_{hor}f\rangle_{\underline{G}}\,\mathrm{dvol}_{\underline{G}}\\
&-2a^{-c\sqrt{\varepsilon}}\int_{T^*M}\left((L-K-1)(v^0)^{2(L-K-2)}\langle v\rangle_G^2+(v^0)^{2(L-K-1)}\right)\cdot\\
&\qquad\cdot\langle\underline{\nabla}^{L-K-2}_{vert}\underline{\nabla}^K_{hor}f,\underline{\nabla}_v\underline{\nabla}^{L-K-2}_{vert}\underline{\nabla}^K_{hor}f\rangle\,\mathrm{dvol}_{\underline{G}}
\end{aligned}
$$

Integrating by parts again for the second term, one obtains

$$
\begin{aligned}
a^{-c\sqrt{\varepsilon}}\mathfrak{E}^{(L-1)}_{1,K}(f,\cdot)\leq&\frac{1}{2}\left(\mathfrak{E}^{(L)}_{1,K}(f,\cdot)+a^{-2c\sqrt{\varepsilon}}\mathfrak{E}^{(L-2)}_{1,K}(f,\cdot)\right)\\
&+\mathcal{K}_{L,K}a^{-c\sqrt{\varepsilon}}\int_{T^*M}(v^0)^{2(L-K-2)}\langle v\rangle_G^2|\underline{\nabla}^{L-K-2}_{vert}\underline{\nabla}^K_{hor}f|^2_{\underline{G}}\,\mathrm{dvol}_{\underline{G}}\\
\lesssim&\,\mathfrak{E}^{(L)}_{1,K}(f,\cdot)+a^{-c\sqrt{\varepsilon}}\mathfrak{E}^{(L-2)}_{1,K}(f,\cdot)\,.
\end{aligned}
$$

For $K=L-1>1$, one can argue identically without an error term after the first integration by parts since $\underline{\nabla}_{hor}v^0=0$, obtaining

$$
a^{-c\sqrt{\varepsilon}}\mathfrak{E}^{(L-1)}_{1,L-1}(f,\cdot)\lesssim\mathfrak{E}^{(L)}_{1,L}(f,\cdot)+a^{-c\sqrt{\varepsilon}}\mathfrak{E}^{(L-2)}_{1,L-2}(f,\cdot).
$$

For $K=0$, one can argue as above replacing f with $f-f_{FLRW}$. Finally, for $L-1=1=K$, we need to perform the following replacement after integrating by parts:

$$
\begin{aligned}
&a^{-3-c\sqrt{\varepsilon}+\frac{5}{2}\omega+\beta}\mathfrak{E}^{(1)}_{1,1}(f,\cdot)\\
=&\,a^{-3-c\sqrt{\varepsilon}+\frac{3}{2}\omega+\beta}\left[-\int_{T^*M}\langle v\rangle_G^2(f-f_{FLRW})\cdot(G^{-1})^{ij}\underline{\nabla}_i\underline{\nabla}_jf\,\mathrm{dvol}_{\underline{G}}\right.\\
&\left.-\int_{T^*M}\langle v\rangle_G^2(f-f_{FLRW})(\underline{G}^{-1})^{ij}\underline{\nabla}_i\underline{\nabla}_jf_{FLRW}+\langle v\rangle_G^2|\underline{\nabla}_{hor}f_{FLRW}|^2_{\underline{G}_0}\,\mathrm{dvol}_G\right]
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim a^{-3+\frac{\omega}{2}}\left(a^{\beta+3\omega}\mathfrak{E}^{(2)}_{1,2}(f,\cdot)+a^{-c\sqrt{\varepsilon}}\cdot a^{\beta+\omega}\mathfrak{E}^{(0)}_{1,0}(f,\cdot)\right)\\
&\quad+a^{-3-c\sqrt{\varepsilon}+\beta+\frac{5}{2}\omega}\|\underline{\nabla}_{hor}f_{FLRW}\|^2_{L^2_{1,\underline{G}}(T^*M)}+a^{3-c\sqrt{\varepsilon}+\beta+\frac{7}{2}\omega}\|\underline{\nabla}^2_{hor}f_{FLRW}\|^2_{L^2_{1,\underline{G}}(T^*M)}\\
&\lesssim a^{-3+\frac{\omega}{2}}\left(a^{\beta+3\omega}\mathfrak{E}^{(2)}_{1,2}(f,\cdot)+a^{-c\sqrt{\varepsilon}}\cdot a^{\beta+\omega}\mathfrak{E}^{(0)}_{1,0}(f,\cdot)\right)\\
&\quad+a^{-3-c\sqrt{\varepsilon}+\beta+2\omega}\left(a^{\frac{\omega}{2}}\|\Gamma-\hat{\Gamma}\|^2_{H^1_G(M)}+a^{\frac{\omega}{2}}\|G^{-1}-\gamma^{-1}\|^2_{H^2_G(M)}\right)
\end{aligned}
$$

The final step follows by applying (3.4.11b). □

Recalling the discussion in Section 3.1.4.2, we would like to construct a total energy as in Chapter 2 that admits a strong bound via the Gronwall lemma. To this end, we essentially take the energy $\mathfrak{E}^{(L)}_{quiesc}$ that describes the quiescent formalism as in Chapter 2 and add the total Vlasov energy to it, which forces us to add further scaled terms to the total energy $\mathfrak{E}^{(L)}_{total}$. Before explaining the precise rationale behind these terms, we introduce these energies.

Definition 3.7.4 (Total energies). We define:

$$
\begin{aligned}
\mathfrak{E}^{(0)}_{total}&=\mathfrak{E}^{(0)}(\phi,\cdot)+\left(\varepsilon^{\frac14}+a^{\frac{\omega}{4}}\right)\left(\mathfrak{E}^{(0)}(W,\cdot)+4\pi C^2\mathfrak{E}^{(0)}(\Sigma,\cdot)\right)\\
&\quad+a^4\mathfrak{E}^{(1)}(\phi,\cdot)+\left(\varepsilon^{\frac12}+a^{\frac{\omega}{2}}\right)a^4\mathfrak{E}^{(1)}(\Sigma,\cdot)\\
&\quad+\mathfrak{E}^{(0)}_{total,Vl}+a^4\mathfrak{E}^{(1)}_{total,Vl}+a^{\frac{\omega}{2}}\|G^{\pm1}-\gamma^{\pm1}\|^2_{L^2_G(M)}+a^{4+\frac{\omega}{2}}\|\Gamma-\hat{\Gamma}\|^2_{L^2_G(M)}
\end{aligned}\tag{3.7.5a}
$$

$$
\mathfrak{E}^{(0)}_{quiesc}=\mathfrak{E}^{(0)}(\phi,\cdot)+\varepsilon^{\frac14}\left(\mathfrak{E}^{(0)}(W,\cdot)+4\pi C^2\mathfrak{E}^{(0)}(\Sigma,\cdot)\right)+a^4\mathfrak{E}^{(1)}(\Sigma,\cdot)\tag{3.7.5b}
$$

and, for $L\in 2\mathbb{N}, 2\leq L\leq 18$

(3.7.5c)

$$
\begin{aligned}
\mathfrak{E}^{(L)}_{total}&=\mathfrak{E}^{(L)}(\phi,\cdot)+\left(\varepsilon^{\frac14}+a^{\frac{\omega}{4}}\right)\left(\mathfrak{E}^{(L)}(W,\cdot)+4\pi C^2\mathfrak{E}^{(L)}(\Sigma,\cdot)\right)+\left(\varepsilon^{\frac12}+a^{\frac{\omega}{2}}\right)\mathfrak{E}^{(L-2)}(\mathrm{Ric},\cdot)\\
&\quad+a^4\mathfrak{E}^{(L+1)}(\phi,\cdot)+\left(\varepsilon^{\frac12}+a^{\frac{\omega}{2}}\right)a^4\mathfrak{E}^{(L+1)}(\Sigma,\cdot)+\left(\varepsilon^{\frac34}+a^{\omega}\right)a^4\mathfrak{E}^{(L-1)}(\mathrm{Ric},\cdot)\\
&\quad+\left(\varepsilon^{\frac34}+a^{\omega}\right)a^8\mathfrak{E}^{(L)}(\mathrm{Ric},\cdot)+\mathfrak{E}^{(L)}_{total,Vl}+a^4\mathfrak{E}^{(L+1)}_{total,Vl}\\
&\quad+\underbrace{a^{\frac{\omega}{2}}\left(\|\Gamma-\hat{\Gamma}\|^2_{H^1_G(M)}+\|G^{\pm1}-\gamma^{\pm1}\|^2_{H^2_G(M)}\right)}_{\text{if } L=2}
\end{aligned}
$$

(3.7.5d)

$$
\begin{aligned}
\mathfrak{E}^{(L)}_{quiesc}&=\mathfrak{E}^{(L)}(\phi,\cdot)+\varepsilon^{\frac14}\left(\mathfrak{E}^{(L)}(W,\cdot)+4\pi C^2\mathfrak{E}^{(L)}(\Sigma,\cdot)\right)+a^4\mathfrak{E}^{(L+1)}(\phi,\cdot)+\varepsilon^{\frac12}a^4\mathfrak{E}^{(L+1)}(\Sigma,\cdot)\\
&\quad+\varepsilon^{\frac12}\mathfrak{E}^{(L-2)}(\mathrm{Ric},\cdot)+\varepsilon^{\frac34}a^4\mathfrak{E}^{(L-1)}(\mathrm{Ric},\cdot)
\end{aligned}
$$

Remark 3.7.5 (On the construction of $\mathfrak{E}^{(L)}_{total}$). Essentially, the total energy can be split into three components – $\mathfrak{E}^{(L)}_{quiesc}$ which describes the core quiescent mechanism as in Chapter 2, $\mathfrak{E}^{(L)}_{total,Vl}$ which is a natural energy for Vlasov matter in near-FLRW regimes, and various scaled energies controlling components of the spacetime metric. We briefly explain why the latter needs to be introduced when one attempts to combine $\mathfrak{E}^{(L)}_{quiesc}$ and $\mathfrak{E}^{(L)}_{total,Vl}$ into a total energy that admits a useful Gronwall estimate:

First, we notice that Vlasov energies occur linearly in energy estimates for the scalar field, see Lemma 3.5.5, and also vice versa, see (3.7.4). Thus, we do not gain smallness in terms of ε, and the Vlasov energy must enter the total energy without ε-weights. Second, Vlasov matter also linearly couples into the geometric evolution equations for the shear and for Bel-Robinson variables, see Lemmas 3.5.14 and 3.5.8, and vice versa, see (3.7.4). However, to obtain a closed energy estimate for the quiescent components, one needs to introduce an ε-weight in $\mathfrak{E}^{(L)}_{quiesc}$ as in Chapter 2. On the other hand, geometric energies occur linearly in (3.7.4). Thus, to obtain closed estimates, we must introduce geometric energies that are not weighted by ε. To nevertheless surpress linear scalar field terms of order t^{-1} in the geometric estimates, we must then introduce scale factor weights on these terms, which must be weaker than those on Vlasov matter. This then extends to Ricci curvature terms as well as low order geometric error terms in the Vlasov equation, which must both exhibit stronger scaling than the second fundamental form, which occurs linearly at order t^{-1} in the evolution equations of either. For the second fundamental form at order $L+1$, the additional scale factor weight must be stronger than the one at order L so that, after applying Lemma 3.5.10, all resulting terms can be absorbed into the left hand side if $a(t_0)$ is sufficiently small. The curvature term at order $L-1$ must, in turn, be scaled more strongly than the second fundamental form at order $L+1$ as before, and the top order curvature term must be similarly strongly weighted so that one can absorb terms resulting from (3.5.19g) into the left hand side.

Proposition 3.7.6 (Total energy bound). *For $L\in 2\mathbb{N}, L\leq 18$, the following holds:*

$$\mathfrak{E}^{(L)}_{total} \lesssim \varepsilon^4 a^{-c\varepsilon^{\frac{1}{8}}} \tag{3.7.6}$$

Proof. We start proving the case of $L=0$ by combining the following energy estimates:

- For the first line, we use (3.5.7a) for $L=0$ for the scalar field energy. For the Bel-Robinson and shear energies, we treat their sum scaled by $\varepsilon^{\frac{1}{4}}$ and their sum scaled by $a^{\frac{\omega}{4}}$ separately. For the former, we apply (3.5.13b) and (3.5.14b) for $L=0$, scaled by $\varepsilon^{\frac{1}{4}}$. For the latter, we apply (3.5.13a) and (3.5.14a) for $L=0$.
- For the second line, we apply (3.5.7b) for $L=1$ and the elliptic shear estimate (3.5.16b) scaled by $\left(\varepsilon^{\frac{1}{2}}+a^{\frac{\omega}{2}}\right)$
- For the third line, we apply (3.6.5a), (3.7.3) and the metric estimate (3.5.17a).

Notice that, by (3.7.1), the indefinite terms on the left hand sides of the analogues of (3.5.13b) and (3.5.14b), respectively (3.5.13a) and (3.5.14a), are nonnegative for $\kappa\leq 0$, and else are so up to a term that can be pulled to right and estimated from above by $\int_t^{t_0} a(s)^{-1}\mathfrak{E}^{(0)}_{total}(s)\,ds$. This leads to the following estimate:

$$\begin{aligned}\mathfrak{E}^{(0)}_{total}(t) \lesssim\, & \varepsilon^4+\int_t^{t_0}\left(\varepsilon^{\frac{1}{8}}a(s)^{-3}+a(s)^{-3+\frac{\omega}{8}}+a(s)^{-2-c\sigma-2\omega}\right)\mathfrak{E}^{(0)}_{total}(s)\,ds\\ & +\left(\varepsilon^{\frac{1}{4}}+a(t)^{\frac{\omega}{4}}\right)\mathfrak{E}^{(0)}_{total}(t)\end{aligned}$$

Recall that we can choose both $\varepsilon > 0$ and $a(t_0)$ to be sufficiently small without loss of generality, see Remark 3.3.8 for the latter. Thus, the final term can be absorbed into the left hand side after adapting implicit constants, and the Gronwall lemma then yields

$$\mathfrak{E}^{(0)}_{total}(t) \lesssim \varepsilon^4 a(t)^{-c\varepsilon^{\frac{1}{8}}} .$$

For $L \geq 2$, we proceed iteratively, assuming the bound to have been shown up to $\mathfrak{E}^{(\leq L-2)}_{total}$. One now similarly combines the following estimates, going through (3.7.5c).

- For the first line: (3.5.7a) for the scalar field energy, (3.5.13b), (3.5.14b) and (3.5.19c) for the Bel-Robinson, shear and Ricci energies scaled by $\varepsilon^{\frac{1}{4}}$, and (3.5.13a), (3.5.14a) and (3.5.19a) for the energies scaled by powers of $a^{\frac{\omega}{4}}$
- For the first two terms in the second line: (3.5.7b) and (3.5.16a)
- For the curvature energies second line: (3.5.19c) and (3.5.19a), respectively (3.5.19f) and (3.5.19e) for $L = 2$, as well as (3.5.19d) and (3.5.19b) for the energies at order $L - 1$
- For the curvature energy in the third line: (3.5.19g) with $l = L$, scaled appropriately
- For the Vlasov energies in the second line: (3.7.4) with $\beta = 0$ at order L and with $\beta = 4$ at order $L + 1$
- For the metric terms if $L = 2$: (3.5.18a) with $l = 2, \beta = 0$ and (3.5.17a) with $l = 2$

Note that, for $L = 2$, the metric and Christoffel symbol norms become a part of the total energy, and else they are absorbed into the lower order terms since they have already been improved.
Altogether, this yields the following:

$$\begin{aligned}\mathfrak{E}^{(L)}_{total}(t) \lesssim \varepsilon^4 + \int_t^{t_0} &\Big[\left(\varepsilon^{\frac{1}{8}} a(s)^{-3} + a(s)^{-3-c\sqrt{\varepsilon}+\frac{\omega}{8}} + a(s)^{-2-c\sigma-(L+2)\omega}\right)\mathfrak{E}^{(L)}_{total}(s)\\ &+\left(\varepsilon^{\frac{1}{8}} a(s)^{-3-c\sqrt{\varepsilon}} + a(s)^{-3-c\sqrt{\varepsilon}+\frac{\omega}{8}} + a(s)^{-2-c\sigma-(L+1)\omega}\right)\mathfrak{E}^{(\leq L-2)}_{total}(s)\Big]\,ds\\ + \left(\varepsilon^{\frac{1}{4}} + a(t)^{\frac{\omega}{4}}\right)\mathfrak{E}^{(L)}_{total}(t) &+ \left(\varepsilon^{\frac{1}{4}} + a(t)^{\frac{\omega}{4}}\right) a(t)^{-c\sqrt{\varepsilon}}\mathfrak{E}^{(\leq L-2)}_{total}(t)\end{aligned}$$

Since the statement has already been shown for $\mathfrak{E}^{(\leq L-2)}_{total}$, this implies after rearranging the first term in the final line as before that

$$\mathfrak{E}^{(L)}_{total}(t) \lesssim \varepsilon^4 a(t)^{-c\varepsilon^{\frac{1}{8}}} + \int_t^{t_0} \left(\varepsilon^{\frac{1}{8}} a(s)^{-3} + a(s)^{-3-c\sqrt{\varepsilon}+\frac{\omega}{8}} + a(s)^{-2-c\sigma-(L+2)\omega}\right)\mathfrak{E}^{(L)}_{total}(s)\,ds$$

and thus

$$\mathfrak{E}^{(L)}_{total}(t) \lesssim \varepsilon^4 a(t)^{-c\varepsilon^{\frac{1}{8}}} .$$

□

Proposition 3.7.7 (Quiescent energy bound). *For $L \in 2\mathbb{N}, L \leq 18$, the following holds:*

$$\mathfrak{E}^{(L)}_{quiesc} \lesssim \varepsilon^4 a^{-c\varepsilon^{\frac{1}{8}}} \tag{3.7.7}$$

Proof. In short, the strategy in this step of the argument is to rerun the argument that yielded (3.7.6), but using this bound to directly estimate all Vlasov terms and metric errors, since it directly implies, for any $l \in \mathbb{N}, l \leq 18$ and any $m \in \mathbb{N}, m \leq l$,

$$a^{(m+1)\omega}\mathfrak{E}^{(l)}_{1,m}(f,\cdot) + a^{\frac{\omega}{2}}\left(\|G^{\pm 1} - \gamma^{\pm 1}\|^2_{H^2_G} + \|\Gamma - \hat{\Gamma}\|^2_{H^1_G}\right) \lesssim \varepsilon^4 a^{-c\varepsilon^{\frac{1}{8}}} \tag{3.7.8}$$

For $L = 0$, we use the analogues of (3.5.7a), (3.5.13b), (3.5.14b) and (3.5.7b) for $L = 0$ and the elliptic shear estimate (3.5.16b). This yields the following:

$$\begin{aligned}
\mathfrak{E}^{(0)}_{quiesc}(t) \lesssim \varepsilon^4 &+ \int_t^{t_0} \left(\varepsilon^{\frac{1}{8}} a(s)^{-3} + a(s)^{-3+\frac{\omega}{4}} + a(s)^{-2-c\sqrt{\varepsilon}}\right) \mathfrak{E}^{(0)}_{quiesc}(s)\,ds \\
&+ \int_t^{t_0} a(s)^{-2-c\sqrt{\varepsilon}-2\omega}\left(\mathfrak{E}^{(0)}_{total,Vl}(s) + a(s)^4\mathfrak{E}^{(1)}_{total,Vl}(s) + a(s)^{\frac{\omega}{2}}\|G^{-1}-\gamma^{-1}\|^2_{L^2_G(M_s)}\right) ds \\
&+ \varepsilon^{\frac{1}{4}}\mathfrak{E}^{(0)}_{quiesc}(t) + a(t)^{2-c\sqrt{\varepsilon}-\omega}\mathfrak{E}^{(0)}_{total,Vl}(t)
\end{aligned}$$

After pulling the first term in the last line to the left and applying (3.7.6), this implies

$$\begin{aligned}
\mathfrak{E}^{(0)}_{quiesc}(t) \lesssim & \varepsilon^4 + \int_t^{t_0} \left(\varepsilon^{\frac{1}{8}} a(s)^{-3} + a(s)^{-3+\frac{\omega}{4}} + a(s)^{-2-c\sqrt{\varepsilon}}\right) \mathfrak{E}^{(0)}_{quiesc}(s) \\
&+ \int_t^{t_0} \varepsilon^4 a(s)^{-2-2\omega-c\varepsilon^{\frac{1}{8}}}\,ds + \varepsilon^4 a(t)^{4-\omega-c\varepsilon^{\frac{1}{8}}}
\end{aligned}$$

Since Lemma 3.2.2 implies that the second line can be bounded by ε^4, the Gronwall lemma leads to $\mathfrak{E}^{(0)}_{quiesc}(t) \lesssim \varepsilon^4 a(t)^{-c\varepsilon^{\frac{1}{8}}}$.
We now proceed iteratively for $L \in 2\mathbb{N}, L \leq 18$, assuming that one has shown

$$\mathfrak{E}^{(\leq L-2)}_{quiesc} \lesssim \varepsilon^4 a^{-c\varepsilon^{\frac{1}{8}}}$$

We combine the following estimates to control the individual terms in (3.7.5d).

- For the first line: (3.5.7a), (3.5.13b), (3.5.14b), (3.5.7b), (3.5.16a)
- For the curvature energies second line: (3.5.19c) (respectively (3.5.19f) for $L = 2$) and (3.5.19d)

This yields the following as before.

$$\begin{aligned}
\mathfrak{E}^{(L)}_{quiesc}(t) \lesssim \varepsilon^4 &+ \int_t^{t_0} \left(\varepsilon^{\frac{1}{8}} a(s)^{-3} + a(s)^{-3+\frac{\omega}{4}} + a(s)^{-2-c\sqrt{\varepsilon}}\right) \mathfrak{E}^{(L)}_{quiesc}(s)\,ds \\
&+ \int_t^{t_0} a(s)^{-2-c\sqrt{\varepsilon}-(L+2)\omega}\left(\mathfrak{E}^{(\leq L)}_{total,Vl}(s) + a(s)^4\mathfrak{E}^{(L+1)}_{total,Vl}(s) + \|G^{-1}-\gamma^{-1}\|^2_{L^2_G(M_s)}\right) ds \\
&+ \int_t^{t_0} \left(\varepsilon^{\frac{1}{8}} a(s)^{-3-c\sqrt{\varepsilon}} + a(s)^{-3+\frac{\omega}{4}}\right) \mathfrak{E}^{(\leq L-2)}_{quiesc}(s)\,ds \\
&+ \varepsilon^{\frac{1}{4}}\mathfrak{E}^{(L)}_{quiesc}(t) + a(t)^{4-c\sqrt{\varepsilon}-(L+1)\omega}\mathfrak{E}^{(L)}_{total,Vl}(t) \\
\lesssim & \varepsilon^4 a(t)^{-c\varepsilon^{\frac{1}{8}}} + \int_t^{t_0} \left(\varepsilon^{\frac{1}{8}} a(s)^{-3} + a(s)^{-3+\frac{\omega}{4}} + a(s)^{-2-c\sqrt{\varepsilon}}\right) \mathfrak{E}^{(L)}_{quiesc}(s)\,ds + \varepsilon^{\frac{1}{4}}\mathfrak{E}^{(L)}_{quiesc}(t)
\end{aligned}$$

The bound (3.7.7) now follows at order L after rearranging and applying the Gronwall lemma, proving the statement. □

Corollary 3.7.8 (Sobolev norm estimates for the solution variables).

$$\|\Psi\|_{H^{18}_G(M)} \lesssim \varepsilon^2 a^{-c\varepsilon^{\frac{1}{8}}} \tag{3.7.9a}$$

$$\|\Sigma\|_{H^{18}_G(M)} \lesssim \varepsilon^{\frac{15}{8}} a^{-c\varepsilon^{\frac{1}{8}}} \tag{3.7.9b}$$

$$\|\mathcal{E}\|_{H^{18}_G(M)} + \|\mathcal{B}\|_{H^{18}_G(M)} \lesssim \varepsilon^{\frac{15}{8}} a^{-c\varepsilon^{\frac{1}{8}}} \tag{3.7.9c}$$

$$\|\mathrm{Ric}[G] - 2\kappa G\|_{H^{16}_G(M)} + \|\Gamma - \hat{\Gamma}\|_{H^{17}_G(M)} + \|G^{\pm1} - \gamma^{\pm1}\|_{H^{18}_G(M)} \lesssim \varepsilon^{\frac{7}{4}} a^{-c\varepsilon^{\frac{1}{8}}} \tag{3.7.9d}$$

$$\|N\|_{H^{16}_G(M)} \lesssim \varepsilon^{\frac{7}{4}} a^{4-c\varepsilon^{\frac{1}{8}}} \tag{3.7.9e}$$

Further, one has the following high order bounds:

$$a^4 \mathfrak{E}^{(17)}(\mathrm{Ric}, \cdot) \lesssim \varepsilon^{\frac{7}{2}} a^{-c\varepsilon^{\frac{1}{8}}} \tag{3.7.10a}$$

$$\|\Gamma - \hat{\Gamma}\|_{H^{17}_G} + a^4 \|\Gamma - \hat{\Gamma}\|_{\dot{H}^{18}_G} \lesssim \varepsilon^{\frac{7}{4}} a^{-c\varepsilon^{\frac{1}{8}}} \tag{3.7.10b}$$

PROOF. The first three bounds as well as the Ricci bounds follow directly from Proposition 3.7.6 using Lemma 2.4.5 to switch bound the respective Sobolev norms up to curvature energies. To obtain the lapse bounds, we apply (3.5.6b) and (3.5.6c), where we observe that, for suitable ω, we can estimate all Vlasov energies using (3.7.8). The metric and Christoffel bounds then follow from (3.5.17b) and (3.5.18b). $\square$

With these improvements in hand, we are now able to derive energy bounds for Vlasov matter that are not scaled:

PROPOSITION 3.7.9 (Vlasov energy bound). *The following bound holds:*

$$\mathfrak{E}^{(\leq 18)}_{1,\leq 18}(f, \cdot) \lesssim \varepsilon^{\frac{7}{2}} a^{-c\varepsilon^{\frac{1}{8}}} \tag{3.7.11}$$

PROOF. As in the proof of Proposition 3.7.6, we again iterate over the number of derivatives, this time in increments of 1 starting from $L = 1$ after treating $L = 0$ as a special case first:
For $L = 0$, note applying the bounds from Corollary 3.7.8 to the energy estimate (3.6.5b) implies

$$\mathfrak{E}^{(0)}_{1,0}(f, t) \lesssim \varepsilon^4 + \int_t^{t_0} \left(a(s)^{-1-c\sigma} + \varepsilon^{\frac{1}{8}} a(s)^{-3} \right) \mathfrak{E}^{(0)}_{1,0}(f, s) + \varepsilon^{\frac{29}{8}} a(s)^{-3-c\varepsilon^{\frac{1}{8}}} + \varepsilon^{\frac{7}{2}} a(s)^{-1-c\varepsilon^{\frac{1}{8}}} \, ds \, .$$

The stated bound then follows with the Gronwall lemma.
For $L = 1$, we start with (3.6.9). Notice that, using (3.7.8), the term containing the Vlasov energy of higher horizontal order $\mathfrak{E}^{(1)}_{1,1}(f, \cdot)$ can be bounded by

$$\int_t^{t_0} a(s)^{-1-2\omega} \cdot a(s)^{2\omega} \mathfrak{E}^{(1)}_{1,1}(f, s) \, ds \lesssim \int_t^{t_0} a(s)^{-1-2\omega} \cdot \varepsilon^4 a(s)^{-c\varepsilon^{\frac{1}{8}}} \, ds \, .$$

Using the bounds from Corollary 3.7.8 on the remaining energies similarly, we obtain

$$\mathfrak{E}^{(1)}_{1,0}(f, t) \lesssim \varepsilon^4 + \int_t^{t_0} \left(a(s)^{-1-c\sigma} + \varepsilon^{\frac{1}{8}} a(s)^{-3} \right) \mathfrak{E}^{(\leq 1)}_{1,0}(f, s) + \varepsilon^{\frac{7}{2}} a(s)^{-1-c\varepsilon^{\frac{1}{8}}} + \varepsilon^{\frac{29}{8}} a(s)^{-3-c\varepsilon^{\frac{1}{8}}} \, ds$$

and thus, again applying the Gronwall lemma,

$$\mathfrak{E}^{(1)}_{1,0}(f, t) \lesssim \varepsilon^{\frac{7}{2}} a(t)^{-c\varepsilon^{\frac{1}{8}}} \, .$$

To deduce the bound on $\mathfrak{E}^{(1)}_{1,1}(f,\cdot)$, we consider (3.6.18) and deduce that the respective integral containing $\mathfrak{E}^{(1)}_{1,0}(f,t)$ therein can be bounded as follows:

$$\int_t^{t_0} \varepsilon^{\frac{15}{8}} a(s)^{-3-c\varepsilon^{\frac{1}{8}}} \mathfrak{E}^{(1)}_{1,0}(f,s)\, ds \lesssim \varepsilon^{\frac{21}{4}} a(t)^{-c\varepsilon^{\frac{1}{8}}}$$

Applying the bounds from Corollary 3.7.8 for the remaining energies proves the bound for $\mathfrak{E}^{(1)}_{1,1}(f,\cdot)$ with the Gronwall lemma as before.
For $1 < L \leq 18$, assume the statement has been shown up to $L-1$. Starting again with the vertical energy estimate (3.6.8) for $\beta = 0$, the bound

$$\mathfrak{E}^{(L)}_{1,0}(f,\cdot) \lesssim \varepsilon^{\frac{7}{2}} a^{-c\varepsilon^{\frac{1}{8}}}$$

follows by the same argument as for $L = 1$, in particular using (3.7.8) to bound $\mathfrak{E}^{(L)}_{1,1}(f,\cdot)$. Next, let $K \in \mathbb{N}, L > K > 0$ and assume the bound to hold for $\mathfrak{E}^{(L)}_{1,\leq K-1}(f,\cdot)$, and consider the intermediate energy estimate (3.6.12). Again, notice that

$$\int_t^{t_0} a(s)^{-1} \mathfrak{E}^{(L)}_{1,K+1}(f,s)\, ds \lesssim \varepsilon^{\frac{7}{2}} a^{-c\varepsilon^{\frac{1}{8}}}$$

holds by (3.7.8), and on the other hand,

$$\int_t^{t_0} \left(\varepsilon^{\frac{7}{8}} a(s)^{-3-c\sqrt{\varepsilon}} + a(s)^{-1-c\sqrt{\varepsilon}}\right) \left(\mathfrak{E}^{(L)}_{1,K-1}(f,s) + \mathfrak{E}^{(\leq L-1)}_{1,\leq K}(f,s)\right) ds \lesssim \varepsilon^{\frac{7}{2}} a^{-c\varepsilon^{\frac{1}{8}}}$$

is satisfied since improved bounds have been established for all energies in the integrand. The shear and scalar field energies can be treated as before, and (3.7.9) leads to similar terms for curvature and metric error terms that are not at top order. At highest order (i.e., $L = 18$), we note that high order curvature terms that occur in (3.7.8) are appropriately scaled by (3.7.10). Altogether, (3.6.12) then leads to the following estimate:

$$\mathfrak{E}^{(L)}_{1,K}(f,\cdot) \lesssim \varepsilon^4 + \int_t^{t_0} \left(a(s)^{-1-c\sigma} + \varepsilon^{\frac{1}{8}} a(s)^{-3}\right) \mathfrak{E}^{(L)}_{1,K}(f,s) + \varepsilon^{\frac{7}{2}} a(s)^{-1-\omega-c\sqrt{\varepsilon}} + \varepsilon^{\frac{29}{8}} a(s)^{-3-c\varepsilon^{\frac{1}{8}}}\, ds,$$

and thus $\mathfrak{E}^{(L)}_{1,\leq K}(f,\cdot) \lesssim \varepsilon^{\frac{7}{2}} a^{-c\varepsilon^{\frac{1}{8}}}$. Having now proven the bound up to $\mathfrak{E}^{(L)}_{1,\leq L-1}(f,\cdot)$ by iterating over K, we can apply the same argument to (3.6.16). Again, note that even at highest order, the metric and curvature bounds (3.7.9) and (3.7.10) are sufficient. This implies the stated bound for $\mathfrak{E}^{(L)}_{1,\leq L}(f,\cdot)$, completing the iteration step and thus proving (3.7.11). □

Now, we are in the position to improve the remaining bootstrap assumption (3.3.13a):

Corollary 3.7.10 (Bootstrap improvement). *For $\mathcal{C}$ as in* (3.3.2c) *and $t \in (t_{Boot}, t_0]$, the following holds:*

$$\mathcal{C}(t) \lesssim \varepsilon^{\frac{7}{4}} a(t)^{-c\varepsilon^{\frac{1}{8}}} \tag{3.7.12}$$

Proof. Regarding the Vlasov matter quantities, note that Lemma 3.4.14 only requires bounds on $\mathfrak{E}^{(L)}_{1,L}(f,\cdot)$ for $L \in \mathbb{N}$. Inserting (3.7.11) and the metric bound from (3.7.9) then implies

$$\|\rho^{Vl} - \rho^{Vl}_{FLRW}\|_{H^{18}_G(M)} + \|\mathfrak{p}^{Vl} - \mathfrak{p}^{Vl}_{FLRW}\|_{H^{18}_G(M)} + \|\jmath^{Vl}\|_{H^{18}_G(M)} + \|(S^{Vl})^{\|}\|_{H^{18}_G(M)} \lesssim \varepsilon^{\frac{7}{4}} a^{-c\varepsilon^{\frac{1}{8}}}$$

Furthermore, applying the above along with (3.7.9a) and (3.7.9b) to Lemma 3.5.6 for $0 \leq l \leq 18$ yields

$$\|\nabla\phi\|_{H^{18}_G(M)} \lesssim \varepsilon^{\frac{7}{4}} a^{-c\varepsilon^{\frac{1}{8}}}$$

after updating $c > 0$. This, along with Corollary 3.7.8, proves From here, the statement follows for all variables measured by $\mathcal{C}$ from the Sobolev embedding $H^{l+2}_\gamma(M) \hookrightarrow C^l_\gamma(M)$ for $l \in \mathbb{N}$, as well as estimates as in Lemma 2.7.4 to relate norms with respect to γ and G to one another. □

Additionally, Proposition 3.7.9 implies the following bound on the distribution function compared to the reference data:

Corollary 3.7.11 (Sobolev norm bound for the Vlasov distribution function). *On the bootstrap interval, one has the following:*

$$\|f - f_{FLRW}\|_{H^{18}_{1,\underline{G}_0}(T^*M_t)} \lesssim \varepsilon^{\frac{7}{4}} a(t)^{-c\varepsilon^{\frac{1}{8}}} \tag{3.7.13}$$

Proof. For $L \leq 18$, we sum (3.4.26b) over $K = 1, \dots, L-1$ as well as all possible derivative combinations represented by $\mathcal{D}_{L,K}$. Note that the curvature error terms in (3.4.26b) only necessitate control of the curvature energy $\mathfrak{E}^{(\leq 16)}(\mathrm{Ric}, \cdot)$ as well as low order norms for the Christoffel symbols and the metric, all of which are covered in Corollary 3.7.8. Thus, inserting the Vlasov energy bound (3.7.11) and Corollary 3.7.8 then implies

$$\|f\|^2_{\dot{H}^L_{1,\underline{G}_0}(T^*M)} - \|\underline{\nabla}^L_{vert} f\|^2_{L^2_{1,\underline{G}_0}(T^*M)} \lesssim \varepsilon^{\frac{7}{2}} a^{-c\varepsilon^{\frac{1}{8}}}$$

Furthermore, by Lemma 3.4.7 and Corollary 3.7.8, one also has

$$\|f_{FLRW}\|^2_{\dot{H}^L_{1,\underline{G}_0}(T^*M)} - \|\underline{\nabla}^L_{vert} f_{FLRW}\|^2_{L^2_{1,\underline{G}_0}(T^*M)} \lesssim \varepsilon^{\frac{7}{2}} a^{-c\varepsilon^{\frac{1}{8}}}$$

The statement then follows by expanding $\|f - f_{FLRW}\|^2_{\dot{H}^L_{1,\underline{G}_0}(T^*M_t)}$ and using (3.7.8) to bound $\mathfrak{E}^{(L)}_{1,0}(f, \cdot)$. □

3.8. The main theorem

We are now in the position to prove the main theorem of this paper:

Theorem 3.8.1 (Past stability of FLRW solutions to the ESFV system). *Consider FLRW solutions of the form* (3.1.2) *to the Einstein scalar-field Vlasov equations* (3.1.1) *for which, in the massless case, f_{FLRW} vanishes on an open neighbourhood of the zero section of P. Let $(M, \mathring{g}, \mathring{k}, \mathring{\pi}, \mathring{\psi}, \mathring{f})$ be CMC initial data to the Einstein scalar-field Vlasov system (see Subsection 3.2.3) that is ε^2-close to the FLRW data on M_{t_0} in the sense of Assumption 3.3.4.*

Then, the past maximal globally hyperbolic development $(\overline{M}, \overline{g}, \phi, \overline{f})$ to the Einstein scalar-field Vlasov system (3.1.1) *admits a foliation by CMC Cauchy hypersurfaces $M_s = t^{-1}(\{s\})$, $s \in (0, t_0]$ such that the solution norms $\mathcal{H}$ and $\mathcal{C}$ (see Definition 3.3.2) satisfy*

$$\mathcal{H}(t) + \mathcal{C}(t) \lesssim \varepsilon^{\frac{7}{4}} a(t)^{-c\varepsilon^{\frac{1}{8}}} \tag{3.8.1a}$$

and one has

$$\|(f - f_{FLRW})\|_{C^{11}_{1,\underline{G}_0}(T^*M_t)} \lesssim \sqrt{\varepsilon}a(t)^{-c\sqrt{\varepsilon}}\,. \tag{3.8.1b}$$

In particular, this implies the following asymptotic behaviours for the solution variables, introducing the Weingarten map $\mathcal{K}^i{}_j = g^{il}k_{lj}$*: There exist a scalar function* $\Psi_{Bang} \in C^{14}_\gamma(M)$*, a* $(1,1)$*-tensor field* $K_{Bang} \in C^{14}_\gamma(M)$*, a volume form* $\mathrm{dvol}_{Bang} \in C^{14}_\gamma(M)$ *and a* $(0,2)$*-tensor field* $M_{Bang} \in C^{14}_\gamma(M)$ *such that the following bounds hold for any* $t \in (0,t_0]$*:*

$$\|n-1\|_{C^{14}_\gamma(M_t)} \lesssim \varepsilon a(t)^{2-c\varepsilon^{\frac{1}{8}}} \tag{3.8.2a}$$

$$\|a^{-3}\mathrm{dvol}_g - \mathrm{dvol}_{Bang}\|_{C^{14}_\gamma(M_t)} \lesssim \varepsilon a(t)^{2-c\varepsilon^{\frac{1}{8}}} \tag{3.8.2b}$$

$$\|a^3\partial_t\phi - (\Psi_{Bang}+C)\|_{C^{14}_\gamma(M_t)} \lesssim \varepsilon a(t)^{2-c\varepsilon^{\frac{1}{8}}} \tag{3.8.2c}$$

$$\left\|\hat\nabla\phi(t,\cdot) - \hat\nabla\phi(t_0,\cdot) + \int_t^{t_0} a(s)^{-3}\,ds\cdot\hat\nabla\Psi_{Bang}\right\|_{C^{14}_\gamma(M_t)} \lesssim \varepsilon a(t)^{2-c\varepsilon^{\frac{1}{8}}} \tag{3.8.2d}$$

$$\|a^3\mathcal{K} - K_{Bang}\|_{C^{14}_\gamma(M_t)} \lesssim \varepsilon a(t)^{2-c\varepsilon^{\frac{1}{8}}} \tag{3.8.2e}$$

$$\left\|g\odot\exp\left[-2\int_t^{t_0} a(s)^{-3}\,ds\cdot K_{Bang}\right] - M_{Bang}\right\|_{C^{12}_\gamma(M_t)} \lesssim \varepsilon a(t)^{2-c\varepsilon^{\frac{1}{8}}} \tag{3.8.2f}$$

Furthermore, one has

$$8\pi(\Psi_{Bang}+C)^2 + |K_{Bang}|^2 = 12\pi C^2, \quad (K_{Bang})^a{}_a = -\sqrt{12\pi}C \tag{3.8.3a}$$

and, denoting $\left(K^{\|}_{Bang}\right)^i{}_j = (K_{Bang})^i{}_j + \sqrt{\frac{4\pi}{3}}C\mathbb{I}^i_j$*,*

$$\|\mathrm{dvol}_\gamma - \mathrm{dvol}_{Bang}\|_{C^{15}_\gamma(M)} + \|M_{Bang}-\gamma\|_{C^{15}_\gamma(M)} + \|K^{\|}_{Bang}\|_{C^{15}_\gamma(M)} + \|\Psi\|_{C^{15}_\gamma(M)} \lesssim \varepsilon\,. \tag{3.8.3b}$$

Additionally, there exists $f_{\mathrm{Bang}} \in C^{10}_{1,\underline{\gamma}_0}(T^*M)$ *such that, for any* $t \in (t,t_0]$*, one has*

$$\|f - f_{\mathrm{Bang}}\|_{C^{10}_{1,\underline{\gamma}_0}(T^*M_t)} \lesssim \sqrt{\varepsilon}a(t)^{2-c\varepsilon^{\frac{1}{8}}} \tag{3.8.4a}$$

where the footprint state f_{Bang} *can be controlled as follows.*

$$\|f_{\mathrm{Bang}} - f_{FLRW}\|_{C^{10}_{1,\underline{\gamma}_0}(T^*M)} \lesssim \sqrt{\varepsilon} \tag{3.8.4b}$$

The momentum support of f_{Bang} *is bounded and, in the massless case, bounded away from the origin with respect to* $\underline{\gamma}|_{vert}$*. The solution norm bound* (3.8.1a) *also implies the following Vlasov matter bounds:*

$$\|f - f_{FLRW}\|_{H^{18}_{1,\underline{G}_0}(T^*M_t)} \lesssim \varepsilon^{\frac{7}{4}}a(t)^{-c\varepsilon^{\frac{1}{8}}} \tag{3.8.4c}$$

$$\left\|T^{Vl}_{00}[\overline{g},f] - T^{Vl}_{00}[\overline{g}_{FLRW},f_{FLRW}]\right\|_{C^{16}_\gamma(M_t)} \lesssim \varepsilon^{\frac{7}{4}}a(t)^{-4-c\varepsilon^{\frac{1}{8}}} \tag{3.8.4d}$$

$$\left\|T^{Vl}_{0(\cdot)}[\overline{g},f] - T^{Vl}_{0(\cdot)}[\overline{g}_{FLRW},f_{FLRW}]\right\|_{C^{16}_\gamma(M_t)} \lesssim \varepsilon^{\frac{7}{4}}a(t)^{-3-c\varepsilon^{\frac{1}{8}}} \tag{3.8.4e}$$

$$\left\|T^{Vl}_{(\cdot)(\cdot)}[\overline{g},f] - T^{Vl}_{(\cdot)(\cdot)}[\overline{g}_{FLRW},f_{FLRW}]\right\|_{C^{16}_\gamma(M_t)} \lesssim \varepsilon^{\frac{7}{4}}a(t)^{-4-c\varepsilon^{\frac{1}{8}}} \tag{3.8.4f}$$

The FLRW solution exhibits stable mean, scalar, Ricci and Kretschmann curvature scalar blow-up toward the Big Bang in the following sense:

$$\left\| a^6 |k|_g^2 - |K_{Bang}|^2 \right\|_{C^0(M_t)} \lesssim \varepsilon a(t)^{2-c\varepsilon^{\frac{1}{8}}} \tag{3.8.5a}$$

$$\left\| a^6 R[\overline{g}] - 8\pi(\Psi_{Bang} + C)^2 \right\|_{C^0(M_t)} \lesssim \varepsilon a(t)^{2-c\varepsilon^{\frac{1}{8}}} \tag{3.8.5b}$$

$$\left\| a^{12} \mathrm{Ric}[\overline{g}]_{\alpha\beta} \mathrm{Ric}[\overline{g}]^{\alpha\beta} - (8\pi)^2(\Psi_{Bang} + C)^4 \right\|_{C^0(M_t)} \lesssim \varepsilon a(t)^{2-c\varepsilon^{\frac{1}{8}}} \tag{3.8.5c}$$

$$\left\| a^{12} \mathcal{K} - \frac{5}{2} \cdot (8\pi)^2 (\Psi + C)^2 \right\|_{C^0(M_t)} \lesssim \varepsilon a(t)^{2-c\varepsilon^{\frac{1}{8}}} \tag{3.8.5d}$$

Finally, $(\overline{M}, \overline{g})$ is causally disconnected and geodesically incomplete toward the past in the same sense as in Theorem 2.8.2.

