DISSERTATION | DOCTORAL THESIS

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Parametric spectral theory with an eye towards applications to contact geometry

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Abstract

This thesis explores parametric arguments in the spectral theory of the curl and Laplace-Beltrami operator, motivated by problems coming from contact geometry and topology.

Contact topology is the study of the topology of totally non-integrable hyperplane fields, and its importance to low dimensional topology is well known. A characterization of contact forms in dimension 3 as nowhere vanishing solutions to the eigenvalue problem of the curl operator opens up a geometric-analytic approach to the subject. One of the aims of this thesis is to build the spectral theoretic foundation for future investigations of this research direction.

To this end, we prove generic spectral simplicity of the curl operator along 1-parameter families of Riemannian metrics, a result we hope will lead to flexibility results for eigenforms of the curl operator. This project is motivated by the fundamental importance 1-parameter families of contact structures in contact topology.

In joint work with Josef Greilhuber, far reaching generalizations of these arguments are used to resolve a conjecture of Vladimir Arnold on the splitting behaviour of multiple eigenvalues of the Laplace-Beltrami operator under perturbation of the metric. Due to this result we now know the codimension of the space of metrics for which the Laplace-Beltrami operator has at least one eigenvalue of multiplicity m.

The fact that contact forms in dimension three are eigenforms of the curl operator and present a rich class of stationary solutions to the three dimensional Euler equations can be exploited to construct exotic solutions to those equations. Combining input from dynamical systems with a good understanding of the perturbation theory of the curl operator allows one to engineer a stationary solution to the three dimensional Euler equations which is isolated in the C^1 topology. This part of the thesis is joint work with Alberto Enciso and Daniel Peralta-Salas.

Another result detailed in this thesis, again obtained in collaboration with Josef Greilhuber, concerns the spectral geometry of the curl operator on smoothly bounded domains in three dimensional Euclidean space. We show that for generic smoothly bounded domains, the spectrum of the curl operator consists of discrete eigenvalues of multiplicity one. As a crucial step we derive a variational formula for the first-order change of eigenvalues of the curl operator along deformations of the underlying smoothly bounded domain.

Kurzfassung

Die vorliegende Dissertation untersucht parametrische Argumente in der Spektraltheorie des Curl-Operators und des Laplace-Beltrami-Operators, motiviert durch Fragestellungen aus der Kontaktgeometrie und -topologie.

Die Kontakttopologie beschäftigt sich mit der Topologie total nicht-integrabler Hyperflächenfelder und ist von großer Bedeutung für die niedrigdimensionale Topologie. Eine Charakterisierung von Kontaktformen in Dimension 3 als nirgends verschwindende Lösungen des Eigenwertproblems des Curl-Operators eröffnet einen geometrisch-analytischen Zugang zu diesem Gebiet. Eines der Ziele der vorliegenden Arbeit ist es, die spektraltheoretische Grundlage für zukünftige Untersuchungen in diese Richtung zu schaffen.

Zu diesem Zweck beweisen wir die generische spektrale Einfachheit des Curl-Operators entlang eindimensionaler Familien Riemannscher Metriken. Wir erhoffen uns dass dieses Resultat zu Flexibilitätsergebnissen für Eigenformen des Curl-Operators führt. Dieses Projekt ist motiviert durch die fundamentale Rolle die eindimensionale Familien von Kontaktstrukturen in der Kontakttopologie einnehmen.

In gemeinsamer Arbeit mit Josef Greilhuber werden weitreichende Verallgemeinerungen dieser Argumente verwendet um eine Vermutung von Vladimir Arnold über das Aufspaltungsverhalten mehrfacher Eigenwerte des Laplace-Beltrami-Operators unter Störungen der Metrik zu lösen. Durch dieses Ergebnis kennen wir nun die Kodimension des Raums der Metriken, für die der Laplace-Beltrami-Operator mindestens einen Eigenwert der Vielfachheit m besitzt.

Die Tatsache, dass Kontaktformen in Dimension drei Eigenfelder des Curl-Operators und damit eine reiche Klasse stationärer Lösungen der dreidimensionalen Euler-Gleichungen darstellen, kann genutzt werden, um exotische Lösungen dieser Gleichungen zu konstruieren. Die Kombination von Erkenntnissen aus der Theorie dynamischer Systeme mit einem guten Verständnis der Störungstheorie des Curl-Operators ermöglicht es, eine stationäre Lösung der dreidimensionalen Euler-Gleichungen zu konstruieren, die bezüglich der C^1-Topologie isoliert ist. Dieses Resultat wurde in gemeinsamer Arbeit mit Alberto Enciso und Daniel Peralta-Salas erzielt.

Ein weiteres in dieser Dissertation dargestelltes Resultat, ebenfalls in Zusammenarbeit mit Josef Greilhuber erlangt, betrifft die Spektralgeometrie des Curl-Operators auf glatt berandeten Gebieten im dreidimensionalen euklidischen Raum. Wir zeigen, dass für generische glatt berandete Gebiete das Spektrum des Curl-Operators aus diskreten Eigenwerten der Vielfachheit eins besteht. Ein entscheidender Schritt dabei ist die Herleitung einer Variationsformel für die Änderung der Eigenwerte des Curl-Operators in erster Ordnung entlang von Deformationen des zugrunde liegenden glatt berandeten Gebiets.

Contents

1 Introduction

Although a large majority of the results obtained in this thesis are of a spectral geometric nature, the initial and continuing motivation for tackling those problems lies in 3-dimensional contact geometry and topology. In the Introduction we therefore strive to provide a rough overview over (metric) contact geometry and topology and their relation to the spectral theory of the curl operator and therefore the remainder of the thesis.

In Section 1.1 we introduce the basic objects in 3-dimensional contact geometry and topology and state some fundamental theorems.

Section 1.2 is devoted to explaining how one can couple Riemannian metrics to contact forms and structures. We mention some of the (few) landmark results exploiting this connection.

The connection between contact forms, the curl operator, and some aspects of topological fluid mechanics in dimension 3 will be described in Section 1.3. The section concludes with an original derivation giving meaning to the eigenvalue of an eigenform of the curl operator in geometric terms.

In Sections 1.4 we recount a metric topology question motivating the investigation of the spectral theory of the curl operator.

Finally, Section 1.5 provides a brief overview of the main results presented in this thesis.

1.1 Elements of contact geometry and topology in dimension 3

In this section we recall the elementary notions and collect some foundational results in the study of contact topology and geometry on 3-dimensional manifolds. The central objects of inquiry in contact topology and geometry are *totally non-integrable* codimension one subbundles of the tangent bundle and the 1-forms whose kernels define those subbundles, respectively. Both contact forms and contact structures arise naturally in a variety of contexts such as geodesic flow on Riemannian manifolds (or Hamiltonian dynamics more generally), low dimensional topology, fluid dynamics, and thermodynamics.

Definition 1.1.1. *Let M be an oriented 3-manifold. A 1-form α on M is called a* ***positive contact form*** *if it satisfies*

$$\alpha \wedge d\alpha > 0$$

on all of M. The pair (M, α) is then referred to as a ***contact (3-) manifold****.*

The condition $\alpha \wedge d\alpha > 0$ is equivalent to asking that $d\alpha$ restricted to the kernel of α be nondegenerate. Put differently, the kernel of $d\alpha$ is 1-dimensional and thus defines a unique vector field up to normalization.

Definition 1.1.2. *Let (M, α) be a contact 3-manifold. Then the unique vector field R_α satisfying*

$$d\alpha(R_\alpha, -) = 0 \quad \textit{and} \quad \alpha(R_\alpha) = 1$$

*is called the **Reeb vectorfield** associated to α.*

Note that while rescaling a positive contact form α by a nowhere vanishing function results in another positive contact form, it drastically changes the kernel of $d\alpha$ and therefore the Reeb vector field. As for any geometric structure there is a natural notion of structure preserving morphism for contact forms.

Definition 1.1.3. *Let (M, α) and (N, α') be contact 3-manifolds. A (local) diffeomorphism $\psi : M \to N$ is called a **strict contactomorphism** if it satisfies $\psi^*\alpha' = \alpha$.*

It is easy to check that strict contactomorphisms preserve the Reeb vector field.

When we restrict our attention to the kernels of contact forms we end up with a significantly more flexible theory. In the following we will treat coorientable plane fields, that is, plane fields which admit a continuous choice of positive normal vector, only.

Definition 1.1.4. *A **coorientable positive contact structure** on a 3-manifold M is a codimension one subbundle ξ of the tangent bundle TM such that there exists a globally defined positive contact form α satisfying that $\ker(\alpha_p) = \xi_p$. The pair (M, ξ) is referred to as a contact manifold as well and it is distinguished from the notion introduced in Definition 1.1.2 by notational conventions only: α or sometimes λ are usually reserved for contact forms, and ξ is reserved for contact structures.*

The fundamental example of a contact structure is given by $(\mathbb{R}^3, \alpha_{st} = dz + \frac{1}{2}r^2 d\theta)$, where (r, θ, z) are cylindrical coordinates on $\mathbb{R}^3$. For an illustration of this contact structure, see Figure 1.1.

Just as in the case of contact forms there is a natural notion of structure preserving maps.

Definition 1.1.5. *Let (M, ξ) and (N, ξ') be contact 3-manifolds. A (local) diffeomorphism $\psi : M \to N$ is called a **contactomorphism** if it satisfies $\psi_* \ker(\alpha) = \ker(\alpha')$. Equivalently, assuming $\xi = \ker(\alpha)$ and $\xi' = \ker(\alpha')$ for α and α' contact forms, ψ satisfies $\psi^*(\alpha') = f\alpha$ for some nowhere vanishing function f.*

Often one considers an even stronger notion of equivalence between contact structures.

Definition 1.1.6. *Two contact structures ξ and ξ' on a manifold M are called **isotopic** if there exists a smooth 1-parameter family of diffeomorphisms ψ_t such that $\psi_0 = id_M$ and $(\psi_1)_*(\xi) = \xi'$. In particular, contact structures which are isotopic are also contactomorphic.*

It is a classical result that any closed orientable 3-manifold admits a contact structure, see Theorem 1.1.9 below. It is thus natural to ask how many different contact structures exist on any given closed oriented 3-manifold up to contactomorphism or contact isotopy.

The well known Darboux theorem says that locally both contact geometry and contact topology are trivial fields of study.

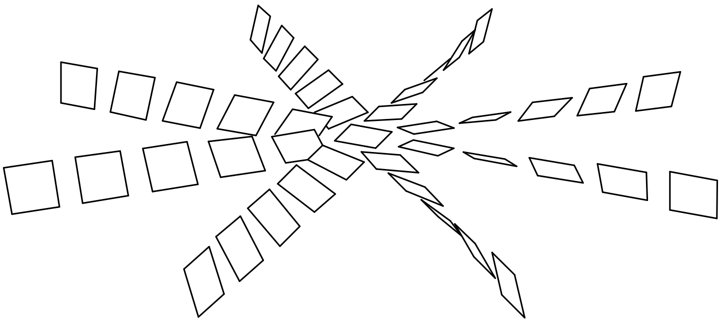

Figure 1.1: A schematic picture of the kernel of $\alpha_{st} = dz + \frac{1}{2}r^2 d\theta$.

Theorem 1.1.7 (Darboux)**.** *[Gei08, Theorem 2.5.1] Let (M, α) be a contact 3-manifold. Then, for all $p \in M$, there exits a neighbourhood U_p containing p, a neighbourhood V_p of 0 in $(\mathbb{R}^3, \alpha_{st})$, and a strict contactomorphism*

$$\psi_p : (U_p, \alpha) \to (V_p, \alpha_{st})$$

such that $\psi_p(p) = 0 \in \mathbb{R}^3$.

Oftentimes the Darboux theorem is summarized as saying that in contact geometry and topology, unlike for example in Riemannian geometry, there are no local invariants. This is the first indication of the remarkable flexibility of the theory, another even more striking one is the so-called Gray stability theorem.

Theorem 1.1.8 (Gray)**.** *[Gei08, Theorem 2.2.2] Let M be closed manifold and let ξ_t be a 1-parameter family of contact structures on M. Then there exists a 1-parameter family of diffeomorphisms $\psi_t : M \to M$ such that $\xi_0 = \psi_t^* \xi_t$*

The upshot of the Gray stability theorem is that 1-parameter families of contact structures preserve the topological types of contact structures.

It is important to note that there is no analogue to this theorem for contact forms ([Gei08, Example 2.2.5]), i.e., 1-parameter families of contact forms are not induced by 1-parameter families of strict contactomorphisms.

At this point we have not yet seen that there is more than one contactomorphism or contact isotopy type of contact structures on any closed oriented 3-manifold. The first such example was found on $\mathbb{R}^3 = \mathbb{S}^3 \setminus \{\text{pt.}\}$: $(\mathbb{R}^3, \alpha_{ot} = \cos(r)dz + r\sin(r)d\theta)$ was

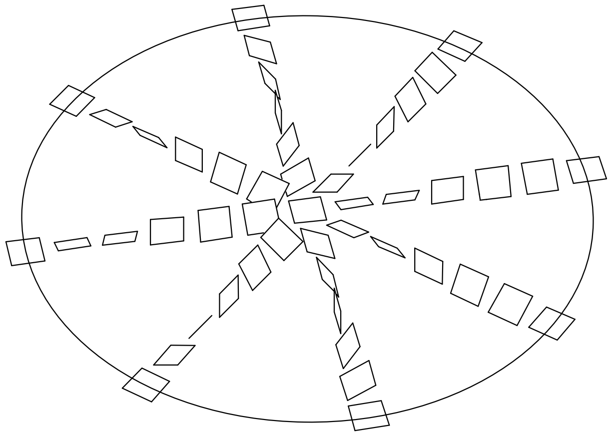

Figure 1.2: A sketch illustrating the kernel of $\alpha_{ot} = \cos(r)dz + r\sin(r)d\theta$ as well as of an overtwisted disk contained in it.

shown not to be contactomorphic to $(\mathbb{R}^3, \alpha_{st})$ by Bennequin in [Ben83], see Figure 1.2 for an illustration of this contact structure. The crucial distinction to $(\mathbb{R}^3, \alpha_{st})$ is that there exists an embedded disk, for example $\mathbb{D} = \{r^2 \leq \pi, z = 0\}$, which has the property that $\xi_{ot} = \ker(\alpha_{ot})$ is tangent to $\mathbb{D}$ along its boundary $\partial\mathbb{D}$. Such a disk is called an **overtwisted disk** and Bennequin proved that $(\mathbb{R}^3, \alpha_{st})$ does not contain any overtwisted disk. While this is not a proof of this fact, we see that the planes defined by α_{st} never quite become tangent the $\{z = 0\}$ plane. This discovery of Bennequin is often seen as the inception of the study of contact topology.

Ever since a plethora of distinct contactomorphism and isotopy classes of contact structures have been found. Indeed, on any closed orientable 3-manifold there always exist countably many distinct isotopy classes of contact structures.

Theorem 1.1.9. *[Gei08, Theorem 4.3.1 (Lutz-Martinet)] Every cooriented plane field on a closed orientable* 3*-manifold is homotopic to a contact structure.*

More generally, there is a fundamental dichotomy in the topology of contact structures of closed 3-manifolds. On the one hand there are contact structures ξ which contain an open ball that is contactomorphic to $(\mathbb{R}^3, \alpha_{ot} = \cos(r)dz + r\sin(r)d\theta)$, which are called **overtwisted**, and those that do not contain any such open ball, termed **tight**. Crucially, tightness respectively overtwistedness are preserved by contactomorphism. Any closed 3-manifold admits countably infinitely many different overtwisted contact structures, but not every closed 3-manifold admits a tight contact structure [EH01]. Indeed, a famous theorem of Eliashberg [Eli89] implies that isotopy classes of overtwisted contact structures are in one-to-one correspondence with homotopy classes of plane fields, so in a strong sense overtwisted contact structures on a manifold M are classified by the algebraic topology of

M. Contrary to this, tight contact structures, when they exist, carry significantly more fine grained topological information about the manifold M, for example about the genus of knots in M [Gei08, Theorem 4.6.34],[Ben83] or whether a given embedded sphere in M is contractible [Hof93, Theorem 8].

As mentioned before, not every closed oriented 3-manifold admits a tight contact structure, and even when they admit one there are only finitely many different isotopy classes of tight contact structures up to Giroux torsion [CGH03], a construction changing the contact structure in the neighbourhood of an incompressible torus. There are several classification results where a complete list of isotopy classes of tight contact structures has been obtained for particular closed oriented 3-manifolds, but the general question of which closed oriented 3-manifolds exactly admit a tight contact structure remains wide open. In particular, the case of the in some sense richest class of closed oriented 3-manifolds, that of hyperbolic manifolds, remains mysterious.

1.2 Compatibility of contact forms with Riemannian metrics

As the crucial distinction between tight and overtwisted contact structures is how much they twist, it makes sense to try to couple contact structures to Riemannian metrics in order to get a quantitative measure of twisting. In particular, since it is the hyperbolic manifolds for which we would like to understand whether they admit tight contact structures or not, the hope is that one can use the existence of the hyperbolic metric in order to construct appropriately compatible contact forms, and in that way prove the existence of tight contact structures on those manifolds. This approach splits into two different problems: First, one has to find a good notion of compatibility between a contact structure and a Riemannian metric and prove that one can find criteria on a compatible Riemannian metric which allows one to draw conclusions about the contact topology of the plane field. This is the topic of the current section. The second part of the problem is to actually construct pairs of contact structures and Riemannian metrics which are compatible, and at this moment in time there is no known way of doing this in general. One motivation for analyzing the curl operator on Riemannian 3-manifolds is to do just that, and we will say more about this issue in the following sections.

We will repeat here the account of instantaneous rotation as introduced in [EKM12a].

Given a cooriented plane field ξ on a Riemannian 3-manifold (M, g), for any point $p \in M$ we may choose a local orthonormal frame (u, v) for ξ in a neighbourhood of p. Denote by n the positive unit normal to ξ, and by Φ_t the flow generated by the locally defined vector field u. As all our arguments will be local in nature we may extend u to all of M by a bump function in order to obtain a family of diffeomorphisms $\Phi_t : M \to M$. Consider the angle between $\Phi_t^* v$ and n at p, i.e.,

$$\cos(\theta(t)) = \frac{g(\Phi_t^* v, n)}{\|\Phi_t^* v\|}$$

Differentiating this expression and evaluating at $t = 0$ gives the so-called instantaneous rotation $\lambda_\xi(p)$ of ξ at p

$$\lambda_\xi(p) := \theta'(0) = -g([u,v],n)$$

It is easy to check that $\lambda_\xi(p)$ only depends on the metric g and on the plane field ξ, but not on the particular choice of orthonormal frame (u,v). Crucially, ξ is a positive contact structure iff $\lambda_\xi > 0$. The instantaneous rotation is a measure of the rotation of the plane field ξ, however we would like to stress that it is nondirectional and its value at a point depends on the behaviour of ξ in an open neighbourhood of that point. In particular, it is difficult to relate λ_ξ to global properties of ξ.

Another natural notion of twisting is that of the second fundamental form of ξ

$$h_\xi(u,v) = \frac{1}{2}g(\nabla_u v + \nabla_v u, n)$$

which has a similar interpretation as the second fundamental form for submanifolds and measures the extrinsic curvature of the plane field ξ, that is, the extent to which geodesics curve away from the plane field distribution. For future use we define the square norm of the second fundamental form $\|h_\xi\|^2 = h_{\xi\,ij} h_\xi^{ij} = \sum_{i=1}^2 \|\nabla_{e_i} n\|^2$ where e_1 and e_2 are an orthonormal basis for ξ.

One way of relating the geometry of a Riemannian metric to the geometry of a contact form and the topology of its underlying plane field is to restrict to a special subclass of metrics.

Definition 1.2.1. *A pair (g,α) is called **weakly compatible** if $R_\alpha \perp_g \ker(\alpha)$. Note that this implies that $n = \frac{R_\alpha}{\|R_\alpha\|_g}$. The pair is called **compatible** if furthermore $\|R_\alpha\|_g = const.$ and $\lambda_\xi = const.$ One refers to the triple (M,g,α) where (g,α) are weakly compatible as a **metric contact manifold** and to the study of these objects as **metric contact geometry** or **metric contact topology**.*

We would like to point out that the particular constants in a compatible pair are for many purposes not that important and some authors have normalized them to $\rho = 1$ and $\lambda = 2$ (e.g. [CH85]), and we will do the same in Chapter 4.

Given a contact form α it is fairly immediate that the set of (weakly) compatible Riemannian metrics is quite large and can be characterized in a convenient way:

Proposition 1.2.2. *[EKM12a, Proposition 2.1.] Let α be a positive contact form on a 3-manifold M and g be a metric on M. We set $\rho = \|R_\alpha\|_g$ and $\xi = \ker(\alpha)$. The following properties are equivalent:*

1. *The pair (g,α) is weakly compatible.*
2. *There is some positive function λ_ξ such that*

$$\star_g d\alpha = \lambda_\xi \alpha,$$

where $\star_g$ is the Hodge star operator associated to g. If this is true then λ_ξ is the instantaneous rotation of ξ with respect to g.

3. *There is an almost complex structure J on ξ that is compatible with $d\alpha|_\xi$, (that is, $d\alpha(Ju, Jv) = d\alpha(u, v)$ for all $u, v \in \xi$ and $d\alpha(v, Jv) > 0$ for all $v \neq 0$ in ξ) such that, using the extension φ of J to TM sending R_α to 0, we have:*
$$g(u, v) = \frac{\rho}{\lambda_\xi} d\alpha(u, \varphi(v)) + \rho^2 \alpha(u)\alpha(v),$$

While Proposition 1.2.2 is elementary it is incredibly convenient. It connects the study of contact forms to eigenforms of the curl operator $\star_g d$ which we will explore further in Section 1.3. Another consequence is that it gives a convenient way of doing perturbation theory for the curl operator *while preserving a given eigenform*, an idea we will rely on in Chapter 5.

This notion of compatibility turns out to be a good one in the sense that the Riemannian geometry of a (weakly) compatible metric captures at least some of the contact geometry of α. This is exemplified by (but not limited to) the following lemma.

Lemma 1.2.3. *[EKM12a, Lemma 2.3.] Let g be a metric that is weakly compatible with a contact form α on the 3-manifold M. Set*
$$n = \frac{1}{\rho} R_\alpha$$
where $\rho = \|R_\alpha\|$ is the norm of the Reeb vector field of α. Then
$$\nabla_n n = -(\nabla \ln \rho)_\xi,$$
where v_ξ denotes the projection of v to ξ along R_α.

Furthermore, we can express the mean curvature of $\ker\alpha$ *as*
$$H = \frac{1}{2} \operatorname{tr} h = -\frac{1}{2} \operatorname{div}_g n = -\frac{1}{2} n \cdot \left(\ln \frac{\rho}{\lambda_\xi}\right).$$
In particular, if (g, α) is a compatible pair, the Reeb field is a volume preserving geodesic vector field.

Lemma 1.2.3 hints at the possibility of connecting the topology of $\ker(\alpha)$ to curvature properties of g. For example, we can use the following Theorem due to Hofer.

Theorem 1.2.4. *[Hof93, Theorem 1] Let (M, α) be a closed contact* 3*-manifold. If* $\ker(\alpha)$ *is overtwisted, then the Reeb vector field R_α has an unknotted, contractible closed orbit.*

With Theorem 1.2.4 in hand we can immediately prove

Proposition 1.2.5. *[EKM12a, Corollary 1.4.] Let (M, g) be a flat Riemannian* 3*-manifold. Any contact form α which is compatible with g defines a tight contact structure.*

Proof. Any flat 3-manifold is universally covered by $\mathbb{R}^3$ equipped with the euclidean metric g_{euc}. Suppose now that $\ker(\alpha)$ is overtwisted. Theorem 1.2.4 implies that there exists a contractible closed Reeb orbit γ, which, by compatibility with the flat metric g, is a contractible closed geodesic on M. This geodesic is then pulled back to a geodesic on $(\mathbb{R}^3, g_{euc})$, so to a straight line. However, since γ is contractible, it is lifted to a contractible closed loop, a contradiction. □

An example of a contact form compatible with a flat metric is given by

$$\alpha = \cos(z)\, dx + \sin(z)\, dy$$

on $\mathbb{T}^3$ equipped with the standard metric $dx \otimes dx + dy \otimes dy + dz \otimes dz$. The proof of the proposition used very little about the Riemannian metric and in principle one could immediately generalize to metrics with nonpositive curvature. However, it is an interesting fact ([Zeg95, Theorem F]) that no closed Riemannian 3-manifold with strictly negative sectional curvature can be foliated by geodesics, so in particular there are no compatible pairs (α, g) where g has strictly negative sectional curvatures. It is unclear whether there exist compatible metrics which are nonnegatively curved but not already flat.

This means that there are two natural ways of continuing the investigation of the interplay of the sectional curvature of Riemannian metrics and contact structures: One can either focus on compatible pairs in positive sectional curvature or relax the degree of compatibility and look at weakly compatible pairs with more general sectional curvature conditions.

As far as the second choice is concerned, if the modulus of compatibility is not too bad one can get good estimates on the so-called tightness radius of a contact manifold.

Definition 1.2.6. *Let (M, ξ) be a 3-dimensional contact manifold and g a Riemannian metric on M. Given a point $p \in M$, the* ***tightness radius*** *$\tau_p(M, \xi)$ of (M, ξ) at p is*

$$\tau_p(M, \xi) = \sup\{r \mid \text{the contact structure on the geodesic ball } B_p(r) \text{ at } p \text{ of radius } r \text{ is tight}\}$$

The ***(global) tightness radius*** *of (M, ξ) then is*

$$\tau(M, \xi) = \inf\{\tau_p(M, \xi) \mid p \in M\}$$

Lemma 1.2.3 indicates the crucial quantity which one should use in order to measure the modulus of compatibility:

$$D_g := \nabla_n n + 2Hn$$

Note that $\nabla_n n \perp 2Hn$, and that D_g is zero whenever (g, α) is a compatible pair. Taking the supremum gives the quantity

$$d_g := \sup\{\|D_g\|(p) \mid p \in M\}$$

The constant d_g carries a surprising amount of information about the contact structure ξ. It can be used to give a quantitative lower bound for the tightness radius [EKM12a, Theorem 1.5] and to give a tightness criterion for a weakly compatible pair (g, α) in nonpositive curvature.

Theorem 1.2.7. *[EKM12a, Theorem 1.9] Let (M, α) be a 3-dimensional contact manifold and suppose that g is a complete Riemannian metric that is weakly compatible with α. If*

$$sec(g) \leq -d_g^2$$

for all $p \in M$ then $\ker(\alpha)$ is a tight contact structure.

In the case of compatible pairs in positive curvature, the most general known result is the so-called contact sphere theorem.

Theorem 1.2.8. *[EKM12a, Theorem 1.1.], [GH16a, Theorem 0.1.] Let (M, α) be a 3-dimensional contact manifold so that α is compatible with a Riemannian metric g whose sectional curvatures satisfy*

$$0 < \frac{1}{4} < sec(g) \leq 1$$

Then the universal cover of M is S^3 and the pullback of $\ker(\alpha)$ to S^3 is tight.

The main ingredients for proving Theorem 1.2.8 are a good understanding of the characteristic foliations of spheres of constant radius $S_r(p)$ centered at a point p, the insight that convexity of these spheres of constant radius implies Levi-pseudoconvexity of the cylinder $\mathbb{R} \times S_r(p)$ in the symplectization of (M, α), Theorem 1.2.4, and the standard arguments in Riemannian geometry needed for the "Riemannian" sphere theorem. It should be noted that only very crude information about J-holomorphic curves is used in order to prove Theorem 1.2.8, namely that J-holomorphic curves cannot leave the subsets $\mathbb{R} \times B_r(p)$ in the symplectization for any r less than the convexity radius of the Riemannian manifold (M, g).

All results relating curvature conditions to the topology of compatible contact forms mentioned thus far were formulated in terms of sectional curvature, and indeed there are not many results which go beyond sectional curvature. This is because most proofs rest on Theorem 1.2.4: One assumes that the contact structure is overtwisted and then uses the sectional curvature conditions and their implications for the behaviour of geodesics (and therefore the Reeb vector field) to derive a contradiction. An exception is [Hoz20] in which Surena Hozoori uses a mixture of sectional and Ricci curvature conditions on the compatible metric in order to obtain constraints on the Conley-Zehnder index of closed flowlines of the Reeb vector field, thus establishing tightness using dynamical properties of the Reeb flow. Another notable example is a result of Patrick Massot [Mas08, Theorem A] in which he proves that a contact structure ξ on a closed Seifert 3-manifold for which there exists a metric g (not necessarily compatible with any contact form defining ξ) so that $\|h_\xi\|^2$ vanishes is tight.

1.3 Connection to the spectral theory of the curl operator and topological hydrodynamics

A large body of research in symplectic dynamics is dedicated to the study of properties of Reeb vector fields. When trying to understand the class of vector fields which are Reeb vector fields for some contact form, it is useful to introduce the following concept.

Definition 1.3.1. *Let (M, g) be an oriented Riemannian 3-manifold. The* ***curl operator*** *on 1-forms is defined by $\star_g d$ where $\star_g$ is the Hodge star operator associated to g.*

The isomorphism $\sharp_g : TM \to T^*M$ induced by a Riemannian metric g via $X \mapsto g(X, -)$ and its inverse $\flat_g$ then give a natural way of defining the curl operator on vector fields. Explicitly, $\nabla \times_g X := (\sharp_g \circ \star_g \circ d \circ \flat_g)X$.

Definition 1.3.2. *A vector field which is parallel to its own curl, that is, one which satisfies $\nabla \times_g X = fX$, for some function f, is called a* ***Beltrami field****.*

A Beltrami field is called ***rotational*** *if the proportionality factor f is nowhere vanishing, and it is called a* ***curl eigenfield*** *if $f = const. \neq 0$*

The following lemma shows that Beltrami fields are a natural and particularly interesting class of objects from the point of view of fluid dynamics. For this, recall that the **incompressible Euler equations** on a Riemannian manifold (M, g) are given by

$$\begin{cases} \partial_t u + \nabla_u u = -\nabla p \\ \operatorname{div}_g u = 0 \end{cases} \tag{1.3.1}$$

where u is a time dependent vector field, p is a time-dependent function, and ∇ denotes the covariant derivative with respect to the Levi-Civita connection associated to g. A vector field u solving Equation 1.3.1 is said to be a **stationary solution** of the incompressible Euler equations if it is independent of time, that is, if $\partial_t u = 0$. Furthermore, we say that a function $f \in C^1$ is a C^1 **first integral** if $df(u) = g(\nabla f, u) = 0$. We want to point out that this condition implies that u is tangent to the (regular) level sets of f.

Lemma 1.3.3. *Let u be a stationary solution to the incompressible Euler equations on a 3-dimensional Riemannian manifold (M, g). If u does not have any C^1 first integrals then u is a curl eigenfield.*

Proof. Using the 3-dimensional vector calculus identity

$$\nabla_u u = u \times_g (\nabla \times_g u) - \frac{1}{2}\|u\|^2$$

the stationary incompressible Euler equations reduce to

$$\begin{cases} u \times_g (\nabla \times_g u) = -\nabla F \\ \operatorname{div}_g u = 0 \end{cases}$$

where $F = p - \frac{1}{2}\|u\|^2$. Taking the inner product of the first equality with u implies that F is a first integral of u, so by assumption F is constant. The system is then easily seen to be equivalent to

$$\begin{cases} \nabla \times_g u = f\, u \\ \operatorname{div}_g u = 0 \end{cases}$$

for some smooth function f. Taking the divergence of the first equation and using the incompressibility of u then means that f is a first integral of u, so once again it is constant. As any curl eigenfield is automatically incompressible, we are done. $\square$

A way of interpreting Lemma 1.3.3 is that those stationary solutions of Equation 1.3.1 which are in some sense very wild have to be curl eigenfields. An example of such wild behaviour of a stationary solution that forces it to be a Beltrami field is the existence of an orbit which is dense in M.

We will now see that nowhere vanishing Beltrami fields are closely connected to Reeb vector fields.

Definition 1.3.4. *A nowhere vanishing vector field X is called* ***Reeb-like*** *if it satisfies $d\alpha(X, -) = 0$ and $\alpha(X) > 0$ for some contact form α, that is, if there exists a positive function f so that $X = f\, R_\alpha$ for some contact form α.*

The following Theorem, proven by Etnyre and Ghrist in [EG00b], posits that in the class of nowhere vanishing vector fields, those vector fields that are Reeb vector fields for some positive contact form are exactly those that are Beltrami fields for some positive proportionality factor, up to rescaling.

Theorem 1.3.5. *[EG00b] Let M be an oriented Riemannian three-manifold. Any nowhere vanishing rotational Beltrami field on M is Reeb-like for some contact form on M. Conversely, given a contact form α on M with Reeb vector field R_α, any nonzero rescaling of R_α is a rotational Beltrami field for some Riemannian metric on M.*

An important consequence of Theorem 1.3.5 is that dynamical properties of nowhere vanishing rotational Beltrami fields can be analyzed with tools from contact geometry. An immediate application of this observation and of the resolution of the Weinstein conjecture by Clifford Taubes [Tau07, Theorem 1.1] is that the flow induced by a nowhere vanishing rotational Beltrami field on a closed oriented 3-manifold always contains a closed orbit.

Also note that by Proposition 1.2.2 the class of nowhere vanishing rotational Beltrami fields coincides with the class of nowhere vanishing curl eigenfields since one can simply adjust the compatible metric so that the instantaneous rotation becomes constant. We will thus focus on eigenforms of the curl operator. Consider the curl operator defined on the Hilbert space $L^2(\Omega^1(M))$

$$\star_g d : L^2(\Omega^1(M)) \to L^2(\Omega^1(M))$$

It is densely defined, self-adjoint and elliptic in the orthogonal complement of its kernel $\mathcal{K}_g = \{\eta \in \Omega^1(M) : \, d\eta = 0\}$ (see for example [Bär19]). Moreover, once restricted to an orthogonal complement $\mathcal{H}_g = \mathcal{K}_g^\perp$ of its kernel, the curl operator has compact resolvent. Taken together, these facts imply that the spectrum of curl on $\mathcal{H}_g$ consists of a discrete set of real eigenvalues with finite multiplicity. All associated eigenforms are smooth provided g is a smooth metric and they satisfy

$$\alpha \wedge d\alpha = \lambda \|\alpha\|^2 \, dvol_g$$

where λ is the curl eigenvalue of α. In other words, eigenforms are contact forms in the complement of their zero set. Work of Alberto Enciso and Daniel Peralta-Salas [EP12b] implies that for generic choice of metric g, all eigenvalues have multiplicity one and the associated eigenforms have transverse zeros only.

For a brief introduction to the physical relevance of curl eigenfields see Chapter 5, Section 1. From a contact topological point of view their relevance is the following. In dimension 3, geometry is still a powerful guiding force and organizing principle for the classification of closed manifolds. All closed oriented 3 manifolds are built from pieces which are naturally associated to a certain Riemannian geometry (for an overview see for example [Mar16, Chapter 12]). As mentioned in Section 1.2, the hope is now that eigenforms of the curl operator may provide a way of building contact forms which are (weakly) compatible with these natural geometries. The reason this is particularly interesting is that the traditional ways of constructing contact structures on 3-manifolds (surgeries, branched coverings) do not in any way reflect the natural geometry of those manifolds. The most serious problem with that approach is that it seems to be very difficult to find criteria which ensure that a given curl eigenform does not have a zero and thus defines a contact structure. Even if this issue could be solved, it is not easy to determine whether the eigenform represents a tight or an overtwisted contact structure.

Low eigenvalue eigenforms, i.e., those that "twist the least", might be the most reasonable candidates when looking for tight contact structures. On (S^3, g_{st}) this intuition appears to be true.

The spectrum of the curl operator on the round sphere consists of eigenvalues $\pm(k+2)$ where k is a nonnegative integer. By symmetry it suffices to concentrate on the positive eigenvalues. The lowest eigenvalues $\lambda = 2$ has multiplicity 3 and the corresponding eigenforms are given by the Hopf fields, whose kernels define the unique tight contact structure on S^3. Moreover, none of these eigenfields have zeros. For the higher eigenvalues, Daniel Peralta-Salas and Radu Slobodeanu proved

Theorem 1.3.6. *[PS21, Theorem 2] Any nonvanishing curl eigenfield on $\mathbb{S}^3$ has even eigenvalue $\lambda = 2m$, where $m \in \{1, 2, \dots\}$. Moreover, for each $m \geq 2$ there exists a nonvanishing curl eigenfield V_m whose associated contact structure is overtwisted.*

It is however not known whether all overtwisted contact structures are realized as a curl eigenform on (S^3, g_{st}) or whether there exists a curl eigenform which represents the tight contact structures when $\lambda > 2$ on (S^3, g_{st}).

Geometric interpretation of the curl eigenvalue

We would like to offer some geometric explanation of the eigenvalue of a curl eigenform.

It is reasonable to expect that for higher eigenvalues we get more wildly oscillating eigenforms, and, indeed, for T^3 with the standard metric this has been proven to be the case at least for random eigenforms by Alberto Enciso, Daniel Peralta-Salas, and Alvaro Romaniega in [EPR23a]. For general Riemannian manifolds the Bochner formula (see, e.g., [Li12, Chapter 3]) indicates that higher eigenvalues may lead to wilder oscillations of the corresponding eigenform. Note that any 1-form satisfying $\star_g d\alpha = \lambda\alpha$ also satisfies $\Delta_g^1\alpha = \lambda^2\alpha$ where $\Delta_g^1 = \star_g d \star_g d + d \star_g d \star_g$ is the Hodge Laplacian on 1-forms in dimension 3, so the Bochner formula for the Laplacian with negative eigenvalues $(-\Delta_g)$, specialized to the curl eigenform α, reads

$$(-\Delta_g)\frac{1}{2}\|\alpha\|^2 = -\lambda^2\|\alpha\|^2 + \|\nabla\alpha\|^2 + Ric(\sharp_g\alpha, \sharp_g\alpha) \tag{1.3.2}$$

In the following Lemma we will analyze the term $\|\nabla\alpha\|^2$.

Lemma 1.3.7. *Let α be a curl eigenform. Then, whenever $\alpha \neq 0$, we have that*

$$\|\nabla\alpha\|^2 = \|\alpha\|^2\Big(\|D_g\|^2 + \|\nabla_n n\|^2 + \|h_\xi\|^2\Big)$$

In particular, the inequalities

$$\|D_g\|^2 + \|h_\xi\|^2 \leq \frac{\|\nabla\alpha\|^2}{\|\alpha\|^2} \leq 2\|D_g\|^2 + \|h_\xi\|^2$$

hold in the complement of the zero set of α.

Proof. First observe that $\alpha = \|\alpha\|\, g(n, -)$, and recall from Lemma 1.2.3 that $H = -\frac{1}{2}\, n \cdot \ln(\rho)$ and $\nabla_n n = (\nabla \ln(\rho))_\xi$. Then, given a local orthonormal frame $\{e_1 = n, e_2, e_3\}$, we compute

$$\begin{aligned}
\|\nabla\alpha\|^2 &= \sum_{i=1}^{3}\|\nabla_{e_i}\alpha\|^2 = \|\nabla_n\alpha\|^2 + \sum_{i=2}^{3}\langle\nabla_{e_i}\alpha, \nabla_{e_i}\alpha\rangle \\
&= (\nabla_n\|\alpha\|)^2 + \|\alpha\|^2\|\nabla_n n\|^2 + \sum_{i=2}^{3}\Big((\nabla_{e_i}\|\alpha\|)^2 + \|\alpha\|^2\|\nabla_{e_i} n\|^2\Big) \\
&= (\nabla_n\|R_\alpha\|^{-1})^2 + \sum_{i=2}^{3}(\nabla_{e_i}\|R_\alpha\|^{-1})^2 + \|\alpha\|^2\big(\|\nabla_n n\|^2 + \|h_\xi\|^2\big) \\
&= \|\alpha\|^2(\nabla_n \ln(\rho))^2 + \|\alpha\|^2\sum_{i=2}^{3}(\nabla_{e_i}\ln(\rho))^2 + \|\alpha\|^2\big(\|\nabla_n n\|^2 + \|h_\xi\|^2\big) \\
&= \|\alpha\|^2(\nabla_n \ln(\rho))^2 + \|\alpha\|^2\sum_{i=2}^{3}(\nabla_{e_i}\ln(\rho))^2 + \|\alpha\|^2\big(\|\nabla_n n\|^2 + \|h_\xi\|^2\big) \\
&= \|\alpha\|^2\big((\nabla_n \ln(\rho))^2 + \|(\nabla\ln(\rho))_\xi\|^2 + \|\nabla_n n\|^2 + \|h_\xi\|^2\big) \\
&= \|\alpha\|^2\big(4H^2 + 2\|\nabla_n n\|^2 + \|h_\xi\|^2\big) \\
&= \|\alpha\|^2\big(\|D_g\|^2 + \|\nabla_n n\|^2 + \|h_\xi\|^2\big)
\end{aligned}$$

The final estimate follows easily from the definition of D_g. □

Remark 1.3.8. *Lemma 1.3.7 gives some interesting insight into nondegenerate zeros of a curl eigenform. A zero p of a 1-form α is called nondegenerate if the map*

$$(\nabla\alpha)(p) : T_pM \to T_p^*M$$

is a linear isomorphism. If α is a curl eigenform then $d\alpha$ vanishes at p as well, implying that $\sharp_g \circ (\nabla\alpha)(p)$ is a symmetric isomorphism of T_pM. Nondegeneracy thus gives the following strict inequality near p

$$0 < \|\sharp_g \circ \nabla\alpha\|^2 = \|\nabla\alpha\|^2 \leq \|\alpha\|^2\big(2\|D_g\|^2 + \|h_\xi\|^2\big)$$

In particular, this inequality holds at p, meaning that, as one approaches, the zero of α, either $\|D_g\|$ or $\|h_\xi\|$ or both blow up. This is consistent with the intuition that there seem to be many overtwisted disks around nondegenerate zeros of curl eigenfields. A precise local model for the contact topology of a neighbourhood of a transverse zero of a curl eigenfield is the subject of future work.

The main content of the next Proposition is to produce a quantity which captures some of the metric contact geometry associated to a (nowhere vanishing) curl eigenform and that is bounded from below by the eigenvalue of that eigenform. For this we take the Bochner formula 1.3.2, evaluate it at a minimum of $\|\alpha\|^2$, use Lemma 1.3.7, and roughly estimate away the Ricci curvature expression.

Proposition 1.3.9. *Let (M, g) be a closed oriented Riemannian 3-manifold, let α be a nowhere vanishing 1-from satisfying $\star_g d\alpha = \lambda\alpha$ for $\lambda \in \mathbb{R} \setminus \{0\}$ and suppose that k and K are lower and upper bounds on the eigenvalues of the Ricci curvature associated to g, respectively. Then there holds*

$$\inf_{p\in M}\|D_g\|^2 + \inf_{p\in M}\|h_\xi\|^2 + k \leq \lambda^2 \leq 2d_g^2 + \sup_{p\in M}\|h_\xi\|^2 + K$$

If the pair (g, α) is compatible we even have that

$$\lambda^2 = \|h_\xi\|^2(p) + Ric(n, n)(p) \qquad \forall p \in M$$

It should be noted that given a contact form α, the space of metrics for which (g, α) is a compatible pair and so that $\ker(\alpha)$ has instantaneous rotation λ is quite large. It is parametrized by the space of almost complex structures on $\ker(\alpha)$. In spite of this, the value of the sum $\|h_\xi\|^2(p) + Ric(n, n)(p)$, a combined measure of the ambient curvature of g and the extrinsic curvature of $\ker(\alpha)$, is constant on that space.

Proof. Let $p_0 \in M$ be a minimum of $\|\alpha\|^2$ which is nonzero by assumption. Then

$$\begin{aligned}
0 &\leq (-\Delta_g)\frac{1}{2}\|\alpha\|^2(p_0) \\
&= -\lambda^2\|\alpha\|^2(p_0) + \|\nabla\alpha\|^2(p_0) + Ric(\sharp_g\alpha, \sharp_g\alpha)(p_0) \\
&\leq \|\alpha\|^2(p_0)(-\lambda^2 + 2\|D_g\|^2(p_0) + \|h_\xi\|^2(p_0) + Ric(n, n)(p_0)) \\
&\leq \|\alpha\|^2(p_0)(-\lambda^2 + 2d_g^2 + \sup_{p\in M}\|h_\xi\|^2 + K)
\end{aligned}$$

Dividing by $\|\alpha\|^2(p_0)$ gives one of the inequalities. The other one follows analogously by taking the Laplacian of $\|\alpha\|^2$ at a maximum. In the compatible case both $(-\Delta_g)\|\alpha\|^2 = 0$ and $D_g = 0$, and so the result follows. □

Notice that for a qualitative interpretation of large curl eigenvalues, the constants K and k do not matter. As a consequence of Proposition 1.3.9 we know that, as the eigenvalue goes to infinity, either the (supremum of the) modulus of compatibility or the (supremum of the) extrinsic curvature of the plane field $\ker(\alpha)$ have to go to infinity as well. In other words, for high eigenvalues, there will always be an open subset of the metric contact manifold (M, g, α) for which either g and α fail to be nicely compatible (when d_g is large) or for which $\ker(\alpha)$ curves very strongly with respect to the ambient geometry (when $\|h\|^2$ is large). The first of these two statements has to do with the contact geometry of α while the second one relates to the contact topology of $\ker(\alpha)$.

Conversely, the eigenvalue λ can be estimated from below by the extrinsic curvature of $\ker(\alpha)$. These considerations may also mean that, given a Riemannian metric g, any compatible contact form tends to be a low eigenvalue curl eigenform. At least for (S^3, g_{st}), this is true ([PS21, Proposition 1]).

1.4 A topological question motivating this thesis

In this section we discuss a natural question which was a driving force in the formation of this thesis. For most of the content of the present document the spectral geometric or fluid dynamical motivations are clear, but we would like to briefly highlight a topological question in the background of many of the perturbation theoretic results which may be less immediately obvious. Of course, there are many more topological questions which motivated this thesis, but the one we choose to present here most clearly connects to the spectral theory developed in the thesis.

The fact that the eigenforms associated to the lowest curl eigenvalue on (S^3, g_{st}) define tight contact structures as well as Theorem 1.2.8 give rise to somewhat optimistic

Question 1.4.1. *Does the lowest eigenvalue curl eigenform on S^3 always define the unique tight contact structure?*

One idea to make progress on Question 1.4.1 is to pick another metric g' on S^3 and to consider a 1-parameter family of metrics g_t so that $g_0 = g'$ and $g_1 = g_{st}$, and to hope that one can solve the eigenvalue problem

$$\star_{g_t} d\alpha_t = \lambda_t \alpha_t$$

for all $t \in [0, 1]$ in such a way that α_t depends continuously on t. (Note that we are ignoring the subtlety that the lowest eigenvalue of the curl operator on (S^3, g_{st}) has multiplicity greater than one). If one can ensure that there exists a path g_t so that the first eigenvalue has multiplicity 1 for all $t \in (0, 1)$ then the resulting 1-parameter family α_t is continuous in t. Furthermore, if there are no zeros of α_t for any value of t, the Gray stability theorem 1.1.8 would imply that α_0 and α_1 define the same isotopy class of contact structures. Question 1.4.1 motivated the work done in Chapter 2 in which we show that the spectrum of the curl operator is simple along generic 1-parameter families of metrics. As for the appearance of zeros, it is entirely unclear (as far as we are aware) under which conditions lowest eigenvalue curl eigenforms can have zeros.

