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Independence in Higher Baire Spaces

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Abstract

The main topic of this thesis is a possible generalisation of independence number denoted $\mathfrak{i}$ to the higher Baire space. The first goal is to show the consistency of $\mathfrak{i} < \mathfrak{c}$. In order to prove this independent families will be introduced. The number $\mathfrak{i}$ will be defined as the minimum cardinality of a maximal independent family. After this, densely maximal and selective independent families will be defined.

The next chapter will focus on the κ-Sacks forcing for some uncountable, inaccessible cardinal κ. Preprocessed conditions will be introduced and will later play a crucial role for the consistency proof. Finally, the countable Sacks forcing will be considered. We will show that countable support iterations of the countable Sacks forcing preserve densely maximal independent families.

In the last chapter the result of the independence number will be generalised to higher Baire space. So we will define κ-independent families for regular, uncountable κ and the κ-independence number $\mathfrak{i}(\kappa)$. Similar to the previous chapter the κ-support product of λ many copies of Sacks forcing posets for some regular uncountable $\lambda > \kappa$ will be used to obtain a model in which $\mathfrak{i}(\kappa) = \kappa^+$ but $2^\kappa > \kappa^+$.

The thesis will be concluded with some remarks about strongly κ-independent families and open questions.

Abstract Deutsch

Das Thema dieser Arbeit ist es, die *independence number* $\mathfrak{i}$ auf höhere Kardinalitäten zu verallgemeinern. Zu Beginn zeigen wir, dass es konsistent ist, dass $\mathfrak{i} < \mathfrak{c}$ gilt. Dafür führen wir *independent families* ein and definieren $\mathfrak{i}$ als die kleinste Kardinalität einer solchen Familie. Schließlich werden *densely maximal* und *selective independent families* definiert.

Im nächsten Kapitel liegt der Fokus auf dem κ-*Sacks forcing*, wobei κ eine überabzählbare, stark unerreichbare Kardinalzahl ist. Anschließend werden *preprocessed conditions* eingeführt. Diese spielen beim anschließenden Konsistenzbeweis eine entscheidende Rolle. Am Ende des Kapitels wird das abzählbare *Sacks forcing* verwendet. Wir zeigen, dass bei Iterationen mit abzählbarem Support, eine *densely maximal independent family* im finalen Modell *densely maximal* bleibt.

Im letzten Kapitel wird nun dieses Resultat für höhere Kardinalitäten verallgemeinert. Dafür werden κ-*independent families* definiert, sowie auch $\mathfrak{i}(\kappa)$, wobei κ regulär und überabzählbar ist. Mithilfe eines Produkts von λ vielen *Sacks forcings* für $\lambda > \kappa$, dessen Support κ ist, erhalten wir ein Modell in dem $\mathfrak{i}(\kappa) = \kappa^+$ aber $2^\kappa > \kappa^+$ gilt.

Am Ende widmen wir uns noch *strongly independent familes* und offenen Fragen.

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Contents

1 Introduction

The independence number was first mentioned in 1935, by Kantorevic and Fichtenholz in [13]. They also proofed the existence of a maximal independent family of size continuum.

Even, ninety years later there are still many open questions, even regarding the countable independence number. For example, is it consistent, that $\mathfrak{i} < \mathfrak{a}$, where $\mathfrak{a}$ is the almost disjointness number? [9] Does ZFC prove the existence of a densely maximal independent family? Is it consistent that there are no selective independent families? [9] Selective independent families were used in the work of Shelah to prove that $\mathfrak{i}$ is consistently less than the ultrafilter number $\mathfrak{u}$ [16]. There it was also shown that they exist under the assumption of CH.

In the following chapter we will introduce the countable independence number. We will repeat one of the many possible proofs that $\mathfrak{i} \leq \mathfrak{c}$. In addition densely maximal independent families will be introduced and some equivalent conditions for an independent family to being densely maximal will be formulated. Then selective independent families will be defined. A selective independent family will be constructed under the assumption of the continuum hypothesis and with the use of forcing. Finally an equivalent condition to the q-set property of a filter will be formulated, which will be relevant for the proof of the consistency of $\mathfrak{i} < \mathfrak{c}$.

The next chapter introduces the κ-Sacks forcing for a regular uncountable cardinal κ. In order to establish similar properties as for the countable Sacks forcing we will have to make assumptions on κ such as $\kappa^{<\kappa} = \kappa$. Two important notions in connection with the Sacks forcing that will play a role in the later proofs are fusion sequences and preprocessed conditions.

In the last part of the third chapter we switch to the countable Sacks forcing. With the use of this and under the assumption of CH in the ground model we will show that a selective independent family remains densely maximal after ω_2 many iterations of the Sacks forcing. This shows that it is consistent that $\mathfrak{i} < \mathfrak{c}$.

The fourth chapter deals with possible generalisations of the independence number to higher Baire spaces. There is more than one way, how one could possibly generalise κ-independence for some uncountable cardinal κ. In this chapter we will define a family $\mathcal{A}$ to be κ-independent if $\bigcap B \setminus (\bigcup C)$ is of size κ for any two disjoint subfamilies B and C of $\mathcal{A}$. This definition makes it possible to again have maximal independent families and that $\mathfrak{i}(\kappa)$ is well-defined. Similar to the second chapter we will construct a densely maximal independent family and (re-)introduce density the filter and the density ideal.

For the construction of the densely maximal independent family we will assume κ to be an inaccessible cardinal, which especially implies that $2^\kappa = \kappa^+$.

This Chapter will be concluded with the proof that the constructed densely maximal independent family remains densely maximal after forcing with the κ-support product of λ many copies of the Sacks forcing. The proof will be essentially similar to the one in the previous chapter. It will make use of properties of the density filter and the density ideal in the ground model. This shows that it is consistent that $\mathfrak{i}(\kappa) < 2^\kappa$.

In the conclusion some remarks on the assumptions made throughout the whole thesis will be stated. In addition the strong κ-independence number which is another possible generalisation of $\mathfrak{i}$ will be discussed.

2 On the countable independence number

2.1 Basic definitions

In this section we will summarise the results about the (countable) independence number and introduce selective independent families. Let us first introduce the independence number.

Definition 2.1. [7]

- A family $\mathcal{I} \subseteq [\omega]^\omega$ is called independent if for all finite sets $A_0, A_1 \subseteq \mathcal{I}$ it holds that the set $(\bigcap_{A \in A_0} A) \cap (\bigcap_{A \in A_1} A^c)$ is infinite.
- An independent family is called maximal independent, if it is not properly contained in any other independent family.
- The independence number $\mathfrak{i}$ is the minimum cardinality of a maximal independent family.

Before proceeding a shorter notation will be introduced. For any set $\mathcal{A} \subseteq [\omega]^\omega$ let $FF(\mathcal{A}) := \{f \mid f : A \to \{0,1\}$ where $A \subseteq \mathcal{A}$ and $|A|$ is finite $\}$. Further, for any $A \in \mathcal{A}$ let $A^0 := A$ and $A^1 := A^c$. This notation will allow us to write $\mathcal{A}^h$ for the term $\bigcap\{A^{h(A)} \mid A \in dom(h)\}$, where $h \in FF(\mathcal{A})$. We will refer to the sets of the form $\mathcal{A}^h$ as boolean combinations. Now Definition 2.1 can be simplified to: "A family $\mathcal{A}$ is called independent if for all $h \in FF(\mathcal{A})$, we have that $\mathcal{A}^h$ is infinite".

Another simplification that might be important later will be the following notation.

Definition 2.2. [7]

- Let $A, B \subseteq \omega$. We say that A is almost contained in B, denoted by $A \subseteq^* B$ if the set $A \setminus B$ is finite.
- Let $F \subseteq \mathcal{P}(\omega)$. A set $G \subseteq w$ is a pseudo-intersection of $\mathcal{F}$ if $G \subseteq^* F$ for all $F \in \mathcal{F}$.

For the rest of this section the independence number will be compared to other cardinals. The first observation will be that it is bounded from above by the cardinality of the real line. The result is due to Kontorovic and Fichtenholz, but we give a different proof than their original one.

Theorem 2.1. [7] There exists a maximal independent family of size $\mathfrak{c}$.

Proof. [10] Let $I := \{(n, A) \mid A \subseteq \mathcal{P}(n)\}$. We want to construct a family of subsets of $\mathfrak{c}$ many subsets of I, such that every finite Boolean combination of those is countably infinite. Since I is a countable set any bijection $I \to \omega$ then gives us an independent family in ω of size 2^ω. This suffices to prove the claim, as this independent family is contained in a maximal independent family, which will be showed later. However any family of subsets of ω cannot have size larger than the one of its powerset which is 2^ω .

Back to the construction: for each $X \subseteq \omega$, we define $B_X := \{(n, A) \in I \mid X \cap n \in A\}$. To show that $\mathcal{F} := \{B_X \mid X \subseteq \omega\}$ is the desired independent family, one has to show two things. Firstly, for all $X \neq X'$, we want to have that $B_X \neq B_{X'}$. Secondly for $S, T \subseteq \mathcal{P}(\omega)$, such that $S \cap T = \emptyset$ and both S and T are finite, it should hold that $(\bigcap_{X \in S} B_X) \cap (\bigcap_{Y \in T} (I \setminus B_Y))$ is infinite.

For the first statement let $X \neq X'$. Then there is some m such that $m \in X$ and $m \notin X'$ (otherwise switch the role of X and X'). Choose some $n > m$. Then $X \cap n \neq X' \cap n$. So $(n, \{X \cap n\})$ is an element of B_X, but it is not one of $B_{X'}$. Hence $B_X \neq B_{X'}$.

To show the second claim let S, T be finite, disjoint subsets of $\mathcal{P}(\omega)$. For any pair (X, Y) such that $X \in S$ and $Y \in T$ let $n_{X,Y}$ be a witness to the fact that $X \neq Y$. So $n_{X,Y}$ is an element of one of the two sets X,Y but not of both. Take $N := \max\{n_{X,Y} \mid X \in S, B \in Y\} + 1$. For all $n > N$, define $A_n := \{X \cap n \mid X \in S\}$. By construction for all $n > N$ (n, A_n) is an element of B_X for $X \in S$ and it is not an element of B_Y for $Y \in T$. So $\{(n, A_n) \mid n > N\} \subseteq (\bigcap_{X \in S} B_X) \cap (\bigcap_{Y \in T} (I \setminus B_Y))$ and since the smaller of this two sets is infinite, so is the larger one.

□

We can also bound the independence number from below. To do so we will first introduce some other cardinal characteristics of the continuum and then formulate some inequalities.

Definition 2.3. [7]

- For two functions $f, g \in^\omega \omega$, we say that g dominates f, denoted by $f \leq^* g$ if there exists some $N \in \omega$ such that for all $n \geq N$ it holds that $f(n) < g(n)$.
- A family $\mathcal{F} \subseteq^\omega \omega$ is called dominating if for every $f \in^\omega \omega$ there exists a $g \in f$ such that $f \leq^* g$.
- The dominating number $\mathfrak{d}$ is the minimum cardinality of a dominating family.

We know that $\omega < \mathfrak{d}$, since for any countable family $\{f_n\}_{n\in\omega}$ of functions $\omega \to \omega$, we can find a function $g \in^\omega \omega$ such that $g \not\leq^* f_n$ for all $n \in \omega$. For instance one can define such a function g by $g(n) := \max_{d \leq n}\{f_d(n)\} + 1$ for all $n \in \omega$ to see that.

One important result is the following.

Theorem 2.2. [7] $\mathfrak{d} \leq \mathfrak{i}$.

In order to prove this statement we first need a lemma.

Lemma 2.1. (similar to [2]) Let $\mathcal{C} := (C_n)_{n<\omega}$ be an almost decreasing chain and let $\mathcal{A}$ be a family of less than $\mathfrak{d}$ many sets such that for every $n \in \omega$ and every $A \in \mathcal{A}$ it holds that $|A \cap C_n| = \omega$. Then $\mathcal{C}$ has a pseudo-intersection which intersects every element of $\mathcal{A}$ on an infinite set.

Proof. At first observe that one can replace every set C_n by $C'_n := \bigcap\limits_{i \leq n} C_i$ and the new family $\mathcal{C}' = (C'_n)_{n<\omega}$ and $\mathcal{A}$ still fulfil the assumptions of the lemma. Furthermore if B is a pseudo-intersection of $\mathcal{C}'$ it is also one of $\mathcal{C}$. So assume without loss of generality that $C_{n+1} \subseteq C_n$ for every $n \in \omega$. Let $f : \omega \to \omega$ be a function. Then $B_f := \bigcup\limits_{n \in \omega} (C_n \cap f(n))$ is a pseudo-intersection of $\mathcal{C}$. Now we want to choose a suitable fuction f such that $B_f \cap A$ is infinite for every $A \in \mathcal{A}$.

In order to achieve this for every $A \in \mathcal{A}$ let $f_A : \omega \to \omega$ be defined such that $f_A(n)$ is the n-th element of the increasing enumeration of $A \cap C_n$. Since $|\mathcal{A}| < \mathfrak{d}$ there is $h : \omega \to \omega$ which is not dominated by any of the functions f_A. Now B_h is a valid choice for the pseudo-intersetion, since for every $A \in \mathcal{A}$ the set $I := \{n : f_A(n) < h(n)\}$ is unbounded. So we get that $B_h \cap A = \bigcup\limits_{n \in \omega} (C_n \cap h(n) \cap A) \supseteq \bigcup\limits_{n \in I} (C_n \cap A \cap h(n)) \supseteq \{f_A(n) \mid n \in I\}$. The last set is unbounded by definition of f_A and thus so is B_h. □

Now back to the proof of the previous theorem.

Proof. (similar to [2]) Let $\mathcal{I}$ be an independent family of size less than $\mathfrak{d}$. We want to show that $\mathcal{I}$ is not maximal. To do so choose a countable subset $\mathcal{D} := (D_n)_{n<\omega}$ of $\mathcal{I}$. For $f : \omega \to \{0, 1\}$ define $\mathcal{C}_f := (C_n)_{n<\omega}$ by $C_n := \bigcap_{k \leq n} D_n^{f(n)}$ and let $\mathcal{A} := \{(\mathcal{I} \setminus \mathcal{D})^h \mid h \in FF(\mathcal{I} \setminus \mathcal{D})\}$. Now $|\mathcal{A}| < \mathfrak{d}$ and $\mathcal{C}_f$ is a decreasing chain of sets that intersect every element of $\mathcal{A}$ on an infinite set. By the previous lemma there is a pseudo-intersection B_f of $\mathcal{C}_f$ such that $A \cap B_f$ is infinite for every $A \in \mathcal{A}$. For every two distinct $f, g : \omega \to 2$ it holds that $B_f \cap B_g$ is finite. We can also assume without loss of generality that they are disjoint. Let $X \subseteq {}^{\omega}2$ be a dense subset, such that ${}^{\omega}2 \setminus X$ is also dense. Define $B := \bigcup\limits_{f \in X} B_f$. Then for every $h \in FF(\mathcal{I})$ it holds that $\mathcal{I}^h \cap B$ and $\mathcal{I}^h \cap (\omega \setminus B)$ are infinite. This then implies that $\mathcal{I}$ is not maximal. Since $\omega \setminus B \supseteq \bigcup\limits_{f \in {}^{w}2 \setminus X} B_f$, we show the statement only for B and for $\omega \setminus B$ it can be done analogously.

Take $h \in FF(\mathcal{I})$ and let $h' := h \restriction (dom(h) \setminus \mathcal{D})$. Then we have that $\mathcal{I}^h \cap B = (\mathcal{I} \setminus \mathcal{D})^{h'} \cap (\bigcap\{D_n^{h(n)} \mid D_n \in dom(h)\}) \cap B \supseteq (\mathcal{I} \setminus \mathcal{D})^{h'} \cap C_n \cap B_h$, but the last set is infinite by the construction of B_h which proves the statement. □

2.2 Dense maximality and selective independent families

If $\mathcal{A} \subseteq [\omega]^\omega$ is a maximal independent family then there exists some $X \in [\omega]^\omega \setminus \mathcal{A}$ such that for some $h \in FF(\mathcal{A})$ the set $\mathcal{A}^h \cap X$ is finite. Then h is a witness to the fact that $\mathcal{A} \cup \{X\}$ is not an independent family. In this chapter we want to consider special maximal independent families which have the property that for every set not in the family the set of witnesses is dense in $FF(\mathcal{A})$. [9]

For the rest of the thesis we have to fix some notation. For $\mathcal{A} \subseteq [\omega]^\omega$ any $h, h' \in FF(\mathcal{A})$ we write $h' \supseteq h$ and say that h' extends h if $\mathrm{dom}(h) \subseteq \mathrm{dom}(h')$ and $h' \restriction \mathrm{dom}(h) = h$.

Definition 2.4. [7] A maximal independent family $\mathcal{A}$ is called densely maximal if for all $X \in [\omega]^\omega \setminus \mathcal{A}$ and for all $h \in FF(\mathcal{A})$ there exists some $h' \in FF(\mathcal{A})$ such that $h' \supseteq h$ and one of the sets $\mathcal{A}^{h'} \cap X$ or $\mathcal{A}^{h'} \cap (\omega \setminus X)$ is finite.

Note that one may equivalently rephrase the above definition and request that for every X and every $h \in FF(\mathcal{A})$ there is an extension h' of h that one of the sets $\mathcal{A}^{h'} \cap X$ or $\mathcal{A}^{h'} \cap (\omega \setminus X)$ has to be empty.

Two other very important notions one needs to define are those of the density filter and the density ideal of an independent family.

Definition 2.5 (Density Filter and Dual Ideal)**.** [7] Let $\mathcal{A}$ be an independent family.

- The density filter of $\mathcal{A}$ is the set $\mathrm{fil}(\mathcal{A}) := \{X \subseteq \omega : \forall h \in FF(\mathcal{A}) \exists h' \supseteq h \text{ s.t } \mathcal{A}^{h'} \subseteq^* X\}$.
- The dual ideal of $\mathrm{fil}(\mathcal{A})$ is the set $\mathrm{id}(\mathcal{A}) := \{X \subseteq w : \omega \setminus X \in \mathrm{fil}(\mathcal{A})\}$. It is also called the density ideal on $\mathcal{A}$.

One has to verify that the above terminology is not misleading, which means that $\mathrm{fil}(\mathcal{A})$ is indeed a filter. But closedness with respect to supersets and finite intersections can be checked easily. By the above definition one can also introduce the density ideal as $id(\mathcal{A}) = \{X \subseteq \omega \mid \forall h \in FF(\mathcal{A}) \exists h' \supseteq h \text{ such that } \mathcal{A}^{h'} \cap X \text{ is finite }\}$. Similar to before the word "finite" in the previous statement can be equivalently replaced by "empty".

An equivalent condition to being densely maximal is stated in the following lemma. A generalisation of this result to the higher Baire space will be proved in Chapter 5.

Lemma 2.2. [7] Let $\mathcal{A}$ be a maximal independent family. Then the following are equivalent:

1. The family $\mathcal{A}$ is densely maximal.
2. For every $h \in FF(\mathcal{A})$ and for every $X \subseteq \mathcal{A}^h$ either $\mathcal{A}^h \setminus X$ is in the density ideal or there exists $h' \supseteq h$ such that $\mathcal{A}^{h'} \subseteq \mathcal{A}^h \setminus X$.
3. For every set $X \in \mathcal{P}(\omega) \setminus fil(\mathcal{A})$ there is $h \in FF(\mathcal{A})$ such that $X \subseteq \omega \setminus \mathcal{A}^h$.

Proof. We will show that 1 $\Leftrightarrow$ 2 and 2 $\Leftrightarrow$ 3. The first implication that will be shown is 1 implies 2.

So assume that $\mathcal{A}$ is densely maximal and let $h \in FF(\mathcal{A})$ and $X \subseteq \mathcal{A}^h$. Further assume that $\mathcal{A} \setminus X \notin id(\mathcal{A})$. We have to show that there is h' extending h such that $\mathcal{A}^{h'} \subseteq \mathcal{A}^h \setminus X$. Since $\mathcal{A}^h \setminus X$ is not in the density ideal there exists $h_1 \in FF(\mathcal{A})$ such that for all h' extending h_1 the set $(\mathcal{A}^h \setminus X) \cap \mathcal{A}^{h'} = \mathcal{A}^h \cap (\omega \setminus X) \cap \mathcal{A}^{h'}$ is infinite. Any such h_1 must be compatible with h, since otherwise already $\mathcal{A}^h \cap \mathcal{A}^{h_1}$ would be empty. So we can fix such an h_1 and take a common extension h_2 of h and h_1. Now for any h' extending h_2 the set $\mathcal{A}^{h'} \cap (\omega \setminus X)$ is infinite. However, since $\mathcal{A}$ is densely maximal h_2 must have an extension h' such that $\mathcal{A}^{h'} \cap X$ is empty. But this means that $\mathcal{A}^{h'} = \mathcal{A}^{h'} \cap (\omega \setminus X) \subseteq \mathcal{A}^h \cap (\omega \setminus X)$.

For the backwards direction assume that 2 holds and let us show 1. To do so let $X \in FF(\mathcal{A})$ and $h \in FF(\mathcal{A})$. We want to find some extension h' of h such that either $\mathcal{A}^{h'} \cap X$ or $\mathcal{A}^{h'} \cap (\omega \setminus X)$ is finite. Take $Y := \mathcal{A}^h$. If $\mathcal{A}^h \setminus Y \in id(\mathcal{A})$, then there is some $h' \supseteq h$ such that $\mathcal{A}^{h'} \cap (\mathcal{A}^h \setminus Y) = \mathcal{A}^{h'} \cap (\omega \setminus X)$ is finite. Otherwise, by 2 there is some h' extending h such that $\mathcal{A}^{h'} \subseteq \mathcal{A}^h \setminus Y$. Since $\mathcal{A}^h \setminus Y = \mathcal{A}^h \setminus X$, we obtain that $\mathcal{A}^{h'} \subseteq \omega \setminus X$ and hence $\mathcal{A}^{h'} \cap X = \emptyset$. Since $h \in FF(\mathcal{A})$ and the set X were chosen arbitrarily, this proves the dense maximality of $\mathcal{A}$.

Next, let us show that 2 implies 3. To do so, assume 2 holds and let $X \in \mathcal{P}(\omega) \setminus fil(\mathcal{A})$. We want to find $h \in FF(\mathcal{A})$ as in 3. Since X is not in the density filter there is some $h_1 \in FF(\mathcal{A})$ such that for any h' that extends it the set $\mathcal{A}^{h'} \setminus X$ is infinite. Let $Y := \mathcal{A}^{h_1} \cap X$. Then $\mathcal{A}^{h_1} \setminus Y$ is not an element of the density ideal by the previous observation. Since 2 holds this means that there is some $h \supseteq h_1$ such that $\mathcal{A}^h \subseteq \mathcal{A}^{h_1} \setminus Y = \mathcal{A}^h \setminus X$. Hence $\mathcal{A}^h \subseteq \omega \setminus X$ and so $X \subseteq \omega \setminus \mathcal{A}^h$ as requested.

Finally, to show that 2 follows from 3 take $h \in FF(\mathcal{A})$ and $X \subseteq \mathcal{A}^h$. Let $Y := \mathcal{A}^h \setminus X$. If $\omega \setminus Y$ belongs to the density filter, then by definition of the density ideal $Y \in id(\mathcal{A})$. Now, if $\omega \setminus Y \notin fil(\mathcal{A})$, then by 3 there is some $h_1 \in FF(\mathcal{A})$ such that $\omega \setminus Y \subseteq \omega \setminus \mathcal{A}^{h_1}$. By definition of Y this means that $\mathcal{A}^{h_1} \subseteq \mathcal{A}^h \setminus X$. So $\mathcal{A}^{h_1} \subseteq \mathcal{A}^h$, which is only possible if h_1 and h agree on their common domain and hence are compatible. Let h' be a common extension of h_1 and h. Then $\mathcal{A}^{h'} \subseteq \mathcal{A}^{h_1} \subseteq \mathcal{A}^h \setminus X$. So in either case one of the two conditions in 2 is satisfied. $\square$

With that theorem one can immediately state another equivalent condition to dense maximality. But to do let us first fix some notation.

Definition 2.6. [6] Let $\mathcal{A} \subseteq \mathcal{P}(\omega)$, then

- The set $\langle \mathcal{A} \rangle_{dn}$ is defined as $\langle \mathcal{A} \rangle_{dn} := \{X \subseteq w \mid \exists A \in \mathcal{A} \text{ such that } (X \subseteq^* A)\}$.
- The set $\langle \mathcal{A} \rangle_{up}$ is defined as $\langle \mathcal{A} \rangle_{up} := \{X \subseteq w \mid \exists A \in \mathcal{A} \text{ such that } (A \subseteq^* X)\}$.

An immediate consequence of the previous theorem is the following.

Corollary 2.1. [7] An independent family $\mathcal{A}$ is densely maximal if and only if $\mathcal{P}(\omega) = fil(\mathcal{A}) \cup \langle \{\omega \setminus \mathcal{A}^h \mid h \in FF(\mathcal{A})\} \rangle_{dn}$.

Proof. To show this let $\mathcal{A}$ be an independent family. Further one can assume that $\mathcal{A}$ is infinite, since any finite family satisfies none of the two above conditions. If $\mathcal{A}$ is densely maximal, then 3 of Lemma 2.2 holds. Hence for any set X that is not in the density filter, there is some $h \in FF(\mathcal{A})$ such that $X \subseteq \omega \setminus \mathcal{A}^h$. But then by definition $X \in \{\omega \setminus \mathcal{A}^h \mid h \in FF(\mathcal{A})\}\rangle_{dn}$.

