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Mathematical aspects of geophysical fluid flows

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Abstract

This thesis builds upon six articles on various topics of mathematical Geophysical Fluid Dynamics (GFD), preceded, in the first chapter, by an introduction which provides the necessary background and an overview of GFD.

The second chapter deals with the oceanic Ekman layer—the relatively thin boundary layer at the top of the ocean, where horizontal pressure-gradient and rotational (Coriolis) forces are balanced by frictional forces due to the wind—in the f-plane approximation, a tangent-plane approximation that permits the use of local Cartesian coordinates. On the one hand, a perturbative approach to the ODE model for steady Ekman flows leads to a quantitative result regarding the angle between the surface current and the generating wind. On the other hand, by means of the Laplace transform, we study the evolution and long-time dynamics of the surface current in the setting of unsteady Ekman flows with time-dependent generating wind and time-dependent (but uniform-in-depth) eddy viscosity.

The third chapter focuses on oceanic flows in spherical coordinates, more specifically on models derived by means of the so-called *thin-shell approximation*, which takes advantage of the small aspect ratio of the ocean to simplify the governing equations while retaining the effects of spherical geometry, and which is applied in two different settings. First, we are concerned with an unsteady model (with constant density) for the Antarctic Circumpolar Current (ACC). After we establish the existence and uniqueness of classical solutions to this model, we show, by using conserved quantities to construct a Lyapunov function, that a certain family of steady solutions—which, under a suitable choice of parameters, closely fits observations of the average velocity field of the ACC—is Lyapunov stable. Then, we investigate steady gyre flows with variable density using elliptic comparison principles, which yield quantitative estimates on the pseudo stream function.

Finally, in the fourth chapter, we discuss the well-posedness of a model—which can be reduced to a semilinear parabolic equation with nonlocal nonlinearities—that describes a remarkable atmospheric phenomenon known as the 'morning glory' cloud pattern. Rewriting the problem as an abstract parabolic problem in an appropriate functional analytic setting, we prove a local strong well-posedness result, then, by means of a Galerkin approximation scheme, we establish the existence of global weak solutions, in a suitable sense, and finally we apply some recent abstract theory for semilinear parabolic equations to prove global well-posedness for the homogenous problem with small initial data.

Zusammenfassung

Diese Arbeit stützt sich auf sechs Artikel zu verschiedenen Themen der mathematischen Geophysikalischen Fluiddynamik (GFD), denen im ersten Kapitel eine Einführung vorausgeht, die den notwendigen Hintergrund und einen Überblick über die GFD liefert.

Das zweite Kapitel befasst sich mit der ozeanischen Ekman-Schicht – der relativ dünnen Grenzschicht an der Oberseite des Ozeans, in der horizontale Druckgradienten- und Rotationskräfte (Corioliskräfte) durch Reibungskräfte aufgrund des Windes ausgeglichen werden – in der f-Ebenen-Approximation, einer Tangenten-Ebenen-Approximation, die die Verwendung lokaler kartesischer Koordinaten erlaubt. Einerseits führt ein perturbativer Ansatz für das ODE-Modell für stetige Ekman-Strömungen zu einem quantitativen Ergebnis bezüglich des Winkels zwischen der Oberflächenströmung und dem erzeugenden Wind. Andererseits untersuchen wir mit Hilfe der Laplace-Transformation die Entwicklung und Langzeitdynamik der Oberflächenströmung im Rahmen instationärer Ekman-Strömungen mit zeitabhängigem erzeugendem Wind und zeitabhängiger (aber in der Tiefe gleichmäßiger) Wirbelviskosität.

Das dritte Kapitel konzentriert sich auf ozeanische Strömungen in Kugelkoordinaten, und zwar auf Modelle, die mit Hilfe der so genannten *Dünnschalennäherung* abgeleitet werden, die das kleine Seitenverhältnis des Ozeans ausnutzt, um die herrschenden Gleichungen zu vereinfachen, während die Auswirkungen der Kugelgeometrie beibehalten werden, und die in zwei verschiedenen Situationen angewendet werden. Zunächst befassen wir uns mit einem nichtstationären Modell (mit konstanter Dichte) für den antarktischen Zirkumpolarstrom (ACC). Nachdem wir die Existenz und Einzigartigkeit klassischer Lösungen dieses Modells nachgewiesen haben, zeigen wir durch die Verwendung konservierter Größen zur Konstruktion einer Lyapunov-Funktion, dass eine bestimmte Familie stetiger Lösungen – die bei geeigneter Wahl der Parameter den Beobachtungen des durchschnittlichen Geschwindigkeitsfeldes des ACC sehr nahe kommt – Lyapunov-stabil ist. Anschließend untersuchen wir stationäre Wirbelströmungen mit variabler Dichte unter Verwendung elliptischer Vergleichsprinzipien, die quantitative Schätzungen der Pseudostromfunktion ergeben.

Im vierten Kapitel schließlich diskutieren wir die Wohlgestelltheit eines Modells (das auf eine semilineare parabolische Gleichung mit nichtlokalen Nichtlinearitäten reduziert werden kann), welches ein bemerkenswertes atmosphärisches Phänomen beschreibt, das als „Morning Glory“-Wolkenmuster bekannt ist. Indem wir das Problem als abstraktes parabolisches Problem in einem geeigneten funktionalanalytischen Rahmen umschreiben, beweisen wir ein lokale starke Wohlgestelltheits, dann zeigen wir mit Hilfe eines Galerkin-Approximationsschemas die Existenz globaler schwacher Lösungen in einem geeigneten

Sinne, und schließlich wenden wir neuere abstrakte Resultate für semilineare parabolische Gleichungen an, um globale Wohlgestelltheit für das homogene Problem mit kleinen Anfangsdaten zu beweisen.

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This thesis is dedicated to my brother Gianluca who, I would like to imagine, in a parallel universe is rejoicing with me on this special occasion.

Luigi Roberti
Vienna, 2025

Contents

Notation

It is convenient to fix some notation which will be used throughout this thesis. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces over $\mathbb{K}$. We denote by $\mathbb{I} : X \to X$ the identity map. We write $\mathcal{L}(X, Y)$ for the space of all continuous linear maps from X to Y, which is a Banach space when endowed with the operator norm

$$\begin{aligned}\|T\|_{\mathcal{L}(X,Y)} &= \sup\left\{\frac{\|Tx\|_Y}{\|x\|_X} : x \in X \setminus \{0\}\right\}\\ &= \sup\big\{\|Tx\|_Y : x \in X \setminus \{0\},\ \|x\|_X = 1\big\}, \quad T \in \mathcal{L}(X, Y).\end{aligned}$$

If $X = Y$, we simply write $\mathcal{L}(X) = \mathcal{L}(X, X)$. Moreover, given $x \in X$ and $r > 0$, we write

$$\mathbb{B}_X(x, r) = \{y \in X : \|x - y\|_X < r\}$$

for the open ball in X with radius r centred at x, and similarly

$$\overline{\mathbb{B}}_X(x, r) = \{y \in X : \|x - y\|_X \leq r\}$$

for its closure. Also, we write $Y \hookrightarrow X$ whenever Y is continuously embedded in X (that is, $Y \subseteq X$, and the natural injection $i : Y \to X$ is continuous); if, in addition, Y is also densely embedded in X, we write $Y \overset{d}{\hookrightarrow} X$. Furthermore, we use $X' = \{\ell : X \to \mathbb{K} \text{ continuous}\}$ for the dual space of X, and, given a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ and an element $x \in X$, we write

$$x_n \to x \quad \text{in } X$$

to mean that x_n converges strongly to x in X, that is, $\|x_n - x\|_X \to 0$ as $n \to \infty$, and

$$x_n \rightharpoonup x \quad \text{in } X$$

to mean that x_n converges weakly to x in X, that is, $\ell(x_n) \to \ell(x)$ in X for each $\ell \in X'$. If $T : X \to X$ (not necessarily continuous), we denote by $\rho(T)$ the resolvent set of T on X, that is, the set of all $\lambda \in \mathbb{C}$ for which $T - \lambda\mathbb{I}$ is invertible with continuous inverse, whereas the spectrum of T on X is denoted by $\sigma(T) = \mathbb{C} \setminus \rho(T)$.

Given an interval $I \subset \mathbb{R}$, an integer $m \in \mathbb{N} = \{1, 2, \ldots\}$, and an exponent $\alpha \in (0, 1)$, we denote by $B(I, X)$, $C(I, X)$, $C^m(I, X)$, and $C^\infty(I, X)$ the spaces of bounded, continuous, m-times differentiable, and infinitely differentiable functions from I to X. The spaces $B(I, X)$ and $C(I, X)$ are endowed with the norm

$$\|f\|_\infty \equiv \|f\|_{B(I,X)} \equiv \|f\|_{C(I,X)} = \sup_{x\in I} \|f(x)\|_X, \quad f \in B(I, X) \text{ (or } C(I, X)).$$

Additionally, we set $C_{\mathrm{b}}(I,X) = B(I,X) \cap C(I,X)$, $C_{\mathrm{b}}^m(I,X) = \{f \in C^m(I,X) : f^(k) \in C_{\mathrm{b}},\, k = 1,\dots,m\}$, and

$$\|f\|_{C^m(I,X)} \equiv \|f\|_{C_{\mathrm{b}}^m(I,X)} = \sum_{k=1}^{m} \|f\|_\infty.$$

The subspace of $C^\infty(I,X)$ consisting of functions whose support is contained in the interior of I is denoted by $C_0^\infty(I,X)$. The Hölder spaces $C^\alpha(I,X)$ and $C^{n+\alpha}(I,X)$ are defined by

$$\begin{aligned} C^\alpha(I,X) &= \left\{ f \in C_{\mathrm{b}}(I,X) : [f]_{C^\alpha(I,X)} = \sup_{s,t\in I,\, s\neq t} \frac{\|f(s)-f(t)\|_X}{|s-t|^\alpha} < \infty \right\}, \\ C^{m+\alpha}(I,X) &= \big\{ f \in C_{\mathrm{b}}^m(I,X) : f^{(m)} \in C^\alpha(I,X) \big\}, \end{aligned}$$

with norms

$$\begin{aligned} \|f\|_{C^\alpha(I,X)} &= \|f\|_\infty + [f]_{C^\alpha(I,X)}, \\ \|f\|_{C^{m+\alpha}(I,X)} &= \|f\|_{C^m(I,X)} + [f^{(m)}]_{C^\alpha(I,X)}. \end{aligned}$$

Moreover, we write $C^{1-}(I,X)$ for the space of all locally Lipschitz continuous functions on I,

$$C^{1-}(I,X) = \left\{ f \in C_{\mathrm{b}}(I,X) : [f]_{C^{1-}(I,X)} = \sup_{s,t\in I,\, s\neq t} \frac{\|f(s)-f(t)\|_X}{|s-t|} < \infty \right\},$$

with norm

$$\|f\|_{C^{1-}(I,X)} = \|f\|_\infty + [f]_{C^{1-}(I,X)}.$$

Now consider an open subset $\Omega \subset \mathbb{R}^n$. Let $\|\cdot\|$ denote the standard Euclidean norm on $\mathbb{R}^n$. The spaces of continuous, m-times differentiable, and infinitely differentiable (real or complex-valued) functions on Ω are denoted by $C(\Omega)$, $C^m(\Omega)$, and $C^\infty(\Omega)$, respectively. We write $C_0^\infty(\Omega)$ to denote the space of all $f \in C^\infty(\Omega)$ whose support is contained in the interior of Ω. By $C(\overline{\Omega})$ we denote the subspace of $C(\Omega)$ whose elements are continuous and bounded on Ω, endowed with the sup norm, while $C^m(\overline{\Omega})$ is the subspace of $C^m(\Omega)$ consisting of functions whose derivatives up to order m are continuous and bounded on Ω (and thus extendable to the boundary $\partial\Omega$ of Ω), endowed with the norm

$$\|f\|_{C^m(\overline{\Omega})} = \sum_{|\beta|\le m} \|\mathrm{D}^\beta f\|_\infty,$$

where, for a multi-index $\beta = (\beta_1,\dots,\beta_n) \in \mathbb{N}_0^n$ (with $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$) of order $|\beta| = \beta_1 + \ldots + \beta_n$, we use the notation

$$\mathrm{D}^\beta f = \frac{\partial^{|\beta|} f}{\partial x^{\beta_1} \cdots \partial x^{\beta_n}}.$$

For an exponent $\alpha \in (0,1)$ and an integer $m \in \mathbb{N}$, we can define the Hölder spaces $C^\alpha(\overline{\Omega})$ and $C^{m+\alpha}(\overline{\Omega})$ as

$$\begin{aligned} C^\alpha(\overline{\Omega}) &= \left\{ f \in C(\overline{\Omega}) : [f]_{C^\alpha(\overline{\Omega})} = \sup_{s,t\in\Omega,\, s\neq t} \frac{|f(s)-f(t)|}{\|s-t\|^\alpha} < \infty \right\}, \\ C^{m+\alpha}(\overline{\Omega}) &= \big\{ f \in C^m(\overline{\Omega}) : \mathrm{D}^\beta f \in C^\alpha(\overline{\Omega}) \text{ for all } \beta \in \mathbb{N}_0^n \text{ with } |\beta| = m \big\}, \end{aligned}$$

with norms

$$\begin{aligned}\|f\|_{C^\alpha(\overline{\Omega})} &= \|f\|_\infty + [f]_{C^\alpha(\overline{\Omega})},\\ \|f\|_{C^{m+\alpha}(\overline{\Omega})} &= \|f\|_{C^m(\overline{\Omega})} + \sum_{|\beta|=m} \left[\mathrm{D}^\beta f\right]_{C^\alpha(\overline{\Omega})}.\end{aligned}$$

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. For $p \in [1, \infty]$ we define the space $L^p(\Omega)$ as

$$L^p(\Omega) = \big\{f : \Omega \to \mathbb{K} : f \text{ is Lebesgue measurable and } \|f\|_{L^p(\Omega)} < \infty\big\},$$

where

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p \,\mathrm{d}x\right)^{1/p} \quad \text{for } 1 \leq p < \infty$$

and

$$\|f\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| = \inf\left\{C \geq 0 : |f(x)| \leq C \text{ for almost every } x \in \Omega\right\},$$

with the *caveat* that we identify functions that coincide almost everywhere (so that each $L^p(\Omega)$ is actually a quotient space of equivalence classes of functions). The space $L^p_{\mathrm{loc}}(\Omega)$ is defined as the space of all functions $f : \Omega \to \mathbb{K}$ such that $f|_K \in L^p(K)$ for all compact subsets $K \subseteq \Omega$. As is well-known, $(L^p(\Omega), \|\cdot\|_{L^p(\Omega)})$ is a Banach space for any $p \in [1, \infty]$, and the space $L^2(\mathbb{R}^n)$ is in fact a Hilbert space with respect to the inner product

$$\langle f, g\rangle_{L^2(\Omega)} = \int_\Omega f(x)\overline{g(x)}\,\mathrm{d}x.$$

Now, let $1 \leq p \leq \infty$ and $s \geq 1$. We denote by $W^{s,p}(\Omega)$ the Sobolev space with exponents s and p. If $s = k \in \mathbb{N}$ is an integer, $W^{k,p}(\Omega)$ consists of all $f \in L^1_{\mathrm{loc}}(\Omega)$ such that for each multi-index β of order $|\beta| \leq k$ the weak partial derivative $\mathrm{D}^\beta f$ exists and belongs to $L^p(\Omega)$, with norm

$$\|f\|_{W^{k,p}(\Omega)} = \begin{cases} \left(\displaystyle\sum_{|\beta|\leq k} \|\mathrm{D}^\beta f\|^p_{L^p(\Omega)}\,\mathrm{d}x\right)^{1/p}, & 1 \leq p < \infty,\\ \displaystyle\sum_{|\beta|\leq k} \|\mathrm{D}^\beta f\|_{L^\infty(\Omega)}, & p = \infty.\end{cases}$$

If $s \in (0, 1)$, the space $W^{s,p}(\Omega)$ is defined as

$$W^{s,p}(\Omega) = \left\{f \in L^p(\Omega) : (x, y) \mapsto \frac{|f(x) - f(y)|}{\|x - y\|^{n/p+s}} \in L^p(\Omega \times \Omega)\right\},$$

with norm

$$\|f\|_{W^{s,p}(\Omega)} = \left(\|f\|^p_{L^p(\Omega)} + \int_\Omega\int_\Omega \frac{|f(x) - f(y)|^p}{\|x - y\|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/p},$$

whereas, if $s > 1$ is not an integer, we can write $s = k + \sigma$ for $k \in \mathbb{N}$ and $\sigma \in (0, 1)$, and in this case $W^{s,p}(\Omega)$ is defined as

$$W^{s,p}(\Omega) = \big\{f \in W^{k,p}(\Omega) : \mathrm{D}^\beta f \in W^{\sigma,p}(\Omega) \text{ for all } \beta \in \mathbb{N}_0^n \text{ with } |\beta| = k\big\},$$

with norm

$$\|f\|_{W^{s,p}(\Omega)} = \left(\|f\|_{W^{k,p}(\Omega)} + \sum_{|\beta|=k} \|\mathrm{D}^\beta f\|^p_{W^{\sigma,p}(\Omega)} \right)^{1/p}.$$

We denote by $W_0^{s,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{s,p}(\Omega)$, and we write $H^s(\Omega) = W^{s,2}(\Omega)$ and $H_0^s(\Omega) = W_0^{s,2}(\Omega)$.

Finally, we will deal with function spaces involving time. To this end, for the Banach space X, the exponent $p \in [1,\infty]$, and $T > 0$, we denote by $L^p(0,T;X)$ the space consisting of all strongly measurable functions $f : [0,T] \to X$ with

$$\|f\|_{L^p(0,T;X)} = \left(\int_0^T \|f(t)\|_X^p \,\mathrm{d}t \right)^{1/p} < \infty$$

if $1 \le p < \infty$, respectively

$$\|f\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{0 \le t \le T} \|f(t)\|_X < \infty$$

if $p = \infty$.

1. Introduction

We begin this thesis by providing some background about the equations of Geophysical Fluid Mechanics, their physical meaning, and the importance of the effects of rotation; this is the subject of Section 1.1. Then, in Section 1.2, we give a brief overview of the topics discussed in the later chapters.

1.1 The equations of Geophysical Fluid Dynamics

The research detailed in this thesis takes place within the field of *Geophysical Fluid Dynamics* (henceforth abbreviated as GFD), which can be broadly defined as the branch of fluid dynamics that deals with large-scale flows on the surface of a rotating planet. Naturally, the most prominent example of such a planet is Earth, whose atmosphere and oceans display a wide range of interesting flows (whether winds, or cloud propagation, or oceanic currents, etc.), but other planets can be the object of research within the scope of GFD as well. In the present work, we will focus exclusively on flows occurring on Earth; the reader interested in the applications of GFD to other celestial bodies may refer, for example, to [26, 27, 28, 41, 65].

1.1.1 The governing equations of fluid dynamics

As always in fluid dynamics, the cornerstone of the discussion consists of the momentum equations, corresponding to Newton's second law, the equation of mass conservation, also referred to as the continuity equation, and appropriate thermodynamic equations. The derivation of these equations can be found in any textbook on fluid dynamics (see, for instance, [12, 23]) and thermodynamics (such as, e.g., [11, 45]); here we simply formulate them for later reference.

In a Cartesian reference system (x, y, z) for $\mathbb{R}^3$, let us denote by $\mathbf{u} = (u, v, w)$ the velocity components of the fluid's velocity field in the x, y and z directions, respectively, by ρ the density, and by p the pressure. In addition, we will denote time by the variable t. The momentum equations—usually known as the *Navier–Stokes equations*—are given, in vector notation, by

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla \mathbf{u} + \mathbf{F}_{\mathrm{v}}(\mathbf{u}) + \mathbf{F}_{\mathrm{b}}, \tag{1.1}$$

where we have used the notation $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})^T$ for the gradient of the differentiable function f, and

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}, u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}, u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial u}{\partial z}\right)^T;$$

in (1.1), $\mathbf{F}_\mathrm{b}$ represents external body forces per unit mass (such as gravity), whereas $\mathbf{F}_\mathrm{v}(\mathbf{u})$ denotes viscous forces per unit volumes. Writing down the latter in a tractable form is typically an arduous task that necessitates of simplifying assumptions on the fluid; so, we will do so explicitly only later on, whenever needed in specific situations. At this point, let us only remark that the most simplified (inexact)—but widely used—form of $\mathbf{F}_\mathrm{v}(\mathbf{u})$, which involves an incompressibility assumption, is $\mathbf{F}_\mathrm{v}(\mathbf{u}) = \mu\Delta\mathbf{u}$, where $\mu > 0$ is called *(kinematic) viscosity*. If the flow considered is inviscid, that is, $\mathbf{F}_\mathrm{v}(\mathbf{u}) = 0$, then the momentum equations (1.1) are called *Euler equations*.

It is convenient to introduce the *material derivative*, denoted

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\partial f}{\partial t} + (\mathbf{u}\cdot\nabla)f;$$

physically, this can be interpreted as the time derivative of f along particle trajectories. The equation of mass conservation (or continuity equation) can be written as

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla\cdot\mathbf{u} = 0, \tag{1.2}$$

or, equivalently,

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0;$$

here, $\nabla\cdot\mathbf{u} = \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$ denotes the *divergence* of the vector field $\mathbf{u}$.

So far, we have four equations (the three components of the momentum equations, plus the continuity equation) for five unknowns (three velocity components, density, pressure). In order to fix this underdeterminacy, we turn to thermodynamics. Let us denote by T the temperature of the fluid. An *equation of state* is an expression that relates the thermodynamic variables (temperature, pressure, density, mass) to each other. Denoting by $\varphi_1,\dots,\varphi_n$ the mass fractions of the various constituents of the fluid (that is, the ratio between the mass of a constituent and the total mass), an equation of state will have the general form

$$p = p(\rho, T, \varphi_1, \dots, \varphi_n);$$

the exact form of this equation will greatly depend on the fluid under consideration. For example, in the case of an ideal gas, the equation of state takes the very simple form

$$p = \rho\mathfrak{R}T, \tag{1.3}$$

where $\mathfrak{R}$ is a gas-specific constant. In more complex situations, such as ocean water, the equation of state is much more complicated and can generally only be written down in approximate form (see, for example, the discussion in [105]). Moreover, now that we have added one more unknown (the temperature), we still need one more equation. For instance, we may use the *first law of thermodynamics* (a statement on conservation of energy which relates energy transfer, thermodynamic work, and heat), which for an ideal gas can be written as

$$c_\mathrm{p}\frac{\mathrm{D}T}{\mathrm{D}t} - \frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} - \kappa\nabla\cdot\nabla T = Q; \tag{1.4}$$

here c_{p} is the (constant) specific heat, κ is the (constant) thermal diffusivity of the gas, and Q is a heat-source term. A more comprehensive discussion on the topic of thermodynamics is beyond the scope of this introduction; we refer the reader to [116, Chapter 1] or [54, Chapter 3], in addition to the references mentioned earlier, for a detailed exposition of thermodynamic principles in GFD. Let us merely point out that in many situations, e.g. when dealing with the ocean's dynamics, the flow can be assumed to an excellent degree to be *incompressible*, i.e. the density is conserved along particle paths; in other words,

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0.$$

This equation, together with the momentum equations (1.1) and the mass continuity (1.2) (which now reduces to $\nabla \cdot \mathbf{u} = 0$), forms a closed system of equations which does not necessitates of the additional thermodynamic equations, hence the discussion is considerably simplified. See, for example, Section 1.9 of [116] for conditions under which incompressibility is a reasonable assumption.

1.1.2 Rotating fluids

One of the main aspects of GFD is the necessity of dealing with *rotating* fluids. In fact, Earth (as well as other planets) is not stationary but rotates about its polar axis at a constant angular speed of approximately $7.29 \times 10^{-5}\,\mathrm{rad\,s^{-1}}$; geophysical fluid flows are therefore conveniently studied in a rotating (rather than stationary) coordinate system where Earth appears to be stationary. Such a coordinate system, while being undoubtedly advantageous, has the drawback of not being inertial, hence apparent forces will appear in the momentum equation.

The following (standard) material is mostly based on Section 2.1 of [116], but see also the discussion in [54, 91]. Let I denote an inertial coordinate system of $\mathbb{R}^3$, and let $\boldsymbol{\Omega} \in \mathbb{R}^3$ be a constant angular velocity vector. Furthermore, suppose that R is a coordinate system that rotates with angular velocity $\boldsymbol{\Omega}$. The subscripts I and R will represent the values of the various quantities in the respective frames. Given a vector $\mathbf{B} \in \mathbb{R}^3$, it can be argued that its rates of change as measured in I and R are related via

$$\left(\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}\right)_I = \left(\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}\right)_R + \boldsymbol{\Omega} \times \mathbf{B}. \tag{1.5}$$

If the vector $\mathbf{r} \in \mathbb{R}^3$ denotes the position of a particle, we can apply (1.5) to obtain

$$\mathbf{v}_I = \mathbf{v}_R + \boldsymbol{\Omega} \times \mathbf{r}, \tag{1.6}$$

where $\mathbf{v}_I = \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_I$ (the *inertial velocity*) and $\mathbf{v}_R = \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_R$ (the *relative velocity*) denote the velocity of the particle measured in the inertial and rotating frame, respectively. Applying (1.5) to $\mathbf{v}_R$, using (1.6), keeping in mind that $\boldsymbol{\Omega}$ is constant, and rearranging terms, we see that

$$\left(\frac{\mathrm{d}\mathbf{v}_I}{\mathrm{d}t}\right)_I = \left(\frac{\mathrm{d}\mathbf{v}_I}{\mathrm{d}t}\right)_R + \boldsymbol{\Omega} \times \mathbf{r} + \boldsymbol{\Omega} \times \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_I,$$

which, in view of (1.5), becomes

$$\left(\frac{\mathrm{d}\mathbf{v}_I}{\mathrm{d}t}\right)_R = \left(\frac{\mathrm{d}\mathbf{v}_I}{\mathrm{d}t}\right)_I - 2\boldsymbol{\Omega} \times \mathbf{v}_R - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}). \tag{1.7}$$

The second and third terms on the right side of (1.7) are called *Coriolis force* and *centrifugal force*, respectively. Neither is a 'real' force—they are apparent forces which arise through the non-inertiality of the rotating reference frame. The Coriolis force plays a crucial role in the dynamics of geophysical flows: clearly, it does not act on bodies which are stationary in the rotating frame, and it deviates moving bodies in the rotating frame at right angles to their direction of motion; in particular, it does not do any work on such moving bodies. On the other hand, writing $\Omega = \|\boldsymbol{\Omega}\|$ and $\mathbf{r} = \mathbf{r}_{\|} + \mathbf{r}_{\perp}$, where $\mathbf{r}_{\|}$ and $\mathbf{r}_{\perp}$ are the components of $\mathbf{r}$ parallel and perpendicular to $\boldsymbol{\Omega}$, respectively, for the centrifugal force it follows that

$$-\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) = \Omega^2 \mathbf{r}_{\perp} = -\nabla \Phi_{\mathrm{ce}},$$

where

$$\Phi_{\mathrm{ce}} = -\frac{\Omega^2 \|\mathbf{r}_{\perp}\|^2}{2} = -\frac{\|\boldsymbol{\Omega} \times \mathbf{r}_{\perp}\|^2}{2}$$

is the *centrifugal potential.*

We further note that, given a scalar field φ, we have

$$\left(\frac{\mathrm{D}\varphi}{\mathrm{D}t}\right)_R = \left(\frac{\mathrm{D}\varphi}{\mathrm{D}t}\right)_I,$$

because the material derivative is simply the time derivative of φ along particle paths and is thus the same in both the inertial and the rotating frame, and

$$\nabla \cdot \mathbf{v}_I = \nabla \cdot (\mathbf{v}_R + \boldsymbol{\Omega} \times \mathbf{r}) = \nabla \cdot \mathbf{v}_R,$$

in view of (1.6) and the fact that $\nabla \cdot (\boldsymbol{\Omega} \times \mathbf{r}) = 0$. Therefore, in a rotating reference frame, the continuity equation (1.2) and thermodynamic equations such as the first law of thermodynamics (1.4) remain unchanged. The momentum equations, however, must include the Coriolis and centrifugal forces; assuming that the only external body force is the gravitational force $\mathbf{F}_{\mathrm{b}} = \mathbf{g}_{\mathrm{grav}} = -\nabla \Phi_{\mathrm{grav}}$, where Φ_{grav} is the *gravitational potential*, and writing $\Phi = \Phi_{\mathrm{grav}} + \Phi_{\mathrm{ce}}$, the Navier–Stokes equations (1.1) in the rotating frame of reference become

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + 2\boldsymbol{\Omega} \times \mathbf{r} = -\frac{1}{\rho}\nabla \mathbf{u} + \mathbf{F}_{\mathrm{v}}(\mathbf{u}) - \nabla \Phi. \tag{1.8}$$

1.1.3 The equations in spherical coordinates

The main technical difficulty encountered in GFD, in addition to the necessity of using a rotating reference frame, is the fact that geophysical flows occur in a layer close to Earth's *curved* surface; therefore, the underlying spherical geometry of these problems must be taken into account if the dynamics is to be described accurately. As a matter of fact, Earth's surface is not most accurately modelled by a sphere, but by an oblate spheroid, namely an ellipsoid obtained through rotation of an ellipse with semi-minor axis of length 6357 km (the polar diameter) and semi-major axis of length 6378 km (the equatorial diameter) around its centre. This is due to the fact that, over time, Earth's shape has adjusted to be very nearly an equipotential surface, with the property that its *effective gravity*

$$\mathbf{g} = -\nabla \Phi = \mathbf{g}_{\mathrm{grav}} + \Omega^2 \mathbf{r}_{\perp},$$

that is, the sum of its gravitational force and the centripetal forces caused by its rotation about its polar axis, is everywhere parallel to the local outward normal vector to the

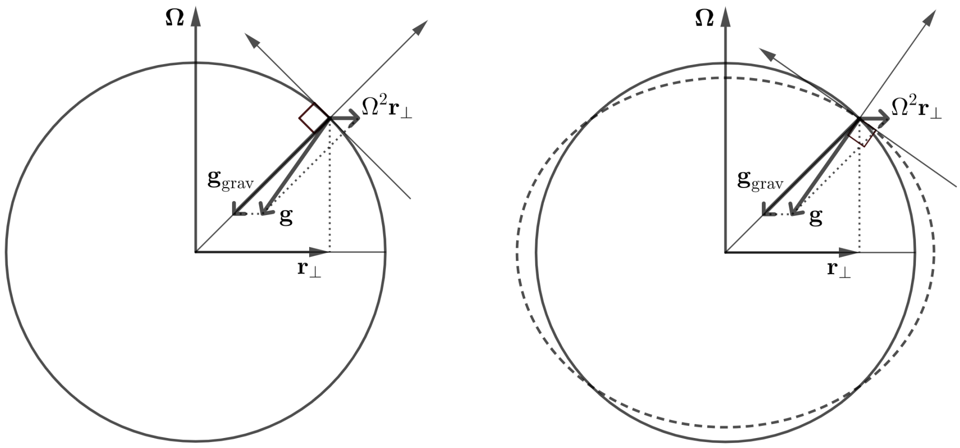

Figure 1.1: The effective gravity $\mathbf{g} = \mathbf{g}_{\mathrm{grav}} + \Omega^2\mathbf{r}_\perp$. If Earth were a perfect sphere (left panel), $\mathbf{g}$ would never be perpendicular to the surface (except at the poles). However, Earth's surface is essentially an equipotential surface for the geopotential, so that $\mathbf{g}$ is indeed normal to the surface (right panel; therein the eccentricity of the ellipsoid is greatly exaggerated). This equipotential surface is often approximated by a sphere, dropping the centrifugal terms to ensure that (effective) gravity points everywhere to the centre of the sphere.

surface, so that Earth's ellipsoidal surface (also known as *geoid*) exhibits a slight equatorial 'bulge'. This deviation of the geoid from a perfect sphere can be taken into account by means of appropriate ellipsoidal coordinates, as detailed in [36, 37]; however, since for most purposes—including the various settings investigated in this thesis—the slight oblateness of the geoid is inconsequential for the dynamics (see, e.g., [36, 121]), we will not pursue this approach, instead treating Earth's surface as a perfect sphere. As we pointed out, Earth's surface is essentially an equipotential surface of the geopotential, so that effective gravity is everywhere perpendicular to it, hence it is customary (although not always the case—a slight inconsistency throughout the literature), when viewing Earth as a perfect sphere, to drop the (very small) centrifugal force from the equations and assume that $\mathbf{g}$ is everywhere normal to the surface, which would not be the case if Earth were truly a perfect sphere with no equatorial bulge; see Figure 1.1 and the discussion in Section 2.2 of [116] for more details.

As in the previous section, we refer the reader to [54, 91, 116] for further details on the derivation of the identities and equations that we are about to discuss. Additionally, see the excellent Appendix B of [99] for a thorough exposition of the technical aspects of the derivation of the Navier–Stokes equations in rotating spherical coordinates.

Given the spherical geometry that characterises the flows in which we are interested, it is a natural choice to introduce, in a rotating frame of reference in which Earth looks stationary, a system of spherical coordinates (φ, θ, r) for $\mathbb{R} \setminus \{(0,0,0)\}$ as in Figure 1.2, where $\varphi \in [0, 2\pi)$ is the *longitude*, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ is the *latitude*, and $r > 0$ is the radial distance from the origin, which coincides with Earth's centre. At each point for which $\theta \notin \{\pm\frac{\pi}{2}\}$, we can define a local orthonormal basis $\{\mathbf{e}_\varphi, \mathbf{e}_\theta, \mathbf{e}_r\}$ of $\mathbb{R}^3$, where $\mathbf{e}_\varphi$ and $\mathbf{e}_\theta$ are the unit tangent vectors to $r\mathbb{S}^2$ (the 2-sphere in $\mathbb{R}^3$ with radius r) in the φ- and θ-direction, respectively, whereas $\mathbf{e}_r$ is the unit outward normal vector to $r\mathbb{S}^2$. The vectors $\mathbf{e}_\varphi$ and $\mathbf{e}_\theta$ are not defined for $\theta \in \{\pm\frac{\pi}{2}\}$, as a consequence of the 'hairy ball theorem', which asserts that the 2-sphere $\mathbb{S}^2$ cannot be equipped with a continuous field of tangent vectors—see, for instance, [59]. Explicitly, the spherical coordinates (φ, θ, r) and the Cartesian coordinates

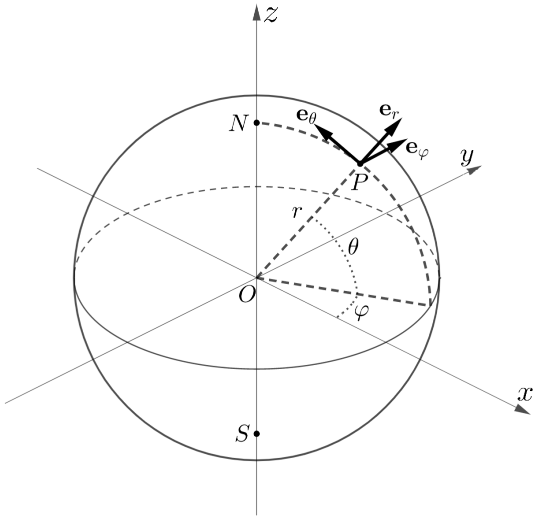

FIGURE 1.2: The rotating system of spherical coordinates.

(x, y, z) in the rotating frame of reference are related as follows:

$$x = r\cos\varphi\cos\theta,$$
$$y = r\sin\varphi\cos\theta,$$
$$z = r\sin\theta.$$

The governing equations can be then expressed in spherical coordinates, once some identities in spherical coordinates are established. Let the scalar field f be sufficiently smooth. Then the material derivative of f can be expressed as

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\partial f}{\partial t} + \frac{u}{r\cos\theta}\frac{\partial f}{\partial\varphi} + \frac{v}{r}\frac{\partial f}{\partial\theta} + w\frac{\partial f}{\partial r}, \tag{1.9}$$

where u, v and w denote the velocity components in the direction of $\mathbf{e}_\varphi$, $\mathbf{e}_\theta$ and $\mathbf{e}_r$, respectively. We will write $\mathbf{u} = u\,\mathbf{e}_\varphi + v\,\mathbf{e}_\theta + w\,\mathbf{e}_r$. If $\mathbf{B} = B^{(\varphi)}\,\mathbf{e}_\varphi + B^{(\theta)}\,\mathbf{e}_\theta + B^{(r)}\,\mathbf{e}_r$ is a sufficiently regular vector field, then the divergence of $\mathbf{B}$ is

$$\nabla\cdot\mathbf{B} = \frac{1}{\cos\theta}\left[\frac{1}{r}\frac{\partial B^{(\varphi)}}{\partial\varphi} + \frac{1}{r}\frac{\partial}{\partial\theta}\big(B^{(\theta)}\cos\theta\big) + \frac{\cos\theta}{r^2}\frac{\partial}{\partial r}\big(r^2B^{(r)}\big)\right]. \tag{1.10}$$

For the gradient of f we have

$$\nabla f = \frac{1}{r\cos\theta}\frac{\partial f}{\partial\varphi}\,\mathbf{e}_\varphi + \frac{1}{r}\frac{\partial f}{\partial\theta}\,\mathbf{e}_\theta + \frac{\partial f}{\partial r}\,\mathbf{e}_r,$$

whereas the Laplacian of f is

$$\Delta f \equiv \nabla^2 f \equiv \nabla\cdot\nabla f = \frac{1}{r^2\cos\theta}\left[\frac{1}{\cos\theta}\frac{\partial^2 f}{\partial\varphi^2} + \frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial f}{\partial\theta}\right) + \cos\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)\right].$$

It is important to note that the direction of the unit vectors $\mathbf{e}_\varphi$, $\mathbf{e}_\theta$ and $\mathbf{e}_r$ changes with location. In fact, for the rate of change of the unit vectors as they move with the flow (that is, their material derivative), we have that

$$
\begin{aligned}
\frac{\mathrm{D}\mathbf{e}_\varphi}{\mathrm{D}t} &= \frac{u}{r\cos\theta}\big(\mathbf{e}_\theta \sin\theta - \mathbf{e}_r\cos\theta\big),\\
\frac{\mathrm{D}\mathbf{e}_\theta}{\mathrm{D}t} &= -\frac{u\tan\theta}{r}\,\mathbf{e}_\varphi - \frac{v}{r}\,\mathbf{e}_r,\\
\frac{\mathrm{D}\mathbf{e}_r}{\mathrm{D}t} &= \frac{u\tan\theta}{r}\,\mathbf{e}_\varphi + \frac{v}{r}\,\mathbf{e}_\theta.
\end{aligned}
$$

These observations make it possible to rewrite the Navier–Stokes equations (1.8) in spherical coordinates; written out in components with respect to $(\mathbf{e}_\varphi, \mathbf{e}_\theta, \mathbf{e}_r)$, they are:

$$
\frac{\mathrm{D}u}{\mathrm{D}t} - \left(2\Omega + \frac{u}{r\cos\theta}\right)(v\sin\theta - w\cos\theta) = F_\mathrm{v}^\varphi(\mathbf{u}) - \frac{1}{\rho r\cos\theta}\frac{\partial p}{\partial\varphi} - \frac{1}{r\cos\theta}\frac{\partial\Phi}{\partial\varphi}, \tag{1.11a}
$$

$$
\frac{\mathrm{D}v}{\mathrm{D}t} + \frac{wv}{r} + \left(2\Omega + \frac{u}{r\cos\theta}\right)u\sin\theta = F_\mathrm{v}^\theta(\mathbf{u}) - \frac{1}{\rho r}\frac{\partial p}{\partial\theta} - \frac{1}{r}\frac{\partial\Phi}{\partial\theta}, \tag{1.11b}
$$

$$
\frac{\mathrm{D}w}{\mathrm{D}t} - \frac{u^2+v^2}{r} - 2\Omega u\cos\theta = F_\mathrm{v}^r(\mathbf{u}) - \frac{1}{\rho}\frac{\partial p}{\partial r} - \frac{\partial\Phi}{\partial r}, \tag{1.11c}
$$

where $\mathbf{F}_\mathrm{v}(\mathbf{u}) = F_\mathrm{v}^\varphi(\mathbf{u})\,\mathbf{e}_\varphi + F_\mathrm{v}^\theta(\mathbf{u})\,\mathbf{e}_\theta + F_\mathrm{v}^r(\mathbf{u})\,\mathbf{e}_r$ denotes the viscous forces per unit volume (which, once again, we are not specifying explicitly here), and Φ is the geopotential. If, as mentioned earlier, the effective gravity is taken to be normal to Earth's spherical surface, then Φ will depend only on r, thus

$$
\Phi(r) = gr, \tag{1.12}
$$

if the gravitational acceleration $g \approx 9.8\,\mathrm{m\,s^{-2}}$ is considered to be constant close to Earth's surface, or, more accurately,

$$
\Phi(r) = -g\frac{R^2}{r},
$$

where $R = 6371\,\mathrm{km}$ is Earth's average radius, so that Φ will not appear in (1.11a) and (1.11b); if the centrifugal terms are retained, then

$$
\Phi(r,\theta) = gr - \frac{1}{2}\Omega^2 r^2\cos^2\theta, \tag{1.13}
$$

or

$$
\Phi(r,\theta) = -g\frac{R^2}{r} - \frac{1}{2}\Omega^2 r^2\cos^2\theta. \tag{1.14}
$$

The mass conservation equation (1.2) can be simply written again as

$$
\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla\cdot\mathbf{u} = 0 \qquad \text{or} \qquad \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0, \tag{1.15}
$$

where the material derivative and the divergence are as in (1.9) and (1.10), respectively.

1.2 Plan of the thesis

The equations of GFD we have just described are intractable in full generality; therefore, some simplifying assumptions are necessary to obtain models which may be amenable to analysis. Such simplifying assumptions usually depend on the properties of the flow at hand and are often dictated by scaling. The remaining sections of this introduction will be devoted to three particular instances where scaling (and, possibly, additional simplifications) leads to models which are appropriate to describe specific geophysical flows.

In **Chapter 2**, we discuss a widely used approximation, the *f-plane approximation*, which is appropriate for flows that are confined to a relatively small region, so that the dynamics can locally be projected onto a tangent plane; this allows us to introduce local Cartesian coordinates. In this approximation, we investigate the oceanic *Ekman layer*, a thin boundary layer in the upper ocean where the effects of viscosity are crucial for the dynamics of wind-induced currents.

In **Chapter 3**, we consider flows of larger size, so that a tangent-plane approximation is no longer feasible. However, we are able to simplify the equations using the *thin-shell approximation*, a method that takes advantage of the small aspect ratio of the ocean (that is, the ratio between the ocean's depth and Earth's radius) to derive asymptotic models that, at leading order, retain the full effects of sphericity. We apply the thin-shell approximation to derive a model that, in its full generality, incorporates density variations. First, taking constant density, we apply this model to the Antarctic Circumpolar Current; after establishing the existence and uniqueness of classical solutions to this problem, we obtain a stability result for a class of zonal (that is, only latitude-dependent) solutions. Then, we take density variations into account when dealing with steady ocean gyres (circular currents occurring in confined regions); by means of elliptic comparison arguments, we can provide bounds on the pseudo stream function for this problem.

Finally, in **Chapter 4**, we consider a model for a fascinating example of an atmospheric flow: the *morning glory cloud pattern*. After briefly sketching the derivation of this model, which can be reduced to studying a semilinear parabolic equation with nonlocal nonlinearity, we discuss results concerning the local well-posedness (in terms of strong solutions) of the problem, as well as the existence of global weak solutions (in an appropriate sense) and the global well-posedness for the homogeneous problem with sufficiently small initial data.

2. The oceanic Ekman layer in the f-plane approximation

In this chapter, we consider the properties of flows in a relatively thin layer of the upper ocean, the so-called *Ekman layer*, where viscosity plays a role through frictional effects due to the action of the wind. In order to do so, we rely on a technique that is widely used in physical oceanography to investigate flows of comparatively small extent, namely the *f-plane approximation*. In Section 2.1, we justify this approximation using a scaling argument. In Sections 2.2 and 2.3, we discuss the main features of the dynamics in the Ekman layer. In Section 2.4, whose results are published in [101], we obtain further insight into the surface deflection angle of Ekman flows by means of a perturbative approach. Finally, in Section 2.5, which is based on the article [103], we investigate the dynamics of unsteady Ekman flows with time-dependent eddy viscosity and wind.

2.1 The f-plane approximation

In order to simplify the momentum equations (1.11), some approximations are required. Let us write $r = R + z$, where R is Earth's average radius. Popular approximations are, for example, the *shallow-fluid approximation*, according to which the variable $r = R + z$ is replaced by R except when used as a differentiating variable, and the *traditional approximation*, which ignores the small Coriolis terms featuring the vertical velocity and the even smaller terms uw/r and vw/r in the horizontal momentum equations (1.11a) and (1.11b). If we make these approximations, and choose the geopotential Φ as in (1.12), equations (1.11a) and (1.11b) become

$$\frac{\mathrm{D}u}{\mathrm{D}t} - 2\Omega v \sin\theta - \frac{uv}{R}\tan\theta = F_{\mathrm{v}}^{\varphi}(\mathbf{u}) - \frac{1}{R\rho\cos\theta}\frac{\partial p}{\partial\varphi}, \tag{2.1a}$$

$$\frac{\mathrm{D}v}{\mathrm{D}t} + 2\Omega u \sin\theta + \frac{u^2}{R}\tan\theta = F_{\mathrm{v}}^{\theta}(\mathbf{u}) - \frac{1}{R\rho}\frac{\partial p}{\partial\theta}. \tag{2.1b}$$

Furthermore, if we look at (1.11c) above, disregard the viscous term, and compare the relative sizes of the various terms for typical large-scale motions, we see that the pressure term and the gravitational term on the right-hand side are several order or magnitudes larger than all terms on the left-hand side, and must therefore be approximately equal in order to 'cancel each other out'; this motivates the widely employed approximation of *hydrostatic balance*:

$$\frac{\partial p}{\partial z} = -\rho g; \tag{2.1c}$$

see Section 2.7 in [116] for extensive comments on the conditions under which hydrostatic balance is a reasonable assumption. The system (2.1a)–(2.1c) is known as the *primitive equations*. Under these approximations, mass conservation reads

$$\frac{\partial \rho}{\partial t} + \frac{1}{R\cos\theta}\frac{\partial}{\partial \varphi}(u\rho) + \frac{1}{R\cos\theta}\frac{\partial}{\partial \theta}(v\rho\cos\theta) + \frac{\partial}{\partial z}(w\rho) = 0.$$

Moreover, if the flow is of relatively limited extent, it is reasonable to expect that geometric effects will only play a marginal role in affecting the dynamics. This intuition motivates the so-called *f-plane approximation*, which allows, for localised flows, to get rid of spherical coordinates and instead introduce a local system of Cartesian coordinates. The idea is to fix a point with spherical coordinates (R, θ_0, φ_0), consider the tangent plane to the surface at that latitude and use a Cartesian system (x, y, z) for the motion with respect to that plane, with z with the local vertical, x in the longitudinal direction, and y in the latitudinal direction. Then, for small excursions, we have that $(x, y, z) \approx (R(\varphi - \varphi_0)\cos\theta_0, R(\theta - \theta_0), r - R)$, and $\mathbf{u} = (u, v, w)$ now denotes the components of the velocity in the tangent plane, and the primitive equations (2.1a)–(2.1c) become

$$\frac{\partial u}{\partial t} + (\mathbf{u}\cdot\nabla)u - fv = F_{\mathrm{v}}^{x}(\mathbf{u}) - \frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{2.2a}$$

$$\frac{\partial v}{\partial t} + (\mathbf{u}\cdot\nabla)v + fu = F_{\mathrm{v}}^{y}(\mathbf{u}) - \frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{2.2b}$$

$$\frac{\partial p}{\partial z} = -\rho g, \tag{2.2c}$$

where $f = 2\Omega\sin\theta_0$ is called the *Coriolis parameter*, and $(F_{\mathrm{v}}^{x}(\mathbf{u}), F_{\mathrm{v}}^{y}(\mathbf{u}), F_{\mathrm{v}}^{z}(\mathbf{u}))$ are the components of the viscous force in Cartesian coordinates.