Remark 3.8.2 (On AVTD behaviour). (3.8.2) demonstrates that the evolution of metric and scalar field are asymptotically velocity term dominated (AVTD) in the sense that $(g, k, n, \partial_0\phi)$ exhibit asymptotic behaviour which, agrees at first order with the solution to the truncated Einstein Vlasov scalar-field system in CMC gauge in which we drop all spatial derivatives from (3.2.20a), (3.2.20c), (3.2.20f) and (3.2.20p). Note that, when considering the evolution of $k^i{}_j$ (or, equivalently, its tracefree part $\hat{k}^i_j$) as well as the lapse equation, Vlasov matter terms are not truncated, leading to lower order terms in the VTD solution – as opposed to the pure scalar-field case, where the VTD solution is simply described by $a^{-3}K_{Bang}$. Similarly truncating the rescaled Vlasov equation (3.2.19a) simply leads to the equation $\partial_t f_{VTD} = 0$, if one views the full Sasaki derivative $\underline{\nabla}_i$ as the horizontal term to be truncated as the form of the isotropic solution suggests. Thus, the Vlasov distribution on the co-mass shell is also AVTD, but one loses control of the derived energy-momentum quantities since the leading order term is influenced by the level of inhomogeneity encoded in even the formal VTD solution via the asymptotics of the spatial metric. Thus, these asymptotics are in line with the critical behaviour of radiation fluids discussed in Section 3.1.3.2, since the corresponding macroscopic quantities degenerate towards the Big Bang, but at controllable rates.

Proof of Theorem 3.8.1. Firstly, we can assume without loss of generality that the initial data is CMC by Remark 3.3.7. If ε is chosen sufficiently small, Lemma 3.3.9 implies that there is a past maximal solution $(\overline{M}, \overline{g}, \phi, \overline{f})$ to the ESFV-system with $\overline{M} \simeq (\mathfrak{t}, t_0] \times M$ for $\mathfrak{t} \in [0, t_0]$. Furthermore, we can assume sufficient regularity for the initial data such that all norms and energies are continuous in time (see Remark 3.3.7)[8].

Thus, for some suitable $K_0 > 0$, the bootstrap assumptions hold on $(t_{Boot}, t_0] \times M$ for $t_{Boot} \in [\mathfrak{t}, t_0)$ (see Assumption 3.3.11). If $\varepsilon > 0$ is, again, chosen to be small enough, the assumption (3.3.13a) is improved by Corollary 3.7.10, and (3.3.13b) is improved by (3.4.16).

[8]In particular, for any $l \in \mathbb{N}$, we again note that the support assumptions on $\mathring{f}$ ensure that $\mathring{f} \in H^l_{1,\underline{\gamma}_0,m}(T^*M_{t_0})$ is equivalent to $\mathring{f} \in H^l_{\mu,\underline{\gamma}_0,m}(T^*M_{t_0})$ for any $\mu \geq 0$.

Thus, (3.8.1a) and (3.8.1b) hold on $(\mathfrak{t}, t_0]$, and consequently (3.8.2) as in the proof of Theorem 2.8.2. In particular, if $\mathfrak{t} > 0$ were to hold, none of the blow-up criteria in Lemma 3.3.9 would be satisfied approaching $\mathfrak{t}$:

(1) By (3.7.12), the eigenvalues of $(G^{-1})^{ij}$ differ from those of $(\gamma^{-1})^{ij}$ by at most $K\varepsilon^{\frac{7}{4}}a^{-c\varepsilon^{\frac{1}{8}}}$ for some $K > 0$, and are thus bounded from above by $1 + \varepsilon^{\frac{7}{4}}a^{-c\varepsilon^{\frac{1}{8}}}$ up to multiplicative constant. Hence, the eigenvalues of G_{ij} are bounded away from 0.
(2) (3.8.2a) prevents (2).
(3) (3.8.2c) prevents (3).
(4) (3.3.11a) and (3.3.11b) are directly prevented by (3.8.1a). Regarding (3.3.11c), we need to verify after scaling transformation that

$$\int_{\mathbb{R}^3} \langle v\rangle_\gamma^{2\cdot 3} \cdot \left(\frac{\langle v\rangle_\gamma}{v^0_\gamma}\right)^{2l} \left(|\nabla^{\leq 1}_{vert} f|^2_{\underline{\gamma}_0}\right) \mu_\gamma^{-1} dv < \infty \tag{3.8.6}$$

holds, where $l \geq 18$ is a sufficiently large natural number. For the massive case, the factor containing l is simply 1. Thus, expanding f into $f_{FLRW} + (f - f_{FLRW})$, and using (3.8.2f), (3.8.1a) and the momentum support bound (3.4.6a), (3.8.6) is bounded by

$$\lesssim \|f_{FLRW}\|^2_{H^1_{3,\underline{\gamma}_0}(T^*M)} + \left(1 + \|G^{-1} - \gamma^{-1}\|_{C^0_G(M)}\right) \cdot (\mathcal{P}^0)^4 \cdot \mathcal{H}^2 \lesssim 1 + \varepsilon^{\frac{7}{2}} a^{-c\varepsilon^{\frac{1}{8}}}\,.$$

In the massless case, (3.4.8) implies

$$\left(\frac{\langle v\rangle_\gamma}{v^0_\gamma}\right)^{2l} = (1 + |v|_\gamma^{-2})^l \lesssim 1\,,$$

and thus the same upper bound holds after updating constants.

Hence, one must have $\mathfrak{t} = 0$. (3.8.3a)-(3.8.3b) follow directly from (3.8.2), the rescaled constraint equations (3.2.20h) and (3.2.20i) and the initial data assumption, as in the proof of (2.8.4)-(2.8.6).

Regarding (3.8.4a), we use (3.8.1a) and (3.8.1b) to obtain

$$\begin{aligned}\left|\frac{d}{dt} f(t, x, v_i)\right| &\lesssim a^{-1-c\varepsilon^{\frac{1}{8}}} \left(\|N\|_{C^1_G(M_t)} + \|f - f_{FLRW}\|_{\dot{C}^1_{1,\underline{G}_0}(T^*M_t)} + \|G^{-1} - \gamma^{-1}\|_{C^1(M_t)}\right)\\ &\lesssim \varepsilon a^{-1-c\varepsilon^{\frac{1}{8}}}\end{aligned}$$

Further, recall that $\langle v\rangle_\gamma = \sqrt{1 + |v|^2_\gamma}$ remains bounded on the momentum support of f, see (3.4.6b). Thus, one has

$$\left|\langle v\rangle_\gamma \frac{d}{dt} f(t, x, v)\right| \lesssim \sqrt{\varepsilon} a^{-1-c\varepsilon^{\frac{1}{8}}}\,.$$

By integrating in time, this implies for any $s, t \in (0, t_0)$ that

$$\sup_{x,v\in T^*M} \langle v\rangle_\gamma \left|f(t, x, v) - f(s, x, v)\right| \lesssim \sqrt{\varepsilon} \left|a(t)^{2-c\varepsilon^{\frac{1}{8}}} - a(s)^{2-c\varepsilon^{\frac{1}{8}}}\right|$$

holds and thus that f converges to a continuous function f_{Bang} in $C^0_{1,\underline{\gamma}_0}(T^*M)$ uniformly when approaching the Big Bang singularity. The bound (3.8.4b) in $C^0_{1,\underline{\gamma}_0}(T^*M)$ then follows directly from above for $t=t_0$ and $s\downarrow 0$. For higher orders, the argument is analogous, instead considering

$$\frac{d}{dt}\left(\langle v\rangle_\gamma^{L-K}\,\hat{\mathcal{D}}_{L,K}f\right)=\langle v\rangle_\gamma^{L-K}\,\hat{\mathcal{D}}_{L,K}\left(\boldsymbol{X}f\right)\,,$$

where $\hat{\mathcal{D}}_{L,K}$ denotes as combination of $L-K$ vertical and K horizontal covariant derivatives with respect to $(T^*M,\underline{\gamma})$. More precisely, it goes through as long as (3.8.1b) can be used to control derivatives of one order higher, i.e., the argument extends to $C^{10}_{1,\underline{\gamma}_0}(T^*M)$. In particular, note that ∂_t and covariant derivatives with respect to $\hat{\underline{\nabla}}$ commute, and commutation errors between Sasaki metrics with respect to γ and G can be controlled sufficiently by (3.8.1a). The momentum support bounds for f_{Bang} are obtained by considering the limit of the support bounds with respect to γ in Lemma 3.4.6.
The remaining Vlasov matter bounds follow directly from (3.8.1a). Everything else is proven identically to Theorem 2.8.2 and [RS18b, Theorem 15.1] – we note the convergence rates in (3.8.5) are weaker than in these results since the presence of Vlasov matter leads to a weaker convergence rate for the lapse (see (3.8.2a)). □

It was not necessary in the analysis above to assume that the momentum support of the initial perturbation was close to that of the FLRW solution, as long as their joint support was uniformly bounded and, in the massless case, bounded away from zero. However, if one additionally requires this, this property is preserved throughout the evolution.

COROLLARY 3.8.3. *Consider the setting of Theorem 3.8.1 and let* $\mathrm{dist}_{\underline{\gamma}}$ *denote the distance function on* T^*M *induced by the Sasaki metric* $\underline{\gamma}$*. If one additionally assumes*

$$\mathrm{dist}_{\underline{\gamma}}(\mathrm{supp}\mathring{f},\mathrm{supp}f_{FLRW})\leq\delta \tag{3.8.7}$$

for some $\delta>0$*, then the following holds for the limiting Vlasov distribution* f_{Bang} *on the Big Bang hypersurface.*

$$\mathrm{dist}_{\underline{\gamma}}(\mathrm{supp}f_{\mathrm{Bang}},\mathrm{supp}f_{FLRW})\lesssim\delta+\varepsilon^{\frac{7}{4}}\,. \tag{3.8.8}$$

PROOF. Let $(\hat{X}_s(t;x,v),\hat{V}_t(s;x,v))$ denote the characteristics of the FLRW solution, emanating from $(x,v)\in T^*M_{t_0}$, i.e., the solution to the following ODE.

$$\frac{d}{dt}\hat{X}^i_s=a^{-1}\,(\gamma^{-1})^{ij}\,\frac{\hat{V}_i}{\sqrt{m^2a^2+|\hat{V}|^2_\gamma}}\qquad\qquad \hat{X}_s(s;x,v)=x \tag{3.8.9a}$$

$$\frac{d}{dt}(\hat{V}_s)_i=a^{-1}\,(\gamma^{-1})^{jk}\,\hat{\Gamma}^l_{ik}\,\frac{\hat{V}_j\hat{V}_l}{\sqrt{m^2a^2+|\hat{V}|^2_\gamma}}\qquad\qquad \hat{V}_s(s;x,v)=v \tag{3.8.9b}$$

Our first goal is to prove that these characteristics are close to the characteristics (X,V) of f given by (3.4.5), i.e., that

$$|X_{t_0}-\hat{X}_{t_0}|_\gamma+|V_{t_0}-\hat{V}_{t_0}|_\gamma\lesssim\varepsilon^{\frac{7}{4}}\,. \tag{3.8.10}$$

holds on $(t,x,v) \in (0,t_0] \times T^*M$. As in the proof of Lemma 3.4.9, the characteristic equations for X and $\hat{X}$ ensure that, for a suitable finite partition $\{(t_j, t_{j-1}]\}_{j=1,\dots,J}$ of $(0,t_0]$, we can choose a finite covering of M such that any characteristic remains contained in a coordinate neighbourhood for each step of the partition. We can also choose this neighbourhood such that Christoffel symbols Γ and $\hat{\Gamma}$ are Lipschitz continuous. Then, it is sufficient to prove

$$|(X_{t_{j-1}} - \hat{X}_{t_{j-1}})(t,x,v)|_\gamma + |(V_{t_{j-1}} - \hat{V}_{t_{j-1}})(t,x,v)|_\gamma \lesssim \varepsilon^{\frac{7}{4}} \tag{3.8.11}$$

for any $t \in (t_j, t_{j-1}]$ and (x,v) in within a coordinate neighbourhood of $(t_j, t_{j-1}] \times T^*M$. We will drop the subscript t_{j-1} for the proof (3.8.11) for notational convenience.
Using (3.8.1a) as well as Lemma 3.4.6, (3.8.9) and (3.4.5) imply the following bound on such a neighbourhood:

$$\begin{aligned}\left|\frac{d}{dt}\left(V-\hat{V}\right)\right| &\lesssim a^{-1-c\varepsilon^{\frac{1}{8}}}\left(|N| + |G^{-1}-\gamma^{-1}|_\gamma + |\Gamma-\hat{\Gamma}|_\gamma + \frac{|V|_\gamma + |\hat{V}|_\gamma}{V^0}|V-\hat{V}|_\gamma\right)\\ &\lesssim a^{-1-c\varepsilon^{\frac{1}{8}}}(\varepsilon^{\frac{7}{4}} + |V-\hat{V}|_\gamma)\end{aligned}$$

Since one has $(V-\hat{V})(t_{j-1},\cdot,\cdot)=0$, the Gronwall lemma implies

$$|(V-\hat{V})_{t_0}(t,x,v)|_\gamma \lesssim \varepsilon^{\frac{7}{4}}$$

using (3.2.6). Consequently, one has

$$\begin{aligned}\left|\frac{d}{dt}\left(X-\hat{X}\right)\right|_\gamma &\lesssim a^{-1-c\varepsilon^{\frac{1}{8}}}\left(|N| + |\Gamma-\hat{\Gamma}|_\gamma + |V-\hat{V}|_\gamma\right)\\ &\lesssim \varepsilon^{\frac{7}{4}} a^{-1-c\varepsilon^{\frac{1}{8}}}\end{aligned}$$

and we obtain (3.8.11) after integrating. By patching coordinate neighbourhoods and iterating over $j=1,\dots,J$, this implies (3.8.10).

Now, note that solutions to (3.8.9) depend on their initial data Lipschitz continuously with respect to γ due to our coordinate choice. Thus, if one has

$$\mathrm{dist}_{\underline{\gamma}}((y,w), \mathrm{supp} f_{FLRW}(t_0,\cdot,\cdot)) \leq \delta$$

for some $(y,w) \in T^*M_{t_0}$, then one has

$$\mathrm{dist}_{\underline{\gamma}}((\hat{X},\hat{V})_{t_0}(0;y,w), \mathrm{supp} f_{FLRW}(0,\cdot,\cdot)) \lesssim \delta \tag{3.8.12}$$

since f_{FLRW} is time-independent. Finally, for any $(x,v) \in \mathrm{supp} f_{\mathrm{Bang}}$, one has $(X_0,V_0)(t_0;x,v) \in \mathrm{supp}\mathring{f}$. By assumption, we can apply (3.8.12) to the characteristic flow of this point with respect to the $(\hat{X},\hat{V})$-characteristics and the distance of the end point of this characterstic to (x,v) is bounded up to a constant by $\varepsilon^{\frac{7}{4}}$ due to (3.8.10). This implies (3.8.8). □

Additionally, the following analogue of (3.8.2f) holds and admits a limiting renormalised metric on the Big Bang hypersurface in the style used in [OGPR23].

COROLLARY 3.8.4. *Let $(\overline{M}, \overline{g}, \phi, f)$ be as in Theorem 3.8.1 and consider the renormalised metrics*

$$\mathcal{G}_t(X,Y) = a(t)^{-2}\,\left(g_t(\mathcal{W}_t(X), \mathcal{W}_t(Y))\right)\,,$$

where

$$\mathcal{W}_{t,x}(w) = \exp\left[-\int_t^{t_0} a(s)^{-3}\,ds\cdot(\Sigma^\sharp)_{t,x}\right]\odot w\,.$$

Then, there exists a Riemannian metric $\mathcal{G}_{Bang}$ on M such that the following holds.

$$\|\mathcal{G}_{Bang}-\gamma\|_{C^{15}_\gamma(M)}\lesssim\varepsilon \tag{3.8.13}$$

$$\|\mathcal{G}-\mathcal{G}_{Bang}\|_{C^{14}_\gamma(M_t)}\lesssim\varepsilon\,a(t)^{2-c\,\varepsilon^{\frac18}} \tag{3.8.14}$$

PROOF. The evolution equation (3.2.20a) implies

$$\begin{aligned}\partial_t\mathcal{G}_{ij} = &-2\,a^{-3}\,(N+1)\,\Sigma_{rs}\,\mathcal{W}^r{}_i\,\mathcal{W}^s{}_j+2\,N\,\frac{\dot a}{a}\,\mathcal{G}_{ij}\\&+2\,a^{-2}\,G_{rs}\,(\Sigma^\sharp)^l{}_i\mathcal{W}^r{}_l\,\mathcal{W}^s{}_j\\&-2\int_{(\cdot)}^{t_0}a(s)^{-3}\,ds\cdot G_{rs}\,(\partial_t\Sigma^\sharp)^l{}_i\,\mathcal{W}^r{}_l\,\mathcal{W}^s{}_j\,.\end{aligned}$$

Using that $\Sigma^\sharp$ and, consequently, $\mathcal{W}$ are self-adjoint with respect to G, the second line cancels with terms in the first, leading to

$$\partial_t\mathcal{G}_{ij}=2\,N\,\frac{\dot a}{a}\,\mathcal{G}_{ij}-2\,a^{-3}\,N\,\Sigma_{rs}\,\mathcal{W}^r{}_i\,\mathcal{W}^s{}_j-2\,G_{rs}\int_{(\cdot)}^{t_0}a(s)^{-3}\,ds\,(\partial_t\Sigma^\sharp)^l{}_i\,\mathcal{W}^r{}_l\,\mathcal{W}^s{}_j\,.$$

Note that Theorem 3.8.1 implies the following bounds for some $c>0$, using Lemma 3.2.2 and (3.2.20c).

$$|\mathcal{W}|_\gamma\lesssim a^{-c\,\varepsilon}$$

$$|\partial_t\Sigma^\sharp|_\gamma\lesssim\varepsilon\,a^{-1-c\,\varepsilon^{\frac18}}$$

Additionally controlling the lapse with (3.8.2a), one has

$$|\partial_t\mathcal{G}|_\gamma\lesssim\varepsilon\,a^{-1-c\,\varepsilon^{\frac18}}\,(|\mathcal{G}|_\gamma+1)\,.$$

The existence of $\mathcal{G}_{Bang}$ as well as the bounds (3.8.13)-(3.8.14) at order 0 now follow as in the proof of Theorem 3.8.1. At higher orders, the respective estimates are obtained similarly. □

REMARK 3.8.5 (Interpreting the asymptotics on the mass shell). One might hope to obtain similar asymptotics for the Vlasov distribution on the mass shell. In particular, when interpreting the Vlasov distribution as a kinetic description of particle trajectories, the velocity of a particle is more naturally understood as a section of the mass shell, while the co-mass shell is the more natural choice from the Hamiltonian perspective. However, even on the level of the VTD solution, understanding the picture on the mass shell becomes difficult: Viewing the Vlasov equation (3.3.12) on the mass shell, it is convenient to view this equation in CMCTC gauge in expansion normalised momentum variables $w^i=a(t)^2q^i$, the characteristics of which are clearly preserved in the FLRW case, and to study

$$f^\sharp_{\mathrm{resc}}(t,x,w) := f^\sharp(t,x,a(t)^{-2}w).$$

Truncating the resulting evolution equation in terms of these variables and denoting the lapse and shear of the VTD solution by n_{VTD} and $k^{\parallel}_{VTD}$, one obtains the following:

$$\partial_t f^\sharp_{VTD} = -2(n_{VTD}-1)\frac{\dot a}{a}w^j - 2n_{VTD}(k^{\parallel}_{VTD})^j{}_i w^i \partial_{w^j} f_{VTD}$$

For initial data $\mathring{f}^\sharp$, this transport equation with initial data at M_{t_0} is solved by $f^\sharp_{VTD}(t,w) = \mathring{f}(x, W_{t_0}(t,x,w))$, where the characteristic W_{t_0} obeys

$$\dot W^j = -2(n_{VTD}-1)\frac{\dot a}{a}W^j + 2(k^{\parallel}_{VTD})^j{}_i W^i\,. \tag{3.8.15}$$

as well as $W_{t_0}(t,x,w) = q$. Since n_{VTD} converges to 1, the leading term in (3.8.15) is given by the leading term of the shear, i.e., one roughly has

$$W_{t_0}(t,x,w) = \exp\left[2\int_t^{t_0} a(s)^{-3}\,ds\cdot (K^{\parallel}_{VTD})_x\right]\odot w + \langle\text{lower order terms}\rangle\,. \tag{3.8.16}$$

When performing the analysis above for $f^\sharp$, one must thus renormalise the particle velocities as follows to account for the VTD behaviour

$$\begin{aligned} f^\sharp_{\mathrm{asymp}}(t,x,w) &= f^\sharp_{\mathrm{resc}}\left(t,x,\exp\left[-2\int_t^{t_0} a(s)^{-3}\,ds\cdot (K^{\parallel}_{Bang})_x\right]\odot w\right)\,. \\ &= f^\sharp\left(t,x,\exp\left[-2\int_t^{t_0} a(s)^{-3}\,ds\cdot (K_{Bang})_x\right]\odot q\right)\,. \end{aligned}$$

Indeed, $f^\sharp_{\mathrm{asymp}}$ converges toward the Big Bang in a similar manner to f, as can be proven by a similar argument to that on the co-mass shell. However, these results can not be directly translated into one another on the Big Bang hypersurface itself since the spatial metric degenerates.

To interpret this asymptotic profile, we analyze $(K^{\parallel}_{Bang})_x$ more closely for an arbitrary point $x\in M$: First, we note that $\Sigma^\sharp$ is self-adjoint with respect to $\mathcal{G}$ on M_t for any $t\in(0,t_0]$ and, consequently, $K^{\parallel}_{Bang}$ is self-adjoint with respect to $\mathcal{G}_{Bang}$ by taking the limit. In particular, $K^{\parallel}_{Bang}$ is diagonalisable. Let $E_{x,+}, E_{x,-}$ and $E_{x,0}$ denote the respective span of eigenspaces associated to positive, zero and negative eigenvalues of $(K^{\parallel}_{Bang})_x$. Note that, if $q\in T^*_xM$ is not orthogonal to $E_{x,+}$, $W_{t_0}(t,x,w(q))$ diverges as $t\downarrow 0$, and thus leaves the support of f if t is sufficiently close to 0. Thus, approaching the Big Bang, the majority of particles have momenta that are orthogonal to $E_{x,+}$. Elements of $E_{x,0}$ are left unchanged by the evolution, while momenta in $E_{x,-}$ approach the origin as $t\downarrow 0$. Thus, Vlasov particle velocities concentrate in the following manner, depending on the signs of eigenvalues of $(K^{\parallel}_{Bang})_x$:

(1) $(+,+,-)$ or $(+,-,-)$: Particle velocities are attracted by $E_{x,-}$.
(2) $(+,0,-)$: Particle velocities concentrate close to the Minkowski sum $E_{x,-}+\left[E_{x,0}\cap\mathrm{supp}\mathring{f}(x,\cdot)\right]$.

(3) $(0,0,0)$: This implies $f^{\sharp}_{\text{asymp}}(t,x,\cdot) = \mathring{f}(x,\cdot)$ for any $t \in (0,t_0)$, and particles are asymptotically distributed close to the FLRW distribution function.

If the momentum support of $\mathring{f}$ is bounded away from the origin, the same interpretations hold, where $E_{x,-}$ can be replaced with $\lim_{r\to\infty} E_{x,-} \cap \{w \in T_xM | |q|_\gamma = r\}$. Recall that we assume this property for f_{FLRW} and $\mathring{f}$ in the massless case.

CHAPTER 4

Quiescent Big Bang formation in $2+1$ dimensions

4.1. Introduction

In this paper, we consider $(2+1)$-dimensional solutions $(\overline{M}=I\times M,\overline{g})$ to the Einstein scalar-field Vlasov (ESFV) system

$$\mathrm{Ric}[\overline{g}]_{\mu\nu}-\frac{1}{2}R[\overline{g}]\,\overline{g}_{\mu\nu}=8\pi\left(T^{SF}_{\mu\nu}+T^{Vl}_{\mu\nu}\right) \tag{4.1.1a}$$

$$T^{SF}_{\mu\nu}=\overline{\nabla}_\mu\phi\,\overline{\nabla}_\nu\phi-\frac{1}{2}\,\overline{g}_{\mu\nu}\,\overline{\nabla}^\alpha\phi\,\overline{\nabla}_\alpha\phi \tag{4.1.1b}$$

$$T^{Vl}_{\mu\nu}=\int_{P_{t,x}}p_\mu\,p_\nu\,f(t,x,p)\,\mathrm{dvol}_{P_{t,x}} \tag{4.1.1c}$$

$$\Box_{\overline{g}}\phi=0 \tag{4.1.1d}$$

$$\mathcal{X}_{\overline{g}}f=0 \tag{4.1.1e}$$

which are close to Friedman-Lemaître-Robertson-Walker (FLRW) solutions in the sense of Lemma 4.2.1. Here, $I\subseteq(0,\infty)$ is an open interval, M is a closed orientable surface of arbitrary genusand the fiber $P_{(t,x)}$ of the co-mass shell P in (4.1.1c) is as in (4.2.9), where we treat Vlasov matter in both the massive ($m=1$) and the massless ($m=0$) case. Furthermore, ϕ is a scalar function on $\overline{M}$, f a nonnegative scalar function on P and $\mathcal{X}_{\overline{g}}$ is the geodesic spray with respect to $\overline{g}$. Our main result can be summarized as follows.

Theorem 4.1.1 (Stability of FLRW solutions to the ESFV system in $2+1$ dimensions). *Let $(M,\mathring{g},\mathring{k},\mathring{\pi},\mathring{\psi},\mathring{f})$ be initial data*[1] *to the ESFV system that is close to FLRW initial data such that the momentum support of $\mathring{f}$ is compact. In the massless case, suppose in addition that one has*

$$\inf\{|p|_{\mathring{g}_x}\,|\,(x,p)\in\mathrm{supp}\mathring{f}\}>0\,.$$

Then, the past maximal globally hyperbolic development of said data in the ESFV system is causally geodesically incomplete. In fact, the Kretschmann scalar exhibits stable blow-up of order t^{-4} as t approaches 0. Furthermore, the solution is asymptotically velocity term dominated[2]*, the Vlasov distribution remains close to its FLRW counterpart, and so does its momentum support if it is close to that of the FLRW distribution function initially.*

The full version of the main theorem can be found in Theorem 4.6.1. We note that, while we will often restrict ourselves to constant mean curvature (CMC) data throughout, this is not a true restriction and can be assumed without loss of generality, see Section

[1] see Section 4.3.2

[2] i.e., the asymptotic behaviour of the solution agrees, at leading order, with a formal analogue of the $(3+1)$-decomposed Einstein equations in which all spatial derivatives are dropped

4.3.3. Additionally, when viewing the Vlasov particle distribution as a function of the mass shell, degeneracies in the metric lead to concentration of particle velocities in the negative eigendirection of the renormalised shear, as is discussed in Remark 4.6.4.

The BKL conjecture (after Belinskiĭ, Khalatinov and Lifshitz, see [LK63; BKL70; BK73; BKL82]) predicts that past asymptotics of cosmological spacetimes are dominated by geometric oscillations for typical matter models, while scalar field matter is predicted to have a stabilizing effect. Theorem 4.1.1 confirms that near-FLRW solutions to the ESFV system in $(2+1)$ dimensions are subject to this stabilization, often referred to as quiescent Big Bang formation. This extends the results of Chapter 3 to three spacetime dimensions. It also turns out that the more rigid geometric structure in the $(2+1)$-dimensional setting allows for an alternative, simpler proof. In particular, that inhomogeneous terms in both matter equations exhibit weaker decay turns out not to be a significant obstruction to our approach, see the discussion in Section 4.1.2.

For $M=\mathbb{S}^2$, the reference FLRW spacetime exhibits a Big Crunch singularity toward the future at some $t=T$ with the same asymptotic properties as the Big Bang singularity, where the scale factor is preserved by the reflection $t\mapsto T-t$. Applying this throughout, Theorem 4.1.1 also shows that near-FLRW data is future stable and causally incomplete for $M=\mathbb{S}^2$ and exhibits entirely analogous asymptotics toward the Big Crunch as toward the Big Bang.

Additionally, as we discuss in Section 4.1.1, Theorem 4.6.1 also implies a stability result for solutions to the $(3+1)$-dimensional Einstein vacuum equations in polarized $U(1)$-symmetry, i.e., vacuum spacetimes that contain a non-degenerate spacelike Killing vector field. Such solutions can be written as spacetimes over $I\times M\times\mathbb{S}^1$ equipped with a metric of the form

$$\breve{g}=e^{-2\sqrt{4\pi}\,\phi}\,\overline{g}+e^{2\sqrt{4\pi}\,\phi}\,(dx^3)^2 \tag{4.1.2}$$

where ϕ is a scalar function on $I\times M$, $\overline{g}$ is spacetime metric on $I\times M$ and $(dx^3)^2$ is the standard coordinate metric on $\mathbb{S}^1$. Using this notation, one obtains the following.

Corollary 4.1.2 (Cosmic Censorship and stable curvature blow-up for polarized $U(1)$-symmetric vacuum solutions). *Consider past maximal solutions to the* $(3+1)$*-dimensional Einstein vacuum equations of the form* (4.1.2) *with*

$$\overline{g}=-dt^2+a^2\,\gamma\ \text{and}\ \partial_t\phi=C\,a^{-2},\ \partial_{x^i}\phi=0$$

where (M,γ) *is a closed orientable surface of constant curvature and* $C<0$.[3] *Let* $(\breve{M},\breve{g})$ *be the past maximal development of nearby polarized* $U(1)$*-symmetric initial data within the Einstein vacuum equations. Then,* $(\breve{M},\breve{g})$ *is a polarized* $U(1)$*-symmetric spacetime that exhibits a crushing singularity with curvature blow-up toward the past boundary and thus is past* C^2*-inextendible. More precisely, the rescaled quantity* $e^{-4\sqrt{4\pi}\,\phi}\,\breve{\mathcal{K}}$ *exhibits stable blow-up, where* $\breve{\mathcal{K}}$ *is the Kretschmann scalar associated with* $\breve{g}$*.*

[3]Requiring $C<0$ excludes extremal Kasner-like spacetimes that do not feature a Big Bang singularity. For further discussion of the reference solutions, see Remark 4.2.3.

The case where $(M,\gamma)\cong(\mathbb{T}^2,(dx^1)^2+(dx^2)^2)$ is also treated in [FRS23, Theorem 6.3]. However, the Kretschmann scalar asymptotics in Corollary 4.1.2 slightly differ from those in [FRS23, Theorem 6.3] since we slice $(\check{M},\check{g})$ differently: The CMC foliation $(M_t)_{t\in(0,t_0]}$ of $(\overline{M},\overline{g})$ underlying Theorem 4.1.1 induces a foliation $(M_t\times\mathbb{S}^1)_{t\in(0,t_0]}$ on $(\check{M},\check{g})$ that leads to Corollary 4.1.2. This foliation is not CMC and, thus, qualitatively different from the CMC slicing used in [FRS23], leading to slightly different expressions of the asymptotics. Nevertheless, Corollary 4.1.2 extends the core result on past C^2-inextendibility and stable curvature blow-up to the cases where (M,γ) is a closed (orientable) constant curvature surface with $\text{genus}(M)=0$ or $\text{genus}(M)\geq 2$.

4.1.1. Previous work. $(2+1)$-dimensional gravity often provides tractable toy models compared to its higher dimensional analogues, especially for the purposes of Quantum Gravity.[4] In these contexts, one often attempts to resolve classical singularities such as the Big Bang singularity with the help of a given Quantum Gravity theory. Nevertheless, it is still useful even from this persective to study the properties of Big Bang singularities in classical $(2+1)$ gravity to understand where precisely deviations between these theories manifest. However, to the author's knowledge, the only results on past asymptotics of cosmological $(2+1)$ spacetimes are either restricted to vacuum solutions or follow implicitly from symmetry-reduced higher dimensional settings.

For $(2+1)$-dimensional cosmological vacuum solutions, the geometric structure of the past singularity is well understood, albeit often without providing concrete asymptotics toward the singularity. Commonly, one tackles these with CMC foliations, and in particular with a CMC gauge that reduces the Einstein vacuum equations to a Hamilton-Jacobi system, see [Mon89; Mon08]. In particular, it was shown by Andersson, Moncrief and Tromba in [AMT97] that any $(2+1)$-dimensional vacuum spacetime containing a CMC hypersurface exhibits a crushing singularity in the direction of contraction. An alternative approach was pioneered by Mess in [Mes07][5], by which these spacetimes can be classified using measured geodesic laminations. This classification, which does not make reference to a time function by itself, was related to the cosmological perspective by Benedetti and Guadagnini in [BG01]. Therein, the authors construct a cosmological time function that measures the geodesic distance of an event from the initial singularity, and the Big Bang can be viewed as a real tree that is in one-to-one correspondence to a measured geodesic lamination. That this description of the initial singularity is independent of the choice of time function and thus also applies to CMC foliations was shown in [And05; Bel14; Bel17]. Similarly, Mondal proved in [Mon23] that, in an appropriate CMC gauge, any vacuum solution curve follows a unique ray in the configuration space of the Hamilton-Jacobi system which approaches a unique point in the space of projective measured laminations toward the initial singularity. To the author's knoweldge, the only extension of such classification approaches to settings containing matter

[4] See [Wit88] and the recent review [Car25]

[5] see also [And+07] for further discussion of this work

occurs in [BG00] which studies certain Einstein solutions coupled to finitely many particles.

Moving on the Einstein scalar-field system and using the notation introduced in (4.1.2), it was shown by Moncrief in [Mon86] that a polarized $U(1)$-symmetric $(3+1)$-dimensional vacuum spacetime is a solution to the Einstein vacuum equations if and only if $(I \times M, \overline{g}, \phi)$ is a solution to the $(2+1)$-dimensional Einstein scalar-field system.[6] Due to precisely this correspondence, one is able to obtain Corollary 4.1.2 from Theorem 4.1.1.
When considering two-dimensional spatial slices with $\text{genus}(M) \geq 2$, the corresponding vacuum Milne solution is nonlinearly stable toward the future within the polarized $U(1)$-symmetry class; see [CBM01]. Toward the past, the first results for the three-dimensional Einstein scalar-field system appear in [IM02] and [CBIM04], in which Fuchsian methods are used to construct families of analytic solutions evolving from the Big Bang singularity which then translate to analytic $(3+1)$-dimensional $U(1)$-symmetric vacuum spacetimes evolving from the past singularity.

In $(3+1)$ dimensions, the original BKL conjecture in [BKL70] predicts that vacuum solutions generally become highly oscillatory as they approach the Big Bang, unless the solution satisfies an integrability condition that holds in polarized $U(1)$-symmetry. It was also predicted therein that this behaviour persists in the presence of most matter sources. However, the authors of the conjecture themselves predicted that scalar field matter should form an exception to this and exhibit quiescent Big Bang formation in which geometric oscillations are dampened; see [BK73]. This quiescent behaviour has been established rigorously in various recent past stability results for cosmological solutions to the Einstein scalar field system in $(d+1)$ dimensions for $d \geq 3$; see, for example, [RS18a; RS18b; BO24a; Spe18; RS22; FU25a; FRS23; OGPR23; BOZ25; Li24b]. In particular, these results on quiescence admit asymptotic control on the solution variables and, consequently, the Kretschmann scalar. This means that the Strong Cosmic Censorship conjecture is verified for these solutions in the following sense: Towards the past, as causal geodesics become incomplete, the spacetime is C^2-inextendible since the Kretschmann scalar blows up.

Of particular relevance to this work, Fournodavlos, Rodnianski and Speck have studied the past asymptotics of the so-called generalized Kasner solutions in [FRS23], i.e., of solutions on $(0,\infty) \times \mathbb{T}^d$ to the Einstein scalar-field and Einstein vacuum systems of the form

$$\overline{g}_{\text{Kas},d} = -dt^2 + \sum_{i=1}^{d} t^{2p_i}\,(dx^i)^2, \quad \sum_{i=1}^{d} p_i = 1, \; \sum_{i=1}^{d} p_i^2 = 1 - 8\pi\, A^2,$$
$$\partial_t \phi_{\text{Kas},d} = A\, t^{-1},\, \partial_{x^i}\phi = 0\,.$$

They showed that non-extremal[7] vacuum Kasner spacetimes for $d = 3$ are nonlinearly stable within the class of polarized $U(1)$-symmetric initial data toward the past. By the same

[6]The factor $\sqrt{4\pi}$ is, in [Mon86], $\sqrt{0.5}$ since the Einstein equations are normalized differently in that work compared to (4.1.1a).
[7]Kasner solutions where none of the Kasner exponents vanish and thus a past singularity forms.

correspondence as discussed above, this immediately implies nonlinear stability of $(2+1)$-dimensional Einstein scalar-field spacetimes of the form

$$\overline{g}_{\mathrm{Kas},2} = -\,S^{2p_3}\,dS^2 + S^{2(p_1+p_3)}\,(dx^1)^2 + S^{2(p_2+p_3)}\,(dx^2)^2,\ \partial_S\phi = p_3\,S^{-1}, \partial_{x^i}\phi = 0\,.$$

After applying the coordinate transformation $dt = S^{p_3}\,dS,\ \tilde{x}_i = (1+p_3)^{-\frac{1}{2}}\,x^i$, this becomes

$$\overline{g}_{\mathrm{Kas},2} = -\,dt^2 + t^{2q}\,(d\tilde{x}^1)^2 + t^{2(1-q)}\,(d\tilde{x}^2)^2,\ \partial_t\phi = \mathrm{sgn}(p_3)\sqrt{q\,(1-q)}\,t^{-1}, \partial_{\tilde{x}^i}\phi = 0,$$

where $q = (p_1+p_3)/(1+p_3)$.

However, it is not obvious that this result extends to $M \times \mathbb{S}^1$ with $\mathrm{genus}(M) \neq 1$: The proof crucially relies on a near-diagonal evolutionary system for the structure coefficients of a Fermi-Walker transported orthonormal frame, in which the only coefficients which may cause instability are automatically zero within the polarized $U(1)$-symmetry class. When perturbing spherical or hyperboloidal spatial metrics, however, these coefficients are no longer perturbed near zero, introducing linear terms of order t^{-1} to the system that cause additional roadblocks to the analysis. Parts of [FRS23] have been extended to non-toroidal spatial geometries in [OGPR23], which provides general conditions for initial data for the Einstein scalar-field system in spatial dimension $d \geq 3$ to form past singularities. However, [OGPR23] does not cover the polarized $U(1)$-symmetric vacuum setting or $d = 2$. Corollary 4.1.2 fills in this gap in results on stable Big Bang formation in the case where $\mathrm{genus}(M) \neq 1$. Since the constant time foliation of $\check{M}$ one obtains from the CMC foliation $(M_t)_{t\in(0,t_0)}$ of $\overline{M}$ is no longer CMC itself, the Kretschmann scalar needs to be rescaled to account for this difference in foliation compared to [FRS23, Theorem 6.3]. Similarly, the induced foliation is also not immediately compatible with the setup in the recent preprint [FG25], which considers the singular initial value problem for quiescent Big Bang scenarios: This work does cover polarized $U(1)$-symmetric initial data on the singularity, but as discussed in [FG25, Remark 1.20], it also only considers locally Gaussian developments emanating from the singularity, which our solutions may not admit. Conversely, the aforementioned rescaling could, at most, yield asymptotic footprints for rescaled analogues of the metric and second fundamental form rather than more natural initial data as discussed above. Even in the $(2+1)$-dimensional scalar-field setting, however, we do not obtain data on the singularity since we cannot take the limit in the momentum constraint, and this issue extends to the vacuum analogue.

To conclude the discussion of Einstein vacuum and scalar-field systems, we briefly discuss the vacuum asymptotics in Gowdy symmetry that effectively reduces the Einstein equations to a $(1+1)$-dimensional problem, see [Gow74; Rin10]. Polarised solutions that are not quotients of Minkowski were shown to be stable in [CIM90]. In [Rin04], it was also shown that a large class of unpolarized $\mathbb{T}^3$ Gowdy symmetric solutions are past stable, including perturbations of Kasner spacetimes with distinct exponents where the directions of symmetry are the directions in which the Kasner exponents are positive. For perturbations of Kasner spacetimes where one of these exponents is negative, solutions undergo a Kasner bounce if they break polarization, making the Kasner solutions generically unstable, see [Rin09a; Li24a].

Consequently, one must also expect to encounter this instability for general non-polarized $U(1)$-symmetric perturbations. Indeed, Corollary 4.1.2 covers Kasner(-like) spacetimes with Kasner exponents $(2/3, 2/3, -1/3)$ where the direction of symmetry is associated with the negative exponent.

Given that scalar fields seem to be very robust in stabilizing past asymptotics, one may wonder whether quiescent Big Bang formation persists when coupling with additional, more physical matter models. This was done for FLRW solutions with spatial geometry $\mathbb{T}^d$ with subcritical Euler fluids in spatial dimension $d \geq 3$ in [BO24b]. Alternatively, since our universe can be modelled as a collisionless self-gravitating gas, Vlasov matter is also a model of cosmological interest; see [BT08, Section 1.2 and 4.1].

Future stability of Einstein-Vlasov solutions with zero cosmological constant is well understood for the Minkowski spacetime and the $(3+1)$-dimensional Milne spacetime; see [Tay17; LT20; FJS21; Big+21] and [AF20] respectively. For accelerated expansion, future stability results can also be found in [AR16] and [Rin13]. The former covers the Einstein-Vlasov system in $\mathbb{T}^3$-Gowdy symmetry, the latter the Einstein Vlasov scalar-field system without symmetry or homgeneity assumptions.

In $(2+1)$ dimensions, isotropic solutions to the massive Einstein-Vlasov system on $I \times M$ of the form

$$-4\,dt^2 + c\,t^2\,\gamma$$

where γ is a homogeneous metric on M and c only depends on the genus of M were shown to be nonlinearly stable in [Faj17a; Faj18], see also [Faj16] for an asymptotic analysis of isotropic solutions. In the negatively curved case, this corresponds to future stability of the vacuum Milne solutions; see Remark 4.1.4.

Toward the past, results had long been restricted to the homogeneous case in $(2+1)$ gravity (see [Faj17b]) or to $(3+1)$ gravity under strong symmetry assumptions (see [Rei96; TR06] in plane, spherical and hyperbolic symmetry, as well as many results on Cosmic Censorship, including [Wea04; DR06; DR16; Smu08; Smu11]). In recent joint work [FU25b] with David Fajman, see also Chapter 3, we have shown the past stability of FLRW solutions to the Einstein scalar-field Vlasov system in $(3+1)$ dimensions, in which the scalar field dominates Vlasov matter and thus the spacetime asymptotics essentially fall in line with the Einstein scalar-field system discussed above. We note that the asymptotic equation of state satisfied by Vlasov matter matches precisely the critical threshold excluded in [BO24b]. This is reflected in the components of the Vlasov energy-momentum tensor only remaining close to their FLRW counterparts up to a leading order offset, which is caused by the interplay of metric degeneracies and momentum characteristics, both in Chapter 3 and in Theorem 4.6.1 below. This effect is seen more strongly when viewing the Vlasov distribution on the mass shell, where they lead to anisotropic velocity concentration, see Remark 4.6.4.

While there are issues in exploiting the conformal structure of $(2+1)$ gravity toward the past as in [Faj17a; Faj18], working in lower spatial dimensions allows one to simplify the proof compared to Chapter 3. In particular, many of the steps taken to control Vlasov matter can be directly extended from Chapter 3. The main contribution of the present paper thus lies in controlling the evolution of geometric variables, as discussed in Section 4.1.2.