On the other hand, if $\mathcal{P}(\omega) = fil(\mathcal{A}) \cup \langle\{\omega \setminus \mathcal{A}^h \mid h \in FF(\mathcal{A})\}\rangle_{dn}$, then we can show, that 3 of Lemma 2.2 is fulfilled. Let $X \in \mathcal{P}(\omega) \setminus fil(\mathcal{A})$. Then $X \in \langle\{\omega \setminus \mathcal{A}^h \mid h \in FF(\mathcal{A})\}\rangle_{dn}$, which means there is some $h \in FF(\mathcal{A})$, such that $X \subseteq^* \omega \setminus \mathcal{A}^h$. We can now construct an extension h' of h such that $X \subseteq \omega \setminus \mathcal{A}^{h'}$. To do so, note that $\mathcal{A}^h \cap X$ is finite and let $n := |\mathcal{A}^h \cap X|$. Further let $\{x_0, \dots x_{n-1}\}$ be an enumeration of $\mathcal{A}^h \cap X$. Now, we construct a sequence $\{h_i\}_{i \leq n} \subseteq FF(\mathcal{A})$ as follows: Let $h_0 := h$. If h_i has already been defined for $0 \leq i < n$ choose $A_i \in \mathcal{A} \setminus dom(h_i)$. Now define $h_{i+1} \in FF(A)$ by letting $dom(h_i + 1) := dom(h) \cup \{A_i\}$, $h_{i+1}(A) = h_i(A)$ for all $A \in dom(h)$. Further, let $h_{i+1}(A_i) = 1$ if $x_i \in A_i$ and otherwise $h_{i+1}(A_i) = 0$. By construction $x_i \notin \mathcal{A}^{h_j}$ for all $0 \leq i < j \leq n$. So especially $x_i \notin \mathcal{A}^{h_n}$ for all $i < n$. By construction $h_n \subseteq h$ and $(\mathcal{A}^{h_n} \cap X) \subseteq (\mathcal{A}^h \cap X) \setminus (\{x_0, \dots x_{n-1}\}) = \emptyset$. So if we take $h' := h_n$, then $X \subseteq \omega \setminus \mathcal{A}^{h'}$ as requested. $\square$

Another equivalent condition for an independent family to being densely maximal is again some property of the density filter. In order to formulate it, the diagonalisation filter has to be introduced.

Definition 2.7. [7] Let $\mathcal{A}$ be an independent family and let $\mathcal{F}$ be a filter on $\mathcal{P}(\omega)$. Then $\mathcal{F}$ is called a diagonalisation filter for $\mathcal{A}$ if and only if:

1. For all $F \in \mathcal{F}$ and all $h \in FF(\mathcal{A})$ it holds that the set $\mathcal{A}^h \cap F$ is infinite.
2. If $\mathcal{G}$ is another filter with that property and $\mathcal{F} \subseteq \mathcal{G}$, then already $\mathcal{F} = \mathcal{G}$.

For any maximal independent family $\mathcal{A}$, the trivial filter $\{\omega\}$ satisfies 1 above. So, using Zorn's Lemma, one can see that for any independent family there is a diagonalisation filter. In fact, there is an even larger filter, that is contained in every diagonalisation filter for $\mathcal{A}$.

Lemma 2.3. [7] Suppose $\mathcal{F}_\mathcal{A}$ is a diagonalisation filter for the independent family $\mathcal{A}$. Then $fil(\mathcal{A}) \subseteq \mathcal{F}_\mathcal{A}$.

Proof. To show this, we would like to show that any set which is not in the diagonalisation filter is also not an element of the density filter. So let $X \subseteq \omega$ be such that $X \notin \mathcal{F}_\mathcal{A}$. Now define $\mathcal{G} := \{A \subseteq \omega \mid \exists F \in \mathcal{F}_\mathcal{A}(A \supseteq F \cap X)\}$. By construction $\mathcal{F}_\mathcal{A} \subseteq \mathcal{G}$. In addition $\mathcal{G}$ is upwards closed and it contains the set X. It is also closed under finite intersection. To see this let $A, B \in \mathcal{G}$. Then there are sets F_A and F_B in $\mathcal{F}_\mathcal{A}$, such that $A \supseteq X \cap F_A$ and $B \supseteq X \cap F_B$. Hence $A \supseteq F_A \cap F_B \cap X$ and $B \supseteq F_A \cap F_B \cap X$. So $A \cap B$ is a superset of $F_A \cap F_B \cap X$. Since $\mathcal{F}_\mathcal{A}$ is a filter $F_A \cap F_B \in \mathcal{F}_\mathcal{A}$. Hence $A \cap B \in \mathcal{G}$.

However by the maximality of $\mathcal{F}_\mathcal{A}$, the set $\mathcal{G}$ must not satisfy both of the properties in Definition 2.7. This means that either

- $\mathcal{G}$ is not a filter
- or there is some $A \in \mathcal{G}$ and some $h \in FF(\mathcal{A})$ such that $\mathcal{A}^h \cap A$ is finite.

We want to show that in either of the above two cases $X \notin fil(\mathcal{A})$.

Since $\mathcal{G}$ is nonempty, upwards closed and closed under finite intersections, the only possibility, how it is not a filter is that $\emptyset \in \mathcal{G}$. This means there is some $F \in \mathcal{F}$ such that $X \cap F = \emptyset$. Pick such a set F and let $h \in FF(\mathcal{A})$. What we want to show is that for no extension h' of h the set $\mathcal{A}^{h'}$ is almost contained in X. This then proves that X does not belong to the density filter. So consider any $h' \in FF(\mathcal{A})$ that extends h. We know by definition of $\mathcal{F}_{\mathcal{A}}$ that $\mathcal{A}^{h'} \cap F$ is an infinite set. Also $\mathcal{A}^{h'} \cap F = \mathcal{A}^{h'} \cap F \cap X \dot{\cup} (\mathcal{A}^{h'} \cap F) \setminus X$. The first part of this disjoint union is the empty set and so $\mathcal{A}^{h'} \cap F = (\mathcal{A}^{h'} \cap F) \setminus X \subseteq \mathcal{A}^{h'} \setminus X$. But this means that $\mathcal{A}^{h'} \setminus X$ is infinite and $\mathcal{A}^{h'} \nsubseteq^* X$.

If on the other hand there is some $A \in \mathcal{G}$ and some $h \in FF(\mathcal{A})$ such that $\mathcal{A}^h \cap A$ is finite, then there is $F \in \mathcal{F}_{\mathcal{A}}$ such that $F \cap X \subseteq A$. So fix such a set A and corresponding $h \in FF(\mathcal{A})$ and $F \in \mathcal{F}_{\mathcal{A}}$. Then $\mathcal{A}^h \cap X \cap F$ is finite. Now assume by contradiction that X was an element of the density filter. Then there is $h' \supseteq h$ such that $\mathcal{A}^{h'} \subseteq^* X$. Now we know that $\mathcal{A}^{h'} \cap F = \mathcal{A}^{h'} \cap F \cap X \dot{\cup} (\mathcal{A}^{h'} \cap F) \setminus X$. By definition of $\mathcal{F}_{\mathcal{A}}$ the set on the left-hand-side of the equation is infinite. By our choice of F and h' both sets on the right-hand-side are finite, as $\mathcal{A}^{h'} \cap F \cap X \subseteq \mathcal{A}^h \cap F \cap X$ and $(\mathcal{A}^{h'} \cap F) \setminus X \subseteq \mathcal{A}^{h'} \setminus X$. But this is a contradiction and so $X \notin fil(\mathcal{A})$. □

This now leads to the final equivalent condition, when an independent family is densely maximal.

Theorem 2.3. [7] An independent family $\mathcal{A}$ is densely maximal if and only if its density filter is a diagonalisation filter.

Proof. Using Corollary 2.1 it suffices to show that $fil(\mathcal{A})$ is a diagonalisation filter if and only if $\mathcal{P}(\omega) = fil(\mathcal{A}) \cup \langle\{\omega \setminus \mathcal{A}^h \mid h \in FF(\mathcal{A})\}\rangle_{dn}$.

For the implication "from left to right" assume that $fil(\mathcal{A})$ is a diagonalisation filter and let $X \notin \langle\{\omega \setminus \mathcal{A}^h \mid h \in FF(\mathcal{A})\}\rangle_{dn}$. We want to show that $X \in fil(\mathcal{A})$. Since X does not belong to $\langle\{\omega \setminus \mathcal{A}^h \mid h \in FF(\mathcal{A})\}\rangle_{dn}$ for every $h \in FF(\mathcal{A})$ it holds that $X \nsubseteq^* \omega \setminus \mathcal{A}^h$. So $\mathcal{A}^h \cap X$ is infinite for every $h \in FF(\mathcal{A})$. Using Zorn's Lemma, we can find a diagonalisation filter $\mathcal{F}_{\mathcal{A}}$ which contains X. By Lemma 2.3 then $fil(\mathcal{A}) \subseteq \mathcal{F}_{\mathcal{A}}$. But $fil(\mathcal{A})$ is a diagonalisation filter and so $fil(\mathcal{A}) = \mathcal{F}_{\mathcal{A}}$. This proves that $X \in fil(\mathcal{A})$.

For the other implication assume $\mathcal{P}(\omega) = fil(\mathcal{A}) \cup \langle\{\omega \setminus \mathcal{A}^h \mid h \in FF(\mathcal{A})\}\rangle_{dn}$. We want to show that $fil(\mathcal{A})$ is a diagonalisation filter. By the previous lemma, we know that the density filter is contained in a diagonalisation filter and so it satisfies property 1 of the definition of a diagonalisation filter. So it remains to show the maximality of $fil(\mathcal{A})$. To do so it suffices to show that for every set $X \notin fil(\mathcal{A})$, there is some $h \in FF(\mathcal{A})$ such that $X \cap \mathcal{A}^h$ is finite. However this is true because for every set X that is not in the density filter there is $h \in FF(\mathcal{A})$ such that $X \subseteq^* \omega \setminus \mathcal{A}^h$. □

For the next chapter we want to introduce selective independent families, which are a special type of densely maximal independent families. To do so, we need two more definitions.

Definition 2.8. [16] Let $\mathcal{F} \subseteq \mathcal{P}(\omega)$ be a filter.

- $\mathcal{F}$ is called a p-set if every subset $\mathcal{F}_0 \subseteq \mathcal{F}$ which is at most countable has a pseudo-intersection K in $\mathcal{F}$.
- It is called a q-set if for every partition $\mathcal{E} = \{E_n\}_{n \in w}$ of ω into finite sets, there is a semi-selector K for E in $\mathcal{F}$, that is $|K \cap E_n| \leq 1$ for all $n \in \omega$.
- The filter $\mathcal{F}$ is called Ramsey if it is a p-set and a q-set.

We will come back to this definition for the general case, however there it will have to be reformulated.

With this definition we can now define a selective independent family.

Definition 2.9 (Selective independent family)**.** [16] An independent family is called selective if it is densely maximal and its density filter is Ramsey.

The goal for the rest of this section will be to show that under the assumption of the Continuum Hypothesis there exists a selective maximal independent family. To show that, such a family $\mathcal{A}$ will be constructed. In the next chapter we will show that this family is indestructible under countable support iterations of Sacks forcing. The construction will be similar to the one, that will be done in higher Baire space and hence some differences will be pointed out later.

In order to obtain a selective independent family, we will start with our ground model V of which we assume that it is a transitive model of ZFC and also satisfies CH. With the help of a forcing poset $\mathbb{P}$, the desired independent family will be in a $\mathbb{P}$-generic extension of V. To start, let us first define that forcing poset and the family which should be selective.

Definition 2.10. [6] Let $\mathbb{P} := \{(\mathcal{A}, A) \mid \mathcal{A} \subseteq \mathcal{P}(\omega)$ is a countable independent family $A \subseteq \omega$ and for all $h \in FF(\mathcal{A}) : |\mathcal{A}^h \cap A| = \omega\}$. A condition $(\mathcal{A}, A) \in \mathbb{P}$ is stronger than another condition $(\mathcal{B}, B)$, denoted $(\mathcal{A}, A) \leq (\mathcal{B}, B)$ as usual if and only if $\mathcal{A} \subseteq \mathcal{B}$ and $A \subseteq^* B$.

Let G be a $V, \mathbb{P}$-generic filter, then we define:

- $\mathcal{A}_G := \bigcup\limits_{\mathcal{A} \in S} \mathcal{A}$, where S is the set of all independent families $\mathcal{A}$, such that there is some $A \subseteq \omega$ with $(\mathcal{A}, A) \in G$ and
- $\mathcal{F}_G := \{A \mid$ there exists an independent family $\mathcal{A}$ such that $(\mathcal{A}, A) \in G\}$

Since V is a countable model such a generic filter G always exists. To show, that the set $\mathcal{A}_G$ is as wanted, we have to show that it is in fact an independent family and that it is densely maximal. Then we have to show that $fil(\mathcal{A}_G)$ is a p-set and a q-set.

Lemma 2.4. [6] The family $\mathcal{A}_G$ is an independent family.

Proof. This follows easily from the fact that for any $h \in FF(\mathcal{A}_G)$ there is some $\mathcal{A} \in G$ such that $dom(h) \subseteq \mathcal{A}$, hence $\mathcal{A}_G^h = \mathcal{A}^{h|\mathcal{A}}$ from which we know that it is an infinite set. $\square$

To show the other properties, we would first like to show a lemma about the forcing poset, which will help a lot.

Lemma 2.5. [6] $\mathbb{P}$ is countably closed and $\aleph_2$-cc.

Proof. Since CH holds in V, there are $\aleph_1$ many subsets of ω. Then all countable independent families are contained in $[\mathcal{P}(\omega)]^\omega$, which is a set of cardinality $(2^\omega)^\omega = 2^{\omega \cdot \omega} = 2^\omega = \omega_1$. So the poset $\mathbb{P}$ is of the same cardinality as $w_1 \times w_1$, which is w_1 and so it is clearly $\aleph_2$-cc.

To show the ω_1-closure, let $\{(\mathcal{A}, A_n)\}_{n\in\omega}$ be a decreasing sequence in $\mathbb{P}$. We can assume that $A_{n+1} \subseteq A_n$, since for any $n \in \omega$ the set $A_{n+1} \setminus \bigcap\limits_{i\leq n} A_i$ is finite and hence the set $\bigcap\limits_{i\leq n} A_n$ still meets every boolean combination of $\mathcal{A}_n$ on an infinite set. If we find a stronger condition for the constructed strictly decreasing sequence, this condition is also stronger than any condition in the original sequence.

Clearly $(\mathcal{A}, A)$, where $\mathcal{A} := \bigcup\limits_{n\in\omega} \mathcal{A}_n$ and A is a pseudo-intersection of the sets A_i for $n \in \omega$, would be a suitable stronger condition. So we just have to construct the pseudo-intersection A. To do so, let $\{h_n\}_{n\in\omega}$ be an enumeration of $FF(\mathcal{A})$, such that every $h \in FF(\mathcal{A})$ occurs infinitely often. Note that for every $h \in FF(\mathcal{A})$ there is some n such that $dom(h) \subseteq \mathcal{A}_n$. Since $A_n \subseteq A_i$ for all $i < n$ and $h \in FF(\mathcal{A}_m)$ for all $m > n$, we have that $\mathcal{A}^h = \mathcal{A}_n^h$ meets every set A_j on an infinite set. Choose $a_0 \in \mathcal{A}^{h_0} \cap A_0$. Now proceed inductively and choose $a_{n+1} \in (\mathcal{A}^{h_n+1} \cap A_{n+1}) \setminus \{a_0, \dots a_n\}$. Define $A := \{a_n \mid n \in \omega\}$. By construction $A_n \setminus A = \{a_0, \dots a_{n-1}\}$ for each $n \in \omega$, so A is indeed a pseudo-intersection of the sets $\{A_n\}_{n\in\omega}$. Since $a_n \in \mathcal{A}^{h_n}$, $a_i \neq a_j$ for $i \neq j$ and every $h \in FF(\mathcal{A})$ occured infinitely often in the enumeration, the set A indeed meets every boolean combination on an infinite set. $\square$

Before showing that the independent family $\mathcal{A}_G$ is densely maximal a few intermediate steps will be needed. The key step for showing dense maximality will be a density argument but to prove this, we will use two technical lemmata.

Lemma 2.6. [6] Let $Y \subseteq w$, $(\mathcal{A}, A)$ be a condition in $\mathbb{P}$ and $h \in FF(\mathcal{A})$. Then there is a subset B of A and an extension h' of h, such that the set $\mathcal{A}^{h'} \cap B$ is either contained in Y or in $\omega \setminus Y$ and in addition $(\mathcal{A}, B) \leq (\mathcal{A}, A)$.

Proof. (analogous to the uncountable in [8]) If there exists an extension $h' \supseteq h$, such that $\mathcal{A}^{h'} \cap A \cap Y$ is finite, then one can choose $B := \mathcal{A}^{h'} \cap A \cap (\omega \setminus Y) \cup (A \setminus \mathcal{A}^{h'})$. By construction $B \subseteq A$ and $A \setminus B$ is a finite set. So B meets every Boolean combination of $\mathcal{A}$ on an infinite set and $\mathcal{A}^{h'} \cap B \subseteq (\omega \setminus Y)$.

Similarly, if $\mathcal{A}^{h'} \cap A \cap (\omega \setminus Y)$ is finite for some h' extending h, then one can choose B to be the set $\mathcal{A}^{h'} \cap A \cap Y \cup (A \setminus \mathcal{A}^{h'})$.

Finally, if for all $h' \supseteq h$ both $\mathcal{A}^{h'} \cap A \cap Y$ and $\mathcal{A}^{h'} \cap A \cap (\omega \setminus Y)$ are unbounded, then one may choose $B := (\mathcal{A}^h \cap A \cap Y) \cup (A \setminus \mathcal{A}^h)$. To see that this set B is as desired let $h' \in FF(\mathcal{A})$. If h and h' have a common extension h_1, then B is a superset of the unbounded set $(\mathcal{A}^{h_1} \cap A \cap Y)$ and hence it meets $\mathcal{A}^{h'}$ on an infinite set. If h and h' are incompatible, then $\mathcal{A}^h \cap \mathcal{A}^{h'} = \emptyset$ and hence $A \cap \mathcal{A}^{h'} \subseteq A \setminus \mathcal{A}^h \subseteq B$. So $\mathcal{A}^{h'} \cap B$ is infinite. $\square$

The next lemma will be an observation on how to increase the independent family in a condition by one element.

Lemma 2.7. [6] For every condition $(\mathcal{A}, A) \in \mathbb{P}$ there is some $B \subseteq A$ such that $(\mathcal{A} \cup \{B\}, A) \leq (\mathcal{A}, A)$.

Proof. (again as in [8]) We only have to show that we can choose B such that $(\mathcal{A} \cup \{B\}, A)$ is a condition in the forcing poset. To do so let $(h_n)_{n \in \omega}$ be an enumeration of $FF(\mathcal{A})$, such that every $h \in FF(\mathcal{A})$ occurs infinitely often. Choose two distinct elements $a_{0,0}$ and $a_{0,1}$ of $A \cap \mathcal{A}^{h_0}$. Proceed iteratively and for each $n \in \omega$ choose two distinct elements $a_{n+1,0}$ and $a_{n+1,1}$ out of $A \cap \mathcal{A}^{h_{n+1}} \setminus (\{a_{i,0} \mid i \leq n\} \cup \{a_{i,1} \mid i \leq n\})$. This is possible, since $\mathcal{A}^{h_n} \cap A$ is infinite for each $n \in \omega$. Now choose $B := \{a_{n,0} \mid n \in \omega\}$. By construction both B and $\omega \setminus B$ meet every set $\mathcal{A}^h \cap A$ on an infinite set for each $h \in FF(\mathcal{A})$. $\square$

With the two lemmata we can prove that certain sets are dense which will prove dense maximality of $\mathcal{A}_G$.

Lemma 2.8. [6] For every $X \subseteq \omega$ the set $\mathcal{D}_X := \{(\mathcal{A}, A) \mid \forall h \in FF(\mathcal{A}) \exists h' \supseteq h \text{ such that } \mathcal{A}^{h'} \subseteq X \text{ or } \mathcal{A}^{h'} \subseteq \omega \setminus X\}$ is dense in $\mathbb{P}$.

Proof. Let $(\mathcal{A}, A) \in \mathbb{P}$ be a condition. We want to construct a condition below it. During the proof that $\mathbb{P}$ is countably closed we showed that for each decreasing sequence $(\mathcal{A}_n, A_n)$, there is some condition $(\mathcal{B}, B)$ below it, such that $\mathcal{B} = \bigcup_{n \in \omega} \mathcal{A}_n$. So if we can construct a decreasing sequence $(\mathcal{A}_n, A_n)_{n \in \omega}$ such that $(\mathcal{A}, A) = (\mathcal{A}_0, A_0)$ and for every $h \in FF(\mathcal{A}_n)$ there is an extension $h' \in \mathcal{A}_{n+1}$, such that $\mathcal{A}_{n+1}^{h'} \subseteq X$ or $\mathcal{A}_{n+1}^{h'} \subseteq \omega \setminus X$, then the condition $(\mathcal{B}, B)$, defined as before is the desired element of $\mathcal{D}_X$ which is stronger than $(\mathcal{A}, A)$.

The only thing which remains to show to construct this sequence is that for any condition $(\mathcal{A}, A)$ there is some $(\mathcal{B}, B)$, such that for all $h \in FF(\mathcal{A})$, there is some $h' \in FF(\mathcal{B})$ such that $h' \supseteq h$ and $\mathcal{B}^{h'} \subseteq X$ or $\mathcal{B}^{h'} \subseteq \omega \setminus X$. To show this pick an enumeration $(h_n)_{n \in \omega}$ of $FF(\mathcal{A})$. Using Lemma 2.6 there is some set $A_0 \subseteq A$, such that $(\mathcal{A}, A_0)$ is a condition and there is $h'' \supseteq h_0$ such that $\mathcal{A}^{h''} \cap A_0 \subseteq X$ or $\mathcal{A}^{h'} \cap A_0 \subseteq \omega \setminus X$. Then Lemma 2.7 yields a set $B_0 \subseteq A_0$, such that $(\mathcal{A} \cup \{B_0\}, A_0)$ is a condition. Let $\mathcal{A}_0 := \mathcal{A} \cup \{B_0\}$. Then $h' := h'' \cup (B_0, 0)$ is the suitable extension of h_0, such that $\mathcal{A}_0^{h_0}$ is either contained in X or in $\omega \setminus X$.

Proceed iteratively for $n \in \omega$ and use lemmata 2.6 and 2.7 to find $(\mathcal{A}_{n+1}, A_{n+1}) \leq (\mathcal{A}_n, A_n)$, such that there is an $h' \in FF(\mathcal{A}_n)$ which extends h_{n+1}, such that $\mathcal{A}_{n+1}^{h'}$ is either contained in X or in $\omega \setminus X$. Again using that $\mathbb{P}$ is countably closed one can find a condition $(\mathcal{B}, B)$ which is stronger than every $(\mathcal{A}_n, A_n)$, with the property that $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{A}_n$. Then this condition has the desired properties.

□

Now we have every ingredient to prove that the family $\mathcal{A}_G$ is densely maximal.

Proposition 2.1. [6] For any generic filter G the independent family $\mathcal{A}_G$ is densely maximal.

Proof. (similar to the one in [8]) It suffices to show that $\mathcal{A}_G$ satisfies 2 from Lemma 2.2. To do so let $h \in FF(\mathcal{A}_G)$ and $X \subseteq \mathcal{A}^h$. Now assume there is no element A in the density ideal of $\mathcal{A}_G$ such that $\mathcal{A}_G^h \setminus X \subseteq A$. This implies that $\mathcal{A}_G^h \setminus X$ cannot be in the density ideal either. So there is some $h_1 \in FF(\mathcal{A}_G)$, such that for any extension h' of h_1 the set $(\mathcal{A}_G^h \setminus X) \cap \mathcal{A}^{h'}$ is infinite.

Since $dom(h)$ is finite there is some condition $(\mathcal{A}, A) \in G$, such that $dom(h) \subseteq \mathcal{A}$. Considering $Y := \mathcal{A}^h \setminus X$ since the set $\mathcal{D}_Y$ as defined in the previous lemma, is dense there is some condition $(\mathcal{B}, B) \in G \cap \mathcal{D}_Y$. And because any two elements is G are compatible there is a condition $(\mathcal{C}, C)$ which is both stronger than $(\mathcal{A}, A)$ and $(\mathcal{B}, B)$. But this condition is again in $\mathcal{D}_Y$ and hence there is some h' extending h in $FF(\mathcal{C})$ and hence also in $FF(\mathcal{A}_G)$, such that $\mathcal{A}_G^{h'}$ is either contained in Y or in $\omega \setminus Y$. By the choice of h it is only possible that $\mathcal{A}_G^{h'} \subseteq Y = \mathcal{A}_G^h \setminus X$, which proves the condition required from the lemma.

□

For the p-set and the q-set property of the density filter we would like to use the set $\mathcal{F}_G$. The first observation will be used later.

Lemma 2.9. [6] The density filter $fil(\mathcal{A}_G)$ is generated by $\mathcal{F}_G$ and the cofinite sets.

Proof. Let Fr denote the set of all cofinite subsets of ω. Since we already showed that $\mathcal{A}_G$ is densely maximal we can use Corollary 2.1 to see that $\mathcal{F}_G \subseteq fil(\mathcal{A}_G)$ and also $Fr \subseteq fil(\mathcal{A}_G)$.

Conversely choose $A \in fil(\mathcal{A})$ arbitrarily. To show that the filter generated by $\mathcal{F}_G \cup Fr$ contains A it suffices to show that $\mathcal{F}_G$ contains a subset of A. We claim that the set $D_A := \{(\mathcal{A}, B) \in \mathbb{P} \mid B \subseteq A \text{ or there is } h \in FF(\mathcal{A}) \text{ such that } |\mathcal{A}^h \setminus A| = \omega\}$ is dense in $\mathbb{P}$. If this holds there is some pair $(\mathcal{A}, B)$ in the generic filter G, with $B \subseteq A$ because otherwise there would be some condition $(\mathcal{A}, C)$ in G such that $\mathcal{A}^h \not\subseteq^* A$, for some $h \in FF(\mathcal{A}) \subseteq FF(\mathcal{A}_G)$ which contradicts A being in the density filter of $\mathcal{A}_G$. Hence this set B is also an element of $\mathcal{F}_G$.