Question 1.4.1 is very far from being answered, but there are a couple of related questions for which the perturbative methods developed in this thesis might be more immediately helpful.

Question 1.4.2. *How can the contact topology of a curl eigenform change along a* 1*-parameter family of metrics (without eigenvalue crossings)? Put differently, is the contact topological type of the k^{th} curl eigenform an invariant associated to the k^{th} curl eigenvalue functional on the space of Riemannian metrics?*

Question 1.4.3. *Can two "essentially different" contact structures be in the same curl eigenspace for some metric?*

These two questions are related to one another: Let α and β be contact forms on a closed oriented 3-manifold M. Suppose that there exists a metric g such that both α and β are curl eigenforms for the k^{th} curl eigenvalue $\lambda_k(g)$. Assuming for simplicity that the multiplicity of the eigenvalue λ is two and that β is L^2-orthogonal to α, we can use Lemma 4.4.4 in Chapter 4 to perturb the metric g and break up the eigenvalue $\lambda_k(g)$ into two eigenvalues of multiplicity one in such a way that α is the eigenform corresponding to $\lambda_k(g')$ and (a small perturbation of) β is the eigenform corresponding to $\lambda_k(g'')$ for metrics g' and g'' close to g. Theorem 2.1.1 ensures that a generic path of metrics g_t connecting g' to g'' will not have eigenvalue crossings, and so we have found an example of a 1-parameter family of metrics such that the contact structures defined by the k^{th} eigenvalue eigenform are connected by a 1-parameter family of (potentially singular) contact structures, see Figure 1.3. In particular, an answer to Question 1.4.3 provides a (partial) answer to Question 1.4.2.

1.5 Overview of thesis

1.5.1 Chapter 2

(Based almost verbatim on [Kep24], with the exception of [Kep24, Section 3.2.] in which a mistake was corrected, leading to a slight restructuring of the section)

The main result of this chapter is that the eigenvalues of the curl operator on a closed oriented Riemannian 3-manifold are simple along 1-parameter families. More precisely we have that

Theorem 1.5.1 (cf. Theorem 2.1.1)**.** *Let $2 \leq \ell < \infty$ and let g_0 and g_1 be two $\mathcal{C}^\ell$ Riemannian metrics. Then, for any $k \geq 1$, there exists a residual set in the space of paths*

$$W^k := \{w \in C^k([0,1], \mathcal{G}^\ell) : w(0) = g_0,\, w(1) = g_1\}$$

connecting these metrics such that the curl operator $*_{g(t)}d$ has simple spectrum for all $0 < t < 1$.

Interestingly, the same is not true the square of the curl operator as eigenvalues coming from positive and negative eigenvalues may cross.

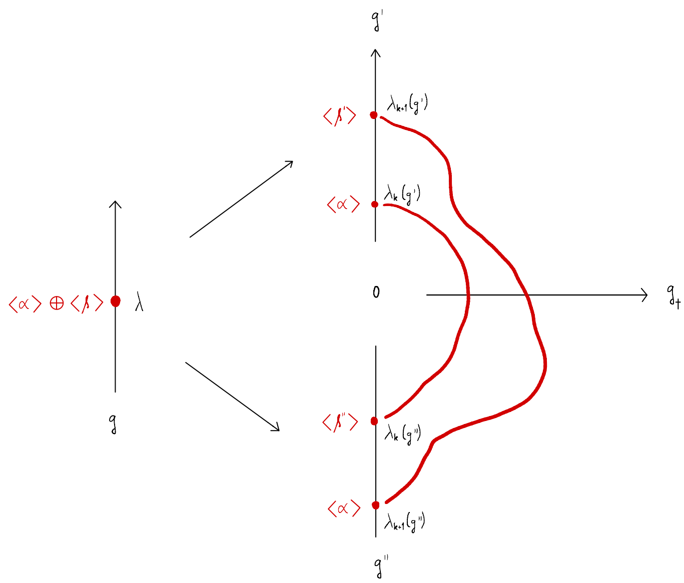

Figure 1.3: An illustration of the perturbative process connecting Question 1.4.3 to Question 1.4.2.

Theorem 1.5.2 (cf. Theorem 2.1.2)**.** *The operator* curl^2 *does not have simple spectrum along generic* 1*-parameter families of Riemannian metrics. Moreover, eigenvalues of the curl operator and of the Laplace-Beltrami operator on functions may cross along generic* 1*-parameter families of metrics.*

One way of interpreting this result is that the set of metrics for which the Hodge Laplacian has simple eigenvalues has codimension 1 (due to [EP12b, Theorem 1.1]), but not codimension 2. However, once decomposed into its action on closed, positive coexact, or negative coexact 1-forms, the Hodge Laplacian will only have non-simple spectrum on a set of codimension 2. This and other similar phenomena in which families of geometric differential operators behave unexpectedly from a spectral geometric point are view are discussed in Chapter 3, Section 3.4.4.

Regarding the Hodge Laplacian acting on 1-forms in dimension 3 there are two possible sources for nonsimple eigenspaces along generic 1-parameter families of Riemannian metrics: Either eigenvalues coming from positive and from negative eigenvalues of the curl operator cross, or a closed and a coclosed eigenvalue cross, and we provide examples for both.

In view of the setup provided in Chapter 3, the results of this Chapter can be upgraded to the setting of smooth metrics as opposed to C^k ones. Moreover, the setup of Chapter 3 provides an alternative way of proving the main theorem of this Chapter as, instead of defining global perturbations which provide the directions in which double eigenvalues break, one could have localized the transversality condition for the curl operator in much the same way as we did in Chapter 3, Proposition 3.2.3 for the Laplace-Beltrami operator.

The transversality theoretic setup of Chapter 3 would then give the desired Theorem 2.1.1.

1.5.2 Chapter 3

(Based almost verbatim on [GK23], joint work with Josef Greilhuber)

In this chapter, Josef Greilhuber and I systematically analyze a certain defining function for the set of Riemannian metrics on a fixed closed smooth manifold M whose associated Laplace-Beltrami operator has an eigenvalue of multiplicity m. This defining function was first introduced by Colin de Verdiére [Col88, Section 1] based on an intuitive description by Vladimir Arnold [Arn72, Section 2.2.]. We demonstrate that this function is C^1 in the sense of Hamilton (cf. Definition 4.5.1), and that it is a submersion except on a very small subset of the Fréchet manifold of smooth Riemannian metrics. This essentially confirms a conjecture of Vladimir Arnold from 1972 [Arn72]: Denote by $\mathcal{G}_m \subset \mathcal{G}$ the set of metrics for which the Laplacian on M has at least one eigenvalue of multiplicity m. Given a "typical" or generic k-parameter S family of metrics, Arnold expected the intersection of S with $\mathcal{G}_m$ to have dimension $k - \left(\frac{m(m+1)}{2} - 1\right)$ [Arn72, Section 2.2.]. This is an immediate consequence of the following

Theorem 1.5.3 (cf. Theorem 3.1.6)**.** *Let M be a closed, connected smooth manifold of dimension at least 2, and $\mathcal{G}(M)$ the set of smooth Riemannian metrics on M. Then the set of metrics $g \in \mathcal{G}(M)$ where Δ_g has an eigenvalue of multiplicity at least m has transversality codimension $\frac{m(m+1)}{2} - 1$. The same holds in any fixed conformal class $\mathcal{G}(M, g_0)$.*

Here transversality codimensionis a notion (introduced in Definition 3.1.4) which makes precise the idea that a certain set has codimension k from a transversality theoretic point of view.

We would like to point out that Arnold's initial formulation of the conjecture was stronger: He believed that the defining function later made precise by Colin de Vérdiere is always a submersion, which is however not true as a counterexample by de Vérdiere [Col88, Théorème 2] as well as several new counterexamples produced by us in Section 3.4 show. Nevertheless our result shows that, at least from a transversality theoretic point of view, the conjecture holds true.

The first step toward proving Theorem 3.1.6 is to find a spectral geometric condition characterizing the failure of transversality for the defining function given to us by Colin de Verdiére. We find a pointwise condition on the eigenfunctions of the Laplace operator for a given metric g and eigenvalue λ of multiplicity m signaling when nearby metrics whose eigenvalue corresponding to λ has multiplicity m form a submanifold in the space of metrics. For this we refer to Definition 3.1.1 and Theorem 3.1.3.

The second part of the proof of Theorem 3.1.6 consists of showing that the set of metrics for which the aforementioned spectral geometric condition is satisfied has codimension infinity. The main tools to prove this are pseudodifferential operator theory and heat kernel asymptotics for the Laplace operator.

Finally, we prove that for all eigenvalues of multiplicity strictly less than seven, the strong version of Arnold's conjecture holds in that de Vérdiere's defining function is a submersion, and show that there is an example of a multiplicity eight eigenvalue for which transversality fails. This is the content of Theorem 3.4.1.

We conclude the Chapter with several observations and potential directions for future research.

1.5.3 Chapter 4

(Based almost verbatim on [EKP25], joint work with Alberto Enciso and Daniel Peralta-Salas)

Explicit stationary solutions to the 2D or 3D Euler equations usually come in parametric families of solutions, and so it was an open question whether there exist isolated stationary solutions. We prove the existence of a closed Riemannian 3-manifold which admits a stationary solution to the Euler equations that is isolated in the C^1-topology.

Theorem 1.5.4 (cf. Theorem 4.1.2)**.** *There exists a closed* 3*-dimensional Riemannian manifold* (M, g) *of class* C^∞ *where the incompressible Euler equations have a smooth steady solution that is isolated in* C^1*. Moreover, the only* C^1 *steady states in a* C^1*-neighborhood of the isolated steady solution are constant multiples of it.*

The crucial observation is the following. Suppose one can find a stationary solution u to the Euler equations in dimension 3 which is so chaotic that not only does it have no C^1 first integrals, but so that there exists an open C^1 neighbourhood U of u in the space of vector fields so that no $v \in U$ has a C^1 first integral. Recalling Lemma 1.3.3, this implies that any solution to the stationary Euler equation in U (including u) is a curl eigenfield. We do indeed find such a chaotic solution and our particular setting allows us to show that the eigenvalue of any other stationary solution to the Euler equations in U would have to have a curl eigenfield *for the same eigenvalue* as u. The main remaining task is thus to prove via some perturbative scheme that one can perturb the curl operator by varying the metric in such a way that u stays an eigenform but the associated eigenspace becomes 1-dimensional. The crucial step is Lemma 4.4.4, and it heavily uses that one can characterize the space of metrics for which a particular contact form is an eigenform of the curl operator (Proposition 1.2.2) in explicit terms.

1.5.4 Chapter 5

(Based almost verbatim on [GK25], joint work with Josef Greilhuber)

The main result of this Chapter is that, for generic smoothly bounded domains in $\mathbb{R}^3$, the spectrum of the curl operator consists of isolated eigenvalues of multiplicity 1. A special feature of this setting is that the space of "good boundary conditions" for the curl operator on a smoothly bounded domain D is quite large. Concretely, it is given by the Grassmannian of Lagrangeans in the first cohomology of the boundary $H^1_{dR}(\partial D; \mathbb{C})$. That is, for each Lagrangean subspace L of $H^1_{dR}(\partial D; \mathbb{C})$, the curl operator curl_L defined on the space of 1-forms which are parallel to ∂D and whose restriction to ∂D is in L is self-adjoint and has compact resolvent. Our main result may then be stated slightly imprecisely as

Theorem (cf. Theorem 5.1.1)**.** *Let $D \subseteq \mathbb{R}^3$ be a smoothly bounded domain and let $L \subset H^1_{dR}(\partial D; \mathbb{C})$ be a Lagrangian subspace. For a comeagre set of elements in the space of smoothly bounded domains modeled on D, the spectrum of the operator* curl_L *consists of simple eigenvalues.*

Note that, in order to state the above Theorem precisely, several pieces of notation need to be introduced, see Theorem 5.1.1 in Chapter 5, Section 1.

A crucial intermediate result is a formula for the first order change of an eigenvalue of the curl operator under perturbation of the underlying domain. Let $\vec{X}$ be a smooth vector field, then the first variation of an eigenvalue λ of the curl operator is given by If we deform the domain D along the flow of a vector field $\vec{X}$, the first variation of an eigenvalue λ of the operator curl_L is given by

$$\dot{\lambda} = -\lambda \int_{\partial D} (\vec{X} \cdot \vec{\nu})\, |u|^2. \tag{1.5.1}$$

where u is an eigenfield of λ.

Another application of this variational formula is a characterization of domains for which the k^{th} eigenvalue functional is critical in terms of the behaviour of the associated eigenforms on the boundary of the domain.

The most challenging part of this project is dealing with the rather complicated boundary conditions required to make the curl operator self-adjoint. These are parametrized by the Grassmannian of Lagrangeans in the first (co-)homology of the domain boundary. An interesting consequence of the treatment of these boundary conditions is the observation that the curl operator depends analytically on these boundary conditions. This raises the question of how the symplectic linear algebra of the space of boundary conditions interacts with the eigenvalue structure, an issue we plan to address in future work.

We want to point out that all results in this paper hold for general Riemannian 3-manifolds provided the first Betti number of M equals one half times the first Betti number of the boundary, see Remark 5.3.2 and the references therein.

2 On spectral simplicity of the Hodge Laplacian and curl operator along paths of metrics

(Based almost verbatim on [Kep24], with the exception of [Kep24, Section 3.2.] in which a mistake was corrected, leading to a slight restructuring of the section)

Abstract

We prove that the curl operator on closed oriented 3-manifolds, i.e., the square root of the Hodge Laplacian on its coexact spectrum, generically has 1-dimensional eigenspaces, even along 1-parameter families of $\mathcal{C}^k$ Riemannian metrics, where $k \geq 2$. We show further that the Hodge Laplacian in dimension 3 has two possible sources for nonsimple eigenspaces along generic 1-parameter families of Riemannian metrics: Either eigenvalues coming from positive and from negative eigenvalues of the curl operator cross, or an exact and a coexact eigenvalue cross. We provide examples for both of these phenomena. In order to prove our results, we generalize a method of Teytel [Tey99], allowing us to compute the meagre codimension of the set of Riemannian metrics for which the curl operator and the Hodge Laplacian have certain eigenvalue multiplicities. A consequence of our results is that while the simplicity of the spectrum of the Hodge Laplacian in dimension 3 is a meagre codimension 1 property with respect to the $\mathcal{C}^k$ topology as proven by work of Enciso and Peralta-Salas in [EP12b], it is not a meagre codimension 2 property.

2.1 Introduction and Statement of Results

While the Hodge Laplacian on differential forms has not received as much attention as the Laplace-Beltrami operator, there has been a substantial amount of interest in this operator in recent years. The importance of its (coexact) spectrum in dimension 3 has been recognized in the study of the geometry of closed 3-manifolds (e.g., [LS18]), low dimensional topology (e.g., [LL21]), and that of its eigenforms (or rather those of its square root, the curl operator) in the study of contact topology (e.g., [EG00a],[EKM12a]) and fluid dynamics (e.g., [Car+21b]). Properties of its spectrum have been studied in various contexts (e.g., [AT12],[Tak03]) and are the subject of open problems [CPR01, Problem 8.24].

Generic properties of the eigenfunctions and eigenvalues of the Laplace-Beltrami operator were established in Uhlenbeck's landmark paper [Uhl76]. In particular, she showed that, given a closed n-dimensional manifold, the eigenvalues of the Laplace-Beltrami operator are generically simple, and that generic 1-parameter families of Riemannian metrics connecting two Riemannian metrics for which the spectrum is simple have simple spectrum throughout. The natural question of whether similar statements could hold for the Hodge Laplacian was soon answered in the negative by Millman [Mil80] who proved that for closed even dimensional manifolds, multiplicities of nonzero eigenvalues of the Hodge Laplacian in the middle degree are always even. This means that a theorem in the generality of Uhlenbeck's cannot hold in this context.

At least in dimension 3, however, Enciso and Peralta-Salas proved that the spectrum of the Hodge Laplacian is generically simple [EP12b]. They did this in several steps, the first of which is to notice that the spectrum of the Hodge Laplacian in dimension 3 is the same as that of its restriction to 1-forms since the Hodge Laplacian $\Delta_g = \delta d + d\delta$ commutes with both the exterior derivative d and the Hodge-star $*_g$. By the Hodge decomposition theorem [Hod89] the eigen-1-forms to nonzero eigenvalues of the Hodge Laplacian split into exact and coexact ones, and so one naturally calls the spectrum of the Hodge Laplacian on the exact and coexact forms the exact and coexact spectrum, respectively. The exact spectrum coincides with the spectrum of the Laplace-Beltrami operator, and so by Uhlenbeck's result we already know that it is generically simple. Analyzing the coexact spectrum however proves much harder. Enciso and Peralta-Salas do this by studying the curl operator $*_g d$, which is the square root of the Hodge Laplacian on coexact forms. They follow the general strategy of Uhlenbeck (which is to apply a Sard-type theorem to the function $\Phi(g, u, \lambda) = (*_g d - \lambda)u)$ except that significant analytical difficulties arise. Those include the PDE $*_g du = \lambda u$ being vector valued and the curl operator not being elliptic (it has infinite dimensional kernel). After showing the generic simplicity of the spectrum of the curl operator, they still needed to break up the symmetric eigenvalues, that is the ones for which $\lambda_i = -\lambda_j$ for some i, j, before showing that the coexact spectrum of the Hodge Laplacian is generically simple. Finally, they examine a variation of the exact and coexact eigenvalues to conclude that even the full Hodge Laplacian has simple spectrum for a residual set in the Banach manifold of $\mathcal{C}^k$ Riemannian metrics with its natural topology.

In this article we want to complement and extend these results. Just like in the work mentioned above we say that a set is *meagre* if it is the countable union of nowhere dense sets, and we say a set is *residual* if its complement is meagre. A property is said to be *generic* if it holds for a residual set. Note that all spaces appearing in this text are Baire spaces, and so residual sets are dense.

Theorem 2.1.1. *Let $2 \leq \ell < \infty$ and let g_0 and g_1 be two $\mathcal{C}^\ell$ Riemannian metrics. Then, for any $k \geq 1$, there exists a residual set in the space of paths*

$$W^k := \{w \in C^k([0,1], \mathcal{G}^\ell) : w(0) = g_0,\, w(1) = g_1\}$$

connecting these metrics such that the curl operator $*_{g(t)}d$ has simple spectrum for all $0 < t < 1$.

That is, the spectrum of the curl operator in dimension 3 is simple along generic 1-parameter families of Riemannian metrics.

Moreover we prove that such a result is false for the Hodge Laplacian itself (unlike for the curl and the Laplace-Beltrami operator) in two different ways. On the one hand, we construct an example of two Riemannian metrics on $\mathbb{S}^3$ so that any path connecting them must have a crossing of eigenvalues, one coming from a positive and one coming from a negative eigenvalue of the associated curl operators, meaning that the Hodge Laplacian restricted to coexact 1-forms does not have simple spectrum along 1-parameter families of Riemannian metrics. On the other hand we also construct an example of two Riemannian metrics on $\mathbb{S}^3$ (different from the previous example) so that any path connecting them has a crossing of an exact and a coexact eigenvalue. Calling those 1-forms in the image of the spectral projector associated to the positive (negative) part of the spectrum of curl positive (negative), we get the following

Theorem 2.1.2. *The Hodge Laplacian does not have simple spectrum along generic* 1*-parameter families of Riemannian metrics. However, it does if one restricts to the closed* 1*-forms, positive coexact* 1*-forms, or negative coexact* 1*-forms.*

This almost immediately implies

Corollary 2.1.3. *The set of Riemannian metrics for which the Hodge Laplacian in dimension* 3 *does not have simple nonzero spectrum is one of meagre codimension* 1 *but not of meagre codimension* 2*.*

which roughly means that simplicity of the spectrum of the Hodge Laplacian is a property that holds for the complement of a codimension 1, but not a codimension 2, set in the space of Riemannian metrics. The notion of meagre codimension is meant to capture the fact that some subset is not exactly a manifold of that codimension but that it it behaves like one under projections to subspaces of the parameter manifold so as to also include countable unions of manifolds of a given codimension. This will be made precise in subsection 2.2.1.

Our approach is different from the method employed in both the Uhlenbeck and the Enciso and Peralta-Salas paper. Instead we adapt an idea that has been pioneered by de Verdière [Col88] and made more user friendly by Teytel [Tey99]. The main insight is that, given a family of self-adjoint operators which differentiably depend on a parameter living in a separable Banach manifold, one can try to find local defining functions for the submanifold of parameter values which have double (or higher multiplicity) eigenvalues provided certain transversality conditions are satisfied. In this fortunate case, we can describe this non-simple subset as a set of meagre codimension 2. Teytel's genericity criterion has found a number of applications, for example for proving generic eigenvalue properties of the Laplace-Neumann operator [GM19].

An issue with this approach is that Teytel proved this theorem for a family of operators that are self-adjoint with respect to *the same* inner product, whereas many classes of geometric operators we care about, such as the Hodge Laplacian, are self-adjoint with respect to the inner product induced by the Riemannian metric for which the operators are defined. It turns out that this difficulty is overcome rather easily as soon as the correct generalizations are chosen. Specifically, we obtain the following variation of Teytel's Theorem [Tey99, Theorem A]

Theorem 2.1.4. *Let $A(q)$ be a family of operators whose resolvents $R_A(q)$ depend Fréchet-differentiably on a parameter q that belongs to a separable Banach manifold $\mathcal{X}$, each densely defined on the same real Banach space $\mathcal{H}$. Furthermore every $A(q)$ is self-adjoint with respect to a differentiable family of inner products $\langle -, - \rangle_q$ defined on $\mathcal{H}$, each inducing norms equivalent to that of $\mathcal{H}$. Assume that the spectrum of each operator $A(q)$ is discrete, of finite multiplicity, and with no finite accumulation points. Assume also that the family $A(q)$ satisfies SAH2. Then the set of all q such that $A(q)$ has a repeated eigenvalue has meagre codimension 2 in $\mathcal{X}$.*

Here the condition SAH2 derives from the "strong Arnold hypothesis" introduced by de Verdière in [Col88] and is essentially the transversality condition needed in order to conclude that we find the local defining functions mentioned above. This will be made precise in section 2.2.2, Definition 2.2.9. This approach is chosen since it would be unclear how to use the Uhlenbeck and Enciso and Peralta-Salas method in order to prove Theorem 2.1.1. Apart from that, however, it gives us stronger statements than we would get by using the former ideas. In particular, the de Verdière-Teytel method allows one to actually determine the (meagre) codimension of the set of Riemannian metric for which some family of geometric operators has eigenvalues of a given multiplicity, and not just prove that it is meagre. Further, a straightforward applicatoin of Uhlenbeck's method only yields that generic 1-parameter families g_t of Riemannian metrics connecting metrics g_0 and g_1 with simple spectrum have simple spectrum for all t, but no such restrictions on g_0 and g_1 exist if one uses Theorem 2.1.4.

We wish to remark that while the curl operator is defined on k forms on manifolds of dimension $2k+1$, it is not true that the results presented here readily generalize to higher odd dimensions. Indeed it was proven by Gier and Hislop [GH16b] that the curl operator in dimension 5 has generically simple spectrum, but because of the skew symmetry of this operator in dimension $4k+1$, this implies that its square, the Hodge Laplacian restricted to the coexact 2-forms in dimension 5, generically has 2-dimensional eigenspaces. The authors of [GH16b] conjecture that the coexact spectrum of the Hodge Laplacian on k forms in dimension $2k+1$ is generically 2 dimensional when k is even and generically 1 dimensional when k is odd.

Even if one aims to prove the generic simplicity of the spectrum of the curl operator in higher dimensions using the techniques presented here one will have to verify the SAH2-condition in a different way as we really do use the fact that the curl operator in dimension 3 is defined on 1-forms.

This papers is organized as follows: We will first recall the notion of meagre codimension as introduced by Teytel in [Tey99] in subsection 2.2.1, then review the setup for the de Verdière-Teytel method and introduce the necessary modifications in 2.2.2. Following this we apply Theorem 2.1.4 to the curl operator, proving that simplicity of the coexact spectrum holds in the complement of a meagre codimension 2 subset. We then use the fact that this approach immediately lends itself to the application of the study of k-parameter families of operators in order to prove Theorem 2.1.1.

Finally we construct counterexamples to simplicity of the Hodge Laplacian along generic 1-parameter families of Riemannian metrics mentioned above and thereby prove Theorem 2.1.2 in Subsection 2.3.2.

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2.2 Preliminaries

2.2.1 Meagre Codimension

In this section, we recall the definition of meagre codimension and some of its basic properties from [Tey99]. We also prove a simple technical lemma that we will need in Section 2.3.2.

Definition 2.2.1. *[Tey99] Let X be a Banach space and $Z \subset X$ be a hyperplane of codimension n. Then a differentiable map $\pi : X \to Z$ is called a nonlinear projection if, for every $x \in X$, $\pi'_x : X \to Z$ is surjection and has kernel of dimension n.*

Using these nonlinear projections allows Teytel to quantify more exactly how small a set is.

Definition 2.2.2. *[Tey99] A subset $Y \subset X$ is said to be of meagre codimension n if $\pi(Y) \subset Z$ is meagre in Z for every hyperplane Z of codimension $n-1$ and every nonlinear projection $\pi : X \to Z$.*

There is a natural way of extending this definition from Banach spaces to Banach manifolds.

Definition 2.2.3. *[Tey99] Given a Banach manifold M, we say that a set $Y \subset M$ has meagre codimension n if for every chart (ϕ, U) of M, $\phi(U \cap Y)$ has meagre codimension n in $\phi(U)$.*

Examples of subsets of meagre codimension n of a Banach space X include codimension n hyperplanes and smooth codimension n submanifolds of X. Clearly, countable unions of sets of meagre codimension n are again sets of meagre codimension n, and sets of meagre codimension n are also sets of meagre codimension m for all $m \leq n$.

We will need the following Lemmas proven by Teytel:

Lemma 2.2.4. *[Tey99] A subset Y of a separable Banach space is of meagre codimension 1 iff it is meagre.*

The proof carries over almost verbatim to subsets of separable Banach manifolds.

Lemma 2.2.5. *[Tey99] Let Y be a set of meagre codimension n and let M be a submanifold of X of codimension $k < n$. Then $Y \cap M$ is a set of meagre codimension $n - k$ in M.*

Lemma 2.2.6. *[Tey99] Let X be a separable Banach manifold and Y be a subset of X. Suppose that for every $q \in Y$ there exists a neighbourhood U_q of q such that $Y \cap U_q$ is of meagre codimension n in U_q. Then Y is of meagre codimension n in X.*

Finally we prove a Lemma we will use for the 1-parameter arguments.

Lemma 2.2.7. *Let X be a Banach manifold, and let $Y \subset X \times \mathbb{R}^k$ be a set of meagre codimension n with $n \geq k$, and let $\pi : X \times \mathbb{R}^k \to X$ be a nonlinear projection. Then $\pi(Y) \subset X$ is a set of meagre codimension $n - k$.*

Proof. Let (U_i, ϕ_i) be a collection of charts covering X, and $(U_i \times \mathbb{R}^k, \psi_i)$ be the associated product charts. Then π_{ψ_i}, the chart representative of the nonlinear projection, is a nonlinear projection from $\psi_i(U_i \times \mathbb{R}^k)$ to $\psi_i(U_i \times \{0\}) = \phi_i(U_i) \times \{0\}$. By the above lemmas, $\pi_{\psi_i}(\psi_i(Y \cap U_i \times \mathbb{R}^k)) \subset \phi_i(U_i)$ is a set of codimension $n - k$. Since this is true for any choice of charts (U_i, ϕ_i) we are done. □

2.2.2 The Genericity Result

We adapt the de Verdière-Teytel construction. Let the parameter space $\mathcal{X}$ be a separable Banach manifold. Let $(\mathcal{H}, \langle -, - \rangle_q)$ be a family of real Hilbert spaces, Fréchet-differentiably parametrized by $q \in \mathcal{X}$ with the same underlying real Banach space $\mathcal{H}$, i.e., all inner products induce equivalent norms. Let furthermore A_q be linear operators satisfying:

- A_q is self-adjoint with respect to the inner product $\langle -, - \rangle_q$ for all $q \in \mathcal{X}$
- the spectrum $\sigma(A_q)$ of A_q consists of countably many discrete eigenvalues, all of which have finite multiplicity for all $q \in \mathcal{X}$
- The resolvents $R_{A_q}(\mu) = (\mu - A_q)^{-1}$, $\mu \in \rho(A_q)$, of $A(q)$ depend Fréchet-differentiably on q

Remark 2.2.8. *Since self adjoint operators are closed, the resolvents R_{A_q} are bounded operators, so it is clear what Fréchet-differentiability means in this context. Teytel required* *the operators A_q to depend Fréchet-differentiably on q with respect to the graph norm, but we find this formulation to be slightly cleaner. Note that, unlike Teytel, we do not need to require the domains $D(A_q)$ to coincide for all $q \in \mathcal{X}$ since the resolvents are defined on all of $\mathcal{H}$.*

Fix some $q_0 \in \mathcal{X}$ and an eigenvalue λ of A_{q_0} with multiplicity m. The general strategy will now be to find defining functions for the submanifold of parameter values q close to q_0 for which the part of the spectrum of A_q near λ is not simple.

Since the spectrum of A_{q_0} is discrete there exists an $\epsilon > 0$ such that λ is the only eigenvalue of A_{q_0} in $(\lambda - \epsilon, \lambda + \epsilon)$. We now consider an open neighbourhood $\mathcal{U}(q_0) \subset \mathcal{X}$ of q_0 such that the spectrum of A_q in $(\lambda - \epsilon, \lambda + \epsilon)$ consists of n eigenvalues whose multiplicities sum to m. Such an open neighbourhood exists because $\sigma(A_q)$ depends continuously on q [RS55, pp. 372-373]. Note that the previous reference only deals with the case of families of operators that are self-adjoint with respect to the same inner product, but the same proof goes through in our case. Continuous dependence on q also follows from the min-max characterization of eigenvalues of selfadjoint operators [Tes14, Theorem 4.10].

We denote the associated sum of eigenspaces by $E(q)$ and define the spectral projection $P : \mathcal{U}(q_0) \times \mathcal{H} \to E(q)$ by

$$P(q) = \frac{1}{2\pi i} \int_\gamma R_{A_q}(\gamma)\, d\gamma$$

Where γ is the simple closed curve $\lambda + \epsilon\, e^{it}$ in the complex plane. Note that, while all the $(\mathcal{H}, \langle -, - \rangle_q)$ are real Hilbert spaces, we can first complexify, then apply the operator $P(q)$, and then restrict to the real elements of the complexification. Furthermore, the projection $P(q_0)$ is an isomorphism from $E(q)$ to $E(q_0)$ for q close enough to q_0. The fact that P depends differentiably on the parameter q follows from an application of dominated convergence.

Using this projection we define the map

$$S(q) = P(q) \circ P(q_0) : E(q_0) \to E(q)$$

and the local defining function $f : \mathcal{U}(q_0) \subset \mathcal{X} \to GL(E(q_0))$ by

$$f(q) = S(q)^{-1} R_A(q) S(q)$$

Here we do not specify the argument of the resolvent since it does not matter and denote it by $R_A(q) = R_{A(q)}$. We do however want the argument to be real since we are working in a real Banach space. One can find a real argument μ so that the resolvent $R_{A(q)}(\mu)$ is defined for all $q \in \mathcal{U}(q_0)$ provided $\mathcal{U}(q_0)$ is chosen small enough.

Note that the preimage of the submanifold $\mathbb{R} \cdot Id \subset GL(E(q_0))$ under f is precisely the set of parameter values for which there is a single eigenvalue in the interval $(\lambda - \epsilon, \lambda + \epsilon)$. This eigenvalue is necessarily of multiplicity m.

To see that this set is a submanifold we have to compute the derivative of f at q_0. Using that $S(q_0) = Id$ one sees that this is given by

$$f'(q_0) = [S'(q_0), R_A(q_0)] + R'_A(q_0)$$

Let $\{v_i\}_{1\leq i\leq m}$ be an orthonormal eigenbasis of $(E(q_0), \langle -,-\rangle_{q_0})$ for the operator A_{q_0}. Given an element $h \in T_{q_0}\mathcal{X}$ we can now express the endomorphism $f'(q_0)[h]$ in this basis and see that $\langle [S'(q_0), R_A(q_0)][h]v_i, v_j\rangle_{q_0} = 0$ by the fact that $R_A(q_0)$ is symmetric with respect to $\langle -,-\rangle_{q_0}$. Thus

$$f'_{ij}(h) := \langle f'(q_0)[h]v_i, v_j\rangle_{q_0} = \langle R'_A(q_0)[h]v_i, v_j\rangle_{q_0}$$

We can now define the condition SAH2

Definition 2.2.9. *[Tey99] Let $\mathcal{H}$ be a real Hilbert space, and λ be an eigenvalue of $A(q)$ of multiplicity $n \geq 2$. We say that the family $A(q)$ satisfies the condition SAH2 if there exist two orthonormal eigenvectors v_1 and v_2 of eigenvalue λ such that the linear functionals $f'_{11} - f'_{12}$ and f'_{12} are linearly independent.*

Remark 2.2.10. *The spectrum of an operator A is simple iff the spectrum of its resolvent R_A is simple. Moreover, whenever the derivative of A_q exists in some suitable sense, $\langle R'_A(q_0)[h]v_i, v_j\rangle_{q_0}$ and $\langle A'(q_0)[h]v_i, v_j\rangle_{q_0}$ only differ by a constant factor so in that case $R_A(q)$ satisfies the condition SAH2 iff $A(q)$ does.*

Remark 2.2.11. *In Teytel's setting, $A'(q_0)$ always maps to $\mathcal{L}(E(q_0))$, the space of symmetric endomorphisms on $E(q_0)$. This means that if the multiplicity of λ is two, the linear functions appearing in condition SAH2 precisely span all directions which are transverse to $\mathbb{R}\cdot Id$. In our setting there is one more direction that $A'(q)$ could potentially map to, however for all examples known to us, $A'(q_0)$ also maps to $\mathcal{L}(E(q_0))$.*

Proof of Theorem 2.1.4. The proof can be copied almost verbatim from the one provided by Teytel in [Tey99, Chapter 3], the only difference being that f maps to $GL(E(q_0))$ as opposed to $\mathcal{L}(E(q_0))$. For the sake of completeness we outline the most important steps leading up to that point. The idea is to filter the set $\mathcal{D} \subset X$ of metrics for which the operator A does not have simple spectrum by the multiplicity of the non-simple eigenvalues, and then show that locally $\mathcal{D}$ has meagre codimension 2, which then allows us to conclude the argument by appealing to Lemma 2.2.6.
Given a finite open interval I, we define the set $\mathcal{D}_{n,I} \subset X$ as the collection of parameters $q \in X$ such that A_q has an eigenvalue λ of multiplicity n but no eigenvalue of higher multiplicity contained in I. Then

$$\mathcal{D} = \bigcup_{N\in\mathbb{N}} \bigcup_{n>1} \mathcal{D}_{n,(-N,N)}$$

and so if we show that for all $n > 1$ and all finite intervals I, $\mathcal{D}_{n.I}$ has meagre codimension 2, we are done since the countable union of sets of meagre codimension 2 is again a set of meagre codimension 2. Fixing $q_0 \in \mathcal{D}$ we note that any finite interval I only

contains finitely many eigenvalues of A_{q_0}. Restricting to those eigenvalues $\lambda_1, \dots, \lambda_s$ with multiplicity $m(\lambda_i) \leq n$ and using the fact that eigenvalues move continuously with the parameter we see that there exists a small neighbourhood U_{q_0} of q_0 so that $D_{n,I} \cap U_{q_0} = \bigcup_{i=1}^{s} D_{n,(a_i,b_i)} \cap U_{q_0}$, where the sum of multiplicities of eigenvalues in (a_i, b_i) is $m(\lambda_i)$ for all A_q with q in U_{q_0}. It thus suffices to show that $U_{q_0} \cap \mathcal{D}_{n,(a,b)}$ with (a, b) containing eigenvalues of total multiplicity n has meagre codimension 2 in U_{q_0}.
When $n = 2$ then A_q satisfying SAH2 gives two linearly independent defining equations, thus implying that $\mathcal{D}_{n,(a,b)} \cap U_{q_0} = f^{-1}(\mathbb{R} \cdot Id) \cap U_{q_0}$ is a submanifold of codimension 2 for U_{q_0} small enough, which is a set of meagre codimension 2.
When $n > 2$ consider the set $E'(\mathbb{R}^n) \subset End(\mathbb{R}^n)$ of all $n \times n$ matrices L with entries $L_{1,1}$, $L_{1,2}$, $L_{2,1}$, and $L_{2,2}$ equal to zero, which is a space of dimension $n^2 - 4$. We define the differentiable map $\bar{f} : E'(\mathbb{R}^n) \times U_{q_0} \to End(\mathbb{R}^n)$ by $\bar{f}(L, q) = L + f(q)$ which is transverse to $\mathbb{R} \cdot Id \subset End(\mathbb{R}^n)$ plus some additional direction $\langle r \rangle$ at $(0, q_0)$ by the SAH2 condition and the fact that the image of f has at most dimension 2. Therefore $\bar{f}^{-1}(\mathbb{R} \cdot Id \oplus \langle r \rangle) \subset L'(\mathbb{R}^n) \times U_{q_0}$ is a submanifold of codimension $n^2 - 2$ in a neighbourhood of $(0, q_0)$. The intersection of this submanifold with U_{q_0} contains $\mathcal{D}_{n,(a,b)} \cap U_{q_0}$, and Lemma 2.2.5 implies that this intersection has meagre codimension $(n^2 - 2) - (n^2 - 4) = 2$ in U_{q_0}. This concludes the proof that $\mathcal{D} \subset X$ is a set of meagre codimension 2. □

Remark 2.2.12. *1. One can easily extend Theorem 2.1.4 to complex Hilbert spaces.*

2. *A reading of Teytel's proof makes clear that it is not the condition SAH2 that really matters, but the number k of linearly independent directions of in $Im(A')$ which are transverse to $\mathbb{R} \cdot Id$ (and whose span does not contain Id). This number k directly translates into non-simplicity of the spectrum being a meagre codimension k property.*

3. *We want to point out that a different approach to adapting Teytel's theorem to the case of differentiably varying inner products on the same Banach space would be to try to locally isometrically trivialize the bundle $(H \to \mathcal{X})$, consisting of fibers $H_q = (\mathcal{H}, \langle -, - \rangle_q)$ over the basepoint q. Supposing $\mathcal{H}$ is separable, an idea for doing this would be to fix a countable orthonormal basis (with respect to any inner product) and then apply the Gram-Schmidt procedure with respect to all $\langle -, - \rangle_q$ and hope that the resulting local frame for H is smooth. Then one could consider the family $A(q)$ in this local bundle chart and try to apply Teytel's Theorem directly. Actually applying this idea to a concrete family of operators however is much more difficult than our approach as one would have to compute the Fréchet derivative of this transformed family of operators which of course depends on the chart one chose, so this does not seem to be useful in practice.*

2.3 Main Results

2.3.1 The curl operator has simple spectrum in the complement of a subset of meagre codimension 2

We now specialize to the case $\mathcal{X} = \mathcal{G}^k$, the set of $\mathcal{C}^k$ Riemannian metrics on a closed oriented 3-manifold M, endowed with the $\mathcal{C}^k$ topology. This constitutes a separable smooth Banach manifold, its tangent space $T_g\mathcal{G}^k$ at a point g can be identified with $\mathcal{S}^k(M)$, the set of symmetric $(0,2)$ tensor fields of differentiability class k on M.

Consider the curl operator $A_g := *_g d$, the square root of Δ_g on the coexact 1-forms. Our strategy is to show that Theorem 2.1.4 applies to the curl operator.
The curl operator is well known to be symmetric with respect to the inner product $\langle \alpha, \beta \rangle_g = \int_M g(\alpha, \beta)\, d\mu_g$ and it admits a self-adjoint extension to a densely defined subspace of $\mathcal{H} = L^2(\Omega^1(M))$. In particular all of its eigenvalues are real and all of its eigenforms are eigenforms of Δ_g and so they are smooth 1-forms.
Moreover its eigenvalues only accumulate at infinity and its eigenspaces except for the eigenvalue 0 are finite dimensional. In order to deal with the infinite dimensional kernel of the curl operator A_g we restrict it to the subspace $\mathcal{H}_g = \ker(d)^{\perp_g}$. All of these Hilbert spaces are equipped with the induced inner products $\langle -, - \rangle_g$ mentioned above and can be identified with $\mathcal{H}/\ker(d)$ as Banach spaces.

We thus see that the curl operator is of the type considered in Theorem 2.1.4 and we are left with analyzing its derivative. This derivative was computed in [EP12b].

Lemma 2.3.1. *[EP12b, Lemma 2.1.] Let u be an eigenform of $A_g = *_g d$ of eigenvalue λ. Then the variation of A_g in direction h is given by*

$$A'_g\,[h]\,u = \lambda\, h(\sharp_g u, -) - \frac{\lambda}{2}\, tr_g(h)\, u$$

where $\sharp_g$ is the canonical isomorphism of the tangent and the cotangent bundle induced by the metric g.

Remark 2.3.2. *We can see that the derivative A'_g of A_g maps to the symmetric operators on $E(g)$. Therefore the best we can hope for is that the subspace of metrics with non-simple spectrum constitute a set of meagre codimension two. The same holds true for Δ_g restricted to the coexact spectrum as we will soon see.*

Now let λ be an eigenvalue of multiplicity 2 and v_1 and v_2 an orthonormal basis for the associated eigenspace $E(g)$ of A_g. Our approach is to make special choices of h and show that A'_g applied to these h already spans a 2-dimensional vector space transverse to $\mathbb{R} \cdot Id \subset \mathcal{L}(E(g))$. It turns out that the symmetric tensor product of the basis vectors, denoted by $v_i \odot v_j$, leads to particularly nice expressions, and so these will be our h.

Using $v_i \odot v_j(\sharp_g v_k) = \frac{1}{2}(g(v_i, v_k)v_j + g(v_j, v_k)v_i)$ and $tr_g(v_i \odot v_j) = g(v_i, v_j)$ we compute

$$\langle A'[v_i \odot v_j]v_k, v_l \rangle =$$
$$\frac{\lambda}{2} \int_M \Big(g(v_i, v_k)\, g(v_j, v_l) + g(v_j, v_k)\, g(v_i, v_l) - g(v_i, v_j)\, g(v_k, v_l) \Big)\, d\mu_g$$

We represent the linear maps $A'[v_1 \odot v_1]$ and $A'[v_1 \odot v_2]$ in this basis and obtain

$$A'[v_1 \odot v_1] = \frac{\lambda}{2} \begin{pmatrix} \int_M \|v_1\|^4 \, d\mu_g & \int_M \|v_1\|^2 \, g(v_1, v_2) \, d\mu_g \\ \int_M \|v_1\|^2 \, g(v_1, v_2) \, d\mu_g & 2 \int_M g(v_1, v_2)^2 \, d\mu_g - \int_M \|v_1\|^2 \, \|v_2\|^2 \, d\mu_g \end{pmatrix}$$

$$A'[v_1 \odot v_2] = \frac{\lambda}{2} \begin{pmatrix} \int_M \|v_1\|^2 \, g(v_1, v_2) \, d\mu_g & \int_M \|v_1\|^2 \, \|v_2\|^2 \, d\mu_g \\ \int_M \|v_1\|^2 \|v_2\|^2 \, d\mu_g & \int_M \|v_2\|^2 \, g(v_1, v_2) \, d\mu_g \end{pmatrix}$$

Lemma 2.3.3. *The matrices Id, $A'[v_1 \odot v_1]$, and $A'[v_1 \odot v_2]$ fail to span $\mathcal{L}(E(g))$ exactly when*

$$\int_M \|v_1\|^4 \, d\mu_g = \left(2 \int_M g(v_1, v_2)^2 \, d\mu_g - \int_M \|v_1\|^2 \|v_2\|^2 \, d\mu_g \right) + \frac{\left(\int_M \|v_1\|^2 g(v_1, v_2) \, d\mu_g \right)^2 - \left(\int_M \|v_1\|^2 g(v_1, v_2) \, d\mu_g \right) \left(\int_M \|v_2\|^2 g(v_1, v_2) \, d\mu_g \right)}{\int_M \|v_1\|^2 \|v_2\|^2 \, d\mu_g}$$

Proof. Identifying $\mathcal{L}(E(g))$ with $\mathbb{R}^3$ in the obvious way we can form the matrix built from the images of Id, $A'[v_1 \odot v_1]$, and $A'[v_1 \odot v_2]$ under this identification

$$\begin{pmatrix} 1 & 0 & 1 \\ \int_M \|v_1\|^2 g(v_1, v_2) \, d\mu_g & \int_M \|v_1\|^2 \|v_2\|^2 \, d\mu_g & \int_M \|v_2\|^2 g(v_1, v_2) \, d\mu_g \\ \int_M \|v_1\|^4 \, d\mu_g & \int_M \|v_1\|^2 g(v_1, v_2) \, d\mu_g & 2 \int_M g(v_1, v_2)^2 \, d\mu_g - \int_M \|v_1\|^2 \|v_2\|^2 \, d\mu_g \end{pmatrix}$$

The vanishing of its determinant gives the condition

$$\begin{aligned} &\left(\int_M \|v_1\|^2 \|v_2\|^2 \, d\mu_g \right) \left(2 \int_M g(v_1, v_2)^2 \, d\mu_g - \int_M \|v_1\|^2 \|v_2\|^2 \, d\mu_g \right) \\ &+ \left(\int_M \|v_1\|^2 g(v_1, v_2) \, d\mu_g \right)^2 - \left(\int_M \|v_1\|^2 \|v_2\|^2 \, d\mu_g \right) \left(\int_M \|v_1\|^4 \, d\mu_g \right) \\ &- \left(\int_M \|v_1\|^2 g(v_1, v_2) \, d\mu_g \right) \left(\int_M \|v_2\|^2 g(v_1, v_2) \, d\mu_g \right) = 0 \end{aligned}$$

Noting that eigenforms of the curl operator are in particular eigenforms of the Hodge Laplacian, we see that they satisfy a unique continuation property ([Aro57],[Kaz88]), i.e. they vanish identically if they vanish on an open subset. It immediately follows that $\int_M \|v_1\|^2 \|v_2\|^2 \, d\mu_g > 0$, and so we are done. □

In order to deal with this degenerate case, we introduce $\tilde{h}_a = v_1 \odot v_1 + a \, tr_g[v_1 \odot v_1] \, g$. We compute once more

$$A'[\tilde{h}_a] = \frac{\lambda}{2} \begin{pmatrix} (1-a) \int_M \|v_1\|^4 \, d\mu_g & (1-a) \int_M \|v_1\|^2 g(v_1, v_2) \, d\mu_g \\ (1-a) \int_M \|v_1\|^2 g(v_1, v_2) \, d\mu_g & 2 \int_M g(v_1, v_2)^2 - (1+a) \int_M \|v_1\|^2 \|v_2\|^2 \, d\mu_g \end{pmatrix}$$

Lemma 2.3.4. *Id, $A'[\tilde{h}_a]$, and $A'[v_1 \odot v_2]$ span $\mathcal{L}(E(g))$ for some choice of a.*

Proof. Setting the determinant associated to these three vectors to zero and solving for $\int_M \|v_1\|^4 \, d\mu_g$ as in the preceding Lemma we get

$$\int_M \|v_1\|^4 \, d\mu_g = \left(\frac{2}{1-a} \int_M g(v_1, v_2)^2 \, d\mu_g - \frac{1+a}{1-a} \int_M \|v_1\|^2 \|v_2\|^2 \, d\mu_g \right) + \\ \frac{\left(\int_M \|v_1\|^2 g(v_1, v_2) \, d\mu_g \right)^2 - \left(\int_M \|v_1\|^2 g(v_1, v_2) \, d\mu_g \right) \left(\int_M \|v_2\|^2 g(v_1, v_2) \, d\mu_g \right)}{\int_M \|v_1\|^2 \|v_2\|^2 \, d\mu_g} \\ = V_a$$

Now suppose that in fact all of these determinants vanish, so the RHS of the above equation does not depend on the choice of $a \in \mathbb{R} \setminus \{1\}$. Then in particular, $V_a = V_0$, and so we get

$$(1-a) \left(2 \int_M g(v_1, v_2)^2 \, d\mu_g - \int_M \|v_1\|^2 \|v_2\|^2 \, d\mu_g \right) \\ = \left(2 \int_M g(v_1, v_2)^2 \, d\mu_g - (1+a) \int_M \|v_1\|^2 \|v_2\|^2 \, d\mu_g \right)$$

which is equivalent to

$$\int_M \left(g(v_1, v_2)^2 - \|v_1\|^2 \|v_2\|^2 \right) d\mu_g = 0$$

meaning that $v_1 \parallel v_2$ for every point in M. We will now show that this is impossible. For this note that the complements of the zero set of v_1 and v_2 are open and dense by the unique continuation property. This means that there exists an open set $U \subset M$ in the complement of these zero sets and a smooth function s such that $v_1 = s \, v_2$ on U. For constant s we immediately get a contradiction since $\langle v_1, v_2 \rangle_g = 0$, so s is some nonconstant function. Plugging this into the eigenform-equation for the curl operator yields

$$\lambda v_1 = A_g v_1 = A_g(s \, v_2) = s \, A_g v_2 + *_g(ds \wedge v_1) = \lambda \, v_1 + *_g(ds \wedge v_1)$$

This means that $ds \parallel v_1$, so $v_1 = f \, ds$ for some smooth function f. Restricting to a subset $\tilde{U} \subset U$ on which ds does not vanish, we see that this leads to the following contradiction

$$0 < v_1 \wedge dv_1 = f \, ds \wedge d(f \, ds) = f^2 \, ds \wedge ds = 0$$

□

Now decompose $\mathcal{H}_g = \mathcal{H}_g^+ \oplus \mathcal{H}_g^-$, where $\mathcal{H}_g^+$ and $\mathcal{H}_g^-$ are the subspaces of $\mathcal{H}_g$ spanned by the eigenforms of A_g corresponding to positive and negative eigenvalues, respectively. This induces the natural splitting $\Delta_g = \Delta_g^+ \oplus \Delta_g^-$. An immediate consequence of the preceding Propositions and Theorem 2.1.4 is the following

Theorem 2.3.5. *The curl operator in dimension 3 has simple spectrum unless g is in a set of meagre codimension 2. Moreover, the same is true for $\Delta_g^{\pm}$.*

Remark 2.3.6. *One may expect that one could simply extend our proof technique to the full Hodge Laplacian Δ_g on one-forms by virtue of the following argument: Given a 2-dimensional eigenspace $E(g)$ of $\Delta_g = A_g^2$ corresponding to an eigenvalue $\lambda^2 > 0$, there exists an orthonormal basis of eigenvectors of A_g spanning $E(g)$. This is true because A_g is an endomorphism of $E(g)$ and since $\mathcal{H}$ admits an orthonormal basis composed of eigenvectors of A_g. Calling this pair of eigenvectors of A_g spanning $E(g)$ v_1 and v_2, we note that the v_i are eigenvectors to potentially different eigenvalues, namely $\pm\lambda$.*
Now, an application of the product rule and the symmetry of the operator A_g with respect to $\langle -, - \rangle_g$ yields

$$\langle (A_g^2)' v_l, v_k \rangle_g = \langle A_g' A_g + A_g A_g' v_l, v_k \rangle_g = 2\,\lambda\, \langle A_g' v_l, v_k \rangle_{q_0}$$

Now if $\lambda_i = -\lambda_j$, we observe that $\Delta_g'[v_1 \odot v_2] = 0$, and $\Delta_g'[v_i \odot v_i] = 2\,\lambda\, A_g'[v_i \odot v_i]$. While this means that we cannot repeat the arguments of the previous two Lemmas for the Hodge Laplacian on coexact one-forms, we do reproduce a result of Enciso and Peralta-Salas [EP12b]: It is not difficult to check that $A_g'[v_1 \odot v_1]$ and $A_g'[v_2 \odot v_2]$ are always nonzero and that they are linearly dependent iff $g(v_1, v_2) = 0$ and $\|v_1\| = \|v_2\|$ pointwise, in which case $A_g'[v_1 \odot v_1]$ is transverse to $\mathbb{R} \cdot Id$. This easily implies that there always exists at least one h such that $\Delta_g'[h]$ is transverse to $\mathbb{R} \cdot Id$. By Lemma 2.2.4, we conclude that the set of Riemannian metrics for which the spectrum of the Hodge Laplacian on coexact one-forms has non-simple spectrum is meagre.