2.2 Geostrophic balance

Let U be a typical value for the horizontal velocity components u and v, and let W be a typical value for the vertical velocity w; furthermore, suppose that L and H are typical length scales for the horizontal and vertical motion, respectively. We will assume that $W/H \leq U/L$, a reasonable assumption for geophysical flows. If we look at the horizontal momentum equations (2.2a) and (2.2b) in the f-plane approximation, we see that the advective terms $(\mathbf{u}\cdot\nabla)u$ and $(\mathbf{u}\cdot\nabla)v$ scale like U^2/L, whereas the Coriolis terms $-fv$ and fu scale like fU; the ratio of these two scalings is called the *Rossby number*,

$$\mathrm{Ro} := \frac{U}{fL}.$$

In the atmosphere and the ocean, the Rossby number is typically very small (cf. the values in [116, p. 88, Table 2.1]), which means that the advection terms are dominated by the Coriolis force and thus are typically disregarded, rendering the equations (2.2a) and (2.2b) linear. As for the time derivatives $\frac{\partial u}{\partial t}$ and $\frac{\partial v}{\partial t}$, suppose that T is a typical time scale for the flow under consideration, then the ratio between the scaling for these terms and the Coriolis terms is

$$\frac{U/T}{fU} = \frac{1}{fT} = \frac{L/U}{T}\mathrm{Ro};$$

in other words, if $T \sim L/U$ (the *advective* time scale) or less, and the Rossby number is small, then time derivatives are dominated by the Coriolis terms as well, and in this case we may drop them from the horizontal momentum equations; for them to be of comparable order of magnitude as the rotational terms, much shorter time scales are necessary. Moreover, if we assume the frictional forces due to viscosity to be small, then, away from the fluid's boundary, the Coriolis terms can be only balanced by pressure-gradient forces, that is,

$$fu \approx -\frac{1}{\rho}\frac{\partial p}{\partial y}, \quad fv \approx \frac{1}{\rho}\frac{\partial p}{\partial x};$$

such a balance is known as *geostrophic balance*, and the *geostrophic velocity* $(u_{\mathrm{g}}, v_{\mathrm{g}})$ is defined as

$$fu_{\mathrm{g}} = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \quad fv_{\mathrm{g}} = \frac{1}{\rho}\frac{\partial p}{\partial x}. \tag{2.3}$$

As is clear from its definition, geostrophic flow is always tangential to isobars (level sets of the pressure field), and if $f > 0$ the flow encircles a local maximum (minimum) of the pressure in (anti-)clockwise direction (and vice-versa if $f < 0$); moreover, if f is constant, as is the case in the f-plane approximation, the geostrophic velocity field is divergence-free,

$$\frac{\partial u_{\mathrm{g}}}{\partial x} + \frac{\partial v_{\mathrm{g}}}{\partial y} = 0.$$

2.3 The oceanic Ekman layer

Let us now consider the effects of viscosity in the ocean. From now on, we will assume the density ρ to be constant. Viscous terms in the momentum equations typically involve second-order derivatives and thus determine the order of the equation and, in particular, the boundary conditions required. In (sea) water, the viscosity is usually very low, so that its effects are essentially negligible away from the boundary; however, in a thin *boundary layer*, the flow must swiftly adjust to different boundary conditions to those that would correspond to an inviscid setting, so that here viscosity plays a crucial role in determining the dynamics. At the surface of the ocean, below a very thin layer—only few millimetres thick—where molecular viscosity is dominant, there is another layer, of depth varying from a few tens of metres to several hundred metres, where the Coriolis force is balanced not just by horizontal pressure-gradient forces, but also by horizontal wind-stress forces that arise through turbulent motion due to the overlying wind. This layer, called the *Ekman layer*, is characterised by the wind stress vanishing below it, so that geostrophic balance holds at greater depths; as discussed previously, we will ignore the nonlinear advective terms and the time derivatives in the horizontal momentum equations (even though, as we mentioned, for short time scales the latter may be retained—see below), whereas in the vertical we are assuming the hydrostatic balance (2.2c). While, due to its turbulent origin, the contribution of the horizontal viscous terms cannot be parametrised exactly, a reasonable approximate parametrisation in terms of the stress $(\tau^x(\mathbf{u}), \tau^y(\mathbf{u}))$ is

$$(F_{\mathrm{v}}^x(\mathbf{u}), F_{\mathrm{v}}^y(\mathbf{u})) = \frac{1}{\rho}\left(\frac{\partial \tau^x(u,v)}{\partial z}, \frac{\partial \tau^y(u,v)}{\partial z}\right);$$

in other words, the horizontal momentum equations (2.2a)–(2.2b) become

$$-fv = \frac{1}{\rho}\frac{\partial \tau^x(u,v)}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{2.4a}$$

$$fu = \frac{1}{\rho}\frac{\partial \tau^y(u,v)}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial y}. \tag{2.4b}$$

We can now exploit the linearity of this system of equations by splitting (u, v) into a geostrophic and an ageostrophic (Ekman) part,

$$(u, v) = (u_{\mathrm{g}}, v_{\mathrm{g}}) + (u_{\mathrm{e}}, v_{\mathrm{e}}),$$

plugging into (2.4a), and using (2.3), to see that the Ekman velocity satisfies

$$-f v_{\mathrm{e}} = \frac{1}{\rho} \frac{\partial \tau^x(u, v)}{\partial z}, \quad f u_{\mathrm{e}} = \frac{1}{\rho} \frac{\partial \tau^y(u, v)}{\partial z}. \tag{2.5}$$

The most common choice for the stress (τ^x, τ^y) is

$$(\tau^x, \tau^y) = \rho K \left(\frac{\partial u_{\mathrm{e}}}{\partial z}, \frac{\partial v_{\mathrm{e}}}{\partial z} \right), \tag{2.6}$$

where $K = K(z)$ is the so-called *eddy viscosity*; see, for instance, [43, 119]. Therefore, denoting a derivative with respect to z by a prime for simplicity, and introducing the complex notation $\mathbf{U} := u_{\mathrm{e}} + \mathrm{i} v_{\mathrm{e}}$, the system (2.5) can be written as the second-order ordinary differential equation

$$\mathrm{i} f \mathbf{U} = (K \mathbf{U}')'. \tag{2.7}$$

The appropriate upper boundary condition at flat sea level $z = 0$, in accordance with (2.6), is

$$\rho K(0) \mathbf{U}'(0) = \boldsymbol{\tau}_0, \tag{2.8}$$

where $\boldsymbol{\tau}_0 \in \mathbb{C}$ is the (given) wind stress at the surface; moreover, the boundary-layer correction $\mathbf{U}$ to the underlying geostrophic flow must vanish at large depth, hence[1]

$$\mathbf{U}(z) \to 0 \quad (z \to -\infty). \tag{2.9}$$

The Ekman layer is named after V. W. Ekman, who, in 1905, in his influential paper [50], proposed the above model to explain observations made by the great Norwegian explorer, Fridtjof Nansen, during his celebrated *Fram* expedition of 1893–1896. Nansen observed that, in Arctic regions, ice patches at the surface of the ocean moved in a direction to the right of the blowing wind, rather than in the same direction. Taking the eddy viscosity K to be constant, and assuming that $f > 0$ (corresponding to latitudes in the Northern Hemisphere), Ekman argued that the corresponding solution of (2.7) subject to the boundary conditions (2.8) and (2.9), namely

$$\mathbf{U}(z) = \frac{\boldsymbol{\tau}_0}{\rho \sqrt{fK}} \, \mathrm{e}^{\lambda z} \, \mathrm{e}^{\mathrm{i}(\lambda z - \pi/4)} = \frac{1 - \mathrm{i}}{2} \frac{\boldsymbol{\tau}_0}{\rho K \lambda} \, \mathrm{e}^{(1+\mathrm{i})\lambda z},$$

where $\lambda = \sqrt{f/(2K)}$, is indeed deflected to the right of $\boldsymbol{\tau}_0$ by an angle of $\pi/4$ for $z = 0$ (it is easy to see that the deflection would be to the left for $f < 0$, that is, in the Southern Hemisphere); moreover, the velocity $\mathbf{U}(z)$ turns clockwise (counter-clockwise for $f < 0$) with increasing depth (an effect now known as *Ekman spiral*), and the average mass transport,

$$\int_{-\infty}^{0} \rho \mathbf{U}(z) \, \mathrm{d}z = -\mathrm{i} \frac{\boldsymbol{\tau}_0}{f},$$

is at a right angle to the originating wind $\boldsymbol{\tau}_0$; see Figure 2.1. While *in situ* measurements have since established the existence of the Ekman spiral and estimated that the mass

[1] While we could consider finite depth, we choose here to work with infinite depth for simplicity.

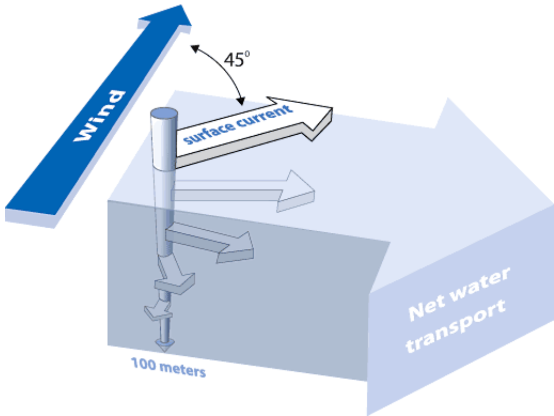

Figure 2.1: Depiction of the Ekman spiral in Ekman's classical solution. Credit: NOAA.

transport is indeed at a right angle to the wind (see, e.g., [92, 100, 110]), there are significant quantitative discrepancies between measurements and Ekman's classical solution (cf. [123]), hence considering eddy viscosity profiles that are allowed to vary with depth is crucial. In [24], it was proved that the qualitative properties of Ekman's solution (surface deflection angle, mass transport, Ekman spiral) remain true for any twice differentiable eddy viscosity K. It is important to note that the deflection of the current to the right (left) of the wind, through the converging or diverging of water masses, drives the formation of vertical motion; this very important mechanism for up- or downwelling in the ocean is known as *Ekman pumping*. On the other hand, the ocean dynamics near the Equator differs significantly from that at higher latitudes. In fact, since the Coriolis force vanishes along the Equator, geostrophic effects are negligible there and nonlinear effects arise at leading order; moreover, the Coriolis force changes sign across the Equator and so acts as a waveguide that gives rise to azimuthal (that is, longitudinal) flow propagation (see, for example, [29, 30, 33, 70]). Therefore, substantially different approaches are required for the description of non-equatorial and equatorial flows. This chapter is devoted to the former aspect of oceanic wind-driven currents; for the latter case, we refer the reader to the above references.

The rigorous mathematical study of the oceanic Ekman layer has attracted considerable attention in recent years; see, for example, [16, 24, 48, 49, 57, 80, 83, 101, 102, 103] for more on the classical Ekman equations in the f-plane, in addition to [25] and [99] for some considerations on nonlinear Ekman-type flows in the f-plane. Moreover, a derivation of equations that describe the dynamics of the Ekman layer in spherical coordinates (without the need for the f-plane approximation) has been achieved in [33] and [39].

Finally, let us briefly remark that the Ekman layer can be found not only at the ocean's surface but also at the ocean's bottom (see, e.g., the discussion in [46]), as well as in the lower atmosphere. The latter is modelled similarly to its oceanic counterpart, but with different boundary conditions; we will not delve into this topic, instead referring the interested reader to [34, 35, 39, 58, 69, 111] and the references therein.

2.4 A perturbative approach for the surface deflection angle

As we saw, investigating the dependence of the surface deflection angle on the profile of the eddy viscosity is crucial to obtain predictions that are in better agreement with observations. In addition to the references mentioned above, it is worth pointing out that, in [120], an application of the WKB approach yields an asymptotic approximation of the solution to the general Ekman model. Furthermore, a result from [24] states that, supposing that $K_0 > 0$ is a constant and $K^\varepsilon(z) = K_0 + \varepsilon K_1(z) + o(\varepsilon)$ as $\varepsilon \to 0^+$, where $K_1 : (-\infty, 0] \to \mathbb{R}$ is a continuously differentiable function, then, for ε sufficiently small, the surface deflection angle of the solution of the perturbed problem for $K(z) = K^\varepsilon(z)$ will be larger (smaller) than $\pi/4$ if and only if

$$\begin{aligned} &\operatorname{Im}\left\{\frac{(1+\mathrm{i})\lambda}{K_0}\int_{-\infty}^{0} \mathrm{e}^{2(1+\mathrm{i})\lambda s}\, K_1(s)\,\mathrm{d}s\right\} \\ &\qquad = \frac{\lambda\sqrt{2}}{K_0}\int_{-\infty}^{0} \mathrm{e}^{2\lambda s}\, K_1(s)\sin\Big(2\lambda s + \frac{\pi}{4}\Big)\,\mathrm{d}s > 0 \quad (<0). \end{aligned} \tag{2.10}$$

The goal of this section is to prove an extension of this fact to the case of perturbations of a non-constant eddy viscosity.

Without loss of generality, we may assume that $\boldsymbol{\tau}_0 = \tau_0 > 0$. It is convenient to non-dimensionalise problem (2.7)–(2.9) by setting

$$\nu = \frac{f\rho}{\tau_0}K, \quad \psi = \frac{\nu(0)}{\sqrt{2\tau_0/\rho}}\mathbf{U}, \quad \zeta = \frac{f}{\sqrt{2\tau_0/\rho}}z.$$

From this we obtain the non-dimensional problem

$$\begin{aligned} (\nu(z)\psi'(\zeta))' - 2\mathrm{i}\psi(\zeta) &= 0 \quad \text{for } \zeta < 0, \\ \psi' &= 1 \quad \text{on } \{\zeta = 0\}, \\ \psi(\zeta) &\to 0 \quad \text{as } \zeta \to -\infty, \end{aligned} \tag{2.11}$$

where the prime now denotes a derivative with respect to ζ. We note that, with the convention that $\arg z \in [0, 2\pi)$ is the argument of $z \in \mathbb{C}$, the surface deflection angle, which is the main objective of this section, is given by $2\pi - \arg\psi(0)$.

Now, suppose that, for the continuous $\nu_0 : (-\infty, 0] \to (0, \infty)$ with $\nu_0(\zeta) = 1$ for $\zeta \le -h$ (for some non-dimensional depth $-h < 0$), the solution ψ_0 to

$$\begin{aligned} (\nu_0(\zeta)\psi_0'(\zeta))' - 2\mathrm{i}\psi_0(\zeta) &= 0 \quad \text{for } \zeta < 0, \\ \psi_0' &= 1 \quad \text{on } \zeta = 0, \\ \psi_0(\zeta) &\to 0 \quad \text{as } \zeta \to -\infty \end{aligned} \tag{2.12}$$

is known. Henceforth we consider a ν of the form

$$\nu^\varepsilon(\zeta) = \nu_0(\zeta) + \varepsilon\nu_1(\zeta) + o(\varepsilon) \quad \text{as } \varepsilon \to 0^+, \tag{2.13}$$

uniformly in $\zeta \le 0$. Moreover, we assume that ν_1 is continuous on $(-\infty, 0]$ and

$$\nu_1(\zeta) = 0, \quad \zeta \le -h.$$

Note that the requirement of a constant ν below some depth is physically reasonable (cf. [48]): indeed, the turbulence is confined to the thin Ekman boundary layer, thus the eddy viscosity at greater depth coincides with that of sea water, which can be normalised to obtain

$$\nu(\zeta) = 1, \quad \zeta \le -h.$$

With this preparation, the main result of this section can be stated as follows [101]:

Theorem 2.4.1. *Let $\nu_0, \nu_1 : (-\infty, 0]$ be continuously differentiable functions such that $\nu_0(\zeta) > 0$ for all $\zeta \le 0$ and $\nu_1(\zeta) = 0$ for $\zeta \le -h$, for some $h > 0$. Let ψ_0 be the solution of (2.12), and suppose that ν^ε is as in (2.13). Then, for ε sufficiently small, the surface deflection angle of the solution of the perturbed problem (2.11) for $\nu(\zeta) = \nu^\varepsilon(\zeta)$ will be larger (smaller) than the surface deflection angle of ψ_0 if and only if*

$$\frac{1}{\nu_0(0)} \operatorname{Im}\left\{ \frac{1}{\psi_0(0)} \int_{-h}^{0} (\psi_0'(s))^2 \nu_1(s) \, \mathrm{d}s \right\} < 0 \quad (> 0). \tag{2.14}$$

Proof. In [48], it is shown that, if q solves the Riccati equation

$$q'(\zeta) + \frac{q(\zeta)^2}{\nu(\zeta)} = 2\mathrm{i}, \quad \zeta \in (-h, 0) \tag{2.15a}$$

$$q(-h) = 1 + \mathrm{i}, \tag{2.15b}$$

then for the restriction of the solution ψ of (2.11) on $(-h, 0)$ we have that $\psi(0) = \nu(0)/q(0)$,

$$\arg \psi(0) = -\arg q(0) \in [0, 2\pi) \tag{2.16}$$

and

$$\psi(\zeta) = \frac{\nu(0)}{q(0)} \exp\left(-\int_\zeta^0 \frac{q(y)}{\nu(y)} \, \mathrm{d}y \right), \quad \zeta \in (-h, 0). \tag{2.17}$$

Following (2.13), let us make the ansatz

$$q^\varepsilon(\zeta) = q_0(\zeta) + \varepsilon q_1(\zeta) + o(\varepsilon) \quad \text{as } \varepsilon \to 0^+, \tag{2.18}$$

uniformly in $\zeta \le 0$, for the solution to (2.15), where q_0 solves

$$q_0'(\zeta) + \frac{q_0(\zeta)^2}{\nu_0(\zeta)} = 2\mathrm{i}, \quad \zeta \in (-h, 0) \tag{2.19a}$$

$$q_0(-h) = 1 + \mathrm{i}, \tag{2.19b}$$

and q_1 is to be determined. Inserting (2.13) and (2.18) into (2.15a), using (2.19a), dividing by ε, and letting $\varepsilon \to 0^+$, we get the following equation for q_1:

$$q_1'(\zeta) = -2\frac{q_0(\zeta)}{\nu_0(\zeta)} q_1(\zeta) + \frac{q_0(\zeta)^2 \nu_1(\zeta)}{\nu_0(\zeta)^2}, \quad \zeta \in (-h, 0), \tag{2.20}$$

with 'initial' condition

$$q_1(-h) = 0, \tag{2.21}$$

in view of (2.15b) and (2.19b). The solution to (2.20)–(2.21) is

$$q_1(\zeta) = \exp\left(-2\int_{-h}^{\zeta} \frac{q_0(y)}{\nu_0(y)} \, \mathrm{d}y \right) \int_{-h}^{\zeta} \exp\left(2\int_{-h}^{s} \frac{q_0(y)}{\nu_0(y)} \, \mathrm{d}y \right) \frac{q_0(s)^2 \nu_1(s)}{\nu_0(s)^2} \, \mathrm{d}s.$$

Thus

$$\begin{aligned} q^\varepsilon(0) &= q_0(0) + \varepsilon \int_{-h}^{0} \exp\left(-2\int_s^0 \frac{q_0(y)}{\nu_0(y)} \, \mathrm{d}y \right) \frac{q_0(s)^2 \nu_1(s)}{\nu_0(s)^2} \, \mathrm{d}s + o(\varepsilon) \\ &= q_0(0) \left[1 + \frac{\varepsilon}{q_0(0)} \int_{-h}^{0} \exp\left(-2\int_s^0 \frac{q_0(y)}{\nu_0(y)} \, \mathrm{d}y \right) \frac{q_0(s)^2 \nu_1(s)}{\nu_0(s)^2} \, \mathrm{d}s \right] + o(\varepsilon). \end{aligned}$$

The formula

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\arg(1+\varepsilon\xi)\Big|_{\varepsilon=0} = \mathrm{Im}(\xi), \quad \xi\in\mathbb{C}$$

(cf. [16, 24]) enables us to express the change of $\arg q^\varepsilon(0)$ due to the perturbation as

$$\mathrm{Im}\left\{\frac{1}{q_0(0)}\int_{-h}^{0}\exp\left(-2\int_s^0\frac{q_0(y)}{\nu_0(y)}\,\mathrm{d}y\right)\frac{q_0(s)^2\nu_1(s)}{\nu_0(s)^2}\,\mathrm{d}s\right\} =: C_{\nu_1}. \tag{2.22}$$

Therefore, by (2.16), a positive (negative) value of C_{ν_1} corresponds to a decrease (increase) of $\arg\psi^\varepsilon(0)$ from the reference value $\arg\psi_0(0)$ (in the counterclockwise direction). Moreover, (2.17) implies

$$\exp\left(-2\int_s^0\frac{q_0(y)}{\nu_0(y)}\,\mathrm{d}y\right) = \frac{\psi_0(s)^2}{\psi_0(0)^2} \quad\text{and}\quad \frac{q_0(s)}{\nu_0(s)} = \frac{\psi_0'(s)}{\psi_0(s)},$$

thus formula (2.22) can be rewritten in terms of ψ_0 as

$$C_{\nu_1} = \frac{1}{\nu_0(0)}\,\mathrm{Im}\left\{\frac{1}{\psi_0(0)}\int_{-h}^{0}(\psi_0'(s))^2\nu_1(s)\,\mathrm{d}s\right\}; \tag{2.23}$$

this gives the claim, since the reference value for the deflection angle is $2\pi-\arg\psi_0(0)$. □

Remark 2.4.2. The easiest situation where formula (2.23) can be applied is the case of constant $\nu_0\equiv 1$ on $(-\infty,0]$ (cf. the discussion in [16, 24]). Here we have

$$\psi_0(\zeta) = \frac{1-\mathrm{i}}{2}\,\mathrm{e}^{(1+\mathrm{i})\zeta},$$

hence, according to (2.16) and (2.23), the change of $\arg\psi(0)$ is given by

$$-\,\mathrm{Im}\left\{(1+\mathrm{i})\int_{-h}^{0}\mathrm{e}^{2(1+\mathrm{i})s}\,\nu_1(s)\,\mathrm{d}s\right\} = -\sqrt{2}\int_{-h}^{0}\mathrm{e}^{2s}\sin\left(2s+\frac{\pi}{4}\right)\nu_1(s)\,\mathrm{d}s,$$

which corresponds, up to a scaling, to the formula (2.10) applied to an upper-layer perturbation that is confined to the interval $(-h,0)$. In fact, returning to the original dimensional variables, the result can be formulated as follows: Let $K_0, K_1 : (-\infty,0]$ be continuously differentiable functions with $K_0(z)>0$ for all $z\le 0$ and $K_1(z)=0$ for $z\le -h$, for some $h>0$. Suppose that $K^\varepsilon(z) = K_0(z)+\varepsilon K_1(z)+o(\varepsilon)$ as $\varepsilon\to 0^+$, and let the solution of (2.7)–(2.9) for $K(z)=K_0(z)$ be denoted by $\mathbf{U}_0(z)$. Then, for ε sufficiently small, the surface deflection angle of the solution of the perturbed problem for $K(z)=K^\varepsilon(z)$ will be larger (smaller) than the surface deflection angle of $\mathbf{U}_0$ if and only if

$$\mathrm{Im}\left\{\frac{\rho}{\boldsymbol{\tau}_0\mathbf{U}_0(0)}\int_{-h}^{0}\left(\mathbf{U}_0'(s)\right)^2K_1(s)\,\mathrm{d}s\right\} > 0 \quad (<0).$$

Remark 2.4.3. It is interesting to note that the presence of a sine function in the integrand is not just a feature of the special case of constant ν_0. In fact, if we write

$$\psi_0(0) = |\psi_0(0)|\,\mathrm{e}^{\mathrm{i}\arg\psi_0(0)} \quad\text{and}\quad \psi_0'(s) = |\psi_0'(s)|\,\mathrm{e}^{\mathrm{i}\arg\psi_0'(s)},$$

then formula (2.23) becomes

$$\begin{aligned} C_{\nu_1} &= \frac{1}{\nu_0(0)|\psi_0(0)|}\mathrm{Im}\left\{\int_{-h}^{0}|\psi_0'(s)|^2\,\mathrm{e}^{\mathrm{i}[2\arg\psi_0'(s)-\arg\psi_0(0)]}\,\nu_1(s)\,\mathrm{d}s\right\}\\ &= \frac{1}{\nu_0(0)|\psi_0(0)|}\int_{-h}^{0}|\psi_0'(s)|^2\sin\left(2\arg\psi_0'(s)-\arg\psi_0(0)\right)\nu_1(s)\,\mathrm{d}s. \end{aligned}$$

From this it is apparent that, in general, it is very difficult to tell at first glance whether a given perturbation ν_1 of ν_0 will lead to a positive or negative increment of the surface deflection angle. For example, suppose that $\nu_1(\zeta) > 0$ for each $\zeta \in (-h, 0)$. Due to the periodicity of the sine, we may assume that $2\arg\psi_0'(s) - \arg\psi_0(0) \in (-\pi, \pi]$ for each $s \in (-h, 0)$. Let us denote

$$\begin{aligned}P &\coloneqq \big\{s \in (-h,0) : 2\arg\psi_0'(s) - \arg\psi_0(0) \in (0,\pi)\big\},\\ N &\coloneqq \big\{s \in (-h,0) : 2\arg\psi_0'(s) - \arg\psi_0(0) \in (-\pi,0)\big\}.\end{aligned}$$

Since $\sin\big(2\arg\psi_0'(s) - \arg\psi_0(0)\big)$ is positive for $s \in P$ and negative for $s \in N$, the sign of C_{ν_1} will depend both on the measures $|P|$ and $|N|$ and on the relative sizes of the values attained by ν_1 on P and N.

Remark 2.4.4. Finally, let us look at one last example that shows how the Riccati variant (2.22) of (2.23) can be useful in its own way, not just as a step in the derivation of (2.23). This is especially true if the solution is found by using the reformulation (2.15) in the first place. In [48] the solution to the Riccati formulation of the problem for the case

$$\nu_0(\zeta) = \big(3(\zeta + h) + 1\big)^{4/3}, \quad \zeta \in (-h, 0)$$

is determined to be

$$q_0(\zeta) = -S(\zeta) - (1-\mathrm{i})S(\zeta)^2 \tan\big((1-\mathrm{i})S(\zeta) + C\big),$$

where $S(\zeta) = \big(3(\zeta + h) + 1\big)^{1/3}$ and

$$C = -1 + \mathrm{i} + \frac{\mathrm{i}}{2}\log\frac{1-\mathrm{i}}{\mathrm{i}-5}.$$

Using the identity

$$\tan\theta = -\frac{\mathrm{d}}{\mathrm{d}\theta}(\log\cos\theta)$$

and the fundamental theorem of calculus, a straightforward computation yields

$$\exp\left(-2\int_s^0 \frac{q_0(y)}{\nu_0(y)}\,\mathrm{d}y\right) = \left(\frac{S(0)\cos\big((1-\mathrm{i})S(s) + C\big)}{S(s)\cos\big((1-\mathrm{i})S(0) + C\big)}\right)^2,$$

which can be directly plugged into (2.22), without first having to determine and differentiate the solution ψ_0.

We conclude this section by saying that formula (2.23) may be relevant especially in view of the fact that, as pointed out in Section 2.3, explicit solutions are hard to find and the few available show an intricate behaviour even for relatively simple eddy viscosity profiles. The perturbation approach that we have pursued here may be useful in bringing about some new insight into this delicate problem, but it would be interesting to obtain further results that do not require the smallness assumption that is intrinsic to the perturbation method. However, it can also be of great importance to bring time dependence into the picture; this is the subject of the next section.

2.5 The surface current of unsteady Ekman flows

As we mentioned in Section 2.2, the inclusion of time evolution in the model can be of interest, at least for small time scales; this was initially done for steady wind and eddy viscosity [77], although the discussion was subsequently extended to the setting of time-dependent winds and eddy viscosity [106, 107], showing that the response of the Ekman layer to time-varying winds is strongly influenced by the time dependence of the eddy viscosity. In the present section, we consider a particular case of a nonzero initial condition, that is, we assume that the flow is initially in the steady state corresponding to the initial wind and eddy viscosity. We tackle the problem as follows. First, we solve the problem by means of Laplace transforms, thereby finding an explicit formula for the surface current; next, we look at the properties and qualitative behaviour of the solution. The analytical results are complemented by some numerical plots that are useful to visualise the most relevant features. The results of this section are published in [103].

For convenience when we are non-dimensionalising, let us denote the time variable by $\tilde{t}$. If we retain the time derivative in the governing equations and, as before, we perform the splitting of the velocity field into a (time-indepedent) geostrophic component and a (time-dependent) Ekman component, we obtain the following time-dependent analogue to (2.7) (cf. [77]):

$$\frac{\partial \mathbf{U}}{\partial t} + \mathrm{i} f \mathbf{U} = \frac{\partial}{\partial z}\left(K \frac{\partial \mathbf{U}}{\partial z}\right),$$

with boundary conditions

$$\begin{aligned}
&\frac{\partial \mathbf{U}}{\partial z}(0,\tilde{t}) = \frac{\boldsymbol{\tau}_0(\tilde{t})}{\rho K(\tilde{t})},\\
&\mathbf{U}(z,\tilde{t}) \to 0 \quad \text{as } z \to -\infty,
\end{aligned}$$

corresponding to the boundary conditions (2.8) and (2.9) for the steady problem.

Suppose that $\tau_0 > 0$ is some typical value of the intensity of the surface wind stress. We can non-dimensionalise the problem by scaling the depth and time variables as

$$\zeta := \frac{zf}{\sqrt{\tau_0/\rho}} \quad \text{and} \quad t := f\tilde{t}$$

(note that the time scale is much smaller than the advective one) and setting

$$\nu(\zeta,t) := \frac{f\rho K(z,\tilde{t})}{\tau_0}, \quad \psi(\zeta,t) := \frac{\mathbf{U}(z,\tilde{t})}{\sqrt{\tau_0/\rho}}, \quad \tau(t) := \frac{\boldsymbol{\tau}_0(\tilde{t})}{\tau_0},$$

to get:

$$\begin{aligned}
&\frac{\partial \psi}{\partial \tilde{t}}(\zeta,\tilde{t}) + \mathrm{i}\psi(\zeta,\tilde{t}) = \frac{\partial}{\partial \zeta}\left(\nu(\zeta,t)\frac{\partial \psi}{\partial \zeta}(\zeta,t)\right), \quad (\zeta,t) \in (-\infty,0)\times(0,\infty),\\
&\frac{\partial \psi}{\partial \zeta}(0,t) = \frac{\tau(t)}{\nu(0,t)},\\
&\psi(\zeta,t) \to 0 \quad \text{as } \zeta \to -\infty,\\
&\psi(\zeta,0) = \psi_0(\zeta),
\end{aligned} \tag{2.24}$$

where ψ_0 is an appropriate initial condition. In what follows, we will make more precise assumptions on ν and ψ_0.

In view of the fact that, as we saw, the steady problem for depth dependent ν is already involved, it should be no surprise that the full, time-dependent problem (2.24), where ν may depend both on time and depth, is also very complicated. In the following, we will focus only on the aspect of time dependence of the eddy viscosity; although this is not the most general case, studying the effect that a time-varying eddy viscosity has on the oceanic response to a time-varying wind is still of great interest. Furthermore, in this case, it is possible to obtain exact analytical results, without being forced to resort to numerical methods. For a (semi)numerical investigation of problems where ν depends on both time and depth, we refer the reader to [106, 107].

From now on we consider problem (2.24) with the simplification that ν is only a function of time; in other words, we investigate

$$\begin{aligned}
&\frac{\partial \psi}{\partial t}(\zeta,t) + \mathrm{i}\psi(\zeta,t) = \nu(t)\frac{\partial^2 \psi}{\partial \zeta^2}(\zeta,t), \quad (\zeta,t) \in (-\infty,0)\times(0,\infty),\\
&\frac{\partial \psi}{\partial \zeta}(0,t) = \frac{\tau(t)}{\nu(t)},\\
&\psi(\zeta,t) \to 0 \quad \text{as } \zeta \to -\infty,\\
&\psi(\zeta,0) = \psi_0(\zeta),
\end{aligned} \tag{2.25}$$

where $\tau : [0,\infty) \to \mathbb{C}$, $\nu : [0,\infty) \to (0,\infty)$ and $\nu(0) = \nu_0 > 0$. As initial condition ψ_0 we take the solution of the ordinary differential problem

$$\begin{aligned}
&\mathrm{i}\psi_0(\zeta) = \nu_0 \frac{\mathrm{d}^2 \psi_0}{\mathrm{d}\zeta^2}(\zeta), \quad \zeta \in (-\infty,0),\\
&\frac{\mathrm{d}\psi_0}{\mathrm{d}\zeta}(0) = \frac{\tau(0)}{\nu_0},\\
&\psi(\zeta) \to 0 \quad \text{as } \zeta \to -\infty,
\end{aligned}$$

that is,

$$\psi_0(\zeta) = \tau(0)\frac{1-\mathrm{i}}{\sqrt{2\nu_0}} \exp\left(\frac{1+\mathrm{i}}{\sqrt{2\nu_0}}\zeta\right), \quad \zeta \in (-\infty,0).$$

Furthermore, we assume that $t \mapsto \nu(t)$ is continuous, bounded and uniformly positive, that is, there exist constants $A > B > 0$ such that

$$A \geq \nu(t) \geq B \quad \text{for all } t \geq 0. \tag{2.26}$$

Physically speaking, the problem (2.25) describes the time evolution of a current that is initially in the steady state corresponding to the constant wind $\tau(0)$ and eddy viscosity ν_0, as the wind and eddy viscosity change over time, while the latter remains constant in depth.

It is convenient to reformulate the problem in the following way. First of all, we introduce the new function

$$\eta(\zeta,t) := \mathrm{e}^{\mathrm{i}t}\,\psi(\zeta,t),$$

which is then a solution of the problem

$$\begin{aligned}
&\frac{\partial \eta}{\partial t}(\zeta,t) = \nu(t)\frac{\partial^2 \eta}{\partial \zeta^2}(\zeta,t), \quad (\zeta,t) \in (-\infty,0)\times(0,\infty),\\
&\frac{\partial \eta}{\partial \zeta}(0,t) = \frac{\mathrm{e}^{\mathrm{i}t}\,\tau(t)}{\nu(t)},\\
&\eta(\zeta,t) \to 0 \quad \text{as } \zeta \to -\infty,\\
&\eta(\zeta,0) = \psi_0(\zeta).
\end{aligned}$$

Next, in order to eliminate the coefficient $\nu(t)$ from the equation, we introduce the change of variable

$$T = \int_0^t \nu(s)\,\mathrm{d}s =: f(t);$$

note that $f(0) = 0$ and that f is monotonically increasing (since $f'(t) = \nu(t) > 0$ for all $t \geq 0$). Moreover, (2.26) implies $\lim_{t\to\infty} f(t) = \infty$, therefore f is a bijection from $[0,\infty)$ onto itself. We also define

$$\varphi(\zeta, T) \coloneqq \eta(\zeta, f^{-1}(T)).$$

Then φ solves

$$\begin{aligned}
&\frac{\partial\varphi}{\partial T}(\zeta,T) = \frac{\partial^2\varphi}{\partial\zeta^2}(\zeta,T), \quad (\zeta,T) \in (-\infty,0)\times(0,\infty),\\
&\frac{\partial\varphi}{\partial\zeta}(0,T) = \frac{\mathrm{e}^{\mathrm{i}f^{-1}(T)}\,\tau(f^{-1}(T))}{\nu(f^{-1}(T))},\\
&\varphi(\zeta,T) \to 0 \quad \text{as } \zeta \to -\infty,\\
&\varphi(\zeta,0) = \psi_0(\zeta).
\end{aligned} \tag{2.27}$$

In order to solve (2.27), we take the Laplace transform of φ with respect to the variable T,

$$\hat\varphi(\zeta,s) \coloneqq \mathscr{L}(\varphi(\zeta,\cdot);s) = \int_0^\infty \mathrm{e}^{-sT}\,\varphi(\zeta,T)\,\mathrm{d}T \quad (\mathrm{Re}(s) > 0),$$

and we compute (using the first equation (2.27), integration by parts and the fourth equation in (2.27)):

$$\frac{\mathrm{d}^2\hat\varphi}{\mathrm{d}\zeta^2}(\zeta,s) = \int_0^\infty \mathrm{e}^{-sT}\,\frac{\partial^2\varphi}{\partial\zeta^2}(\zeta,T)\,\mathrm{d}T = \int_0^\infty \mathrm{e}^{-sT}\,\frac{\partial\varphi}{\partial t}(\zeta,T)\,\mathrm{d}T = -\psi_0(\zeta) + s\hat\varphi(\zeta,s).$$

The boundary conditions in (2.27) transform to

$$\frac{\mathrm{d}\hat\varphi}{\mathrm{d}\zeta}(0,s) = \int_0^\infty \mathrm{e}^{-sT}\,\frac{\partial\varphi}{\partial\zeta}(0,T)\,\mathrm{d}T = \mathscr{L}\left(\frac{\mathrm{e}^{\mathrm{i}f^{-1}(\cdot)}\,\tau(f^{-1}(\cdot))}{\nu(f^{-1}(\cdot))};s\right)$$

and

$$\hat\varphi(\zeta,s) \to 0 \quad (\zeta \to -\infty),$$

respectively. Let us introduce the shorter notation

$$F(s) \coloneqq \mathscr{L}\left(\frac{\mathrm{e}^{\mathrm{i}f^{-1}(\cdot)}\,\tau(f^{-1}(\cdot))}{\nu(f^{-1}(\cdot))};s\right).$$

We are then looking at the ordinary differential problem

$$\begin{aligned}
&\frac{\mathrm{d}^2\hat\varphi}{\mathrm{d}\zeta^2}(\zeta,s) - s\hat\varphi(\zeta,s) = -\psi_0(\zeta), \quad \zeta \in (-\infty,0),\\
&\frac{\mathrm{d}\hat\varphi}{\mathrm{d}\zeta}(0,s) = F(s),\\
&\hat\varphi(\zeta,s) \to 0 \quad \text{as } \zeta \to -\infty.
\end{aligned} \tag{2.28}$$

The general solution of the first equation in (2.28) is (see [93])

$$\hat\varphi(\zeta,s) = A\,\mathrm{e}^{\sqrt{s}\zeta} + B\,\mathrm{e}^{-\sqrt{s}\zeta} + \frac{\mathrm{e}^{-\sqrt{s}\zeta}}{2\sqrt{s}}\int_0^\zeta \mathrm{e}^{\sqrt{s}y}\,\psi_0(y)\,\mathrm{d}y - \frac{\mathrm{e}^{\sqrt{s}\zeta}}{2\sqrt{s}}\int_0^\zeta \mathrm{e}^{-\sqrt{s}y}\,\psi_0(y)\,\mathrm{d}y, \tag{2.29}$$

where the constants $A, B \in \mathbb{C}$ have to be determined from the boundary conditions. Since

$$\int_0^\zeta \mathrm{e}^{\pm\sqrt{s}y}\,\psi_0(y)\,\mathrm{d}y = \frac{C}{\frac{1+\mathrm{i}}{\sqrt{2\nu_0}} \pm \sqrt{s}}\left[\mathrm{e}^{\pm\sqrt{s}\zeta}\exp\left(\frac{1+\mathrm{i}}{\sqrt{2\nu_0}}\zeta\right) - 1\right],$$

where $C = \tau(0)(1-\mathrm{i})/\sqrt{2\nu_0}$, we can rewrite (2.29) as

$$\hat{\varphi}(\zeta, s) = \left(A + \frac{C}{2\sqrt{s}\left(\frac{1+\mathrm{i}}{\sqrt{2\nu_0}} - \sqrt{s}\right)}\right)\mathrm{e}^{\sqrt{s}\zeta} + \left(B - \frac{C}{2\sqrt{s}\left(\frac{1+\mathrm{i}}{\sqrt{2\nu_0}} + \sqrt{s}\right)}\right)\mathrm{e}^{-\sqrt{s}\zeta} + \frac{\psi_0(\zeta)}{s - \frac{\mathrm{i}}{\nu_0}}.$$

The boundary conditions in (2.28) imply

$$\begin{aligned} B - \frac{C}{2\sqrt{s}\left(\frac{1+\mathrm{i}}{\sqrt{2\nu_0}} + \sqrt{s}\right)} &= 0, \\ A + \frac{C}{2\sqrt{s}\left(\frac{1+\mathrm{i}}{\sqrt{2\nu_0}} - \sqrt{s}\right)} &= \left[F(s) - \frac{\tau(0)}{\nu_0\left(s - \frac{\mathrm{i}}{\nu_0}\right)}\right]\frac{1}{\sqrt{s}}; \end{aligned}$$

therefore, the solution of (2.28) is

$$\hat{\varphi}(\zeta, s) = \left[F(s) - \frac{\tau(0)}{\nu_0\left(s - \frac{\mathrm{i}}{\nu_0}\right)}\right]\frac{\mathrm{e}^{\sqrt{s}\zeta}}{\sqrt{s}} + \frac{\psi_0(\zeta)}{s - \frac{\mathrm{i}}{\nu_0}}. \tag{2.30}$$

The inverse Laplace transform of (2.30) for $\zeta < 0$ cannot be computed explicitly in terms of elementary or special functions. Nevertheless, the Laplace inversion can be performed in the case $\zeta = 0$, yielding a formula for the surface current. In fact, we can exploit the following well-known property of the Laplace transform (see [56]):

$$\mathscr{L}(f * g; s) = \mathscr{L}(f; s)\mathscr{L}(g; s); \tag{2.31}$$

here, $f * g$ denotes the Laplace convolution of the functions f and g:

$$(f * g)(x) := \int_0^x f(x-\xi)g(\xi)\,\mathrm{d}\xi = \int_0^x f(\xi)g(x-\xi)\,\mathrm{d}\xi.$$

Since we have (see [56])

$$\frac{1}{\sqrt{s}} = \mathscr{L}\left(\frac{1}{\sqrt{\pi(\cdot)}}; s\right), \quad F(s) = \mathscr{L}\left(\frac{\mathrm{e}^{\mathrm{i}f^{-1}(\cdot)}\,\tau(f^{-1}(\cdot))}{\nu(f^{-1}(\cdot))}; s\right) \quad \text{and} \quad \frac{1}{s - \frac{\mathrm{i}}{\nu_0}} = \mathscr{L}\left(\mathrm{e}^{\mathrm{i}(\cdot)/\nu_0}; s\right),$$

it follows from (2.31) that

$$\varphi(0, T) = \int_0^T \left(\frac{\mathrm{e}^{\mathrm{i}f^{-1}(s)}\,\tau(f^{-1}(s))}{\nu(f^{-1}(s))} - \tau(0)\frac{\mathrm{e}^{\mathrm{i}s/\nu_0}}{\nu_0}\right)\frac{1}{\sqrt{\pi(T-s)}}\,\mathrm{d}s + \tau(0)\frac{1-\mathrm{i}}{\sqrt{2\nu_0}}\,\mathrm{e}^{\mathrm{i}T/\nu_0}$$

and therefore (after the substitution $\sigma = f^{-1}(s)$)

$$\begin{aligned} \psi(0, t) = &\frac{1}{\sqrt{\pi}}\int_0^t \left[\tau(\sigma)\,\mathrm{e}^{-\mathrm{i}(t-\sigma)} - \tau(0)\,\mathrm{e}^{-\mathrm{i}t}\,\frac{\nu(\sigma)}{\nu_0}\exp\left(\frac{\mathrm{i}}{\nu_0}\int_0^\sigma \nu(s)\,\mathrm{d}s\right)\right] \times \\ &\times \left(\int_\sigma^t \nu(s)\,\mathrm{d}s\right)^{-1/2}\mathrm{d}\sigma + \tau(0)\frac{1-\mathrm{i}}{\sqrt{2\nu_0}}\,\mathrm{e}^{-\mathrm{i}t}\exp\left(\frac{\mathrm{i}}{\nu_0}\int_0^t \nu(s)\,\mathrm{d}s\right). \end{aligned} \tag{2.32}$$

Given the physical interpretation of (2.25), we expect that, for constant $\nu(t) = \nu_0$ and $\tau(t) = \tau(0)$ for all $t \geq 0$, the current will not leave its initial state, which is already the steady state for the eddy viscosity ν_0 and the shear wind stress $\tau(0)$. Let us check this: we have

$$\int_{t_1}^{t_2} \nu(s)\,\mathrm{d}s = \nu_0(t_2 - t_1)$$

and thus

$$\psi(0,t) = \frac{\tau(0)}{\sqrt{\pi}} \int_0^t \frac{\mathrm{e}^{-\mathrm{i}(t-\sigma)} - \mathrm{e}^{-\mathrm{i}t}\,\mathrm{e}^{\mathrm{i}\sigma}}{\sqrt{\nu_0(t-\sigma)}}\,\mathrm{d}\sigma + \tau(0)\frac{1-\mathrm{i}}{\sqrt{2\nu_0}}\,\mathrm{e}^{-\mathrm{i}t}\,\mathrm{e}^{\mathrm{i}t} = \tau(0)\frac{1-\mathrm{i}}{\sqrt{2\nu_0}} = \psi_0(0)$$

for all $t \geq 0$, as expected.

Another reasonable prediction is that, if the eddy viscosity approaches a fixed value as time increases, the current should settle towards the corresponding steady state. This is less straightforward to check than the previous case, but in fact it can be proven under mild assumptions on the convergence rate of $\nu(t)$, as we will show shortly. Before that, it is convenient to rewrite (2.32) in a different form. Note that

$$\left(\int_\sigma^t \nu(s)\,\mathrm{d}s\right)^{-1/2} = -\frac{\sqrt{2\pi\nu_0}}{\nu(\sigma)(1+\mathrm{i})} \exp\left(\frac{\mathrm{i}}{\nu_0}\int_\sigma^t \nu(s)\,\mathrm{d}s\right) \frac{\mathrm{d}}{\mathrm{d}\sigma}\mathrm{erf}\left(\frac{1+\mathrm{i}}{\sqrt{2\nu_0}}\left(\int_\sigma^t \nu(s)\,\mathrm{d}s\right)^{1/2}\right),$$

where erf denotes the complex error function. From this we obtain

$$\begin{aligned}
&\frac{1}{\sqrt{\pi}}\int_0^t \frac{\nu(\sigma)}{\nu_0}\exp\left(\frac{\mathrm{i}}{\nu_0}\int_0^\sigma \nu(s)\,\mathrm{d}s\right)\left(\int_\sigma^t \nu(s)\,\mathrm{d}s\right)^{-1/2}\mathrm{d}\sigma \\
&\quad= -\frac{1-\mathrm{i}}{\sqrt{2\nu_0}}\exp\left(\frac{\mathrm{i}}{\nu_0}\int_0^t \nu(s)\,\mathrm{d}s\right)\int_0^t \frac{\mathrm{d}}{\mathrm{d}\sigma}\mathrm{erf}\left(\frac{1+\mathrm{i}}{\sqrt{2\nu_0}}\left(\int_\sigma^t \nu(s)\,\mathrm{d}s\right)^{1/2}\right)\mathrm{d}\sigma \\
&\quad= \frac{1-\mathrm{i}}{\sqrt{2\nu_0}}\exp\left(\frac{\mathrm{i}}{\nu_0}\int_0^t \nu(s)\,\mathrm{d}s\right)\mathrm{erf}\left(\frac{1+\mathrm{i}}{\sqrt{2\nu_0}}\left(\int_0^t \nu(s)\,\mathrm{d}s\right)^{1/2}\right)
\end{aligned}$$

and thus

$$\begin{aligned}
\psi(0,t) &= \tau(0)\frac{1-\mathrm{i}}{\sqrt{2\nu_0}}\,\mathrm{e}^{-\mathrm{i}t}\exp\left(\frac{\mathrm{i}}{\nu_0}\int_0^t \nu(s)\,\mathrm{d}s\right)\left[1 - \mathrm{erf}\left(\frac{1+\mathrm{i}}{\sqrt{2\nu_0}}\left(\int_0^t \nu(s)\,\mathrm{d}s\right)^{1/2}\right)\right] \\
&\quad+ \frac{1}{\sqrt{\pi}}\int_0^t \tau(\sigma)\,\mathrm{e}^{-\mathrm{i}(t-\sigma)}\left(\int_\sigma^t \nu(s)\,\mathrm{d}s\right)^{-1/2}\mathrm{d}\sigma.
\end{aligned}$$

Since

$$\lim_{t\to\infty}\left[1 - \mathrm{erf}\left(\frac{1+\mathrm{i}}{\sqrt{2\nu_0}}\left(\int_0^t \nu(s)\,\mathrm{d}s\right)^{1/2}\right)\right] = 0,$$

if $\lim_{t\to\infty}\psi(0,t)$ exists we must necessarily have

$$\lim_{t\to\infty}\psi(0,t) = \lim_{t\to\infty}\frac{1}{\sqrt{\pi}}\int_0^t \tau(\sigma)\,\mathrm{e}^{-\mathrm{i}(t-\sigma)}\left(\int_\sigma^t \nu(s)\,\mathrm{d}s\right)^{-1/2}\mathrm{d}\sigma. \tag{2.33}$$

Now we prove two auxiliary results that we are going to need soon.