To show that D_A is a dense set let $(\mathcal{C}, C)$ be an arbitrary condition in $\mathbb{P}$. If some Boolean combination of $\mathcal{C}$ is not almost contained in A, then $(\mathcal{C}, C) \in D_A$. Otherwise $B := C \cap A$ meets every set $\mathcal{C}^h$ for $h \in FF(\mathcal{C})$ on an infinite set as $\mathcal{C}^h \cap C \subseteq \mathcal{C}^h \subseteq^* A$ and hence $(\mathcal{C}, B) \leq (\mathcal{C}, C)$. $\square$

Notice that if a filter $\mathcal{F}$ is generated by a set B, then $\mathcal{F} = \{A \mid \exists B_0, \dots, B_n \in B \text{ such that } A \subseteq \bigcap_{i \leq n} B_i\}$. So by the above lemma any element of $fil(\mathcal{A}_G)$ is a superset of a finite intersection of sets in $\mathcal{F}_G$ and cofinite sets. By the properties of the extension relation of $\mathbb{P}$ and G being generic for any sets $A, B \in \mathcal{F}_G$ there is some pseudo-intersection C of A and B in $\mathcal{F}_G$. Combining these two statements, for any set $A \in fil(\mathcal{A}_G)$ there is some $F \in \mathcal{F}_G$ such that $F \subseteq^* A$. This observation will now help to prove the p-set and q-set property of $fil(\mathcal{A}_G)$.

Lemma 2.10. [6] The filter $fil(\mathcal{A}_G)$ is a p-set.

Proof. (very similar to the one in [8]) We have to show that any countable subset of $fil(\mathcal{A}_G)$ has a pseudo-intersection. By the observation before, it is sufficient to show that there is a pseudo-intersection in $\mathcal{F}_G$ for every countable subset of $\mathcal{F}_G$.

Assume not, this means $V[G] \models \exists \mathcal{H} \in [\mathcal{F}_G]^\omega (\forall X \in \mathcal{F}_G (\exists H \in \mathcal{H}(X \not\subseteq^* H)))$ Let $\mathcal{H}$ be a subset of $\mathcal{F}_G$ that witnesses the previous existential formula. By the forcing theorem there is some condition $p := (\mathcal{A}, A) \in G$, such that $p \Vdash \forall X \in \mathcal{F}_G (\exists H \in \mathcal{H}(X \not\subseteq^* H))$. Now let $\{H_n\}_{n \in \omega}$ be an enumeration of the set $\mathcal{H}$. Then for each $n \in \omega$ there is a countable independent family $\mathcal{A}_i$ such that $(\mathcal{A}_i, H_i) \in G$. Since for any two conditions in G there is a condition in G that is below both of them we can construct a decreasing sequence of conditions in G as follows. Let q_0 be a condition in G, such that $q_0 \leq p$ and $q_0 \leq (\mathcal{A}_0, H_0)$. Now for each $n \in \omega$ iteratively let $q_{n+1} \in G$ be stronger than both q_n and $(\mathcal{A}_{n+1}, H_{n+1})$. Since $\mathbb{P}$ is countably closed there is some $q := (\mathcal{B}, B) \in \mathbb{P}$ such that $q \leq q_n$ for all $n \in \omega$. By the filter existence lemma, there is a generic filter G', such that $q \in G'$. Now in $V[G']$ the set B is a pseudo-intersection of the sets in $\mathcal{H}$. By the forcing theorem there is some $r \leq q \in G'$, such that r forces that $\mathcal{H}$ has a pseudo-intersection. But this is a contradiction, because $r \leq q \leq p$ and p forces the opposite. So any countable subset of $\mathcal{F}_G$ has a pseudo-intersection in $\mathcal{F}_G$. $\square$

Now, let us proceed with the q-set property.

Lemma 2.11. [6] The density filter $fil(\mathcal{A}_G)$ is a q-set.

Proof. So we have to show that for every partition $(A_n)_{n \in \omega}$ of ω into finite sets, there is some $E \in fil(\mathcal{A})$, such that $|A_n \cap E| \leq 1$ for every $n \in \omega$.

To show this let $\mathcal{P} := (A_n)_{n \in \omega}$ be such a partition. We would like to find some semi-selector in $\mathcal{F}_G$ and hence in $fil(\mathcal{A})$ for this partition using a density argument.

Let $D_{\mathcal{P}} := \{(\mathcal{A}, A) \in \mathbb{P} \mid A \text{ is a semi-selector for } \mathcal{P}\}$. We want to show that the set $D_{\mathcal{P}}$ is dense. Take $(\mathcal{A}, A) \in \mathbb{P}$ arbitrarily. To find some condition in $D_{\mathcal{P}}$ which is below that condition, first fix an enumeration $\{h_n\}_{n \in \omega}$ of $FF(\mathcal{A})$, in which each $h \in FF(\mathcal{A})$ occurs infinitely often. Let i_0 be the minimal $n \in \omega$ such that $\mathcal{A}^{h_0} \cap A \cap A_n \neq \emptyset$. Such an i_0

must exist, because $\mathcal{A} \cap h_0 \cap A$ is infinite. Now choose a_0 to be the minimal element n in $\mathcal{A}^{h_0} \cap A \cap A_{i_0}$. Now proceed iteratively and for $k \in \omega$ let i_{k+1} be the minimal element in $\omega \setminus i_k$, such that $|\mathcal{A}^{h_{k+1}} \cap A \cap A_{i_n}| \neq \emptyset$. Then choose a_{k+1} to be the minimal element of $\mathcal{A}^{h_{k+1}} \cap A \cap A_{i_{k+1}}$. Let $E := \{a_n \mid n \in \omega\}$. By construction this set is an infinite subset of A and it intersects every set A_n for $n \in \omega$ at most once. In addition since every $h \in FF(\mathcal{A})$ occures infinitely often in the enumeration E meets every boolean combination of $\mathcal{A}$ on an infinite set. So $(\mathcal{A}, E) \leq (\mathcal{A}, A)$. Since we chose an arbitrary condition this means that $D_{\mathcal{P}}$ a is dense and the set $\mathcal{F}_G$ contains a semi-selector for $\mathcal{P}$. □

If one would like to generalise the definition of a selective independent family to independent families of subsets of some uncountable cardinal, one would have to think carefully about the q-set property. For the countable setting there is an equivalent condition to being a q-set, which could be generalised more easily.

Lemma 2.12. [3] Let $\mathcal{F}$ be a filter. Then $\mathcal{F}$ is a q-set if and only if for any increasing function $f : \omega \to \omega$ there is a set $\{k_n \mid n \in \omega\} \in F$, such that for every $n \in \omega$ it holds that $f(k_n) < k_{n+1}$.

Proof. Let $\mathcal{F}$ be a filter and assume that $\mathcal{F}$ is a q-set. Further let $f : \omega \to \omega$ be an increasing function. In order to construct the desired sequence $(k_n)_{n\in\omega}$ we first define a sequence $(n_l)_{n\in\omega}$ as follows: $n_0 := 0$ and for every $l \in \omega$ we choose $n_{l+1} := min\{n \mid n > n_l \text{ and } \forall m \leq n_l : f(m) \leq n\}$. Then $\{[n_{3l}, n_{3l+3})\}_{l\in\omega}$ is a partition of ω. By assumption there is a semi-selector C for that partition.

Now we can define a relation R_1 on C_1 in the way that $xR_1Y \iff (x = y) \vee (x < y \leq f(x)) \vee (y < x \leq f(y))$. By construction R_1 is reflexive and symmetric. It is also transitive. To show that, we show that there are no three pairwise distinct elements $x, y, z \in C$ such that xR_1y and yR_1z hold.

In order to do so, we can distinguish three cases, namely either

1. y is the largest element of $\{x, y, z\}$ or
2. y is the smallest element of $\{x, y, z\}$ or
3. It holds that $x < y < z$ or $z < y < x$.

It the first two cases we obtain after renaming the variables that we have three elements a, b, c such that $a < b < c < f(a)$. But this leads to a contradiction: Choose $l_i \in \omega$ for $i = 0, 1, 2$ such that $a \in [n_{3l_0}, n_{3l_0+3})$, $b \in [n_{3l_1}, n_{3l_1+3})$ and $c \in [n_{3l_2}, n_{3l_2+3})$. Then we know that $a < n_{3l_0+3} \leq n_{3l_1}$. Hence $f(a) \leq n_{3l_1+1} < n_{3l_1+3} \leq n_{3l_2} \leq c$ contradicting $c \leq f(a)$.

For the third case assume without loss of generality $x < y < z$. By assumption we know that xR_1y which means $x < y \leq f(x)$ and analogously $y < z \leq f(y)$. As in the previous case we can choose l_0, l_1, l_2 such that $x \in [n_{3l_0}, n_{3l_0+3})$, $y \in [n_{3l_1}, n_{3l_1+3})$ and $z \in [n_{3l_2}, n_{3l_2+3})$. But then since $x < n_{3l_0+3} \leq n_{3l_1}$, we know that $f(x) \leq n_{3l_1+1}$ and so

$y \leq f_(x) \leq n_{3l_1+1}$. This implies that $f(y) \leq n_{3l_1+2} < n_{3l_1+3} \leq n_{3l_2} \leq z$, contradicting $z \leq f(y)$.

Since either case leads to a contradiction, there are no pairwise distinct x, y, z such that xR_1y and yR_1z. So R_1 is an equivalence relation on C_1 which partitions C_1 in equivalence classes that contain at most two elements. If we found a semi-selector for these equivalence classes in $\mathcal{F}$ this would finish the proof. The only issue is that the domain of R_1 is not ω. So we define a relation R_2 as follows: for all $x, y \in \omega$ let $xR_2y \iff (x = y) \vee (x, y \in C_1 \wedge xR_1y)$. By construction R_2 is an equivalence relation that partitions ω into equivalence classes of size at most 2. We can find a semi-selector $C_2 \in \mathcal{F}$ for that partition. Now take $C := C_1 \cap C_2$. $\mathcal{F}$ is a filter, so $C \in \mathcal{F}$. In addition for any two elements $x, y \in C$ such that $x < y$ we know that $\neg xR_2y$ and so especially $\neg xR_1y$. This means that $f(x) < y$. So choose $\{k_n\}_{n\in\omega}$ to be the increasing enumeration of C then this is the desired sequence.

For the other direction suppose that for every increasing function f there is a sequence $\{k_n\}_{n\in\omega} \in \mathcal{F}$ such that $f(k_n) < k_{n+1}$. Let $\mathcal{E}$ be a partition of ω into finite sets. We want to find a semi-selector for $\mathcal{E}$ in $\mathcal{F}$. Define $f : \omega \to \omega$ by $f(n) := max\{m \mid m \in \bigcup_{(i\leq n)\wedge(i\in E)} E\}$. Then by assumption there is a sequence $\{k_n : n \in \omega\} \in \mathcal{F}$ such that $f(k_n) < k_{n+1}$. Now $\{k_n : n \in \omega\}$ is a semi-selector for $\mathcal{E}$, because for $n \in \omega$ it holds that $k_n \leq f(k_n) < k_{n+1}$. Hence k_{n+1} is not an element of the union of sets in $\mathcal{E}$ that contain an element less or equal than k_n, so k_{n+1} is in a different set than all k_i for $i \leq n$. $\square$

3 An Introduction to Sacks forcing

In this chapter, the goal is to define the Sacks forcing. In particular, we are interested in the κ-support product of λ-Sacks forcing. For this whole chapter, we assume κ to be a inaccessible cardinal, so note especially that κ is measurable, $2^\kappa = \kappa^+$ and $2^{<\kappa} = \kappa$. One could also establish the key properties of the forcing using $\diamondsuit(\kappa)$ as it is done in [15], but since the assumption of κ being inaccessible is also needed in Chapter 5, we may already use it here.

Before starting with the κ-Sacks forcing we briefly introduce the countable Sacks forcing.

3.1 The countable Sacks forcing

At first we need to define the forcing poset. In the following "$\frown$ " will always denote the concatenation of strings. We take the definition from [11] but stick to the notation of [15], that we later use for the κ-Sacks forcing.

Definition 3.1. [11] A subset of ${}^{<\omega}2$ which is closed with respect to initial segments is called a tree. Given a tree T we say an element $s \in T$ splits in T if both $s^\frown 0$ and $s^\frown 1$ are elements of T. A tree T is called perfect if for every $s \in T$ there is some $t \in T$, such that $t \subseteq s$ and t splits in T. The Sacks forcing $\mathbb{S}$ consists of all perfect trees $\subseteq {}^{<\omega}2$. For two elements $p, q \in \mathbb{S}$ it holds that $q \leq p$ if and only if $q \subseteq p$.

The poset $\mathbb{S}$ is not ω-closed but there is a similar result that we will use often. For that we have to define fusion orderings. We define the stem of a tree T to be the largest node of T, that is comparable with any other node of T.

Definition 3.2. (similar to [15])

- For a condition $p \in \mathbb{S}$, let $split_0(p) := \{stem(p)\}$ and define $split_{n+1}(p) := \bigcup\{stem(p_{\sigma^\frown i}) \mid \sigma \in split_n(p), i \in \{0,1\}\}$.
- We write $p \leq_n q$ for two conditions $p, q \in \mathbb{S}$ if $p \leq q$ and $split_n(p) = split_n(q)$.
- A sequence $(p_n)_{n\in\omega} \subseteq \mathbb{S}$ such that $p_{n+1} \leq_n p_n$ is called a fusion sequence.

With that definition we obtain the following

Lemma 3.1. [15] Let $(p_n)_{n\in\omega}$ be a fusion sequence in $\mathbb{S}$. Then there is some $p \in \mathbb{S}$ such that $p \leq p_n$ for all $n \in \omega$.

The proof is similar to the one of Lemma 3.3 which is the analogous result for κ-Sacks forcing , so we will not write it here.

In the last section of this chapter we focus on the preservation of selective independent families under the countable Sacks forcing. In order to see that in our final model after iterations of Sacks forcing it holds that $\mathfrak{i} = \omega_1 < \mathfrak{c}$, we have to check whether the Sacks forcing preserves cardinals greater or equal than $\aleph_1$. This is not clear right away, since $\mathbb{S}$ is not ccc. [11]

What one can show is that the Sacks forcing preserves $\aleph_1$ because it satisfies Baumgartner's Axiom A. [5]

Definition 3.3 (Baumgartner's Axiom A)**.** [11] Let $\mathbb{P}$ be a forcing poset. Then $\mathbb{P}$ satisfies Axiom A if and only if there is a descreasing sequence $(\leq_n)_{n\in\omega}$ of partial orders on the set $\mathbb{P}$, such that $\leq_0 = \leq_{\mathbb{P}}$ and $\leq_{n+1} \subseteq \leq_n$ and the following hold.

1. If $(p_n)_{n<\omega} \subseteq \mathbb{P}$ is a sequence such that for all $n < \omega$ it holds that $p_{n+1} \leq_n p_n$, then there is a condition $p \in \mathbb{P}$, such that $p \leq p_n$ for all $n \in \omega$.
2. For every condition $p \in \mathbb{P}$, every antichain $A \subseteq \mathbb{P}$ and every $n \in \omega$ there is a $q \leq_n p$, such that the set of conditions in A which are compatible with q is at most countable.

Theorem 3.1. [11] A forcing notion which satisfies Axiom A preserves $\aleph_1$.

A proof of this can be found in [11].

Lemma 3.2. [11] The Sacks forcing satisfies Axiom A.

Proof. Take $\leq_n$ to be the fusion orderings defined in this chapter. We will show 1 of the axiom when proving the fusion lemmata.

So it remains to show 2 of the definition.

Let $n \in \omega$, let $p \in \mathbb{S}$ and let A be an antichain in $\mathbb{S}$. For each $t \in split_n(p)$ take an element $q^t \leq p_t$, such that q^t is compatible with at most one element of A. This is possible, since if p_t is incompatible with every $a \in A$ just take $q^t := p_t$. Otherwise choose $a \in A$, such that p_t and a are compatible. Let q^t be a common extension of a and p_t. Then q^t is incompatible with every other $b \in A$, because A is an antichain.

Let $q := \bigcup\{q^t \mid t \in split_n(p)\}$. By construction $q \leq_n p$. Assume there is $a \in A$, such that a and q are compatible. We want to show that there is some $t \in split_n(p)$, such that q^t and a are compatible. Let r be a common extension of q and a. If $r \leq q^t$ for some $t \in split_n(p)$, clearly q^t and a are compatible. Otherwise there is some $t \in split_n(p)$ such that q^t and r are compatible. But this implies that q^t and a are compatible.

Since every q^t is compatible with at most one element of A, we know that q is compatible with at most $|split_n(p)| = 2^n$ elements of A. $\square$

3.2 Definition and the generalised fusion lemma

In this section we first define the κ-Sacks forcing. The definition is very similar to the countable. The only difference is that we now have to pay attention to limit steps.

Definition 3.4. [15] A set $p \subseteq^{<\kappa} 2$ that is closed under initial segments is called a tree. Such a tree is called perfect if for every $\alpha < \kappa$ and every sequence $(u_\beta)_{\beta<\alpha}$ such that $u_\beta \subseteq u_\gamma$ for every $\beta < \gamma$, also $\bigcup\{u_\beta \mid \beta < \alpha\}$ is an element of the tree.

The term "perfect" is from [15], where a tree with the above closure property is referred to as a "perfect set". We will need another definition that is the one of splitting nodes.

Definition 3.5. [15] For a tree p we call a node $u \in p$ splitting or say u splits in p if both $u^\frown 0$ and $u^\frown 1$ are in p. Here " $^\frown$" means the concatenation of sequences.

Within these definitions we have all ingredients for the Sacks-forcing.

Definition 3.6. [15] The κ-Sacks forcing is the poset $\mathbb{S}_\kappa$, which consists of all perfect trees $p \subseteq^{<\kappa} 2$ with the following properties:

- For every $u \in p$ there is some $t \in p$, such that t splits in p and $u = t$ or u is an initial segment of t.
- For any limit ordinal $\gamma < \kappa$ and every node $u \in^\delta 2$, such that the set of ordinals β with $u \restriction \beta$ splits in p is unbounded in δ, also u splits in p.

The extension relation is given via $p \leq q$ if and only if $p \subseteq q$.

Before giving further definitions let us fix some notation. By $u \trianglelefteq t$ for some nodes u, t of a tree, we will denote that either $u = t$ or u is an initial segment of t. For a tree p and some node s in it p_s will denote the subtree that consists of all nodes $t \in p$ such that either $t \subseteq s$ or $s \subseteq t$. The next definition that is needed will be the one of splitting levels.

Definition 3.7. [15] For a condition $p \in \mathbb{S}_\kappa$ its stem is denoted $\mathrm{stem}(p)$ and defined as the largest node s such that for all $t \in p$ either $s \subseteq t$ or $t \subseteq s$. For an ordinal $\alpha < \kappa$ the α-th splitting level of p is denoted $\mathrm{split}_\alpha(p)$ and is defined recursively as follows:

- $\mathrm{split}_0(p) := \{stem(p)\}$
- For $\alpha = S(\gamma)$, we have that $\mathrm{split}_\alpha := \{\mathrm{stem}(p_{u^\frown i}) \mid u \in \mathrm{split}(\gamma) \text{ and } i \in \{0, 1\}\}$.
- If α is a limit ordinal, then $\mathrm{split}_\alpha(p)$ is the set of all nodes $s \in p$, which are a limit of nodes in $\bigcup\limits_{\gamma<\alpha} \mathrm{split}_\gamma(p)$.
- For any splitting node $t \in p$, $sl(t)$ will denote the ordinal α, such that $t \in split_\alpha(p)$. Further $split(p)$, will denote the set of all splitting nodes in p.

Before starting with the product forcing the notion of a fusion sequence will be introduced.

Definition 3.8. [8] We will write $p \leq_\alpha q$ for two conditions p and q if and only if $p \leq q$ and $\text{split}_\alpha(p) = \text{split}_\alpha(q)$. A sequence $(p_\alpha)_{\alpha<\kappa}$ of conditions in the Sacks forcing is called a fusion sequence if $p_{S(\alpha)} \leq_\alpha p_\alpha$ for all $\alpha < \kappa$ and for every limit ordinal $\delta < \kappa$ and every $\alpha < \delta$ it holds that $p_\delta \leq_\alpha p_\alpha$.

The forcing notion used in the next chapters is the one of the κ-support product of λ many copies of the Sacks forcing. Here λ is some regular uncountable cardinal. We will denote that forcing notion by $\mathbb{S}_\kappa^\lambda$. In this setting also some kind of fusion sequence will be introduced.

Definition 3.9. [8] For an ordinal β and set of conditions $\{p_\alpha \mid \alpha < \beta\} \subseteq \mathbb{S}_\kappa^\lambda$ the expression $\bigwedge\limits_{\alpha<\beta} p_\alpha$ will denote a condition p such that $dom(p) = \bigcup\limits_{\alpha<\beta} dom(p_\alpha)$ and $p(\gamma) = \bigcap\{p_\alpha(\gamma) \mid \gamma \in dom(p_\alpha)\}$ for all $\gamma \in dom(p)$.

The above condition p does not necessarily have to be defined. If the sequence of the conditions $(p_\alpha)_{\alpha<\beta}$ is not decreasing after any reordering, then p is certainly undefined. Also if the resulting condition has support of size greater than κ it is not defined.

In order to answer the question, when the above definition is reasonable, some results about closedness will be needed, to see when a decreasing sequence of conditions in the Sacks forcing does have a common stronger condition.

Lemma 3.3. [15] The poset $\mathbb{S}_\kappa$ is $< \kappa$-closed,in particular for any ordinal $\beta < \kappa$ and a decreasing sequence $(p_\alpha)_{\alpha<\beta}$ the set $p := \bigcap\limits_{\alpha<\beta} p_\alpha$ is a condition in the κ-Sacks forcing.

Proof. If β is finite the condition p is equal to the last condition in the sequence and there is nothing to do. So assume β is infinite.

One can see that p is a perfect tree, because since any condition p_α is closed under initial segments and under limits of increasing sequences of conditions so is p. So it remains to check the properties of Definition 3.6. The closure of the splitting nodes again follows in the same way as the closure under limits. So we only have to care about the fact whether every sequence in p has an extension which splits in p.

Let $s \in p$. We want to show that there is some $t \subseteq s$ such that t splits in p. At first we want to find some $f \in^\kappa 2$, such that $f \subseteq p$ and s is an initial segment of f. Since we know that p is closed under limits it suffices to show that for any $t \in p$ either $t^\frown 0$ or $t^\frown 1$ also belongs to p. We know that for $\alpha < \beta$ the sequence t has got some (not necessarily proper) extension which splits in p_α. This implies especially that every tree p_α for $\alpha < \beta$ contains either $t^\frown 0$ or $t^\frown 1$. Since β is infinite for one of the two sequences there are β many ordinals α such that the sequence occurs in p_α. But this means it has to occur in all the p_α and hence in p.

Now pick such a cofinal sequence $f \supseteq s$ in p. Iteratively construct a sequence $\gamma_\alpha \subseteq \kappa$ for $\alpha \leq \beta$. This sequence satisfies that $s = f \restriction \gamma_0$ and for any $\alpha < \beta$ the sequence $f \restriction \gamma_{S(\alpha)}$ splits in p_α. This is possible because $f \subseteq p_\alpha$ for every α and there we will have some extension of $f \restriction \gamma_\alpha$ which splits in p_α. The shortest such sequence will be an initial segment of f. In addition we request that for any limit ordinal $\delta \leq \beta$ the ordinal $\gamma_\delta := \bigcup_{\alpha \leq \delta} \gamma_\alpha$. Since κ is a regular cardinal, we know that $\gamma_\beta < \kappa$. Now $f \restriction \gamma_\beta \in p$ and it is a node that splits in every p_α, because by construction it is either defined to be a splitting node or it is a limit of splitting nodes and hence a splitting node. So $f \restriction \gamma_\beta$ splits in p. □

Similar to the notation from before we will write $p \leq_{F,\alpha} q$ for $p, q \in \mathbb{S}^\lambda_\kappa$ and a set $F \subseteq dom(q)$ with $|F| \leq \kappa$ if we have that $p \leq q$ and for every $\beta \in F$, $p(\beta) \leq_\alpha q(\beta)$. This notation will lead to the definition of a generalised fusion sequence in the product forcing. But before that we would like to show a closure result of fusion sequences in $\mathbb{S}_\kappa$.

Lemma 3.4. [15] Let $(p_\alpha)_{\alpha<\kappa}$ be a sequence in the Sacks forcing, such that $p_{S(\alpha)} \leq_\alpha p_\alpha$ for every $\alpha < \kappa$ and $p_\delta = \bigcap_{\alpha<\delta} p_\alpha$ for every limit ordinal $\delta < \kappa$. Then $p := \bigcap_{\alpha<\kappa} p_\alpha$ is a condition in $\mathbb{S}_\kappa$ which is stronger than every p_α by construction.

Proof. Similar to the previous proof any intersection of perfect trees is again a perfect tree and in addition the splitting nodes of p are closed. So as before we have to check that for any $s \in p$ there is some $t \subseteq s$, which splits in p.

So take $s \in p$. Analogous to the previous proof, just replacing β by κ we can find a cofinal branch $f \in^\kappa 2$ through p, such that s is an initial segment of f. Similar to before, we construct a sequence $(\gamma_n)_{n\in\omega}$, such that $f \restriction \gamma_0 = s$ and $f \restriction \gamma_{n+1}$ splits in p_{γ_n} for every $n \in \omega$. Let $\gamma := \bigcup_{n\in\omega} \gamma_n$. Then $\gamma < \kappa$ is a limit and hence $p_\gamma = \bigcap_{\alpha<\gamma} p_\alpha$. By construction and the previous lemma $f \restriction \gamma$ splits in p_γ. For any pair of conditions q and r and any ordinal $\alpha < \kappa$ it holds that $split_\alpha(q) = split_\alpha(r)$ implies that $q \cap^{S(\alpha)} 2 = p \cap^{S(\alpha)} 2$. By definition of the sequence this implies that $f \restriction \gamma$ splits in every p_α for $\alpha \geq \gamma$. But this implies that $f \restriction \gamma$ splits in p.

□

Definition 3.10. [8] Let $(p_\alpha)_{\alpha<\kappa}$ be a sequence in $\mathbb{S}^\lambda_\kappa$ and let $F_\alpha \subseteq \lambda$ for all $\alpha < \kappa$. Further assume $p_{S(\alpha)} \leq_{F_\alpha,\alpha} p_\alpha$ for all $\alpha < \kappa$ and $p_\delta = \bigwedge_{\alpha<\delta} p_\alpha$ for every limit ordinal $\delta < \kappa$. In addition for every $\alpha < \kappa$ the cardinality of F_α is less than κ, $F_\alpha \subseteq F_{\alpha+1}$,$F_\delta = \bigcup_{\alpha<\delta} F_\alpha$ for every limit ordinal $\delta < \kappa$ and $\bigcup_{\alpha<\kappa} F_\alpha = \bigcup_{\alpha<\kappa} dom(p_\alpha)$.