The proof of Corollary 2.1.3 will show that this is as far as we can go for the Hodge Laplacian on coexact 1-forms.

2.3.2 The 1-parameter family arguments

To prove Theorem 2.1.1 we pick a new parameter space. We fix two Riemannian metrics g_0 and g_1 and define $W^k := \{w \in C^k([0,1], \mathcal{G}^\ell) : w(0) = g_0,\ w(1) = g_1\}$, where $2 \leq \ell < \infty$. We use $\mathcal{X} = W^k \times (0,1)$ and define the family of operators $A^*_{(w,t)} = A_{w(t)}$. Having made these definitions we are ready to prove Theorem 2.1.1.

Proof of Theorem 2.1.1. The plan is to once again use Theorem 2.1.4. To this end we note that $(A^*)'_{(w,t)}[(u,s)] = A'_{w(t)}[w'(t)s + u(t)]$. Thus given a parameter value (w_0, t_0) for which an eigenspace of A^* is 2-dimensional, we evaluate the derivative of A^* at (w_0, t_0) and apply it to $(u = w_0, s = 0)$. This shows that $Im((A^*)'_{(w_0,t_0)}) \supseteq Im(A'_{w_0(t_0)})$, and so we may conclude that the non-simple set N has meagre codimension 2 in $W^k \times (0,1)$. Lemma 2.2.7 tells us that $\pi : (w,t) \mapsto w$ maps N to a set of meagre codimension 1 in W^k, which is a meagre set by Lemma 2.2.4. Since the complement of $\pi(N)$ in W^k are precisely those 1-parameter families which have simple coexact spectrum for all values of t, we are done. ☐

We can also prove, however, that an analogous statement for the nonzero spectrum of the Hodge Laplacian is false. As mentioned in the introduction, generic simplicity of the Hodge Laplacian in dimension 3 fails in two different ways: Eigenvalues corresponding to positive and negative eigenvalues of the curl operator may cross, as may exact and coexact eigenvalues. We will need the following lemmas to prove this.

Lemma 2.3.7. *Let $T : X \to Y$ be a bounded operator between Banach spaces X and Y, and let furthermore $Z \subset X$, $W \subset Y$ be closed subspaces. Suppose T vanishes on Z. Then*

$$\|\tilde{T}\|_{X/Z \to Y/W} \leq \|T\|_{X \to Y}$$

where $\tilde{T}$ is the projection of T to the quotient X/Z.

Proof. We compute

$$\begin{aligned}\|\tilde{T}(u+Z)\|_{Y/W} &= \inf_{w \in W} \|\tilde{T}(u+Z) + w\|_Y \leq \|\tilde{T}(u+Z)\|_Y = \|T(u)\|_Y \\ &\leq \|T\|_{X \to Y} \cdot \inf_{v \in \{u+Z\}} \|v\|_X = \|T\|_{X \to Y} \cdot \|u\|_{X/Z}\end{aligned}$$

and the result follows. □

The next lemma will tell us that no eigenvalues of the Hodge Laplacian can "disappear into its kernel" along 1-parameter families of metrics.

Lemma 2.3.8. *Let $(g_t)_{t \in [0,1]}$ be a continuous 1-parameter family of Riemannian metrics. Then the lowest nonzero eigenvalue $\lambda_1(t)$ of the Hodge Laplacian is bounded away from 0.*

Proof. It suffices to prove this Lemma for the curl operator as the claim for the exact part of the spectrum follows analogously.

The curl operator $A_g : H^1(M) \to L^2(M)$ is bounded and descends to a bounded operator between the Banach spaces $\tilde{A}_g : H^1(M)/ker(d) \to L^2(M)/ker(d)$. The continuous dependence of the operator norm on g factors to the quotient by Lemma 2.3.7. Calling $H^1(M)/ker(d) = X$ and $L^2(M)/ker(d) = Y$ we know that for every metric g, the lowest (positive or negative) eigenvalue is greater than zero and so

$$\|A_g u\|_Y \geq C_g \|u\|_X$$

Fix a Riemannian metric g_0. Then there exists a neighbourhood U of g_0 such that $\|A_g u\|_Y > \frac{C_{g_0}}{2} \|u\|_X$.
Indeed, choose U so that $\|A_g - A_{g_0}\|_{X \to Y} < \frac{C_{g_0}}{2}$. Then

$$\begin{aligned}\|A_g u\|_Y &\geq \|A_{g_0} u\|_Y - \|(A_g - A_{g_0})u\|_Y \\ &\geq C_{g_0}\|u\|_X - \|A_g - A_{g_0}\|_{X \to Y} \cdot \|u\|_Y \\ &\geq \frac{C_{g_0}}{2} \|u\|_X\end{aligned}$$

Now suppose that the family $\lambda_1(t)$ goes to zero as $t \to t_0$. Then $C_{g_0} > 0$, but C_{g_t} goes to 0 as t goes to t_0, which is a contradiction to the above bound. □

It should be noted that the proof of Lemma 2.3.8 uses the fact that all A_g have the same kernel. With these preparatory results out of the way we can prove

Proposition 2.3.9. *There exist Riemannian metrics g_0 and g_1 on $\mathbb{S}^3$ such that the Hodge Laplacian has a crossing of an exact and a coexact eigenvalue along any continuous curve $g(t)$ with $g(0) = g_0$ and $g(1) = g_1$. Similarly, there exist Riemannian metrics g_0' and g_1' on $\mathbb{S}^3$ such that the Hodge Laplacian has a crossing of a positive and a negative curl eigenvalue along any continuous curve $g(t)$ with $g(0) = g_0'$ and $g(1) = g_1'$.*

Proof. The idea of both statements is the same, we introduce a combinatorial invariant associated to the spectrum of the Hodge Laplacian for some metric g which can only change along 1-parameter families of metrics if a closed and a coclosed eigenvalue of the Hodge Laplacian cross (or, for the second statement, if a positive and a negative curl eigenvalue cross). Then we exhibit two metrics for which this invariant is different, thus yielding the result. We will start with the statement on exact and coexact eigenvalues.

Given a Riemannian metric g so that Δ_g has simple spectrum, we colour eigenvalues red or blue depending on whether they come from an exact or a coexact eigenvalue of the Hodge Laplacian, respectively. See Figure 2.1 for an illustration of this. Clearly the corresponding sequence of red and blue dots on the positive half line does not change along 1-parameter families of metrics unless there is a crossing of a red and a blue dot.

We note that Lemma 2.3.8 tells us that (nonzero) eigenvalues of the Hodge Laplacian never go to 0 along 1-parameter families of Riemannian metrics.

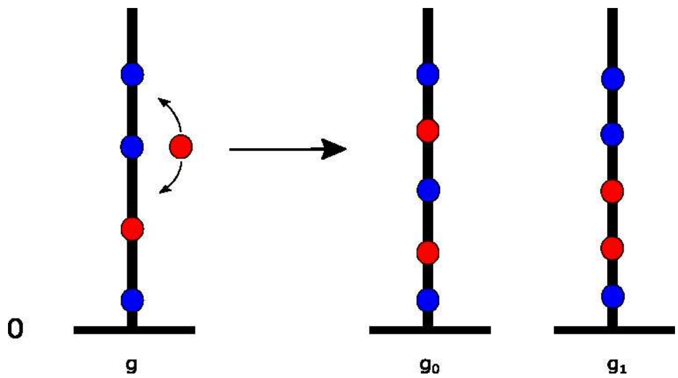

Figure 2.1: Changing the combinatorial structure of the spectrum.

Consequently, if we find two Riemannian metrics g_0 and g_1 for which the Hodge Laplacian has simple spectrum but whose spectrum defines a different sequence of colored points, we have proven the first part of the Lemma, see Figure 2.1 for an illustration of this process.

The main ingredient for constructing two metrics whose associated spectra of the Hodge Laplacian define different sequences of coloured points is a particular deformation of the round metric g_{st} on odd dimensional spheres considered by Tanno in [Tan79] and[Tan83].

We will specialize the discussion to the 3-dimensional setting. Let α be a Hopf field on S^3, then consider the 1-parameter family of Riemannian metrics given by

$$g_t = t^{-1} g_{st} + (t^{-2} - t^{-1}) \alpha \otimes \alpha$$

Using the fact that Δ_{g_t} and $\Delta_{g_{st}}$ on functions commute, Tanno managed to compute the full effect of this deformation on the spectrum of the Laplacian on functions in [Tan79, Lemma 4.1. and Proposition 4.2.]. In particular, he proved that the first eigenvalue of the Laplacian on functions is given by

$$\lambda_1^0(t) = \begin{cases} 2t + t^{-2} & \text{if } t^{-3} \leq 6 \\ 8t & \text{if } t^{-3} \geq 6 \end{cases}$$

Note that at $t = 1$ (the round metric), the first eigenvalue of the Laplacian on functions is 3, wheres the square of the first curl eigenvalue is 4. Thus, the combinatorial structure of the first few eigenvalues of $g_1 = g_{st}$ is that the first four eigenvalues (counted with multiplicity) belong to the exact spectrum, and the next 6 eigenvalues (counted with multiplicity) belong to the coexact spectrum. While the Hodge Laplacian restricted to the coexact spectrum does not have the nice commutation relation descried above, Tanno proved in [Tan83, Theorem 4.1.] that there is a coexact eigenvalue of the Hodge Laplacian which goes below $\lambda_1^0(t)$, the crossing happening at $t = \left(\frac{1}{4} + \frac{\sqrt{5}}{4}\right)^{\frac{1}{3}}$, thus changing the combinatorial structure of the spectrum of the Hodge Laplacian to one where the first eigenvalue is a coexact one. This concludes the proof of the first claim.

For the second claim we consider the same family of metrics but concentrate on the deformation of the 6-dimensional eigenspace of the Hodge Laplacian associated to the eigenvalue 4 with respect to the round metric $g_1 = g_{st}$. This eigenspace is composed of the Hopf fields and the anti-Hopf fields (obtained from the Hopf fields via an application of the orientation reversing involution $(x_1, y_1, x_2, y_2) \mapsto (x_1, y_1, x_2, -y_2)$ restricted to $\mathbb{S}^3$, see for example [PS21, Remark 6]). Tanno proves [Tan83, Theorem 4.1] that, along the family of metrics g_t, this eigenspace splits into three eigenspaces associated to the three eigenvalues:

1. $4t^2$
2. $2t^{-2}$
3. $8t + t(t^6 + 8t^3)^{\frac{1}{2}} + t^4$

The first of these corresponds to the Hopf field α, the second one corresponds to the remaining two Hopf fields, and the third one corresponds to the three anti-Hopf fields and therefore the first negative eigenvalue of the curl operator g_t for t close enough to 1. Therefore we have that along the transition from $t = 1 - \varepsilon$ to $t = 1 + \varepsilon$ the combinatorial structure of the coexact spectrum changes from one where the first eigenvalue is positive, followed by three negative eigenvalues (counted with multiplicity) to one where the first two eigenvalues (counted with multiplicity) are positive followed by three negative ones. Breaking up these higher multiplicities with a generic perturbation completes the proof. □

We now get Theorem 2.1.2 and Corollary 2.1.3 as easy consequences of everything that has been discussed in section 2.3.

Proof of Theorem 2.1.2. Proposition 2.3.9 implies that the spectrum of the Hodge Laplacian is not simple along generic 1-parameter families of Riemannian metrics. The result by Uhlenbeck on the simplicity of the Laplace-Beltrami operator along 1-parameter families of Riemannian metrics, Lemma 2.3.4 and the proof of Theorem 2.1.1 allow us to conclude simplicity of the positive and negative coexact spectrum along generic 1-parameter families. □

An immediate consequence of Proposition 2.3.9 is the proof of Corollary 2.1.3.

Proof of Corollary 2.1.3. By Lemma 2.2.4, a set has meagre codimension 1 iff it is meagre. The fact that the non-simple set of Riemannian metrics for which the Hodge Laplacian in dimension 3 does not have simple nonzero spectrum has meagre codimension 1 was proven in [EP12b].
Now by Proposition 2.3.9, the non-simple set cannot have meagre codimension 2 since otherwise we would get generic simplicity for 1-parameter families. □

3 On Arnold's Transversality Conjecture for the Laplace-Beltrami Operator

(Based almost verbatim on [GK23], joint work with Josef Greilhuber)

Abstract

This paper is concerned with the structure of the set of Riemannian metrics on a connected manifold such that the corresponding Laplace-Beltrami operator has an eigenvalue of a given multiplicity. We introduce a simple geometric condition on metrics and their corresponding Laplace eigenfunctions, and show it is equivalent to the strong Arnold hypothesis. This hypothesis essentially posits that multiple Laplace eigenvalues split up under perturbation like those of symmetric matrices. We prove our condition is satisfied except on a set of infinite codimension, and use this to obtain non-crossing rules for Laplace eigenvalues. Furthermore, we show that our condition is satisfied on all metrics admitting eigenvalues of multiplicity at most six, and exhibit examples of metrics violating it.

3.1 Introduction

A famous result of Wigner and von Neumann [NW29] called the "non-crossing rule" states that energy levels of quantum mechanical systems do not cross along generic 1-parameter families of Hamiltonians. Inspired by this, Arnold [Arn72] proposed a transversality condition for sufficiently general families of operators, later called the *strong Arnold hypothesis* (SAH) by de Verdière [Col88]: operators in the family with an eigenvalue of multiplicity m should locally form a manifold of codimension $\frac{m(m+1)}{2} - 1$, given by a natural defining function. Arnold also noted that generic k-parameter families of such operators avoid m-fold eigenvalues to exactly the extent one would expect by the codimension count. We will say that such a family of self-adjoint operators satisfies a *maximal non-crossing rule*.

The non-crossing rule for 1-parameter families of Laplace-Beltrami operators was proven by Uhlenbeck [Uhl76] using a Sard-type theorem, as opposed to a local structure theorem for the set of operators with multiple eigenvalues that Arnold had in mind. Arnold had conjectured that the family of Laplace operators over the set of Riemannian metrics on a given manifold satisfies the SAH, but counterexamples were exhibited by de Verdière in [Col88]. Soon afterwards, Lupo and Micheletti [LM93] independently discovered Arnold's approach to eigenvalue multiplicities in families of self-adjoint operators. Micheletti and Pistoia [MP98] later showed that the SAH is satisfied for Laplace operators on 2-dimensional manifolds for eigenvalues of multiplicity 2, and proved a precursor to Theorem 3.1.3 of the present text. Around the same time, Teytel introduced a useful criterion for establishing 1-parameter non-crossing rules [Tey99, Theorem A]. In a similar vein, Porter and Schwarz [PS17] established very general theorems on avoidance of intersection with countable unions of submanifolds in a Banach space.

To the best of our knowledge, there has not been further progress on understanding the local structure of the set of Riemannian metrics for which the Laplace operator has non-simple spectrum. In particular, no non-crossing rule in multiplicity greater than two has been established for the Laplacian (nor for any other natural, infinite dimensional family of operators). In this paper we prove that the set of metrics for which the SAH does not hold has infinite codimension. This means that Arnold's transversality conjecture "morally" holds for this family of self-adjoint operators, because it allows for a maximal non-crossing rule: Generic k-parameter families of metrics avoid Laplace operators with multiple eigenvalues to exactly the extent that Arnold expected.

We would like to point out that there are very general families of operators for which Arnold's transversality hypothesis does not hold, even in this restricted sense. Arnold himself was aware that the hypothesis fails if, for example, certain symmetry conditions are imposed on the family of operators, which forces some clusters of eigenvalues to collide along 1-parameter families of operators [VWA97, Appendix 10, subsection D]. This will be discussed in Section 3.4.4.

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3.1.1 Statement of the results.

We begin by introducing a condition on eigenfunctions of the Laplace operator which will turn out to be equivalent to the SAH (Theorem 3.1.3) but is significantly simpler than all previously known conditions.

Definition 3.1.1. *Let (M,g) be a closed Riemannian manifold and Δ_g its Laplace-Beltrami operator. We call a nonzero eigenvalue $\lambda \in \sigma(\Delta_g)$ of multiplicity $m \geq 2$* degenerate *if there exist k linearly independent eigenfunctions $u_1,\dots,u_k$ corresponding to λ such that, for some choice of signs $\epsilon_1,\dots,\epsilon_k \in \{-1,1\}$,*

$$\sum_{j=1}^{k} \epsilon_j u_j^2 = 0, \tag{3.1.1}$$

$$\sum_{j=1}^{k} \epsilon_j \, du_j \otimes du_j = 0. \tag{3.1.2}$$

If there are linearly independent eigenfunctions such that (3.1.1) holds, but not necessarily (3.1.2), we call the eigenvalue conformally degenerate.

Remark 3.1.2. *In fact, (3.1.2) already implies (3.1.1), as is shown in Proposition 3.2.5, Section 3.2. Thus, one could drop (3.1.1) from the definition of a degenerate eigenvalue. We included it to make the relation to conformal degeneracy clear.*

This definition is motivated by the behaviour of the eigenvalue λ under general and under conformal perturbation of the metric, respectively. We will see that, under perturbation by a symmetric $(0,2)$-tensor field, λ splits up in the same manner as an m-fold eigenvalue of a symmetric matrix if and only if λ is nondegenerate. If one instead considers only conformal perturbations, the same holds if and only if λ is conformally nondegenerate, which is a stronger condition. The notion of conformal nondegeneracy in multiplicity 2 has previously been recognized as central to the splitting behaviour of double eigenvalues for conformally covariant operators by Canzani [Can14].

Definition 3.1.1 is superficially remarkably similar to the condition characterizing *extremal* eigenvalues in [EI08a], i.e., those eigenvalues which necessarily decrease in absolute value under perturbation of the metric, see Question 3.4.8 in Section 3.4.4.

To formulate a precise statement, we introduce some notation. Let M be a closed, smooth manifold, which we will always assume to be of dimension at least 2. Denote the Fréchet manifold of smooth Riemannian metrics on M by $\mathcal{G}(M)$. Fixing $g_0 \in \mathcal{G}(M)$, we also consider the space $\mathcal{G}(M,g_0)$ of smooth conformal multiples of g_0. Finally, we will denote the space of real symmetric $m \times m$ matrices with $\mathbb{R}^{\frac{m(m+1)}{2}}$.

Theorem 3.1.3. *Let g be a Riemannian metric on a closed smooth manifold M of dimension at least 2. Suppose λ is a nondegenerate eigenvalue of Δ_g with multiplicity m. Then there exist an open neighborhood $\mathcal{U}$ of g in $\mathcal{G}(M)$, a submersion $\pi : \mathcal{U} \to \mathbb{R}^{\frac{m(m+1)}{2}}$ and $\varepsilon > 0$ such that*

$$\sigma(\Delta_{g'}) \cap (\lambda - \varepsilon, \lambda + \varepsilon) = \sigma(\pi(g')), \tag{3.1.3}$$

counted with multiplicity, for all $g' \in \mathcal{U}$.

If λ is conformally nondegenerate, the analogous statement with $\mathcal{G}(M)$ replaced by $\mathcal{G}(M,g)$ holds as well.

Next, we prove that the set of (conformally) degenerate metrics is small. In order to make this precise we introduce the following notion.

Definition 3.1.4. *A subset $\mathcal{D}$ of a Fréchet manifold $\mathcal{X}$ is said to have transversality codimension k if for every smooth manifold M of dimension less than k, the set of maps in $C^\infty(M, \mathcal{X})$ avoiding $\mathcal{D}$ is residual. We say a subset $\mathcal{D}$ of $\mathcal{X}$ has transversality codimension infinity if it has transversality codimension k for all $k \in \mathbb{N}$.*

This definition is meant to capture sets that behave like submanifolds from a transversality theoretic point of view but which might be much more irregular. Indeed, we will see that the set of metrics for which the Laplacian has non-simple spectrum has a fairly complicated structure. Using this definition, we can make precise the sense in which metrics admitting a conformally degenerate eigenvalue (and therefore also those admitting a degenerate eigenvalue) have infinite codimension.

Theorem 3.1.5. *Let M be a closed, connected smooth manifold of dimension at least 2, and $\mathcal{G}(M)$ the set of smooth Riemannian metrics on M. Then the set of metrics $g \in \mathcal{G}(M)$ such that Δ_g has a conformally degenerate eigenvalue has infinite transversality codimension. The same holds in any fixed conformal class $\mathcal{G}(M, g_0)$.*

Theorem 3.1.3 and Theorem 3.1.5 together imply a sweeping statement on the set of metrics admitting eigenvalues of a given multiplicity. It shows that a maximal non-crossing rule indeed holds in our setting.

Theorem 3.1.6. *Let M be a closed, connected smooth manifold of dimension at least 2, and $\mathcal{G}(M)$ the set of smooth Riemannian metrics on M. Then the set of metrics $g \in \mathcal{G}(M)$ where Δ_g has an eigenvalue of multiplicity at least m has transversality codimension $\frac{m(m+1)}{2} - 1$. The same holds in any fixed conformal class $\mathcal{G}(M, g_0)$.*

Remark 3.1.7. *Throughout this paper we work with C^∞ metrics. In previous work on the subject of non-crossing rules ([Uhl76],[Tey99]), the underlying spaces of metrics had finite regularity and thus a Banach manifold structure. However, all relevant maps occurring in the present paper have finite dimensional domains or codomains, allowing us to carry over the requisite transversality theory. Smooth regularity is crucial for our proof of Theorem 3.1.5. We will nevertheless be able to conclude that the maximal non-crossing rule holds for C^s metrics for all $s \geq 0$, see Corollary 3.3.4. This corollary is proven from Theorem 3.1.6, using only continuous dependence of the eigenvalues, and holds in lower regularity than previous statements in the literature, which were obtained via perturbation theory in C^s-regularity, $s \geq 2$.*

The structure of the paper is as follows: In Section 2 we set the stage for the transversality theoretic results: We introduce the strong Arnold hypothesis in detail, and prove a structure theorem similar to [MP98, Theorem 1.8]. We then specialize it to the case of the Laplace operator and prove Theorem 3.1.3. Section 3 is the technical heart of the paper, establishing Theorems 3.1.5 and 3.1.6. Section 4 is concerned with a closer examination of degenerate eigenvalues, containing a proof that no such eigenvalues exist below multiplicity 7 (Theorem 3.4.1), and examples of (conformally) degenerate eigenvalues. We conclude the paper with a number of open questions.

3.1.2 Preliminaries

In this paper, Einstein's summation convention is used, meaning that any index occuring twice in an expression is automatically summed over. We will often write ∂_j for $\frac{\partial}{\partial x^j}$ in coordinate expressions.

Let (M, g) be a closed, smooth Riemannian manifold. Its Laplace-Beltrami operator Δ_g is given in coordinates by

$$\Delta_g u = |g|^{-\frac{1}{2}}\partial_j(|g|^{\frac{1}{2}} g^{jk}\partial_k u),$$

where, as usual, $|g|$ denotes $\det g$ and g^{jk} the components of g^{-1}. Of course, this expression arises necessarily if we wish the identity

$$\int_M u\Delta_g v d\mu_g = -\int_M g(\nabla_g u, \nabla_g v)\, d\mu_g$$

to hold for all $u, v \in C^\infty(M)$, where ∇_g and μ_g denote the gradient with respect to g and the volume form, respectively. Let $H^2(M)$ and $L^2(M)$ denote the respective Sobolev spaces, defined with respect to a fixed reference metric on M. It is well known that Δ_g is a bounded operator from $H^2(M)$ to $L^2(M)$. It can be interpreted as a closed unbounded operator on $L^2(M)$, which is self-adjoint with respect to the inner product induced by g. Its spectrum consists of one zero eigenvalue and a countable, discrete set of negative eigenvalues with finite multiplicity.

It will be crucial in our proof of Theorem 3.1.5 that eigenfunctions satisfy a unique continuation theorem: If M is connected, eigenfunctions of Δ_g cannot vanish to infinite order anywhere. This follows from Aronszajn's unique continuation theorem for general elliptic operators of second order [Aro57].

Following Hamilton's book [Ham82], we call a map $F : \mathcal{F} \to \mathcal{F}'$ between Fréchet spaces $\mathcal{F}$ and $\mathcal{F}'$ *continuously differentiable*, or C^1, if

$$DF(q)[h] = \lim_{t\to 0} \frac{F(q + th) - F(q)}{t}$$

exists everywhere, is linear in h, and *jointly continuous* as a function of q and h. This notion is stronger than Gateaux differentiability, but much weaker than the definition of continuous differentiability one might expect: The derivative DF need *not* be a continuous map from $\mathcal{F}$ into the space of continuous linear operators from $\mathcal{F}$ to $\mathcal{F}'$. In particular, we never specify a topology on that space.

Let X, Y and Z be Banach spaces, and denote by $\mathcal{L}(X, Y)$ and $\mathcal{L}(Y, Z)$ the spaces of continous linear maps from X to Y and from Y to Z, respectively, equipped with the operator norm. Suppose $A : \mathcal{F} \to \mathcal{L}(Y, Z)$ and $B : \mathcal{F} \to \mathcal{L}(X, Y)$ are C^1 maps. Then the following version of the product rule holds: For all $q, h \in \mathcal{F}$,

$$D(A \circ B)(q)[h] = DA(q)[h] \circ B(q) + A(q) \circ DB(q)[h].$$

It is proven in the same way as the product rule in one-variable calculus, since only a directional derivative is taken. The right hand side is clearly jointly continuous in q and h, hence $A \circ B : \mathcal{F} \to \mathcal{L}(X, Z)$ is C^1.

All maps occuring in this paper have either finite dimensional domain or finite dimensional codomain. Thus, none of the usual problems which beset differential geometry in Fréchet spaces occur. Borrowing terminology from Freyn [Fre15], we call a subset $\mathcal{M} \subseteq \mathcal{X}$ of a Fréchet manifold $\mathcal{X}$ a *cofinite Fréchet submanifold* of codimension $N \in \mathbb{N}$ if, for all $p \in \mathcal{M}$, there exists a neighborhood $\mathcal{U} \subseteq \mathcal{X}$ of p and a C^1 submersion $\pi : \mathcal{U} \to \mathbb{R}^N$ such that $\mathcal{M} \cap \mathcal{U} = \pi^{-1}(0)$.

The standard proof of the implicit function theorem then yields that a cofinite Fréchet submanifold is locally a graph over its tangent space at any point $p \in \mathcal{M}$, and therefore a Fréchet manifold modelled on the Fréchet space $\ker DF(p) \subseteq T_p\mathcal{X}$. No Nash-Moser iteration is needed in this construction. Lacking a reference for this simple observation, we provide a proof in Section 3.5.

Thom's transversality theory can be carried over with minimal modifications to the setting of smooth maps $f : \mathbb{R}^K \to \mathcal{X}$ and cofinite submanifolds $\mathcal{M} \subseteq \mathcal{X}$. The necessary statements are provided and proven in Section 3.6.

The family of operators $\Delta_g : H^2(M) \to L^2(M)$ gives rise to a C^1 map

$$\Delta_{(\cdot)} : \mathcal{G}(M) \to \mathcal{L}(H^2(M), L^2(M)),$$

into the Banach space of bounded linear operators from $H^2(M)$ to $L^2(M)$. Its resolvent $(\lambda - \Delta_g)^{-1}$ is likewise a C^1 map from a dense open subset of $\mathbb{C} \times \mathcal{G}(M)$ to $\mathcal{L}(L^2(M), H^2(M))$ (and thus also, if needed, into $\mathcal{L}(L^2(M))$). Finally, fixing a circle γ in $\mathbb{C}$, the corresponding Riesz projection

$$P_\gamma(g) = \frac{1}{2\pi i} \int_\gamma (\zeta - \Delta_g)^{-1} d\zeta$$

is a C^1 map into $\mathcal{L}(L^2(M))$ on the open subset of metrics admitting no Laplace eigenvalue on γ. These statements are proven in Section 3.7.

3.2 A local structure theorem for metrics with Laplace eigenvalues of higher multiplicity

The goal of the present section is to prove Theorem 3.1.3. We initially formulate the setting rather generally in order to facilitate the application of our ideas to different problems. In particular, we do not require the operator family of interest to have a fixed domain, independent of the parameter, and work with the resolvent instead. This is not needed in the present situation, where one can view the Laplacian as a family of bounded operators from $H^2(M)$ to $L^2(M)$. However, it enables an application of our theorem to operator families with varying boundary conditions.

3.2.1 General setup for the construction of local defining functions and parameter transversality

The construction described in this section is inspired by similar setups in [Col88], [LM93], and [MP98], all of whom worked in finite regularity and could thus rely on Banach manifold theory. Another notable difference lies in our explicit construction of the *matrix-valued* function π which exactly describes the behaviour of an eigenvalue of higher multiplicity under perturbation: [LM93] proceeds from a purely geometric perspective, and while [Col88] introduces a similar defining function, it is valued in quadratic forms, assuming a fixed inner product on the underlying space. Our result is directly applicable to families of operators which are self-adjoint with respect to a variable inner product, without having to find an explicit family of isometries which conjugates the problem into one on a fixed Hilbert space, as in [Col88]. At the end of the following discussion, Theorem 3.2.1 provides a summary of our general setup.

Suppose we are given a separable Fréchet manifold $\mathcal{X}$, a C^1 family of equivalent inner products $\{\langle -, - \rangle_q\}_{q\in\mathcal{X}}$ on a fixed real Hilbert space H, and a family of unbounded operators $\{A_q\}_{q\in\mathcal{X}}$, each self-adjoint with respect to $\langle -, - \rangle_q$ and with compact resolvent. Assume furthermore that the resolvents $(z - A_q)^{-1}$ form a C^1 family of bounded operators defined on a dense open subset of $\mathbb{C} \times \mathcal{X}$. Now fix a parameter value q_0 such that A_{q_0} has an eigenvalue λ of multiplicity m. Since the eigenvalues of A move continuously with respect to the parameter by the variational characterization of eigenvalues [Tes14, Theorem 4.10], there exist $\epsilon > 0$ and an open neighbourhood $\mathcal{U}$ of q_0 such that the rank of the spectral projection

$$P(q) = \frac{1}{2\pi i}\int_\gamma (\zeta - A_q)^{-1} d\zeta, \quad \gamma(t) = \lambda + \epsilon e^{it}$$

is m for all $q \in \mathcal{U}$. Define $E_\gamma(q) := \operatorname{Im} P(q)$. Consider the family of linear maps $S(q) := P(q_0) \circ P(q)|_{E_\gamma(q_0)} : E_\gamma(q_0) \to E_\gamma(q_0)$. After possibly shrinking $\mathcal{U}$, $S(q)$ is an isomorphism for all $q \in \mathcal{U}$, since $P(q)$ varies continuously with q, and $P(q_0)|_{E_\gamma(q_0)} = \mathrm{id}|_{E_\gamma(q_0)}$. By construction, $S(q)^{-1} \circ P(q_0) : H \to E_\gamma(q_0)$ is a C^1 family of left inverses to $P(q)|_{E_\gamma(q_0)}$. Fix $\mu \in \mathbb{R}$ outside of the spectrum of all $A_q, q \in \mathcal{U}$, and define the map

$$\begin{aligned} f : \mathcal{U} &\to GL(E_{q_0}) \\ q &\mapsto S(q)^{-1} \circ P(q_0) \circ (\mu - A_q)^{-1} \circ \left(P(q)|_{E_\gamma(q_0)}\right). \end{aligned}$$

The eigenvalues of $f(q)$ are precisely the eigenvalues of $(\mu - A_q)^{-1}$ corresponding to eigenvalues of A_q encircled by γ. Thus, $f^{-1}(\mathbb{R} \cdot \mathrm{id})$ contains precisely those $q \in \mathcal{U}$ for which A_q has an eigenvalue of multiplicity m in $(\lambda - \epsilon, \lambda + \epsilon)$.

We want to use the map f as a local defining function for the set of parameter values $q \in \mathcal{U}$ so that A_q has an eigenvalue of multiplicity m that is close to λ. To achieve this, f has to be modified further to account for the fact that $f(q)$ is symmetric with respect to the inner product $\langle P(q)\,\cdot\,, P(q)\,\cdot\,\rangle_q$, which of course depends on q. Since this inner product is continuously differentiable in q, we may perform Gram-Schmidt orthonormalization to obtain a C^1 family of linear isomorphisms $Q(q) : \mathbb{R}^m \to E_{q_0}$ such that $\langle P(q)Q(q)e_k, P(q)Q(q)e_\ell\rangle_q = \delta_{k\ell}$. Then, symmetry of $(\mu - A_q)^{-1}$ with respect to $\langle\,\cdot\,,\,\cdot\,\rangle_q$ shows that $Q(q)^{-1} \circ f(q) \circ Q(q)$ is a symmetric $m \times m$ matrix, allowing us to define

$$\begin{aligned}\pi_\mu : \mathcal{U} &\to \mathbb{R}^{\frac{m(m+1)}{2}}\\ q &\mapsto Q(q)^{-1} \circ f(q) \circ Q(q).\end{aligned}$$

As a composition of C^1 families of linear maps and their inverses (which are C^1, e.g, by [Ham82, Theorem II.3.1.1]), π_μ is C^1. Denote $R(q) = (\mu - A_q)^{-1}$ and $\tilde{P}(q) = P(q)|_{E_\gamma(q_0)}$. Given $h \in T_{q_0}\mathcal{X}$, we first calculate

$$\begin{aligned}Df(q_0)[h] =& - S(q_0)^{-1} \circ DS(q_0)[h] \circ S(q_0)^{-1} \circ P(q_0) \circ R(q_0) \circ \tilde{P}(q_0)\\ &+ S(q_0)^{-1} \circ P(q_0) \circ DR(q_0)[h] \circ \tilde{P}(q_0)\\ &+ S(q_0)^{-1} \circ P(q_0) \circ R(q_0) \circ D\tilde{P}(q_0)[h]\\ &= -\tfrac{1}{\mu-\lambda} DS(q_0)[h] + P(q_0) \circ DR(q_0)[h] + P(q_0) \circ R(q_0) \circ D\tilde{P}(q_0)[h],\end{aligned}$$

where we used $S(q_0) = \mathrm{id}$ and that $R(q_0)$ acts on $E_\gamma(q_0)$ via multiplication by $\frac{1}{\mu-\lambda}$. Furthermore, $P(q_0) \circ R(q_0) = \frac{1}{\mu-\lambda} P(q_0)$. This allows us to further simplify

$$\begin{aligned}Df(q_0)[h] &= P(q_0) \circ DR(q_0)[h] - \tfrac{1}{\mu-\lambda} \Big(DS(q_0)[h] - P(q_0) \circ D\tilde{P}(q_0)[h] \Big)\\ &= P(q_0) \circ DR(q_0)[h],\end{aligned}$$

because the term in brackets vanishes by the product rule. Now the derivative of π_μ at q_0 is computed to be

$$\begin{aligned}D\pi_\mu(q_0)[h] &= -Q(q_0)^{-1} \circ DQ(q_0)[h] \circ Q(q_0)^{-1} \circ f(q_0) \circ Q(q_0)\\ &+ Q(q_0)^{-1} \circ Df(q_0)[h] \circ Q(q_0) + Q(q_0)^{-1} \circ f(q_0) \circ DQ(q_0)[h]\\ &= Q(q_0)^{-1} \circ Df(q_0)[h] \circ Q(q_0) = Q(q_0)^{-1} \circ P(q_0) \circ DR(q_0)[h] \circ Q(q_0),\end{aligned}$$

since $f(q_0) = \frac{\mathrm{id}}{\mu-\lambda}$. To obtain the entries of the matrix $D\pi_\mu(q_0)[h]$, let $\{e_k\}_{k=1}^m$ denote the standard basis of $\mathbb{R}^m$, so that $\big\{Q(q_0)e_k\big\}_{k=1}^m$ is an orthonormal basis of the eigenspace $E_\gamma(q_0)$. For $1 \leq k, \ell \leq m$, we have

$$\begin{aligned}\langle D\pi_\mu(q_0)[h]e_k, e_\ell\rangle_{\mathbb{R}^m} &= \langle P(q_0) \circ DR(q_0)[h] \circ Q(q_0)e_k, Q(q_0)e_\ell\rangle_{q_0}\\ &= \langle DR(q_0)[h] \circ Q(q_0)e_k, Q(q_0)e_\ell\rangle_{q_0}.\end{aligned}$$

It remains to remove the dependence on the arbitrary choice of resolvent. Note that, for all $q \in \mathcal{U}$, $A_q|_{E_\gamma(q)} = \mu - R(q)^{-1}$. Thus, if we define $\pi(q) = \mu - \pi_\mu(q)^{-1}$, then the spectrum of A_q encircled by γ agrees with that of the matrix $\pi(q)$. Since $\pi_\mu(q_0) = \frac{\mathrm{id}}{\mu-\lambda}$, the quotient rule implies

$$\langle D\pi(q_0)[h]e_k, e_\ell\rangle_{\mathbb{R}^m} = \big\langle (\mu-\lambda)^2 DR(q_0)[h] \circ Q(q_0)e_k, Q(q_0)e_\ell \big\rangle_{q_0}.$$

This matrix is, by construction, independent of the choice of μ. Alternatively, one may also differentiate the first resolvent identity to show that the operator $(\mu - \lambda)^2 P(q_0) \circ DR(q_0)[h]|_{E_\gamma(q_0)}$ is independent of this choice.

In the case that all operators A_q share the same domain (which will be true for the Laplace-Beltrami operators on a closed Riemannian manifold M), one can arrive at the expression $\left\langle(\mu-\lambda)^2 DR(q_0)[h]\circ Q(q_0)e_k, Q(q_0)e_\ell\right\rangle_{q_0}$ directly, without considering the resolvent. More precisely, suppose there is a fixed Banach space D which embeds continuously into H, and $A : \mathcal{X}\to\mathcal{L}(D,H)$ is a C^1 family of bounded linear operators. Differentiating the identity $R(q)\circ(\mu-A_q)=\mathrm{id}_D$ yields

$$(\mu-\lambda)DR(q_0)[h]|_{E_\gamma(q_0)} = -R(q_0)DA(q_0)[h]|_{E_\gamma(q_0)},$$

from which, since $P(q_0)\circ R(q_0)=(\mu-\lambda)^{-1}P(q_0)$, it follows that

$$(\mu-\lambda)^2 P(q_0)\circ DR(q_0)[h]|_{E_\gamma(q_0)} = -P(q_0)\circ DA(q_0)[h]|_{E_\gamma(q_0)},$$

If we denote the orthonormal basis $\{Q(q_0)e_k\}_{k=1}^m$ by $\{u_k\}_{k=1}^m$, we can thus write $D\pi(q_0)$ in a very convenient way:

$$D\pi(q_0)[h]_{k\ell} = \langle DA(q_0)[h]u_k, u_\ell\rangle_{q_0} \tag{3.2.1}$$

It is this expression that we will work with in the following sections.

Surjectivity of $D\pi(q)$ is an open condition, since π has a finite dimensional codomain. Thus, if $D\pi(q_0)$ is surjective, we can find an open neighborhood $\mathcal{U}$ of q_0 such that $\pi|_{\mathcal{U}}$ is a submersion. In particular, as mentioned in Section 3.1.2 (Preliminaries), this means that the set of parameters q such that that the eigenvalue λ does not split up is a cofinite Fréchet submanifold, because it is given as the preimage $\pi^{-1}(\mathbb{R}\cdot\mathrm{id})$, i.e., the preimage of a manifold under a submersion.

Let us collect the results of the preceding discussion in a theorem, which we must stress is similar to [MP98, Theorem 1.8], but more versatile, as it describes the full eigenvalue splitting behaviour, and adapted to a Fréchet manifold setting.

Theorem 3.2.1. *Let $\mathcal{X}$ be a Fréchet manifold, H a Hilbert space, $\{\langle -,-\rangle_q\}_{q\in\mathcal{X}}$ a C^1 family of inner products on H, and $\{A_q\}_{q\in\mathcal{X}}$ a family of unbounded operators, each self-adjoint with respect to $\langle -,-\rangle_q$ and with compact resolvent. Assume that the resolvent $(z-A_q)^{-1}$ is a C^1 family of bounded operators defined on a dense open subset of $\mathbb{C}\times\mathcal{X}$.*

Suppose λ is an eigenvalue with multiplicity m of A_{q_0} for some $q_0\in\mathcal{X}$. Denote $R(q)=(\mu-A_q)^{-1}$, for some $\mu\in\rho(A_{q_0})\cap\mathbb{R}$, and choose an orthonormal basis $\{u_k\}_{k=1}^m$ of the λ-eigenspace of A_{q_0}. If the linear map

$$h\mapsto \big(\langle DR(q_0)[h]u_k, u_\ell\rangle\big)_{k,\ell=1}^m$$

from $T_{q_0}\mathcal{X}$ to $\mathbb{R}^{\frac{m(m+1)}{2}}$ is surjective, there exist an open neighborhood $\mathcal{U}$ of q_0 in $\mathcal{X}$, a C^1 submersion $\pi:\mathcal{U}\to\mathbb{R}^{\frac{m(m+1)}{2}}$, and $\varepsilon>0$, such that

$$\sigma(A_q)\cap(\lambda-\varepsilon,\lambda+\varepsilon)=\sigma(\pi(q)), \tag{3.2.2}$$

counted with multiplicity, for all $q\in\mathcal{U}$. If, additionally, all operators A_q share the same domain D, the linear map

$$h\mapsto \big(\langle DA(q_0)[h]u_k, u_\ell\rangle\big)_{k,\ell=1}^m$$

is well-defined and it is sufficient to check its surjectivity instead.
The preimage $\pi^{-1}(\mathbb{R}\cdot\mathrm{id})$ *is a Fréchet submanifold of codimension* $\frac{m(m+1)}{2}-1$.

In view of this theorem and the preceding discussion, we can formulate the strong Arnold hypothesis introduced by Colin de Verdière [Col88] very conveniently.

Definition 3.2.2. *The family of operators* $\{A_q\}_{q\in\mathcal{X}}$ *is said to satisfy the strong Arnold hypothesis (SAH) if* π *is a submersion for all* q *and* λ. *We say it satisfies the SAH at* q *and* λ *if* π *is a submersion at* q *for the eigenvalue* λ.

If the family A_q satisfies the strong Arnold hypothesis, one can prove a non-crossing rule akin to Theorem 3.1.6 in this very abstract setting provided that $\mathcal{X}$ is separable. This hypothesis is needed in several places to extract countable open covers, using the fact that a separable metric space is Lindelöf (i.e, every open cover admits a countable subcover). The family of Laplace-Beltrami operators does not satisfy the SAH, and so a proof of Theorem 3.1.6 must wait until the end of Section 3.3. Assuming the SAH, the proof of Theorem 3.1.6 can be carried over to the abstract setting of Theorem 3.2.1. We omit this to avoid overloading this article.