Lemma 2.5.1. *Suppose that there exist positive constants $a, \nu_\infty > 0$ and $\delta > \frac{1}{2}$ such that*

$$|\nu(t) - \nu_\infty| \leq \frac{a}{1+t^\delta} \quad \textit{for all } t \geq 0. \tag{2.34}$$

Then for all $0 < \sigma < t$ we have

$$\left(\nu_\infty - \frac{1}{t}\int_0^t \frac{a}{1+s^\delta}\,\mathrm{d}s\right)(t-\sigma) < \int_\sigma^t \nu(s)\,\mathrm{d}s < \left(\nu_\infty + \frac{1}{t}\int_0^t \frac{a}{1+s^\delta}\,\mathrm{d}s\right)(t-\sigma).$$

Remark 2.5.2. As will be evident from the proof, in the statement of Lemma 2.5.1 it would suffice to require $\delta > 0$. However, the assumption $\delta > \frac{1}{2}$ will be necessary for the proof of Theorem 2.5.4.

Proof. The assumption (2.34) implies that, for $0 < \sigma < t$, we have

$$\left(\nu_\infty - \frac{1}{t-\sigma}\int_\sigma^t \frac{a}{1+s^\delta}\,\mathrm{d}s\right)(t-\sigma) \leq \int_\sigma^t \nu(s)\,\mathrm{d}s \leq \left(\nu_\infty + \frac{1}{t-\sigma}\int_\sigma^t \frac{a}{1+s^\delta}\,\mathrm{d}s\right)(t-\sigma).$$

Moreover, the function $\sigma \mapsto \frac{1}{t-\sigma}\int_\sigma^t \frac{a}{1+s^\delta}\,\mathrm{d}s$ is strictly monotonically decreasing in $[0,t]$. In fact, by the mean value theorem there exists $\omega \in [\sigma, t]$ such that

$$\frac{1}{t-\sigma}\int_\sigma^t \frac{a}{1+s^\delta}\,\mathrm{d}s = \frac{a}{1+\omega^\delta};$$

since $s \mapsto a/(1+s^\delta)$ is strictly monotonically decreasing on $(0,\infty)$, it follows $\omega \in (\sigma, t)$ and

$$\frac{a}{1+t^\delta} < \frac{a}{1+\omega^\delta} = \frac{1}{t-\sigma}\int_\sigma^t \frac{a}{1+s^\delta}\,\mathrm{d}s < \frac{a}{1+\sigma^\delta};$$

therefore,

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{1}{t-\sigma}\int_\sigma^t \frac{a}{1+s^\delta}\,\mathrm{d}s\right) = \frac{1}{t-\sigma}\left(\frac{1}{t-\sigma}\int_\sigma^t \frac{a}{1+s^\delta}\,\mathrm{d}s - \frac{a}{1+\sigma^\delta}\right) < 0.$$

In particular, we get the estimate

$$\frac{1}{t-\sigma}\int_\sigma^t \frac{a}{1+s^\delta}\,\mathrm{d}s < \frac{1}{t}\int_0^t \frac{a}{1+s^\delta}\,\mathrm{d}s \quad \text{for all } \sigma \in (0,t),$$

and thus the claim follows. □

Lemma 2.5.3. *Suppose that there exist constants $b > 0$, $\gamma > 1$, and $\tau_\infty \in \mathbb{C}$, such that*

$$|\tau(t) - \tau_\infty| \leq \frac{b}{1+t^\gamma} \quad \textit{for all } t \geq 0. \tag{2.35}$$

Then

$$\lim_{t\to\infty} \frac{1}{\sqrt{\pi}}\int_0^t \tau(\sigma)\frac{\mathrm{e}^{-\mathrm{i}(t-\sigma)}}{\sqrt{t-\sigma}}\,\mathrm{d}\sigma = \tau_\infty\frac{1-\mathrm{i}}{\sqrt{2}}.$$

Proof. Since (with the substitution $\xi = t - \sigma$)

$$\lim_{t\to\infty} \frac{1}{\sqrt{\pi}}\int_0^t \frac{\mathrm{e}^{-\mathrm{i}(t-\sigma)}}{\sqrt{t-\sigma}}\,\mathrm{d}\sigma = \lim_{t\to\infty} \frac{1}{\sqrt{\pi}}\int_0^t \frac{\mathrm{e}^{-\mathrm{i}\xi}}{\sqrt{\xi}}\,\mathrm{d}\xi = \frac{1-\mathrm{i}}{\sqrt{2}}$$

(see [56, p. 436, No. 3.757]), it is enough to show

$$\lim_{t\to\infty}\left|\int_0^t \tau(\sigma)\frac{\mathrm{e}^{-\mathrm{i}(t-\sigma)}}{\sqrt{t-\sigma}}\,\mathrm{d}\sigma-\tau_\infty\int_0^t\frac{\mathrm{e}^{-\mathrm{i}(t-\sigma)}}{\sqrt{t-\sigma}}\,\mathrm{d}\sigma\right|=0.$$

Using the triangle inequality and (2.35), we see that

$$\begin{aligned}\left|\int_0^t \tau(\sigma)\frac{\mathrm{e}^{-\mathrm{i}(t-\sigma)}}{\sqrt{t-\sigma}}\,\mathrm{d}\sigma-\tau_\infty\int_0^t\frac{\mathrm{e}^{-\mathrm{i}(t-\sigma)}}{\sqrt{t-\sigma}}\,\mathrm{d}\sigma\right| &\le \int_0^t\frac{|\tau(\sigma)-\tau_\infty|}{\sqrt{t-\sigma}}\,\mathrm{d}\sigma\\ \le \int_0^t \frac{b}{1+\sigma^\gamma}\frac{1}{\sqrt{t-\sigma}}\,\mathrm{d}\sigma &= \int_0^{t/2}\frac{b}{1+\sigma^\gamma}\frac{1}{\sqrt{t-\sigma}}\,\mathrm{d}\sigma+\int_{t/2}^t\frac{b}{1+\sigma^\gamma}\frac{1}{\sqrt{t-\sigma}}\,\mathrm{d}\sigma.\end{aligned}$$

Let $C:=\int_0^\infty \frac{b}{1+\sigma^\gamma}\,\mathrm{d}\sigma\in(0,\infty)$. The first integral on the right-hand side can be estimated as follows:

$$\int_0^{t/2}\frac{b}{1+\sigma^\gamma}\frac{1}{\sqrt{t-\sigma}}\,\mathrm{d}\sigma\le\sqrt{\frac{2}{t}}\int_0^{t/2}\frac{b}{1+\sigma^\gamma}\,\mathrm{d}\sigma\le C\sqrt{\frac{2}{t}}.$$

On the other hand, for the second integral we find

$$\int_{t/2}^t\frac{b}{1+\sigma^\gamma}\frac{1}{\sqrt{t-\sigma}}\,\mathrm{d}\sigma\le\frac{b}{1+\left(\frac{t}{2}\right)^\gamma}\int_{t/2}^t\frac{1}{\sqrt{t-\sigma}}\,\mathrm{d}\sigma=\frac{b\sqrt{2t}}{1+\left(\frac{t}{2}\right)^\gamma}.$$

Therefore

$$\left|\int_0^t \tau(\sigma)\frac{\mathrm{e}^{-\mathrm{i}(t-\sigma)}}{\sqrt{t-\sigma}}\,\mathrm{d}\sigma-\tau_\infty\int_0^t\frac{\mathrm{e}^{-\mathrm{i}(t-\sigma)}}{\sqrt{t-\sigma}}\,\mathrm{d}\sigma\right|\le C\sqrt{\frac{2}{t}}+\frac{b\sqrt{2t}}{1+\left(\frac{t}{2}\right)^\gamma}\to 0\quad\text{as } t\to\infty,$$

as we wanted to show. ☐

With the aid of these two lemmas, we can prove the following result.

Theorem 2.5.4. *Suppose that ν and τ satisfy the assumptions of Lemma 2.5.1 and Lemma 2.5.3, respectively. Then*

$$\lim_{t\to\infty}\psi(0,t)=\tau_\infty\frac{1-\mathrm{i}}{\sqrt{2\nu_\infty}}.$$

Proof. Since ν is bounded on the interval $[0,\infty)$, without loss of generality we may assume that $\frac{1}{2}<\delta<1$. We show that

$$\lim_{t\to\infty}\left|\int_0^t\tau(\sigma)\,\mathrm{e}^{-\mathrm{i}(t-\sigma)}\left(\int_\sigma^t\nu(s)\,\mathrm{d}s\right)^{-1/2}\mathrm{d}\sigma-\frac{1}{\sqrt{\nu_\infty}}\int_0^t\tau(\sigma)\frac{\mathrm{e}^{-\mathrm{i}(t-\sigma)}}{\sqrt{t-\sigma}}\,\mathrm{d}\sigma\right|=0;$$

then, in view of (2.33), the statement follows from Lemma 2.5.3.

Applying the triangle inequality and using (2.35) yields

$$\begin{aligned}&\left|\int_0^t\tau(\sigma)\,\mathrm{e}^{-\mathrm{i}(t-\sigma)}\left[\left(\int_\sigma^t\nu(s)\,\mathrm{d}s\right)^{-1/2}-\frac{1}{\sqrt{\nu_\infty(t-\sigma)}}\right]\mathrm{d}\sigma\right|\\ &\qquad\le\int_0^t|\tau(\sigma)|\left|\left(\int_\sigma^t\nu(s)\,\mathrm{d}s\right)^{-1/2}-\frac{1}{\sqrt{\nu_\infty(t-\sigma)}}\right|\mathrm{d}\sigma\\ &\qquad\le(|\tau_\infty|+b)\int_0^t\left|\left(\int_\sigma^t\nu(s)\,\mathrm{d}s\right)^{-1/2}-\frac{1}{\sqrt{\nu_\infty(t-\sigma)}}\right|\mathrm{d}\sigma.\end{aligned}$$

Note that

$$\frac{1}{t}\int_0^t \frac{a}{1+s^\delta}\,\mathrm{d}s < \frac{1}{t}\int_0^t \frac{a}{s^\delta}\,\mathrm{d}s = \frac{a}{b(1-\delta)t^\delta} \to 0 \quad \text{as } t\to\infty,$$

hence there exists $t_0 > 0$ such that

$$\nu_\infty - \frac{1}{t}\int_0^t \frac{a}{1+s^\delta}\,\mathrm{d}s > 0 \quad \text{for all } t \geq t_0.$$

For all such $t \geq t_0$, Lemma 2.5.1 implies that, for $0 < \sigma < t$,

$$\begin{aligned}\left(\nu_\infty + \frac{1}{t}\int_0^t \frac{a}{1+s^\delta}\,\mathrm{d}s\right)^{-1/2} &\frac{1}{\sqrt{t-\sigma}} - \frac{1}{\sqrt{\nu_\infty(t-\sigma)}}\\ &< \left(\int_\sigma^t \nu(s)\,\mathrm{d}s\right)^{-1/2} - \frac{1}{\sqrt{\nu_\infty(t-\sigma)}}\\ &< \left(\nu_\infty - \frac{1}{t}\int_0^t \frac{a}{1+s^\delta}\,\mathrm{d}s\right)^{-1/2} \frac{1}{\sqrt{t-\sigma}} - \frac{1}{\sqrt{\nu_\infty(t-\sigma)}};\end{aligned}$$

here, the lower bound is negative and the upper one is positive, hence it follows that

$$\left|\left(\int_\sigma^t \nu(s)\,\mathrm{d}s\right)^{-1/2} - \frac{1}{\sqrt{\nu_\infty(t-\sigma)}}\right| < \alpha(t)\frac{1}{\sqrt{\nu_\infty(t-\sigma)}},$$

where

$$\alpha(t) := \max\left\{1 - \left(1 + \frac{1}{\nu_\infty t}\int_0^t \frac{a}{1+s^\delta}\,\mathrm{d}s\right)^{-1/2}, \left(1 - \frac{1}{\nu_\infty t}\int_0^t \frac{a}{1+s^\delta}\,\mathrm{d}s\right)^{-1/2} - 1\right\}.$$

Thus

$$\int_0^t \left|\left(\int_\sigma^t \nu(s)\,\mathrm{d}s\right)^{-1/2} - \frac{1}{\sqrt{\nu_\infty(t-\sigma)}}\right|\mathrm{d}\sigma < \alpha(t)\int_0^t \frac{1}{\sqrt{\nu_\infty(t-\sigma)}}\,\mathrm{d}\sigma = \frac{2\alpha(t)\sqrt{t}}{\sqrt{\nu_\infty}}.$$

Finally, let us take a closer look at $\alpha(t)$. Since $\delta < 1$, we have

$$\frac{1}{\nu_\infty t}\int_0^t \frac{a}{1+s^\delta}\,\mathrm{d}s < \frac{1}{\nu_\infty t}\int_0^t \frac{a}{s^\delta}\,\mathrm{d}s = Ct^{-\delta},$$

with $C = a/(\nu_\infty(1-\delta)) > 0$. Thus

$$\alpha(t) < \max\left\{1 - (1+Ct^{-\delta})^{-1/2}, (1-Ct^{-\delta})^{-1/2} - 1\right\}$$

for t sufficiently large (so that $1 - Ct^{-\delta} > 0$.) Furthermore, as $t \to \infty$ it holds that (see [56])

$$1 - \left(1+Ct^{-\delta}\right)^{-1/2} = 2Ct^{-\delta} + O\left(t^{-2\delta}\right) \quad \text{and} \quad \left(1-Ct^{-\delta}\right)^{-1/2} - 1 = 2Ct^{-\delta} + O\left(t^{-2\delta}\right),$$

so

$$\begin{aligned}\alpha(t)\sqrt{t} &< \max\left\{1 - \left(1+Ct^{-\delta}\right)^{-1/2}, \left(1-Ct^{-\delta}\right)^{-1/2} - 1\right\}\sqrt{t}\\ &= 2Ct^{\frac{1}{2}-\delta} + O\left(t^{\frac{1}{2}-2\delta}\right) \quad \text{as } t\to\infty,\end{aligned}$$

which implies

$$\lim_{t\to\infty} \alpha(t)\sqrt{t} = 0.$$

This concludes the proof. □

Since, for general τ and ν, formula (2.32) looks quite intricate, we illustrate it by means of some simple numerical plots for particular choices of ν and τ: this allows us to get at least some insight into the qualitative behaviour of the surface current as a function of time. In particular, the time evolution of the surface deflection angle (that is, the angle between the wind and the velocity of the surface current) may be of interest. In the steady state, as we saw, the surface current is deflected to the right of the wind direction by an angle of amplitude $\pi/4$; the amplitude of the surface deflection angle $\theta_0(t)$ of $\psi(0,t)$ is determined via the relation

$$\tan\theta_0(t) = -\frac{\operatorname{Im}\psi(0,t)}{\operatorname{Re}\psi(0,t)},$$

where $\operatorname{Re} w$ and $\operatorname{Im} w$ denote the real and the imaginary part of the complex number $w \in \mathbb{C}$, respectively (the minus sign in the formula above accounts for the fact that the current is deflected to the right of the wind direction).

In principle, $\nu(t)$ and $\tau(t)$ could be arbitrary functions of t (as long as ν is continuous and satisfies (2.26)); however, it turns out (see the discussion in [95]) that in practice it is reasonable to assume that the (dimensional) eddy viscosity $K(\tilde{t})$ and the (dimensional) surface wind stress $\boldsymbol{\tau}_0(\tilde{t})$ are related via

$$K(\tilde{t}) = c\frac{|\boldsymbol{\tau}_0(\tilde{t})|}{\rho f},$$

where $c = 0.01$; according to the scaling performed earlier, this translates to the non-dimensional identity

$$\nu(t) = c|\tau(t)|.$$

For simplicity, let us restrict ourselves to the situation $\tau(t) \in \mathbb{R}$, which means that the wind is changing in intensity but not in direction.

Example 2.5.5. As a first example, consider

$$\tau(t) = 2 - \mathrm{e}^{-t^2},$$

so that the wind intensity (and the eddy viscosity) increase from their initial values, reaching a plateau for large times. The left panel of Figure 2.2 shows the time evolution of $\tan(\theta_0(t))$, while the right panel is a plot of the parametric curve

$$t \mapsto \left(\frac{\operatorname{Re}\psi(0,t)}{|\operatorname{Re}\psi_0(0)|}, \frac{\operatorname{Im}\psi(0,t)}{|\operatorname{Im}\psi_0(0)|}\right) \in \mathbb{R}^2,$$

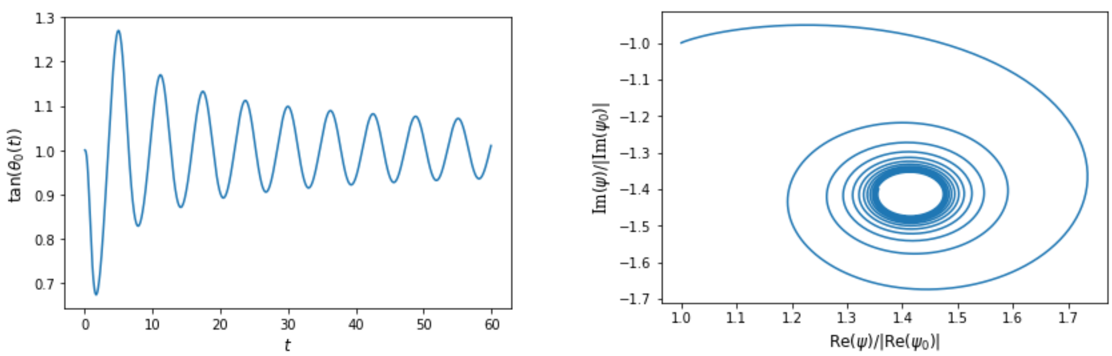

FIGURE 2.2: Plots of $\tan\theta_0(t)$ (left) and the (normalised) velocity components (right) for $\nu(t) = c|\tau(t)|$ and $\tau(t) = 2 - \mathrm{e}^{-t^2}$.

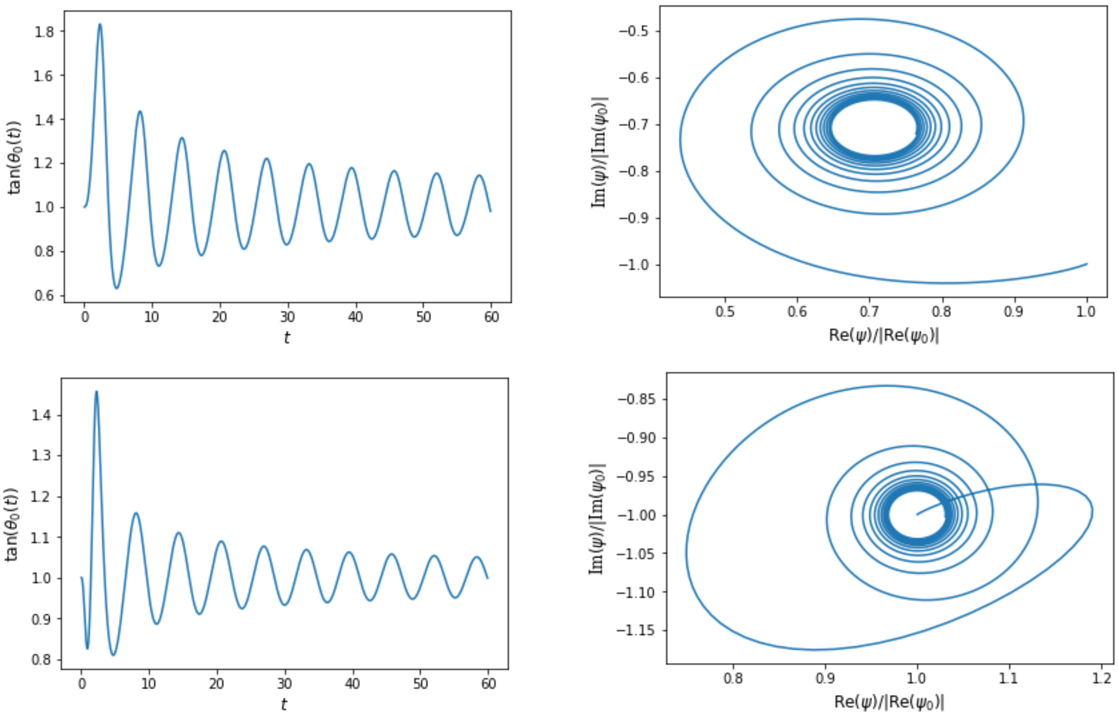

Figure 2.3: Plots of $\tan\theta_0(t)$ (top left) and the (normalised) velocity components (top right) for $\tau(t) = 1 + \mathrm{e}^{-t^2}$, as well as $\tan\theta_0(t)$ (bottom left) and the (normalised) velocity components (bottom right) for $\tau(t) = 1 + 2t^2\left(1 - \frac{3}{4}t^2\right)\mathrm{e}^{-t^2}$.

where each component of the velocity field is normalised with respect to the corresponding component of the initial velocity (so that the initial state is situated at $(1, -1)$). We see that the surface current deviates significantly from the initial steady state, both in magnitude and direction (even though the wind direction does not change): the deflection angle takes values between approximately 34° and 52°. Initially, the solution leaves the initial steady state and, as the wind and eddy viscosity approach their new constant values, the flow slowly converges towards the corresponding steady state through dampened *inertial oscillations* around the final steady state, which is $(\sqrt{2}, -\sqrt{2})$ in this particular case.

Example 2.5.6. As a second example, let us look at a wind whose intensity decreases with time:

$$\tau(t) = 1 + \mathrm{e}^{-t^2}.$$

We see from Figure 2.3 (top panels) that this time the inertial oscillations occur in the opposite direction to that in the previous case, around the asymptotic steady state $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. Here, the amplitude of the deflection angle is between 33° and 62°.

Example 2.5.7. Finally, consider an example where the wind and the eddy viscosity are not monotonic in time, but reach a maximum and a minimum before returning to the initial state:

$$\tau(t) = 1 + 2t^2\left(1 - \frac{3}{4}t^2\right)\mathrm{e}^{-t^2};$$

the coefficients are chosen so that the maximum and the minimum values are approximately the same distance from the initial value $\tau(0) = 1$. The corresponding plots are shown in Figure 2.3 (bottom panels). The maximum deflection angle in this case is about 55°, whereas the minimum is approximately 39°.

In summary, on the analytic side, if the wind and eddy viscosity converge to constant values as time goes to infinity, we can provide sufficient conditions on the convergence rates to ensure that the solution converges to the steady state corresponding to the limit values of the wind and eddy viscosity. Moreover, by plotting the solution for specific choices of wind and eddy viscosity, we saw that even if the wind changes only in intensity, but not in direction, this still leads to significant deviations of the direction of the surface current from its steady-state direction: if the wind and the eddy viscosity approach constant values as time grows, then the solution approaches the corresponding steady state in slowly shrinking inertial oscillations.

Of course, assuming that the eddy viscosity does not vary with depth is a strong restriction; in order to obtain a more accurate description of real-life ocean currents, effects of stratification—which can be modelled via a depth-varying eddy viscosity, as in Section 2.4—should be take into account. In [106] and [107], the authors investigate particular situations of an eddy viscosity that depends both on time and depth, applying a method based on Laplace transforms analogous to the one we employed earlier in this paper; however, performing the final Laplace inversion explicitly (as we did to derive the formula (2.32) for the surface current) is generally not possible, so very little can be said from this solution formula by analytical means; instead, the analysis has to rely entirely upon numerical methods. Although this approach does offer some insight into the dynamics, in order to understand and appreciate the whole complexity of the implications of these models a better theoretical and analytical understanding would be of key importance and should be the subject of further investigation in the future.

3. Oceanic flows in the thin-shell approximation

In this chapter, we turn our attention to large-scale ocean currents. As we pointed out earlier, density variations in the ocean are typically very small, a fact that we will take advantage of to further simplify the equations. In Section 3.2, based on the article [104] and devoted to the investigation of a model describing the Antarctic Circumpolar Current, we will assume constant density; however, we will see in Section 3.3, based on the paper [47], how variable density can be dealt with by means of incompressibility, a reasonable assumption in the ocean. The models of both sections are derived through the so-called *thin-shell approximation*, which exploits the small aspect ratio of the ocean to obtain asymptotic models which retain the effects of sphericity at leading order.

3.1 The thin-shell approximation

In this section, which builds upon the derivation performed in [47], we will show how the strategy from, for example, [32, 42, 82] can be adapted to the setting of variable density (assuming incompressibility) to derive an asymptotic model in spherical coordinates through the smallness of the ocean's aspect ratio. The idea is to write down the Euler equations in rotating spherical coordinates, namely the equations (1.11) with $\mathbf{F}_{\mathrm{v}}(\mathbf{u}) = 0$ and geopotential given by (1.13), and adding primes to all variables and constants to underline that they are dimensional quantities:

$$
\begin{aligned}
&\frac{\mathrm{D}u'}{\mathrm{D}t'} + \frac{u'w' - u'v'\tan\theta}{r'} - 2\Omega'(v'\sin\theta - w'\cos\theta) = -\frac{1}{r'\rho'\cos\theta}\frac{\partial p'}{\partial\varphi},\\
&\frac{\mathrm{D}v'}{\mathrm{D}t'} + \frac{v'w' + u'^2\tan\theta}{r'} + 2\Omega' u'\sin\theta + \Omega'^2 r'\sin\theta\cos\theta = -\frac{1}{r'\rho'}\frac{\partial p'}{\partial\theta},\\
&\frac{\mathrm{D}w'}{\mathrm{D}t'} - \frac{u'^2 + v'^2}{r'} - 2\Omega' u'\cos\theta - \Omega'^2 r'\cos^2\theta = -\frac{1}{\rho'}\frac{\partial p'}{\partial r'} - g',
\end{aligned}
$$

together with the equation of mass conservation

$$
\frac{\mathrm{D}\rho'}{\mathrm{D}t'} + \rho'\left(\frac{1}{r'\cos\theta}\frac{\partial u'}{\partial\varphi} + \frac{1}{r'\cos\theta}\frac{\partial}{\partial\theta}(v'\cos\theta) + \frac{1}{r'^2}\frac{\partial}{\partial r'}(r'^2 w')\right) = 0, \tag{3.1}
$$

where

$$
\frac{\mathrm{D}}{\mathrm{D}t'} = \frac{\partial}{\partial t'} + \frac{u'}{r'\cos\theta}\frac{\partial}{\partial\varphi} + \frac{v'}{r'}\frac{\partial}{\partial\theta} + w'\frac{\partial}{\partial r'},
$$

as in (1.9). Additionally, we assume the flow to be incompressible (a reasonable assumption in the ocean), that is,

$$\frac{1}{r'\cos\theta}\frac{\partial u'}{\partial\varphi}+\frac{1}{r'\cos\theta}\frac{\partial}{\partial\theta}(v'\cos\theta)+\frac{1}{r'^2}\frac{\partial}{\partial r'}(r'^2w')=0; \tag{3.2}$$

the mass conservation (3.1) thus simplifies to

$$\frac{\mathrm{D}\rho'}{\mathrm{D}t'}=0. \tag{3.3}$$

We then non-dimensionalise: we choose a vertical length scale H' and a suitable speed scale U', and let $R'=6371\,\mathrm{km}$ be Earth's radius, then set

$$\varepsilon=\frac{H'}{R'}\quad\text{and}\quad\omega=\frac{\Omega'R'}{U'};$$

here, ε is the aspect ratio of the ocean, a very small number. Define the non-dimensional variables t,z,u,v,w,ρ and P by

$$t'=\frac{R'}{U'}t,\quad r'=R'+H'z,\quad \big(u',v',w'\big)=U'(u,v,kw),\quad \rho'=\overline{\rho}'\rho,\quad p'=\overline{\rho}'U'^2P, \tag{3.4}$$

where U' is a horizontal velocity scale, $H'=4\,\mathrm{km}$ is the mean depth of the ocean, $\overline{\rho}'\approx 1.03\,\mathrm{kg\,m^{-3}}$ is the average ocean water density, and $g=g'H'/U'^2$. The parameter k is a scaling factor associated with the vertical velocity and is typically very small, of the order of 10^{-4} (see [3, 67, 117]). In ocean currents, it is typically observed that horizontal velocities are of order $10^{-1}\,\mathrm{m\,s^{-1}}$, so that we may set $U'=10^{-1}\,\mathrm{m\,s^{-1}}$; in this way, $\omega=\Omega'R'/U'\approx 4650$. With this choice for the scaling, the Euler equations become

$$\frac{\hat{\mathrm{D}}u}{\hat{\mathrm{D}}t}+\frac{kuw-uv\tan\theta}{1+\varepsilon z}-2\omega(v\sin\theta-kw\cos\theta)=-\frac{1}{(1+\varepsilon z)\rho\cos\theta}\frac{\partial P}{\partial\varphi}, \tag{3.5a}$$

$$\frac{\hat{\mathrm{D}}v}{\hat{\mathrm{D}}t}+\frac{kvw+u^2\tan\theta}{1+\varepsilon z}+2\omega u\sin\theta+\omega^2(1+\varepsilon z)\sin\theta\cos\theta=-\frac{1}{(1+\varepsilon z)\rho}\frac{\partial P}{\partial\theta}, \tag{3.5b}$$

$$k\varepsilon\frac{\hat{\mathrm{D}}w}{\hat{\mathrm{D}}t}-\varepsilon\frac{u^2+v^2}{1+\varepsilon z}-2\varepsilon\omega u\cos\theta-\varepsilon\omega^2(1+\varepsilon z)^2\cos^2\theta=-\frac{1}{\rho}\frac{\partial P}{\partial z}-g, \tag{3.5c}$$

whereas from (3.3) we obtain the non-dimensional equation of mass conservation

$$\frac{\hat{\mathrm{D}}\rho}{\hat{\mathrm{D}}t}=0, \tag{3.6}$$

where

$$\frac{\hat{\mathrm{D}}}{\hat{\mathrm{D}}t}=\frac{\partial}{\partial t}+\frac{u}{(1+\varepsilon z)\cos\theta}\frac{\partial}{\partial\varphi}+\frac{v}{1+\varepsilon z}\frac{\partial}{\partial\theta}+\frac{k}{\varepsilon}w\frac{\partial}{\partial z};$$

the incompressibility condition (3.2) becomes

$$\frac{\partial u}{\partial\varphi}+\frac{\partial}{\partial\theta}(v\cos\theta)+\frac{k}{\varepsilon}\frac{1}{1+\varepsilon z}\frac{\partial}{\partial z}\big((1+\varepsilon z)^2w\big)=0. \tag{3.7}$$

Since, typically, $k = O(\varepsilon^2)$ (see the discussion in [32]), the leading-order problem from the non-dimensional equations (3.5)–(3.7) in the limit $\varepsilon \to 0$ is given by

$$\frac{\partial u}{\partial t} + \frac{u}{\cos\theta}\frac{\partial u}{\partial \varphi} + v\frac{\partial u}{\partial \theta} - uv\tan\theta - 2\omega v\sin\theta = -\frac{1}{\rho\cos\theta}\frac{\partial P}{\partial \varphi}, \tag{3.8a}$$

$$\frac{\partial v}{\partial t} + \frac{u}{\cos\theta}\frac{\partial v}{\partial \varphi} + v\frac{\partial v}{\partial \theta} + u^2\tan\theta + 2\omega u\sin\theta + \omega^2\sin\theta\cos\theta = -\frac{1}{\rho}\frac{\partial P}{\partial \theta}, \tag{3.8b}$$

$$\frac{\partial P}{\partial z} = -\rho g, \tag{3.8c}$$

$$\frac{\partial \rho}{\partial t} + \frac{u}{\cos\theta}\frac{\partial \rho}{\partial \varphi} + v\frac{\partial \rho}{\partial \theta} = 0, \tag{3.8d}$$

$$\frac{\partial u}{\partial \varphi} + \frac{\partial}{\partial \theta}(v\cos\theta) = 0. \tag{3.8e}$$

The equations must be complemented by appropriate boundary conditions, which will depend on the problem under consideration.

3.2 The Antarctic Circumpolar Current

The Antarctic Circumpolar Current (ACC) is one of Earth's most important currents. It transports approximately 140 million cubic metres of water per second over a distance of roughly 24,000 km. Furthermore, it is the only current that flows all around the globe, unhindered by land masses, and connects the Atlantic, Pacific, and Indian basins, thereby playing a crucial role in the global climate dynamics; in particular, it isolates Antarctica from warmer waters to the north, allowing the formation of an essentially circular ice cover over the ocean surface during winter. It is situated roughly between 40°S and 60°S and has a typical width of about 2000 km, its narrowest point (approximately 700 km) being Drake Passage at the southern tip of South America; see, for example, [54, 116], as well as Figures 3.1 and 3.2.

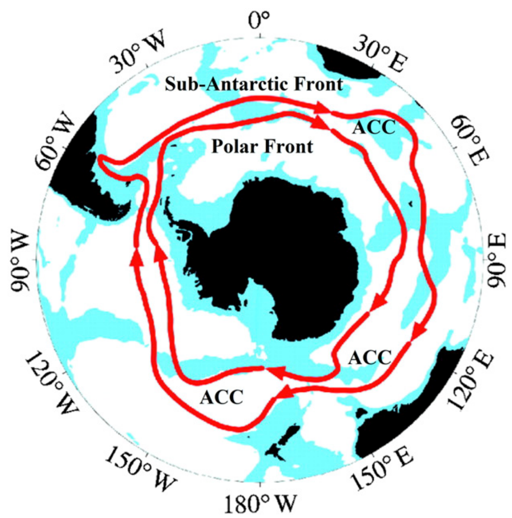

Figure 3.1: Sketch of the ACC. It is bounded by the inner Polar Front and the outer Subantarctic Front (red curves). Oceanic regions with depths shallower than 3.5 km are shaded blue. Image from [31]. CC BY 4.0 (`https://creativecommons.org/licenses/by/4.0`).

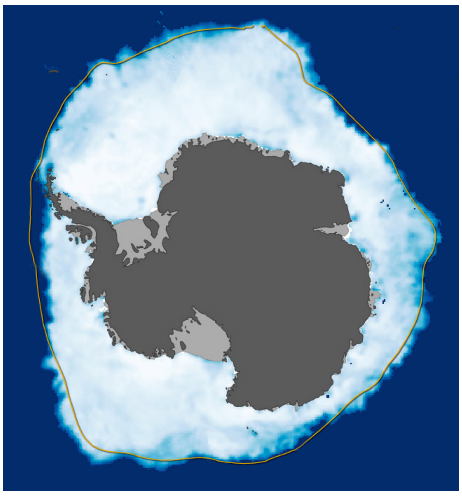

Figure 3.2: Map showing the sea ice extent around Antarctica on 26 September 2012; almost all of the sea ice formed during the Antarctic winter melts during the summer. The ACC shields Antarctica from warmer waters to the north, therefore ice forms in a nearly symmetric circular pattern. Image credit: NASA Earth Observatory image by Jesse Allen, using DMPS SSMIS ice concentration data provided courtesy of the National Snow and Ice Data Center.

The ACC has very peculiar features. First, its meridional (North-South) velocity component is observed to be significantly smaller than the azimuthal (East-West) component, while the vertical velocity is altogether negligible. Moreover, unlike most other currents, it is not a single flow but instead comprises several localised, vertically coherent high-speed jets (typically 40–50 km wide, with speeds in excess of 1 m s^{-1}), reaching the seafloor, separated by regions of low-speed flow, with average speeds below 20 cm s^{-1} [31, 74]. To summarise, the ACC is in general a deep-reaching, jet-structured flow that is mostly propagating zonally to the East around Antarctica, except where it is occasionally deviated by the underlying bottom topography, such as the Scotia Ridge (between 55°W and 40°W, downstream of South America), the Southwest Indian Ridge (between 20°E and 30°E) and the Pacific-Antarctic Ridge (between 145°W and 120°W)—see [2]. It is also worth noting that density stratification is much weaker in the Southern Ocean than in equatorial regions; this contributes to the vertical coherence of the ACC flow [115].

Although the thin-shell method described in Section 3.1 was originally devised as a means to describe ocean gyres [32, 42], it has since been applied to other oceanic flows [39, 64, 66], including the Antarctic Circumpolar Current (as well as atmospheric flows, via a slightly different scaling—see [27, 36, 38] and Section 4.1 below). Nevertheless, as a consequence of the thin-shell approximation, the motion is essentially two-dimensional, as the vertical motion vanishes at leading order. This means that this model is only appropriate to describe the near-surface flow; still, in spite of this, it is a useful tool to gain new insights, and as such has been studied extensively in the past few years. In particular, several authors showed how the stereographic projection may be a useful tool by which to rewrite the problem in a way amenable to analysis—see, for instance, [66, 82], where this approach is formalised, as well as [20, 21], where the reformulation is utilised to construct zonal solutions with certain properties.

Until recently, all results concerning this model of the ACC only dealt with steady (that is, time-independent) flows, with nearly exclusive focus on purely zonal solutions (with the only exception of the study [64], which discusses the existence, uniqueness, and regularity of solutions to an elliptic partial differential equation which arises by applying the Mercator projection to equation (3.15) below for particular choices of the vorticity). In this section, whose results are presented in the preprint [104], we will show how the analysis can be extended to the time-dependent case, establishing well-posedness in a classical sense as well as long-time stability of certain steady zonal flows. The stereographic projection will be useful when tackling the issue of existence and uniqueness of classical solutions, but the stability properties are more conveniently dealt with in the original spherical coordinates. For recent developments concerning models that rely on approximations other than thin-shell, the reader is referred to [2, 31], where the authors construct explicit steady azimuthal depth-dependent solutions.

It is also worth pointing out that the issues of stability of stationary solutions and existence of nonzonal stationary solutions for the Euler equations on a rotating sphere have been addressed in the papers [27, 112]; however, these studies deal with flows on the whole sphere (with applications to stratospheric flows on planets, in the case of [27] and [28]), whereas our work focusses on solutions defined on a proper subset of the sphere, which requires a different approach due to the presence of boundary conditions.

3.2.1 Preliminaries

Throughout this section, we assume constant density ρ, which, according to the scaling (3.4), takes the value $\rho = 1$. Then, discarding the equation (3.8c) of hydrostatic balance, the system (3.8) reduces to

$$\frac{\partial u}{\partial t} + \frac{u}{\cos\theta}\frac{\partial u}{\partial \varphi} + v\frac{\partial u}{\partial \theta} - uv\tan\theta - 2\omega v\sin\theta = -\frac{1}{\cos\theta}\frac{\partial p}{\partial \varphi}, \tag{3.9a}$$

$$\frac{\partial v}{\partial t} + \frac{u}{\cos\theta}\frac{\partial v}{\partial \varphi} + v\frac{\partial v}{\partial \theta} + u^2\tan\theta + 2\omega u\sin\theta = -\frac{\partial p}{\partial \theta}, \tag{3.9b}$$

$$\frac{\partial u}{\partial \varphi} + \frac{\partial}{\partial \theta}(v\cos\theta) = 0, \tag{3.9c}$$

where

$$p = P + \frac{1}{2}\omega^2\sin^2\theta.$$

To model the ACC, we are interested in an annular region $\mathcal{C}$ on the sphere situated in the Southern hemisphere and delimited by fixed latitudes $-\frac{\pi}{2} < \theta_1 < \theta_2 < 0$:

$$\mathcal{C} = \big\{(\varphi,\theta) \in [0,2\pi) \times (\theta_1,\theta_2)\big\}. \tag{3.10}$$

As boundary conditions, we take

$$v(\varphi,\theta_1,t) = v(\varphi,\theta_2,t) = 0 \quad \text{for all } \varphi \in [0,2\pi) \text{ and all } t \geq 0. \tag{3.11}$$

The existence of classical solutions to this problem is established in Section 3.2.3 below.

We will denote by Ω the vorticity associated to (u,v), namely,

$$\Omega = \frac{1}{\cos\theta}\left[\frac{\partial v}{\partial \varphi} - \frac{\partial}{\partial \theta}(u\cos\theta)\right].$$

Since the domain $\mathcal{C}$ is doubly connected and the velocity field $u\,\mathbf{e}_\varphi + v\,\mathbf{e}_\theta$ is tangential to the boundary $\partial\mathcal{C} = \{\theta = \theta_1\} \cup \{\theta = \theta_2\}$ of $\mathcal{C}$, equation (3.9c) implies the existence of a stream function ψ with the property that

$$u(\varphi,\theta,t) = -\frac{\partial\psi}{\partial\theta}(\varphi,\theta,t) \quad \text{and} \quad v(\varphi,\theta,t) = \frac{1}{\cos\theta}\frac{\partial\psi}{\partial\varphi}(\varphi,\theta,t). \tag{3.12}$$

At each fixed t, the stream function ψ is constant on each of the two circles that $\partial\mathcal{C}$ is comprised of (see [62]) and is defined up to an additive constant (depending on t), hence we may determine ψ uniquely by prescribing, for instance, a fixed value $\psi_2 \in \mathbb{R}$ to be attained by ψ on $\{\theta = \theta_2\}$, so that

$$\psi(\varphi,\theta_1,t) = \psi_1(t) \quad \text{and} \quad \psi(\varphi,\theta_2,t) = \psi_2 \quad \text{for all } \varphi \in [0,2\pi) \text{ and all } t \geq 0. \tag{3.13}$$

In terms of ψ, the equations of motion (3.9) can be reformulated (by eliminating the pressure p; cf. [27]) as

$$\frac{\partial}{\partial t}(\Delta_{\mathbb{S}^2}\psi) + \frac{1}{\cos\theta}\left[\frac{\partial\psi}{\partial\varphi}\frac{\partial}{\partial\theta} - \frac{\partial\psi}{\partial\theta}\frac{\partial}{\partial\varphi}\right]\big(\Delta_{\mathbb{S}^2}\psi + 2\omega\sin\theta\big) = 0 \quad \text{in } \mathcal{C}, \tag{3.14}$$

with boundary conditions (3.13), where

$$\Delta_{\mathbb{S}^2} = \frac{1}{\cos^2\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{\cos\theta}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial}{\partial\theta}\right)$$

is the Laplace–Beltrami operator on the surface of the unit sphere. Note that, by the rank theorem (see [1]), stationary (that is, time-independent) solutions to (3.14) are given by the solutions to the equation

$$\Delta_{\mathbb{S}^2}\psi + 2\omega\sin\theta = F(\psi) \quad \text{in } \mathcal{C}, \tag{3.15}$$

as long as the gradient of ψ does not vanish anywhere in the annulus, where F is an arbitrary C^1 function on $\mathbb{R}$. Physically, the meaning of this equation is that the total vorticity Ω (which is easily checked to be equal to $\Delta_{\mathbb{S}^2}\psi$) is given by the sum of two components: one due to the rotation of the Earth (namely, $-2\omega\sin\theta$) and another due to the motion of the ocean ($F(\psi)$, the so-called *oceanic vorticity*). In later sections, we will consider affine oceanic vorticities of the form

$$F(\psi) = -\lambda\psi + \Upsilon \tag{3.16}$$

for real parameters $\lambda, \Upsilon \in \mathbb{R}$. This choice is not only for mathematical convenience, but also physically motivated. In fact, the main sources of oceanic vorticity are wind stress and tidal currents, which are modelled by constant nonzero vorticity, whose sign determines whether the underlying shear currents correspond to ebb or flow tide; see, e.g., the discussion in [113]. Since it can be of interest to investigate perturbations of constant vorticities as well, the first, most natural generalisation is to consider linear vorticities, which would be the next order in the Taylor expansion of a general vorticity function.

3.2.2 Notation and main results

To formulate the existence result for our problem, we will need to introduce Hölder spaces on the set $\mathcal{C}$ defined in (3.10). Let us write

$$\overline{\mathcal{C}} = \{(\varphi,\theta) \in [0,2\pi) \times [\theta_1,\theta_2]\}.$$

Given $f : \mathcal{C} \to \mathbb{R}$ and $\gamma \in (0,1)$, we say that $f \in C^{0+\gamma}(\overline{\mathcal{C}})$ if f is bounded and continuous on $\mathcal{C}$ and there exists $C > 0$ such that

$$|f(\varphi,\theta) - f(\phi,\vartheta)| \le C\, d((\varphi,\theta),(\phi,\vartheta))^{\gamma} \quad \text{for all } (\varphi,\theta),(\phi,\vartheta) \in \overline{\mathcal{C}},$$

where $d(\cdot,\cdot)$ denotes the geodesic length on the sphere (that is, $d((\varphi,\theta),(\phi,\vartheta))$ is the length of the shortest arc on the sphere connecting the points with the spherical coordinates (φ,θ) and (ϕ,ϑ)). Given $k \in \mathbb{N}$, we say that $f \in C^{k+\gamma}(\overline{\mathcal{C}})$ if $f \in C^k(\mathcal{C})$ with bounded derivatives and all partial derivatives of f of order k with respect of (φ,θ) are in $C^{0+\gamma}(\overline{\mathcal{C}})$. A vector-valued function $\mathbf{f} : \mathcal{C} \to \bigcup_{X\in\mathcal{C}} T_X\mathbb{S}^2$, $\mathbf{f}(\varphi,\theta) = f(\varphi,\theta)\,\mathbf{e}_\varphi + g(\varphi,\theta)\,\mathbf{e}_\theta$ is said to be in $C^{k+\gamma}(\overline{\mathcal{C}})$ if both f and g are.

Let us now summarise the main results of this section. First, we will deal with the issue of the existence and uniqueness of classical solutions to our problem.

Theorem 3.2.1 (Global classical solutions). *Suppose that $\mathbf{u}_0 = u_0\,\mathbf{e}_\varphi + v_0\,\mathbf{e}_\theta \in C^{1+\gamma}(\overline{\mathcal{C}})$ for some $\gamma \in (0,1)$, $\frac{\partial u_0}{\partial\varphi} + \frac{\partial}{\partial\theta}(v_0\cos\theta) = 0$ in $\mathcal{C}$, and $v_0 = 0$ on the boundary $\partial\mathcal{C} = \{\theta=\theta_1\} \cup \{\theta=\theta_2\}$ of $\mathcal{C}$. Furthermore, let $T > 0$. Then, there exists a solution $(u\,\mathbf{e}_\varphi + v\,\mathbf{e}_\theta, p)$ to (3.9)–(3.11), with $(u\,\mathbf{e}_\varphi + v\,\mathbf{e}_\theta)|_{t=0} = \mathbf{u}_0$, such that u, v, p, and all their derivatives as appear in (3.9) belong to $C(\overline{\mathcal{C}} \times [0,T])$. This solution is unique up to an arbitrary function of t which may be added to p.*

The strategy, carried out in Section 3.2.3, is to reformulate the problem (3.9)–(3.11) more conveniently in Cartesian coordinates by means of the stereographic projection, which allows us to apply classical arguments. The Cartesian version of Theorem 3.2.1 is formulated in Theorem 3.2.3 below.

The central result of this section concerns the stability of certain zonal steady flows, denoted by $\Psi_{\lambda,\Upsilon}$, which solve (3.13)–(3.15) with F of the form (3.16). These flows are constructed in Section 3.2.4 using Sturm–Liouville theory. The stability result, proved in Section 3.2.4 by exploiting some conservation laws to construct a Lyapunov function, reads as follows.