Then $(p_\alpha)_{\alpha<\kappa}$ is called a generalised fusion sequence.

The next lemma is a result about κ-closure of generalised fusion sequences.

Lemma 3.5. [15] Let $(p_\alpha)_{\alpha<\kappa}$ be a generalised fusion sequence. Then $p := \bigwedge_{\alpha<\kappa} p_\alpha \in \mathbb{S}^\lambda_\kappa$.

Proof. At first one has to check, that $dom(p)$ has cardinality less or equal than κ. But this is true since $dom(p) = \bigcup_{\alpha<\kappa} dom(p_\alpha)$, which is a union of κ many sets of size κ. In order to show that p is really a condition, we have to show that $p(\beta) \in \mathbb{S}_\kappa$ for every

$\beta \in dom(p)$. So take $\beta \in dom(p)$. Then there is some $\alpha < \kappa$, such that $\beta \in F_\alpha$. Since $F_\alpha \subseteq F_\gamma$ for every $\gamma > \alpha$, this implies that $p_{S(\gamma)}(\beta) \leq_\gamma p_\gamma$ for every $\gamma \geq \alpha$. In addition $p_\delta(\beta) = \bigcap\limits_{\gamma<\delta} p_\gamma(\beta)$ for every limit ordinal δ by definition of the sequence. Now one can apply Lemma 3.3 to the sequence $(q_\gamma)_{\gamma<\kappa} \subseteq \mathbb{S}_\kappa$, where $q_\gamma = p_\gamma(\beta)$ for $\gamma \geq \alpha$ and $q_\gamma = p_\alpha(\beta)$ otherwise. This yields that $\bigcap\limits_{\gamma<\kappa} q_\gamma \in \mathbb{S}_\kappa$. But $\bigcap\limits_{\gamma<\kappa} q_\gamma = \bigcap\limits_{\gamma<\kappa} p_\gamma(\beta)$, which proves what we wanted to show. $\square$

3.3 Preprocessed conditions and outer hulls

The idea of preprocessed conditions for names for sets in some generic extension is that we can find an approximation of the characteristic function of that set in the ground model, witnessed by a condition. For the final proof that the maximal independent family constructed in the next chapter is preserved under the Sacks forcing these approximations will be needed. Firstly, we will have to introduce some notation.

Definition 3.11. [8] Let $p \in \mathbb{S}_\kappa^\lambda$ be a condition, $\alpha < \kappa$ and $F \subseteq supp(p)$ of cardinality less than κ. Then we define $\Lambda_\alpha^F(p) := \prod\limits_{i\in F} split_\alpha(p(i))$. Further for every tuple $\overline{\sigma} = (\sigma_i)_{i\in F} \in \Lambda_\alpha^F(p)$ define the condition $p_{\overline{\sigma}}$ such that $p_{\overline{\sigma}}(i) = p(i)_{\sigma_i}$ for each $i \in F$ and $p_{\overline{\sigma}}(i) = p(i)$ otherwise. By construction $p_{\overline{\sigma}} \leq p$.

Similarly, for every $h \in^F 2$ and any $\overline{\sigma} \in \Lambda_\alpha^F(p)$ define a condition $p_{\overline{\sigma}}^h$, such that $supp(p_{\overline{\sigma}}^h) = supp(p)$ in the way that $p_{\overline{\sigma}}^h(i) = (p(i))_{\sigma_i^\frown h(i)}$ for all $i \in F$ and $p_{\overline{\sigma}}^h(i) = p(i)$ otherwise.

With these definitions we can now define what it means for a condition to be preprocessed for a name of a set.

Definition 3.12. [8]

- Let $p \in \mathbb{S}_\kappa$ and $\dot{X}$ be an $\mathbb{S}_\kappa$-name for a subset of κ. Then p is preprocessed for $\dot{X}$ if for every $\alpha < \kappa$ and for every $t \in split_\alpha(p)$ there is a function $x_t : \alpha \to 2$ in the ground model such that $p_t \Vdash \chi_{\dot{X}} \restriction \alpha = \check{x}_t$.
- Similarly let $\dot{X}$ be a $\mathbb{S}_\kappa^\lambda$ name for a subset of κ, let p be a condition in $\mathbb{S}_\kappa^\lambda$ and $F \subseteq supp(p)$ such that $|F| < \kappa$. Then p is preprocessed for the pair $(F, \dot{X})$ if for all $\alpha < \kappa$ and all $\overline{\sigma} \in \Lambda_\alpha^F(p)$ there are $F' \supseteq F$, such that F is contained in $dom(p)$ and of size less than κ and $\overline{\tau}_{\overline{\sigma}} \in \Lambda_\alpha^{F'}(p)$ and $x_{\overline{\sigma}} \in^\alpha 2$ such that $(\tau_{\overline{\sigma}})_i = \sigma_i$ for all $i \in F$ and $p_{\overline{\tau}_{\overline{\sigma}}} \Vdash \chi_{\dot{X}} \restriction \alpha = \check{x}_{\overline{\sigma}}$.
- Let $p \in \mathbb{S}_\kappa^\lambda$ and $\dot{X}$ be a name for a subset of κ. Then p is preprocessed for $\dot{X}$ if for all $F \subseteq dom(p)$ with $|F| < \kappa$ the condition p is preprocessed for $(\dot{X}, F)$.

The next goal before introducing the outer hull will be to show that the set of preprocessed conditions in $\mathbb{S}_\kappa^\lambda$ is dense. At first let us show that the set of preprocessed conditions in $\mathbb{S}_\kappa$ is dense.

Lemma 3.6. [8] Let $\dot{X}$ be an $\mathbb{S}_\kappa$-name for a subset of κ and let $p \in \mathbb{S}_\kappa$. Then there is $q \leq p$ such that q is preprocessed for $\dot{X}$.

Proof. If we can construct a fusion sequence $(q_\alpha)_{\alpha<\kappa}$ below p, such that for every $\alpha < \kappa$ (and $\alpha > 0$) there is some $t \in split_\alpha(q_\alpha)$ and a function $x_t : \alpha \to 2$, such that $q_\alpha \Vdash \chi_{\dot{X}} \restriction \alpha = \check{X}_t$) then the condition q below the fusion sequence is as desired.

So start by letting $q_0 := p$. In order to construct q_1 let $t := stem(q_0)$. Now for each $i \in \{0,1\}$ there is a condition $w_{t,i}$ extending $p_{t^\frown i}$ and some $x(t,i) \in \{0,1\}$, such that $w_{t,i} \Vdash \chi_{\dot{X}}(0) = \check{x}(t,i)$. Let $q_1 := w_{t,0} \cup w_{t,1}$. Clearly $q_1 \leq q_0$. Also for every $s \in split_1(q_1)$ there is $x_s \in^1 2$, such that $(q_1)_s \Vdash \chi_{\dot{X}}(0) = \check{x}(t,i)$. Indeed, for $s \in split_1(q_1)$ it holds that $(q_1)_s \leq w_{t,i}$ for some i and forcing is inherited by stronger conditions. So $(q_1)_s \Vdash \chi_{\dot{X}} \restriction 1 = \check{x}_s$, where $x_s := ((0, x(s,i)))$.

Proceed inductively and suppose q_α has been defined such that for every $t \in split_\alpha(q_\alpha)$ there is a function $x_t \in^\alpha 2$ such that $(q_\alpha)_s \Vdash \chi_{\dot{X}} \restriction \alpha = \check{x}_t$. Similar to before for each $t \in split_\alpha(q_\alpha)$ and each $i \in \{0,1\}$ find a condition $w_{t,i} \leq (q_\alpha)_{t^\frown i}$ and $x(t,i) \in \{0,1\}$, such that $w_{t,i} \Vdash \chi_{\dot{X}}(\alpha) = \check{X}(t,i)$. Further let $q_{\alpha+1} := \bigcup\{w_{t,i} \mid t \in split_\alpha(q_\alpha), i \in \{0,1\}\}$. Now $q_{\alpha+1} \leq_\alpha q_\alpha$ because $split_\alpha(q_{\alpha+1}) = split_\alpha(q_\alpha)$.

If we choose $t \in split_\alpha(q_{\alpha+1})$ then t is an extension of $r^\frown i$ for some $r \in split_\alpha(q_\alpha)$ and $i \in \{0,1\}$. By assumption we already have that there is some $x_r \in^\alpha 2$ such that $(q_\alpha)_r \Vdash \chi_{\dot{X}} \restriction \alpha = \check{x}_r$. This implies since $(q_{\alpha+1})_t \leq (q_\alpha)_r$ that $(q_{\alpha+1}) \Vdash \chi_{\dot{X}} \restriction \alpha = \check{x}_r$. In addition we know that $(q_{\alpha+1}) \leq w_{r,i}$ and so $q_{\alpha+1} \Vdash \chi_{\dot{X}}(\alpha) = \check{X}(r,i)$. Define $x_t := x_r \cup \{(\alpha, x(r,i))\}$, then $(q_{\alpha+1}) \Vdash \chi_{\dot{X}} \restriction \alpha + 1 = \check{x}_t$ as requested.

Assume now, that α is a limit ordinal and q_β has been defined for every $\beta < \alpha$, such that for every $t \in split_\beta(q_\beta)$ there is $x_t : \beta \to 2$, such that $(q_\beta)_t \Vdash \chi_{\dot{X}} \restriction \beta = \check{x}_t$. Take $q_\alpha := \bigcap\limits_{\beta<\alpha} q_\beta$. Then for any $t \in split_\alpha(q_\alpha)$ there is a cofinal sequence $(\alpha_\beta)_{\beta<\alpha}$ with $t \restriction \alpha_\beta \in split_\beta(q_\beta)$ and functions $x_{t\restriction\alpha_\beta}$, such that $(q_{\alpha_\beta}) \Vdash \chi_{\dot{X}} \restriction \beta = x_{t\restriction\alpha_\beta}$. So choose $x_t := \bigcup\limits_{\beta<\alpha} x_{t\restriction\alpha_\beta}$. Then $(q_\alpha)_t \Vdash \chi_{\dot{X}} \restriction \alpha = \check{x}_t$. $\square$

A similar result can be obtained for the forcing poset $\mathbb{S}_\kappa^\lambda$ and we will do it in two steps.

Lemma 3.7. [8] Let $p \in \mathbb{S}_\kappa^\lambda$ and let $\dot{X}$ be a name for a subset of κ. Then for any $F \in supp(p)$ with $|F| < \kappa$ and any $\gamma < \kappa$ there is a condition $q \leq_{F,\gamma} p$ such that q is preprocessed for $(F, \dot{X})$.

Proof. Fix p, F and γ as above. We will construct a sets $(F_\alpha)_{\alpha<\kappa}$ and a fusion sequence $(q_\alpha)_{\alpha<\kappa}$ such that

1. $q_{\alpha+1} \leq_{F_\alpha,\gamma+\alpha} q_\alpha$ for all $\alpha < \kappa$
2. For every $\overline{\sigma} \in \Lambda^{F_\alpha}_{\gamma+\alpha}$ there is a function $x_{\overline{\sigma}} \in^\alpha 2$ such that $(q_{\alpha+1})_{\overline{\sigma}} \Vdash \chi_{\dot{X}} \restriction \alpha = \check{x}_{\overline{\sigma}}$.

Given such a fusion sequence we will show that $q := \bigwedge\limits_{\alpha<\kappa} q_\alpha$ is condition which has all the requested properties.

So let us construct the fusion sequence. Let $\beta < \kappa$ be some indecomposable ordinal. For $\alpha < \beta$ let $q_\alpha := p$ and $F_\alpha := F$. Then at stage β we will start with the construction as follows.

For limit steps δ we proceed as in the definition of a fusion sequence and let $F_\delta := \bigcup_{\alpha<\delta} F_\alpha$ and $q_\delta := \bigwedge_{\alpha<\delta} q_\alpha$. So we only have to explain how to obtain $q_{\alpha+1}$ when q_α has already been constructed. To do so choose an enumeration $\{\gamma_l = (\overline{\sigma}_l, h_l) \mid l < \rho\}$ of all pairs $(\overline{\sigma}, h)$ such that $\overline{\sigma} \in \Lambda_\alpha^{F_\alpha}(q_\alpha)$ and $h \in^{F_\alpha} 2$. We know that $\rho = 2^{|F_\alpha|} \cdot 2^{|F_\alpha|} < \kappa$.

In order to find $q_{\alpha+1}$ as requested we construct a sequence $(r_l)_{l<\rho}$ such that $r_0 := q_\alpha$ and with the following properties

1. If l is a limit ordinal then $r_l := \bigwedge_{k<l} r_k$.
2. For every $l < \rho$ it holds that $r_{l+1} <_{F_\alpha,\gamma+\alpha} r_l$ and
3. $(r_{l+1})^{h_l}_{\overline{\sigma}_l} \Vdash \chi_{\dot{X}} \restriction \alpha = \check{x}(\alpha, l)$ for a function $x(\alpha, l) \in^\alpha 2$ in the ground model.

In order to ensure 3 we have to explain how r_{l+1} is obtained from r_l. Assume r_l has already been defined for some $l < \rho$. Then there is some condition $w_l < (r_l)^{h_l}_{\overline{\sigma}_l}$, such that $w_l \Vdash \chi_{\dot{X}} \restriction \alpha = \check{x}(\overline{\sigma}_l, l)$ for some ground model function $x(\overline{\sigma}_l, l) : \alpha \to 2$. The condition w_l does not satisfy $w_l \leq_{F_\alpha,\gamma+\alpha} r_l$ since it is missing some splitting nodes. So, we define r_{l+1} in a way that we add those: $r_{l+1}(i) = w_l(i) \cup \{(q_\alpha(i))_{\tau^\frown j} \mid \tau \in \mathit{split}_\alpha(r_l(i)) \setminus (\sigma_l)_i \text{ or } j \neq h(i), j \in \{0,1\}\}$ if $i \in F_\alpha$ and $r_{l+1}(i) = w_l(i)$ otherwise.

Now r_{l+1} is as requested. We define $q_{\alpha+1} := \bigwedge_{l<\rho} r_l$ and $F_{\alpha+1} := F_\alpha \cup \{\mathit{min}(\mathit{supp}(q_{\alpha+1}) \setminus F_\alpha)\}$. By construction of the fusion sequence we ensured that $q \leq_{F,\gamma} p$. So it remains to check that q is preprocessed for the pair (F, X). Take $\alpha < \kappa$ and $\overline{\sigma} \in \Lambda^F_\alpha(q)$. By construction $F_\alpha \supseteq F$. And $F_\alpha \in \mathit{dom}(p)$. We can extend $\overline{\sigma}$ to a condition $\overline{\sigma}' \in \Lambda^{F_\alpha}_\alpha$ by adding for each $i \in F_\alpha \setminus F$ the nodes in $\mathit{split}_\alpha(q_i)$ to the sequences in $\overline{\sigma}$. By construction of the sequence $(q_\alpha)_{\alpha<\kappa}$ we have that $q \leq_{F_\alpha,\alpha} q_\alpha$ and so $\Lambda^{F_\alpha}_\alpha(q) = \Lambda^{F_\alpha}_\alpha(q_\alpha)$. Again by construction there is $x_{\overline{\sigma}'} \in^\alpha 2$ such that $(q_{\alpha+1})_{\overline{\sigma}'} \Vdash \chi_{\dot{X}} \restriction \alpha = \check{x}_{\overline{\sigma}'}$. But now $q_{\overline{\sigma}'} \leq (q_{\alpha+1})_{\overline{\sigma}'}$ and forcing is inherited by stronger conditions, so $q_{\overline{\sigma}'} \Vdash \chi_{\dot{X}} \restriction \alpha = \check{x}_{\overline{\sigma}'}$. □

With the help of this lemma we can show a result for $\mathbb{S}^\lambda_\kappa$ analogous to Lemma 3.6.

Lemma 3.8. [8] Let $\dot{X}$ be an $\mathbb{S}^\lambda_\kappa$-name for a subset of κ and let $p \in \mathbb{S}^\lambda_\kappa$. Then there is a condition $q \leq p$ such that q is preprocessed for $\dot{X}$.

Proof. Fix $\dot{X}$ and let $(G_\alpha)_{\alpha<\kappa}$ be an enumeration of all subsets of $\mathit{dom}(p)$ of size less than κ. We want to construct a fusion sequence $(q_\alpha)_{\alpha<\kappa}$ with sets $(F_\alpha)_{\alpha<\kappa}$ such that $q_0 = p$, $q_{\alpha+1} \leq_{F_\alpha,\alpha} q_\alpha$ and $q_{\alpha+1}$ is preprocessed for $(G_\alpha, \dot{X})$ for every $\alpha < \kappa$. The limit stages of the construction of the sequence follow from the definition of a fusion sequence. For the successor step assume we already defined the condition q_α and the set F_α. Then the previous lemma yields a condition $r \leq q_\alpha$ which is preprocessed for (G_α, X). Then choose $q_{\alpha+1} := r$ and $F_{\alpha+1} := F_\alpha \cup G_\alpha$. Proceed iteratively. The condition $q := \bigwedge_{\alpha<\kappa} q_\alpha$ which is stronger than every q_α is now preprocessed for $\dot{X}$. $\square$

For the preservation results in this and the next chapter we only need one further notion that is the one of an outer hull.

Definition 3.13. [8]

1. Let $p \in \mathbb{S}_\kappa$ and let $\dot{X}$ be an $\mathbb{S}_\kappa$-name for a subset of κ. For any splitting node t of p the set Y_t is called the outer hull of $\dot{X}$ below p_t. It is defined as $Y_t := \{\beta \in \kappa \mid p_t \nVdash \check{\beta} \notin \dot{X}\}$. If there is $q \leq p_t$, such that $q \Vdash \check{\beta} \in \dot{X}$ for some β then q is called a witness to $\beta \in Y_t$.
2. Similarly for a condition $p \in \mathbb{S}^\lambda_\kappa$, an $\mathbb{S}^\lambda_\kappa$-name $\dot{X}$ for a subset of κ, $\alpha < \kappa$, $F \subseteq supp(p)$ such that $|F| < \kappa$ and $\overline{\sigma} \in \Lambda^F_\alpha$ the outer hull of $\dot{X}$ below $p_{\overline{\sigma}}$ is the set $Y_\sigma := \{\beta \in \kappa \mid p_{\overline{\sigma}} \nVdash \check{\beta} \notin \dot{X}\}$. As before a condition $q \leq p_{\overline{\sigma}}$ such that $q \Vdash \check{X} \in \dot{X}$ is a witness to $\beta \in Y_{\overline{\sigma}}$.

Before concluding this section we will establish a connection between preprocessed conditions and the outer hull.

Lemma 3.9. [8] Let $\dot{X}$ be an $\mathbb{S}_\kappa$-name for a subset of κ and let $p \in \mathbb{S}_\kappa$ be preprocessed for $\dot{X}$. Then for every $\alpha < \kappa$, every $\beta \in Y_t$ and every $t \in split_\alpha(p)$ there is $r(t, \beta) \in p$ such that $r(t, \beta) \trianglerighteq t$ and $p_{r(t,\beta)} \Vdash \check{\beta} \in \dot{X}$.

Proof. Fix α, t and $\beta \in Y_t$ as above. By the definition of the outer hull and the forcing theorem there is $q \leq p_t$ such that $q \Vdash \check{\beta} \in \dot{X}$. Now take $r(t, \beta) =: r \in split_{\beta+1}(q)$. Then since $q \leq p_t$ $r \lhd t$ and there is some $\beta' \geq \beta + 1$ such that $r(t, \beta) \in split_{\beta'}(p)$. By the definition of being preprocessed there is $x_r \in^{\beta'} 2$ such that $p_r \Vdash \dot{X} \restriction \beta' = \check{x}_r$. But since $q_r \leq p_r$ and $q_r \Vdash \check{\beta} \in \dot{X}$ we musst get that $p_r \Vdash \check{x}_r(\beta) = 1$. $\square$

In the above proof we may also choose $r \in split_{\beta+1}(p)$, since for any $r^* \trianglelefteq r$ (where r is the tuple from above) such that $r^* \in split_{\beta+1}(p)$ there is some $x_{r^*} \in^{\beta+1} 2$ such that $p_{r^*} \Vdash \chi_{\dot{X}} \restriction \beta + 1 = \check{x}_{r^*}$. Again using that forcing is inherited by stronger conditions one obtains that $p_{r^*} \Vdash \check{x}_{r^*}(\beta) = 1$. Similarly one can see that for $\beta < sl(t, p)$ even $p_t \Vdash \check{\beta} \in \dot{X}$. A similar result to the previous lemma but for the product forcing will be the next lemma.

Lemma 3.10. [8] Let $\dot{X}$ be an $\mathbb{S}^\lambda_\kappa$-name for a subset of κ. Further let $p \in \mathbb{S}^\lambda_\kappa$ and $F \subseteq supp(p)$ be of size less than κ such that p is preprocessed for $(F, \dot{X})$. For each $\alpha < \kappa$ and each $\overline{\sigma} \in \Lambda^F_\alpha(p)$ let Y_σ be the outer hull of $\dot{X}$ below $p_{\overline{\sigma}}$. Then for each α and $\overline{\sigma}$ and for each $\beta \in Y_{\overline{\sigma}}$ there are $F' \supseteq F$ with $|F'| < \kappa$, $\alpha' \geq \alpha$ and $\overline{\tau}' \in \Lambda^{F'}_{\alpha'}$ such that $\overline{\tau}'$ is of the form $\overline{\tau}' = (r_i(\overline{\sigma}, \beta))_{i \in F'}$ such that for each $i \in F'$ it holds that $r_i(\overline{\sigma}, \beta) \in p(i)$. Furthermore σ_i is an initial segment of $r_i(\overline{\sigma}, \beta)$ for every $i \in F$ and $p_{\overline{\tau}'} \Vdash \check{\beta} \in \dot{X}$.

Proof. Fix α, F and $\overline{\sigma} \in \Lambda^F_\alpha(p)$ as above. By the definition of $Y_{\overline{\sigma}}$ there is $q \leq p_{\overline{\sigma}}$ such that $q_\alpha \Vdash \check{\beta} \in \dot{X}$. For each $i \in F$ choose $r_i := r_i(\overline{\sigma}, \beta)$ in $split_{\beta+1}(q(i))$. Observe that this implies that $\sigma_i \trianglerighteq r_i$ for every $i \in F$. Now since $|F| < \kappa$ we can find some $\beta' \geq \beta$

such that $r_i \in split_{\beta'}(p(i))$ for each $i \in F$. Choose $\overline{\tau} := (r_i(\overline{\sigma}, \beta))_{i \in F}$. Using the fact that p is preprocessed for $(F, \dot{X})$ one can find $F' \subseteq F$, $\overline{\tau}' \in \Lambda_\alpha^{F'}(p)$ and $x_{\overline{\tau}'} \in^{\beta'} 2$ such that $p_{\overline{\tau}'} \Vdash \chi_{\dot{X}} \restriction \beta' = \check{x}_{\overline{\tau}'}$. But $q_{\overline{\tau}'} \leq p_{\overline{\tau}'}$ and $q_{\overline{\tau}'} \Vdash \check{\beta} \in \dot{X}$. So it must hold that $p_{\overline{\tau}'} \Vdash (\chi_{\dot{X}} \restriction \beta')(\beta) = 1$ and hence $p_{\overline{\tau}'} \Vdash \check{\beta} \in \dot{X}$. □

Similar to before one may choose the $r_i(\overline{\sigma}, \beta)$ in $split_{\beta+1}(p)(i)$ for each $i \in F$. The two lemmata we just showed give us another expression for the outer hull. In order to write that notation more briefly for $p \in \mathbb{S}_\kappa^\lambda$ and $F \subseteq supp(p)$ with $|F| < \kappa$ let $\Lambda(F) := \bigcup_{\alpha < \kappa} \Lambda_\alpha^F(p)$ and write $\overline{\sigma} \trianglerighteq \overline{\tau}$ for $\overline{\sigma}, \overline{\tau} \in \Lambda(F)$ if $\overline{\sigma}(i) \trianglerighteq \overline{\tau}(i)$ for each $i \in F$.

Corollary 3.1. [8]

- Let $\dot{X}$ be an $\mathbb{S}_\kappa$ name for a subset of κ and $p \in \mathbb{S}_\kappa$ be preprocessed for $\dot{X}$. For $t \in p$ let Y_t be the outer hull of $\dot{X}$ below p_t. Then $Y_t = \{\beta \in \kappa \mid \exists r \in split(p) \text{ such that } t \trianglerighteq r \text{ and } p_r \Vdash \check{\beta} \in \dot{X}\}$.
- Similarly let $\dot{X}$ be an $\mathbb{S}_\kappa^\lambda$ name for a subset of κ and $p \in \mathbb{S}_\kappa$ with $F \subseteq supp(p)$ and $|F| < \kappa$ such that p is preprocessed for $F, \dot{X}$. Let $\overline{\sigma} \in \Lambda(F)$ and $Y_{\overline{\sigma}}$ be the outer hull of $\dot{X}$ below $p_{\overline{\sigma}}$. Then $Y_{\overline{\sigma}} = \{\beta \in \kappa \mid \exists \overline{\tau} \in \Lambda(F') \text{ for some } F' \supseteq F, |F'| < \kappa \text{ such that } \overline{\sigma} \trianglerighteq \overline{\tau} \text{ and } p_{\overline{\tau}} \Vdash \check{\beta} \in \dot{X}\}$.

Before starting with the preservation of maximal independent families under κ-Sacks forcing we need one last result.

Theorem 3.2. [15] Assume that κ is strongly inaccessible. Then $\mathbb{S}_\kappa$ is κ^{++}-cc and hence preserves cardinals greater or equal than κ^{++}. In addition $\mathbb{S}_\kappa$ preserves κ^+.

That $\mathbb{S}_\kappa$ is κ^{++}-cc follows from the fact that under the assumption of $2^\kappa = \kappa^+$ one has that $|\mathbb{S}_\kappa| = \kappa^+$. The preservation of κ^+ under the assumption of κ being inaccessible or of $\diamondsuit(\kappa)$, can be found in Kanamori's paper [15].

Now, finally we have all ingredients to state and prove results about preservation of maximal independent families under iterations and products of Sacks forcing.

3.4 Preservation of selective independent families under countable support Sacks forcing

In this section, we would like to show the preservation of selective independent families under countable support iteration of the Sacks forcing. We will give a proof of the preservation of densely maximal independent families and later formulate a general result for selective ones. Some of the steps here can be generalised to the uncountable as it will be done later.