4.1.2. Comments on the proof and outline. In this section, I outline the structure of this paper and comment on the main features of the proof of Theorem 4.1.1, the full version of which is given in Theorem 4.6.1.

This work relies on a $(2+1)$ decomposition of the ESFV system using constant mean curvature (CMC) gauge with zero shift, often referred to as a constant mean curvature transported coordinate (CMCTC) gauge. While this means we cannot make use of the conformal structure of $(2+1)$ gravity used in prior work (see Section 4.1.1), using CMCTC gauge seems difficult to avoid: Given that the AVTD picture means we have to expect the asymptotic results (4.6.3) and (4.6.13), the spatial metric will degenerate toward the Big Bang. However, the shift vector is determined up to a conformal Killing field via an elliptic equation in the underlying gauge in which metric terms explicitly occur and thus, as in Remark 2.1.4, would inherit any such degeneracy; see also [Mon86, p. 137, (3.28)] for the conformal $(2+1)$ gauge. This would cause linear terms of order t^{-1} or worse to appear in the evolution equations and prevent one from closing energy estimates.

In Section 4.2, we state the ESFV equations in CMCTC gauge and collect the reference solutions as well as some facts regarding their scale factors. The considerations that need to be taken to ensure local well-posedness and to obtain sufficient continuation criteria are sketched in Section 4.3, along with setting up the necessary norms and energies to state the initial data assumptions and set up the bootstrap argument. I also discuss the analogous background solutions and initial data notions for the polarized $U(1)$-symmetric vacuum setting in Remark 4.2.3 and Remark 4.3.4.

Sections 4.4 and 4.5 contain low order a priori bounds and energy estimates for the solution variables[8], respectively. Proofs are kept brief when they closely follow the arguments in Chapter 3. The essential differences in the approach compared to Chapter 3 occur in Sections 4.5.2 and 4.5.3. Here, the fact that the Ricci tensor $\mathrm{Ric}[G]$ is pure trace means that any curvature terms in the evolution of the shear $\hat{k}^\sharp$ (see (4.2.14b)) can be ignored in the evolution of $\mathfrak{E}^{(L)}(\hat{k},\cdot)$. This significantly simplifies the energy estimates for the geometry as a whole, with two exceptions at order zero and at top order to deal with metric error terms that Vlasov matter introduces.

In order to control Vlasov matter at order zero, one needs to control derivatives of the reference distribution, which introduces terms containing $\Gamma-\hat{\Gamma}$. To then not lose derivatives in the zero order energy estimate, we need a scaled low order energy estimate for the first order energy of $\hat{k}$, which we establish in Lemma 4.5.3 using the Codazzi constraint. On the

[8]i.e., the shear, the lapse, the scalar field and the Vlasov distribution, along with the spatial metric and the associated Ricci curvature

other hand, since Vlasov matter is controlled using the Sasaki metric (4.2.12) which contains first order terms in the spatial metric, one incurs high order curvature terms that do not effectively cancel as they do in the geometric evolution. Since these arise from horizontal derivative terms in the Vlasov equation, i.e., terms including lifts of spatial derivatives to the co-mass shell which are scaled favourably, the scaled top order curvature energy estimate (4.5.12b) is sufficient to close the estimates. Since we only have to control the scalar curvature, one can use the momentum constraint to obtain this estimate directly without requiring additional high order estimates for the shear. Altogether, this allows us to establish a combined total energy estimate in Section 4.5.5 for the total energy (4.3.2h), see (4.5.23).

However, as in Chapter 3, (4.5.23) on its own only provides weak control on components of the spacetime metric and on Vlasov matter since all of the respective energies enter into the total energy (4.3.2h) with a time-dependent weight. For the total Vlasov energy, these rates are introduced to mitigate the effect of borderline and linear shear terms in the commuted Vlasov equation, as well as of metric errors from the linearisation near f_{FLRW}, which would otherwise prevent us from closing the bootstrap argument. The additional weighted shear and curvature energies are then needed to obtain a closed estimate from the total energy, since Vlasov matter appears linearly in the evolution equations of all other variables. Since the Vlasov terms that enter the geometric evolution do so at a rate that is strictly weaker than t^{-1}, this is still sufficient to obtain an appropriate total energy estimate. In turn, the resulting total energy bound controls Vlasov matter sufficiently well to yield a strong bound on the quiescent system, see (4.5.24).

To control Vlasov matter, one can use a hierarchy between horizontal and vertical derivatives in the rescaled Vlasov equation (4.2.14j) analogous to that in Chapter 3. This allows us to introduce scaled Vlasov energies into the total energy in a way that admits a closed total estimate and to obtain the non-scaled Vlasov energy estimates (4.5.20), improving the remaining Vlasov bootstrap assumptions.

We note that one might expect three potential roadblocks from differences in decay rates compared to Chapter 3:

(1) In the background solution, the Vlasov density ρ^{Vl}_{FLRW} behaves as $t^{-\frac{3}{2}}$ instead of $t^{-\frac{4}{3}}$ toward the Big Bang, thus potentially having a larger impact on the spacetime asymptotics compared to the scalar field whose matter density behaves like t^{-2} in both cases.
(2) In the rescaled wave equation, spatial derivative terms are of order 1 instead of order $t^{\frac{1}{3}}$; see (4.2.14i) and (3.2.20p).
(3) In the rescaled Vlasov equation, the horizontal term diverges at order $t^{-\frac{1}{2}}$ instead of $t^{-\frac{1}{3}}$; see (4.2.14j) and (3.2.19a).

The first point does lead to weaker convergence rates for the lapse and consequently all other variables in Theorem 4.6.1 due to its effect on the lapse equations; see (4.2.14g). However, the Vlasov density and pressure are still strictly asymptotically dominated by scalar field matter, which suffices for our purposes. The other two points could only become problematic at top order, since at low orders, it similarly is sufficient that either of these rates are integrable in time to obtain the bounds in Section 4.4. At top order, to deal with issues that do arise from the third point, the scaled energy estimate for the scalar curvature remains sufficient, see Lemma 4.5.5 and (4.5.26).

Once the energy bounds in Corollary 4.5.8 are obtained, the main theorem in Section 4.6 follows by standard arguments as in, for example, [RS18b; Spe18] and Chapter 2. The asymptotic behaviour of the dual Vlasov distribution function $f^\sharp$ on the mass shell is related to that of the spatial metric since the renormalisation of velocities is similar to that used for the renormalised spatial metric $\mathcal{G}$ in Corollary 4.6.2, which converges toward the Big Bang hypersurface. The metric $\mathcal{G}$ in Corollary 4.6.2 also corresponds to the spatial metrics used in the framework of [OGPR23].

Finally, translating these results to the polarized $U(1)$ symmetric vacuum setting, Corollary 4.6.5 follows by taking the asymptotic control on the solution variables established in Theorem 4.6.1 and, with these in hand, computing the norm of the second fundamental form as well as the Kretschmann scalar with respect to the induced foliation. The necessary curvature formulas are collected in Section 4.7.2.

4.1.3. Further applications. To conclude the introduction, I briefly discuss two potential extensions of this work:

Remark 4.1.3 (Polarized $U(1)$-symmetric Einstein-Vlasov solutions). Given Theorem 4.1.1 and Corollary 4.1.2, one is led to ask whether our approach can be applied to polarized $U(1)$-symmetric solutions to the Einstein-Vlasov system in (3+1) dimensions, especially given that the scalar field driving quiescence in $(2+1)$ gravity for the ESFV system dominates Vlasov matter terms. When performing the same decomposition for the Einstein-Vlasov system as in [Mon86], one clearly obtains a different system from the one in Section 4.2.4, since the Vlasov and wave equations become directly coupled. However, since we essentially deal with the scalar field and Vlasov matter components separately (see Section 4.1.2), one might be able to apply similar ideas if Vlasov matter does not determine the spacetime geometry of the reference solution. Specifically, one might be able to apply similar ideas to spacetimes of the form

$$\left(\overline{M} = I \times \mathbb{S}^1 \times M,\ \overline{g} = -e^{2\mu(s,r)}dr^2 + e^{2\lambda(s,r)}dx^2 + r^2\gamma, f = 0\right)$$

within polarized $U(1)$-symmetric solutions to the Einstein-Vlasov equations. In particular, after reducing to the corresponding $(2+1)$ system, the estimates for the geometry in Sections 4.5.2 and 4.5.3 should largely extend. If similar matter estimates to the ones in this work were to still go through, one would likely be able to show that the Kretschmann scalar blow-up exhibited by these specific vacuum Kasner-like solutions is stable within polarized $U(1)$-symmetric solutions to the Einstein equations, extending similar results in surface symmetry in [Rei96] to polarized $U(1)$-symmetry. Furthermore, the resulting quiescent asymptotics would lead f to asymptote to a small limiting distribution on the cotangent bundle of the

Big Bang hypersurface. However, the matter estimates in the $(2+1)$ system would need to be fully reworked as they are now directly and fully coupled, while it is essential in this work that we can ensure quiescence before sharply controlling Vlasov matter. This is left as an open problem for future work.

REMARK 4.1.4 (Theorem 4.1.1 and global stability). For genus$(M) = 0$, as discussed after Theorem 4.1.1, time reflection symmetry implies that the results also show stability of the Big Crunch. A straightforward Cauchy stability argument as in Section 2.10 shows that the past and future stability regimes can be connected to obtain a global stability result. Furthermore, one can likely also obtain global stability results for genus$(M) \geq 2$:

Future global stability for the vacuum solutions on $(0, \infty) \times M \times \mathbb{S}^1$, where M is of genus genus$(M) \geq 2$, was shown in [CBM01] for small polarized $U(1)$-symmetric data.[9] This was done precisely by showing that the $(2+1)$ Milne solution is future stable within the Einstein scalar-field equations, using the correspondence from [Mon86] discussed above.

When all Vlasov terms are set to zero, this can be combined with Theorem 4.1.1 (and, consequently, Corollary 4.1.2) to a global stability result for the $(2+1)$-dimensional Einstein scalar-field system (respectively, a global stability result within the polarized $U(1)$-symmetry class of $(3+1)$ vacuum solutions): The difference in gauge choice between [CBM01] and this work essentially only comes down to whether the shift determines $\partial_t \sigma$ or vice versa (see [CBM01, p. 1014]), where σ is conformal to the evolved spatial metric g with $R[\sigma] = -1$. Thus, one can straightforwardly transform between these gauge choices on a given initial data hypersurface. From there, the regimes of past and future stability can again be connected using Cauchy stability.

On the other hand, [Faj17a] builds upon the approach to control geometry in [CBM01] with an energy formalism for massive Vlasov matter to show future stability of Milne spacetimes in the $(2+1)$-dimensional Einstein-Vlasov system. Thus, one can likely combine both results with Theorem 4.1.1 to obtain global stability for near-FLRW data to the massive ESFV system with negatively curved spatial geometry, but we omit details to focus on Big Bang asymptotics.

4.2. Preliminaries

4.2.1. Notation. We will denote the Levi-Civita connection of a Semi-Riemannian metric $\mathfrak{g}$ by $\nabla[\mathfrak{g}]$ and its Christoffel symbols by $\Gamma[\mathfrak{g}]$. The associated volume form is denoted by $\mathrm{dvol}_{\mathfrak{g}}$, the associated trace operator by $\mathrm{tr}_{\mathfrak{g}}$ and the divergence with respect to $\nabla[\mathfrak{g}]$ by $\mathrm{div}_{\mathfrak{g}}$. In the case where $\mathfrak{g}$ is Riemannian, we denote the induced inner product and pointwise norm by $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ and $|\cdot|_{\mathfrak{g}}$ respectively and write $\langle \cdot \rangle_{\mathfrak{g}} = \sqrt{1 + |\cdot|_{\mathfrak{g}}^2}$. For the Riemann curvature tensor $\mathrm{Riem}[\mathfrak{g}]$ of $\mathfrak{g}$, we use the sign convention

$$\mathrm{Riem}[\mathfrak{g}](X, Y)Z = \left(\nabla[\mathfrak{g}]_X \nabla[\mathfrak{g}]_Y - \nabla[\mathfrak{g}]_Y \nabla[\mathfrak{g}]_X - \nabla[\mathfrak{g}]_{[X,Y]}\right) Z$$

[9] See also the future stability of $(3+1)$ Milne spacetimes proven in [AM04].

for vector fields X, Y and Z. The Ricci and scalar curvature are denoted by $\mathrm{Ric}[\mathfrak{g}]$ and $R[\mathfrak{g}]$ respectively. Recall that, when $\mathfrak{g}$ is a two-dimensional Riemannian metric, one has

$$\mathrm{Riem}[\mathfrak{g}]_{ijkl} = \frac{R[\mathfrak{g}]}{2}\left(\mathfrak{g}_{il}\,\mathfrak{g}_{jk} - \mathfrak{g}_{ik}\,\mathfrak{g}_{jl}\right) \text{ and } \mathrm{Ric}[\mathfrak{g}]_{ij} = \frac{R[\mathfrak{g}]}{2}\mathfrak{g}_{ij}\,.$$

For most of this paper, we work with a $(2+1)$-dimensional spacetime $(\overline{M}, \overline{g})$ that admits a time function $t \in C^\infty(\overline{M})$ whose level sets $M_s = t^{-1}(\{s\}),\ s \in t(\overline{M})$ are diffeomorphic to a closed orientable surface M and have constant mean curvature for all $s \in t(\overline{M})$. In particular, $\overline{M}$ is diffeomorphic to $I \times M$ where $I \subset \mathbb{R}$ is an open interval.

In constant mean curvature transported coordinate (CMCTC) gauge, $\overline{g}$ takes the form

$$\overline{g} = -n^2\,dt^2 + g \tag{4.2.1}$$

where g is a time-dependent Riemannian metric on M. We will abbreviate $\nabla \equiv \nabla[g]$ and $\overline{\nabla} \equiv \nabla[\overline{g}]$. In our convention, the lapse n is positive, i.e..,

$$n = (-\overline{g}(\overline{\nabla}t, \overline{\nabla}t))^{-\frac{1}{2}}\,.$$

The future directed timelike unit normal e_0 to M_t is given by $e_0 = n^{-1}\partial_t$.

We will often work with the time-dependent rescaled Riemannian metric $G = a(t)^{-2}\,g$ on M_t where the scale factor $a \in C^\infty(I)$ is introduced in Lemma 4.2.1. Note that, for a time-dependent vector field X and a time-dependent tensor field $\mathfrak{T}$ on M, one has $\nabla[G]_X\mathfrak{T} = \nabla_X\mathfrak{T}$. We use Δ_G to denote the Laplace-Beltrami operator with respect to G.

The second fundamental form is a symmetric $(0,2)$-tensor on M_t given by

$$k(X,Y) = -\overline{g}(\overline{\nabla}_X e_0, Y)\,.$$

We will assume that the mean curvature $\tau := \mathrm{tr}_g k$ of M_t is given by

$$\tau = -2\,\frac{\dot{a}}{a}\,. \tag{4.2.2}$$

In the following, Latin indices $i, j, \dots$ refer to indices with respect to a given local coordinate system $(x^i)_{i=1,2}$ on M or to the momentum coordinates $(p_i)_{i=1,2}$ on the co-mass shell induced by such coordinates; see Section 4.2.3.

Lowercase Greek letters are spacetime indices with respect to $\overline{M}$, running from 0 to 2. The index 0 always denotes tensor components relative to e_0. We use Einstein summation convention if not stated otherwise.

We raise and lower indices with respect to $\overline{g}$ for tensors on $\overline{M}$ and g for those on M without comment. When we raise indices of a tensor $\mathfrak{T}$ on M with respect to G, the result is denoted by $\mathfrak{T}^\sharp$. To simplify notation, we will occasionally write $\mathfrak{T}_1 * \mathfrak{T}_2$ for an arbitrary contraction of time-dependent tensor fields on M with respect to G up to multiplicative constant and $\nabla^I\mathfrak{T}$ for $I \in \{0,1,2,\dots\}$ when I covariant derivatives of a time-dependent tensor $\mathfrak{T}$ are taken.

We will also work with polarized $U(1)$-symmetric $(3+1)$-dimensional spacetimes $(\check{M} \cong \overline{M} \times \mathbb{S}^1, \check{g})$ where ∂_{x^3} refers to the extended coordinate derivative on $\mathbb{S}^1$ and the index 3 to components with respect to ∂_{x^3}.

Finally, for real functions ζ_1, ζ_2 with $\zeta_2 \geq 0$, we write $\zeta_1 \lesssim \zeta_2$ if there exists a constant $K > 0$ such that $\max\{\zeta_1, 0\} \leq K\zeta_2$. If ζ_1 is also nonnegative, we write $\zeta_1 \simeq \zeta_2$ if $\zeta_1 \lesssim \zeta_2$ and $\zeta_2 \lesssim \zeta_1$ holds.

4.2.2. The reference solutions. Our main result deals with perturbations of FLRW backgrounds with isotropic scalar field and Vlasov matter. Using standard formulas for warped product spacetimes as in, for example, [O'N83, p. 345], one shows that such backgrounds take the following form in the ESFV system (4.1.1).

LEMMA 4.2.1 (FLRW solutions to the ESFV system). *Let $(\overline{M}, \overline{g}_{FLRW})$ be an FLRW spacetime of the form*

$$\overline{M} = (0, T) \times M, \quad \overline{g}_{FLRW} = -dt^2 + a(t)^2\, \gamma$$

where $T \in (0, \infty]$, $a \in C^\infty(0, T)$ is a positive function and (M, γ) is a closed orientable surface with constant curvature $\kappa \in \mathbb{R}$. Further, let ϕ_{FLRW} and f_{FLRW} be spatially homogeneous and isotropic solutions to (4.1.1d) *and* (4.1.1e) *for $\overline{g} = \overline{g}_{FLRW}$, i.e., functions such that*

$$\partial_t \phi_{FLRW} = C\, a(t)^{-2}, \quad \nabla \phi_{FLRW} = 0, \tag{4.2.3a}$$

$$f_{FLRW}(t, x, p_i) = \mathcal{F}\left((\gamma^{-1})^{ij}\, p_i\, p_j\right) \tag{4.2.3b}$$

for some $C \in \mathbb{R}$ and $\mathcal{F} \in C_c^\infty([0, \infty), [0, \infty))$. We define

$$\rho^{Vl}_{FLRW}(t) := 2\pi \int_0^\infty R\sqrt{m^2\, a(t)^2 + R^2}\, \mathcal{F}(R^2)\, dR\,, \tag{4.2.3c}$$

$$\mathfrak{p}^{Vl}_{FLRW}(t) := \pi \int_0^\infty \frac{R^3}{\sqrt{m^2\, a(t)^2 + R^2}}\, \mathcal{F}(R^2)\, dR\,, \tag{4.2.3d}$$

where $m = 0$ in the massless case and $m = 1$ in the massive case.

Then, $(\overline{M}, \overline{g}_{FLRW}, \phi_{FLRW}, f_{FLRW})$ is a solution to the ESFV system (4.1.1) *if and only if the Friedman equation is satisfied:*

$$\dot{a}^2 = 4\pi\, C^2\, a^{-2} + 4\pi\, a^{-1}\left(\rho^{Vl}_{FLRW} + 2\, \mathfrak{p}^{Vl}_{FLRW}\right) - \kappa \tag{4.2.3e}$$

Note that ρ^{Vl}_{FLRW} and $\mathfrak{p}^{Vl}_{FLRW}$ are rescalings of the non-zero components of $T^{Vl}[\overline{g}_{FLRW}, f_{FLRW}]$, i.e.,

$$\rho^{Vl}_{FLRW} = a^3\, T^{Vl}_{00}[\overline{g}_{FLRW}, f_{FLRW}] \quad \text{and} \quad \mathfrak{p}^{Vl}_{FLRW} = \frac{1}{2}\, a^3\, \mathrm{tr}_{a^2\gamma}\left(T^{Vl}[\overline{g}_{FLRW}, f_{FLRW}]\right)\,.$$

Moreover, observe that $(\overline{M}, \overline{g})$ is of the form (4.2.1) and that the mean curvature τ of the constant time surfaces M_t satisfies (4.2.2). In particular, (4.2.3e) implies the following identities for the mean curvature τ.

$$\tau^2 = 16\pi\, C^2\, a^{-4} + 16\pi\, a^{-3}\left(\rho^{Vl}_{FLRW} + 2\, \mathfrak{p}^{Vl}_{FLRW}\right) - 4\,\kappa\, a^{-2} \tag{4.2.4a}$$

$$\partial_t\tau = 16\pi\, C^2\, a^{-4} + 8\pi\, a^{-3}\left(\rho_{FLRW}^{Vl} + 4\,\mathfrak{p}_{FLRW}^{Vl}\right) - 2\,\kappa\, a^{-2} \tag{4.2.4b}$$

Throughout, we assume $a(0)=0$, that a is positive somewhere and that the scalar field is non-trivial, i.e., that $C \neq 0$. Under these assumptions, one obtains the following result as in Lemma 3.2.2 and [Spe18, Lemma 3.7].

LEMMA 4.2.2 (Properties of the scale factor). *The singular initial value problem* (4.2.3e) *with $a(0)=0$ and $C\neq 0$ has a unique nonnegative solution on the maximal interval of existence $(0,T)$ for some $T\in(0,\infty]$. For $\kappa\leq 0$, one has $T=\infty$. Else, one has $T<\infty$, $\left(-\frac{T}{2},\frac{T}{2}\right)\ni t\mapsto a\left(t+\frac{T}{2}\right)$ is even and the scale factor can be continuously extended to $a(T)=0$.*

As $t\downarrow 0$, one has $a(t)\simeq\sqrt{t}$. Further, given $t_0<\frac{T}{2}$, $t\in(0,t_0)$ and $q\in(0,\infty)$, there exists a constant K independent of t, t_0 and q such that the following estimates hold.

$$\sqrt{4\pi}\,|C|\,a(t)^{-1} \leq \dot{a}(t) + \max\{0,\kappa\} \tag{4.2.5a}$$

$$\int_t^{t_0} a(s)^{-2-q}\,ds \leq \frac{K}{q}\,a(t)^{-q} \tag{4.2.5b}$$

We recall that, for $\kappa>0$, the time reflection symmetry of a implies that the entirety of Theorem 4.6.1 also applies toward the future for $t_0'\in(\frac{T}{2},T)$: Similar to how one argues in Section 4.3.3, one can use Cauchy stability in harmonic gauge to extend past $t=\frac{T}{2}$ while controlling the perturbation size and can then find a nearby CMC hypersurface to restart the stability analysis.

REMARK 4.2.3 (Reference solutions for Corollary 4.1.2). For $f_{FLRW}=0$, the FLRW solutions above correspond to the following polarized $U(1)$-symmetric $(3+1)$-dimensional vacuum solutions for some $t_1>0$.

$$\breve{g} = \exp\left(2\sqrt{4\pi}\,C\int_t^{t_1} a(s)^{-2}\,ds\right)\left(-dt^2 + a(t)^2\,\gamma\right) + \exp\left(-2\sqrt{4\pi}\,C\int_t^{t_1} a(s)^{-2}\,ds\right)(dx^3)^2 \tag{4.2.6}$$

Using the curvature formulas in Section 4.7 for $\nabla\phi=0$, $\hat{k}=0$ and $n=1$ as well as the Friedman equation (4.2.3e), one computes that the Kretschmann scalar

$$\breve{\mathcal{K}} = \mathrm{Riem}[\breve{g}]_{\alpha\beta\gamma\delta}\,\mathrm{Riem}[\breve{g}]^{\alpha\beta\gamma\delta}$$

satisfies

$$\begin{aligned}\exp\left(4\sqrt{4\pi}C\int_t^{t_1} a(s)^{-2}\right)\breve{\mathcal{K}} &= 128\pi\, C^2\, a(t)^{-4}\left(\frac{\tau(t)}{2} + \sqrt{4\pi}\, C\, a(t)^{-2}\right)\\ &= 128\pi\, C^2\, a(t)^{-8}\left(\sqrt{4\pi}\,C - \sqrt{4\pi\, C^2 - \kappa\, a(t)^2}\right)^2 .\end{aligned} \tag{4.2.7}$$

To better understand these solutions, we transform the spacetime metric $\breve{g}$ into the more familiar Kasner-like form, depending on the signs of κ and C.

For $\kappa=0$, the Friedman equation (4.2.3e) with $a(0)=0$ is solved by $a(t)=2\,\pi^{\frac14}\,|C|^{\frac12}\,t^{\frac12}$. In particular, there exists $t_1\in(0,\infty)$ such that $a(t_1)=1$.

Assuming in addition that $C < 0$, we use (4.2.3e) to introduce the new time coordinate S.

$$\dot{S}(t) = \exp\left(\sqrt{4\pi}\, C \int_t^{t_1} a(s)^{-2}\, ds\right) = a(t),\ S(0) = 0$$

It follows that $S(t)$ is proportional to $t^{\frac{3}{2}}$ and, consequently, that $\breve{g}$ corresponds to the Kasner solution

$$\breve{g} = -dS^2 + S^{\frac{4}{3}}\,\left((dx^1)^2 + (dx^2)^2\right) + S^{-\frac{2}{3}}\,(dx^3)^2\,,$$

possibly upon rescaling the spatial coordinates by constants. Furthermore, one then has

$$a(t)^4\,\check{\mathcal{K}} = \exp\left(4\sqrt{4\pi}\, C \int_t^{t_1} a(s)^{-2}\, ds\right)\check{\mathcal{K}} = 2048\,\pi^2\, C^4\, a(t)^{-8}$$

and thus that $\check{\mathcal{K}}$ is proportional to S^{-4}.

For $C > 0$ and $\kappa = 0$, one finds that the solution can be transformed to the following extremal Kasner solution:

$$\breve{g} = -\,dS^2 + (d\tilde{x}^1)^2 + (d\tilde{x}^2)^2 + S^2(d\tilde{x}^3)^2$$

Here, S is proportional to a and $\tilde{x}^i$ are time-independent rescalings of the original coordinates. One also sees from (4.2.7) that the Kretschmann scalar vanishes. Indeed, this simply is the Minkowski metric expressed in an unusual coordinate system.

For $\kappa \neq 0$, one obtains that the spacetime metric can be written as

$$\breve{g} = -dS^2 + b(S)^2\,\gamma + b_3(S)^2\,(dx^3)^2\,. \tag{4.2.8}$$

where the scale factors asymptotically correspond to their spatially flat counterparts at leading order, as we compute in Section 4.7.1.

For $C < 0$, one has $b(S) \simeq S^{\frac{2}{3}}$ and $b_3(S) \simeq S^{-\frac{1}{3}}$ as S approaches 0 and the leading order terms in $e^{4\sqrt{4\pi}\,\phi_{FLRW}}\check{\mathcal{K}}$ are also analogous to the spatially flat case.

For $C > 0$, we obtain $b(S) \simeq 1$ and $b_3(S) \simeq S$ as S approaches 0. However, unlike in the spatially flat setting, $(\check{M}, \breve{g})$ exhibits a curvature singularity of order S^{-2}. Since the metric still asymptotes towards an extremal Kasner-like metric, one cannot expect such singularities to be stable. In fact, they appear to be unstable within polarized $U(1)$-symmetric solutions; see Remark 4.6.8. This is why we do not consider $C > 0$ (i.e., $\phi > 0$) in Corollary 4.1.2.

4.2.3. Rescaled variables and Sasaki metrics. Before declaring our normalised solution variables, we have to introduce canonical local coordinates on the co-mass shell

$$P = \bigsqcup_{t\in I} P_t\,,\quad P_t = \left\{(t,x,p) \in T^*\overline{M} \mid x \in M,\ p \in T^*_{(t,x)}\overline{M},\ p_\mu\, p^\mu = -m^2,\ p^0 > 0\right\}. \tag{4.2.9}$$

Let $(U, (x^i)_{i=1,2})$ be a coordinate neighbourhood of M and

$$V := \{(t,x,p) \in P \mid x \in U\}\,.$$

Then, for $w = (s, y, q) \in V$, we define $y^0(w) = s$ and $y^i(w) = x^i(y)$ for $i = 1, 2$. Furthermore, we introduce the canonical momentum coordinates $(p_\mu)_{\mu=0,1,2}$ by

$$q = p_0(w)\,\theta^0\big|_y + p_i(w)\,dx^i\big|_y \ ,$$

where θ^0 is the dual coframe to the future directed unit normal e_0. Note that the co-mass shell relation and (4.2.1) imply

$$p^0 = -p_0 = \sqrt{m^2 + |p|_g^2}\,.$$

Then, $(y^0, y^1, y^2, p_1, p_2)$ is a local coordinate system on V canonically derived from coordinates (x^1, x^2) on U such that (y^i, p_i) are time-independent. In all that follows, we will use such canonically induced coordinates on the co-mass shell without further comment. We will write t instead of $y^0(t, x, p)$ and x^i instead of $y^i(t, x, p)$.

Now, we can introduce the following rescaled solution variables.

Definition 4.2.4 (Expansion normalised variables). Consider a solution $(\overline{M}, \overline{g}, \phi, \overline{f})$ of the ESFV system (4.1.1) in CMCTC gauge, i.e., where $\overline{g}$ is of the form (4.2.1) and $\overline{M}$ is foliated by constant time slices M_t with constant mean curvature τ. We define the following normalised variables as

$$G_{ij} = a^{-2} g_{ij}, \quad N = n - 1, \ \Psi = a^2 e_0\phi - C \tag{4.2.10a}$$

We decompose the second fundamental form k_{ij} as

$$k_{ij} = \frac{\tau}{2}\, g_{ij} + \hat{k}_{ij} \tag{4.2.10b}$$

and refer to the tracefree spatial tensor $\hat{k}$ as the shear. Additionally, we define the rescaled momentum variables

$$v_i = p_i \quad \text{and} \quad v^0 = a\,p^0 = \sqrt{m^2\,a^2 + |v|_G^2}\,. \tag{4.2.10c}$$

The rescaled Vlasov energy-momentum components are defined as follows:

$$\rho^{Vl} = a^3\, T^{Vl}_{00}, \ \jmath^{Vl}_i = a^2\, T^{Vl}_{0i}, \ \mathfrak{p}^{Vl} = \frac{1}{2}\, a^3\, \mathrm{tr}_g T^{Vl}, \ S^{Vl}_{ij} = a^3\, T^{Vl}_{ij}$$
$$S^{Vl,\parallel}_{ij} = S^{Vl}_{ij} - \mathfrak{p}^{Vl} G_{ij}$$

Note that $S^{Vl,\parallel}$ vanishes if the Vlasov distribution is isotropic.

The rescaled metric G on M induces the Sasaki metric $\underline{G}$ on T^*M as follows: For the local vector field frame $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{B}^1, \mathbf{B}^2\}$ given by

$$\mathbf{A}_i = \partial_{x^i} + v_l\,\Gamma[G]^l_{ij}\,\partial_{v_j} \text{ and} \tag{4.2.11a}$$

$$\mathbf{B}^i = \partial_{v_i}\,, \tag{4.2.11b}$$

we set

$$\underline{G}(\mathbf{A}_i, \mathbf{A}_j) = G_{ij}, \ \underline{G}(\mathbf{A}_i, \mathbf{B}^j) = 0, \ \underline{G}(\mathbf{B}^i, \mathbf{B}^j) = (G^{-1})^{ij} \text{ for } i, j = 1, 2\,. \tag{4.2.12}$$

The Levi-Civita connection with respect to $\underline{G}$ is denoted by $\underline{\nabla} \equiv \nabla[\underline{G}]$. Such metrics were first studied on the tangent bundle by Sasaki in [Sas58]. We refer the recent review article [CGS22] for a more detailed overview of this equivalent formulation on the cotangent bundle and co-mass shell and its application to relativistic kinetic theory.

Note that the horizontal derivatives $\{\mathbf{A}_1, \mathbf{A}_2\}$ are orthogonal to the vertical derivatives $\{\mathbf{B}^1, \mathbf{B}^2\}$ with respect to $\underline{G}$. The latter spans an integrable distribution on T^*M whose leaves are the tangent spaces ${}_{\bullet}T^*_x M,\ x \in M$. Thus, we can distinguish between covariant derivatives taken in vertical directions, denoted by $\underline{\nabla}_{\mathrm{vert}}$ and those taken in horizontal directions, denoted by $\underline{\nabla}_{\mathrm{hor}}$, without making reference to local coordinates.
Similarly, this induces the Riemannian metric $\underline{G}|_{\mathrm{vert}}$ on the vertical distribution by

$$\underline{G}|_{\mathrm{vert}}(\mathbf{B}^i, \mathbf{B}^j) = (G^{-1})^{ij}\,.$$

Viewing (G_{ij}) in transported local coordinates as a matrix and denoting its determinant by $\det G$, note that one has

$$\mathrm{dvol}_{(\underline{G}|_{\mathrm{vert}})_x} = \sqrt{\det G_x^{-1}}\, dv^1 \wedge dv^2 = \mathrm{dvol}_{G_x^{-1}} \tag{4.2.13}$$

In the following, we will write $\underline{\nabla}_i = \underline{\nabla}_{\mathbf{A}_i}$ and $\underline{\nabla}^{i+2} = \underline{\nabla}_{\mathbf{B}^i}$ for $i = 1, 2$. We will also use the momentum-rescaled Sasaki metric $\underline{G}_0$ which is determined by

$$\underline{G}_0(\mathbf{A}_i, \mathbf{A}_j) = G_{ij},\ \underline{G}_0(\mathbf{A}_i, \mathbf{B}^j) = 0,\ \underline{G}_0(\mathbf{B}^i, \mathbf{B}^j) = (v^0)^{-2}\,(G^{-1})^{ij}\,.$$

The reference metrics $\underline{\gamma}$ and $\underline{\gamma}_0$ on P_t are defined analogously for $i, j = 1, 2$.

$$\begin{aligned}
\underline{\gamma}\left(\partial_{x^i} + v_m \Gamma[\gamma]^m_{li}\, \partial_{v_l}, \partial_{x^j} + v_s \Gamma[\gamma]^s_{rj}\, \partial_{v_r}\right) &= \gamma_{ij},\\
\underline{\gamma}(\partial_{x^i} + \Gamma[\gamma]^m_{li}\, \partial_{v_l}, \partial_{v_j}) &= 0\\
\underline{\gamma}(\partial_{v_i}, \partial_{v_j}) &= (\gamma^{-1})^{ij}
\end{aligned}$$

$$\begin{aligned}
\underline{\gamma}_0\left(\partial_{x^i} + v_m \Gamma[\gamma]^m_{li}\, \partial_{v_l}, \partial_{x^j} + v_s \Gamma[\gamma]^s_{rj}\, \partial_{v_r}\right) &= \gamma_{ij},\\
\underline{\gamma}_0(\partial_{x^i} + v_m\Gamma[\gamma]^m_{li}\, \partial_{v_l}, \partial_{v_j}) &= 0\\
\underline{\gamma}_0(\partial_{v_i}, \partial_{v_j}) &= (m^2 + |v|^2_\gamma)^{-1}\,\gamma_{ij}
\end{aligned}$$

4.2.4. The expansion normalised $2+1$ system in CMCTC gauge. The rescaled metric and shear satisfy the following evolution equations.

(4.2.14a)

$$\partial_t G_{ij} = -\,2\,a^{-2}\,(N+1)\,\hat{k}_{ij} + 2\,N\,\frac{\dot{a}}{a}\,G_{ij}$$

(4.2.14b)

$$\begin{aligned}
\partial_t (\hat{k}^\sharp)^i{}_j = &\frac{1}{2}\,(N+1)\,R[G]\,\mathbb{I}^i_j + \left(4\pi\,a^{-1}\,\rho^{Vl}_{FLRW} - 2\,\kappa\right)\mathbb{I}^i_j + \tau\,N\,(\hat{k}^\sharp)^i{}_j - \nabla^{\sharp i}\nabla_j N\\
&- 8\pi\nabla^{\sharp i}\phi\,\nabla_j\phi - 8\pi\,a^{-1}\left((S^{Vl,\|})^{\sharp i}{}_j + \mathbb{I}^i_j \int_{T^*_\bullet M} \frac{m^2\,a^2 + \frac{1}{2}\,|v|^2_G}{v^0}\, f\,\mathrm{dvol}_{(\underline{G}|_{vert})_\bullet}\right)
\end{aligned}$$

The Christoffel symbols and the scalar curvature with respect to G evolve as follows:

$$\partial_t \Gamma[G]^i_{jl} = -a^{-2}\left[\nabla_j\left((N+1)\,(\hat{k}^\sharp)^i{}_l\right) + \nabla_l\left((N+1)\,(\hat{k}^\sharp)^i{}_j\right) - \nabla^{\sharp i}\left((N+1)\,\hat{k}_{jl}\right)\right] + \frac{\dot{a}}{a}\left[\mathbb{I}^i_l\,\nabla_j N + \mathbb{I}^i_j\,\nabla_l N - G_{jl}\,\nabla^{\sharp i} N\right] \tag{4.2.14c}$$

$$\partial_t R[G] = -2a^{-2}\,\mathrm{div}_G\mathrm{div}_G\left((N+1)\,\hat{k}\right) - 4\,N\,\frac{\dot{a}}{a}\,(R[G]-2\kappa) - 8\,\kappa\,N\,\frac{\dot{a}}{a} \tag{4.2.14d}$$

The Hamiltonian and momentum constraints take the form

$$R[G] - 2\,\kappa = a^{-2}\,|\hat{k}|^2_G + 8\pi\,a^{-2}\left[|\Psi|^2 + 2\,C\,\Psi\right] + 16\pi\,a^{-1}\left(\rho^{Vl} - \rho^{Vl}_{FLRW}\right), \tag{4.2.14e}$$

$$(\mathrm{div}_G\hat{k})_i = -8\pi\,(\Psi + C)\,\nabla_i\phi - 8\pi\,\jmath^{Vl}_i. \tag{4.2.14f}$$

Using (4.2.4a)-(4.2.4b), the lapse equations can be written as

$$\mathcal{L}N = H, \quad \tilde{\mathcal{L}}N = \tilde{H} \tag{4.2.14g}$$

with the elliptic operators

$$\mathcal{L}\zeta = a^2\Delta_G\zeta - h\,\zeta \quad \text{and} \quad \tilde{\mathcal{L}}\zeta = a^2\,\Delta_G\zeta - \tilde{h}\,\zeta\,,$$

where

$$\begin{aligned}
h &= \left[16\pi\,C^2 + 16\pi\,a\left(\rho^{Vl}_{FLRW} + \mathfrak{p}^{Vl}_{FLRW}\right) - 2\,\kappa\,a^2\right] + H\,,\\
\tilde{h} &= \left[16\pi\,C^2 + 8\pi a\left(\rho^{Vl}_{FLRW} + 2\,\mathfrak{p}^{Vl}_{FLRW}\right) - 2\,\kappa\,a^2\right] + \tilde{H}\,,\\
H &= |\hat{k}|^2_G + 8\pi\,\Psi^2 + 16\pi\,C\,\Psi + 8\pi\,a\left(\rho^{Vl} - \rho^{Vl}_{FLRW}\right) \text{ and}\\
\tilde{H} &= a^2\,(R[G]-2\kappa) - 8\pi\,a^2\,|\nabla\phi|^2_G - 8\pi\,a\left(\rho^{Vl} - \rho^{Vl}_{FLRW}\right).
\end{aligned}$$

The wave equation takes the form

$$\partial_t\nabla\phi = a^{-2}\,(\Psi + C)\,\nabla N + a^{-2}\,(N+1)\,\nabla\Psi\,, \tag{4.2.14h}$$

$$\partial_t\Psi = \mathrm{div}_G\left((N+1)\,\nabla\phi\right) + 2\,N\,\frac{\dot{a}}{a}\,(\Psi + C)\,. \tag{4.2.14i}$$

The Vlasov equation can be written as

$$\partial_t f = -a^{-1}\,\frac{v^{\sharp j}}{v^0}\,(N+1)\,\mathbf{A}_j f + a^{-1}\,v^0\,\nabla_j N\,\mathbf{B}^j f. \tag{4.2.14j}$$

To extend the Vlasov equation to higher orders, we introduce the following differential operator $\boldsymbol{X}$ acting on time-dependent tensors $\mathfrak{T}$ on T^*M.

$$\boldsymbol{X}\mathfrak{T} := -a^{-1}\,\frac{v^j}{v^0}\,(N+1)\,\underline{\nabla}_j\mathfrak{T} + a^{-1}\,v^0\,\nabla_j N\,\mathbf{B}^j\mathfrak{T} \tag{4.2.14k}$$

Additionally, for time-dependent functions ζ on M and ξ on TM, (4.2.14a) implies the following formulas that we will use below without further comment.

$$\partial_t\int_M \zeta\,\mathrm{dvol}_G = \int_M (\partial_t\zeta - N\tau\zeta)\,\mathrm{dvol}_G$$

$$\partial_t \int_{T^*M} \xi \,\mathrm{dvol}_{\underline{G}} = \int_{T^*M} \partial_t \xi \,\mathrm{dvol}_{\underline{G}}$$

Finally, note

$$\nabla_{i_1} \dots \nabla_{i_\ell} \int_{T^*_\bullet M} \xi \,\mathrm{dvol}_{(\underline{G}|_{vert})_\bullet} = \int_{T^*_\bullet M} \underline{\nabla}_{i_1} \dots \underline{\nabla}_{i_\ell} \xi \,\mathrm{dvol}_{(\underline{G}|_{vert})_\bullet} . \tag{4.2.15}$$

4.3. Initial data, local well-posedness and bootstrap assumptions

In this section, we state the initial data assumptions, describe how they initialize a unique maximal globally hyperbolic development toward the past and state the bootstrap assumptions.

4.3.1. Solution norms and energies. Let $\ell \geq 0$ be an integer and $\mu \in [0, \infty)$. For functions ζ on $\overline{M}$ and ξ on the mass shell P, we introduce the following Sobolev and supremum norms:

$$\|\zeta\|_{H^\ell_G(M_t)} = \left(\sum_{I=0}^{\ell} \int_{M_t} |\nabla^I \zeta|^2_G \,\mathrm{dvol}_G \right)^{\frac{1}{2}} \quad \|\xi\|_{H^\ell_{\mu,\underline{G}_0}(T^*M_t)} = \left(\sum_{I=0}^{\ell} \int_{T^*M_t} \langle v \rangle_G^{2\mu} \, |\underline{\nabla}^I \xi|^2_{\underline{G}_0} \,\mathrm{dvol}_{\underline{G}} \right)^{\frac{1}{2}}$$

$$\|\zeta\|_{C^\ell_G(M_t)} = \sum_{I=0}^{\ell} \sup_{x \in M_t} |\nabla^I \zeta(t,x)|_G \qquad \|\xi\|_{C^\ell_{\mu,\underline{G}_0}(T^*M_t)} = \sum_{I=0}^{\ell} \sup_{(x,v)\in T^*M_t} \langle v \rangle_G^{\mu} \, \left|\underline{\nabla}^I \xi(t,x,v)\right|_{\underline{G}_0}$$

$$\|\xi\|_{C^\ell_{\mu,\underline{\gamma}_0}(T^*M_t)} = \sum_{I=0}^{\ell} \sup_{(x,v)\in T^*M_t} \langle v \rangle_\gamma^{\mu} \, \left|\nabla[\underline{\gamma}]^I \xi(t,x,v)\right|_{\underline{\gamma}_0}$$

Using the rescaled variables introduced above, we can now define our main solution norms:

Definition 4.3.1 (Solution norms).