Theorem 3.2.2 (Stability of zonal flows). *Given $\lambda \le 0$ and $\Upsilon \in \mathbb{R}$, let $\Psi_{\lambda,\Upsilon}$ be the solution of (3.33) below, whose existence and uniqueness is shown in Proposition 3.2.16; let $\mathbf{u}^*_{\lambda,\Upsilon} = u^*_{\lambda,\Upsilon}\mathbf{e}_\varphi$ denote the corresponding steady zonal solution to (3.9) and (3.11) (via (3.12)), with associated vorticity $\Omega^*_{\lambda,\Upsilon}$. Suppose that $\mathbf{u}(t) = u(t)\mathbf{e}_\varphi + v(t)\mathbf{e}_\theta$ is a (time-dependent) solution to (3.9) and (3.11) with initial data $\mathbf{u}|_{t=0} = \mathbf{u}_0 = u_0\mathbf{e}_\varphi + v_0\mathbf{e}_\theta$, and let $\Omega(t)$ denote the associated vorticity at time t, with initial value $\Omega|_{t=0} = \Omega_0$. Then*

$$-\lambda\|\mathbf{u}(t)-\mathbf{u}^*_{\lambda,\Upsilon}\|^2_{L^2(\mathcal{C})} + \|\Omega(t)-\Omega^*_{\lambda,\Upsilon}\|^2_{L^2(\mathcal{C})} = -\lambda\|\mathbf{u}_0-\mathbf{u}^*_{\lambda,\Upsilon}\|^2_{L^2(\mathcal{C})} + \|\Omega_0-\Omega^*_{\lambda,\Upsilon}\|^2_{L^2(\mathcal{C})} \quad (3.17)$$

for all $t \ge 0$.

3.2.3 A Cartesian reformulation and classical solutions

The issue of the existence of classical solutions to the problem (3.9)–(3.11) in the domain $\mathcal{C}$ can be discussed most conveniently by transferring the problem into Cartesian coordinates in the plane; this will enable us to rely on classical arguments for the well-posedness of the Euler equations in bounded two-dimensional domains.

Let us fix some additional notation that will be needed throughout this section. Let (x,y) denote the Cartesian coordinates in $\mathbb{R}^2$. The corresponding standard unit vectors are denoted $\mathbf{e}_x = (1,0)$ and $\mathbf{e}_y = (0,1)$. Given a domain $D \subseteq \mathbb{R}^2$, with a sufficiently smooth boundary, and $T > 0$, we will write $Q_T = D \times [0,T]$ and $\overline{Q}_T = \overline{D} \times [0,T]$. For a scalar function $h = h(\mathbf{x})$ (where $\mathbf{x} = (x,y) \in D$), we will write

$$\nabla h = \frac{\partial h}{\partial x}\mathbf{e}_x + \frac{\partial h}{\partial y}\mathbf{e}_y \quad \text{and} \quad \nabla^{\perp} h = \frac{\partial h}{\partial y}\mathbf{e}_x - \frac{\partial h}{\partial x}\mathbf{e}_y,$$

whereas for a vector function $\mathbf{f} = f_1\,\mathbf{e}_x + f_2\,\mathbf{e}_y$ we will denote

$$\nabla \cdot \mathbf{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} \quad \text{and} \quad \nabla \times \mathbf{f} = \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y},$$

the former being the divergence and the latter being the (scalar) curl. It follows, for the Laplacian in Cartesian coordinates,

$$\Delta h = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \nabla \cdot \nabla h = -\nabla \times \nabla^{\perp} h.$$

We begin by projecting the problem onto a planar region on the equatorial plane by means of a stereographic projection with the centre of projection situated at the North Pole. We introduce the change of coordinates

$$x = \frac{\cos\theta}{1-\sin\theta}\cos\varphi, \quad y = \frac{\cos\theta}{1-\sin\theta}\sin\varphi,$$

(x,y) being the Cartesian coordinates in the equatorial plane. This defines a bijection from the domain $\mathcal{C}$ onto the annulus

$$D = \big\{(x,y) \in \mathbb{R}^2 : r_1^2 < x^2 + y^2 < r_2^2\big\},$$

where $r_i = \cos\theta_i/(1-\sin\theta_i)$, $i \in \{1,2\}$. The inverse of this coordinate transformation is easily computed by observing that

$$\tan\varphi = \frac{y}{x} \quad \text{and} \quad \sin\theta = -\frac{1-x^2-y^2}{1+x^2+y^2};$$

in particular, we have

$$\begin{aligned}
\frac{\partial}{\partial\varphi} &= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y},\\
\frac{\partial}{\partial\theta} &= \frac{1}{\cos\theta}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) = \frac{1+x^2+y^2}{2\sqrt{x^2+y^2}}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right).
\end{aligned}$$

While this coordinate change might seem cumbersome at first, it will, in fact, allow us to rewrite the equations (3.9) in simple Cartesian form. To this end, we introduce the new unknowns $\mathbf{U} = (U,V)$ and Π as

$$\begin{aligned}
U(x,y,t) &= (1-\sin\theta)[v(\varphi,\theta,t)\cos\varphi - u(\varphi,\theta,t)\sin\varphi]\\
&= \frac{2[xv(\varphi(x,y),\theta(x,y),t) - yu(\varphi(x,y),\theta(x,y),t)]}{(1+x^2+y^2)\sqrt{x^2+y^2}}, \qquad (3.18\text{a})\\
V(x,y,t) &= (1-\sin\theta)[u(\varphi,\theta,t)\cos\varphi + v(\varphi,\theta,t)\sin\varphi],\\
&= \frac{2[xu(\varphi(x,y),\theta(x,y),t) + yv(\varphi(x,y),\theta(x,y),t)]}{(1+x^2+y^2)\sqrt{x^2+y^2}}, \qquad (3.18\text{b})\\
\Pi(x,y,t) &= p(\varphi(x,y),\theta(x,y),t). \qquad (3.18\text{c})
\end{aligned}$$

This definition of U and V can be motivated as follows. Considering the standard polar coordinates (φ, r) in the plane (where, for us, $r = \cos\theta/(1-\sin\theta)$) and denoting by $\mathbf{e}_r, \mathbf{e}_\varphi$ the standard polar unit vectors in the plane, we have, at each point (φ, r),

$$\begin{aligned} \mathbf{e}_r &= \cos\varphi\, \mathbf{e}_x + \sin\varphi\, \mathbf{e}_y, \\ \mathbf{e}_\varphi &= -\sin\varphi\, \mathbf{e}_x + \cos\varphi\, \mathbf{e}_y; \end{aligned} \tag{3.19}$$

in particular, for any vector $\mathbf{v} = v^\varphi \mathbf{e}_\varphi + v^r \mathbf{e}_r = v^x \mathbf{e}_x + v^y \mathbf{e}_y$, its polar components (v^φ, v^r) and Cartesian components (v^x, v^y) are related via

$$\begin{aligned} v^x &= v^r \cos\varphi - v^\varphi \sin\varphi, \\ v^y &= v^\varphi \cos\varphi + v^r \sin\varphi. \end{aligned}$$

The additional factor $(1-\sin\theta)$ in (3.18b) and (3.18c) is simply a scaling that turns out to be convenient to write the equations more compactly.

Let us introduce for brevity the functions

$$\alpha(x, y) = \frac{(1 + x^2 + y^2)^2}{4} \quad \text{and} \quad \beta(x, y) = 2\omega \frac{1 - x^2 - y^2}{1 + x^2 + y^2}$$

(note that $\alpha, 1/\alpha, \beta \in C^\infty(\overline{D})$) and the linear map $J : \mathbb{R}^2 \to \mathbb{R}^2$ for which $J\mathbf{e}_x = \mathbf{e}_y$ and $J\mathbf{e}_y = -\mathbf{e}_x$. Plugging (3.19) into (3.9), rearranging terms, and using (3.18), an elementary but rather lengthy calculation eventually yields the 2D Euler-like equation

$$\mathbf{U}_t + (\alpha \mathbf{U} \cdot \nabla)\mathbf{U} + \frac{|\mathbf{U}|^2}{2} \nabla\alpha + \beta J \mathbf{U} = -\nabla \Pi, \tag{3.20a}$$

$$\nabla \cdot \mathbf{U} = 0 \tag{3.20b}$$

in D, with boundary condition

$$\mathbf{U} \cdot \mathbf{n} = 0 \quad \text{on } \partial D; \tag{3.21}$$

here, $\mathbf{n}$ denotes the outward unit normal vector on the boundary ∂D of D. The accordingly transformed initial data will be denoted by

$$\mathbf{U}|_{t=0} = \mathbf{U}_0, \tag{3.22}$$

for some sufficiently regular function $\mathbf{U}_0 : D \to \mathbb{R}^2$. We then have the following result concerning the existence and uniqueness of classical solutions:

Theorem 3.2.3. *Suppose that $\mathbf{U}_0 \in C^{1+\gamma}(\overline{D})$ for some $\gamma \in (0,1)$, $\nabla \cdot \mathbf{U}_0 = 0$, and $\mathbf{U}_0 \cdot \mathbf{n} = 0$. Furthermore, let $T > 0$. Then there exists a solution $(\mathbf{U}, \Pi)$ to (3.20)–(3.22) such that $\mathbf{U}$, Π, and all their derivatives as appear in (3.20) belong to $C(\overline{Q_T})$. This solution is unique up to an arbitrary function of t which may be added to Π.*

Before we discuss the proof of Theorem 3.2.3, we note that Theorem 3.2.1 is now an immediate consequence of it, in view of some simple observations. Indeed, denoting by (x, y) the Cartesian coordinates in the projection plane corresponding to the spherical coordinates (θ, φ), we have

$$\begin{aligned} U_0(x, y) &= \delta(x, y)[x v_0(\theta, \varphi) - y u_0(\theta, \varphi)], \\ V_0(x, y) &= \delta(x, y)[x u_0(\theta, \varphi) + y v_0(\theta, \varphi)], \end{aligned}$$

where $\delta(x, y) = 2\big[(1 + x^2 + y^2)\sqrt{x^2 + y^2}\big]^{-1}$. The claim follows in view of the fact that $x\delta, y\delta \in C^\infty(\overline{D})$ (thus in particular $x\delta, y\delta \in C^{1+\gamma}(\overline{D})$), and from the following lemma, since then $\mathbf{u}_0 \in C^{1+\gamma}(\overline{\mathcal{C}})$ implies $\mathbf{U}_0 \in C^{1+\gamma}(\overline{D})$.

Lemma 3.2.4. *There exists a constant $C > 0$ (depending only on $\mathcal{C}$, or equivalently D) such that*

$$d((\phi_1, \vartheta_1), (\phi_2, \vartheta_2)) \leq C \|(x_1, y_1) - (x_2, y_2)\|$$

for all $(\phi_1, \vartheta_1), (\phi_2, \vartheta_2) \in \mathcal{C}$, with corresponding projections $(x_1, y_1), (x_2, y_2) \in D$ (where, as before, $\|\cdot\|$ denotes the Euclidean norm).

Proof. Let $(X_1, Y_1, Z_1), (X_2, Y_2, Z_2) \in \mathbb{R}^3$ be the Cartesian coordinates of points of $\mathbb{S}^2$ corresponding to $(\phi_1, \vartheta_1), (\phi_2, \vartheta_2) \in \mathcal{C}$. Then

$$(X_k, Y_k, Z_k) = \left(\frac{2x_k}{1 + x_k^2 + y_k^2}, \frac{2y_k}{1 + x_k^2 + y_k^2}, \frac{-1 + x_k^2 + y_k^2}{1 + x_k^2 + y_k^2} \right), \quad k \in \{1, 2\}.$$

Some straightforward calculations yield

$$\|(X_1, Y_1, Z_1) - (X_2, Y_2, Z_2)\|_1 \leq \frac{4(1 + 2r_2^2)}{(1 + r_1^2)^2} \|(x_1, y_1) - (x_2, y_2)\|_1,$$

where $\|\cdot\|_1$ denotes the ℓ^1-norm in $\mathbb{R}^n$, $n \in \{2, 3\}$. The equivalence of norms on $\mathbb{R}^n$ now implies

$$\|(X_1, Y_1, Z_1) - (X_2, Y_2, Z_2)\| \leq \tilde{C} \|(x_1, y_1) - (x_2, y_2)\|$$

for some $\tilde{C} > 0$. Since

$$\|(X_1, Y_1, Z_1) - (X_2, Y_2, Z_2)\| \leq d((\phi_1, \vartheta_1), (\phi_2, \vartheta_2)) \leq \frac{\pi}{2} \|(X_1, Y_1, Z_1) - (X_2, Y_2, Z_2)\|$$

for all $(\phi_1, \vartheta_1), (\phi_2, \vartheta_2) \in \mathcal{C}$, the claim follows with $C = \frac{\pi}{2}\tilde{C}$. □

The proof of Theorem 3.2.3

In proving the global well-posedness result stated in Theorem 3.2.3, we will follow closely the classical paper [72]; since numerous proofs—especially of technical lemmas—carry over almost (if not entirely) word for word to our situation, we shall often omit them and instead simply refer the reader to that work. Nevertheless, some care is needed at several places, where the difference between our equations (3.20) and the 'standard' Euler equations in the plane (which would correspond to having $\alpha = 1$ and $\beta = 0$ in (3.20a)) will play a role; this will be the main focus of the subsequent discussion. The domain D, the initial condition $\mathbf{U}_0$, and $T > 0$ are supposed to be fixed.

Some more notation will be needed for this proof. If $j, k \in \mathbb{N} \cup \{0\}$ and $\varepsilon, \delta \in [0, 1)$, the class $C^{j,k}(\overline{Q}_T)$ is the set of all (scalar- or vector-valued) functions whose derivatives of order p in $\mathbf{x}$ and of order q in t exist and are continuous for all integers $0 \leq p \leq j$ and $0 \leq q \leq k$, whereas $C^{j+\varepsilon,k+\delta}(\overline{Q}_T)$ is the set of all (scalar- or vector-valued) functions whose derivatives of order p in $\mathbf{x}$ and of order q in t exist and are Hölder continuous with exponent ε in $\mathbf{x}$ (if $\varepsilon > 0$) or with exponent δ in t (if $\delta > 0$), or both (if $\varepsilon > 0$ and $\delta > 0$) for all integers $0 \leq p \leq j$ and $0 \leq q \leq k$.

The Euclidean scalar product of two vectors $\mathbf{f}, \mathbf{g}$ is denoted by $\mathbf{f} \cdot \mathbf{g}$; we denote by $\|\mathbf{f}\| = \sqrt{\mathbf{f} \cdot \mathbf{f}}$ the Euclidean norm, as before. Moreover, we will write

$$(\varphi, \psi) = \int_D \varphi(x, y)\psi(x, y)\, \mathrm{d}x\, \mathrm{d}y \quad \text{and} \quad (\mathbf{f}, \mathbf{g}) = \int_D \mathbf{f}(x, y) \cdot \mathbf{g}(x, y)\, \mathrm{d}x\, \mathrm{d}y$$

for scalar functions φ, ψ and vector functions $\mathbf{f}, \mathbf{g}$.

A vector-valued function $\mathbf{f}$ with $\nabla \cdot \mathbf{f} = 0$ will be referred to as a *flow*. If, in addition, $\mathbf{f} \cdot \mathbf{n} = 0$ on ∂D (where $\mathbf{n}$ is the outward normal derivative on ∂D), we will call $\mathbf{f}$ a *tangential flow*.

Following the notation in [72], we will denote by $L_1(z), L_2(z), \ldots$ monotone increasing functions of $z \geq 0$, which may depend only on D, $\mathbf{U}_0$, and T.

Technical preparation

We begin by stating and proving a lemma (which is a slight generalisation of Lemma 1.1 in [72]) that will be useful later on.

Lemma 3.2.5. *Let $\mathbf{U} \in C^1(\overline{D})$ be a flow and set $\varphi = \nabla \times \mathbf{U}$; also, let $f \in C^1(\overline{D})$. Then*

$$(f\varphi, \mathbf{U} \cdot \nabla\Phi) + ((f\mathbf{U} \cdot \nabla)\mathbf{U}, \nabla^{\perp}\Phi) + \left(\frac{|\mathbf{U}|^2}{2}\nabla f, \nabla^{\perp}\Phi\right) = 0$$

for all $\Phi \in C^1(\overline{D})$ with $\Phi|_{\partial D} = 0$.

Proof. If $\mathbf{U} \in C^2(\overline{D})$, the claim is a consequence of the easily verified identity

$$\begin{aligned} f\varphi\mathbf{U} \cdot \nabla\Phi + ((f\mathbf{U} \cdot \nabla)\mathbf{U}) \cdot \nabla^{\perp}\Phi &= \nabla \cdot (f\varphi\Phi\mathbf{U}) - \nabla \times ((\Phi f\mathbf{U} \cdot \nabla)\mathbf{U}) \\ &\quad - \varphi\Phi\mathbf{U} \cdot \nabla f - ((\Phi\mathbf{U} \cdot \nabla)\mathbf{U}) \cdot \nabla^{\perp} f, \end{aligned}$$

together with the observation that

$$\begin{aligned} (\varphi\Phi\mathbf{U}, \nabla f) + ((\Phi\mathbf{U} \cdot \nabla)\mathbf{U}), \nabla^{\perp} f) &= \left(\nabla\left(\Phi\frac{|\mathbf{U}^2|}{2}\right) - \frac{|\mathbf{U}|^2}{2}\nabla\Phi, \nabla^{\perp} f\right) \\ &= \left(\frac{|\mathbf{U}|^2}{2}\nabla f, \nabla^{\perp}\Phi\right). \end{aligned}$$

The result for $\mathbf{U} \in C^1(\overline{D})$ follows from here by approximating $\mathbf{U}$ by means of the usual mollifiers (analogously to [72, Lemma 1.1]). □

We may eliminate the pressure Π from (3.20a)–(3.20b), to formally obtain the vorticity equation

$$\xi_t + \mathbf{U} \cdot \nabla(\alpha\xi + \beta) = \xi_t + \nabla \cdot [(\alpha\xi + \beta)\mathbf{U}] = 0, \tag{3.23}$$

where

$$\xi = \nabla \times \mathbf{U}.$$

It is in fact convenient to multiply (3.23) by α and use the time-independence of α and β, to rewrite (3.23) as

$$\zeta_t + (\alpha\mathbf{U}) \cdot \nabla\zeta = 0, \tag{3.24}$$

where

$$\zeta \coloneqq \alpha\xi + \beta.$$

We will denote by $g : \mathbb{R}^2 \times \mathbb{R}^2 \supseteq D \times D \to \mathbb{R}$ the Green function of the domain D. The integral operator with kernel g is denoted by G:

$$G\varphi(\mathbf{x}) = \int_D g(\mathbf{x}, \tilde{\mathbf{x}})\varphi(\tilde{\mathbf{x}})\,\mathrm{d}\tilde{\mathbf{x}}, \quad \mathbf{x} \in D.$$

Then, as is known (see, for example, [51, 53]), $\psi = G\varphi$ is the solution of the Poisson boundary value problem $\Delta_{\mathbb{S}^2}\psi = -\varphi$, $\psi|_{\partial D} = 0$. Recall some useful estimates and regularity properties for g and G:

Lemma 3.2.6. *(i) Let $\varepsilon, \delta \in (0,1)$. The map $C^{0+\delta}(\overline{D}) \to C^{1+\delta}(\overline{D})$, $\varphi \mapsto \nabla^\perp G\varphi$ is continuous. If $\varphi \in C^{0+\delta,0}(\overline{Q}_T)$, then $\nabla^\perp G\varphi \in C^{1+\delta',0}(\overline{Q}_T)$ for all $\delta' \in (0,\delta)$. If $\varphi \in C^{0+\delta,0+\varepsilon}(\overline{Q}_T)$, then $\nabla^\perp G\varphi \in C^{1+\delta',\varepsilon'}(\overline{Q}_T)$ for all $\delta' \in (0,\delta)$ and $\varepsilon' \in (0,\varepsilon)$.*

(ii) If $\varphi \in L^\infty(D)$, then $G\varphi \in C^1(\overline{D})$ and $\mathbf{U} = \nabla^\perp G\varphi$ is a tangential flow with

$$\|\mathbf{U}\|_\infty \leq C\|\varphi\|_\infty,$$

where $C > 0$ depends only on D.

The first step will be to show that, for all scalar functions $\varphi = \varphi(\mathbf{x}, t)$ in a certain class S', we can construct a unique globally-in-time tangential flow $\mathbf{U} = U\mathbf{e}_x + V\mathbf{e}_y$ such that

$$\nabla \times \mathbf{U} = \varphi, \tag{3.25}$$

$$\partial_t(\mathbf{U}, \mathbf{U}^*) + \left((\alpha\mathbf{U} \cdot \nabla)\mathbf{U} + \frac{|\mathbf{U}|^2}{2}\nabla\alpha + \beta J\mathbf{U}, \mathbf{U}^*\right) = 0, \tag{3.26}$$

$$(\mathbf{U}|_{t=0} - \mathbf{U}_0, \mathbf{U}^*) = 0, \tag{3.27}$$

where $\mathbf{U}^* \in C^1(\overline{D})$ is defined as

$$\mathbf{U}^* = \frac{\nabla^\perp\psi^*}{(\nabla^\perp\psi^*, \nabla^\perp\psi^*)}, \tag{3.28}$$

$\psi^* \in C^2(\overline{D})$ being the unique harmonic function with $\psi^*|_{\Gamma_1} = 1$ and $\psi^*|_{\Gamma_2} = 0$ (with $\Gamma_1 = \{(x,y) : x^2 + y^2 = r_1^2\}$ and $\Gamma_2 = \{(x,y) : x^2 + y^2 = r_2^2\}$).

The next technical auxiliary result, which will be needed in the sequel, is [72, Lemma 1.6].

Lemma 3.2.7. *Let $\mathbf{f} \in C(\overline{D})$ be a vector-valued function such that*

$$(\mathbf{f}, \mathbf{U}^*) = 0 \quad \textit{and} \quad (\mathbf{f}, \nabla^\perp\Phi) = 0$$

for all $\Phi \in C_0^\infty(D)$ (where $\mathbf{U}^*$ is as in (3.28)). Then there exists a function $p \in C^1(\overline{D})$ such that $\mathbf{f} = \nabla p$. If $\mathbf{f} \in C(\overline{Q}_T)$ and the above conditions are satisfied for each $t \in [0,T]$, then p can be chosen from $C^{1,0}(\overline{Q}_T)$.

We now define the class

$$S' = \bigcup_{\varepsilon \in (0,1)} C^{\varepsilon,0}(\overline{Q}_T).$$

Lemma 3.2.8. *For all $\varphi \in S'$ there exists a unique $\mathbf{U} \in C^{1,0}(\overline{Q}_T)$ satisfying (3.25)–(3.27); $\mathbf{U}$ is a tangential flow and can be written as $\mathbf{U} = \overline{\mathbf{U}} + \mathbf{U}'$, with*

$$\overline{\mathbf{U}} = \nabla^\perp G\varphi \quad \textit{and} \quad \mathbf{U}' = \lambda(t)\mathbf{U}^*, \tag{3.29}$$

where $|\lambda(t)| \leq L_1(\|\varphi\|_\infty)$.

Proof. Since $\varphi \in C^{\varepsilon,0}(\overline{D})$ for some $\varepsilon > 0$, by Lemma 3.2.6 we have $G\varphi \in C^{2,0}(\overline{Q}_T)$ and $\overline{\mathbf{U}} = \nabla^\perp G\varphi \in C^{1,0}(\overline{Q}_T)$. Clearly, $\nabla \times \overline{\mathbf{U}} = -\Delta_{\mathbb{S}^2}G\varphi = \varphi$ and $\nabla \times \mathbf{U}^* = 0$, thus (3.25) is satisfied by $\mathbf{U}$ as in (3.29), for any choice of $\lambda(t)$. We now determine $\lambda(t)$ so that (3.26) and (3.27) are satisfied as well. Note that $(\overline{\mathbf{U}}, \mathbf{U}^*) = 0$, as is seen by integrating by parts and by $\nabla \times \mathbf{U}^* = 0$ and $G\varphi|_{\partial D} = 0$. Plugging $\mathbf{U} = \overline{\mathbf{U}} + \mathbf{U}'$ into (3.26) and (3.27) yields for $\lambda(t)$ the ordinary differential equation

$$\lambda'(t) + \mu_1(t)\lambda(t)^2 + \mu_2\lambda(t)^2 + \mu_3(t) = 0,$$

with initial condition

$$\lambda(0) = (\mathbf{U}_0, \mathbf{U}^*);$$

here, μ_2 is a real constant, whereas $\mu_1(t)$ and $\mu_3(t)$ are continuous functions of t. Standard ODE theory (see, e.g., [114]) yields the existence of a *local* solution $\lambda(t)$. Next we show that this solution can be extended to the whole interval $[0,T]$. To this end, let $[0,\tau)$ denote the maximal interval of existence, for an appropriate $\tau > 0$. We multiply (3.26) by $\lambda(t)$, so that

$$(\partial_t \mathbf{U}, \mathbf{U}') + \left((\alpha \mathbf{U}\cdot\nabla)\mathbf{U} + \frac{|\mathbf{U}|^2}{2}\nabla\alpha + \beta J\mathbf{U}, \mathbf{U}'\right) = 0.$$

Plugging in $\mathbf{U} = \overline{\mathbf{U}} + \mathbf{U}'$, and using again $(\overline{\mathbf{U}}, \mathbf{U}^*) = 0$ and the identities

$$\begin{aligned}((\alpha\mathbf{U}\cdot\nabla)\mathbf{U}', \mathbf{U}') &= -\frac{1}{2}(\mathbf{U}\cdot\nabla\alpha, |\mathbf{U}'|^2),\\ ((\alpha\mathbf{U}\cdot\nabla)\overline{\mathbf{U}}, \mathbf{U}') &= -((\alpha\mathbf{U}\cdot\nabla)\mathbf{U}', \overline{\mathbf{U}}) - (\mathbf{U}\cdot\nabla\alpha, \mathbf{U}'\cdot\overline{\mathbf{U}}),\end{aligned}$$

we obtain the equation

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\lambda(t)^2] + \gamma_1(t)\lambda(t) + \gamma_2(t)\lambda(t)^2 = 0,$$

where the coefficients[1]

$$\gamma_1(t) = -((\alpha\overline{\mathbf{U}}\cdot\nabla)\mathbf{U}^*, \overline{\mathbf{U}}) - (\overline{\mathbf{U}}\cdot\nabla\alpha, \mathbf{U}^*\cdot\overline{\mathbf{U}}) + \frac{1}{2}(|\overline{\mathbf{U}}|^2\nabla\alpha, \mathbf{U}^*) + (\beta J\overline{\mathbf{U}}, \mathbf{U}^*)$$

and

$$\begin{aligned}\gamma_2(t) = &-((\alpha\mathbf{U}^*\cdot\nabla)\mathbf{U}^*, \overline{\mathbf{U}}) - (\mathbf{U}^*\cdot\nabla\alpha, \mathbf{U}^*\cdot\overline{\mathbf{U}}) - \frac{1}{2}(\overline{\mathbf{U}}\cdot\nabla\alpha, |\mathbf{U}^*|^2)\\ &+ ((\overline{\mathbf{U}}\cdot\mathbf{U}^*)\nabla\alpha, \mathbf{U}^*) + (\beta J\mathbf{U}^*, \mathbf{U}^*)\end{aligned}$$

depend on $\overline{\mathbf{U}}$, but not on its derivatives, and—in view of Lemma 3.2.6—are therefore readily estimated in terms of the norm $\|\varphi\|_\infty$:

$$|\gamma_1(t)| \le L_2(\|\varphi\|_\infty), \quad |\gamma_2(t)| \le L_3(\|\varphi\|_\infty).$$

Hence, with $C(\|\varphi\|_\infty) = \max\{L_2(\|\varphi\|_\infty), L_3(\|\varphi\|_\infty)\}$,

$$\begin{aligned}\lambda(t)^2 &= \lambda(0)^2 - 2\int_0^t \left[\gamma_1(s)\lambda(s) + \gamma_2(s)\lambda(s)^2\right]\mathrm{d}s\\ &\le \lambda(0)^2 + 2C(\|\varphi\|_\infty)\int_0^t \left[|\lambda(s)| + |\lambda(s)|^2\right]\mathrm{d}s\\ &\le \lambda(0)^2 + C(\|\varphi\|_\infty)t + 3C(\|\varphi\|_\infty)\int_0^t \lambda(s)^2\,\mathrm{d}s,\end{aligned}$$

and Gronwall's inequality now implies

$$\lambda(t)^2 \le (\lambda(0)^2 + C(\|\varphi\|_\infty)\tau)\,\mathrm{e}^{3C(\|\varphi\|_\infty)\tau} \quad \text{for all } t\in[0,\tau).$$

Therefore, $\lambda(t)$ is uniformly bounded on the interval $[0,\tau)$, for any $\tau\in[0,T]$, and so can be extended to the whole of $[0,T]$. Setting

$$L_1(\|\varphi\|_\infty) = [(\mathbf{U}_0,\mathbf{U}^*)^2 + C(\|\varphi\|_\infty)T]^{1/2}\,\mathrm{e}^{3C(\|\varphi\|_\infty)T/2}$$

finishes the proof. □

[1]The terms which would make up the coefficient of $\lambda(t)^3$ cancel exactly.

Next, given $\varphi \in S'$ and $\mathbf{U}$ defined according to Lemma 3.2.8, we turn to solving the initial value problem for the transport equation

$$\begin{aligned} &\zeta_t + (\alpha \mathbf{U}) \cdot \nabla \zeta = 0, \\ &\zeta|_{t=0} = \zeta_0, \end{aligned} \tag{3.30}$$

where $\zeta_0 = \alpha \nabla \times \mathbf{U}_0 + \beta$. This is solved via a straightforward application of the method of characteristics (see [51]). Given $t \in [0, T]$ and $\mathbf{x} = (x, y) \in D$, we will denote by

$$s \mapsto \mathbf{X}^t(\mathbf{x}; s) = (X^t(x, y; s), Y^t(x, y; s))$$

the characteristic line for which $\mathbf{X}^t(\mathbf{x}; t) = \mathbf{x}$. In other words, $\mathbf{X}^t(\mathbf{x}; \cdot)$ solves the 'initial' value problem

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{X}^t(\mathbf{x}; s) = \alpha(\mathbf{X}^t(\mathbf{x}; s))\mathbf{U}(\mathbf{X}^t(\mathbf{x}; s), s), \\ &\mathbf{X}^t(\mathbf{x}; t) = \mathbf{x}. \end{aligned} \tag{3.31}$$

Since $\alpha \mathbf{U} \in C^{1,0}(Q_T)$, existence of a local unique solution is guaranteed by standard theory. Due to the fact that $\alpha \mathbf{U}$ has zero normal component on ∂D, the solution can be extended to the whole interval $s \in [0, T]$, and has useful regularity properties:

Lemma 3.2.9. *Given $t \in [0, T]$ and $\mathbf{x} \in D$, the problem (3.31) has a unique solution $\mathbf{X}^t(\mathbf{x}; s)$ defined for all $s \in [0, T]$; this solution is continuously differentiable in all three arguments. For fixed t and s, $\mathbf{X}^t(\cdot; s)$ is a one-to-one measure-preserving map of $\overline{D}$ onto itself, with its Jacobian determinant equal to 1, where ∂D is mapped onto itself. $\mathbf{X}^s(\cdot; s)$ is the identity map, and $\mathbf{X}^s(\cdot; t)$ is the inverse map of $\mathbf{X}^t(\cdot; s)$.*

Proof. Due to the fact that $\alpha \mathbf{U} \in C^{1,0}(Q_T)$ is tangential, the proof is word for word identical to those of [72, Lemma 2.2 & Lemma 2.3]. □

Given the characteristic lines $\mathbf{X}^t(\mathbf{x}; s)$, the solution of (3.30) at $(x, y, t) \in Q_T$ is then simply given by

$$\zeta(x, y, t) = \zeta_0(X^t(x, y; 0), Y^t(x, y; 0)), \tag{3.32}$$

or, in terms of $\xi = \frac{1}{\alpha}\zeta - \frac{\beta}{\alpha}$,

$$\xi(x, y, t) = \frac{\zeta_0(X^t(x, y; 0), Y^t(x, y; 0))}{\alpha(x, y)} - \frac{\beta(x, y)}{\alpha(x, y)}.$$

Since $\mathbf{X}^t(\mathbf{x}; s)$ is determined by $\mathbf{U}$, which is determined by φ, this means that ξ is determined by φ as well, so we may write

$$\xi = F[\varphi],$$

F being a map from S' to $C(\overline{Q}_T)$. It is important to point out that, as is known, ζ given by (3.32) is a classical solution of (3.30) only if $\zeta_0 \in C^1(\overline{D})$; however, if this is not the case, then ζ is still a weak solution, in the following sense (cf. [72, Lemma 2.4]):

Lemma 3.2.10. *For all $\Phi \in C^1(\overline{D})$, we have*

$$\partial_t(\zeta, \Phi) = (\zeta, \alpha \mathbf{U} \cdot \nabla \Phi).$$

Let us now list some more estimates that will be needed later on. The proof is essentially identical to the proofs of Lemmas 2.5, 2.6, and 2.7 in [72] (up to occasional trivial adjustments to account for α and β) and is therefore omitted.

Lemma 3.2.11. *Given $\varphi \in S'$, $\xi = F[\varphi]$ satisfies the inequalities*

$$|\xi(\mathbf{x}, t)| \leq A\|\zeta_0\|_\infty + B,$$

where

$$A = \sup_{\mathbf{x}\in D} \frac{1}{\alpha(\mathbf{x})} = \frac{4}{(1+r_1^2)^2} \quad \text{and} \quad B = \sup_{\mathbf{x}\in D} \frac{\beta(\mathbf{x})}{\alpha(\mathbf{x})} = 8\omega \frac{1-r_1^2}{(1+r_1^2)^3},$$

and

$$|\xi(\mathbf{x}_1, t) - \xi(\mathbf{x}_2, t)| \leq L_4(\|\varphi\|_\infty)(\|\mathbf{x}_1 - \mathbf{x}_2\|_2^\delta + |t-s|^\delta),$$

where $\delta^{-1} = L_5(\|\varphi\|_\infty)$, whenever $\|\mathbf{x}_1 - \mathbf{x}_2\|_2 \leq 1$ and $|t-s| \leq 1$.

We are now in a position to define a new class S that will allow us to apply Schauder's fixed-point theorem.

Definition 3.2.12. For $M = A\|\zeta_0\|_\infty + B$, we define the class S as the set of all functions $\varphi : D \times [0, T] \to \mathbb{R}$ such that $\|\varphi\|_\infty \leq M$ and

$$|\varphi(\mathbf{x}_1, t) - \varphi(\mathbf{x}_2, t)| \leq L(\|\mathbf{x}_1 - \mathbf{x}_2\|_2^\delta + |t-s|^\delta) \quad \text{for } \|\mathbf{x}_1 - \mathbf{x}_2\|_2 \leq 1,\ |t-s| \leq 1,$$

where $L = L_4(\|\varphi\|_\infty)$ and $\delta^{-1} = L_5(\|\varphi\|_\infty)$ (here it is crucial that L and Δ depend on $\|\varphi\|_\infty$ but not on the Hölder exponents of φ).

Clearly, $S \subseteq S'$, and, by Lemma 3.2.11, F maps S into itself. It is also clear that S is a compact convex subset of the Banach space $C(\overline{Q}_T)$. Moreover, we have (entirely analogously to [72, Lemma 2.8]):

Lemma 3.2.13. *The map F is continuous on S in the topology of $C(\overline{Q}_T)$.*

Schauder's fixed-point theorem now yields the existence of a fixed point $\varphi^* \in S$ of F:

$$\xi = F[\varphi^*] = \varphi^*.$$

Proof of Theorem 3.2.3

With all this preparation, we are ready to prove Theorem 3.2.3. The regularity of the derivatives of $\mathbf{U}$ is established exactly as in [72, Lemma 3.2 & Lemma 3.3]. Let us write

$$\mathbf{W} = \mathbf{U}_t + (\alpha\mathbf{U} \cdot \nabla)\mathbf{U} + \frac{|\mathbf{U}|^2}{2}\nabla\alpha + \beta J\mathbf{U}$$

as a short-hand notation. Let $\Phi \in C^1(\overline{D})$ with $\Phi|_{\partial D} = 0$. Then, using Lemmas 3.2.5 and 3.2.10, (3.23), and the fact that $\xi = \varphi^*$, we have

$$\begin{aligned}\partial_t(\mathbf{U}, \nabla^\perp\Phi) &= \partial_t(\nabla \times \mathbf{U}, \Phi) = \partial_t(\varphi^*, \Phi) = (\alpha\varphi^* + \beta, \mathbf{U} \cdot \nabla\Phi) \\ &= -((\alpha\mathbf{U} \cdot \nabla)\mathbf{U}, \nabla^\perp\Phi) - \left(\frac{|\mathbf{U}|^2}{2}\nabla\alpha, \nabla^\perp\Phi\right) - (\beta J\mathbf{U}, \nabla^\perp\Phi),\end{aligned}$$

that is, $(\mathbf{W}, \nabla^\perp\Phi) = 0$ for each such Φ. On the other hand, $(\mathbf{W}, \mathbf{U}^*) = 0$ by (3.26). Lemma 3.2.7 now yields the existence of $\Pi \in C^{1,0}(\overline{Q}_T)$ such that $W = -\nabla\Pi$; this is

(3.20a). Equations (3.20b) and (3.21) are automatically satisfied by construction of $\mathbf{U}$ (cf. Lemma 3.2.8). It remains to show (3.22). Set $\mathbf{W}_0 := \mathbf{U}|_{t=0} - \mathbf{U}_0$; then

$$\nabla \times \mathbf{W}_0 = \nabla \times \mathbf{U}|_{t=0} - \nabla \times \mathbf{U}_0 = \varphi^*|_{t=0} - \nabla \times \mathbf{U}_0 = 0,$$

because $\varphi^* = \xi$ and $\xi|_{t=0} = \nabla \times \mathbf{U}_0$. Thus

$$(\mathbf{W}_0, \nabla^\perp \Phi) = (\nabla \times \mathbf{U}_0, \Phi) = 0$$

for all $\Phi \in C^1(\overline{D})$ with $\Phi|_{t=0} = 0$. Moreover, $(\mathbf{W}, \mathbf{U}^*) = 0$ by (3.27). Therefore, by Lemma 3.2.7 (and since $\mathbf{W}_0 \in C^1(\overline{D})$) there is $q \in C^2(\overline{D})$ such that $\mathbf{W}_0 = \nabla q$. But $\mathbf{W}_0$ is a tangential flow, q is harmonic in D and $\nabla q \cdot \mathbf{n} = 0$, thus q is constant and $\mathbf{W}_0 = 0$, as desired.

For the uniqueness part, suppose that $(\mathbf{U}, \Pi)$ and $(\tilde{\mathbf{U}}, \tilde{\Pi})$ solve (3.20)–(3.22). Set $\hat{\mathbf{U}} := \tilde{\mathbf{U}} - \mathbf{U}$ and $Q := \tilde{\Pi} - \Pi$; then $\hat{\mathbf{U}} \cdot \mathbf{n} = 0$ on ∂D and $\hat{\mathbf{U}}|_{t=0} = 0$. Subtracting the equations for $(\mathbf{U}, P)$ and $(\tilde{\mathbf{U}}, \tilde{P})$, we obtain

$$\hat{\mathbf{U}}_t + (\alpha\tilde{\mathbf{U}} \cdot \nabla)\hat{\mathbf{U}} + (\alpha\hat{\mathbf{U}} \cdot \nabla)\mathbf{U} + \frac{1}{2}(|\tilde{\mathbf{U}}|^2 - |\mathbf{U}|^2)\nabla\alpha + \beta J\hat{\mathbf{U}} = -\nabla Q.$$

Multiplying by $\hat{U}$, integrating over D, and noticing that

$$((\alpha\tilde{\mathbf{U}} \cdot \nabla)\hat{\mathbf{U}}, \hat{\mathbf{U}}) = -\frac{1}{2}(\tilde{\mathbf{U}} \cdot \nabla\alpha, |\hat{\mathbf{U}}|^2)$$

(as in the proof of Lemma 3.2.8) and $(\nabla Q, \hat{\mathbf{U}}) = 0$ (because $\hat{\mathbf{U}}$ is a tangential flow), we see that

$$\partial_t(\hat{\mathbf{U}}, \hat{\mathbf{U}}) = \frac{1}{2}(\tilde{\mathbf{U}} \cdot \nabla\alpha, |\hat{\mathbf{U}}|^2) - ((\alpha\hat{\mathbf{U}} \cdot \nabla)\mathbf{U}, \hat{\mathbf{U}}) - \frac{1}{2}(((\tilde{\mathbf{U}} + \mathbf{U}) \cdot \hat{\mathbf{U}})\nabla\alpha, \hat{\mathbf{U}}).$$

Using the fact that $\alpha, \mathbf{U}, \tilde{\mathbf{U}}$ are continuously differentiable, we can find a real (not necessarily positive) constant $C \in \mathbb{R}$ such that

$$\partial_t(\hat{\mathbf{U}}, \hat{\mathbf{U}}) \leq C(\hat{\mathbf{U}}, \hat{\mathbf{U}}).$$

By Gronwall's inequality and the initial condition $\hat{\mathbf{U}}|_{t=0} = 0$, it follows that $\hat{\mathbf{U}} = 0$ and thus $\nabla Q = 0$.

3.2.4 Existence and stability of a class of zonal solutions

In this section, we will seek zonal solutions of equation (3.15) (that is, solutions that depend solely on θ) in the case where F is given by (3.16). In other words, for given $\lambda, \Upsilon \in \mathbb{R}$, we are looking for a function $\Psi_{\lambda,\Upsilon} = \Psi_{\lambda,\Upsilon}(\theta)$ which solves the following ODE problem (where a prime denotes a derivative with respect to θ):

$$\begin{aligned} &\left(\Psi'_{\lambda,\Upsilon}(\theta)\cos\theta\right)' = -\lambda\Psi_{\lambda,\Upsilon}(\theta)\cos\theta + \Upsilon\cos\theta - \omega\sin 2\theta, \quad \theta \in (\theta_1, \theta_2), \\ &\Psi_{\lambda,\Upsilon}(\theta_1) = \psi_1, \;\; \Psi_{\lambda,\Upsilon}(\theta_2) = \psi_2, \end{aligned} \tag{3.33}$$

for $\psi_1, \psi_2 \in \mathbb{R}$.

Zonal solutions via Sturm–Liouville

Let us recollect some well-known facts about Sturm–Liouville problems (see, e.g., [114] for proofs):

Lemma 3.2.14. *Let $a, b \in \mathbb{R}$, $a < b$, and suppose that the functions $p \in C^1([a,b])$ and $w \in C([a,b])$ are both positive: $p(x) > 0$ and $w(x) > 0$ for all $x \in [a,b]$. Also, let $q \in C([a,b])$ and $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ with $(\alpha,\beta) \neq (0,0) \neq (\gamma,\delta)$. Then, there exists a sequence $\mu_1 < \mu_2 < \ldots \to +\infty$ and, for each $n \in \mathbb{N}$, a function $y_n \in C^2([a,b])$ such that*

$$\begin{aligned}
&(p(x)y_n'(x))' + q(x)y_n(x) = -\mu_n w(x) y_n(x), \quad x \in (a,b),\\
&\alpha y_n(a) + \beta y_n'(a) = 0,\\
&\gamma y_n(b) + \delta y_n'(b) = 0.
\end{aligned}$$

In addition, all solutions to this problem are a multiple of y_n (i.e., each eigenvalue μ_n is simple), and y_n has exactly $n-1$ zeros. Moreover, the set $\{y_n : n \in \mathbb{N}\}$ is an orthonormal basis of the weighted L^2-space $L^2((a,b); w(x)\,\mathrm{d}x)$ with respect to the inner product

$$\langle f, g\rangle = \int_a^b f(x)g(x)w(x)\,\mathrm{d}x$$

(that is, $\langle y_n, y_m\rangle = \delta_{mn}$). The eigenfunction expansion

$$f = \sum_{n=1}^{\infty} \langle f, y_n\rangle y_n$$

converges uniformly for all $f \in C^2([a,b])$ with $f(a) = f(b) = 0$. Finally, each eigenvalue μ_n can be recovered from a corresponding eigenfunction y_n via the Rayleigh quotient*:*

$$\mu_n = \frac{-p y_n y_n'|_b^a + \int_a^b \left[p(y_n')^2 - q y_n^2\right]\mathrm{d}x}{\langle y_n, y_n\rangle}. \tag{3.34}$$

In several places throughout the rest of this section we will additionally require some knowledge about inhomogeneous Sturm–Liouville problems; this is the subject of the next lemma.

Lemma 3.2.15. *Let $a, b \in \mathbb{R}$, $a < b$, and suppose that the functions $p \in C^1([a,b])$ and $w \in C([a,b])$ are both positive: $p(x) > 0$ and $w(x) > 0$ for all $x \in [a,b]$. Also, let $q, h \in C([a,b])$, and $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ with $(\alpha,\beta) \neq (0,0) \neq (\gamma,\delta)$. For $\mu \in \mathbb{R}$, consider the inhomogeneous Sturm–Liouville problem*

$$(p(x)y'(x))' + q(x)y(x) = -\mu w(x)y(x) + h(x), \quad x \in (a,b), \tag{3.35a}$$
$$\alpha y(a) + \beta y'(a) = 0, \tag{3.35b}$$
$$\gamma y(b) + \delta y'(b) = 0. \tag{3.35c}$$

Let $\mu_1 < \mu_2 < \ldots$ be the eigenvalues of the homogeneous problem ($h \equiv 0$), with corresponding eigenfunctions y_n ($n \in \mathbb{N}$), scaled so that $\langle y_n, y_n\rangle = 1$ for all $n \in \mathbb{N}$. Then the inhomogeneous problem (3.35) has a unique solution if and only if $\mu \notin \{\mu_n : n \in \mathbb{N}\}$. If $\mu = \mu_k$ for some $k \in \mathbb{N}$, then:

(a) either $\langle \frac{h}{w}, y_k\rangle \neq 0$, in which case there is no solution, or

(b) $\langle \frac{h}{w}, y_k\rangle = 0$, *in which case there are infinitely many solutions, which are of the form*

$$y(x) = \sum_{n=1}^{\infty} b_n y_n(x),$$

where

$$b_n = \frac{\langle \frac{h}{w}, y_k\rangle}{\mu - \mu_n} \quad \text{if } n \neq k$$

and b_k *is an arbitrary real number.*

Proof. Suppose that $\mu \notin \{\mu_n : n \in \mathbb{N}\}$. Let $c_n = \langle \frac{h}{w}, y_n\rangle$, thus

$$\frac{h(x)}{w(x)} = \sum_{n=1}^{\infty} c_n y_n(x)$$

(with convergence in the Hilbert space $L^2((a,b); w(x)\,\mathrm{d}x)$). Any solution y of the inhomogeneous problem (3.35) must have an eigenfunction expansion

$$y = \sum_{n=1}^{\infty} b_n y_n;$$

we now show that appropriate coefficients $(b_n)_{n\in\mathbb{N}}$ can indeed be uniquely determined. Since

$$(p(x)y_n'(x))' + q(x)y_n(x) + \mu w(x)y_n(x) = (\mu - \mu_n)w(x)y_n(x),$$

plugging into (3.35a) yields

$$\frac{h(x)}{w(x)} = \sum_{n=1}^{\infty} b_n(\mu - \mu_n)y_n(x),$$

hence

$$(\mu - \mu_n)b_n = c_n, \quad n \in \mathbb{N}, \tag{3.36}$$

and thus $b_n = \frac{c_n}{\mu - \mu_n}$ for all $n \in \mathbb{N}$.

If, on the other hand, $\mu = \mu_k$ for some $k \in \mathbb{N}$, and $c_k \neq 0$, then the equality (3.36) cannot be satisfied for $n = k$, so there is no solution in this case. Finally, if $\mu = \mu_k$ for some $k \in \mathbb{N}$ and $c_k = 0$, the equality (3.36) is satisfied for any choice of the coefficient b_k; this concludes the proof. □

Combining the previous two lemmas, we obtain a family of zonal solutions to (3.33).

Proposition 3.2.16. *Let* $0 < \lambda_1 < \lambda_2 < \ldots \to +\infty$ *be the sequence of eigenvalues (provided by Lemma 3.2.14; note that* $\lambda_1 > 0$ *follows from (3.34)) of the problem*

$$\begin{aligned} &\big(\Psi'(\theta)\cos\theta\big)' = -\lambda\Psi(\theta)\cos\theta, \quad \theta \in (\theta_1, \theta_2), \\ &\Psi(\theta_1) = \Psi(\theta_2) = 0. \end{aligned} \tag{3.37}$$

Then the problem (3.33) has a unique solution $\Psi_{\lambda,\Upsilon}$ *for all* $\lambda \notin \{\lambda_n : n \in \mathbb{N}\}$.