First one has to notice that all the proofs of the previous sections about fusion sequences and preprocessed conditions in $\mathbb{S}_\kappa$ can be adapted to $\mathbb{S}_\omega$ easily. Since the conditions in $\mathbb{S}_\omega$ are subsets of ${}^{<\omega}2$, we will not have to worry about limits of sequences of splitting nodes or any limit steps and only need to rephrase the successor steps of the proofs from before. For better notation let us write $\mathbb{S}$ for $\mathbb{S}_\omega$.

For the main statement of this section we have to define a certain type of forcing posets.

Definition 3.14. [12] Let V denote the ground model. A forcing poset $\mathbb{P}$ is called ${}^\omega\omega$-bounding if for every generic extension $V[G]$ and for every $g \in {}^\omega\omega \cap V[G]$ there is some $f \in {}^\omega\omega \cap V$, such that $g \leq f$.

With the use of this definition let us formulate a preservation theorem which will be used for the proof of the main theorem of this section.

Theorem 3.3. [16] Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta, \alpha < \delta, \beta < \delta \rangle$ be a countable support iteration of proper ${}^\omega\omega$ bounding posets. Let $\mathcal{F} \subseteq \mathcal{P}(\omega)$ be Ramsey and let $\mathcal{H}$ be a subset of $\mathcal{P}(\omega) \setminus \langle \mathcal{F} \rangle_{up}$. Assume for every $\alpha < \delta$ it holds that $V^{\mathbb{P}_\alpha} \models \langle \mathcal{F} \rangle_{up} \cup \langle \mathcal{H} \rangle_{dn} = \mathcal{P}(\omega)$. Then also $V^{\mathbb{P}_\delta} \models \langle \mathcal{F} \rangle_{up} \cup \langle \mathcal{H} \rangle_{dn} = \mathcal{P}(\omega)$.

A proof of this can be found in [16].

Before proceeding with the main theorem another lemma will be needed.

Lemma 3.11. [5] Again let V denote the ground model and assume $(CH)^V$. Let $\mathcal{A}$ be an independent family and let G be $\mathbb{S}$-generic. Then the density ideal $(id(\mathcal{A}))^{V[G]}$ is generated by $id(\mathcal{A}) \cap V$. In other words for every set $X \cap id(\mathcal{A}) \cap V[G]$ there is some $Y \in id(\mathcal{A}) \cap V$, such that $X \subseteq Y$.

Proof. Let $\dot{X}$ be a name for a set in $id(\mathcal{A})$. Then there is some $p \in \mathbb{S}$, such that $p \Vdash \dot{X} \in id(\mathcal{A})$. Starting with a finite subset of $\mathcal{A}$ and iteratively adding sets, using a fusion sequence we can find some countable $\mathcal{B} \subseteq \mathcal{A}$, and a condition $q \leq p$ such that $q \Vdash \dot{X} \in id(\mathcal{B})$. Now any bijection $f : \mathcal{B} \to \omega$ leads to a bijection $F : FF(\mathcal{B}) \to {}^{<\omega}2$, such that for all $h, h' \in FF(\mathcal{B})$, $h' \supseteq h$ if and only if $F(h') \subseteq F(h)$.

Remember that for all $h \in FF(\mathcal{B})$ there is $h' \supseteq h$ such that $X \cap \mathcal{B}^{h'} = \emptyset$. With that in mind we can find a dense subset $\dot{D}$ of ${}^{<\omega}2$ such that $q \Vdash h \in \dot{D}$ if and only if $\mathcal{B}^{F^{-1}(h)} \cap X = \emptyset$. Next we want to show that there is some dense set $K \in {}^{<\omega}2 \cap V$ such that the set $\Delta = \{r \in \mathbb{S} \mid r \Vdash \check{K} \subseteq \dot{X}\}$ is dense below q.

In order to see that let $(s_n)_{n<\omega}$ be an enumeration of ${}^{<\omega}2$. We want to construct a fusion sequence $(S_n)_{n\in\omega} \subseteq \mathbb{S}$ and a sequence $(t_n)_{n<\omega} \subseteq {}^{<\omega}2$ such that $t_n \supseteq s_n$ for every n and such that for every $n \in \omega$ any $R \leq S_n$ forces that $t_n \in \dot{D}$.

Start with $S_0 = q$. Now assume we already constructed S_n and t_n as above. In order to construct S_{n+1} and t_{n+1} let $\{u_i \mid i < 2^n\}$ be an enumeration of $split_n(S_n)$. Now $(S_{u_0}) \Vdash$ "$\dot{D}$ is dense" . Hence there is some $t'_0 \supseteq s_{n+1}$ and $U_0 \leq (S_n)_{u_0}$, such that $U_0 \Vdash t'_0 \in \dot{D}$. Proceed iteratively, if t'_i has already been defined for some $i < 2^n - 1$ and find $U_{i+1} \leq (S_n)_{u_{i+1}}$ and $t'_{i+1} \supseteq t'_i$ such that $U_{i+1} \Vdash t'_{i+1} \in \dot{D}$. Finally take $t_{n+1} := t'_{2^n-1}$ and $S_{n+1} := \bigcup_{i\in 2^n} U_i$. By construction every $R \leq S_{n+1}$ satisfies $R \Vdash t_{n+1} \in \dot{D}$.

Now pick r to be stronger than every condition of $(S_n)_{n\in\omega}$ and let $K := \{t_n\}_{n\in\omega}$. Then K is dense in ${}^{<\omega}2$ and $r \Vdash \check{K} \subseteq \dot{X}$. Since we can replace q by any stronger condition and go through the same procedure we obtain the desired set Δ.

To finish this proof pick any dense ground model set K as above. $G \cap \Delta \neq \emptyset$ and so $V[G] \models K \subseteq \dot{D}[G]$. Let $Y : \bigcap_{t\in K} (\omega \setminus \mathcal{A}^{F^{-1}(t)})$. Then $Y \in id(\mathcal{A}) \cap V$. Now for any $h \in FF(\mathcal{B})$, there is some $t \in K$, such that $V[G](\models t \supseteq F(h)) \wedge (\dot{X} \subseteq \omega \setminus \mathcal{B}^{F^{-1}(t)})$. Hence $V[G] \models \dot{X}[G] \subseteq Y$. □

Now we have every ingredient to prove the main theorem.

Theorem 3.4. [5] Assume that (CH) holds in V and let $\mathcal{A}$ be an independent family which is selective in the ground model. Further let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta, \alpha < \omega_2, \beta < \omega_2\rangle$ be a countable support iteration with $\dot{\mathbb{Q}}_\alpha = \mathbb{S}$ for all $\alpha < \omega_2$. Then $\mathcal{A}$ is densely maximal in $V^{\mathbb{P}_{\omega_2}}$.

Proof. We want to show that $(\mathcal{A} \text{ is densely maximal })^{V^{\mathbb{P}_\alpha}}$ for each $\alpha < \omega_2$ using induction on α. To do so let us treat successor and limit steps separately. For any step we want to show the equivalent condition to dense maximality as showed in Corollary 2.1.

At first consider the limit step. Let α be a limit ordinal and suppose for every $\beta < \alpha$ already $V^{\mathbb{P}_\beta} \models$ "$\mathcal{A}$ is densely maximal" . By Lemma 3.11 we have that $(fil(\mathcal{A}))^{V_\beta} = \langle fil(\mathcal{A}) \cap V\rangle_{up}$. So $V^{\mathbb{P}_\beta} \models (\mathcal{P}(\omega) = fil(\mathcal{A}) \cup \langle\{\omega \setminus \mathcal{A}^h \mid h \in FF(\mathcal{A})\}\rangle_{dn})$ or equivalently $(\mathcal{P}(\omega) = \langle fil(\mathcal{A}) \cap V\rangle_{up} \cup \langle\{\omega \setminus \mathcal{A}^h \mid h \in FF(\mathcal{A})\}\rangle_{dn})^{V^{\mathbb{P}_\beta}}$. But then Theorem 3.3 implies that $(\mathcal{P}(\omega) = \langle fil(\mathcal{A}) \cap V\rangle_{up} \cup \langle\{\omega \setminus \mathcal{A}^h \mid h \in FF(\mathcal{A})\}\rangle_{dn})^{V^{\mathbb{P}_\delta}}$. And so $\mathcal{A}$ is densely maximal in $V^{\mathbb{P}_\delta}$.

For the successor step assume we already showed that $\mathcal{A}$ is densely maximal in $V^{\mathbb{P}_\alpha}$ for some $\alpha < \omega_2$ and let us show that it is also densely maximal in $V^{\mathbb{P}_{\alpha+1}}$. Assume not. Again with corollary 2.1 this means there is some $X \notin fil(\mathcal{A})$, such that $X \notin \langle\{\omega \setminus \mathcal{A}^h \mid h \in FF(\mathcal{A})\}\rangle_{dn}$. Let $\dot{X}$ be a name for such a set and let $p \in \mathbb{Q}_\alpha = \mathbb{S}$ such that for all $h \in FF(\mathcal{A})$, $p \Vdash |\mathcal{A}^h \cap \dot{X}| = \omega$ and $p \Vdash \dot{X} \notin fil(\mathcal{A})$.

Using Lemma 3.6, we can assume that p is preprocessed. For each $t \in split(p)$ let Y_t be the outer hull of $\dot{X}$ below p_t. For every $t \in split(p)$ and every $m \in Y_t$, let $r(t,m)$ be the extension of t of least splitting level, such that $p_{r(t,m)} \Vdash \check{m} \in Y_t$. We already showed that then $sl(r(t,m)) = max\{sl(t), m+1\}$. Now define $h : \omega \to \omega$ by $h(n) = max\{n+1\} \cup \{sl(r(t,m)) \mid m \leq n, t \in split_n(p)\}$.

One can easily see, that $X[G] \subseteq Y_t$ for an $\mathbb{S}$-generic G. If $m \in X[G]$, there is some $q \leq p$, such that $q \Vdash \check{m} \in \dot{X}$. But then q is a witness that $m \in Y_t$. Next we want to show that for every $n \in \omega$ and every $t \in split_n(p)$ there is a set $B_t \in id(\mathcal{A}) \cap V^{\mathbb{P}_\alpha}$ such that $\omega \setminus Y_t \in B_t$. Assume not. Then especially $\omega \setminus Y_t \notin id(\mathcal{A}) \cap V^{\mathbb{P}_\alpha}$ and so $Y_t \notin fil(\mathcal{A}) \cap V^{\mathbb{P}_\alpha}$. But since $\mathcal{A}$ is densely maximal in $V^{\mathbb{P}_\alpha}$ this means that there is some $h \in FF(\mathcal{A})$ such that $Y_t \subseteq \omega \setminus \mathcal{A}^h$. But since $p \Vdash \dot{X} \subseteq \check{Y}_t$ this would imply that $p \Vdash \dot{X} \cap \mathcal{A}^h = \emptyset$ which is a contradiction to the choice of p.

Now that we know that $\omega \setminus B_t \subseteq Y_t$ and $\omega \setminus B_t \in fil(\mathcal{A}) \cap V^{\mathbb{P}_\alpha} = (\langle fil(\mathcal{A}) \cap V \rangle_{up})^{V^{\mathbb{P}_\alpha}}$ and the ground model density filter is a p-set, there is a set $C \in fil(\mathcal{A}) \cap V$, such that $C \subseteq^* Y_t$ for all $n \in \omega$ and all $t \in split_n(p)$.

So for every $n < \omega$ and $t \in split_n(p)$ the set $C \setminus Y_t$ is finite. Since $split_n(p)$ is finite as well we can find $f \in {}^\omega \omega$, such that $C \setminus Y_t \subseteq f(n)$ for every $t \in split_n(p)$. We can assume that $h(n) \leq f(n)$ for every $n \in \omega$ by choosing $f(n)$ to be larger if necessary. Since $\mathbb{P}_\alpha$ is ${}^\omega\omega$-bounding, one can assume that f is in the ground model. Here we will assume without loss of generality that f is strictly increasing. Let $g : \omega \to \omega$ be defined by $g(n) = f(f(n))$ for all $n \in \omega$.

Using Lemma 2.12 and the fact that $fil(\mathcal{A}) \cap V$ is a q-set, we can find a set C^* in $fil(\mathcal{A}) \cap V$, such that for its increasing enumeration $(k_n)_{n\in\omega}$ it holds that $g(k_n) < k_{n+1}$ for all $n \in \omega$. Now choose $C_0 := C \cap C^*$. Again $C_0 \in fil(\mathcal{A})$. Our goal now is to construct a fusion sequence $(q_n)_{n\in\omega}$ below p, such that $q_{n+1} \Vdash \check{k}_n \in \dot{X}$.

We start with $q_0 := p$. Consider $t \in stem(p)$, for each $i \in \{0, 1\}$, there is $r_i(t, k_0)$ extending $t^\frown i$, such that $p_{r_i(t,k_0)} \Vdash \check{k}_0 \in \dot{X}$. Let $q_1 := p_{r_0(t,k_0)} \cup p_{r_1(t,k_0)}$. The $q_1 \leq_0 q_0$ and $q_1 \Vdash \check{k}_0 \in \dot{X}$. Observe in addition that $sl(r_i(t, k_0)) \leq h(k_0)$ for each $i \in \{0, 1\}$.

Assume, we already constructed the sequence up to q_n for some $n \geq 1$, such that $q_n \Vdash \check{k}_{n-1} \in \dot{X}$ and each $t \in split_n(q_n)$ is in $split_N(p)$ for some $N \leq h(k_{n-1})$. Then we construct q_{n+1} as follows.

Take $t \in split_n(q_n)$, then $t \in split_N(p)$ for an $N \leq h(k_{n-1})$. But then $Y_t \setminus C_0 \subseteq Y_t \setminus C^* \subseteq f(h(k_{n-1})) \subseteq f(f(k_{n-1}))$. We know that $f(h(k_{n-1})) \leq f(f(k_{n-1})) < k_n$ and so $k_n \in Y_t$. For each $i \in \{0, 1\}$ there is $r_i(t, k_n) \subseteq t^\frown i$, such that $p_{r_i(t,k_n)} \Vdash \check{k}_n \in \dot{X}$. Let $q_{n+1} := \bigcup\{p_{r_i(t,k_n)} \mid t \in split_n(q_n), i \in \{0, 1\}\}$. Then $q_{n+1} \Vdash \check{k}_n \in \dot{X}$, $q_{n+1} \leq_n q_n$ and every $t \in split_{n+1}(q_{n+1})$ is in $split_M(p)$ for some $M \leq h(k_n)$.

Now that we constructed the fusion sequence $\{q_n\}_{n\in\omega}$, choose q such that $q \leq q_n$ for all $n \in \omega$. Then $q \Vdash \check{C}_0^* \subseteq \dot{X}$. But this implies that $q \Vdash \dot{X} \in \langle fil(\mathcal{A}) \cap V \rangle_{up}$. But this contradicts the choice of p. $\square$

Note that in the above proof we only showed that the selective independent family remains densely maximal. The p-set property also holds in the final model since $(fil(\mathcal{A}))^{V^{\mathbb{P}_\alpha}} = \langle fil(\mathcal{A}) \cap V \rangle_{up}$. We did not pay attention to the q-set property of the density filter in the new models. However, one can show that selective independent families are preserved under countable support iterations of Sacks forcing. Let us formulate an even more general result. To do so another definition is needed.

Definition 3.15. [9] A forcing notion $\mathbb{P}$ is called Cohen preserving if for every $\mathbb{P}$-name $\dot{D}$ and for every condition $p \in \mathbb{P}$, such that $p \Vdash$ "$\dot{D} \subseteq^{<\omega} 2$ is dense and open" there is a ground model set E and some $q \leq p$, such that $q \Vdash \check{E} \subseteq \dot{D}$.

We already showed in the proof of Lemma 3.11 that $\mathbb{S}$ is Cohen preserving. Now a more general preservation theorem is the following.

Theorem 3.5. [9] Let δ be an ordinal and $\mathcal{I}$ be a selective independent family in the ground model. Further let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha, \beta < \delta \rangle$ be a countable support iteration of proper forcing notions such that for all $\alpha < \delta \Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ is Cohen preserving". If for all $\alpha < \delta$ also $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ preserves the dense maximality of $\mathcal{I}$" then $((\mathcal{I})$ is selective $)^{\mathbb{P}_\delta}$.

A proof of this result can be found in [9]. By the remark above the theorem implies especially that countable support iterations of the Sacks forcing preserve selective independent families.

Corollary 3.2. (summarised in [9] even though the singe results were known earlier) A selective independent family remains selective after countable support iterations of

- Sacks forcing,
- Miller partition forcing,
- Miller lite forcing,
- and any combination of those.

An immediate consequence of the preservation of selective independent families under ω_2 many iterations of the Sacks forcing is that consistently $\mathfrak{i} < \mathfrak{c}$, since during ω_2 many iterations of the Sacks forcing at least ω_2 many Sacks reals are added.

Theorem 3.6. [5] It is consistent that the independence number $\mathfrak{i}$ is strictly less than 2^ω.

4 The higher independence

In this chapter we want to focus on results of the generalised independence number. The main goal will be to show that it is consistent that the generalised independence number is less than the continuum of κ. To achieve this we are going to show that under certain assumptions a maximal κ-independent family remains maximal after the κ-support of λ many copies of κ-Sacks-forcing. [8]

However, one has to think about the generalisation of the countable case and what assumptions one has to make on the cardinal κ in order to obtain certain results.

4.1 On the general setting

Firstly, we have to define independence in higher Baire spaces. Similar to the second chapter, for any family $\mathcal{A} \subseteq [\kappa]^\kappa$, we denote by $FF(\mathcal{A})$ the set $\{f \mid f : A \to \{0,1\}$ where $A \subseteq \mathcal{A}$ is a finite subset $\}$. Sticking to the same notation as before we again write $\mathcal{A}^h$ for $\bigcap\{A^{h(A)} \mid A \in dom(h)\}$. Here again $h \in FF(\mathcal{A})$ and for any set $A \in \mathcal{A}$ we define A^0 to be the set A itself and A^1 to be its complement.

Definition 4.1. [8]

- Let κ be a cardinal. A family $\mathcal{A} \subseteq [\kappa]^\kappa$ is called κ-independent if for any $h \in FF(\mathcal{A})$, it holds that $|\mathcal{A}^h| = \kappa$.
- A κ-independent family is called maximal κ-independent if it is maximal with respect to set inclusion.
- The number $\mathfrak{i}(\kappa)$ is the least size of a maximal κ-independent family.

There are also other ideas, how to generalise independence for larger cardinals than ω. One would be for instance the notion of strong κ-independence. In that a family would be strongly κ-independent if every Boolean combination of less than κ many elements of that family is of size κ. There are many open questions regarding strong κ-independence, also regarding the existence of a maximal strongly κ-independent family. A few results and problems of strong κ-independent families will be discussed at the end of the thesis. However, in this chapter by "independent" we will always mean κ-independent and "maximal independent" will refer to maximal κ-independent.

Another definition that is important for the rest of this chapter is the one of dense maximality. In the following definition the notation $f \supseteq g$ for two functions f and g

means that $\mathrm{dom}(g) \subseteq \mathrm{dom}(f)$ and that the two functions coincide on their common domain.

Definition 4.2. [8] An independent family $\mathcal{A}$ is called densely maximal if and only if for every set $X \subseteq \kappa$ that is not an element of $\mathcal{A}$ and for every $h \in FF(\mathcal{A})$ there is some $h' \supseteq h$ such that $\mathcal{A}^{h'} \cap X = \emptyset$ or $\mathcal{A}^{h'} \cap (\kappa \setminus X) = \emptyset$.

Clearly, every densely maximal independent family is a maximal independent family. In addition one may interpret the term "dense" here in that sense that for any set X not in $\mathcal{A}$, the set of Boolean combinations that are a witness to not adding X to $\mathcal{A}$ and still having a maximal independent family, is dense in the boolean combinations.

It is yet open whether the existence of a densely maximal independent family follows from ZFC [8]. For $\kappa = \omega$ such a family exists assuming the continuum hypothesis. Still one can show the existence of a maximal independent family.

Proposition 4.1. [10] Every independent family is contained in a maximal independent family.

Proof. The statement follows immediately from Zorn's Lemma: Let $(\mathcal{A}_\alpha)_{\alpha<\gamma}$ be an increasing chain of κ-independent families. Then its union $\mathcal{A} := \bigcup\limits_{\alpha<\gamma} \mathcal{A}_\alpha$ is an independent family. To see this take any $h \in FF(\mathcal{A})$. Then there is some $\alpha < \gamma$ such that $\mathrm{dom}(h) \subseteq \mathcal{A}_\alpha$. Since $\mathcal{A}^h = \mathcal{A}_\alpha^{h\restriction\mathcal{A}_\alpha}$, which is of cardinality κ by assumption, this proves the claim. Hence any increasing chain has an upper bound so by Zorn's Lemma there are maximal elements. $\square$

Similar to the results for the countable setting, $\mathfrak{i}(\kappa)$ can be bounded from above. This time another proof will be used.

Theorem 4.1. [10] There exists a maximal independent family of cardinality 2^κ.

Proof. [10] This proof is similar to the one about ω-independence in the first chapter.

Let C be the set of all functions $s : dom(s) \to \{0, 1\}$, such that $dom(s)$ is a finite subset of κ. Let $D := \{a \subseteq C \mid |a| < \omega\}$. Now D has cardinality κ and we will construct an independent family of subsets of D of cardinality 2^κ. Using a bijection $D \to \kappa$ and Zorn's Lemma will then lead to the statement of the theorem.

For every $f : \kappa \to 2$ let $A_f := \{a \mid \exists s \in a \text{ such that } f \restriction dom(s) = s\}$. Clearly for two distinct $f, g : \kappa \to 2$ it holds that $A_f \neq A_g$. Then $\mathcal{A} := \{A_f \mid f : \kappa \to 2\}$ is as wanted. Since $|\mathcal{A}| = 2^\kappa$, it remains to show that for any pair S, T of finite disjoint subsets of $\mathcal{A}$ it holds that $\bigcap\limits_{f\in S} A_f \setminus (\bigcup\limits_{g\in T} A_g)$ is of cardinality κ.

To do so fix $S = \{f_0, \dots f_{n-1}\}$ and $T = \{g_0, \dots, g_{m-1}\}$ two finite disjoint subsets of $\mathcal{A}$ of size $n < \omega$ and $m < \omega$. Now fix $i < n$. For each $j < m$ there is an α_j, such that $f_i(\alpha_j) \neq g_j(\alpha_j)$. Then define a partial function s_i, such that $dom(s_i) = \{\alpha_j \mid j < m\}$ and $s(\alpha_j) = f_i(\alpha_j)$. Then f_i extends s_i but none of the functions g_j is an extension of s_i.

Define $a := \{s_i \mid i < n\}$. By construction $a \in \bigcap_{f\in S} A_f \setminus (\bigcup_{g\in T} A_g)$. In addition we know that $M := \kappa \setminus (\bigcup_{s\in a} dom(s))$ is a set of size κ. For $\alpha \in M$ define $a_\alpha := \{s_0 \cup \{(\alpha, f_0(\alpha))\}\} \cup \{s_i \mid 1 \leq i < n\}$. Then $a_\alpha \neq a_\beta$ for $\alpha \neq \beta$ and $\{a_\alpha \mid \alpha \in M\} \subseteq \bigcap_{f\in S} A_f \setminus (\bigcup_{g\in T} A_g)$. This proves that this is a set of size κ. $\square$

We will formulate a result about the relationship of $\mathfrak{d}(\kappa)$ and $\mathfrak{i}(\kappa)$ in our final model. Firstly let us reformulate the definition of the dominating number.

Definition 4.3. [2] Let f, g be two functions in κ^κ. We write $f \leq^* g$ if there exists an $\alpha < \kappa$ such that for all $\beta > \alpha$ it holds that $f(\beta) \leq g(\beta)$. A family $\mathcal{F} \subseteq \kappa^\kappa$ is called *dominating* if for all $g \in \kappa^\kappa$ there exist an $f \in \mathcal{F}$ such that $g \leq^* f$.

Then we define the generalised dominating number $\mathfrak{d}$ as follows:

$\mathfrak{d}(\kappa) = \min\{|\mathcal{F}| : \mathcal{F}$ is a dominating family in $\kappa^\kappa\}$.

In the countable we know that $\mathfrak{d} \leq \mathfrak{i}$, in the higher Baire space only a similar result for some very large cardinals κ is known.

In contrast the relationship between $\mathfrak{r}(\kappa)$ and $\mathfrak{i}(\kappa)$ can be showed easily.

Definition 4.4. [2] Let $A, B \subseteq \kappa$ we say that A splits B if $|B \cap A| = |B \setminus A| = \kappa$. Then $\mathfrak{r}(\kappa)$ denotes the minimum cardinality of a family $\mathcal{S}$ of unbounded subsets of κ such that there is no $A \subseteq \kappa$ which splits every $S \in \mathcal{S}$.

The connection of $\mathfrak{r}(\kappa)$ and $\mathfrak{i}(\kappa)$ is the following.

Proposition 4.2. [8] It holds that $\mathfrak{r}(\kappa) \leq \mathfrak{i}(\kappa)$.

Proof. Let $\mathcal{A}$ be an independent family that consists of less than $\mathfrak{r}(\kappa)$ many sets. We want to show that $\mathcal{A}$ is not maximal. To do so let $\mathcal{S} := \{\mathcal{A}^h \mid h \in FF(\mathcal{A})\}$. Then $|\mathcal{S}| = |\mathcal{A}^{<\omega}| = |\mathcal{A}|$. So there is some $A \subseteq \kappa$ which splits every $S \in \mathcal{S}$. But then $\mathcal{A} \cup \{A\}$ is again an independent family. $\square$

Before moving on to the main result let us specify the general setting. For the next two sections we want to have certain restrictions on κ. In particular the cardinal κ should be inaccessible, which means that $2^\kappa = \kappa^+$, κ is regular and for all $\lambda < \kappa$, we have that $2^\lambda < \kappa$. In particular we want to make use of κ being a mesurable cardinal or equivalently that there exists a non-principal κ-complete ultrafilter on κ.[14]

4.2 Construction of a densely maximal κ-independent family

In this section we aim to construct the densely maximal independent family we want to remain maximal after λ many copies of the Sacks-forcing. With the assumptions above let $\mathcal{U}$ be a normal measure on κ. The filter $\mathcal{U}$ is supposed to be non-principal and as observed in [14] it does not contain any set of cardinality less than κ. Another observation that will be important later is that $\mathcal{U}$ contains every club set $C \subseteq \kappa$. This can be checked easily let C be a closed unbounded set in κ and let $\{c_\alpha \mid \alpha < \kappa\}$ be an increasing enumeration of it. We know that every set $C_\alpha := \kappa \setminus c_\alpha$ is contained in the normal measure. For $\alpha > \beta$ it holds that $c_\alpha \in C_\beta$. Hence for every $\alpha < \kappa$ we know that $c_\alpha \in \bigcap\limits_{\beta<\alpha} C_\beta$. This implies that the set C is the diagonal intersection of the sets $(C_\alpha)_{\alpha<\kappa}$. So it also belongs to $\mathcal{U}$.