(4.3.1a)

$$\begin{aligned}\mathcal{H}(t) = &\|\Psi\|_{H^{10}_G(M_t)} + \|\nabla\phi\|_{H^9_G(M_t)} + a(t)\,\|\nabla\phi\|_{\dot{H}^{10}_G(M_t)} + \|\hat{k}\|_{H^{10}_G(M_t)} + \|G-\gamma\|_{H^{10}_G(M_t)}\\ &+ \|R[G] - 2\,\kappa\|_{H^8_G(M_t)} + a(t)\,\|R[G]-2\,\kappa\|_{\dot{H}^9_G(M_t)} + \|f - f_{FLRW}\|_{H^{10}_{1,\underline{G}_0}(T^*M_t)}\\ &+ a(t)^{-1}\,\|N\|_{H^8_G(M_t)}\end{aligned}$$

(4.3.1b)

$$\begin{aligned}\mathcal{C}(t) = &\|\Psi\|_{C^8_G(M_t)} + \|\nabla\phi\|_{C^7_G(M_t)} + a(t)\,\|\nabla\phi\|_{C^8_G(M_t)} + \|\hat{k}\|_{C^8_G(M_t)} + \|G-\gamma\|_{C^8_G(M_t)}\\ &+ \|R[G] - 2\,\kappa\|_{C^6_G(M_t)} + \|\rho^{Vl} - \rho^{Vl}_{FLRW}\|_{C^8_G(M_t)} + \|S^{Vl,\parallel}\|_{C^8_G(M_t)}\\ &+ a(t)^{-1}\,\|N\|_{C^6_G(M_t)}\end{aligned}$$

We will see in the proof of Theorem 4.6.1 that $\mathcal{H}$ essentially controls $\mathcal{C}$, as in Corollary 3.7.10. Thus, the goal is to control $\mathcal{H}$ in a bootstrap argument, using the following associated energies:

Definition 4.3.2 (Energies). Let K, L be integers with $L \geq K \geq 0$. We define the following energies for the scalar field, the shear, the lapse, the scalar curvature and the Vlasov distribution.

$$\mathfrak{E}^{(L)}(\phi, \cdot) = (-1)^L \int_M \left[\Psi\,\Delta_G^L \Psi - a^2\,\phi\,\Delta_G^{L+1}\phi\right] \mathrm{dvol}_G \tag{4.3.2a}$$

$$\mathfrak{E}^{(L)}(\hat{k},\cdot) = (-1)^L \int_M \langle \hat{k}, \Delta_G^L \hat{k}\rangle_G \,\mathrm{dvol}_G \tag{4.3.2b}$$

$$\mathfrak{E}^{(L)}(N,\cdot) = (-1)^L \int_M N\Delta_G^L N \,\mathrm{dvol}_G \tag{4.3.2c}$$

$$\mathfrak{E}^{(L)}(R,\cdot) = (-1)^L \int_M (R[G]-2\,\kappa)\,\Delta_G^L(R[G]-2\,\kappa)\,\mathrm{dvol}_G \tag{4.3.2d}$$

$$\mathfrak{E}^{(L)}_{\mu,0}(f,\cdot) = \int_{T^*M_t} \langle v\rangle_G^{2\mu}\,|\underline{\nabla}^L_{vert}(f-f_{FLRW})|^2_{\underline{G}_0}\,\mathrm{dvol}_{\underline{G}} \tag{4.3.2e}$$

$$\mathfrak{E}^{(L)}_{\mu,K}(f,\cdot) = \int_{T^*M_t} \langle v\rangle_G^{2\mu}\,|\underline{\nabla}^{L-K}_{\mathrm{vert}}\underline{\nabla}^K_{\mathrm{hor}} f|^2_{\underline{G}_0}\,\mathrm{dvol}_{\underline{G}} \tag{4.3.2f}$$

$$\mathfrak{E}^{(\leq L)}_{\mu,\leq K}(f,\cdot) = \sum_{R=1}^{L}\sum_{S=0}^{\min\{K,R\}} \mathfrak{E}^{(R)}_{\mu,S}(f,\cdot) \tag{4.3.2g}$$

Let $\varepsilon>0$ denote the square root of the perturbation size, see Assumption 4.3.3, and let $\omega>0$ be sufficiently small, as discussed below. Then, we define the following total energies:

$$\begin{aligned}\mathfrak{E}^{(L)}_{\mathrm{total}} &= \mathfrak{E}^{(L)}(\phi,\cdot) + \left(\varepsilon^{\frac14}+a^{\frac{\omega}{4}}\right)\mathfrak{E}^{(L)}(\hat{k},\cdot) + \left(\varepsilon^{\frac12}+a^{\frac{\omega}{2}}\right)\mathfrak{E}^{(L-2)}(R,\cdot)\\ &\quad + \mathfrak{E}^{(L)}_{total,Vl} + a^{2+\frac{\omega}{4}}\,\mathfrak{E}^{(L-1)}(R,\cdot)\\ &\quad + \begin{cases} a^{\frac{\omega}{2}}\|G^{\pm1}-\gamma^{\pm1}\|^2_{L^2_G} + a^{2+\frac{\omega}{2}}\,\|\Gamma-\hat{\Gamma}\|^2_{L^2_G} & L=0\\ a^{\frac{\omega}{2}}\|G^{\pm1}-\gamma^{\pm1}\|^2_{H^2_G} + a^{\frac{\omega}{2}}\,\|\Gamma-\hat{\Gamma}\|^2_{H^1_G} & L=2\\ 0 & \text{else}\end{cases}\end{aligned} \tag{4.3.2h}$$

$$\mathfrak{E}^{(L)}_{\mathrm{total,Vl}} = \sum_{K=0}^{L} a^{(K+1)\omega}\,\mathfrak{E}^{(L)}_{1,K}(f,\cdot) \tag{4.3.2i}$$

$$\mathfrak{E}^{(L)}_{\mathrm{quiesc}} = \mathfrak{E}^{(L)}(\phi,\cdot) + \varepsilon^{\frac14}\,\mathfrak{E}^{(L)}(\hat{k},\cdot) + \varepsilon^{\frac12}\,\mathfrak{E}^{(L)}(R,\cdot) \tag{4.3.2j}$$

The total Vlasov energy is effectively weighted by a^ω to mitigate terms in the Vlasov integral estimates that would otherwise diverge at order t^{-1} or slightly worse and prevent one from being able to obtain strong bounds from energy estimates. Essentially, one is able to show weak control on Vlasov matter by obtaining improved estimates for $\mathfrak{E}^{(L)}_{\mathrm{total}}$, which suffices to obtain a strong bound on $\mathfrak{E}^{(L)}_{\mathrm{quiesc}}$ since the scalar field dominates Vlasov matter toward the Big Bang. In turn, this strong bound on the quiescent variables yields a strong bound on Vlasov matter. However, as in Chapter 3, one needs to introduce time-weighted geometric energies and metric error terms to ensure that the Vlasov energy hierarchy is compatible with the mechanism in $\mathfrak{E}^{(L)}_{quiesc}$. We note that, in contrast to Chapter 3, we do not require an ε-weight on the high order curvature term in $\mathfrak{E}^{(L)}_{\mathrm{total}}$, since such a term now no longer occurs naturally in the quiescent energy due to the simplified energy estimates for $\hat{k}$ and R, see Section 4.5.

In the following, ω will be small enough, but significantly larger than the perturbation size ε^2 and the bootstrap parameter η; see Assumption 4.3.7. If η and ε are taken to be sufficiently small, one can choose, for example, $\omega = \frac{1}{100}$.

Furthermore, we will often assume L to be an even integer in the sequel to simplify the exposition. For the energies defined in (4.3.2a)-(4.3.2d), integration by parts immediately implies that, for $c > 0$,

$$\mathfrak{E}^{(L-1)} \leq \frac{1}{2}\left(\mathfrak{E}^{(L)} + \mathfrak{E}^{(L-2)}\right), \quad \text{and} \quad a^{-c\sqrt{\varepsilon}}\,\mathfrak{E}^{(L-1)} \leq \frac{1}{2}\left(\mathfrak{E}^{(L)} + a^{-2c\sqrt{\varepsilon}}\mathfrak{E}^{(L-2)}\right). \tag{4.3.3}$$

Due to the weights in (4.3.2e)-(4.3.2f), this is only approximately true for Vlasov energies, but this will still be sufficient to improve (4.3.2h); see (4.4.6b) for the Vlasov energy analogue. We also note that (4.4.6b) is the reason why one needs to consider metric error terms of up to second order in $\mathfrak{E}^{(2)}_{\text{total}}$.

4.3.2. Initial data. Initial data for the $(2+1)$-dimensional ESFV system (4.1.1) takes the form $(M, \mathring{g}, \mathring{k}, \mathring{\pi}, \mathring{\psi}, \mathring{f})$, where $\mathring{g}$ and $\mathring{k}$ are symmetric $(0,2)$-tensors on M, $\mathring{\pi}$ is an exact (0,1)-tensor and $\mathring{\psi}$ and $\mathring{f}$ are scalar fields on M and T^*M, respectively and which satisfy the constraint equations

$$\begin{aligned} R[\mathring{g}] + (\mathrm{tr}_{\mathring{g}}\mathring{k})^2 - |\mathring{k}|^2_{\mathring{g}} &= 8\pi\left[\mathring{\psi}^2 + |\mathring{\pi}|^2_{\mathring{g}}\right] + 16\pi \int_{T^*_\bullet M} \sqrt{m^2 + |p|^2_{\mathring{g}}}\,\mathring{f}(\cdot,p)\,\mathrm{dvol}_{\mathring{g}^{-1}_\bullet}\,, \\ \mathrm{div}_{\mathring{g}}\mathring{k} &= -8\pi\,\mathring{\psi}\,\nabla[\mathring{g}]\mathring{\pi} - 8\pi \int_{T^*_\bullet M} p\,\mathring{f}(\cdot,p)\,\mathrm{dvol}_{\mathring{g}^{-1}_\bullet}\,. \end{aligned}$$

We also assume that $\mathring{f}$ is nonnegative and that the momentum support of $\mathring{f}$ is compact, i.e., that

$$\sup\{|p|_{\mathring{g}} \,|\, (x,p) \in \mathrm{supp}\mathring{f}\} < \infty\,.$$

In the massless case ($m = 0$), we additionally assume that

$$\inf\{|p|_{\mathring{g}} \,|\, (x,p) \in \mathrm{supp}\mathring{f}\} > 0\,. \tag{4.3.4}$$

This is required to ensure well-posedness, see Section 4.3.3. The CMC condition is enforced by requiring

$$\mathrm{tr}_{\mathring{g}}\mathring{k} = -2\,\frac{\dot{a}(t_0)}{a(t_0)}\,.$$

That such data, embedded into a spacetime $(\overline{M}, \overline{g})$ solving (4.1.1), admits unique maximal globally hyperbolic developments follows from standard arguments as in [FB52; CBG69].

We assume such data $(g, k, \nabla\phi, e_0\phi, f)$ to be close to FLRW data in the following sense:

Assumption 4.3.3 (Initial data assumption). *At the initial time $t_0 \in (0, T/2)$ (see Lemma 4.2.2) and $\varepsilon \in (0,1)$ small enough, we take initial data for the ESFV system to be close to FLRW data in the following sense:*

$$\mathcal{H}(t_0) + \mathcal{C}(t_0) \leq \varepsilon^2 \tag{4.3.5a}$$

$$\|f - f_{FLRW}\|_{C^5_{1,\underline{G}_0}(T^*M_{t_0})} \leq \varepsilon^2 \tag{4.3.5b}$$

Note that (4.3.5a) implies that all energies in Definition 4.3.2 of order $L \leq 10$ are bounded by ε^4 at $t = t_0$.

REMARK 4.3.4 (Polarized $U(1)$-symmetric initial data for the $(3+1)$ Einstein vacuum equations). For Corollary 4.1.2, we consider initial data $(M\times\mathbb{S}^1, \mathring{\tilde{g}}, \mathring{\tilde{k}})$ to the vacuum equations that is polarized and $U(1)$-symmetric, i.e., $\mathring{\tilde{g}}$ and $\mathring{\tilde{k}}$ satisfy

$$\mathring{\tilde{g}}_{13} = \mathring{\tilde{g}}_{23} = \mathring{\tilde{k}}_{13} = \mathring{\tilde{k}}_{23} = 0$$

on top of the vacuum constraint equations. Given the correspondence (4.1.2), this induces initial data for the scalar field given by

$$\mathring{\pi}_i = \frac{1}{2\sqrt{4\pi}}\,\partial_i \log(\mathring{\tilde{g}}_{33}), \quad \mathring{\psi} = -\frac{1}{\sqrt{4\pi}}\left(\mathring{\tilde{g}}_{33}\right)^{-\frac{3}{2}}\mathring{\tilde{k}}_{33}, \tag{4.3.6a}$$

and consequently data for the metric and the second fundamental form by

$$\mathring{g}_{ij} = \left(\mathring{\tilde{g}}_{33}\right)^{-1}\mathring{\tilde{g}}_{ij}, \quad \mathring{k}_{ij} = \sqrt{\left(\mathring{\tilde{g}}_{33}\right)}\,\mathring{\tilde{k}}_{ij} - \sqrt{4\pi}\,\mathring{\psi}\,\mathring{g}_{ij}\,. \tag{4.3.6b}$$

4.3.3. Local well-posedness. A local well-posedness theory for the $(2+1)$-dimensional Einstein-Vlasov-scalar-system in CMCTC gauge can be developed following the same arguments as in Section 3.3.3. More precisely, one first combines the well-posedness results [Rin13, Corollary 24.10] and [Sve12, Paper B, Lemma 9.2] in harmonic gauge with the approach in [FK20] to find a CMC hypersurface close to general near-FLRW data, allowing one to assume that any initial data close to FLRW data is CMC without loss of generality. Then, one follows the proof of [RS18b, Theorem 14.1] to obtain local well-posedness results in CMCTC gauge, along with continuation criteria. We note, as in Chapter 3, that a direct extension of the arguments in [Rin13; Sve12] requires working with the dual Vlasov distribution $f^\sharp$ in terms of canonical coordinates on the mass shell, but the spacetime is sufficiently regular that one can translate between both frameworks at no additional cost.

The new nuance compared to the $(3+1)$-dimensional case arises when considering the results in [Sve12, Paper B], which are needed to obtain an analogue of the harmonic gauge well-posedness result in Lemma 3.3.6 for $m=0$, since [Sve12, Paper B] is restricted to odd spatial dimensions. However, as noted in [Sve12, Remark 5.2], all of the results apply to even spatial dimensions if one adds an additional momentum weight $\langle p\rangle_\gamma^\alpha |p|_\gamma^{-\alpha}$ for $\alpha\in\mathbb{R}\backslash\mathbb{Z}$, $\alpha>0$ to all weighted norms for the Vlasov distribution function throughout the work and adapts the function spaces accordingly. Consequently, analogous additional momentum weights would have to enter our regularity assumptions and continuation criteria a priori. However, since we assume (4.3.4) anyhow in the massless case, one verifies that these weights are uniformly bounded from above and below using the method of characteristics, as in Lemma 4.4.2. Similarly, adding to the continuation criteria that

$$t\mapsto \inf\{|p|_g \,|\, x\in M,\ (t,x,p)\in \operatorname{supp} f\}$$

is uniformly bounded from below by a positive constant is sufficient to resolve this technical hurdle.

In summary, the following holds.

LEMMA 4.3.5 (Local well-posedness and continuation criteria for the (2+1) ESFV system). *For sufficiently small $\varepsilon > 0$, the $(2+1)$-dimensional ESFV system* (4.1.1) *in CMCTC gauge with initial data as in Assumption 4.3.3 is locally well-posed, launching a foliation $(M_t)_{t\in(t_-,t_0]}$ of $\overline{M}$ for some $t_- > 0$. The solution can be extended to $t = t_-$ as long as, approaching t_-, the following continuation criteria hold for some $Q > 0$:*

(1) *The eigenvalues of $g_{t,x}$ remain bounded from below by Q.*
(2) *One has $n \geq Q$.*
(3) *One has $|e_0\phi|^2 + |\nabla\phi|_g^2 \geq Q$.*
(4) *For $m = 0$, one has*

$$\inf_{t\in(t_-,t_0]} \inf\{|p|_g \,|\, x \in M,\ (t,x,p) \in \operatorname{supp} f\} \geq Q\,.$$

(5) *The following quantities remain uniformly bounded.*

$$\|g\|_{C^2_\gamma(M_t)},\ \|\hat{k}\|_{C^1_\gamma(M_t)},\ \|e_0\phi\|_{C^1_\gamma(M_t)},\ \|\nabla\phi\|_{C^1_\gamma(M_t)},\ \|\rho^{Vl}\|_{C^1_\gamma(M_t)},\ \|\jmath^{Vl}\|_{C^1_\gamma(M_t)},\ \|S^{Vl}\|_{C^1_\gamma(M_t)},$$
$$\sup_{x\in M}\int_{T^*_xM} \langle p\rangle_\gamma^2\ \left[|f(t,x,p)|^2 + (m^2 + |p|^2_\gamma)\,|\partial_{p_i} f(t,x,p)|^2_\gamma\right]\ \mathrm{dvol}_{\mathring{g}_x^{-1}}$$

With regards to the final bound for the integrated Vlasov distribution, the equivalent criterion (3.3.11c) in the higher dimensional setting of Chapter 3 contains the additional weight $\langle p\rangle_\gamma^{2\ell}(p^0_\gamma)^{-2\ell}$, where the integer ℓ depends on the regularity of the initial data. However, in the massive case, this weight is identical to 1 and in the massless case, it is uniformly bounded from above and below on the support of f and can thus be dropped due to criterion (4).

In addition, one can choose the data to be sufficiently regular such that all energies used within this paper are continuously differentiable in time, which we tacitly assume on top of Assumption 4.3.3 without loss of generality. This does not impact the continuation criteria (1)–(5).

REMARK 4.3.6 (On local well-posedness for polarized $U(1)$-symmetric vacuum solutions). For the sake of this work, we refrain from separately establishing a local well-posedness statement for the vacuum solutions arising from polarized $U(1)$-symmetric vacuum data as discussed in Remark 4.3.4. Instead, the local well-posedness result for the Einstein (Vlasov) scalar-field system in CMCTC gauge and $(2+1)$-dimensions is sufficient for our purposes: Polarized $U(1)$-symmetric vacuum data induces data for the Einstein scalar-field system; see Remark 4.3.4. The maximal Einstein scalar-field solution this launches then induces a globally hyperbolic development of the original vacuum data by the correspondence (4.1.2). $U(1)$-symmetry and polarization are then propagated in the induced foliation $(M_t\times\mathbb{S}^1)_{t\in(t_-,t_0]}$ of $\check{M}$ by construction. While this development is not a priori past maximal, as evidenced by the extremal Kasner solution discussed in Remark 4.2.3, for the solutions in Corollary 4.1.2, we show that the Kretschmann scalar of the vacuum solution blows up as $t\downarrow 0$, ensuring that the full past development is obtained.

4.3.4. Bootstrap assumptions. The bootstrap interval will be denoted by $(t_{Boot}, t_0]$, $t_{Boot} \in [0, t_0)$. On this interval, we make the following bootstrap assumptions:

Assumption 4.3.7 (Bootstrap assumptions). *For $\eta \gg \varepsilon^{\frac{1}{8}} > 0$ sufficiently small, one has*

$$\mathcal{C}(t) \lesssim \varepsilon\, a(t)^{-c\eta} \quad \textit{and} \tag{4.3.7a}$$

$$\|f - f_{FLRW}\|_{C^5_{1,\underline{G}_0}(T^*M_t)} \lesssim \varepsilon^{\frac{1}{4}}\, a(t)^{-c\eta} \tag{4.3.7b}$$

for any $t \in (t_{Boot}, t_0]$.

One may take $\eta = \varepsilon^{\frac{1}{16}}$. The goal of Sections 4.4 and 4.5 is to obtain (4.6.1) for some $c' > 0$, which improves the bootstrap assumptions for sufficiently small $\varepsilon > 0$. Consequently, we may implicitly update the constant c from line to line throughout the bootstrap argument.

Throughout the bootstrap argument in Section 4.4 and 4.5, we always assume $t \in (t_{Boot}, t_0]$.

4.4. A priori estimates

4.4.1. Strong low order pointwise bounds. In this section, we collect strong bounds in C^ℓ_G for $\ell \leq 6$ that immediately follow from the bootstrap assumption. For the most part, these bounds can be derived as in Section 3.4, so we only sketch these arguments.

Lemma 4.4.1 (Zero order bounds for kinetic variables).

$$\|\hat{k}\|_{C^0_G(M_t)} \lesssim \varepsilon \tag{4.4.1a}$$

$$\|\Psi\|_{C^0(M_t)} \lesssim \varepsilon \tag{4.4.1b}$$

Proof. Using (4.2.14b) and that the $\hat{k}$ is tracefree, we compute

$$\begin{aligned}\frac{1}{2}\,\partial_t\left(|\hat{k}|^2_G\right) &= \frac{1}{2}\,\partial_t\left((\hat{k}^\sharp)^b{}_c\,(\hat{k}^\sharp)^c{}_b\right)\\ &= -\,\langle \nabla^2 N, \hat{k}\rangle_G - 8\pi\,\langle \nabla\phi\,\nabla\phi, \hat{k}\rangle_G - 8\pi\,a^{-1}\,\langle S^{Vl,\parallel}, \hat{k}\rangle_G - N\,\tau\,|\hat{k}|^2_G\,.\end{aligned}$$

The bootstrap assumptions (4.3.7a) imply

$$|\hat{k}|_G(t) \lesssim |\hat{k}|_G(t_0) + \int_t^{t_0} \varepsilon\, a(s)^{-1-c\eta}\, ds + \int_t^{t_0} \varepsilon\, a(s)^{-1-c\sqrt{\varepsilon}}\, |\hat{k}|_G(s)\, ds\,.$$

Estimate (4.4.1a) now follows from the Gronwall lemma, Lemma 4.2.2 and the initial data assumption (4.3.5a). An analogous argument proves (4.4.1b). □

As a direct consequence, we obtain that the momentum support remains compact and, in the massless case, bounded away from the origin:

Lemma 4.4.2 (Momentum support bounds). *The following momentum support bound holds.*

$$\sup_{x\in M}\ \sup_{v\in \mathrm{supp} f(t,x,\cdot)} v^0 \lesssim a(t)^{-c\sqrt{\varepsilon}}, \quad \sup_{x\in M}\ \sup_{v\in \mathrm{supp} f(t,x,\cdot)} |v|_\gamma \lesssim 1 \tag{4.4.2a}$$

For $m = 0$, one additionally has

$$\sup_{x\in M}\ \sup_{v\in \mathrm{supp} f(t,x,\cdot)} |v|_G^{-1} \lesssim a(t)^{-c\sqrt{\varepsilon}}, \quad \sup_{x\in M}\ \sup_{v\in \mathrm{supp} f(t,x,\cdot)} |v|_\gamma^{-1} \lesssim 1\,. \tag{4.4.2b}$$

PROOF. Let $x \in M_t$ and $v \in \operatorname{supp} f(t,x,\cdot)$. By the method of characteristics, the solution of (4.2.14j) is given by

$$f(t,x,v) = \mathring{f}(t, X(t;x,v), V(t;x,v)),$$

where, writing $V^0 = \sqrt{m^2\, a^2 + |v|^2_G}$, V solves the initial value problem

$$\begin{aligned}\dot{V}_i &= -\, a^{-1}\,(N+1)\,(G^{-1})^{jk}\Gamma[G]^l_{ij}\,\frac{V_k V_l}{V^0} + a^{-1} V^0\,\nabla^{\sharp i} N\\ V(t_0;x,v) &= v\end{aligned}$$

and X is given by

$$\begin{aligned}\dot{X}^i &= a^{-1}\,(N+1)\,\frac{(G^{-1})^{ij}V_j}{V^0}\,,\\ X(t_0;x,v) &= x\,.\end{aligned}$$

The bootstrap assumption (4.3.7a) then yields

$$|\dot{V}|_G \lesssim a^{-1-c\sqrt{\varepsilon}}\, V^0\,.$$

The strong low order bound (4.4.1a), along with (4.3.7a), implies $|\partial_t G|_G \lesssim \varepsilon a^{-2}$ using (4.2.14a). Thus, (4.4.2a) follows from these bounds using the Gronwall lemma and that the momentum support is bounded initially. (4.4.2b) is proven analogously, noting that $\partial_t \gamma = 0$. □

From this, one obtains the following bounds, which in turn control low order nonlinear error terms in the energy estimates:

LEMMA 4.4.3 (Strong low order C_G-bounds).

$$\|\hat{k}\|_{C^6_G(M_t)} + \|\Psi\|_{C^6_G(M_t)} \lesssim \varepsilon\, a(t)^{-c\sqrt{\varepsilon}} \tag{4.4.3a}$$

$$\|\nabla\phi\|_{C^5_G(M_t)} + \|G-\gamma\|_{C^6_G(M_t)} + \|R[G] - 2\,\kappa\|_{C^4_G(M_t)} \lesssim \sqrt{\varepsilon}\, a(t)^{-c\sqrt{\varepsilon}} \tag{4.4.3b}$$

$$\|f - f_{FLRW}\|_{C^5_{1,\underline{G}_0}(T^*M_t)} \lesssim \sqrt{\varepsilon}\, a(t)^{-c\sqrt{\varepsilon}} \tag{4.4.3c}$$

$$\|\rho^{Vl} - \rho^{Vl}_{FLRW}\|_{C^5_G(M_t)} + \|S^{Vl,\parallel}\|_{C^5_G(M_t)} + \|\jmath^{Vl}\|_{C^5_G(M_t)} \lesssim \sqrt{\varepsilon}\, a(t)^{-c\sqrt{\varepsilon}} \tag{4.4.3d}$$

PROOF. (4.4.3a)-(4.4.3b) are proven as in Lemma 3.4.2, (4.4.3c) as in Lemmas 3.4.8–3.4.9 and (4.4.3d) as in Corollary 3.4.10. Considering the shear bound in (4.4.3a), for example, note that the tracefree part $(\partial_t\hat{k})^{\parallel}$ of $\partial_t\hat{k}$ satisfies the following bound for $J \in \{1,2,\dots,6\}$; see (4.2.14b).

$$\left\|(\partial_t\hat{k}^\sharp)^{\parallel}\right\|_{C^J_G(M_t)} \lesssim \tau(t)\,\|N\|_{C^J_G(M_t)}\,\|\hat{k}\|_{C^J_G(M_t)} + \|N\|_{C^{J+2}_G(M_t)} + \|\nabla\phi\|^2_{C^J_G(M_t)} + a(t)^{-1}\|S^{Vl,\parallel}\|_{C^J_G(M_t)}$$

One also obtains that

$$\begin{aligned}\left\|[\partial_t, \nabla^J]\hat{k}^\sharp\right\|_{C^0_G(M_t)} \lesssim{}& \tau(t)\,\|N\|_{C^J_G(M_t)}\,\|\hat{k}\|_{C^J_G(M_t)} + a(t)^{-2}\,\|\hat{k}\|_{C^0_G(M_t)}\,\|\hat{k}\|_{\dot{C}^J_G(M_t)}\\ &+ a(t)^{-2}\,\|N+1\|_{C^J_G(M_t)}\,\|\hat{k}\|^2_{C^{J-1}_G(M_t)}\,.\end{aligned}$$

Assume that the bound $\|\hat{k}\|_{C^{J-1}_G(M_t)} \lesssim \varepsilon\, a(t)^{-c\sqrt{\varepsilon}}$ is already established – for $J=1$, this follows from (4.4.1a). The bootstrap assumption (4.3.7a) and the zero order (4.4.1a) then imply

$$-\partial_t \|\nabla^J \hat{k}\|^2_{\dot{C}^J_G(M_t)} \lesssim \varepsilon\, a(t)^{-2}\, \|\hat{k}\|^2_{\dot{C}^J_G(M_t)} + \left(\varepsilon^2\, a(t)^{-2-c\sqrt{\varepsilon}} + \varepsilon\, a(t)^{-1-c\eta}\right) \|\hat{k}\|_{\dot{C}^J_G(M_t)}\,.$$

This is sufficient to prove the bound at order J using the Gronwall lemma. Completing the iteration, we obtain the shear bound in (4.4.3a). The remaining bound in (4.4.3a) follows analogously using the already obtained shear bound and (4.4.3b) along with (4.4.3a).

Regarding (4.4.3c), one considers the rescaled Vlasov equation (4.2.14j) as an inhomogeneous transport equation for $f - f_{FLRW}$, where the inhomogeneities arise from reference terms depending on derivatives of f_{FLRW}, which can be bounded using (4.4.3a)-(4.4.3b). At order 0, the bound follows by controlling $f - f_{FLRW}$ along the characteristics, similar to the proof of Lemma 4.4.2.

From here, one proceeds iteratively after commuting the Vlasov equation with $\mathbf{A}^K\mathbf{B}^{J-K}$ for increasing $K = 0, \dots, J$.[10] For $J = 1, K = 0$, using (4.4.1a) to control the borderline shear term and (4.3.7b) to control the horizontal derivative terms, one can apply the Gronwall lemma along with (4.4.2a) to obtain

$$(v^0)|\mathbf{B}(f - f_{FLRW})|_{\underline{G}|_{vert}} \lesssim \varepsilon^{\frac{1}{4}}\,.$$

Using this bound for $J = 1, K = 1$, one locally obtains

$$|\mathbf{A}(f - f_{FLRW})|_G \lesssim \sqrt{\varepsilon}\, a^{-c\sqrt{\varepsilon}}$$

and then (4.4.3c) for $C^1_{1,\underline{G}_0}(T^*M_t)$ by repeating the first step with this improved bound in place of the bootstrap assumption and combining the local estimates. For $J = 2, 3, 4$ and 5, one proceeds analogously. Finally, (4.4.3d) is an immediate consequence of (4.2.15), (4.4.2a) and (4.4.3c). $\square$

4.4.2. Relating Sobolev norms to energies. With these low order estimates in hand, we can now show that our energies are almost coercive:

PROPOSITION 4.4.4 (Near-coercivity of energies). *Let ℓ be an integer with $0 \leq \ell \leq 9$. Then, the following estimates hold, where energies with superscript $\ell - 2$ or $\ell - 3$ only appear when $\ell \geq 5$:*

(4.4.4a)
$$\|\Psi\|_{H^\ell_G(M_t)} + a(t)\,\|\nabla\phi\|_{H^\ell_G(M_t)} \lesssim \sqrt{\mathfrak{E}^{(\ell)}(\phi,t)} + a(t)^{-c\sqrt{\varepsilon}}\left(\sqrt{\mathfrak{E}^{(\leq \ell-2)}(\phi,t)} + \varepsilon\sqrt{\mathfrak{E}^{(\leq \ell-2)}(R,t)}\right)$$

(4.4.4b)
$$\|\hat{k}\|_{H^\ell_G(M_t)} \lesssim \sqrt{\mathfrak{E}^{(\ell)}(\hat{k},t)} + a(t)^{-c\sqrt{\varepsilon}}\left(\sqrt{\mathfrak{E}^{(\leq \ell-2)}(\hat{k},t)} + \varepsilon^2\sqrt{\mathfrak{E}^{(\leq \ell-3)}(R,t)}\right)$$

(4.4.4c)
$$\|N\|_{H^\ell_G(M_t)} \lesssim \sqrt{\mathfrak{E}^{(\ell)}(N,t)} + a(t)^{-c\sqrt{\varepsilon}}\left(\sqrt{\mathfrak{E}^{(\leq \ell-3)}(N,t)} + \varepsilon\sqrt{\mathfrak{E}^{(\leq \ell-3)}(R,t)}\right)$$

[10] Since M is compact and f has compact momentum support, one can restrict the rest of this proof to coordinate neighbourhoods.

(4.4.4d)
$$\|R[G]-2\,\kappa\|_{H^\ell_G(M_t)}\lesssim\sqrt{\mathfrak{E}^{(\ell)}(R,t)}+a(t)^{-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq\ell-2)}(R,t)}$$

(4.4.4e)
$$\|\rho^{Vl}-\rho^{Vl}_{FLRW}\|_{H^\ell_G(M_t)}\lesssim a(t)^{-c\sqrt{\varepsilon}}\left(\sqrt{\mathfrak{E}^{(\leq\ell)}_{1,\leq\ell}(f,t)}+\|G^{-1}-\gamma^{-1}\|_{L^2_G(M_t)}\right)$$

(4.4.4f)
$$\|S^{Vl,\|}\|_{H^\ell_G(M_t)}\lesssim a(t)^{-c\sqrt{\varepsilon}}\left(\sqrt{\mathfrak{E}^{(\leq\ell)}_{1,\leq\ell}(f,t)}+\|G^{-1}-\gamma^{-1}\|_{L^2_G(M_t)}\right)$$

(4.4.4g)
$$\|\jmath^{Vl}\|_{H^\ell_G(M_t)}\lesssim a(t)^{-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq\ell)}_{1,\leq\ell}(f,t)}$$

(4.4.4h)
$$\begin{aligned}\|f-f_{FLRW}\|_{H^\ell_{1,\underline{G}_0}(T^*M_t)}\lesssim&\sqrt{\mathfrak{E}^{(\ell)}_{1,\leq\ell}(f,t)}+a(t)^{-c\sqrt{\varepsilon}}\left(\sqrt{\mathfrak{E}^{(\leq\ell-1)}_{1,\leq\ell-1}(f,t)}+\varepsilon\sqrt{\mathfrak{E}^{(\leq\ell-2)}(R,t)}\right)\\&+a(t)^{-c\sqrt{\varepsilon}}\left(\|\Gamma-\hat\Gamma\|_{H^{\ell-1}_G(M_t)}+\|G^{\pm1}-\gamma^{\pm1}\|_{H^{\ell-1}_G(M_t)}\right)\end{aligned}$$

We also have the following bound

(4.4.5)
$$\|\nabla\phi\|_{H^9(M_t)}\lesssim a(t)^{-c\sqrt{\varepsilon}}\left(\sqrt{\mathfrak{E}^{(\leq10)}(\hat k,t)}+\sqrt{\varepsilon}\sqrt{\mathfrak{E}^{(\leq9)}(\phi,t)}+\sqrt{\mathfrak{E}^{(\leq9)}_{1,\leq9}(f,t)}+\sqrt{\mathfrak{E}^{(\leq8)}(R,t)}\right)$$

Proof. For (4.4.4a)-(4.4.4d), the proof proceeds as in Lemma 2.4.5 by integration by parts and commuting derivatives, where low order terms in the resulting commutator errors are controlled using Lemma 4.4.3. Conversely, (4.4.4e)-(4.4.4g) follow as in Lemma 3.4.14, using (4.2.15), (4.4.2a) as well as Jensen's inequality. The zero order metric error terms in (4.4.4e)-(4.4.4g) arise from the reference quantity in the zero order estimate. The Vlasov distribution bound (4.4.4h) is proven as in Lemma 3.4.13. In particular, higher order metric terms in the second line only arise from computing derivatives of the FLRW reference distribution and not from rearranging derivatives into the specific order required by the Vlasov energy.

Finally, by rearranging the momentum constraint (4.2.14f), differentiating and applying Lemma (4.4.3a) and (4.4.3d) to control lower order terms, one obtains
$$\|\nabla\phi\|_{H^9(M_t)}\lesssim a(t)^{-c\sqrt{\varepsilon}}\left(\|\hat k\|_{H^{10}_G(M_t)}+\|\jmath^{Vl}\|_{H^9_G(M_t)}\right)+\sqrt{\varepsilon}\,a(t)^{-c\sqrt{\varepsilon}}\,\|\Psi\|_{H^9_G(M_t)}\,.$$
The bound (4.4.5) then follows from applying (4.4.4b), (4.4.4g) and (4.4.4a). □

Finally, we can prove the following lemma that allows us to extend (4.3.3) to energies of the Vlasov distribution except for one special case.

Lemma 4.4.5. *Let K,L, with $0<K<L$ and $(K,L)\neq(1,2)$. Then, the following estimates hold.*

(4.4.6a)
$$a(t)^{-c\sqrt{\varepsilon}}\,\mathfrak{E}^{(L-1)}_{1,K}(f,t)\lesssim\mathfrak{E}^{(L)}_{1,K}(f,t)+a(t)^{-2c\sqrt{\varepsilon}}\,\mathfrak{E}^{(L-2)}_{1,\min\{K,L-2\}}(f,t)$$

Moreover, given $\omega > 0$, one has

(4.4.6b)

$$a(t)^{2\omega-c\sqrt{\varepsilon}}\mathfrak{E}^{(1)}_{1,1}(f,t) \lesssim a(t)^{3\omega}\,\mathfrak{E}^{(2)}_{1,2}(f,t) + a(t)^{\omega-2c\sqrt{\varepsilon}}\,\mathfrak{E}^{(0)}_{1,0}(f,t)$$
$$+ a(t)^{\frac{\omega}{2}}\left(a(t)^{\frac{\omega}{2}}\,\|\Gamma-\hat{\Gamma}\|^2_{H^1_G(M_t)} + a(t)^{\frac{\omega}{2}}\,\|G^{\pm1}-\gamma^{\pm1}\|^2_{H^2_G(M_t)}\right).$$

Proof. The first bound follows from a straightforward integration by parts, applying Young's inequality and rearranging. For (4.4.6b), we proceed similarly. Here, we expand f into $(f - f_{FLRW}) + f_{FLRW}$ and use

$$\underline{\nabla}_{\mathrm{hor}} f_{FLRW} = v * v * \gamma^{\pm1} * (\Gamma-\hat{\Gamma}) * \mathcal{F}'(|v|^2_\gamma)$$

as well as (4.4.2a) and (4.4.3b) to bound the momentum supports of f and f_{FLRW} respectively by $a^{-c\sqrt{\varepsilon}}$ up to constant. For details, we refer to the proof of Lemma 3.7.3. □

4.5. Energy estimates

For all of the estimates in this section, take $L \in \{0,2,4,6,8,10\}$ and energies of "negative order" are understood to vanish.

The core simplification in our argument compared to that in Chapters 2 and 3 occurs in Section 4.5.2, which is the core geometric estimate. To handle high order curvature terms arising from Vlasov matter, we additionally need to prove a new scaled integral energy estimate in Lemma 4.5.5 using the momentum constraint. The remaining parts of the argument are similar to those in the previous chapters; we will keep their discussions relatively brief.

4.5.1. Elliptic lapse estimate. First, we derive elliptic bounds for the lapse. The former is required to estimate high order lapse terms through the evolution equations, while the latter is needed to obtain sharp bounds on the lapse and improve the bootstrap assumptions.

Lemma 4.5.1 (Estimates for the lapse). *The following energy estimates hold.*

(4.5.1)

$$\begin{aligned} &a(t)^4\,\mathfrak{E}^{(L+2)}(N,t) + a(t)^2\,\mathfrak{E}^{(L+1)}(N,t) + \mathfrak{E}^{(L)}(N,t)\\ &\quad\lesssim \varepsilon^2\,\mathfrak{E}^{(L)}(\hat{k},t) + \mathfrak{E}^{(L)}(\phi,t) + a(t)^{2-c\sqrt{\varepsilon}}\left(\mathfrak{E}^{(\le L)}_{1,\le L}(f,t) + \|G^{-1}-\gamma^{-1}\|^2_{L^2_G(M_t)}\right)\\ &\qquad + \varepsilon^2\,a(t)^{-c\sqrt{\varepsilon}}\left(\mathfrak{E}^{(\le L-2)}(\hat{k},t) + \mathfrak{E}^{(\le L-2)}(\phi,t)\right) + \varepsilon\,a(t)^{-c\sqrt{\varepsilon}}\,\mathfrak{E}^{(\le L-3)}(R,t)\end{aligned}$$

(4.5.2)

$$\begin{aligned} &a(t)^4\,\mathfrak{E}^{(L+2)}(N,t) + a(t)^2\,\mathfrak{E}^{(L+1)}(N,t) + \mathfrak{E}^{(L)}(N,t)\\ &\quad\lesssim a(t)^{4-c\sqrt{\varepsilon}}\,\mathfrak{E}^{(\le L)}(R,t) + \varepsilon\,a(t)^{2-c\sqrt{\varepsilon}}\,\mathfrak{E}^{(\le L)}(\phi,t)\\ &\qquad + a(t)^{2-(L+1)\omega-c\sqrt{\varepsilon}}\,a(t)^{(L+1)\omega}\mathfrak{E}^{(\le L)}_{1,\le L}(f,t) + a(t)^{2-\frac{\omega}{2}-c\sqrt{\varepsilon}}\,a(t)^{\frac{\omega}{2}}\|G^{-1}-\gamma^{-1}\|^2_{L^2_G(M_t)}\end{aligned}$$

Proof. To obtain these estimates from (4.2.14g), we verify that solutions to the equation $\mathcal{L}\zeta = Z$ (respectively, $\tilde{\mathcal{L}}\zeta = Z$) satisfy

$$a^2\,\|\Delta_G\zeta\|_{L^2_G} + a\,\|\nabla\zeta\|_{L^2_G} + \|\zeta\|_{L^2_G} \lesssim \|Z\|_{L^2_G}$$

since the estimates then follow as in Lemma 3.5.4. To this end, it suffices to check that h (respectively, $\tilde{h}$) is bounded from below by a positive constant and that $|\nabla h|_G \lesssim \sqrt{\varepsilon}$ (respectively, $|\nabla \tilde{h}|_G \lesssim \sqrt{\varepsilon}$). The former follows from (4.4.1b) and (4.4.1a) (respectively, (4.4.3b)) as well as (4.4.3d), the latter from (4.4.3a) (respectively, (4.4.3b)) and (4.4.3d). □

4.5.2. Estimates for the second fundamental form. Using that the curvature tensor is pure trace, we obtain a simple energy estimate for the second fundamental form.

LEMMA 4.5.2 (Energy estimate for second fundamental form).

(4.5.3a)

$$\begin{aligned}\mathfrak{E}^{(L)}(\hat{k},t) \lesssim{}& \varepsilon^4 + \int_t^{t_0} \left(\varepsilon^{\frac{1}{8}}\, a(s)^{-2} + a(s)^{-1-c\sqrt{\varepsilon}}\right) \mathfrak{E}^{(L)}(\hat{k},s) + \varepsilon^{-\frac{1}{8}}\, a(s)^{-2}\mathfrak{E}^{(L)}(\phi,s)\, ds\\ &+ \int_t^{t_0} \varepsilon^{\frac{3}{8}}\, a(s)^{-2}\cdot \varepsilon^{\frac{1}{2}}\,\mathfrak{E}^{(L-2)}(R,s) + a(s)^{-1-(L+1)\omega-c\sqrt{\varepsilon}}\, \mathfrak{E}^{(\leq L)}_{total,Vl}(s)\, ds\\ &+ \int_t^{t_0} \varepsilon\, a(s)^{-2-c\sqrt{\varepsilon}}\, \mathfrak{E}^{(\leq L-2)}(\hat{k},s) + \varepsilon\, a(s)^{-2-c\sqrt{\varepsilon}}\, \mathfrak{E}^{(\leq L-2)}(\phi,s)\, ds\\ &+ \int_t^{t_0} \varepsilon^{\frac{7}{8}}\, a(s)^{-2-c\sqrt{\varepsilon}}\, \mathfrak{E}^{(\leq L-4)}(R,s) + a(s)^{-1-\frac{\omega}{2}-c\sqrt{\varepsilon}}\, a(s)^{\frac{\omega}{2}}\, \|G^{-1}-\gamma^{-1}\|^2_{L^2_G(M_s)}\, ds\end{aligned}$$

(4.5.3b)

$$\begin{aligned}a(t)^{\frac{\omega}{4}}\,\mathfrak{E}^{(L)}(\hat{k},\cdot) \lesssim{}& \varepsilon^4 + \int_t^{t_0} \left(\varepsilon^{\frac{1}{8}}\, a(s)^{-2} + a(s)^{-2+\frac{\omega}{8}} + a(s)^{-1-c\sqrt{\varepsilon}}\right) a(s)^{\frac{\omega}{4}}\,\mathfrak{E}^{(L)}(\hat{k},s)\, ds\\ &+ \int_t^{t_0} a(s)^{-2+\frac{\omega}{8}}\,\mathfrak{E}^{(L)}(\phi,s) + \varepsilon^{\frac{1}{2}} a(s)^{-2+\frac{\omega}{8}}\cdot \varepsilon^{\frac{1}{2}}\,\mathfrak{E}^{(L-2)}(\mathrm{Ric},s)\, ds\\ &+ \int_t^{t_0} a(s)^{-1-(L+1)\omega-c\sqrt{\varepsilon}}\mathfrak{E}^{(\leq L)}_{total,Vl}(s) + a(s)^{-1-\frac{\omega}{2}-c\sqrt{\varepsilon}}\cdot a(s)^{\frac{\omega}{2}}\,\|G^{-1}-\gamma^{-1}\|^2_{L^2_G(M_s)}\, ds\\ &+ \int_t^{t_0} \varepsilon a(s)^{-2-c\sqrt{\varepsilon}}\cdot a(s)^{\frac{\omega}{4}}\,\mathfrak{E}^{(\leq L-2)}(\hat{k},s) + \varepsilon a(s)^{-2-c\sqrt{\varepsilon}+\frac{\omega}{8}}\,\mathfrak{E}^{(\leq L-2)}(\phi,s)\, ds\\ &+ \int_t^{t_0} \varepsilon^{\frac{3}{2}} a(s)^{-2-c\sqrt{\varepsilon}+\frac{\omega}{8}}\cdot \varepsilon^{\frac{1}{2}}\,\mathfrak{E}^{(\leq L-4)}(R,s)\, ds\end{aligned}$$

PROOF. We first prove (4.5.3a) for $L \geq 4$ and sketch the necessary adaptions for (4.5.3b), the estimates for $L \leq 2$ are proven entirely analogously. Commuting (4.2.14b) with $\Delta_G^{\frac{L}{2}}$, we obtain

$$\partial_t \Delta_G^{\frac{L}{2}} (\hat{k}^\sharp)^i{}_j = \tau\left((\hat{k}^\sharp)^i{}_j\, \Delta_G^{\frac{L}{2}} N + N\, \Delta_G^{\frac{L}{2}} (\hat{k}^\sharp)^i{}_j\right) - \nabla^{\sharp i}\nabla_j \Delta_G^{\frac{L}{2}} N \tag{4.5.4a}$$

$$- 16\pi\left(\nabla \Delta_G^{\frac{L}{2}}\phi \otimes_{\mathrm{symm}} \nabla\phi\right)^{\sharp i}_{\ \ j} - 8\pi\, a^{-1}\, \Delta_G^{\frac{L}{2}} (S^{Vl,\|})^{\sharp i}{}_j \tag{4.5.4b}$$

$$+ \mathbb{I}^i_j\, \Delta_G^{\frac{L}{2}}\mathfrak{h} + [\partial_t, \Delta_G^{\frac{L}{2}}](\hat{k}^\sharp)^i{}_j + (\mathfrak{K}_{\mathrm{junk}})^i{}_j\,, \tag{4.5.4c}$$

where $\mathfrak{h}$ is a scalar function depending on G, $R[G]$, N and f as well as reference quantities, and $\mathfrak{K}_{\mathrm{junk}}$ collects lower order error terms arising from the product rule.