Proof. By setting

$$\tilde{\Psi}(\theta) = \Psi_{\lambda,\Upsilon}(\theta) - (a\theta + b),$$

where

$$a = \frac{\psi_2 - \psi_1}{\theta_2 - \theta_1} \quad \text{and} \quad b = \frac{\theta_2\psi_1 - \theta_1\psi_2}{\theta_2 - \theta_1},$$

problem (3.33) transforms to

$$\begin{aligned} &\left(\tilde{\Psi}'(\theta)\cos\theta\right)' = -\lambda\tilde{\Psi}(\theta)\cos\theta + a\sin\theta + [\Upsilon - \lambda(a\theta + b)]\cos\theta - \omega\sin 2\theta, \quad \theta \in (\theta_1, \theta_2), \\ &\tilde{\Psi}(\theta_1) = \tilde{\Psi}(\theta_2) = 0, \end{aligned}$$

which has homogeneous Dirichlet boundary conditions. The claim then follows from Lemma 3.2.15. □

Remark 3.2.17. Note that, by the weak maximum principle (see, e.g., [53]), solutions to problem (3.15) are unique if $\lambda \leq 0$. Therefore, the radial solutions for $\lambda \leq 0$ are in fact *all* solutions for such values of λ.

Remark 3.2.18. It is possible to infer that $\lambda \notin \{\lambda_n : n \in \mathbb{N}\}$ if the annular domain $\mathcal{C}$ is sufficiently 'narrow'. In fact, via a fixed-point argument, we can show existence of solutions to (3.33) for all $\lambda \in \mathbb{R}$, that is, it can be guaranteed that λ is not an eigenvalue of (3.37), provided that the region $\mathcal{C}$ be sufficiently 'narrow' (depending on $|\lambda|$). The result is formulated more conveniently after applying a stereographic projection centred at the North Pole; in other words, as in Section 3.2.3, we perform the change of variable

$$r = \frac{\cos\theta}{1 - \sin\theta},$$

after which, setting $\tilde{\Psi}_\lambda(r) = \Psi_{\lambda,\Upsilon}(\theta)$, problem (3.33) becomes

$$\frac{1}{r}\left(r\tilde{\Psi}_\lambda'(r)\right)' = -\frac{4\lambda}{(1+r^2)^2}\tilde{\Psi}_\lambda(r) + 8\omega\frac{1-r^2}{(1+r^2)^3}, \quad r \in (r_1, r_2), \tag{3.38a}$$

$$\tilde{\Psi}_\lambda(r_1) = \psi_1, \ \tilde{\Psi}_\lambda(r_2) = \psi_2, \tag{3.38b}$$

where $r_1 = \cos\theta_1/(1 - \sin\theta_1)$ and $r_2 = \cos\theta_2/(1 - \sin\theta_2)$ (so that $0 < r_1 < r_2 < 1$).

Proposition 3.2.19. *Let $\lambda \in \mathbb{R}$ and suppose that*

$$\frac{r_2}{r_1} \leq \mathrm{e}^{(2|\lambda|)^{-1/2}}. \tag{3.39}$$

Then there exists a unique solution $\tilde{\Psi}_\lambda \in C^2([r_1, r_2])$ of problem (3.38).

Proof. Following [19], we the change variables according to

$$r = \mathrm{e}^{-\tau}, \quad u(\tau) = \tilde{\Psi}_\lambda(r) = \tilde{\Psi}_\lambda(\mathrm{e}^{-\tau}).$$

Setting

$$\tau_1 = \log\frac{1}{r_2}, \quad \tau_2 = \log\frac{1}{r_1}$$

(so that $\tau_1 < \tau_2$), equation (3.38a) becomes

$$u''(\tau) = -\frac{\lambda u(\tau)}{\cosh^2(\tau)} + 2\omega\frac{\sinh(\tau)}{\cosh^3(\tau)}, \quad \tau \in (\tau_1, \tau_2) \tag{3.40}$$

(where the prime now denotes a derivative with respect to τ), whereas the boundary conditions (3.38b) become simply

$$u(\tau_1) = \psi_2, \quad u(\tau_2) = \psi_1. \tag{3.41}$$

Integrating (3.40) twice on $[\tau_1, \tau]$, where $\tau \in [\tau_1, \tau_2]$, yields the integral equation

$$u(\tau) = \psi_2 - \psi_1 + \mu(u)(\tau - \tau_1) - \lambda \int_{\tau_1}^{\tau} \frac{\tau - s}{\cosh^2(s)} u(s)\,\mathrm{d}s + 2\omega \int_{\tau_1}^{\tau} (\tau - s)\frac{\sinh(s)}{\cosh^3(s)}\,\mathrm{d}s, \tag{3.42}$$

$\mu(u) \in \mathbb{R}$ being an integration constant for which we must require

$$\mu(u) = \frac{1}{\tau_2 - \tau_1}\left\{\psi_1 - \psi_2 + \lambda \int_{\tau_1}^{\tau_2} \frac{\tau_2 - s}{\cosh^2(s)} u(s)\,\mathrm{d}s - 2\omega \int_{\tau_1}^{\tau_2} (\tau_2 - s)\frac{\sinh(s)}{\cosh^3(s)}\,\mathrm{d}s\right\}$$

for the boundary conditions (3.41) to be satisfied. Let us now define

$$X = \{f \in C([\tau_1, \tau_2]) : f(\tau_1) = \psi_2,\ f(\tau_2) = \psi_1\},$$

which is a closed subset of the space $C([\tau_1, \tau_2])$ of continuous functions on $[\tau_1, \tau_2]$. We define the map $\mathcal{T} : X \to X$ as

$$\mathcal{T}(u)(t) = \psi_2 - \psi_1 + \mu(u)(\tau - \tau_1) - \lambda \int_{\tau_1}^{\tau} \frac{\tau - s}{\cosh^2(s)} u(s)\,\mathrm{d}s + 2\omega \int_{\tau_1}^{\tau} (\tau - s)\frac{\sinh(s)}{\cosh^3(s)}\,\mathrm{d}s;$$

it is easily checked that indeed $\mathcal{T}(X) \subseteq X$, so that $\mathcal{T}$ is well-defined. We now show that $\mathcal{T}$ is a contraction on X. Let $u, v \in X$; then

$$\begin{aligned} |\mathcal{T}(u)(\tau) - \mathcal{T}(v)(\tau)| &\le 2|\lambda|(\tau_2 - \tau_1) \int_{\tau_1}^{\tau_2} |u(s) - v(s)|\,\mathrm{d}s \\ &\le 2|\lambda|(\tau_2 - \tau_1)^2 \|u - v\|_{L^\infty(\tau_1,\tau_2)} \quad \text{for all } \tau \in [\tau_1, \tau_2], \end{aligned}$$

which, in view of (3.39), yields the asserted contraction property of $\mathcal{T}$ on X. Banach's fixed-point theorem now yields the existence of a unique fixed point $u \in X$ of $\mathcal{T}$; the fixed point u is thus a solution of the integral equation (3.42) satisfying the boundary conditions (3.41). Finally, note that, u being continuous, the right-hand side of (3.42) is twice continuously differentiable; this implies that, in fact, $u \in C^2([\tau_1, \tau_2])$. Transforming back to the original variable $r = \mathrm{e}^{-\tau}$ then completes the proof. □

Stability

We begin the discussion of stability by identifying certain conserved quantities. To simplify notation, we denote derivatives by subscripts in what follows.

Proposition 3.2.20. *Let $f \in C^1(\mathbb{R})$ be arbitrary. Then, the quantities*

$$\frac{1}{2}\int_{\mathcal{C}} \left(\psi_\theta^2(\varphi, \theta, t) + \frac{1}{\cos^2\theta}\psi_\varphi^2(\varphi, \theta, t)\right)\mathrm{d}\sigma, \tag{3.43a}$$

$$\int_0^{2\pi} \psi_\theta(\varphi, \theta_1, t)\,\mathrm{d}\varphi, \qquad \int_0^{2\pi} \psi_\theta(\varphi, \theta_2, t)\,\mathrm{d}\varphi \tag{3.43b}$$

and

$$\int_{\mathcal{C}} f(\Delta_{\mathbb{S}^2}\psi(\varphi, \theta, t) + 2\omega \sin\theta)\,\mathrm{d}\sigma \tag{3.43c}$$

are conserved by smooth solutions of (3.14) with boundary condition (3.13).

Remark 3.2.21. The expression (3.43a) is the kinetic energy, the quantities in (3.43b) are the circulations along the 'lower' and 'upper' boundary, and (3.43c) corresponds to the so-called *Casimir invariants* (see [27]).

Proof. First, evaluating (3.9a) on $\{\theta = \theta_i\}$, $i \in \{1, 2\}$, and taking (3.11) into account, we see that

$$\psi_{\theta t}|_{\theta=\theta_i} = -u_t|_{\theta=\theta_i} = \left(\frac{u^2}{2\cos\theta}\bigg|_{r=r_1} + \frac{p}{\cos\theta}\bigg|_{r=r_1}\right)_\varphi,$$

hence the quantities in (3.43b) are conserved. Consequently, using (3.14), the boundary condition (3.13) (which implies that $\psi_\varphi = 0$ on the boundary), and the 2π-periodicity in the φ-argument, we see via integration by parts that

$$\begin{aligned}
\partial_t \int_{\mathcal{C}} \frac{1}{2}\left(\psi_\theta^2 + \frac{1}{\cos^2\theta}\psi_\varphi^2\right) \mathrm{d}\sigma &= \int_{\mathcal{C}} \left(\psi_\theta\psi_{\theta t} + \frac{1}{\cos^2\theta}\psi_\varphi\psi_{\varphi t}\right) \mathrm{d}\sigma \\
&= \psi_2\cos\theta_2 \int_0^{2\pi} \psi_{\theta t}|_{r=r_2}\,\mathrm{d}\varphi - \psi_1\cos\theta_1 \int_0^{2\pi} \psi_{\theta t}|_{r=r_1}\,\mathrm{d}\varphi \\
&\quad - \int_{\mathcal{C}} \psi(\Delta_{\mathbb{S}^2}\psi)_t\,\mathrm{d}\sigma \\
&= \int_0^{2\pi}\int_{\theta_1}^{\theta_2} \psi\big[\psi_\varphi\partial_\theta - \psi_\theta\partial_\varphi\big](\Delta_{\mathbb{S}^2}\psi + 2\omega\sin\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi = 0;
\end{aligned}$$

this establishes the claim for (3.43a). For (3.43c), using (3.14) and (3.13), it follows

$$\begin{aligned}
&\partial_t \int_{\mathcal{C}} f(\Delta_{\mathbb{S}^2}\psi + 2\omega\sin\theta)\,\mathrm{d}\sigma \\
&\quad = \int_0^{2\pi}\int_{\theta_1}^{\theta_2} \big\{\psi_\theta\big[f(\Delta_{\mathbb{S}^2}\psi + 2\omega\sin\theta)\big]_\varphi - \psi_\varphi\big[f(\Delta_{\mathbb{S}^2}\psi + 2\omega\sin\theta)\big]_\theta\big\}\,\mathrm{d}\theta\,\mathrm{d}\varphi = 0;
\end{aligned}$$

this completes the proof. $\square$

In view of Proposition 3.2.20, we see that the functional

$$\begin{aligned}
\mathcal{E}_{\alpha,\beta}(\psi) = \frac{1}{2}\int_{\mathcal{C}} \left[-\lambda\left(\psi_\theta^2 + \frac{\psi_\varphi^2}{\cos^2\theta}\right) + (\Delta_{\mathbb{S}^2}\psi + 2\omega\sin\theta - \Upsilon)^2\right] \mathrm{d}\sigma \\
+ \lambda\left\{\alpha\int_0^{2\pi} \psi_\theta|_{\theta=\theta_2}\,\mathrm{d}\varphi + \beta\int_0^{2\pi} \psi_\theta|_{\theta=\theta_1}\,\mathrm{d}\varphi\right\}
\end{aligned}$$

is constant along smooth solutions to (3.14) and (3.13) for any $\alpha, \beta \in \mathbb{R}$. For such a solution ψ, we may compute the first and second variation of $\mathcal{E}_{\alpha,\beta}(\psi)$, to find

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}_{\alpha,\beta}(\psi + s\xi)\bigg|_{s=0} &= \int_{\mathcal{C}} (\lambda\psi - \Upsilon + \Delta_{\mathbb{S}^2}\psi + 2\omega\sin\theta)\Delta_{\mathbb{S}^2}\xi\,\mathrm{d}\sigma \\
&\quad + \lambda\bigg\{(\alpha - \psi_2\cos\theta_2)\int_0^{2\pi} \xi_\theta|_{\theta=\theta_2}\,\mathrm{d}\varphi \\
&\qquad + (\beta + \psi_1(t)\cos\theta_1)\int_0^{2\pi} \xi_\theta|_{\theta=\theta_1}\,\mathrm{d}\varphi\bigg\}
\end{aligned}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathcal{E}_{\alpha,\beta}(\psi + s\xi)\bigg|_{s=0} = \int_{\mathcal{C}} \left[-\lambda\left(\xi_\theta^2 + \frac{\xi_\varphi^2}{\cos^2\theta}\right) + (\Delta_{\mathbb{S}^2}\xi)^2\right] \mathrm{d}\sigma.$$

Therefore, wee see that, choosing

$$\alpha = \psi_2 \cos\theta_2 \quad \text{and} \quad \beta = -\psi_1 \cos\theta_1, \tag{3.44}$$

the zonal solution $\psi = \Psi_{\lambda,\Upsilon}$ to (3.33) is a critical point of the functional $\mathcal{E}_{\alpha,\beta}$, with nonnegative second variation if $\lambda \leq 0$. These observations naturally lead us to the stability result stated in Theorem 3.2.2. Let us point out that, as is readily seen upon inspection of the upcoming proof, the conclusion (3.17) of Theorem 3.2.2 actually holds for all values of λ for which $\Psi_{\lambda,\Upsilon}$ is defined, although clearly this implies stability only for $\lambda \leq 0$ (as suggested by the expression for the second variation of $\mathcal{E}_{\alpha,\beta}$).

Proof of Theorem 3.2.2. The idea of this proof is similar to [81, pp. 104–109]), but we present it nonetheless for the sake of completeness. We choose α and β as in (3.44). For notational convenience, we will write $\mathcal{E}_{\alpha,\beta} = \mathcal{E}$ and $\psi^* = \Psi_{\lambda,\Upsilon}$ throughout the proof. Let ψ be the stream function corresponding to (u, v) (as in (3.12)) solving (3.14)–(3.13). It is immediate to check that

$$\Omega = \Delta_{\mathbb{S}^2}\psi.$$

Using the identity $\frac{1}{2}(a^2 - b^2) = \frac{1}{2}(a-b)^2 + b(a-b)$, valid for all $a, b \in \mathbb{R}$, we see that

$$\begin{aligned}
\mathcal{E}(\psi) - \mathcal{E}(\psi^*) &= -\frac{\lambda}{2}\int_{\mathcal{C}} \left[(\psi_\theta - \psi_\theta^*)^2 + \frac{1}{\cos^2\theta}(\psi_\varphi - \psi_\varphi^*)^2\right] \mathrm{d}\sigma \\
&\quad - \lambda \int_{\mathcal{C}} \left[\psi_\theta^*(\psi_\theta - \psi_\theta^*) + \frac{1}{\cos^2\theta}\psi_\varphi^*(\psi_\varphi - \psi_\varphi^*)\right] \mathrm{d}\sigma \\
&\quad + \frac{1}{2}\int_{\mathcal{C}} (\Delta_{\mathbb{S}^2}\psi - \Delta_{\mathbb{S}^2}\psi^*)^2 \,\mathrm{d}\sigma \\
&\quad + \int_{\mathcal{C}} (\Delta_{\mathbb{S}^2}\psi^* + 2\omega\sin\theta - \Upsilon)(\Delta_{\mathbb{S}^2}\psi - \Delta_{\mathbb{S}^2}\psi^*)\,\mathrm{d}\sigma \\
&\quad + \lambda\psi_2\cos\theta_2 \int_0^{2\pi} (\psi_\theta - \psi_\theta^*)|_{\theta=\theta_1}\,\mathrm{d}\varphi - \lambda\psi_1\cos\theta_1 \int_0^{2\pi} (\psi_\theta - \psi_\theta^*)|_{\theta=\theta_2}\,\mathrm{d}\varphi \\
&= -\frac{\lambda}{2}\|\mathbf{u} - \mathbf{u}_{\lambda,\Upsilon}^*\|_{L^2(\mathcal{C})}^2 + \frac{1}{2}\|\Delta_{\mathbb{S}^2}\psi - \Delta_{\mathbb{S}^2}\psi^*\|_{L^2(\mathcal{C})}^2,
\end{aligned}$$

in view of (3.12). Therefore, since $\mathcal{E}$ is a first integral for (3.9)–(3.11),

$$\begin{aligned}
-\lambda\|\mathbf{u}(t) - \mathbf{u}_{\lambda,\Upsilon}^*\|_{L^2(\mathcal{C})}^2 + \|\Omega(t) - \Omega_{\lambda,\Upsilon}^*\|_{L^2(\mathcal{C})}^2 &= 2(\mathcal{E}(\psi(t)) - \mathcal{E}(\psi^*)) \\
&= 2(\mathcal{E}(\psi_0) - \mathcal{E}(\psi^*)) \\
&= -\lambda\|\mathbf{u}_0 - \mathbf{u}_{\lambda,\Upsilon}^*\|_{L^2(\mathcal{C})}^2 + \|\Omega_0 - \Omega_{\lambda,\Upsilon}^*\|_{L^2(\mathcal{C})}^2,
\end{aligned}$$

as claimed. □

Remark 3.2.22. In particular, taking $\lambda = 0$, we have stability (albeit exclusively in terms of the vorticity) of the solution corresponding to the stream function $\Psi_{0,\Upsilon}$ which solves the problem

$$\big(\Psi_{0,\Upsilon}'(\theta)\cos\theta\big)' = \Upsilon\cos\theta - \omega\sin 2\theta, \quad \theta \in (\theta_1, \theta_2), \tag{3.45a}$$

$$\Psi_{0,\Upsilon}(\theta_1) = \psi_1,\ \Psi_{0,\Upsilon}(\theta_2) = \psi_2 \tag{3.45b}$$

(existence and uniqueness of $\Psi_{0,\Upsilon}$ are guaranteed by Proposition 3.2.16). It is interesting to note that, in this special case, one can find infinitely many Lyapunov functions. Indeed, for $n \in \mathbb{N}$, consider the functional

$$\mathcal{E}_n(\psi) = \int_{\mathcal{C}} \left[-\Upsilon \frac{n+1}{n} (\Delta_{\mathbb{S}^2}\psi + 2\omega \sin\theta)^n + (\Delta_{\mathbb{S}^2}\psi + 2\omega \sin\theta)^{n+1} \right] \mathrm{d}\sigma,$$

which is conserved along solutions of (3.14)–(3.13) by Proposition 3.2.20. A simple calculation yields

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}_n(\Psi_{0,\Upsilon} + s\xi)\bigg|_{s=0} = 0 \quad \text{and} \quad \frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathcal{E}_n(\Psi_{0,\Upsilon} + s\xi)\bigg|_{s=0} = (n+1)\Upsilon^{n-1}\int_{\mathcal{C}} (\Delta_{\mathbb{S}^2}\xi)^2 \,\mathrm{d}\sigma,$$

thus $\Psi_{0,\Upsilon}$ is a critical point of $\mathcal{E}_n$ for each $n \in \mathbb{N}$ and the second variation is non-negative provided that n be even and $\Upsilon \geq 0$, or n odd and Υ of any sign. The solution of (3.45) can in fact be computed explicitly by integrating (3.45a) twice; this gives

$$\Psi_{0,\Upsilon}(\theta) = \zeta(\theta) + c_1\eta(\theta) + c_2,$$

where

$$\zeta(\theta) = -\Upsilon \log\cos\theta + \omega\sin\theta - \frac{\omega}{2}\tanh^{-1}(\sin\theta),$$
$$\eta(\theta) = \log\left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right) - \log\left(\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right)$$

and the integration constants c_1, c_2 are determined through the boundary values (3.45b) as

$$c_1 = \frac{\psi_2 - \psi_1 + \zeta(\theta_1) - \zeta(\theta_2)}{\eta(\theta_2) - \eta(\theta_1)},$$
$$c_2 = \frac{\psi_1\eta(\theta_2) - 2\psi_1\eta(\theta_1) + \psi_2\eta(\theta_1) - \zeta(\theta_1)\eta(\theta_2) + 2\eta(\theta_1)\zeta(\theta_1) - \zeta(\theta_2)\eta(\theta_1)}{\eta(\theta_2) - \eta(\theta_1)}.$$

The stability properties of the zonal solutions $\Psi_{\lambda,\Upsilon}$ that we discussed may contribute to explaining why the current pattern observed in the ACC is persistent. To this end, it

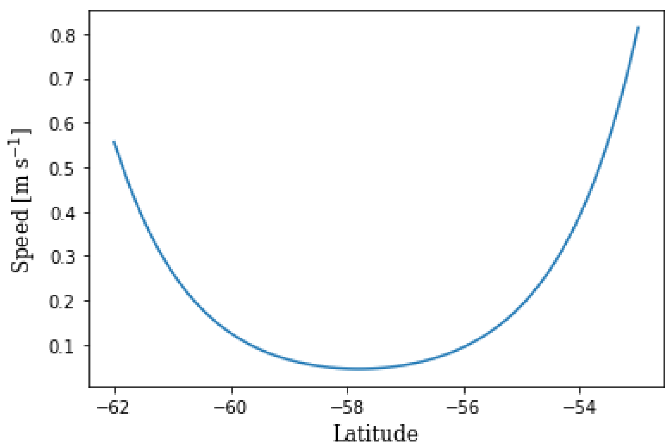

FIGURE 3.3: The (dimensional) zonal velocity corresponding to the (nondimensional) stream function $\Psi_{\lambda,\Upsilon}$ for the values $\lambda = -3000$, $\Upsilon = 30000$, $\psi_1 = -5$, and $\psi_2 = -25$.

is interesting to ask whether the large family of solutions $\Psi_{\lambda,\Upsilon}$ of (3.33) contains solutions (for some $\lambda \leq 0$) that exhibit at least some features that emerge from *in situ* measurement of the ACC. As mentioned in the Introduction, the ACC is in fact composed of fast and vertically coherent jets, with speeds that can exceed $1\,\mathrm{m\,s^{-1}}$, separated by zones of low-speed flow. In particular, the ACC is delimited by two such jets, namely the inner Polar Front and the outer Subantarctic Front, which are situated at approximate latitudes of 60°S and 50°S. The presence of such jets at the boundary of the region, separated by a low-speed region in the middle, is captured by $\Psi_{\lambda,\Upsilon}$ for appropriate choices of λ, ψ_1 and ψ_2; see Figure 3.3.

3.3 Ocean gyres with variable density

In ocean dynamics, gyres are wind-driven currents that are confined to a closed region, where a slow spiralling motion takes place about a central point; such regions are not necessarily circular (their shape is usually more irregular) and occur in a variety of sizes, which can be in excess of several hundreds of thousands of kilometres. The five largest gyres are observed in the Atlantic and Pacific Ocean—both in the Northern and Southern Hemisphere—and in the Indian Ocean [116]; smaller gyres have been observed, for instance, off the coast of Alaska (the Beaufort Gyre—see [40]). Due to the effect of the Coriolis force, gyres in the Northern Hemisphere usually rotate clockwise, whereas southern gyres spin counterclockwise. At the equator, the Coriolis force changes sign and thus effectively acts as a waveguide, so that equatorial flows have a typical zonal structure that prevents the formation of gyres (see [29, 54]).

Due to gyres' large extension, one should expect their dynamics to be significantly influenced not only by changes in the Coriolis force across the gyre region but also by the effects of Earth's curvature. The simplest tangent-plane approximation, namely the f-plane approximation discussed in Section 2.1, fails to take either of these aspects into account; on the other hand, the β-plane approximation (a slight generalisation of the f-plane which accounts for small variations of the Coriolis parameter in the azimuthal direction—see [116, Section 2.3.2]) is only consistent in equatorial regions [89], where gyres are absent. It is therefore appropriate to work in the spherical coordinates described in Section 1.1.3, the equations being those of the thin-shell approximation derived in Section 3.1. In particular, in this section, we will retain density variations, in contrast to Section 3.2.

This section, which is based on the article [47], is structured as follows. First, we derive from (3.8) an elliptic vorticity equation for a pseudo stream function ψ, whose existence can be inferred from (3.8d) and (3.8e). Next, we rewrite the equation for ψ using a modified stereographic projection, whereby the centre of projection is situated at the South Pole; this is a more convenient setting than the usual stereographic projection centred at the North Pole when dealing with regions situated in the Northern Hemisphere. In the absence of terrestrial rotation, the equation can be solved explicitly, as shown in [44]; some examples illustrate the significance of such solutions for the descriptions of arctic gyres. Finally, following the strategy developed in [42], we show that, for gyre regions whose diameter is of the order of hundreds of kilometres, at leading order the explicit solution for the fixed sphere is an approximation of the corresponding solution for the rotating sphere, in a sense that is formulated precisely in Theorem 2.5.4 below. The same procedure is applied to the case of gyres in the Southern Hemisphere by means of the 'classical' stereographic projection, whereby the centre of projection is situated at the

North Pole.

For reference, let us write down the thin-shell equations (3.8) we have derived in Section 3.1, suppressing the time dependence and neglecting the equation (3.8c) of hydrostatic balance:

$$\frac{u}{\cos\theta}\frac{\partial u}{\partial\varphi}+v\frac{\partial u}{\partial\theta}-uv\tan\theta-2\omega v\sin\theta=-\frac{1}{\rho\cos\theta}\frac{\partial P}{\partial\varphi}, \tag{3.46a}$$

$$\frac{u}{\cos\theta}\frac{\partial v}{\partial\varphi}+v\frac{\partial v}{\partial\theta}+u^2\tan\theta+2\omega u\sin\theta+\omega^2\sin\theta\cos\theta=-\frac{1}{\rho}\frac{\partial P}{\partial\theta}, \tag{3.46b}$$

$$\frac{u}{\cos\theta}\frac{\partial \rho}{\partial\varphi}+v\frac{\partial \rho}{\partial\theta}=0, \tag{3.46c}$$

$$\frac{\partial u}{\partial\varphi}+\frac{\partial}{\partial\theta}(v\cos\theta)=0. \tag{3.46d}$$

Note that, in simply connected regions of the sphere $\mathbb{S}^2$, equations (3.46d) and (3.46c) imply the existence of a (pseudo) stream function $\psi(\varphi,\theta)$ for which

$$\sqrt{\rho}u=-\frac{\partial\psi}{\partial\theta}\quad\text{and}\quad\sqrt{\rho}v=\frac{1}{\cos\theta}\frac{\partial\psi}{\partial\varphi};$$

moreover, it follows from (3.46c) that, as long as $\nabla_{\mathbb{S}^2}\psi\neq 0$, where

$$\nabla_{\mathbb{S}^2}f=\frac{1}{\cos\theta}\frac{\partial f}{\partial\varphi}\,\mathbf{e}_\varphi+\frac{\partial f}{\partial\theta}\,\mathbf{e}_\theta$$

denotes the gradient of the differentiable function f on $\mathbb{S}^2$, there exists a function ϱ such that

$$\rho(\varphi,\theta)=\varrho(\psi(\varphi,\theta))$$

for all (φ,θ); see, for instance, the discussion in [118] and [122]. We shall always think of ϱ as being given. Now we may eliminate P in (3.46a) and (3.46b) by multiplying (3.46a) and (3.46b) by $\varrho(\psi)\cos\theta$ and $\varrho(\psi)$, respectively, and cross-differentiating; this leads us to the vorticity equation

$$\left[\frac{\partial\psi}{\partial\varphi}\frac{\partial}{\partial\theta}-\frac{\partial\psi}{\partial\theta}\frac{\partial}{\partial\varphi}\right]\left(\Delta_{\mathbb{S}^2}\psi+2\omega\sqrt{\varrho(\psi)}\sin\theta+\frac{\omega^2}{2}\varrho'(\psi)\sin^2\theta\right)=0,$$

where

$$\Delta_{\mathbb{S}^2}f=\frac{1}{\cos\theta}\left[\frac{1}{\cos\theta}\frac{\partial^2 f}{\partial\varphi^2}+\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial f}{\partial\theta}\right)\right]$$

is the Laplace-Beltrami operator on the surface of the unit sphere $\mathbb{S}^2$. Finally, throughout regions where $\nabla_{\mathbb{S}^2}\psi\neq 0$, the rank theorem [1] yields

$$\Delta_{\mathbb{S}^2}\psi=F(\psi)-2\omega\sqrt{\varrho(\psi)}\sin\theta-\frac{\omega^2}{2}\varrho'(\psi)\sin^2\theta, \tag{3.47}$$

where F is an arbitrary smooth function. Clearly, if ϱ is a constant (and equal to 1, in view of the scaling (3.4)), (3.47) reduces to (3.15). In what follows, we will investigate the case of a particular choice of F, namely

$$F(\psi)=a\,\mathrm{e}^{b\psi}+c, \tag{3.48}$$

where a,b,c are real constants; in this case, explicit solutions (in the form of Stuart-type vortices) can be constructed in the case of constant density ϱ (see [44]). In general, even for constant density, explicit solutions of this elliptic problem are remarkably rare; see the discussion in [27].

3.3.1 Gyres in the Northern Hemisphere

Since we are interested in the description of polar arctic gyres, located in a region Ω in the Northern Hemisphere delimited by a closed streamline, it is convenient to employ a modified stereographic projection, where the centre of projection is situated at the South Pole instead of at the North Pole; see Figure 3.4. The coordinate transformation reads

$$\xi = r\,\mathrm{e}^{\mathrm{i}\varphi} \quad \text{with} \quad r = \frac{\cos\theta}{1+\sin\theta}, \tag{3.49}$$

where $(r,\varphi) \in [0,\infty) \times [-\pi,\pi)$ are the polar coordinates in the equatorial plane. Thus, we have

$$\sin\theta = \frac{1-\xi\bar{\xi}}{\xi\bar{\xi}+1}, \quad \cos\theta = \frac{2\sqrt{\xi\bar{\xi}}}{\xi\bar{\xi}+1}, \quad \frac{\partial}{\partial\theta} = -\frac{\xi}{\cos\theta}\frac{\partial}{\partial\xi} - \frac{\bar{\xi}}{\cos\theta}\frac{\partial}{\partial\bar{\xi}}, \quad \frac{\partial}{\partial\varphi} = \mathrm{i}\xi\frac{\partial}{\partial\xi} - \mathrm{i}\bar{\xi}\frac{\partial}{\partial\bar{\xi}}.$$

Some manipulations of (3.47) yield the equation

$$\frac{\partial^2\psi}{\partial\xi\partial\bar{\xi}} + 2\omega\sqrt{\varrho(\psi)}\frac{1-\xi\bar{\xi}}{(1+\xi\bar{\xi})^3} + \frac{\omega^2}{2}\varrho'(\psi)\frac{(1-\xi\bar{\xi})^2}{(1+\xi\bar{\xi})^4} - \frac{F(\psi)}{(1+\xi\bar{\xi})^2} = 0. \tag{3.50}$$

Then, setting

$$\psi(\xi,\bar{\xi}) = \zeta(\xi,\bar{\xi}) + A\log(1+\xi\bar{\xi}),$$

where $A \in \mathbb{R}$ is to be determined, and noting that

$$\frac{\partial^2\psi}{\partial\xi\partial\bar{\xi}} = \frac{\partial^2\zeta}{\partial\xi\partial\bar{\xi}} + \frac{A}{(1+\xi\bar{\xi})^2} \quad \text{and} \quad \mathrm{e}^{b\psi} = (1+\xi\bar{\xi})^{Ab}\,\mathrm{e}^{b\zeta},$$

by choosing

$$A = \frac{2}{b} \quad \text{and} \quad c = \frac{2}{b} + C\omega,$$

where $C \in \mathbb{R}\setminus\{0\}$ is an arbitrary constant, equation (3.50) transforms to

$$\frac{\partial^2\zeta}{\partial\xi\partial\bar{\xi}} = a\,\mathrm{e}^{b\zeta} - 2\omega\sqrt{\varrho(\zeta+g)}\frac{1-\xi\bar{\xi}}{(1+\xi\bar{\xi})^3} - \frac{\omega^2}{2}\varrho'(\zeta+g)\frac{(1-\xi\bar{\xi})^2}{(1+\xi\bar{\xi})^4} + \frac{C\omega}{(1+\xi\bar{\xi})^2}, \tag{3.51}$$

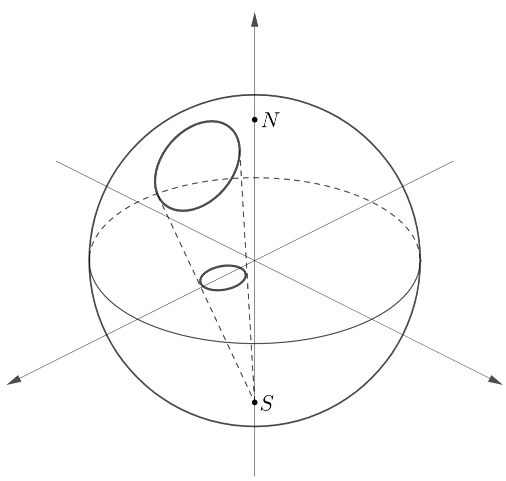

Figure 3.4: The stereographic projection centred at the South Pole, defined in (3.49).

where

$$g(\xi,\bar{\xi}) \coloneqq \frac{2}{b}\log(1+\xi\bar{\xi}).$$

If we set $\omega = 0$ in this equation, we thus obtain the Liouville equation

$$\frac{\partial^2\zeta}{\partial\xi\partial\bar{\xi}} = a\,\mathrm{e}^{b\zeta}, \tag{3.52}$$

which is exactly solvable: its general solution is namely given by

$$\zeta_0(\xi,\bar{\xi}) = \frac{1}{b}\log\left(\frac{4|f'(\xi)|^2}{(2-ab|f(\xi)|^2)^2}\right), \tag{3.53}$$

where f is a meromorphic function with $f' \neq 0$, $|f| \neq \sqrt{2}/\sqrt{ab}$ and having at most isolated singularities in the domain in which the equation is to be solved (see [44]). For a particular choice of f, this yields a solution ζ_0 to (3.51) for $\omega = 0$ and thus a stream function

$$\psi_0(\xi,\bar{\xi}) = \zeta_0(\xi,\bar{\xi}) + \frac{2}{b}\log(1+\xi\bar{\xi}) \tag{3.54}$$

that solves (3.50) for F as in (3.48) and $\omega = 0$.

Example 3.3.1. Let us consider a particular choice of f that should exemplify the link to the description of arctic gyres. For simplicity, we set $b = 1$. Let

$$f(\xi) = \xi^2 + 1.$$

Working in Cartesian coordinates $(x,y) = (r\cos\varphi, r\sin\varphi) = (\frac{\xi+\bar{\xi}}{2}, \frac{\xi-\bar{\xi}}{2\mathrm{i}})$ in the projection plane, it is easy to see that the streamline corresponding to $\psi = c$ for a real constant c (that is, the level set $\psi(\xi,\bar{\xi}) = c$) corresponds to the set of all points $(x,y) \in \mathbb{R}^2$ that satisfy the equation

$$16(x^2+y^2)(1+x^2+y^2)^2 = k[2 - a((x^2+y^2)^2 + 2(x^2-y^2)+1))]^2,$$

where $k = \mathrm{e}^c$; for k small enough, we obtain a closed streamline around the North Pole (see the left panel of Figure 3.5).

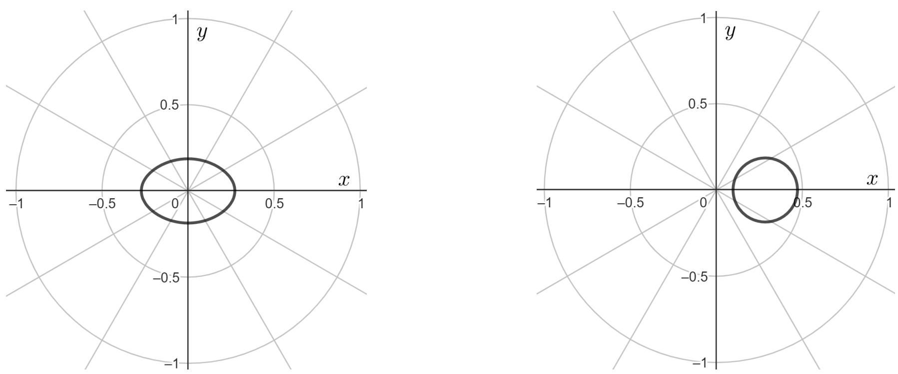

FIGURE 3.5: The streamlines from Example 3.3.1 (left panel) and Example 3.3.2 (right panel) in the Cartesian projection plane for $a = 4$, $b = 1$, $k = 0.2$ and $p = 0.3$. The origin corresponds to the North Pole; the circle of radius 1 corresponds to the Equator.

Example 3.3.2. As a second example, let us take $b = 1$ as in the previous example and

$$f(\xi) = (\xi - p)^2,$$

where $p \in (0, 1)$. Then streamlines $\psi = c$ correspond to points in the Cartesian projection plane whose coordinates (x, y) satisfy the equation

$$16((x-p)^2 + y^2)(1 + x^2 + y^2)^2 = k[2 - a((x-p)^2 + y^2)^2]^2,$$

where $k = \mathrm{e}^c$; for k small enough, this yields a closed streamline in the Northern Hemisphere below the North Pole (see the right panel of Figure 3.5).

We now turn our attention to the situation where $\omega > 0$ in (3.51). Given a solution ζ_0 of (3.52) in a bounded region Ω of the sphere with smooth boundary in the Northern Hemisphere (such as, for instance, in Example 3.3.1 and 3.3.2), we will denote by O the projection of Ω onto the complex ξ-plane under the modified stereographic projection introduced in the previous section. Furthermore, for given $\omega > 0$, let ζ solve (3.51) with boundary condition

$$\zeta = \zeta_0 \quad \text{on } \partial O.$$

Our aim is to show that such a solution ζ exists and that ζ_0 is an approximation of ζ in O in a certain sense. In the sequel, we will make use of the following lemma (for the proof, we refer to [42]).

Lemma 3.3.3. *Let $O \subset \mathbb{R}^2$ be a bounded open subset with smooth boundary, and let $F_0 \in C(\overline{O})$. Then the Dirichlet problem*

$$\begin{aligned} \Delta U_0 &= F_0 \quad \text{in } O, \\ U_0 &= 0 \quad \text{on } \partial O \end{aligned}$$

has a unique solution $U_0 \in C^2(O) \cap C(\overline{O})$. Moreover, if $F_0 < 0$ or $F_0 > 0$ throughout O, then we have $U_0 > 0$ respectively $U_0 < 0$ throughout O, and the estimate

$$|U_0| \le \frac{1}{16} M D^2$$

holds in all of O, where D is the diameter of O and

$$0 \le M = \max_{(x,y) \in \overline{O}} |F_0(x, y)|.$$

Also, the following technical result about the stereographic projection will be needed later on.

Lemma 3.3.4. *Suppose that $\mathcal{C}$ is a (closed) spherical cap contained in the Northern Hemisphere. Then, the projection of $\mathcal{C}$ onto the (Cartesian) projection plane is a circle whose diameter is given as follows:*

(a) If $\mathcal{C}$ does not contain the North Pole, then the diameter of its projection is

$$D_{\mathcal{C}} = \frac{2 \sin\left(\frac{\theta_N - \theta_S}{2}\right)}{\cos\left(\frac{\theta_N - \theta_S}{2}\right) + \sin\left(\frac{\theta_N + \theta_S}{2}\right)}, \tag{3.55}$$

where θ_N and θ_S are the latitude of the northern and the southern tip of $\mathcal{C}$, respectively.

(b) If $\mathcal{C}$ contains the North Pole, then let θ_N and θ_S be the latitudes of the northernmost and southernmost tip of the edge of $\mathcal{C}$, respectively (unless $\mathcal{C}$ is centred at the North Pole, there are always exactly two such points of different latitudes, in which case we set $\theta_N > \theta_S$). Then the diameter of the projection of $\mathcal{C}$ is

$$D_{\mathcal{C}} = \frac{2\cos\left(\frac{\theta_N+\theta_S}{2}\right)}{\cos\left(\frac{\theta_N-\theta_S}{2}\right) + \sin\left(\frac{\theta_N+\theta_S}{2}\right)}. \tag{3.56}$$

Proof. It is a well-known fact (see, for instance, [52]) that the stereographic projection maps a circle on the sphere that does not pass through the centre of projection onto circles of the plane, whereas circles that go through the centre of projection are mapped onto straight lines.

Let us consider case (a). We can determine the diameter of the projected circle by computing the distance D between the projection of the points with spherical coordinates $(\hat{\varphi}, \theta_N)$ and $(\hat{\varphi}, \theta_S)$, where $\hat{\varphi}$ is the longitude of the meridian about which $\mathcal{C}$ is symmetric. We thus find

$$\begin{aligned} D^2 &= \left(\frac{\cos\theta_N}{1+\sin\theta_N}\cos\hat{\varphi} - \frac{\cos\theta_S}{1+\sin\theta_S}\cos\hat{\varphi}\right)^2 + \left(\frac{\cos\theta_N}{1+\sin\theta_N}\sin\hat{\varphi} - \frac{\cos\theta_S}{1+\sin\theta_S}\sin\hat{\varphi}\right)^2 \\ &= \left(\frac{\cos\theta_N}{1+\sin\theta_N} - \frac{\cos\theta_S}{1+\sin\theta_S}\right)^2 = \frac{4\sin^2\left(\frac{\theta_N-\theta_S}{2}\right)}{\left[\cos\left(\frac{\theta_N-\theta_S}{2}\right) + \sin\left(\frac{\theta_N+\theta_S}{2}\right)\right]^2}; \end{aligned}$$

since $\theta_N, \theta_S \in [0, \frac{\pi}{2})$, we may take the square root on both sides to obtain (3.55).

The expression (3.56) in case (b) follows directly from (3.55) and the identity

$$\frac{\sin\left(\frac{\pi}{4}-\alpha\right)}{\cos\left(\frac{\pi}{4}-\alpha\right)+\sin\left(\frac{\pi}{4}+\alpha\right)} + \frac{\sin\left(\frac{\pi}{4}-\beta\right)}{\cos\left(\frac{\pi}{4}-\beta\right)+\sin\left(\frac{\pi}{4}+\beta\right)} = \frac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)+\sin(\alpha+\beta)}$$

for any $\alpha, \beta \in \mathbb{R}$, which is easily verified using standard trigonometric identities. □

We first formulate and prove our main result for $C > 0$. It will be convenient to work in Cartesian coordinates $(x, y) = (r\cos\varphi, r\sin\varphi)$ in the projection plane.

Theorem 3.3.5. *Let ψ_0 be a solution of (3.47) for $\omega = 0$ in a bounded region Ω with smooth boundary that can be enclosed in a closed spherical cap $\mathcal{C}$ contained in the Northern Hemisphere. Let O be the image of Ω in the Cartesian projection plane under the stereographic projection defined in (3.49). Suppose that we have $C > 0$ in (3.51) and $ab > 0$ in (3.48). Furthermore, assume that there exist constants $m_1, m_2, M_1, M_2 > 0$ with*

$$-m_1 \le \varrho' \le M_1 \quad \text{and} \quad m_2 \le \varrho \le M_2 \tag{3.57}$$

everywhere and such that the inequalities

$$C \ge \frac{(1+T^2)(1-t^2)}{2(1+t^2)^3}\left[4\sqrt{M_2} + \omega M_1 \frac{1-t^2}{1+t^2}\right] \tag{3.58}$$

and

$$m_1 \le \frac{4}{\omega}\frac{(1+t^2)^4}{(1-t^2)^2}\frac{1-T^2}{(1+T^2)^3}\sqrt{m_2} \tag{3.59}$$

hold, where

$$T = \max_{(x,y)\in\overline{O}} \sqrt{x^2+y^2} \quad \text{and} \quad t = \min_{(x,y)\in\overline{O}} \sqrt{x^2+y^2}.$$

Then, there exists a solution ψ of (3.47) for $\omega > 0$ in Ω with

$$\psi = \psi_0 \quad \text{on } \partial\Omega$$

such that

$$0 \le \psi_0 - \psi \le \frac{C D_{\mathcal{C}}^2 \omega}{16}(1+\sin\vartheta)^2, \tag{3.60}$$

where $D_{\mathcal{C}}$ is as in Lemma 3.3.4 and ϑ is the latitude of the northernmost point of Ω.

Remark 3.3.6. (i) Inspecting the proof of Theorem (2.5.4), one sees that the condition (3.57) is stronger than needed: in fact, it suffices to assume

$$\varrho'(\zeta_0 - U_0 + g) \ge -m_1, \quad \varrho'(\zeta_0 + g) \le M_1, \quad \varrho(\zeta_0 - U_0 + g) \ge m_2, \quad \varrho(\zeta_0 + g) \le M_2$$

throughout O, where U_0 is as in Lemma 3.3.3 for

$$F_0(x,y) = -\frac{4C\omega}{(1+x^2+y^2)^2} < 0, \quad (x,y) \in O.$$

(ii) Let us briefly comment on the assumptions (3.58) and (3.59). First of all, (3.58) can always be satisfied by choosing $C > 0$ large enough; indeed, this gives a lower bound on C which will appear in (3.60). For (3.59), since we expect m_2 and M_2 to be on the order of about 1, it follows from (3.59) that m_1 has to be small; this is physically reasonable, since variations in the density of ocean water are typically very small. See the discussion in Section 3.3.3 below for more information on this aspect.