To construct our independent family with the desired properties we want to define a forcing poset $\mathbb{P}_{\mathcal{U}}$, where any $\mathbb{P}_U$-generic filter then adds such a family.

Definition 4.5. [8] Let $\mathbb{P}_{\mathcal{U}}$ consist of all pairs $(\mathcal{A}, A)$ such that $\mathcal{A}$ is a κ-independent family of size κ and A is an element of $\mathcal{U}$ such that for every $h \in FF(\mathcal{A})$ we have that $|\mathcal{A}^h \cap A| = \kappa$. Furthermore $\mathbb{P}_{\mathcal{U}}$ is a poset with the extension relation $(\mathcal{A}, A) \leq (\mathcal{B}, B)$ if $\mathcal{A} \supseteq \mathcal{B}$ and $|A \setminus B| < \kappa$.

Our goal is to show that for a $\mathbb{P}_{\mathcal{U}}$-generic filter G the set $\mathcal{A}_G$, defined as in Theorem 4.2 is a densely maximal independent family. We will call such a family obtained by the poset $\mathbb{P}_{\mathcal{U}}$ and any subfamily, $\mathcal{U}$-supported. The first step will be to show that $\mathcal{A}_G$ is a maximal independent family.

4.2.1 Maximality of $\mathcal{A}_G$

In this section let G be $\mathbb{P}_{\mathcal{U}}$ generic. To achieve our goal that is to show that $\mathcal{A}_G$ is maximal independent we want to prove two statements first. The first one will be used more often.

Lemma 4.1. [8] The poset $\mathbb{P}_{\mathcal{U}}$ is κ^+-closed.

Proof. First notice, that we use the assumption that $2^\kappa = \kappa^+$. Now let $S := \{(\mathcal{A}_\alpha, A_\alpha)\}_{\alpha<\kappa}$ be a decreasing sequence in $\mathbb{P}_{\mathcal{U}}$. We want to show that there is $(\mathcal{A}, A)$ such that $(\mathcal{A}, A)$ is stronger than $(\mathcal{A}_\alpha, A_\alpha)$ for every $\alpha < \kappa$. To do so it suffices to show the claim for $S' := \{(\mathcal{A}_\alpha, B_\alpha)\}$. Here each B_α is defined as follows: We let $B_0 := A_0$, clearly $B_0 \in \mathcal{U}$. We proceed inductively. If we constructed all sets $B_\alpha \in U$ as desired for all $\alpha < \beta$ for some successor ordinal $\beta = S(\delta)$, then we let $B_\beta := A_\beta \cap B_\delta$. By construction $A_\beta \subseteq^* B_\beta$ because, hence it still intersects $\mathcal{A}^h_\beta$ on a set of size κ for every $h \in FF(\mathcal{A}_\alpha)$. If now β is a limit ordinal and we constructed all sets B_α for $\alpha < \beta$, we let $C := \bigcup\limits_{\alpha<\beta} (B_\alpha \setminus A_\beta)$. Since $A_\beta \subseteq^* A_\alpha \subseteq^* B_\alpha$ for all $\alpha < \beta$ the set C has cardinality less than κ. Hence $\kappa \setminus C$ is in $\mathcal{U}$. So we can define $B_\beta := A_\beta \cap (\kappa \setminus C)$. Then $B_\beta \in \mathcal{U}$ and again A_β is almost contained in B_β We proceed inductively and in the end $B_\alpha \subseteq B_\beta$ for each $\alpha > \beta$.

Clearly a tuple that is stronger than any pair in S' is also stronger than any pair in S. First let $\mathcal{A} := \bigcup\limits_{\alpha<\kappa} \mathcal{A}_\alpha$. Then $\mathcal{A}$ is an independent family of cardinality κ. To construct a pseudo-intersection B of the sets B_α for $\alpha < \kappa$ such that $|\mathcal{A}^h \cap B| = \kappa$ for all $h \in FF(\mathcal{A})$ we proceed inductively. For each $\alpha < \kappa$ let us enumerate the set $FF(\mathcal{A}_\alpha)$

as $\{h_{\alpha,\beta}\}_{\beta<\kappa}$. Then $\{h_{\alpha,\beta}\}_{\alpha,\beta<\kappa}$, where for each $h \in FF(\mathcal{A})$ α is the least ordinal γ such that $dom(h) \subseteq \mathcal{A}_\gamma$ and β is the position of $h \restriction \mathcal{A}_\alpha$ in the enumeration of $FF(\mathcal{A}_\alpha)$ an enumeration of $FF(\mathcal{A})$. Now for each $\alpha < \kappa$ and each $\beta, \gamma < \alpha$ choose an element $b_{(\alpha,\beta,\gamma)}$ such that $b_{(\alpha,\beta,\gamma)} \in B_\alpha \cap \mathcal{A}_\alpha^{h_{\beta,\gamma}}$. Since the sequence of the B_α is decreasing we have that $b_{(\alpha,\beta,\gamma)} \in B_\beta$ for every β and every α and γ such that $\gamma, \beta < \alpha < \kappa$. Let $B := \{b_{(\alpha,\beta,\gamma)}\}_{\alpha<\kappa;\beta,\gamma<\alpha}$. Then $B \setminus B_\delta \subseteq \{b_{(\alpha,\beta,\gamma)} \mid \alpha < \delta \text{ and } \beta, \gamma < \alpha\}$, which implies that $|B \setminus\ B_\delta| < max(|\alpha|, \aleph_0) < \kappa$. So $B \subseteq^* B_\alpha$ for each $\alpha < \kappa$ and in addition by construction $|B \cap \mathcal{A}^h| = \kappa$. However, we cannot guarantee that $B \in \mathcal{U}$. To achieve that let $C := \Delta_{\alpha<\kappa} B_\alpha = \{\alpha \mid \alpha < \kappa \text{ and } \alpha \in \bigcap_{\beta<\alpha} B_\beta\}$, which is the diagonal intersection of the sets B_α. We know that $C \in \mathcal{U}$. So if we now let $A := B \cup C$, then $A \supseteq C$ and hence $A \in \mathcal{U}$. Then the pair $(\mathcal{A}, A)$ is as wanted.

$\square$

The second statement will also be used in the proof our claim.

Lemma 4.2. [8] Let $(\mathcal{A}, A) \in \mathbb{P}_\mathcal{U}$. Then there is a subset B of A, such that $B \notin \mathcal{A}$ and the family $\mathcal{A} \cup \{B\}$ is an independent family. In addition $(\mathcal{A} \cup \{B\}, A) \in \mathbb{P}_\mathcal{U}$.

Proof. We need to ensure that we pick B such that for every $h \in FF(\mathcal{A})$ the sets $\mathcal{A}^h \cap B$, $\mathcal{A}^h \cap (\kappa \setminus B)$, $\mathcal{A}^h \cap B \cap A$ and $\mathcal{A}^h \cap (\kappa \setminus B) \cap A$ are all of size κ. Then B is as requested.

To construct the set that way let $\{h_\alpha\}_{\alpha<\kappa}$ be an enumeration of $FF(\mathcal{A})$. Now for every $\alpha < \kappa$ we iteratively choose two distinct elements $a_{\alpha,0}$ and $a_{\alpha,1}$ such that $a_{\alpha,i} \in (\mathcal{A}^{h_\alpha} \cap A) \setminus \{a_{\beta,j} \mid \beta < \alpha, j \in \{0,1\}\}$ for $i \in \{0,1\}$. This is possible because $\mathcal{A}^{h_\alpha} \cap A$ is is of size κ for every $\alpha < \kappa$. Now we let $B := \{a_{\alpha,0} \mid \alpha < \kappa\}$. By our construction $B \notin \mathcal{A}$ since it intersects every set $\kappa \setminus A$ for $A \in \mathcal{A}$. Also for any $h \in FF(\mathcal{A})$ the four sets mentioned above are of size κ, hence $\mathcal{A} \cup \{B\}$ is an independent family and A meets every boolean combination of sets of that family on a set of size κ. $\square$

With these two lemmata we can prove then main result of this section.

Theorem 4.2. [8] Let G be a filter that is $\mathbb{P}_\mathcal{U}$-generic. Then the family $\mathcal{A}_G := \bigcup_{\mathcal{A} \in \mathcal{B}_G} \mathcal{A}$, where $\mathcal{B}_G$ is defined as $\mathcal{B}_G := \{\mathcal{A} \mid \exists A \in \mathcal{U} \text{ such that } (\mathcal{A}, A) \in G\}$, is a maximal independent family.

Proof. Suppose not and let $X \in [\kappa]^\kappa \setminus \mathcal{A}_G$ be such that $\mathcal{A}_G \cup \{X\}$ is an independent family. Then we can use the Truth-Lemma to find some $(\mathcal{A}, A) \in G$ such that $(\mathcal{A}, A) \Vdash (X \notin \mathcal{A}_G$ and "$\mathcal{A}_G \cup \{X\}$ is independent"). The set X also belongs to the ground model because $\mathbb{P}_\mathcal{U}$ is κ-closed by the Lemma 4.1.

Assume now that for the set A from above and for every $h \in FF(\mathcal{A})$ both the sets $\mathcal{A}^h \cap X \cap A$ and $\mathcal{A}^h \cap (\kappa \setminus X) \cap A$ are of size κ. Then we have that $(\mathcal{A} \cup \{X\}, A) \in \mathbb{P}_U$ and $(\mathcal{A} \cup \{X\}, A) \leq (\mathcal{A}, A)$. We know that there is some $\mathbb{P}_\mathcal{U}$-generic filter H such that $(\mathcal{A} \cup \{X\}, A) \in H$ and hence $X \in \mathcal{A}_H$. Thus, $(\mathcal{A} \cup \{X\}, A) \Vdash$ "$X \notin \mathcal{A}_G$". This contradicts our previous assumption.

The other possibilities is that there is some $h \in FF(\mathcal{A})$ such that $\mathcal{A}^h \cap X \cap A$ or $\mathcal{A}^h \cap (\kappa \setminus X) \cap A$ has size less than κ. Using the previous lemma we can find $B \subseteq A$, such that $B \notin \mathcal{A}$ and $(\mathcal{A} \cup \{B\}, A) \in \mathbb{P}_{\mathcal{U}}$. Hence $(\mathcal{A} \cup \{B\}, A) \leq (\mathcal{A}, A)$. So there is some $h \in FF(\mathcal{A} \cup \{B\})$, such that $(\mathcal{A} \cup B)^h \cup X$ or $(\mathcal{A} \cup B)^h \cup (\kappa \setminus X)$ is bounded in κ. Analogous to the previous case then $(\mathcal{A} \cup \{B\}, A) \Vdash$ "$\mathcal{A}_G \cup \{X\}$ is not independent". This is a contradiction, since forcing is inherited by stronger conditions. □

4.2.2 Density filter and density ideal

Before we can show dense maximality, we have to introduce some new concepts. Part of it will be similar to the countable case which is discussed in the first chapter.

Definition 4.6. [8] For any $\mathcal{U}$-supported independent family $\mathcal{A}$ let $\mathcal{U}^*$ be the dual filter of $\mathcal{U}$ and define $id_{<\omega,\kappa}(\mathcal{A}) := \{X \mid X \in \mathcal{U}^* \text{ and for all } h \in FF(\mathcal{A}) \exists h' \supseteq h \text{ such that } \mathcal{A}^{h'} \cap X = \emptyset\}$. We refer to $id_{<\omega,\kappa}(\mathcal{A})$ as the density ideal of $\mathcal{A}$ and write $id(\mathcal{A})$. The dual filter of $id(\mathcal{A})$ will be denoted $fil(\mathcal{A})$ and will be referred to as the density filter of $\mathcal{A}$.

The main difference to the countable definition in the first chapter is the relationship to the filter $\mathcal{U}$. To justify the above terms one should check whether $id(\mathcal{A})$ is really an ideal. It is downward closed since for any $X \in id(\mathcal{A})$ and any $h \in FF(\mathcal{A})$ such that $\mathcal{A}^h \cap X = \emptyset$, it also holds that $\mathcal{A}^h \cap Y = \emptyset$ for all $X \subseteq Y$. For two sets $X, Y \in id(\mathcal{A})$, we need to check whether for any $h \in FF(\mathcal{A})$ there is some extension h' such that $\mathcal{A}^{h'} \cap (X \cup Y) = \emptyset$. Fix X, Y and h as before. Then there is $h_1 \supseteq h$ such that $\mathcal{A}^{h_1} \cap X = \emptyset$. Now there is $h' \supseteq h_1$ such that $\mathcal{A}^{h'} \cap Y = \emptyset$. Now $\mathcal{A}^{h'} \cap (X \cup Y) = \emptyset$ and hence $X \cup Y \in id(\mathcal{A})$.

Using the definition of the density ideal we can prove the dense maximality of the maximal independent family $\mathcal{A}_G$ for any $\mathbb{P}_{\mathcal{U}}$-generic filter G. Before we do this we need three lemmata. One of them is an equivalent condition to dense maximality, the other two will follow now. The first one is the key idea for showing the second one.

Lemma 4.3. [8] Let Y be an unbounded subset of κ. Then for every $(\mathcal{A}, A) \in \mathbb{P}_{\mathcal{U}}$ and for every $h \in FF(\mathcal{A})$ there are some $B \subseteq A$, so that $(\mathcal{A}, B) \leq (\mathcal{A}, A)$ and some h' extending h such that $\mathcal{A}^{h'} \cap B$ is either contained in Y or in its complement.

Proof. Fix $(\mathcal{A}, A)$ and h. If there is some $h' \supseteq h$, such that $A^{h'} \cap A \cap Y$ is of cardinality less than κ, then $A^{h'} \cap A \subseteq^* A^{h'} \cap A \cap (\kappa \setminus Y)$. If we choose $B := (\mathcal{A}^{h'} \cap Y \cap (\kappa \setminus Y)) \cup (A \setminus \mathcal{A}^{h'})$, then $A \subseteq^* B$ and $B \subseteq^* A$. Hence B is an element of $\mathcal{U}$ that has an unbounded intersection with every Boolean combination of $\mathcal{A}$. Analogously, if there is some extension h' of h such that $\mathcal{A}^{h'} \cap A \cap (\kappa \setminus Y)$ is bounded, we can take $B := (\mathcal{A}^{h'} \cap A \cap Y) \cup (A \setminus \mathcal{A}^{h'})$.

Now assume that for every h' extending h , $\mathcal{A}^{h'} \cap A$ has an unbounded intersection with both Y and $\kappa \setminus Y$. If $A \setminus \mathcal{A}^h \in \mathcal{U}$, we can take $B := (A \setminus \mathcal{A}^h) \cup (A \cup \mathcal{A}^h \cup Y)$. Otherwise, we now that $A \cap \mathcal{A}^h \in \mathcal{U}$. This implies that so is either $A \cap \mathcal{A}^h \cap Y$ or $A \cap \mathcal{A}^h \cap (\kappa \setminus Y)$. So we can take B to be the union of $A \setminus \mathcal{A}^h$ and the one of these two sets that is in $\mathcal{U}$. □

Lemma 4.4. [8] For every $Y \subseteq \kappa$ the set $D_Y := \{(\mathcal{A}, A) \in \mathbb{P}_{\mathcal{U}} \mid \forall h \in FF(\mathcal{A}) \exists h' \supseteq h \text{ such that } \mathcal{A}^{h'} \text{ is either contained in } Y \text{ or in } \kappa \setminus Y\}$ is dense in $\mathbb{P}_{\mathcal{U}}$.

Proof. For Y of size less than κ the statement follows immediately as then $\kappa \setminus Y$ is an element of $\mathcal{U}$.

For the general case let $(\mathcal{A}, A) \in \mathbb{P}_{\mathcal{U}}$. We want to construct some stronger condition in D_Y. To do so, we want to use the κ^+-closedness of $\mathbb{P}_{\mathcal{U}}$ some times. In the proof of Lemma 4.1, we showed something even stronger, namely that for any decreasing chain $\{(\mathcal{A}_\alpha), A_\alpha\}_{\alpha<\kappa}$, there exists a condition $(\mathcal{B}, B)$ that is stronger than all these conditions and has the property that $\mathcal{B} = \bigcup_{\alpha<\kappa} \mathcal{A}_\alpha$. So whenever we use Lemma 4.1 in this proof we want to obtain a condition of that form.

With that knowledge we want to construct a sequence $(\mathcal{A}_n, A_n)_{n\in\omega}$ such that $(\mathcal{A}_{n+1}, A_{n+1})$ is stronger than $(\mathcal{A}_n, A_n)$ for all $n \in \omega$, $(\mathcal{A}_0, A_0) = (\mathcal{A}, A)$ and for every $h \in FF(A_n)$, there is some $h' \in FF(A_{n+1})$ such that $h' \supseteq h$, if h is viewed a function with domain contained in $\mathcal{A}_{n+1}$, and $\mathcal{A}^{h'}$ is either contained in Y or in $\kappa \setminus Y$. By the observation from before the common extension $(\mathcal{B}, B)$ of these conditions $(\mathcal{A}_n, A_n)$ such that $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{A}_n$ is an element of D_Y such that $(\mathcal{B}, B) \leq (\mathcal{A}, A)$.

How to construct such a sequence? Given $(\mathcal{C}, C) \in \mathbb{P}_{\mathcal{U}}$, we want to construct some $(\mathcal{D}, D) \leq (\mathcal{C}, C)$ such that for all $h \in FF(\mathcal{C})$ there is some $h' \supseteq h \in FF(\mathcal{D})$ such that $\mathcal{D}^{h'}$ is contained in Y or in $\kappa \setminus Y$. Let $\{h_\alpha\}_{\alpha<\kappa}$ be an enumeration of $FF(\mathcal{C})$. We want to construct a sequence $\{(\mathcal{C}_\alpha, C_\alpha)\}_{\alpha<\kappa}$ such that $(\mathcal{C}_\alpha, C_\alpha) \leq (\mathcal{C}, C)$, $(\mathcal{C}_\beta, C_\beta) \leq (\mathcal{C}_\alpha, C_\alpha)$ for $\alpha < \beta$ and for every $\alpha < \kappa$ and there exist an $h' \in FF(\mathcal{C}_\alpha)$ such that $h' \supseteq h$ and $\mathcal{C}_\alpha^{h'}$ is either contained in Y or in $\kappa \setminus Y$. Application of the Lemma 4.1, then yields the suitable $(\mathcal{D}, D)$, where $\mathcal{D} = \bigcup_{\alpha<\kappa} \mathcal{C}_\alpha$

So, let us proceed with the construction. By the previous lemma there is some $E \subseteq C$ such that $(\mathcal{C}, E) \leq (\mathcal{C}, C)$ and there is $h' \supseteq h_0$ such that $\mathcal{C}^{h'} \cap E$ is either contained in Y or in its complement. Using Lemma 4.2, we can find $F \subseteq E$, such that $(\mathcal{C} \cup \{F\}, E)$ is an extension of $\mathcal{C}, E$. In that extension we can choose $h'' := h \cup (F, 0)$, which is the desired extension for h. So let $(\mathcal{C}_0, C_0) := (\mathcal{C} \cup \{F\}, E)$. We construct the sequence iteratively. Every successor step is analogous to the construction of $(\mathcal{C}_0, C_0)$. For the limit step, for some limit ordinal α let first $(\mathcal{C}^*_\alpha, C^*_\alpha)$ be a common extension of every $(\mathcal{C}_\beta, C_\beta)$ with $\beta < \alpha$ such that $\mathcal{C}^*_\alpha = \bigcup_{\beta<\alpha} \mathcal{C}_\beta$. Obtain $(\mathcal{C}_\alpha, C_\alpha) < (\mathcal{C}^*_\alpha, C^*_\alpha)$, where one has some $h' \supseteq h_\alpha$ such that $\mathcal{C}_\alpha^{h'})$ is contained in Y or its complement in the same way as in the construction of $(\mathcal{C}_0, C_0)$.

□

With the help of these two lemmata we can show dense maximality. In fact we want to show an equivalent condition, which will be formulated below.

Lemma 4.5. [8] Let $\mathcal{A}$ be an independent family. Then $\mathcal{A}$ is densely maximal if and only if for all $h \in FF(\mathcal{A})$ and for all $X \subseteq \mathcal{A}^h$ there is some $B \in id(\mathcal{A})$ with $\mathcal{A}^h \setminus X \subseteq B$ or there exists an extension h' of h such that $\mathcal{A}^{h'} \subseteq \mathcal{A}^h \setminus X$.

Proof. Take an independent family $\mathcal{A}$ and suppose that it is densely maximal. We want to show that it satisfies the above property. So let $h \in FF(\mathcal{A})$ and $X \subseteq \mathcal{A}^h$. Assume that there is no $B \in id(\mathcal{A})$ such that $\mathcal{A}^h \setminus X \subseteq B$. Then $\mathcal{A}^h \setminus X$ is not an element of the density ideal either. This means that there is some $h_1 \in FF(\mathcal{A})$ such that for all h' extending h_1 it holds that $\mathcal{A}^{h'} \cap (\mathcal{A}^{h'} \setminus X) \neq \emptyset$. If h and h_1 do not have a common extension, then $\mathcal{A}^{h_1} \cap (\mathcal{A}^h \setminus X) \subseteq \mathcal{A}^{h_1} \cap \mathcal{A}^h = \emptyset$. This would contradict the assumption from before. Take a common extension h' of both h and h_1. Since $\mathcal{A}$ is densely maximal h' has an extension h'' such that $\mathcal{A}^{h''} \cap X = \emptyset$. This shows that $\mathcal{A}^{h''} \subseteq \mathcal{A}^{h'} \setminus X \subseteq \mathcal{A}^h$, which proves the first direction of the statement.

Now assume the above property holds for the independent family $\mathcal{A}$ and let us show that $\mathcal{A}$ is densely maximal. Fix $Y \subseteq \kappa$ and $h \in FF(\mathcal{A})$. We want to find an extension h' of h such that $\mathcal{A}^{h'} \cap Y = \emptyset$. Let $X := Y \cap \mathcal{A}^h$. If there is some $B \in id(\mathcal{A})$ such that $\mathcal{A}^h \setminus Y \subseteq B$, then $\mathcal{A}^h \setminus Y$ is in the density ideal as well and so is $\mathcal{A}^h \setminus X$. Hence there is some $h' \supseteq h$ such that $\emptyset = \mathcal{A}^{h'} \cap (\mathcal{A}^h \setminus X) = \mathcal{A}^{h'} \setminus X$. Otherwise there exists some h' extending h such that $\mathcal{A}^{h'} \subseteq \mathcal{A}^h \setminus Y = \mathcal{A}^h \setminus X$ and so $\mathcal{A}^{h'} \cap X = \emptyset$. So $\mathcal{A}$ is densely maximal. $\square$

Now we can show the main result of this section, which is that the family $\mathcal{A}_G$ is densely maximal.

Theorem 4.3. [8] For any $\mathbb{P}_{\mathcal{U}}$-generic filter G, the family $\mathcal{A}_G$ is densely maximal.

Proof. In fact we would like to prove the equivalent condition from Lemma 4.5. To do so take $h \in FF(\mathcal{A}_G)$ and $X \subseteq \mathcal{A}_G^h$. Assume now as in the proof before that there is no $B \in id(\mathcal{A}_G)$ such that $\mathcal{A}_G^h \setminus X \subseteq B$. So especially $\mathcal{A}_G^h \setminus X$ is not in the ideal. Analogous to before we find h' extending h such that for all $h'' \supseteq h$, we have that $\mathcal{A}^{h''} \cap (\mathcal{A}^h \setminus X) \neq \emptyset$. Now take $Y := \mathcal{A}^h \setminus X$. Using Lemma 4.4, we can find $h'' \supseteq h'$, such that $\mathcal{A}^{h''}$ is either contained in Y or in $\kappa \setminus Y$. If $\mathcal{A}^{h''} \subseteq \kappa \setminus Y$, then $\mathcal{A}^{h''} \cap (\mathcal{A}^h \setminus X) = \emptyset$, which is a contradiction to how we defined h'. So $\mathcal{A}^{h''} \subseteq Y = \mathcal{A}^h \setminus X$, which is exactly the second case in the previous lemma. $\square$

4.2.3 More on density filter and density ideal

The main goal of this section is to develop some results that will be helpful to argue about the generic extension in the next section. A lot of these observations will be about the generators of density ideal and density filter. The first goal will be to show that the density filter is a p-set. As in the countable setting this statement can be shown more easily considering a set which generates the density filter. These generators are similar to the countable setting and can be found with the help of a few lemmata. The first one is just a useful observation.

Lemma 4.6. [8] For two independent families $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \subseteq \mathcal{B}$, it holds that $id(\mathcal{A}) \subseteq id(\mathcal{B})$.

Proof. This follows from the definition of the density ideal. Let $X \in id(\mathcal{A})$, and $h \in FF(\mathcal{B})$. We want to find some extension h' of h such that $\mathcal{B}^{h'} \cap X = \emptyset$. Consider $h_1 := h \restriction \mathcal{A}$. Then there exists some $h_2 \supseteq h_1$ such that $\mathcal{A}^{h_2} \cap X = \emptyset$. Now take $h' := h_2 \cup (h \setminus h_1)$. Then $h' \in FF(\mathcal{B})$ is well defined and $\mathcal{B}^{h'} \cap X \subseteq \mathcal{B}^{h_2} \cap X = \emptyset$. Since X was chosen arbitrarily this proves the claim. $\square$

The next lemmata will be there to establish a set of generators of $id(\mathcal{A}_G)$, from which we will easily get a set of generators of $fil(\mathcal{A}_G)$.