One has

$$\partial_t \mathfrak{E}^{(L)}(\hat{k},\cdot) = \int_M N\,\tau\,|\Delta_G^{\frac{L}{2}}\hat{k}|_G^2 + 2\,\langle \partial_t \Delta_G^{\frac{L}{2}}\hat{k}^\sharp, \Delta_G^{\frac{L}{2}}\hat{k}^\sharp\rangle_G \,\mathrm{dvol}_G\,.$$

Consequently, the pure trace term in (4.5.4c) cancels when inserting the evolution equation in the second term. Note that this includes high order curvature terms in (4.5.4), i.e., those that don't arise from commutator errors. Abbreviating

$$\left(\partial_t \Delta_G^{\frac{L}{2}}\hat{k}^\sharp\right)^{\parallel} = \partial_t \Delta_G^{\frac{L}{2}}\hat{k}^\sharp - \mathbb{I}\,\Delta_G^{\frac{L}{2}}\mathfrak{h}\,,$$

the bootstrap assumption (4.3.7a) implies

$$-\partial_t \mathfrak{E}^{(L)}(\hat{k},t) \lesssim \varepsilon\, a(t)^{-1-c\eta}\,\mathfrak{E}^{(L)}(\hat{k},t) + \left\|\left(\partial_t \Delta_G^{\frac{L}{2}}\hat{k}^\sharp\right)^{\parallel}\right\|_{L^2_G(M_t)} \sqrt{\mathfrak{E}^{(L)}(\hat{k},t)}\,. \tag{4.5.5}$$

Next, we estimate $\left(\partial_t \Delta_G^{\frac{L}{2}}\hat{k}^\sharp\right)^{\parallel}$ by going through the remaining terms in (4.5.4). We apply the bootstrap assumption (4.3.7a) as well as (4.4.1a) to bound the right hand side of (4.5.4a) in L^2_G by

$$\lesssim \varepsilon\, a(t)^{-1-c\eta}\sqrt{\mathfrak{E}^{(L)}(\hat{k},t)} + a(t)^{-2}\left(a(t)^{-2}\sqrt{a(t)^4\,\mathfrak{E}^{(L+2)}(N,t)} + \varepsilon\sqrt{\mathfrak{E}^{(L)}(N,t)}\right)\,. \tag{4.5.6}$$

Regarding (4.5.4b), we use (4.4.3b) and (4.4.4f) to obtain the bound

$$\lesssim \sqrt{\varepsilon}\, a(t)^{-1-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(L)}(\phi,t)} + a(t)^{-1-c\sqrt{\varepsilon}}\left(\sqrt{\mathfrak{E}^{(\leq L)}_{1,\leq L}(f,t)} + \|G^{-1}-\gamma^{-1}\|_{L^2_G(M)}\right)\,.$$

In combination, we obtain

$$\begin{aligned}\left\|\left(\partial_t \Delta_G^{\frac{L}{2}}\hat{k}^\sharp\right)^{\parallel}\right\|_{L^2_G(M_t)} &\lesssim a(t)^{-2}\left(\sqrt{a(t)^4\,\mathfrak{E}^{(L+2)}(N,t)} + \sqrt{\mathfrak{E}^{(L)}(N,t)}\right)\\ &\quad + \varepsilon\, a(t)^{-1-c\eta}\sqrt{\mathfrak{E}^{(L)}(\hat{k},t)}\\ &\quad + \sqrt{\varepsilon}\, a(t)^{-1-c\sqrt{\varepsilon}}\left(\sqrt{\mathfrak{E}^{(L)}(\phi,t)} + \sqrt{\mathfrak{E}^{(L)}_{1,L}(f,t)} + \|G^{-1}-\gamma^{-1}\|_{L^2_G(M_t)}\right)\\ &\quad + \|[\partial_t,\Delta_G^{\frac{L}{2}}]\hat{k}\|_{L^2_G(M_t)} + \|\mathfrak{K}_{\mathrm{junk}}\|_{L^2_G(M_t)}\,.\end{aligned}$$

Applying the elliptic estimate (4.5.1) for the lapse terms arising from (4.5.6), the following bound thus follows from (4.5.5).

$$-\partial_t \mathfrak{E}^{(L)}(\hat{k},t) \lesssim \varepsilon\, a(t)^{-1-c\,\eta}\mathfrak{E}^{(L)}(\hat{k},t) \tag{4.5.7a}$$

$$+\sqrt{\mathfrak{E}^{(L)}(\hat{k},t)}\left\{a(t)^{-2}\left(\varepsilon\sqrt{\mathfrak{E}^{(L)}(\hat{k},t)} + \sqrt{\mathfrak{E}^{(L)}(\phi,t)}\right)\right. \tag{4.5.7b}$$

$$+\,a(t)^{-1-c\sqrt{\varepsilon}}\left(\sqrt{\mathfrak{E}^{(\leq L)}_{1,\leq L}(f,t)} + \|G^{-1}-\gamma^{-1}\|_{L^2_G(M_t)}\right) \tag{4.5.7c}$$

$$+\,\varepsilon\, a(t)^{-2-c\sqrt{\varepsilon}}\left(\sqrt{\mathfrak{E}^{(\leq L-2)}(\hat{k},t)} + \sqrt{\mathfrak{E}^{(\leq L-2)}(\phi,t)}\right) \tag{4.5.7d}$$

$$+\sqrt{\varepsilon}\, a(t)^{-2-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-3)}(R,t)} \tag{4.5.7e}$$

$$+ \|[\partial_t, \Delta_G^{\frac{L}{2}}]\hat{k}\|_{L^2_G(M_t)} + \|\mathfrak{K}_{\text{junk}}\|_{L^2_G(M_t)} \Big\} \tag{4.5.7f}$$

The curvature terms in the penultimate line arise from estimating commutator errors when proving (4.5.1).

A naive estimate of (4.5.7b) would lead to the borderline, unscaled term $a(t)^{-2}\,\mathfrak{E}^{(L)}(\hat{k},t)$, which would prevent us from obtaining bootstrap improvements down the line. Thus, we apply Young's inequality as follows

$$a^{-2}\sqrt{\mathfrak{E}^{(L)}(\hat{k},\cdot)}\sqrt{\mathfrak{E}^{(L)}(\phi,\cdot)} \lesssim \varepsilon^{\frac{1}{8}}\,a^{-2}\,\mathfrak{E}^{(L)}(\hat{k},\cdot) + \varepsilon^{-\frac{1}{8}}\,a^{-2}\,\mathfrak{E}^{(L)}(\phi,\cdot)\,.$$

Applying Young's inequality as well as (4.3.3) in a similar manner for the remaining terms, the estimate below follows.

$$
\begin{aligned}
&-\partial_t\mathfrak{E}^{(L)}(\hat{k},t)\\
&\lesssim \left(\varepsilon^{\frac{1}{8}}\,a(t)^{-2} + a(t)^{-1-c\sqrt{\varepsilon}}\right)\mathfrak{E}^{(L)}(\hat{k},t) + \varepsilon^{-\frac{1}{8}}\,a(t)^{-2}\,\mathfrak{E}^{(L)}(\phi,t) && \text{(4.5.8a)}\\
&\quad + a(t)^{-1-(L+1)\omega-c\sqrt{\varepsilon}}\,a(t)^{(L+1)\omega}\mathfrak{E}^{(L)}_{1,L}(f,t) + a(t)^{-1-\frac{\omega}{2}-c\sqrt{\varepsilon}}\,a(t)^{\frac{\omega}{2}}\|G^{-1}-\gamma^{-1}\|^2_{L^2_G(M_t)} && \text{(4.5.8b)}\\
&\quad + \varepsilon^{\frac{7}{8}}\,a(t)^{-2}\,\mathfrak{E}^{(\leq L-2)}(R,t) + \varepsilon^{\frac{15}{8}}\,a(t)^{-2-c\sqrt{\varepsilon}}\left(\mathfrak{E}^{(\leq L-2)}(\hat{k},t) + \mathfrak{E}^{(\leq L-2)}(\phi,t)\right) && \text{(4.5.8c)}\\
&\quad + \varepsilon^{\frac{7}{8}}\,a(t)^{-2-c\sqrt{\varepsilon}}\,\mathfrak{E}^{(\leq L-4)}(R,t) + \left(\|[\partial_t, \Delta_G^{\frac{L}{2}}]\hat{k}\|_{L^2_G(M_t)} + \|\mathfrak{K}_{\text{junk}}\|_{L^2_G(M_t)}\right)\sqrt{\mathfrak{E}^{(L)}(\hat{k},t)} && \text{(4.5.8d)}
\end{aligned}
$$

In particular, the highest order curvature term arises from applying (4.3.3) to the curvature term in (4.5.7e). It remains to check that the final terms in (4.5.8d) are error terms dominated by what is already present in (4.5.8a)-(4.5.8c) as well as the first term in (4.5.8d).

After some straightforward computations and applications of Lemma 4.4.4 to estimate Sobolev norms by energies, one sees that the commutator term satisfies

$$
\begin{aligned}
\|[\partial_t, \Delta_G^{\frac{L}{2}}]\hat{k}\|_{L^2_G(M_t)} \lesssim{}& \varepsilon\,a(t)^{-2}\sqrt{\mathfrak{E}^{(\leq L)}(\hat{k},\cdot)} + \varepsilon\,a(t)^{-2-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-2)}(\hat{k},\cdot)}\\
&+ \varepsilon\,a(t)^{-2-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-2)}(N,\cdot)} + \varepsilon^2\,a(t)^{-2-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-3)}(R,\cdot)}\,.
\end{aligned}
$$

Moreover, $\mathfrak{K}_{\text{junk}}$ can be bounded as follows.

$$
\begin{aligned}
\|\mathfrak{K}_{\text{junk}}\|_{L^2_G} \lesssim{}& \varepsilon\,a(t)^{-1-c\eta}\sqrt{\mathfrak{E}^{(\leq L-1)}(\hat{k},\cdot)} + \varepsilon\,a(t)^{-2-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-1)}(N,\cdot)}\\
&+ \sqrt{\varepsilon}\,a(t)^{-1-c\sqrt{\varepsilon}}\sqrt{\mathfrak{E}^{(\leq L-1)}(\phi,\cdot)} + \varepsilon\,a(t)^{-1-c\eta}\sqrt{\mathfrak{E}^{(\leq L-3)}(R,\cdot)}
\end{aligned}
$$

After applying the same steps as above to these error upper bounds, one sees that the final term in (4.5.8d) indeed only contributes negligible error terms to the right hand side of (4.5.8). The statement then follows upon integrating (4.5.8) over $[t,t_0]$ and rearranging.

For (4.5.3b), one similarly computes the analogue of (4.5.8), dropping $-(\partial_t a^{\frac{\omega}{4}})\mathfrak{E}^{(L)}(\hat{k},\cdot)$ in an upper estimate, and immediately distribute weights as needed for borderline energy terms, i.e., for shear, scalar field and curvature energies

$$
\begin{aligned}
-\partial_t \left(a^{\frac{\omega}{4}}\,\mathfrak{E}^{(L)}(\hat{k},\cdot)\right) \lesssim\, & \varepsilon a(t)^{-1-c\eta}\,\mathfrak{E}^{(L)}(\hat{k},\cdot)\\
&+\sqrt{a^{\frac{\omega}{4}}\,\mathfrak{E}^{(L)}(\hat{k},\cdot)}\Big\{\varepsilon a^{-2}\sqrt{a^{\frac{\omega}{4}}\,\mathfrak{E}^{(L)}(\hat{k},\cdot)}+a^{-2+\frac{\omega}{8}}\sqrt{\mathfrak{E}^{(L)}(\phi,\cdot)}\\
&\qquad+a^{-1-c\sqrt{\varepsilon}+\frac{\omega}{8}}\left(\sqrt{\mathfrak{E}^{(\leq L)}_{1,\leq L}(f,\cdot)}+\|G-\gamma\|_{L^2_G}\right)\\
&\qquad+\varepsilon a^{-2-c\sqrt{\varepsilon}}\sqrt{a^{\frac{\omega}{4}}\,\mathfrak{E}^{(\leq L-2)}(\hat{k},\cdot)}+\varepsilon a^{-2-c\sqrt{\varepsilon}+\frac{\omega}{8}}\sqrt{\mathfrak{E}^{(\leq L-2)}(\phi,\cdot)}\\
&\qquad+a^{-2-c\sqrt{\varepsilon}+\frac{\omega}{8}}\sqrt{\varepsilon\mathfrak{E}^{(\leq L-3)}(R,\cdot)}\Big\}\\
&+\sqrt{a^{\frac{\omega}{4}}\,\mathfrak{E}^{(L)}(\hat{k},\cdot)}\cdot a^{\frac{\omega}{8}}\left(\|[\partial_t,\Delta_G^{\frac{L}{2}}]\hat{k}\|_{L^2_G}+\|\mathfrak{K}_{\text{junk}}\|_{L^2_G}\right)
\end{aligned}
$$

Again, one observes that terms introduced by the last line are strictly dominated by the remaining right hand side up to a constant, and applies the Young inequality to the remaining lines. The scaled bound (4.5.3b) then follows by integrating and inserting the initial data assumption. □

To overcome the technical difficulty that the zero order Vlasov energy estimate will involve first order terms in the metric, we also require the following elliptic estimate for the shear.

LEMMA 4.5.3 (Scaled first order energy estimate for the second fundamental form).

$$
a^2\,\mathfrak{E}^{(1)}(\hat{k},\cdot)\lesssim\mathfrak{E}^{(0)}(\phi,\cdot)+a^{2-c\sqrt{\varepsilon}}\,\mathfrak{E}^{(0)}(\hat{k},\cdot)+a^{2-c\sqrt{\varepsilon}}\,\mathfrak{E}^{(0)}_{1,0}(f,\cdot) \tag{4.5.9}
$$

PROOF. Let u be an arbitrary tracefree M_t-tangent $(0,2)$-tensor and define the exterior covariant derivative

$$
(d^{\nabla}u)_{ijl}=\nabla_l u_{ij}-\nabla_j u_{il}
$$

which satisfies

$$
\int_M |d^{\nabla}u|_g^2\,\mathrm{dvol}_g=2\int_M |\nabla u|_g^2-\nabla_j u_{il}\,\nabla^l u^{ij}\,\mathrm{dvol}_g\,.
$$

By integration by parts and using $\mathrm{Riem}[g]_{ijlm}=\frac{R[g]}{2}\,(g_{il}\,g_{jm}-g_{im}\,g_{jl})$ as well as $\mathrm{tr}_g u=0$, one obtains

$$
\begin{aligned}
\int_M \nabla_j u_{il}\,\nabla^l u^{ij}\,\mathrm{dvol}_g&=\int_M\left[|\mathrm{div}_g u|_g^2+\frac{R[g]}{2}\,(g_{jm}\,g_{il}-g_{ji}\,g_{lm})\,u^{il}\,u^{mj}+\frac{R[g]}{2}\,\mathbb{I}^l_j\,u^{ij}\,u_{il}\right]\mathrm{dvol}_g\\
&=\int_M\left[|\mathrm{div}_g u|_g^2+R[g]\,|u|_g^2\right]\,\mathrm{dvol}_g\,.
\end{aligned}
$$

After rearranging and rescaling, this becomes

$$
\int_M |\nabla u|_G^2\,\mathrm{dvol}_G=\int_M\left[\frac{1}{2}\,|d^{\nabla}u|_G^2+|\mathrm{div}_G u|_G^2+\frac{3}{4}\,R[G]\,|u|_G^2\right]\,\mathrm{dvol}_G\,. \tag{4.5.10}
$$

Now considering $u=\hat{k}$ explicitly, note that the Codazzi equation for the $(2+1)$-dimensional spacetime metric $\overline{g}$ implies

$$
d^{\nabla}\hat{k}_{ijl}=\mathrm{Riem}[\overline{g}]_{0ijl}=\mathrm{Ric}[\overline{g}]_{0j}\,g_{il}-\mathrm{Ric}[\overline{g}]_{0l}\,g_{ij}
$$

since the shift vector vanishes in our transported coordinates. Now applying (4.1.1a), this yields

$$d^\nabla \hat{k}_{ijl} = 8\pi (\Psi + C) \left[G_{il}\nabla_j\phi - G_{ij}\nabla_l\phi\right] + 8\pi \left(G_{il}\,\jmath_j^{Vl} - G_{ij}\,\jmath_l^{Vl}\right)$$

Thus, also applying (4.2.14f) to compute $\operatorname{div}_G\hat{k}$, (4.5.10) becomes

$$\mathfrak{E}^{(1)}(\hat{k},\cdot) \lesssim \int_M \left[|\Psi + C|^2\,|\nabla\phi|_G^2 + |\jmath^{Vl}|_G^2 + |R[G]|\,|\hat{k}|_G^2\right] \mathrm{dvol}_G$$

The a priori estimates (4.4.1b), (4.4.4g) and (4.4.3b) then imply (4.5.9). □

4.5.3. Estimates for metric variables. In this section, we collect various estimates for norms and energies associated with the metric, the tensor $\Gamma - \hat{\Gamma}$ and the scalar curvature. In particular, we need to derive integral estimates for $\Gamma - \hat{\Gamma}$ at low orders to deal with error terms arising from the Vlasov reference distribution as well as high order energy estimates for the curvature to control commutator errors from the Vlasov equation. In both cases, the key ingredient is the Codazzi equation – for the former in the form of the elliptic estimate (4.5.9) and for the latter by directly applying the momentum constraint (4.2.14f).

LEMMA 4.5.4 (Norm estimates for metric quantities). *Let I be an integer with $0 \le I \le 10$. Then, the following estimates hold:*

(4.5.11a)
$$\begin{aligned} a(t)^{\frac{\omega}{2}}\|G^{\pm1} - \gamma^{\pm1}\|^2_{H^I_G(M_t)} &\lesssim \varepsilon^4 + \int_t^{t_0}\left(a(s)^{-2+\frac{\omega}{8}-c\sqrt{\varepsilon}} + a(s)^{-1-c\eta}\right) a(s)^{\frac{\omega}{2}}\,\|G^{\pm1} - \gamma^{\pm1}\|^2_{H^I_G(M_s)}\,ds \\ &\quad + \int_t^{t_0} a(s)^{-2+\frac{\omega}{8}}\left(a(s)^{\frac{\omega}{2}}\,\mathfrak{E}^{(\le I)}(\hat{k},s) + \mathfrak{E}^{(\le I)}(\phi,s) + \varepsilon^{\frac12}\cdot\varepsilon^{\frac12}\,\mathfrak{E}^{(\le I-2)}(R,s)\right) ds \\ &\quad + \int_t^{t_0} a(s)^{-1-c\sqrt{\varepsilon}-\omega(I+\frac12)}\cdot a(s)^{(I+1)\omega}\,\mathfrak{E}^{(\le I)}_{1,\le I}(f,s)\,ds \end{aligned}$$

(4.5.11b)
$$\|G^{\pm1} - \gamma^{\pm1}\|^2_{H^I_G(M_t)} \lesssim a(t)^{-c\varepsilon^{\frac18}}\left(\varepsilon^4 + \varepsilon^{-\frac14}\sup_{s\in(t,t_0)}\left(\|N\|^2_{H^I_G(M_s)} + \|\hat{k}\|^2_{H^I_G(M_s)}\right)\right)$$

Additionally, one has the following low order integral estimates:

(4.5.11c)
$$\begin{aligned} a(t)^{2+\frac{\omega}{2}}\|\Gamma - \hat{\Gamma}\|^2_{L^2_G(M_t)} &\lesssim \varepsilon^4 + \int_t^{t_0}\left(a(s)^{-2+\frac{\omega}{8}} + a(s)^{-1-c\sqrt{\varepsilon}-\omega}\right)\cdot a(s)^{2+\frac{\omega}{2}}\|\Gamma - \hat{\Gamma}\|^2_{L^2_G(M_t)}\,ds \\ &\quad + \int_t^{t_0} a(s)^{-2+\frac{\omega}{8}-c\sqrt{\varepsilon}}\left(\mathfrak{E}^{(0)}(\phi,s) + a(s)^{\frac{\omega}{4}}\,\mathfrak{E}^{(0)}(\hat{k},s)\right) ds \\ &\quad + \int_t^{t_0} a(s)^{-1-c\sqrt{\varepsilon}-\frac{\omega}{2}}\left(a(s)^{\omega}\,\mathfrak{E}^{(0)}_{1,0}(f,s) + a(s)^{\frac{\omega}{2}}\|G^{-1} - \gamma^{-1}\|^2_{L^2_G(M_s)}\right) ds \end{aligned}$$

(4.5.11d)
$$a(t)^{\frac{\omega}{2}}\,\|\Gamma - \hat{\Gamma}\|^2_{H^1_G(M_t)} \lesssim \varepsilon^4 + \int_t^{t_0}\left(a(s)^{-2+\frac{\omega}{8}} + a(s)^{-1-c\sqrt{\varepsilon}-\frac{\omega}{2}}\right) a(s)^{\frac{\omega}{2}}\,\|\Gamma - \hat{\Gamma}\|^2_{H^1_G(M_s)}ds$$

$$+\int_t^{t_0} a(s)^{-2+\frac{\omega}{8}-c\sqrt{\varepsilon}}\left(a(s)^{\frac{\omega}{4}}\,\mathfrak{E}^{(\le 2)}(\hat{k},s)+\mathfrak{E}^{(\le 2)}(\phi,s)\right)\,ds$$
$$+\int_t^{t_0} a(s)^{-1-c\sqrt{\varepsilon}-\omega}\left(a(s)^{2\omega}\,\mathfrak{E}^{(\le 1)}_{1,\le 1}(f,s)+a(s)^{\frac{\omega}{2}}\,\|G^{-1}-\gamma^{-1}\|^2_{L^2_G(M_s)}\right)\,ds$$

Proof. (4.5.11a) and (4.5.11b) follow as in Lemma 2.6.14 by commuting (4.2.14a) with ∇^I and applying the lapse estimate (4.5.1) as well as the near-coercivity estimates from Lemma 4.4.4. Regarding (4.5.11c), the evolution equation (4.2.14c) implies, using the bootstrap assumption (4.3.7a) for the lapse,

$$-\partial_t\left(a^{2+\frac{\omega}{2}}\|\Gamma-\hat{\Gamma}\|^2_{L^2_G(M_\cdot)}\right)\lesssim \varepsilon a^{-c\eta}\cdot a^{2+\frac{\omega}{2}}\|\Gamma-\hat{\Gamma}\|^2_{L^2_G(M_\cdot)}$$
$$\underbrace{+a^{2+\frac{\omega}{2}}\left(a^{-2}\sqrt{\mathfrak{E}^{(\le 1)}(\hat{k},\cdot)}+a^{-2}\sqrt{\mathfrak{E}^{(1)}(N,\cdot)}\right)\|\Gamma-\hat{\Gamma}\|_{L^2_G(M_\cdot)}}_{(*)}$$

Regarding $(*)$, one can apply shear estimate in Lemma 4.5.3 and the lapse estimate (4.5.1) for $L=0$ to obtain the following estimate

$$\begin{aligned}(*) &= a^{-2+\frac{\omega}{8}}\left(\sqrt{a^{2+\frac{\omega}{4}}\,\mathfrak{E}^{(\le 1)}(\hat{k},\cdot)}+\sqrt{a^{2+\frac{\omega}{4}}\,\mathfrak{E}^{(\le 1)}(N,\cdot)}\right)\sqrt{a^{2+\frac{\omega}{2}}\|\Gamma-\hat{\Gamma}\|^2_{L^2_G(M_\cdot)}}\\
&\lesssim \Bigg(a^{-2+\frac{\omega}{4}}\sqrt{\mathfrak{E}^{(0)}(\phi,\cdot)}+\left(\varepsilon a^{-2+\frac{\omega}{8}}+a^{-1-c\sqrt{\varepsilon}}\right)\sqrt{a(s)^{\frac{\omega}{4}}\,\mathfrak{E}^{(0)}(\hat{k},\cdot)}\\
&\quad +a^{-1-c\sqrt{\varepsilon}-\frac{\omega}{2}}\sqrt{a^{\omega}\mathfrak{E}^{(0)}_{1,0}(f,\cdot)}+a^{-1-c\sqrt{\varepsilon}+\frac{\omega}{4}}\|G^{-1}-\gamma^{-1}\|_{L^2_G(M_\cdot)}\Bigg)\sqrt{a^{2+\frac{\omega}{2}}\|\Gamma-\hat{\Gamma}\|^2_{L^2_G(M_\cdot)}}\end{aligned}$$

(4.5.11c) then follows by applying the Young inequality.
(4.5.11d) is proven identically, except that one applies the Young inequality to estimate the shear term directly and inserts (4.5.1) at order $L=2$. □

To control high order curvature terms arising from the upcoming Vlasov energy estimates, it will be necessary to control curvature terms at order $L-1$ to control the other solution variables at order L. Since we only need to control the scalar curvature, the momentum constraint admits a convenient integral estimate that immediately regains this potential loss in regularity:

Lemma 4.5.5 (Energy estimate for the curvature).

(4.5.12a)

$$\begin{aligned}\mathfrak{E}^{(L-2)}(R,t) &\lesssim \varepsilon^4+\int_t^{t_0}\left(\varepsilon^{\frac{1}{8}}\,a(s)^{-2}+a(s)^{-1-c\sqrt{\varepsilon}}\right)\mathfrak{E}^{(L-2)}(R,s)\,ds\\
&\quad+\int_t^{t_0}\Big\{\varepsilon^{-\frac{1}{8}}\,a(s)^{-2}\left(\mathfrak{E}^{(L)}(\phi,s)+\mathfrak{E}^{(L)}(\hat{k},s)\right)\\
&\qquad+a(s)^{-1-c\sqrt{\varepsilon}-(L+1)\omega}\left(a(s)^{(L+1)\omega}\,\mathfrak{E}^{(\le L)}_{1,\le L}(f,s)+a(s)^{\frac{\omega}{2}}\,\|G^{-1}-\gamma^{-1}\|^2_{L^2_G(M_s)}\right)\\
&\qquad+\varepsilon^{-\frac{1}{8}}\,a(s)^{-2-c\sqrt{\varepsilon}}\left(\mathfrak{E}^{(\le L-2)}(\phi,s)+\mathfrak{E}^{(\le L-2)}(\hat{k},s)\right)\\
&\qquad+\varepsilon^{\frac{7}{8}}\,a(s)^{-2-c\sqrt{\varepsilon}}\,\mathfrak{E}^{(\le L-4)}(R,s)\Big\}\,ds,\end{aligned}$$

(4.5.12b)

$$\begin{aligned}
a(t)^2\mathfrak{E}^{(L-1)}(R,t) \lesssim{}& \varepsilon^4 + \int_t^{t_0} \left(\varepsilon^{\frac{1}{8}}\, a(s)^{-2} + a(s)^{-1-c\eta}\right) a(s)^2\, \mathfrak{E}^{(L-1)}(R,s)\, ds \\
&+ \int_t^{t_0} \Big\{ \left(\varepsilon^{\frac{15}{8}}\, a(s)^{-2} + a(s)^{-1-c\sqrt{\varepsilon}}\right) \mathfrak{E}^{(L)}(\hat{k},s) \\
&\quad + \left(\varepsilon^{-\frac{1}{8}}\, a(s)^{-2} + a(s)^{-1-c\sqrt{\varepsilon}}\right) \mathfrak{E}^{(L)}(\phi,s) \\
&\quad + a(s)^{-1-c\sqrt{\varepsilon}-(L+1)\omega} \left(a(s)^{(L+1)\omega}\, \mathfrak{E}^{(\leq L)}_{1,\leq L}(f,s) + a(s)^{\frac{\omega}{2}}\, \|G^{-1}-\gamma^{-1}\|^2_{L^2_G(M_s)}\right) \\
&\quad + \varepsilon\, a(s)^{-2}\, \mathfrak{E}^{(L-2)}(R,s) + \varepsilon\, a(s)^{-2-c\sqrt{\varepsilon}}\, \mathfrak{E}^{(\leq L-4)}(R,s) \\
&\quad + \left(\varepsilon^{\frac{15}{8}}\, a(s)^{-2-c\sqrt{\varepsilon}} + a(s)^{-1-c\sqrt{\varepsilon}}\right) \left(\mathfrak{E}^{(\leq L-2)}(\phi,s) + \mathfrak{E}^{(\leq L-2)}(\hat{k},s)\right)\Big\}\, ds
\end{aligned}$$

(4.5.12c)

$$\begin{aligned}
a(t)^{\frac{\omega}{2}}\, \mathfrak{E}^{(L-2)}(R,t) \lesssim{}& \varepsilon^4 + \int_t^{t_0} \left(\varepsilon^{\frac{1}{8}} a(s)^{-2} + a(s)^{-2+\frac{\omega}{8}} + a(s)^{-1-c\sqrt{\varepsilon}}\right) \mathfrak{E}^{(L-2)}(R,s)\, ds \\
&+ \int_t^{t_0} \Big\{ a(s)^{-2+\frac{\omega}{8}} \left(\mathfrak{E}^{(L)}(\phi,s) + \mathfrak{E}^{(L)}(\hat{k},s)\right) \\
&\quad + a(s)^{-1-c\sqrt{\varepsilon}-(L+1)\omega} \left(\mathfrak{E}^{(\leq L)}_{total,Vl}(s) + a(s)^{\frac{\omega}{2}} \|G^{-1}-\gamma^{-1}\|^2_{L^2_G(M_s)}\right) \\
&\quad + a(s)^{-2-c\sqrt{\varepsilon}+\frac{\omega}{4}} \left(\mathfrak{E}^{(\leq L-2)}(\phi,s) + a(s)^{\frac{\omega}{4}}\, \mathfrak{E}^{(\leq L-2)}(\hat{k},s)\right) \\
&\quad + \varepsilon^{\frac{7}{8}} a(s)^{-2-c\sqrt{\varepsilon}} \mathfrak{E}^{(\leq L-4)}(R,s)\Big\}\, ds
\end{aligned}$$

(4.5.12d)

$$\begin{aligned}
a(t)^{2+\frac{\omega}{4}}\, \mathfrak{E}^{(L-1)}(R,t) \lesssim{}& \varepsilon^4 + \int_t^{t_0} \left(\varepsilon^{\frac{1}{8}} a(s)^{-2} + a(s)^{-2+\frac{\omega}{8}} + a(s)^{-1-c\eta}\right) a(s)^{2+\frac{\omega}{4}}\, \mathfrak{E}^{(L-1)}(R,s)\, ds \\
&+ \int_t^{t_0} \Big\{ \left(\varepsilon a(s)^{-2} + a(s)^{-1-c\sqrt{\varepsilon}}\right) \cdot a(s)^{\frac{\omega}{4}}\, \mathfrak{E}^{(L)}(\hat{k},s) \\
&\quad + \left(a(s)^{-2+\frac{\omega}{8}} + a(s)^{-1-c\sqrt{\varepsilon}}\right) \mathfrak{E}^{(L)}(\phi,s) \\
&\quad + a(s)^{-1-c\sqrt{\varepsilon}+(L+1)\omega} \left(\mathfrak{E}^{(\leq L)}_{total,Vl}(s) + a(s)^{\frac{\omega}{2}} \|G^{-1}-\gamma^{-1}\|^2_{L^2_G}\right) \\
&\quad + \varepsilon a(s)^{-2} \cdot a(s)^{\frac{\omega}{4}}\, \mathfrak{E}^{(L-2)}(R,s) + \varepsilon a(s)^{-2-c\sqrt{\varepsilon}} \cdot a(s)^{\frac{\omega}{4}}\, \mathfrak{E}^{(\leq L-4)}(R,s) \\
&\quad + \left(\varepsilon^2 a(s)^{-2-c\sqrt{\varepsilon}} + a(s)^{-1-c\sqrt{\varepsilon}}\right) \left(\mathfrak{E}^{(\leq L-2)}(\phi,s) + a(s)^{\frac{\omega}{4}}\, \mathfrak{E}^{(\leq L-2)}(\hat{k},s)\right)\Big\}\, ds
\end{aligned}$$

Proof. We focus on proving (4.5.12d) and (4.5.12c), since (4.5.12c) and (4.5.12a) follow along similar, but simpler lines by commuting (4.2.14d) with $\Delta_G^{\frac{L}{2}}$. To prove (4.5.12b), we first compute, using bootstrap assumption (4.3.7a), that

$$\begin{aligned}
-\partial_t \left(a^2\, \mathfrak{E}^{(L-1)}(R,\cdot)\right) \lesssim{}& \varepsilon\, a^{-1-c\eta}\, a^2\, \mathfrak{E}^{(L-1)}(R,\cdot) \\
&+ \|a\, \partial_t \nabla \Delta_G^{\frac{L}{2}-1} R[G]\|_{L^2_G(M_\cdot)} \sqrt{a^2\, \mathfrak{E}^{(L-1)}(R,\cdot)}\,.
\end{aligned} \tag{4.5.13}$$

To obtain a convenient expression for $a\, \partial_t \nabla \Delta_G^{\frac{L}{2}-1} R[G]$, applying (4.2.14f) to (4.2.14d) yields

$$a\, \partial_t R[G] = 2\, N\, \tau\, a\, (R[G] - 2\,\kappa) - 8\, \kappa\, \dot{a}\, N - 2\, a^{-1}\, \langle \hat{k}, \nabla^2 N\rangle_G$$

$$+ 16\pi\, a^{-1} \langle \nabla N, (\Psi + C)\, \nabla\phi + \jmath^{Vl} \rangle_G$$
$$+ 16\pi\, a^{-1} (N+1) \left[(\Psi + C)\, \Delta_G \phi + \langle \nabla\phi, \nabla\Psi \rangle_G + \mathrm{div}_G \jmath^{Vl} \right] .$$

Commuting the above with $\nabla \Delta_G^{\frac{L}{2}-1}$, this schematically becomes the following.

(4.5.14a)
$$a\, \partial_t \nabla\, \Delta_G^{\frac{L}{2}-1} R[G] = -4\, N\, \dot{a} \nabla \Delta_G^{\frac{L}{2}-1} (R[G] - 2\,\kappa) - 4\, \dot{a}\, R[G]\, \nabla \Delta_G^{\frac{L}{2}-1} N$$

(4.5.14b) $$+ a^{-1} \left(\hat{k} * \nabla^{L+1} N + \nabla \hat{k} * \nabla^L N + \nabla^2 \hat{k} * \nabla^{L-1} N \right)$$

(4.5.14c) $$+ 16\pi\, C\, a^{-1} \nabla \Delta_G^{\frac{L}{2}} \phi + a^{-1} \left(\nabla\Psi * \nabla^2 \Delta_G^{\frac{L}{2}-1} \phi + \nabla^2 \phi * \nabla \Delta_G^{\frac{L}{2}-1} \Psi + \nabla\phi * \nabla^L \Psi \right)$$

(4.5.14d) $$+ 16\pi\, a^{-1} \nabla \Delta_G^{\frac{L}{2}} \jmath^{Vl} + a\, [\partial_t, \nabla \Delta_G^{\frac{L}{2}-1}] R[G] + a\, \mathfrak{R}_{\mathrm{junk},L-1}$$

Herein, $\mathfrak{R}_{\mathrm{junk},L-1}$ collects lower order error terms. Using the low order bounds (4.4.1a), (4.4.3a) and (4.4.3b) as well as the bootstrap bound (4.3.7a) for the lapse, the L^2_G-norm of the right hand side of (4.5.14a)-(4.5.14b) is bounded by

$$\lesssim \varepsilon\, a^{-1-c\eta} \sqrt{a^2\, \mathfrak{E}^{(L-1)}(R,\cdot)} + \varepsilon\, a^{-2} \sqrt{a^2\, \mathfrak{E}^{(L+1)}(N,\cdot)}$$
$$+ \varepsilon\, a^{-1-c\sqrt{\varepsilon}} \sqrt{\mathfrak{E}^{(L)}(N,\cdot)} + a^{-1-c\sqrt{\varepsilon}} \sqrt{\mathfrak{E}^{(\leq L-1)}(N,\cdot)} .$$

Using the low order estimates (4.4.3a) and (4.4.3b) as well as (4.3.3), that of (4.5.14c) is bounded by

$$\lesssim a^{-2} \sqrt{\mathfrak{E}^{(L)}(\phi,\cdot)} + \left(\varepsilon\, a^{-2-c\sqrt{\varepsilon}} + \sqrt{\varepsilon}\, a^{-1-c\sqrt{\varepsilon}} \right) \sqrt{\mathfrak{E}^{(\leq L-2)}(\phi,\cdot)} ,$$

while the Vlasov term in (4.5.14d) can be estimated directly using (4.4.4g). The remaining error terms in (4.5.14d) can be bounded as follows:

$$a \left\| [\partial_t, \nabla \Delta_G^{\frac{L}{2}-1}] R[G] \right\|_{L^2_G} \lesssim \varepsilon\, a^{-2} \sqrt{a^2\, \mathfrak{E}^{(L-1)}(R,\cdot)} + \varepsilon\, a^{-1-c\sqrt{\varepsilon}} \sqrt{\mathfrak{E}^{(\leq L-2)}(R,\cdot)}$$
$$+ \varepsilon\, a^{-1-c\sqrt{\varepsilon}} \left(\sqrt{\mathfrak{E}^{(\leq L-2)}(\hat{k},\cdot)} + \sqrt{\mathfrak{E}^{(\leq L-2)}(N,\cdot)} \right)$$
$$a \|\mathfrak{R}_{\mathrm{junk},L-1}\|_{L^2_G} \lesssim \varepsilon\, a^{-1-c\sqrt{\varepsilon}} \left(\sqrt{\mathfrak{E}^{(\leq L-1)}(\hat{k},\cdot)} + \sqrt{\mathfrak{E}^{(\leq L-2)}(N,\cdot)} \right)$$
$$+ \varepsilon\, a^{-c\eta} \sqrt{\mathfrak{E}^{(\leq L-2)}(R,\cdot)}$$

Returning to (4.5.13) with these estimates in hand, one obtains the following, using the Young inequality and (4.3.3).

$$-\partial_t \left(a^2\, \mathfrak{E}^{(L-1)}(R,\cdot) \right) \lesssim \left(\varepsilon^{\frac{1}{8}}\, a^{-2} + a^{-1-c\eta} \right) \left(a^2 \mathfrak{E}^{(L-1)}(R,\cdot) \right) + \varepsilon\, a^{-2} \left(a^2\, \mathfrak{E}^{(L+1)}(N,\cdot) \right)$$
$$+ a^{-1-c\sqrt{\varepsilon}}\, \mathfrak{E}^{(L)}(N,\cdot) + \varepsilon^{-\frac{1}{8}}\, a^{-2}\, \mathfrak{E}^{(L)}(\phi,\cdot)$$
$$+ \varepsilon\, a^{-1-c\sqrt{\varepsilon}} \left(\mathfrak{E}^{(L)}(\hat{k},\cdot) + \mathfrak{E}^{(L-2)}(R,\cdot) \right)$$
$$+ a^{-1-c\sqrt{\varepsilon}}\, \mathfrak{E}^{(\leq L)}_{1,\leq L}(f,\cdot) + \varepsilon\, a^{-1-c\sqrt{\varepsilon}}\, \mathfrak{E}^{(\leq L-2)}(N,\cdot)$$

$$+\left(\varepsilon\, a^{-2-c\sqrt{\varepsilon}}+\sqrt{\varepsilon}\, a^{-1-c\sqrt{\varepsilon}}\right)\mathfrak{E}^{(\leq L-2)}(\phi,\cdot)$$
$$+\,\varepsilon\, a^{-1-c\sqrt{\varepsilon}}\left(\mathfrak{E}^{(\leq L-2)}(\hat{k},\cdot)+\mathfrak{E}^{(\leq L-4)}(R,\cdot)\right)$$

The estimate (4.5.12b) now follows from inserting the lapse estimate (4.5.1) up to order L (which, in particular, also controls $a^2\mathfrak{E}^{(L+1)}(N,\cdot)$ sufficiently) and then integrating in time.

Regarding (4.5.12d), the individual estimates lead to the following bound by distributing the scale factor weight between individual factors.

$$\begin{aligned}
-\partial_t\big(a^{2+\frac{\omega}{4}}\,\mathfrak{E}^{(L-1)}(R,\cdot)\big) \lesssim{}& \left(\varepsilon^{\frac18}\, a(s)^{-2}+a(s)^{-1-c\eta}\right)\left(a^2\,\mathfrak{E}^{(L-1)}(R,\cdot)\right)\\
&+\sqrt{a^{2+\frac{\omega}{4}}\,\mathfrak{E}^{(L-1)}(R,\cdot)}\,\Bigg\{\varepsilon a^{-2}\sqrt{a^{2+\frac{\omega}{4}}\,\mathfrak{E}^{(L+1)}(N,\cdot)}+a^{-1-c\sqrt{\varepsilon}+\frac{\omega}{8}}\sqrt{\mathfrak{E}^{(L)}(N,\cdot)}+\\
&\quad+a^{-2+\frac{\omega}{8}}\sqrt{\mathfrak{E}^{(L)}(\phi,\cdot)}+\varepsilon a^{-1-c\sqrt{\varepsilon}}\left(\sqrt{a^{\frac{\omega}{4}}\,\mathfrak{E}^{(L)}(\hat{k},\cdot)}+\sqrt{a^{\frac{\omega}{4}}\,\mathfrak{E}^{(L-2)}(R,\cdot)}\right)\\
&\quad+\left(\varepsilon a^{-2-c\sqrt{\varepsilon}+\frac{\omega}{8}}+a^{-1-c\sqrt{\varepsilon}+\frac{\omega}{8}}\right)\sqrt{\mathfrak{E}^{(\leq L-2)}(\phi,\cdot)}\\
&\quad+\varepsilon a^{-1-c\sqrt{\varepsilon}}\left(\sqrt{a^{\frac{\omega}{4}}\,\mathfrak{E}^{(\leq L-2)}(\hat{k},\cdot)}+\sqrt{a^{\frac{\omega}{4}}\,\mathfrak{E}^{(\leq L-4)}(R,\cdot)}\right)\Bigg\}
\end{aligned}$$

One again inserts the lapse estimate (4.5.1), distributing weights for the resulting terms in a similar manner. In particular, the leading order shear term from $a^2\mathfrak{E}^{(L+1)}(N,\cdot)$ enters as follows.

$$\begin{aligned}
&\left(\varepsilon a^{-2}+a^{-1-c\sqrt{\varepsilon}+\frac{\omega}{8}}\right)\sqrt{a^{2+\frac{\omega}{4}}\,\mathfrak{E}^{(L-1)}(R,\cdot)}\sqrt{\varepsilon^2 a^{\frac{\omega}{4}}\,\mathfrak{E}^{(L)}(\hat{k},\cdot)}\\
\lesssim{}&\varepsilon^{\frac18}a^{-2}a^{2+\frac{\omega}{4}}\,\mathfrak{E}^{(L-1)}(R,\cdot)+\varepsilon^2 a^{-2}\,a^{\frac{\omega}{4}}\,\mathfrak{E}^{(L)}(\hat{k},\cdot)
\end{aligned}$$

Using Young's inequality as well as (4.3.3) and integrating in time, this then implies (4.5.12d). □

4.5.4. Energy estimates for matter components. Herein, we collect the energy estimates for scalar field and Vlasov matter. The proof for the former goes along similar lines as in Lemma 2.6.2 and Lemma 3.5.5, but we verify that the delicate cancellations therein which rely on the respective Friedman equations still extend to this setting, including to $C<0$.