3.2.2 The case of the Laplace-Beltrami operator: Proof of Theorem 3.1.3

We now specialize the results of the preceding section to $\mathcal{X}=\mathcal{G}(M)$, the space of smooth metrics on a closed manifold M, or a fixed conformal class $\mathcal{X}=\mathcal{G}(M,g_0)$, and $A_q=\Delta_g$, the Laplace-Beltrami operator.

Proposition 3.2.3. *Let* M *be a closed smooth manifold of dimension* $n\geq 2$. *The following are equivalent:*

1. $\{\Delta_g\}_{g\in\mathcal{G}(M)}$ *does not satisfy the SAH at* g *and* λ.
2. *There exist a basis* $\{u_\alpha\}$ *of* E_λ *and* $A\in\mathbb{R}^{\frac{m(m+1)}{2}}\setminus\{0\}$ *such that*
$$A^{\alpha\beta}\left(\frac{\lambda}{2}u_\alpha u_\beta\, g+\frac{1}{2}g(\nabla u_\alpha,\nabla u_\beta)\, g-du_\alpha\otimes du_\beta\right)\equiv 0. \tag{3.2.3}$$
3. *There exist a basis* $\{u_\alpha\}$ *of* E_λ *and* $A\in\mathbb{R}^{\frac{m(m+1)}{2}}\setminus\{0\}$ *such that*
$$A^{\alpha\beta}u_\alpha u_\beta\equiv 0$$
and
$$A^{\alpha\beta}du_\alpha\otimes du_\beta\equiv 0.$$
4. λ *is a degenerate eigenvalue of* Δ_g.

Similarly, in the conformal setting, the following are equivalent:

1. $\{\Delta_g\}_{g\in\mathcal{G}(M,g_0)}$ *does not satisfy the SAH at* g *and* λ.
2. *There exist a basis* $\{u_\alpha\}$ *of* E_λ *and* $A \in \mathbb{R}^{\frac{m(m+1)}{2}} \setminus \{0\}$ *such that*
$$A^{\alpha\beta}\left(\lambda u_\alpha u_\beta + \frac{n-2}{n} g(\nabla u_\alpha, \nabla u_\beta)\right) \equiv 0.$$
3. *There exist a basis* $\{u_\alpha\}$ *of* E_λ *and* $A \in \mathbb{R}^{\frac{m(m+1)}{2}} \setminus \{0\}$ *such that*
$$A^{\alpha\beta} u_\alpha u_\beta \equiv 0.$$
4. λ *is a conformally degenerate eigenvalue of* Δ_g.

Proof. **(1)** $\iff$ **(2)** We express the Laplace operator in local coordinates:
$$\Delta_g u = |g|^{-\frac{1}{2}} \partial_j (|g|^{\frac{1}{2}} g^{jk} \partial_k u)$$
where by $|g|$ we mean the determinant of g. Given a symmetric $(0,2)$-tensor field h and using the identities
$$D_h |g|^s = s \cdot tr_g(h)|g|^s, \quad D_h g^{ij} = -h^{ij},$$
one can compute the variation of Δ_g in direction h:
$$\begin{aligned} D_h \Delta_g u =& -\frac{1}{2} tr_g(h)\Delta_g u + \frac{1}{2}|g|^{-\frac{1}{2}}\partial_j(tr_g(h)|g|^{\frac{1}{2}} g^{jk}\partial_k u) - |g|^{-\frac{1}{2}}\partial_j(|g|^{\frac{1}{2}} h^{jk}\partial_k u) \\ =& \frac{1}{2}\partial_j(tr_g(h))g^{jk}\partial_k u - |g|^{-\frac{1}{2}}\partial_j(|g|^{\frac{1}{2}} h^{jk}\partial_k u) \end{aligned}$$
We now compute the L^2 inner product
$$\begin{aligned} \langle u, D_h\Delta_g v\rangle =& \frac{1}{2}\int_M \partial_j(tr_g(h))g^{jk}(\partial_k u)v\, d\mu_g - \int_M |g|^{-\frac{1}{2}}\partial_j(|g|^{\frac{1}{2}} h^{jk}\partial_k u)v\, d\mu_g \\ =& -\frac{1}{2}\int_M \Big(tr_g(h)(\Delta_g u)v + tr_g(h)g(\nabla u, \nabla v) - 2h^{jk}[du\otimes dv]_{jk}\Big)\, d\mu_g \\ =& \int_M h^{jk}\left(-\frac{1}{2}g_{jk}(\Delta_g u)v - \frac{1}{2}g_{jk}g(\nabla u,\nabla v) + [du\otimes dv]_{jk}\right) d\mu_g \end{aligned}$$

Pick an arbitrary basis $\{u_\alpha\}$ of E_λ. If the SAH is not satisfied at g, then Theorem 3.2.1 implies there exists a nonzero symmetric matrix $A \in \mathbb{R}^{\frac{m(m+1)}{2}}$ such that for all h, $A^{\alpha\beta}\langle u_\alpha, D_h\Delta_g u_\beta\rangle = 0$. We obtain
$$0 = \int_M h^{jk}A^{\alpha\beta}\left(\frac{\lambda}{2}g_{jk}u_\alpha u_\beta + \frac{1}{2}g_{jk}g(\nabla u_\alpha, \nabla u_\beta) - [du_\alpha \otimes du_\beta]_{jk}\right) d\mu_g$$

for all symmetric $(0,2)$-tensor fields h. Since the expression in brackets is itself a symmetric $(0,2)$-tensor field, this is equivalent to

$$0 \equiv A^{\alpha\beta}\left(\frac{\lambda}{2}u_\alpha u_\beta\, g + \frac{1}{2}g(\nabla u_\alpha, \nabla u_\beta)\, g - du_\alpha \otimes du_\beta\right).$$

(2) $\iff$ (3) The converse direction is clear. For the other direction, we contract equation 3.2.3 with the metric g

$$A^{\alpha\beta}\left(\frac{n\lambda}{2}u_\alpha u_\beta + \frac{n-2}{2}g(\nabla u_\alpha, \nabla u_\beta)\right) = 0 \tag{3.2.4}$$

In dimension $n = 2$, this implies that $A^{\alpha\beta}u_\alpha u_\beta = 0$. Applying the Laplacian to this identity yields

$$0 = \Delta_g A^{\alpha\beta}u_\alpha u_\beta = 2A^{\alpha\beta}u_\alpha \Delta_g u_\beta + 2A^{\alpha\beta}g(\nabla u_\alpha, \nabla u_\beta) = 2A^{\alpha\beta}g(\nabla u_\alpha, \nabla u_\beta)$$

Reinserting these two expressions into equation 3.2.3 gives the desired conclusion.
In dimension $n \geq 3$, consider the function $F = A^{\alpha\beta}u_\alpha u_\beta$ and apply the Laplacian

$$\begin{aligned}\Delta_g F &= 2A^{\alpha\beta}u_\alpha \Delta_g(u_\beta) + 2A^{\alpha\beta}g(\nabla u_\alpha, \nabla u_\beta)\\ &= (2\lambda - 2\lambda\frac{n}{n-2})A^{\alpha\beta}u_\alpha u_\beta\\ &= 2\lambda\big(1 - \frac{n}{n-2}\big)F\end{aligned}$$

where we used equation 3.2.4. Note that the coefficient of F on the right is positive. It follows that $F \equiv 0$.

(3) $\iff$ (4)

The matrix A is symmetric, and hence, it can be diagonalized. Let B^γ_δ be a change of basis such that $A^{\alpha\beta} = B^\alpha_\gamma D^{\gamma\delta}B^\beta_\delta$, where D is diagonal. Then

$$A^{\alpha\beta}u_\alpha u_\beta = B^\alpha_\gamma D^{\gamma\delta}B^\beta_\delta u_\alpha u_\beta = D^{\gamma\delta}v_\gamma v_\delta$$

where $v_\gamma = B^\alpha_\gamma u_\alpha$. Upon rescaling the basis vectors $\{v_\gamma\}$ we may assume that the diagonal entries of D are valued in $\{-1, 0, 1\}$. The basis $\{v_\alpha\}$ may be chosen to be orthogonal: One may choose the original basis $\{u_\alpha\}$ orthonormal, and diagonalize the symmetric matrix $A^{\alpha\beta}$ by an orthogonal change of basis.

The proof for the conformal case is very similar. We can again use integration by parts to show that π is a submersion if and only if there is no basis $\{u_\alpha\}$ and $A \in \mathbb{R}^{\frac{m(m+1)}{2}}$ so that the traced version of equation 3.2.3 holds. Proving the equivalence of (2) and (3) again works by applying the Laplacian to $A^{\alpha\beta}u_\alpha u_\beta$. $\square$

Remark 3.2.4. *We would like to point out that a version of the first equivalence in Proposition 3.2.3 always works for self-adjoint differential operators on some L^2 as long as the space of variations is closed under multiplication with bump functions. One can* *use integration by parts to isolate h in $D_h A$ and then localize. The resulting expression may be unwieldy (this is the case, e.g, for the Hodge Laplacian).*

A particularly interesting class of examples is that of conformally covariant operators on vector bundles equipped with a metric g as treated by Canzani [Can14]. An inspection of the conformal variation computed in [Can14, Proposition 5.4] allows us to conclude that the failure of the SAH at a metric g for an eigenvalue λ in the conformal class is equivalent to the following: There exists a basis $\{u_\alpha\}$ of E_λ and an element $A \in \mathbb{R}^{\frac{m(m+1)}{2}}$ so that

$$A^{\alpha\beta} g(u_\alpha, u_\beta) \equiv 0.$$

Specializing to conformally covariant operators acting on smooth functions, this becomes

$$A^{\alpha\beta} u_\alpha u_\beta \equiv 0.$$

just like for the Laplace-Beltrami operator. Canzani proved that this condition is never satisfied in multiplicity 2, i.e., that there are no conformally degenerate eigenvalues of multiplicity 2 in this setting [Can14, Proposition 5.6]. Actually her proof shows something stronger, namely that even in higher multiplicity there can never be a pair of eigenfunctions u_1 and u_2 such that the above identity is satisfied. Inspecting the proof of Theorem 3.2.1 it is not hard to see that this implies a 1-parameter non-crossing rule for all conformally covariant operators acting on smooth functions.

The preceding observations and the proof of Theorem 3.1.5 (Section 3.3), which develops techniques for showing that relations between eigenfunctions of the form $A^{\alpha\beta} u_\alpha u_\beta \equiv 0$ are rarely satisfied, suggest that our methods can be applied to a great variety of operators.

Theorem 3.1.3 follows directly by combining Theorem 3.2.1 and Proposition 3.2.3.

Theorem 3.1.3, together with the cofinite transversality theory presented in Section 3.6, already implies that Theorem 3.1.6 holds in the complement of those metrics admitting a degenerate eigenvalue. In order to establish Theorem 3.1.6 in full we still have to show that the set of such metrics is small. A careful proof of Theorem 3.1.6 is therefore presented at the end of section 3.3.

Before proceeding with our analysis of degenerate eigenvalues, we show that they are in fact fully characterized by the condition $\sum_{j=1}^{k} \epsilon_j \, du_j \otimes du_j = 0$ (Equation (3.1.2) in Definition 3.1.1), since this condition already implies $\sum_{j=1}^{k} \epsilon_j u_j^2 = 0$ (Equation (3.1.1) in Definition 3.1.1). This fact will not be used in the rest of the text.

Proposition 3.2.5. *Let (M, g) be a closed Riemannian manifold. Suppose there exist m eigenfunctions $\{u_\alpha\}_{\alpha=1}^{m}$ of Δ_g, corresponding to an eigenvalue $\lambda < 0$, and a symmetric matrix $A^{\alpha\beta} \in \mathbb{R}^{\frac{m(m+1)}{2}}$, such that*

$$A^{\alpha\beta} \, du_\alpha \otimes du_\beta = 0.$$

Then $A^{\alpha\beta} u_\alpha u_\beta = 0$ as well.

Proof. In coordinates, $A^{\alpha\beta}\, du_\alpha \otimes du_\beta = 0$ corresponds to the identity

$$A^{\alpha\beta}\partial_k u_\alpha \partial_\ell u_\beta = 0,$$

for all $k, \ell = 1, \ldots, n$. Taking the derivative of this expression in the ∂_j-direction and contracting with $g^{k\ell}$ yields

$$A^{\alpha\beta} g^{k\ell} \partial_j \partial_k u_\alpha \partial_\ell u_\beta + A^{\alpha\beta} g^{k\ell} \partial_k u_\alpha \partial_j \partial_\ell u_\beta = 0.$$

On the other hand, because both A and g are symmetric,

$$A^{\alpha\beta} g^{k\ell} \partial_j \partial_k u_\alpha \partial_\ell u_\beta = A^{\beta\alpha} g^{\ell k} \partial_j \partial_\ell u_\beta \partial_k u_\alpha = A^{\alpha\beta} g^{k\ell} \partial_k u_\alpha \partial_j \partial_\ell u_\beta,$$

so $A^{\alpha\beta} g^{k\ell} \partial_j \partial_k u_\alpha \partial_\ell u_\beta = 0$. This allows us to reduce the expression which results from taking the divergence of $0 = A^{\alpha\beta}\, \nabla u_\alpha \otimes \nabla u_\beta$ in the first slot. In coordinates, this amounts to the following calculation, valid for all $\ell = 1, \ldots, n$:

$$\begin{aligned} 0 &= A^{\alpha\beta} |g|^{-\frac{1}{2}} \partial_j \left(|g|^{\frac{1}{2}} g^{jk} \partial_k u_\alpha \partial_\ell u_\beta \right) \\ &= A^{\alpha\beta} \left((\Delta_g u_\alpha) \partial_\ell u_\beta + g^{jk} \partial_k u_\alpha \partial_j \partial_\ell u_\beta \right) = \lambda A^{\alpha\beta} u_\alpha \partial_\ell u_\beta \end{aligned}$$

Now, we take the divergence in the second slot and obtain

$$\begin{aligned} 0 &= A^{\alpha\beta} |g|^{-\frac{1}{2}} \partial_j \left(u_\alpha |g|^{\frac{1}{2}} g^{j\ell} \partial_\ell u_\beta \right) \\ &= A^{\alpha\beta} \left(u_\alpha \Delta u_\beta + g^{j\ell} \partial_j u_\alpha \partial_\ell u_\beta \right) = \lambda A^{\alpha\beta} u_\alpha u_\beta, \end{aligned}$$

as desired. □

3.3 Metrics with a (conformally) degenerate eigenvalue are rare

By the results of the preceding section, the behavior of non-degenerate multiple eigenvalues under perturbation of the metric parallels that of multiple eigenvalues of symmetric matrices. To get an unconditional result like Theorem 3.1.6, one must still prove that metrics with a degenerate eigenvalue are rare. Examples of metrics with a degenerate eigenvalue indeed exist, as is shown in Proposition 3.4.3 below. The aim of this section is to prove Theorem 3.1.5, and thus establish Theorem 3.1.6.

The proof of Theorem 3.1.5 takes up most of this section, and proceeds roughly as follows: We first stratify the set of metrics admitting conformally degenerate eigenvalues into several pieces which can be covered locally by cofinite Fréchet submanifolds of arbitrarily high codimension. The key distinction turns out to be whether the fundamental solution to the Helmholtz equation $(\lambda - \Delta)u = \delta_p$ is a quotient of smooth functions whose denominator does not vanish to infinite order anywhere. Those strata where this is not the case are shown to be of infinite transversality codimension using basic microlocal analysis (Proposition 3.3.1), and the other case is dealt with using heat kernel asymptotics (Lemma 3.3.3). Finally, we give a careful proof of Theorem 3.1.6, which at the same time provides the final arguments necessary to establish Theorem 3.1.5.

It is convenient to introduce the ring of quotients of smooth functions whose denominator does not vanish to infinite order anywhere. We shall denote it by $\mathcal{Q}^\infty(M)$, and sometimes write $\frac{\varphi}{\psi} \in \mathcal{Q}^\infty(M)$ for its elements, where the denominator $\psi \in C^\infty(M)$ has finite vanishing order everywhere.

We begin by stratifying the set $\mathcal{S} \subseteq \mathcal{G}(M)$ of metrics admitting a conformally degenerate eigenvalue.

(i) Let $\mathcal{S}_\infty(\lambda)$ denote those metrics for which $\lambda \in \sigma(\Delta_g)$, and the Schwartz kernel of the operator L defined by $L(\lambda - \Delta_g) = (\lambda - \Delta_g)L = \mathbb{I} - P_\lambda$ is an element of $\mathcal{Q}^\infty(M \times M)$.

(ii) For $m \in \mathbb{N}$, $k \in \{1, \dots, \frac{m(m+1)}{2}\}$ and $\lambda \in (-\infty, 0)$, let $\mathcal{S}_{m,k}(\lambda)$ denote the set of $g \in \mathcal{G}(M) \setminus \mathcal{S}_\infty(\lambda)$ such that $\lambda \in \sigma(\Delta_g)$ has multiplicity m, and, given a basis $\{u_\alpha\}_{\alpha=1}^m$ of E_λ, the equation $A^{\alpha\beta}u_\alpha(x)u_\beta(x) \equiv 0$ has exactly k linearly independent solutions $A \in \mathbb{R}^{\frac{m(m+1)}{2}}$.

The next proposition shows that the union of strata $\mathcal{S}_{m,k}(\lambda)$ can be covered locally by Fréchet manifolds of finite, but arbitrarily high, codimension, if λ is constrained in a suitably small interval.

Proposition 3.3.1. *Let M be a closed, connected smooth manifold of dimension at least 2. Consider a metric $g \in \mathcal{S}_{m,k}(\lambda)$, for some $m, k \in \mathbb{N}$, $\lambda \in (-\infty, 0)$. Let $\gamma : \mathbb{S}^1 \to \mathbb{C}$ be a smooth, positively oriented Jordan curve encircling λ, but no other point of $\sigma(\Delta_g)$. Let $\mathcal{U}$ be a neighborhood of g in $\mathcal{G}(M)$ such that for each $\tilde{g} \in \mathcal{U}$, exactly m eigenvalues of $\Delta_{\tilde{g}}$, counted with multiplicity, are encircled by γ. For $\tilde{g} \in \mathcal{U}$, denote the sum of the eigenspaces of these m eigenvalues by $E_\gamma(\tilde{g})$.*

Let $\widetilde{\mathcal{S}}_{m,k} \subseteq \mathcal{U}$ denote the set of all $\tilde{g} \in \mathcal{U}$ for which there exists a basis $\{\tilde{u}_\alpha\}_{\alpha=1}^m$ of $E_\gamma(\tilde{g})$ and at least k linearly independent matrices $\tilde{A} \in \mathbb{R}^{\frac{m(m+1)}{2}}$ with $\tilde{A}^{\alpha\beta}\tilde{u}_\alpha\tilde{u}_\beta = 0$. Then, for any $N \in \mathbb{N}$, the set $\widetilde{\mathcal{S}}_{m,k}$ is contained in a Fréchet submanifold of $\mathcal{G}(M)$ of codimension N, after possibly shrinking $\mathcal{U}$. An identical statement also holds in any conformal class $\mathcal{G}(M, g_0)$.

Before commencing with the proof of Proposition 3.3.1, it is important to observe that the condition defining $\widetilde{\mathcal{S}}_{m,k}$ is independent of the choice of basis for $E_\gamma(\tilde{g})$, since changing to a different basis just amounts to conjugating the matrix $\tilde{A}$. The key point of the above, somewhat technical, definition is that $\widetilde{\mathcal{S}}_{m,k}$ can be analyzed via the implicit function theorem, and $\mathcal{S}_{m,k}(\tilde{\lambda}) \cap \mathcal{U} \subseteq \widetilde{\mathcal{S}}_{m,k}$ for all $\tilde{\lambda}$ encircled by γ.

Lemma 3.3.2. *1. Possibly after shrinking $\mathcal{U}$, one continuously differentiable choice $(\tilde{u}_\alpha)_{\alpha=1}^m$ of an eigenbasis for $E_\gamma(\tilde{g})$ is given by $\tilde{u}_\alpha = P(\tilde{g})u_\alpha$, where $P(\tilde{g})$ denotes the spectral projection onto $E_\gamma(\tilde{g})$.*

2. *Define* $F : \mathcal{U} \times \mathbb{R}^{\frac{m(m+1)}{2}} \to C^\infty(M)$ *by*

$$F(\tilde{g}, \tilde{A}) = \tilde{A}^{\alpha\beta} P_\gamma(\tilde{g}) u_\alpha P_\gamma(\tilde{g}) u_\beta.$$

Then F *is continuously differentiable. Its derivative at* (g, A) *with respect to the metric is explicitly given by*

$$D_{\tilde{g}} F(g, A)[h] = 2A^{\alpha\beta} (L\Delta'_h u_\alpha) u_\beta,$$

where Δ'_h *is shorthand for* $D\Delta(g)[h]$ *and* $L : L^2(M) \to H^2(M)$ *is the unique operator satisfying* $(\lambda - \Delta_g) \circ L = \mathbb{I} - P_\lambda(g)$ *and* $L \circ (\lambda - \Delta_g) = \mathbb{I} - P_\lambda(g)$ *on* $L^2(M)$ *and* $H^2(M)$*, respectively.*

3. *Let* $V \subseteq \mathbb{R}^{\frac{m(m+1)}{2}}$ *be a subspace of codimension* $k - 1$ *intersecting* $\ker F(g, \cdot)$ *transversally. Let* $\mathcal{A} := \{A \in V : \operatorname{tr}(A^T A) = 1\}$*, and* $F_1 = F|_{\mathcal{U} \times \mathcal{A}}$*. Then* $\widetilde{\mathcal{S}}_{m,k} \subseteq \pi(F_1^{-1}(\{0\}))$*, where* $\pi : \mathcal{U} \times \mathcal{A} \to \mathcal{U}$ *denotes the natural projection.*

Proof. We begin by differentiating under the contour integral for the spectral projection, justified by the dominated convergence theorem for Bochner integrals.

$$D_g P_\gamma = D_g \left(\frac{1}{2\pi i} \int_\gamma (z - \Delta)^{-1} dz \right) = \frac{1}{2\pi i} \int_\gamma (z - \Delta)^{-1} \Delta' (z - \Delta)^{-1} dz.$$

If $\Delta_g u = \lambda u$, we have

$$D_g P_\gamma(g) u = \frac{1}{2\pi i} \left(\int_\gamma (z - \lambda)^{-1} (z - \Delta)^{-1} dz \right) \Delta' u.$$

By the Riesz-Dunford calculus, $L = \frac{1}{2\pi i} \int_\gamma (z - \lambda)^{-1} (z - \Delta)^{-1} dz$ is the unique operator satisfying $L(\lambda - \Delta_g) = (\lambda - \Delta_g) L = \mathbb{I} - P_\lambda(g)$. The map $\tilde{u}_\alpha = P(\tilde{g}) u_\alpha$ is thus continuously differentiable, with derivative $D\tilde{u}_\alpha(g)[h] = L\Delta'_h u_\alpha$. Since linear independence is an open condition, $\{P(\tilde{g}) u_\alpha\}_{\alpha=1}^m$ forms a basis of E_γ for all $\tilde{g}$ in some neighborhood $\mathcal{U}$ of g. Calculating $D_g F(g, A)$ from the expression $D\tilde{u}_\alpha(g)[h] = L\Delta'_h u_\alpha$ is straightforward. If $\tilde{g} \in \mathcal{S}_{m,k}$, then $\dim \ker F(\tilde{g}, \cdot) \geq k$, and hence $V \cap \ker F(\tilde{g}, \cdot) \neq \{0\}$. Thus, there exists $\tilde{A} \in \mathcal{A}$ with $F_1(\tilde{g}, \tilde{A}) = 0$. □

Proof of Proposition 3.3.1. Let $A \in \mathbb{R}^{\frac{m(m+1)}{2}}$ be any nonzero symmetric matrix such that $F(g, A) = 0$, with F defined as in Lemma 3.3.2. We will first show that $D_g F(g, A)$ is a pseudodifferential operator with non-smooth Schwartz kernel, because g is not contained in the exceptional set $\mathcal{S}_\infty(\lambda)$. Pseudodifferential operators of *finite rank* are smoothing operators, and therefore have smooth Schwartz kernels. [1] Thus, $D_g F(g, A)$ must have infinite rank. Therefore, $\ker D_g F(g, A)$ has infinite codimension, which we then show implies $\widetilde{\mathcal{S}}_{m,k}$ has infinite codimension as well.

[1]Indeed, suppose A is a ΨDO such that $A(C^\infty(M)) \subseteq C^\infty(M)$ is finite dimensional. As $C^\infty(M)$ is dense in $\mathcal{D}'(M)$, $A(C^\infty(M))$ must be dense in $A(\mathcal{D}'(M))$. Since $A(C^\infty(M))$ is finite dimensional, now $A(\mathcal{D}'(M)) = A(C^\infty(M)) \subseteq C^\infty(M)$, so A is a smoothing operator.

For a fixed $u \in E_\lambda(g)$, $L \circ \Delta'_{[\cdot]}u : \Gamma(S^2(T^*M)) \to C^\infty(M)$ is a pseudodifferential operator of order at most -1. This follows from the fact that $h \mapsto \Delta'_h u$ is a differential operator of order 1, and L is a pseudodifferential operator of order -2, as is easily seen from the functional calculus of pseudodifferential operators (see, e.g, [Tay81, Chapter XII, Theorem 1.3]). In a slight abuse of notation, let $L \in \mathcal{D}'(M \times M)$ also denote the Schwartz kernel of the operator L. It satisfies the identity $\Delta_x L(x,y) = \lambda L(x,y) - \delta(x,y) + P(x,y)$, where $\delta(x,y)$ and $P(x,y)$ are the Schwartz kernels of the identity and the spectral projection onto E_λ, respectively.

We now compute the Schwartz kernel of $L \circ \Delta'_{[\cdot]}u$ for a fixed function $u \in C^\infty(M)$. This kernel is a distributional section of the bundle $\pi_2^*(\mathcal{S}^2(T^*M)) \to M \times M$, i.e., the pull-back of the bundle of symmetric $(0,2)$-tensors on M along the projection to the second factor of $M \times M$. Its computation is most easily expressed in local coordinates, which suffices since we will only be interested in its behavior near the diagonal. In the following, we assume h to be compactly supported in a coordinate patch.

$$\begin{aligned} L \circ \Delta'_h u &= \int_M L(x,y) \left(\tfrac{1}{2} g^{ij} \partial_{y^i} \left(g_{k\ell} h^{k\ell} \right) \partial_{y^j} u - |g|^{-\frac{1}{2}} \partial_{y^k} \left(|g|^{\frac{1}{2}} h^{k\ell} \partial_{y^\ell} u \right) \right) |g|^{\frac{1}{2}} dy \\ &= \int_M \left[\partial_{y^k} L(x,y) \partial_{y^\ell} u - \tfrac{1}{2} \big(L(x,y) \Delta u + g(\nabla_y L(x,y), \nabla u) \big) g_{k\ell} \right] h^{k\ell} |g|^{\frac{1}{2}} dy \end{aligned}$$

With this expression in hand, the Schwartz kernel of $D_g F(g, A)$ is readily computed as the distributional section K of $\pi_2^*(\mathcal{S}^2(T^*M)) \to M \times M$ given by

$$\begin{aligned} K(x,y)_{k\ell} = {} & A^{\alpha\beta} \left(\partial_{y^k} L(x,y) \partial_{y^\ell} u_\alpha(y) + \partial_{y^\ell} L(x,y) \partial_{y^k} u_\alpha(y) \right) u_\beta(x) \\ & - A^{\alpha\beta} \big(\lambda L(x,y) u_\alpha(y) u_\beta(x) + g(\nabla_y L(x,y), \nabla u_\alpha(y)) u_\beta(x) \big) g_{k\ell} \end{aligned} \tag{3.3.1}$$

Smoothness of $K(x,y)$ would imply $\rho(x,y) := A^{\alpha\beta} L(x,y) u_\alpha(y) u_\beta(x)$ was smooth as well. If $\dim M = 2$ this follows directly from taking traces (in y):

$$-g^{k\ell} K(x,y)_{k\ell} = n\lambda \, A^{\alpha\beta} L(x,y) u_\alpha(y) u_\beta(x).$$

In dimension 3 and above, we first apply the Laplacian in y to $\rho(x,y)$, which yields

$$\begin{aligned} \Delta_y \rho(x,y) = A^{\alpha\beta} \big[& \, 2\lambda L(x,y) u_\alpha(y) u_\beta(x) + 2 g(\nabla_y L(x,y), \nabla u_\alpha(y)) u_\beta(x) \\ & - \delta(x,y) u_\alpha(y) u_\beta(x) + P(x,y) u_\alpha(y) u_\beta(x) \, \big]. \end{aligned}$$

The third term is zero because $A^{\alpha\beta} u_\alpha(y) u_\beta(x)$ vanishes at the diagonal. Taking the trace of K, we obtain

$$g^{k\ell} K(x,y)_{k\ell} = (2-n) g(\nabla_y L(x,y), \nabla u_\alpha(y)) u_\beta(x) - n\lambda \rho(x,y).$$

Inserting this into the expression above shows that

$$\left(\tfrac{4\lambda}{n-2} + \Delta_y \right) \rho(x,y) = A^{\alpha\beta} P(x,y) u_\alpha(y) u_\beta(x) - \tfrac{2}{n-2} g^{k\ell} K(x,y)_{k\ell}.$$

Elliptic regularity implies $\rho(x, y)$ is smooth in y for fixed x. In fact, $\rho \in C^\infty(M \times M)$, since $\frac{4\lambda}{n-2} + \Delta_y$ is invertible on $C^\infty(M)$, and $C^\infty(M \times M) = C^\infty(M, C^\infty(M))$.

For fixed $x \in M$, the function $A^{\alpha\beta}u_\alpha(y)u_\beta(x)$ is an eigenfunction of Δ_g. Because it does not vanish identically on $M \times M$, there exists $x \in M$ where $A^{\alpha\beta}u_\alpha(y)u_\beta(x)$ is nonzero. Since M is connected, this means $A^{\alpha\beta}u_\alpha(y)u_\beta(x)$ does not vanish to infinite order at $x = y$. If $\rho(x, y)$ was smooth, $L(x, y) = \rho(x, y)(A^{\alpha\beta}u_\alpha(y)u_\beta(x))^{-1}$ is an element of $\mathcal{Q}^\infty(M)$, which would contradict the assumption $g \in \mathcal{S}_{m,k}(\lambda)$. Thus, $\rho(x, y)$ is not smooth, which shows $K(x, y)$ is non-smooth as well. We conclude that $D_g F(g, A)$ has infinite rank.

Consider the restricted map $F_1 : \mathcal{U} \times \mathcal{A} \to C^\infty(M)$ defined in Lemma 3.3.2(3). Up to sign, there exists a unique $A \in \mathcal{A}$ such that $F_1(g, A) = 0$. Since $D_g F_1(g, A)$ has infinite rank, for any $N \in \mathbb{N}$ we may choose a linear map $P_N : C^\infty(M) \to \mathbb{R}^N$ such that $P_N \circ D_g F_1(g_0, A_0)$ is onto, and $\ker P_N \cap \operatorname{Im} D_A F_1(g, A) = \{0\}$. By the implicit function theorem (as in Appendix A, Proposition 3.5.2), the set $(P_N \circ F)^{-1}(\{0\})$ is locally a Fréchet submanifold of $\mathcal{U} \times \mathcal{A}$, which has codimension N. Its image under the projection $\pi : \mathcal{U} \times \mathcal{A} \to \mathcal{U}$ is still a manifold if its tangent space intersects $\ker D\pi$ trivially. This is true by construction, since $B \in \ker D\pi \cap \ker P_N \circ D_A F_1(g, A)$ would solve $F_1(g, B) = 0$ and be orthogonal to A, which was excluded by restriction to $\mathcal{A}$. Thus, after possibly shrinking $\mathcal{U}$, $\widetilde{\mathcal{S}}_{m,k}$ is contained in a Fréchet submanifold of codimension $N - \frac{m(m+1)}{2} + 1$. Since N was arbitrary, this concludes the proof in the non-conformal setting.

The proof in the conformal case is almost identical, because only conformal perturbations were used to show $D_g F(g, A)$ has infinite rank. The key point of the preceding paragraphs was that $K(x, y)_{k\ell}$ is not smooth (unless $g \in \mathcal{S}^\infty(\lambda)$) because $g^{k\ell}K(x, y)_{k\ell}$ is not a smooth function, and the latter is precisely the Schwartz kernel of $D_g F(g, A)$ when restricted to conformal multiples of g. □

Next, we show that $\mathcal{S}_\infty = \bigcup_{\lambda \in (-\infty, 0)} \mathcal{S}_\infty(\lambda)$ has infinite codimension. In fact, if $n = 2$ or $n \geq 3$ is odd, this set is empty. If $n \geq 4$ is even, it may well be empty, too, but our methods only allow us to show it is very small.

Lemma 3.3.3. *Let $L(x, y)$ denote the Schwartz kernel of the pseudodifferential operator L defined by $L(\lambda - \Delta_g) = (\lambda - \Delta_g)L = \mathbb{I} - P_\lambda$. Near the diagonal, it takes the form $L(x, y) = d(x, y)^{2-n}L_1(x, y) + \mathcal{P}(\lambda, g)\log(d(x, y)) + L_2(x, y)$, where*

(i) $L_1(x, y)$ is smooth and nonvanishing at the diagonal,

(ii) $\mathcal{P}(\lambda, g)$ is a universal polynomial of λ and a finite jet of g at x, which does not vanish identically on any conformal class, and

(iii) $L_2(x, y)$ is bounded.

In particular, $\mathcal{S}_\infty = \varnothing$ if $n = 2$ or $n \geq 3$ is odd. If $n \geq 4$ is even, $g \in \mathcal{S}_\infty$ means $\mathcal{P}(\lambda, g) \equiv 0$ for some $\lambda \in \sigma(\Delta_g)$. The set of g for which this occurs has, at least, infinite transversality codimension, even when fixing the conformal class.

Proof. Denote by $P_{[2\lambda,0]}$ the spectral projection onto all eigenvalues contained in the interval $[2\lambda, 0] \subseteq (-\infty, 0]$. Consider the operator $R_\lambda = (\lambda-\Delta)^{-1}(\mathbb{I}-P_{[2\lambda,0]})$, which differs from the operator L we are interested in by a smoothing operator. We will obtain the asymptotics of its Schwartz kernel by considering the heat propagator $e^{t\Delta}(\mathbb{I}-P_{[2\lambda,0]})$. This operator has very good decay properties in the limit $t \to \infty$: $\|e^{t\Delta}(\mathbb{I}-P_{[2\lambda,0]})\| \lesssim e^{-2t(|\lambda|-\varepsilon)}$ in all reasonable norms, including L^∞-bounds on the kernel. One then easily obtains the operator identity

$$R_\lambda = \int_0^\infty e^{t(\Delta-\lambda)}(\mathbb{I} - P_{[2\lambda,0]})dt,$$

as the integrand is bounded as $t \to 0$ and decays exponentially as $t \to \infty$.

For $T > 0$, the operator $\int_T^\infty e^{t(\Delta-\lambda)}(\mathbb{I} - P_{[2\lambda,0]})dt$ is a smoothing operator, as its kernel is smooth. The operator $\int_0^T e^{t(\Delta-\lambda)}P_{[2\lambda,0]}dt$ is likewise a smoothing operator, as it maps into $E_{[2\lambda,0]}$, which is a finite dimensional space of smooth functions. Thus, up to a smoothing operator,

$$R_\lambda \sim \int_0^T e^{t(\Delta-\lambda)}dt = \int_0^T \sum_{k=0}^\infty \frac{|\lambda|^k}{k!} t^k e^{t\Delta}dt.$$

The heat kernel has an asymptotic expression of the form

$$k(t,p,q) = (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{d(p,q)^2}{4t}\right) \sum_{j=0}^\infty a_j(p,q)t^j,$$

where $a_j(p,q) \in C^\infty(M \times M)$, and furthermore $a_j(p,p)$ is a polynomial in finitely many derivatives of the metric g. One obtains this e.g., from specializing Theorem 7.15 in [Roe88, pg. 101] to the operator $d + d^*$ on $\Lambda(T^*M)$. We collect powers of t in the kernel of $e^{t(\Delta-\lambda)}$:

$$\begin{aligned}
\sum_{k=0}^\infty \frac{|\lambda|^k}{k!} t^k k(t,p,q) &= (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{d(p,q)^2}{4t}\right) \sum_{k=0}^\infty \sum_{j=0}^\infty \frac{|\lambda|^k}{k!} a_j(p,q) t^{k+j} \\
&= (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{d(p,q)^2}{4t}\right) \sum_{k=0}^\infty \left(\sum_{j=0}^k \frac{|\lambda|^j}{j!} a_{k-j}(p,q)\right) t^k
\end{aligned}$$

For the kernel $R(p,q)$ of R_λ, one gets the following asymptotic expression by plugging in the heat kernel asymptotics into $\int_0^T e^{t(\Delta-\lambda)}dt$, and performing the change of variables $s = \frac{d(p,q)^2}{4t}$.

$$R(p,q) \sim \sum_{k=0}^\infty 4^{-k-1}\pi^{-\frac{n}{2}} \left(\sum_{j=0}^k \frac{|\lambda|^j}{j!} a_{k-j}(p,q)\right) d(p,q)^{-n+2+2k} \int_{\frac{d(p,q)^2}{4T}}^\infty s^{\frac{n}{2}-k-2} e^{-s} ds$$

Since $\int_{\frac{d(p,q)^2}{4T}}^{\infty} s^{\frac{n}{2}-k-2}e^{-s}ds \sim \Gamma(\frac{n}{2}-k-1)$ up to order $n-2k-2$, the singularity of $R(p,q)$ near the diagonal is given by

$$R(p,q) \sim \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} 4^{-k-1}\pi^{-\frac{n}{2}}\Gamma(\frac{n}{2}-k-1)\left(\sum_{j=0}^{k}\frac{|\lambda|^j}{j!}a_{k-j}(p,q)\right)d(p,q)^{-n+2+2k}$$
$$+\frac{1+(-1)^n}{2^{n+3}\pi^{\frac{n}{2}}}\left(\sum_{j=0}^{n/2}\frac{|\lambda|^j}{j!}a_{k-j}(p,q)\right)\log(d(p,q)),$$

with the logarithmic term present if n is even. The power singularities are thus of the form $d(p,q)^{2-n}L_1(p,q)$ for some function $L_1 \in C^\infty(M\times M)$. The logarithmic term vanishes only if $\mathcal{P}(\lambda,g)(p) := \sum_{j=0}^{n/2}\frac{|\lambda|^j}{j!}a_{k-j}(p,p)$ vanishes identically.

We now show that, unless $\mathcal{P}(\lambda,g)\equiv 0$, the kernel L cannot be a quotient of smooth functions. Recall the ring $\mathcal{Q}^\infty(M)$ of quotients of smooth functions whose denominator does not vanish to infinite order. We wish to show there exists $x\in M$ such that $L(x,y)\notin \mathcal{Q}^\infty(M)$. Consider an arbitrary fraction $\frac{\varphi}{\psi}\in Q^\infty(M)$. Along a curve $\gamma:(-\varepsilon,\varepsilon)\to M$ where $\psi\circ\gamma$ does not vanish to infinite order, $\left(\frac{\varphi}{\psi}\circ\gamma\right)(t) = t^k h(t)$ for some $k\in\mathbb{Z}$ and $h\in C^\infty(-\varepsilon,\varepsilon)$ with $h(0)\neq 0$.

Thus, if $n=2$ or $n\geq 3$ is odd, its leading order singularity already precludes $L(x,y)$ from lying in $\mathcal{Q}^\infty(M)$. If $n\geq 4$ is even, $d(x,y)^{2-n}L_1(x,y)\in\mathcal{Q}^\infty(M)$, and hence $L(x,y)\in\mathcal{Q}^\infty(M)$ if and only if $\mathcal{P}(\lambda,g)\log(d(x,y))+L_2(x,y)\in Q^\infty(M)$. Thus, $L(x,y)\in\mathcal{Q}^\infty(M)$ implies that $\mathcal{P}(\lambda,g)$ vanishes identically on M. Since $\mathcal{P}(\lambda,g)$ is a purely local quantity, which is generically nonzero, the set of metrics where $\mathcal{P}(\lambda,g)$ vanishes identically has infinite transversality codimension, and thus also $\mathcal{S}^\infty$.

The only caveat is that $\mathcal{P}(\lambda,g)$ might vanish *as a polynomial*, which we can exclude by exhibiting a single metric where $\mathcal{P}(\lambda,g)\neq 0$. The euclidean metric already suffices: The fundamental solution to Helmholtz' equation in Euclidean space, $(1+\Delta_{\mathrm{Euc}})u=\delta_0$, is given by $u(x)=c_n|x|^{1-\frac{n}{2}}Y_{\frac{n}{2}-1}(|x|)$, where $Y_{\frac{n}{2}-1}$ is a Bessel function of the second kind and c_n a nonzero dimensional constant. If n is even, the full asymptotic expansion of $Y_m(z)$ for small arguments given in [AS72, Eq. 9.1.11] implies

$$Y_m(z) = -\frac{\left(\frac{1}{2}z\right)^{-m}}{\pi}\sum_{k=0}^{m-1}\frac{(m-k-1)!}{k!}(\tfrac{1}{4}z^2)^k+\frac{2}{\pi}\log(\tfrac{1}{2}z)J_m(z)+\mathcal{O}(1),$$

where J_m is a Bessel function of the first kind. Since $J_{\frac{n}{2}-1}$ vanishes to order $\frac{n}{2}-1$, and $u(x)=c_n|x|^{1-\frac{n}{2}}Y_{\frac{n}{2}-1}(|x|)$, we conclude that $\mathcal{P}(\lambda,g_{\mathrm{Euc}})$ is a nonzero constant.

For the last claim, fix a conformal class $\mathcal{G}(M,g_0)$. We must show that $\mathcal{P}(\lambda,g)$ does not vanish identically on $\mathcal{G}(M,g_0)$. Take an arbitrary point $p\in M$ and consider $g_\varepsilon=\varphi_\varepsilon g_0$, where $\varphi_\varepsilon\equiv\varepsilon^{-2}$ on $\mathbb{B}_\varepsilon^{g_0}(p)$ and $\varphi_\varepsilon\equiv 1$ outside $\mathbb{B}_{2\varepsilon}^{g_0}(p)$. In normal coordinates at p, g_ε and all its derivatives converge to the Euclidean metric as ε approaches 0. Since $\mathcal{P}(\lambda,g_{\mathrm{Euc}})\neq 0$, there must be some ε such that $\mathcal{P}(\lambda,g_\varepsilon)$ does not vanish. Hence, $\mathcal{P}(\lambda,g)$ does not vanish identically on any conformal class. □

The full non-crossing rule (Theorem 3.1.6) follows by "stitching together" the local codimension bounds from Proposition 3.3.1 and Lemma 3.3.3 (which, along the way, yields Theorem 3.1.5) with the local structure theorem near nondegerate eigenvalues, Theorem 3.1.3.

Proof of Theorem 3.1.5 and Theorem 3.1.6. By Lemma 3.3.3, the exceptional set $\mathcal{S}_\infty$ has infinite transversality codimension. It remains to show the same for $\mathcal{S}_{m,k} = \bigcup_{\lambda\in(-\infty,0)} S_{m,k}(\lambda)$.

Fix $N \in \mathbb{N}$. We now show that $\mathcal{S}_{m,k}$ has transversality codimension N, and thus, as N was arbitrary, of infinite transversality codimension. Let us order (m,k) lexicographically, i.e., $(m',k') \succeq (m,k)$ if $m' > m$ or $m' = m$, $k' \geq k$. For $R \in \mathbb{N}$, let $\overline{\mathcal{S}}_{m,k;R}$ denote the set of metrics g such that $g \in \mathcal{S}_{m,k}(\lambda)$ for some $\lambda \in [-R,0)$, but there exist no $(m',k') \succeq (m,k)$ such that $g \in \mathcal{S}_{m',k'}(\lambda')$ for some $\lambda' \in [-R,0)$. Since both m and k are dimensions of the null space of some linear system varying continuously with g, and hence upper semicontinuous, this definition achieves the following: For any $g \in \overline{\mathcal{S}}_{m,k;R}$, there exists an open neighborhood $\mathcal{U}_0$ of g with $\mathcal{U}_0 \cap \overline{\mathcal{S}}_{m',k';R} = \varnothing$ for any $(m',k') \succeq (m,k)$.

Consider $g \in \overline{\mathcal{S}}_{m,k;R}$ and $\mathcal{U}_0$ as above. Let $\lambda_1,\ldots,\lambda_\ell \in [-R,0)$ denote all eigenvalues for which $g \in \mathcal{S}_{m,k}(\lambda)$. Proposition 3.3.1 yields neighborhoods $\mathcal{U}_j$ of g, $j = 1,\ldots,\ell$, and corresponding submanifolds $\widetilde{\mathcal{S}}_j \subseteq \mathcal{U}_j$ of codimension N such that $\left(\bigcap_{j=0}^k \mathcal{U}_j\right) \cap \overline{\mathcal{S}}_{m,k;R} \subseteq \bigcup_{j=1}^k \widetilde{\mathcal{S}}_j$. Thus, by Lemma 3.6.5, $\overline{\mathcal{S}}_{m,k;R}$ has transversality codimension N. The set $\mathcal{S}_{m,k}$ is contained in a countable union of such sets:

$$\mathcal{S}_{m,k} \subseteq \bigcup_{\substack{(m',k')\succeq(m,k),\\ R\in\mathbb{N}}} \overline{\mathcal{S}}_{m',k';R}.$$

As N was arbitrary, this shows $\mathcal{S}_{m,k}$ has infinite transversality codimension, and so is the set $\mathcal{S} = \mathcal{S}_\infty \cup \left(\bigcup_{m,k} \mathcal{S}_{m,k}\right)$ of all metrics admitting a degenerate eigenvalue. Thus, Theorem 3.1.5 is proven.

Nondegenerate eigenvalues are dealt with in much the same way. Introduce $\overline{\mathcal{T}}_{m;R}$, the set of metrics admitting a nondegenerate eigenvalue $\lambda \in [-R,0)$ of multiplicity exactly m, but no $\lambda \in [-R,0)\cap\sigma(\Delta_g)$ which is degenerate, or nondegenerate but of higher multiplicity than m. Analogous to before, there exist a neighborhood $\mathcal{U}$ around any $g \in \overline{\mathcal{T}}_{m;R}$, and submanifolds $\mathcal{T}_1,\ldots,\mathcal{T}_\ell \subseteq \mathcal{U}$ of codimension $\frac{m(m+1)}{2} - 1$, such that $\mathcal{U} \cap \overline{\mathcal{T}}_{m;R} = \bigcup_{j=1}^\ell \mathcal{T}_j$. Lemma 3.6.5 implies $\overline{\mathcal{T}}_{m;R}$ has transversality codimension $\frac{m(m+1)}{2} - 1$. The set of metrics admitting an eigenvalue of multiplicity at least m is covered by the union of $\mathcal{S}$ and the sets $\overline{\mathcal{T}}_{m';R}$, $m' \geq m, R \in \mathbb{N}$. It therefore has transversality codimension $\frac{m(m+1)}{2} - 1$. $\square$

The maximal non-crossing rule in C^∞ regularity implies a maximal non-crossing rule in finite regularity. Recall Courant's min-max characterization of the k^{th} eigenvalue [Lab15, Theorem 4.5.12]:

$$-\lambda_k = \min_{\substack{V\subseteq H^1(M)\\ \dim V=k}} \max_{u\in V\setminus\{0\}} \frac{\int_M g(du,du)d\mu_g}{\int_M u^2 d\mu_g} \tag{3.3.2}$$

This definition is meaningful for continuous metrics, and the eigenvalues thus obtained agree with those of the operator $\Delta_g : H^2(M) \to L^2(M)$ if g is continuously differentiable. For $\ell \geq 0$, let $\mathcal{G}^\ell(M)$ denote the Banach manifold of C^ℓ metrics on M. The eigenvalue functional λ_k is continuous on $\mathcal{G}^\ell(M)$ for all $\ell \geq 0$. This follows from Lemma 4.5.13 in [Lab15], which is stated in smooth regularity, but works just as well in $\mathcal{G}^0(M)$.