Lemma 4.7. [8] $\Vdash_{\mathbb{P}_{\mathcal{U}}} id(\mathcal{A}_G) = \bigcup\limits_{\mathcal{A} \in \mathcal{B}_G} id(\mathcal{A})$, where $\mathcal{B}_G = \{\mathcal{A} \mid \exists A \text{ such that } (\mathcal{A}, A) \in G\}$

Proof. To show this let G be $\mathbb{P}_{\mathcal{U}}$-generic. Since $A_G = \bigcup\{\mathcal{A} \mid \exists A : (\mathcal{A}, A) \in G\}$ and $id(\mathcal{A} \subseteq id(\mathcal{A}_G))$ by the previous lemma, $V[G] \models \bigcup\limits_{\mathcal{A} \in \mathcal{B}_G} id(\mathcal{A}) \subseteq id(\mathcal{A}_G)$.

For other the set inclusion let X be a set in $V[G]$, such that $V[G] \models X \in id(\mathcal{A}_G)$. Then there are a name $\dot{X}$ for X and a condition $p = (\mathcal{A}, A) \in G$, such that $p \Vdash \dot{X} \in id(\mathcal{A}_G)$. We want to find a condition below p which forces that $\dot{X} \in \bigcup\limits_{\mathcal{A} \in \mathcal{B}_G} id(\mathcal{A})$. To do so let $h \in FF(\mathcal{A})$ then p forces that there is an extension h' of h in $FF(\mathcal{A}_G)$, such that $\mathcal{A}_G^{h'} \cap X = \emptyset$.. By the definition of $\mathcal{A}_G$ there is some $(\mathcal{A}', A') \leq (\mathcal{A}, A)$ in G, such that $dom(h') \subseteq \mathcal{A}'$. We can proceed inductively similar to the proof of Lemma 4.4 to obtain some condition $(\mathcal{B}, B) \leq (\mathcal{A}, A)$ such that for all $h \in FF(\mathcal{B})$ there is some $h' \subseteq h$ in $FF(\mathcal{B})$, such that $\mathcal{B}^{h'} \cap X = \emptyset$. But this means that $X \in id(\mathcal{B})$. So there is some condition q below $(\mathcal{B}, B)$ and hence below p such that $q \Vdash \dot{X} \in \bigcup_{\mathcal{A} \in \mathcal{B}_G} id(\mathcal{A})$. $\square$

Before taking a look at the generators of $id(\mathcal{A}_G)$, we will need another short lemma.

Lemma 4.8. [8] Let $(\mathcal{A}, A) \in \mathbb{P}_{\mathcal{U}}$ and let $X \in id(\mathcal{A})$. Then $(\mathcal{A}, A \setminus X)$ is a condition.

Proof. Since $X \in id(\mathcal{A}) \subseteq \mathcal{U}^*$, we know that $\kappa \setminus \in \mathcal{U}$ and so is $\mathcal{A} \setminus X$. So it suffices to show that $A \setminus X$ meets every boolean combination of $\mathcal{A}$ on a set of size κ. To do so let $h \in FF(\mathcal{A})$. Then there is some h' extending h such that $\mathcal{A}^{h'} \cap X = \emptyset$. So $\mathcal{A}^{h'} \cap A = \mathcal{A}^{h'} \cap A \cap X \cup \mathcal{A}^{h'} \cap (A \setminus X) = \mathcal{A}^{h'} \cap (A \setminus X) \subseteq \mathcal{A}^{h} \cap (A \setminus X)$. This means that $\mathcal{A}^{h} \cap (A \setminus X)$ contains the set $\mathcal{A}^{h'} \cap A$ which is of size κ. $\square$

Now back to the generators of the density ideal.

Lemma 4.9. [8] For any generic filter G the density ideal $id(\mathcal{A}_G)$ in $V[G]$ is generated by the sets $\{\kappa \setminus A \mid \exists \mathcal{A} : (\mathcal{A}, A) \in G\}$.

Proof. Let $\mathcal{I}_G$ be the ideal generated by the sets $\kappa \setminus A$ for each condition $(\mathcal{A}, A) \in G$. At first we show that $id(\mathcal{A}_G) \subseteq \mathcal{I}_G$. So let $X \in id(\mathcal{A}_G)$. Using Lemma 4.7 there is some condition $(\mathcal{A}, A) \in G$, such that $X \in id(\mathcal{A})$. Using Lemma 4.8 and that for any $(\mathcal{B}, B) \leq (\mathcal{A}, A)$ one has that $id(\mathcal{A}) \subseteq id(\mathcal{B})$, one can see that the set $D := \{(\mathcal{B}, B) \mid B \cap X = \emptyset\}$ is dense below $(\mathcal{A}, A)$. So $G \cap D$ contains an element $(\mathcal{B}, B)$ and $X \subseteq \kappa \setminus B$. Hence $X \in \mathcal{I}_G$.

To show that $\mathcal{I}_G \subseteq id(\mathcal{A}_G)$, let $X \in \mathcal{I}_G$. By the definition of this ideal this means that there is $n \in \omega$ and there are some conditions $((\mathcal{A}_i, A_i))_{i \leq n}$ in G, such that $X \subseteq \bigcup_{i \leq n} (\kappa \setminus A_i)$. Since G is a filter the condition $(\bigcup_{i \leq n} \mathcal{A}_i, \bigcap_{i \leq n} A_i) =: (\mathcal{B}, B)$ is also an element of the generic filter and $X \subseteq \kappa \setminus B$. Using Lemma 4.4 and the fact that $B \subseteq \kappa \setminus X$, we can see that the set $D' := \{(\mathcal{C}, C) \mid \forall h \in FF(\mathcal{C}) \exists h' \supseteq h \text{ such that } \mathcal{C}^{h'} \cap X = \emptyset\}$ is dense below $(\mathcal{B}, B)$. By G being generic $G \cap D'$ has an element $(\mathcal{C}, C)$. Now $X \in id(\mathcal{C})$ and this means using Lemma 4.7 again that $X \in \mathcal{A}_G$. □

An immediate consequence of this result is the following:

Corollary 4.1. [8] The density filter $fil(\mathcal{A}_G)$ is generated by $\mathcal{F}_G := \{A \mid \exists \mathcal{A} \text{ such that } (\mathcal{A}, A) \in G\}$.

For the preservation under the Sacks forcing we want the family $\mathcal{A}_G$ to have properties that are similar to being a selective independent family in the countable setting. The p-set property of a selective independent family however can be generalised easily.

Definition 4.7. [8] Let $\mathcal{F}$ be a filter. Then $\mathcal{F}$ is a (κ-)p-set iff every family $\mathcal{H} \subseteq \mathcal{F}$ such that $|\mathcal{H}| < \kappa$ has a pseudo-intersection in $\mathcal{F}$.

For the rest of this chapter we will write 'p-set' for κ-p-set. Analogous to what was done in the first chapter we can show that $fil(\mathcal{A}_G)$ is indeed a p-set.

Lemma 4.10. [8] The density filter $fil(\mathcal{A}_G)$ is a p-set.

Proof. Since any set in $fil(\mathcal{A}_G)$ contains a finite intersection of sets in $\mathcal{F}_G$ it suffices to show that any subset of $\mathcal{F}_G$ of size $\leq \kappa$ has a pseudo-intersection in $\mathcal{F}_G$.

Now suppose this does not hold in $V[G]$. So there is some $p = (\mathcal{A}, A) \in \mathbb{P}_{\mathcal{U}}$ such that $p \Vdash \exists \mathcal{H} \subseteq [\mathcal{F}_G]^{\kappa} (\forall F \in \mathcal{F}_G \exists H \in \mathcal{H} (F \not\subseteq^* H))$. Since $\mathbb{P}_{\mathcal{U}}$ is κ^+-closed there is a family $\mathcal{H}' := \{A_\alpha\}_{\alpha < \kappa}$ in $\mathcal{F}_G \cap V$ that does not have a pseudo-intersection in $\mathcal{F}_G$. By the definition of $\mathcal{F}_G$ and the set $\mathcal{A}_G$, we can find for each $\alpha < \kappa$ an independent family $\mathcal{A}_\alpha$, such that $(\mathcal{A}_\alpha, A_\alpha) \in G$. Since any two conditions in G are compatible, we may assume $((\mathcal{A}_\alpha, A_\alpha))_{\alpha < \kappa}$ to be a decreasing chain of conditions below p. But this then has a common lower bound $(\mathcal{B}, B) \in \mathbb{P}_{\mathcal{U}}$. But that means $(\mathcal{B}, B) \leq p$ and $(\mathcal{B}, B) \Vdash$ "B is a pseudo-intersection of $\{A_\alpha\}_{\alpha < \kappa}$". But this contradicts the fact that forcing is inherited by stronger conditions. □

The q-set property of filters on $\mathcal{P}(\omega)$ is harder to generalise. What comes close to the q-set property is the existence of sets in the filter such that for any increasing function $f : \kappa \to \kappa$ there is a set in the filter the enumeration function of which grows way much faster than f In the next two lemmata we show this property for $fil(\mathcal{A}_G)$.

Lemma 4.11. [8] Let $(\mathcal{A}, A) \in \mathbb{P}_{\mathcal{U}}$ be a condition and $f : \kappa \to \kappa$ be strictly increasing. Then there is $A^* \subseteq A$ such that $(\mathcal{A}, A^*) \leq (\mathcal{A}, A)$ and if $\{a_\alpha\}_{\alpha < \kappa}$ is the increasing enumeration of A^* then $f(a_\alpha) < a_{S(\alpha)}$ for all $\alpha < \kappa$.

Proof. Fix $(\mathcal{A}, A)$ and f and let $\{h_\xi\}_{\xi < \kappa}$ be an enumeration of $FF(\mathcal{A})$, in which each $h \in FF(\mathcal{A})$ occurs unboundedly often. We want to construct the set A^* iteratively. To do so let $C_f := \{\xi \mid \forall\zeta < \xi : f(\zeta) < \xi\}$. Then C_f is a closed and unbounded set and therefore in $\mathcal{U}$. Define $E := C_f \cap A$. Hence $E \in \mathcal{U}$, since the normal measure is closed under finite intersections.

By construction for all $a < b$ which are in E it holds that $f(a) < b$. Since E does not necessarily intersect every Boolean combination on a set of size κ, we want to add some witnesses out of A which ensure this and remove some elements of E, such that the first property remains satisfied.

For the construction define $s_E : \kappa \to \kappa$ as $s_E(\alpha) := min\{\beta \in E \mid \alpha < \beta\}$. Further let $g(\alpha) := s_E(s_E(\alpha))$. To abbreviate notation write (α, β) for the set $\{\gamma \mid \alpha < \gamma < \beta\}$.

We want to construct a sequence $\{B_\xi\}_{\xi<\kappa}$, such that $A^* = \bigcup_{\xi<\kappa} B_\xi$. Start with the choice of B_0. If $\mathcal{A}^{h_0} \cap E \neq \emptyset$ choose $a_0 := min(\mathcal{A}^{h_0} \cap E)$. Otherwise choose it to be $a_0 := min(\mathcal{A}^{h_0} \cap A)$. Then define $B_0 := \{a_0\} \cup ((a_0, g(f(a_0)) \cap E))$.

Proceed iteratively. If we already defined a_ξ and B_ξ for some ordinal ξ then we do the following. If $(\mathcal{A}^{h_{\xi+1}} \cap E) \setminus (g(f(a_\xi))) \neq \emptyset$ choose $a_{\xi+1} := min((\mathcal{A}^{h_{\xi+1}} \cap E) \setminus (g(f(a_\xi))))$. Otherwise take $a_{\xi+1} := min((\mathcal{A}^{h_{\xi+1}} \cap A) \setminus (g(f(a_\xi))))$. Then define $B_{\xi+1} := B_\xi \cup \{a_{\xi+1}\} \cup (a_{\xi+1}, g(f(a_{\xi+1}))) \cap E$.

If ξ is a limit and a_γ and B_γ have already been defined for $\gamma < \xi$, then let $B^*_\xi := \bigcup_{\gamma<\xi} B_\gamma$. Let $m := supB^*_\xi$. Again if $(\mathcal{A}^{h_\xi} \cap E) \setminus (m \cup \{m\}) \neq \emptyset$ choose $a_\xi := min((\mathcal{A}^{h_\xi} \cap E) \setminus (m \cup \{m\}))$. Otherwise take $a_\xi := min((\mathcal{A}^{h_\xi} \cap A) \setminus (m \cup \{m\}))$. Define $B_\xi := B^*_\xi \cup a_\xi \cup (a_\xi, g(f(a_\xi))) \cap E$.

By the choice of the sequence $\{a_\xi\}_{\xi<\kappa}$, the set A^* intersects every $\mathcal{A}^h$ for $h \in FF(\mathcal{A})$ on a set of size κ. In addition $A^* \subseteq A$. Firstly we still have to check that for any $a, b \in A^*$, such that $a < b$ it holds that $f(a) < b$. We know that this holds if $b \in E$. If $b \notin E$, then $b = a_\xi$ for some $\xi < \kappa$. If $\xi = \gamma + 1$ is a successor, then we know that $a \in B_\gamma$ and since $a < g(f(a_\gamma))$ we know that $f(a) < g(f(a_\gamma) < a_{\gamma+1}$. If ξ is a limit ordinal, then $a \in B_\gamma$ for some $\gamma < \xi$ and because $a < g(f(a_\gamma))$ also $f(a) < g(f(a_\gamma)) < a_\xi$.

What still needs to be shown is that $A^* \in \mathcal{U}$. To do so let $C' := C_f \setminus ([0, a_0) \cup \bigcup_{\xi<\kappa} ([g(f(a_\xi)), a_{\xi+1})) = \bigcup_{\xi<\kappa} (C_f \cap [a_\xi, g(f(a_\xi)))$. We want to show that C' is a club because then $A^* \supseteq C' \cap E$ and hence it is in the normal measure. C' is unbounded because so is the sequence $\{a_\xi\}_{\xi<\kappa}$ and $a_\xi < s_E(f(a_\xi)) < g(f(a_\xi))$ and hence $s_E(f(a_\xi)) \in C_f$. To show that C' is closed let $\lambda < \kappa$ be a limit ordinal and let $\{b_\gamma\}_{\gamma<\lambda} \subseteq C'$. Let $b := sup\{b_\gamma : \gamma < \lambda\}$. We want to show that $b \in C'$. Let $\eta := sup\{\xi \mid a_\xi < b\}$, then $a_\eta \leq b$. If $b = a_\eta$, we are done, as then $b \in C_f \cap A$. Otherwise we want to show that $b \in (a_\eta, g(f(a_\eta))) \cap C_f$. Because C_f is a club, we know that $b \in C_f$. Since $b > a_\eta$ there is some β, such that for all $\gamma \geq \beta$ it holds that $b_\gamma \in (a_\eta, g(f(a_\eta))) \cap C_f$. Hence $b \leq g(f(a_\eta))$. But it cannot be the case that $b = g(f(a_\eta))$, because $b = g(f(a_\eta)) > s_E(f(a_\eta))$ and hence there is some γ such that $b_\gamma > s_E(f(a_\eta))$, contradicting the definition of g and s_E. $\square$

From this density result we get the existence of such a set A^* for every f in $fil(\mathcal{A}_G)$ easily.

Lemma 4.12. [8] Let $f : \kappa \to \kappa$ be a strictly increasing function. Then there is an element $A \in fil(\mathcal{A}_G)$, such that if $\{a_\alpha\}_{\alpha<\kappa}$ is the increasing enumeration of A for every $\alpha < \kappa$ it holds that $f(a_\alpha) < a_{S(\alpha)}$.

Proof. Let f be a strictly increasing function. By the previous lemma the set $D_f := \{(\mathcal{A}, A) \mid$ for the increasing enumeration $\{a_\alpha\}_{\alpha<\kappa}$ of A it holds that $f(a_\alpha) < a_{S(\alpha)} \forall \alpha < \kappa\}$ is dense in $\mathbb{P}_\mathcal{U}$. So there is some condition $(\mathcal{A}, A) \in D_f \cap G$. Since $fil(\mathcal{A}_G)$ is generated by the set $\mathcal{F}_G$, which contains A, we know $A \in fil(\mathcal{A}_G)$. $\square$

4.3 Preservation under κ-Sacks-forcing

In this section we want to show that the maximal independent family added by a $\mathbb{P}_\mathcal{U}$-generic filter G remains maximal after forcing with the poset $\mathbb{S}_\kappa^\lambda$. As the proof is involved we will first show a weaker result which is that if V_0 is the ground model and $V := V_0[G]$ for a $\mathbb{P}_\mathcal{U}$-generic filter G, then $\mathcal{A}_G$ is densely maximal in any $\mathbb{S}_\kappa$-generic extension $V^{\mathbb{S}_\kappa}$.

Theorem 4.4. [8] Let κ be a measurable cardinal and $\mathcal{U}$ be a normal measure on κ. Let V_0, V and $V^{\mathbb{S}_\kappa}$ be defined as above. Then $V^{\mathbb{S}_\kappa} \models$ "$\mathcal{A}_G$ is densely maximal" .

Proof. Abbreviate $\mathcal{A}_G$ by $\mathcal{A}$ during this proof. We already showed that $\mathcal{A}$ is densely maximal in V. To show the same in $V^{\mathbb{S}_\kappa}$ we want to use Lemma 4.5. So we have to show that for any $h \in FF(\mathcal{A})$ and for any $X \subseteq \mathcal{A}^h \cap \kappa$ there is either some $B \in id(\mathcal{A})$ such that $\mathcal{A}^h \setminus X \subseteq B$ or there is an extension h' of h such that $\mathcal{A}^{h'} \subseteq \mathcal{A}^h \setminus X$. Assume not, then there are $h \in FF(\mathcal{A})$ and $X \subseteq \kappa$ such that $V^{\mathbb{S}_\kappa} \models (X \subseteq \mathcal{A}^h) \wedge (\mathcal{A}^h \setminus X \notin id(\mathcal{A})) \wedge (\forall h' \supseteq h(\mathcal{A}^{h'} \cap X \neq \emptyset))$.

Let $\dot{X}$ be a name for X and let $p \in \mathbb{S}_\kappa$, such that $p \Vdash (\dot{X} \subseteq \mathcal{A}^h) \wedge (\mathcal{A}^h \setminus \dot{X} \notin id(\mathcal{A})) \wedge (\forall h' \supseteq h(\mathcal{A}^{h'} \cap \dot{X} \neq \emptyset))$. Using Lemma 3.8 we can replace p by a stronger condition which is preprocessed for $\dot{X}$, so assume without loss of generality that p is already preprocessed for $\dot{X}$. For each $t \in split(p)$ let Y_t be the outer hull of $\dot{X}$ below p_t. By Corollary 3.1 then $Y_t = \{\beta < \kappa \mid \exists r \in split(p), t \trianglerighteq r \text{ or } r \trianglerighteq t \text{ and } p_r \Vdash \check{\beta} \in \dot{X}\}$.

At first observe that $Y_t \subseteq \mathcal{A}^h$ for each t. This can be easily seen if one picks $m \in Y_t$. Then there has to be a condition $q_m \leq p_t$ such that $q_m \Vdash \check{m} \in Y_t$. But then $q_m \leq p_t \leq p \Vdash \dot{X} \subseteq \mathcal{A}^h$ and hence $q_m \Vdash \check{m} \in \mathcal{A}^h$.

Our goal now ist to use the outer hulls, the p-set and the "similar to q-set" property of $fil(\mathcal{A})$ to obtain a contradiction to the existence of the above X. In one part of the argument we want to build a fusion sequence in which we gather "smallest witnesses" to an ordinal being in the outer hull. Before showing this construction let us define the setup for these minimal witnesses.

For every $t \in split(p)$ and $\beta \in Y_t$ let $r(t, \beta)$ be the witness r below p_t of least splitting level such that $p_r \Vdash \beta \in \dot{X}$. Further define $H : \kappa \to \kappa$ by $H(\gamma) := sup\{S(\gamma)\} \cup \{sl(r(t,\beta)) \mid t \in split_\gamma(p), \beta < \gamma\}$. Here $sl(r(t,\beta))$ is defined to be 0 if $\beta \notin X$.

Now pick $t \in split(p)$ since $\mathcal{A}$ is densely maximal in V we can use Lemma 4.5 again to see that either there is $B_t \in id(\mathcal{A})$ sucht that $\mathcal{A}^h \setminus Y_t \subseteq B_t$ or there is some $h' \in FF(\mathcal{A})$, such that $\mathcal{A}^{h'} \subseteq \mathcal{A}^h \setminus Y_t$. However, if the second case would apply, then this would imply that $p \Vdash \emptyset = \mathcal{A}^{h'} \supseteq \mathcal{A}^{h'} \cap X$, contradicting the choice of p.

So the first case applies which means for any $t \in split(p)$ one has that $\mathcal{A}^h \setminus Y_t \subseteq B_t \in id(\mathcal{A})$ and hence $\mathcal{A}^h \setminus Y \in id(\mathcal{A})$. Then, by definition $Y_t \cup \kappa \setminus \mathcal{A}^h = \kappa \setminus (\mathcal{A}^h \setminus Y_t) \in fil(\mathcal{A})$ for all $t \in split(p)$.

We know that $|split(p)| \leq |p| \leq 2^{<\kappa} \leq \kappa$. Using this and the fact that $fil(\mathcal{A})$ is a p-set one can find a set $C \in fil(\mathcal{A})$, such that $C \subseteq^* Y_t \cup \kappa \setminus \mathcal{A}^h$ for every $t \in split(p)$. From this set inclusion one gets that $C \cap \mathcal{A}^h \subseteq^* Y_t$ for all $t \in split(p)$ and hence $|(C \cap \mathcal{A}^h) \setminus Y| < \kappa$.

Especially $|\bigcup\{(C \cap \mathcal{A}^h) \mid t \in split_{\alpha+1}(p)\} \setminus Y_t| < \kappa$. Thus one can define $f \in^\kappa \kappa$ such that for all $\alpha < \kappa$ and all $t \in split_{\alpha+1}(p)$ it holds that $(C \cap \mathcal{A}^h) \setminus Y_t \subseteq f(\alpha)$. Without loss of generality we choose f to be strictly increasing and such that $H(\gamma+1) < f(\gamma)$ for all $\gamma < \kappa$. Without loss of generality also assume that f is a function in V because $\mathbb{S}_\kappa$ is ${}^\kappa\kappa$-bounding.

Now let $g := f \circ f$. Since $g \in^\kappa \kappa$ is strictly increasing by Lemma 4.12 there is a set $C^* \in fil(\mathcal{A})$ such that if $(k_\alpha)_{\alpha<\kappa}$ is the increasing enumeration of $C*$ then $g(k_\alpha) < k_{\alpha+1}$ for all $\alpha < \kappa$. Now choose $C' := (C \cap C^*) \setminus f(0)$. Then $C' \in fil(\mathcal{A})$ and let $(k_\alpha)_{\alpha<\kappa}$ be the increasing enumeration of $C' \cap \mathcal{A}^h$.

Now we would like to construct a fusion sequence $(q_\alpha)_{\alpha<\kappa}$ such that $q_0 = p$ and for every $\alpha < \kappa$, $q_{\alpha+1} \Vdash \check{k}_\alpha \in \dot{X}$. Then the limit q of the fusion sequence forces $C' \cap \mathcal{A}^h \subseteq \dot{X}$.

This, however would be a contradiction because the above implies $q \Vdash \mathcal{A}^h \setminus \dot{X} \subseteq \mathcal{A}^h \setminus C'$. Since $\mathcal{A}^h \setminus C' \subseteq \kappa \setminus C' \in id(\mathcal{A})$ this means that $q \Vdash \mathcal{A}^h \setminus \dot{X} \in id(\mathcal{A})$, which is a contradiction to $q \leq p$ and the definition of p.

So let us construct the fusion sequence. In the limit steps as always we will take intersections. So let us construct q_1 separately and then explain, how the successor step is built.

We know that for each $t \in split_1(p)$, we have that $C' \setminus Y_t \subseteq f(0)$ and $k_0 > f(0)$. Since p is preprocessed for $\dot{X}$, we can find $r_i(t,k_0) \in split_{H(k_0)}(p)$ for each $t \in split_0(p)$ and $i \in \{0,1\}$ such that $p_{r_i(t,k_0)} \Vdash \check{k}_0 \in \dot{X}$. Now take $q_1 := \bigcup\{p_{r_i(t,k_0)} \mid t \in split_0(p), i \in \{0,1\}\}$.

Assume now we already defined the fusion sequence up to some q_α in the way that $q_{\gamma+1} = \bigcup\{p_{r_i(t,k(\gamma))} \mid t \in split_\gamma(q_\gamma), i \in \{0,1\}\}$. Now let us define $q_{\alpha+1}$. To do so let $t \in split_\alpha(q_\alpha)$ and distinguish two cases.

For the first case assume that $\alpha = \beta + 1$ is a successor. Then by inductive hypothesis $t \in split_\delta(p)$ for $\delta \leq H(k_\beta)$. But then $f(\delta) \leq f(H(k_\beta)) \leq f(f(k_\beta)) = g(k_\beta) < k_\alpha$ and hence $k_\alpha \in Y_t$. As before we can find $r_i(t, k_\alpha)$, such that $p_{r_i(t,k_\alpha)} \Vdash \check{k}_\alpha \in \dot{X}$ for $i \in \{0,1\}$. Again take $q_{\alpha+1} = \bigcup\{p_{r_i(t,k(\alpha))} \mid t \in split_\gamma(q_\gamma), i \in \{0,1\}\}$.

The second case if α is a limit ordinal is nearly analogous only for the limit step observe that for $t \in split_\alpha(q_\alpha)$ by construction $t \in split_\delta(p)$ for some $\delta < \bigcup_{\gamma<\alpha} H(k_\gamma)$, as for the construction of $q_{\gamma+1}$ only nodes of splitting level less or equal than $H(k_\gamma)$ and the ones above those were removed. But then there is some $\beta < \alpha$ such that $\delta < H(k_\beta)$. Now one can proceed as in the first step.

□

As we now know how the preservation for the single step works, we can do the proof for the product forcing which works quite similar.

Theorem 4.5. [8] Again let V_0 be the ground model in which GCH holds. Let κ be an inaccessible cardinal and let $V = V[G]$ for a $\mathbb{P}_{\mathcal{U}}$-generic filter G. Let $V^{\mathbb{S}_\kappa^\lambda}$ be an $\mathbb{S}_\kappa^\lambda$-generic extension of V. Further let $\mathcal{A} := \mathcal{A}_G$ be the independent family added by G. Then $\mathcal{A}$ remains densely maximal in $\mathbb{S}_\kappa^\lambda$.