LEMMA 4.5.6 (Scalar field energy estimate).

$$\begin{aligned}
&\mathfrak{E}^{(L)}(\phi,t)+\int_t^{t_0}\dot a(s)\,a(s)\,\mathfrak{E}^{(L+1)}(N,s)+\frac{\dot a(s)}{a(s)}\,\mathfrak{E}^{(L)}(N,s)\,ds \qquad (4.5.15)\\
\lesssim{}&\varepsilon^4+\int_t^{t_0}\left(\varepsilon\, a(s)^{-2}+a(s)^{-1-c\sqrt{\varepsilon}}\right)\mathfrak{E}^{(L)}(\phi,s)\,ds\\
&+\int_t^{t_0}\Big\{\varepsilon\, a(s)^{-2}\mathfrak{E}^{(L)}(\hat{k},s)+\varepsilon^{\frac32}\,a(s)^{-2}\,\mathfrak{E}^{(L-2)}(R,s)
\end{aligned}$$

$$+ a(s)^{-1-c\sqrt{\varepsilon}-(L+1)\omega} \cdot \left(a(s)^{(L+1)\omega}\, \mathfrak{E}^{(\leq L)}_{1,\leq L}(f,s) + a(s)^{\frac{\omega}{2}}\, \|G-\gamma\|^2_{L^2_G(M_s)} \right)$$

$$+ \sqrt{\varepsilon}\, a(s)^{-2-c\sqrt{\varepsilon}} \mathfrak{E}^{(\leq L-2)}(\phi,s) + \varepsilon\, a(s)^{-2-c\sqrt{\varepsilon}}\, \mathfrak{E}^{(\leq L-2)}(\hat{k},s)$$

$$+ \varepsilon^{\frac{3}{2}}\, a(s)^{-2-c\sqrt{\varepsilon}} \mathfrak{E}^{(\leq L-4)}(R,s) \Big\}\, ds$$

Proof. Schematically, after commuting (4.2.14h) and (4.2.14i) with $\Delta_G^{\frac{L}{2}}$ and applying integration by parts to cancel the highest order terms in $\nabla\phi$, one obtains the following.

$$-\partial_t \mathfrak{E}^{(L)}(\phi,\cdot) = \int_M \Big\{ 4\, C\, \frac{\dot{a}}{a}\, \Delta_G^{\frac{L}{2}} N\, \Delta^{\frac{L}{2}} \Psi + 2\langle \nabla \Delta_G^{\frac{L}{2}} N, \nabla\phi\rangle_G\, \Delta_G^{\frac{L}{2}} \Psi$$
$$-2\, C\, \langle \nabla \Delta_G^{\frac{L}{2}} N, \nabla \Delta_G^{\frac{L}{2}} \phi\rangle_G - 2\, \dot{a}\, a\, |\nabla \Delta_G^{\frac{L}{2}} \phi|^2_G \Big\}\, \mathrm{dvol}_G + \langle \text{error terms}\rangle$$

The first two terms arise from (4.2.14i), the third from (4.2.14h) and the fourth from differentiating the weight within the energy. Regarding the first term, applying $\Delta_G^{\frac{L}{2}}$ to both sides of (4.2.14g) implies

$$0 = -\frac{1}{4\pi}\, \dot{a}\, a \int_M \mathrm{div}_G \left(\nabla \Delta_G^{\frac{L}{2}} N\, \Delta_G^{\frac{L}{2}} N \right) \mathrm{dvol}_G$$
$$= \int_M \left[-\frac{1}{4\pi}\, \dot{a}\, a\, |\nabla \Delta_G^{\frac{L}{2}} N|^2_G - \frac{1}{4\pi}\frac{\dot{a}}{a}\, \Delta_G^{\frac{L}{2}} (H + hN)\, \Delta_G^{\frac{L}{2}} N \right] \mathrm{dvol}_G$$

(4.5.16a)
$$= \int_M \Big[-\frac{1}{4\pi}\, \dot{a}\, a\, |\nabla \Delta_G^{\frac{L}{2}} N|^2_G - \left(4\, C^2\, \frac{\dot{a}}{a} + 4\, \dot{a} \left(\rho^{Vl}_{FLRW} + 2\, \mathfrak{p}^{Vl}_{FLRW} \right) - \frac{\kappa}{2\pi}\, \dot{a}\, a \right) |\Delta_G^{\frac{L}{2}} N|^2_G$$

(4.5.16b)
$$- 4\, C\, \frac{\dot{a}}{a}\, \Delta_G^{\frac{L}{2}} \Psi\, \Delta_G^{\frac{L}{2}} N - 4\, \dot{a} \Delta^{\frac{L}{2}} \left(\rho^{Vl} - \rho^{Vl}_{FLRW} \right) \Delta_G^{\frac{L}{2}} N + \langle \text{nonlinear error terms}\rangle \Big]\, \mathrm{dvol}_G\,.$$

Inserting this zero into (4.5.17), the first two terms of (4.5.16a) have definite sign and lead to the terms on the left hand side of (4.5.15) after absorbing some more terms discussed below, while the remaining ones in (4.5.16a) are bounded by $a^{-1}\, \mathfrak{E}^{(L)}(N,\cdot)$. The first term in the (4.5.16b) cancels the first term on the right hand side of (4.5.17), while the latter can be bounded using (4.4.4e). Altogether, this means that the below bound holds for an appropriate constant $K > 0$.

(4.5.17)
$$\partial_t \mathfrak{E}^{(L)}(\phi,\cdot) + \frac{1}{4\pi}\, \dot{a}\, a\, \mathfrak{E}^{(L+1)}(N,\cdot) + 4\, C^2 \frac{\dot{a}}{a}\, \mathfrak{E}^{(L)}(N,\cdot)$$
$$\leq \int_M \Big\{ 2\, \langle \nabla \Delta_G^{\frac{L}{2}} N, \nabla\phi\rangle_G\, \Delta_G^{\frac{L}{2}} \Psi - 2\, C\, \langle \nabla \Delta_G^{\frac{L}{2}} N, \nabla \Delta_G^{\frac{L}{2}} \phi\rangle_G - 2\, \dot{a}\, a\, |\nabla \Delta_G^{\frac{L}{2}} \phi|^2_G \Big\}\, \mathrm{dvol}_G$$
$$+ K \left(a^{-1}\, \mathfrak{E}^{(L)}(N,\cdot) + a^{-1-c\sqrt{\varepsilon}} \left(\mathfrak{E}^{(L)}_{1,L}(f,\cdot) + \|G^{-1} - \gamma^{-1}\|^2_{L^2_G} \right) \right) + \langle \text{error terms}\rangle$$

Since, by (4.4.3b), the first term on the right hand side of (4.5.17) is bounded in absolute value by

$$\lesssim \int_M \sqrt{\varepsilon}\, a^{-c\sqrt{\varepsilon}}\, |\nabla \Delta_G^{\frac{L}{2}} N|_G\, |\Delta_G^{\frac{L}{2}} \Psi|_G\, \mathrm{dvol}_G \lesssim \int_M \varepsilon\, \dot{a}\, a\, |\nabla \Delta_G^{\frac{L}{2}} N|^2_G\, \mathrm{dvol}_G + a^{-c\sqrt{\varepsilon}}\, \mathfrak{E}^{(L)}(\phi,\cdot)\,,$$

we can absorb this resulting lapse term into the left hand side of (4.5.17) by updating prefactors, while we simply keep the scalar field term on the right hand side.

Moving on to the second term on the right hand side of (4.5.17), (4.2.5a) and Young's inequality imply

$$\begin{aligned}\left|2\,C\,\langle\nabla\Delta_G^{\frac{L}{2}}N,\nabla\Delta_G^{\frac{L}{2}}\phi\rangle_G\right| &\leq \frac{1}{\sqrt{\pi}}\left(\dot{a}\,a+\sqrt{\max\{0,\kappa\}}\,a\right)|\nabla\Delta_G^{\frac{L}{2}}N|_G\,|\nabla\Delta_G^{\frac{L}{2}}\phi|_G\\ &\leq\left(\dot{a}\,a+\sqrt{\max\{0,\kappa\}}\,a\right)\left(2\,|\nabla\Delta_G^{\frac{L}{2}}\phi|_G^2+\frac{1}{8\pi}\,|\nabla\Delta_G^{\frac{L}{2}}N|_G^2\right)\,,\end{aligned}$$

The first term, combined with the third term in (4.5.17), only leaves

$$\sqrt{\max\{0,\kappa\}}\int_M a\,|\nabla\Delta_G^{\frac{L}{2}}\phi|_G^2\,\mathrm{dvol}_G\lesssim a^{-1}\,\mathfrak{E}^{(L)}(\phi,\cdot)\,.$$

The second term can also be absorbed into the second term on the left of (4.5.17) up to the error term $a^{-1}\mathfrak{E}^{(L)}(N,\cdot)$ that we keep on the right.

In summary, this shows

$$\begin{aligned}-\partial_t\mathfrak{E}^{(L)}(\phi,\cdot)\;+\dot{a}\,a\,\mathfrak{E}^{(L+1)}(N,\cdot)+\frac{\dot{a}}{a}\,\mathfrak{E}^{(L)}(N,\cdot)&\\ \lesssim a^{-1-c\sqrt{\varepsilon}}\left(\mathfrak{E}^{(L)}(\phi,\cdot)+\mathfrak{E}^{(L)}(N,\cdot)+\mathfrak{E}^{(\leq L)}_{1,\leq L}(f,\cdot)+\|G^{-1}-\gamma^{-1}\|^2_{L^2_G(M_\cdot)}\right)&\\ +\langle\text{collected borderline and junk error terms}\rangle&\end{aligned}$$

From here, (4.5.15) follows by estimating the error terms with the help of the low order bounds in Section 4.4, the elliptic lapse estimate Lemma 4.5.1 and the near-coercivity estimate Lemma 4.4.4. □

Regarding the estimates for Vlasov matter, the structural features exploited in Section 3.6 fully survive in (4.2.14j) and thus allow us to extract the energy estimates in Lemma 4.5.7 as in the higher-dimensional setting. We provide estimates (4.5.18)-(4.5.19), which are necessary for obtaining bounds on $\mathfrak{E}^{(L)}_{\text{total}}$, as well as the unscaled estimates (4.5.20), which are needed to improve the bootstrap assumptions on Vlasov matter.

LEMMA 4.5.7 (Total Vlasov energy estimate). *At order* 0*, the following bound holds:*

(4.5.18)

$$\begin{aligned}a(t)^\omega\,\mathfrak{E}^{(0)}_{1,0}(f,t)\lesssim\varepsilon^4&+\int_t^{t_0}\left(\varepsilon a(s)^{-2}+a^{-2+\frac{\omega}{4}}\right)\,a(s)^\omega\mathfrak{E}^{(0)}_{1,0}(f,s)\,ds\\ &+\int_t^{t_0}a(s)^{-2-c\sqrt{\varepsilon}+\frac{\omega}{4}}\left(\mathfrak{E}^{(0)}(\phi,s)+\varepsilon^2\,\mathfrak{E}^{(0)}(\hat{k},s)+a(s)^{2+\frac{\omega}{2}}\|\Gamma-\hat{\Gamma}\|^2_{L^2_G(M_s)}\right)\,ds\\ &+\int_t^{t_0}a(s)^{-1-c\sqrt{\varepsilon}+\frac{\omega}{4}}\,a(s)^{\frac{\omega}{2}}\,\|G^{-1}-\gamma^{-1}\|^2_{L^2_G(M_s)}\,ds\end{aligned}$$

For $L\in\{2,4,6,8,10\}$, the total Vlasov energy $\mathfrak{E}^{(L)}_{\text{total,Vl}}$ as defined in Definition 4.3.2 obeys the following energy estimate.

(4.5.19a)

$$\mathfrak{E}^{(L)}_{\text{total,Vl}}(t)\lesssim\varepsilon^4+\int_t^{t_0}\left(a(s)^{-1-\omega}+a(s)^{-2-c\sqrt{\varepsilon}+\frac{\omega}{2}}+\varepsilon^{\frac{1}{8}}\,a(s)^{-2}\right)\mathfrak{E}^{(L)}_{\text{total,Vl}}(s)\,ds$$

$$+\int_t^{t_0} a(s)^{-2-c\sqrt{\varepsilon}+L\omega}\left(\mathfrak{E}^{(L)}(\phi,s)+a(s)^{\frac{\omega}{4}}\mathfrak{E}^{(L)}(\hat{k},s)\right)ds$$
$$+\int_t^{t_0} a(s)^{-2-c\sqrt{\varepsilon}+L\omega}a(s)^{2+\frac{\omega}{4}}\mathfrak{E}^{(L-1)}(R,s)+a(s)^{-2-c\sqrt{\varepsilon}+L\omega}a(s)^{\frac{\omega}{2}}\mathfrak{E}^{(L-2)}(R,s)\,ds$$
$$+\int_t^{t_0}\left(a(s)^{-1-\omega-c\eta}+a(s)^{-2-c\sqrt{\varepsilon}+\frac{\omega}{2}}\right)\mathfrak{E}^{(\leq L-2)}_{\text{total,Vl}}(f,s)\,ds$$
$$+\int_t^{t_0} a(s)^{-2-c\sqrt{\varepsilon}+\omega}\left(\mathfrak{E}^{(\leq L-2)}(\phi,s)+a(s)^{\frac{\omega}{4}}\mathfrak{E}^{(\leq L-2)}(\hat{k},s)\right)ds$$
$$+\int_t^{t_0} a(s)^{-2-c\sqrt{\varepsilon}+\omega}a(s)^{\frac{\omega}{2}}\mathfrak{E}^{(\leq L-4)}(R,s)\,ds$$
$$+\int_t^{t_0} a(s)^{-1-c\sqrt{\varepsilon}}\cdot a(s)^{\frac{\omega}{2}}\left(\|\Gamma-\hat{\Gamma}\|^2_{H^1_G(M_s)}+\|G^{-1}-\gamma^{-1}\|^2_{H^2_G(M_s)}\right)ds$$

Recalling $\mathfrak{E}^{(L)}_{\text{total}}$ from Definition 4.3.2, this becomes

$$\mathfrak{E}^{(L)}_{\text{total,Vl}}(t)\lesssim\varepsilon^4+\int_t^{t_0}\left(a(s)^{-1-\omega-c\sqrt{\varepsilon}}+a(s)^{-2-c\sqrt{\varepsilon}+\frac{\omega}{4}}+\varepsilon^{\frac{1}{8}}a(s)^{-2}\right)\mathfrak{E}^{(L)}_{\text{total}}(s)\,ds$$
$$+\int_t^{t_0}\left(a(s)^{-1-\omega-c\eta}+a(s)^{-2-c\sqrt{\varepsilon}+\frac{\omega}{4}}\right)\mathfrak{E}^{(\leq L-2)}_{\text{total}}(s)\,ds\,.\tag{4.5.19b}$$

Additionally, the following individual unscaled estimates hold for any integers J,K with $0\leq K\leq J\leq 10$:

(4.5.20)

$$\mathfrak{E}^{(J)}_{1,K}(f,t)\lesssim\varepsilon^4+\int_t^{t_0}\left(a(s)^{-1-c\eta}+\varepsilon^{\frac{1}{8}}a(s)^{-2}\right)\mathfrak{E}^{(J)}_{1,K}(f,s)\,ds$$
$$+\underbrace{\int_t^{t_0} a(s)^{-1-c\eta-(K+2)\omega}a(s)^{(K+2)\omega}\mathfrak{E}^{(J)}_{1,K+1}(f,s)\,ds}_{\text{if } K<J}$$
$$+\int_t^{t_0}\Big\{\varepsilon^{\frac{7}{8}}a(s)^{-2-c\sqrt{\varepsilon}}\left(\mathfrak{E}^{(J)}_{1,\leq K-1}(f,s)+\mathfrak{E}^{(\leq J-1)}_{1,\leq K}(f,s)\right)$$
$$+\varepsilon^{-\frac{1}{8}}a(s)^{-2-c\sqrt{\varepsilon}}\left(\mathfrak{E}^{(\leq J)}(\phi,s)+\mathfrak{E}^{(\leq J)}(\hat{k},s)+\mathfrak{E}^{(\leq J-2)}(R,s)+a(s)^2\mathfrak{E}^{(J-1)}(R,s)\right)$$
$$+a(s)^{-1-c\sqrt{\varepsilon}}\left(\|\Gamma-\hat{\Gamma}\|^2_{H^1_G(M_s)}+\|G^{-1}-\gamma^{-1}\|^2_{H^2_G(M_s)}\right)\Big\}\,ds$$

PROOF. Again, we only provide an outline: Firstly, recalling the Vlasov operator from (4.2.14k), one has that, for $K>0$,

$$a^{(K+1)\omega}\left|\int_{T^*M}\langle v\rangle^2_G\langle \boldsymbol{X}\underline{\nabla}^K_{\text{hor}}\underline{\nabla}^{J-K}_{\text{vert}}f,\underline{\nabla}^K_{\text{hor}}\underline{\nabla}^{J-K}_{\text{vert}}f\rangle_{\underline{G}_0}\,\text{dvol}_{\underline{G}}\right|\lesssim\varepsilon\,a^{1-c\eta}\,a^{(K+1)\omega}\mathfrak{E}^{(L)}_{1,K}(f,\cdot)$$

and the analogous bound for $K=0$ replacing f with $f-f_{FLRW}$, see Lemma 3.6.2. This is shown by integrating by parts and using (4.4.2a), (4.3.7a) and (4.4.1a) to bound terms that arise when derivatives hit variables other than f. This provides the core energy conservation mechanism and leads directly to (4.5.18) after bounding error terms arising from $\boldsymbol{X}f_{FLRW}$ as well as various error terms that arise when the time derivative hits momentum weights or metric terms. These can be controlled using the same tools along with (4.4.3b). The final line of (4.5.19a) occurs from reference terms similar to (4.4.6b).

For higher orders, we consider the evolution of $a^{(K+1)\omega}\mathfrak{E}^{(L)}_{1,K}(f,\cdot)$ for $L > K > 0$ as an example:

$$\begin{aligned}
&- a^{(K+1)\omega}\partial_t\mathfrak{E}^{(L)}_{1,K}(f,\cdot)\\
&= - a^{(K+1)\omega}\int_{T^*M}\langle v\rangle_G^2\left(2\frac{\partial_t\langle v\rangle_G}{\langle v\rangle_G}+2(L-K)\frac{\partial_t v^0}{v^0}\right)|\underline{\nabla}_{vert}^{L-K}\underline{\nabla}_{hor}^K f|^2_{\underline{G}_0}\,\mathrm{dvol}_{\underline{G}} \quad (4.5.21a)\\
&\quad - a^{(K+1)\omega}\int_{T^*M}\langle v\rangle_G^2\,\partial_t\underline{G}^{-1}*_{\underline{G}}\underline{\nabla}_{vert}^{L-K}\underline{\nabla}_{hor}^K f*_{\underline{G}}\underline{\nabla}_{vert}^{L-K}\underline{\nabla}_{hor}^K f\,\mathrm{dvol}_{\underline{G}} \quad (4.5.21b)\\
&\quad - a^{(K+1)\omega}\int_{T^*M}\langle v\rangle_G^2\langle[\partial_t,\underline{\nabla}_{vert}^{L-K}\underline{\nabla}_{hor}^K]f,\underline{\nabla}_{vert}^{L-K}\underline{\nabla}_{hor}^K]f\rangle_{\underline{G}_0}\,\mathrm{dvol}_{\underline{G}} \quad (4.5.21c)\\
&\quad - a^{(K+1)\omega}\int_{T^*M}\langle v\rangle_G^2\langle[\underline{\nabla}_{vert}^{L-K}\underline{\nabla}_{hor}^K,\boldsymbol{X}]f,\underline{\nabla}_{vert}^{L-K}\underline{\nabla}_{hor}^K]f\rangle_{\underline{G}_0}\,\mathrm{dvol}_{\underline{G}} \quad (4.5.21d)\\
&\quad + \langle\text{term controlled via (4.5.20)}\rangle
\end{aligned}$$

The first two lines can be controlled in absolute value by $\lesssim \varepsilon a^{-2}\mathfrak{E}^{(L)}_{1,K}(f,\cdot)$ by inserting the evolution of the spatial metric (4.2.14a) (and that of its inverse) and controlling resulting zero order shear and lapse terms with (4.4.1a) and (4.3.7a) respectively. (4.5.21c) can be controlled by bounds on $\|\partial_t R[G]\|_{H^{K-2}}$ and $\|\partial_t\Gamma[G]\|_{H^{K-1}}$, see (4.2.14d) and (4.2.14c), as well as using Lemma 4.4.3 to estimate low order terms and Lemma 4.4.2 to estimate momentum weights where necessary. This leads to

$$\begin{aligned}
&a^{\frac{(K+1)\omega}{2}}\left\|[\partial_t,\underline{\nabla}_{\mathrm{hor}}^K\underline{\nabla}_{\mathrm{vert}}^{L-K}]f\right\|_{L^2_{\underline{G}_0}(T^*M.)}\\
&\lesssim a^{-2+\frac{\omega}{2}-c\sqrt{\varepsilon}}\left(\sqrt{a^{K\omega}\mathfrak{E}^{(\le K)}(\hat{k},\cdot)}+\sqrt{a^{K\omega}\mathfrak{E}^{(\le K)}(N,\cdot)}+\varepsilon\sqrt{a^{K\omega}\mathfrak{E}^{(\le K-2)}(\mathrm{Ric},\cdot)}\right)\\
&\quad+\varepsilon\, a^{-2+\frac{\omega}{2}-c\sqrt{\varepsilon}}\left(\sqrt{a^{K\omega}\mathfrak{E}^{(L)}_{1,K-1}(f,\cdot)}+\sqrt{a^{K\omega}\mathfrak{E}^{(\le L-1)}_{1,\le K-1}(f,\cdot)}\right).
\end{aligned}$$

After applying Young's inequality, (4.3.3) and (4.4.6b) as usual, the resulting terms are controlled by the first four lines of (4.5.19a).

Regarding (4.5.21d), the two main considerations need to be taken when commuting with the horizontal term and the shear term in $\boldsymbol{X}$. When commuting with the horizontal term, one obtains the following high order terms

$$a^{-1}\,\nabla^{K-1}R[G]*\frac{v*v}{v^0}*\underline{\nabla}_{\mathrm{vert}}^{L-K+1}f+a^{-1}\left(\frac{1}{v^0}+\frac{v*v}{(v^0)^3}\right)*\underline{\nabla}_{\mathrm{vert}}^{L-K-1}\underline{\nabla}_{\mathrm{hor}}^{K+1}f\,, \quad (4.5.22)$$

among some lower order nonlinear terms. The first term in (4.5.22) can be controlled in $L^2_{1,\underline{G}_0}$ by $a^{-2-c\sqrt{\varepsilon}}\sqrt{a^2\,\mathfrak{E}^{(K-1)}(R,\cdot)}$ and is the reason we need to include this high order scaled energy in the third line of (4.5.19a). The second term can be bounded by $a^{-1}\sqrt{\mathfrak{E}^{(L)}_{1,K+1}(f,\cdot)}$, taking the momentum weights in $\underline{G}_0$ into account. Thus, these two leading order terms from

(4.5.22) in (4.5.21d) can be controlled by

$$\lesssim \left(a^{-2+\frac{\omega}{2}} + a^{-1}\right) a^{(K+1)\omega} \mathfrak{E}^{(L)}_{1,K}(f,\cdot) + a^{-2+K\omega-c\sqrt{\varepsilon}} \mathfrak{E}^{(K-1)}(R,\cdot) + a^{-1-\omega}\, a^{(K+2)\omega} \mathfrak{E}^{(L)}_{1,K+1}(f,\cdot)\,.$$

Note that the final term can be bounded by $a^{-1-\omega}\, \mathfrak{E}^{(L)}_{\text{total,Vl}}$, and thus is absorbed into the first line of (4.5.19a). Regarding the vertical derivative term in the Vlasov operator, high order lapse terms can be controlled using Lemma 4.5.1 and (4.4.3c), and high order Vlasov terms have negligible effect since ∇N converges at low orders due to the bootstrap assumption (4.3.7a). In summary, integrating (4.5.21) for each K leads to right hand sides controlled by (4.5.19a), and after summing over K, to (4.5.19a)-(4.5.19b).

The bound (4.5.20) is extracted similarly, where instead of exploiting a scaling hierarchy, one simply keeps energy terms that would otherwise barely fail to be integrable, at the cost of the weight $\varepsilon^{-\frac{1}{8}}$ when applying Young's inequality. Note that, once these spacetime quantities are sufficiently well controlled via estimates for $\mathfrak{E}^{(L)}_{\text{total}}$, these are simply inhomogeneous error terms. Thus, these estimates will be sufficient to close the Vlasov energy argument. □

4.5.5. Total energy estimate. Finally, we can combine the energy estimates from this section to obtain the following total energy bounds.

Corollary 4.5.8 (Total energy bounds). *For any $L \in \{0,2,4,6,8,10\}$ and sufficiently small $\omega \gg \eta$ (e.g., $\omega = \frac{1}{100}$ and η sufficiently small), the total energy satisfies the following bound.*

$$\mathfrak{E}^{(\leq L)}_{\text{total}}(t) \lesssim \varepsilon^4\, a(t)^{-c\varepsilon^{\frac{1}{8}}} \tag{4.5.23}$$

Furthermore, one has

$$\mathfrak{E}^{(\leq L)}_{\text{quiesc}}(t) \lesssim \varepsilon^4\, a(t)^{-c\varepsilon^{\frac{1}{8}}} \tag{4.5.24}$$

as well as the following energy bounds for the lapse and curvature.

$$a^4\, \mathfrak{E}^{(12)}(N,t) + a^2\, \mathfrak{E}^{(11)}(N,t) + \mathfrak{E}^{(\leq 10)}(N,t) \lesssim \varepsilon^{\frac{15}{4}}\, a^{2-c\varepsilon^{\frac{1}{8}}} \tag{4.5.25}$$

$$a^2 \mathfrak{E}^{(9)}(R,\cdot) \lesssim \varepsilon^{\frac{15}{4}} a(t)^{2-c\varepsilon^{\frac{1}{8}}} \tag{4.5.26}$$

Furthermore, for integers J, K with $0 \leq K \leq J \leq 10$, one has

$$\mathfrak{E}^{(\leq J)}_{1,\leq K}(f,t) \lesssim \varepsilon^{\frac{7}{2}}\, a(t)^{-c\varepsilon^{\frac{1}{8}}}\,. \tag{4.5.27}$$

Proof. First, we combine the energy estimates within this section into the following total estimate.

$$\mathfrak{E}^{(L)}_{\text{total}}(t) \lesssim \varepsilon^4 + \int_t^{t_0} \left(\varepsilon^{\frac{1}{8}}\, a(s)^{-2} + a(s)^{-1-c\eta-(L+1)\omega} + a(s)^{-2-c\sqrt{\varepsilon}+\frac{\omega}{8}}\right) \mathfrak{E}^{(L)}_{\text{total}}(s)\, ds + \int_t^{t_0} \left(\varepsilon^{\frac{3}{8}}\, a(s)^{-2-c\sqrt{\varepsilon}} + a(s)^{-1-c\eta-(L+1)\omega}\right) \mathfrak{E}^{(\leq L-2)}_{\text{total}}(s)\, ds \tag{4.5.28}$$

More precisely, this is obtained by estimating terms in the following manner, in the same order as in (4.3.2h).

- For any L, use the following estimates to control the first line of (4.3.2h).
  - Apply (4.5.15) to the scalar field energy.
  - Apply (4.5.3a) and (4.5.3b) to the shear energy, with the former to the summand scaled by $\varepsilon^{\frac{1}{4}}$ and the latter to the one scaled by $a^{\frac{\omega}{4}}$.
  - Apply (4.5.12a) and (4.5.12c) to the curvature energies at order $L-2$ analogously.
- The terms in the second line of (4.3.2h) are treated as follows.
  - To control the Vlasov energy, apply (4.5.18) at order zero and (4.5.19b) for $L \geq 2$.
  - Apply (4.5.12b) and (4.5.12d) to the curvature energies at order $L-1$.
- Moreover, the metric errors in the third line are controlled as follows:
  - For $L = 0$, one uses (4.5.11a) and (4.5.11c).
  - For $L = 2$, one uses (4.5.11a) and (4.5.11d).

Applying the Gronwall lemma to (4.5.28) then immediately implies (4.5.23) for $L = 0$, since the second line does not occur. Assuming (4.5.23) has been shown for $L-2$, (4.5.28) implies

$$\begin{aligned}\mathfrak{E}^{(L)}_{\text{total}}(t) \lesssim & \int_t^{t_0} \left(\varepsilon^{\frac{1}{8}}\, a(s)^{-2} + a(s)^{-2-c\sqrt{\varepsilon}+\frac{\omega}{8}} + a(s)^{-1-c\eta-(L+1)\omega}\right) \mathfrak{E}^{(L)}_{\text{total}}(s)\, ds \\ & + \varepsilon^4 \left[1 + \int_t^{t_0} \left(\varepsilon^{\frac{1}{8}} a(s)^{-2-c\varepsilon^{\frac{1}{8}}} + a(s)^{-1-(L+1)\omega-c\eta}\right) ds\right]\end{aligned}$$

and thus (4.5.23) at order L after applying (4.2.5b) to the second line as well as the Gronwall lemma. After completing this iteration, (4.5.23) is proven for all L in the statement.

Moving on to $\mathfrak{E}^{(L)}_{\text{quiesc}}$, one obtains the following estimate combining (4.5.15), (4.5.3a) and (4.5.12a).

$$\begin{aligned}\mathfrak{E}^{(L)}_{\text{quiesc}}(t) \lesssim\, & \varepsilon^4 + \int_t^{t_0} \left(\varepsilon^{\frac{1}{8}} a(s)^{-2} + a(s)^{-1-c\eta}\right) \mathfrak{E}^{(L)}_{\text{quiesc}}(s)\, ds \\ & + \int_t^{t_0} \left(\varepsilon^{\frac{3}{8}}\, a(s)^{-2-c\sqrt{\varepsilon}} + a(s)^{-1-c\sqrt{\varepsilon}}\right) \mathfrak{E}^{(\leq L-2)}_{\text{quiesc}}(s) + a(s)^{-1-c\sqrt{\varepsilon}-(L+1)\omega}\, \mathfrak{E}^{(\leq L)}_{\text{total,Vl}}(s)\, ds \\ & + \int_t^{t_0} a(s)^{-1-c\sqrt{\varepsilon}-(L+1)\omega} \cdot a(s)^{\frac{\omega}{2}} \left(\|G^{-1}-\gamma^{-1}\|^2_{H^2_G(M_s)} + \|\Gamma - \hat{\Gamma}\|^2_{H^1_G(M_s)}\right) ds\end{aligned}$$

In the final line, we collect all possible metric error terms across different orders for the sake of simplicity. Using (4.5.23) for the final line, and assuming that the bound (4.5.24) has been obtained for order $L-2$ or lower, one again obtains

$$\mathfrak{E}^{(L)}_{\text{quiesc}}(t) \lesssim \varepsilon^4 a(t)^{-c\varepsilon^{\frac{1}{8}}} + \int_t^{t_0} \left(\varepsilon^{\frac{1}{8}} a(s)^{-2} + a(s)^{-1-c\eta}\right) \mathfrak{E}^{(L)}_{\text{quiesc}}(s)\, ds$$

and thus (4.5.24) at order L via the Gronwall lemma. This proves the bound for $L \in 2\mathbb{N}$, $L \leq 10$ by iteration. Furthermore, (4.5.25) follows from inserting (4.5.23) and (4.5.24) into (4.5.2), using the former to bound Vlasov energies and metric errors and the latter for any remaining terms. Similarly, applying (4.5.23) and (4.5.24) to (4.5.12b) for $L = 10$ implies

$$a(t)^2\mathfrak{E}^{(9)}(R,t) \lesssim \varepsilon^4 + \varepsilon^{\frac{15}{4}} a(t)^{-c\varepsilon^{\frac{1}{8}}} + \int_t^{t_0} \left(\varepsilon^{\frac{1}{8}} a(s)^{-2} + a(s)^{-1-c\eta}\right) a(s)^2\mathfrak{E}^{(L-1)}(R,s)\, ds$$

and thus (4.5.26) by the Gronwall lemma.

Finally, we turn to (4.5.20) to prove (4.5.27). Note that we can use (4.5.23) to bound the second line of (4.5.20) by

$$\leq \int_t^{t_0} a(s)^{-1-c\eta-(K+2)\omega}\, \mathfrak{E}^{(J)}_{\mathrm{total}}(s)\, ds \lesssim \int_t^{t_0} \varepsilon^4 a(s)^{-1-(K+2)\omega-c\eta}\, ds\,.$$

Additionally, one can use (4.5.24), along with (4.5.26) for the top order curvature term, to bound the penultimate line of (4.5.20), as well as (4.5.23) to bound the metric errors in the final line. This yields

$$\begin{aligned}
\mathfrak{E}^{(J)}_{1,K}(f,t) \lesssim \varepsilon^4 &+ \int_t^{t_0} \left(a(s)^{-1-c\eta-(K+2)\omega} + \varepsilon^{\frac{1}{8}}\, a(s)^{-2}\right) \mathfrak{E}^{(J)}_{1,K}(f,s)\, ds\\
&+ \int_t^{t_0} \varepsilon^{\frac{7}{8}}\, a(s)^{-2-c\sqrt{\varepsilon}} \left(\mathfrak{E}^{(J)}_{1,\leq K-1}(f,s) + \mathfrak{E}^{(\leq J-1)}_{1,\leq K}(f,s)\right) ds\\
&+ \int_t^{t_0} \varepsilon^{\frac{29}{8}}\, a(s)^{-2-c\varepsilon^{\frac{1}{8}}} + \varepsilon^4 a(s)^{-1-(K+1)\omega-c\eta}\, ds\,.
\end{aligned}$$

For $\omega < \frac{1}{24}$ and $\eta \ll \omega$, the last line is simply bounded by $\varepsilon^4 a(t)^{-c\varepsilon^{\frac{1}{8}}}$ up to a constant. Again, (4.5.27) for $J = K = 0$ now follows directly from the Gronwall lemma, since the second line does not appear, and the full statement follows by iterating over K for each $J > 0$ and then over J. □

4.6. Past Stability

4.6.1. The $(2+1)$-dimensional ESFV system. Corollary 4.5.8 now puts us in the position to close the bootstrap argument and prove the main result of this paper:

THEOREM 4.6.1 (Past stability of FLRW solutions to the $(2+1)$-dimensional ESFV system). *Let $(M, \mathring{g}, \mathring{k}, \mathring{\pi}, \mathring{\psi}, \mathring{f})$ be CMC initial data for the Einstein scalar-field Vlasov system as in Section 4.3.2 that is close to the FLRW solution from Lemma 4.2.1 in the sense of Assumption 4.3.3. Further, assume $\mathring{f}$ to have compact momentum support. If $m = 0$, additionally assume the momentum support of $\mathring{f}$ to be bounded away from the origin.*

Then, the past maximal globally hyperbolic development $(\overline{M}, \overline{g}, \phi, f)$ admits a CMC foliation $(M_t)_{t\in(0,t_0]}$ along which the following estimates hold for some $c' > 0$.

$$\mathcal{H} + \mathcal{C} \lesssim \varepsilon^{\frac{7}{4}}\, a(t)^{-c'\varepsilon^{\frac{1}{8}}}, \quad \|f - f_{FLRW}\|_{C^5_{1,\gamma_0}(T^*M)} \lesssim \sqrt{\varepsilon}\, a^{-c'\sqrt{\varepsilon}} \tag{4.6.1}$$

Consequently, the solution is asymptotically velocity term dominated in the following sense: There exist a scalar function Ψ_{Bang}, a tracefree $(1,1)$-tensor field $\hat{K}_{\mathrm{Bang}}$ and a $(0,2)$-tensor

H_{Bang} *on the Big Bang hypersurface* M_0*, which we identify with* M *from here on out, satisfying*

$$\|\Psi_{\mathrm{Bang}}\|_{C^7_\gamma(M)} + \|\hat{K}_{\mathrm{Bang}}\|_{C^7_\gamma(M)} + \|H_{\mathrm{Bang}} - \gamma\|_{C^7_\gamma(M)} \lesssim \varepsilon \tag{4.6.2}$$

such that the following estimates hold.

(4.6.3)

$$\begin{aligned}\|n-1\|_{C^6_\gamma(M_t)} + \|a^2\,\partial_t\phi - (\Psi_{\mathrm{Bang}} + C)\|_{C^6_\gamma(M_t)} + \|a^2\,\hat{k}^\sharp - \hat{K}_{\mathrm{Bang}}\|_{C^6_\gamma(M_t)}& \\ + \left\|a^{-2}\,g\odot\exp\left[-2\int_t^{t_0} a(s)^{-2}\,ds\cdot\hat{K}_{\mathrm{Bang}}\right] - H_{\mathrm{Bang}}\right\|_{C^6_\gamma(M_t)} &\lesssim \varepsilon\, a(t)^{1-c'\varepsilon^{\frac{1}{8}}}\end{aligned}$$

$$\left\|\phi - \phi(t_0,\cdot) + \int_t^{t_0} a(s)^{-2}\,ds\cdot(\Psi_{\mathrm{Bang}} + C)\right\|_{C^6_\gamma(M_t)} \lesssim \varepsilon\, a(t)^{1-c'\varepsilon^{\frac{1}{8}}} \tag{4.6.4}$$

Furthermore, one has

$$-8\pi\,\Psi_{\mathrm{Bang}}\,(\Psi_{\mathrm{Bang}} + 2\,C) = \hat{K}_{\mathrm{Bang}}\cdot\hat{K}_{\mathrm{Bang}} \tag{4.6.5}$$

where, for $(1,1)$ *tensor fields* $\mathfrak{S},\mathfrak{T}$ *on* M*, we write* $\mathfrak{S}\cdot\mathfrak{T} = \mathfrak{S}^i{}_j\mathfrak{T}^j{}_i$*. Additionally, there exists* $f_{\mathrm{Bang}} \in C^4_{1,\underline{\gamma}_0}(T^*M)$ *satisfying*

$$\|f_{\mathrm{Bang}} - \mathring{f}\|_{C^4_{1,\underline{\gamma}_0}(T^*M)} \lesssim \sqrt{\varepsilon} \tag{4.6.6}$$

such that, one has

$$\|f - f_{\mathrm{Bang}}\|_{C^4_{1,\underline{\gamma}_0}(T^*M_t)} \lesssim \sqrt{\varepsilon}\, a(t)^{1-c'\varepsilon^{\frac{1}{8}}}\,. \tag{4.6.7}$$

If, on top of Assumption 4.3.3, one assumes

$$\mathrm{dist}_{\underline{\gamma}}(\mathrm{supp}\mathring{f}, \mathrm{supp}f_{FLRW}) \le \varepsilon^2\,, \tag{4.6.8}$$

where $\mathrm{dist}_{\underline{\gamma}}$ *denotes the metric on* T^*M *induced by the Riemannian metric* $\underline{\gamma}$*, then one also has*

$$\mathrm{dist}_{\underline{\gamma}}(\mathrm{supp}f_{\mathrm{Bang}}, \mathrm{supp}f_{FLRW}) \lesssim \varepsilon^{\frac{7}{4}}\,. \tag{4.6.9}$$

Finally, $(\overline{M},\overline{g})$ *is geodesically past incomplete, forming a stable crushing singularity as* $t\downarrow 0$ *at which the Kretschmann scalar* $\mathcal{K} = \mathrm{Riem}[\overline{g}]_{\alpha\beta\gamma\delta}\,\mathrm{Riem}[\overline{g}]^{\alpha\beta\gamma\delta}$ *exhibits stable blow-up:*

$$\|a^4\,|k|_g^2 - K^2_{\mathrm{Bang}}\|_{C^0_\gamma(M_t)} + \left\|a^8\,\mathcal{K} - 6(8\pi)^2\,(\Psi_{\mathrm{Bang}} + C)^4\right\|_{C^0_\gamma(M_t)} \lesssim \varepsilon\, a(t)^{1-c\varepsilon^{\frac{1}{8}}} \tag{4.6.10}$$

As explained after Lemma 4.2.2, if $M \cong \mathbb{S}^2$, the FLRW solutions admit a time reflection symmetry so that a fully analogous stability statment holds toward the future and, consequently, that these solutions are globally stable.

Proof. On the bootstrap interval $(t_{Boot}, t_0]$, the improved estimate (4.6.1) follows directly for all terms in $\mathcal{H}$ except for $\|G-\gamma\|_{H^{10}_G(M_t)}$ by inserting the improved energy estimates obtained in Corollary 4.5.8 into the near-coercivity estimates in Lemma 4.4.4. Regarding the metric term, the preceding argument gives

$$\|N\|^2_{H^{10}_G(M_t)} + \|\hat{k}\|^2_{H^{10}_G(M_t)} \lesssim \varepsilon^{\frac{15}{4}}\, a^{-c\varepsilon^{\frac{1}{8}}} .$$

Thus the improved bound for $\|G-\gamma\|_{H^{10}_G(M_t)}$ follows from (4.5.11b). In particular, Lemma 4.4.4 also implies

$$\|\rho - \rho^{Vl}\|_{H^{10}_G(M_t)} + \|S^{Vl,\parallel}\|_{H^{10}_G(M_t)} + \|\jmath^{Vl}\|_{H^{10}_G(M_t)} \lesssim \varepsilon^{\frac{7}{4}}\, a(t)^{-c\varepsilon^{\frac{1}{8}}} .$$

Using the improved bound on $\|G-\gamma\|_{H^{10}_G(M_t)}$, one straightforwardly computes that all bounds extend to the respective Sobolev spaces with respect to the reference metric γ after updating constants. The bounds for $\mathcal{C}$ in (4.6.1) then follow by Sobolev embedding and similarly changing back from γ to G, proving the first estimate in (4.6.1).

Recalling the low order Vlasov bound in (4.6.1) from (4.4.3c), this improves the bootstrap assumptions; see Assumption 4.3.7. Along with the momentum support bounds from Lemma 4.4.2, one also checks that all continuation criteria from Section 4.3.3 hold as $t \downarrow t_{Boot}$. Hence, the solution can be extended beyond $t = t_{Boot}$ and consequently to $(M_t)_{t\in(0,t_0]}$ now that the bootstrap argument has been completed and (4.6.1) holds throughout.