Corollary 3.3.4. *Let $\ell \geq 0$, and consider the Banach manifold $\mathcal{G}^\ell(M)$ of C^ℓ metrics on M. The set of metrics which have at least one eigenvalue of multiplicity at least m has transversality codimension $\frac{m(m+1)}{2} - 1$.*

Proof. Denote the set of metrics admitting an eigenvalue $\lambda \in [-R,0)\cap\sigma(\Delta_g)$ of multiplicity at least m by $\mathcal{D}_{m;R} \subset \mathcal{G}^\ell(M)$. This set is closed, since eigenvalues move continuously under deformations of the metric. Consequently, the set of smooth maps from a compact k-dimensional manifold N to $\mathcal{G}^\ell(M)$ that avoids $\mathcal{D}_{m;R}$ is open.

We now show it is dense if $k < \frac{m(m+1)}{2} - 1$. For this, let $S_\varepsilon : \mathcal{G}^\ell(M) \to \mathcal{G}^\ell(M)$ be a family of smoothing operators converging to the identity in the strong operator topology. Since N is compact, we can ensure that $\|f - S_\varepsilon f\|_{C^\ell}$ is arbitrarily small. Now, $S_\varepsilon f$ can be perturbed in $\mathcal{G}^\infty(M)$ to avoid $\mathcal{D}_{m;R}$ by Theorem 3.1.6.

The set of smooth maps from N to $\mathcal{G}^\ell(M)$ which avoid eigenvalues of multiplicity at least m is therefore a countable intersection of dense open sets, and hence Baire generic. For non-compact N, the result follows by a countable compact exhaustion. □

3.4 Low multiplicities and examples

This section is concerned with natural questions and examples arising from the degeneracy condition (Definition 3.1.1). The simplest examples are flat tori and the round sphere. Thus, already in 1988, Colin de Verdière classified the degenerate and conformally degenerate eigenvalues on the square torus and the two-dimensional round sphere while investigating the strong Arnold hypothesis [Col88]. We discuss these examples, in two *and* higher dimensions, in sections 3.4.2 and 3.4.3, respectively. Section 3.4.1 is concerned with eigenvalues of low multiplicity, and Section 3.4.4 lists further examples and comparisons.

3.4.1 Nonexistence of degenerate eigenvalues of low multiplicity

Degenerate eigenvalues must have fairly high multiplicity. This observation goes back at least to [MP98, Theorem 4.3], which essentially says that there are no conformally degenerate eigenvalues of multiplicity 2 in two dimensions. It is natural to ask for the lowest possible multiplicity of a (conformally) degenerate eigenvalue. The following theorem gives an almost complete answer.

Theorem 3.4.1. *On a connected, closed, smooth Riemannian manifold, there exist no conformally degenerate eigenvalues of multiplicity less than four, and no degenerate eigenvalues of multiplicity less than seven.*

In Section 3.4.2, we will see examples of conformally degenerate eigenvalues of multiplicity 4 and degenerate eigenvalues of multiplicity 8. It is an interesting question whether degenerate eigenvalues of multiplicity 7 exist (see Question 3.4.5).

Proof of Theorem 3.4.1. The first claim will follow from the fact that, on a connected Riemannian manifold, the product of two linearly independent real valued Laplace eigenfunctions cannot be nonnegative. To see this, let $u, v \in C^\infty(M)$ be two nonzero eigenfunctions such that $u(x)v(x) \geq 0$ for all $x \in M$. Near any $x \in u^{-1}(\{0\})$, u must take both positive and negative values by the maximum principle, which implies that $v(x) = 0$ as well. Thus, u and v share the same zero set. Let Ω be a connected component of $u^{-1}(\mathbb{R} \setminus \{0\})$. The functions $u|_\Omega$ and $v|_\Omega$ are positive solutions to the Dirichlet eigenvalue problem on $\overline{\Omega}$. Any such solution corresponds to the lowest eigenvalue, which is always simple, so $u = tv$ for some $t > 0$.

Consider a conformally degenerate eigenvalue $\lambda \in \sigma(\Delta_g)$, and associated linearly independent eigenfunctions $u_1, \dots, u_k \in E_\lambda$ such that $\sum_{j=1}^k \epsilon_j u_j^2 = 0$ for some $\epsilon_1, \dots, \epsilon_k \in \{-1, 1\}$. Not all ϵ_j can be 1, so $k \geq 2$. If $k = 2$, $u_1^2 = u_2^2$, which by unique continuation implies $u_1 = \pm u_2$, a contradiction. Suppose now that $k = 3$. Without loss of generality, $u_1^2 = u_2^2 + u_3^2$. Then $(u_1 - u_2)(u_1 + u_2) = u_3^2 \geq 0$, which is also impossible, as shown in the first paragraph. A conformally degenerate eigenvalue must therefore be of multiplicity at least 4.

If $\lambda \in \sigma(\Delta_g)$ is degenerate, there exist linearly independent eigenfunctions $u_1, \dots, u_k$ such that $\sum_{j=1}^k \epsilon_j u_j^2 = 0$ and $\sum_{j=1}^k \epsilon_j \, du_j \otimes du_j = 0$. Without loss of generality, $\epsilon_1 = \ldots = \epsilon_\ell = -1$, and $\epsilon_{\ell+1} = \ldots = \epsilon_k = 1$. The same argument as before shows $\min(\ell, k - \ell) \geq 2$. Consider the maps $\Phi : M \to \mathbb{R}^\ell$, $\Phi = (u_1, \dots, u_\ell)$ and $\Psi : M \to \mathbb{R}^{k-\ell}$, $\Psi = (u_{\ell+1}, \dots, u_k)$. Then $|\Phi(x)| = |\Psi(x)|$ for all $x \in M$, as well as $\Phi^* g_{\mathbb{R}^\ell} = \Psi^* g_{\mathbb{R}^{k-\ell}}$, with $g_{\mathbb{R}^\ell}$ and $g_{\mathbb{R}^{k-\ell}}$ denoting the standard metrics on $\mathbb{R}^\ell$ and $\mathbb{R}^{k-\ell}$, respectively. Both conditions together imply that the normalized maps $\tilde{\Phi} : M \to \mathbb{S}^{\ell-1}$, $\tilde{\Phi}(x) = |\Phi(x)|^{-1}\Phi(x)$ and $\tilde{\Psi} : M \to \mathbb{S}^{k-\ell-1}$, $\tilde{\Psi}(x) = |\Psi(x)|^{-1}\Psi(x)$ satisfy a similar pull-back metric condition: $\tilde{\Phi}^* g_{\mathbb{S}^{\ell-1}} = \tilde{\Psi}^* g_{\mathbb{S}^{k-\ell-1}}$.

To show the last claim, consider a vector field $V \in \Gamma(TM)$. Our assumptions on Φ and Ψ imply $|V\Phi|^2 = |V\Psi|^2$, and furthermore

$$\Phi \cdot (V\Phi) = V\left(\tfrac{1}{2}|\Phi|^2\right) = V\left(\tfrac{1}{2}|\Psi|^2\right) = \Psi \cdot (V\Psi).$$

These identities imply the desired claim, via polarization, as

$$\begin{aligned}\left(\tilde{\Phi}^* g_{\mathbb{S}^{\ell-1}}\right)(V, V) &= |\Phi|^{-2} \left| V\Phi - \left(|\Phi|^{-2}\, \Phi \cdot (V\Phi)\right) \Phi \right|^2 \\ &= |\Phi|^{-2} \, |V\Phi|^2 - |\Phi|^{-4} \left(\Phi \cdot (V\Phi)\right)^2 = \left(\tilde{\Psi}^* g_{\mathbb{S}^{k-\ell-1}}\right)(V, V).\end{aligned}$$

Of course, it may happen that $\Phi(x) = \Psi(x) = 0$ for some $x \in M$, and hence, $\tilde{\Phi}$ and $\tilde{\Psi}$ are not defined. However, they are defined on a dense open subset. Since all following arguments are local, this is sufficient.

Suppose there exists an open set U where $\operatorname{rk} D\tilde{\Phi} = 1$. Possibly after shrinking U, the rank theorem implies there exists $\tau \in C^\infty(U)$ and a function $F : \mathbb{R} \to \mathbb{S}^{\ell-1}$ such that $\tilde{\Phi} = F \circ \tau$. For any $V \in \ker D\tau$, we have $D\tilde{\Phi}(V) = 0$, and thus also $D\tilde{\Psi}(V) = 0$, because $\tilde{\Phi}^* g_{\mathbb{S}^{\ell-1}} = \tilde{\Psi}^* g_{\mathbb{S}^{k-\ell-1}}$. Hence, there also exists $G : \mathbb{R} \to \mathbb{S}^{k-\ell-1}$ with $\tilde{\Psi} = G \circ \tau$. Denote $\rho = |\Phi| = |\Psi|$. Then the preceding calculation shows there exist functions $g_1, \dots, g_k$ such that $u_j = (g_j \circ \tau)\rho$ for $j = 1, \dots, k$. The eigenfunction equation becomes

$$\begin{aligned} 0 &= (\lambda - \Delta)\big((g_j \circ \tau)\rho\big) \\ &= (g_j \circ \tau)(\lambda\rho - \Delta\rho) - (g_j'' \circ \tau)|\nabla\tau|^2\rho - (g_j' \circ \tau)(\rho\Delta\tau + \nabla\rho \cdot \nabla\tau). \end{aligned}$$

Note that $|\nabla\tau|^2\rho$ does not vanish, by assumption. There can be at most 2 linearly independent functions g_j which satisfy this equation. Indeed, consider a curve $\gamma : (-\varepsilon, \varepsilon) \to M$ such that $\tau \circ \gamma = \mathrm{id}$. Then g_j satisfies the linear ordinary differential equation $g_j''(t) = A(t)g_j(t) - B(t)g_j'(t)$, where

$$A = \left(\frac{\lambda\rho - \Delta\rho}{|\nabla\tau|^2\rho}\right) \circ \gamma, \; B = \left(\frac{\rho\Delta\tau + \nabla\rho \cdot \nabla\tau}{|\nabla\tau|^2\rho}\right) \circ \gamma.$$

If $k \geq 3$, this would imply a linear dependence between $u_1, \dots, u_k$ along γ, which is determined solely by the values of $u_1, \dots, u_k$ and their first derivatives along γ at 0. Thus, it extends to all curves which agree up to first order with γ at 0, and hence, to an open subset of U. By unique continuation, the linear dependence now extends to the whole of M. This contradicts our assumption that $u_1, \dots, u_k$ are linearly independent. The case $k \leq 2$ was already excluded earlier, so there exists no open set U where $\operatorname{rk} D\tilde{\Phi} = 1$.

We deduce that $\ell \geq 3$ and $k \geq \ell + 3$, and that there exists an open set $U \subseteq M$ where $D\tilde{\Phi}$ and $D\tilde{\Psi}$ are of rank at least 2. Consider the case $\ell = k - \ell = 3$. On U, $\tilde{\Phi}$ and $\tilde{\Psi}$ are submersions with $\tilde{\Phi}^* g_{\mathbb{S}^2} = \tilde{\Psi}^* g_{\mathbb{S}^2}$, so there exists a local isometry F of $\mathbb{S}^2$ with $\tilde{\Phi} = F \circ \tilde{\Psi}$. Since all local isometries of $\mathbb{S}^2$ are linear, there exists $A \in O(3)$ such that $\tilde{\Phi} = A\tilde{\Psi}$ on U. Recalling that $|\Phi| = |\Psi|$, we find that $\Phi = A\Psi$ as well, which, by the unique continuation principle, is a contradiction, as $u_1, \dots, u_k$ are linearly independent.

In conclusion, we have shown that if $u_1, \dots, u_\ell, \dots, u_k \in E_\lambda$ are linearly independent eigenfunctions with $\sum_{j=1}^{\ell} u_j^2 = \sum_{j=\ell+1}^{k} u_j^2$, then $\min(\ell, k - \ell) \geq 2$, and if additionally $\sum_{j=1}^{\ell} du_j \otimes du_j = \sum_{j=\ell+1}^{k} du_j \otimes du_j$, then $\min(\ell, k - \ell) \geq 3$ and $\max(\ell, k - \ell) \geq 4$. $\square$

Remark 3.4.2. *In the proof of Theorem 3.4.1, the exact multiplicity of the eigenvalue was never used. This suggests to call an eigenvalue* conformally (ℓ, m)-degenerate, *if there exist corresponding eigenfunctions $u_1, \dots, u_\ell, u_{\ell+1}, \dots, u_{\ell+m}$ which are linearly independent and satisfy $\sum_{j=1}^{\ell} u_j^2 = \sum_{j=\ell+1}^{\ell+m} u_j^2$, and* (ℓ, m)-degenerate *if, additionally, $\sum_{j=1}^{\ell} du_j \otimes du_j = \sum_{j=\ell+1}^{\ell+m} du_j \otimes du_j$. Then there are no conformally* $(0, m)$- *or* $(1, m)$-*degenerate eigenvalues, and no* $(2, m)$- *or* $(3, 3)$-*degenerate eigenvalues.*

This allows for an alternative proof of Theorem 3.1.6 for multiplicities up to 6, which works in a C^k setting just as well as in the present C^∞ setting. Smoothness is only needed *for the proof that the set of degenerate eigenvalues has infinite codimension (Section 3.3), which relies on microlocal analysis.*

3.4.2 Examples with many degenerate eigenvalues: Flat tori

A lattice of full rank $\Lambda \subseteq \mathbb{R}^n$ gives rise to a torus, $\mathbb{T}^n = \mathbb{R}^n/\Lambda$, which inherits its metric from $\mathbb{R}^n$. A full basis of eigenfunctions of its Laplace operator is given by $u_\kappa(x) = e^{2\pi i \kappa \cdot x}$, where κ ranges over all elements of $\Lambda^* = \{\kappa \in \mathbb{R}^n : \kappa \cdot v \in \mathbb{Z} \text{ for all } v \in \Lambda\}$, the dual lattice. The corresponding eigenvalues are $-|\kappa|^2$, $\kappa \in \Lambda^*$. Thus, all eigenvalues have even multiplicity. There are many lattices that give rise to tori with eigenvalues of arbitrarily high multiplicity, e.g., the square lattice $\mathbb{Z} \oplus \mathbb{Z}$, or the triangular lattice $\mathbb{Z} \oplus e^{\frac{2i\pi}{3}}\mathbb{Z}$. There are examples of flat tori where some eigenvalues have multiplicity 2, but there are also eigenvalues of higher multiplicity, e.g., if $\Lambda = \mathbb{Z} \oplus 2\mathbb{Z}$. In this case, there will be both conformally degenerate and nondegenerate eigenvalues (as a consequence of the following proposition).

Proposition 3.4.3. *Consider the torus $\mathbb{T}^n = \mathbb{R}^n/\Lambda$, where Λ is a lattice of full rank. An eigenvalue of its Laplace-Beltrami operator on the corresponding torus is degenerate if (and in dimension 2 only if) its multiplicity is at least* $n(n+1)+2$. *It is conformally degenerate if and only if its multiplicity is at least* 4.

Proof. Let λ be a Laplace-Beltrami eigenvalue of $\mathbb{T}^n$. If it has multiplicity 2, it cannot be conformally degenerate, by Theorem 3.4.1. If its multiplicity is at least 4, there exist linearly independent elements $\kappa_1, \kappa_2 \in \Lambda^*$ with $|\kappa_1|^2 = |\kappa_1|^2$, which correspond to eigenfunctions $\cos(\kappa_j \cdot x)$ and $\sin(\kappa_j \cdot x)$, $j = 1, 2$. Since $\cos(\kappa_1 \cdot x)^2 + \sin(\kappa_1 \cdot x)^2 = \cos(\kappa_2 \cdot x)^2 + \sin(\kappa_2 \cdot x)^2 = 1$, λ is conformally degenerate.

To show the first part of the statement, let λ be an eigenvalue of multiplicity $2m$. The corresponding eigenfunctions are $\cos(\kappa_j \cdot x), \sin(\kappa_j \cdot x)$, where $\kappa_j \in \Lambda^*$ are lattice elements with $-|\kappa_j|^2 = \lambda$, and $j = 1, \ldots, m$. We will look for a nontrivial choice of coefficients $\mu_1, \ldots, \mu_m$ such that

$$\sum_{j=1}^{m} \mu_j \left(\cos(\kappa_j \cdot x)^2 + \sin(\kappa_j \cdot x)^2\right) = 0 \text{ and}$$
$$\sum_{j=1}^{m} \mu_j \left((d\cos(\kappa_j \cdot x))^2 + (d\sin(\kappa_j \cdot x))^2\right) = 0,$$

which, by Proposition 3.2.3, would imply that λ is degenerate. The first line is equivalent to $\sum_{j=1}^{m} \mu_j = 0$, while the second line reduces to $\sum_{j=1}^{m} \mu_j \kappa_j \kappa_j^T = 0$, a system of $\frac{n(n+1)}{2}$ equations. In view of Proposition 3.2.5 it is not surprising that the first equation is linearly dependent on the remaining ones: $\operatorname{tr}(\kappa_j \kappa_j^T) = |\kappa_j|^2 = -\lambda$. Thus, if $m \geq \frac{n(n+1)}{2} + 1$, there exists a nontrivial solution to this system, and λ is degenerate.

It may be that not all components of the system $\sum_{j=1}^{m} \mu_j \kappa_j \kappa_j^T = 0$ are linearly independent, in which case λ is degenerate even if its multiplicity is less than $n(n+1)+2$. However, in dimension two, an eigenvalue of multiplicity less than 8 has multiplicity 6, and therefore, due to Theorem 3.4.1, is not degenerate. $\square$

3.4.3 An example with no degenerate eigenvalues: The round 2-sphere

A homogeneous space exhibiting very different behaviour is $\mathbb{S}^2$, with its standard metric. Its eigenvalues $0 = \lambda_0 > \lambda_1 > \lambda_2 > \ldots$ are given by $\lambda_\ell = -\ell(\ell+1)$, $\ell \in \mathbb{N}$, and have multiplicity $2\ell+1$. The corresponding eigenspace comprises all homogeneous harmonic polynomials of degree ℓ, restricted to $\mathbb{S}^2$. It may seem, due to the high multiplicities occurring in this example, that some or all of these eigenvalues could be degenerate. However, this is not the case, as was shown by Colin de Verdière [Col88]. Our proof is largely identical, but we nevertheless include it here, because it is instrumental in understanding the higher dimensional case, which was not discussed in [Col88].

Proposition 3.4.4 ([Col88, pg. 186])**.** *All nonzero eigenvalues of the Laplace-Beltrami operator on $\mathbb{S}^2$ with its standard metric are conformally nondegenerate.*

This statement is related to the Clebsch-Gordan theory of $SO(3)$, for which see [Hal16, Appendix C]. However, our proof only uses basic properties of spherical harmonics and elementary representation theory. A particularly neat overview of all necessary facts is given in [Kos22, Chapter 7].

Proof. Consider the ℓ^{th} eigenvalue, $\lambda = -\ell(\ell+1)$, of the Laplace-Beltrami operator on $\mathbb{S}^2$. The isometry group of the sphere, $SO(3)$, is irreducibly represented on the corresponding eigenspace E_λ via $g \cdot u = (g^{-1})^* u$. Denote the set of harmonic homogeneous polynomials of degree ℓ in three variables by $\mathcal{H}_\ell$, so that $E_\lambda = \mathcal{H}_\ell|_{\mathbb{S}^2}$.

Consider the set of even homogeneous polynomials of degree 2ℓ, which we shall call $\mathcal{E}_{2\ell}$. It is likewise a representation of $SO(3)$, which decomposes into

$$\mathcal{E}_{2\ell}(\mathbb{R}^3) = \oplus_{j=0}^{\ell} \|x\|^{2(\ell-j)} \mathcal{H}_{2j} \cong \oplus_{j=0}^{\ell} \mathcal{H}_{2j}$$

The product of two eigenfunctions $u, v \in E_\lambda$ is an element of $\mathcal{E}_{2\ell}|_{\mathbb{S}^2}$. This suggests to consider the linear map $\Phi : \vee^2 \mathcal{H}_\ell \to \mathcal{E}_{2\ell}(\mathbb{R}^3)$ which sends $\frac{1}{2}(u \otimes v + v \otimes u)$ to the polynomial $u(x)v(x)$. Conformal nondegeneracy of λ is equivalent to $\ker \Phi = \{0\}$. A quick dimension count reveals that $\dim \vee^2 \mathcal{H}_\ell = (\ell+1)(2\ell+1) = \dim \mathcal{E}_{2\ell}$, hence $\ker \Phi = \{0\}$ if and only if Φ is surjective.

By Schur's Lemma it is sufficient to show that each irreducible component $\mathcal{H}_{2j}$ of $\mathcal{E}_{2\ell}(\mathbb{R}^3)$ intersects $\operatorname{Im} \Phi$ nontrivially. To show this, consider the subspace $\mathcal{Z} \subseteq \mathcal{E}_{2\ell}$ of polynomials invariant under rotation around the z-axis. It has dimension $\ell+1$ and intersects each $\mathcal{H}_{2j}$ in a one-dimensional space generated by the "zonal" spherical harmonic of degree $2j$. Thus, it suffices to show $\mathcal{Z} \subseteq \operatorname{Im} \Phi$.

One basis of $\mathcal{H}_\ell$ is given by *Laplace's spherical harmonics*: For $m \in \{0, \ldots, \ell\}$, let $Y_\ell^m(x,y,z) = c(m,\ell)(x+iy)^m Q_\ell^m(z)$, where $Q_\ell^m(z)$ is the respective associated Legendre polynomial, divided by $(1-z^2)^{\frac{m}{2}}$, and $c(m,\ell)$ is a normalization constant. For $m \in \{-\ell \ldots, 0\}$, instead $Y_\ell^m(x,y,z) = c(|m|,\ell)(x-iy)^{|m|} Q_\ell^{|m|}(z)$. The products

$$Y_\ell^m(x,y,z) Y_\ell^{-m}(x,y,z) = c(m,\ell)^2 (x^2+y^2)^m Q_\ell^m(z)^2$$

are linearly independent as m ranges from 0 to ℓ, since they are all of distinct degree in z. Because these are $\ell+1$ linearly independent functions in $\mathcal{Z} \cap \operatorname{Im} \Phi$, we conclude that $\mathcal{Z} \subseteq \operatorname{Im} \Phi$ as desired. $\square$

The situation changes drastically in higher dimensions. On $\mathbb{S}^n$ with its standard metric, the eigenvalues of the Laplacian are $\lambda_\ell = -\ell(\ell+n-1)$, $\ell \in \mathbb{N}$, and have multiplicity

$$\binom{n+\ell}{n} - \binom{n+\ell-2}{n}.$$

The proof of Proposition 3.4.4 could be repeated under some minor changes, if not for one crucial issue: The dimension of $\vee^2\mathcal{H}_\ell$ is a polynomial of degree $2(n-1)$ in ℓ. On the other hand $\dim \mathcal{E}_{2\ell}$ is merely of degree n in ℓ. Hence, if $n \geq 3$, the map $\Phi : \vee^2\mathcal{H}_\ell \to \mathcal{E}_{2\ell}$, which assigns the polynomial $uv \in E_{2\ell}$ to the tensor $\frac{1}{2}(u \otimes v + v \otimes u)$, must have a nontrivial kernel for all large enough ℓ. For these ℓ, the eigenvalue λ_ℓ is conformally degenerate. A slightly more involved dimension comparison shows that λ_ℓ is in fact degenerate for all large enough ℓ.

Let us obtain a precise upper bound for the degree of the smallest degenerate eigenvalue on $\mathbb{S}^3$. Denote the spaces of homogeneous polynomials in 4 variables of degree ℓ by $\mathcal{E}_\ell$, and the space of harmonic homogeneous polynomials by $\mathcal{H}_\ell$. Consider the map $\vee^2\mathcal{H}_\ell \to \mathcal{E}_{2\ell-2}^{\frac{4\times 5}{2}}$ which sends $\frac{1}{2}(u \otimes v + v \otimes u)$ to $\left(\frac{du}{dx^j}(x)\frac{dv}{dx^k}(x)\right)_{j,k=1}^4$. By Proposition 3.2.5, degeneracy of λ_ℓ is equivalent to this map having a nontrivial kernel, which is certainly the case if

$$\dim \vee^2\mathcal{H}_\ell \geq 10 \dim \mathcal{E}_{2\ell-2},$$

i.e., if the following inequality of binomial coefficients holds:

$$\frac{1}{2}\left(\binom{3+\ell}{3} - \binom{1+\ell}{3} + 1\right)\left(\binom{3+\ell}{3} - \binom{1+\ell}{3}\right) \geq 10\binom{1+2\ell}{3},$$

For $\ell > 1$, this inequality reduces to

$$\ell^3 - \frac{65}{3}\ell^2 - \frac{44}{3}\ell - 2 \geq 0,$$

which is satisfied when $\ell \geq 23$ (both reduced and solved via *Mathematica* [Mat]). Thus, any eigenvalue of degree at least 23 of $\mathbb{S}^3$ is degenerate. Since any solution to the degeneracy equation in four variables lifts to one in $n \geq 4$ variables, we conclude that the same holds for all $\mathbb{S}^n$, $n \geq 3$.

3.4.4 Further examples and observations

Products of homogeneous spaces

Let M be a connected, closed Riemannian manifold of volume 1 such that its isometry group, $\mathrm{Aut}(G)$, acts transitively on M. On any eigenspace E_λ of its Laplace-Beltrami operator, the isometry group acts by orthogonal transformations. A well known consequence (see e.g., [EI02, Corollary 2.3]) is that $\sum_{\alpha=1}^m u_\alpha^2 = m$ for any real valued orthonormal basis $\{u_\alpha\}_{\alpha=1}^m$ of E_λ.

Thus, if $\lambda < 0$ is a Laplace-Beltrami eigenvalue for two such homogeneous spaces M and M', then λ becomes a conformally degenerate eigenvalue on the product $M \times M'$. After all, given orthonormal bases $\{u_\alpha\}_{\alpha=1}^m$ and $\{v_\alpha\}_{\alpha=1}^{m'}$ of the corresponding eigenspaces on M and M', then $u_1 \otimes 1, \ldots, u_m \otimes 1, 1 \otimes v_1, \ldots, 1 \otimes v_{m'}$ is an orthonormal set of eigenfunctions on $M \times M'$, and hence,

$$\frac{1}{m}\sum_{\alpha=1}^{m}(u_\alpha \otimes 1)^2 - \frac{1}{m'}\sum_{\alpha=1}^{m'}(1 \otimes v_\alpha)^2 = 0$$

implies that λ is conformally degenerate, by Proposition 3.2.3. However, it need not be degenerate, as the example of the torus shows (see Proposition 3.4.3).

Disconnected Manifolds

It is instructive to observe that connectedness of M is absolutely necessary for Theorem 3.1.6 to hold. To see this, suppose M is the disjoint union of closed Riemannian manifolds M_1 and M_2 such that $\lambda > 0$ is a Laplace-Beltrami eigenvalue of both M_1 and M_2 with corresponding eigenfunctions u_1 and u_2. Extended by zero on the other component, these functions are also eigenfunctions on M. Since $(u_1 - u_2)^2 = (u_1 + u_2)^2$ and $(du_1 - du_2)^2 = (du_1 + du_2)^2$, λ is a degenerate eigenvalue of M. In this way, a degenerate eigenvalue on M arises every time an eigenvalue on M_1 coincides with an eigenvalue on M_2. In particular, a crossing of simple eigenvalues on M_1 and M_2 leads to a smooth hypersurface in $\mathcal{G}(M)$ of metrics admitting a degenerate eigenvalue of multiplicity 2.

Connectedness of M is used in the proof of Theorem 3.1.6 exactly once, to conclude via unique continuation that, for an orthonormal basis $\{u_\alpha\}_{\alpha=1}^m$ of E_λ and nonzero $A \in \mathbb{R}^{\frac{m(m+1)}{2}}$, the function $A^{\alpha\beta}u_\alpha(x)u_\beta(y)$ does not vanish to infinite order along the diagonal in $M \times M$. The failure of Theorem 3.1.6 for disconnected M stems from the failure of unique continuation for eigenfunctions in this setting.

Comparison with the Hodge Laplacian

Comparing the degenerate eigenvalues of the Laplace-Beltrami operator to those of the Hodge Laplacian is particularly interesting. We call an eigenvalue of the Hodge Laplacian degenerate it does not satisfy the SAH. This has a corresponding "localized" form analogous to the one derived for the Laplace-Beltrami operator in Proposition 3.2.3. It was shown in [Kep22] that the spectrum of the 3-dimensional Hodge Laplacian in degree 1 has a more complicated structure than that of the Laplace-Beltrami operator: eigenvalues corresponding to the exact, positive coexact, and negative coexact spectrum may collide and form degenerate eigenvalues. Indeed it is proven that metrics with degenerate eigenvalues are a codimension 1 occurence, that is one cannot hope to avoid them (or double eigenvalues more generally) along 1-parameter families of Riemannian metrics. In particular, the lowest eigenvalue $\lambda = 4$ of the Hodge Laplacian restricted to the coexact spectrum on the round 3-sphere has multiplicity 6 and is degenerate.

This degeneracy is resolved by passing to the curl operator, the square root of the Hodge Laplacian on the coexact spectrum. It is not hard to prove that degenerate eigenvalues for the curl operator must have multiplicity at least 4. This implies that the two curl eigenvalues of lowest absolute value on the round 3-sphere are non-degenerate, as both have multiplicity 3. It is worth pointing out that conformally degenerate eigenvalues of the curl operator occur already in multiplicity 2, for example in Berger's sphere.

Decomposability

The observations 3.4.4 and 3.4.4 in this subsection show that Arnold's transversality hypothesis or a maximal non-crossing rule do not always hold even for very large families of operators. Some thought reveals that all of the aforementioned situations have a common problem: the family of operators is in a certain sense decomposable. More precisely, there exist smooth families of projections

$$\pi_i(q) : H \to H_i(q)$$

which commute with the operator $A(q)$ for all $q \in \mathcal{X}$, where H_i are nontrivial closed subspaces so that $H = \bigoplus_i H_i(q)$. For the Laplace-Beltrami operators on disconnected manifolds these projections are given naturally by the restriction to any of the connected components, in the case of the Hodge Laplacian we get (metric dependent) projections to the closed subspaces associated with the exact, positive coexact, and negative coexact spectrum. If the spectra of the $A(q)$ restricted to the closed subspaces $H(q)$ behave sufficiently independently one should expect collisions of eigenvalues coming from different $H(q)$.

Equivariance

There is a different but related issue regarding generic simplicity of the spectrum. Sometimes a family of operators is obstructed from having simple spectrum because of an internal symmetry of the operators themselves, i.e., if there exists an action on the domain of the operator which commutes with the operator. The simplest example known to us is the Hodge Laplacian on 1-forms in dimension 2 where eigenvalues coming from functions and those coming from 2-forms coincide, independent of the choice of metric. This is a consequence of the fact that the Hodge Laplacian commutes with the Hodge star operator which is an almost complex structure on the space of 1-forms in dimension 2.

This is a more general effect in dimension $2n$ for forms of degree n, see the work of Millman [Mil80]. Interestingly, a similar phenomenon occurs for the middle degree $2n$ in dimensions $4n+1$ as shown by Gier and Hislop [GH16b, Section 3.1].

Thus, it seems that equivariance with respect to some group action is an obstruction to generic simplicity, and decomposability one to generic simplicity along 1*-parameter families.*

Arnold exhibited an example of a family of operators in which both phenomena occur at the same time [VWA97, Appendix 10, subsection D]. He pointed out that Dirichlet eigenvalues of the Laplacian on domains with a $\frac{1}{3}$-rotational symmetry come in two types: those of multiplicity 1, whose associated eigenfunction is symmetric, and those of multiplicity 2, whose associated eigenfunctions are antisymmetric with respect to the $\mathbb{Z}_3$ action. By varying the domain while preserving the symmetry one can achieve that symmetric and antisymmetric eigenvalues collide (or pass through one another), but eigenvalues of the same type will never merge. The equivariance of this family of operators leads to non-simple spectrum, and the decomposability, given by projections to the symmetric and antisymmetric eigenspaces, leads to collisions among these different types of eigenvalues. It is worth pointing out that there are no more collision events once one restricts to either symmetric or antisymmetric eigenvalues.

3.4.5 Open questions

We conclude our paper with a list of natural questions, and some ideas for how one might resolve them.

Question 3.4.5. *Do there exist degenerate eigenvalues of multiplicity* 7*?*

Theorem 3.4.1 leaves an interesting gap: Degenerate eigenvalues cannot have multiplicity less than seven, but all examples known to the authors have multiplicity at least eight. In the proof of Theorem 3.4.1, nonexistence of degenerate eigenvalues of multiplicity six follows from the rigidity of local isometries of $\mathbb{S}^2$. Local isometric immersions of $\mathbb{S}^2$ into $\mathbb{S}^3$ are no longer fully rigid, but they do exhibit an interesting partial rigidity phenomenon, as was shown by O'Neill and Stiel [OS63]: As in the case of local isometric immersions of $\mathbb{R}^2$ into $\mathbb{R}^3$, their image is ruled by geodesics. One might use this to show degenerate eigenvalues of multiplicity seven do not exist, but this seems to go beyond the scope of the present paper. Let us also note that there exists a certain left-invariant metric on $SU(2) \cong \mathbb{S}^3$ admitting a first eigenvalue of multiplicity 7 (as shown by Urakawa in [Ura79]), and thus a plausible candidate for a degenerate eigenvalue of this multiplicity. However, a closer examination reveals that this eigenvalue is nondegenerate.

Question 3.4.6. *What is the local behaviour of the spectrum near degenerate metrics?*

According to Theorem 3.2.1, given a metric g admitting a nondegenerate eigenvalue of multiplicity m, the set of perturbations which do not split up this eigenvalue is a smooth manifold of codimension $\frac{m(m+1)}{2} - 1$. Understanding the behaviour of a *degenerate* eigenvalue under perturbation is more difficult. The natural defining function π from Section 3.2 is no longer a submersion, but it could still cut out a submanifold (much like the function $y^2 = 0$ cuts out a submanifold of $\mathbb{R}^2_{x,y}$). However, this seems unlikely. It would be interesting, but potentially difficult, to exhibit a rigorously treated example.

The paper [Arn72] contains a discussion of the complicated shape of the singularity of the set of symmetric matrices with an eigenvalue of multiplicity at least 2 along the lower dimensional submanifold of symmetric matrices with an eigenvalue of multiplicity 3 (Remark 1.4 in [Arn72]). By Theorem 3.2.1, near a nondegenerate eigenvalue of multiplicity 3, the set of metrics splitting off exactly one eigenvalue inherits the shape of this singularity, up to a trivial factor of finite codimension. The shape of the tangent cone of the analogous singularity near a *degenerate* eigenvalue is presumably topologically different, but one would likely need to take at least a second variation to compute it exactly.

Question 3.4.7. *Does there exist a "natural" parameter space containing the Laplace-Beltrami operators with respect to which there are no degenerate eigenvalues?*

By enlarging the parameter space, more and more "directions" in which eigenvalues can split up become available. Thus, it is easy to find degenerate eigenvalues with respect to variations in a fixed conformal class, but already harder if one allows variations among all possible metrics. A natural question is whether there exists a good parameter space containing all Laplace-Beltrami operators, e.g., elliptic differential operators of degree 2, where all eigenvalues satisfy the strong Arnold hypothesis.

Question 3.4.8. *What is the geometric meaning of the degeneracy condition?*

As briefly mentioned in the introduction, the (conformal) degeneracy condition looks very similar to the one obtained by El Soufi and Ilias in [EI08a] characterizing metrics which extremize the k-th eigenvalue functional. They show that a metric g extremizes the eigenvalue functional λ_k if there exists a basis of eigenfunctions $\{u_j\}_{j=1}^m$ associated the same eigenvalue and coefficients ϵ_j valued in $\{0, 1\}$ satisfying

$$\sum_j \epsilon_j \, du_j \otimes du_j = g \tag{3.4.1}$$

and the converse is true provided one considers the greatest or smallest of the eigenvalue functionals coinciding at g. The analogous extremality condition in a conformal class is

$$\sum_j \epsilon_j u_j^2 = 1$$

Using a result of Takahashi [Tak66] one can reinterpret the extremality condition as saying that the eigenfunctions $\{u_j\}$ give a minimal isometric immersion of (M, g) into the sphere $\mathbb{S}^{d-1}(\sqrt{\frac{n}{|\lambda_k|}})$, where d is the number of nonzero coefficients ϵ_j. Conversely, every such minimal immersion gives rise to an extremal metric.

It would be interesting to see whether the nondegeneracy condition admits a similar interpretation, perhaps as a map with special geometric properties into the null cone of a semi-Riemannian $\mathbb{R}^{s,t}$.

We would also like to remark that, exactly as in the extremality condition, the condition on the square sum of the eigenfunctions in the definition of a degenerate eigenvalue is in fact implied by the condition on the square sum of their gradients (Proposition 3.2.5).

Question 3.4.9. *What is the topology of the set of metrics which exhibit at least one eigenvalue of multiplicity m? Is it connected?*

To answer this question, one must presumably combine Theorem 3.1.3 with a new global argument, possibly a generalized non-crossing rule conditioned to preserve the multiplicity of a given eigenvalue.

3.5 Appendix: Cofinite Fréchet submanifolds

Our first appendix establishes the "cofinite" version of the implicit function theorem used throughout the text. It is analogous to [Fre15, Corollary 5.5], but has a significantly simpler proof. Recall the definition of a C^1 map between Fréchet spaces [Ham82, Definition 3.1.1.]: A map $F : \mathcal{F} \supseteq \mathcal{U} \to \mathcal{G}$ is called *continuously differentiable*, or just C^1, if

$$DF(x)u = \lim_{t \to 0} \frac{F(x+tu) - F(x)}{t}$$

exists for all $x \in \mathcal{U}$, $u \in \mathcal{F}$, and furthermore, $DF(x)u$ is linear in u and *jointly continuous* in x and u.

Lemma 3.5.1. *Let $\mathcal{U} \subseteq \mathcal{F}$ and $V \subseteq \mathbb{R}^m$ be open neighborhoods of the origin in a Fréchet space $\mathcal{F}$ and in $\mathbb{R}^m$, respectively. Suppose $F : \mathcal{U} \times V \to \mathbb{R}^m$ is C^1 in the sense introduced above, $F(0,0) = 0$, and $D_2F(0,0)$ is invertible. Then there exists a neighborhood $\mathcal{U}' \times V' \subseteq \mathcal{U} \times V$ of $(0,0)$ and a C^1 function $G : \mathcal{U}' \to \mathbb{R}^m$ such that for all $(x,y) \in \mathcal{U}' \times V'$, we have $F(x,y) = 0$ if and only if $y = G(x)$.*

Proof. This is a matter of checking that the usual fixed point iteration works under the present, weaker than usual, assumptions on $\mathcal{F}$. Define $R : \mathcal{U} \times V \to \mathbb{R}^m$,

$$R(x,y) = y - D_2F(0,0)^{-1}F(x,y).$$

Clearly, $F(x,y) = 0$ if and only if $R(x,y) = y$. We can rewrite

$$R(x,y) = \left(\int_0^1 \left(\mathrm{id}_{\mathbb{R}^m} - D_2F(0,0)^{-1}D_2F(x,ty) \right) dt \right) y$$

By assumption, the expression $D_2F(x,y)h$ is jointly continuous in x, y and h. Since we assumed V is *finite-dimensional*, this means that the operator $D_2F(x,y) : \mathbb{R}^m \to \mathbb{R}^m$ is continuous in x, y with respect to the norm topology. Thus, there exists a neighborhood $\mathcal{U}' \times V' \subseteq \mathcal{U} \times V$ of $(0,0)$ such that

$$\|\mathrm{id}_{\mathbb{R}^m} - D_2F(0,0)^{-1}D_2F(x,y)\| < \frac{1}{2}$$

for all $x, y \in \mathcal{U}' \times V'$, and hence $R(\mathcal{U}' \times V') \subseteq V'$. Analogously, we have that $R(x,y)$ is a contraction on V' for any fixed $x \in \mathcal{U}'$. Indeed, $R(x,y_1) - R(x,y_2)$

$$= \left(\int_0^1 \left(\mathrm{id}_{\mathbb{R}^m} - D_2F(0,0)^{-1}D_2F(x,(1-t)y_1 + ty_2) \right) dt \right) (y_2 - y_1).$$

Thus, given $x \in \mathcal{U}'$, there is a unique solution $y \in V'$ to the equation $F(x,y) = 0$.

To prove that F admits all directional derivatives is straightforward. Let $x \in \mathcal{U}'$, and $u \in \mathcal{F}$. Then the finite dimensional implicit function theorem, applied to the restriction of F to $\langle x, u \rangle \times \mathbb{R}^m$ shows that $DG(x)[u] = \lim_{t \to 0} \frac{G(x+tu) - G(x)}{t}$ exists and

$$DG(x)[u] = D_yF(x, G(x))^{-1} D_xF(x, G(x))u.$$

This expression is clearly continuous in x and u, and hence, G is C^1. □

Proposition 3.5.2. *Let $\mathcal{U} \subseteq \mathcal{F}$ be an open subset of a Fréchet space $\mathcal{F}$. Suppose a C^1 map $F : \mathcal{U} \to \mathbb{R}^m$ is such that $DF(p) : \mathcal{F} \to \mathbb{R}^m$ is surjective at a point $p \in \mathcal{U} \cap F^{-1}(0)$. Then there exists a neighborhood $\mathcal{U}' \subseteq \mathcal{U}$ of p, an open neighborhood $\mathcal{V} \subseteq \mathcal{F}'$ of the origin in a Fréchet space $\mathcal{F}'$, and a C^1 map $G : \mathcal{V} \to \mathcal{U}'$ such that $F^{-1}(0) \cap \mathcal{U}' = G(\mathcal{V})$. If $\mathcal{F}$ was equipped with a grading, G is furthermore tame.*

Proof. Since $DF(p)$ is onto, we may choose $v_1, \dots, v_m \in \mathcal{F}$ such that $\{DF(p)v_j\}_{j=1}^m$ form a basis of $\mathbb{R}^m$. By continuity of $q \mapsto DF(q)v_j$, there exists a neighborhood $\mathcal{U}_1$ of p such that $DF(q)v_1, \dots, DF(q)v_m$ are linearly independent for all $q \in \mathcal{U}_1$. Note that $\mathcal{F}' := \ker DF(p)$ is a closed complement to $V := \langle v_1, \dots, v_m \rangle$. Let $\mathcal{V}_1 \subseteq \ker DF(p)$ and $U_1 \subseteq V$ be open neighborhoods of the origin with $p + \mathcal{V}_1 \oplus U_1 \subseteq \mathcal{U}_1$. The map $\tilde{F} : \mathcal{V}_1 \times U_1 \to \mathbb{R}^m$, $\tilde{F}(x, y) = F(p + x + y)$ satisfies the hypothesis of Lemma 3.5.1, which yields a neighborhood $\mathcal{V} \oplus U \subseteq \mathcal{V}_1 \oplus U_1$ of the origin and a C^1 map $\tilde{G} : \mathcal{V} \to U$ such that $F^{-1}(0) \cap (p + \mathcal{V} \oplus U)$ is parametrized by $G : \mathcal{V} \to \mathcal{F}$,

$$G(x) = p + x + \tilde{G}(x).$$

The first claim follows after we denote $\mathcal{U}' := p + \mathcal{V} \oplus U$.

The map G is tame with respect to any grading on $\mathcal{F}$ (and the induced grading on $\mathcal{F}'$), since it is the sum of an isometric embedding and a map of finite rank. □

A statement akin to Proposition 3.5.2 for submersions valued in a Banach space is obtained in [Fre15], albeit with the added complication that surjectivity of DF at a single point no longer implies F is a submersion. This theorem implies Proposition 3.5.2 immediately (Corollary 5.5 in [Fre15]), but its proof invokes the Nash-Moser implicit function theorem. We found it noteworthy that cofinite Fréchet submanifolds can be parametrized using the standard implicit function theorem.

3.6 Appendix: Transversality theory in finite codimension

The aim of this appendix is to provide the necessary transversality-theoretic background needed throughout the paper.

Definition 3.6.1. *A subset $\mathcal{D}$ of a Fréchet manifold $\mathcal{X}$ is said to have transversality codimension k if for any manifold N of dimension less than k, the set of maps in $C^\infty(N, \mathcal{X})$ that avoids $\mathcal{D}$ is residual.*

It is clear from the definition that countable unions of sets of transversality codimension k_i have transversality codimension $\min_i k_i$.

Before we provide a nontrivial example of a set of transversality codimension k we will first need to prove a variant of the parametric transversality theorem. Due to the fact that we consider finite codimension manifolds only, its proof is essentially the same as in finite dimensions (see for example [GP10]), but for completeness we will include it here.

Lemma 3.6.2. *[Parametric Transversality] Let N be a smooth compact manifold of dimension n, $\mathcal{X}$ a Fréchet manifold, and $Z \subset \mathcal{X}$ the level set of a C^1 submersion $\phi : \mathcal{X} \to \mathbb{R}^k$, where $n \leq k$. Given a finite dimensional manifold S of dimension s and a smooth map $F : N \times S \to \mathcal{X}$ such that both F and ∂F are transverse to Z, F_s and ∂F_s are transverse to Z for generic s.*

Proof. Using the chain rule and that ϕ is a submersion we see that $W := F^{-1}(Z)$ is a C^1 submanifold of $N \times S$ with boundary $\partial W = W \cap \partial(N \times S)$. Denote by $\pi : N \times S \to S$ the projection to the second factor. The conclusion of this Lemma will follow by an application of Sard's theorem once we prove that $s \in S$ being a regular value for the projection π restricted to W and ∂W implies that $F_s \pitchfork Z$ and $\partial F_s \pitchfork Z$, respectively. Sard's theorem for C^1 maps holds provided the dimension of the source is less than or equal to the dimension of the target, which is where the dimensional restriction in the statement of the Lemma comes from.
Assume now that s is a regular value of π restricted to W (the boundary case works in exactly the same way). For $x \in W$, denote $F_s(x) = z \in Z$. Transversality of F implies that

$$\mathrm{Im}(TF)_{(x,s)} + T_z Z = T_z \mathcal{X}$$

This means that for any $a \in T_z\mathcal{X}$ there exists $(b, c) \in T_{(x,s)}\big(N \times S\big)$ such that $(TF)_{(x,s)}(b, c) = a$. We need to find $v \in T_s S$ such that $F_s(v) = a + d$ for some $d \in T_z Z$. To this end note that the fact that s is regular guarantees the existence of a $u \in T_{(x,s)}W$ so that $T\pi_s(u) = c$. Noting that $(TF)(u, e) \in T_z Z$ we set $v = b - u$ and compute

$$(TF_s)_{(x,s)}(v) = (TF)_{(x,s)}(v, e) = a - (TF)(u, e)$$

and so (TF_s) hits a up to an element in $T_z Z$ and we are done. □

Remark 3.6.3.
1. *For the purposes of proving a maximal non-crossing rule, the above Lemma is sufficient. If one wishes to control generic intersections of images of smooth maps from N to $\mathcal{X}$ with Z when $n > k$ via Sard's theorem one would have to prove smooth versions of Proposition 3.5.2 and Lemma 3.7.1.*
2. *The above proof clearly works when M has corners.*
3. *It also works if we prescribe the boundary ∂M and only vary F on the interior of M.*

The following Lemma tells us that transversality codimension is a local concept in separable Fréchet spaces.