Proof of Theorem 3.3.5. Let us define the difference

$$\gamma := \zeta_0 - \zeta = \psi_0 - \psi;$$

then, noting that $e^{b\zeta_0} - e^{b\zeta} = e^{b\zeta_0}(1 - e^{-b\gamma})$, it follows that γ solves the equation

$$\frac{\partial^2\gamma}{\partial\xi\partial\bar\xi} = a\, e^{b\zeta_0}(1-e^{-b\gamma}) + 2\omega\sqrt{\varrho(\zeta_0-\gamma+g)}\frac{1-\xi\bar\xi}{(1+\xi\bar\xi)^3} + \frac{\omega^2}{2}\varrho'(\zeta_0-\gamma+g)\frac{(1-\xi\bar\xi)^2}{(1+\xi\bar\xi)^4} - \frac{C\omega}{(1+\xi\bar\xi)^2}$$

with homogeneous Dirichlet boundary conditions

$$\gamma = 0 \quad \text{on } \partial O.$$

In Cartesian coordinates $(x,y) = (r\cos\varphi, r\sin\varphi)$ in the projected plane, this produces the elliptic equation

$$\begin{aligned} -\Delta_{(x,y)}\gamma + 4a\, e^{b\zeta_0}(1-e^{-b\gamma}) + 8\omega\sqrt{\varrho(\zeta_0-\gamma+g)}\frac{1-x^2-y^2}{(1+x^2+y^2)^3} \\ + 2\omega^2\varrho'(\zeta_0-\gamma+g)\frac{(1-x^2-y^2)^2}{(1+x^2+y^2)^4} - \frac{4C\omega}{(1+x^2+y^2)^2} = 0. \end{aligned} \tag{3.61}$$

Now, functions $\gamma_*, \gamma^* \in C^2(O)\cap C(\overline{O})$ that vanish on ∂O are said to be a subsolution and a supersolution if

$$\begin{aligned} -\Delta_{(x,y)}\gamma_* + 4a\, e^{b\zeta_0}(1-e^{-b\gamma_*}) + 8\omega\sqrt{\varrho(\zeta_0-\gamma_*+g)}\frac{1-x^2-y^2}{(1+x^2+y^2)^3} \\ + 2\omega^2\varrho'(\zeta_0-\gamma_*+g)\frac{(1-x^2-y^2)^2}{(1+x^2+y^2)^4} - \frac{4C\omega}{(1+x^2+y^2)^2} \le 0 \quad \text{in } O \end{aligned}$$

and

$$-\Delta_{(x,y)}\gamma^* + 4a\,\mathrm{e}^{b\zeta_0}(1-\mathrm{e}^{-b\gamma^*}) + 8\omega\sqrt{\varrho(\zeta_0-\gamma^*+g)}\frac{1-x^2-y^2}{(1+x^2+y^2)^3}$$
$$+ 2\omega^2\varrho'(\zeta_0-\gamma^*+g)\frac{(1-x^2-y^2)^2}{(1+x^2+y^2)^4} - \frac{4C\omega}{(1+x^2+y^2)^2} \geq 0 \quad \text{in } O$$

respectively. Because the nonlinearity in (3.61) is smooth, the existence of a subsolution and a supersolution would imply the existence of a solution $\gamma \in C^2(O) \cap C(\overline{O})$ of (3.61) with

$$\gamma_* \leq \gamma \leq \gamma^* \quad \text{in } O;$$

see [53]. Now, let U_0 be as in Lemma 3.3.3 for

$$F_0(x,y) = -\frac{4C\omega}{(1+x^2+y^2)^2} < 0, \quad (x,y) \in O;$$

then U_0 is positive in O. Since $ab > 0$, with the assumptions (3.57)–(3.59) it follows that $\gamma_* = 0$ is a subsolution and $\gamma^* = U_0$ is a supersolution. Indeed, we have

$$8\omega\sqrt{\varrho(\zeta_0+g)}\frac{1-x^2-y^2}{(1+x^2+y^2)^3} + 2\omega^2\varrho'(\zeta_0+g)\frac{(1-x^2-y^2)^2}{(1+x^2+y^2)^4} - \frac{4C\omega}{(1+x^2+y^2)^2}$$
$$\leq 8\omega\sqrt{M_2}\frac{1-t^2}{(1+t^2)^3} + 2\omega^2 M_1\frac{(1-t^2)^2}{(1+t^2)^4} - \frac{4C\omega}{(1+T^2)^2} \leq 0$$

and

$$4a\,\mathrm{e}^{b\zeta_0}(1-\mathrm{e}^{-b\gamma^*}) + 8\omega\sqrt{\varrho(\zeta_0-\gamma^*+g)}\frac{1-x^2-y^2}{(1+x^2+y^2)^3} + 2\omega^2\varrho'(\zeta-\gamma^*+g)\frac{(1-x^2-y^2)^2}{(1+x^2+y^2)^4}$$
$$\geq 8\omega\sqrt{m_2}\frac{1-T^2}{(1+T^2)^3} - 2\omega^2 m_1\frac{(1-t^2)^2}{(1+t^2)^4} \geq 0.$$

Thus, the method of sub- and supersolutions yields the existence of a solution $\gamma \in C^2(O) \cap C(\overline{O})$ of (3.61) with

$$0 \leq \gamma \leq U_0 \leq \frac{1}{16}MD^2 \quad \text{in } O,$$

where, as in Lemma 3.3.3, D is the diameter of O and

$$0 \leq M = \max_{(x,y)\in\overline{O}}[-F_0(x,y)].$$

Now, let $\mathcal{C}$ be the smallest closed spherical cap that contains Ω; then O will be contained in the stereographic projection of $\mathcal{C}$, therefore D will be bounded above by the right-hand side of either (3.55) or (3.56), depending on whether $\mathcal{C}$ does or does not contain the North Pole. Finally, because of the straightforward estimate

$$M \leq \frac{4C\omega}{(1+t^2)^2},$$

we obtain

$$0 \leq \psi_0 - \psi \leq \frac{C\omega}{4} \cdot \frac{D_{\mathcal{C}}^2}{(1+t^2)^2}.$$

The claim now follows from (3.49). □

A similar result holds in the case $C < 0$, although we have to modify the assumptions on ϱ. In particular, we must require that ϱ' be negative.

Theorem 3.3.7. *Let ψ_0 be a solution of (3.47) for $\omega = 0$ in a bounded region Ω with a smooth boundary enclosed in a closed spherical cap $\mathcal{C}$ contained in the Northern Hemisphere. Let O be the image of Ω in the Cartesian projection plane under the stereographic projection defined in (3.49). Suppose that we have $C < 0$ in (3.51) and $ab > 0$ in (3.48). Furthermore, assume that there exist constants $m_1, m_2, M_1, M_2 > 0$ with*

$$-m_1 \leq \varrho' \leq -M_1 \quad \textit{and} \quad m_2 \leq \varrho \leq M_2$$

everywhere and such that the inequalities

$$m_1 \leq \frac{2}{\omega} \frac{(1+t^2)^4}{(1-t^2)^2(1+T^2)^2} \left(2\,\frac{1-T^2}{1+T^2}\sqrt{m_2} - C \right)$$

and

$$M_1 \geq \frac{4}{\omega} \frac{(1+T^2)^4}{(1-T^2)^2} \frac{1-t^2}{(1+t^2)^3} \sqrt{M_2}$$

hold, where T and t are as in Theorem 3.3.5. Then, there exists a solution ψ of (3.47) for $\omega > 0$ in Ω with

$$\psi = \psi_0 \quad \textit{on } \partial\Omega$$

such that

$$\frac{C D_{\mathcal{C}}^2 \omega}{16} (1+\sin\vartheta)^2 \leq \psi_0 - \psi \leq 0,$$

where $D_{\mathcal{C}}$ is as in Lemma 3.3.4 and ϑ is the latitude of the northernmost point of Ω.

Proof. The proof is analogous to that of Theorem 3.3.5, the only difference being that now (using the same notation as in the previous proof) $F_0 > 0$ in O, so that $U_0 < 0$ in O and thus it follows, with the modified assumptions on ϱ, that now a supersolution is given by $\gamma^* = 0$, whereas $\gamma_* = U_0$ is a subsolution. The remaining estimates then follow as before. Note that, similar to Theorem 3.3.5, it is in fact enough to require

$$\varrho'(\zeta_0 + g) \geq -m_1, \quad \varrho'(\zeta_0 - U_0 + g) \leq -M_1, \quad \varrho(\zeta_0 + g) \geq m_2, \quad \varrho(\zeta_0 - U_0 + g) \leq M_2$$

throughout O. □

3.3.2 Gyres in the Southern Hemisphere

When considering regions that are confined to the Southern Hemisphere, it is convenient to employ the standard stereographic projection, that is, to place the centre of projection at the North Pole. The coordinate transformation is then given by

$$\xi = r\,\mathrm{e}^{\mathrm{i}\varphi} \quad \text{with} \quad r = \frac{\cos\theta}{1-\sin\theta}, \tag{3.62}$$

where $(r, \varphi) \in [0,\infty) \times [-\pi, \pi)$ are the polar coordinates in the equatorial plane. One then easily computes

$$\sin\theta = \frac{\xi\bar{\xi} - 1}{\xi\bar{\xi} + 1}, \quad \cos\theta = \frac{2\sqrt{\xi\bar{\xi}}}{\xi\bar{\xi} + 1}, \quad \frac{\partial}{\partial\theta} = \frac{\xi}{\cos\theta}\frac{\partial}{\partial\xi} + \frac{\bar{\xi}}{\cos\theta}\frac{\partial}{\partial\bar{\xi}}, \quad \frac{\partial}{\partial\varphi} = \mathrm{i}\xi\frac{\partial}{\partial\xi} - \mathrm{i}\bar{\xi}\frac{\partial}{\partial\bar{\xi}},$$

and (3.47) becomes

$$\frac{\partial^2\psi}{\partial\xi\partial\bar{\xi}} - 2\omega\sqrt{\varrho(\psi)}\frac{1-\xi\bar{\xi}}{(1+\xi\bar{\xi})^3} + \frac{\omega^2}{2}\varrho'(\psi)\frac{(1-\xi\bar{\xi})^2}{(1+\xi\bar{\xi})^4} - \frac{F(\psi)}{(1+\xi\bar{\xi})^2} = 0. \tag{3.63}$$

Setting as before

$$\psi(\xi,\bar{\xi}) = \zeta(\xi,\bar{\xi}) + g(\xi,\bar{\xi}) \quad \text{and} \quad c = \frac{2}{b} + C\omega,$$

where $C \in \mathbb{R}\setminus\{0\}$ and $g(\xi,\bar{\xi}) = \frac{2}{b}\log(1+\xi\bar{\xi})$, equation (3.63) transforms to

$$\frac{\partial^2\zeta}{\partial\xi\partial\bar{\xi}} = a\,\mathrm{e}^{b\zeta} + 2\omega\sqrt{\varrho(\zeta+g)}\frac{1-\xi\bar{\xi}}{(1+\xi\bar{\xi})^3} - \frac{\omega^2}{2}\varrho'(\zeta+g)\frac{(1-\xi\bar{\xi})^2}{(1+\xi\bar{\xi})^4} + \frac{C\omega}{(1+\xi\bar{\xi})^2}. \tag{3.64}$$

As mentioned above, the general solution of (3.64) for $\omega = 0$, that is,

$$\frac{\partial^2\zeta}{\partial\xi\partial\bar{\xi}} = a\,\mathrm{e}^{b\zeta},$$

is given by (3.53).

For later reference, let us formulate a version of Lemma 3.3.4 in this new setting. The proof is entirely analogous to that of Lemma 3.3.4.

Lemma 3.3.8. *Suppose that $\mathcal{C}$ is a (closed) spherical cap contained in the Southern Hemisphere. Then the projection of $\mathcal{C}$ onto the (Cartesian) projection plane is a circle whose diameter is given as follows:*

(a) If $\mathcal{C}$ does not contain the South Pole, then the diameter of its projection is

$$D_{\mathcal{C}} = \frac{2\sin\left(\frac{\theta_N-\theta_S}{2}\right)}{\cos\left(\frac{\theta_N-\theta_S}{2}\right) - \sin\left(\frac{\theta_N+\theta_S}{2}\right)}, \tag{3.65}$$

where θ_N and θ_S are the latitude of the northern and the southern tip of $\mathcal{C}$, respectively.

(b) If $\mathcal{C}$ contains the North Pole, then let θ_N and θ_S the latitudes of the northernmost and southernmost tip of the edge of $\mathcal{C}$, respectively (unless $\mathcal{C}$ is centred at the North Pole, there are always exactly two such points, of different latitudes, in which case we set $\theta_N > \theta_S$). Then, the diameter of the projection of $\mathcal{C}$ is

$$D_{\mathcal{C}} = \frac{2\cos\left(\frac{\theta_N+\theta_S}{2}\right)}{\cos\left(\frac{\theta_N-\theta_S}{2}\right) - \sin\left(\frac{\theta_N+\theta_S}{2}\right)}. \tag{3.66}$$

Let us now prove analogous results to those for the Northern Hemisphere.

Theorem 3.3.9. *Let ψ_0 be a solution of (3.47) for $\omega = 0$ in a bounded region Ω with smooth boundary which is enclosed in a closed spherical cap $\mathcal{C}$ that is contained in the Southern Hemisphere. Let O be the image of Ω in the Cartesian projection plane under the stereographic projection defined in (3.62). Suppose that we have $ab > 0$ in (3.48). Furthermore, assume that there exist constants $m_1, m_2, M_1, M_2 > 0$ with the following properties (where T and t are as in Theorem 3.3.5):*

- *If $C > 0$, then*
$$m_1 \le \varrho' \le M_1 \quad \text{and} \quad m_2 \le \varrho \le M_2$$
everywhere, and the inequalities
$$M_1 \le \frac{2}{\omega} \frac{(1+t^2)^4}{(1-t^2)^2(1+T^2)^2} \left(2\frac{1-T^2}{1+T^2}\sqrt{m_2} + C \right)$$
and
$$m_1 \ge \frac{4}{\omega} \frac{(1+T^2)^4}{(1-T^2)^2} \frac{1-t^2}{(1+t^2)^3} \sqrt{M_2}$$
are satisfied.
- *If $C < 0$, then*
$$-m_1 \le \varrho' \le M_1 \quad \text{and} \quad m_2 \le \varrho \le M_2$$
everywhere, with the inequalities
$$-C \ge \frac{(1+T^2)^2}{2} \frac{1-t^2}{(1+t^2)^3} \left(4\sqrt{M_2} + \omega m_1 \frac{1-t^2}{1+t^2} \right)$$
and
$$M_1 \le \frac{4}{\omega} \frac{(1+t^2)^4}{(1-t^2)^2} \frac{1-T^2}{(1+T^2)^3} \sqrt{m_2}.$$

Then, there exists a solution ψ of (3.47) for $\omega > 0$ in Ω with
$$\psi = \psi_0 \quad \text{on } \partial\Omega$$
such that
$$0 \le \operatorname{sign}(C)(\psi_0 - \psi) \le \frac{|C| D_C^2 \omega}{16} (1 - \sin\vartheta)^2, \tag{3.67}$$
where D_C is as in Lemma 3.3.8 and ϑ is the latitude of the southernmost point of Ω.

Proof. In Cartesian coordinates $(x, y) = (r\cos\varphi, r\sin\varphi)$, equation (3.64) becomes
$$-\Delta(x,y)\gamma + 4a\,\mathrm{e}^{b\zeta_0}(1 - \mathrm{e}^{-b\gamma}) - 8\omega\sqrt{\varrho(\zeta_0 - \gamma + g)}\frac{1-x^2-y^2}{(1+x^2+y^2)^3}$$
$$+ 2\omega^2\varrho'(\zeta_0 - \gamma + g)\frac{(1-x^2-y^2)^2}{(1+x^2+y^2)^4} - \frac{4C\omega}{(1+x^2+y^2)^2} = 0;$$
from here we proceed analogously to the proofs of Theorems 3.3.5 and 3.3.7. □

Remark 3.3.10. Similarly to Theorem 3.3.5 and Theorem 3.3.7, the conditions on the density can be made sharper. If $C > 0$ it is enough to assume
$$\varrho'(\zeta_0 - U_0 + g) \ge m_1, \quad \varrho'(\zeta_0 + g) \le M_1, \quad \varrho(\zeta_0 + g) \ge m_2, \quad \varrho(\zeta_0 - U_0 + g) \le M_2,$$
whereas if $C < 0$ the requirements
$$\varrho'(\zeta_0 + g) \ge -m_1, \quad \varrho'(\zeta_0 - U_0 + g) \le M_1, \quad \varrho(\zeta_0 - U_0 + g) \ge m_2, \quad \varrho(\zeta_0 + g) \le M_2$$
suffice. Here, as before, U_0 is the function from Lemma 3.3.3 for
$$F_0(x,y) = -\frac{4C\omega}{(1+x^2+y^2)^2}, \quad (x,y) \in O.$$

3.3.3 Some explicit values

For completeness, we have discussed the cases $C > 0$ and $C < 0$ for gyre regions in both the Northern and the Southern Hemisphere; however, it is clear from the assumptions on the density that the special case of constant density can be included only by taking $C > 0$ in the Northern Hemisphere and $C < 0$ in the Southern Hemisphere. As a concrete numerical example for the bound given in Theorem 3.3.5, let us look at the Ierapetra gyre (a small gyre in the Mediterranean that appears early in late summer—see [10]), for which $\theta_N = \vartheta \approx 34.5°$ and $\theta_S \approx 33.5°$. The condition (3.58) gives a lower bound for the constant C that appears in (3.60); clearly, the larger M_1 is, the larger C must be as well. If the density is constant, then we can take $M_1 = 0$, so that C must be no smaller than approximately 1.2 and the bound (3.60) is about 0.1. If the density is non-constant but very slowly varying, say $M_1 = 0.01$, then we must choose $C \approx 8.8$ and the bound on $\psi_0 - \psi$ becomes approximately 0.8. Taking $M_1 = 0.1$ results in a bound of about 6.9. Note that condition (3.59) implies that m_1 cannot exceed 0.0014. In addition, the extension of the gyre also plays a role in the smallness of the bound (owing to the form of (3.55)). For example, for the Beaufort Gyre (for which $\theta_N = \vartheta \approx 84°$ and $\theta_S \approx 76°$) we find that already for $M_1 = 0.001$ the bound in (3.60) exceeds 22. In summary, the bound (3.60) is only relevant as long as density variations are small and the gyre is not too large. Clearly, the same applies for the bound (3.67) (for $C < 0$).

4. The ‘morning glory’ atmospheric pattern

In the final chapter of this thesis, we leave the ocean and turn our attention to a particular example of an atmospheric flow: the ‘morning glory’ cloud pattern. First, in Section 4.1, we briefly discuss the formation mechanism and sketch the derivation of an asymptotic model for its description. In Section 4.2, we provide a brief overview of some technical concepts and results which are needed in the subsequent sections. In Section 4.3, we prove the local well-posedness of the model for the morning glory pattern, and, in Section 4.4, we construct global weak solutions of this model. The results of Section 4.3 and Section 4.4 are published in [84]. Finally, in Section 4.5 (whose results are the subject of the article [85]), global well-posedness is established for the homogeneous problem under the assumption that the initial datum is sufficiently small (in H^1).

4.1 Formation mechanism and modelling

The ‘morning glory’ cloud pattern is a spectacular atmospheric phenomenon that can occasionally be observed in coastal regions of particular shape, especially in Australia in the Gulf of Carpentaria, but also over the English Channel, Germany, Russia, Canada, Mexico, Brazil, and more [18, 109]; see Figure 4.1. It consists of a train of elongated, thin tubular clouds that propagate for hundreds of kilometres in the horizontal direction perpendicular to the cloud line while appearing to be rolling, and whose longitudinal length can be greater than 1000 km; occasionally they can appear as a single roll cloud, but they are more often observed in wave trains of transversal width rarely exceeding 30 km [14, 94]. The motion is thus essentially two-dimensional.

The mechanism driving the formation of such roll clouds can be summarised as follows (cf. Figure 4.2). The setup consists of a cold, shallow breeze incoming from the sea and a warmer, deeper breeze coming in opposite directions from the inland. When the two breezes collide head-on, they produce a thermal inversion layer: the warmer and less dense breeze is lifted on top of the colder, denser breeze, thereby remaining sandwiched between the layer of colder air below and the much colder air above it—in the troposphere, temperature decreases monotonically with height. As the warmer breeze is pushed on top of the colder air, disturbances form at the top of the thermal inversion layer and propagate in the direction of the originating warmer breeze; the cooling updraft at the front of each wave causes condensation of the ascending moist air, resulting in the formation of clouds. In other words, strictly speaking, it is not the clouds themselves that are propagating, but rather the oscillatory pattern induced by the collision of the two breezes: each cloud roll is constantly being formed at the front of each disturbance and being eroded at its back,

Figure 4.1: (Left panel) Morning glory cloud over Goondiwindi on 5 July 2017. Image credit: https://www.couriermail.com.au/news/queensland/warwick/morning-glory-clouds-go-viral-as-world-looks-to-our-skies/news-story/2d5950ae8e55c6ffa090cfb985396111. (Right panel) Morning glory cloud over the Gulf of Carpentaria on 10 August 2009. Image credit: Mike Petroff. CC BY-SA 4.0 (https://creativecommons.org/licenses/by-sa/4.0).

and this makes it appear as if the cloud is rolling about its longitudinal axis; see [13] for more details.

Many attempts have been made to model this phenomenon, such as, for instance, in [22, 61, 71, 88]; however, these are all *ad hoc* models, designed to fit the qualitative properties of the flow, but not derived consistently from first principles. The first consistent derivation of an asymptotic model from the general governing equations was proposed in [38]. Here, we will briefly highlight (largely in words) the key ingredients of the derivation of this model, without delving into the full details, with the sole purpose of giving an overview of the background of the equation that has been studied in the articles [84] and [85], whose findings are discussed in Sections 4.3, 4.4 and 4.5; for the full derivation, whose calculations are long and tedious, we refer the reader to [38].

The derivation begins with the following set of general governing equations:

- the Navier–Stokes equations in rotating spherical coordinates,
- the equation of mass conservation,
- an equation of state for the air, and
- the first law of thermodynamics.

These are nothing other than the equations (1.11), (1.15), (1.3), and (1.4), with the only difference that we will write all variables and constants with a prime to denote that they

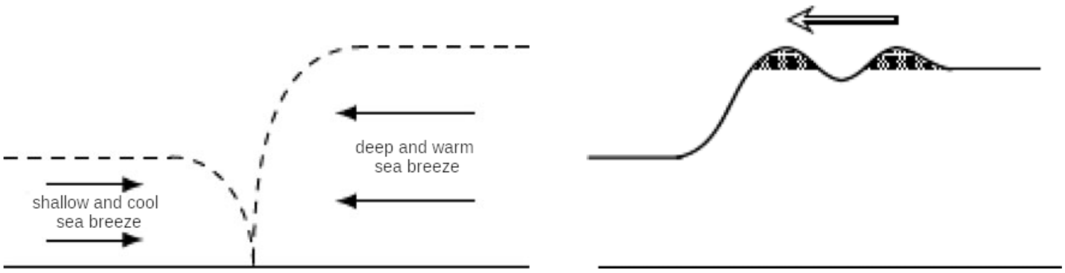

Figure 4.2: The formation mechanism of morning glory clouds. Image from [38]. CC BY 4.0 (https://creativecommons.org/licenses/by/4.0).

are dimensional quantities; for example, the first law of thermodynamics will be written

$$c_{\mathrm{p}}'\frac{\mathrm{D}T'}{\mathrm{D}t'} - \frac{1}{\rho'}\frac{\mathrm{D}\rho'}{\mathrm{D}t'} - \kappa'\nabla'\cdot\nabla' T' = Q'.$$

As geopotential we take (1.14) (with appropriate primes on the various variables and constants), whereas the viscous terms in the momentum equations (1.11) are

$$\begin{aligned}
(F_{\mathrm{v}}^{\varphi\prime}(\mathbf{u}'), F_{\mathrm{v}}^{\theta\prime}(\mathbf{u}'), F_{\mathrm{v}}^{r'\prime}(\mathbf{u}')) = {} & \nabla_\mu'^2(u',v',w') \\
& - \frac{1}{3}\left(\frac{1}{r'\cos\theta}\left(\frac{\mu_{\mathrm{h}}'}{\rho'}\frac{\mathrm{D}\rho'}{\mathrm{D}t'}\right)_\varphi, \frac{1}{r'}\left(\frac{\mu_{\mathrm{h}}'}{\rho'}\frac{\mathrm{D}\rho'}{\mathrm{D}t'}\right)_\theta, \left(\frac{\mu_{\mathrm{v}}'}{\rho'}\frac{\mathrm{D}\rho'}{\mathrm{D}t'}\right)_r\right) \\
& - \frac{1}{r'^2\cos^2\theta}\big(\mu_{\mathrm{h}}'u', \mu_{\mathrm{h}}'v', 2\mu_{\mathrm{v}}'w'(\cos^2\theta - v'\sin\theta\cos\theta)\big) \\
& + \frac{2\mu_{\mathrm{h}}'}{r'^2\cos\theta}(w_\varphi' - v_\varphi'\tan\theta, u_\varphi'\tan\theta, -u_\varphi') \\
& + \frac{2\mu_{\mathrm{h}}'}{r'^2}(0, w_\theta'. - v_\theta') + \frac{\mathrm{d}\mu_{\mathrm{v}}'}{\mathrm{d}r'}r'\frac{\partial}{\partial r}\left(\frac{u'}{r}, \frac{v'}{r'}, \frac{w'}{r'}\right) \\
& + \frac{1}{r'}\left(\frac{1}{r'\cos\theta}\frac{\mathrm{d}\mu_{\mathrm{h}}'}{\mathrm{d}r'}\frac{\partial}{\partial\varphi}, \frac{1}{r'}\frac{\mathrm{d}\mu_{\mathrm{h}}'}{\mathrm{d}r'}\frac{\partial}{\partial\theta}, \frac{\mathrm{d}\mu_{\mathrm{v}}'}{\mathrm{d}r'}\frac{\partial}{\partial r'}\right)(r'w') \\
& + \frac{\mathrm{d}\mu_{\mathrm{v}}'}{\mathrm{d}r'}\left(0, 0, \frac{u_\varphi'}{r'\cos\theta} + \frac{(v'\cos\theta)_\theta}{r'\cos\theta} + \frac{(r'^2w')_{r'}}{r'^2}\right),
\end{aligned}$$

where

$$\nabla_\mu'^2 := \mu_{\mathrm{v}}'\left(\frac{\partial^2}{\partial r'^2} + \frac{2}{r'}\frac{\partial}{\partial r'}\right) + \frac{\mu_{\mathrm{h}}'}{r'^2}\left(\frac{1}{\cos^2\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial\theta^2} - \tan\theta\frac{\partial}{\partial\theta}\right)$$

and μ_{h}' and μ_{v}' are the horizontal and vertical dynamic eddy viscosity coefficients, respectively, each assumed to be a function of r' only. Given these equations, we then perform the following steps.

(i) We non-dimensionalise the equations. To this end, let R' be Earth's average radius, H' the height of the troposphere, and $\bar{\rho}'$ an average density value, $\bar{\mu}'$ an average dynamic eddy viscosity, $\varepsilon = H'/R'$ (the *aspect ratio*). We define the non-dimensional variables

$$r' = R'(1+\varepsilon z), \quad t = \Omega' t', \quad \rho' = \bar{\rho}'\rho, \quad p' = \bar{\rho}'(\Omega' R')^2 p, \quad T' = \frac{(\Omega' R')^2}{\mathfrak{R}'}T,$$
$$(u',v',w') = \Omega' H'(u,v,w), \quad \mu_{\mathrm{v}}'(r') = \bar{\mu}' m(z), \quad \mu_{\mathrm{h}}'(r') = \bar{\mu}' M(z),$$

and the non-dimensional parameters

$$R_{\mathrm{e}} = \frac{\bar{\rho}'\Omega' H'}{\bar{\mu}'}, \quad g = \frac{g' H'}{\Omega'^2 R'^2}, \quad c_{\mathrm{p}} = \frac{c_{\mathrm{p}}'}{\mathfrak{R}'}, \quad \kappa = \frac{\kappa'}{\mathfrak{R}'\Omega' H'^2}.$$

(ii) We rotate (φ,θ) through a fixed angle α about a point (φ_0,θ_0), to produce new coordinates (Φ,Θ) such that the cloud line is perpendicular to the Θ-direction. The change of coordinates is

$$\varphi - \varphi_0 = \Phi\cos\alpha - \varepsilon\Theta\sin\alpha, \quad \theta - \theta_0 = \Phi\sin\alpha + \varepsilon\Theta\cos\alpha,$$

with corresponding tangential velocity components (U,V), where

$$u = U\cos\alpha - V\sin\alpha, \quad v = U\sin\alpha + V\cos\alpha.$$

The factor ε effectively 'zooms in' with respect to the coordinate Θ only; this takes into account the different length scales of the cloud perpendicularly and parallel to the cloud line.

(iii) In the equations thus derived, we take advantage of the smallness of the parameter ε to perform an *asymptotic expansion* (this is to all effects a thin-shell approximation, similarly in spirit to what we saw in Chapter 3 for oceanic flows): we write

$$q(\Phi,\Theta,z,t,\varepsilon) \sim \sum_{n\geq 0} \varepsilon^n q_n(\Phi,\Theta,z,t)$$

where q (and correspondingly the coefficients q_n) represents each of the variables p, ρ, T, U, V, w, and plug this expansion into the equations, collecting all terms at same order in ε.

(iv) Isolating the equations at order $O(1)$, $O(\varepsilon)$ and $O(\varepsilon^2)$, and assuming $U_0 = 0$ (which means that, at leading order, there is no motion in a direction parallel to the cloud line, in agreement with observations), after lengthy calculations we can fully determine the dynamics at leading order. In particular, we can infer the existence of a stream function Ψ such that

$$\left(\cos^2\alpha + \frac{\sin^2\alpha}{C}\right)\rho_0 V_0 = \frac{\partial\Psi}{\partial z}, \quad \rho_0 w_0 = -\frac{\partial\Psi}{\partial\Theta}$$

and

$$\begin{aligned}&\frac{\partial^2\Psi}{\partial z\partial t} + \frac{1}{\rho_0}\frac{\partial\Psi}{\partial z}\frac{\partial^2\Psi}{\partial z\partial\Theta} - \frac{\partial\Psi}{\partial\Theta}\frac{\partial}{\partial z}\left(\frac{1}{\rho_0}\frac{\partial\Psi}{\partial z}\right)\\ &\quad - \sigma\rho_0\left(S\frac{\partial\Psi}{\partial z} + C\cos\alpha\frac{\partial\Psi}{\partial\Theta}\right) - \nu\rho_0\Delta_0\left(\frac{1}{\rho_0}\frac{\partial\Psi}{\partial z}\right) = K(\Phi,z,t),\end{aligned} \tag{4.1}$$

where

$$S = \sin(\theta_0 + \Phi\sin\alpha), \quad C = \cos(\theta_0 + \Phi\sin\alpha), \quad \sigma = \frac{2(\sin^2\alpha + C\cos^2\alpha)}{(1-C)\sin\alpha\cos\alpha},$$

$\nu = R_e^{-1}$ is the non-dimensional viscosity (inverse Rossby number), K is an arbitrary thermodynamic forcing, and $\rho_0 > 0$ depends only on $gz + \frac{1}{2}S^2$; here,

$$\Delta_0 := \frac{\partial}{\partial z}\left(m(z)\frac{\partial}{\partial z}\right) + M(z)\left(\cos^2\alpha + \frac{\sin^2\alpha}{C}\right)\frac{\partial^2}{\partial\Theta^2}.$$

(v) Defining the new variable

$$Z = \int_0^z \rho_0\left(gs + \frac{1}{2}S^2\right)\mathrm{d}s,$$

equation (4.1) transforms to

$$\begin{aligned}&\frac{\partial^2\Psi}{\partial Z\partial t} + \frac{\partial\Psi}{\partial Z}\frac{\partial^2\Psi}{\partial Z\partial\Theta} - \sigma\left(S\frac{\partial\Psi}{\partial Z} + \frac{C\cos\alpha}{d_0}\frac{\partial\Psi}{\partial\Theta}\right)\\ &\quad - \nu\left[\frac{\partial}{\partial Z}\left(\hat{m}\frac{\partial}{\partial z}\right) + \hat{M}\left(\cos^2\alpha + \frac{\sin^2\alpha}{C}\right)\frac{\partial^2}{\partial\Theta^2}\right]\frac{\partial\Psi}{\partial Z} = \hat{K}(\Phi,Z,t),\end{aligned} \tag{4.2}$$

where

$$\rho_0 = d_0(\Phi,Z), \quad m\rho_0 = \hat{m}(\Phi,Z), \quad \frac{M}{\rho_0} = \hat{M}(\Phi,Z), \quad \frac{K}{\rho_0} = \hat{K}(\Phi,Z,t).$$

In [38], oscillatory solutions of equation (4.2) were constructed explicitly under some extra simplifying assumptions, showing that the model possesses solutions that exhibit (at least some of) the qualitative properties we expect from a flow that should model the morning glory cloud pattern. However, a natural question is whether equation (4.2), subject to appropriate initial and boundary conditions, is well-posed, that is, it possesses a unique solution that depends continuously on the initial data in an appropriate sense. The paper [84], presented below, sets out to establish some preliminary results in this direction. To this end, let us assume for simplicity that the coefficients d_0, $\hat{m} > 0$ and $\hat{M} > 0$ are constants; moreover, we will assume

$$\hat{m} = \hat{M}\left(\cos^2\alpha + \frac{\sin^2\alpha}{C}\right),$$

since the relative size of these coefficients is immaterial for our analysis, as it does not affect the ellipticity of the corresponding second-order differential operator appearing in (4.2). We then define $\eta = S\sigma$, $\beta = C\sigma\cos\alpha/d_0$, $\mu = \nu\hat{m} > 0$, and[1]

$$u = \frac{\partial\Psi}{\partial Z}, \quad v = -\frac{\partial\Psi}{\partial\Theta}; \tag{4.3}$$

we also introduce the new variables

$$(x, y) = (\Theta, Z).$$

Then, dropping the hat on $\hat{K}$, the equations (4.2) and (4.3) yield the problem

$$\left.\begin{aligned} u_t + uu_x + vu_y &= \nu\Delta u + \eta u + \beta v + K \\ u_x + v_y &= 0 \end{aligned}\right\} \quad \text{in } (0,\infty)\times\Omega \tag{4.4a}$$

in the unbounded horizontal strip

$$\Omega := \{(x, y) \in \mathbb{R}^2 \, : x \in \mathbb{R},\ 0 < y < 1\},$$

where

$$\mu \in (0,\infty) \quad \text{and} \quad \eta, \beta \in \mathbb{R}.$$

As initial and boundary conditions we take

$$u = 0 \quad \text{on } (0,\infty)\times\partial\Omega, \tag{4.4b}$$

$$v = 0 \quad \text{on } (0,\infty)\times\{y = 0\}, \tag{4.4c}$$

$$u|_{t=0} = u_0 \quad \text{in } \Omega, \tag{4.4d}$$

where $u_0 : \Omega \to \mathbb{R}$ is a given function that, in an appropriate sense, decays at infinity; see the statements of Theorem 4.3.2 and Theorem 4.4.1 below for the precise assumptions on u_0. In what follows, we will first establish local strong well-posedness of the problem in H^s for $s \in (1, 2)$ by writing the problem as an abstract parabolic problem in a suitable functional analytical setting, for which powerful results from abstract theory can be applied to obtain the conclusion. Then, by means of a Galerkin approximation scheme, we will prove the existence of global weak solutions (in an appropriate sense). It is worth noting that these results have been extended in [85], where some new abstract results on the

[1] We have used the letters u and v already, to indicate the nondimensional tangential velocity components. However, since there is no danger of confusion, we will use them here again with a different meaning.

well-posedness of certain quasilinear and semilinear problems can be applied, among other examples, to our problem at hand, yielding global strong well-posedness in H^1 for the *homogeneous* problem ($K \equiv 0$) for sufficiently small initial data.

An important observation with respect to the evolution problem (4.4) is that the second equation in (4.4a) and the boundary condition (4.4c) may be used to eliminate the unknown v from the problem, but only at the price of having to include some nonlocal terms that involve u instead. Indeed, given $z := (x, y) \in \Omega$, the fundamental theorem of calculus yields

$$v(z) = \int_0^y v_y(x,s)\,\mathrm{d}s = -\int_0^y u_x(x,s)\,\mathrm{d}s =: -Tu_x(z),$$

where, given $u \in L^2(\Omega)$, we define $Tu : \Omega \to \mathbb{R}$ by

$$Tu(x,y) := \int_0^y u(x,s)\,\mathrm{d}s, \quad (x,y) \in \Omega. \tag{4.5}$$

We also mention the related model consisting of the viscous primitive equations of large-scale ocean and atmosphere dynamics considered in [17, 60] (and the references therein) where, similarly as in our context, the last velocity component is expressed via the fundamental theorem of calculus in terms of the other velocity components. This observation enables us to rewrite (4.4a) in the following form:

$$u_t = \mu\Delta u - uu_x + u_y Tu_x + \eta u - \beta Tu_x + K \quad \text{in } \Omega,\ t > 0, \tag{4.6a}$$

subject to the Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial\Omega, \quad t > 0, \tag{4.6b}$$

and the initial condition

$$u(0) = u_0. \tag{4.6c}$$

4.2 Some mathematical background

In this section, we collect some definitions and results about analytic semigroups and interpolation theory that will be needed later in this chapter.

4.2.1 Analytic semigroups

We begin by reviewing some basic semigroup theory which will be needed later. For proofs and more details, we refer the reader, for example, to [55, 90].

Definition 4.2.1. Let $(X, \|\cdot\|_X)$ be a Banach space. A mapping $S : [0,\infty) \to \mathcal{L}(X)$ is called a *strongly continuous semigroup* (also a C_0-*semigroup*) if it has the following properties:

(i) $S(0) = \mathbb{I}$,

(ii) $S(t+s) = S(t)S(s)$ for all $t, s \in [0,\infty)$, and

(iii) the mapping $[0,\infty) \to X$, $t \mapsto S(t)u$ is continuous for any $u \in X$.

If, in addition,

$$\|S(t)\|_{\mathcal{L}(X)} \le 1 \quad \text{for all } t \ge 0$$

(in other words, $\|S(t)u\|_X \le \|u\|_X$ for all $t \ge 0$ and all $u \in X$), then S is called a *contraction semigroup*.

One of the most basic properties of semigroups is the following estimate on their growth/decay:

Lemma 4.2.2. *Let S be a strongly continuous semigroup on a Banach space X. Then there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that*

$$\|S(t)\| \leq M\,\mathrm{e}^{\omega t} \quad \text{for all } t \in [0,\infty).$$

Next, we introduce the concept of *generator* of a strongly continuous semigroup.

Definition 4.2.3. Let S be a strongly continuous semigroup on X. Set

$$\mathcal{D}(A) := \left\{ u \in X : \lim_{t\to 0^+} \frac{S(t)u - u}{t} \text{ exists} \right\}.$$

The operator

$$A : X \supset \mathcal{D}(A) \to X, \quad u \mapsto Au := \lim_{t\to 0^+} \frac{S(t)u - u}{t}$$

is called *generator* (also *infinitesimal generator*) of S. It is customary to write

$$S(t) = \mathrm{e}^{tA}, \quad t \geq 0.$$

The next proposition summarises some basic properties of strongly continuous semigroups.

Proposition 4.2.4. *Let S be a strongly continuous semigroup on X and A a generator of S. Let $u \in \mathcal{D}(A)$. Then:*

(i) $S(t)u \in \mathcal{D}(A)$ *for all* $t \in [0,\infty)$.
(ii) $AS(t)u = S(t)Au$ *for all* $t \in [0,\infty)$.
(iii) *The mapping* $[0,\infty) \to X$, $t \mapsto S(t)u$ *is differentiable on* $[0,\infty)$.
(iv) $\dfrac{\mathrm{d}}{\mathrm{d}t} S(t)u = AS(t)u$ *for all* $t \in [0,\infty)$.
(vi) $\mathcal{D}(A) \subset X$ *is dense in* X *and* A *is closed.*

Much can be said about strongly continuous semigroups; in particular, it is possible to provide sufficient conditions for an operator A to be the infinitesimal generator of a strongly continuous semigroup (Hille–Yosida theorem and Lumer–Phillips theorem). However, this will not be needed for us; instead, we now introduce a special class of strongly continuous semigroups.

Definition 4.2.5. A strongly continuous semigroup $S(t) = \mathrm{e}^{tA}$ on X with infinitesimal generator A is called an *analytic semigroup* if it has the following properties:

(i) there exists $\theta \in (0, \pi/2)$ such that e^{tA} can be extended to $t \in \Delta_\theta$, where

$$\Delta_\theta = \{t \in \mathbb{C} : |\arg(t)| < \theta\} \cup \{0\} \subseteq \mathbb{C},$$

and the semigroup properties (i)–(iii) in Definition 4.2.1 hold also on Δ_θ;

(ii) the map $t \mapsto \mathrm{e}^{tA}$ is analytic (in the sense of the uniform operator topology) on the open set $\Delta_\theta \setminus \{0\}$.

It is also convenient to introduce some notation that we will use later. To this end, let E_0 and E_1 Banach spaces with $E_1 \overset{d}{\hookrightarrow} E_0$. We denote by

$$\mathcal{H}(E_1, E_0)$$

the set of all $A \in \mathcal{L}(E_1, E_0)$ such that $-A$, viewed as a linear operator in E_0 with domain $\mathcal{D}(A) = E_1$, is the infinitesimal generator of an analytic semigroup $\{\mathrm{e}^{-tA} : t \geq 0\}$ on E_0, that is, in $\mathcal{L}(E_0)$.

The set $\mathcal{H}(E_1, E_0)$ can be characterised in the following way. Given $\kappa \geq 1$ and $\omega > 0$, we write

$$A \in \mathcal{H}(E_1, E_0, \kappa, \omega)$$

if $\omega + A \in \mathcal{L}(E_1, E_0)$ is invertible with continuous inverse and

$$\kappa^{-1} \leq \frac{\|(\lambda + A)x\|_0}{|\lambda|\|x\|_0 + \|x\|_{L^1(\Omega)}} \leq \kappa, \quad x \in E_1 \setminus \{0\}, \operatorname{Re} \lambda \geq \omega,$$

where $\|\cdot\|_j$ is the norm in E_j, $j \in \{0, 1\}$. Then we can describe $\mathcal{H}(E_1, E_0)$ as follows:

Lemma 4.2.6. *We have that*

$$\mathcal{H}(E_1, E_0) = \bigcup_{\kappa \geq 1,\, \omega > 0} \mathcal{H}(E_1, E_0, \kappa, \omega).$$

We will also need the following perturbation result.

Lemma 4.2.7. *Let $\kappa \geq 1$ and $\omega > 0$.*

(i) The set $\mathcal{H}(E_1, E_0)$ is open in $\mathcal{L}(E_1, E_0)$. In fact, given $r \in (0, \kappa^{-1})$ and $A \in \mathcal{H}(E_1, E_0, \kappa, \omega)$,

$$\overline{\mathbb{B}}_{\mathcal{L}(E-1, E_0)}(A, r) \subseteq \mathcal{H}(E_1, E_0, \kappa/(1 - \kappa r), \omega).$$

(ii) Let $A \in \mathcal{H}(E_1, E_0, \kappa, \omega)$, $0 < r < \kappa^{-1}$ and $\beta \geq 0$. Then, given $B \in \mathcal{L}(E_1, E_0)$ satisfying

$$\|Bx\|_0 \leq r\|x\|_{L^1(\Omega)} + \beta\|x\|_0, \quad x \in E_1,$$

it follows that

$$A + B \in \mathcal{H}(E_1, E_0, \kappa/(1 - \kappa r), \max\{\omega, \beta/r\}).$$

For proofs of the preceding lemmas and a detailed discussion of further properties of analytic semigroups, see [9, Section I.1].

4.2.2 Interpolation spaces

Next, we discuss the concepts of real and complex interpolation spaces, which will be necessary later on. We restrict ourselves to some definitions and statements of basic properties; for a more comprehensive discussion (including proofs), see, for example, [9, 79].

First, we need the definition of intermediate spaces and interpolation spaces.

Definition 4.2.8. Let $(E_0, \|\cdot\|_0)$ and $(E_1, \|\cdot\|_1)$ be (real or complex) Banach spaces.

(i) We say that (E_0, E_1) is an *interpolation couple* if there exists a locally convex vector space V with $E_0 \hookrightarrow V$ and $E_1 \hookrightarrow V$. In this case, the intersection $E_0 \cap E_1$ and the sum $E_0 + E_1$ are well-defined linear subspaces of V, which are Banach spaces when endowed with the norms

$$\|v\|_{E_0 \cap E_1} = \max\{\|v\|_0, \|v\|_1\}$$

and

$$\|v\|_{E_0+E_1} = \inf\{\|e_0\|_0 + \|e_1\|_1 : v = e_0 + e_1,\ e_0 \in E_0,\ e_1 \in E_1\},$$

respectively.

(ii) If (E_0, E_1) is an interpolation couple, a Banach space E is called an *intermediate space* if

$$E_0 \cap E_1 \hookrightarrow E \hookrightarrow E_0 + E_1.$$

If, in addition, for every $T \in \mathcal{L}(E_0) \cap \mathcal{L}(E_1)$ (that is, for every $T \in \mathcal{L}(E_0 + E_1)$ such that $T|_{E_j} \in \mathcal{L}(E_j)$, $j \in \{0, 1\}$) it follows that $T|_E \in \mathcal{L}(E)$, then E is called an *interpolation space* for the interpolation couple (E_0, E_1).

Real interpolation

We present here the definition of real interpolation spaces based on the so-called K-*method*. Another approach to introducing real interpolation spaces is via the *trace method*, which we will not pursue here; the reader is referred to [79] for details of the trace method.

Definition 4.2.9. Let (E_0, E_1) be an interpolation couple. For every $x \in E_0 + E_1$ and $t > 0$, set

$$K(t, x, E_0, E_1) = \inf\big\{\|e_0\|_0 + t\|e_1\|_1 : x = e_0 + e_1,\ e_0 \in E_0,\ e_1 \in E_1\big\}.$$

For $\theta \in (0, 1)$ and $p \in [1, \infty]$, we define the space

$$(E_0, E_1)_{\theta,p} = \big\{x \in E_0 + E_1 : t \mapsto t^{-\theta-1/p} K(t, x, E_0, E_1) \in L^p(0, \infty)\big\},$$

endowed with the norm

$$\|x\|_{\theta,p} = \big\|t^{-\theta-1/p} K(t, x, E_0, E_1)\big\|_{L^p(0,\infty)},$$

as well as the space

$$(E_0, E_1)_\theta = \Big\{x \in E_0 + E_1 : \lim_{t \to 0^+} t^{-\theta} K(t, x, E_0, E_1) = 0\Big\}.$$

Proposition 4.2.10. *Let (E_0, E_1) be an interpolation couple, $\theta \in (0, 1)$, $p \in [1, \infty]$.*

(i) For $1 \le p_1 \le p_2 \le \infty$, we have

$$E_0 \cap E_1 \hookrightarrow (E_0, E_1)_{\theta,p_1} \hookrightarrow (E_0, E_1)_{\theta,p_2} \hookrightarrow (E_0, E_1)_\theta \hookrightarrow (E_0, E_1)_{\theta,\infty} \hookrightarrow E_0 + E_1.$$

Moreover,

$$(E_0, E_1)_{\theta,\infty} \hookrightarrow \overline{E}_0 \cap \overline{E}_1,$$

where $\overline{E}_0$ and $\overline{E}_1$ are the closures of E_0 and E_1 in $E_0 + E_1$.

(ii) *If, in addition,* $E_1 \hookrightarrow E_0$*, for* $0 < \theta_1 < \theta_2 < 1$ *we have*

$$(E_0, E_1)_{\theta_1,\infty} \hookrightarrow (E_0, E_1)_{\theta_2,1}.$$

In particular, $(E_0, E_1)_{\theta_2,p} \hookrightarrow (E_0, E_1)_{\theta_2,q}$ *for all* $p, q \in [1, \infty]$*.*

(iii) *For* $\theta \in (0, 1)$ *and* $p \in [1, \infty]$*, the spaces* $((E_0, E_1)_{\theta,p}, \|\cdot\|_{\theta,p})$ *and* $((E_0, E_1)_\theta, \|\cdot\|_{\theta,\infty})$ *are Banach spaces.*

(iv) *Let* (F_0, F_1) *be another interpolation couple. If* $T \in \mathcal{L}(E_0, E_1) \cap \mathcal{L}(F_0, F_1)$*, then* $T \in \mathcal{L}((E_0, E_1)_{\theta,p}, (F_0, F_1)_{\theta,p}) \cap \mathcal{L}((E_0, E_1)_\theta, (F_0, F_1)_\theta)$*. Moreover,*

$$\|T\|_{\mathcal{L}((E_0,E_1)_{\theta,p},(F_0,F_1)_{\theta,p})} \le \|T\|^{1-\theta}_{\mathcal{L}(E_0,F_0)} \|T\|^{\theta}_{\mathcal{L}(E_1,F_1)}.$$

In particular, $(E_0, E_1)_{\theta,p}$ *and* $(E_0, E_1)_\theta$ *are interpolation spaces for* (E_0, E_1)*.*

(v) *There is* $c(\theta, p) > 0$ *such that*

$$\|y\|_{\theta,p} \le c(\theta, p) \|y\|_0^{1-\theta} \|y\|_1^{\theta} \quad \textit{for all } y \in E_0 \cap E_1.$$

Remark 4.2.11. The spaces $(E_0, E_1)_{\theta,p}$ and $(E_0, E_1)_\theta$ are called *real interpolation spaces* with exponent θ (and parameter p) for E_0 and E_1, and $(\cdot,\cdot)_{\theta,p}$ and $(\cdot,\cdot)_\theta$ are often referred to as the *real interpolation functors* with exponent θ (and parameter p).