The remainder of the theorem largely follows as in [RS18b, Theorem 15.1] and Theorem 3.8.1 as well as Corollary 3.8.3. While more details can be found in these works, we demonstrate the spirit of these arguments by proving (4.6.7) at order 0 with (4.6.3) already proven; the other asymptotic bounds are obtained similarly and (4.6.10) follows from these by straightforward computations.
The rescaled Vlasov equation (4.2.14j) can be rewritten as follows, explicitly computing $\mathbf{A}_j f_{FLRW}$ in the third term:

$$\begin{aligned}\partial_t f = &- N\, a^{-1}\, \frac{v^j}{v^0}\, \mathbf{A}_j f \\ &- a^{-1}\, \frac{v^j}{v^0} \left[\mathbf{A}_j(f - f_{FLRW}) + 2\mathcal{F}'\left(|v|_\gamma\right)^2 \left(\Gamma[G]^m_{ij} - \Gamma[\gamma]^m_{ij}\right)(\gamma^{-1})^{jl}\, v_m\, v_l\right] \\ &+ a^{-1}\, v^0\, \nabla^{\sharp j} N\, \mathbf{B}_j f\end{aligned} \tag{4.6.11}$$

Now applying (4.6.1) and (4.4.2a) throughout, as well as (4.6.3) for the third line, we obtain

$$\left|\frac{d}{dt}\left(\langle v\rangle_\gamma f\right)\right| \lesssim a^{-1-c\varepsilon^{\frac{1}{8}}} .$$

and consequently, for any $s,t \in (0,t_0]$,

$$\sup_{x,v\in TM} \langle v\rangle_\gamma |f(t,x,v) - f(s,x,v)| \lesssim \sqrt{\varepsilon}\left(a(t)^{1-c\varepsilon^{\frac{1}{8}}} - a(s)^{1-c\varepsilon^{\frac{1}{8}}}\right) .$$

Hence, $f(t,\cdot,\cdot)$ uniformly converges to a limit f_{Bang} in $C^0_{1,\underline{\gamma}_0}(T^*M)$ as $t \downarrow 0$ and (4.6.7) holds at order 0. Note that we lose one order of control on the Vlasov distribution compared to (4.6.1) since, when applying (4.6.11) and higher order analogues, we need to estimate

$f - f_{FLRW}$ at order $\ell+1$ to control f_{asymp} at order ℓ. Further, note that evaluating (4.6.7) at $t = t_0$ yields (4.6.6). $\square$

We formulate the asymptotic behaviour of the spatial metric g in (4.6.3) to be consistent with the FLRW Big Bang stability works [RS18a; Spe18] as well as Chapter 2. This expression is in line with the formal solution

$$(g_{VTD})_{ij} = a(t)^2 \left\{ \exp\left(2\int_t^{t_0} a(s)^{-2}\, ds \cdot \hat{K}_{VTD}\right) \odot H_{VTD} \right\}_{ij}$$

that one obtains by solving the velocity term dominated equations; see also [Spe18, Remark 19.2]. Alternatively, one can renormalise the spatial metric as in [OGPR23, Definition 6] such that the resulting metric converges toward the Big Bang.

Corollary 4.6.2. *Let $(\overline{M}, \overline{g}, \phi, f)$ be as in Theorem 4.6.1 and consider the time-dependent family of endomorphisms*

$$\mathcal{W}_{t,\cdot} : T.M_t \longrightarrow T.M_t, \qquad \mathcal{W}_{t,x}(w) = \exp\left[-\int_t^{t_0} a(s)^{-2}\, ds \cdot (\hat{k}^\sharp)_{t,x}\right] \odot w$$

as well as the family of renormalised metrics

$$\mathcal{G}_{t,\cdot} : T.M_t \times T.M_t \to \mathbb{R}, \quad \mathcal{G}_{t,x}(X,Y) = a(t)^{-2}\, \left(g_{t,x}(\mathcal{W}_{t,x}(X), \mathcal{W}_{t,x}(Y))\right).$$

Then, there exists a symmetric $(0,2)$-tensor field $\mathcal{G}_{Bang}$ on M such that the following bounds hold.

$$\|\mathcal{G}_{Bang} - \gamma\|_{C^7_\gamma(M)} \lesssim \varepsilon \tag{4.6.12}$$

$$\|\mathcal{G} - \mathcal{G}_{Bang}\|_{C^6_\gamma(M_t)} \lesssim \varepsilon\, a(t)^{1-c'\,\varepsilon^{\frac{1}{8}}} \tag{4.6.13}$$

Moreover, $\hat{K}_{Bang}$ is self-adjoint with respect to $\mathcal{G}_{Bang}$.

Proof. The asymptotics in (4.6.13) are proven identically to the $(3+1)$-dimensional analogue in Corollary 3.8.4, using the evolution equations (4.2.14a) and (4.2.14b) along with Theorem 4.6.1 to obtain

$$|\partial_t \mathcal{G}|_\gamma \lesssim \varepsilon\, a^{-1-c\,\varepsilon^{\frac{1}{8}}}\, (|\mathcal{G}|_\gamma + 1)$$

and then extracting a Big Bang limit of $\mathcal{G}$ as for the Vlasov distribution above. The bound (4.6.12) follows directly from (4.6.13) and the initial data assumption, and $\hat{K}_{Bang}$ is self-adjoint with respect to $\mathcal{G}_{Bang}$ since $\hat{k}^\sharp$ is self-adjoint with respect to G and, consequently, $\mathcal{G}$ on any M_t for $t \in (0, t_0]$. $\square$

Remark 4.6.3 (Asymptotics of the constraint equations). Recall that (4.6.5) is the limit of the renormalised Hamiltonian constraint. Unfortunately, one cannot obtain a similar limit for the momentum constraint since the Levi-Civita connection associated to the spatial metric degenerates along with the metric itself, preventing a limit on the divergence of the shear, along with causing degeneracies in $\jmath^{Vl}$. While it might be possible to recast both of these terms with the help of $\mathcal{G}$, this would require tracking eigendirections of the shear along the evolution. Even works that use such frames explicitly, including [OGPR23], lose control of this frame towards the past, preventing an asymptotic momentum constraint. In light of [FG25], where one can ensure that a Gaussian approximate VTD solution satisfies the momentum constraint given Einstein nonlinear scalar-field data on the Big Bang hypersurface unlike for the VTD solution in CMCTC gauge, this might be an issue with the foliation itself. We leave this question open to future work.

REMARK 4.6.4 (Concentration of Vlasov particle velocities). Note that (4.6.7) implies that the Vlasov distribution is asymptotically velocity term dominated along with the other solution variables if viewed on the co-mass shell. However, as in Remark 3.8.5, the picture changes when one tries to analyse the velocity of Vlasov particles rather than their (conjugate) momenta and views the distribution function on the mass shell. In short, "lifting" to the mass shell means one must renormalize the spatial metric as indicated by Corollary 4.6.2. The resulting velocity characteristics are, consequently, dominated by their VTD component

$$Q_{t,x}(q) = \exp\left[-2\int_t^{t_0} a(s)^{-2}\,ds\cdot\left(\hat{K}_{Bang}\right)_x\right]\odot a(t)^{-2}q.$$

Thus and in line with Corollary 4.6.2, it is useful to consider the FLRW renormalised particle velocities $w^i = a(t)^{-2}q^i$ and the a posteriori expansion-renormalised velocities

$$W_{t,x}(w) = \exp\left[2\int_t^{t_0} a(s)^{-2}\,ds\cdot\left(\hat{K}_{Bang}\right)_x\right]\odot w.$$

Indeed, one can prove by similar means as in Theorem 4.6.1 that, for

$$f^\sharp_{\mathrm{asymp}}(t,x,w) := f^\sharp(t,x,W^{-1}_{t,x}(w))\,,$$

the support of $f^\sharp_{\mathrm{asymp}}(t,\cdot,\cdot)$ also remains compact and that this renormalised function converges to an asymptotic footprint $f^\sharp_{\mathrm{Bang}}$. In particular, one has

$$\mathrm{supp} f^\sharp(t,x,\cdot) = \left\{W_{t,x}(w)\,|\,w\in\mathrm{supp} f^\sharp_{\mathrm{asymp}}(t,x,\cdot)\right\}$$

by definition of $f^\sharp_{\mathrm{asymp}}$. When approaching the Big Bang, particle velocities thus concentrate in directions w where $\lim_{t\to 0} W_{t,x}(q)$ remains bounded.

To identify these directions, first consider the massive case ($m=1$) where particles can be initially at rest, i.e., consider $x\in M$ such that $\mathring{f}(x,0)\neq 0$. In the case where $(\hat{K}_{\mathrm{Bang}})_x = 0$, $f^\sharp_{\mathrm{asymp}}$ simply agrees with the (FLRW renormalised) initial distribution and $f^\sharp$ converges to an asymptotic profile f_{Bang} that is close to the FLRW distribution. Otherwise, the tracefree and self-adjoint operator $(\hat{K}_{\mathrm{Bang}})_x$ has precisely one positive and one negative eigendirection. Denoting these eigenspaces by $E_{x,\pm}$, it follows that $\lim_{t\to 0} W_{t,x}(q) = 0$ is satisfied if one takes $w\in E_{x,-}$. Otherwise, $\lim_{t\to 0} W_{t,x}(q) = \infty$ holds and w lies outside of $\mathrm{supp} f^\sharp(t,x,\cdot)$ for small enough $t>0$. Thus, (4.6.7) implies the following pointwise limit for any $w\in T^*_xM$.

$$\lim_{t\to 0} f^\sharp(t,x,q) = f^\sharp_{\mathrm{Bang}}(x,0)\,\chi_{E_{x,-}}(w(q))$$

If $f^\sharp_{\mathrm{Bang}}(\cdot,0)\neq 0$, which holds for sufficiently small $\varepsilon>0$ due to (4.6.6), the velocity support thus concentrates toward $E_{x,-}$ and any element of $E_{x,-}$ lies in the support.

For $m=0$, or more generally when the initial particle distribution vanishes in the neighbourhood of the zero section for some $x\in M$, $f^\sharp_{\mathrm{Bang}}$ may also vanish at the origin for some

$x \in M$. In that case, for any $q \in T_x^*M$, the Vlasov distribution vanishes both if $W_{t,x}(q)$ becomes too large or too small. Denoting the orthogonal projectors onto $E_{x,\pm}$ by $\mathbb{P}_\pm$, one has

$$\sup_{q\in\mathrm{supp}f^\sharp(t,x,\cdot)} |\mathbb{P}_+ w(q)|_\gamma \to 0 \quad \text{as} \quad t\downarrow 0$$

as well as

$$\inf_{q\in\mathrm{supp}f^\sharp(t,x,\cdot)} |\mathbb{P}_- w(q)|_\gamma \to \infty \quad \text{as} \quad t\downarrow 0$$

while the momentum support still remains bounded for any $t > 0$; see (4.4.2a). In other words, particle velocities concentrate in the direction of $E_{x,-}$ toward the Big Bang, but become arbitrarily large.

Finally, one expects that $\hat{K}_{Bang} = 0$ is a non-generic case, and thus that velocity concentration is generic. However, this is not immediately apparent from Theorem 4.6.1, as we do not obtain a scattering result. However, nearby Kasner solutions must be contained in the solutions covered by Theorem 4.6.1 for which $\hat{K}_{Bang}$ only vanishes if it coincides with the FLRW solution. Since the construction above also ensures that the footprint states depend on initial data continuously, it follows that the set of solutions for which velocities concentrate is, at least, of positive measure. In light of the scattering result [Li24b] for the linearized Einstein scalar-field system, it further is reasonable to conjecture that $\hat{K}_{Bang}$ vanishing entirely is truly non-generic, but this still remains open even in the full Einstein scalar-field setting.

Additionally, we can now state the full version of Corollary 4.1.2 and show how Theorem 4.6.1 can be translated to the polarized $U(1)$-symmetric vacuum setting:

COROLLARY 4.6.5 (Past stability of non-extremal polarized $U(1)$-symmetric $(3+1)$-dimensional vacuum solutions). *Consider $3+1$ vacuum spacetimes of the form*

$$\left(\check{M} = (0,t_0] \times M \times \mathbb{S}^1,\ \check{g}_{ref} = e^{-2\sqrt{4\pi}\,\phi_{\mathrm{ref}}}\left(-dt^2 + a(t)^2\gamma\right) + e^{2\sqrt{4\pi}\,\phi_{\mathrm{ref}}}(dx^3)^2\right), \tag{4.6.14}$$

where ϕ_{ref} is a spatially homogeneous function with $\partial_t\phi_{\mathrm{ref}} = C\,a^{-2}$ and $C < 0$ and (M,γ) is a surface of constant curvature; see Remark 4.2.3 and Appendix 4.7.1. On a constant time slice $\{t = t_0\}$ of (4.6.14), take polarized $U(1)$-symmetric initial data $(M \times \mathbb{S}^1, \mathring{\tilde{g}}, \mathring{\tilde{k}})$ that is close to that of (4.6.14) in the sense of Assumption 4.3.3 and the initial data correspondence in Remark 4.3.4. Additionally, assume without loss of generality that the induced mean curvature $\mathring{k}$ on M_{t_0} has constant trace; see (4.3.6).

Then, such a solution admits a past maximal global hyperbolic development $(\check{M}, \check{g})$ that is polarized $U(1)$-symmetric in the sense that the transported coordinate derivative ∂_{x^3} on $\mathbb{S}^1$ acts as a Killing vector field. Writing

$$\check{g} = e^{-2\sqrt{4\pi}\,\phi}\left(-n^2dt^2 + g\right) + e^{2\sqrt{4\pi}\,\phi}(dx^3)^2 \tag{4.6.15}$$

for $\phi : I\times M \to \mathbb{R}$, (g,k,n,ϕ) satisfy the asymptotic bounds of Theorem 4.6.1 with $f \equiv 0$ along the constant time foliation $(M_t \times \mathbb{S}^1)_{t\in(0,t_0]}$. Moreover, let $\tilde{k}$ be the second fundamental form with respect to the foliation $(M_t \times \mathbb{S}^1, \tilde{g} = \check{g}|_{M_t\times\mathbb{S}^1})_{t\in(0,t_0]}$ and let $\check{\mathcal{K}}$ denote the Kretschmann *scalar with respect to $\breve{g}$. Then, there exist positive continuous functions $Y_{\mathrm{Bang}}, Z_{\mathrm{Bang}}$ on M such that the following bounds hold.*

$$\|e^{-2\sqrt{4\pi}\,\phi}\, a^4\, |\tilde{k}|^2_{\tilde{g}} - Y_{\mathrm{Bang}}\|_{C^0(M_t)} \lesssim \varepsilon\, a(t)^{2-c\varepsilon^{\frac{1}{8}}} \tag{4.6.16a}$$

$$\|e^{-4\sqrt{4\pi}\,\phi}\, a^8\, \breve{\mathcal{K}} - Z_{\mathrm{Bang}}\|_{C^0(M_t)} \lesssim \varepsilon\, a(t)^{4-c\varepsilon^{\frac{1}{8}}} \tag{4.6.16b}$$

In particular, the $(\breve{M}, \breve{g})$ is C^2-inextendible toward the past.

Remark 4.6.6 (Curvature blow-up). Applying the Friedman equation (4.2.3e), note that one has

$$e^{\sqrt{4\pi}\,\phi_{\mathrm{ref}}(t)} = \mathfrak{c}_\kappa(t)\, e^{\sqrt{4\pi} a(t_0)}\, a(t)^{-1}\,,$$

where $\mathfrak{c}_\kappa \in C^\infty([0,t_0])$ is identical to 1 for $\kappa = 0$ and converges to a positive constant approaching $t = 0$ otherwise. Combining this with (4.6.4), this implies that $|\tilde{k}|^2_g$ and $\breve{\mathcal{K}}$ blow up at order close to a^{-6} and a^{-12} respectively, up to a factor of $a^{\pm c\varepsilon}$. Using the identities (4.6.17) below, the components of $\breve{g}$ can be bounded similarly.

Proof. Following the initial data correspondence in Remark 4.2.3, Theorem 4.6.1 yields the past maximal $(2+1)$-dimensional Einstein scalar field solution $(\overline{M}, \overline{g}, \phi)$ that corresponds to $(3+1)$-dimensional vacuum solution

$$\left(\breve{M} = \overline{M} \times \mathbb{S}^1, \breve{g} = e^{-2\sqrt{4\pi}\,\phi}\, \overline{g} + e^{2\sqrt{4\pi}\,\phi}\, (dx^3)^2\right)$$

launched by $(M \times \mathbb{S}^1, \mathring{\tilde{g}}, \mathring{\tilde{k}})$. Since $\overline{g}_{0i} = 0$ for $i = 1, 2$, we can write

$$\breve{g} = -n^2\, e^{-2\sqrt{4\pi}\,\phi}\, dt^2 + \tilde{g},$$

where $\tilde{g}$ is a time-dependent Riemannian metric on $M \times \mathbb{S}^1$. Additionally writing the second fundamental form and shear with respect to this foliation as $\tilde{k}$ and $\hat{\tilde{k}}$, as well as $\psi = n^{-1}\,\partial_t\phi$, the components of $\tilde{g}$ and $\tilde{k}$ are as follows.

$$\tilde{g}_{ij} = e^{-2\sqrt{4\pi}\,\phi}\, g_{ij}, \quad \tilde{g}_{33} = e^{2\sqrt{4\pi}\,\phi} \tag{4.6.17a}$$

$$\tilde{k}_{ij} = e^{-\sqrt{4\pi}\,\phi}\left(k_{ij} + \sqrt{4\pi}\,\psi\, g_{ij}\right), \quad \tilde{k}_{33} = -\sqrt{4\pi}\, e^{3\sqrt{4\pi}\,\phi}\,\psi \tag{4.6.17b}$$

$$\tilde{g}_{13} = \tilde{g}_{23} = \tilde{k}_{13} = \tilde{k}_{23} = 0 \tag{4.6.17c}$$

From this, we directly compute, using the Friedman equation (4.2.3e), that

$$\begin{aligned}
|\tilde{k}|^2_{\tilde{g}} &= e^{2\sqrt{4\pi}\,\phi}\left[\left|\hat{k} + \left(\frac{\tau}{2} + \sqrt{4\pi}\,\psi\right) g\right|^2_g + 4\pi\,\psi^2\right] \\
&= e^{2\sqrt{4\pi}\,\phi}\, a^{-4}\left[|\hat{k}|^2_G + 2\left(\sqrt{4\pi}\, C - \sqrt{4\pi\, C^2 - \kappa\, a^2} + \sqrt{4\pi}\,\Psi\right)^2 + 4\pi\,(C+\Psi)^2\right].
\end{aligned}$$

Now, (4.6.16a) follows from (4.6.3) and (4.6.5), defining

$$\begin{aligned}
Y_{\mathrm{Bang}} &:= \hat{K}_{\mathrm{Bang}} \cdot \hat{K}_{\mathrm{Bang}} + 8\pi\,(\Psi_{\mathrm{Bang}} + 2\,C)^2 + 4\pi\,(\Psi_{\mathrm{Bang}} + C)^2 \\
&= 16\pi\, C\,(\Psi_{\mathrm{Bang}} + 2\,C) + 4\pi\,(\Psi_{\mathrm{Bang}} + C)^2 \\
&= 4\pi\,(\Psi_{\mathrm{Bang}} + 3\,C)^2\,.
\end{aligned}$$

After some lengthy computations, for which we defer to Section 4.7.2, and again applying the Friedman equation (4.2.3e) in the second line below, one also sees that the Kretschmann scalar $\check{\mathcal{K}}$ takes the following form.

(4.6.18)
$$\begin{aligned} e^{-4\sqrt{4\pi}\,\phi}\,\check{\mathcal{K}} &= 32\cdot 4\pi\,\psi^2\left(\frac{\tau}{2}+\sqrt{4\pi}\,\psi\right)^2+8\cdot 4\pi\,\psi^2\,|\hat{k}|_g^2+\langle\mathrm{Err}\rangle\\ &= 32\pi\,a^{-8}\,(C+\Psi)^2\left[4\left(\sqrt{4\pi}\,C-\sqrt{4\pi\,C^2-\kappa\,a^2}+\sqrt{4\pi}\,\Psi\right)^2+|\hat{k}|_G^2\right]+\langle\mathrm{Err}\rangle \end{aligned}$$

The error term $\langle\mathrm{Err}\rangle$ can be bounded as below using Theorem 4.6.1.

$$\|\langle\mathrm{Err}\rangle\|_{C^0_\gamma(M_t)}\lesssim\varepsilon a(t)^{-4-c\varepsilon^{\frac{1}{8}}}$$

The bound (4.6.16b) again follows from (4.6.3) and (4.6.5) for

$$\begin{aligned} Z_{\mathrm{Bang}} &:= 32\pi\,(\Psi_{\mathrm{Bang}}+C)^2\left(16\pi(\Psi_{\mathrm{Bang}}+2\,C)^2+\hat{K}_{\mathrm{Bang}}\cdot\hat{K}_{\mathrm{Bang}}\right)\\ &= 256\pi^2\,(\Psi_{\mathrm{Bang}}+C)^2\,(\Psi_{\mathrm{Bang}}+2\,C)(\Psi_{\mathrm{Bang}}+4\,C)\,. \end{aligned}$$

Since (4.6.16b) shows that the Kretschmann scalar blows up as noted in Remark 4.6.6, C^2-inextendibility follows immediately. □

Remark 4.6.7 (The induced foliation is not CMC). As (4.6.17) shows, the foliation $(M_t\times\mathbb{S}^1)_{t\in(0,t_0)}$ is no longer a constant mean curvature foliation. More precisely, one has

$$\mathrm{tr}_{\tilde{g}}\tilde{k}=e^{\sqrt{4\pi}\,\phi}\,(\tau+\sqrt{4\pi}\,\psi)$$

and thus

$$e^{-\sqrt{4\pi}\,\phi_{\mathrm{ref}}(t)}\,|\nabla\mathrm{tr}_{\tilde{g}}\tilde{k}|_{(\gamma+(dx^3)^2)}\lesssim\varepsilon\,|\log(t)|$$

by (4.6.4). This slight difference in foliation leads to differences in asymptotic bounds of our Kretschmann scalar in the spatially flat case compared to [FRS23], even if one were to express $\breve{g}$ in a coordinate system in which the lapse converges to 1.

Remark 4.6.8 (Instability of extremal Kasner(-like) solutions). In principle, one can attempt to redo the argument of Corollary 4.6.5 corresponding to reference solutions with $\partial_t\phi_{\mathrm{ref}}=C\,a^{-2}$, $C>0$. As noted in Remark 4.2.3, this corresponds to extremal Kasner-like spacetimes. Since one has

$$\frac{\tau}{2}+\sqrt{4\pi}\,\partial_t\phi_{\mathrm{ref}}=a^{-2}\left(\sqrt{4\pi\,C^2}-\sqrt{4\pi\,C^2+\kappa\,a^2}\right)=\mathcal{O}\left(\kappa\,a^{-1}\right)\,,$$

the reference solution does not exhibit curvature blow-up in the spatially flat case by the first line of (4.6.18). For $\kappa\neq 0$, since $e^{-4\sqrt{4\pi}\,\phi_{\mathrm{ref}}}$ is now proportional to a^{-4} at leading order, the Kretschmann scalar blows up at order $\mathcal{O}\left(a^{-2}\right)$. However, for the development of nearby initial data, (4.6.18) becomes

$$e^{-4\sqrt{4\pi}\,\phi}\,\check{\mathcal{K}}=32\pi\,a^{-8}\,(C+\Psi)^2\left(|\hat{k}|_G^2+\Psi^2\right)+\langle\text{lower order terms}\rangle\,.$$

Consequently, if Ψ_{Bang} and $\hat{K}_{\mathrm{Bang}}$ do not vanish,[11] the Kretschmann scalar blows up at an order above $a^{-4+c\varepsilon^{\frac{1}{8}}}$ for some $c > 0$. While instability of these extremal solutions is to be expected simply by considering the Kasner family itself, this seems to indicate that this instability is generic. To make this rigorous, one would need to consider the scattering theory for initial data near that of extremal solutions in detail to find positive measure initial data sets for which Ψ_{Bang} does not vanish or only vanishes on a null set.

4.7. Appendix: Formulas for polarized $U(1)$-symmetric vacuum spacetimes

4.7.1. Kasner-like form of reference metrics for $\kappa \neq 0$. In this section, we collect the necessary transformations to bring the reference solutions described by (4.2.6) into Kasner-like form

$$-dS^2 + b(S)^2\,\gamma + b_3(S)^2\,(dx^3)^2$$

for $\kappa \neq 0$, as discussed in Remark 4.2.3. Recall from Remark 4.2.3 that, for some $t_1 > 0$, the time variable $S \equiv S(t)$ is determined by

$$\frac{dS}{dt} = \exp\left(\sqrt{4\pi}\,C\int_t^{t_1} a(s)^{-2}\,ds\right),\ S(0) = 0\,. \tag{4.7.1}$$

Note that $S : [0, T/2) \to [0, \infty)$ is invertible and we can view $t \equiv t(S)$ as a function of S.

The non-negative scale factors $b(S)$ and $b_3(S)$ are then given by

$$b(S)^2 = a(t(S))^2\,\exp\left(2\sqrt{4\pi}\,C\int_{t(S)}^{t_1} a(s)^{-2}\,ds\right) \tag{4.7.2a}$$

$$b_3(S)^2 = \exp\left(-2\sqrt{4\pi}\,C\int_{t(S)}^{t_1} a(s)^{-2}\,ds\right) \tag{4.7.2b}$$

Without loss of generality, we assume $\kappa = \pm 1$ for this section; the remaining cases can be reduced to these by an appropriate rescaling of γ, which can be achieved by choosing $t_1 > 0$ appropriately.

To obtain the asymptotic behaviours of b and b_3 as S approaches 0, we will first compute expressions for these functions in terms of $t(S)$ and then expand S in terms of t.

First, rearranging (4.2.3e) with $\rho_{FLRW} = \mathfrak{p}_{FLRW} = 0$ and integrating over the interval $(0, t)$, we obtain

$$\begin{aligned} t &= \int_0^{a(t)} \left(4\pi\,C^2\,x^{-2} - \kappa\right)^{-\frac{1}{2}}\,dx \\ &= 2\,\sqrt{\pi}\,|C| \int_0^{\frac{a(t)}{2\sqrt{\pi}\,|C|}} \frac{1}{(y^{-2} - \kappa)^{-\frac{1}{2}}}\,dy \\ &= \sqrt{\pi}\,|C| \int_0^{\frac{a(t)}{2\sqrt{\pi}\,|C|}} \frac{2y}{\sqrt{1 - \kappa\,y^2}}\,dy \end{aligned}$$

[11] By (4.6.5) and since $\hat{K}_{\mathrm{Bang}}$ is self-adjoint, if $\varepsilon > 0$ is sufficiently small, $\Psi_{\mathrm{Bang}}(x) = 0$ is satisfied for some $x \in M$ if and only if $\hat{K}_{\mathrm{Bang}}(x) = 0$.

$$= -\kappa\left(\sqrt{4\pi\, C^2 - \kappa\, a(t)^2} - 2\sqrt{\pi}\,|C|\right).$$

After rearranging, this yields

$$a(t)^2 = 4\sqrt{\pi}\,|C|\,t - \kappa\,t^2 = t\,(4\sqrt{\pi}\,|C|\, - \kappa\,t)\,. \tag{4.7.3}$$

Below, we write $D := (4\sqrt{\pi}\,|C|)^{-1}$.

We now solve the inital value problem (4.7.1), depending on the signs of C and κ. Note that (4.7.3) implies

$$\begin{aligned} \exp\left(\sqrt{4\pi}\,C\int_t^{t_1} a(s)^{-2}\,ds\right) &= \exp\left(\sqrt{4\pi}\,C\;\;D\;\left[\log(s) - \log(D^{-1} - \kappa\,s)\right]\Big|_{s=t}^{t_1}\right) \\ &= \left[\frac{t}{D^{-1} - \kappa\,t}\,\frac{D^{-1} - \kappa\,t_1}{t_1}\right]^{-\frac{\mathrm{sgn}(C)}{2}} \end{aligned} \tag{4.7.4}$$

For $C < 0$ and $\kappa = -1$, one thus obtains the following for $K_{t_1} := t_1^{\frac{\mathrm{sgn}(C)}{2}}\;\left(D^{-1} - \kappa\,t_1\right)^{-\frac{\mathrm{sgn}(C)}{2}}$.

$$\begin{aligned} S(t) =& K_{t_1}\int_0^t \sqrt{\frac{s}{D^{-1}+s}}\,ds = \frac{1}{D}\int_1^{1+D\,t}\sqrt{1-\frac{1}{u}}\,du \\ =& 2\,K_{t_1}\,D^{-1}\int_0^{\operatorname{arcosh}(\sqrt{1-D\,t})}\sinh(x)^2\,dx \\ =& K_{t_1}\,D^{-1}\;\left[\sinh(x)\,\cosh(x) - x\right]\Big|_{x=0}^{\operatorname{arsinh}\left(\sqrt{D\,t}\right)} \\ =& K_{t_1}\,D^{-1}\;\left(\sqrt{D\,t}\,\sqrt{D\,t+1} - \operatorname{arsinh}(\sqrt{D\,t})\right)\,. \end{aligned}$$

Note that both summands are analytic in $\sqrt{D\,t}$ near $\sqrt{D\,t} = 0$. Thus, computing the Taylor expansion of both summands in $\sqrt{D\,t}$, one observes

$$\begin{aligned} S(t) =& K_{t_1}\,D^{-1}\left((D\,t)^{\frac{1}{2}} + \frac{1}{2}(D\,t)^{\frac{3}{2}} + \mathcal{O}\left((D\,t)^{\frac{5}{2}}\right)\right) \\ &- K_{t_1}\,D^{-1}\left((D\,t)^{\frac{1}{2}} - \frac{1}{6}(D\,t)^{\frac{3}{2}} + \mathcal{O}\left((D\,t)^{\frac{5}{2}}\right)\right) \\ =& \frac{2}{3}\,K_{t_1}\,D^{\frac{1}{2}}\,t^{\frac{3}{2}} + \mathcal{O}\left(t^{\frac{5}{2}}\right) \quad \text{as } t\downarrow 0 \end{aligned}$$

For $C < 0$ and $\kappa = 1$, one analogously computes

$$\begin{aligned} S(t) =& K_{t_1}\int_0^t\sqrt{\frac{s}{D^{-1}-s}}\,ds \\ =& 2K_{t_1}\,D^{-1}\int_0^{\arccos(\sqrt{1+D\,t})}\sin(x)^2\,dx \\ =& -K_{t_1}\,D^{-1}\,\left(\sqrt{D\,t}\,\sqrt{1-D\,t} + \arcsin\left(\sqrt{D\,t}\right)\right) = \frac{2}{3}\,K_{t_1}\,D^{\frac{1}{2}}\,t^{\frac{3}{2}} + \mathcal{O}\left(t^{\frac{5}{2}}\right) \quad \text{as } t\downarrow 0 \end{aligned}$$

In both cases, this implies

$$t(S) = \left(\frac{3}{2}\right)^{\frac{2}{3}}\;K_{t_1}^{-\frac{2}{3}}\,D^{-\frac{1}{3}}\,S^{\frac{2}{3}} + \mathcal{O}\left(S\right)\,.$$

Plugging this back into (4.7.2), along with using (4.7.3) and (4.7.4), implies the following expansions for $b(S)^2$ and $b_3(S)^2$ as S approaches 0.

$$\begin{aligned}
b(S)^2 &= a(t(S))^2 \exp\left(4\sqrt{\pi}\, C \int_{t(S)}^{t_1} a(s)^{-2}\, ds\right) \\
&= K_{t_1}^2 \,\left(4\sqrt{\pi}\, |C|\, t(S) - \kappa\, t(S)^2\right) \frac{t(S)}{4\sqrt{\pi}\, |C| - \kappa\, t(S)} \\
&= K_{t_1}^2\, t(S)^2 + \mathcal{O}\left(t(S)^3\right) \\
&= \left(\frac{3}{2}\right)^{\frac{4}{3}} K_{t_1}^{\frac{2}{3}}\, D^{-\frac{2}{3}}\, S^{\frac{4}{3}} + \mathcal{O}\left(S^2\right) \\
b_3(S)^2 &= \exp\left(-2\sqrt{4\pi}\, C \int_{t(S)}^{t_1} a(s)^{-2}\, ds\right) \\
&= K_{t_1}^{-2}\, D\, t(S)^{-1} + \mathcal{O}\left(1\right) \\
&= \left(\frac{3}{2}\right)^{-\frac{2}{3}} K_{t_1}^{-\frac{1}{3}}\, D^{\frac{4}{3}}\, S^{-\frac{2}{3}} + \mathcal{O}\left(1\right)
\end{aligned}$$

Turning now to the case where $C > 0$, we first compute for $\kappa = -1$ that

$$\begin{aligned}
S(t) &= K_{t_1} \int_0^t \sqrt{\frac{D^{-1} + s}{s}}\, ds = K_{t_1}\, D^{-1} \int_0^{D\,t} \sqrt{1 + \frac{1}{u}}\, du \\
&= 2K_{t_1}\, D^{-1} \int_0^{\operatorname{arsinh}(\sqrt{D\,t})} \cosh(x)^2\, dx \\
&= K_{t_1}\, D^{-1} \left(\sqrt{D\,t}\sqrt{1 + D\,t} + \operatorname{arsinh}\left(\sqrt{D\,t}\right)\right) \\
&= 2K_{t_1} \sqrt{\frac{t}{D}} + \mathcal{O}\left(t^{\frac{3}{2}}\right) \quad \text{as } t \downarrow 0\,.
\end{aligned}$$

For $\kappa = 1$, one has

$$\begin{aligned}
S(t) &= K_{t_1} \int_0^t \sqrt{\frac{D^{-1} - s}{s}}\, ds = D^{-1} \int_0^t \sqrt{\frac{1}{u} - 1}\, du \\
&= 2K_{t_1}\, D^{-1} \int_0^{\arcsin(\sqrt{D\,t})} \sin(x)^2\, ds \\
&= K_{t_1}\, D^{-1} \left[\sqrt{D\,t}\sqrt{1 - D\,t} + \arcsin(\sqrt{D\,t})\right] \\
&= 2K_{t_1} \sqrt{\frac{t}{D}} + \mathcal{O}\left(t^{\frac{3}{2}}\right) \quad \text{as } t \downarrow 0\,.
\end{aligned}$$

Thus, as S approaches 0, one now has

$$\begin{aligned}
b(S)^2 &= K_{t_1}^2 \,\left(4\sqrt{\pi}\, C\, t(S) - \kappa\, t(S)^2\right) \frac{4\sqrt{\pi}\, C - \kappa\, t(S)}{t(S)} \\
&= 16\pi\, C^2\, K_{t_1} + \mathcal{O}\left(t(S)\right)
\end{aligned}$$

$$
\begin{aligned}
&= 16\pi\, C^2\, K_{t_1} + \mathcal{O}\left(S^2\right) \quad \text{and} \\
b_3(S)^2 &= K_{t_1}^{-2}\, \frac{t(S)}{4\sqrt{\pi}\, C + t(S)} = K_{t_1}^{-2}\, \frac{t(S)}{4\sqrt{\pi}\, C} + \mathcal{O}\left(t(S)^2\right) \\
&= 4\pi\, C^2\, K_{t_1}^{-4}\, S^2 + \mathcal{O}\left(S^3\right) .
\end{aligned}
$$

4.7.2. Christoffel symbols and curvature components. In this section, we collect formulas for the Christoffel symbols and Riemann curvature tensor for spacetime metrics under consideration in Corollary 4.6.5. Therein, the spacetime metric on $I \times M \times \mathbb{S}^1$ takes the form

$$
\check{g} = e^{-2\sqrt{4\pi}\,\phi}\left(-n^2\, dt^2 + g\right) + e^{2\sqrt{4\pi}\,\phi}\,(dx^3)^2 ,
$$

where g is a Riemannian metric on M. Additionally, if k denotes the second fundamental form with respect to the metric g along $(M_t)_{t\in(0,t_0]}$, $\mathrm{tr}_g k = \tau$ only depends on time. Below, the index t corresponds to the vector field ∂_t and we write $\hat{k}_{ij} = k_{ij} - \frac{\tau}{2}\, g_{ij}$ as well as $\psi = n^{-1}\,\partial_t\phi$.

The Christoffel symbols $\check{\Gamma}$ with respect to $\check{g}$ take the following form:

$$
\begin{aligned}
\check{\Gamma}^t_{tt} &= \partial_t n - \sqrt{4\pi}\, n\, \psi, & \check{\Gamma}^t_{ti} &= \partial_i \log(n) - \sqrt{4\pi}\,\partial_i\phi \\
\check{\Gamma}^l_{tt} &= n\, g^{lm}\,\partial_m n + \sqrt{4\pi}\, n^2\, g^{lm}\,\partial_m\phi & \check{\Gamma}^l_{tj} &= -\, n\left(k^l{}_j + \sqrt{4\pi}\,\psi\,\mathbb{I}^l_j\right) \\
\check{\Gamma}^t_{ij} &= -\, n^{-1}\left(k_{ij} + \sqrt{4\pi}\,\psi\, g_{ij}\right) & \check{\Gamma}^l_{ij} &= \Gamma^l_{ij} - \sqrt{4\pi}\,\partial_i\phi\,\mathbb{I}^l_j - \sqrt{4\pi}\,\partial_j\phi\,\mathbb{I}^l_i + \sqrt{4\pi}\,\partial_m\phi\, g^{lm} g_{ij} \\
\check{\Gamma}^3_{t3} &= \sqrt{4\pi}\, n\,\psi & \check{\Gamma}^3_{3i} &= \sqrt{4\pi}\,\partial_i\phi \\
\check{\Gamma}^t_{33} &= \sqrt{4\pi}\, n^{-1}\, e^{4\sqrt{4\pi}\phi}\,\psi & \check{\Gamma}^i_{33} &= -\,\sqrt{4\pi}\, e^{4\sqrt{4\pi}\phi}\, g^{ij}\,\partial_j\phi \\
\check{\Gamma}^3_{\mu\nu} &= \check{\Gamma}^\rho_{3\nu} = \check{\Gamma}^3_{33} = 0
\end{aligned}
$$

With these, one can compute the curvature components listed below. For components in which the index 3 does not occur, it can simplify computation to view $\check{g}|_{I\times M} = e^{-2\sqrt{4\pi}\,\phi}\,\overline{g}$ as a conformal transformation and use standard formulas that then relate $\mathrm{Riem}[\check{g}]$ with $\mathrm{Riem}[\overline{g}]$. Additionally, we use $\square_{\overline{g}}\phi = 0$, i.e.,

$$
e_0\psi = n^{-1}\,\partial_t\psi = \Delta_G\phi + \langle\nabla\log(n), \nabla\phi\rangle_g + \tau\,\psi ,
$$

as well as $\mathrm{Ric}[\overline{g}]_{\mu\nu} = 8\pi\,\overline{\nabla}_\mu\phi\,\overline{\nabla}_\nu\phi$ and that the Weyl tensor $W[\overline{g}]$ vanishes.

The Riemann curvature tensor is fully described by the following components:

$$
\begin{aligned}
\mathrm{Riem}[\check{g}]^t{}_{itj} &= \mathrm{Riem}[\overline{g}]^t{}_{itj} + \sqrt{4\pi}\,\nabla_i\nabla_j\phi + \sqrt{4\pi}\, n\, k_{ij}\,\psi + 4\pi\,\nabla_i\phi\,\nabla_j\phi \\
&\quad + \left(-\sqrt{4\pi}\, e_0\psi + \sqrt{4\pi}\,\langle\nabla\log(n), \nabla\phi\rangle_g + 4\pi\,|\nabla\phi|^2_g\right) g_{ij} \\
&= -\,\sqrt{4\pi}\left(\sqrt{4\pi}\,\psi + \frac{\tau}{2}\right)\psi\, g_{ij} + \sqrt{4\pi}\, n\,\hat{k}_{ij}\psi + \left(-\sqrt{4\pi}\,\Delta_G\phi - 4\pi\,|\nabla\phi|^2_g\right) g_{ij} \\
&\quad + \sqrt{4\pi}\,\nabla_i\nabla_j\phi \\
\mathrm{Riem}[\check{g}]^t{}_{ijl} &= \mathrm{Riem}[\overline{g}]^t{}_{ijl} + \left(\sqrt{4\pi}\, n^{-1}\,\nabla_j\psi - \sqrt{4\pi}\, k^r{}_l\,\nabla_r\phi - 8\pi\, n\,\psi\,\nabla_l\phi\right) g_{ij} \\
&\quad - \left(\sqrt{4\pi}\, n^{-1}\,\nabla_j\psi - \sqrt{4\pi}\, k^r{}_j\,\nabla_r\phi - 8\pi\, n\,\psi\,\nabla_j\phi\right) g_{il}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\sqrt{4\pi}\, n^{-1}\, \nabla_l \psi - \sqrt{4\pi}\, k^r{}_l\, \nabla_r \phi\right) g_{ij} - \left(\sqrt{4\pi}\, n^{-1}\, \nabla_j \psi - \sqrt{4\pi}\, k^r{}_j\, \nabla_r \phi\right) g_{il} \\
\mathrm{Riem}[\breve{g}]^{12}{}_{12} &= \mathrm{Riem}[\overline{g}]^{12}{}_{12} + \left(\sqrt{4\pi}\, \nabla^1 \nabla_1 \phi - 4\pi\, \nabla^1 \phi\, \nabla_1 \phi - \sqrt{4\pi}\, k^1{}_1\, \psi\right) \\
&\quad + \left(\sqrt{4\pi}\, \nabla^2 \nabla_2 \phi + 4\pi\, \nabla^2 \phi\, \nabla_2 \phi + \sqrt{4\pi}\, k^2{}_2\, \psi\right) - 4\pi\left(-\psi^2 + |\nabla\phi|^2\right) \\
&= 2\sqrt{4\pi}\, \psi \left(\frac{\tau}{2} + \sqrt{4\pi}\, \psi\right) + \sqrt{4\pi}\, \Delta_G \phi \\
\mathrm{Riem}[\breve{g}]^3{}_{t3t} &= -\sqrt{4\pi}\, n^2\, e_0 \psi - 8\pi\, (n\, \psi)^2 + \sqrt{4\pi}\, n\, \langle \nabla\phi, \nabla n\rangle_g + 4\pi\, n^2\, |\nabla\phi|_g^2 \\
&= -2\sqrt{4\pi}\, n^2\, \psi\, \left(\frac{\tau}{2} + \sqrt{4\pi}\, \psi\right) - \sqrt{4\pi}\, n^2\, \Delta_G \phi + \sqrt{4\pi}\, n\, \langle \nabla\phi, \nabla n\rangle_g \\
&\quad + 4\pi\, n^2\, |\nabla\phi|_g^2 \\
\mathrm{Riem}[\breve{g}]^3{}_{i3j} &= -\sqrt{4\pi}\, \nabla_i \nabla_j \phi - 12\pi\, n\, \psi\, \nabla_i \phi - \sqrt{4\pi}\, \psi\, \left(\hat{k}_{ij} + \left(\frac{\tau}{2} + \sqrt{4\pi}\psi\right) g_{ij}\right) \\
&\quad + 4\pi\, |\nabla\phi|_g^2\, g_{ij} \\
\mathrm{Riem}[\breve{g}]^3{}_{i3t} &= -\sqrt{4\pi}\, \nabla_i (n\, \psi) + \sqrt{4\pi}\psi\, \nabla_i n - 12\pi\, n\, \psi\, \nabla_i \phi - \sqrt{4\pi}\, n\, \hat{k}^r{}_i\, \nabla_r \phi \\
&\quad - \sqrt{4\pi}\, n\, \tau\, \nabla_i \phi \\
\mathrm{Riem}[\breve{g}]^3{}_{\mu\nu\rho} &= 0
\end{aligned}
$$

Bibliography

[ADM62] Richard Arnowitt, Stanley Deser, and Charles W. Misner. "The Dynamics of General Relativity". *in Gravitation: An Introduction to Current Research (Chap. 7). Edited by Louis Witten. John Wiley & Sons Inc* (1962). DOI: 10.1007/s10714-008-0661-1.

[AF20] Lars Andersson and David Fajman. "Nonlinear stability of the Milne model with matter". *Comm. Math. Phys.* 378.1 (2020), pp. 261–298. ISSN: 0010-3616. DOI: 10.1007/s00220-020-03745-w.

[AFF19] Artur Alho, Grigorios Fournodavlos, and Anne T. Franzen. "The wave equation near flat Friedmann-Lemaître-Robertson-Walker and Kasner Big Bang singularities". *J. Hyperbolic Differ. Equ.* 16.2 (2019), pp. 379–400. ISSN: 0219-8916. DOI: 10.1142/S0219891619500140.

[AM03] Lars Andersson and Vincent Moncrief. "Elliptic-hyperbolic systems and the Einstein equations". *Ann. Henri Poincaré* 4.1 (2003), pp. 1–34. ISSN: 1424-0637. DOI: 10.1007/s00023-003-0120-1.