Lemma 3.6.4. *Let $\mathcal{D}$ be a subset of a separable Fréchet manifold $\mathcal{X}$. If for all $x \in \mathcal{D}$ there exists an open set $U_x \subset \mathcal{X}$ such that $\mathcal{D} \cap U_x$ has transversality codimension k in $\mathcal{X}$, then $\mathcal{D}$ has transversality codimension k in $\mathcal{X}$.*

Proof. The union $\mathcal{O} = \bigcup_{x \in \mathcal{D}} U_x$ is an open set in $\mathcal{X}$ and therefore itself a separable Fréchet manifold, so one may extract a countable subcover $\{U_{x_n}\}$ of $\mathcal{O}$. We conclude by noting that $\mathcal{D}$ is the countable union of the sets $U_n \cap \mathcal{D}$ all of which have transversality codimension k. □

Lemma 3.6.5. *Let $\mathcal{X}$ be an open subset of a separable Fréchet space. Let $\mathcal{D} \subset \mathcal{X}$ be a subset for which there exist an open cover $(U_i)_{i \in I}$ and submersions $\phi_j : U_i \to \mathbb{R}^k$ with the property that, for each $p \in \mathcal{D}$, there is a neighborhood V of p and a countable set $J \subseteq I$ with $\mathcal{D} \cap V \subseteq \bigcup_{j \in J} \phi_j^{-1}(\{0\})$. Then $\mathcal{D}$ has transversality codimension k.*

Proof. We need to prove that, for any manifold N of dimension less that k, the set of smooth maps $C^\infty(N, \mathcal{X})$ from N to $\mathcal{X}$ which avoids $\mathcal{D}$ is residual. Assume without loss of generality that N is compact, as the general case follows by a compact exhaustion argument. We will show that a generic set in $C^\infty(N, \mathcal{X})$ avoids $\phi_j^{-1}(\{0\})$ for any j. The strategy is to show that every point $z \in \phi_j^{-1}(\{0\})$ has a neighbourhood V_z so that $V_z \cap \phi_j^{-1}(\{0\})$ has transversality codimension k. The claim then follows by an application of Lemma 3.6.4. Let B_z be some neighbourhood of $z \in \phi_j^{-1}(\{0\})$ and $f \in C^\infty(N, \mathcal{X})$ which is not transverse to $B_z \cap \phi_j^{-1}(\{0\})$ at the point z. We define $F_z : M \times \mathbb{R}^k \to \mathcal{X}$ by $F_z(x, s) = f(x) + s_i h^i(f(x))$ where $f(x) = z$ and $h^i(f(x))$ a basis of a subspace $K_x \pitchfork \ker(T\phi_j)$. After potentially restricting B_z we may assume that F_z is transverse to $B_z \cap \phi_j^{-1}(\{0\})$. The parametric transversality Lemma 3.6.2 now implies that the set of s such that F_s is transverse to $B_z \cap \mathcal{D}$ is residual which implies density in $\mathbb{R}^k$, meaning that f can be approximated arbitrarily well in C^∞ with smooth functions that are transverse to $B_z \cap \phi_j^{-1}(\{0\})$.

Now pick an open subset V_z of B_z satisfying that $\overline{V_z \cap \phi_j^{-1}(\{0\})} \subset B_z$. Clearly, the set of maps $C^\infty(N, \mathcal{X})$ which avoids $\overline{V_z \cap \phi_j^{-1}(\{0\})}$ is open, and the above argument shows that it is dense as well. Since subsets of sets of transversality codimension k are themselves of transversality codimension k, we have proven that $\phi_j^{-1}(\{0\})$ has transversality codimension k.

This immediately implies that $\mathcal{D} \cap V$ has transversality codimension k, and another application of Lemma 3.6.4 allows us to conclude that $\mathcal{D}$ itself has transversality codimension k. □

Remark 3.6.6. *The Fréchet manifold of smooth metrics is an open subset of the Fréchet space of smooth $(0, 2)$-tensor fields, and so the setting of Lemma 3.6.5 is good enough for the purposes of Theorem 3.1.6. A more general setting in which one can run a similar argument is the class of separable Fréchet manifolds that admit smooth bump functions, i.e., those modelled on C^∞-regular separable Fréchet spaces, see [KM97, Chapter 3, Subsection 14]. In this case, one can define F as above in a chart and extend it to a globally defined map via a bump function.*

3.7 Appendix: Continuous differentiability in the metric of the Laplacian and its spectral projections

Lemma 3.7.1. *Let M be a closed, smooth manifold, and $\mathcal{G}(M)$ the Fréchet manifold of smooth Riemannian metrics on M.*

1. *The family $\Delta : \mathcal{G}(M) \to \mathcal{L}(H^2(M), L^2(M))$ of Laplace-Beltrami operators is continuously differentiable. In local coordinates,*
$$D\Delta(g)[h]u = \tfrac{1}{2} g^{jk}\partial_j \mathrm{tr}_g h\, \partial_k u - |g|^{-\frac{1}{2}}\partial_j(|g|^{\frac{1}{2}} h^{jk}\partial_k u). \tag{3.7.1}$$

2. *Fix $g \in \mathcal{G}(M)$ and $\lambda \in \rho(\Delta_g)$. Then there exists a neighborhood $\mathcal{U} \subseteq \mathcal{G}(M)$ of g and $\varepsilon > 0$ such that $(z - \Delta)^{-1} : \mathbb{B}_\varepsilon(\lambda) \times \mathcal{G}(M) \to \mathcal{L}(L^2(M), H^2(M))$ is defined, and continuously differentiable.*

3. *Fix $g \in \mathcal{G}(M)$ and $\gamma : \mathbb{S}^1 \to \rho(\Delta_g)$, a smooth, closed, positively oriented Jordan curve. For $\tilde{g} \in \mathcal{G}(M)$, let $P_\gamma(\tilde{g})$ denote the spectral projection to the eigenspaces of $\Delta_{\tilde{g}}$ corresponding to eigenvalues encircled by γ. Then there exists a neighborhood $\mathcal{U} \subseteq \mathcal{G}(M)$ of g such that $P_\gamma : \mathcal{U} \to \mathcal{L}(L^2(M), H^2(M))$ is continuously differentiable.*

Proof. To allow computation in coordinates, choose a smooth partition of unity $\sum_{j=1}^{\infty} \chi_i = 1$ adapted to a coordinate atlas. It is then sufficient to check $\chi_i \Delta(g)$ is smooth in g for all i. The operator $\chi_i\Delta(g)$ can be written as a sum of compositions of smooth vector fields ∂_j, $j = 1, \dots, n$ and multiplication operators which depend in a continuously differentiable fashion on g:

$$\chi_i\Delta(g) = \chi_i \circ |g|^{-\frac{1}{2}} \circ \partial_j \circ |g|^{\frac{1}{2}} \circ g^{jk} \circ \partial_k$$

This composition should be read as follows:

1. ∂_k is a bounded linear operator $\mathcal{L}(H^2(M), H^1(M))$, independent of g,
2. $|g|^{\frac{1}{2}} \circ g^{jk}$ is a C^1 family of bounded operators on $H^1(M)$,
3. ∂_j is a fixed bounded linear operator $\mathcal{L}(H^1(M), L^2(M))$, and
4. $\chi_i \circ |g|^{-\frac{1}{2}}$ is a C^1 family of bounded operators on $L^2(M)$.

For continuous differentiability of the multiplication operator $|g|^{\frac{1}{2}} \circ g^{jk}$ on $H^1(M)$ it is crucial that g has at least one derivative, since the derivative may hit the metric. This is more than guaranteed by working with $g \in \mathcal{G}(M)$. The formula for $D\Delta(g)[h]$ follows from the chain and product rule as well as the identities

$$D|g|[h] = |g|\mathrm{tr}_g h, \qquad Dg^{jk}[h] = -h^{jk}.$$

By the same argument, $\lambda - \Delta(g)$ is a family of bounded operators from $H^2(M)$ to $L^2(M)$ which depends continuously on both λ and g.

Now fix $g \in \mathcal{G}(M)$ and $\lambda \in \rho(M)$. From the continuous dependence of the eigenvalues on the metric [Lab15, Lemma 4.5.13], it follows that there exists a neighborhood $\mathcal{U}$ of g (which can even be chosen to be a C^0-norm ball), and $\varepsilon > 0$ such that $(z - \Delta(\tilde{g}))^{-1} \in \mathcal{L}(L^2(M), H^2(M))$ is defined for all $\tilde{g} \in \mathcal{U}$ and $z \in \mathbb{B}_\varepsilon(\lambda)$. By [Ham82, Theorem II.3.1.1], it follows that $(z - \Delta)^{-1} : \mathbb{B}_\varepsilon(\lambda) \times \mathcal{U} \to \mathcal{L}(L^2(M), H^2(M)$ is continuously differentiable.

Finally, fix a metric $g \in \mathcal{G}(M)$ and a curve γ in the resolvent set of g as in the statement of the lemma. The spectral projection P_γ is given by

$$P_\gamma(\tilde{g}) = \frac{1}{2\pi i} \int_\gamma (z - \Delta(\tilde{g}))^{-1} dz,$$

which is defined whenever γ avoids the spectrum on $\tilde{g}$. In particular, there exists a neighborhood $\mathcal{U} \subseteq \mathcal{G}(M)$ of g (which again may be taken as a C^0-ball) such that $P_\gamma : \mathcal{U} \to \mathcal{L}(L^2(M), H^2(M))$ exists. Continuous differentiability follows from differentiating under the integral using the dominated convergence theorem. □

4 Isolated steady solutions of the 3D Euler equations

(Based almost verbatim on [EKP25], joint work with Alberto Enciso and Daniel Peralta-Salas)

Abstract

We show that there exist closed three-dimensional Riemannian manifolds where the incompressible Euler equations exhibit smooth steady solutions that are isolated in the C^1-topology. The proof of this fact combines ideas from dynamical systems, which appear naturally because these isolated states have strongly chaotic dynamics, with techniques from spectral geometry and contact topology, which can be effectively used to analyze the steady Euler equations on carefully chosen Riemannian manifolds. Interestingly, much of this strategy carries over to the Euler equations in Euclidean space, leading to the weaker result that there exist analytic steady solutions on $\mathbb{T}^3$ such that the only analytic steady Euler flows in a C^1-neighborhood must belong to a certain linear space of dimension six. For comparison, note that in any C^k-neighborhood of a shear flow there are infinitely many linearly independent analytic shears.

4.1 Introduction

In the study of steady states of the incompressible Euler equations, one often finds a subtle interplay between flexibility and rigidity properties, that is, between the existence of a wealth of steady Euler flows and the significant constraints that they must nonetheless satisfy. To elaborate on this idea, for the time being we shall focus on the two dimensional case, where our understanding is much more complete.

As is well known, in $\mathbb{R}^2$, the velocity field can be written in terms of the perpendicular gradient of the stream function as $v := \nabla^\perp \psi = (\partial_2\psi, -\partial_1\psi)$, whereas in all dimensions the pressure is determined by the velocity through the formula $p := -\Delta^{-1}\operatorname{div}(v\cdot\nabla v)$. The steady Euler equations

$$v\cdot\nabla v = -\nabla p\,, \qquad \operatorname{div} v = 0\,, \tag{4.1.1}$$

can then be equivalently written as

$$\nabla^\perp\psi\cdot\nabla\Delta\psi = 0\,.$$

While this is generally not a well behaved equation from the point of view of the analysis of PDEs, it does certainly ensure the existence of many steady states. Specifically, on the plane $\mathbb{R}^2$ (or on $\mathbb{T}^2$ in the case of periodic conditions, with $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, or more generally on any two-dimensional Riemannian manifold), any function satisfying an elliptic equation of the form

$$\Delta\psi = f(\psi) \tag{4.1.2}$$

defines a steady Euler flow as $v := \nabla^\perp\psi$. Note that not all steady states can be constructed this way, as the vorticity $\Delta\psi$ and the stream function do not generally satisfy an equation of this form globally.

It is a truly infinite-dimensional feature of the Euler equations that steady solutions of the form (4.1.2) are often embedded in rich families of solutions. Indeed, Constantin, Drivas and Ginsberg [CDG21] have recently shown that, under suitable technical hypotheses, these solutions are structurally stable. Also, given a steady state on an annular domain for which the linearization of (4.1.2) is positive definite, Choffrut and Šverák [CŠ12] completely characterized the nearby steady states by showing that they are in one-to-one correspondence with their distribution functions, so that there exists a unique stationary solution on the corresponding orbit in the group of area-preserving diffeomorphisms.

On the rigidity side, it is known that, under suitable technical hypotheses, steady Euler flows must inherit the symmetries of their vessel, understood as a two-dimensional Riemannian manifold possibly with boundary. This is the case when the flow does not have any stagnation points if the domain is a disk, an annulus or a periodic channel [HN23; HN17; HN19] or if the solution is compactly supported and satisfies certain sign conditions [Góm+21; Rui23]. In the particular case of the periodic channel $\mathbb{T}\times\mathbb{R}$, for instance, this result ensures that the only steady Euler flows without stagnation points are shear flows $v(x) := (V(x_2), 0)$. Similar results also hold when the steady flow is dynamically stable, satisfies (4.1.2) and the linearized operator is positive [CDG21].

Results such as [HN23; HN17; HN19] ensure that steady Euler flows without stagnation points are isolated from non-shears in the C^1-norm. The dependence of this property on the absence of stagnation points and on the topology is remarkable [CEW23; LZ11]; for instance [CEW23], the vanishing shear flow $(\sin x_2, 0)$ is not isolated from non-shears even in the analytic topology, while the shear flow $(x_2^2, 0)$, which also has stagnation points, is isolated from non-shears in H^s, $s > 5$.

In three dimensions, the interplay between rigidity and flexibility is similar, but our understanding is much more limited. The steady Euler equations can be equivalently written as

$$v \times \operatorname{curl} v = \nabla F\,, \tag{4.1.3}$$

where the so-called *Bernoulli function* F is arbitrary, and the closest analog to (4.1.2) is

$$\operatorname{curl} v = f v\,, \qquad \operatorname{div} v = 0\,,$$

which defines the so-called *Beltrami fields* with proportionality factor f (called the *Beltrami function*). Although these equations do have a rich space of solutions, and the fact that f is arbitrary provides some additional flexibility, whenever the function f is nonconstant the equations exhibit some surprising rigidity phenomena [EP16; CK20]. As a rule of thumb, in 3D, an advantage is that divergence-free fields can have a richer behavior, and a drawback is that the equations are notoriously harder to analyze.

A recent survey by Drivas and Elgindi [DE23] draws the attention to the fact that none of the known steady 2D Euler flows are isolated in C^1. Note this is also true for the shear flows studied in [HN23; HN17; HN19; CEW23; LZ11], as one can find other shear flows in any C^1-neighborhood of the solution. In fact, [DE23, Problem 1] is to show that no continuously differentiable steady 2D Euler flow is isolated in C^1. In three dimensions the question is also wide open. We recall that isolated steady states are defined as follows:

Definition 4.1.1. *A steady solution v_0 of the incompressible Euler equations is* isolated in C^1 *if it is the only C^1 steady Euler flow in a C^1-neighborhood of v_0, modulo a constant scaling of v_0 or a conjugation of v_0 by an isometry of the ambient space.*

Our objective in this paper is to construct a steady 3D Euler flow which is isolated in C^1. To grant us some leeway, we shall consider the equations on a closed (i.e., compact boundaryless) oriented manifold M of dimension 3 endowed with a Riemannian metric g. Our main result is that indeed one can construct Riemannian manifolds so that the Euler equations admit an isolated steady state:

Theorem 4.1.2. *There exists a closed 3-dimensional Riemannian manifold (M, g) of class C^∞ where the incompressible Euler equations have a smooth steady solution that is isolated in C^1. Moreover, the only C^1 steady states in a C^1-neighborhood of the isolated steady solution are constant multiples of it.*

Remark 4.1.3. *There is a broad range of closed 3-manifolds that admit (for a suitable metric) steady states that are isolated in C^1. This includes the unit tangent bundle of any negatively curved surface and, more generally, any hyperbolic 3-manifold. See Remark 4.2.3.*

Roughly speaking, the result hinges on having a steady state with robust strongly chaotic dynamics. Thus we start off by letting v_0 be an Anosov flow on M for which we can find a contact form whose Reeb field is precisely v_0. This easily yields a Riemannian metric g where v_0 is a Beltrami field (i.e., $\operatorname{curl}_g v_0 = 2v_0$), so that v_0 is, in particular, a steady Euler flow. By further tweaking the metric, one can ensure that 2 is a simple eigenvalue of the curl operator. To conclude, we then use the robust chaotic dynamics of the field to show that any C^1-small perturbation v of v_0 that is a steady state must also satisfy the equation $\operatorname{curl}_g v = 2v$, essentially as a consequence of the fact that v does not admit any (nonconstant) continuous first integrals. The theorem then follows from the simplicity of the eigenvalue. The proof of this theorem is given in Section 4.2. Interestingly, the most technically challenging part of the argument is the simplicity of the curl eigenvalue, so this point is presented separately in Section 4.4, and some parts of the proof are relegated to an Appendix. A direct consequence of the proof of Theorem 4.1.2 is that, in a suitably chosen metric, any Anosov Reeb flow on a 3-manifold defines a steady Euler flow which is isolated up to a finite-dimensional family (see Remark 4.2.3).

Much of this strategy carries over to the Euler equations on the usual Euclidean space setting, say on the flat torus $\mathbb{T}^3 := (\mathbb{R}/2\pi\mathbb{Z})^3$. To see this, let $\mathcal{V}$ be the space of eigenfields of the curl operator on $\mathbb{T}^3$ with eigenvalue 1, which is the smallest positive curl eigenvalue, see, e.g., [PS23]. It is well known that $\mathcal{V}$ is a six-dimensional vector space, invariant under rigid motions. Moreover, it contains the family of *ABC flows* [AK21], i.e., the fields

$$u_{ABC}(x) := \big(A\sin x_3 + C\cos x_2,\ B\sin x_1 + A\cos x_3,\ C\sin x_2 + B\cos x_1\big) \qquad (4.1.4)$$

defined by arbitrary real constants A, B and C.

Our key observation is that one can identify a nonvanishing ABC flow u_0 with positive topological entropy. Since this property is preserved by C^1-small perturbations and fields with positive topological entropy cannot have a (nonconstant) analytic first integral (this fact is specific to 3-manifolds), in Section 4.3 we show how to establish the following weaker analog on $\mathbb{T}^3$ of Theorem 4.1.2. For comparison, note that in a C^k-neighborhood of a shear flow, for any k, there are infinitely many linearly independent analytic shears.

Theorem 4.1.4. *There exist ABC flows u_0 on $\mathbb{T}^3$ such that the only analytic steady Euler flows in a C^1-neighborhood of u_0 belong to the six-dimensional space $\mathcal{V}$.*

A minor comment is that, for the purposes of this result, there is nothing really special about ABC flows. In fact, essentially the same argument shows that “most” Beltrami fields on $\mathbb{T}^3$ are isolated in C^1 within the set of analytic steady Euler flows, up to the finite-dimensional family of Beltrami fields with the same curl eigenvalue. A precise statement of this fact is given in Remark 4.3.1 below.

4.2 Anosov steady states: proof of Theorem 4.1.2

Key to proving Theorem 4.1.2 is the concept of Anosov flows. Recall [FH19] that a vector field X on a closed 3-manifold M is said to be *Anosov* if there is a constant $0 < \Lambda < 1$ and line subbundles E^s and E^u of TM such that $TM = E^s \oplus E^u \oplus \operatorname{span}\{X\}$, and the flow φ_t of X satisfies:

- Invariance: $d\varphi_t\,(E^s) = E^s$ and $d\varphi_t\,(E^u) = E^u$ for every $t \in \mathbb{R}$.
- Hyperbolicity: $|\, d\varphi_t|_{E^s}\,| \leq \Lambda^t$ and $|\, d\varphi_{-t}|_{E^u}\,| \leq \Lambda^t$ for all $t > 0$. Here the operator norm $|\cdot|$ is induced by some Riemannian metric on M (whose choice is irrelevant).

We shall consider Anosov flows that are also Reeb flows for some contact structure. Let α be a contact form on M, i.e., a 1-form α such that $\alpha \wedge d\alpha > 0$ on M. A Riemannian metric g on M is called *compatible* with α if

$$|\alpha|_g = 1\,, \qquad \star_g d\alpha = 2\alpha\,,$$

where $\star_g$ is the Hodge star operator computed with the metric g, see, e.g., [EKM12b, Proposition 2.1] and Lemma 4.4.1 below.

A contact manifold (M, α) defines a Reeb vector field R as the unique solution to

$$i_R d\alpha = 0\,, \qquad \alpha(R) = 1\,.$$

It is straightforward to check that for any compatible metric g we have

$$g(R, \cdot) = \alpha(\cdot)\,, \qquad |R|_g = 1\,.$$

It then follows that R is a Beltrami field (as defined in the Introduction) with constant factor 2, and hence a steady state for the incompressible Euler equations on (M, g). See e.g. [PS23] and references therein for a recent account on Beltrami fields and contact geometry.

The simplest example of an Anosov Reeb flow is the geodesic flow on the spherical cotangent bundle of a compact Riemannian surface of genus > 1 with constant negative curvature, but this does not exhaust all the examples of Anosov Reeb flows [FH13].

It is obvious that Theorem 4.1.2 is a consequence of the following result:

Theorem 4.2.1. *Assume that a closed 3-manifold M admits an Anosov Reeb flow R. Then there exists a Riemannian metric g such that R is a steady state of the incompressible Euler equations on (M, g) and any other C^1 steady state v that is C^1-close to R is of the form $v = cR$ for some constant $c \in \mathbb{R}$.*

Proof. Let (M, α) be a contact 3-manifold, and R its associated Reeb field, which is Anosov by assumption. As mentioned above, any metric g compatible with α makes R a Beltrami field with constant factor 2, i.e.,

$$\operatorname{curl}_g R = 2R\,.$$

By Theorem 4.4.2, there exists a compatible metric g such that 2 is a simple eigenvalue of curl_g, and hence R is the only Beltrami field with factor 2 up to a constant multiple.

Consider a C^1-neighborhood $\mathcal{B}_\varepsilon(R)$ of R and let $v \in \mathcal{B}_\varepsilon(R)$ be a steady solution of the incompressible Euler equations on (M, g). If $\varepsilon > 0$ is small enough, the stability of Anosov dynamics implies that v is an Anosov flow as well [FH19, Corollary 5.1.9]. Since any codimension one Anosov volume preserving flow on a compact manifold is ergodic ([Ver74, Corollary 1.1], [FH19, Theorem 7.1.26]), any continuous first integral of v must be constant on M. Accordingly, the Bernoulli function associated with v must be constant, and therefore it immediately follows from the stationary Euler equations that the vector fields $\operatorname{curl}_g v$ and v are collinear on M. Since $|v|_g^2 > 0$ on M because v is C^1-close to the nonvanishing field R, we conclude that v satisfies the equations

$$\operatorname{curl}_g v = fv\,, \qquad \operatorname{div}_g v = 0\,,$$

with factor

$$f := \frac{\operatorname{curl}_g v \cdot v}{|v|_g^2}\,.$$

Taking the divergence in the Beltrami equation above, we infer that

$$\nabla_g f \cdot v = 0\,,$$

so f is a first integral of v. Reasoning as above, we conclude that f is constant on M. Since 2 is a simple eigenvalue of curl_g, the C^1-closeness of v to R implies that $f = 2$ and hence $v = cR$ for some constant $c \in \mathbb{R}$. This completes the proof of the theorem. $\square$

Remark 4.2.2. *A crucial property used in the proof of Theorem 4.2.1 is the fact that Anosov flows are robustly transitive (actually, they are C^1-structurally stable, i.e., any C^1-close flow is topologically equivalent, which is a stronger condition). Note that any volume preserving flow on a 3-manifold that is robustly transitive is Anosov [BR10; AM07], so the ideas in the proof of Theorem 4.2.1 cannot be extended to deal with more general vector fields.*

Remark 4.2.3. *Let R be an Anosov Reeb flow on a closed 3-manifold M. There is a broad range of closed 3-manifolds supporting Anosov Reeb flows, see [FH19]. The proof of Theorem 4.2.1 shows that R is a stationary solution of the Euler equations on (M, g) for any compatible metric g, and it is isolated in C^1, up to a finite dimensional family of steady states, whose dimension is the multiplicity of the eigenvalue 2 of curl_g. The most involved part of the proof of Theorem 4.2.1 consists in proving that the compatible metric g can be chosen so that the multiplicity is 1.*

4.3 ABC flows: proof of Theorem 4.1.4

In [Chi95] it was proven that an ABC flow (4.1.4) with $A = 1$, $B \in (0, 1)$ and $C > 0$ small enough, which we henceforth denote by v_0, exhibits a transverse homoclinic intersection, which implies that the field has positive topological entropy [GH90, Theorem 5.3.5]. We also observe that v_0 does not vanish at any point of $\mathbb{T}^3$. Indeed, when $A = 1$ and $C = 0$, the ABC flow takes the form

$$(\sin x_3,\ B \sin x_1 + \cos x_3,\ B \cos x_1)\,,$$

which is never zero provided that $B \in (0, 1)$. By continuity, this property holds if C is small enough, depending on B.

Let v be an analytic steady state in a C^1-neighbourhood of v_0. By (4.1.3), the Bernoulli function F is the unique solution to the equation

$$\Delta F = \mathrm{div}(v \times \mathrm{curl}\, v)\,,$$

up to an additive constant. Since v is analytic, F is analytic as well by elliptic regularity (see e.g. [Nar85, Theorem 3.8.12]). As the existence of transverse homoclinic intersections is robust under C^1-small perturbations, we infer that v also has positive topological entropy. Then by [GH90, Corollary 4.8.5] the steady state v does not admit a nontrivial analytic first integral, so F is necessarily a constant on $\mathbb{T}^3$.

The previous discussion shows that v is a Beltrami field with factor

$$f := \frac{v \cdot \mathrm{curl}\, v}{|v|^2}\,,$$

which is an analytic function because v is and v does not vanish at any point of $\mathbb{T}^3$ provided that it is close enough to v_0 (which is nonvanishing). As before, f is then a constant on $\mathbb{T}^3$, so necessarily $f = 1$, so v is a curl eigenfield with eigenvalue 1, as we wanted to prove.

Remark 4.3.1. *Reference [EPR23b] introduced the parametric Gaussian ensemble of random Beltrami fields (u_λ) on $\mathbb{T}^3$, where the corresponding curl eigenvalue λ ranges over the set of admissible eigenvalues (i.e., the set of square roots of integers that are congruent with* 1, 2, 3, 5 *or* 6 *modulo* 8*, whose multiplicity N_λ tends to infinity for large λ). It was proved that, with a probability that tends to* 1 *as $\lambda \to \infty$, u_λ has positive topological entropy. The proof of Theorem 4.1.4 then shows that, with a probability that tends to 1 as $\lambda \to \infty$ along the admissible eigenvalues, the random Beltrami field u_λ is isolated in C^1 within the class of analytic steady Euler flows, up to the N_λ-dimensional family of Beltrami fields with eigenvalue λ.*

4.4 Curl generic simplicity in the class of compatible metrics

Let (M, α) be a contact 3-manifold. We denote by $\xi := ker(\alpha)$ the contact structure defined by α. Any Riemannian metric g compatible with α has the orthogonal decomposition

$$g = g_\xi + \alpha \otimes \alpha \tag{4.4.1}$$

where g_ξ is a smooth degenerate quadratic form on TM satisfying $g_\xi(R, \cdot) = 0$ that is positive definite on ξ. We denote the space of smooth metrics which are compatible with α by $\mathcal{G}_\alpha$ and consider it as a topological subspace of the space of smooth metrics $\mathcal{G}$ on M. The following lemma is elementary, see, e.g., [EKM12b, Proposition 2.1]:

Lemma 4.4.1. *A metric g of the form* (4.4.1) *is in $\mathcal{G}_\alpha$ if and only if its associated Riemannian volume is equal to $\frac{1}{2}\alpha \wedge d\alpha$.*

The main result of this section is the following theorem, which shows that for a generic choice of compatible metric, the eigenvalue 2 of $\star_g d$ has multiplicity 1:

Theorem 4.4.2. *Let (M, α) be a contact* 3*-manifold. There exists an open and dense subset V of the space of compatible metrics $\mathcal{G}_\alpha$ such that for all $g \in V$ the eigenvalue* 2 *of $\star_g d$ is simple.*

We denote by E_2 the eigenspace of $\star_g d$ with eigenvalue 2. We observe that 2 is an eigenvalue (with a corresponding eigenform α) for any compatible metric, so we will use the same notation E_2 independently of the particular compatible metric.

Before proceeding we will briefly introduce the functional analytic setup of this problem, that will allow us to employ the perturbation-theoretic tools described in the Appendix. For any Riemannian metric g, the curl operator is an unbounded self-adjoint operator

$$\star_g d : L^2(\Omega^1(M), g) \to L^2(\Omega^1(M), g)$$

with dense domain $H^1(\Omega^1(M))$. Crucially, this domain does not depend on the particular choice of Riemannian metric g. It is well known that the curl operator restricts to a self-adjoint operator with compact inverse on the g-orthogonal complement of the L^2-closed 1-forms on M, i.e., on the space $H^1(\Omega^1(M)) \cap \{u \in L^2(M) : du = 0\}^{\perp_g}$. In particular, this means that all eigenvalues are discrete and have finite multiplicity. However we will not work with the curl operator on these domains since they depend on the metric g. Instead, we consider the curl operator on the metric independent space $H^1(\Omega^1(M))$ which only changes its spectrum by adding an infinite dimensional kernel.

In summary, we have the following situation. The family of curl operators parametrized by the Fréchet manifold $\mathcal{G}$ is C^1 by arguments perfectly analogous to those presented in [GK24, Lemma C.1]. Furthermore, all curl operators are densely defined on the same domain $H^1(\Omega^1(M))$, their spectrum consists of a discrete set of real eigenvalues, all of which have finite multiplicity except the eigenvalue 0, hence Theorem 4.5 may be used in the analysis of the eigenvalue 2.

To prove Theorem 4.4.2 one approach would be to show that $\mathcal{G}_\alpha$ is a Fréchet manifold in its own right and to apply the techniques outlined in the Appendix to $\mathcal{G}_\alpha$. We will sidestep this technicality, viewing $\mathcal{G}_\alpha$ as a subset of the Fréchet manifold $\mathcal{G}$ and argue in the following way: Given a compatible metric g for α, we know that α is contained in the eigenspace of $\star_g d$ associated to the eigenvalue 2. Assume now that this eigenspace has dimension $k > 1$. According to the method described in the Appendix we will then construct a codimension 1 Fréchet-submanifold S of an open set $\mathcal{U}(g) \subset \mathcal{G}$ containing g such that S contains all $g' \in \mathcal{U}(g)$ for which the eigenvalue $\lambda = 2$ of $\star_{g'} d$ has multiplicity k. Moreover we will construct a smooth 1-parameter family of metrics g_t with $g = g_0$ which is transverse to S and stays strictly inside $\mathcal{G}_\alpha$. This proves that following g_t will break up the k-dimensional eigenspace containing α, thus strictly reducing its dimension, while also making sure that α is an eigenform for $\star_{g_t} d$ with eigenvalue 2 for all t. It is clear that this idea can then be used to prove density of compatible metrics for which the eigenvalue 2 is simple. These arguments are similar to the ones used in [Kep24] and [GK24].

4.4.1 Proof of Theorem 4.4.2

Let $V \subset \mathcal{G}_\alpha$ be the set of compatible metrics such that 2 is a simple eigenvalue of $\star_g d$. By the continuous dependence of eigenvalues of $\star_g d$ on the Riemannian metric [Tes14, Theorem 4.10] it is clear that V is open, so it remains to prove that V is dense in $\mathcal{G}_\alpha$. For this we want to find a perturbation within the space of compatible metrics that breaks up the multiplicity of the eigenvalue 2.

Given $g \in \mathcal{G}_\alpha \setminus V$, the eigenspace E_2 has finite dimension $k > 1$. We can thus pick β in E_2 such that:

$$\int_M \alpha \wedge \star_g \beta = 0 \quad \text{and} \quad \int_M \beta \wedge \star_g \beta = 1 \,.$$

Let $\hat{M} := \{p \in M : \alpha(p) \wedge \beta(p) \neq 0\}$ be the subset of M on which α and β are not collinear. The following lemma shows that this set is generic:

Lemma 4.4.3. *$\hat{M}$ is open and dense in M.*

Proof. The proof of this result is part of the proof of [Kep24, Lemma 3.3] (Lemma 2.3.4 in Chapter 2 of this thesis). □

Denote by β_ξ the projection of β to ξ along R, i.e.,

$$\beta_\xi := \beta - \beta(R)\alpha\,.$$

Notice that β_ξ is nowhere vanishing on $\hat{M}$. Let $\hat{\beta}_\xi := \frac{\beta_\xi}{|\beta_\xi|_g}$ and a 1-form w such that

$$(\alpha, \hat{\beta}_\xi, w) \tag{4.4.2}$$

is a positively oriented orthonormal coframe for g on $\hat{M}$. By construction,

$$\alpha \wedge \hat{\beta}_\xi \wedge w = \mathrm{Vol}_g = \frac{1}{2}\alpha \wedge d\alpha\,.$$

Without any loss of generality we shall assume that the total volume is normalized, i.e., $\int_M \mathrm{Vol}_g = 1$.

Consider the following perturbation of the metric g

$$g_\varepsilon := g + \varepsilon\, \beta_\xi \otimes \beta_\xi + \left[\left(1 + \frac{\varepsilon^2}{4}|\beta_\xi|_g^4\right)^{\frac{1}{2}} - \frac{\varepsilon}{2}|\beta_\xi|_g^2 - 1\right] g_\xi\,.$$

Notice that g_ε is C^∞ on the whole M because $|\beta_\xi|_g^2$ is the square norm of the smooth 1-form β_ξ (and this is a C^∞ function on M), $|\beta_\xi|_g^4$ is an even power of it, and the function $1 + \frac{\varepsilon^2}{4}|\beta_\xi|_g^4$ in the square root is non vanishing. It is clear that this is a metric on M of the form (4.4.1). To compute the Riemannian volume associated to g_ε we consider the following coframe,

$$\left(\alpha, \left[\left(1 + \frac{\varepsilon^2}{4}\,|\beta_\xi|_g^4\right)^{\frac{1}{2}} + \frac{\varepsilon}{2}\,|\beta_\xi|_g^2\right]^{1/2} \hat{\beta}_\xi, \left[\left(1 + \frac{\varepsilon^2}{4}\,|\beta_\xi|_g^4\right)^{\frac{1}{2}} - \frac{\varepsilon}{2}\,|\beta_\xi|_g^2\right]^{1/2} w\right).$$

It is easy to check that it is orthonormal with respect to g_ε. As before, it is defined on $\hat{M}$. Clearly

$$\mathrm{Vol}_{g_\varepsilon} = \mathrm{Vol}_g$$

on $\hat{M}$, and then on the whole of M because $\hat{M}$ is open and dense. Therefore g_ε is a metric compatible with α by Lemma 4.4.1.

Let us now consider the linearization of the perturbation g_ε at the metric g, i.e.,

$$h := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} g_\varepsilon = \beta_\xi \otimes \beta_\xi - \frac{1}{2}\,|\beta_\xi|_g^2\, g_\xi\,,$$

which is, of course, a smooth symmetric $(0,2)$ tensor field on M. The variation of the operator $\star_g d$ with respect to h is given by the following lemma [EP12b, Lemma 2.1]:

Lemma 4.4.4. *Let $h \in \mathcal{T}^{0,2}$ be a smooth symmetric $(0,2)$ tensor field and let α_1 be an eigenform of $\star_g d$ corresponding to the eigenvalue λ. The variation $\delta_h(\star_g d)$ of $\star_g d$ in direction h satisfies*

$$g(\alpha_2^\sharp, \left(\delta_h(\star_g d)\alpha_1\right)^\sharp) = \lambda\, h(\alpha_2^\sharp, \alpha_1^\sharp) - \frac{\lambda}{2}(\operatorname{tr}_g h)\, g(\alpha_2^\sharp, \alpha_1^\sharp)$$

for any 1-form α_2. As usual, $\sharp$ denotes the metric dual with respect to g, and $\operatorname{tr}_g h$ is the trace of the tensor h computed with respect to the metric g.

Using the orthonormal coframe (4.4.2), we easily get that

$$\operatorname{tr}_g h = 0$$

on $\hat{M}$, and hence on the whole M. Applying Lemma 4.4.4 to our case, we immediately obtain:

$$g(\alpha^\sharp, (\delta_h(\star_g d)\alpha)^\sharp) = 0\,,$$
$$g(\beta^\sharp, (\delta_h(\star_g d)\beta)^\sharp) = |\beta_\xi|_g^4\,.$$

Let $\mathcal{U}_g$ be a C^∞-neighborhood of the compatible metric g in $\mathcal{G}$. Noticing that the spectrum of $\star_g d$ is discrete and the operator $\star_g d$ defines a C^1 function (in the sense of Definition 4.5.1) on the space of metrics, arguing as in [GK24, Section 2.1.] (Section 3.2.1. in Chapter 3 of this thesis) one can build a C^1 function

$$\pi : \mathcal{U}_g \to \operatorname{Sym}(k, \mathbb{R})\,,$$

where $\operatorname{Sym}(k,\mathbb{R})$ is the space of $k \times k$ symmetric matrices with real entries. The definition of π is rather involved and can be found in the aforementioned reference, but its essential properties are summarized in the Appendix. Crucially, the subset of metrics in $\mathcal{U}_g$ for which the k-dimensional eigenspace E_2 does not split up is contained in

$$\pi^{-1}(\mathbb{R} \cdot \mathrm{Id})\,.$$

Next, we use the following elementary lemma:

Lemma 4.4.5. *If there exists a variation of the metric $h \in T_g\mathcal{G}$ such that the derivative $\pi'(h)$ of π applied to h is not contained in $\mathbb{R} \cdot \mathrm{Id}$, then the set $\pi^{-1}(\mathbb{R} \cdot \mathrm{Id})$ is contained in a codimension 1 Fréchet-submanifold of some neighborhood of g in $\mathcal{U}_g$.*

Proof. Indeed, set $v := \pi'(h)$ and take any linear projection $p_v : \operatorname{Sym}(k,\mathbb{R}) \to \mathbb{R} \cdot v$ to the span of v that contains $\mathbb{R} \cdot \mathrm{Id}$ in its kernel (this obviously exists by the choice of h). The composition

$$p_v \circ \pi : \mathcal{U}_g \to \mathbb{R}$$

is a C^1 map in the sense of Hamilton whose zero set contains $\pi^{-1}(\mathbb{R} \cdot \mathrm{Id})$. Moreover, the derivative of $p_v \circ \pi$ is $p_v \circ \pi'$ so it follows by construction that $(p_v \circ \pi')(h) = v \neq 0$, so the map $p_v \circ \pi$ is a submersion. We can thus apply Proposition 4.5.3 in the Appendix to conclude that, after possibly restricting $\mathcal{U}_g$, all metrics in $\pi^{-1}(\mathbb{R} \cdot \mathrm{Id})$ are contained in a codimension 1 Fréchet submanifold of $\mathcal{U}_g$. □

Now we prove that the variation h we constructed above satisfies these requirements, i.e., $\pi'(h)$ is not a multiple of the identity matrix. Extend the pair α and β to an L^2-orthonormal basis $(u_i)_{i=1}^k$ of E_2 where $u_1 = \alpha$ and $u_2 = \beta$. Theorem 4.5.2 in the Appendix tells us the expression of the map π':

$$\pi' : h \mapsto \left(\langle u_i, \delta_h(\star_g d)u_j\rangle_{L^2}\right)_{i,j=1}^k .$$

If $\pi'(h)$ were contained in the span of the identity matrix then, in particular,

$$-\langle u_1, \delta_h(\star_g d)u_1\rangle_{L^2} + \langle u_2, \delta_h(\star_g d)u_2\rangle_{L^2} = 0 .$$

Taking the variation h we constructed above, we obtain

$$-\langle \alpha, \delta_h(\star_g d)\alpha\rangle_{L^2} + \langle \beta, \delta_h(\star_g d)\beta\rangle_{L^2} = \int_M |\beta_\xi|_g^4 \operatorname{Vol}_g ,$$

which cannot be 0 by Lemma 4.4.3.

The previous arguments show that there exists a neighborhood $\mathcal{U}_g$ of g so that the set of compatible metrics in $\mathcal{U}_g$ for which the eigenspace E_2 has dimension k is contained in a Fréchet submanifold S of codimension 1, and furthermore the 1-parameter family of metrics g_ε is transverse to S, that is g_ε is not contained in S for all $\varepsilon \neq 0$ provided $|\varepsilon|$ is small enough. This transversality is a consequence of the equation

$$\pi(g_\varepsilon) = \pi(g) + \varepsilon\pi'(h) + o(\varepsilon) ,$$

the fact that $\pi'(h)$ is not a multiple of the identity matrix, and that S is precisely the set of metrics $\pi^{-1}(\mathbb{R} \cdot \mathrm{Id})$ in the neighborhood $\mathcal{U}_g$.

Since the dimension of eigenspaces is lower semi-continuous, by perturbing in direction h we therefore get a compatible metric g' for which E_2 has dimension strictly less than k, and after repeating the above argument finitely many times we end up with a 1-dimensional eigenspace. As the perturbations are arbitrarily small, we get density of compatible metrics for which the eigenspace E_2 is simple.

Remark 4.4.6. *It was proven in [Kep24] that the curl operator satisfies spectral simplicity along generic* 1*-parameter families of Riemannian metrics, and it is natural to ask whether the same statement holds in the space $\mathcal{G}_\alpha$. Using the example of Berger spheres, however, it is easy to see that generic simplicity is the best one can hope for in this setting.*

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4.5 Appendix: Splitting of eigenvalues

For families of self-adjoint operators whose spectrum consists of discrete eigenvalues of finite multiplicity there is a powerful tool, first introduced by Colin de Verdière [Col88] and developed further in [Tey99], [Kep24], as well as in [GK24] (Chapters 2 and 3 of this thesis, respectively), to analyze the splitting behavior of multiple eigenvalues. We will outline the main idea of this technique after fixing some notation and defining the following notion of differentiability introduced by Hamilton [Ham82].

Definition 4.5.1. *A map $F : \mathcal{F} \to \mathcal{F}'$ between Fréchet spaces $\mathcal{F}$ and $\mathcal{F}'$ is called C^1 if the directional derivative*

$$DF(x)u = \lim_{t\to 0} \frac{F(x+tu) - F(x)}{t}$$

exists for all $x \in \mathcal{F}$, $u \in T_x\mathcal{F}$ and, furthermore, $DF(x)u$ is linear in u and jointly continuous in x and u.

Let $\mathcal{X}$ be a Fréchet manifold, H a Banach space, $\{\langle \cdot\,,\,\cdot \rangle_q\}_{q\in\mathcal{X}}$ a C^1 family of inner products on H, and $\{A_q\}_{q\in\mathcal{X}}$ a C^1 family of unbounded operators from some fixed dense domain $D \subset H$ to H, each self-adjoint with respect to $\langle \cdot\,,\,\cdot \rangle_q$ and with compact resolvent. Suppose that λ is an eigenvalue with finite multiplicity k of A_{q_0} for some $q_0 \in \mathcal{X}$.

The compactness of the resolvent of A_{q_0} implies that λ is isolated and we can encircle it by some simple closed loop γ in $\mathbb{C}$ which does not enclose any other eigenvalues of A_{q_0}. By the variational characterization of eigenvalues of self-adjoint operators [Tes14], the eigenvalues of A_q move "continuously" with respect to the parameter q in the sense that there exists a neighborhood $\mathcal{U}(q_0)$ of q_0 such that the number of eigenvalues of A_q, when counted with multiplicity, encircled by γ is exactly k for all $q \in \mathcal{U}(q_0)$. One can associate to γ a spectral projection $P_\gamma(q)$

$$P_\gamma(q) := \frac{1}{2\pi i} \int_\gamma (z - A_q)^{-1} dz\,,$$

which maps the Banach space H to the sum of eigenspaces associated to the eigenvalues of A_q encircled by γ. We will denote the image set of $P_\gamma(q)$ by $E(q)$. The fact that the family of operators is C^1 implies that P_γ is C^1 as well, see the proof of [GK24, Lemma C.1]. One can restrict $P_\gamma(q)$ to a linear bijection

$$P_\gamma(q)|_{E(q_0)} : E(q_0) \to E(q)\,,$$

which is invertible, and this family of inverses $(q \mapsto P_\gamma(q)|_{E(q_0)}^{-1})$ is C^1 as well [Ham82, Theorem II.3.1.]. Using this we can build the C^1 function

$$\tilde{\pi} = (P_\gamma|_{E(q_0)})^{-1} \circ A_{(\,\cdot\,)} \circ P_\gamma|_{E(q_0)} : \mathcal{U}(q_0) \to \mathrm{End}(E(q_0))\,.$$

With some more work $\tilde{\pi}$ can be modified to a different map π such that $\pi(q)$ is a symmetric endomorphism on $E(q_0)$ with respect to $\langle\,\cdot\,,\,\cdot\,\rangle_{q_0}$ for all $q \in \mathcal{U}(q_0)$. This involves a parametric version of the Gram-Schmidt orthonormalization procedure and is done in detail in [GK24, Section 2.1]. The final result is a map

$$\pi : \mathcal{U}(q_0) \to \mathrm{Sym}(E(q_0)) \cong \mathrm{Sym}(k, \mathbb{R})\,,$$

where we denote the symmetric endomorphisms of $E(q_0)$ with respect to $\langle\,\cdot\,,\,\cdot\,\rangle_{q_0}$ by $\mathrm{Sym}(E(q_0))$ (i.e., the space of $k \times k$ symmetric matrices with real entries). The essential properties of π are summarized in the following Theorem.