Complex interpolation

Next, we turn to complex interpolation, for which some preparation is required. If (E_0, E_1) is an interpolation couple of *complex* Banach spaces and $S = \{z = x + \mathrm{i}y \in \mathbb{C} : 0 \le x \le 1\}$, we denote by $\mathcal{F}(E_0, E_1)$ the space of all functions $f : S \to E_0 + E_1$ such that

- f is holomorphic in the interior of S and continuous up to its boundary, and
- $t \mapsto f(\mathrm{i}t) \in C(\mathbb{R}, E_0)$, $t \mapsto f(1 + \mathrm{i}t) \in C(\mathbb{R}, E_1)$, and

$$\|f\|_{\mathcal{F}(E_0,E_1)} := \max\left\{ \sup_{t\in\mathbb{R}} \|f(\mathrm{i}t)\|_0, \sup_{t\in\mathbb{R}} \|f(1+\mathrm{i}t)\|_1 \right\} < \infty.$$

Moreover, we denote by $\mathcal{F}_0(E_0, E_1)$ the subspace of $\mathcal{F}(E_0, E_1)$ consisting of all functions $f \in \mathcal{F}(E_0, E_1)$ such that

$$\lim_{|t|\to\infty} \|f(\mathrm{i}t)\|_0 = 0 \quad \text{and} \quad \lim_{|t|\to\infty} \|f(1+\mathrm{i}t)\|_1 = 0.$$

It is easily checked that $\mathcal{F}(E_0, E_1)$ and $\mathcal{F}_0(E_0, E_1)$ are Banach spaces.

Definition 4.2.12. Let (E_0, E_1) be a complex interpolation couple. For $\theta \in [0, 1]$, we define

$$[E_0, E_1]_\theta = \{f(\theta) : f \in \mathcal{F}(E_0, E_1)\}$$

and

$$\|x\|_{[E_0,E_1]_\theta} = \inf\left\{\|f\|_{\mathcal{F}(E_0,E_1)} : f \in \mathcal{F}(E_0, E_1),\ f(\theta) = x\right\}.$$

In the next proposition, we collect some of the main properties of the spaces $[E_0, E_1]_\theta$.

Proposition 4.2.13. *Let* (E_0, E_1) *be a complex interpolation couple,* $\theta \in (0, 1)$*. Then:*

(i) $E_0 \cap E_1 \hookrightarrow [E_0, E_1]_\theta \hookrightarrow E_0 + E_1$*. In fact,* $E_0 \cap E_1$ *is densely and continuously embedded in* $[E_0, E_1]_\theta$*.*

(ii) $([E_0, E_1]_\theta, \|\cdot\|_{[E_0,E_1]_\theta})$ *is a Banach space.*

(iii) Let (F_0, F_1) be another complex interpolation couple. If $T \in \mathcal{L}(E_0, F_0) \cap \mathcal{L}(E_1, F_1)$, then $T|_{[E_0,E_1]_\theta} \in \mathcal{L}([E_0, E_1]_\theta, [F_0, F_1]_\theta)$, and

$$\|T\|_{\mathcal{L}([E_0,E_1]_\theta,[F_0,F_1]_\theta)} \leq \|T\|^{1-\theta}_{\mathcal{L}(E_0,F_0)} \|T\|^{\theta}_{\mathcal{L}(E_1,F_1)}.$$

In particular, $[E_0, E_1]_\theta$ is an interpolation space for (E_0, E_1).

(iv) We have

$$\|y\|_{[E_0,E_1]_\theta} \leq \|y\|_0^{1-\theta} \|y\|_1^{\theta} \quad \text{for all } y \in E_0 \cap E_1.$$

Remark 4.2.14. The spaces $[E_0, E_1]_\theta$ are called *complex interpolation spaces* with exponent θ for the complex interpolation couple (E_0, E_1), and $[\cdot, \cdot]_\theta$ is often referred to as the *complex interpolation functor* with exponent θ.

4.3 Local well-posedness

Let us introduce some additional notation. For $s \in [0, 2]$ we set

$$H^s_{\mathrm{D}}(\Omega) = \begin{cases} H^s(\Omega), & \text{if } s \in (0, 1/2), \\ \{u \in H^s(\Omega) \,:\, u = 0 \text{ on } \partial\Omega\}, & \text{if } s \in (1/2, 2]. \end{cases}$$

Letting $[\cdot, \cdot]_\theta$ be the complex interpolation functor with exponent $\theta \in [0, 1]$ from Definition 4.2.12, it follows from [4, Theorem 13.3] that

$$[L^2(\Omega), H^2_{\mathrm{D}}(\Omega)]_\theta = H^{2\theta}_{\mathrm{D}}(\Omega). \tag{4.7}$$

We will also make repeated use of the following definition:

Definition 4.3.1. Let X denote a metric space and $\mathcal{T} \in (0, \infty]$. A map $\Sigma : [0, \mathcal{T}) \times X \to X$ is called a *(continuous) semiflow* if it has the following properties:

(i) $\Sigma(0, x) = x$ for all $x \in X$;

(ii) $\Sigma(t, \Sigma(s, x)) = \Sigma(t + s, x)$ for all $x \in X$ and all $s, t \in [0, \mathcal{T})$ with $t + s \in [0, \mathcal{T})$;

(iii) Σ is continuous.

The goal of this section is to establish the existence and uniqueness of classical solutions to (4.6) for sufficiently regular initial data and under the assumption that the function K is locally Lipschitz continuous with respect to time in $H^r_{\mathrm{D}}(\Omega)$ for some small $r > 0$, that is,

$$K \in C^{1-}([0, \infty), H^r_{\mathrm{D}}(\Omega)), \tag{4.8}$$

as stated in the next theorem.

Theorem 4.3.2 (Local well-posedness). *Let $s \in (1, 2)$, $r \in (0, s) \setminus \{1/2\}$, $\alpha := s/2$, and assume that $K = K(t, z)$ satisfies (4.8). Then, for each $u_0 \in H^s_{\mathrm{D}}(\Omega)$, the Cauchy problem (4.6) possesses a unique maximal classical solution*

$$u \in C([0, T^+), H^s_{\mathrm{D}}(\Omega)) \cap C((0, T^+), H^2_{\mathrm{D}}(\Omega)) \cap C^1((0, T^+), L^2(\Omega)) \cap C^\alpha([0, T^+), L^2(\Omega)),$$

where $T^+ = T^+(u_0) > 0$ is the maximal time of existence of the solution. Moreover, the map $(t, u_0) \mapsto u(t; u_0)$ is a semiflow on $H^s_{\mathrm{D}}(\Omega)$.

To prove Theorem 4.3.2, we reformulate (4.6) in a suitable functional analytic framework as a quasilinear evolution problem and apply abstract theory for these types of problem from [8] (see also [86]). Although $(4.6)_1$ has a semilinear structure, the application of the quasilinear theory enables us to consider more general initial data. To this end, we first introduce the function spaces which we are going to use in our analysis and provide some useful estimates for the operator T defined in (4.5).

4.3.1 The abstract formulation

In order to prove Theorem 4.3.2, we establish some lemmas as preparation. The first provides an important mapping property of the operator T.

Lemma 4.3.3. *Given $s \in [0,2]$, then $T \in \mathcal{L}(H^s(\Omega))$ with $\|T\|_{\mathcal{L}(H^s(\Omega))} \leq 1$.*

Proof. First, we take $u \in C_0^\infty(\Omega)$ and compute, in view of Hölder's inequality and Tonelli's theorem, that

$$\begin{aligned} \|Tu\|_{L^2(\Omega)} &= \left(\int_\Omega \left| \int_0^y u(x,s)\,\mathrm{d}s \right|^2 \mathrm{d}x\,\mathrm{d}y \right)^{1/2} \\ &\leq \left(\int_{\mathbb{R}} \int_0^1 u^2(x,s)\,\mathrm{d}s\,\mathrm{d}x \right)^{1/2} = \|u\|_{L^2(\Omega)}. \end{aligned} \tag{4.9}$$

Since $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$, this proves the claim for $s=0$.

In order to demonstrate the claim for $s=1$, we use the fact that $C^\infty(\overline{\Omega}) \cap H^1(\Omega)$ is dense in $H^1(\Omega)$, the estimate (4.9), and the observation that, given $u \in C^\infty(\overline{\Omega}) \cap H^1(\Omega)$, we have

$$(Tu)_x = Tu_x \quad \text{and} \quad (Tu)_y = u,$$

to obtain

$$\|Tu\|^2_{H^1(\Omega)} \leq 2\|w\|_{L^2(\Omega)} + \|w_x\|_{L^2(\Omega)} \leq 2\|w\|^2_{H^1(\Omega)}.$$

The claim for $s=2$ is now easily inferred from the result for $s=1$. In view of the interpolation property

$$[L^2(\Omega), H^2(\Omega)]_\theta = H^{2\theta}(\Omega), \quad \theta \in [0,1]$$

(see, for example, [4, Corollary 11.4]), the desired claim is a consequence of the result obtained for $s \in \{0,2\}$. □

Let $E_0 := L^2(\Omega)$, $E_1 := H^2_{\mathrm{D}}(\Omega)$, and set $E_\theta := [E_0, E_1]_\theta$ for $\theta \in (0,1)$. We can now reformulate (4.6) as the following evolution problem:

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = A(u(t))u(t) + K(t), \quad t > 0, \qquad u(0) = u_0, \tag{4.10}$$

where $K \in C^{1-}([0,\infty), E_{r/2})$ (see our assumption (4.8), as well as (4.7)), while the operator $A : E_{s/2} \to \mathcal{L}(E_1, E_0)$, $s \in (1,2)$, is defined by

$$A(u)v := \mu\Delta v + \eta v - uv_x + u_y T v_x - \beta T v_x, \quad u \in E_{s/2},\ v \in E_1.$$

The next lemma shows that A is well-defined and smooth with respect to its arguments.

Lemma 4.3.4. *Given $s \in (1,2)$, we have:*

(i) $[(u,v) \mapsto uv] : H^s(\Omega) \times H^{s-1}(\Omega) \to L^2(\Omega)$ *is continuous;*

(ii) $[(u,v) \mapsto uTv] : H^{s-1}(\Omega) \times H^{(3-s)/2}(\Omega) \to L^2(\Omega)$ *is continuous.*

Moreover, $A \in C^\infty(E_{s/2}, \mathcal{L}(E_1, E_0))$.

Proof. In view of Lemma 4.3.3 we have $T \in \mathcal{L}(H^{3/2-s}(\Omega))$. Therefore, it suffices to show that the bilinear operator $[(u,v) \mapsto uv]$ is continuous from $H^s(\Omega) \times H^{s-1}(\Omega)$, respectively from $H^{s-1}(\Omega) \times H^{3/2-s}(\Omega)$, to $L^2(\Omega)$. These properties are straightforward consequences of the multiplication result stated in [7, Theorem 4.1]; this proves (i) and (ii). Finally, the property $A \in C^\infty(E_{s/2}, \mathcal{L}(E_1, E_0))$ is a direct consequence of these results and (4.7). □

In order to apply the theory from [8] to the setting of (4.10), we are still bound to show that $-A(u)$ generates an analytic semigroup in $\mathcal{L}(E_0)$ for all $u \in E_{s/2}$ and $s \in (1, 2)$.

Lemma 4.3.5. *Given $s \in (1, 2)$ and $u \in E_{s/2}$, we have $-A(u) \in \mathcal{H}(E_1, E_0)$.*

Proof. According to Lemma 4.3.3 and Lemma 4.3.4, we have

$$\begin{aligned}&[v \mapsto u_y T v_x] \in \mathcal{L}(E_{(5-s)/4}, E_0),\\&[v \mapsto u v_x] \in \mathcal{L}(E_{s/2}, E_0),\\&[v \mapsto \eta v - \beta T v_x] \in \mathcal{L}(E_{1/2}, E_0).\end{aligned}$$

The generator property $-\Delta \in \mathcal{H}(E_1, E_0)$ of the Laplacian is established in [7, Theorem 13.4]. Moreover, the operators considered above can be treated as perturbations, and the desired generator result follows through Lemma 4.2.7(ii). □

4.3.2 The proof of Theorem 4.3.2

Let $\widetilde{K}$ denote the even reflection of K with respect to the origin. Then, according to (4.8), we have $\widetilde{K} \in C^{1-}(\mathbb{R}, E_{r/2})$. Setting $U := (t, u)$, the problem (4.10) is equivalent to the autonomous quasilinear evolution problem

$$\frac{\mathrm{d}U(t)}{\mathrm{d}t} = B(U(t))U(t) + f(U(t)), \quad t > 0, \qquad U(0) = (0, u_0), \tag{4.11}$$

where

$$B(U) = \begin{pmatrix} 0 & 0 \\ 0 & A(u) \end{pmatrix} \quad \text{and} \quad f(U) := \begin{pmatrix} 1 \\ \widetilde{K}(t) \end{pmatrix}.$$

Let $s \in (1, 2)$, $r \in (0, s) \setminus \{1/2\}$, and choose $\overline{s} \in (1, s)$. Moreover, define $F_i := \mathbb{R} \times E_i$ for $i \in \{0,\ 1\}$ and set

$$F_\theta := [F_0, F_1]_\theta = \mathbb{R} \times E_\theta, \quad \theta \in (0, 1),$$

see [9, Proposition I.2.3.3]. Our assumption (4.8), Lemma 4.3.4, Lemma 4.3.5, and [9, Corollary I.1.6.3], imply that

$$-B \in C^\infty(F_{\overline{s}/2}, \mathcal{H}(F_1, F_0)) \quad \text{and} \quad f \in C^{1-}(F_{\overline{s}/2}, F_{r/2}).$$

Therefore, we are in a position to apply the quasilinear parabolic theory from [9] (see [9, Theorem 12.1] or [86, Theorem 1.1]) to (4.11) and thus establish the claims of Theorem 4.3.2.

4.4 Existence of global weak solutions

In this section, we show that, for each $u_0 \in L^2(\Omega)$, and under the assumption

$$K \in L^2(0, t; L^2(\Omega)) \quad \text{for all } t > 0, \tag{4.12}$$

the evolution problem (4.6) possesses a global weak solution, as stated in Theorem 4.4.1. We tackle the problem through a two-step approximation strategy and on an equivalent formulation of (4.6) in terms of the new variable

$$w(t, x, y) := u(t, x, y)\, \mathrm{e}^{-\gamma t}, \quad t \geq 0,\ (x, y) \in \Omega \tag{4.13}$$

(see the system (4.17) below), where the real constant γ is defined as

$$\gamma := 1 + \eta + \frac{\beta^2}{2\mu}. \tag{4.14}$$

The reason for this change of variables is motivated by the fact that the L^2-functional

$$\mathcal{E}(u) := \frac{1}{2}\int_\Omega u^2,$$

does not have the desirable property of being non-increasing along solutions of (4.6) for $K \equiv 0$, as we have

$$\begin{aligned}\frac{\mathrm{d}\mathcal{E}(u(t))}{\mathrm{d}t} &= \int_\Omega u(t)\partial_t u(t)\\ &= \int_\Omega u(t)\big[\mu\Delta u - uu_x + u_y T u_x + \eta u - \beta T u_x + K\big](t)\\ &= \int_\Omega \big[-\mu|\nabla u|^2 + \eta u^2 - \beta u T u_x + Ku\big](t)\end{aligned}$$

for $t > 0$. However, it is important to note that the nonlinear terms vanish in the energy balance. This motivates the substitution (4.13), which is a trick which finds application, for example, in the proof of the weak parabolic maximum principle. With γ chosen as in (4.14), it is easy to see that the functional $\mathcal{E}$ is non-increasing along w (if $K \equiv 0$), since

$$\frac{\mathrm{d}\mathcal{E}(w(t))}{\mathrm{d}t} + \frac{1}{2}\int_\Omega \big[\mu|\nabla w(t)|^2 + w(t)^2\big](t) \le \frac{1}{2}\int_\Omega K(t)^2, \quad t > 0; \tag{4.15}$$

see Section 4.4. The estimate (4.15) is essential for the proof of Theorem 4.4.1.

Theorem 4.4.1. *Given $u_0 \in L^2(\Omega)$ and $K = K(t,x,y)$ that satisfies (4.12), there exists a global weak solution u to (4.6) with the following properties:*

(i) for all $t > 0$, we have $u \in L^\infty(0,t;L^2(\Omega)) \cap L^2(0,t;H^1(\Omega))$;

(ii) for all $t > 0$ and all $\phi \in C^\infty([0,t]\times\Omega)$ with the property that there exists $M > 0$ with $\operatorname{supp}\phi(\tau) \subset (-2^M, 2^M)\times(0,1)$ for all $\tau \in [0,t]$, we have

$$\begin{aligned}&\int_\Omega u(t)\phi(t) - \int_\Omega u_0\phi(0) - \int_0^t \left(\int_\Omega u(\tau)\partial_t\phi(\tau)\right)\mathrm{d}\tau\\ &\qquad + \int_0^t \left(\int_\Omega \big[\mu\nabla u\cdot\nabla\phi + uT\partial_x u\partial_y\phi + \phi\big(2u\partial_x u - \eta u + \beta T\partial_x u - K\big)\big](\tau)\right)\mathrm{d}\tau = 0;\end{aligned}$$

(iii) for almost all $t > 0$,

$$\begin{aligned}\mathcal{E}\big(u(t)\,\mathrm{e}^{-\gamma t}\big) + \frac{1}{2}\int_0^t \left(\int_\Omega \mathrm{e}^{-2\gamma\tau}\left(\mu|\nabla u(\tau)|^2 + u(\tau)^2\right)\right)\mathrm{d}\tau&\\ \le \mathcal{E}(u_0) + \frac{1}{2}\int_0^t \left(\int_\Omega K(\tau)^2\right)\mathrm{d}\tau.&\end{aligned} \tag{4.16}$$

The proof of this result is presented in Section 4.4. After reformulating (4.6) in terms of the new variable w, see (4.17), we establish (4.15) and then consider the problem (4.17) on the rectangle $\Omega^N := (-2^N, 2^N)\times(0,1)$ with $N \ge 1$, a regularised K, and homogeneous Dirichlet boundary conditions on $\partial\Omega^N$, see (4.24). Using a Galerkin scheme, we prove in Section 4.4.1 that problem (4.24) possesses a weak solution w^N, see Proposition 4.4.4. In a second step, we prove that the sequence of weak solutions (w^N) converges, in a suitable sense, towards a weak solution to (4.17), see Theorem 4.4.2. Then, in a final step, we return to the original unknown u and deduce from Theorem 4.4.2 that u is a weak solution to (4.6), as stated in Theorem 4.4.1.

In terms of the variable w defined in (4.13), the problem (4.6) may be reformulated as the system

$$w_t = \mu\Delta w - \mathrm{e}^{\gamma t} ww_x + \mathrm{e}^{\gamma t} w_y Tw_x + (\eta-\gamma)w - \beta Tw_x + \mathrm{e}^{-\gamma t} K \quad \text{in } \Omega,\ t>0, \tag{4.17a}$$

$$w = 0 \quad \text{on } \partial\Omega,\quad t>0, \tag{4.17b}$$

$$w(0) = u_0. \tag{4.17c}$$

We note that the constant γ is defined in (4.14) in such a way that the L^2-energy functional is non-increasing along (classical) solutions to (4.17) when $K \equiv 0$. Indeed, for $t>0$, we have

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{E}(w(t))}{\mathrm{d}t} &= \int_\Omega w(t)\partial_t w(t) \\
&= \int_\Omega w(t)\big[\mu\Delta w - \mathrm{e}^{\gamma t} ww_x + \mathrm{e}^{\gamma t} w_y Tw_x + (\eta-\gamma)w - \beta Tw_x + \mathrm{e}^{-\gamma t} K\big](t) \\
&= \int_\Omega \big[-\mu|\nabla w|^2 + (\eta-\gamma)w^2 - \beta wTw_x + \mathrm{e}^{-\gamma t} Kw\big](t).
\end{aligned}$$

If $\beta \neq 0$, in view of Lemma 4.3.3, Hölder's inequality, and Young's inequality, we estimate

$$\begin{aligned}
\|wTw_x\|_{L^1(\Omega)} &\le \|w\|_{L^2(\Omega)}\|Tw_x\|_{L^2(\Omega)} \le \|w\|_{L^2(\Omega)}\|\nabla w\|_{L^2(\Omega)} \\
&\le \frac{|\beta|\|w\|^2_{L^2(\Omega)}}{2\mu} + \frac{\mu\|\nabla w\|^2_{L^2(\Omega)}}{2|\beta|},
\end{aligned}$$

and

$$\|\mathrm{e}^{-\gamma t} Kw\|_{L^1(\Omega)} \le \frac{\|w\|^2_{L^2(\Omega)}}{2} + \frac{\|K\|^2_{L^2(\Omega)}}{2},$$

from which we conclude that

$$\frac{\mathrm{d}\mathcal{E}(w(t))}{\mathrm{d}t} + \frac{1}{2}\int_\Omega \big[\mu|\nabla w|^2 + w^2\big](t) \le \frac{1}{2}\int_\Omega K(t)^2.$$

This energy inequality is at the base of our construction of weak solutions to (4.17), as stated below.

Theorem 4.4.2. *Given $u_0 \in L^2(\Omega)$ and $K = K(t,x,y)$ that satisfies (4.12), there exists a global weak solution w to (4.17) with the following properties:*

(i) for all $t>0$, we have $w \in L^\infty(0,t;L^2(\Omega)) \cap L^2(0,t;H^1(\Omega))$;

(ii) for all $t>0$ and all $\xi \in C_0^\infty(\Omega)$, it holds that

$$\begin{aligned}
&\int_\Omega w(t)\xi - \int_\Omega u_0\xi + \int_0^t \bigg(\int_\Omega \big[\mu\nabla w(\tau)\cdot\nabla\xi + \mathrm{e}^{\gamma\tau}\partial_y\xi w(\tau)T\partial_x w(\tau) \\
&\quad + 2\,\mathrm{e}^{\gamma\tau}\xi w(\tau)\partial_x w(\tau) - (\eta-\gamma)w(\tau)\xi + \beta\xi T\partial_x w(\tau) - \mathrm{e}^{-\gamma\tau} K(\tau)\xi\big]\bigg)\,\mathrm{d}\tau = 0;
\end{aligned} \tag{4.18}$$

(iii) for almost all $t > 0$,

$$\mathcal{E}(w(t)) + \frac{1}{2}\int_0^t \bigg(\int_\Omega \big[\mu|\nabla w(\tau)|^2 + w(\tau)^2\big] \bigg) \,\mathrm{d}\tau \leq \mathcal{E}(u_0) + \frac{1}{2}\int_0^t \bigg(\int_\Omega K(\tau)^2 \bigg) \,\mathrm{d}\tau. \tag{4.19}$$

In order to prove Theorem 4.4.2, in the sequel we fix $u_0 \in L^2(\Omega)$ and $K = K(t, x, y)$, which satisfies (4.12), and we set

$$k(t) := \|K\|^2_{L^2(0,t+1;L^2(\Omega))}, \quad t \geq 0. \tag{4.20}$$

For later purposes, we need to regularise K. To this end, we set

$$\widetilde{K}(t, z) := \begin{cases} K(t, x, y), & (t, x, y) \in [0, \infty) \times \Omega, \\ 0, & (t, x, y) \in \mathbb{R}^3 \setminus \big([0, \infty) \times \Omega\big). \end{cases}$$

Then, $\widetilde{K} \in L^2((-t, t) \times \mathbb{R}^2)$ for all $t > 0$. Given $n \in \mathbb{N}$, we further set

$$K_n := \widetilde{K} * \rho_{1/n}, \tag{4.21}$$

where $(\rho_\varepsilon)_{\varepsilon>0}$ is a standard mollifier in $\mathbb{R}^3$. Then, $K_n \in C^\infty(\mathbb{R}^3)$ satisfies

$$K_n \to K \quad \text{in } L^2(0, t; L^2(\Omega)) \text{ for all } t > 0 \tag{4.22}$$

and

$$\|K_n\|^2_{L^2(0,t;L^2(\Omega))} \leq k(t) \quad \text{for all } t > 0 \text{ and } n \in \mathbb{N}. \tag{4.23}$$

In a first step, we consider the problem (4.17) on the rectangle $\Omega^N := (-2^N, 2^N) \times (0, 1)$ with $N \in \mathbb{N}$ (see (4.24)) and construct, through a Galerkin scheme, a weak solution w^N to (4.24) (see Proposition 4.4.4 below). Then, in Section 4.4.2, we prove that the sequence $(w^N)_N$ of weak solutions to (4.24) converges, in a suitable sense, towards a weak solution to (4.17). To perform these steps, we will need more than once the following refinement of the Aubin–Lions Lemma, whose proof can be found in [108, Corollary 4]:

Lemma 4.4.3. *Let X, B and Y be Banach spaces with continuous embeddings*

$$X \hookrightarrow B \hookrightarrow Y,$$

and assume additionally that the embedding $X \hookrightarrow B$ is compact. Let $p \in [1, \infty)$, $\mathcal{T} > 0$, and suppose that the subset $F \subseteq L^p(0, \mathcal{T}; X)$ is bounded in $L^p(0, \mathcal{T}; X)$ and that the set $\partial_t F := \{\partial_t f : f \in F\}$ is bounded in $L^1(0, \mathcal{T}; Y)$. Then F is relatively compact in $L^p(0, \mathcal{T}; B)$.

4.4.1 A Galerkin scheme in bounded domains

In this section, we consider the problem

$$w_t = \mu\Delta w - \mathrm{e}^{\gamma t}\, w w_x + \mathrm{e}^{\gamma t}\, w_y T w_x + (\eta - \gamma) w - \beta T w_x + \mathrm{e}^{-\gamma t}\, K \quad \text{in } \Omega^N,\ t > 0, \tag{4.24a}$$

$$w = 0 \quad \text{on } \partial\Omega^N, \quad t > 0, \tag{4.24b}$$

$$w(0) = u_0^N := u_0|_{\Omega^N}, \tag{4.24c}$$

where Ω^N is the rectangle

$$\Omega^N := (-2^N, 2^N) \times (0,1), \quad N \in \mathbb{N},$$

and we use a Galerkin scheme to construct a global weak solution w^N to (4.24), as stated in Proposition 4.4.4. Before stating this result, we introduce the energy functional $\mathcal{E}_N$ by the formula

$$\mathcal{E}_N(u) := \frac{1}{2}\int_{\Omega^N} u^2. \tag{4.25}$$

Proposition 4.4.4. *Given $u_0 \in L^2(\Omega)$, $K = K(t,x,y)$ satisfying (4.12), and $N \in \mathbb{N}$, there exists a global weak solution w^N to (4.24) with the following properties:*

(i) for all $t > 0$,

$$w^N \in L^\infty\big(0,t;L^2(\Omega^N)\big) \cap L^2\big(0,t;H^1(\Omega^N)\big) \cap W^{1,4/3}\big(0,t;H^2_{\mathrm{D}}(\Omega^N)'\big),$$

where $H^2_{\mathrm{D}}(\Omega^N) := \big\{u \in H^2(\Omega^N) : u = 0 \text{ on } \partial\Omega^N\big\}$;

(ii) for all $t > 0$ and all $\xi \in H^2_{\mathrm{D}}(\Omega^N)$, we have

$$\begin{aligned}\int_{\Omega^N} w^N(t)\xi = \int_{\Omega^N} u_0^N \xi& \\ + \int_0^t \bigg(\int_{\Omega^N} \big[&-\mu\nabla w^N(\tau)\cdot\nabla\xi - \mathrm{e}^{\gamma\tau}\,\partial_y\xi w^N(\tau)T\partial_x w^N(\tau) \\ &- 2\,\mathrm{e}^{\gamma\tau}\,\xi w^N(\tau)\partial_x w^N(\tau) + (\eta-\gamma)w^N(\tau)\xi \\ &- \beta\xi T\partial_x w^N(\tau) + \mathrm{e}^{-\gamma\tau}\,K(\tau)\xi\big]\bigg)\,\mathrm{d}\tau;\end{aligned} \tag{4.26}$$

(iii) for almost all $t > 0$, we have

$$\begin{aligned}\mathcal{E}_N(w^N(t)) + \frac{1}{2}\int_0^t \bigg(\int_{\Omega^N} \big[\mu\big|\nabla w^N(\tau)\big|^2 + \big|w^N(\tau)\big|^2\bigg)\,\mathrm{d}\tau& \\ \le \mathcal{E}(u_0) + \frac{1}{2}\int_0^t\bigg(\int_{\Omega^N} K(\tau)^2\bigg)\,\mathrm{d}\tau&.\end{aligned} \tag{4.27}$$

The proof of Proposition 4.4.4 is postponed to the end of this section, as it requires some preparation. We first collect in Lemma 4.4.5 some classical results related to the spectrum of the Laplace operator with homogeneous Dirichlet boundary conditions.

Lemma 4.4.5. *Given $N \in \mathbb{N}$, let $-\lambda_k^N$, $k \in \mathbb{N}$, with*

$$0 < \lambda_1^N \le \lambda_2^N \le \dots$$

denote the eigenvalues of the Dirichlet Laplacian, and let ϕ_k^N denote an eigenfunction corresponding to the eigenvalue λ_k^N, $k \ge 1$. Then $\lambda_k^N \to \infty$ for $k \to \infty$, and the eigenfunctions can be chosen such that the set $\{\phi_k^N : k \ge 1\}$ is an orthonormal basis of $L^2(\Omega)$. Furthermore, letting

$$u_n := \sum_{k=1}^n \langle u, \phi_k^N\rangle_{L^2(\Omega^N)}\phi_k^N,$$

where $\langle\cdot,\cdot\rangle_{L^2(\Omega^N)}$ is the $L^2(\Omega^N)$-scalar product, it holds that

$$u_n \underset{n\to\infty}{\to} u \quad \text{in } H^2(\Omega^N) \tag{4.28}$$

provided that $u \in H^2_{\mathrm{D}}(\Omega^N)$.

Proof. It is well known that $u_n \to u$ in $L^2(\Omega^N)$. Hence, in view of $\Delta u \in L^2(\Omega^N)$, we also have that

$$(\Delta u)_n = \sum_{k=1}^{n} \langle \Delta u, \phi_k^N \rangle_{L^2(\Omega^N)} \phi_k^N \underset{n\to\infty}{\to} \Delta u \quad \text{in } L^2(\Omega^N).$$

Since $u \in H^2_{\mathrm{D}}(\Omega^N)$, Stokes' theorem yields

$$\langle \Delta u, \phi_k^N \rangle_{L^2(\Omega^N)} = \langle u, \Delta \phi_k^N \rangle_{L^2(\Omega^N)} = -\lambda_k^N \langle u, \phi_k^N \rangle_{L^2(\Omega^N)}, \quad k \geq 1,$$

which shows that $(\Delta u)_n = \Delta u_n$. Hence $(1-\Delta)u_n \to (1-\Delta)u$ for $n \to \infty$ in $L^2(\Omega^N)$. Moreover,since Ω^N is convex, $1-\Delta : H^2_{\mathrm{D}}(\Omega^N) \to L^2(\Omega^N)$ is an isomorphism (see, for example, [15, Regularity Theorem 7.2]), and we may conclude that (4.28) is true. □

The operator T has properties similar to those of Lemma 4.3.3 when acting on functions defined on Ω^N.

Lemma 4.4.6. *Given $N \geq 1$, we have $\|T\|_{\mathcal{L}(L^2(\Omega^N))} \leq 1$.*

Proof. The claim follows by arguing as in Lemma 4.3.3. □

In the following, $N \in \mathbb{N}$ is fixed. Given $n \in \mathbb{N}$, we set $V_n \coloneqq \operatorname{span}\{\phi_1^N, \dots, \phi_n^N\}$ and define

$$u_{0,n} \coloneqq \sum_{k=1}^{n} \omega_k \phi_k^N \in V_n, \quad \text{where} \quad \omega_k \coloneqq \langle u_0, \phi_k^N \rangle_{L^2(\Omega^N)}, \quad k \leq n.$$

We then have

$$\|u_{0,n}\|_{L^2(\Omega^N)} \leq \|u_0\|_{L^2(\Omega^N)} \leq \|u_0\|_{L^2(\Omega)}, \quad n \in \mathbb{N}, \tag{4.29}$$

and, moreover,

$$u_{0,n} \to u_0^N \quad \text{in } L^2(\Omega^N). \tag{4.30}$$

We next look for a solution $w_n \coloneqq w_n^N$ to (4.24) of the form

$$w_n(t,x,y) = \sum_{k=1}^{n} F_k^N(t) \phi_k^N(x,y), \qquad t \geq 0,\ z \in \Omega^N,$$

such that $(F_1^N, \dots, F_n^N) : [0,\infty) \to \mathbb{R}^n$ is a continuously differentiable function with

$$(F_1^N, \dots, F_n^N)(0) = \omega \coloneqq (\omega_1, \dots, \omega_n) \tag{4.31}$$

and such that w_n satisfies the equation (4.24a) with K replaced by K_n—see (4.21)—at each time $t \geq 0$ in a weak sense, that is, when testing the equation with functions from V_n. We note that (4.31) is equivalent to the equation $w_n(0) = u_{0,n}$. In order to establish the existence of the solution w_n, we test (4.24a) at $t \geq 0$ with ϕ_ℓ^N, $1 \leq \ell \leq N$, and obtain that

$$\begin{aligned} \frac{\mathrm{d}F_\ell^N(t)}{\mathrm{d}t} &= (\eta - \gamma - \mu\lambda_\ell^N) F_\ell^N(t) \\ &\quad + \int_{\Omega^N} \phi_\ell^N(x,y) \Big[-\mathrm{e}^{\gamma t} \Big(\sum_{k=1}^{n} F_k^N(t) \phi_k^N(x,y) \Big) \Big(\sum_{k=1}^{n} F_k^N(t) \partial_x \phi_k^N(x,y) \Big) \\ &\qquad + \mathrm{e}^{\gamma t} \Big(\sum_{k=1}^{n} F_k^N(t) \partial_y \phi_k^N(x,y) \Big) \Big(\sum_{k=1}^{n} F_k^N(t) (T\partial_x \phi_k^N)(x,y) \Big) \\ &\qquad - \beta \sum_{k=1}^{n} F_k^N(t) (T\partial_x \phi_k^N)(x,y) + \mathrm{e}^{-\gamma t} K_n(t,z) \sum_{k=1}^{n} F_k^N(t) \phi_k^N(x,y) \Big]. \end{aligned}$$

Therefore, $\xi := (F_1^N, \dots, F_n^N)$ solves an initial value problem for an ordinary differential equation of the form

$$\xi'(t) = \Psi^n(t, \xi(t)), \qquad \xi(0) = \omega, \tag{4.32}$$

where, since K_n is smooth, it holds that $\Psi^n \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R}^n)$. By standard theory (see, for example, [6, 97]), there exists a maximal solution $(F_1^N, \dots, F_n^N) \in C^1([0, T_n^+), \mathbb{R}^n)$ to (4.32) with the maximal existence time $T_n^+ \in (0, \infty]$.

Next, we prove that the solution is bounded on each time interval $[0, T] \cap [0, T_n^+)$, with $T > 0$, which ensures that $T_n^+ = \infty$. Testing the equation (4.24a) with $w_n(t) \in V_n$ and $t \in [0, T_n^+)$, we obtain that

$$\begin{aligned} \frac{\mathrm{d}\mathcal{E}_N(w_n(t))}{\mathrm{d}t} &= \int_{\Omega^N} w_n(t)\partial_t w_n(t) \\ &= \int_{\Omega^N} \big[-\mu|\nabla w_n|^2 + (\eta - \gamma)w_n^2 - \beta w_n T \partial_x w_n + \mathrm{e}^{-\gamma t} K_n w_n \big](t). \end{aligned}$$

Recalling Lemma 4.4.6, (4.14), and the estimates (4.23) and (4.29), we may argue as in the derivation of (4.15) to obtain, after integration on $[0, t]$, that

$$\begin{aligned} \mathcal{E}_N(w_n(t)) + \frac{1}{2}\int_0^t \bigg(\int_{\Omega^N} \big[\mu|\nabla w_n(\tau)|^2 + |w_n(\tau)|^2\big] \bigg) \mathrm{d}\tau &\le \mathcal{E}(u_0) + \frac{1}{2}\int_0^t \bigg(\int_{\Omega^N} K_n(\tau)^2 \bigg) \mathrm{d}\tau \\ &\le \mathcal{E}(u_0) + \frac{k(t)}{2} \end{aligned} \tag{4.33}$$

for all $t \in [0, T_n^+)$. Since

$$\mathcal{E}_N(w_n(t)) = \frac{1}{2}\sum_{k=1}^n \big(F_k^N(t)\big)^2, \quad t \in [0, T_n^+),$$

the estimate (4.33) implies that the solution $(F_1^N, \dots, F_n^N)$ is globally defined.

In the remainder of this section, we denote by C positive constants which may depend only on η, β, and μ.

Lemma 4.4.7 (A priori estimates). *There exists a positive constant C such that, for all $t \ge 0$ and $n \in \mathbb{N}$, we have*

$$\int_0^t \|w_n(\tau)\|^2_{H^1(\Omega^N)}\,\mathrm{d}\tau \le C\big(\mathcal{E}(u_0) + k(t)\big) \tag{4.34}$$

and

$$\int_0^t \|\partial_t w_n(\tau)\|^{4/3}_{H^2_{\mathrm{D}}(\Omega^N)'}\,\mathrm{d}\tau \le C\big(1 + \mathcal{E}(u_0) + k(t)\big)^2 \mathrm{e}^{Ct}. \tag{4.35}$$

Proof. The estimate (4.34) follows immediately from (4.33), so we only need to prove (4.35). To do this, let $\xi \in H^2_{\mathrm{D}}(\Omega^N)$. According to Lemma 4.4.5, we may decompose ξ as $\xi = \xi_n + \xi_0$, where $\xi_n \in V_n$ and $\xi_0 := \xi - \xi_n \in H^2_{\mathrm{D}}(\Omega^N)$ are orthogonal in $H^2(\Omega^N)$; thus

$$\|\xi_n\|_{H^2(\Omega^N)} \le \|\xi\|_{H^2(\Omega^N)} \quad \text{for all } n \in \mathbb{N}.$$

Hence, given $t \ge 0$, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^N} w_n(t)\xi &= \int_{\Omega^N} \partial_t w_n(t)\xi = \int_{\Omega^N} \partial_t w_n(t)\xi_n \\ &= \int_{\Omega^N} \big[\mu\Delta w_n - \mathrm{e}^{\gamma t} w_n \partial_x w_n + \mathrm{e}^{\gamma t} \partial_y w_n T \partial_x w_n \\ &\qquad + (\eta - \gamma)w_n - \beta T \partial_x w_n + \mathrm{e}^{-\gamma t} K_n\big](t)\xi_n. \end{aligned}$$

Stokes' theorem, together with the observation that $\partial_y T u = u$ for $u \in H^1(\Omega^N)$, leads us to the identity

$$\int_{\Omega^N} \xi_n \partial_y w_n(t) T \partial_x w_n(t) = -\int_{\Omega^N} \left[\partial_y \xi_n w_n(t) T \partial_x w_n(t) + \xi_n w_n(t) \partial_x w_n(t)\right],$$

hence we have

$$\begin{aligned}\int_{\Omega^N} \partial_t w_n(t)\xi = \int_{\Omega^N} \big[-\mu \nabla w_n(t)\cdot\nabla\xi_n - \mathrm{e}^{\gamma t}\,\partial_y \xi_n w_n(t) T\partial_x w_n(t) + (\eta-\gamma) w_n(t)\xi_n \\ - 2\,\mathrm{e}^{\gamma t}\,\xi_n w_n(t)\partial_x w_n(t) - \beta \xi_n T\partial_x w_n(t) + \mathrm{e}^{-\gamma t} K_n(t)\xi_n\big].\end{aligned} \tag{4.36}$$

The relation (4.36), combined with Lemma 4.4.6, now yields

$$\begin{aligned}\left|\int_{\Omega^N} \partial_t w_n(t)\xi\right| \le C\big(\|w_n(t)\|_{H^1(\Omega)}\|\xi_n\|_{H^1(\Omega)} + \mathrm{e}^{\gamma t}\,\|\xi_n\|_{W^1_4}\|w_n(t)\|_{L^4(\Omega)}\|w_n(t)\|_{H^1(\Omega)} \\ + \|K_n(t)\|_{L^2(\Omega)}\|\xi_n\|_{L^2(\Omega)}\big).\end{aligned}$$

The embeddings $H^2(\Omega^N) \hookrightarrow W^{1,4}(\Omega^N)$, $H^{1/2}(\Omega^N) \hookrightarrow L^4(\Omega^N)$, the estimate

$$\|u\|_{H^{1/2}(\Omega)} \le C\|u\|_{L^2(\Omega)}^{1/2}\|u\|_{H^1(\Omega)}^{1/2}, \quad u \in H^1(\Omega^N),$$

and the bounds (4.23) and (4.33) imply that there exists a positive constant C such that

$$\begin{aligned}\left|\int_{\Omega^N} \partial_t w_n(t)\xi\right| &\le C\Big[1 + k(t)^{1/2} + \big(1 + k(t) + \mathcal{E}(u_0)\big)^{1/2}\mathrm{e}^{Ct}\,\|w_n(t)\|_{H^1(\Omega)}^{3/2}\Big]\|\xi_n\|_{H^2(\Omega)} \\ &\le C\big(1 + k(t) + \mathcal{E}(u_0)\big)^{1/2}\mathrm{e}^{Ct}\Big[1 + \|w_n(t)\|_{H^1(\Omega)}^{3/2}\Big]\|\xi\|_{H^2(\Omega)}.\end{aligned}$$

Therefore

$$\|\partial_t w_n(t)\|_{H^2_{\mathrm{D}}(\Omega^N)'} \le C\big(1 + k(t) + \mathcal{E}(u_0)\big)^{1/2}\mathrm{e}^{Ct}\Big[1 + \|w_n(t)\|_{H^1(\Omega)}^{3/2}\Big] \quad \text{for all } t \ge 0,$$

and the desired claim (4.35) follows by applying (4.34). □

The bounds provided in Lemma 4.4.7 enable us to obtain some compactness results for the sequence $(w_n)_n$. In fact, taking into account that

$$H^1(\Omega^N) \hookrightarrow H^{1/2}(\Omega^N) \hookrightarrow H^2_{\mathrm{D}}(\Omega^N)',$$

with compact embedding $H^1(\Omega^N) \hookrightarrow H^{1/2}(\Omega^N)$, we infer from (4.34), (4.35), and Lemma 4.4.3 that the sequence $(w_n)_n$ is relatively compact in $L^2(0,t;H^{1/2}(\Omega^N))$. Moreover, a standard Cantor's diagonal argument allows us to conclude that there exists a subsequence of $(w_n)_n$ (not relabelled) and a function $w^N : [0,\infty)\times\Omega^N \to \mathbb{R}$ such that

$$w_n \to w^N \quad \text{in } L^2\big(0,t;H^{1/2}(\Omega^N)\big) \text{ for all } t > 0, \tag{4.37}$$

$$w_n(t) \to w^N(t) \quad \text{in } L^2(\Omega^N) \text{ for almost all } t > 0, \tag{4.38}$$

$$\nabla w_n \rightharpoonup \nabla w^N \quad \text{in } L^2\big((0,t)\times\Omega^N\big) \text{ for all } t > 0. \tag{4.39}$$

Recalling (4.35), we also have

$$\partial_t w_n \rightharpoonup \partial_t w^N \quad \text{in } L^{4/3}\big(0,t;H^2_{\mathrm{D}}(\Omega^N)'\big) \text{ for all } t > 0. \tag{4.40}$$

The convergences (4.37)–(4.40) enable us to prove Proposition 4.4.4.

Proof of Proposition 4.4.4. We show that the function w^N identified above by the convergences (4.37)–(4.40) satisfies all the properties stated in Proposition 4.4.4. The property (i) follows from (4.37)–(4.40) by passing to lim inf in (4.33) and (4.35). Moreover, passing to lim inf in (4.33), we deduce from (4.22) and (4.37)–(4.39) that the energy estimate (4.27) is valid.