[AM04] Lars Andersson and Vincent Moncrief. "Future Complete Vacuum Spacetimes". In: *The Einstein Equations and the Large Scale Behavior of Gravitational Fields.* Ed. by Piotr T. Chruściel and Helmut Friedrich. Basel: Birkhäuser Basel, 2004, pp. 299–330. DOI: 10.1007/978-3-0348-7953-8_8.

[AM11] Lars Andersson and Vincent Moncrief. "Einstein spaces as attractors for the Einstein flow". *J. Differential Geom.* 89.1 (2011), pp. 1–47. ISSN: 0022-040X. DOI: 10.4310/jdg/1324476750.

[AMT97] Lars Andersson, Vincent Moncrief, and Anthony J. Tromba. "On the global evolution problem in $2 + 1$ gravity". *J. Geom. Phys.* 23.3-4 (1997), pp. 191–205. ISSN: 0393-0440. DOI: 10.1016/S0393-0440(97)87804-7.

[And05] Lars Andersson. "Constant mean curvature foliations of simplicial flat spacetimes". *Communications in Analysis and Geometry* 13.5 (2005), pp. 963–979. DOI: 10.4310/CAG.2005.v13.n5.a6.

[And+07] Lars Andersson et al. "Notes on a paper of Mess". *Geometriae Dedicata* 126 (2007), pp. 47–70. DOI: 10.1007/s10711-007-9164-6.

[And11] Håkan Andréasson. "The Einstein-Vlasov system/Kinetic Theory". *Living Reviews in Relativity* 14 (2011), pp. 1–55. DOI: doi.org/10.12942/lrr-2011-4.

[AR01] Lars Andersson and Alan D. Rendall. "Quiescent cosmological singularities". *Communications in Mathematical Physics* 218.3 (2001), pp. 479–511. ISSN: 0010-3616. DOI: 10.1007/s002200100406.

[AR10] Paul T. Allen and Alan D. Rendall. "Asymptotics of linearized cosmological perturbations". *J. Hyperbolic Differ. Equ.* 7.2 (2010), pp. 255–277. ISSN: 0219-8916. DOI: 10.1142/S0219891610002141.

[AR16] Håkan Andréasson and Hans Ringström. "Proof of the cosmic no-hair conjecture in the $\mathbb{T}^3$-Gowdy symmetric Einstein–Vlasov setting". *Journal of the European Mathematical Society* 18.7 (2016), pp. 1565–1650. DOI: 10.4171/JEMS/623.

[Bac19] Alain Bachelot. "Wave Asymptotics at a Cosmological Time-Singularity: Classical and Quantum Scalar Fields". *Comm. Math. Phys.* 369.3 (2019), pp. 973–1020. ISSN: 0010-3616,1432-0916. DOI: 10.1007/s00220-019-03356-0.

[Ban73] Daniel Bancel. "Problème de Cauchy pour l'équation de Boltzmann en relativité générale". In: *Annales de l'institut Henri Poincaré. Section A, Physique Théorique.* Vol. 18. 3. 1973, pp. 263–284.

[Bar78] John D. Barrow. "Quiescent cosmology". *Nature* 272 (1978), pp. 211–215. DOI: 10.1038/272211a0.

[BD23] François Béguin and Tom Dutilleul. "Chaotic dynamics of spatially homogeneous spacetimes". *Comm. Math. Phys.* 399.2 (2023), pp. 737–927. ISSN: 0010-3616,1432-0916. DOI: 10.1007/s00220-022-04583-8.

[Bel14] Mehdi Belraouti. "Sur la géométrie de la singularité initiale des espaces-temps plats globalement hyperboliques". *Ann. Inst. Fourier (Grenoble)* 64.2 (2014), pp. 457–466. ISSN: 0373-0956,1777-5310. DOI: 10.5802/aif.2854.

[Bel17] Mehdi Belraouti. "Asymptotic behavior of Cauchy hypersurfaces in constant curvature space-times". *Geom. Dedicata* 190 (2017), pp. 103–133. ISSN: 0046-5755,1572-9168. DOI: 10.1007/s10711-017-0230-4.

[Ber03] Nicolas Bergeron. "Lefschetz properties for arithmetic real and complex hyperbolic manifolds". *Int. Math. Res. Not.* 20 (2003), pp. 1089–1122. ISSN: 1073-7928,1687-0247. DOI: 10.1155/S1073792803212253.

[BF22] Hamed Barzegar and David Fajman. "Stable cosmologies with collisionless charged matter". *J. Hyperbolic Differ. Equ.* 19.4 (2022), pp. 587–634. ISSN: 0219-8916,1793-6993. DOI: 10.1142/S0219891622500175.

[BFK19] Volker Branding, David Fajman, and Klaus Kröncke. "Stable cosmological Kaluza-Klein spacetimes". *Comm. Math. Phys.* 368.3 (2019), pp. 1087–1120. ISSN: 0010-3616. DOI: 10.1007/s00220-019-03319-5.

[BG00] Riccardo Benedetti and Enore Guadagnini. "Geometric cone surfaces and (2+1)-gravity coupled to particles". *Nuclear Phys. B* 588.1-2 (2000), pp. 436–450. ISSN: 0550-3213,1873-1562. DOI: 10.1016/S0550-3213(00)00484-3.

[BG01] Riccardo Benedetti and Enore Guadagnini. "Cosmological time in (2+1)-gravity". *Nuclear Physics B* 613.1 (2001), pp. 330–352. ISSN: 0550-3213. DOI: 10.1016/S0550-3213(01)00386-8.

[Big+21] Léo Bigorgne, David Fajman, Jérémie Joudioux, Jacques Smulevici, and Maximilian Thaller. "Asymptotic stability of Minkowski space-time with non-compactly supported massless Vlasov matter". *Arch. Ration. Mech. Anal.* 242.1 (2021), pp. 1–147. ISSN: 0003-9527,1432-0673. DOI: 10.1007/s00205-021-01639-2.

[BK71] Vladimir A. Belinskiĭ and Isaak M. Khalatnikov. "General solution of the gravitational equations with a physical oscillatory singularity". *Soviet Journal of Experimental and Theoretical Physics* 32 (1971), pp. 169–172.

[BK73] Vladimir A. Belinskiĭ and Isaak M. Khalatnikov. "Effect of scalar and vector fields on the nature of the cosmological singularity". *Soviet Journal of Experimental and Theoretical Physics* 36 (1973), p. 591.

[BKL70] Vladimir A. Belinskiĭ, Isaak M. Khalatnikov, and Evgeny M. Lifshitz. "Oscillatory approach to a singular point in the relativistic cosmology". *Advances in Physics* 19.80 (1970), pp. 525–573.

[BKL82] Vladimir A. Belinskiĭ, Isaak M. Khalatnikov, and Evgeny M. Lifshitz. "A general solution of the Einstein equations with a time singularity". *Advances in Physics* 31.6 (1982), pp. 639–667.

[BL17] Florian Beyer and Philippe G. LeFloch. "Self-gravitating fluid flows with Gowdy symmetry near cosmological singularities". *Comm. Partial Differential Equations* 42.8 (2017), pp. 1199–1248. ISSN: 0360-5302,1532-4133.
DOI: `10.1080/03605302.2017.1345938`.

[BMP25] James Bonifacio, Dalimil Mazáč, and Sridip Pal. "Spectral Bounds on Hyperbolic 3-Manifolds: Associativity and the Trace Formula". *Comm. Math. Phys.* 406.3 (2025), Paper No. 51. ISSN: 0010-3616,1432-0916.
DOI: `10.1007/s00220-024-05222-0`.

[BO24a] Florian Beyer and Todd A. Oliynyk. "Localized Big Bang Stability for the Einstein-Scalar Field Equations". *Arch. Ration. Mech. Anal.* 248.1 (2024), p. 3. ISSN: 0003-9527,1432-0673. DOI: `10.1007/s00205-023-01939-9`.

[BO24b] Florian Beyer and Todd A. Oliynyk. "Past stability of FLRW solutions to the Einstein-Euler-scalar field equations and their Big Bang singularites". *Beijing Journal of Pure and Applied Mathematics* 1.2 (2024), pp. 515–637.
DOI: `10.4310/BPAM.2024.v1.n2.a4`.

[BO24c] Florian Beyer and Todd A. Oliynyk. "Relativistic perfect fluids near Kasner singularities". *Comm. Anal. Geom.* 32.6 (2024), pp. 1701–1794. ISSN: 1019-8385,1944-9992. DOI: `10.4310/cag.241204004223`.

[Bol68] Ludwig Boltzmann. "Studien über das Gleichgewicht der lebendigen Kraft zwischen bewegten materiellen Punkten". *Wiener Berichte* 58 (1868), pp. 517–560.

[BOZ25] Florian Beyer, Todd A. Oliynyk, and Weihang Zheng. *Localized past stability of the subcritical Kasner-scalar field spacetimes.* 2025.
DOI: `10.48550/arXiv.2502.09210`.

[BS07] Andrew R. Booker and Andreas Strömbergsson. "Numerical computations with the trace formula and the Selberg eigenvalue conjecture". *J. Reine Angew. Math.* 607 (2007), pp. 113–161. ISSN: 0075-4102,1435-5345.
DOI: `10.1515/CRELLE.2007.047`.

[BT08] James Binney and Scott Tremaine. *Galactic Dynamics, Second Edition.* Princeton: Princeton University Press, 2008. ISBN: 9781400828722.
DOI: `10.1515/9781400828722`.

[BT79] John D. Barrow and Frank J. Tipler. “Analysis of the generic singularity studies by Belinskii, Khalatnikov, and Lifschitz”. *Physics Reports* 56.7 (1979), pp. 371–402. ISSN: 0370-1573. DOI: 10.1016/0370-1573(79)90097-8.

[Cal94] Patrick J. Callahan. *Spectral geometry of hyperbolic 3-manifolds.* Thesis (Ph.D.)–University of Illinois at Urbana-Champaign. ProQuest LLC, Ann Arbor, MI, 1994, p. 40.

[Car25] Steven J. Carlip. “Quantum Gravity in 2+1 Dimensions”. In: *Encyclopedia of Mathematical Physics (Second Edition).* Ed. by Richard Szabo and Martin Bojowald. Second Edition. Oxford: Academic Press, 2025, pp. 595–611. ISBN: 978-0-323-95706-9. DOI: 10.1016/B978-0-323-95703-8.00256-1.

[CBC02] Yvonne Choquet-Bruhat and Spiros Cotsakis. “Global hyperbolicity and completeness”. *J. Geom. Phys.* 43.4 (2002), pp. 345–350. ISSN: 0393-0440. DOI: 10.1016/S0393-0440(02)00028-1.

[CBG69] Yvonne Choquet-Bruhat and Robert P. Geroch. “Global aspects of the Cauchy problem in general relativity”. *Communications in Mathematical Physics* 14 (1969), pp. 329–335. DOI: 10.1007/BF01645389.

[CBIM04] Yvonne Choquet-Bruhat, James Isenberg, and Vincent Moncrief. “Topologically general U(1) symmetric vacuum space-times with AVTD behavior”. *Nuovo Cimento Soc. Ital. Fis. B* 119.7-9 (2004), pp. 625–638. ISSN: 1594-9982. DOI: 10.1393/ncb/i2004-10174-x.

[CBM01] Yvonne Choquet-Bruhat and Vincent Moncrief. “Future global in time Einsteinian spacetimes with U(1) isometry group”. *Ann. Henri Poincaré* 2.6 (2001), pp. 1007–1064. ISSN: 1424-0637. DOI: 10.1007/s00023-001-8602-5.

[CGS22] Rubén O. Acuña Cárdenas, Carlos Gabarrete, and Olivier Sarbach. “An introduction to the relativistic kinetic theory on curved spacetimes”. *Gen. Relativity Gravitation* 54.3 (2022), Paper No. 23, 120. ISSN: 0001-7701,1572-9532. DOI: 10.1007/s10714-022-02908-5.

[CH03] Alan A. Coley and Sigbjørn Hervik. “Braneworld singularities”. *Classical and Quantum Gravity* 20.14 (June 2003), p. 3061. DOI: 10.1088/0264-9381/20/14/308.

[Che63] Nikolay A. Chernikov. “The Relativistic Gas in the Gravitational Field”. *Acta Physica Polonica* XXIII (1963), p. 629.

[Chr07] Demetrios Christodoulou. *The formation of shocks in 3-dimensional fluids.* Vol. 2. European Mathematical Society, 2007. DOI: 10.4171/031.

[CI93] Piotr T. Chruściel and James Isenberg. “Nonisometric vacuum extensions of vacuum maximal globally hyperbolic spacetimes”. *Phys. Rev. D (3)* 48.4 (1993), pp. 1616–1628. ISSN: 0556-2821. DOI: 10.1103/PhysRevD.48.1616.

[Cic+24] Michele Cicoli et al. “String Cosmology: From the Early Universe to Today”. *Physics Reports* 1059 (2024), pp. 1–155. DOI: 10.1016/j.physrep.2024.01.002.

[CIM90] Piotr T. Chruściel, James Isenberg, and Vincent Moncrief. “Strong cosmic censorship in polarised Gowdy spacetimes”. *Classical Quantum Gravity* 7.10 (1990),

pp. 1671–1680. ISSN: 0264-9381,1361-6382.
DOI: 10.1088/0264-9381/7/10/003.

[CK90] Demetrios Christodoulou and Sergiu Klainerman. "Asymptotic properties of linear field equations in Minkowski space". *Comm. Pure Appl. Math.* 43 (1990), pp. 137–199. DOI: 10.1002/cpa.3160430202.

[CK93] Demetrios Christodoulou and Sergiu Klainerman. *The global nonlinear stability of the Minkowski space.* Vol. 41. Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 1993, pp. x+514. ISBN: 0-691-08777-6.

[CLN06] Bennet Chow, Peng Lu, and Lei Ni. *Hamilton's Ricci Flow.* Graduate studies in mathematics. American Mathematical Society/Science Press, 2006. ISBN: 9780821842317.

[Col+20] Planck Collaboration et al. "Planck 2018 results. VI. Cosmological parameters". *Astronomy & Astrophysics* 641, A6 (Sept. 2020), A6.
DOI: 10.1051/0004-6361/201833910.

[CR95] Piotr T. Chrusciel and Alan D. Rendall. "Strong Cosmic Censorship in Vacuum Space-Times with Compact, Locally Homogeneous Cauchy Surfaces". *Annals of Physics* 242.2 (Sept. 1995), 349–385. ISSN: 0003-4916.
DOI: 10.1006/aphy.1995.1084.

[CS99] Neil J. Cornish and David N. Spergel. "On the eigenmodes of compact hyperbolic 3-manifolds" (1999). DOI: 10.48550/ARXIV.MATH/9906017.

[CZ95] Xu Cheng and Detang Zhou. "First eigenvalue estimate on Riemannian manifolds". *Hokk. Math. J.* 24.3 (1995), pp. 453 –472.
DOI: 10.14492/hokmj/1380892604.

[Dam+02] Thibault Damour, Marc Henneaux, Alan D. Rendall, and Marsha Weaver. "Kasner-Like Behaviour for Subcritical Einstein-Matter Systems". *Ann. Henri Poincaré* 3 (2002), pp. 1049–1111. DOI: 10.1007/s000230200000.

[Dar27] Georges Darmois. *Les équations de la gravitation einsteinienne.* 25. Gauthier-Villars, 1927.

[DL17] Mihalis Dafermos and Jonathan Luk. "The interior of dynamical vacuum black holes I: the C^0-stability of the Kerr Cauchy horizon" (2017). To appear in *Annals of Mathematics.* DOI: 10.48550/arXiv.1710.01722.

[DR06] Mihalis Dafermos and Alan D. Rendall. *Strong cosmic censorship for T^2-symmetric cosmological spacetimes with collisionless matter.* 2006.
DOI: 10.48550/arXiv.gr-qc/0610075.

[DR16] Mihalis Dafermos and Alan D. Rendall. "Strong cosmic censorship for surface-symmetric cosmological spacetimes with collisionless matter". *Comm. Pure Appl. Math.* 69.5 (2016), pp. 815–908. ISSN: 0010-3640,1097-0312.
DOI: 10.1002/cpa.21628.

[Dra03] Sever Silvestru Dragomir. *Some Gronwall type inequalities and applications.* Nova Science Publishers, Inc., Hauppauge, NY, 2003, pp. viii+193. ISBN: 1-59033-827-8.

[DS17] Willem De Sitter. "Einstein's theory of gravitation and its astronomical consequences. Third paper". *Monthly Notices of the Royal Astronomical Society, Vol. 78, p. 3-28* 78 (1917), pp. 3–28.

[Ehl73] Jürgen Ehlers. "Survey of general relativity theory". In: *Relativity, Astrophysics and Cosmology: Proceedings of the Summer School Held, 14–26 August, 1972 at the Banff Centre, Banff, Alberta.* Springer, 1973, pp. 1–125.

[Ein16] Albert Einstein. "Die Grundlage der allgemeinen Relativitätstheorie". *Annalen der Physik* 354.7 (1916), pp. 769–822.
DOI: `https://doi.org/10.1002/andp.2005517S151`.

[Ein17] Albert Einstein. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften* (1917), pp. 142–152.

[ELS72] Douglas M. Eardley, Edison P. Liang, and Rainer K. Sachs. "Velocity-Dominated Singularities in Irrotational Dust Cosmologies". *Journal of Mathematical Physics* 13.1 (Jan. 1972), pp. 99–107. ISSN: 0022-2488. DOI: `10.1063/1.1665859`.

[Eva98] Lawrence C. Evans. *Partial Differential Equations.* 1st edition. American Mathematical Society, 1998. ISBN: 978-0-821-80772-9.

[Faj16] David Fajman. "Future asymptotic behavior of three-dimensional spacetimes with massive particles". *Classical Quantum Gravity* 33.11 (2016), 11LT01, 9. ISSN: 0264-9381,1361-6382. DOI: `10.1088/0264-9381/33/11/11LT01`.

[Faj17a] David Fajman. "The nonvacuum Einstein flow on surfaces of negative curvature and nonlinear stability". *Comm. Math. Phys.* 353.2 (2017), pp. 905–961. ISSN: 0010-3616. DOI: `10.1007/s00220-017-2842-9`.

[Faj17b] David Fajman. "Topology and incompleteness for 2 + 1-dimensional cosmological spacetimes". *Lett. Math. Phys.* 107.6 (2017), pp. 1157–1176. ISSN: 0377-9017,1573-0530. DOI: `10.1007/s11005-016-0932-9`.

[Faj18] David Fajman. "The nonvacuum Einstein flow on surfaces of nonnegative curvature". *Comm. Partial Differential Equations* 43.3 (2018), pp. 364–402. ISSN: 0360-5302,1532-4133. DOI: `10.1080/03605302.2018.1446159`.

[Faj20] David Fajman. "Future attractors in 2 + 1 dimensional Λ gravity". *Phys. Rev. Lett.* 125.12 (2020), pp. 121102, 6. ISSN: 0031-9007.
DOI: `10.1103/physrevlett.125.121102`.

[FB52] Yvonne Fourès-Bruhat. "Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires". *Acta Mathematica* 88 (1952), pp. 141–225.
DOI: `10.1007/BF02392131`.

[FB56] Yvonne Fourès-Bruhat. "Sur L'Intégration des Équations de la Relativité Générale". fre. *Journal of Rational Mechanics and Analysis* 5.6 (1956), pp. 951–966. ISSN: 1943-5282.

[FG25] Andrés Franco-Grisales. "Developments of initial data on big bang singularities for the Einstein-nonlinear scalar field equations" (2025).
DOI: `10.48550/arXiv.2409.17065`.

[FJS21] David Fajman, Jérémie Joudioux, and Jacques Smulevici. "The stability of the Minkowski space for the Einstein–Vlasov system". *Analysis & PDE* 14.2 (2021), pp. 425 –531. DOI: `10.2140/apde.2021.14.425`.

[FK20] David Fajman and Klaus Kröncke. "Stable fixed points of the Einstein flow with positive cosmological constant". *Comm. Anal. Geom.* 28.7 (2020), pp. 1533–1576. ISSN: 1019-8385. DOI: `10.4310/CAG.2020.v28.n7.a2`.

[FL23] Grigorios Fournodavlos and Jonathan Luk. "Asymptotically Kasner-like singularities". *Amer. J. Math.* 145.4 (2023), pp. 1183–1272. ISSN: 0002-9327,1080-6377. DOI: `10.1353/ajm.2023.a902957`.

[FOW24] David Fajman, Maximilian Ofner, and Zoe Wyatt. "Slowly expanding stable dust spacetimes". *Arch. Ration. Mech. Anal.* 248.5 (2024), Paper No. 83, 66. ISSN: 0003-9527,1432-0673. DOI: `10.1007/s00205-024-02030-7`.

[Fri22] Alexander Friedman. "Über die Krümmung des Raumes". *Zeitschrift für Physik* 10 (1922), pp. 377–386. DOI: `10.1007/BF01332580`.

[Fri24] Alexander Friedman. "Über die Möglichkeit einer Welt mit konstanter negativer Krümmung des Raumes". *Zeitschrift für Physik* 21.1 (1924), pp. 326–332. DOI: `10.1007/BF01328280`.

[Fri86] Helmut Friedrich. "On the existence of n-geodesically complete or future complete solutions of Einstein's field equations with smooth asymptotic structure". *Communications in Mathematical Physics* 107 (1986), pp. 587–609. DOI: `10.1007/BF01205488`.

[FRS23] Grigorios Fournodavlos, Igor Rodnianski, and Jared Speck. "Stable big bang formation for Einstein's equations: the complete sub-critical regime". *Journal of the American Mathematical Society* 36.3 (2023), pp. 827–916. ISSN: 0894-0347,1088-6834. DOI: `10.1090/jams/1015`.

[FU22] David Fajman and Liam Urban. "Blow-up of waves on singular spacetimes with generic spatial metrics". *Lett. Math. Phys.* 112.2 (2022), Paper No. 42, 31. ISSN: 0377-9017,1573-0530. DOI: `10.1007/s11005-022-01522-5`.

[FU25a] David Fajman and Liam Urban. "Cosmic censorship near FLRW spacetimes with negative spatial curvature". *Analysis & PDE* 18.7 (2025), pp. 1615 –1713. DOI: `10.2140/apde.2025.18.1615`.

[FU25b] David Fajman and Liam Urban. "On the past maximal development of near-FLRW data for the Einstein scalar-field Vlasov system" (2025). Revision submitted to *Communications in Mathematical Physics* on June 24, 2025. 78 pages. DOI: `10.48550/arXiv.2402.08544`.

[FW21] David Fajman and Zoe Wyatt. "Attractors of the Einstein-Klein-Gordon system". *Comm. Part. Diff. Eq.* 46.1 (2021), pp. 1–30. ISSN: 0360-5302. DOI: `10.1080/03605302.2020.1817072`.

[GaNS19] Pedro M. Girão, José Natário, and Jorge Drumond Silva. "Solutions of the wave equation bounded at the Big Bang". *Classical Quantum Gravity* 36.7 (2019), pp. 075016, 10. ISSN: 0264-9381. DOI: `10.1088/1361-6382/ab09b2`.

[Gow74] Robert H. Gowdy. "Vacuum spacetimes with two-parameter spacelike isometry groups and compact invariant hypersurfaces: Topologies and boundary conditions". *Annals of Physics* 83.1 (1974), pp. 203–241. ISSN: 0003-4916. DOI: 10.1016/0003-4916(74)90384-4.

[Haw66a] Stephen W. Hawking. "The occurrence of singularities in cosmology". *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences* 294.1439 (1966), pp. 511–521. DOI: 10.1098/rspa.1966.0221.

[Haw66b] Stephen W. Hawking. "The occurrence of singularities in cosmology. II". *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences* 295.1443 (1966), pp. 490–493. DOI: 10.1098/rspa.1966.0255.

[Haw67] Stephen W. Hawking. "The Occurrence of Singularities in Cosmology. III. Causality and Singularities". *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences* 300.1461 (1967), pp. 187–201. ISSN: 00804630. DOI: 10.1098/rspa.1967.0164.

[HE68] Stephen W. Hawking and George F.R. Ellis. "The cosmic black-body radiation and the existence of singularities in our universe". *Astrophysical Journal, vol. 152, p. 25* 152 (1968), p. 25. DOI: 10.1086/149520.

[Hin+13] Gary Hinshaw, Davin E. Larson, Eiichiro Komatsu, David N. Spergel, et al. "Nine-year Wilkinson Microwave Anisotropy Probe (WMAP) observations: cosmological parameter results". *The Astrophysical Journal Supplement Series* 208.2 (2013), p. 19. DOI: 10.1088/0067-0049/208/2/19.

[HP70] Stephen W. Hawking and Roger Penrose. "The singularities of gravitational collapse and cosmology". *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences* 314.1519 (1970), pp. 529–548. DOI: 10.1098/rspa.1970.0021.

[IM02] James Isenberg and Vincent Moncrief. "Asymptotic behaviour in polarized and half-polarized U(1) symmetric vacuum spacetimes". *Classical Quantum Gravity* 19.21 (2002), pp. 5361–5386. ISSN: 0264-9381. DOI: 10.1088/0264-9381/19/21/305.

[Ino01] Kaiki Taro Inoue. "Numerical study of length spectra and low-lying eigenvalue spectra of compact hyperbolic 3-manifolds". *Classical Quantum Gravity* 18.4 (2001), pp. 629–652. ISSN: 0264-9381. DOI: 10.1088/0264-9381/18/4/305.

[Isr63] Werner Israel. "Relativistic kinetic theory of a simple gas". *Journal of Mathematical Physics* 4.9 (1963), pp. 1163–1181. DOI: 10.1063/1.1704047.

[Jüt11] Ferencz Jüttner. "Das Maxwellsche Gesetz der Geschwindigkeitsverteilung in der Relativtheorie". *Annalen der Physik* 339.5 (1911), pp. 856–882. DOI: 10.1002/andp.19113390503.

[Kas21] Edward Kasner. "Geometrical Theorems on Einstein's Cosmological Equations". *American Journal of Mathematics* 43.4 (1921), pp. 217–221. ISSN: 00029327, 10806377. DOI: 10.2307/2370192.

[Krö15] Klaus Kröncke. "On the stability of Einstein manifolds". *Ann. Global Anal. Geom.* 47.1 (2015), pp. 81–98. ISSN: 0232-704X. DOI: 10.1007/s10455-014-9436-y.

[Lan21] Klaas Landsman. "Singularities, black holes, and cosmic censorship: A tribute to Roger Penrose". *Foundations of Physics* 51 (2021), pp. 1–38. DOI: 10.1007/s10701-021-00432-1.

[Lem27] Abbé G. Lemaître. "Un univers homogène de masse constante et de rayon croissant rendant compte de la vitesse radiale des nébuleuses extra-galactiques". *Annales de la Société scientifique de Bruxelles* 47 (1927), 49–56.

[Lem31a] Abbé G. Lemaître. "A Homogeneous Universe of Constant Mass and Increasing Radius accounting for the Radial Velocity of Extra-galactic Nebulæ". *Monthly Notices of the Royal Astronomical Society* 91.5 (Mar. 1931), pp. 483–490. ISSN: 0035-8711. DOI: 10.1093/mnras/91.5.483.

[Lem31b] Abbé G. Lemaître. "L'expansion de l'espace". *Publications du Laboratoire d'Astronomie et de Geodesie de l'Universite de Louvain* 8 (Jan. 1931), pp. 101–120.

[Li24a] Warren Li. "BKL bounces outside homogeneity: Gowdy symmetric spacetimes" (2024). DOI: 10.48550/arXiv:2408.12427v1.

[Li24b] Warren Li. *Scattering towards the singularity for the wave equation and the linearized Einstein-scalar field system in Kasner spacetimes.* arXiv:2401.08437. 2024.

[Lin66] Richard W. Lindquist. "Relativistic transport theory". *Annals of Physics* 37.3 (1966), pp. 487–518. DOI: 10.1016/0003-4916(66)90207-7.

[Lio38] Joseph Liouville. "Note sur la théorie de la variation des constantes arbitraires". *Journal de mathématiques pures et appliquées* 3 (1838), pp. 342–349.

[LK61a] Evgeny M. Lifshitz and Isaak M. Khalatnikov. "On the Singularities of Cosmological Solutions of the Gravitational Equations". *Journal of Experimental and Theoretical Physics* 12.3 (1961), pp. 558–563.

[LK61b] Evgeny M. Lifshitz and Isaak M. Khalatnikov. "On the Singularities of Cosmological Solutions of the Gravitational Equations I". *Journal of Experimental and Theoretical Physics* 12.1 (1961), pp. 108–113.

[LK63] Evgeny M. Lifshitz and Isaak M. Khalatnikov. "Investigations in relativistic cosmology". *Advances in Physics* 12.46 (1963), pp. 185–249. DOI: 10.1080/00018736300101283.

[LL22] Francesco Lin and Michael Lipnowski. "The Seiberg-Witten equations and the length spectrum of hyperbolic three-manifolds". *J. Amer. Math. Soc.* 35.1 (2022), pp. 233–293. ISSN: 0894-0347,1088-6834. DOI: 10.1090/jams/982.

[LL24] Francesco Lin and Michael Lipnowski. "Closed geodesics and Frøyshov invariants of hyperbolic three-manifolds". *Journal of the European Mathematical Society* (2024). published online first. DOI: 10.4171/JEMS/1452.

[LSK61] Evgeny M. Lifshitz, V.V. Sudakov, and Isaak M. Khlalatnikov. "Singularities of Cosmological Solutions of Gravitational Equations III". *Journal of Experimental and Theoretical Physics* 13.6 (1961), pp. 1298–1305.

[LT20] Hans Lindblad and Martin Taylor. “Global stability of Minkowski space for the Einstein-Vlasov system in the harmonic gauge”. *Arch. Ration. Mech. Anal.* 235.1 (2020), pp. 517–633. ISSN: 0003-9527,1432-0673. DOI: `10.1007/s00205-019-01425-1`.

[Mes07] Geoffrey Mess. “Lorentz spacetimes of constant curvature”. *Geometriae Dedicata* 126 (2007), pp. 3–45. DOI: `10.1007/s10711-007-9155-7`.

[Mis67] Charles W. Misner. “Taub-NUT Space as a Counterexample to almost anything”. In: *Relativity Theory and Astrophysics. Vol.1: Relativity and Cosmology.* Ed. by Jürgen Ehlers. Vol. 8. 1967, p. 160.

[Mis69] Charles W. Misner. “Mixmaster universe”. *Physical Review Letters* 22.20 (1969), p. 1071. DOI: `10.1103/PhysRevLett.22.1071`.

[Mon08] Vincent Moncrief. “Relativistic Teichmüller theory—a Hamilton-Jacobi approach to $2+1$-dimensional Einstein gravity”. In: *Surveys in differential geometry. Vol. XII. Geometric flows.* Vol. 12. Surv. Differ. Geom. Int. Press, Somerville, MA, 2008, pp. 203–249. ISBN: 978-1-57146-118-6. DOI: `10.4310/SDG.2007.v12.n1.a6`.

[Mon23] Puskar Mondal. “Big-bang limit of $2+1$ gravity and Thurston boundary of Teichmüller space”. *Journal of Mathematical Physics* 64.11 (Nov. 2023), p. 112501. ISSN: 0022-2488. DOI: `10.1063/5.0136631`.

[Mon81] Vincent Moncrief. “Global properties of Gowdy spacetimes with $T^3 \times \mathbb{R}$ topology”. *Annals of Physics* 132.1 (1981), pp. 87–107. ISSN: 0003-4916. DOI: `10.1016/0003-4916(81)90270-0`.

[Mon86] Vincent Moncrief. “Reduction of Einstein’s equations for vacuum space-times with spacelike U(1) isometry groups”. *Ann. Physics* 167.1 (1986), pp. 118–142. ISSN: 0003-4916,1096-035X. DOI: `10.1016/S0003-4916(86)80009-4`.

[Mon89] Vincent Moncrief. “Reduction of the Einstein equations in $2+1$ dimensions to a Hamiltonian system over Teichmüller space”. *J. Math. Phys.* 30.12 (1989), pp. 2907–2914. ISSN: 0022-2488,1089-7658. DOI: `10.1063/1.528475`.

[MT69] Charles W. Misner and Abraham H. Taub. “A singularity-free empty universe”. *Sov. Phys. JETP* 28.122 (1969), p. 2.

[OGPR23] Hans Oude Groeniger, Oliver Petersen, and Hans Ringström. “Formation of quiescent big bang singularities” (2023). DOI: `10.48550/arXiv.2309.11370`.

[O’N83] Barrett O’Neill. *Semi-Riemannian geometry with applications to relativity.* Academic Press, 1983. ISBN: 0-12-526740-1.

[Pen65] Roger Penrose. “Gravitational Collapse and Space-Time Singularities”. *Phys. Rev. Lett.* 14 (3 Jan. 1965), pp. 57–59. DOI: `10.1103/PhysRevLett.14.57`.

[Pen79] Roger Penrose. “Singularities and time-asymmetry”. In: *General relativity: An Einstein Centenary Survey.* Ed. by Stephen W. Hawking and Werner Israel. Cambridge University Press, 1979, pp. 581–639.

[Rei96] Gerhard Rein. “Cosmological solutions of the Vlasov-Einstein system with spherical, plane, and hyperbolic symmetry”. *Math. Proc. Cambridge Philos. Soc.* 119.4

(1996), pp. 739–762. ISSN: 0305-0041,1469-8064.
DOI: 10.1017/S0305004100074569.

[Ren06] Alan D. Rendall. "Mathematical properties of cosmological models with accelerated expansion". In: *Analytical and numerical approaches to mathematical relativity.* 1st edition. Springer, 2006, pp. 141–155. ISBN: 978-3-540-33484-2. DOI: 10.1007/b11550259.

[Ren08] Alan D. Rendall. *Partial differential equations in general relativity.* Vol. 16. Oxford Graduate Texts in Mathematics. Oxford University Press, Oxford, 2008, pp. xvi+279. ISBN: 978-0-19-921540-9; 978-0-19-921541-6.

[Rin00] Hans Ringström. "Curvature blow up in Bianchi VIII and IX vacuum spacetimes". *Classical and quantum gravity* 17.4 (2000), p. 713. DOI: 10.1088/0264-9381/17/4/301.

[Rin01] Hans Ringström. "The Bianchi IX attractor". *Ann. Henri Poincaré* 2.3 (2001), pp. 405–500. ISSN: 1424-0637,1424-0661. DOI: 10.1007/PL00001041.

[Rin04] Hans Ringström. "On Gowdy vacuum spacetimes". *Mathematical Proceedings of the Cambridge Philosophical Society* 136.2 (2004), 485–512. DOI: 10.1017/S0305004103007321.

[Rin08] Hans Ringström. "Future stability of the Einstein-non-linear scalar field system". *Invent. Math.* 173.1 (2008), pp. 123–208. ISSN: 0020-9910. DOI: 10.1007/s00222-008-0117-y.

[Rin09a] Hans Ringström. "Strong cosmic censorship in T^3-Gowdy spacetimes". *Ann. of Math. (2)* 170.3 (2009), pp. 1181–1240. ISSN: 0003-486X,1939-8980. DOI: 10.4007/annals.2009.170.1181.

[Rin09b] Hans Ringström. *The Cauchy problem in general relativity.* ESI Lectures in Mathematics and Physics. European Mathematical Society (EMS), Zürich, 2009, pp. xiv+294. ISBN: 978-3-03719-053-1. DOI: 10.4171/053.

[Rin10] Hans Ringström. "Cosmic Censorship for Gowdy Spacetimes". *Living Reviews in Relativity* 13.2 (2010). cited April 22, 2025. DOI: 10.12942/lrr-2010-2.

[Rin13] Hans Ringström. *On the topology and future stability of the universe.* Oxford Mathematical Monographs. Oxford University Press, Oxford, 2013, pp. xiv+718. ISBN: 978-0-19-968029-0. DOI: 10.1093/acprof:oso/9780199680290.001.0001.

[Rin19] Hans Ringström. "A unified approach to the Klein-Gordon equation on Bianchi backgrounds". *Comm. Math. Phys.* 372.2 (2019), pp. 599–656. ISSN: 0010-3616. DOI: 10.1007/s00220-019-03325-7.

[Rin20] Hans Ringström. "Linear systems of wave equations on cosmological backgrounds with convergent asymptotics". *Astérisque* 420 (2020), pp. xi + 510. ISSN: 0303-1179. DOI: 10.24033/ast.

[Rin21] Hans Ringström. *Wave equations on silent big bang backgrounds.* To appear in *Memoirs of the American Mathematical Society.* 2021. DOI: 10.48550/arXiv.2101.04939.

[Rin25] Hans Ringström. "Cosmology, the big bang and the BKL conjecture". *Comptes Rendus. Mécanique* 353.G1 (2025), pp. 53–78. DOI: 10.5802/crmeca.277.

[Rob35] Howard P. Robertson. "Kinematics and World-Structure". *Astrophysical Journal* 82 (Nov. 1935), p. 284. DOI: 10.1086/143681.

[RP09] Matteo Realdi and Giulio Peruzzi. "Einstein, de Sitter and the beginning of relativistic cosmology in 1917". *General Relativity and Gravitation* 41 (2009), pp. 225–247. DOI: 10.1007/s10714-008-0664-y.

[RS18a] Igor Rodnianski and Jared Speck. "A regime of linear stability for the Einstein-scalar field system with applications to nonlinear Big Bang formation". *Ann. of Math. (2)* 187.1 (2018), pp. 65–156. ISSN: 0003-486X. DOI: 10.4007/annals.2018.187.1.2.

[RS18b] Igor Rodnianski and Jared Speck. "Stable big bang formation in near-FLRW solutions to the Einstein-scalar field and Einstein-stiff fluid systems". *Selecta Math. (N.S.)* 24.5 (2018), pp. 4293–4459. ISSN: 1022-1824. DOI: 10.1007/s00029-018-0437-8.

[RS22] Igor Rodnianski and Jared Speck. "On the nature of Hawking's incompleteness for the Einstein-vacuum equations: the regime of moderately spatially anisotropic initial data". *J. Eur. Math. Soc. (JEMS)* 24.1 (2022), pp. 167–263. ISSN: 1435-9855,1435-9863. DOI: 10.4171/jems/1092.

[Ryd16] Barbara Ryden. *Introduction to Cosmology*. 2nd ed. Cambridge University Press, 2016. DOI: 10.1017/9781316651087.

[Sas58] Shigeo Sasaki. "On the differential geometry of tangent bundles of Riemannian manifolds". *Tohoku Mathematical Journal* 10.3 (1958), pp. 338 –354. DOI: 10.2748/tmj/1178244668.

[Sbi22] Jan Sbierski. "On holonomy singularities in general relativity and the $C^{0,1}_{\mathrm{loc}}$-inextendibility of space-times". *Duke Math. J.* 171.14 (2022), pp. 2881–2942. ISSN: 0012-7094,1547-7398. DOI: 10.1215/00127094-2022-0040.

[Sbi23] Jan Sbierski. "The C^0-inextendibility of a class of FLRW spacetimes" (2023). DOI: 10.48550/arXiv.2312.07443.

[Sha11] Arick Shao. "On breakdown criteria for nonvacuum Einstein equations". *Ann. Henri Poincaré* 12.2 (2011), pp. 205–277. ISSN: 1424-0637. DOI: 10.1007/s00023-011-0082-7.

[SM58] John L. Synge and Philip M. Morse. "The Relativistic Gas". *Physics Today* 11.12 (Dec. 1958), pp. 56–56. ISSN: 0031-9228. DOI: 10.1063/1.3062345.

[Smu08] Jacques Smulevici. "Strong cosmic censorship for T^2-symmetric spacetimes with cosmological constant and matter". *Ann. Henri Poincaré* 9.8 (2008), pp. 1425–1453. ISSN: 1424-0637,1424-0661. DOI: 10.1007/s00023-008-0391-7.

[Smu11] Jacques Smulevici. "On the area of the symmetry orbits of cosmological spacetimes with toroidal or hyperbolic symmetry". *Anal. PDE* 4.2 (2011), pp. 191–245. ISSN: 2157-5045,1948-206X. DOI: 10.2140/apde.2011.4.191.

[Spe18] Jared Speck. "The Maximal Development of Near-FLRW Data for the Einstein-Scalar Field System with Spatial Topology $\mathbb{S}^3$". *Communications in Mathematical Physics* 364.3 (Oct. 2018), 879–979. ISSN: 1432-0916. DOI: 10.1007/s00220-018-3272-z.

[Sve12] Christopher Svedberg. "Future stability of the Einstein-Maxwell-Scalar field system and non-linear wave equations coupled to generalized massive-massless Vlasov equations". QC 20120503. PhD thesis. KTH, Mathematics (Div.), 2012, pp. vi, 19. ISBN: 978-91-7501-294-0.

[Tay17] Martin Taylor. "The global nonlinear stability of Minkowski space for the massless Einstein-Vlasov system". *Ann. PDE* 3.1 (2017), Paper No. 9, 177. ISSN: 2524-5317,2199-2576. DOI: 10.1007/s40818-017-0026-8.

[Tch04] Sophonie Blaise Tchapnda. "Structure of solutions near the initial singularity for the surface-symmetric Einstein–Vlasov system". *Classical and Quantum Gravity* 21.23 (Nov. 2004), p. 5333. DOI: 10.1088/0264-9381/21/23/003.

[TNR04] David Tegankong, Norbert Noutchegueme, and Alan D. Rendall. "Local existence and continuation criteria for solutions of the Einstein-Vlasov-scalar field system with surface symmetry". *J. Hyperbolic Differ. Equ.* 1.4 (2004), pp. 691–724. ISSN: 0219-8916,1793-6993. DOI: 10.1142/S0219891604000305.

[TR06] David Tegankong and Alan D. Rendall. "On the nature of initial singularities for solutions of the Einstein-Vlasov-scalar field system with surface symmetry". *Math. Proc. Cambridge Philos. Soc.* 141.3 (2006), pp. 547–562. ISSN: 0305-0041,1469-8064. DOI: 10.1017/S0305004106009571.

[TW61] Gerald E. Tauber and Joseph W. Weinberg. "Internal state of a gravitating gas". *Physical Review* 122.4 (1961), p. 1342. DOI: 10.1103/PhysRev.122.1342.

[Urb24] Liam Urban. *Quiescent Big Bang formation in* $2+1$ *dimensions.* Revised version submitted to *Annales Henri Poincaré* on May 26, 2025. 50 pages. 2024. DOI: 10.48550/arXiv.2412.03396.

[Vla61] Anatoliĭ Aleksandrovich Vlasov. "Many-particle theory and its application to plasma". *Gordon and Breach, New York* (1961).

[Vla68] Anatoliĭ Aleksandrovich Vlasov. "The Vibrational Properties of an Electron Gas". *Soviet Physics Uspekhi* 10.6 (June 1968), p. 721. DOI: 10.1070/PU1968v010n06ABEH003709.

[Wal37] Arthur G. Walker. "On Milne's Theory of World-Structure". *Proceedings of the London Mathematical Society* 2.1 (1937), pp. 90–127. DOI: 10.1112/plms/s2-42.1.90.

[Wan19] Jinhua Wang. "Future stability of the $1+3$ Milne model for the Einstein-Klein-Gordon system". *Classical Quantum Gravity* 36.22 (2019), pp. 225010, 65. ISSN: 0264-9381. DOI: 10.1088/1361-6382/ab4dd3.

[Wan21] Jinhua Wang. "Nonlinear wave equation in a cosmological Kaluza Klein spacetime". *J. Math. Phys.* 62.6 (2021), Paper No. 062504, 15. ISSN: 0022-2488. DOI: 10.1063/5.0046459.

[WE97] John Wainwright and George Ellis. *Dynamical Systems in Cosmology*. Cambridge University Press, 1997. DOI: 10.1017/CBO9780511524660.

[Wea04] Marsha Weaver. "On the area of the symmetry orbits in T^2 symmetric spacetimes with Vlasov matter". *Classical Quantum Gravity* 21.4 (2004), pp. 1079–1097. ISSN: 0264-9381,1361-6382. DOI: 10.1088/0264-9381/21/4/023.

[Wit88] Edward Witten. "2+1 dimensional gravity as an exactly soluble system". *Nuclear Physics B* 311.1 (1988), pp. 46–78. DOI: 10.1016/0550-3213(88)90143-5.