Proof. The idea of this proof is very similar to the one for the single step. We assume that $\mathcal{A}$ is not densely maximal in $V^{\mathbb{S}_\kappa^\lambda}$. Again using Lemma 4.5 this means that there is some $h \in FF(\mathcal{A})$ and some set X, such that $X \subseteq \mathcal{A}^h$ but neither $(\mathcal{A}^h \setminus X) \in id(\mathcal{A})$ holds nor is there an h' extending h such that $\mathcal{A}^{h'} \cap X = \emptyset$.

Let $h \in FF(\mathcal{A})$ as above and let$\dot{X}$ be a name for such a set and let $p \in \mathbb{S}_\kappa^\lambda$ be a condition such that $p \Vdash (\dot{X} \subseteq \mathcal{A}^h) \wedge (\mathcal{A}^h \setminus \dot{X} \notin id(\mathcal{A})) \wedge (\forall h' \supseteq h(\mathcal{A}^{h'} \cap \dot{X} \neq \emptyset))$. By Lemma 3.7, we can assume without loss of generality that p is prepossessed for $\dot{X}$.

Now one can observe that if we take a subset $F \subseteq supp(p)$ such that $|F| < \kappa$, then $|\Lambda_\alpha^F(p)| < \kappa$ for every $\alpha < \kappa$. This holds since $|split_\alpha(p(i))| \leq 2^{|\alpha|} < \kappa$ for every $i \in F$ and so $|\Lambda_\alpha^F(p)| \leq (2^\alpha)^{|F|} = 2^{\alpha \cdot |F|} < \kappa$.

Similar to the single step proof if we fix $\alpha < \kappa$ and $F \subseteq supp(p)$, such that $|F| < \kappa$. Then for every $\overline{\sigma} \in \Lambda_\alpha^F$ it holds that the outer hull $Y_{\overline{\sigma}}$ is a subset of $\mathcal{A}^h$. To see this, similar to before let $m \in Y_{\overline{\sigma}}$. Then there are $F' \subseteq F$ with $|F'| < \kappa$ and $\overline{\tau} \subseteq \overline{\sigma}$ such that $p_{\overline{\tau}} \Vdash \check{m} \in \dot{X}$. But since $p \Vdash \dot{X} \subseteq \mathcal{A}^h$, we get that $p_{\overline{\tau}} \Vdash \check{m} \in \mathcal{A}^h$.

For each $\overline{\sigma} \in \Lambda_\alpha^F$ and each $\beta \in Y_{\overline{\sigma}}$ let $r(\overline{\sigma}, \beta)$ be the witness to $\beta \in Y_{\overline{\sigma}}$, such that for each $i \in F$ it holds that $stem(r(\overline{\sigma}, \beta)(i))$ is minimal.

Analog to before we will define a function that should be the maximum of the splitting levels of the minimal witnesses, which is $H(\gamma) := sup\{\gamma + 1\} \cup \{sl(stem(r(\overline{\sigma}, \beta))) \mid \beta < \gamma, F \subseteq supp(p) \cap \gamma, |F| < \kappa, \overline{\sigma} \in \Lambda_\gamma^F\}$.

Again following the lines of the previous proof for every α and F as before and every $\overline{\sigma} \in \Lambda_\alpha^F(p)$ there is some $B_\sigma \in id(\mathcal{A})$, such that $\mathcal{A}^h \setminus Y_\sigma \subseteq B_{\overline{\sigma}}$. This follows directly from the dense maximality in the model V. Either there is such a set $B_{\overline{\sigma}}$ or there is some

$h' \supseteq h$, such that $\mathcal{A}^{h'} \cap Y_t = \emptyset$. But this would mean $\mathcal{A}^{h'} \cap X = \emptyset$, contradicting the choice of X and p.

So we have that $\kappa \setminus B_{\overline{\sigma}} \subseteq \kappa \setminus \mathcal{A}^h \cup Y_{\overline{\sigma}}$. But now, since $\kappa \setminus B_{\overline{\sigma}} \in fil(\mathcal{A})$, so is any superset of it. As discussed before $|\Lambda^F_\alpha(p)| < \kappa$ and the density filter is a κ-p-set. So we can find a pseudo-intersection C of the sets $Y_{\overline{\sigma}} \cup (\kappa \setminus \mathcal{A}^h)$. This means that $\mathcal{A}^h \cap C \subseteq^* Y_{\overline{\sigma}}$ for every $\overline{\sigma} \in \Lambda^F_\alpha$.

Using that $|\Lambda^F_\beta(p)|$ is less than κ for every $\beta < \kappa$, we can find a function $f : \kappa \to \kappa$ such that for all $\alpha < \kappa$, all $|F| \subseteq supp(p)$, such that $|F| < \kappa$ and all $\overline{\sigma} \in \lambda$ it holds that $(C \cap \mathcal{A}^h) \setminus Y_{\overline{\sigma}} \subseteq f(\alpha)$. As in the previous proof we may assume now, that f is strictly increasing and $f(\alpha) > max(H(\alpha+1), \alpha+2)$ for all $\alpha < \kappa$. Again assume that f is a function in V.

Using Lemma 4.12 again we can find a set C^*, such that if $(l_\alpha)_{\alpha<\kappa}$ is its increasing enumeration, then $f(f(l_\alpha)) < l_{\alpha+1}$. Let $C' := (C \cap C^*) \setminus f(0)$ and notice that $C' \in fil(\mathcal{A})$. Let $(k_\alpha)_{\alpha<\kappa}$ be the increasing enumeration of C'.

As in the previous proof we want to construct a fusion sequence $\langle q_\alpha, F_\alpha \mid \alpha < \kappa \rangle$ such that $q_{\alpha+1} \leq_{F_\alpha,\alpha} q_\alpha$ and $q_{\alpha+1} \Vdash \check{k}_\alpha \in \dot{X}$ for every $\alpha < \kappa$.

We start with $q_0 := p$ and $F_0 := \{min(supp(p))\}$. Let $\overline{\sigma} := \langle t \rangle$, with $t \in stem(min(supp(p)))$. Since p is preprocessed, we know that for every $\overline{\tau} \in \Lambda^{F_0}_1$, $C' \setminus Y_{\overline{\tau}} \subseteq f(0) < k_0$. Hence $k_0 \in Y_\tau$.

Now for every $h \in^{F_0} 2$ there is exactly one $\overline{\tau} \in \Lambda^F_1(p)$, such that $\tau_i \subseteq \sigma_i^\frown h(i)$, for $i \in F_0$. Since $k_0 \in Y_{\overline{\tau}}$ there are $F_{0,h} \subseteq F_0$ and $\tau' \in \Lambda^{F_{0,h}}_1(p)$, such that $p_{\overline{\tau}'} \Vdash \check{k}_0 \in \dot{X}$. Now set $r_h(\overline{\sigma}, k_0) := \tau'$. Further define $q_1 := \bigcup\{r_h(\overline{\sigma}, k_0) \mid h \in^{F_0} 2\}$. Then $q_1 \leq_{0,F_0} q_0$ and $q_1 \Vdash \check{k}_0 \in \dot{X}$. Further define $F'_0 := \bigcup\{F_{0,h} \mid h \in^{F_0} 2\}$. Let $F_1 := F'_0 \cup \{min(supp(q_1) \setminus F'_0)\}$.

Now we proceed inductively. The limit steps are defined by how a fusion sequence looks like. So we just explain the successor steps.

Suppose now, we constructed q_α and F_α for some α. Further $split_\alpha(q_\alpha) \subseteq split_\delta(p)$ for some $\delta \leq H(k_\beta)$ with $\beta < \alpha$. Let $\{(\overline{\sigma}_l, h_l)\}_{l<\rho}$ be an enumeration of all tuples $(\overline{\sigma}, h)$, such that $\overline{\sigma} \in \Lambda^{F_\alpha}_\alpha$ and $h : F_\alpha \to 2$. We will now construct a sequence $\{w_l \mid l \leq \rho\}$, such that $w_0 = q_\alpha$, $w_{l+1} \leq_{F_\alpha,\alpha} w_l$, for limit ordinals $\delta < \rho$ it holds that $w_\delta = \bigwedge\limits_{l<\delta} w_l$ and $(w_l)^{h_l}_{\overline{\sigma}_l} \Vdash \check{k}_\alpha \in \dot{X}$. Clearly $\rho = |split_\alpha(q_\alpha)| \cdot 2^{F_\alpha} < \kappa$.

In order to achieve this, we have to explain the construction of the successor steps. Assume we already constructed r_l for some $l < \rho$. Consider $\overline{\tau} \in \Lambda^{F_\alpha}_{\alpha+1}(q_\alpha)$ such that for all $i \in F_\alpha$ $\tau_i \subseteq (\sigma_l)_i^\frown h(i)$. Then $\overline{\tau} \in \Lambda^{F_\alpha}_{\delta+1}(p)$. Now $\delta + 1 \leq H(k_\beta) + 1$ and $f(H(k_\beta) + 1) \leq f(f(k_\beta)) < k_\alpha$ and so $k_\alpha \in Y_{\overline{\tau}}$. Now there are $F_{\alpha,l} \supseteq F_\alpha$, $\overline{\tau}' \in \Lambda^{F_{\alpha,l}}_\alpha$ and $r_h(\overline{\tau}', k_\alpha)$, such that $r_h(\overline{\tau}', k_\alpha) \Vdash \check{k}_\alpha \in \dot{X}$.

Now it might not be the case that $split_\alpha(r_h(\overline{\tau}', k_\alpha)(i)) = split_\alpha(w_l(i))$ for all $i \in F_\alpha$. So we define $w_{l+1}(i) := r_h(\overline{\tau}', k_\alpha) \cup \{(q_\alpha(i))_{t^\frown j} \mid t \in split_\alpha(r_l(i)) \setminus \sigma_i \text{ or } j = 1 - h_l(i)\}$

for all $i \in F_{\alpha,l}$ and $w_{l+1}(i) = r_h(\overline{\tau}', k_\alpha)(i)$ if $i \notin F_{\alpha,l}$. Finally define $F'_\alpha := \bigcup_{l<\rho} F_{\alpha,l}$ and $F_{\alpha+1} := F'_\alpha \cup \{min(supp(q_\alpha) \setminus F'_\alpha)\}$. Let $q_{\alpha+1} := \bigwedge_{l<\rho} w_l$. Then $|F_{\alpha+1}| < \kappa$ and $q_{\alpha+1} \Vdash \check{k}_\alpha \in \dot{X}$ because $(q_{\alpha+1})^h_{\overline{\sigma}} \Vdash \check{k}_\alpha \in \dot{X}$ for all $\overline{\sigma} \in \Lambda^{F_\alpha}_\alpha$ and all $h \in^{F_\alpha} 2$.

Analogous to the proof for the single step let $q \leq q_\alpha$ for every $\alpha < \kappa$. Then $q \Vdash \mathcal{A}^h \cap \check{C}' \subseteq \dot{X}$. As before then $\mathcal{A}^h \setminus X \subseteq \mathcal{A}^h \setminus C' \subseteq \kappa \setminus C' \in id(\mathcal{A})$. So $q \Vdash \mathcal{A}^h \setminus \dot{X} \in id(\mathcal{A})$ contradicting the choice of p.

□

This theorem leads to some consequences. Using the facts that $\mathbb{S}^\lambda_\kappa$ is ${}^\kappa\kappa$-bounding and that if $\mathfrak{d}(\kappa) = \kappa^+$, then $\mathfrak{a}(\kappa) = \kappa^+$ [1] one obtains the following.

Theorem 4.6. [8] In the just constructed model $V^{\mathbb{S}^\lambda_\kappa}$ it holds that $\mathfrak{b}(\kappa) = \mathfrak{d}(\kappa) = \mathfrak{a}(\kappa) = \mathfrak{r}(\kappa) = \mathfrak{i}(\kappa) = \kappa^+ < 2^\kappa$.

Proof. We know that $\mathfrak{d}(\kappa) = \kappa^+$ in V and the family of functions in ${}^\kappa\kappa \cap V$ is still dominating in $V^{\mathbb{S}^\lambda_\kappa}$ because $\mathbb{S}^\lambda_\kappa$ is ${}^\kappa\kappa$-bounding. So $\mathfrak{d}(\kappa) = \kappa^+$ holds in $V^{\mathbb{S}^\lambda_\kappa}$. Because $\mathfrak{d}(\kappa) = \kappa^+$ also $\mathfrak{a}(\kappa) = \kappa^+$ holds in the final model. In addition $\kappa^+ \leq \mathfrak{b}(\kappa) \leq \mathfrak{d}(\kappa)$ and $\kappa^+ \leq \mathfrak{r}(\kappa) \leq \mathfrak{i}(\kappa)$ hold in any model of ZFC. Finally we showed that our densely maximal independent family $\mathcal{A}$ in V remained maximal in $V^{\mathbb{S}^\lambda_\kappa}$. Since $|\mathcal{A}| = \kappa^+$ this implies that $\mathfrak{i}(\kappa) = \kappa^+$ in $V^{\mathbb{S}^\lambda_\kappa}$, which means that also $\mathfrak{r}(\kappa) = \kappa^+$. □

We assumed κ to be inaccessible and needed at least GCH below κ^+ in order to construct a densely maximal independent family. An open question is whether ZFC proves the existence of κ-densely maximal independent family.[8] In addition in our model $\mathfrak{i}(\kappa)$ was equal to κ^+. One may ask now, if it is consistent that $\kappa^+ < \mathfrak{i}(\kappa) < 2^\kappa$. [8]

4.4 On the strong independence number

As mentioned at the beginning of this chapter one may generalise the definition of $\mathfrak{i}$ to higher Baire space also in other ways. In this section we take a closer look at another generalisation.

Definition 4.8. [8] A family $\mathcal{A} \subseteq [\kappa]^\kappa$ is called strongly (κ)-independent if for any two disjoint subfamilies $\mathcal{B}$ and $\mathcal{C}$, both of size less than κ the set $\bigcap_{A\in\mathcal{B}} A \setminus (\bigcup_{A\in\mathcal{C}} (\kappa \setminus A))$ is of size κ.

As before a strongly κ-independent family is called maximal if it is not properly contained in any other strongly κ-independent family. $\mathfrak{i}_s(\kappa)$ is defined to be the minimum cardinality of a maximal strongly independent family.

The main problem of this definition is that $\mathfrak{i}_s(\kappa)$ might not be well defined. Remember that the proof that every κ-independent family is contained in a maximal one relied on Zorn's Lemma. We showed that the union of an increasing chain of independent families is again an independent family. In order to show this we picked a finite Boolean combination and argued that there is one family in the increasing chain, that contains all sets of the Boolean combination. This argument does not work anymore when some $h \in FF(\mathcal{A})$ might have infinite domain.

Another issue is the question whether there is some ZFC relation between $\mathfrak{i}_s(\kappa)$ and $\mathfrak{i}(\kappa)$. [8] Clearly any strongly κ-independent family is also κ-independent. But it is not clear whether a maximal strongly κ-independent family is also maximal κ-independent. An open question is whether there is some ZFC-relation between $\mathfrak{i}_s(\kappa)$ and $\mathfrak{i}(\kappa)$.

However there are also results on strong independence which one can prove and that are analogous to the countable. One is the following.

Proposition 4.3. [8] Suppose that for $\mathfrak{d}(\kappa)$ it holds that for any $\gamma < \mathfrak{d}(\kappa)$, $\gamma^{<\kappa} < \mathfrak{d}(\kappa)$. Let $\mathcal{A}$ be a strongly independent family of less than $\mathfrak{d}(\kappa)$ many sets, then $\mathcal{A}$ is not maximal.

In order to show this, we first need a lemma.

Lemma 4.13. [2] Let $\mathcal{A}$ be a family of subsets of κ such that $|\mathcal{A}| < \mathfrak{d}(\kappa)$. Further, let $\mathcal{C} = \{C_\alpha : \alpha < \kappa\}$ be a $\subseteq^*$- decreasing chain of subsets of κ of size κ such that $|C_\alpha \cap A| = \kappa$ for all $A \in \mathcal{A}$ and all $\alpha < \kappa$. Then there exists a pseudo-intersection B of $\mathcal{C}$ such that $|A \cap B| < \kappa$ holds for all $A \in \mathcal{A}$.

Proof. Firstly observe, that we can construct a sequence $\tilde{C}_\alpha$ which is decreasing such that $\tilde{C}_\alpha \subseteq^* C_\alpha$. This can be done by letting $\tilde{C}_0 = C_0$ and defining $\tilde{C}_\alpha = C_\alpha \setminus \hat{C}_\alpha$ where $\hat{C}_\alpha := C_\alpha \setminus (\bigcap_{\gamma<\alpha} C_\gamma)$. Note that $|\hat{C}_\alpha| < \kappa$ for all α. By the construction any pseudo-intersection of $\tilde{\mathcal{C}} := \{\tilde{C}_\alpha : \alpha < \kappa\}$ is also a pseudo-intersection of the original family $\mathcal{C}$.

So let us assume without loss of generality that $\mathcal{C}$ is $\subseteq$- decreasing. Let $h \in \kappa^\kappa$. Then the set $B_h := \bigcup_{\alpha<\kappa} (C_\alpha \cap h(\alpha))$ is a pseudo-intersection of $\mathcal{C}$. (To check that observe that $B_h \setminus C_\beta = \bigcup_{\alpha<\kappa} ((C_\alpha \setminus C_\beta) \cap h(\alpha)) = \bigcup_{\alpha<\beta} ((C_\alpha \setminus C_\beta) \cap h(\alpha))$ which has size less than κ.)

The next goal is to find $h \in \kappa^\kappa$ such that $B_h \cap A$ has size κ for all $A \in \mathcal{A}$. For all $A \in \mathcal{A}$ and all $\beta < \kappa$ we define $f_A(\beta)$ to be the β-th element of $C_\beta \cap A$. Since $|\mathcal{A}| < \mathfrak{d}(\kappa)$ there is a function $h \in \kappa^\kappa$, such that $h \not\leq^* f_A$ for all $A \in \mathcal{A}$. This means the set $X_A := \{\alpha : f_A(\alpha) < h(\alpha), \alpha < \kappa\}$ has size κ.

Then B_h does what we want since $B_h \cap A = \bigcup_{\alpha<\kappa} (C_\alpha \cap A \cap h(\alpha)) \supseteq \bigcup_{\alpha\in X_A} (C_\alpha \cap A \cap h(\alpha)) \supseteq \bigcup_{\alpha\in X_A} (C_\alpha \cap A \cap f_A(\alpha))$ which is unbounded in κ.

□

Proof of Proposition 4.3 [2]. Let $\mathcal{A}$ be an independent family of size $< \mathfrak{d}(\kappa)$. We want to show that it is not maximal.

Let $\mathcal{D} := \{D_n : n < \omega\}$ be a subfamily of $\mathcal{A}$ and define $\mathcal{A}' := \mathcal{A} \setminus \mathcal{D}$.

For every function $f : \omega \to \{0,1\}$ we define $C_n^f := \bigcap\limits_{m<n} D_m^{f(m)}$.

Then let $\mathcal{F} := \{(\mathcal{A}')^h \mid h \in FF(\mathcal{A}')\}$. Then $|A \cap C_n^f| = \kappa$ for any $A \in \mathcal{F}$ and any $n < \omega$, since $\mathcal{A}$ is an independent family. Due to the lemma the family $\{C_n^f : n < \omega\}$ has a pseudo-intersection B_f which intersects every $A \in \mathcal{F}$ in a set of size κ.

For two different functions $f \neq g : \kappa \to \{0,1\}$, there is an $\alpha < \kappa$ such that $f(\alpha) \neq g(\alpha)$. In particular this implies for $n > \alpha$ that $C_n^f \cap C_n^g = \emptyset$ since $C_n^f \subseteq D_\alpha^{f(\alpha)}$ and $C_n^g \subseteq D_\alpha^{g(\alpha)}$. As $B_f \subseteq^* C_n^f$ and $B_g \subseteq^* C_n^g$ this implies $|B_f \cap B_g| < \kappa$. We can also assume without loss of generality that $B_f \cap B_g = \emptyset$ for every $f \neq g$.

Using this assumption let X and X' be two disjoint and dense subsets of ${}^\kappa 2$. Further define $Y := \bigcup\limits_{f \in X} B_f$ and $Y' := \bigcup\limits_{f \in X'} B_f$. Then $Y \cap Y' = \emptyset$. If we show that both $|Y \cap \mathcal{A}^h| = \kappa$ and $|Y' \cap \mathcal{A}^h| = \kappa$ holds for all $h \in FF(\mathcal{A})$, then we could add Y (or Y') to $\mathcal{A}$ and it still remained independent which contradicts the assumption of maximality.

To show this let $h \in FF(\mathcal{A})$. Define $h' \in FF(\mathcal{A}')$ by $h' := h \restriction dom(h) \cap \mathcal{A}'$. And choose $f \in X$ such that for $D_n \in dom(h)$ it holds that $f(n) = h(D_n)$ Then we have that $Y \cap \mathcal{A}^h = Y \cap (\mathcal{A}')^{h'} \cap (\bigcap D_n \in dom(h) D_n^{f(n)})^* \supseteq Y \cap (\mathcal{A}')^{h'} \cap B_f = (\mathcal{A}')^{h'} \cap B_f$, which is a set of size κ by construction.

The argument for $|Y' \cap A| = \kappa$ is completely analogous.

□

In addition there is some result about the existence of a strongly κ-independent family of size κ.

Proposition 4.4. [8] Assume that κ is strongly inaccessible. Then there is a strongly independent family of size 2^κ.

Proof. The construction is again similar to the ones for the countable and the κ-independent families. We again find a set of cardinality κ and take an independent family of subsets of that.

Let $\mathcal{C} := \{(\alpha, A) \mid \alpha < \kappa \text{ and } A \subseteq \mathcal{P}(\alpha)\}$. Because κ is inaccessible, we know that $|\mathcal{C}| = \bigcup\limits_{\alpha<\kappa} 2^\alpha = \kappa$. For every subset X of κ let $A_X := \{(\alpha, A) \in \mathcal{C} \mid X \cap \alpha \in A\}$. Now $\{A_X : X \subseteq \kappa\}$ is the desired independent family.

To see this, let $I, J \subseteq [\kappa]^{\leq\kappa}$ of size less than κ with $I \cap J = \emptyset$. Then $\bigcap\limits_{X \in I} A_X \setminus (\bigcup\limits_{Y \in J} A_Y)$ is of size κ.

Indeed, for any two distinct $X, Y \in I \cup J$ let $\alpha_{X,Y}$ be the least ordinal such that $X \cap \alpha_{X,Y} \neq Y \cap \alpha_{X,Y}$. Then $\alpha^* := \bigcup\{\alpha_{X,Y} \mid X, Y \in I \cup J\}$ is less than κ, because

κ is regular. For every $\alpha \geq \alpha^*$ define $A_\alpha := \{X \cap \alpha \mid X \in I\}$. By construction $(\alpha, A_\alpha) \in \bigcap_{X \in I} A_X \setminus (\bigcup_{Y \in J} A_Y)$. □

Note that the existence of a strongly independent family of a given size does not tell us whether $\mathfrak{i}_s(\kappa)$ is defined, because as discussed before it does not have to be the case that any independent family is contained in a maximal one.

5 Conclusion

The main results showed in this thesis are the consistency of $\mathfrak{i} < 2^\omega$ as well as the consistency of $\mathfrak{i}(\kappa) < 2^\kappa$. Both proofs do have similar steps, which we compare now. In addition some assumptions on the ground model were used throughout both proofs. We conclude with a discussion of these assumptions.

The idea of the construction of a model in which $\mathfrak{i} < \mathfrak{c}$ is to first find a model in which the continuum hypothesis holds and there is a selective independent family $\mathcal{A}$.

The first time we needed the continuum hypothesis was to construct the selective independent family. In order to preserve cardinals greater or equal than ω_1, we needed the forcing poset to be ccc, which we guaranteed by CH.

After ω_2 many iterations of the Sacks forcing this independent family remains densely maximal. Since it has size ω_1 in the ground model and the Sacks forcing preserves ω_1, it still has that size in the final model. In order to guarantee that the continuum is really larger than ω_1, we must have that all forcing posets $\mathbb{S}$ during the iteration are ω_2-cc. This is the reason why exactly ω_2 iterations were made. In any intermediate model for $\alpha < \kappa$ the forcing poset $\mathbb{Q}_\alpha$ has size ω_1 and thus preserves cardinals greater or equal than ω_2. Thus in the final model we added at least ω_2 many Sacks reals but we also preserved cardinals greater or equal than ω_2, so $2^\omega \geq \omega_2 > \omega_1 = \mathfrak{i}$ in the final model. So we needed CH again here for the Sacks forcing.

The proof mainly relied on the p-set and the q-set property. We defined a filter to be a q-set if for every partition of ω into finite sets there is a semiselector in the filter. Then we could show by a density argument that $fil(\mathcal{A})$ is a q-set using this definition. Afterwards we showed that being a q-set is equivalent to the fact that for every strictly increasing function there is a set in the filter which "grows even faster". This property was the one, that we needed for the final argument of the proof.

In the uncountable there is no analogous generalisation to the q-set property. That was why we needed to make an additional assumption to still get those "very fast growing" sets in the density filter. In the higher Baire spaces we used nearly the same poset as in the countable setting but we restricted it to a normal measure $\mathcal{U}$. The dense maximality and the p-set property of the added family could have been showed without that restriction. For the sets that grow faster than a given strictly increasing function, we used a club C_f. Without the normal measure $\mathcal{U}$, we would not have had a guarantee that for any condition $(\mathcal{A}, A)$ the set C_f would even intersect A. Thus the proof would have failed.

In addition we needed that κ is inaccessible for the Sacks forcing to preserve cardinals greater or equal than κ^+. We used product forcing instead of iterated forcing. One issue with iterated forcing would have been that we would have had to think about the limit steps. In the countable Theorem 3.3 solved that for us.

As mentioned before, it is yet open whether ZFC proves the existence of a selective independent family as well as of a κ-densely maximal independent family. In addition in our model we had $\kappa^+ = \mathfrak{i}(\kappa) < 2^\kappa$. Beyond of the results discussed in the thesis, as well as [4], the problem of controlling the spectrum of maximal independent families in the higher Baire spaces is widely open.

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