It remains to verify (ii). To this end, for given $\xi \in H^2_{\mathrm{D}}(\Omega^N)$, we integrate the identity (4.36) over $[0,t]$, to deduce that, for all $n \in \mathbb{N}$, we have

$$\begin{aligned}\int_{\Omega^N} w_n(t)\xi_n = \int_{\Omega^N} u_{0,n}\xi_n \\ + \int_0^t \bigg(\int_{\Omega^N} \big[-\mu \nabla w_n(\tau)\cdot\nabla\xi_n - \mathrm{e}^{\gamma\tau}\,\partial_y\xi_n w_n(\tau) T\partial_x w_n(\tau) \\ - 2\,\mathrm{e}^{\gamma\tau}\,\xi_n w_n(\tau)\partial_x w_n(\tau) + (\eta-\gamma)w_n(\tau)\xi_n \\ - \beta\xi_n T\partial_x w_n(\tau) + \mathrm{e}^{-\gamma\tau}\,K_n(\tau)\xi_n\big]\bigg)\,\mathrm{d}\tau.\end{aligned} \tag{4.41}$$

In view of (4.28), (4.30), and (4.38), we have for almost all $t > 0$ that

$$\int_{\Omega^N} w_n(t)\xi_n \to \int_{\Omega^N} w^N(t)\xi \quad \text{and} \quad \int_{\Omega^N} u_{0,n}\xi_n \to \int_{\Omega^N} u_0^N\xi.$$

Moreover, since $\xi_n \to \xi$ in $H^1(\Omega^N)$, the convergences (4.22), (4.37), and (4.39) ensure that

$$\begin{aligned}&\int_0^t \bigg(\int_{\Omega^N} \big[-\mu\nabla w_n(\tau)\cdot\nabla\xi_n + (\eta-\gamma)w_n(\tau)\xi_n + \mathrm{e}^{-\gamma\tau}\,K_n(\tau)\xi_n\big]\bigg)\,\mathrm{d}\tau \\ &\quad\to \int_0^t \bigg(\int_{\Omega^N} \big[-\mu\nabla w^N(\tau)\cdot\nabla\xi + (\eta-\gamma)w^N(\tau)\xi + \mathrm{e}^{-\gamma\tau}\,K(\tau)\xi\big]\bigg)\,\mathrm{d}\tau \quad \text{for all } t>0.\end{aligned}$$

Next, we observe that, since $H^{1/2}(\Omega^N) \hookrightarrow L^4(\Omega^N)$, the relations (4.28) and (4.37) lead to $\xi_n \to \xi$ in $L^\infty(0,t;L^4(\Omega^N))$ and $w_n \to w^N$ in $L^2(0,t;L^4(\Omega^N))$, hence

$$w_n\xi_n \to w^N\xi \quad \text{in } L^2((0,t)\times\Omega^N) \text{ for all } t>0. \tag{4.42}$$

Combining the convergences (4.39) and (4.42), we arrive at

$$\int_0^t \bigg(\int_{\Omega^N} \mathrm{e}^{\gamma\tau}\,\xi_n w_n(\tau)\partial_x w_n(\tau)\bigg)\,\mathrm{d}\tau \to \int_0^t \bigg(\int_{\Omega^N} \mathrm{e}^{\gamma\tau}\,\xi w^N(\tau)\partial_x w^N(\tau)\bigg)\,\mathrm{d}\tau.$$

It remains to pass to the limit in the terms that involve the nonlocal operator T. We define the functions

$$\begin{aligned}\zeta_n(t,x,s) &:= \int_s^1 (w_n(t)\partial_y\xi_n + \beta\xi_n)(x,y)\,\mathrm{d}y, \\ \zeta(t,x,s) &:= \int_s^1 (w^N(t)\partial_y\xi + \beta\xi)(x,y)\,\mathrm{d}y\end{aligned}$$

for $t \geq 0$ and $(x,s) \in \Omega^N$, and note that, in view of Fubini's theorem, we have

$$\begin{aligned}&\int_0^t \bigg(\int_{\Omega^N} \mathrm{e}^{\gamma\tau}(w_n(\tau)\partial_y\xi_n + \beta\xi_n)T\partial_x w_n\bigg)\,\mathrm{d}\tau \\ &\quad= \int_0^t \bigg(\int_{\Omega^N} \mathrm{e}^{\gamma\tau}\,\partial_x w_n(\tau,x,s)\zeta_n(\tau,x,s)\,\mathrm{d}(x,s)\bigg)\,\mathrm{d}\tau.\end{aligned} \tag{4.43}$$

Using Hölder's inequality, we then find

$$
\begin{aligned}
\|\zeta_n(t)-\zeta(t)\|^2_{L^2(\Omega^N)} &\le \int_{\Omega^N}\int_0^1 |w_n(t)\partial_y\xi_n+\beta\xi_n-w^N(t)\partial_y\xi-\beta\xi|^2(x,y)\,\mathrm{d}y\,\mathrm{d}(x,s)\\
&\le \|w_n(t)\partial_y\xi_n-w^N(t)\partial_y\xi\|^2_{L^2(\Omega^N)}+\beta^2\|\xi_n-\xi\|^2_{L^2(\Omega^N)}\\
&\le \|w_n(t)-w^N(t)\|^2_{L^4(\Omega^N)}\|\partial_y\xi_n\|^2_{L^4(\Omega^N)}\\
&\quad+\|w^N(t)\|^2_{L^4(\Omega^N)}\|\partial_y\xi_n-\partial_y\xi\|^2_{L^4(\Omega^N)}\\
&\quad+\beta^2\|\xi_n-\xi\|^2_{L^2(\Omega^N)}
\end{aligned}
$$

for $t\ge 0$. Since $H^2(\Omega^N)\hookrightarrow W^{1,4}(\Omega^N)$, the relation (4.28) guarantees that $\partial_y\xi_n\to\partial_y\xi$ in $L^\infty(0,t;L^4(\Omega^N))$, and with $w_n\to w^N$ in $L^2(0,t;L^4(\Omega^N))$ we conclude that

$$
\zeta_n\to\zeta\quad\text{in } L^2\big((0,t)\times\Omega^N\big)\text{ for all } t>0.
$$

The latter property, together with (4.39), enables us to pass to the limit $n\to\infty$ in (4.43) to arrive at

$$
\begin{aligned}
&\int_0^t\left(\int_{\Omega^N}\mathrm{e}^{\gamma\tau}(w_n(\tau)\partial_y\xi_n+\beta\xi_n)T\partial_x w_n(\tau)\right)\mathrm{d}\tau\\
&\qquad\to\int_0^t\left(\int_{\Omega^N}\mathrm{e}^{\gamma\tau}(w^N(\tau)\partial_y\xi+\beta\xi)T\partial_x w^N(\tau)\right)\mathrm{d}\tau.
\end{aligned}
$$

Finally, observing that both sides of (4.26) are continuous with respect to t, we conclude that the identity (4.26) is satisfied for all $t>0$, and the proof is complete. □

4.4.2 The proof of Theorem 4.4.2

Now, we prove that the sequence $(w^N)_N$ provided by Proposition 4.4.4 is relatively compact in $L^2(0,t;H^s(\Omega^M))$ for all $s\in(0,1)$, $M\ge 1$, and $t>0$. This is one of the main ingredients that will allow us to pass to the limit $N\to\infty$ in the relations (4.26) and (4.27) and thus establish our main result, namely Theorem 4.4.2. Thus, let $M\in\mathbb{N}$ be given. Since $\Omega^M\subset\Omega^N$ for $N\ge M$, we deduce from (4.27) that

$$
(w^N)_{N\ge M}\text{ is bounded in } L^\infty\big(0,t;L^2(\Omega^M)\big)\cap L^2\big(0,t;H^1(\Omega^M)\big)\text{ for all } t>0. \tag{4.44}
$$

We further note that the identity (4.26) entails that

$$
\begin{aligned}
&\langle\partial_t w^N(t),\xi\rangle_{H^2_{\mathrm{D}}(\Omega^N)',H^2_{\mathrm{D}}(\Omega^N)}\\
&\quad=\int_{\Omega^N}\big[-\mu\nabla w^N(t)\cdot\nabla\xi-\mathrm{e}^{\gamma t}\,\partial_y\xi w^N(t)T\partial_x w^N(t)-2\,\mathrm{e}^{\gamma t}\,\xi w^N(t)\partial_x w^N(t)\\
&\qquad+(\eta-\gamma)w^N(t)\xi-\beta\xi T\partial_x w^N(t)+\mathrm{e}^{-\gamma t}\,K(t)\xi\big]
\end{aligned}
$$

for all $N\in\mathbb{N}$, $\xi\in H^2_{\mathrm{D}}(\Omega^N)$, and almost all $t>0$. Let $H^2_0(\Omega^M)$ denote the closure of the set $C^\infty_0(\Omega^M)$ in $H^2(\Omega^M)$. Given $N\ge M$, we clearly have $H^2_0(\Omega^M)\hookrightarrow H^2_{\mathrm{D}}(\Omega^N)$, and therefore $\partial_t w^N(t)\in H^2_{\mathrm{D}}(\Omega^N)'\hookrightarrow H^2_0(\Omega^M)'$, where, arguing as in the proof of Lemma 4.4.7, it holds that

$$
\begin{aligned}
&\big|\langle \partial_t w^N(t), \xi\rangle_{H_0^2(\Omega^M)', H_0^2(\Omega^M)}\big| \\
&\quad = \bigg| \int_{\Omega^M} \big[-\mu \nabla w^N(t)\cdot\nabla\xi - \mathrm{e}^{\gamma t}\,\partial_y \xi w^N(t) T\partial_x w^N(t) - 2\,\mathrm{e}^{\gamma t}\,\xi w^N(t)\partial_x w^N(t) \\
&\qquad\quad + (\eta-\gamma) w^N(t)\xi - \beta\xi T\partial_x w^N(t) + \mathrm{e}^{-\gamma t} K(t)\xi\big] \bigg| \\
&\quad \le C\bigg[1 + \|K(t)\|_{L^2(\Omega)} \\
&\qquad\quad + \bigg(1 + \mathcal{E}(u_0) + \int_0^t \Big(\int_\Omega K(\tau)^2\Big)\,\mathrm{d}\tau\bigg)^{1/2} \mathrm{e}^{Ct}\,\|w^N(t)\|_{H^1(\Omega)}^{3/2}\bigg] \|\xi\|_{H^2(\Omega)}.
\end{aligned}
$$

Consequently, for almost all $t > 0$ we have that

$$
\begin{aligned}
\|\partial_t w^N(t)\|_{H_0^2(\Omega^M)'} \le C\bigg[& 1 + \|K(t)\|_{L^2(\Omega)} \\
& + \bigg(1 + \mathcal{E}(u_0) + \int_0^t \Big(\int_\Omega K(\tau)^2\Big)\,\mathrm{d}\tau\bigg)^{1/2} \mathrm{e}^{Ct}\,\|w^N(t)\|_{H^1(\Omega)}^{3/2}\bigg],
\end{aligned}
$$

and with (4.44) we deduce that

$$
(\partial_t w^N)_{N\ge M} \text{ is bounded in } L^{4/3}\big(0,t; H_0^2(\Omega^M)'\big) \text{ for all } t>0. \tag{4.45}
$$

Due to the embeddings

$$
H^1(\Omega^M) \hookrightarrow H^{1/2}(\Omega^M) \hookrightarrow H_0^2(\Omega^M)'
$$

(where the first embedding is compact), the bounds (4.44)–(4.45), and Lemma 4.4.3 we conclude that the sequence $(w^N)_{N\ge M}$ is relatively compact in $L^2(0,t;H^{1/2}(\Omega^M))$. Moreover, a standard Cantor's diagonal argument allows us to conclude that there exists a subsequence of $(w^N)_N$ (not relabelled) and a function $w : [0,\infty)\times\Omega \to \mathbb{R}$ such that

$$
\begin{aligned}
w^N &\to w && \text{in } L^2\big(0,t;H^{1/2}(\Omega^M)\big) \text{ for all } t>0,\ M\ge 1, && (4.46)\\
w^N(t) &\to w(t) && \text{in } L^2(\Omega^M) \text{ for almost all } t>0 \text{ and all } M\ge 1, && (4.47)\\
\nabla w^N &\rightharpoonup \nabla w && \text{in } L^2\big((0,t)\times\Omega^M\big) \text{ for all } t>0,\ M\ge 1, && (4.48)\\
\partial_t w^N &\rightharpoonup \partial_t w && \text{in } L^{4/3}\big(0,t;H_0^2(\Omega^M)'\big) \text{ for all } t>0,\ M\ge 1. && (4.49)
\end{aligned}
$$

Hence we infer from (4.27) that, for $N\ge M$ and for almost all $t>0$, we have

$$
\int_{\Omega^M} w^N(t)^2 + \int_0^t \bigg(\int_{\Omega^M} \big[\mu\big|\nabla w^N(\tau)\big|^2 + \big|w^N(\tau)\big|^2\big]\bigg)\,\mathrm{d}\tau \le 2\mathcal{E}(u_0) + \int_0^t \Big(\int_\Omega K(\tau)^2\Big)\,\mathrm{d}\tau.
$$

Passing to lim inf in the latter estimate, we find, in view of (4.46)–(4.48), that

$$
\int_{\Omega^M} w(t)^2 + \int_0^t \bigg(\int_{\Omega^M} \big[\mu\big|\nabla w(\tau)\big|^2 + \big|w(\tau)\big|^2\big]\bigg)\,\mathrm{d}\tau \le 2\mathcal{E}(u_0) + \int_0^t \Big(\int_\Omega K(\tau)^2\Big)\,\mathrm{d}\tau.
$$

By the monotone convergence theorem, we may pass to the limit $M\to\infty$ in the latter relation and conclude that (4.19) is satisfied, which proves also the claim (i).

It remains to demonstrate the property (4.18). However, this follows by letting $N\to\infty$ in (4.27) (with $\xi\in C_0^\infty(\Omega)$), in view of the convergences (4.46)–(4.49). Since the details are similar to those in Proposition 4.4.4, we omit them here.

4.4.3 The proof of Theorem 4.4.1

It is straightforward to deduce from the identity (4.18) that, for $0 \le s < t$ and $\xi \in C_0^\infty(\Omega)$, we have

$$\int_\Omega w(t)\xi - \int_\Omega w(s)\xi + \int_s^t \left(\int_\Omega \left[\mu \nabla w(\tau) \cdot \nabla \xi + J_1(\tau)\partial_y \xi + J_2(\tau)\xi \right] \right) \mathrm{d}\tau = 0, \tag{4.50}$$

where

$$J_1(\tau) := \mathrm{e}^{\gamma\tau}\, w(\tau) T \partial_x w(\tau)$$

and

$$J_2(\tau) := 2\,\mathrm{e}^{\gamma\tau}\, w(\tau)\partial_x w(\tau) - (\eta - \gamma) w(\tau) + \beta T \partial_x w(\tau) - \mathrm{e}^{-\gamma\tau}\, K(\tau).$$

Let $t > 0$ be given and choose $\eta \in C^\infty([0,t] \times \Omega)$ with the property that there exists $M > 0$ with $\operatorname{supp} \eta(\tau) \subset \Omega^M$ for all $\tau \in [0,t]$, where as before we set $\Omega^M := (-2^M, 2^M) \times (0,1)$. Moreover, let Δ_n, $n \in \mathbb{N}$, be the uniform partition $0 = t_0 < t_1 < \ldots < t_n = t$ of the interval $[0,t]$. Then, it follows from (4.50) that

$$\begin{aligned} &\int_{\Omega^M} w(t)\eta(t) - \int_{\Omega^M} u_0 \eta(0) + \int_{\Omega^M} \sum_{i=1}^n \big(w(t_i)\eta(t_{i-1}) - w(t_{i-1})\eta(t_i) \big) \\ &\quad + \int_0^t \left(\int_{\Omega^M} \left[\mu \nabla w(\tau) \cdot \nabla \varphi_n(\tau) + J_1(\tau)\partial_y \varphi_n(\tau) + J_2(\tau)\varphi_n(\tau) \right] \right) \mathrm{d}\tau = 0, \end{aligned} \tag{4.51}$$

where

$$\varphi_n(\tau, x, y) := \sum_{i=1}^n (\eta(t_{i-1}, x, y) + \eta(t_i, x, y)) \mathbf{1}_{(t_{i-1}, t_i]}(\tau), \qquad (\tau, x, y) \in [0,t] \times \Omega.$$

Next, we pass to the limit $n \to \infty$ in all terms of (4.51). To this end, we first observe that

$$\left. \begin{array}{l} \varphi_n \to 2\eta \\ \nabla \varphi_n \to 2\nabla \eta \end{array} \right\} \quad \text{as } n \to \infty \text{ in } L^\infty((0,t) \times \Omega^M).$$

Since ∇w, J_1, $J_2 \in L^1((0,t) \times \Omega^M)$, we obtain that

$$\begin{aligned} &\int_0^t \left(\int_{\Omega^M} \left[\mu \nabla w(\tau) \cdot \nabla \varphi_n(\tau) + J_1(\tau)\partial_y \varphi_n(\tau) + J_2(\tau)\varphi_n(\tau) \right] \right) \mathrm{d}\tau \\ &\quad \to 2 \int_0^t \left(\int_\Omega \left[\mu \nabla w(\tau) \cdot \nabla \eta(\tau) + J_1(\tau)\partial_y \eta(\tau) + J_2(\tau)(\tau)\eta(\tau) \right] \right) \mathrm{d}\tau. \end{aligned} \tag{4.52}$$

Finally, we pass to the limit $n \to \infty$ in

$$\begin{aligned} \int_{\Omega^M} \sum_{i=1}^n \big(w(t_i)\eta(t_{i-1}) - w(t_{i-1})\eta(t_i) \big) &= \int_{\Omega^M} w(t)\eta(t_{n-1}) - u_0 \eta(t_1) \\ &\quad + \int_{\Omega^M} \sum_{i=1}^{n-1} w(t_i)(\eta(t_{i-1}) - \eta(t_{i+1})). \end{aligned}$$

Recalling (4.49), we have that $w \in C([0,t], H_0^2(\Omega^M)')$, hence

$$\int_{\Omega^M} w(t)\eta(t_{n-1}) - u_0\eta(t_1) \to \int_{\Omega^M} w(t)\eta(t) - u_0\eta(0) \quad \text{as } n \to \infty. \tag{4.53}$$

Furthermore, using that $w \in C([0,t], H_0^2(\Omega^M)')$, we also have

$$\begin{aligned}
&\sum_{i=1}^{n-1} \int_{\Omega^M} w(t_i)(\eta(t_{i+1}) - \eta(t_{i-1})) \\
&\qquad = \sum_{i=1}^{n-1} \int_{\Omega^M} w(t_i)(\eta(t_{i+1}) - \eta(t_i)) + \sum_{i=1}^{n-1} \int_{\Omega^M} w(t_i)(\eta(t_i) - \eta(t_{i-1})) \\
&\qquad \to 2\int_0^t \Big(\int_{\Omega^M} w(\tau)\partial_t\eta(\tau) \Big) \,\mathrm{d}\tau \quad \text{as } n \to \infty.
\end{aligned}$$

The latter convergence, together with (4.52)–(4.53), yields from (4.51) the integral identity

$$\begin{aligned}
&\int_\Omega w(t)\eta(t) - \int_\Omega u_0\eta(0) - \int_0^t \Big(\int_\Omega w(\tau)\partial_t\eta(\tau) \Big) \,\mathrm{d}\tau \\
&\qquad + \int_0^t \Big(\int_\Omega \big[\mu\nabla w(\tau)\cdot\nabla\eta(\tau) + J_1(\tau)\partial_y\eta(\tau) + J_2(\tau)\eta(\tau) \big] \Big) \,\mathrm{d}\tau = 0.
\end{aligned} \tag{4.54}$$

Setting $u(t,x,y) := \mathrm{e}^{\gamma t}\, w(t,x,y)$ and $\phi(t,x,y) := \mathrm{e}^{-\gamma t}\,\eta(t,x,y)$ for $t \geq 0$ and $(x,y) \in \Omega$, it follows from Theorem 4.4.2 and (4.54) that u is a weak solution to (4.6) as stated in Theorem 4.4.1 (i) and (ii). Finally, the energy estimate (4.16) follows directly from (4.19).

4.5 Global well-posedness for the homogeneous problem

The local well-posedness from Section 4.3 can be improved to global well-posedness for the homogeneous problem (that is, $K \equiv 0$) and small initial data. This is achieved through an application of abstract theory for quasi- and semilinear parabolic problems developed in [85].

First, let us set up the functional analytic stage. Let $(E_0, \|\cdot\|_0)$ and $(E_1, \|\cdot\|_1)$ be Banach spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ with continuous and dense embedding $E_1 \hookrightarrow E_0$. For each $\theta \in (0,1)$, we denote by $(\cdot,\cdot)_\theta$ either the real interpolation functor $(\cdot,\cdot)_{\theta,p}$ (for an arbitrary $p \in [1,\infty)$) from Definition 4.2.9, or the complex interpolation functor $[\cdot,\cdot]_\theta$ from Definition 4.2.12, and define $E_\theta := (E_0, E_1)_\theta$ as the corresponding interpolation space with norm denoted by $\|\cdot\|_\theta$.

We fix exponents

$$0 < \gamma < \beta < \alpha < \xi < 1 \quad \text{with} \quad q := \frac{1+\gamma-\alpha}{\xi-\alpha} > 1. \tag{4.55a}$$

For the quasilinear part, we assume for some open subset O_β of E_β that

$$-A \in C^{1-}\big(O_\beta, \mathcal{H}(E_1, E_0)\big). \tag{4.55b}$$

Moreover, denoting by O_ξ the open subset of E_ξ defined by $O_\xi := O_\beta \cap E_\xi$, we assume that $f : O_\xi \to E_\gamma$ is locally Lipschitz continuous in the sense that, for each $R > 0$, there is a constant $N(R) > 0$ such that

$$\|f(w)-f(v)\|_\gamma \leq N(R)\Big[1+\|w\|_\xi^{q-1}+\|v\|_\xi^{q-1}\Big]\big[\big(1+\|w\|_\xi+\|v\|_\xi\big)\|w-v\|_\beta+\|w-v\|_\xi\big] \tag{4.55c}$$

for w, $v \in O_\xi \cap \bar{\mathbb{B}}_{E_\beta}(0,R)$, where $q > 1$ is defined in (4.55a). Of course, (4.55c) includes the case that

$$\|f(w)-f(v)\|_\gamma \leq N(R)\Big[1+\|w\|_\xi^{q-1}+\|v\|_\xi^{q-1}\Big]\|w-v\|_\xi$$

for $w,\, v \in O_\xi \cap \bar{\mathbb{B}}_{E_\beta}(0,R)$.

Under assumptions (4.55), it turns out that the problem

$$u'(t) = A(u(t))u(t) + f(u(t)), \quad t > 0, \qquad u(0) = u_0, \tag{4.56}$$

is well-posed in $O_\alpha := O_\beta \cap E_\alpha$. We emphasise that the semilinear part f need not be defined on O_α and possibly requires more regularity than the quasilinear part A since $E_\xi \hookrightarrow E_\alpha \hookrightarrow E_\beta$ by (4.55a). Before stating the main result precisely, let us comment on the imposed assumptions. If the nonlinearities A and f are defined and Lipschitz continuous on the same set O_β, a general and powerful result on the well-posedness of (4.56) is [8, Theorem 12.1] (for proofs, see [5] and [86]). It states that, if

$$(-A, f) \in C^{1-}\big(O_\beta, \mathcal{H}(E_1, E_0) \times E_\gamma\big), \quad u_0 \in O_\alpha, \quad 0 < \gamma \le \beta < \alpha < 1,$$

then problem (4.56) admits a unique maximal strong solution and the solution map induces a semiflow on O_α. Theorems 4.5.2 and 4.5.3 below extend the work from [87]—where the semiflow property is proven when the second part of assumption (4.55a) is replaced by the assumption that the strict inequality

$$1 \le q < \frac{1+\gamma-\alpha}{\xi-\alpha} \tag{4.57}$$

holds—to the critical case $q(\xi-\alpha) = 1+\gamma-\alpha$. The latter covers important applications and yields a semiflow in spaces that are optimal in a certain sense. As pointed out in [68, 96, 98], for evolution equations possessed of a scaling invariance, the space E_α corresponding to the critical case $q(\xi-\alpha) = 1+\gamma-\alpha$ may be invariant under this scaling and is therefore in this sense a critical space. In [96, 98] the quasilinear problem (4.56) is thoroughly investigated in the context of maximal L^p-regularity for the critical case (4.55a), with an additional structural condition on the Banach spaces E_0 and E_1 (that is, E_0 must be a UMD space and $H^1_p(\mathbb{R}^+, E_0) \cap L^p(\mathbb{R}^+, E_1) \hookrightarrow H^{1-\theta}_p(\mathbb{R}^+, E_\theta)$ for each $\theta \in [0,1]$). Recently, it has been shown that these structural conditions are in fact not needed for the critical case in the maximal L^p-regularity setting; see [68, Section 18.2] for details and also [68, Section 18.5] for a discussion of the history of the critical case. It is worth emphasising that Theorems 4.5.2 and 4.5.3 cover situations for which maximal regularity does not seem to be available.

Remark 4.5.1. It is possible to extend the results to a larger class of interpolation functors (see [9, Section I.1.2.1] for a general definition of interpolation functors). In fact, is suffices to require that, for $\theta \in (0,1)$, $(\cdot,\cdot)_\theta$ be an arbitrary admissible interpolation functor, and to assume that there are interpolation functors $\{\cdot,\cdot\}_{\alpha/\xi}$ and $\{\cdot,\cdot\}_{\gamma/\eta}$ of exponents α/ξ and γ/η for $\eta \in \{\alpha,\beta,\xi\}$, respectively, such that

$$E_\alpha \doteq \{E_0, E_\xi\}_{\alpha/\xi}, \qquad E_\gamma \doteq \{E_0, E_\eta\}_{\gamma/\eta}, \quad \eta \in \{\beta,\alpha,\xi\}. \tag{4.58}$$

Assumption (4.58) is automatically satisfied if $(\cdot,\cdot)_\theta = [\cdot,\cdot]_\theta$ is the complex interpolation functor for every $\theta \in \{\gamma,\beta,\alpha,\xi\}$, or if $(\cdot,\cdot)_\theta = (\cdot,\cdot)_{\theta,p}$ is the real interpolation functor with parameter $p \in [1,\infty]$ for every $\theta \in \{\gamma,\beta,\alpha,\xi\}$, or if $(\cdot,\cdot)_\theta = (\cdot,\cdot)^0_{\theta,\infty}$ is the continuous interpolation functor for every $\theta \in \{\gamma,\beta,\alpha,\xi\}$. This is explicitly stated in [9, Remarks I.2.11.2(b)] and is a consequence of the reiteration theorems for the complex and the real methods (see [9, Section I.2.8]).

For the subcritical regime (4.57), well-posedness of (4.56) is shown in [75] in time-weighted L^p-spaces in the framework of maximal L^p-regularity assuming that f satisfies a local Lipschitz continuity property similar to (4.55c) in real interpolation spaces with $\gamma = 0$. We also refer to [76] for a result in the same spirit based on the concept of continuous maximal regularity in time-weighted spaces and assuming (4.55c) with strict inequality (4.57) in the scale of continuous interpolation spaces.

We now state our main Theorem 4.5.2 that extends our previous research [87] on the subcritical case (4.57) to the critical case (4.55a). It is comparable to the aforementioned results [68, 76, 96, 98] for the critical case (4.55a) outside the setting of maximal regularity and for (almost) arbitrary admissible interpolation functors:

Theorem 4.5.2. *Suppose (4.55).*

(i) (Existence) *Given any $u_0 \in O_\alpha$, there is $t^+ = t^+(u_0) \in (0,\infty]$ such that the quasilinear Cauchy problem (4.56) possesses a maximal strong solution*

$$u(\cdot;u_0) \in C^1\big((0,t^+),E_0\big) \cap C\big((0,t^+),E_1\big) \cap C\big([0,t^+),O_\alpha\big) \cap C^{\alpha-\beta}\big([0,t^+),E_\beta\big) \tag{4.59}$$

with

$$\lim_{t\to 0} t^{\xi-\alpha}\|u(t;u_0)\|_\xi = 0.$$

(ii) (Uniqueness) *If*

$$\widetilde{u} \in C^1\big((0,T],E_0\big) \cap C\big((0,T],E_1\big) \cap C^\vartheta\big([0,T],O_\beta\big)$$

with

$$\lim_{t\to 0} t^{\xi-\alpha}\|\widetilde{u}(t)\|_\xi = 0$$

solves (4.56) for some $T > 0$ and $\vartheta \in (0,1)$, then $T < t^+(u_0)$ and $\widetilde{u} = u(\cdot;u_0)$ on $[0,T]$.

(iii) (Continuous dependence) *The map $(t,u^0) \mapsto u(t;u_0)$ is a semiflow on O_α.*

(iv) (Global existence) *If the set $u([0,t^+(u_0));u_0)$ is relatively compact in O_α, then $t^+(u_0) = \infty$.*

(v) (Blow-up criteria) *Let $u_0 \in O_\alpha$ be such that $t^+(u_0) < \infty$.*

(a) If $u(\cdot;u_0) : [0,t^+(u_0)) \to E_\alpha$ is uniformly continuous, then

$$\lim_{t\nearrow t^+(u_0)} \operatorname{dist}_{E_\alpha}\big(u(t;u_0),\partial O_\alpha\big) = 0. \tag{4.60}$$

(b) If $u(\cdot;u_0) : [0,t^+(u_0)) \to E_\beta$ is uniformly Hölder continuous, then

$$\limsup_{t\nearrow t^+(u_0)} \|f(u(t;u_0))\|_0 = \infty \quad \textit{or} \quad \lim_{t\nearrow t^+(u_0)} \operatorname{dist}_{E_\beta}\big(u(t;u_0),\partial O_\beta\big) = 0. \tag{4.61}$$

(c) If E_1 is compactly embedded in E_0, then

$$\limsup_{t\nearrow t^+(u_0)} \|u(t;u_0)\|_\theta = \infty \quad \textit{or} \quad \lim_{t\nearrow t^+(u_0)} \operatorname{dist}_{E_\alpha}\big(u([0,t];u_0),\partial O_\alpha\big) = 0 \tag{4.62}$$

for each $\theta \in (\alpha,1)$.

The local existence and uniqueness result of Theorem 4.5.2 is based on Banach's fixed-point theorem, where we point out the technical difficulties induced by the singularity at $t = 0$ of $t \mapsto f(u(t))$. These are handled by working in a framework of time-weighted spaces as previously in [87] for the subcritical case (4.57).

Of course, Theorem 4.5.2 is also valid in the particular case of semilinear parabolic problems

$$u' = Au + f(u), \quad t > 0, \qquad u(0) = u_0, \tag{4.63}$$

with

$$-A \in \mathcal{H}(E_1, E_0). \tag{4.64a}$$

However, for semilinear problems, Theorem 4.5.2 can be sharpened. In fact, now we impose

$$0 \le \gamma < \alpha < \xi \le 1, \quad q := \frac{1+\gamma-\alpha}{\xi-\alpha} > 1. \tag{4.64b}$$

Note that (4.64b) implies, in particular, that $(\gamma, \xi) \neq (0,1)$. Let O_α be an open subset of E_α and put $O_\xi := O_\alpha \cap E_\xi$. The nonlinearity $f : O_\xi \to E_\gamma$ is supposed to be locally Lipschitz continuous in the sense that for each $R > 0$ there exists a positive constant $N(R)$ such that

$$\|f(w)-f(v)\|_\gamma \le N(R)\Big[1+\|w\|_\xi^{q-1}+\|v\|_\xi^{q-1}\Big]\big[\big(1+\|w\|_\xi+\|v\|_\xi\big)\|w-v\|_\alpha+\|w-v\|_\xi\big] \tag{4.64c}$$

for all w, $v \in O_\xi \cap \bar{\mathbb{B}}_{E_\alpha}(0,R)$. The well-posedness result in the framework of the semilinear parabolic problem (4.63) reads as follows:

Theorem 4.5.3. *Suppose (4.64).*

(i) (Existence and uniqueness) *Given any $u_0 \in O_\alpha$, there is $t^+ = t^+(u_0) \in (0, \infty]$ such that the semilinear Cauchy problem (4.63) possesses a unique maximal strong solution*

$$u(\cdot; u_0) \in C^1\big((0,t^+), E_0\big) \cap C\big((0,t^+), E_1\big) \cap C\big([0,t^+), O_\alpha\big) \tag{4.65}$$

with

$$\lim_{t\to 0} t^{\xi-\alpha}\|u(t;u_0)\|_\xi = 0.$$

(ii) (Continuous dependence) *The map $(t, u^0) \mapsto u(t; u_0)$ is a semiflow on O_α.*

(iii) (Global existence) *If the set $u([0,t^+(u_0)); u_0)$ is relatively compact in O_α, then $t^+(u_0) = \infty$.*

(iv) (Blow-up criteria) *Let $u_0 \in O_\alpha$ be such that $t^+(u_0) < \infty$.*

(a) If $u(\cdot; u_0) : [0, t^+(u_0)) \to E_\alpha$ is uniformly continuous, then

$$\lim_{t \nearrow t^+(u_0)} \operatorname{dist}_{E_\alpha}\big(u(t;u_0), \partial O_\alpha\big) = 0.$$

(b) If

$$\limsup_{t \nearrow t^+(u_0)} \|f(u(t;u_0))\|_0 < \infty,$$

then

$$\lim_{t \nearrow t^+(u_0)} \operatorname{dist}_{E_\alpha}\big(u(t;u_0), \partial O_\alpha\big) = 0.$$

Lastly, let us assume, additionally to (4.64), that

$$0 \in O_\alpha \tag{4.66a}$$

and that there exist constants $c_0 > 0$, $\delta \in (0,1)$, and $q_* > 1$ with $q_*(\xi - \alpha) \leq 1 + \gamma - \alpha$ such that

$$\|f(v)\|_\gamma \leq c_0 \|v\|_\xi^{q_*}, \qquad v \in O_\xi \cap \bar{\mathbb{B}}_{E_\alpha}(0,\delta). \tag{4.66b}$$

In particular, (4.66) ensures that 0 is a stationary solution to (4.63). In Corollary 4.5.4 we establish, by exploiting the superlinear behaviour of f near zero in (4.66b), the exponential stability of this equilibrium in the case when the spectral bound

$$s(A) := \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}$$

of A is negative.

Corollary 4.5.4. *Suppose (4.64) and (4.66), assume that $s(A) < 0$, and choose $\varpi \in (0, -s(A))$. Then, there exists an open neighbourhood U_α of 0 in E_α and a constant $M \geq 1$ such that for each $u_0 \in U_\alpha$ the maximal solution $u(\cdot; u_0)$ to (4.63) is globally defined, that is, $t^+(u_0) = \infty$, and*

$$\|u(t;u_0)\|_\alpha + t^{\xi-\alpha}\|u(t;u_0)\|_\xi \leq M\,\mathrm{e}^{-\varpi t}\,\|u_0\|_\alpha, \quad t > 0,\ u_0 \in U_\alpha.$$

For the proofs of Theorems 4.5.2 and 4.5.3 and Corollary 4.5.4, we refer the reader to [85].

The main objective of this section is to show how Theorem 4.5.3 and Corollary 4.5.4 can be applied to problem (4.6). We consider both the periodic and the non-periodic setting simultaneously and use the abstract results in Theorem 4.5.3 and Corollary 4.5.4 to prove that problem (4.6) generates a semiflow in $H^1_{\mathrm{D}}(\Omega)$, which is a critical case that cannot be covered by the results from [87]. Moreover, under suitable assumptions on the parameters η, β, ν, it is shown that solutions that are initially small in $H^1_{\mathrm{D}}(\Omega)$ converge at an exponential rate to the zero solution as time elapses.

Theorem 4.5.5. *Let $\mu \in (0,\infty)$ and η, $\beta \in \mathbb{R}$. Then, given $u_0 \in H^1_{\mathrm{D}}(\Omega)$, there exists a unique maximal strong solution $u = u(\cdot; u_0)$ to problem (4.6) with*

$$u \in C\big((0,t^+), H^2(\Omega)\big) \cap C^1\big((0,t^+), L^2(\Omega)\big) \cap C\big([0,t^+), H^1_{\mathrm{D}}(\Omega)\big) \cap C_{1/4}\big((0,T], H^{3/2}(\Omega)\big)$$

for all $0 < T < t^+$, where $t^+ := t^+(u_0) \in (0,\infty]$ is the maximal existence time of the solution. Moreover, if

$$\Omega = \mathbb{R} \times (0,1) \quad \text{and} \quad \eta + \frac{|\beta|\pi}{2} + \max\left\{\frac{|\beta|}{2\mu}, \pi\right\}\left(\frac{|\beta|}{2} - \pi\mu\right) < 0,$$

or

$$\Omega = \mathbb{S} \times (0,1) \quad \text{and} \quad \eta + \frac{\beta^2}{16\mu} < \pi^2\mu,$$

where we write $\mathbb{S} = [0, 2\pi]$ to indicate that solutions are 2π-periodic, then there exist constants r, $\varpi \in (0,1)$ and $M \geq 1$ such that for all $\|u_0\|_{H^1(\Omega)} \leq r$ the solution $u = u(\cdot; u_0)$ to (4.6) is globally defined, that is, $t^+(u_0) = \infty$, and

$$\|u(t)\|_{H^1(\Omega)} + t^{1/4}\|u(t)\|_{H^{3/2}} \leq M\,\mathrm{e}^{-\varpi t}\,\|u_0\|_{H^1(\Omega)}, \quad t > 0.$$

The proof of Theorem 4.5.5 is based on the auxiliary results of Lemmas 4.5.6–4.5.8 below and is presented at the end of this section.

To recast the Cauchy problem (4.6) into an appropriate functional analytic framework, we use the well-known fact

$$-\Delta_{\mathrm{D}} := -\Delta|_{H^2_{\mathrm{D}}(\Omega)} \in \mathcal{H}\big(H^2_{\mathrm{D}}(\Omega), L^2(\Omega)\big),$$

see e.g. [4, Theorem 13.4]. Moreover, letting $[\cdot,\cdot]_\theta$ be the complex interpolation functor, we have

$$\big[H^{s_1}_{\mathrm{D}}(\Omega), H^{s_2}_{\mathrm{D}}(\Omega)\big]_\theta \doteq H^s_{\mathrm{D}}(\Omega), \quad s_1,\, s_2 \in [0,2], \quad s := (1-\theta)s_1 + \theta s_2 \neq 1/2, \tag{4.67}$$

see [4, Theorem 13.3]. Now, we set

$$E_0 := L^2(\Omega), \quad E_1 := H^2_{\mathrm{D}}(\Omega), \quad E_\theta := [E_0, E_1]_\theta \quad \text{for } \theta \in [0,1],$$

and fix

$$q = 2, \quad 0 =: \gamma < \alpha := \frac{1}{2} < \xi := \frac{3}{4}. \tag{4.68}$$

We then have

$$q(\xi - \alpha) = 1 + \gamma - \alpha, \tag{4.69}$$

and, recalling (4.67),

$$E_\xi = H^{3/2}_{\mathrm{D}}(\Omega) \hookrightarrow E_\alpha = H^1_{\mathrm{D}}(\Omega).$$

Defining

$$Au := \mu\Delta_{\mathrm{D}} u + \eta u - \beta T u_x,$$

the perturbation result [9, I.Theorem 1.3.1 (ii)] and Lemma 4.3.3 ensure that

$$-A \in \mathcal{H}(H^2_{\mathrm{D}}(\Omega), L^2(\Omega)). \tag{4.70}$$

Introducing the nonlinear function

$$f : E_\xi \to E_0, \quad f(u) := u_y T u_x - u u_x, \tag{4.71}$$

we may formulate (4.6) as the semilinear evolution problem

$$u'(t) = Au(t) + f(u(t)), \quad t > 0, \qquad u(0) = u_0. \tag{4.72}$$

We now provide sufficient conditions on the constants η, β, and μ guaranteeing that the semigroup $(\mathrm{e}^{tA})_{t\geq 0}$ generated by A on $E_0 = L^2(\Omega)$ exhibits exponential decay.

Lemma 4.5.6 (Non-periodic case). *If* $\Omega = \mathbb{R} \times (0,1)$*, then*

$$s(A) \leq \eta + \frac{|\beta|\pi}{2} + \max\left\{\frac{|\beta|}{2\mu}, \pi\right\}\left(\frac{|\beta|}{2} - \pi\mu\right). \tag{4.73}$$

Proof. Given $u_0 \in E_0$, set $u(t) := \mathrm{e}^{tA}\, u_0$, $t \geq 0$. Then, since

$$u \in C^1((0,\infty), E_1) \cap C([0,\infty), E_0),$$

Hölder's inequality, Young's inequality, Poincaré's inequality

$$\|u\|^2_{L^2(\Omega)} \leq \frac{1}{\pi^2}\|\nabla u\|^2_{L^2(\Omega)}, \quad u \in H^1_{\mathrm{D}}(\Omega),$$

for the strip-like domain Ω (the optimal Poincaré constant of Ω being the same as that of the interval $[0,1]$), and Lemma 4.3.3 ensure for $t>0$ and each $\delta \in (0,1]$ that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\|u(t)\|^2_{L^2(\Omega)}}{2} &= \int_\Omega [u(\mu\Delta u + \eta u - \beta T u_x)](t) \\
&= \int_\Omega \big[-\mu|\nabla u|^2(t) + \eta|u|^2(t) - [\beta u(Tu_x)](t)\big] \\
&\le -\mu\|\nabla u(t)\|^2_{L^2(\Omega)} + \eta\|u(t)\|^2_{L^2(\Omega)} + |\beta|\,\|u(t)\|_{L^2(\Omega)}\|Tu_x(t)\|_{L^2(\Omega)} \\
&\le \left(\eta + \frac{\beta^2}{4\delta\mu} - (1-\delta)\pi^2\mu\right)\|u(t)\|^2_{L^2(\Omega)}.
\end{aligned}$$

In other words, $\|\,\mathrm{e}^{tA}\,\|_{\mathcal{L}(E_0)} \le \mathrm{e}^{\omega(\delta)t}$ for all $t \ge 0$, where

$$\omega(\delta) \coloneqq \eta + \frac{\beta^2}{4\delta\mu} - (1-\delta)\pi^2\mu, \quad \delta \in (0,1].$$

Consequently, we infer from [78, Corollary 2.3.2] that

$$\begin{aligned}
s(A) \le \min\big\{\omega(\delta) : \delta \in (0,1]\big\} &= \begin{cases} \omega(1), & d \coloneqq \frac{|\beta|}{2\pi\mu} \ge 1, \\ \omega(d), & d = \frac{|\beta|}{2\pi\mu} \le 1, \end{cases} \\
&= \eta + \frac{|\beta|\pi}{2} + \max\left\{\frac{|\beta|}{2\mu}, \pi\right\}\left(\frac{|\beta|}{2} - \pi\mu\right)
\end{aligned}$$

as is easily checked. □

In the periodic case, we obtain a slightly weaker constraint on the constants than in (4.73). The proof exploits the fact that, since $-A \in \mathcal{H}(E_1, E_0)$ and the embedding $E_1 \hookrightarrow E_0$ is compact, the spectrum $\sigma(A)$ of A consists entirely of isolated eigenvalues with finite algebraic multiplicities [73, Theorem III.6.29].

Lemma 4.5.7 (Periodic case). *If $\Omega = \mathbb{S} \times (0,1)$ and*

$$\eta + \frac{\beta^2}{16\mu} < \pi^2\mu, \tag{4.74}$$

then the spectral bound $s(A)$ of A is negative.

Proof. Note that $\lambda \in \mathbb{C}$ is an eigenvalue of A if and only if there exists a solution $0 \ne u \in H^2_{\mathrm{D}}(\Omega)$ to

$$\begin{aligned} \Delta u + au + bTu_x &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{4.75}$$

where we set

$$a = \frac{\eta - \lambda}{\mu} = a_1 + \mathrm{i}a_2, \quad a_1, a_2 \in \mathbb{R}, \quad \text{and} \quad b = -\frac{\beta}{\mu}.$$

Then, standard elliptic regularity, e.g. [53, Theorem 8.13], ensures that $u \in C^\infty(\overline{\Omega})$ and therefore, for each $n \in \mathbb{Z}$, the function $u_n : [0,1] \to \mathbb{C}$ with

$$u_n(y) \coloneqq \int_0^{2\pi} u(x,y)\,\mathrm{e}^{inx}\,\mathrm{d}x, \quad y \in [0,1],$$

satisfies $u_n \in C^\infty([0,1])$. Moreover, since $\{e^{inx} : n \in \mathbb{Z}\}$ is an orthogonal basis of $L^2(\mathbb{S})$, problem (4.75) is equivalent to

$$\left.\begin{aligned} &u_n'' + (a-n^2)u_n - inbFu_n = 0 \quad \text{in } (0,1),\\ &u_n(0) = u_n(1) = 0 \end{aligned}\right\} \quad \text{for all } n \in \mathbb{Z}, \tag{4.76}$$

where, given $f \in C([0,1])$, we set

$$Ff(y) := \int_0^y f(s)\,\mathrm{d}s, \quad y \in [0,1].$$

If $n = 0$, the Sturm-Liouville problem (4.76) may posses a nontrivial solution u_0 only if $a \in \mathbb{R}$. Moreover, if $a \in \mathbb{R}$, then (4.76) has only the trivial solution $u_0 = 0$ for $a < \pi^2$.

Assume now $n \in \mathbb{Z}\setminus\{0\}$ and let $f_n, g_n \in C^\infty([0,1],\mathbb{R})$ denote the real and imaginary parts of u_n, that is, $u_n = f_n + ig_n$. Then, f_n and g_n solve the coupled problem

$$f_n'' + (a_1 - n^2)f_n - a_2 g_n + nbFg_n = 0 \qquad \text{in } (0,1), \tag{4.77a}$$

$$g_n'' + (a_1 - n^2)g_n + a_2 f_n - nbFf_n = 0 \qquad \text{in } (0,1), \tag{4.77b}$$

$$f_n(0) = g_n(0) = f_n(1) = g_n(1) = 0. \tag{4.77c}$$

Testing (4.77a) and (4.77b) with f_n and g_n, respectively, and taking the sum of the resulting identities, we end up with

$$nb\int_0^1 \big(f_nFg_n - g_nFf_n\big) = (n^2 - a_1)\int_0^1 |u_n|^2 + \int_0^1 |u_n'|^2. \tag{4.78}$$

The boundary conditions (4.77c) enable us to use Wirtinger's inequality for functions, see for example [63, Section 7.7], to estimate

$$\int_0^1 |u_n'|^2 \geq \pi^2 \int_0^1 |u_n|^2.$$

Moreover, Fubini's theorem and Hölder's inequality imply

$$\begin{aligned} nb\int_0^1 \big(f_nFg_n - g_nFf_n\big) &\leq |nb|\int_0^1 \big(|f_n|\,F|g_n| + |g_n|\,F|f_n|\big)\\ &= |nb|\bigg(\int_0^1 |f_n|\bigg)\bigg(\int_0^1 |g_n|\bigg) \leq |nb|\,\|f_n\|_{L^2(\Omega)}\,\|g_n\|_{L^2(\Omega)}\\ &\leq \frac{|nb|}{2}\int_0^1 |u_n|^2. \end{aligned}$$

These estimates, together with (4.78), lead us to

$$\bigg(n^2 + \pi^2 - a_1 - \frac{|nb|}{2}\bigg)\int_0^1 |u_n|^2 \leq 0.$$

It is however not difficult to verify that the condition (4.74) ensures that, for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq 0$ and $n \in \mathbb{Z}$, it holds that $n^2 + \pi^2 > a_1 + |nb|/2$, hence $u_n = 0$ for all $n \in \mathbb{Z}$. Consequently, all eigenvalues of A have a negative real part and the claim follows. □

Regarding the nonlinearity $f(u) = u_yTu_x - uu_x$ defined in (4.71), we establish the following result. Recall that $E_\xi = H^{3/2}_{\mathrm{D}}(\Omega)$.

Lemma 4.5.8. *The function $f : E_\xi \to E_0$ satisfies $f(0) = 0$ and is locally Lipschitz continuous in the sense there exists a constant $C > 0$ such that for all u_1, $u_2 \in E_\xi$ we have*

$$\|f(u_1) - f(u_2)\|_0 \leq C(\|u_1\|_\xi + \|u_2\|_\xi)\|u_1 - u_2\|_\xi. \tag{4.79}$$

Proof. Lemma 4.3.3 together with the embedding $H^{1/2}(\Omega) \hookrightarrow L^4(\Omega)$ yields

$$\begin{aligned}\|f(u)\|_{L^2(\Omega)} &\leq \|u_y\|_{L^4(\Omega)}\|Tu_x\|_{L^4(\Omega)} + \|u\|_{L^4(\Omega)}\|u_x\|_{L^4(\Omega)} \\ &\leq C\big(\|u_y\|_{H^{1/2}(\Omega)}\|Tu_x\|_{H^{1/2}(\Omega)} + \|u\|_{H^{1/2}}\|u_x\|_{H^{1/2}(\Omega)}\big) \\ &\leq C\|u\|_\xi^2,\end{aligned}$$

and the claim is an immediate consequence of this estimate. □

We are now in a position to establish Theorem 4.5.5.

Proof of Theorem 4.5.5. Recalling (4.67)–(4.69), (4.70), Lemma 4.5.6, Lemma 4.5.7, and Lemma 4.5.8, we may apply Theorem 4.5.3 and Corollary 4.5.4 to the semilinear evolution problem (4.72) and obtain the desired claim. □

Complete list of publications and talks

Since, as we have mentioned, the present thesis contains only a selection of the candidate's work, we will provide here a full list of publications. The first paper is, in fact, anterior to the beginning of the doctoral studies, but we include it here for the sake of completeness.

1. On the decrease of velocity with depth in irrotational periodic water waves. *Monatsh. Math.* **193**(3) (2020), 671–682.
2. Perturbation analysis for the surface deflection angle of Ekman-type flows with variable eddy viscosity. *J. Math. Fluid Mech.* **23**(3) (2021), No. 57.
3. The Ekman spiral for piecewise-constant eddy viscosity. *Appl. Anal.* **101**(15) (2022), 5528–5536.
4. The surface current of Ekman flows with time-dependent eddy viscosity. *Comm. Pure Appl. Anal.* **21**(7), 2463–2477.
5. (With B.-V. Matioc) Weak and classical solutions to an asymptotic model for atmospheric flows. *J. Differ. Equ.* **367** (2023), 603–624.
6. (With Q. Ding) Stratified ocean gyres with Stuart-type vortices. *Ann. Mat. Pura Appl.* **203** (2024), 2847–2862.
7. (With E. Stefanescu) Global-in-time existence, uniqueness and stability of solutions to a model of the Antarctic Circumpolar Current. *Preprint* (2024), available at arXiv:2409.17013.
8. (With B.-V. Matioc and C. Walker) Quasilinear parabolic equations with superlinear nonlinearities in critical spaces. *J. Differ. Equ.* **429** (2025), 283–317.

Several of these scientific works have been presented at conferences, workshops or seminars. Here is a complete list of all scientific talks given by the candidate over the course of his studies.

1. The oceanic Ekman layer.
   Talk in *Seminar Applied and Computational PDE*, University of Vienna (2022).
2. On the surface deflection angle of Ekman flows with varying eddy viscosity.
   Talk at conference *DMV-Jahrestagung 2022*, Freie Universität Berlin (2022).
3. Weak and classical solutions to an asymptotic model for atmospheric flows.

Talk at conference *ÖMG Tagung 2023*, TU Graz (2023).

4. Geophysical Fluid Dynamics: An overview and some recent advancements.
   Invited talk at *MCMP Seminar*, University of Vienna (2024).

5. Classical well-posedness and stability for a model of the Antarctic Circumpolar Current.
   Talk in *PDE Afternoon Seminar*, University of Vienna (2024).

6. Existence, uniqueness and stability for a model of the Antarctic Circumpolar Current.
   Talk at workshop *Mathematical Theory of Water Waves*, Lund University (2024).

7. Well-posedness of a semilinear parabolic equation arising from the modelling of atmospheric flows.
   Invited talk in *Mathematical Sciences Seminar*, University College Cork (2024).

8. Well-Posedness of a semilinear parabolic equation arising from the modelling of atmospheric flows.
   Invited talk at a minisymposium at *Conference on Mathematics of Wave Phenomena*, Karlsruhe Institute of Technology (2025).

In addition, the following events were attended:

1. *Workshop on Nonlinear Dispersive Waves* (online attendance).
   University College Cork (2023).

2. *Aspects of Nonlinear Evolution. Conference in Honour of Joachim Escher's 60th Birthday.*
   Leibniz Universität Hannover (2023).

3. *Fluid Flows – Analysis and Modelling. Conference in Honour of Robin S. Johnson's 80th Birthday.*
   University of Vienna (2024).

4. *EWM-EMS Summer School: Water Waves and Nonlinear Dispersive Equations.*
   Mittag-Leffler-Institut (2024).
   At this summer school, a poster giving an overview of selected works from the candidate's research was presented.

5. *Mathematical Advances in Geophysical Fluid Dynamics.*
   Mathematisches Forschungsinstitut Oberwolfach (2025).

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