Theorem 4.5.2 ([GK24, Theorem 2.1])**.** *Let $\mathcal{X}$ be a Fréchet manifold, H a Banach space, $\{\langle\,\cdot\,,\,\cdot\,\rangle_q\}_{q\in\mathcal{X}}$ a C^1 family of inner products on H, and $\{A_q\}_{q\in\mathcal{X}}$ a C^1 family of unbounded operators from some fixed dense domain $D \subset H$ to H, each self-adjoint with respect to $\langle\,\cdot\,,\,\cdot\,\rangle_q$ and with compact resolvent. Suppose λ is an eigenvalue with multiplicity k of A_{q_0} for some $q_0 \in \mathcal{X}$.*

Then, for $\varepsilon > 0$ small enough, there exist an open neighborhood $\mathcal{U}(q_0)$ of q_0 and a C^1 function $\pi : \mathcal{U}(q_0) \to \mathrm{Sym}(k, \mathbb{R})$ satisfying

$$\sigma(A_q) \cap (\lambda - \varepsilon, \lambda + \varepsilon) = \sigma(\pi(q))$$

for all $q \in \mathcal{U}(q_0)$, where σ denotes the spectrum of the corresponding operator. In particular, $\pi^{-1}(\mathbb{R} \cdot \mathrm{Id})$ is the set of parameters $q \in \mathcal{U}(q_0)$ such that $\lambda(q)$ has multiplicity k. Furthermore the derivative of π at q_0 is given by

$$\pi' : T_{q_0}\mathcal{U}(q_0) \to T_{\mathrm{Id}}\,\mathrm{Sym}(k, \mathbb{R}) \cong \mathrm{Sym}(k, \mathbb{R})\,,$$
$$h \mapsto \big(\langle DA_{q_0}[h]u_m, u_\ell\rangle\big)_{m,\ell=1}^{k}\,,$$

where $\{u_m\}_{m=1}^k$ is an orthonormal basis of the λ-eigenspace of A_{q_0}.

This result means that the map π measures how the multiple eigenvalue λ breaks up as A_{q_0} is perturbed. Remarkably, a full understanding of π gives a precise picture of this behaviour, not just one up to first order. Nevertheless it is of great interest to understand the derivative of π at q_0. In Section 4.4.1 we have used the derivative of π together with the following proposition to construct a submanifold containing all those parameters q in some open neighborhood of q_0 for which the k-dimensional eigenspace corresponding to λ is not broken up.

Proposition 4.5.3 ([GK24, Proposition A.2])**.** *Let $\mathcal{U} \subseteq \mathcal{F}$ be an open subset of a Fréchet space $\mathcal{F}$. Suppose a C^1 map $F : \mathcal{U} \to \mathbb{R}^m$ is such that $DF(p) : \mathcal{F} \to \mathbb{R}^m$ is surjective at a point $p \in \mathcal{U} \cap F^{-1}(0)$. Then there exists a neighborhood $\mathcal{U}' \subseteq \mathcal{U}$ of p, an open neighborhood $\mathcal{V} \subseteq \mathcal{F}'$ of the origin in some Fréchet space $\mathcal{F}'$, and a C^1 map $G : \mathcal{V} \to \mathcal{U}'$ such that $F^{-1}(0) \cap \mathcal{U}' = G(\mathcal{V})$.*

Proposition 4.5.3 implies that preimages of regular values of C^1 maps from Fréchet manifolds to $\mathbb{R}^m$ are submanifolds of codimension m. It is noteworthy that in this finite codimension case one can prove Proposition 4.5.3 and the associated implicit function theorem [GK24, Lemma A.1] without using Nash–Moser iteration.

5 Spectral geometry of the curl operator on smoothly bounded domains

(Based almost verbatim on [GK25], joint work with Josef Greilhuber)

Abstract

We show that the spectrum of the curl operator on a generic smoothly bounded domain in three-dimensional Euclidean space consists of simple eigenvalues. The main new ingredient in our proof is a formula for the variation of curl eigenvalues under a perturbation of the domain, reminiscent of Hadamard's formula for the variation of Laplace eigenvalues under Dirichlet boundary conditions. As another application of this variational formula, we simplify the derivation of a well-known necessary condition for a domain to minimize the first curl eigenvalue functional among domains of a given volume and derive similar necessary conditions for a domain extremizing higher eigenvalue functionals.

5.1 Introduction

This paper is concerned with the spectrum of the operator

$$\operatorname{curl} u = \nabla \times u,$$

which acts on vector fields u defined on smoothly bounded domains in $\mathbb{R}^3$, i.e., bounded domains $D \subseteq \mathbb{R}^3$ with the property that for each point $p \in \partial D$ there exists an open neighborhood $U \subseteq \mathbb{R}^3$ of p and a function $\rho \in C^\infty(U)$ with $U \cap D = \rho^{-1}((-\infty, 0))$ and $d\rho(p) \neq 0$. Under appropriately chosen boundary conditions, the curl operator is self-adjoint with compact resolvent, so its spectrum consists of a discrete set of real eigenvalues of finite multiplicity.

The corresponding eigenfields describe magnetic fields in plasmas in which no Lorentz force acts on the charged particles constituting the plasma and are thus often referred to as "force-free magnetic fields" in the plasma physics literature. They can be shown to be energy-minimal magnetic fields (within a fixed helicity class of magnetic fields) and have been proposed to be the natural end configurations of dissipative magnetic fluids in which the magnetic forces dominate ([Wol58],[LA91]).

More generally, curl eigenfields lie at the intersection of a number of mathematical subdisciplines. On the one hand, they are a particularly rich example class of solutions to the Euler equations, and, on the other hand, their orthogonal complements define contact structures [EG00b, Theorem 2.1]. This confluence is interesting from both an analytic as well as a contact geometric and topological point of view and has led to some remarkable flexibility results for the study of the Euler equations (e.g. [EG00b],[EG00a],[Car+21a]) and rigidity results in contact topology (e.g. [EKM12a]).

While there do exist studies of the spectral theory of the curl operator on domains in specific settings (e.g. [Can+00b],[EP23]), there have been few systematic treatments of the curl operator as a self-adjoint operator on domains.

5.1.1 The setup

By imposing boundary conditions on the curl operator, we may obtain a Fredholm operator or an unbounded self-adjoint operator. One especially natural choice, introduced by Giga and Yoshida in [YG90], can be described as follows: Let $D \subset \mathbb{R}^3$ be a smoothly bounded domain and $\vec{\nu}$ its outward pointing unit normal. Define

$$L^2_\partial(D) = \left\{\vec{u} \in L^2(D, \mathbb{C}^3) : \nabla \cdot \vec{u} = 0, \vec{u} \cdot \vec{\nu} = 0 \text{ on } \partial D\right\},$$

the space of boundary-parallel, divergence free, and square integrable vector fields, where the divergence is defined in the weak sense. (It adds no difficulty and is helpful in the spectral theory to work over $\mathbb{C}$, which will be the convention for the rest of this paper.) Let

$$H^1_{\partial\partial}(D) = \left\{\vec{u} \in L^2_\partial(D) : \operatorname{curl} \vec{u} \in L^2_\partial(D)\right\},$$

where curl is understood in the sense of distributions. It is shown in [YG90] that curl is a Fredholm operator from $H^1_{\partial\partial}(D)$ to $L^2_\partial(D)$, whose spectrum is however all of $\mathbb{C}$ unless ∂D has trivial first homology, i.e., ∂D consists of a union of spheres. A more meaningful spectrum is obtained by introducing the spaces

$$\begin{aligned} L^2_\Sigma(D) &= \left\{\vec{u} \in L^2_\partial(D) : \textstyle\int_S \vec{u} \cdot \vec{\nu}_S = 0\right\}, \\ H^1_{\Sigma\Sigma}(D) &= \left\{\vec{u} \in L^2_\Sigma : \operatorname{curl} \vec{u} \in L^2_\Sigma(D)\right\}. \end{aligned}$$

where S ranges over all compact embedded orientable surfaces in D with $\partial S \subset \partial D$, and, given S, $\vec{\nu}_S$ is a unit normal vector field of S. The main result of [YG90] is that curl is an unbounded self-adjoint operator with compact resolvent on $L^2_\Sigma(D)$ with domain $H^1_{\Sigma\Sigma}(D)$.

There are other ways to restrict the space $H^1_{\partial\partial}(D)$ in order to obtain self-adjoint operators with compact resolvent. It follows from the work of Hiptmair, Kotiuga, and Tordeux [HKT12] that these are indexed by Lagrangian subspaces of the first cohomology of the domain's boundary, denoted by $H^1_{dR}(\partial D, \mathbb{C})$. The symplectic structure on $H^1_{dR}(\partial D, \mathbb{C})$ is induced by the wedge product on 1-forms, extended as a sesquilinear, antisymmetric form from $H^1_{dR}(\partial D, \mathbb{R})$. We will discuss these operators, from which Giga–Yoshida's curl operator arises as a special case, in detail in Section 5.2.2. Given a Lagrangian subspace $L \subset H^1_{dR}(\partial D, \mathbb{C})$ we will denote the associated curl operator by curl_L. We wish to point out that Lagrangian boundary conditions encompass not just Giga–Yoshida's zero flux boundary conditions but also the "Amperian" boundary conditions proposed by Cantarella [Can99].

5.1.2 Statement of results

In this paper, we answer the question whether the spectrum of the curl operator with Lagrangian boundary conditions is generically simple. To state our result precisely, we formalize the "space of domains" as follows.

Let $D_0 \subseteq \mathbb{R}^3$ be a smoothly bounded domain. We consider the space $\mathcal{X}(D_0)$ of smooth embeddings of $\bar{D}_0$ into $\mathbb{R}^3$. As an open subset of the space of smooth maps from $\bar{D}_0$ to $\mathbb{R}^3$ this space naturally inherits the structure of a smooth Frechét manifold. Given a diffeomorphism $\Phi \in \mathcal{X}(D_0)$, we write D_Φ for $\Phi(D_0)$. Given a Lagrangian $L \subseteq H^1_{dR}(\partial D_0, \mathbb{C})$, we write L_Φ for the image of L under $(\Phi^{-1})^* : H^1_{dR}(\partial D_0, \mathbb{C}) \to H^1_{dR}(\partial D_\Phi, \mathbb{C})$. In the following theorem, we additionally require the Lagrangian $L \subseteq H^1_{dR}(\partial D_0, \mathbb{C})$ to be *real*, i.e. invariant under complex conjugation, which just means that L arises from a Lagrangian in $H^1_{dR}(\partial D, \mathbb{R})$ by complexification. This ensures that there exists a basis of real-valued eigenforms for curl_L, an important element in our proof.

Theorem 5.1.1. *Let $D_0 \subseteq \mathbb{R}^3$ be a smoothly bounded domain and let $L \subset H^1_{dR}(\partial D_0, \mathbb{R})$ be a Lagrangian subspace. For a comeagre subset of diffeomorphisms $\Phi \in \mathcal{X}(D_0)$ the spectrum of the operator curl_{L_Φ} on D_Φ consists of simple eigenvalues.*

Note that L_Φ is locally constant on the set of diffeomorphisms $\Phi \in \mathcal{X}(D_0)$ with a given image $\bar{D} \subseteq \mathbb{R}^3$, since any two isotopic diffeomorphisms $\Phi_1 : \bar{D}_0 \to \bar{D}$ and $\Phi_2 : \bar{D}_0 \to \bar{D}$ induce the same map on cohomology. Thus, the operator curl_{L_Φ} and hence also its spectrum locally depend on D, but not on the choice of embedding $\Phi : \bar{D}_0 \to \bar{D}$. Therefore, Theorem 5.1.1 really is a statement on generic *domains* as opposed to generic embeddings $\Phi : \bar{D}_0 \to \mathbb{R}^3$.

We would like to point out that there have been no partial results in this direction, that is, this result has not been known for any boundary conditions or any domains.

The main ingredient in Theorem 5.1.1 is a formula reminiscent of Hadamard's famous rule for the variation of eigenvalues of the Laplace operator with Dirichlet boundary conditions. If we deform the domain D along the flow of a vector field $\vec{X}$, the first variation of an eigenvalue λ of the curl_L operator is given by

$$\dot{\lambda} = -\lambda \int_{\partial D} (\vec{X} \cdot \vec{\nu})\, |u|^2. \tag{5.1.1}$$

where u is an eigenfield of λ. See Theorem 5.4.1 for a precise statement.

The proof of Theorem 5.1.1 is informed by methods the authors have previously used in [GK23] for spectral genericity results. It rests on the idea that sufficiently large families of operators with compact resolvent which do not satisfy generic spectral simplicity have eigenfunctions that satisfy strong pointwise constraints derived with the help of the Hadamard rule (5.1.1). These constraints are then shown to lead to a contradiction, thus establishing Theorem 5.1.1.

To the best of our knowledge, formula (5.1.1) is new even for the curl operator on domains diffeomorphic to the solid torus, where shape optimization problems for certain choices of Lagrangian boundary conditions have been considered previously in [Can+00a] and [EP23], using variational formulas for the Rayleigh quotient outwardly similar to

(5.1.1). Remarkably, the Hadamard rule for the curl operator bears a strong similarity to the corresponding formula, recently derived by Lamberti and Zaccharon [LZ21], for the Maxwell system with various boundary conditions (see also Lamberti–Pauli–Zaccharon [LPZ25]). (Note that the boundary conditions of the Maxwell system, which is based on the differential operator curl^2, preclude it from being the square of any self-adjoint curl operator [HKT12, Section 8].)

A second application of the Hadamard rule is to significantly simplify certain calculations appearing in the treatment of optimal domains for the first curl eigenvalue, a topic which has recently attracted substantial interest ([EP23], [Ger23], [EGP24]), as well as to generalize those results to arbitrary Lagrangian boundary conditions. We also deduce necessary conditions for a domain to be a local extremum of the k^{th} eigenvalue functional.

As is customary, we fix the volume of D to obtain a meaningful variational problem: Write $\mathcal{X}_1(D_0)$ for the subset of diffeomorphisms in $\mathcal{X}(D_0)$ with $|D_\Phi| = 1$. Let $L \in H^1_{dR}(\partial D_0; \mathbb{C})$ be a Lagrangian. For $\Phi \in \mathcal{X}_1(D_0)$ and $k \in \mathbb{N}$, let $\lambda^L_k(\Phi)$ and $\lambda^L_{-k}(\Phi)$ denote the k^{th} positive and negative eigenvalue of curl_{L_Φ}, respectively.

Theorem 5.1.2. *Let $D_0 \subseteq \mathbb{R}^3$ and $L \subset H^1_{dR}(\partial D_0, \mathbb{C})$. If $\Phi \in \mathcal{X}_1(D_0)$ is a local extremum of the k^{th} eigenvalue functional λ^L_k, then there exists a family of pairwise orthogonal curl_{L_Φ}-eigenfields $(u_j)_{j=1}^{\ell}$ corresponding to the eigenvalue $\lambda^L_k(\Phi)$ which satisfy*

$$|u_1|^2 + \ldots + |u_\ell|^2 = 1$$

on the boundary of the domain D_Φ. At local minima of λ^L_1 and local maxima of λ^L_{-1}, respectively, that eigenvalue is simple and the corresponding (suitably rescaled) eigenform u satisfies $|u|^2 = 1$ on the boundary.

The last claim of Theorem 5.1.2 exists in the literature in certain special cases regarding the domain and boundary conditions ([Can+00a], [EP23], [Ger23]). We provide it here again to show that it holds for any domain and Lagrangian boundary conditions. It should be noted that no domains which minimize λ^L_1 or maximize λ^L_{-1} are known. In fact, it has been proven that there are no smooth axisymmetric domains (satisfying a certain additional geometric assumption on their boundary) which minimize λ_1 or maximize λ_{-1} for Giga–Yoshida boundary conditions, see [EP23]. However, the existence of optimal domains within the class of convex domains was established in [EGP24]. (In this case, all Lagrangian boundary conditions coincide as the boundary topology is that of a sphere.) As far as the authors are aware, nothing is known about the existence of domains which extremize higher eigenvalues of the curl operator.

5.1.3 Organization of this paper.

In Section 5.2 we review the self-adjoint realizations of the curl operator on smoothly bounded domains in the language of differential 2-forms, which turns out to be advantageous for the perturbation theory employed later on. In particular, we relate the setting for the analysis of the curl operator of Giga and Yoshida [YG90] to that of Hiptmair, Kotiuga, and Tordeux [HKT12] in Subsection 5.2.2. Section 5.3 is devoted to proving a multi-parameter analytic dependence lemma (Lemma 5.3.1) for curl_L. This will be used both in the proof of the Hadamard rule for curl_L (Theorem 5.4.1) in Section 5.4 as well as for that of Theorem 5.1.1 in Section 5.5. Finally, Theorem 5.1.2 is proven in Section 5.6.

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5.2 Preliminary observations

Although we are interested in the curl operator on domains in $\mathbb{R}^3$, much of the analysis of this situation is best expressed through the curl operator with respect to general Riemannian metrics on a fixed smoothly bounded domain. The curl operator arises from the exterior derivative on 1-forms on three-dimensional smooth manifolds. With the Hodge-star operator induced by a Riemannian metric g, one obtains a degree-preserving operator, e.g. the operator $\star \circ d$ acting on 1-forms as in [HKT12]. For variational arguments, we require a function space which is independent of the metric. This leads us instead to consider the operator $d \circ \star$ on 2-forms: After identifying vector fields with 2-forms via $u(\vec{v}, \vec{w}) := \det_g(\vec{u}, \vec{v}, \vec{w})$, $d \circ \star$ corresponds to the curl operator on vector fields, and the conditions $\nabla \cdot u = 0$ and $\vec{u} \cdot \vec{\nu} = 0$ take the metric-independent and diffeomorphism-invariant shape $du = 0$ and $\iota_{\partial D}^* u = 0$. (Here and in the rest of this paper, $\iota_{\partial D}$ denotes the embedding $\partial D \hookrightarrow D$.)

5.2.1 Various domains for the curl operator

Giga and Yoshida's curl operator, when expressed in the language of 2-forms on a compact Riemannian three-manifold (D, g) with boundary, is an unbounded self-adjoint operator on the space

$$L^2_\Sigma(D) = \left\{ u \in L^2(D, \Lambda^2 T^* D) : du = 0, \iota_{\partial D}^* u = 0, \textstyle\int_S u = 0 \right\}.$$

Here, S ranges over all compact orientable surfaces in D with boundary in ∂D. The exterior derivative in this definition must be understood in the weak sense. The functionals $u \mapsto \iota_{\partial D}^* u$ and $u \mapsto \int_S u$ are bounded on the space of weakly closed L^2-forms, as follows from the Weyl decomposition [Sch95, Chapter 2]. Hence, $L^2_\Sigma(D)$ is indeed a Banach space. Note that $du = 0$ and $\iota_{\partial D}^* u = 0$ imply that $\int_S u$ depends only on the relative homology class of S in $H_2(\bar{D}, \partial D; \mathbb{C})$. Thus, the condition $\int_S u = 0$ adds only a finite number of linear constraints.

The domain of Giga–Yoshida's curl operator is the space

$$H^1_{\Sigma\Sigma}(D, g) = \left\{ u \in L^2_\Sigma(D) : d\star u \in L^2_\Sigma(D) \right\}.$$

This space, unlike $L^2_\Sigma(D)$ itself, depends *as a set* on the metric g, since the operator $\star$ occurs in its definition and involves the metric. We may write $H^1_{\Sigma\Sigma}(D, g)$ and $L^2_\Sigma(D)$ to emphasize this distinction and will keep to this convention in Section 5.2 and Section 5.3. One should think of $L^2_\Sigma(D)$ as a Banach space equipped with a fixed reference norm equivalent to any norm arising from an inner product corresponding to a choice of smooth metric on D.

Dropping the *zero-flux* condition $\int_S u = 0$ in the definition of $L^2_\Sigma(D)$ results in the space

$$L^2_\partial(D) = \left\{ u \in L^2(D, \Lambda^2 T^* D) : du = 0, \iota_{\partial D}^* u = 0 \right\},$$

which is the direct sum of $L^2_\Sigma(D)$ and the finite-dimensional space of harmonic 2-forms with vanishing tangential component, see [Sch95, Corollary 2.6.2]. The space of harmonic 2-forms will be denoted by $\mathcal{H}_\partial(D, g)$. Extending the domain of curl to

$$H^1_{\Sigma\partial}(D, g) = \left\{ u \in L^2_\Sigma(D) : d\star u \in L^2_\partial(D) \right\},$$

we obtain a Fredholm operator with the same spectrum as before, which is, however, no longer self-adjoint [YG90, Remark 2]. Extending the domain further to

$$H^1_{\partial\partial}(D, g) = \left\{ u \in L^2_\partial(D) : d\star u \in L^2_\partial(D) \right\} = H^1_{\Sigma\partial}(D, g) \oplus \mathcal{H}_\partial(D, g),$$

we obtain yet another Fredholm operator whose (point) spectrum, however, is all of $\mathbb{C}$, unless the space of harmonic forms $\mathcal{H}_\partial(D, g)$ is trivial [YG90, Theorem 2] which is true if and only if ∂D is a union of spheres. (Our nomenclature for these spaces is essentially that of [YG90], except that we use the symbol ∂ instead of σ to indicate tangential boundary conditions.)

To deal with variations in the metric, it is convenient to analyze the curl operator on a domain that does not depend on the metric g. This is achieved by considering the spaces

$H^1_\partial(D) = \{u \in H^1(D, \Omega^2(D)) : du = 0, \iota_{\partial\Omega}^* u = 0\}$,
$H^1_\Sigma(D) = \{u \in H^1_\partial(D) : \int_S u = 0\}$ and
$L^2_{cl}(D) = \{u \in L^2(D, \Omega^2(D)) : du = 0, \int_C u = 0$ for all connected components $C \subseteq \partial D\}$.

Note that the functional $u \mapsto \int_C u$ is bounded on the space of *closed* L^2 forms (which is itself a closed subspace of $L^2(D, \Omega^2(D))$). Hence $L^2_{cl}(D)$ as defined above is a closed subspace of $L^2(D, \Omega^2(D))$. It follows e.g. from [EGP18, Theorem 1.1] that curl is a Fredholm operator from $H^1_\partial(D)$ to $L^2_{cl}(D)$, that its kernel is $\mathcal{H}\partial(D,g)$, and that it is invertible from $H^1_\Sigma(D)$ to $L^2_{cl}(D)$.

For the convenience of the reader, let us collect the function spaces mentioned above with their definitions here once more.

$$\begin{aligned}
L^2_{cl}(D) &= \{u \in L^2(D, \Omega^2(D)) : du = 0, \textstyle\int_C u = 0 \text{ for all connected components } C \text{ of } \partial D\} \\
L^2_\partial(D) &= \{u \in L^2_{cl}(D) : \iota^*_{\partial D} u = 0\} \\
L^2_\Sigma(D) &= \{u \in L^2_\partial(D) : \textstyle\int_S u = 0 \text{ for all } S \subset D \text{ with } \partial S \subset \partial D\} \\
H^1_\partial(D) &= \{u \in H^1(D, \Omega^2(D)) : du = 0, \iota^*_{\partial\Omega} u = 0\} \\
H^1_\Sigma(D) &= \{u \in H^1_\partial(D) : \textstyle\int_S u = 0 \text{ for all } S \subset D \text{ with } \partial S \subset \partial D\} \\
H^1_{\Sigma\partial}(D,g) &= \{u \in H^1_\Sigma(D) : d\star u \in L^2_\partial(D)\} \\
H^1_{\Sigma\Sigma}(D,g) &= \{u \in H^1_\Sigma(D) : d\star u \in L^2_\Sigma(D)\} \\
\mathcal{H}\partial(D,g) &= \{u \in L^2_\partial(D) : d\star u = 0\} \\
H^1_{\partial\partial}(D,g) &= H^1_{\Sigma\partial}(D,g) \oplus \mathcal{H}\partial(D,g)
\end{aligned}$$

5.2.2 Self-adjoint realizations of the curl operator

By restricting the domain of the operator $\operatorname{curl} : H^1_{\partial\partial}(D,g) \to L^2_\partial(D)$ to carefully chosen subspaces of $H^1_{\partial\partial}(D,g)$, a wealth of self-adjoint curl operators arises. These operators correspond to the self-adjoint extensions of curl based on closed traces, as presented in [HKT12], but restricted to the space of boundary-parallel, closed 2-forms. Let us briefly recall the discussion in [HKT12] as it applies in our setting.

We begin by considering the following identity, valid for all $u, v \in H^1_{\partial\partial}(D,g)$:

$$(\operatorname{curl} u, v)_{L^2_\partial(D,g)} - (u, \operatorname{curl} v)_{L^2_\partial(D,g)} = \int_{\partial D} \star u \wedge \star \bar{v} \tag{5.2.1}$$

Since $d\star u \in L^2_\partial(D)$, $\iota^*_{\partial D}(\star u)$ is a closed 1-form on ∂D. Let ℓ denote the genus of ∂D. Then the de Rham cohomology group $H^1_{dR}(D, \mathbb{C})$ is of rank 2ℓ, and the right hand side of (5.2.1) descends to a sesquilinear symplectic form ω_D on $H^1_{dR}(D, \mathbb{C})$.

Consider a Lagrangian subspace $L \subseteq H^1_{dR}(\partial D, \mathbb{C})$, i.e., an ℓ-dimensional complex subspace such that $\omega_D(X,Y) = 0$ for all $X, Y \in L$. By Hiptmair–Kotiuga–Tordeux' work, restricting curl to the closed subspace

$$H^1_L(D,g) := \{u \in H^1_{\partial\partial}(D,g) : [\star u] \in L\},$$

yields an unbounded self-adjoint operator on $L^2_\partial(D,g)$. In fact, their work shows all subspaces of $H^1_{\partial\partial}(D,g)$ to which curl restricts to an unbounded self-adjoint operator on $L^2_\partial(D)$ are of this form for some Lagrangian $L \subseteq H^1_{dR}(D, \mathbb{C})$ (See [HKT12, Section 6], especially [HKT12, Theorem 6.4]).

Reinterpreting the zero-flux boundary condition

The zero-flux boundary conditions can be seen as a special case of this setup as follows: Any surface $\Sigma \subset D$ with boundary on ∂D gives rise to a cycle $\partial\Sigma$ in ∂D. The space $L_\Sigma \subseteq H^1_{dR}(\partial D;\mathbb{C})$ cut out by all linear equations of the form $\int_{\partial\Sigma} u = 0$ arising from surfaces $\Sigma \subset D$ with boundary in ∂D is Lagrangian. Indeed, using standard methods (see, for example, [Kot87]) one can show that there exists a basis $[\Sigma_1], \dots, [\Sigma_{b_2(D,\partial D)}]$ of the relative homology group $H_2(\bar{D}, \partial D; \mathbb{C})$ such that the homology classes $[\Sigma_i]$ are represented by compact embedded orientable surfaces Σ_i in D with boundary in ∂D. The image of the boundary map $\partial : H_2(\bar{D}, \partial D; \mathbb{C}) \to H_1(\partial D; \mathbb{C})$ is a Lagrangian subspace ([Mar16, Proposition 9.1.4.]) with respect to the homological intersection number and so we may conclude that the collection of homology classes represented by $\partial\Sigma_1, \dots, \partial\Sigma_{b_2(D,\partial D)} \subset \partial D$ generate a Lagrangian subspace $\tilde{L}_\Sigma$ of $H_1(\partial D, \mathbb{C})$. As both the Poincaré duality isomorphism and the de Rham isomorphism preserve the intersection form [War83, Theorem 5.45.], the annihilator $L_\Sigma = \{[u] \in H^1_{dR}(\partial D, \mathbb{C}) : \int_{\partial\Sigma} u = 0\}$ of $\tilde{L}_\Sigma$ is then a Lagrangian subspace in $H^1_{dR}(\partial D, \mathbb{C})$. It gives rise to the curl operator domain

$$H^1_{L_\Sigma}(D, g) = \left\{u \in H^1_{\partial\partial}(D, g) : \textstyle\int_S \operatorname{curl} u = 0\right\} = H^1_{\Sigma\Sigma}(D, g) \oplus \mathcal{H}_\partial(D, g).$$

We conclude that the curl operator obtained from L_Σ differs from Giga–Yoshida's curl operator only by the presence of its kernel, $\mathcal{H}_\partial(D, g)$, which is excised by the full zero-flux boundary conditions (which additionally require $\int_\Sigma u = 0$). While Giga–Yoshida's curl operator is an unbounded self-adjoint operator on $L^2_\Sigma(D, g)$, the curl operator obtained from L_Σ is an unbounded self-adjoint operator on the larger space $L^2_\partial(D, g)$.

5.3 Analytic dependence of the resolvent

In this section, we show that resolvent of curl_L varies (real)-analytically along analytic perturbations of the domain and also depends analytically on the Lagrangian L. It is convenient to prove these statements in the more general setting of a smoothly bounded domain D equipped with a metric g_t which depends analytically on a parameter $t \in (-\varepsilon, \varepsilon)^k$. The case of a k-parameter variation of the domain boundary is encompassed by this: If Φ_t is an analytic k-parameter family of diffeomorphisms, then $g_t = \Phi_t^* g$ is an analytic k-parameter family of smooth metrics on D and curl on $D_t = \Phi(D)$ is conjugate to $\operatorname{curl}_{g_t}$ on D via the diffeomorphism Φ_t. Furthermore, we also prove Lipschitz-continuous dependence of the resolvent with respect to C^2-smooth perturbations of the domain.

Let us stress once more that the domain $H^1_L(D, g_t)$ of $\operatorname{curl}_{t,L}$ depends (as a set) on the metric g_t and on the Lagrangian L, but the codomain, $L^2_\partial(D)$, is independent of both. Let $\iota : H^1_L(D, g_t) \to L^2_\partial(D)$ denote the canonical embedding, let $\sigma(\operatorname{curl}_{t,L})$ denote the spectrum of $\operatorname{curl}_{t,L}$ and let $\lambda \in \mathbb{C}$. The inverse of $(\lambda\iota - \operatorname{curl}_{t,L}) : H^1_L(D, g_t) \to L^2_\partial(D)$ exists whenever $\lambda \notin \sigma(\operatorname{curl}_{t,L})$. The operators $R_{t,L}(\lambda) = \iota \circ (\lambda\iota - \operatorname{curl}_{t,L})^{-1}$ then form a family of bounded operators on a *fixed* space. This family is analytic in t and L (interpreting L as a point in the Grassmanian $\operatorname{LGr}_\ell(H^1_{dR}(\partial D, \mathbb{C}))$ of complex Lagrangians in $H^1_{dR}(\partial D, \mathbb{C})$, which is a real-analytic submanifold of the complex Grassmannian $\operatorname{Gr}_\ell(H^1_{dR}(\partial D, \mathbb{C}))$, as follows from [EM99, Theorem 3]).

Lemma 5.3.1. *Let $D \subseteq \mathbb{R}^3$ be a smoothly bounded domain. Let g_t be a family of smooth metrics on $\overline{D}$ depending real-analytically on $t \in (-\varepsilon, \varepsilon)^k$. To each $t \in (-\varepsilon, \varepsilon)^k$ and Lagrangian $L \in \mathrm{LGr}(H^1_{dR}(\partial D, \mathbb{C}))$, we associate the operator* $\mathrm{curl}_{t,L} : H^1_L(D, g_t) \to L^2_\partial(D)$.

Then the resolvents $R_{t,L}(\lambda) : L^2_\partial(D) \to L^2_\partial(D)$ of $\mathrm{curl}_{t,L} : H^1_L(D, g_t) \to L^2_\partial(D)$ form a family of compact operators which is meromorphic in λ for fixed t and L and real-analytic in t and L near any $(t, L, \lambda) \in (-\varepsilon, \varepsilon)^k \times \mathrm{LGr}(H^1_{dR}(\partial D, \mathbb{C})) \times \mathbb{C}$ where $R_{t,L}(\lambda)$ is defined.

Proof. The Hodge star operators $\star_t : H^1(\Omega^2(D)) \to H^1(\Omega^1(D))$ form a real-analytic family of bounded operators. It extends to a holomorphic family $\star_z$ of bounded operators on some neighborhood U of $(-\varepsilon, \varepsilon)^k$ in $\mathbb{C}^k$. The operators $\mathrm{curl}_z := d \circ \star_z$ thus form a holomorphic family of bounded operators from $H^1_\partial(D)$ to $L^2_{cl}(D)$.

Let us also choose a system of cut surfaces $\Sigma_1, \ldots, \Sigma_\ell \subset D$ which form a basis of the relative homology $H_2(\bar{D}, \partial D)$ (cf. Section 5.2.2) and consider the map flux : $H^1_\partial(D) \to \mathbb{C}^\ell$ given by

$$\mathrm{flux}(u) = \left(\int_{\Sigma_j} u \right)_{j=1}^{\ell},$$

which is bounded by the trace lemma. The restriction of smooth functions on D to Σ_j extends to a bounded linear functional from $H^1(D)$ to $L^2(\Sigma_j)$. Note that $\mathrm{flux}(u)$ is independent of g_t and the choices of Σ_j in $[\Sigma_j]$, the latter since u is closed and $\iota^*_{\partial D} u = 0$.

Lastly, we introduce boundary conditions of the form

$$\sum_{j=1}^{\ell} \left(a_j \int_{\alpha_j} \star u + b_j \int_{\beta_j} \star u \right) = 0,$$

where $a_j, b_j \in \mathbb{C}$ and α_j, β_j form a basis of $H_1(\partial D)$. (See Figure 5.1 for an illustration of the curve system α_j, β_k.) A minor inconvenience which occurs here is that $H^1(\Omega^2(D))$ offers insufficient regularity to ensure continuity of the functional $u \mapsto \int_{\alpha_j} \star u$ since the curves α_j and β_j are of codimension 2. We fix this by considering a tubular neighborhood of each curve α_j in ∂D which is foliated by curves $\{\alpha_j^\tau, \tau \in [0,1]\}$ and introducing the functional $u \to \int_0^1 \big(\int_{\alpha_j^\tau} \star u \big) d\tau$, which is bounded by the trace lemma. For $u \in H_{\partial\partial}(D)$, these two functionals agree as $\iota^*_{\partial D} \star u$ is closed. Let ℓ denote the genus of ∂D. Define

$$B : H^1(\Omega^2(D)) \to \mathbb{C}^{2\ell}$$
$$B(u) = \left(\int_0^1 \left(\int_{\alpha_j^\tau} \star u \right) d\tau, \int_0^1 \left(\int_{\beta_j^\tau} \star u \right) d\tau \right)_{j=1}^{\ell}$$

Let $f_1, \ldots, f_\ell \in (\mathbb{C}^{2\ell})^*$ be ℓ linearly independent linear functionals such that $[\star u]$ lies in the Lagrangian L if and only if $f_j \circ B(u) = 0$ for $j = 1, \ldots, \ell$. Write $B_L : H^2_\partial(D) \to \mathbb{C}^\ell$, $B_L = (f_j \circ B)_{j=1}^{\ell}$, so that $H^1_L(D) = H^1_{\partial\partial}(D) \cap B_L^{-1}(\{0\})$. On a sufficiently small chart of $\mathrm{LGr}_\ell(H^1_{dR}(\partial D, \mathbb{C}))$ we may choose $f_1, \ldots, f_\ell$ which depend real-analytically on L. With this choice, the operator B_L then depends real-analytically on L. Let $V \subseteq \mathbb{C}^{\dim_{\mathbb{R}} \mathrm{LGr}_\ell(H^1_{dR}(\partial D,\mathbb{C}))}$ denote a neighborhood of the chart to which B_L extends holomorphically.

It follows e.g. from [EGP18, Theorem 1.1] that $\mathrm{curl}_t : H^1_\Sigma(D) \to L^2_{cl}(D)$ is invertible for real t. Since $H^1_\partial(D) = H^1_\Sigma(D) \oplus \mathcal{H}_\partial(D, g_t)$ and $\mathrm{flux} : \mathcal{H}_\partial(D, g_t) \to \mathbb{C}^\ell$ is bijective ([Sch95, Corollary 2.6.2],[CDG02, Theorem 1]), $\mathrm{curl}_t \oplus \mathrm{flux}$ has an inverse $A_t : L^2_{cl}(D) \oplus \mathbb{C}^\ell \to H^1_\partial(D)$. Let $\iota : H^1_\partial(D) \to L^2_{cl}(D)$ denote the natural embedding. The formula

$$\begin{aligned} &A_z : L^2_{cl}(D) \to H^1_\Sigma(D) \\ &A_z := A_t(\mathbb{1} - (\mathrm{curl}_z \oplus \mathrm{flux} - \mathrm{curl}_t \oplus \mathrm{flux}) \circ A_t)^{-1} \end{aligned}$$

provides an inverse to $\mathrm{curl}_z \oplus \mathrm{flux}$ for all $z \in U$ where $\mathbb{1} - (\mathrm{curl}_z \oplus \mathrm{flux} - \mathrm{curl}_t \oplus \mathrm{flux}) \circ A_t$ is invertible. After restricting U to a smaller neighborhood of $(-\varepsilon, \varepsilon)^k$, we may assume this is the case on all of U, since curl_z depends continuously on z in the norm topology on $\mathcal{L}(H^1_\partial(D), L^2_{cl}(D))$. Hence, $(A_z)_{z\in U}$ is an analytic family of operators on $L^2_{cl}(D)$. Next, consider the family of operators

$$\begin{aligned} &R(s,t,L,\lambda) : L^2_{cl}(D) \oplus \mathbb{C}^\ell \to L^2_{cl}(D) \oplus \mathbb{C}^\ell \\ &R(s,t,L,\lambda) = \mathbb{1} - \big(\lambda\iota \oplus (s\,\mathrm{flux} - sB_L)\big) \circ A_t, \end{aligned}$$

which is analytic in $t \in U$, $L \in V$ and $s, \lambda \in \mathbb{C}$. Note that $\lambda\iota \oplus (s\,\mathrm{flux} - sB_L)$ is a compact operator from $H^1_\partial(D)$ to $L^2_{cl}(D) \oplus \mathbb{C}^\ell$. Since $R(0,t,L,0) = \mathbb{1}$ is invertible, the hypotheses of a multivariable version of the Analytic Fredholm Theorem are met [SYZ11, Theorem 3]. It follows that there exists an analytic variety $\mathcal{S} \subseteq \mathbb{C} \times U \times V \times \mathbb{C}$ such that $R(s,t,L,\lambda)$ is invertible if $(s,t,L,\lambda) \notin \mathcal{S}$ and such that $R(s,t,L,\lambda)^{-1}$ is analytic on the complement of $\mathcal{S}$.

Furthermore, the points $(s,t,L,\lambda) \in \mathcal{S}$ are precisely those where $R(s,t,L,\lambda)$ has a finite-dimensional kernel. In this case, there exist $(v,f) \in L^2_{cl}(D) \oplus \mathbb{C}^\ell$ such that $v = \lambda u$ and $f = s\,\mathrm{flux}\,u - sB_L(u)$, where $u = A_t(v,f)$ is the unique form satisfying $\mathrm{flux}\,u = f$ and $\mathrm{curl}_t\, u = v$. If $s = 1$, the form u is then an eigenform of $\mathrm{curl}_{t,L}$ in $H^1_L(D, g_t)$ corresponding to the eigenvalue λ.

Next, we analyze the complement of $\mathcal{S}$, i.e. those points where $R(1,t,L,\lambda)$ is invertible. Consider $(v,f) \in L^2_{cl}(D) \oplus \mathbb{C}^\ell$ and write

$$(v', f') = R(1,t,L,\lambda)^{-1}(v,f), \;\; A_t(v',f') = u'$$

Then $R(1,t,L,\lambda)(v',f') = (v,f)$ implies

$$\begin{aligned} v' - \lambda u' = \mathrm{curl}\, u' - \lambda u' = v, \\ f' - f' + B_L(u') = B_L(u') = f. \end{aligned}$$

Thus, $(\mathrm{curl} - \lambda\iota) \oplus B_L$ has an explicit inverse in

$$A_t \circ R(1,t,L,\lambda)^{-1},$$

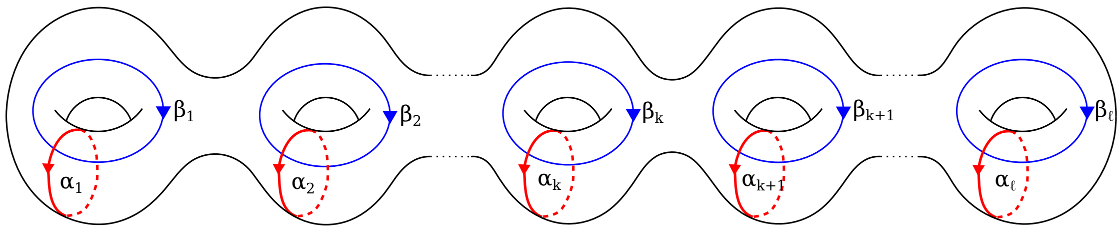

Figure 5.1: The curves α_j and β_j. Note that the surface ∂D need not be connected.

whenever λ is not contained in the spectrum of curl_L. By construction, this operator is analytic in t, L and λ. The resolvent of $\mathrm{curl}_{t,L} : H^1_L(D, g_t) \to L^2_{cl}(D)$, interpreted as a bounded linear operator on $L^2_{cl}(D)$, is given by restricting

$$R_{t,L}(\lambda) := \iota \circ A_t \circ R(1, t, L, \lambda)^{-1}$$

to $L^2_\partial(D) \simeq L^2_\partial(D) \oplus \{0\} \subseteq L^2_{cl}(D)$. Thus, the resolvent $R_{t,L}(\lambda)$ is meromorphic in λ and real-analytic in t and L near any (t, L, λ) where it is defined. □

Remark 5.3.2. *In the proof of Lemma 5.3.1 we used that the map* $\mathrm{flux} : \mathcal{H}_\partial(D, g_t) \to \mathbb{C}^\ell$ *is a linear isomorphism. In particular, the dimension of the space of boundary parallel harmonic vector fields equals the genus of ∂D. Equivalently, the equality $b_1(D) = \frac{1}{2}b_1(\partial D)$ holds for domains in $\mathbb{R}^3$ (see [CDG02, Theorem 1]). Here b_1 denotes the first Betti number, the dimension of the first (Co-) Homology group. For a general compact 3-manifold M with boundary one only has that $b_1(M) \geq \frac{1}{2}b_1(\partial M)$ [Mar16, Corollary 9.1.5]. It is easy to see that the inequality becomes strict after taking a connected sum with any closed manifold that is not a rational homology sphere.*

A straightforward modification of the proof of Lemma 5.3.1 also yields continuous dependence of the resolvent (in the norm topology) on the metric with respect to the C^1-norm. We write $R_{g,L}(\lambda)$ for the resolvent of $\mathrm{curl}_{g,L} : H^1_L(D, g) \to L^2_\partial(D)$.

Lemma 5.3.3. *Let $D \subseteq \mathbb{R}^3$ be a smoothly bounded domain with boundary of genus ℓ. Let g_0 be a smooth metric on D, let $L \in \mathrm{LGr}(\mathbb{C}^{2\ell})$ and let $\lambda \in \mathbb{C}$ such that $R_{g_0,L}(\lambda)$ exists. Then there exist $\varepsilon > 0$ and $C > 0$ such that for all g with $\|g - g_0\|_{C^1} \leq \varepsilon$, the resolvent $R_{g,L}(\lambda)$ exists and such that*

$$\|R_{g,L}(\lambda) - R_{g_0,L}(\lambda)\|_{L^2_\partial(D) \to H^1(\Omega^2(D))} \leq C\|g - g_0\|_{C^1} \tag{5.3.1}$$

hold for all such metrics g.

Proof. Since $\star_g : H^1(\Omega^2(D)) \to H^1(\Omega^1(D))$ arises from a vector bundle isomorphism which, in local coordinates, has coefficients that are smooth functions of components of the metric and its inverse, it depends Lipschitz-continuously on g in the following manner: Given a smooth metric g_0 on D, there exist $\varepsilon > 0$ and $C > 0$ such that

$$\| \star_g - \star_{g_0} \|_{H^1(\Omega^2(D)) \to H^1(\Omega^1(D))} \leq C\|g - g_0\|_{C^1} \tag{5.3.2}$$

for all g with $\|g-g_0\|_{C^1} < \varepsilon$. Since $d : H^1(\Omega^2(D)) \to L^2(\Omega^2(D))$ and flux : $H^1(\Omega^2(D)) \to \mathbb{C}^\ell$ are bounded and independent of g, it follows that

$$\| \operatorname{curl}_g \oplus \operatorname{flux} - \operatorname{curl}_{g_0} \oplus \operatorname{flux} \|_{H^1(\Omega^2(D)) \to L^2(\Omega^2(D)) \oplus \mathbb{C}^\ell} \leq C \|g - g_0\|_{C^1}. \tag{5.3.3}$$

Consider the inverse $A_g : L^2_{cl}(D) \to H^1_\partial(D)$ to the operator $\operatorname{curl}_g \oplus \operatorname{flux}$, as constructed in the proof of Lemma 5.3.1. It is expressed by

$$A_g := A_{g_0}(\mathbb{1} - (\operatorname{curl}_g \oplus \operatorname{flux} - \operatorname{curl}_{g_0} \oplus \operatorname{flux}) \circ A_{g_0})^{-1},$$

whenever $\mathbb{1} - (\operatorname{curl}_g \oplus \operatorname{flux} - \operatorname{curl}_{g_0} \oplus \operatorname{flux}) \circ A_{g_0}$ is invertible. By (5.3.3) this holds for all g with $\|g - g_0\|_{C^1} < \varepsilon$ after possibly shrinking ε. It follows that there exists $C' > 0$ (depending on C and $\|A_{g_0}\|_{L^2_{cl}(D) \oplus \mathbb{C}^\ell \to H^1_\partial(D)}$) such that

$$\|A_g - A_{g_0}\|_{L^2_{cl}(D) \oplus \mathbb{C}^\ell \to H^1_\partial(D)} \leq C' \|g - g_0\|_{C^1}$$

for all g with $\|g - g_0\|_{C^1} < \varepsilon$. Next, consider the Fredholm operator

$$R(1, g, L, \lambda) = \mathbb{1} - \big(\lambda\iota \oplus (\operatorname{flux} - B_L)\big) \circ A_g$$

from the proof of Lemma 5.3.1. It was shown there that $R(1, g, L, \lambda)$ is invertible if and only if $\lambda \notin \sigma(\operatorname{curl}_{g,L})$, so $R(1, g_0, L, \lambda)$ is invertible.

Observe that

$$\begin{aligned} &\|R(1, g, L, \lambda) - R(1, g_0, L, \lambda)\|_{L^2_{cl}(D) \oplus \mathbb{C}^\ell \to L^2_{cl}(D) \oplus \mathbb{C}^\ell} \\ &= \|\big(\lambda\iota \oplus (\operatorname{flux} - B_L)\big) \circ (A_g - A_{g_0})\|_{L^2_{cl}(D) \oplus \mathbb{C}^\ell \to L^2_{cl}(D) \oplus \mathbb{C}^\ell} \leq C'' \|g - g_0\|_{C^1}, \end{aligned}$$

for some $C'' > 0$. Since invertibility of Fredholm operators is an open condition, $R(1, g, L, \lambda)^{-1}$ is thus invertible for all g with $\|g - g_0\|_{C^1} < \varepsilon$ after perhaps shrinking $\varepsilon > 0$ once more, and

$$\|A_g \circ R(1, g, L, \lambda)^{-1} - A_{g_0} \circ R(1, g_0, L, \lambda)^{-1}\|_{L^2_{cl}(D) \oplus \mathbb{C}^\ell \to H^1_\partial(D)} \leq C''' \|g - g_0\|_{C^1}.$$

Since $R_{g,L}(\lambda)$ is the restriction of $\iota \circ A_g \circ R(1, g, L, \lambda)^{-1}$ to $L^2_\partial(D) \simeq L^2_\partial(D) \oplus \{0\}$, this establishes the claimed Lipschitz-continuous dependence of $R_{g,L}(\lambda)$ on g. □

Basic perturbation theory of operators with compact resolvent now shows that the spectrum of $\operatorname{curl}_{g,L}$ depends continuously on g with respect to the C^1-norm. We record the proof for the convenience of the reader.

Corollary 5.3.4. *Let D, g_0 and L be as in Lemma 5.3.3. Let λ be an eigenvalue of $\operatorname{curl}_{g_0,L}$ of multiplicity $m \in \mathbb{N}$. Then there exist $\varepsilon > 0$, an open neighborhood $\mathcal{U}$ of g_0 in the space of smooth metrics on D and m continuous functions $\lambda_1, \ldots, \lambda_m : \mathcal{U} \to \mathbb{R}$ such that*

$$\sigma(\operatorname{curl}_{g,L}) \cap (\lambda - \varepsilon, \lambda + \varepsilon) = \{\lambda_1(g), \ldots, \lambda_m(g)\}$$

for every $g \in \mathcal{U}$.

Proof. We reduce the problem to a finite-dimensional one as follows. Let $\Gamma \subseteq \mathbb{C}$ be a closed curve encircling λ, but no other point of $\sigma(\mathrm{curl}_{g_0,L})$. Since Γ is compact, there exists $\delta > 0$ such that $R_{g,L}(\zeta)$ exists for all $\zeta \in \Gamma$ and g with $\|g - g_0\|_{C^1} < \delta$. Let $\mathcal{U}$ be the set of metrics with $\|g - g_0\|_{C^1} < \delta$. By Lemma 5.3.3 and the dominated convergence theorem, the spectral projection

$$P_\Gamma(g) := \frac{1}{2\pi i} \int_\Gamma R_{g,L}(\zeta) d\zeta$$

depends continuously on $g \in \mathcal{U}$. Since $P_\Gamma(g)$ is a continuously varying projection, its rank is constant, so $\mathrm{rk} P_\Gamma(g) = \mathrm{rk} P_\Gamma(g_0) = m$.

Let $E_\Gamma(g) \subseteq L^2_\partial(D)$ denote the image of P_Γ, i.e., the sum of the eigenspaces corresponding to eigenvalues encircled by Γ. The operator $P_\Gamma(g_0)P_\Gamma(g) : E_\Gamma(g_0) \to E_\Gamma(g_0)$ depends continuously on g and acts as the identity when $g = g_0$. Hence one may shrink $\mathcal{U}$ so $(P_\Gamma(g_0)P_\Gamma(g))^{-1}$ exists on $\mathcal{U}$. Consider the family of operators on the finite-dimensional space $E_\Gamma(g_0)$ given by

$$A(g) := (P_\Gamma(g_0)P_\Gamma(g))^{-1} \circ P_\Gamma(g_0) \circ \mathrm{curl}_g \circ P_\Gamma(g).$$

Note that $A(g)$ is a concatenation of *bounded* linear operators.
All operators depend continuously on g. Plugging in the eigenbasis of curl_g shows that $A(g)$ shares its spectrum with curl_g:

$$\sigma(A(g)) = \sigma(\mathrm{curl}_{g,L} \cap (\lambda - \varepsilon, \lambda + \varepsilon)$$

Since $A(g)$ depends continuously on g (in fact Lipschitz continuously with respect to the C^1 norm), the continuous dependence of the spectrum follows from the perturbation theory of finite-dimensional operators, see, e.g., [Kat66, Theorem 5.1 and Theorem 5.2] as well as the discussion leading up to [Kat66, Theorem 5.2]. □

Remark 5.3.5. *In the following, our metrics g will be of the form $g = \Phi^* g_0$ for a fixed ambient metric g_0 and a diffeomorphism Φ. Corollary 5.3.4 implies that the spectrum of $\mathrm{curl}_{\Phi^* g_0, L}$ depends continuously on Φ with respect to the C^2-norm. By conjugation, the spectrum of the curl operator on $\Phi(D)$ with domain $H^1_L(\Phi(D), g_0)$ depends continuously on Φ in the same way.*

As a consequence of Lemma 5.3.1 and (a generalization of) Rellich's theorem for analytic *one-parameter* families of operators, we also obtain the following corollary.

Corollary 5.3.6. *Let D and L be as in Lemma 5.3.3. Let g_t, $t \in (-\varepsilon, \varepsilon)$ be a real-analytic one-parameter family of smooth metrics on D. Consider an eigenvalue λ of $\mathrm{curl}_{g_0,L}$ of multiplicity m with corresponding eigenspace E_λ. Then there exist analytic functions $\lambda_1(t), \dots, \lambda_m(t)$ and analytic one-parameter families $u_1(t), \dots, u_m(t)$ in $L^2_\partial(D)$ such that*

1. *$u_j(t) \in H^1_L(D, g_t)$ and $\mathrm{curl}_{g_t,L}\, u_j(t) = \lambda_j(t)\, u_j(t)$ for all $j \in \{1, \dots, m\}$,*
2. *$(u_j(0))_{j=1}^m$ forms an orthonormal basis of E_λ, and*

3. $(u_j(t))_{j=1}^m$ *is an orthonormal set with respect to* g_t *for any* $t \in (-\varepsilon, \varepsilon)$*.*

Proof. Fix an arbitrary point $t_0 \in (-\varepsilon, \varepsilon)$. The first two paragraphs of the proof of Corollary 5.3.4 carry over to the analytic setting. After possibly shrinking the interval $(-\varepsilon, \varepsilon)$ to a smaller interval centered around t_0 to ensure invertibility of $(P_\Gamma(g_{t_0})P_\Gamma(g_t))^{-1}$, we thus obtain an analytic one-parameter family

$$A(g_t) := (P_\Gamma(g_{t_0})P_\Gamma(g_t))^{-1} \circ P_\Gamma(g_{t_0}) \circ \operatorname{curl}_{g_t} \circ P_\Gamma(g_t).$$

of operators on the finite-dimensional space $E(g_{t_0})$. Note that $A(g_t)$ is symmetric with respect to the inner product $(\cdot,\cdot)_t := (P_\Gamma(g_t)\cdot, P_\Gamma(g_t)\cdot)_{g_t}$, but not necessarily with respect to a fixed inner product on $E(g_{t_0})$. Performing Gram–Schmidt orthogonalization on some fixed choice of real basis of $E(g_{t_0})$ with respect to $(\cdot,\cdot)_t$ yields a real-analytic family of maps $Q_t : \mathbb{C}^m \to E(g_{t_0})$ intertwining the standard inner product on $\mathbb{C}^m$ and $(\cdot,\cdot)_t$. Setting

$$B_t = Q_t^{-1} \circ A(g_t) \circ Q_t$$

yields a real-analytic one-parameter family of symmetric $m \times m$-matrices. By Rellich's theorem for analytic one-parameter families of self-adjoint matrices [Rel37, Theorem 2], there exist real-analytic functions $\lambda_1(t), \dots, \lambda_m(t) \in \mathbb{R}$ and real-analytic families $v_1(t), \dots, v_m(t) \in \mathbb{C}^m$ such that the following hold:

(1) $\sigma(B_t) = \{\lambda_1(t), \dots, \lambda_m(t)\}$, counted with multiplicity,

(2) $B_t v_j(t) = \lambda_j(t) v_j(t)$ for all $j \in \{1, \dots, m\}$,

(3) $v_j \cdot v_k = 1$ if $j = k$ and 0 if $j \neq k$.

Setting $u_j(t) = P_\Gamma(g_t) Q_t v_j(t)$ and tracing back through the definitions of $A(g_t)$ and B_t completes the proof. □

This observation will let us use our Hadamard rule, which is proven in the following section, in the analysis of eigenvalues of higher multiplicity as well as of simple ones.

5.4 The Hadamard rule

In this section, we prove formula (5.1.1) for the variation of a curl eigenvalue under domain deformations. In its statement, $(u \cdot v)$ denotes the (sesquilinear) pointwise inner product of two-forms, which is of course the same as the inner products $(\star u \cdot \star v)$ and $(\star u^\# \cdot \star v^\#)$ of the corresponding one-forms and their vector proxies, respectively.

Theorem 5.4.1 (Hadamard rule). *Let* $D \subseteq \mathbb{R}^3$ *be a smoothly bounded domain and* $\Phi_t : D \to \mathbb{R}^3$ *a smooth one-parameter family of embeddings of* D*. Write* $D_t = \Phi_t(D)$*, and* $X_t = \frac{d\Phi_t}{dt}$*. Furthermore, fix a Lagrangian* $L \subseteq H^1_{dR}(\partial D; \mathbb{C})$*.*

(a) *Suppose that u_t is a smooth* 1*-parameter family of closed, boundary-parallel* 2*-forms which solves $d(\star u_t) = \lambda_t u_t$ on D_t and satisfies $[\Phi_t^*(\star u_t)] \in L$ for all t. Then*

$$\dot{\lambda} = -\lambda \int_{\partial D} (X \cdot \nu)\, |u|^2 \, d\sigma, \tag{5.4.1}$$

where ν is the outward pointing unit normal of ∂D and $d\sigma$ is the induced surface measure on ∂D.

(b) *Let v_t be another smooth* 1*-parameter family of closed, boundary-parallel* 2*-forms solving $d(\star v_t) = \mu_t v_t$ satisfying $[\Phi_t^*(\star v_t)] \in L$ for all t. Suppose $\mu_0 = \lambda_0 \neq 0$ and $(u_t, v_t)_{L^2(D_t)} = 0$ for all t. Then*

$$0 = \int_{\partial D} (X \cdot \nu)\, (u \cdot v) \, d\sigma. \tag{5.4.2}$$

Proof. We will first prove the identities (5.4.3)–(5.4.5) below, which follow from L^2-normalization, Lagrangian boundary conditions and parallel boundary conditions, respectively. In these, w_t is a 1-parameter family of eigenforms which may stand for both u_t and v_t. We write ρ_t as a placeholder for λ_t and μ_t, so that $d(\star w_t) = \rho_t w_t$. Then

$$\int_{\partial D} (X \cdot \nu)\, (\star u \cdot \star w) \, d\sigma + \int_D \star\dot{u} \wedge \bar{w} + \int_D u \wedge \star\dot{\bar{w}} = 0 \tag{5.4.3}$$

$$\int_{\partial D} \star\dot{u} \wedge \star\bar{w} = -\lambda \int_{\partial D} (X \lrcorner\, u) \wedge \star\bar{w} \tag{5.4.4}$$

$$\int_{\partial D} (X \lrcorner\, u) \wedge \star\bar{w} = \int_{\partial D} (X \cdot \nu)(u \cdot w) \, d\sigma \tag{5.4.5}$$

Identity (5.4.3) follows from differentiating the expression

$$\langle u_t, w_t \rangle_{D_t} = \int_{D_t} u_t \wedge \star\bar{w}_t = \int_{D_t} (\star u_t \cdot \star w_t) \; dV_{\mathbb{R}^3},$$

which is constantly 1 if $w_t = u_t$ or constantly 0 if $w_t = v_t$. Thus,

$$\begin{aligned} 0 = \frac{d}{dt}|_{t=0} \int_{D_t} u_t \wedge \star\bar{w}_t &= \int_{\partial D} (X \cdot \nu)(\star u \cdot \star w) + \int_D \dot{u} \wedge \star\bar{w} + \int_D u \wedge \star\dot{\bar{w}} \\ &= \int_{\partial D} (X \cdot \nu)(\star u \cdot \star w) + \int_D \star\dot{u} \wedge \bar{w} + \int_D u \wedge \star\dot{\bar{w}}, \end{aligned}$$

noting that $\int_D \dot{u} \wedge \star\bar{w} = \int_D (\star\dot{u}, \star w) \, dV_{\mathbb{R}^3} = \int_D \star\dot{u} \wedge \bar{w}$.

We begin the proof of (5.4.4) by differentiating

$$\frac{d}{dt}|_{t=0} \Phi_t^*(\star u_t) = \star\dot{u} + X \lrcorner\, (d\star u) + d(X \lrcorner\, \star u)$$

Since $[\star w]$ and $[\Phi_t^*(\star u_t)]$ both belong to the same Lagrangian $L \subseteq H^1_{dR}(\partial D; \mathbb{C})$,

$$0 = \int_{\partial D} \Phi_t^*(\star u_t) \wedge \star \bar{w}.$$

Differentiating this identity at $t = 0$ and applying Stokes' theorem on ∂D yields

$$\begin{aligned} 0 &= \int_{\partial D} \star\dot{u} \wedge \star\bar{w} + \lambda \int_{\partial D} (X \lrcorner\, u) \wedge \star\bar{w} + \int_{\partial D} d(X \lrcorner \star u) \wedge \star\bar{w} \\ &= \int_{\partial D} \star\dot{u} \wedge \star\bar{w} + \lambda \int_{\partial D} (X \lrcorner\, u) \wedge \star\bar{w} - \int_{\partial D} (X \lrcorner \star u) \wedge (d\star\bar{w}) \\ &= \int_{\partial D} \star\dot{u} \wedge \star\bar{w} + \lambda \int_{\partial D} (X \lrcorner\, u) \wedge \star\bar{w}, \end{aligned}$$

where the last term in the second line vanished since $\iota^*_{\partial D}(d\star\bar{w}) = \rho\, \iota^*_{\partial D}\bar{w} = 0$.

For identity (5.4.5), let $X^\perp = (X \cdot \nu)\,\nu$ and $X^T = X - X^\perp$. The graded product rule for the interior product yields

$$\begin{aligned} (X \lrcorner\, u) \wedge \star\bar{w} &= X \lrcorner\, (u \wedge \star\bar{w}) - u \wedge (X \lrcorner \star\bar{w}) \\ &= X^\perp \lrcorner\, (u \wedge \star\bar{w}) + X^T \lrcorner\, (u \wedge \star\bar{w}) - u \wedge (X \lrcorner \star\bar{w}) \end{aligned}$$

When pulled back to ∂D, the terms $X^T \lrcorner\, (u \wedge \star\bar{w})$ and $u \wedge (X \lrcorner \star\bar{w})$ both vanish – the former because X^T is parallel to ∂D and the latter because $\iota^*_{\partial D} u = 0$. Hence,

$$\begin{aligned} \int_{\partial D} (X \lrcorner\, u) \wedge \star\bar{w} &= \int_{\partial D} X^\perp \lrcorner\, (u \wedge \star\bar{w}) \\ &= \int_{\partial D} (X \cdot \nu)\,(u \cdot w)\,(\nu \lrcorner\, dV_{\mathbb{R}^3}) \\ &= \int_{\partial D} (X \cdot \nu)\,(u \cdot w)\, d\sigma \end{aligned}$$

With the identities 5.4.3-5.4.5 in hand we compute

$$\begin{aligned} \dot{\lambda} &= \frac{d}{dt}\Big|_{t=0} \int_{D_t} d(\star u_t) \wedge \star\bar{u}_t \\ &= \lambda \int_{\partial D} \big(X \cdot \nu\big)\, |\star u|^2\, d\sigma + \int_D d(\star\dot{u}) \wedge \star\bar{u} + \lambda \int_D u \wedge \star\dot{\bar{u}} \\ &= \lambda \int_{\partial D} \big(X \cdot \nu\big)\, |\star u|^2\, d\sigma + \int_{\partial D} \star\dot{u} \wedge \star\bar{u} + \lambda \int_D \star\dot{u} \wedge \bar{u} + \lambda \int_D u \wedge \star\dot{\bar{u}} \\ &\overset{(5.4.3)}{=} \int_{\partial D} \star\dot{u} \wedge \star\bar{u} \overset{(5.4.4)}{=} -\lambda \int_{\partial D} (X \lrcorner\, u) \wedge \star\bar{u} \\ &\overset{(5.4.5)}{=} -\lambda \int_{\partial D} (X \cdot \nu) |u|^2\, d\sigma, \end{aligned}$$

yielding our Hadamard rule, part (a). The third line resulted from an application of Stokes' theorem to $\int_{\partial D} \star\dot{u} \wedge \star\bar{u}$, keeping in mind that $d\star\bar{u} = \lambda\bar{u}$ since λ is real.

For part (b) we essentially repeat the same computations, keeping in mind that $\mu_0 = \lambda_0$.

$$\begin{aligned}
0 &= \frac{d}{dt}|_{t=0} \int_{D_t} d(\star u_t) \wedge \star \bar{v}_t \\
&= \lambda \int_{\partial D} \big(X \cdot \nu\big) \left(\star u \cdot \star v\right) d\sigma + \int_D d(\star \dot{u}) \wedge \star \bar{v} + \lambda \int_D u \wedge \star \dot{\bar{v}} \\
&= \lambda \int_{\partial D} \big(X \cdot \nu\big) \left(\star u \cdot \star v\right) d\sigma + \int_{\partial D} \star \dot{u} \wedge \star \bar{v} + \mu \int_D \star \dot{u} \wedge \bar{v} + \lambda \int_D u \wedge \star \dot{\bar{v}} \\
&\overset{(5.4.3)}{=} \int_{\partial D} \star \dot{u} \wedge \star \bar{v} \overset{(5.4.4)}{=} -\lambda \int_{\partial D} (X \lrcorner\, u) \wedge \star \bar{v} \\
&\overset{(5.4.5)}{=} -\lambda \int_{\partial D} (X \cdot \nu)\,(u \cdot v)\, d\sigma
\end{aligned}$$

Note that no assumptions on the connectedness of ∂D were made. $\square$

Theorem 5.4.1 provides a way of computing the variation of a curl eigenvalue in the direction of arbitrary deformation velocities. Given a domain D_{Φ_0} and $f \in C^\infty(\partial D_{\Phi_0})$, we can construct an an analytic one-parameter family of diffeomorphisms Φ_t extending Φ_0 such that $\left\langle \frac{d}{dt}|_{t=0}\Phi_t, \nu\right\rangle = f$. This can be achieved by extending $f\nu$ to a vector field X on D_{Φ_0} and setting $\Phi_t = \Phi_0 + tX \circ \Phi_0$, which yields a diffeomorphism onto its image for all t small enough. Since $\Phi_t^* g_{\mathbb{R}^3}$ is then a one-parameter family of metrics which is analytic in t, Corollary 5.3.6 then gives the existence of an analytic one-parameter family of curl eigenforms as required in the statement of Theorem 5.4.1.

5.5 Generic simplicity of the spectrum

The following basic lemma implies that, for a residual set in the space of embeddings of a smoothly bounded domain D into $\mathbb{R}^3$, the multiplicity of each associated eigenvalue of curl_L is locally constant. This will be useful in proving Theorem 5.5.2.

Lemma 5.5.1. *Let $\mathcal{X}$ be a topological space, and $(\lambda_k)_{k\in\mathbb{Z}}$ a sequence of continuous functions such that $\lambda_k \leq \lambda_{k'}$ for any $k \leq k'$, and such that there does not exist a point where infinitely many λ_k take the same value. The set of all $x \in \mathcal{X}$ with the property that, for each $k \in \mathbb{Z}$, there exist integers $k_1 \leq k \leq k_2$ and a neighborhood U of x where*

$$\lambda_{k_1-1} < \lambda_{k_1} = \ldots = \lambda_k = \ldots = \lambda_{k_2} < \lambda_{k_2+1},$$

is a countable intersection of dense open sets.

Proof. For each pair $k_1 \leq k_2$, denote $\mathcal{K}_{k_1,k_2} = (\lambda_{k_1} - \lambda_{k_2})^{-1}\{0\}$. This set is closed, since $\lambda_{k_1} - \lambda_{k_2}$ is continuous. Fix $k \in \mathbb{Z}$ and consider

$$\mathcal{U}_k := \bigcup_{k_1 \leq k \leq k_2} \left(\mathcal{K}^c_{k_1-1,k_1} \cap \mathcal{K}^o_{k_1,k_2} \cap \mathcal{K}^c_{k_2,k_2+1}\right),$$

where o and c denote the interior and complement of a set, respectively. As a union of open sets, $\mathcal{U}_k$ is open. To show that it is also dense, let $x \in \mathcal{X}$ be an arbitrary point. Let k_1 and k_2 be the smallest and largest index, respectively, such that there exists a neighborhood of x on which $\lambda_{k_1} = \lambda_k$ and $\lambda_k = \lambda_{k_2}$. Then

$$x \in \overline{\mathcal{K}^c_{k_1-1,k_1}} \cap \mathcal{K}^o_{k_1,k_2} \cap \overline{\mathcal{K}^c_{k_2,k_2+1}} \subseteq \overline{\mathcal{U}_k},$$

which shows $\mathcal{U}_k$ is dense, as claimed. The set of interest is the (countable) intersection of all $\mathcal{U}_k$, $k \in \mathbb{Z}$. □

We split the proof of Theorem 5.1.1 into two parts. First, we use the Hadamard rule to derive strong consequences from the assumption that the spectrum of curl is not generically simple. We then show these consequences lead to a contradiction, hence proving Theorem 5.1.1.

Recall that in the formulation of Theorem 5.1.1, we fixed a domain $D_0 \subseteq \mathbb{R}^3$ and a Lagrangian $L \subseteq H^1_{dR}(\partial D_0; \mathbb{R})$. We then considered the Frechét manifold $\mathcal{X}(D_0)$ of smooth embeddings $\Phi : \overline{D_0} \to \mathbb{R}^3$, and associated to each $\Phi \in \mathcal{X}$ its image $D_\Phi := \Phi(D_0)$ and the Lagrangian $L_\Phi := (\Phi^{-1})^* L$. To formulate our intermediate result, we denote the nonzero eigenvalues of the curl operator $\mathrm{curl}_{L_\Phi} : H^1_{L_\Phi}(D_\Phi) \to L^2_\partial(D_\Phi)$, counted with multiplicity, by $\lambda^L_k(\Phi)$, where $k \in \mathbb{Z} \setminus \{0\}$.

Proposition 5.5.2. *Let $D_0 \subseteq \mathbb{R}^3$ and $L \subseteq H^1_{dR}(\partial D_0; \mathbb{R})$ be as in Theorem 5.1.1. Then for a comeagre subset of diffeomorphisms $\Phi \in \mathcal{X}(D_0)$, the following holds:*

1. *For each $k \in \mathbb{Z}\setminus\{0\}$, the multiplicity of $\lambda^L_k(\Phi')$ is constant for all Φ' in a neighborhood $\mathcal{U}_k \subseteq \mathcal{X}(D_0)$ of Φ, and equals either one or two.*
2. *In the latter case, the components of any orthonormal basis $(u_1, u_2) \in H^1_{\partial\partial}(D_{\Phi'})$ of the eigenspace corresponding to $\lambda^L_k(\Phi')$ are pointwise orthogonal and of equal length on $\partial D_{\Phi'}$ for any $\Phi' \in \mathcal{U}_k$. Furthermore, $\star u_1$ and $\star u_2$ pull back to harmonic forms on $\partial D_{\Phi'}$ with its induced metric.*

Proof. Fix any $\Phi_0 \in \mathcal{X}(D_0)$ and choose a real number μ not in the spectrum of $\mathrm{curl}_{L_{\Phi_0}}$ on D_{Φ_0}. Since the spectrum of curl_{L_Φ} depends continuously on Φ by Corollary 5.3.4, we may choose a neighborhood $\mathcal{U} \subseteq \mathcal{X}(D_0)$ of Φ_0 small enough such that μ does not lie in the spectrum of curl_{L_Φ} on D_Φ for all $\Phi \in \mathcal{U}$. This allows us to define the family of operators

$$\begin{aligned} R_\Phi &: L^2_\partial(D_0) \to L^2_\partial(D_0), \\ R_\Phi\, u &= \Phi^* \circ (\mu - \mathrm{curl}_{L_\Phi})^{-1} \circ (\Phi^{-1})^*\, u. \end{aligned}$$

The operator R_Φ coincides with the resolvent of the curl operator

$$d \circ \star_{\Phi^* g} : H^1_L(D_0, \Phi^* g) \to L^2_\partial(D_0).$$

It is self-adjoint with respect to the inner product induced by $\Phi^* g$.

Since the eigenvalues of R_Φ, and thus also the eigenvalues of curl_{L_Φ} on D_Φ, depend continuously on Φ by Corollary 5.3.4, we may apply Lemma 5.5.1 to conclude that there exists a comeagre subset of $\mathcal{U}$ where the multiplicity of each eigenvalue $\lambda_k^L(\Phi)$ is locally constant.

Let $\Phi_1 \in \mathcal{U}$ be one such map on which the multiplicity of every eigenvalue $\lambda_k^L(\Phi)$ is locally constant. Fix $k \in \mathbb{Z}\backslash\{0\}$ such that $\lambda_{k-1}^L(\Phi_1) < \lambda_k^L(\Phi)$ and let $\mathcal{U}_1$ be a neighborhood of Φ_1 on which

$$\lambda_{k-1}^L(\Phi) < \lambda_k^L(\Phi) = \ldots = \lambda_{k+m}^L(\Phi) < \lambda_{k+m+1}^L(\Phi)$$

for some $m \in \mathbb{N}$ and all $\Phi \in \mathcal{U}_1$. Consider any eigenfunction $u \in L^2_\partial(D_0)$ of R_{Φ_1} corresponding to the curl eigenvalue $\lambda = \lambda_k^L(\Phi_1)$. Choose a small circle $C \subseteq \mathbb{C}$ encircling $(\mu-\lambda)^{-1}$, but no other eigenvalue of R_{Φ_1}. After possibly shrinking $\mathcal{U}_1$, the Riesz projection

$$P_\Phi = \frac{1}{2\pi i}\int_C (\zeta - R_\Phi)^{-1}$$

is defined and depends continuously on $\Phi \in \mathcal{U}_1$, see the proof of Corollary 5.3.4. Since the multiplicity of λ_k^L is assumed to be constant on $\mathcal{U}_1$, the range of P_Φ consists of eigenfunctions of R_Φ for all $\Phi \in \mathcal{U}_1$. We may construct a family of eigenfunctions $u_\Phi \in L^2_\partial(D_0)$ satisfying

$$R_\Phi u_\Phi = (\mu - \lambda_k^L(\Phi))^{-1} u_\Phi, \qquad \|u\|_{L^2(D_0,\Phi^* g)} = 1, \qquad u_{\Phi_1} = u,$$

by simply setting $u_\Phi = \|P_\Phi u\|^{-1}_{L^2(D_0,\Phi^* g)} P_\Phi u$, after possibly shrinking $\mathcal{U}_1$ to ensure $P_\Phi u \neq 0$.

Write $D := D_{\Phi_1}$. Given $f \in C^\infty(\partial D)$, we consider an analytic 1-parameter family of diffeomorphisms $\Phi(t)$ in $\mathcal{U}_1$ with $\Phi(0) = \Phi_1$ and $\left\langle \frac{d}{dt}|_{t=0}\Phi, \nu \right\rangle = f$, giving rise to the analytic 1-parameter family of metrics $\Phi(t)^* g_{\mathbb{R}^3}$ on D_0. Lemma 5.3.1 then implies that $P_{\Phi(t)}$ depends analytically on t, and Theorem 5.4.1 yields

$$\frac{d}{dt}|_{t=0}\, \lambda_k^L(\Phi(t)) = -\lambda \int_{\partial D} f|u|^2\, d\sigma.$$

For simplicity, denote $\frac{d}{dt}|_{t=0}\lambda_k^L(\Phi(t))$ by $\dot\lambda_f$. By polarization, we find that

$$\int_{\partial D} f\,(u_j \cdot u_k)\; d\sigma = -\frac{\dot\lambda_f}{\lambda}\,\delta_{j,k}$$

for any real valued orthonormal basis $(u_j)_{j=1}^m$ of E_λ (note that such a basis always exists because curl_L commutes with complex conjugation). Indeed,

$$\begin{aligned}\int_{\partial D} f\,(u_j \cdot u_k)\; d\sigma &= \frac{1}{2}\left(\int_{\partial D} f\left|\tfrac{u_1+u_2}{\sqrt{2}}\right|^2 d\sigma + \int_{\partial D} f\left|\tfrac{u_1-u_2}{\sqrt{2}}\right|^2 d\sigma - \int_{\partial D} f|u_1|^2\, d\sigma - \int_{\partial D} f|u_2|^2\, d\sigma\right)\\ &= -\frac{1}{2}\left(\frac{\dot\lambda_f}{\lambda} + \frac{\dot\lambda_f}{\lambda} - \frac{\dot\lambda_f}{\lambda} - \frac{\dot\lambda_f}{\lambda}\right) = 0\end{aligned}$$

and

$$\int_{\partial D} f|u_1|^2\,d\sigma = -\frac{\dot{\lambda}_f}{\lambda} = \int_{\partial D} f|u_2|^2\,d\sigma.$$

Since these identities hold for any $f \in C^\infty(\partial D)$, the matrix $(u_i \cdot u_j)_{i,j=1}^m$ is a multiple of the identity at every point $p \in \partial D$. This conclusion in fact holds for any $\Phi \in \mathcal{U}_1$, since the preceding argument only used local constancy of the eigenvalue multiplicity and can therefore be repeated at any $\Phi \in \mathcal{U}_1$.

There does not exist a curl eigenform u which vanishes identically on ∂D (cf. [EP23, proof of Proposition 2.1]). Consequently, there exists an open subset of ∂D on which all u_i are nonvanishing and pointwise orthogonal to one another. As the u_i are boundary parallel, it follows that $m \leq 2$. The eigenspace E_λ is thus either one-dimensional or spanned by two eigenforms u_1 and u_2 which are orthogonal and of equal length at each point $p \in \partial D$.

The assumption that the Lagrangian L is *real*, which was not used until now, helps to deal with the latter case. If L is real, the operator curl_L commutes with complex conjugation, hence all eigenforms in this proof can be taken to be real valued. This allows us to show $\star u_1$ and $\star u_2$ are in fact harmonic with respect to the induced metric on ∂D. Write $\alpha = \iota_{\partial D}^* \star u_1$, $\beta = \iota_{\partial D}^* \star u_2$, and let $\star'$ denote the Hodge-star on ∂D with its induced metric. Consider a connected open set $V \subseteq \partial D$ where $|\alpha| \neq 0$. Since β and α are orthogonal and of equal length, $\beta = \pm \star' \alpha$ on V, with the sign constant throughout V by continuity. Without loss of generality, assume that $\beta = \star'\alpha$. Part of the boundary condition $d \star u_1 \in L^2_\partial(D)$ is that $\iota_{\partial D}^*(d \star u_1) = 0$, and hence $d\alpha = 0$. Analogously, $d\beta = 0$. Together, we have $d\beta = d \star' \beta = 0$, i.e. β is harmonic on V with its induced metric, and analogously α as well. By continuity, α and β are harmonic on $\overline{V}$. Repeating this argument on all connected components of the set $\{p \in \partial D : |u_1| \neq 0\}$ shows that α and β are harmonic on its closure. In the interior of its complement, α and β vanish identically. Hence, both are in fact harmonic on all of ∂D. It now follows that on each connected component M of ∂D, either $\beta = \star'\alpha$ or $\beta = - \star' \alpha$ holds. This is because at least one of the harmonic forms $\beta - \star'\alpha$ or $\beta + \star'\alpha$ vanishes on an open subset of M and therefore on all of M by the unique continuation theorem for harmonic differential forms [AKS62, Theorem 1]. □

We thank Daniel Peralta-Salas for the important observation that boundary parallel curl eigenforms which are orthogonal and of equal length on the boundary must restrict to harmonic forms on the boundary with respect to its induced metric. The proof of Theorem 5.1.1 in the case of disconnected boundary rests on this observation.

Proof of Theorem 5.1.1. According to Proposition 5.5.2, we are presented with a dichotomy: Either the spectrum of curl_L is simple for generic domains diffeomorphic to D_0 or there exists an integer k, an embedding $\Phi_1 : D_0 \to \mathbb{R}^3$, an open subset $\mathcal{U}_k$ in the space of smooth maps from D_0 to $\mathbb{R}^3$ containing Φ_1 and a family of *real-valued* orthonormal bases $u_1(\Phi)$ and $u_2(\Phi)$ of the eigenspace associated to $\lambda_k^L(\Phi)$ such that the 1-forms $\star u_i(\Phi)$ restrict to harmonic forms on ∂D_Φ with respect to the induced metric for all $\Phi \in \mathcal{U}_k$ and satisfy $\star u_2 = \star'(\star u_1)$, where $\star'$ is the Hodge-star on ∂D. We will show that the second alternative is impossible. For this we distinguish two cases, depending on whether ∂D_0 is connected.

Case 1: ∂D is connected. Here, we use the self-adjointness of curl_L to obtain

$$\begin{aligned} 0 &= |\langle \operatorname{curl} u_1, u_2\rangle - \langle u_1, \operatorname{curl} u_2\rangle| \\ &= \left| \int_{\partial D} \star u_1 \wedge \star u_2 \right| \\ &= \left| \int_{\partial D} \star u_1 \wedge \star'(\star u_1) \right| \\ &= \int_{\partial D} |\star u_1|^2 d\sigma, \end{aligned}$$

which implies that u_1 vanishes identically on ∂D. As already mentioned, this possibility is ruled out by an argument presented in [EP23, Proof of Proposition 2.3]. If ∂D is connected, it is therefore impossible for two real valued curl eigenforms u_1 and u_2 of the same eigenvalue to be orthogonal and of equal length on the boundary.

Case 2: ∂D is not connected. Here, the argument above fails, since it is possible that $\star u_1 = \star'(\star u_2)$ on one boundary component and $\star u_1 = -\star'(\star u_2)$ on another. Instead, we will make an argument that plays the finite-dimensionality of the space of harmonic forms on a fixed boundary component of ∂D against the infinite-dimensionality of the space of perturbations of the remaining boundary components via the Cauchy–Kovalevskaya theorem for curl eigenforms [EP12a, Theorem 3.1]. We will make this precise in the next paragraph.

Fix a boundary component $M \subseteq \partial D$ and let $(\Phi_t)_{t\in[-1,1]^k}$ be a real-analytic m-parameter family of diffeomorphisms in $\mathcal{U}_1$ (m to be fixed later) such that $M \subseteq \partial D_{\Phi_t}$ for all $t \in [-1,1]^m$ and $\partial D_{\Phi_{t_1}} \cap \partial D_{\Phi_{t_2}}$ contains no open subset of $\partial D_{\Phi_{t_1}}$ but M if $t_1 \neq t_2$. Since the multiplicity of $\lambda(t) := \lambda_k(D_{\Phi_t})$ is constantly equal to two, by the same argument as in the proof of Proposition 5.5.2 we obtain analytic m-parameter families of real-valued eigenforms $u_1(t)$ and $u_2(t)$ such that

$$\begin{aligned} &\operatorname{curl} u_j(t) = \lambda(t) u_j(t), \\ &\langle u_i(t), u_j(t)\rangle_{L^2(D_t)} = \delta_{i,j} \end{aligned}$$

for $i, j \in \{1,2\}$. Furthermore, by Proposition 5.5.2, $\iota_M^*(\star u_1(t))$ and $\iota_M^*(\star u_2(t))$ are pointwise orthogonal harmonic 1-forms on M for all $t \in [-1,1]^m$. Let ℓ' denote the genus of M. The map $F : [-1,1]^m \to \mathbb{R}^{1+2\ell'}$ given by

$$F(t) = \Big(\lambda(t), [\iota_M^*(\star u_1(t))]_{H^1_{dR}(M)} \Big)$$

is real analytic. If $m = 3 + 2\ell'$, by Sard's theorem, a generic level set of F is a submanifold of codimension $1 + 2\ell'$, and hence two-dimensional. Let us henceforth restrict t to such a level set $\mathcal{S}$. Then $[\iota_M^*(\star u_1(t))]_{H^1_{dR}(M)}$ is constant, and since $\iota_M^*(\star u_1(t))$ is harmonic, this means $\iota_M^*(\star u_1(t))$ is constant. As $\lambda(t)$ is constant as well, the uniqueness part of the Cauchy-Kovalevskaya theorem for curl eigenforms obtained in [EP12a, Theorem 3.1] implies that $\star u_1(t_1) = \star u_1(t_2)$ on $D_{t_1} \cap D_{t_2}$ for all $t_1, t_2 \in \mathcal{S}$. On the domain $D_{\mathcal{S}} = \bigcup_{t \in \mathcal{S}} D_t$, we can thus find $u_1^{\mathcal{S}}$ such that $\star u_1^{\mathcal{S}}|_x = \star u_1(t)|_x$ whenever $x \in D_t$. Define $u_2^{\mathcal{S}}$ analogously. The parallel boundary conditions imply that the vector proxies $(\star u_1^{\mathcal{S}})^\sharp$ and $(\star u_2^{\mathcal{S}})^\sharp$ are tangential to ∂D_t for all $t \in \mathcal{S}$. Equivalently, ∂D_t integrates the (possibly singular) plane field $\ker \star(\star u_1^{\mathcal{S}} \wedge \star u_2^{\mathcal{S}})$. As a consequence, $\partial D_{t_1} \cap \partial D_{t_2} \setminus M$ contains an open subset of ∂D_{t_1} if $\partial D_{t_1} \cap \partial D_{t_2} \setminus M$ contains a single point $p \in D_{\mathcal{S}}$ where $u_1^{\mathcal{S}}|_p \neq 0$. The vanishing set of $u_1^{\mathcal{S}}$ is at most two-dimensional and so cannot contain ∂D_t for all $t \in \mathcal{S}$. Thus, there exists an open set $O \subseteq D_{\mathcal{S}}$ where $u_1^{\mathcal{S}}$ does not vanish, and a nonempty open set $\mathcal{S}' \subseteq \mathcal{S}$ such that ∂D_t intersects O nontrivially for all $t \in \mathcal{S}'$. Since ∂D_t and $\mathcal{S}'$ are two-dimensional, and O is only three-dimensional, there must exist $t_1, t_2 \in \mathcal{S}'$ such that $\partial D_{t_1} \cap \partial D_{t_2} \cap O \neq \emptyset$. Then, $\partial D_{t_1} \cap \partial D_{t_2}$ contains a nontrivial open subset of ∂D_1 other than M, contradicting our assumption on the family D_t, and hence finishing our argument. $\square$

Remark 5.5.3. *The proof of Theorem 5.1.1 in the case of connected boundary applies verbatim to any family of domains large enough so that the space of "deformation velocities" $\langle X, \nu \rangle$ spans a dense subspace of $L^2(\partial D)$. If ∂D is disconnected, we additionally need to be able to to perturb ∂D while keeping one component fixed. Both of these properties are satisfied, for example, if one restricts attention to domains with analytic boundary.*

5.6 Local extrema of the eigenvalue functionals

This section is concerned with proving Theorem 5.1.2, i.e. with the characterization of local extrema for the k^{th} eigenvalue functionals λ_k^L over the space $\mathcal{X}_1(D_0)$ of domains diffeomorphic to D_0 with volume 1.

Local extrema of λ_k^L often arise when an eigenvalue cluster has the property that along any volume-preserving perturbation, the lowest eigenvalue splitting off the cluster is non-increasing, while the highest eigenvalue is non-decreasing. This picture provides the intuition behind the first part of Theorem 5.1.2, which parallels a theorem obtained by El Soufi and Ilias [EI08b] for the Laplace-Beltrami operator acting on functions on a closed Riemannian manifold, and is proven using the same general strategy.

Proof of Theorem 5.1.2. Suppose Φ_0 is a local extremum of λ_k^L in $\mathcal{X}_1(D_0)$. Equivalently, Φ is a local extremum for dimensionless functional $|D_\Phi|^{\frac{1}{3}} \lambda_k^L(\Phi)$ over the entire space $\mathcal{X}(D_0)$. For the rest of the proof, we write $D = D_{\Phi_0}$ and recenter $\mathcal{X}(D_0)$ at D via the isomorphism $\mathcal{X}(D_0) \to \mathcal{X}(D)$, $\Phi \mapsto \Phi \circ \Phi_0^{-1}$.

Assume that $\lambda := \lambda_k^L(\mathrm{id})$ has multiplicity m. Consider any function $f \in C^\infty(\partial D)$. As in the proof of Theorem 5.1.1, let Φ_t be an analytic 1-parameter family of diffeomorphisms defined on a neighborhood of D, such that $\Phi_0 = \mathrm{id}$ and $\langle X, \nu \rangle = f$, where X denotes the vector field $\frac{d}{dt}|_{t=0} \Phi_t$ and ν the outward unit normal to ∂D. Corollary 5.3.6 yields $\Lambda_1(t), \ldots, \Lambda_m(t)$ and $u_1(t), \ldots, u_m(t) \in H^1_{L_{\Phi_t}}(D_{\Phi_t})$ such that $(u_j(0))_{j=1}^m$ is an orthonormal basis of the eigenspace E_λ and

$$\operatorname{curl} u_j(t) = \Lambda_j(t) u_j(t).$$

Theorem 5.4.1 then implies that

$$\Lambda_j'(0) = -\lambda \int_{\partial D} f|u_j|^2 \text{ and } 0 = \int_{\partial D} f(u_j \cdot u_k).$$

Suppose first that the (finite dimensional) space spanned by $\{|u|^2|_{\partial D} : u \in E_\lambda\}$ does not contain the constant function 1. Then there exists a function $f \in C^\infty(\partial D)$ with $\int_{\partial D} f|u|^2 d\sigma = 0$ for all $u \in E_\lambda$ but $\int_{\partial D} f\, d\sigma = 1$. Construct an analytic 1-parameter family Φ_t as above which satisfies $\langle X, \nu\rangle = f$. Then there exist $u_1, \ldots, u_m \in E_\lambda$ with

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} \left(|D_t|^{\frac{1}{3}} \Lambda_j(t)\right) &= \Lambda_j'(0)|D_t|^{\frac{1}{3}} + \frac{1}{3}\Lambda_j(0)|D_t|^{-\frac{2}{3}} \frac{d}{dt}|D_t| \\ &= -\lambda |D|^{\frac{1}{3}} \int_{\partial D} f|u_j|^2 d\sigma + \frac{1}{3}\lambda |D|^{-\frac{2}{3}} \int_{\partial D} f\, d\sigma \end{aligned} \tag{5.6.1}$$

for all $j \in \{1, \ldots, m\}$. Thus, all normalized eigenvalues $|D_t|^{\frac{1}{3}}\Lambda_j(t)$ are greater in absolute value than λ for some $t > 0$, and smaller in absolute value for some $t < 0$, contradicting the assumption that D was a local extremum for λ_k^L.

If $\operatorname{span}_{\mathbb{R}}(\{|u|^2|_{\partial D} : u \in E_\lambda\})$ contains 1 but the convex cone spanned by $\{|u|^2|_{\partial D} : u \in E_\lambda\}$ does not, we may choose $f \in \operatorname{span}_{\mathbb{R}}(\{|u|^2|_{\partial D} : u \in E_\lambda\})$ with $\int_{\partial D} f|u|^2 d\sigma \leq 0$ for all $u \in E_\lambda$, but $\int_{\partial D} f\, d\sigma > 0$, and arrive at the same contradiction as in the previous paragraph.

Hence, 1 lies in the cone spanned by $\{|u|^2|_{\partial D} : u \in E_\lambda\}$. An elementary linear algebra calculation, which we isolate in Lemma 5.6.1 below, shows that there exists an orthonormal basis $(u_j)_{j=1}^m$ of E_λ and $\tau_1, \ldots, \tau_m \geq 0$ such that $|\tau_1 u_1|^2 + \ldots + |\tau_m u_m|^2 = 1$ on ∂D. After discarding $\tau_j u_j$ whenever $\tau_j = 0$, we have found an orthogonal set of eigenforms with the claimed property.

It remains to consider the special case of the lowest positive eigenvalue. (The case of the highest negative eigenvalue is analogous). Here, we must have $\frac{d}{dt}\big|_{t=0}\left(|D_t|^{\frac{1}{3}}\Lambda_j(t)\right) = 0$ for all $j \in \{1, \ldots, m\}$ along any 1-paramater family ϕ_t, since otherwise we could further decrease the lowest eigenvalue of the eigenvalue cluster $\Lambda_1, \ldots, \Lambda_m$, which by definition decreases λ_1^L. Hence,

$$\int_{\partial D} f|u_j|^2 d\sigma = \frac{1}{3}|D|^{-1} \int_{\partial D} f\, d\sigma$$

for all $f \in C^\infty(\partial D)$. Part (b) of Theorem 5.4.1 furthermore asserts that

$$\int_{\partial D} f(u_j \cdot u_k) d\sigma = 0$$

for all $j \neq k$. It follows that

$$\int_{\partial D} f|u|^2 d\sigma = \frac{1}{3}|D|^{-1} \int_{\partial D} f\, d\sigma$$

for any L^2-normalized $u \in E_\lambda$. Since $f \in C^\infty(\partial D)$ was arbitrary, $|u|^2 = \frac{1}{3}|D|^{-1}$ on ∂D.

Having established this, one may proceed as in the proof of [Ger23, Proposition 3.6] to show that $\dim E_\lambda = 1$. Let us note that the key idea there is to show by direct calculation that $\dim E_\lambda \geq 2$ would imply the induced metric on ∂D is flat, which is not possible for a smoothly bounded domain $D \subseteq \mathbb{R}^3$. $\square$

Lemma 5.6.1. *Let $B : V \times V \to W$ be a symmetric bilinear map between a complex, finite dimensional inner product space V and another complex vector space W. For any element c in the cone $\mathcal{C}$ spanned by $\{B(v, \bar{v}), v \in \mathbb{R}^m\}$, there exists an orthonormal basis $(v_j)_{j=1}^m$ of V and coefficients $\tau_1, \dots, \tau_m \geq 0$ such that $c = \sum_{j=1}^m B(\tau_j v_j, \tau_j \bar{v}_j)$.*

Proof. Since $c \in \mathcal{C}$, there exist $\eta_1, \dots, \eta_N \geq 0$ and $u_1, \dots, u_N \in V$ such that

$$c = \sum_{k=1}^N \eta_k B(u_k, \bar{u}_k).$$

Pick an orthonormal basis $(w_j)_{j=1}^m$ of V. Write $u_k = \sum_{j=1}^m \nu_k^j w_j$. Then

$$c = \sum_{k=1}^N \eta_k \sum_{i,j=1}^m \nu_k^i \bar{\nu}_k^j B(w_i, \bar{w}_j) = \sum_{i,j=1}^m \left(\sum_{k=1}^N \eta_k \nu_k^i \bar{\nu}_k^j \right) B(w_i, \bar{w}_j).$$

Denote $\alpha_{ij} = \eta_k \nu_k^i \bar{\nu}_k^j$. Then α is a hermitian, positive semidefinite $m \times m$ matrix. Thus, there exists a unitary matrix β and a diagonal matrix $\tau = \operatorname{diag}(\tau_1, \dots, \tau_m)$ with nonnegative entries such that $\beta \tau \beta^* = \alpha$. In other words,

$$\sum_{k=1}^m \tau_k \beta_{i,k} \overline{\beta_{j,k}} = \alpha_{i,j}.$$

Setting $v_k = \sum_{j=1}^m \beta_{j,k} w_j$ for $k = 1, \dots, m$ yields

$$\begin{aligned}
\sum_{k=1}^m \tau_k B(v_k, \bar{v}_k) &= \sum_{k=1}^m \tau_k \sum_{i,j=1}^m \beta_{i,k} \overline{\beta_{j,k}} B(w_i, \bar{w}_j) \\
&= \sum_{i,j=1}^m \left(\sum_{k=1}^m \tau_k \beta_{i,k} \overline{\beta_{j,k}} \right) B(w_i, \bar{w}_j) = \sum_{i,j=1}^m \alpha_{i,j} B(w_i, \bar{w}_j) = c,
\end{aligned}$$

and thus $c = \sum_{k=1}^m B(\sqrt{\tau_k} v_k, \sqrt{\tau_k} \bar{v}_k)$. Normalizing $(v_k)_{k=1}^m$ yields the result. $\square$

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